Quasiminimality and Coercivity in the Calculus of Variations

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To my family

Abstract

Throughout this thesis, we consider the variational integral

$$\mathcal{F}(u,\Omega) := \int_\Omega F\Big(x, u(x), Du(x)\Big)\,\mathrm{d}x$$

where $F\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is a Carathéodory integrand satisfying some p-coercivity and p-growth conditions.

The main contribution of this work is the characterization of mean coercivity in terms of quasiconvexity. Briefly, mean coercivity is a notion of coercivity in the integral mean and under certain p-growth condition, a continuous function F is mean p-coercive if and only if F is strongly p-quasiconvex at some $\xi \in \mathbb{R}^{N\times n}$. Known for its relation to one of the conditions that lead to partial regularity, we also prove the stability property of strong quasiconvexity, meaning that if F is strongly 2-quasiconvex everywhere and strongly p-quasiconvex somewhere, then it is indeed strongly p-quasiconvex everywhere.

The second main contribution is to establish that strong quasiminimizers are also minimizers of non-trivial variational integrals. We first establish the existence of good minimizing sequences of the variational integral containing quasiminimizers of an inhomogeneous p-Dirichlet integral and then prove higher integrability of the gradients of these quasiminimizers. Given that u is a strong quasiminimizer of an inhomogeneous p-Dirichlet integral, we show that it is in fact also a true minimizer of an integral $\int_\Omega F^{qc}\big(x, Du(x)\big)\,\mathrm{d}x$, where F is an explicit integrand. This further strengthens the versatility of quasiminimizers in treating different problems, and in our case minimizers of the variational problem. The result remains true for real-valued u when we consider strong superquasiminimizers and superminimizers instead.

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Contents

Chapter 1

Introduction

Over the past 60 years, there has been renewed interest in the calculus of variations, motivated partly by Morrey's notion of quasiconvexity [102] and more recently by ongoing research in materials science. This leads to steady development in the direct method which deals with proving the existence of minimizers of variational problems in an enlarged space of admissible functions and then sometimes followed by proving more regularity of the minimizer. The existence of minimizers relies on certain growth, coercivity and convexity conditions, which often fail to be satisfied in many interesting problems, particularly many problems in materials science are found to be non-(quasi)convex and thus the importance of the relaxation theorem in this context.

This work is in line with two of Hilbert's problems from his famous list. In connection with the 20^{th} problem that concerns the existence of minimizers for variational problems, we establish a characterization of mean coercivity which is intimately related to quasiconvexity. With regards to the regularity of minimizers that is contained in the 19^{th} problem, we study additional properties of quasiminimizers that include regularity on the whole set and its relation to the general existence theorem.

Here we give an outline of our main results. We postpone definitions, underlying results and references (whenever possible) to the subsequent chapters.

1.1 Quasiconvexity and mean coercivity in the calculus of variations

We consider the variational problem

$$\inf\left\{\mathcal{F}(u,\Omega)=\int_\Omega F\big(x,u(x),Du(x)\big)\,\mathrm{d}x\colon u\in X\right\} \tag{1.1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, $u\colon \Omega \to \mathbb{R}^N$ is a vector-valued function, Du is the Jacobian matrix of u defined as

$$Du = \left(\frac{\partial u^\alpha}{\partial x_i}\right)_{1\le i\le n}^{1\le \alpha\le N} \in \mathbb{R}^{N\times n},$$

and $F\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is a Carathéodory integrand (measurable in its first and jointly continuous in its second and third arguments). Here X is the space of admissible functions and to guarantee the existence of minimizer to the variational problem, a natural choice of X might at first seem to be either $C^1(\Omega;\mathbb{R}^N)$ or $C^2(\Omega;\mathbb{R}^N)$, but these spaces are too small and one has in general no existence of minimizers in such spaces. Instead a better choice of X turns out to be an appropriate Sobolev space.

In this work, we consider

$$X = g + W_0^{1,p}(\Omega;\mathbb{R}^N)$$

where g is a given function in the Sobolev space $W^{1,p}(\Omega;\mathbb{R}^N)$. When we say $u \in g+W_0^{1,p}(\Omega;\mathbb{R}^N)$ (or written as $u \in W_g^{1,p}(\Omega;\mathbb{R}^N)$), it simply means, up to technicality, that $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ and $u = g$ on $\partial\Omega$.

The general existence theorem in the above space relies on two properties. The first is the assumptions on the integrand F, usually assumed to satisfy some pointwise coercivity and growth conditions, for instance

$$c_1|\xi|^p - c_2 \le F(x,u,\xi) \le c_3(|\xi|^p + 1)$$

for almost every $x \in \Omega$ and for all $(u,\xi) \in \mathbb{R}^N \times \mathbb{R}^{N\times n}$ where constants $c_3 \ge c_1 > 0$ and $c_2 < \infty$. Meanwhile the second property is sequential weak lower semicontinuity of the functional $\mathcal{F}$, meaning that

$$\liminf_{j\to\infty} \mathcal{F}(u_j,\Omega) \ge \mathcal{F}(\bar{u},\Omega) \quad \text{whenever} \quad u_j \rightharpoonup \bar{u} \text{ in } W^{1,p}(\Omega;\mathbb{R}^N)$$

where $\rightharpoonup$ denotes convergence in the weak topology of $W^{1,p}$ or simply known as weak convergence. When $p = \infty$, the above is replaced by sequential weak* lower semicontinuity with weak* convergence denoted by $\overset{*}{\rightharpoonup}$.

Intimately related to sequential weak lower semicontinuity are the notions of convexity and quasiconvexity. In the scalar case when $\min\{N,n\} = 1$, convexity of the integrand F with respect to the gradient variable ξ is known to be a necessary and sufficient condition for sequential weak lower semicontinuity of the functional $\mathcal{F}$ (Marcellini-Sbordone [91]). However, convexity fails to be a necessary condition in the vectorial case. It turns out that quasiconvexity fits into the role as a necessary and sufficient condition for a functional $\mathcal{F}$ to be sequentially weakly lower semicontinuous in the vectorial case. Quasiconvexity was introduced by Morrey [102]: A function $F\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is said to be quasiconvex in the ξ variable if

$$F(x,u,\xi) \leq \frac{1}{\operatorname{meas}\Omega} \int_\Omega F\big(x,u,\xi + D\varphi(x)\big)\,\mathrm{d}x$$

for every bounded open set $\Omega \subset \mathbb{R}^n$, for every $(x,u,\xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$ and for every piecewise affine $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^N)$.

The existence theorem can be summarized as follows: If an integrand $F\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n} \to \mathbb{R}$ satisfying the coercivity and growth conditions is quasiconvex, then the variational problem (1.1.1) has at least one minimizer.

In reality, the integrand F often fails to be quasiconvex and hence the existence of minimizer is not guaranteed. This is considered to be a crucial problem, for instance in materials science where the minimization of energy may not be achieved. Therefore, a good understanding of the classical and generalized equilibrium solutions for non-quasiconvex energies is required. The need to relax the problem leads to what we recognize today as the *relaxation theorem* that allows us to define a generalized solution for a variational problem that does not have a classical solution:

$$\inf\left\{\mathcal{F}(u,\Omega)\colon u \in W_g^{1,p}(\Omega;\mathbb{R}^N)\right\} = \inf\left\{\mathcal{F}^{qc}(u,\Omega)\colon u \in W_g^{1,p}(\Omega;\mathbb{R}^N)\right\} \tag{1.1.2}$$

where

$$\mathcal{F}^{qc}(u,\Omega) = \int_\Omega F^{qc}\big(x,u(x),Du(x)\big)\,\mathrm{d}x$$

and F^{qc} is the quasiconvexification of F in the gradient argument defined as

$$F^{qc}(x,u,\cdot) = \sup\left\{G(x,u,\cdot)\colon G \leq F \text{ and } G \text{ quasiconvex}\right\}.$$

We also call F^{qc} the quasiconvex envelope.

The result stated in (1.1.2) was first established by Dacorogna [25] under slightly stronger assumptions (he assumed the integrand was continuous and bounded below) for the case without lower order terms where he also presented the characterization of the quasiconvex envelope of a function $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$:

$$F^{qc}(\xi) = \inf\left\{\frac{1}{\operatorname{meas}\Omega}\int_\Omega F\big(\xi + D\varphi(x)\big)\,\mathrm{d}x\colon \varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^N) \text{ piecewise affine}\right\}$$

for all $\xi \in \mathbb{R}^{N\times n}$ where $\Omega \subset \mathbb{R}^n$ is a bounded open set with $\operatorname{meas}(\partial\Omega) = 0$. Moreover, the infimum in the formula is independent of the choice of Ω. This characterization is also known as

Dacorogna's relaxation formula and we adapt the proof of Dacorogna [29] to show that it persists for functions F that are not bounded from below in Theorem 3.1.3 of Chapter 3.

Of course the convexification in the one-dimensional case is much older and dates back from the first systematic development by Bogolubov [19] in 1930. Briefly speaking, he recognized the need to construct a lower semicontinuous extension of the functional $\mathcal{F}$ and when $\mathcal{F}$ is coercive on $W^{1,p}$, this extension is actually the sequential lower semicontinuous envelope of $\mathcal{F}$ given by the formula

$$\bar{\mathcal{F}}(u,\Omega) = \inf_{\{u_j\}} \left\{ \liminf_{j\to\infty} \mathcal{F}(u_j,\Omega) \colon u_j \rightharpoonup u \text{ in } W^{1,p}(\Omega;\mathbb{R}^N) \right\}.$$

He went on to show that $\bar{\mathcal{F}}$ is equal to

$$\mathcal{F}^c(u,\Omega) = \int_\Omega F^c\big(x,u(x),Du(x)\big)\,\mathrm{d}x$$

where F^c is the convex envelope of F with respect to the gradient variable. A more detailed historical discussion can be found in Chapter 3.

A standard problem that is known to be not sequentially weakly lower semicontinuous (and hence not quasiconvex) is the optimal design problem in electrostatics. Introducing a model function for the problem, Kohn-Strang [76] showed that it is possible to write the quasiconvexification formula explicitly and that the relaxation result (1.1.2) remains true. The original model function $F\colon \mathbb{R}^{N\times 2} \to \mathbb{R}_+$ for the optimal design problem is in the form

$$F(\xi) = \Phi(|\xi|) = \begin{cases} |\xi|^2 + 1 & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0 \end{cases}$$

where $|\cdot|$ denotes the Euclidean norm. The particular explicit formula for quasiconvex envelope obtained for the function above is called the Kohn-Strang relaxation formula. A detailed study of this explicit relaxation formula is included in Chapter 3 and we follow the derivation of Zhang [133].

The proof of the relaxation theorem in the sense of Bogolubov [19] can be improved by the use of Young measures. Originally introduced by Young [130] as *generalized curves*, this theory bears a similarity to the relaxation theorem in its purpose, that is to treat problems in the calculus of variations and optimal control that could not be solved using classical methods. However, contrary to the relaxation theorem that requires seeking a lower semicontinuous envelope of the functional $\mathcal{F}$, these functionals are always lower semicontinuous in the class of Young measures. Its wide range of application, particularly in calculus of variations, non-linear partial differential equations and materials science, also leads to its remarkable development over the past 40 years or so. A main breakthrough in this aspect is of course by Kinderlehrer-Pedegral [70] in which they gave a characterization of gradient p-Young measure so as to prove the relaxation theorem in a simpler way. This was further developed by Sychev [115] and we follow his work to give a detailed report in Chapter 3.

So far, we have only discussed on the general existence theorem for non-quasiconvex functions. Our interest is in fact on the coercivity condition but we will not be able to establish our result without first mentioning the above. Recall that the existence of minimizer is guaranteed when the integrand F satisfies the growth, coercivity and quasiconvexity conditions. Similar to the quasiconvexity condition, coercivity should not be taken for granted. For example, if we let $F : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be defined by

$$F(\xi) = \det \xi + \varepsilon|\xi|^2$$

for all $\xi \in \mathbb{R}^{2\times 2}$ where $\varepsilon \in (0, \frac{1}{2})$, then F is not coercive in a pointwise sense. However, it is mean coercive, which is thus a weaker condition than coercivity.

From the pointwise coercivity condition that we considered above

$$F(\xi) \geq c_1|\xi|^p - c_2$$

where constants $c_1 > 0$ and $c_2 < \infty$, we infer that for all $u \in W^{1,p}_g(\Omega;\mathbb{R}^N)$

$$\int_\Omega F(Du)\,\mathrm{d}x \geq c_1 \int_\Omega |Du|^p\,\mathrm{d}x - \tilde{c}_2.$$

This type of inequality we often refer to as a Gårding's inequality and a function F satisfying this inequality is said to be mean coercive. Since our variational problem (1.1.1) is considered in the space $W^{1,p}_g(\Omega;\mathbb{R}^N)$, it is natural to consider also the boundary condition in the mean coercivity condition: We say that a function F is mean coercive if there exist $c > 0$ and $\tilde{c} < \infty$ such that for all $u, g \in W^{1,p}(\Omega;\mathbb{R}^N)$ with $u \in W^{1,p}_g(\Omega;\mathbb{R}^N)$ it holds that

$$\int_\Omega F(Du)\,\mathrm{d}x \geq c\int_\Omega |Du|^p\,\mathrm{d}x - \tilde{c}\int_\Omega (|Dg|^p + 1)\,\mathrm{d}x.$$

It is clear that pointwise p-coercivity implies mean p-coercivity while the converse is false.

One of the main results of this thesis is the characterization of mean coercivity in terms of quasiconvexity:

Theorem 1.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Suppose $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is a continuous integrand satisfying for some $c_1 < \infty$ and $p \in (1,\infty)$ the growth condition*

$$|F(\xi)| \leq c_1(|\xi|^p + 1)$$

for all $\xi \in \mathbb{R}^{N\times n}$*. Then, there exists a* $c_2 > 0$ *such that*

$$(F - c_2|\cdot|^p)^{qc}(0) > -\infty, \tag{1.1.3}$$

where for a function G *we denote by* G^{qc} *its quasiconvex envelope if and only if there exist constants* $c_3 > 0$ *and* $c_4 < \infty$*, such that for all* $u,\ g \in W^{1,p}(\Omega;\mathbb{R}^N)$ *with* $u \in W^{1,p}_g(\Omega;\mathbb{R}^N)$

$$\int_\Omega F(Du)\,\mathrm{d}x \geq c_3 \int_\Omega |Du|^p\,\mathrm{d}x - c_4 \int_\Omega (|Dg|^p + 1)\,\mathrm{d}x. \tag{1.1.4}$$

In turn, (1.1.3) and (1.1.4) are both equivalent to the existence of a $c_5 > 0$ *such that* $F - c_5|\cdot|^p$ *is quasiconvex at some* $\xi \in \mathbb{R}^{N\times n}$*.*

The theorem clearly tells us that under p-growth condition, a continuous function F is mean coercive if and only if there exists $\varepsilon > 0$ such that $F - \varepsilon|\cdot|^p$ is quasiconvex at some $\xi \in \mathbb{R}^{N\times n}$. The latter is closely related to one of the conditions leading to partial regularity of minimizers and any continuous function F satisfying this condition is said to be strongly p-quasiconvex.

In fact, we prove the result for a more general class of open sets, refer to Theorem 6.1.1 in Chapter 6. The first part of the theorem can be proved by using some well-known inequalities and some results on quasiconvex envelopes, including Dacorogna's relaxation formula. Meanwhile the proof of the second part, still heavily relies on Dacorogna's relaxation formula, has a deep application of Young measure theory.

We also show that strong quasiconvexity possesses a stability property for $p \in (2,\infty)$ as stated in the following theorem:

Theorem 1.1.2. *Let* $B \subset \mathbb{R}^n$ *be a bounded open ball and* $m > 0$*. Let* $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ *be a* C^2 *integrand satisfying for some* $c_1 < \infty$ *and* $p \in (2,\infty)$ *the growth condition*

$$|F(\xi)| \leq c_1(|\xi|^p + 1)$$

for all $\xi \in \mathbb{R}^{N\times n}$*. Suppose that there exists* $c_2 > 0$ *such that*

$$F - c_2|\cdot|^2$$

is quasiconvex and there exists $c_3 > 0$ *such that*

$$F - c_3|\cdot|^p$$

is quasiconvex at ξ_0 *where* $|\xi_0| \leq m$*. Then for each* m*, there exists* $c = c(m) > 0$ *such that*

$$F - c|\cdot|^p$$

is quasiconvex for all ξ *satisfying* $|\xi| \leq m$*.*

Proving the above theorem requires only elementary results such as Young's inequality, Chebychev's inequality, consequence of Hölder's inequality and a version of fundamental theorem of calculus. We give the details in Section 6.2 of Chapter 6.

1.2 Quasiminimality and relaxation for a inhomogeneous Dirichlet integral

We now turn our attention to the second issue, namely additional properties of quasiminimizers and its relation to the general existence theorem. Quasiminimizers were first introduced by Giaquinta-Giusti [55]: Let $F\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ be a Carathéodory integrand satisfying the inequality

$$0 \le F(x,u,\xi) \le L|\xi|^p + b(x)|u|^\gamma + a(x)$$

where $1 < p \le \gamma < p^* = \frac{np}{n-p}$ and let $Q \ge 1$. We say that $u \in W^{1,p}_{\text{loc}}(\Omega;\mathbb{R}^N)$ is a quasiminimizer of the functional $\mathcal{F}(u,\Omega) = \int_\Omega F\big(x,u(x),Du(x)\big)\,\mathrm{d}x$ if

$$\mathcal{F}(u,K) \le Q\mathcal{F}(v,K)$$

for all $v \in W^{1,p}_{\text{loc}}(\Omega;\mathbb{R}^N)$ with $K =: \operatorname{supp}(v-u) \Subset \Omega$.

If $Q = 1$ in the above definition, then u is a standard minimizer of the functional $\mathcal{F}$. When $N = 1$ so that u and v are real-valued, we may consider superquasiminimizer in which the quasiminimality inequality holds for all $v \ge u$. Similar definition holds for subquasiminimizer where $v \le u$.

The notion of quasiminimizer is seen as a unifying tool in treating different problems due to its wide range of application that leads to a broad and flexible class of maps under general circumstances . The theory of quasiminimizer has been extended to minimizers of variational integrals, weak solutions of elliptic systems, quasi-regular mappings, potential theory, to name a few. The most attractive feature of quasiminimizer is its regularity properties which include in the scalar case Hölder continuity, weak maximum principle and Harnack inequality, and for the vectorial case higher integrability in the interior. Moreover, quasiminimizers are more flexible than solutions of differential equations and minimizers under perturbation.

The first goal in our study of quasiminimality is to establish the role played by quasiminimizers in the variational problem (1.1.1) that satisfies the mean coercivity and growth conditions. It turns out that if the variational problem has a minimizing sequence, then there exists another minimizing sequence consisting of quasiminimizers for the inhomogeneous p-Dirichlet integral. This result generalizes Yan-Zhou's construction [128] in various aspects and includes local functionals of p-growth: We say that a functional $\mathcal{F}\colon W^{1,p}(\Omega;\mathbb{R}^N)\times\mathcal{O}(\Omega)\to\mathbb{R}$ is local on $\mathcal{O}(\Omega)$ if for $u,v\in W^{1,p}(\Omega;\mathbb{R}^N)$

$$\mathcal{F}(u,\omega)=\mathcal{F}(v,\omega) \text{ whenever } u=v \text{ a.e. on } \omega\in\mathcal{O}(\Omega).$$

We are now ready to state the theorem which is essentially Theorem 5.1.2 in Chapter 5:

Theorem 1.2.1. *Let $\mathcal{F}\colon W^{1,p}(\Omega;\mathbb{R}^N)\times\mathcal{O}(\Omega)\to\mathbb{R}$ be a local functional that is $W^{1,1}$-lower semicontinuous on $\mathcal{O}(\Omega)$ and assume that*

$$|\mathcal{F}(u,\omega)|\leq c_1\int_\omega(|Du|^p+1)\,\mathrm{d}x$$

for all $u\in W^{1,p}(\omega;\mathbb{R}^N)$ and for all $\omega\in\mathcal{O}(\Omega)$ where $c_1<\infty$ and $p>1$ are constants. Assume furthermore that

$$\mathcal{F}(u,\omega)+c_2\int_\omega(|Dg|^p+1)\,\mathrm{d}x\geq c_3\int_\omega|Du|^p\,\mathrm{d}x$$

for all $g\in W^{1,p}(\omega;\mathbb{R}^N)$, $u\in W^{1,p}_g(\omega;\mathbb{R}^N)$ and $\omega\in\mathcal{O}(\Omega)$ where constants $c_2<\infty$ and $c_3>0$. Then there exists a constant $Q=Q(c_1,c_2,c_3,p)<\infty$ such that if $\{u_j\}$ is a minimizing sequence for the variational problem

$$\inf\left\{\mathcal{F}(u,\Omega):\ u\in W^{1,p}_g(\Omega;\mathbb{R}^N)\right\},$$

then there exists another minimizing sequence $\{v_j\}$ for the variational problem consisting of quasiminimizers of the inhomogeneous p-Dirichlet integral

$$\int_\Omega(|Du|^p+1)\,\mathrm{d}x$$

and such that

$$\int_\Omega|Dv_j-Du_j|\,\mathrm{d}x\to 0.$$

Following the framework of Yan-Zhou [128], the minimizing sequences $\{v_j\}$ are constructed using Ekeland's variational principle [39]. The quasiminimizers within these minimizing sequences are then obtained by using the mean coercivity and growth conditions. As fascinating as it is, we even show that these quasiminimizers are globally higher integrable and not only in the interior.

Theorem 1.2.2. *Let $\Omega\subset\mathbb{R}^n$ be bounded Lipschitz domain. Let $g\in W^{1,\tilde{p}}(\Omega;\mathbb{R}^N)$ for some $\tilde{p}>p$. If $u\in W^{1,p}_g(\Omega;\mathbb{R}^N)$ is a quasiminimizer of the inhomogeneous p-Dirichlet integral*

$$\int_\Omega(|Dv|^p+1)\,\mathrm{d}x,$$

then there exists a constant $\delta_0 = \delta_0(n, N, p, Q, c_0) \in (0, \tilde{p} - p]$ *such that* $u \in W^{1,p+\delta}(\Omega;\mathbb{R}^N)$ *whenever* $\delta \in (0, \delta_0)$. *In particular,* $|Du| \in L^{p+\delta}(\Omega)$ *for all* $\delta \in (0, \delta_0)$ *and*

$$\left(\int_\Omega |Du|^{p+\delta}\,\mathrm{d}x\right)^{\frac{1}{p+\delta}} \leq c\left(\int_\Omega |Du|^p\,\mathrm{d}x\right)^{\frac{1}{p}} + c\left(\int_\Omega |Dg|^{p+\delta}\,\mathrm{d}x\right)^{\frac{1}{p+\delta}} + c$$

where constant $c = c(n, N, p, Q, c_0) \in (0, \infty)$.

In general, one cannot obtain higher regularity of a minimizer of a variational problem. The results above show not only that we can find another minimizing sequence for the variational problem, but the minimizing sequence also contains quasiminimizers of the inhomogeneous p-Dirichlet integral and moreover they are higher integrable. We will see in Chapter 6 how these nice properties of quasiminimizers are used in the proof of the characterization of mean coercivity.

To prove Theorem 1.2.2, we have to identify some regularity properties that work already for quasiminimizers. For this reason, we include a list of results such as Caccioppoli's inequality in the interior and Gehring's lemma for the quasiminimizers in Chapter 5 before proceeding to give the proof of our theorem. Combining these results with fact that Ω is bounded, we conclude that higher integrability of the gradients of these quasiminimizers can be extended to the whole of Ω and hence we have global higher integrability. Similar to Theorem 1.1.1, we prove the result for a more general class of open sets and we refer to Theorem 5.3.1 in Chapter 5 for the statement and proof.

Returning to Theorem 1.2.1, we can clearly see that a minimizer of an integrand satisfying some coercivity and growth conditions is indeed a quasiminimizer of the corresponding inhomogeneous p-Dirichlet integral which appears to be much simpler. One could wonder if the converse holds. For this, we introduce strong quasiminimizer where the definition is similar to that of quasiminimizer with the condition that it is taken over the set of $\{Dv - Du \neq 0\}$. It turns out that strong quasiminimizers of a inhomogeneous p-Dirichlet integral are absolute minimizers of a quasiconvex x-dependent integral:

Theorem 1.2.3. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set,* $Q > 1$, $p \in (1, \infty)$ *and* $a \in L^1(\Omega)^+$. *Then* $\bar{u} \in W^{1,p}(\Omega;\mathbb{R}^N)$ *is a strong quasiminimizer for the integral* $\int_\Omega \big(|Du(x)|^p + a(x)\big)\,\mathrm{d}x$ *if and only if* $\bar{u}$ *is an absolute minimizer for the integral* $\int_\Omega F^{qc}\big(x, Du(x)\big)\,\mathrm{d}x$, *where for each* $x \in \Omega$, $F^{qc}(x, \cdot)$ *is the quasiconvex envelope of*

$$F(x,\xi) = \begin{cases} Q\big(|\xi|^p + a(x)\big) & \text{if } \xi \neq D\bar{u}(x) \\ |D\bar{u}(x)|^p + a(x) & \text{if } \xi = D\bar{u}(x) \end{cases}. \tag{1.2.1}$$

This result, which is contained in Chapter 7 (see Theorem 7.0.6), provides us with another aspect on how quasiminimizers can be extended to minimizers of variational integrals and hence the advantage of the theory of quasiminimizer in unifying the treatment of different problems. We deduce from Theorem 1.2.2 that these strong quasiminimizers are indeed higher integrable in Ω and so does the minimizer of the x-dependent quasiconvex integral. We then use the Kohn-Strang relaxation formula to write down the integral functional explicitly that the strong quasiminimizer $\bar{u}$ is minimizing when $p = 2$.

Scaling down our problem to real-valued functions u, we even have an equivalent result for strong superquasiminimizer and superminimizer, in which both notions are of general importance in potential theory.

Theorem 1.2.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $Q > 1$, $p \in (1, \infty)$ and $a \in L^1(\Omega)^+$. Then $\bar{u} \in W^{1,p}(\Omega)$ is a strong superquasiminimizer of the integral $\int_\Omega \big(|Du(x)|^p + a(x)\big)\,\mathrm{d}x$ if and only if $\bar{u}$ is an superminimizer of the integral $\int_\Omega F^c\big(x, Du(x)\big)\,\mathrm{d}x$, where for each $x \in \Omega$, $F^c(x, \cdot\,)$ is the convex envelope of (1.2.1).*

At first instance, it does not seem completely trivial if the one-sided relaxation as above works. However, the next theorem, which we also prove in Chapter 7 (see Theorem 7.2.1), provides us with a positive answer:

Theorem 1.2.5. *Let $F\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a Carathéodory integrand satisfying the coercivity and growth conditions*

$$\frac{1}{c}|\xi|^p - a(x) \le F(x, \xi) \le c|\xi|^p + a(x)$$

for all $\xi \in \mathbb{R}^n$ and a.e. $x \in \Omega$, where constant $c > 0$ and $a \in L^1(\Omega)^+$. Assume that $\bar{u} \in W^{1,p}(\Omega)$ is a superminimizer of $\mathcal{F}(u, \Omega) = \int_\Omega F\big(x, Du(x)\big)\,\mathrm{d}x$:

$$\mathcal{F}(\bar{u}, \Omega) \le \mathcal{F}(\bar{u} + \varphi, \Omega)$$

for all non-negative $\varphi \in W_0^{1,p}(\Omega)$. Then the integrand $F(x, \cdot)$ is convex at $D\bar{u}(x)$ for a.e. $x \in \Omega$.

1.3 Overview

The thesis is organized as follows: Chapter 2 is basically a detailed literature on the direct method in the calculus of variations, beginning with the motivation, followed by sequential weak lower semicontinuity and its corresponding weak topology, notions of convexity and finally lower semicontinuity and existence theorems.

The discussion on relaxation theorem is contained in Chapter 3, in which we include a historical account of the relaxation theorem and associate with convexification and quasiconvexification, the duality and Dacorogna's relaxation formula, respectively. We also include the derivation of Kohn-Strang relaxation formula as given by Zhang [133]. A big part of Chapter 3 is also dedicated to a rather self-contained study of Sychev's approach to Young measure theory [115] that proved the relaxation theorem in a more elegant way. Since there are too much materials for the Young measure theory, the proofs are omitted in this chapter but included in Appendix B for reference of interested readers.

Continuing with more literature, Chapter 4 is a compact chapter on quasiminimizers containing different examples of quasiminimizers as well as regularity properties that have been established whenever they are scalar-valued.

We leave the discussion on regularity properties for vector-valued quasiminimizers in Chapter 5, which we start by establishing the first of our main results, namely Theorem 1.2.1, and then focus on the quasiminimizer for the inhomogeneous p-Dirichlet integral to obtain the usual higher integrability in the interior and subsequently global higher integrability as in Theorem 1.2.2.

Using the properties obtained in Chapter 5, we prove Theorem 1.1.1 on the characterization of mean coercivity in Chapter 6. Related to mean coercivity is the strong quasiconvexity, and so we prove the stability of the latter in the same chapter.

Meanwhile, Chapter 7 deals with strong quasiminimizers of the inhomogeneous p-Dirichlet integral being absolute minimizers of the quasiconvex x-dependent integral of (1.2.1), followed by the relaxation of (1.2.1) when $p = 2$ and we conclude the chapter with a discussion on strong superquasiminimizers and superminimizers.

Chapter 2

The Direct Method in the Calculus of Variations

The direct method is a well-established way to prove existence of minimizers in variational problems. It relies on the following elementary observation: Suppose we are given a non-empty set S and a functional $\mathcal{F}\colon S \to (-\infty, \infty]$. If there exists a topology τ on S such that

(i) $\mathcal{F}$ is τ–lower semicontinuous, that is, the sub-level sets $\{s \in S : \mathcal{F}(s) \leq t\}$, where $t \in \mathbb{R}$, are all τ–closed;

(ii) $\mathcal{F}$ is τ–coercive, that is, the sub-level sets $\{s \in S : \mathcal{F}(s) \leq t\}$, where $t \in \mathbb{R}$, are all relatively τ–compact (τ–compact closures);

then $\mathcal{F}$ has a minimizer on S: there exists $\overline{s} \in S$ such that $\mathcal{F}(\overline{s}) \leq \mathcal{F}(s)$ for all $s \in S$. In fact, the existence of a minimizer is equivalent to the existence of a topology τ on S with the properties (i) and (ii) (see for instance Holmes [64], Exercise 2.43). In the calculus of variations, a typical task is to minimize an integral functional over a class of Sobolev maps, and in such situations the topology τ is taken to be a suitable weak topology (or more precisely its sequential version, see for instance Buttazzo [22], Remark 1.1.4).

We consider the variational integral

$$\mathcal{F}(u, \Omega) = \int_\Omega F\big(x, u(x), Du(x)\big)\,\mathrm{d}x, \tag{2.0.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded and open set, $u\colon \Omega \to \mathbb{R}^N$ is a vector-valued function of a suitable Sobolev class, Du is the Jacobian matrix of u defined as

$$Du = \left(\frac{\partial u^\alpha}{\partial x_i}\right)_{1\le i\le n}^{1\le\alpha\le N} \in \mathbb{R}^{N\times n},$$

and $F\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is a Carathéodory integrand, and more precisely:

Definition 2.0.1. *An $\mathcal{L}^n$-Carathéodory integrand is a function $F\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ such that*

(i) $(u,\xi) \mapsto F(x,u,\xi)$ is continuous for $\mathcal{L}^n$-almost every $x \in \Omega$,

(ii) $x \mapsto F(x,u,\xi)$ is $\mathcal{L}^n$-measurable for every $(u,\xi) \in \mathbb{R}^N \times \mathbb{R}^{N\times n}$.

This condition guarantees that the composed function $x \mapsto F\big(x,u(x),Du(x)\big)$ is measurable whenever u is a Sobolev function. In order to ensure that the variational integral $\mathcal{F}(u,\Omega)$ is well-defined for Sobolev functions $u \in W^{1,p}(\Omega;\mathbb{R}^N)$, where $p \in [1,\infty)$, we shall assume that the integrand satisfies a standard p-growth condition here:

$$|F(x,u,\xi)| \le c(|\xi|^p + 1) \tag{2.0.2}$$

for almost every $x \in \Omega$ and for all $(u,\xi) \in \mathbb{R}^n \times \mathbb{R}^{N\times n}$, where $c \in (0,\infty)$ is a constant.

We usually denote integrands by F and the associated functionals by $\mathcal{F}$. Apart from that, we also refer to $\min\{N,n\} = 1$ when we use the term *scalar case* and $\min\{N,n\} > 1$ the *vectorial case*.

Given such a functional $\mathcal{F}$, we consider the variational problem

$$\inf\left\{\mathcal{F}(u,\Omega)\colon u - g \in W_0^{1,p}(\Omega;\mathbb{R}^N)\right\}, \tag{P}$$

where $g \in W^{1,p}(\Omega;\mathbb{R}^N)$ is a given function and for $p \in [1,\infty)$, $W_0^{1,p}$ is defined as the closure of C_0^∞ functions in $W^{1,p}$. For $p = \infty$, we have $W_0^{1,\infty} = W^{1,\infty} \cap W_0^{1,1}$. We shall henceforth write $u - g \in W_0^{1,p}$ as $u \in W_g^{1,p}$. More precisely, we wish to find $\bar{u} \in W_g^{1,p}(\Omega;\mathbb{R}^N)$ such that

$$\mathcal{F}(\bar{u},\Omega) \le \mathcal{F}(u,\Omega) \text{ for every } u \in W_g^{1,p}(\Omega;\mathbb{R}^N).$$

Such a function is called an F-minimizer (or an $\mathcal{F}$-minimizer or simply a minimizer when F or $\mathcal{F}$ is clear from context).

The existence problem in the calculus of variations was recognized by Hilbert [63] in his famous list of problems where it is contained in his 20th problem. In fact, the existence problem was not considered to be an issue during early stages of calculus of variations, which at the least dates back to the beginning of infinitesimal calculus. In the one-dimensional case and under ideal conditions, it is known that one can find the critical points of a function by letting its derivative to be equal to zero. Analogously, this can be done for solutions of smooth variational problems with vanishing functional derivative. This leads to solving the associated Euler-Lagrange equation:

$$\frac{\partial}{\partial x_i}\frac{\partial F}{\partial \xi_i^\alpha}\big(x,u(x),Du(x)\big)-\frac{\partial F}{\partial u^\alpha}\big(x,u(x),Du(x)\big)=0, \tag{2.0.3}$$

where the usual summation convention is in force and we sum over $i=1,\dots,n$, and (2.0.3) is a system of N quasilinear coupleds partial differential equations, one for each $\alpha=1,\dots,N$. If there exists a minimizer u for the functional $\mathcal{F}$ under suitable growth and differentiability assumptions on the integrand F, then being a solution to the Euler-Lagrange equation (2.0.3) is a necessary condition for u to minimize $\mathcal{F}$. Sufficiency is also achieved if, in addition, $(u,\xi)\mapsto F(x,u,\xi)$ is convex for almost every $x\in\Omega$, or more generally if the functional $\mathcal{F}$ is convex on the Dirichlet class $W_g^{1,p}(\Omega;\mathbb{R}^N)$. Sometimes, the weak form of Euler-Lagrange equation,

$$\int_\Omega\left(\frac{\partial F}{\partial \xi_i^\alpha}\big(x,u(x),Du(x)\big)D_i\varphi^\alpha+\frac{\partial F}{\partial u^\alpha}\big(x,u(x),Du(x)\big)\varphi^\alpha\right)\mathrm{d}x=0, \tag{2.0.4}$$

is considered under weaker differentiability assumptions.

The direct method in the calculus of variations is a powerful abstract method for proving the existence of minimizers. This approach grew out of many people's work, in particular its early development is often attributed to the works of Riemann, Hilbert, Weierstrass and Tonelli. The essence of the direct method is to split the problem into two parts, namely to enlarge the space of admissible functions which then gives a general existence theorem and to prove some regularity results for the minimizer of problem (*P*). The introduction of Sobolev space makes it possible to prove general existence theorems, which in turn relies on sequential weak lower semicontinuity of the functional $\mathcal{F}$ with respect to the weak topology on $W^{1,p}$, and discussion along this line is given in this chapter. The second issue will be considered in Chapter 4.

2.1 Weak topology on Sobolev spaces

To understand the motivation for using sequential weak lower semicontinuity, we will first consider some basic definitions and results of the weak topology on $W^{1,p}$. For $p\in(1,\infty)$, we recall that the dual of the Lebesgue space L^p is L^q where $\frac{1}{p}+\frac{1}{q}=1$, and L^∞ is the dual space of L^1, whereas L^1 is not the dual of any space.

Definition 2.1.1 (Strong and weak convergence in L^p). *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and let $u_j, u \in L^p(\Omega;\mathbb{R}^N)$, $p \in [1,\infty]$.*

(i) We say that u_j converges strongly to u (written as $u_j \to u$) in $L^p(\Omega;\mathbb{R}^N)$ if

$$\|u_j - u\|_{L^p(\Omega;\mathbb{R}^N)} \to 0.$$

(ii) We say that u_j converges weakly to u (written as $u_j \rightharpoonup u$) in $L^p(\Omega;\mathbb{R}^N)$ if

$$\int_\Omega u_j(x)\cdot v(x)\,\mathrm{d}x \to \int_\Omega u(x)\cdot v(x)\,\mathrm{d}x$$

for every $v \in L^q(\Omega;\mathbb{R}^N)$, where $\frac{1}{p}+\frac{1}{q}=1$ and $p \in [1,\infty)$.
We say that u_j converges weakly to u (written as $u_j \overset{*}{\rightharpoonup} u$) in $L^\infty(\Omega;\mathbb{R}^N)$ if*

$$\int_\Omega u_j(x)\cdot v(x)\,\mathrm{d}x \to \int_\Omega u(x)\cdot v(x)\,\mathrm{d}x$$

for every $v \in L^1(\Omega;\mathbb{R}^N)$.

In our investigation, we are interested in the weak convergence in $W^{1,p}(\Omega;\mathbb{R}^N)$, where now $\Omega \subset \mathbb{R}^n$ is assumed to be open: Assume first that $p \in [1,\infty)$. Then we say a sequence $\{u_j\}_{j=1}^\infty \subset W^{1,p}(\Omega;\mathbb{R}^N)$ converges weakly to $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ (written as $u_j \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^N)$) if

$$\begin{cases} u_j \rightharpoonup u \text{ in } L^p(\Omega;\mathbb{R}^N), \\ Du_j \rightharpoonup Du \text{ in } L^p(\Omega;\mathbb{R}^N). \end{cases}$$

When $p=\infty$, we say that $u_j \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(\Omega;\mathbb{R}^N)$ if the above holds with weak* convergence in $L^\infty(\Omega;\mathbb{R}^N)$. We will henceforth write $\{u_j\}$ instead of $\{u_j\}_{j=1}^\infty$. A standard result for weak convergence (or weak* convergence) is its compactness result given as follows:

Theorem 2.1.2 (Characterization of weak convergence in L^p, $p \in (1,\infty]$). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $\{u_j\}$ be a bounded sequence in $L^p(\Omega;\mathbb{R}^N)$, $p \in (1,\infty)$. Then there exists a subsequence (not relabeled) $\{u_j\}$ and a $u \in L^p(\Omega;\mathbb{R}^N)$ such that $u_j \rightharpoonup u$ in $L^p(\Omega;\mathbb{R}^N)$. If $p=\infty$, then the result still holds with $\rightharpoonup$ replaced by $\overset{*}{\rightharpoonup}$.*

In general, one cannot pass from weak convergence in $W^{1,p}(\Omega;\mathbb{R}^N)$ to strong convergence in $W^{1,p}(\Omega;\mathbb{R}^N)$ (written as $u_j \to u$ in $W^{1,p}(\Omega;\mathbb{R}^N)$). When the n-dimensional Lebesgue measure of Ω is finite or simply $\operatorname{meas}\Omega < \infty$, the obstacle to strong convergence is due to oscillation or concentration (or both). When $\operatorname{meas}\Omega = \infty$, a third phenomenon can obstruct the weak convergence from being strong, namely mass escaping to infinity (lack of tightness). As we shall always work under the assumption that Ω is bounded, and hence that $\operatorname{meas}\Omega < \infty$, we will here focus solely on oscillation and concentration phenomena.

Definition 2.1.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded measurable set. Let $u_j, u\colon \Omega \to \mathbb{R}^N$ be measurable functions.*

(i) We say $\{u_j\}$ converges to u in measure (or simply $u_j \to u$ in measure) if for any $\varepsilon > 0$

$$\lim_{j\to\infty} \operatorname{meas}\big(\{x \in \Omega\colon |u_j(x) - u(x)| > \varepsilon\}\big) = 0.$$

When $u_j \rightharpoonup u$ in $L^p(\Omega;\mathbb{R}^N)$ but $u_j \not\to u$ in measure, we say that $\{u_j\}$ oscillates. Here $p \in [1,\infty]$ is allowed and for $p = \infty$, we of course mean weak convergence.*

(ii) Let $\mathbb{F} \subset L^1(\Omega;\mathbb{R}^N)$ be a family of integrable functions $u\colon \Omega \to \mathbb{R}^N$. Then $\mathbb{F}$ is equiintegrable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_E |u(x)|\,\mathrm{d}x < \varepsilon$$

for all $u \in \mathbb{F}$ and all measurable sets $E \subset \Omega$ with $\operatorname{meas} E < \delta$. For $p \in (1,\infty)$, we say that $\mathbb{F}$ is p-equiintegrable if the family of functions $\big\{|u|^p\colon u \in \mathbb{F}\big\}$ is equiintegrable. When $u_j \rightharpoonup u$ in $L^p(\Omega;\mathbb{R}^N)$ but $\{u_j\}$ is not p-equiintegrable, we say that $\{u_j\}$ concentrates in $L^p(\Omega;\mathbb{R}^N)$.

Example 2.1.4.

(i) Let $\Omega = (0,1)$, $N = 1$ and $u_j(x) = \sin(jx)$.
It is easy to see that $u_j \overset{*}{\rightharpoonup} 0$ in $L^\infty(0,1)$. Indeed it is a special case of the Riemann-Lebesgue lemma (see Dacorogna [28], Theorem 1.22). However, it is clear that $\{u_j\}$ does not converge in the strong sense in any L^p. In particular, u_j oscillates between -1 and 1 in a smaller and smaller scale as $j \to \infty$. Hence, the sequence $\{u_j\}$ oscillates.

(ii) Let $\Omega = (0,1)$, $N = 1$ and $u_j(x) = j^{\frac{1}{2}}\mathbb{1}_{(0,\frac{1}{j})}(x)$, where $\mathbb{1}_A$ denotes the indicator function for the set A.
One can easily see that u_j is bounded in $L^2(0,1)$ and that $u_j \rightharpoonup 0$ in $L^2(0,1)$. However, $\{u_j\}$ does not converge strongly in $L^2(0,1)$ since $\|u_j\|_{L^2} = 1$. The problem here is that the 2-equiintegrability assumption in the definition above is not satisfied. Therefore, the sequence $\{u_j\}$ concentrates in $L^2(0,1)$. It is worth noting that $u_j \to 0$ in $L^p(0,1)$ for each $p \in [1,2)$.

We can see the involvement of strong convergence in the above examples. Indeed if a sequence does not potray any of those bad behaviours, we can pass to strong convergence and this is clearly stated in Vitali's convergence theorem (see for example Fonseca-Leoni [49], Theorem 2.24).

Theorem 2.1.5 (Vitali's convergence theorem). *Let $p \in [1, \infty)$ and $u_j, u \in L^p(\Omega; \mathbb{R}^N)$ where $\operatorname{meas} \Omega < \infty$. Then $u_j \to u$ in $L^p(\Omega; \mathbb{R}^N)$ if and only if*

(i) $\{u_j\}$ is p-equiintegrable,

(ii) $u_j \to u$ in measure.

In order to understand the essence of the p-equiintegrability condition in Vitali's convergence theorem, we recall the De la Vallée-Poussin criterion (see for example Fonseca-Leoni [49], Theorem 2.29).

Lemma 2.1.6 (De la Vallée-Poussin criterion for equiintegrability). *Assume that $\operatorname{meas} \Omega < \infty$. A family $\mathbb{F} \subset L^1(\Omega; \mathbb{R}^N)$ is equiintegrable if and only if there exists a Borel function $\Phi \colon [0, \infty) \to [0, \infty]$ such that $\frac{\Phi(t)}{t} \to \infty$ as $t \to \infty$ and*

$$\sup_{u \in \mathbb{F}} \int_\Omega \Phi \circ |u(x)| \, \mathrm{d}x < \infty.$$

It can be shown that the equiintegrability condition above is also equivalent to

$$\sup_{u \in \mathbb{F}} \int_{\Omega \cap \{|u| > t\}} |u(x)| \, \mathrm{d}x \to 0 \quad \text{as} \quad t \to \infty.$$

Another important fact about equiintegrability is the following result that we quote from Theorem 2.54 of Fonseca-Leoni [49]:

Theorem 2.1.7 (Dunford-Pettis theorem). *A family $\mathbb{F} \subset L^1(\Omega; \mathbb{R}^N)$ is relatively sequentially weakly compact if and only if $\mathbb{F}$ is equiintegrable.*

Here a family $\mathbb{F} \subset L^1(\Omega; \mathbb{R}^N)$ of integrable functions $u \colon \Omega \to \mathbb{R}^N$ is said to be relatively sequentially weakly compact if any sequence $\{u_j\} \subset \mathbb{F}$ admits a weakly convergent subsequence (the weak limit $u \in L^1(\Omega; \mathbb{R}^N)$ need not belong to $\mathbb{F}$). In connection with Vitali's convergence theorem, we see that if $u_j \rightharpoonup u$ in $L^1(\Omega; \mathbb{R}^N)$, then

$$u_j \to u \text{ in } L^1(\Omega; \mathbb{R}^N) \quad \Longleftrightarrow \quad u_j \to u \text{ in measure.}$$

It may seem, to the readers, that the above discussions about oscillation and concentration are irrelevant in the study of sequential weak lower semicontinuity. We will see below how these two phenomena play a role in encouraging the use of sequential weak lower semicontinuity, and moreover these notions are again considered in Chapter 3.

To understand the motivation behind the study of sequential weak lower semicontinuity, we first set

$$m = \inf \left\{ \mathcal{F}(u, \Omega) \colon u \in W_g^{1,p}(\Omega; \mathbb{R}^N) \right\} \tag{2.1.1}$$

as in problem (P) and choose a minimizing sequence $u_j \in W_g^{1,p}(\Omega; \mathbb{R}^N)$ such that

$$\mathcal{F}(u_j, \Omega) \to m \ \text{ as } \ j \to \infty. \tag{2.1.2}$$

The aim is to show that there exists some convergent subsequence $\{u_{j_k}\}_{k=1}^{\infty}$ that converges to a map $\bar{u}$ that we hope can be shown to be a minimizer.

A main difficulty is that $\mathcal{F}$ is usually not sequentially continuous with respect to the weak topology, meaning that we cannot deduce that

$$\mathcal{F}(\bar{u}, \Omega) = \lim_{k \to \infty} \mathcal{F}(u_{j_k}, \Omega) \tag{2.1.3}$$

and hence we cannot guarantee that $\bar{u}$ is indeed a minimizer. This is due to the fact that we cannot control the behaviour of the weakly convergent sequences, meaning that the gradients can be either highly oscillatory or concentrate as $j_k \to \infty$.

The advantage of weak lower semicontinuity is that we do not need such a strong condition as (2.1.3) and what that would entail on F. In fact, if we only have

$$\mathcal{F}(\bar{u}, \Omega) \leq \liminf_{k \to \infty} \mathcal{F}(u_{j_k}, \Omega),$$

we can still deduce from (2.1.2) that $\mathcal{F}(\bar{u}, \Omega) \leq m$ and from (2.1.1) since $\bar{u} \in W_g^{1,p}(\Omega; \mathbb{R}^N)$ by general theory of Sobolev space that $\mathcal{F}(\bar{u}, \Omega) \geq m$. It then follows that $\bar{u}$ is a minimizer.

Definition 2.1.8. *Let $p \in [1, \infty)$. We say that the functional $\mathcal{F}$ is sequentially weakly lower semicontinuous in $W^{1,p}(\Omega; \mathbb{R}^N)$ if*

$$\liminf_{j \to \infty} \mathcal{F}(u_j, \Omega) \geq \mathcal{F}(\bar{u}, \Omega) \quad \textit{whenever} \quad u_j \rightharpoonup \bar{u} \textit{ in } W^{1,p}(\Omega; \mathbb{R}^N).$$

When $p = \infty$, we say that the functional $\mathcal{F}$ is sequentially weakly lower semicontinuous in $W^{1,\infty}(\Omega; \mathbb{R}^N)$ if*

$$\liminf_{j \to \infty} \mathcal{F}(u_j, \Omega) \geq \mathcal{F}(\bar{u}, \Omega) \quad \textit{whenever} \quad u_j \overset{*}{\rightharpoonup} \bar{u} \textit{ in } W^{1,\infty}(\Omega; \mathbb{R}^N).$$

Now we are ready to outline the standard direct method in this context which starts by taking a minimizing sequence u_j that approximates the infimum m of the problem (P), meaning that $\mathcal{F}(u_j,\Omega) \to m$. Then, we show that $m > -\infty$ and that there exists a convergent subsequence $u_{j_k} \rightharpoonup \bar{u}$ in $W^{1,p}_g(\Omega;\mathbb{R}^N)$ with respect to the $W^{1,p}$-weak topology on $W^{1,p}_g(\Omega;\mathbb{R}^N)$. The final step is to show that $\mathcal{F}$ is sequentially weakly (or weakly*) lower semicontinuous with respect to the same topology and hence we conclude that $\bar{u}$ is a minimizer of $\mathcal{F}$ on $W^{1,p}_g(\Omega;\mathbb{R}^N)$.

The first step is usually established by showing that $\mathcal{F}$ is coercive, meaning that

$$\lim_{\|u\|_{W^{1,p}}\to\infty} \mathcal{F}(u,\Omega) = +\infty,$$

and normally follows from estimates on $\mathcal{F}$ from below. In many cases, the coerciveness of $\mathcal{F}$ can be guaranteed by imposing the pointwise coercivity condition on the integrand

$$F(x,u,\xi) \geq c_1|\xi|^p - c_2, \tag{2.1.4}$$

for almost every $x \in \Omega$ and for all $(u,\xi) \in \mathbb{R}^N \times \mathbb{R}^{N\times n}$ for some constants $c_1 > 0$ and $c_2 < \infty$. Recall that we also imposed the p-growth condition (2.0.2) on the integrand F. We may consider other coercivity and growth conditions in some parts of the thesis, but these will be mentioned beforehand.

The key notions in the study of sequential weak lower semicontinuity are convexity and quasiconvexity. It is known that convexity of the integrand F is a necessary and sufficient condition to ensure weak lower semicontinuity in the scalar case. Convexity remains a sufficient condition in the vectorial case but is far from being a necessary condition. In this connection, Morrey [102] introduced quasiconvexity which plays a central role in vector-valued calculus of variations.

2.2 Notions of convexity

Convexity plays an important role in many areas of mathematics. In the calculus of variations, it is indispensible in the study of lower semicontinuity in the scalar case. Recall the definition of convexity:

Definition 2.2.1. *A function $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}\cup\{+\infty\}$ is said to be convex if*

$$F\big(t\xi + (1-t)\eta\big) \leq tF(\xi) + (1-t)F(\eta)$$

for every $\xi,\eta \in \mathbb{R}^{N\times n}$ and for all $t \in [0,1]$.

Convex analysis will not be discussed in detail here and we shall freely make use of standard results. Among useful references in convex analysis are Ekeland-Témam [40] and Rockafellar [110]. We remark that, for simplicity, the standard definitions and basic properties are given without lower order terms.

When one passes to the vectorial case, convexity is no longer a necessary condition for lower semicontinuity. The introduction of quasiconvexity in the groundbreaking work of Morrey [102] led to the advancement of vector-valued calculus of variations. We will see in Section 2.3 that quasiconvexity takes up the role of convexity as a necessary and sufficient condition for sequential weak lower semicontinuity in vector-valued calculus of variations, provided that the integrand F satisfies the coercivity (2.1.4) and growth (2.0.2) conditions. These results are attributed to the works of Acerbi-Fusco [1], Marcellini [90], Meyers [99] and Morrey [102, 103]. The following is a variant of Morrey's definition [102].

Definition 2.2.2. *A function $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is said to be quasiconvex at $\xi \in \mathbb{R}^{N\times n}$ if*

$$F(\xi) \leq \frac{1}{\operatorname{meas}\Omega} \int_{\Omega} F\big(\xi + D\varphi(x)\big)\,\mathrm{d}x \tag{2.2.1}$$

for every bounded open set $\Omega \subset \mathbb{R}^n$ and for every piecewise affine $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^N)$. It is said to be quasiconvex if it is quasiconvex at all $\xi \in \mathbb{R}^{N\times n}$.

We shall henceforth write integral means as dashed integrals, and so in particular

$$⨍_{\Omega} F\big(\xi + D\varphi(x)\big)\,\mathrm{d}x = \frac{1}{\operatorname{meas}\Omega} \int_{\Omega} F\big(\xi + D\varphi(x)\big)\,\mathrm{d}x.$$

It is important to realize that the definition of quasiconvexity is independent of the choice of Ω. Note that we do not need any condition on F in the definition above since we only test (2.2.1) with piecewise affine φ. We remark that in fact once (2.2.1) holds for all piecewise affine φ, it will also hold for *generalized piecewise affine maps* meaning that maps $\varphi \in W_0^{1,\infty}(\Omega_1;\mathbb{R}^N)$ such that $D\varphi$ has a finite range. This is used in the proof of Proposition 2.2.3 below, which in turn is closely related to Proposition 5.11 in [29].

Proposition 2.2.3. *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a function and fix $\xi \in \mathbb{R}^{N\times n}$. Let $\Omega_1 \subset \mathbb{R}^n$ be a bounded open set and assume the inequality*

$$F(\xi) \leq ⨍_{\Omega_1} F\big(\xi + D\varphi(x)\big)\,\mathrm{d}x$$

holds for every piecewise affine $\varphi \in W_0^{1,\infty}(\Omega_1;\mathbb{R}^N)$*. Then the inequality*

$$F(\xi) \leq \fint_{\Omega_2} F\big(\xi + D\psi(x)\big)\,\mathrm{d}x$$

holds for every bounded open set $\Omega_2 \subset \mathbb{R}^n$ *with* $\operatorname{meas}(\partial\Omega_2) = 0$ *and for every piecewise affine* $\psi \in W_0^{1,\infty}(\Omega_2;\mathbb{R}^N)$.

Proof. Let $\Omega_2 \subset \mathbb{R}^n$ be another bounded open set and let $\psi \in W_0^{1,\infty}(\Omega_2;\mathbb{R}^N)$ be a piecewise affine function. By the exhaustion argument (see Proposition A.2.1), we can write Ω_1 as

$$\Omega_1 = \bigcup_{i\in I}(x_i + \lambda_i\Omega_2) \cup N$$

where N denotes the null set with $\operatorname{meas} N = 0$ (since $\operatorname{meas}(\partial\Omega_2) = 0$), $I \subseteq \mathbb{N}$ and $\{x_i + \lambda_i\Omega_2\}_{i\in I}$ disjoint. Define

$$\varphi(x) = \begin{cases} \lambda_i\psi\left(\frac{x-x_i}{\lambda_i}\right) & \text{if } x \in x_i + \lambda_i\Omega_2,\, i \in I \\ 0 & \text{if } x \in N \end{cases}. \tag{2.2.2}$$

Note that, due to disjointness and using translation invariance of Lebesgue measure, we get

$$\begin{aligned} \operatorname{meas}\Omega_1 = \operatorname{meas}\Big(\bigcup_{i\in I}(x_i + \lambda_i\Omega_2)\Big) &= \sum_{i\in I}\operatorname{meas}(x_i + \lambda_i\Omega_2) \\ &= \sum_{i\in I}\operatorname{meas}(\lambda_i\Omega_2) \\ &= \sum_{i\in I}\lambda_i^n \operatorname{meas}\Omega_2 \end{aligned}$$

and hence

$$\sum_{i\in I}\lambda_i{}^n = \frac{\operatorname{meas}\Omega_1}{\operatorname{meas}\Omega_2}. \tag{2.2.3}$$

Then $\varphi \in W_0^{1,\infty}(\Omega_1;\mathbb{R}^N)$, also piecewise affine, and hence by our assumption

$$\begin{aligned} F(\xi)\operatorname{meas}\Omega_1 &\leq \int_{\Omega_1} F\big(\xi + D\varphi(x)\big)\,\mathrm{d}x \\ &= \sum_{i\in I}\int_{x_i+\lambda_i\Omega_2} F\left(\xi + D\psi\left(\frac{x-x_i}{\lambda_i}\right)\right)\mathrm{d}x \\ &= \sum_{i\in I}\lambda_i{}^n\int_{\Omega_2} F\big(\xi + D\psi(x)\big)\,\mathrm{d}x \\ &= \frac{\operatorname{meas}\Omega_1}{\operatorname{meas}\Omega_2}\int_{\Omega_2} F\big(\xi + D\psi(x)\big)\,\mathrm{d}x. \end{aligned}$$

$\square$

We will see later in Remark 2.2.20 that we can even drop the condition on the boundary of Ω_2. It is not an easy task to verify that a given integrand F is quasiconvex. In this connection, other auxiliary convexity conditions have been introduced, namely rank one convexity and polyconvexity.

Definition 2.2.4. *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}\cup\{\infty\}$ be a function.*

(i) F is said to be rank one convex if

$$F\big(t\xi + (1-t)\eta\big) \leq tF(\xi) + (1-t)F(\eta)$$

for every $\xi, \eta \in \mathbb{R}^{N\times n}$ with $\operatorname{rank}(\xi-\eta) \leq 1$ and for every $t \in [0,1]$.

(ii) F is said to be polyconvex if there exists $\tilde{F}\colon \mathbb{R}^{\tau(n,N)} \to \mathbb{R}\cup\{\infty\}$ convex, such that

$$F(\xi) = \tilde{F}\big(T(\xi)\big),$$

where $T\colon \mathbb{R}^{N\times n} \to \mathbb{R}^{\tau(n,N)}$ is such that

$$T(\xi) := (\xi, \operatorname{adj}_2 \xi, \cdots, \operatorname{adj}_{n\wedge N} \xi).$$

Here $\operatorname{adj}_s \xi \in \mathbb{R}^{\binom{N}{s}\times\binom{n}{s}}$ is the adjugate matrix of order s, $2 \leq s \leq n\wedge N = \min\{n, N\}$ and

$$\tau(n,N) := \sum_{s=1}^{n\wedge N} \sigma(s), \text{ where } \sigma_s := \binom{N}{s}\binom{n}{s} = \frac{N!n!}{(s!)^2(N-s)!(n-s)!}.$$

Remark 2.2.5.

(i) We have only defined quasiconvexity for real-valued functions F for technical reasons. Anyway in most of the thesis, we even impose growth conditions on F.

(ii) It is convenient to use tensor notation for rank one matrices. For $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^n$, we denote

$$a\otimes b = (a^\alpha b_i)^{1\leq\alpha\leq N}_{1\leq i\leq n},$$

whereby

$$(a\otimes b)x = (b\cdot x)a \text{ for } x \in \mathbb{R}^n$$

and

$$|a\otimes b| = |a||b|.$$

(iii) We refer to Section 5.4 of [29] for more details about adjugate matrices but note here that for $N = n = 2$, F is polyconvex if there exists a convex function $\tilde{F}\colon \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ such that

$$F(\xi) = \tilde{F}(\xi, \det \xi)$$

and for $N = n = 3$, F is polyconvex if there exists a convex function $\tilde{F}\colon \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3} \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ such that

$$F(\xi) = \tilde{F}\big(\xi, \operatorname{cof} \xi, \det \xi\big).$$

With the introduction of the auxiliary notions of convexity, we will now discuss how they are related to quasiconvexity, beginning with rank one convexity which is a necessary condition for quasiconvexity:

Lemma 2.2.6 (Morrey [102])**.** *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a quasiconvex function. Then F is rank one convex.*

Proof. Our aim is to show that

$$F\big(t\xi + (1-t)\eta\big) \le tF(\xi) + (1-t)F(\eta)$$

for every $\xi, \eta \in \mathbb{R}^{N\times n}$ with $\operatorname{rank}(\xi - \eta) \le 1$ and $t \in [0,1]$. Let $\varepsilon > 0$ and apply Lemma A.2.2 and Theorem A.2.4 to find disjoint open sets $\Omega_\xi, \Omega_\eta, \Omega_i \subset \Omega$ and piecewise affine $\varphi_\varepsilon \in W_0^{1,\infty}(\Omega; \mathbb{R}^N)$ such that

$$\begin{cases} \|D\varphi_\varepsilon\|_{L^\infty(\Omega)} < m, \\ D\varphi_\varepsilon(x) = \begin{cases} (1-t)(\xi - \eta) & \text{if } x \in \Omega_\xi \\ -t(\xi-\eta) & \text{if } x \in \Omega_\eta \\ a \otimes b_i & \text{if } x \in \Omega_i,\ i = 3, \dots, k \end{cases}, \\ \operatorname{meas} \Omega_\xi > (1-\varepsilon) t \operatorname{meas} \Omega, \\ \operatorname{meas} \Omega_\eta > (1-\varepsilon)(1-t) \operatorname{meas} \Omega, \end{cases}$$

where constant $m = m(\xi, \eta)$ and the number, $k-2$, of auxiliary matrices $a \otimes b_i$ are independent of ε. We use the quasiconvexity of F and the above approximation to obtain

$$\begin{aligned} F\big(t\xi + (1-t)\eta\big) &\le \fint_\Omega F\big(t\xi + (1-t)\eta + D\varphi_\varepsilon(x)\big)\,\mathrm{d}x \\ &= \frac{1}{\operatorname{meas}\Omega} \int_{\Omega_\xi} F(\xi)\,\mathrm{d}x + \frac{1}{\operatorname{meas}\Omega} \int_{\Omega_\eta} F(\eta)\,\mathrm{d}x \\ &\quad + \frac{1}{\operatorname{meas}\Omega} \sum_{i=3}^{k} \int_{\Omega_i} F\left(\xi + a \otimes b_i\right)\,\mathrm{d}x \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that F is rank one convex. □

It is easy to see that the rank one convexity of a function $F \in C^2(\mathbb{R}^{N\times n})$ is equivalent to the Legendre-Hadamard condition:

$$\sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n}\frac{\partial^2 F(\xi)}{\partial \xi_i^\alpha \partial \xi_j^\beta}\lambda^\alpha\lambda^\beta\mu_i\mu_j \geq 0$$

for every $\lambda \in \mathbb{R}^N$, $\mu \in \mathbb{R}^n$ and $\xi = (\xi_i^\alpha)_{1\leq i\leq n}^{1\leq\alpha\leq N} \in \mathbb{R}^{N\times n}$. Note that the Legendre-Hadamard condition, also known as the ellipticity condition in its strict form, is the usual inequality required for the Euler-Lagrange system of equations (see Agmon-Douglis-Nirenberg [2]).

On the other hand, polyconvexity is a stronger condition as compared to quasiconvexity and it is trivial that a convex function is also polyconvex. The notion of polyconvexity was introduced by Ball [9] while Morrey had already observed that related conditions were sufficient for quasiconvexity in [102, 103]. In fact, the function T in the definition of polyconvexity has coordinate functions that are quasiaffine and rank one affine.

Definition 2.2.7. *A function F is said to be polyaffine, quasiaffine or rank one affine if both F and $-F$ are polyconvex, quasiconvex or rank one convex, respectively.*

Lemma 2.2.8 (Dacorogna [29], Lemma 5.5)**.** *Let $\xi \in \mathbb{R}^{N\times n}$ and $T(\xi)$ be defined as in Definition 2.2.4. Then we have the following:*

(i) For every bounded open $\Omega \subset \mathbb{R}^n$, for every $\xi \in \mathbb{R}^{N\times n}$ and for every $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^N)$, T is quasiaffine:

$$T(\xi) = \fint_\Omega T\big(\xi + D\varphi(x)\big)\,\mathrm{d}x.$$

(ii) For every $\xi,\eta \in \mathbb{R}^{N\times n}$ with $\operatorname{rank}(\xi-\eta) \leq 1$ and for every $t \in [0,1]$, T is rank one affine:

$$T\big(t\xi + (1-t)\eta\big) = tT(\xi) + (1-t)T(\xi).$$

We omit the proof of this which can be found in Dacorogna [29], pages 255 and 409, respectively. Ball [9] established the equivalence of polyaffine, quasiaffine and rank one affine functions, thereby completing a long series of results (refer to the list in Dacorogna [29], page 179).

Theorem 2.2.9 (Dacorogna [29], Theorem 5.20)**.** *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$. The following conditions are equivalent:*

(i) F is quasiaffine.

(ii) F is rank one affine.

(ii') For every $\xi \in \mathbb{R}^{N\times n}$, $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^n$, the function $F \in C^1$ satisfies

$$F(\xi + a \otimes b) = F(\xi) + \langle DF(\xi); a \otimes b\rangle,$$

where $\langle\cdot;\cdot\rangle$ denotes the scalar product in $\mathbb{R}^{N\times n}$.

(iii) F is polyaffine.

(iii') There exists $\Gamma \in \mathbb{R}^{\tau(n,N)}$ such that for every $\xi \in \mathbb{R}^{N\times n}$,

$$F(\xi) = F(0) + \langle\Gamma; T(\xi)\rangle$$

where $\langle\cdot;\cdot\rangle$ denotes the scalar product in $\mathbb{R}^{\tau(n,N)}$ and T is as in Definition 2.2.4.

Interestingly, if the function F depends on a quasiaffine function, then all the notions of convexity are equivalent in the following sense:

Theorem 2.2.10 (Dacorogna [26], Corollary 2.5 and Corollary 3.2)**.** *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$, let $\Phi\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a quasiaffine function that is not identically constant and $G\colon \mathbb{R} \to \mathbb{R}$ be such that*

$$F(\xi) = G\big(\Phi(\xi)\big)$$

(in particular, if $N = n$, take $\Phi(\xi) = \det \xi$). Then

$$F \textit{ polyconvex} \Leftrightarrow F \textit{ quasiconvex} \Leftrightarrow F \textit{ rank one convex} \Leftrightarrow G \textit{ convex}.$$

We will now use the nice property of the minors in Lemma 2.2.8 to show sufficiency of polyconvexity for quasiconvexity and in the extended case, for rank one convexity.

Lemma 2.2.11 (Dacorogna [29], Theorem 5.3)**.**

(i) Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a polyconvex function. Then F is quasiconvex.

(ii) Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\}$ be a polyconvex function. Then F is rank one convex.

Proof.

(i) Since F is polyconvex, there exists $\tilde{F}\colon \mathbb{R}^{\tau(n,N)} \to \mathbb{R}$ convex, such that

$$F(\xi) = \tilde{F}\big(T(\xi)\big),$$

where $T\colon \mathbb{R}^{N\times n}\to\mathbb{R}^{\tau(n,N)}$ is given in Definition 2.2.4. We now use Jensen's inequality and the fact that T is quasiaffine to obtain, for every bounded open $\Omega\subset\mathbb{R}^n$, for every $\xi\in\mathbb{R}^{N\times n}$ and for every piecewise affine $\varphi\in W_0^{1,\infty}(\Omega;\mathbb{R}^N)$,

$$\begin{aligned}
\fint_\Omega F\big(\xi+D\varphi(x)\big)\,\mathrm{d}x &= \fint_\Omega \tilde F\Big(T\big(\xi+D\varphi(x)\big)\Big)\,\mathrm{d}x\\
&\geq \tilde F\left(\fint_\Omega T\big(\xi+D\varphi(x)\big)\,\mathrm{d}x\right)\\
&= \tilde F\big(T(\xi)\big)=F(\xi),
\end{aligned}$$

which is precisely the quasiconvexity of F at ξ.

(ii) Once again, we have

$$F(\xi)=\tilde F\big(T(\xi)\big),$$

as above. Let $t\in[0,1]$, $\xi,\eta\in\mathbb{R}^{N\times n}$ with $\operatorname{rank}(\xi-\eta)\leq 1$. We now use the fact that T is rank one affine to find

$$\begin{aligned}
F\big(t\xi+(1-t)\eta\big) &= \tilde F\Big(T\big(t\xi+(1-t)\xi\big)\Big)\\
&= \tilde F\big(tT(\xi)+(1-t)\eta\big)\\
&\leq t\tilde F\big(T(\xi)\big)+(1-t)\tilde F\big(T(\eta)\big)\\
&= tF(\xi)+(1-t)F(\eta)
\end{aligned}$$

which is precisely the rank one convexity of F.

$\square$

We can summarize the above results by the following string of implications:

$$F\text{ convex}\ \Rightarrow\ F\text{ polyconvex}\ \Rightarrow\ F\text{ quasiconvex}\ \Rightarrow\ F\text{ rank one convex} \tag{2.2.4}$$

when $F\colon\mathbb{R}^{N\times n}\to\mathbb{R}$ and

$$F\text{ convex}\ \Rightarrow\ F\text{ polyconvex}\ \Rightarrow\ F\text{ rank one convex}$$

when $F\colon\mathbb{R}^{N\times n}\to\mathbb{R}\cup\{+\infty\}$. We note that all the notions coincide in the scalar case. Otherwise, the converse implications are all false in general.

If $\min\{N,n\}\geq 2$ and if we let F to be a 2×2 minor of ξ, then F is polyconvex but not convex. The issue of whether rank one convexity implies quasiconvexity was open for a long time. In the dimensions $N \geq 3$, $n \geq 2$, Šverák presented in [114] a remarkable counter example: a quartic polynomial which is rank one convex but is not quasiconvex. The question however remains open in the dimensions $N = 2$, $n \geq 2$.

In connection with the non-equivalence of quasiconvexity and polyconvexity, there are several examples that show the former does not imply the latter. One of the well-known examples was given by Terpstra [120] which was subsequently investigated by Ball [10] and Serre [111] (see (ii) in the theorem below). The example is a quadratic form and one must require $\min\{N, n\} \geq 3$ for the dimensions. Quadratic forms play an important role in the calculus of variations and we quote without proof the following results:

Theorem 2.2.12 (Dacorogna [29], Theorem 5.25)**.** *Let $M : \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ be a linear self-adjoint transformation and put*

$$F(\xi) := \langle M\xi; \xi\rangle$$

where $\xi \in \mathbb{R}^{N\times n}$ and $\langle\cdot;\cdot\rangle$ denotes the usual scalar product in $\mathbb{R}^{N\times n}$. Then the following statements hold:

(i) F is rank one convex if and only if F is quasiconvex.

(ii) If $\min\{N, n\} \geq 3$, then in general

$$F \textit{ rank one convex } \nRightarrow F \textit{ polyconvex.}$$

(iii) If $\min\{N, n\} = 2$, then

$$F \textit{ polyconvex } \Leftrightarrow F \textit{ quasiconvex } \Leftrightarrow F \textit{ rank one convex.}$$

Note that the example above assumes that F is quadratic, and hence the associated Euler-Lagrange equations are linear (Terpstra [120]). An interesting fact about these results is that some of them have been considered or established even before the introduction of quasiconvexity. For instance, the proof of statement (i) was already implicitly known by Van Hove [125, 126]. We have also included the case for $\min\{N, n\} = 2$ to show the equivalence of polyconvexity, quasiconvexity and rank one convexity for quadratic forms. The proof of this is contained in the works of Serre [111], Terpstra [120] and Uhlig [123] following considerable attention from several authors since it was first raised by Bliss in 1937.

Apart from that, Šverák [113] proved that the exist quasiconvex functions $F\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ with subquadratic growth that are not polyconvex. Adapting Šverák's approach, Zhang [132] then developed a method for constructing non-trivial quasiconvex functions with linear growth at infinity and Müller [107] showed that there exist quasiconvex functions which are homogeneous of degree one but not convex.

Another well-known example for the case $N = n = 2$ by Alibert-Dacorogna-Marcellini (see [4], [32]) clearly illustrates the non-equivalence of these different notions of convexity.

Theorem 2.2.13 (Dacorogna [29], Theorem 5.51)**.** *Let $\gamma \in \mathbb{R}$ and let $F_\gamma\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ be defined as*

$$F_\gamma(\xi) = |\xi|^2 \left(|\xi|^2 - 2\gamma \det \xi\right).$$

Then

$$\begin{aligned}
F_\gamma \text{ is convex} \quad &\Leftrightarrow \quad |\gamma| \le \gamma_c = \frac{2}{3}\sqrt{2}, \\
F_\gamma \text{ is polyconvex} \quad &\Leftrightarrow \quad |\gamma| \le \gamma_{pc} = 1, \\
F_\gamma \text{ is quasiconvex} \quad &\Leftrightarrow \quad |\gamma| \le \gamma_{qc} = 1 \text{ and } \gamma_{qc} > 1, \\
F_\gamma \text{ is rank one convex} \quad &\Leftrightarrow \quad |\gamma| \le \gamma_{rc} = \frac{2}{\sqrt{3}}.
\end{aligned}$$

The last result and that of polyconvexity implies $|\gamma| \le 1$ were established by Dacorogna-Marcellini [32]. The other results were subsequently proven by Alibert-Dacorogna [4] and later Iwaniec-Kristensen [65] recovered the result of $\gamma_{qc} > 1$ in a simpler way. While the value of γ_{qc} remains an open problem, numerical evidences of Dacorogna-Douchet-Gangbo-Rappaz [30], Dacorogna-Haeberly [31] and Gremaud [61] tend to indicate $\gamma_{qc} = \frac{2}{\sqrt{3}}$. If it can be proven that $\gamma_{qc} < \frac{2}{\sqrt{3}}$, then it shows that rank one convex functions need not be quasiconvex even in the 2-by-2 case.

Of course, there are also examples in which some converse implications in (2.2.4) can be shown to be true, for instance the quasiaffine functions considered in Theorem 2.2.10 and the quadratic functions in Theorem 2.2.12. Another interesting example that relates all the notions of convexity is when the integrand takes on the value in the Euclidean norm:

Theorem 2.2.14 (Dacorogna [29], Theorem 5.58)**.** *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ and let $|\cdot|$ denote the Euclidean norm given by, for all $\xi \in \mathbb{R}^{N\times n}$,*

$$|\xi| := \left(\sum_{i=1}^{n}\sum_{\alpha=1}^{N} (\xi_i^\alpha)^2\right)^{\frac{1}{2}}.$$

(i) (Dacorogna [27]) *Let* $G\colon \mathbb{R}_+ \to \mathbb{R}$ *be such that* $F(\xi) = G(|\xi|)$. *Then*

$$F \text{ convex } \Leftrightarrow F \text{ polyconvex } \Leftrightarrow F \text{ quasiconvex } \Leftrightarrow F \text{ rank one convex}$$
$$\Leftrightarrow G \text{ convex and } G(0) = \inf\{G(x)\colon x \geq 0\}.$$

(ii) (Ball-Murat [15], Theorem 4.1 and Lemma 4.3) *Let* $N = n$, $1 \leq \beta < 2n$, $H\colon \mathbb{R} \to \mathbb{R}$ *be such that*

$$F(\xi) = |\xi|^\beta + H(\det \xi).$$

Then

$$F \text{ polyconvex } \Leftrightarrow F \text{ quasiconvex } \Leftrightarrow F \text{ rank one convex } \Leftrightarrow H \text{ convex}.$$

We can clearly see that even though rank one convexity is a weaker notion than quasiconvexity, it does play a key role in quasiconvex analysis. In some cases, we may even consider separately convex functions:

Definition 2.2.15. *A function* $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}\cup\{+\infty\}$ *is said to be separately convex if it is convex in each variable. In particular, the function*

$$\xi_i^\alpha \to F(\xi) \text{ is convex for every } i = 1,\dots,n \text{ and } \alpha = 1,\dots,N$$

while the other arguments are fixed.

It is easy to see that if a function $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is rank one convex, then it is also separately convex. Since all separately convex functions are locally Lipschitz as shown below, it is clear that any function that is convex, polyconvex, quasiconvex or rank one convex is also locally Lipschitz. We present the detailed proof of the following result as it plays an important role in our work.

Theorem 2.2.16 (Dacorogna [29], Theorem 2.31)**.** *Let* $F\colon \mathbb{R}^N \to \mathbb{R}\cup\{+\infty\}$ *be separately convex and* $F \not\equiv +\infty$. *Then* F *is locally Lipschitz, and hence continuous, on* int (dom F).

Proof. The proof is to be divided into three steps.

Step 1. We need to first prove that if $x \in$ int (dom F), then F is bounded above in a neighbourhood of x. Without loss of generality, suppose that $x_0 = 0$. Since $x_0 = 0 \in$ int (dom F), then there exists $\varepsilon > 0$ such that

$$X_\varepsilon = \left\{x = (x_1, ..., x_N) \in \mathbb{R}^N\colon \|x\|_\infty \leq \varepsilon\right\} \subset \text{dom } F, \tag{2.2.5}$$

where $\|x\|_\infty := \max\{|x_i| \colon i = 1, ..., N\}$. Let

$$a := \max\Big\{F(x) \colon x_i \in \{-\varepsilon, 0, \varepsilon\} \text{ for every } i = 1, ..., N\Big\}.$$

We would like to prove that

$$\|x\|_\infty \leq \varepsilon \quad \Rightarrow \quad F(x) \leq a. \tag{2.2.6}$$

Let $x, z \in X_\varepsilon$ with $z_i \in \{-\varepsilon, 0, \varepsilon\}$, $i = 1, \ldots, N$. If $x_N \neq 0$, we write

$$x_N = \frac{|x_N|}{\varepsilon}(\text{sgn}\, x_N)\,\varepsilon + \left(1 - \frac{|x_N|}{\varepsilon}\right) 0.$$

If $z_i \in \{-\varepsilon, 0, \varepsilon\}$ for $i = 1, \ldots, N-1$, we get by separate convexity of F that

$$\begin{aligned}
& F(z_1, ..., z_{N-1}, x_N) \\
\leq \; & \frac{|x_N|}{\varepsilon} F\Big(z_1, ..., z_{N-1}, (\text{sgn}\, x_N)\,\varepsilon\Big) + \left(1 - \frac{|x_N|}{\varepsilon}\right) F(z_1, ..., z_{N-1}, 0) \\
\leq \; & \frac{|x_N|}{\varepsilon} a + \left(1 - \frac{|x_N|}{\varepsilon}\right) a = a.
\end{aligned}$$

The same inequality holds if $x_N = 0$. Similarly, if $x_{N-1} \neq 0$, then we get by separate convexity once again that

$$\begin{aligned}
& F(z_1, ..., z_{N-2}, x_{N-1}, x_N) \\
\leq \; & \frac{|x_{N-1}|}{\varepsilon} F\Big(z_1, ..., z_{N-2}, (\text{sgn}\, x_N)\,\varepsilon, x_N\Big) + \left(1 - \frac{|x_{N-1}|}{\varepsilon}\right) F(z_1, ..., z_{N-2}, 0, x_N) \\
\leq \; & a.
\end{aligned}$$

Repeating the process with respect to all the variables for $x_i \in \{-\varepsilon, 0, \varepsilon\}$ gives us (2.2.6) and hence concluding Step 1.

Step 2. Next, we show that if $x \in \text{int}\,(\text{dom}\, F)$, then F is continuous at x. Once again, there is no loss of generality if we assume that $x = 0$ and $F(0) = 0$. By the previous step, we have shown that F is bounded above in a neighbourhood of $x = 0$, hence there exist $\lambda > 0$ and $a > 0$ such that

$$\|x\|_\infty \leq \lambda \quad \Rightarrow \quad F(x) \leq a. \tag{2.2.7}$$

Now fix $\varepsilon > 0$ and there is no loss of generality in assuming that $\varepsilon \leq aN2^N$. We would like to show that

$$\|x\|_\infty \leq \frac{\varepsilon}{aN2^N}\lambda \quad \Rightarrow \quad |F(x)| \leq \varepsilon. \tag{2.2.8}$$

Let

$$\delta := \frac{\varepsilon}{aN2^N} \leq 1.$$

Since F is separately convex, we obtain

$$\begin{aligned} F(x) = F(x_1, ..., x_N) &= F\left(\delta(\frac{x_1}{\delta}, x_2, ..., x_N) + (1-\delta)(0, x_2, ..., x_N)\right) \\ &\leq \delta F\left(\frac{x_1}{\delta}, x_2, ..., x_N\right) + (1-\delta)F(0, x_2, ..., x_N). \end{aligned}$$

Repeat the process with the second variable gives us

$$\begin{aligned} F(x) &\leq \delta F\left(\frac{x_1}{\delta}, x_2, ..., x_N\right) + (1-\delta)F(0, x_2, ..., x_N) \\ &= \delta F\left(\frac{x_1}{\delta}, x_2, ..., x_N\right) + (1-\delta)F\left(\delta(0, \frac{x_2}{\delta}, x_3, ..., x_N) + (1-\delta)(0, 0, x_3, ..., x_N)\right) \\ &\leq \delta F\left(\frac{x_1}{\delta}, x_2, ..., x_N\right) + (1-\delta)\delta F\left(0, \frac{x_2}{\delta}, x_3, ..., x_N\right) + (1-\delta)^2 F(0, 0, x_3, ..., x_N). \end{aligned}$$

Iterating the process with each of the variable gives us

$$F(x) \leq \delta \sum_{i=1}^{N} (1-\delta)^{i-1} F\left(0, ..., 0, \frac{x_i}{\delta}, x_{i+1}, ..., x_N\right) + (1-\delta)^N F(0, ..., 0).$$

Assuming that

$$\|x\|_\infty \leq \delta\lambda = \frac{\varepsilon}{aN2^N}\lambda \leq \lambda,$$

from (2.2.7) and since $F(0) = 0$, we obtain

$$F(x) \leq \delta a \sum_{i=1}^{N} (1-\delta)^{i-1} \leq \delta a N \leq \varepsilon.$$

Hence, we infer that

$$\|x\|_\infty \leq \frac{\varepsilon}{aN2^N}\lambda \quad \Rightarrow \quad F(x) \leq \varepsilon.$$

What is left to be shown is that

$$\|x\|_\infty \leq \frac{\varepsilon}{aN2^N}\lambda \quad \Rightarrow \quad F(x) \geq -\varepsilon.$$

By convexity of F with respect to the last variable, we have

$$\begin{aligned} 0 = F(0, ..., 0) &= F\left(\frac{1}{1+\delta}(0, ..., 0, x_N) + \frac{\delta}{1+\delta}(0, ..., 0, -\frac{x_N}{\delta})\right) \\ &\leq \frac{1}{1+\delta}F(0, ..., 0, x_N) + \frac{\delta}{1+\delta}F\left(0, ..., 0, -\frac{x_N}{\delta}\right). \end{aligned}$$

Following from the inequality above, we proceed similarly with variable x_{N-1}:

$$\begin{aligned} F(0, ..., 0, x_N) &= F\left(\frac{1}{1+\delta}(0, ..., 0, x_{N-1}, x_N) + \frac{\delta}{1+\delta}(0, ..., 0, -\frac{x_{N-1}}{\delta}, x_N)\right) \\ &\leq \frac{1}{1+\delta}F(0, ..., 0, x_{N-1}, x_N) + \frac{\delta}{1+\delta}F\left(0, ..., 0, -\frac{x_{N-1}}{\delta}, x_N\right). \end{aligned}$$

Substituting this into the previous estimate, we obtain

$$\begin{aligned} 0 \quad &\leq \quad \frac{1}{(1+\delta)^2}F(0,...,0,x_{N-1},x_N) + \frac{\delta}{(1+\delta)^2}F\left(0,...,0,-\frac{x_{N-1}}{\delta},x_N\right) \\ &\quad + \frac{\delta}{1+\delta}F\left(0,...,0,-\frac{x_N}{\delta}\right). \end{aligned}$$

Iterating the process as above, we deduce that

$$0 \leq \frac{1}{(1+\delta)^N}F(x_1,...,x_N) + \sum_{i=1}^{N}\frac{\delta}{(1+\delta)^{N-i+1}}F\left(0,...,0,-\frac{x_i}{\delta},x_{i+1},...,x_N\right)$$

and hence if we assume that

$$\|x\|_\infty \leq \delta\lambda = \frac{\varepsilon}{aN2^N}\lambda \leq \lambda,$$

and from (2.2.7), we have

$$\begin{aligned} F(x_1,...,x_N) \quad &\geq \quad -\delta\sum_{i=1}^{N}(1-\delta)^{i-1}F\left(0,...,0,-\frac{x_i}{\delta},x_{i+1},...,x_N\right) \\ &\geq \quad -\delta a\sum_{i=1}^{N}(1-\delta)^{i-1} \\ &\geq \quad -\delta aN2^N = -\varepsilon, \end{aligned}$$

as required. Thus it has been proven that (2.2.8) holds and that F is continuous at $x = 0$.

Step 3. Lastly, we will show that F is locally Lipschitz on int (dom F). Let $x \in$ int (dom F) and by continuity of F at x, there exist $\alpha, \beta > 0$ such that

$$\|y - x\|_\infty \leq 2\beta \quad \Rightarrow \quad |F(y)| \leq \alpha < \infty. \tag{2.2.9}$$

Letting z and z_1 be such that

$$\|z_1 - z\|_\infty, \|z_1 - x\|_\infty \leq \beta, \tag{2.2.10}$$

we can see that (2.2.10) implies that $\|z - x\|_\infty \leq 2\beta$. Combining this with (2.2.9) and (2.2.10), we find that

$$\|z_1 - z\|_\infty, \|z_1 - x\|_\infty \leq \beta \quad \Rightarrow \quad F(z) - F(z_1) \leq 2\alpha. \tag{2.2.11}$$

Let $\varepsilon > 0$. By (2.2.7), (2.2.8) and (2.2.11), we deduce that

$$\|z_1 - z\|_\infty, \|z_1 - x\|_\infty \leq \frac{\beta\varepsilon}{2\alpha N2^N} \quad \Rightarrow \quad |F(z) - F(z_1)| \leq \varepsilon. \tag{2.2.12}$$

Choosing

$$\varepsilon := \frac{2\alpha N2^N}{\beta}\|z_1 - z\|_\infty,$$

we infer from (2.2.12) that

$$\|z_1 - z\|_\infty, \|z_1 - x\|_\infty \le \beta \quad \Rightarrow \quad |F(z) - F(z_1)| \le \frac{2\alpha N 2^N}{\beta} \|z_1 - z\|_\infty. \tag{2.2.13}$$

In a similar manner, let z_2 be such that $\|z_2 - x\|_\infty \le \beta$. Denoting by $[z_1, z_2]$ the segment in $\mathbb{R}^N$ with endpoints z_1 and z_2, let

$$u_1, ..., u_M \in [z_1, z_2]$$

be such that

$$u_1 = z_1, u_2, ..., u_m = z_2 \text{ and } |u_m - u_{m+1}|_\infty \le \beta, \;\; m = 1, ..., M-1.$$

By (2.2.10), we can find that

$$\|u_m - x\|_\infty \le 2\beta, \; m = 1, ..., M.$$

Hence using (2.2.13), we obtain

$$\|u_m - u_{m+1}\|_\infty \le \beta \quad \Rightarrow \quad |F(u_m) - F(u_{m+1})| \le \frac{2\alpha N 2^N}{\beta} \|u_m - u_{m+1}\|_\infty. \tag{2.2.14}$$

Summing the inequalities in (2.2.14) for $m = 1, ..., M$, we immediately get

$$\|z_1 - x\|_\infty, \|z_2 - x\|_\infty \le \beta \quad \Rightarrow \quad |F(z_1) - F(z_2)| \le \frac{2\alpha N 2^N}{\beta} \|z_1 - z_2\|_\infty$$

and whence the result. □

Ball-Kirchheim-Kristensen [14] gave a variant of the result in which the Lipschitz constant of separately convex functions can be estimated explicitly:

Lemma 2.2.17 (Ball-Kirchheim-Kristensen [14], Lemma 2.2)**.** *Let $B(\xi_0, r)$ be an open ball with centre at ξ_0 and radius r and let $F\colon B(\xi_0, 2r) \to \mathbb{R}$ be a separately convex function. Then*

$$\operatorname{lip}\big(F; B(\xi_0, r)\big) \le \sqrt{Nn} \frac{\operatorname{osc}\big(F; B(\xi_0, 2r)\big)}{r} \tag{2.2.15}$$

where for a set S,

$$\operatorname{lip}(F; S) = \sup_{\xi,\eta \in S} \frac{|F(\xi) - F(\eta)|}{|\xi - \eta|}$$

and

$$\operatorname{osc}(F; S) := \sup \{|F(\xi) - F(\eta)| \colon \xi, \eta \in S)\}.$$

In particular, a separately convex function is locally Lipschitz on int (dom F).

A better Lipschitz constant applies if F is rank one convex in which $\sqrt{\min\{N,n\}}$ takes the place of $\sqrt{Nn}$ in (2.2.15) (see Ball-Kirchheim-Kristensen [14], Remark on page 6). In fact if F is convex, the estimate is even sharper since it does not even depend on the dimension (see Fonseca-Leoni [49], Remark 4.37).

As a consequence of the previous theorem, we can even obtain a global Lipschitz condition for separately convex functions satisfying growth condition (2.0.2) - see Fusco [50], Marcellini [90] and Morrey [103].

Proposition 2.2.18 (Dacorogna [29], Proposition 2.32)**.** *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a separately convex function satisfying*

$$|F(\xi)| \leq c\left(|\xi|^p + 1\right)$$

for every $\xi \in \mathbb{R}^{N\times n}$, where constant $c > 0$ and $p \geq 1$. Then there exists $\tilde{c} \geq 0$ such that for every $\xi, \eta \in \mathbb{R}^{N\times n}$,

$$|F(\xi) - F(\eta)| \leq \tilde{c}\left(|\xi|^{p-1} + |\eta|^{p-1} + 1\right)|\xi - \eta|. \tag{2.2.16}$$

Proof. Assume that $n = 1$. The proof is divided into 3 steps.

Step 1. We prove that for every $\lambda > \mu > 0$ and $t \in \mathbb{R}$, every convex function $G \in \mathbb{R} \to \mathbb{R}$ satisfies

$$\frac{G(t\pm\mu) - G(t)}{\mu} \leq \frac{G(t\pm\lambda) - G(t)}{\lambda}. \tag{2.2.17}$$

Note that since G is convex, we may write

$$G(t\pm\mu) = G\left(\frac{\mu}{\lambda}(t\pm\lambda) + \left(1 - \frac{\mu}{\lambda}\right)t\right) \leq \frac{\mu}{\lambda}G(t\pm\lambda) + \left(1 - \frac{\mu}{\lambda}\right)G(t)$$

and hence the required (2.2.17).

Step 2. By fixing $\hat{\xi}_1 = (\xi_2, \ldots, \xi_N) \in \mathbb{R}^{N-1}$ and defining for $t \in \mathbb{R}$

$$G(t) := F(t, \hat{\xi}_1), \tag{2.2.18}$$

we show that there exists $c_1 > 0$ such that

$$|G(\xi_1) - G(\eta_1)| \leq c_1\left(|\xi|^{p-1} + |\eta|^{p-1} + 1\right)|\xi_1 - \eta_1|. \tag{2.2.19}$$

Without loss of generality, for $\xi_1, \eta_1 \in \mathbb{R}$, let $\xi_1 < \eta_1$. Take λ and μ as in Step 1 and let

$$\lambda = |\xi| + |\eta| + 1 \quad \text{and} \quad \mu = |\xi_1| - |\eta_1|.$$

Now, we have

$$G(\eta_1) - G(\xi_1) = G\big(\xi_1 + (\eta_1 - \xi_1)\big) - G(\xi_1) \le (\eta_1 - \xi_1)\frac{G(\xi_1 + |\xi| + |\eta| + 1) - G(\xi_1)}{|\xi| + |\eta| + 1}$$

and

$$G(\xi_1) - G(\eta_1) = G\big(\eta_1 - (\eta_1 - \xi_1)\big) - G(\eta_1) \le (\eta_1 - \xi_1)\frac{G\big(\eta_1 - (|\xi| + |\eta| + 1)\big) - G(\eta_1)}{|\xi| + |\eta| + 1}.$$

By writing both of the above as functions of F as in (2.2.18) and then applying the growth condition, we conclude that (2.2.19) holds.

Step 3. We show that the result holds for the function F. We first write

$$\begin{aligned} F(\xi) - F(\eta) \quad = \quad & F(\xi_1, \xi_2, \ldots, \xi_N) - F(\eta_1, \xi_2, \ldots, \xi_N) \\ & + \sum_{i=1}^{N-2} [F(\eta_1, \ldots, \eta_i, \xi_{i+1}, \ldots, \xi_N) - F(\eta_1, \ldots, \eta_i, \eta_{i+1}, \xi_{i+2}, \ldots, \xi_N)] \\ & + [F(\eta_1, \ldots, \eta_{N-1}, \xi_N) - F(\eta_1, \ldots, \eta_{N-1}, \eta_N)] \end{aligned}$$

and we may apply (2.2.19) to each term to obtain the required inequality (2.2.16). □

One can easily see that when F is assumed to be continuous, then we can replace the set of piecewise affine test functions in $W_0^{1,\infty}$ by test functions in C_0^∞ or in $W_0^{1,\infty}$ in the definition of quasiconvexity. In fact, the definition is also true for test functions in $\varphi \in W_0^{1,p}$, provided that the continuous integrand F satisties the growth condition (2.0.2).

Lemma 2.2.19. *If $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is quasiconvex and satisfies the growth condition (2.0.2) for all $\xi \in \mathbb{R}^{N\times n}$, then (2.2.1) holds for all $\xi \in \mathbb{R}^{N\times n}$ and for all $\varphi \in W_0^{1,p}(\Omega; \mathbb{R}^N)$.*

Proof. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $\xi \in \mathbb{R}^{N\times n}$ and $\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^N)$. Let F be a quasiconvex function and recall that $W_0^{1,p}$ by definition is the closure of C_0^∞ in $W^{1,p}$. Then there exists $\varphi_j \in C_0^\infty(\Omega;\mathbb{R}^N)$ such that $\varphi_j \to \varphi$ in $W^{1,p}(\Omega;\mathbb{R}^N)$. We use Vitali's convergence theorem. Firstly, note that since $D\varphi_j \to D\varphi$ in measure and F is locally Lipschitz, we have $F(\xi + D\varphi_j) \to F(\xi + D\varphi)$ in measure. Since F satisfies the growth condition for all $\xi \in \mathbb{R}^{N\times n}$, we write

$$|F(\xi + D\varphi_j)| \le c(|\xi + D\varphi_j|^p + 1) \le \tilde{c}(|\xi|^p + |D\varphi_j|^p + 1)$$

where constants $c, \tilde{c} > 0$. Since $D\varphi_j$ is p-equiintegrable, so is $F(\xi + D\varphi_j)$. The result follows from the quasiconvexity of F:

$$\int_\Omega F\big(\xi + D\varphi(x)\big)\,\mathrm{d}x = \lim_{j\to\infty}\int_\Omega F\big(\xi + D\varphi_j(x)\big)\,\mathrm{d}x \ge F(\xi)\,\mathrm{meas}\,\Omega.$$

□

Remark 2.2.20. *Since a quasiconvex function $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ by Theorem 2.2.16 is locally Lipschitz, it is possible to remove the condition on the boundary of Ω in the Jensen's inequality (see Proposition 2.2.3). More precisely, let $\Omega \subset \mathbb{R}^n$ be a bounded and open set (possibly with $\operatorname{meas}(\partial\Omega) > 0$). Then when $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is quasiconvex, we have for all $\xi \in \mathbb{R}^{N\times n}$ that*

$$\fint_{\Omega} F\big(\xi + D\varphi(x)\big)\,\mathrm{d}x \geq F(\xi) \tag{2.2.20}$$

holds for all $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^N)$.

Indeed fix $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^N)$. Then by definition of $W_0^{1,\infty}$, we can find a sequence $\{\varphi_j\}$ in $C_0^\infty(\Omega;\mathbb{R}^N)$ such that $\varphi_j \to \varphi$ in $W^{1,1}(\Omega;\mathbb{R}^N)$ and $\sup_j \|D\varphi_j\|_{L^\infty} < \infty$. By Theorem 2.2.16 and the dominated convergence theorem, we have

$$\lim_{j\to\infty} \fint_{\Omega} F\big(\xi + D\varphi_j(x)\big)\,\mathrm{d}x = \fint_{\Omega} F\big(\xi + D\varphi(x)\big)\,\mathrm{d}x.$$

Consider $\varphi_j \in C_0^\infty(\Omega;\mathbb{R}^N)$. We may take $\Omega_j \Subset \Omega$ such that $\operatorname{supp}\varphi_j \subset \Omega_j$, $\operatorname{meas}(\partial\Omega_j) = 0$ and $\operatorname{meas}(\Omega \setminus \bar{\Omega}_j) < \frac{1}{j}$. Since φ_j can be approximated in the $W^{1,\infty}$-sense by piecewise affine functions (see for instance Appendix A), it follows by quasiconvexity of F at ξ and a straightforward application of the dominated convergence theorem and Theorem 2.2.16 (as above) that also

$$\fint_{\Omega_j} F\big(\xi + D\varphi_j(x)\big)\,\mathrm{d}x \geq F(\xi).$$

We easily deduce that (2.2.20) from this.

Furthermore, if F is separately convex and bounded from above, then it is also bounded from below, as one can see from the lemma below:

Lemma 2.2.21 (Kristensen [83], Lemma 2.5)**.** *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be separately convex and satisfy*

$$\limsup_{\xi\to\infty} \frac{F(\xi)}{|\xi|^p} < \infty$$

for some $p \in [1,\infty)$. Then

$$\limsup_{\xi\to\infty} \frac{|F(\xi)|}{|\xi|^p} < \infty.$$

2.3 Weak lower semicontinuity and existence theorems

As mentioned earlier, convexity and quasiconvexity play a key role in the semicontinuity problem in the scalar and vectorial cases, respectively. The connection between convexity and lower semicontinuity was first established by Tonelli [122] under the assumption that F is a function of single variable and twice differentiable. Many other authors have since proved the theorem under less restrictive assumptions. The result of Serrin [112] which managed to drop the condition $F \in C^2$ for the first time has been made into a model in semicontinuity problem by other authors. One of the most notable results in the scalar case is due to Marcellini-Sbordone [91] which shows that convexity is a necessary and sufficient condition for $\mathcal{F}$ to be sequentially weakly lower semicontinuous.

Theorem 2.3.1 (Marcellini-Sbordone [91])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $F\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a non-negative Carathéodory integrand. Then the functional $\mathcal{F}$ is sequentially weakly lower semicontinuous in $W^{1,p}(\Omega)$, $p \in [1, \infty]$, if and only if F is convex in ξ.*

Note that the functionals $\mathcal{F}$ will only in very special cases be sequentially lower continuous. Fortunately as noted in an earlier discussion, it suffices to have only sequential weak lower semicontinuity of $\mathcal{F}$ to ensure the existence of a minimizer. The sequential weak lower semicontinuity of $\mathcal{F}$ is closely related to convexity properties of the integrand F.

The lower semicontinuity result in the vectorial case was first established by Morrey [102]: under strong regularity assumptions, the functional $\mathcal{F}$ is sequentially weakly* lower semicontinuous on $W^{1,\infty}(\Omega; \mathbb{R}^N)$ if and only if the integrand F is quasiconvex. Meyers [99] extended Morrey's result to the semicontinuity on $W^{k,\infty}(\Omega; \mathbb{R}^N)$. Acerbi-Fusco [1] proved an equivalent result of Theorem 2.3.1:

Theorem 2.3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. If $F\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is a Carathéodory integrand satisfying*

$$0 \leq F(x, u, \xi) \leq a(x) + c(|u|^p + |\xi|^p)$$

where $p \geq 1$, $a \in L^1(\Omega)^+$ and constant $c > 0$, then the functional $\mathcal{F}$ is sequentially weakly lower semicontinuous in $W^{1,p}(\Omega; \mathbb{R}^N)$ if and only if $\xi \mapsto F(x, u, \xi)$ is quasiconvex in ξ for almost every $x \in \Omega$ and for every $u \in \mathbb{R}^N$.

The result was further refined by Marcellini [90] with a slightly less restrictive coercivity and growth conditions

$$-c_1|\xi|^r - c_2|u|^q - c_3(x) \leq F(x, u, \xi) \leq G(x, u)(|\xi|^p + 1) \tag{2.3.1}$$

where constants $c_1, c_2 > 0$, $c_3 \in L^1(\Omega)$ and $G \geq 0$ is a Carathéodory function, and exponents $p \geq 1$, $1 \leq r < p$ ($r = 1$ if $p = 1$) and $1 \leq q < \frac{np}{n-p}$ ($r \geq 1$ if $p \geq n$).

Summing up the results from above, we conclude that if $F\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is a quasiconvex function satisfying (2.3.1), then the problem (P) admits a minimizer $u \in W^{1,p}_g(\Omega; \mathbb{R}^N)$.

Chapter 3

Relaxation in the Calculus of Variations

In this chapter, we discuss the problem of existence of a minimizer for our problem (P) when F is not necessarily convex (in scalar case) or quasiconvex (in vectorial case). This kind of problem has received much attention leading to remarkable developments in recent years, partly motivated by advances in materials science. The study of equilibrium in the materials science problems such as microstructures and phase transition within certain alloys (Ball-James [12], [13]) leads to the search for minimizers of the energy $\mathcal{F}$ but the fact that F is not quasiconvex may prevent the existence of classical solution. In view of this, a good understanding of the existence of minimizers for integrals with non-quasiconvex integrands is essential. Other contemporary issues considered in the study of materials include optimal design for thin films (Fonseca-Francfort [47]) and nucleation (Fonseca-Leoni [48]).

We will see that though the infimum in the variational problem (P) is not necessarily attained in a classical set up (since F is not quasiconvex), we can still prove an existence theorem, with some extra assumptions on F, using the so-called relaxed variational problem

$$\inf\left\{\mathcal{F}^{qc}(u,\Omega)=\int_\Omega F^{qc}\big(x,u(x),Du(x)\big)\,\mathrm{d}x\colon\, u-g\in W_0^{1,p}(\Omega;\mathbb{R}^N)\right\}, \tag{P^{qc}}$$

where F^{qc} is the quasiconvex envelope of $F\big(x,u(x),\cdot\big)$. In the scalar case, we consider the corresponding convex envelope F^c and variational problem (P^c). In particular, we shall see that for every minimizer $\bar{u}$ of (P^{qc}), there exists a minimizing sequence $\{u_j\}_{j=1}^\infty$ of (P) such that

$$\begin{cases} u_j \rightharpoonup \bar{u} \text{ in } W_0^{1,p}(\Omega;\mathbb{R}^N)\\ \mathcal{F}(u_j,\Omega)\to\mathcal{F}^{qc}(\bar{u},\Omega)\end{cases}$$

which gives us

$$\inf(\mathcal{F})=\inf(\mathcal{F}^{qc}).$$

In the scalar case where $\mathcal{F}^{qc} = \mathcal{F}^{c}$, we have

$$\inf(\mathcal{F}) = \inf(\mathcal{F}^{qc}) = \inf(\mathcal{F}^{c}) = \inf(\mathcal{F}^{**}).$$

This result is called the *relaxation theorem*.

This chapter is divided into two main sections. The first section includes a brief history of the relaxation theorem, for which convex and quasiconvex envelopes are heavily used. It is, therefore, essential to discuss these envelopes, and we do that in the subsections that follow. We even give an example where the envelopes can be calculated explicitly, namely the so-called Kohn-Strang relaxation formula [77]. The paper [77] concerned the relaxation of a problem in optimal design.

The second section is a rather 'self-contained' study of Young measure based in part on the work of Sychev [115]. The use of Young measures in this context is motivated by its advantage in giving a direct proof to the the relaxation theorem and also for a lower semicontinuity theorem where the assumptions on F (compared to Acerbi-Fusco [1]) can be weakened.

3.1 Relaxation of non-quasiconvex integrands

For a given function $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}\cup\{\infty\}$ we consider the convex F^c, polyconvex F^{pc}, quasiconvex F^{qc} and rank one convex F^{rc} envelopes, which are defined as the largest convex, polyconvex, quasiconvex and rank one convex functions below F, respectively:

$$\begin{aligned}
F^{c}(\xi) &= \sup\left\{G(\xi)\colon G \leq F \text{ and } G \text{ convex}\right\},\\
F^{pc}(\xi) &= \sup\left\{G(\xi)\colon G \leq F \text{ and } G \text{ polyconvex}\right\},\\
F^{qc}(\xi) &= \sup\left\{G(\xi)\colon G \leq F \text{ and } G \text{ quasiconvex}\right\},\\
F^{rc}(\xi) &= \sup\left\{G(\xi)\colon G \leq F \text{ and } G \text{ rank one convex}\right\},
\end{aligned}$$

for each $\xi \in \mathbb{R}^{N\times n}$. If there is no such G for the definitions above, then we invoke the usual convention that $\sup\emptyset = -\infty$.

One important property of the definitions above is that the envelopes of a real-valued F are either also real-valued *for all* $\xi \in \mathbb{R}^{N\times n}$ or equal to $-\infty$ *for all* $\xi \in \mathbb{R}^{N\times n}$. Take the definition of quasiconvex envelope F^{qc} for example, where the supremum is taken over the set of functions $G\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ that are quasiconvex and satisfy $G(\xi) \leq F(\xi)$ for all $\xi \in \mathbb{R}^{N\times n}$. If there is no such function, we use the convention that $\sup\emptyset = -\infty$ and conclude that in this instance $F^{qc}(\xi) = -\infty$ for all $\xi \in \mathbb{R}^{N\times n}$. Next, assume that $F^{qc}(\xi_0) > -\infty$ for some $\xi_0 \in \mathbb{R}^{N\times n}$. By the definition of

F^{qc}, we have

$$F^{qc}(\xi_0) = \sup\{G(\xi_0) \colon G \le F \text{ and } G \text{ quasiconvex}\} > -\infty,$$

so there must exist a real-valued and quasiconvex G_0 such that $G_0 \le F$. Taking G_0 into the definition of F^{qc}, we have

$$F^{qc}(\xi) \ge G_0(\xi) > -\infty$$

for all $\xi \in \mathbb{R}^{N\times n}$. Recall that for real-valued functions G,

$$G \text{ convex} \;\Rightarrow\; G \text{ polyconvex} \;\Rightarrow\; G \text{ quasiconvex} \;\Rightarrow\; G \text{ rank one convex},$$

and consequently for real-valued functions F,

$$F^c \le F^{pc} \le F^{qc} \le F^{rc}. \tag{3.1.1}$$

Note that all the envelopes coincide with the convex envelope in the scalar case, since all matrices in $\mathbb{R}^{N\times n}$ have rank at most 1 when $\min\{N, n\} = 1$.

The first systematic development of relaxation theory is due to Bogolubov [19] in the one-dimensional case, in which he suggested an explicit construction of a lower semicontinuous extension for the functional $\mathcal{F}$. When $\mathcal{F}$ is coercive on $W^{1,p}$, then the extension is indeed the lower semicontinuous envelope of $\mathcal{F}$ and is given by the formula

$$\bar{\mathcal{F}}(u,\Omega) = \inf_{\{u_j\}} \left\{ \liminf_{j\to\infty} \mathcal{F}(u_j,\Omega) \colon u_j \rightharpoonup u \text{ in } W^{1,p}(\Omega;\mathbb{R}^N) \right\}.$$

Note that the definition of lower semicontinuous envelope is as follows:

$$\bar{\mathcal{F}}(u,\Omega) = \sup\left\{ \mathcal{G}(u,\Omega) \colon \mathcal{G} \le \mathcal{F} \text{ and } G \text{ weakly lower semicontinuous on } W^{1,p}(\Omega;\mathbb{R}^N) \right\}.$$

Bogolubov showed that $\bar{\mathcal{F}}$ is also an integral functional with the integrand F^c, the convexification of the integrand F in the gradient variable:

$$\bar{\mathcal{F}}(u,\Omega) = \int_\Omega F^c\big(x, u(x), Du(x)\big)\,\mathrm{d}x. \tag{3.1.2}$$

Thus by Carathéodory's theorem, we may write

$$F^c(x,u,\cdot) = \inf\left\{ \sum_{i=1}^{N+1} \alpha_i F(x,u,\xi_i) \colon \sum_{i=1}^{N+1} \alpha_i\xi_i = \xi_i, \alpha_i \ge 0 \text{ with } \sum_{i=1}^{N+1} \alpha_i = 1 \right\} \tag{3.1.3}$$

for all x and u when F has superlinear growth

$$c_1|\xi|^p - c_2 \le F(x,u,\xi) \le \frac{1}{c_1}|\xi|^p + c_2, \tag{3.1.4}$$

where $c_1 \in (0,1)$, $c_2 > 0$ and $p \in (1,\infty)$.

The original problem (P) can now be solved by searching for a solution u to the relaxed problem (P^c) and then check that

$$F\big(x,u(x),Du(x)\big) = F^c\big(x,u(x),Du(x)\big)$$

holds a.e. in Ω. Of course if we have the *weak-strong convergence* property at a minimizer for the relaxed problem, i.e. if

$$u_j \rightharpoonup u \text{ in } W^{1,p}(\Omega;\mathbb{R}^N),\ \mathcal{F}(u_j,\Omega) \to \mathcal{F}(u,\Omega) \quad \Rightarrow \quad u_j \to u \text{ in } W^{1,p}(\Omega;\mathbb{R}^N),$$

then that minimizer is also a minimizer for the original problem.

Ekeland-Temam [40] proved the integral representation formula (3.1.2) for integrands satisfying p-growth in the scalar case. With the introduction of quasiconvexity and its rising importance in vector-valued calculus of variations, there was a need to obtain the result also for non-quasiconvex functionals. In this connection, Dacorogna [25] showed that the result remains valid for quasiconvexification of $F = F(Du)$. Later on, Acerbi-Fusco [1] generalized the result to $F = F\big(x,u(x),Du(x)\big)$ with an added coercivity condition on the integrand F. Finally Sychev [115] showed, by the use of Young measures, that the relaxation theorem can be achieved for Carathéodory integrands F satisfying some coercivity and growth conditions.

The study of the relaxation theorem will not be complete without a discussion on convex and quasiconvex envelopes.

Relaxation in the scalar case: convex envelope

As usual, we always begin with convex envelope. We have seen a characterization of convex envelope by the use of Carathéodory's theorem.

Another approach to the characterization of convex envelope is by Ekeland-Temam [41], in which they associated the study of variational problems with duality (Fenchel [45], Moreau [101]). Duality plays a central role in convex analysis and this association enables us to define the generalized solution of a variational problem that has no classical solution. This result was established by Ekeland-Temam [40] and Marcellini-Sbordone [91]. A detailed discussion on relaxation in the scalar case can be found in the book of Ekeland-Temam [41], and we shall be brief since the relaxation theorem for non-quasiconvex case covers the non-convex case. In fact all the envelopes coincide with the convex envelope when $\min\{N,n\} = 1$. Henceforth we will assume that $n = 1$

throughout this subsection. Let us first recall some of the basic properties of convex analysis from [41].

We will give a brief account on how duality works in convex analysis, starting with the definition of dual and bidual functions.

Definition 3.1.1. *Let $\langle \cdot ; \cdot \rangle$ denote the usual scalar product in $\mathbb{R}^N$ and $F \colon \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ with $F \not\equiv +\infty$.*

(i) The dual function of F is the function $F^* \colon \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ defined by

$$F^*(x^*) := \sup_{x \in \mathbb{R}^N} \Big\{ \langle x; x^* \rangle - F(x) \Big\}.$$

(ii) The bidual function of F is the function $F^{**} \colon \mathbb{R}^N \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$F^{**}(x) := \sup_{x^* \in \mathbb{R}^N} \Big\{ \langle x, x^* \rangle - F^*(x^*) \Big\}.$$

We will see later that, under certain condition,

$$F^{**} = F^c = \bar{F},$$

where $\bar{F}$ is the lower semicontinuous envelope of F defined in a similar way as $\bar{\mathcal{F}}$, and we may write

$$\bar{F}(x) = \inf_{\{x_j\}} \Big\{ \liminf_{x_j \to x} F(x_j) \Big\}$$

and observe that this is possible whenever the underlying topological space (here $\mathbb{R}^N$) is first countable.

Note that, even if $F \not\equiv -\infty$, the bidual F^{**} and convex envelope F^c may still take the value $-\infty$. Moreover, if $F \equiv +\infty$, then $F^* = -\infty$ and $F^{**} \equiv +\infty$.

The next theorem shows how bidual function F^{**}, convex envelope F^c and lower semicontinuous envelope $\bar{F}$ are related to each other. This is a summary found in Dacorogna [29] (see Theorem 2.43), and they are standard results going back to Fenchel [45] and Moreau [101].

Theorem 3.1.2. *Let $F \colon \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$. Then the following statements hold:*

(i) F^* is convex and lower semicontinuous.

(ii) If F is convex and lower semicontinuous, then $F^* \not\equiv +\infty$.

(iii) In general, we have

$$F^{**} \leq F^{c} \leq F.$$

In addition, if F^c is lower semicontinuous and $F^c > -\infty$, then

$$F^{**} = F^{c},$$

and if F is convex and lower semicontinuous, then

$$F^{**} = F^{c} = F.$$

(iv) The identity $F^{***} = F^{*}$ is always valid.

We can clearly see that F^{**} is also a convex and lower semicontinuous envelope of F, and hence the importance of duality in the existence theorem for non-convex problem. Later we will also give an example in which, for a particular function, we can even obtain an explicit formula for convex and quasiconvex envelopes.

Relaxation in the vectorial case: quasiconvex envelope

The following theorem is a well-known representation formula for the quasiconvex envelope due to Dacorogna [25] (see Theorem 5). Note that most authors in the literature assume a lower bound for the function F, hence in such cases the quasiconvex envelope is always real-valued. Here, we provide the proof of the very same theorem, with the exception that F is allowed to be unbounded from below. The essence of the proof follows from the proof of Theorem 6.9 by Dacorogna [29]. It is worth noting that Kinderlehrer-Pedegral [70] (see Appendix there) gave a proof of a similar quasiconvexification formula for F continuous.

Theorem 3.1.3 (Dacorogna's relaxation formula). *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a function. Recall that the quasiconvex envelope of F is given as*

$$F^{qc}(\xi) = \sup \left\{ G(\xi) \colon G \leq F \text{ and } G \text{ quasiconvex} \right\}. \tag{3.1.5}$$

Then, for every $\xi \in \mathbb{R}^{N\times n}$, we have

$$F^{qc}(\xi) = \inf \left\{ \fint_{\Omega} F\big(\xi + D\varphi(x)\big)\,\mathrm{d}x \colon \varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^N) \text{ piecewise affine} \right\} \tag{3.1.6}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set with $\operatorname{meas}(\partial\Omega) = 0$ and it is not excluded that both sides take on the value $-\infty$. In addition, the infimum in the formula is independent of the particular choice of the set Ω.

Proof. Note that by (3.1.5), we have either

$$F^{qc}(\xi) > -\infty, \quad \forall \xi \in \mathbb{R}^{N\times n}$$

or

$$F^{qc}(\xi) = -\infty, \quad \forall \xi \in \mathbb{R}^{N\times n}.$$

Since F is unbounded from below, then the lower semicontinuous envelope $\bar{F}\colon \mathbb{R}^{N\times n} \to \mathbb{R}\cup\{-\infty\}$ is given by

$$\bar{F}(\xi) = \liminf_{\eta\to\xi} F(\eta).$$

Then $\bar{F}$ is lower semicontinuous and if $\bar{F} = -\infty$ for some $\xi \in \mathbb{R}^{N\times n}$, then it is also true that $F^{qc}(\xi) = -\infty$ and hence $F^{qc}(\xi) = -\infty$ for all $\xi \in \mathbb{R}^{N\times n}$.

When $\xi \in \mathbb{R}^{N\times n}$ and $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^N)$, then by our assumptions, $x \mapsto F\big(\xi + D\varphi(x)\big)$ is bounded and measurable, hence in particular integrable and the right hand side of (3.1.6) is well-defined. Next, denote for $\Omega \subset \mathbb{R}^n$ a bounded open set

$$\tilde{F}^{qc}_{\Omega}(\xi) := \inf\left\{ ⨍_{\Omega} F\big(\xi + D\varphi(x)\big)\,\mathrm{d}x \colon \varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^N)\right\},$$

which is precisely the expression given on the right hand side of (3.1.6). The proof is divided into three steps with the first two aimed at showing that $\tilde{F}^{qc}_{\Omega}$ is quasiconvex and the last step involves showing the equivalence of F^{qc} and $\tilde{F}^{qc}_{\Omega}$.

Step 1. We aim to show that the definition $\tilde{F}^{qc}_{\Omega}$ is independent of the choice of Ω; i.e. given two such sets Ω_1 and Ω_2 with $\operatorname{meas}(\partial\Omega_1) = \operatorname{meas}(\partial\Omega_2) = 0$, we have

$$\tilde{F}^{qc}_{\Omega_1} = \tilde{F}^{qc}_{\Omega_2}. \tag{3.1.7}$$

We proceed as in the proof of Proposition 2.2.3. Let $\xi \in \mathbb{R}^{N\times n}$ and $\psi \in W_0^{1,\infty}(\Omega_2;\mathbb{R}^N)$ and define $\varphi \in W_0^{1,\infty}(\Omega_1;\mathbb{R}^N)$ by the formula (2.2.2). Then

$$\begin{aligned}
\int_{\Omega_1} F\big(\xi + D\varphi(x)\big)\,\mathrm{d}x &= \sum_{i\in I}\int_{x_i+\lambda_i\Omega_2} F\left(\xi + D\psi\left(\frac{x-x_i}{\lambda_i}\right)\right)\mathrm{d}x\\
&= \sum_{i\in I}\lambda_i^n \int_{\Omega_2} F\big(\xi + D\psi(x)\big)\,\mathrm{d}x\\
&= \frac{\operatorname{meas}\Omega_1}{\operatorname{meas}\Omega_2}\int_{\Omega_2} F\big(\xi + D\psi(x)\big)\,\mathrm{d}x\\
&\geq \tilde{F}^{qc}_{\Omega_2}(\xi)\operatorname{meas}\Omega_1,
\end{aligned}$$

hence $\tilde{F}^{qc}_{\Omega_1}(\xi) \geq \tilde{F}^{qc}_{\Omega_2}(\xi)$. Repeating the same procedure with Ω_1 and Ω_2 swapped gives the reverse inequality and we conclude that the definition $\tilde{F}^{qc}_{\Omega}$ is independent of the choice of Ω. Henceforth we write simply $\tilde{F}^{qc}$ instead of $\tilde{F}^{qc}_{\Omega}$.

Step 2. We wish to show that $\tilde{F}^{qc}$ is quasiconvex by first establishing that

$$\int_{\Omega} \tilde{F}^{qc}\big(\xi + D\varphi(x)\big)\,\mathrm{d}x \geq \tilde{F}^{qc}(\xi)\ \operatorname{meas}\Omega \tag{3.1.8}$$

for every $\xi \in \mathbb{R}^{N\times n}$ and $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^N)$. Firstly, we will prove that either

$$\tilde{F}^{qc}(\xi) > -\infty, \quad \forall \xi \in \mathbb{R}^{N\times n}$$

or

$$\tilde{F}^{qc}(\xi) = -\infty, \quad \forall \xi \in \mathbb{R}^{N\times n}. \tag{3.1.9}$$

The case for $\tilde{F}^{qc}(\xi) > -\infty$ for all $\xi \in \mathbb{R}^{N\times n}$ is obvious. Therefore, we will only prove the second statement (3.1.9). Note that we wish to show that given $\xi_0 \neq \xi$, and if $\tilde{F}^{qc}(\xi_0) = -\infty$, then $\tilde{F}^{qc}(\xi) = -\infty$ for all $\xi \in \mathbb{R}^{N\times n}$. Let $m \in \mathbb{R}$. Since $\tilde{F}^{qc}(\xi_0) = -\infty$, we can take $\varphi = \varphi_m \in W_0^{1,\infty}\left((0,1)^n;\mathbb{R}^N\right)$ such that

$$\int_{(0,1)^n} F\big(\xi_0 + D\varphi(x)\big)\,\mathrm{d}x < m.$$

Now, take $\rho \in W_0^{1,\infty}\big((-1,2)^n;\mathbb{R}^N\big), 0 \leq \rho \leq 1$, such that $\rho = 1$ on $(0,1)^n$ and $\rho = 0$ near $\partial(-1,2)^n$. Putting

$$\psi(x) := \rho(x)\big((\xi_o - \xi)x + \varphi(x)\big), \quad x \in (-1,2)^n,$$

we have $\psi \in W_0^{1,\infty}\left((-1,2)^n;\mathbb{R}^N\right)$ and thus

$$\begin{aligned}
\tilde{F}^{qc}(\xi) &\leq \frac{1}{\operatorname{meas}\big((-1,2)^n\big)} \int_{(-1,2)^n} F\big(\xi + D\psi(x)\big)\,\mathrm{d}x \\
&= 3^{-n}\left[\int_{(0,1)^n} F\big(\xi_0 + D\varphi(x)\big)\,\mathrm{d}x + \int_{(-1,2)^n\setminus(0,1)^n} F\Big(\xi + D\big(\rho(x)(\xi_0 - \xi)x\big)\Big)\,\mathrm{d}x\right] \\
&< 3^{-n}(m + c)
\end{aligned}$$

where c is a constant. As m is arbitrary and c is independent of m, we have $\tilde{F}^{qc}(\xi) = -\infty$ and hence this establishes the alternative (3.1.9).

Now, we are ready to prove the quasiconvexity of $\tilde{F}^{qc}(\xi)$. Since there will be nothing to prove when $\tilde{F}^{qc}(\xi) = -\infty$ for all $\xi \in \mathbb{R}^{N\times n}$, we assume that $\tilde{F}^{qc}(\xi) > -\infty$ for all $\xi \in \mathbb{R}^{N\times n}$. Fix $\xi \in \mathbb{R}^{N\times n}$ and a piecewise affine $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^N)$. Let us assume

$$\Omega = \bigcup_{i=1}^{k} S_i$$

where $S_1, \ldots, S_k$ are non-overlapping simplices intersected with Ω and the restriction of φ to each S_i is affine. Now $x \mapsto \tilde{F}^{qc}\big(\xi + D\varphi(x)\big)$ is a simple measurable function with

$$\int_\Omega \tilde{F}^{qc}\big(\xi + D\varphi(x)\big)\,\mathrm{d}x = \sum_{i=1}^k \tilde{F}^{qc}(\xi + \xi_i)\,\mathrm{meas}\, S_i$$

where $D\varphi(x) = \xi_i$ a.e. on S_i. For a given $\varepsilon > 0$, we find by the definition of $\tilde{F}^{qc}$ for each $i \in \{1, \ldots, k\}$ a $\psi_i \in W_0^{1,\infty}(S_i; \mathbb{R}^N)$ such that

$$\mathrm{meas}\, S_i\big(\tilde{F}^{qc}(\xi + \xi_i) + \varepsilon\big)\,\mathrm{meas}\, S_i > \int_{S_i} F\big(\xi + \xi_i + D\psi_i(x)\big)\,\mathrm{d}x.$$

Extend each ψ_i to $\Omega \setminus S_i$ by zero and note that hereby

$$\psi := \varphi + \sum_{i=1}^k \psi_i \in W_0^{1,\infty}(\Omega; \mathbb{R}^N)$$

with

$$\begin{aligned} \int_\Omega \tilde{F}^{qc}\big(\xi + D\varphi(x)\big)\,\mathrm{d}x &> -\varepsilon\,\mathrm{meas}\,\Omega + \int_\Omega F\big(\xi + D\psi(x)\big)\,\mathrm{d}x \\ &\geq -\varepsilon\,\mathrm{meas}\,\Omega + \tilde{F}^{qc}(\xi)\,\mathrm{meas}\,\Omega. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have established (3.1.8) holds for all piecewise affine functions $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^N)$. Consequently, by Lemma 2.2.6, $\tilde{F}^{qc}$ is a rank one convex function. It then follows from Theorem 2.2.16 that $\tilde{F}^{qc}$ is locally Lipschitz, and so in particular continuous. Because a general $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^N)$ can be approximated by piecewise affine functions (see Theorem A.2.4), we can conclude that $\tilde{F}^{qc}$ is indeed quasiconvex.

Step 3. In this final step, we will deduce that $\tilde{F}^{qc} = F^{qc}$. If G is quasiconvex, then

$$G(\xi) \leq \fint_\Omega G\big(\xi + D\varphi(x)\big)\,\mathrm{d}x$$

for all piecewise affine $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^N)$. Using the definition of $\tilde{G}^{qc}$ on the quasiconvexity of G above gives us $G = \tilde{G}^{qc}$. Therefore, let $G \leq F$ be quasiconvex and so we can deduce that

$$G = \tilde{G}^{qc} \leq \tilde{F}^{qc} \leq F.$$

We now use the definition of quasiconvex envelope and the fact that $\tilde{F}^{qc}$ is quasiconvex to obtain

$$F^{qc} \leq \tilde{F}^{qc} = F^{qc}$$

which establishes the result. $\square$

An example: Kohn-Strang relaxation formula

The Kohn-Strang formula, originally obtained in order to solve an optimal design problem in electrostatics, is a well-known example of an explicit formula for a non-trivial quasiconvex envelope. While Kohn-Strang [77, 78, 79] gave a full account of the connection between relaxation and optimal design, the result was first mapped out by the very same authors in [76]. Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}_+$ be in the following form:

$$F(\xi) = \Phi(|\xi|) = \begin{cases} |\xi|^2 + 1 & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0 \end{cases}, \tag{3.1.10}$$

where $|\cdot|$ denotes the Euclidean norm. The Kohn-Strang function (3.1.10) is not quasiconvex, and hence the existence of minimizer for the corresponding integral functional is not guaranteed. This problem is tackled by the use of Kohn-Strang relaxation formula [77], in which the formula of convex and quasiconvex envelopes can be obtained explicitly:

Theorem 3.1.4 (Kohn-Strang relaxation formula). *Let* $\min\{N, n\} = 2$ *and, for* $\xi \in \mathbb{R}^{N\times 2}$*, the function* $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}_+$ *is given by (3.1.10). If* $\theta\colon \mathbb{R} \to \mathbb{R}_+$ *is such that*

$$\theta(t) := \begin{cases} t^2 + 1 & \text{if } t \geq 1 \\ 2t & \text{if } 0 \leq t \leq 1 \end{cases},$$

then

$$F^c(\xi) = \theta(|\xi|)$$

and

$$F^{pc}(\xi) = F^{qc}(\xi) = F^{rc}(\xi) = \theta\Big((|\xi|^2 + 2|\operatorname{adj}_2 \xi|^{\frac{1}{2}}\Big) - 2|\operatorname{adj}_2 \xi|, \tag{3.1.11}$$

where $\operatorname{adj}_2 \xi$ *is the* $\frac{N(N-1)}{2}$ *vector of the* 2×2 *minors of* $\xi \in \mathbb{R}^{N\times 2}$.

The result for convex envelope in the scalar case was obtained by Ekeland-Temam [40] in the following way: The graph of F is given by $F(\xi) = |\xi|^2 + 1$ with discontinuity at $\xi = 0$ where $F(0) = 0$. For simplicity, we let $t = |\xi|$. Differentiating F with respect to t gives us $F'(t) = 2t$. We need the convexification of F to grow linearly from $t = 0$ until a point where it intersects the tangent of $F(t) = t^2 + 1$. Let this intersection point be t_0. Then, the function of the linear graph is $G(t) = 2t_0 t$. To find the value of t_0, we need $t_0^2 + 1 = 2t_0^2$ which gives us $t_0 = \pm 1$. Hence, the convexification of F is given by

$$F^c(\xi) = \begin{cases} |\xi|^2 + 1 & \text{if } |\xi| \geq 1 \\ 2|\xi| & \text{if } |\xi| \leq 1 \end{cases}.$$

While the convexification result might seem to be elementary, that is not the case if we consider vector-valued functions. The rest of the result in Theorem 3.1.4 can be found in the work of Kohn-Strang [77], which was proved using homogenization. Since then, the vectorial case has been studied extensively by Allaire-Francfort [5] and Allaire-Lods [6]. Both recovered F^{qc} by using methods from homogenization, G-closure and bounds of effective moduli.

The approach used by Zhang [133] is by far the simplest and only requires linear algebra together with the application of optimal translation method [46]. Generally, the translation lower bound for a continuous function $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ by any (quasi)-convex function G is given by

$$H := (F-G)^c + G \leq F,$$

where for a function F we denote by F^c its convex envelope. Considering G as a rank one convex quadratic form, we see that H is the sum of a convex function and a rank one convex quadratic form, hence H is quasiconvex (see Theorem 2.2.12).

The main idea of Zhang's approach is to decompose the function F into simpler functions such that the supremum of these functions is essentially F. The quasiconvex envelopes of these simpler functions can be easily found and taking the supremum of the quasiconvex envelopes then gives us the quasiconvex envelope of function F.

Before we move on, we present some of the notations from linear algebra which is the main tool used in the derivations. Let $\mathbb{R}^{N\times n}$ be the space of all real $N\times n$ matrices with Euclidean inner product $\xi\cdot\eta = \mathrm{tr}(\xi^T\eta)$ where tr denotes the trace operator and ξ^T the transpose of ξ. For any $\xi\in\mathbb{R}^{N\times n}$, define $[\xi] = \sqrt{\xi^T\xi}$ where $\xi^T\xi$ is a symmetric and positive semidefinite $n\times n$ matrix.

We now consider a function $F\colon \mathbb{R}^{N\times n}\to\mathbb{R}_+$, for $\lambda>0$ and $p>1$, in the following form:

$$F(\xi) = \Phi(|\xi|) = \begin{cases} |\xi|^p + \lambda & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0 \end{cases}. \tag{3.1.12}$$

Note that we can also write out an explicit convexification formula for (3.1.12) in a similar way as above and therefore

$$F^c(\xi) = \begin{cases} |\xi|^p + \lambda & \text{if } |\xi| \geq \left(\frac{\lambda}{p-1}\right)^{\frac{1}{p}} \\ p\left(\frac{\lambda}{p-1}\right)^{\frac{p-1}{p}} |\xi| & \text{if } |\xi| \leq \left(\frac{\lambda}{p-1}\right)^{\frac{1}{p}} \end{cases}. \tag{3.1.13}$$

When $p=2$, (3.1.12) is called the generalized Kohn-Strang function. Henceforth, whenever we say generalized Kohn-Strang function, we refer to $F\colon\mathbb{R}^{N\times n}\to\mathbb{R}_+$ in the form of

$$F(\xi) = \Phi(|\xi|) = \begin{cases} |\xi|^2 + \lambda & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0 \end{cases} \tag{3.1.14}$$

where $\lambda > 0$ and its quasiconvex envelope (when $\min\{N, n\} \geq 2$) obtained by Allaire-Francfort [5] is in the following form:

$$F^{qc}(\xi) = \begin{cases} |\xi|^2 + \lambda & \text{if } \sum_{i=1}^n a_i \geq \sqrt{\lambda} \\ |\xi|^2 - (\sum_{i=1}^n a_i)^2 + \sqrt{\lambda}\sum_{i=1}^n a_i & \text{if } \sum_{i=1}^n a_i \leq \sqrt{\lambda} \end{cases} \tag{3.1.15}$$

where $a_1 \geq a_2 \geq \ldots a_n \geq 0$ are the eigenvalues of $[\xi] = \sqrt{\xi^T\xi}$.

Back to Zhang's derivation, we begin by assuming $N = n = 2$. Let $P_{E_+} : \mathbb{R}^{2\times 2} \to E_+$ and $P_{E_-} : \mathbb{R}^{2\times 2} \to E_-$ be the orthogonal projections where

$$E_+ = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, a, b \in \mathbb{R} \right\} \quad \text{and} \quad E_- = \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, a, b \in \mathbb{R} \right\}$$

are the subspaces of conformal and anti-conformal matrices in $\mathbb{R}^{2\times 2}$, respectively.

Remark 3.1.5. *For*

$$\xi = \begin{pmatrix} \xi_1^1 & \xi_2^1 \\ \xi_1^2 & \xi_2^2 \end{pmatrix},$$

we associate $\tilde{\xi}$ in the following way

$$\tilde{\xi} = \begin{pmatrix} \xi_2^2 & -\xi_1^2 \\ -\xi_2^1 & \xi_1^1 \end{pmatrix}$$

We can decompose the matrix into its conformal ξ_+ and anti-conformal ξ_- parts,

$$\xi_+ = \frac{1}{2}(\xi + \tilde{\xi}) \quad \textit{and} \quad \xi_- = \frac{1}{2}(\xi - \tilde{\xi}).$$

It then follows that

$$2\det \xi_+ = |\xi_+|^2 \quad \textit{and} \quad 2\det \xi_- = -|\xi_-|^2,$$
$$|\xi_+|^2 + |\xi_-|^2 = |\xi|^2 \quad \textit{and} \quad |\xi_+|^2 - |\xi_-|^2 = 2\det \xi.$$

Theorem 3.1.6 (Zhang [133], Theorem 1.1)**.** *Let F be given by (3.1.14) with $N = n = 2$. Then*

$$F^{qc}(\xi) = \max\{F_+^{qc}(\xi), F_-^{qc}(\xi)\}, \tag{3.1.16}$$

with

$$F_+ = \Phi(|P_{E_+}(\xi)|) + |P_{E_-}(\xi)|^2, \quad F_- = \Phi(|P_{E_-}(\xi)|) + |P_{E_+}(\xi)|^2, \tag{3.1.17}$$

and

$$F_+^{qc} = g(|P_{E_+}(\xi)|) - 2\det \xi, \quad F_-^{qc} = g(|P_{E_-}(\xi)|) + 2\det \xi, \tag{3.1.18}$$

where

$$g(t) = C(\Phi(t) + t^2) = \begin{cases} 2t^2 + \lambda & \text{if } t \geq \sqrt{\frac{\lambda}{2}} \\ 4t\sqrt{\frac{\lambda}{2}} & \text{if } 0 \leq t \leq \sqrt{\frac{\lambda}{2}} \end{cases} \tag{3.1.19}$$

and for a function h, we denote Ch as its convex envelope.

In the theorem above, we can clearly see the essence of Zhang's approach: Firstly, the function F is decomposed into F_+ and F_- so that their respective quasiconvex envelopes F_+^{qc} and F_-^{qc} can be obtained in an easier way. Noting that each of these quasiconvex envelopes are essentially the sum of a convex envelope and a rank one convex quadratic form, it is clear that F_+^{qc} and F_-^{qc} are quasiconvex and hence taking the maximum of these two gives the quasiconvex envelope of F. The proof of the theorem is also interesting and requires two standard results, one on linear algebra and another on rank one convexification. We begin by presenting the result of the former.

Lemma 3.1.7 (Zhang [133], Lemma 3.1)**.** *Let $\Lambda = \{a \otimes b \colon a, b \in \mathbb{R}^2\}$. Then both $P_{E_+} \colon \Lambda \to E_+$ and $P_{E_-} \colon \Lambda \to E_-$ are onto mappings.*

Proof. We begin with P_{E_+} in which the matrices in

$$E_+ = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, a, b \in \mathbb{R} \right\}$$

can be written as

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = aI + b \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where I is the identity matrix. It is clear that $\{I, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\}$ forms a basis for $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$. Hence $\left\{E_1 = \frac{\sqrt{2}}{2} I, E_2 = \frac{\sqrt{2}}{2}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right\}$ forms an orthonormal basis of E_+. By letting $a = (1, 0)$ and $b = \sqrt{2}(t, s)$, we have $P_{E_+}(a \otimes b) = tE_1 + sE_2$ which proves that $P_{E_+} \colon \Lambda \to E_+$ is an onto mapping.

The proof for P_{E_-} is similar with $\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\}$ forming a basis for $\left(\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right)$ and hence $\left\{E_3 = \frac{\sqrt{2}}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), E_4 = \frac{\sqrt{2}}{2}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right\}$ forming an orthonormal basis of E_-. Similarly, we let $a = (1, 0)$ and $b = \sqrt{2}(t, s)$ and therefore $P_{E_-}(a \otimes b) = tE_3 + sE_4$. □

Recall that rank one convexity plays a key role in quasiconvex analysis and particularly in this section. The following iteration method used by Kohn-Strang [78] will be proven to be useful in the proof of Theorem 3.1.6. Assume that $F \colon \mathbb{R}^{N \times n} \to \mathbb{R}$ is not rank one convex and we are interested in calculating F^{rc} for a lower semicontinuous function F bounded below, namely

$$\begin{cases} F^{(0)} = F, \\ F^{(k+1)}(\xi) = \inf\left\{\alpha F^{(k)}(\xi_1) + (1-\alpha)F^{(k)}(\xi_2) \colon {\scriptstyle \alpha\xi_1+(1-\alpha)\xi_2=\xi, \atop \operatorname{rank}(\xi_1-\xi_2)\le 1,\ 0\le\alpha\le 1}\right\} \end{cases} \tag{3.1.20}$$

Since F is not rank one convex, there must be a counterexample where there exist $\xi_1, \xi_2 \in \mathbb{R}^{N\times n}$ such that

$$\begin{cases} \operatorname{rank}(\xi_1-\xi_2) = 1, \\ \xi = t\xi_1 + (1-t)\xi_2 \text{ for } t \in [0,1], \\ F(\xi) > tF(\xi_1) + (1-t)F(\xi_2). \end{cases}$$

Choosing for each ξ an extremal counterexample gives us a new function below F but no smaller that F^{rc}, namely

$$F(\xi) > tF(\xi_1) + (1-t)F(\xi_2) > F^{rc}(\xi).$$

Take $F^{(0)} = F$ as in (3.1.20). At each step, a new function $F^{(k+1)}$ is constructed from the extremal counter example to the rank one convexity of $F^{(k)}$ as shown in (3.1.20). Then $F^{(k+1)}(\xi) \le F^{(k)}(\xi)$, with equality only if F^{rc} is rank one convex. It is obvious that $F^{(k+1)}$ is a non-increasing function and the limit of this non-increasing function is indeed a rank one convex function. Hence

$$\lim_{k\to\infty} F^{(k)}(\xi) = F^{rc}(\xi).$$

Particularly, when $N = 1$ or $n = 1$, F^{rc} coincides with F^c.

Proof of Theorem 3.1.6. We divide the proof into 3 steps.

Step 1. We first show that (3.1.14) can be written as (3.1.17). From (3.1.14), we have

$$\begin{aligned} F(\xi) = \Phi(|\xi|) &= \begin{cases} \lambda + |\xi|^2 & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0 \end{cases} \\ &= \begin{cases} \lambda + |P_{E_+}(\xi)|^2 + |P_{E_-}(\xi)|^2 & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0 \end{cases} \\ &= |P_{E_-}(\xi)|^2 + \begin{cases} \lambda + |P_{E_+}(\xi)|^2 & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0 \end{cases} \\ &= |P_{E_-}(\xi)|^2 + \Phi(|P_{E_+}(\xi)|), \end{aligned}$$

thus obtaining $F_+ = \Phi\big(|P_{E_+}(\xi)|\big) + |P_{E_-}(\xi)|^2$, the first equation in (3.1.17). Similarly, we also have $F_- = \Phi\big(|P_{E_-}(\xi)|\big) + |P_{E_+}(\xi)|^2$ and hence concluding Step 1.

Step 2. We will now show that given (3.1.17), the quasiconvex envelopes are given by (3.1.18), respectively. By letting $h\big(P_{E_+}(\xi)\big) = \Phi(|P_{E_+}(\xi)|) + |P_{E_+}(\xi)|^2$, wee see that (3.1.19) can be written as

$$g(|P_{E_+}(\xi)|) = Ch\big(P_{E_+}(\xi)\big).$$

From (3.1.17) and the fact that $|P_{E_+}(\xi)|^2 - |P_{E_-}(\xi)|^2 = 2\det\xi$, it is obvious that

$$F_+(\xi) = h\big(P_{E_+}(\xi)\big) - 2\det\xi.$$

By the definition of convex envelope, we always have

$$F_+(\xi) \geq C_{E_+}\Big(h\big(P_{E_+}(\xi)\big) - 2\det\xi\Big)$$

and since $\det\xi$ is affine, we write

$$F_+(\xi) \geq C_{E_+}h\big(P_{E_+}(\xi)\big) - 2\det\xi = g(|P_{E_+}(\xi)|) - 2\det\xi.$$

This, together with (3.1.1), leads us to $F^{rc}(\xi) \geq F^{qc}(\xi) \geq g(|P_{E_+}(\xi)|) - 2\det\xi$.

We will need the reverse inequality to complete Step 2. We first consider $F_+^{(1)}(\xi)$ as in the iteration method for rank one convexification. Let $B_1, B_2 \in E_+$ be such that $tB_1 + (1-t)B_2 = P_{E_+}(\xi)$ where $t \in [0,1]$. Further let $B = B_1 - B_2$ and invoking Lemma 3.1.7, there exist some $a \otimes b \in \Lambda$ such that $P_{E_+}(a \otimes b) = B$. We set $\xi_1 = \xi + (1-t)a \otimes b$ and $\xi_2 = \xi - ta \otimes b$ to get

$$\begin{cases} t\xi_1 + (1-t)\xi_2 = \xi, \\ \xi_1 - \xi_2 = a \otimes b, \\ P_{E_+}(\xi_1) = B_1, \\ P_{E_+}(\xi_2) = B_2. \end{cases}$$

Invoking the iteration method for rank one convexification, we have

$$\begin{aligned} F_+^{(1)}(\xi) &\leq tF_+(\xi_1) + (1-t)F_+(\xi_2) \\ &= th\big(P_{E_+}(\xi_1)\big) - 2t\det\xi_1 + (1-t)h\big(P_{E_+}(\xi_2)\big) - 2(1-t)\det\xi_2 \\ &= th(B_1) + (1-t)h(B_2) - 2\det\xi. \end{aligned}$$

Taking the infimum on $B_1, B_2 \in E_+$ with $tB_1 + (1-t)B_2 = P_{E_+}(\xi)$ for some $t \in [0,1]$, we see that

$$F_+^{(1)}(\xi) \leq C_1 h\left(P_{E_+}(\xi)\right) - 2\det\xi.$$

Repeating this process gives us

$$F_+^{(k)}(\xi) \le C_k h\left(P_{E_+}(\xi)\right) - 2\det\xi$$

for $k = 1, 2, \dots$ and passing to the limit as $k \to \infty$, we obtain

$$F_+^{rc}(\xi) \le C_{E_+} h\left(P_{E_+}(\xi)\right) - 2\det\xi.$$

Finally, we have the equality, namely

$$F_+^{rc}(\xi) = F_+^{qc}(\xi) = g(|P_{E_+}(\xi)|) - 2\det\xi.$$

We can prove, in a similar way, for F_-^{rc} to see that

$$F_-^{rc}(\xi) = F_-^{qc}(\xi) = g(|P_{E_-}(\xi)|) + 2\det\xi.$$

Step 3. In this final step, we will show that we have (3.1.16). If $F_+^{qc}(\xi) \le F_-^{qc}(\xi)$, we have

$$\begin{aligned} & g(|P_{E_+}(\xi)|) - 2\det\xi \le g(|P_{E_-}(\xi)|) + 2\det\xi \\ \Longrightarrow\quad & g(|P_{E_+}(\xi)|) - 2|P_{E_+}(\xi)|^2 \le g(|P_{E_-}(\xi)|) - 2|P_{E_-}(\xi)|^2. \end{aligned}$$

This is an increasing function, hence we have that

$$F_+^{qc}(\xi) \le F_-^{qc}(\xi) \quad \Leftrightarrow \quad |P_{E_+}(\xi)| \le |P_{E_-}(\xi)|.$$

Suppose now that $|P_{E_+}(\xi)| \le |P_{E_-}(\xi)|$, then

$$\begin{aligned} \max\left\{F_+^{qc}(\xi), F_-^{qc}(\xi)\right\} &= F_-^{qc}(\xi) \\ &= g(|P_{E_-}(\xi)|) + 2\det\xi \\ &= C\left(\Phi(|P_{E_-}(\xi)|) + |P_{E_-}(\xi)|^2\right) + 2\det\xi \\ &= C\left(\Phi(|P_{E_+}(\xi)|) - 2\det\xi + |P_{E_+}(\xi)|^2 - 2\det\xi\right) + 2\det\xi \\ &= C\left(\Phi(|P_{E_+}(\xi)|) + |P_{E_+}(\xi)|^2\right) - 2\det\xi \\ &= g(|P_{E_+}(\xi)|) - 2\det\xi = F^{qc}(\xi). \end{aligned}$$

Otherwise, if $|P_{E_-}(\xi)| \le |P_{E_+}(\xi)|$, we also obtain

$$\max\left\{F_+^{qc}(\xi), F_-^{qc}(\xi)\right\} = F_+^{qc}(\xi) = g(|P_{E_+}(\xi)|) - 2\det\xi = g(|P_{E_-}(\xi)|) + 2\det\xi = F^{qc}(\xi)$$

and hence the conclusion of the theorem. $\square$

The formula is more complicated when we deal with general $N \times n$ matrices. Let $X \in \mathbb{R}^{N\times n}$ with $|X| = 1$ and let $x_1 \in (0,1)$ be the largest eigenvalue of $[X]$. The eigenvalues of $[X]$ are often called the singular values of X. Further let the orthogonal projections onto the one-dimensional space $\operatorname{span}(X)$ and its orthogonal complement be denoted by P_X and $P_{X^\perp}$, respectively. If X is a rank one matrix, let $\lambda_X = 0$, otherwise define

$$\frac{1}{\lambda_X} = \max_{|a|,|b|=1} \frac{|P_X(a\otimes b)|^2}{|P_{X^\perp}(a\otimes b)|^2} < \infty,$$

where $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^n$. It is clear that $\frac{1}{\lambda_X}$ is finite since

$$\begin{aligned}
|P_{X^\perp}(a\otimes b)|^2 &= |a\otimes b|^2 - |P_X(a\otimes b)|^2 \\
&= |a|^2|b|^2 - \left|\frac{\langle a\otimes b, X\rangle}{|X|^2} X\right|^2 \\
&= |a|^2|b|^2 - \frac{\Big||a\otimes b||X|\cos\theta\Big|^2}{|X|^2} \\
&= |a|^2|b|^2 - |a|^2|b|^2\cos^2\theta \\
&= (1-\cos^2\theta)|a|^2|b|^2 \\
&\geq \big(1 - C(X)\big)|a|^2|b|^2 \\
&= c(X)|a|^2|b|^2
\end{aligned}$$

for some constant $c(X) > 0$ when $\operatorname{rank}(X) > 1$. We now give the result of the characterization of λ_X.

Proposition 3.1.8 (Zhang [133], Proposition 1.3)**.** *Suppose* $\operatorname{rank}(X) > 1$ *with* $|X| = 1$. *Then* $\lambda_X = \frac{1}{x_1^2} - 1 > 0$, *where* $x_1 \in (0,1)$ *is the largest eigenvalue of* $[X] = \sqrt{X^T X}$.

Proof. Let $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$ be the eigenvalues of $[X]$. The definition of λ_X gives us

$$\begin{aligned}
\frac{1}{\lambda_X} &= \max_{|a|,|b|=1} \frac{|P_X(a\otimes b)|^2}{|P_{X^\perp}(a\otimes b)|^2} \\
&= \max_{|a|,|b|=1} \frac{|P_X(a\otimes b)|^2}{|a\otimes b|^2 - |P_X(a\otimes b)|^2} \\
&= \max_{|a|,|b|=1} \frac{|P_X(a\otimes b)|^2}{1 - |P_X(a\otimes b)|^2}
\end{aligned}$$

and hence

$$\lambda_X = \frac{1}{\max_{|a|,|b|=1}|P_X(a\otimes b)|^2} - 1.$$

Let u be a unit eigenvector of $[X]$ corresponding to the largest eigenvector x_1. Further let $a = \frac{Xu}{x_1}$ and $b = u$. Then

$$|P_X(a \otimes b)|^2 = (a^T X b)^2 = \left(\frac{Xu}{x_1} Xu\right)^2 = \left(\frac{u^T X^T X u}{x_1}\right)^2 = x_1^2$$

and hence $\lambda_X = \frac{1}{x_1^2} - 1$. $\square$

We need the following result of von Neumann (see Mirsky [100]):

Proposition 3.1.9 (Zhang [133], Proposition 2.1)**.** *Suppose $\xi, \eta \in \mathbb{R}^{N\times n}$ and $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ and $b_1 \geq b_2 \geq \cdots \geq b_n \geq 0$ are the eigenvalues of $[\xi]$ and $[\eta]$, respectively. Then*

$$\left|\mathrm{tr}(\xi^T \eta)\right| \leq \sum_{i=1}^{n} a_i b_i.$$

Proof. By [100], if $\xi, \eta \in \mathbb{R}^{n\times n}$, we have that

$$\left|\mathrm{tr}(\xi \eta)\right| \leq \sum_{i=1}^{n} a_i b_i.$$

Note that, for $\xi \in \mathbb{R}^{N\times n}$, $\xi^T\xi$ and $\xi\xi^T$ have the same non-zero eigenvalues. Without loss of generality, we can always enlarge ξ and η to be square matrices by adding $n - N$ rows with zero entries if $n > N$ or $N - n$ columns if $N > n$:

$$\hat{\xi} = \begin{pmatrix} \xi \\ 0 \end{pmatrix}, \quad \hat{\eta} = \begin{pmatrix} \eta \\ 0 \end{pmatrix} \quad \text{if } n > N,$$
$$\hat{\xi} = (\xi \ \ 0), \quad \hat{\eta} = (\eta \ \ 0) \quad \text{if } N > n.$$

In both cases, we see that the non-zero eigenvalues of $[\hat{\xi}]$ and $[\xi]$, and those of $[\hat{\eta}]$ and $[\eta]$ are the same, respectively. Moreover, $\mathrm{tr}(\hat{\xi}^T\hat{\eta}) = \mathrm{tr}(\xi^T\eta)$ and hence the conclusion of the proposition. $\square$

Proposition 3.1.10 (Zhang [133], Proposition 1.4)**.** *Let $\xi \in \mathbb{R}^{N\times n}$ and let $\mathcal{R}(\xi)$ and $\mathcal{R}([\xi])$ be the range of the linear mappings $\xi\colon \mathbb{R}^n \to \mathbb{R}^N$ and $[\xi]\colon \mathbb{R}^n \to \mathbb{R}^n$, respectively. Then there is a partial isometry $R_\xi \in \mathbb{R}^{N\times n}$ from $\mathcal{R}([\xi])$ to $\mathcal{R}(\xi)$ such that*

$$\begin{cases} |R_\xi(y)| = |y| & \text{if } y \in \mathcal{R}([\xi]), \\ R_\xi(y) = 0 & \text{if } y \in \left(\mathcal{R}([\xi])\right)^\perp. \end{cases}$$

Furthermore, $\mathrm{rank}(R_\xi) = \mathrm{rank}([\xi]) = \mathrm{rank}(\xi)$ *and $R_\xi^T R_\xi = P_{\mathcal{R}([\xi])}$ - the orthogonal projection from $\mathbb{R}^n$ to $\mathcal{R}([\xi])$.*

Proof. The case $n \times n$ is easy since we have $|[\xi]x| = |\xi x|$ for every $x \in \mathbb{R}^n$.

We now define the mapping $R_\xi : \mathcal{R}([\xi]) \to \mathcal{R}(\xi)$, that is to say if $y = [\xi]x$, we let $R_\xi(y) = \xi x$. This mapping is well-defined since

$$y = [\xi]x_1 = [\xi]x_2 \quad \Rightarrow \quad R_\xi y = \xi x_1 = \xi x_2.$$

Since $\xi : \mathbb{R}^n \to \mathbb{R}^N$ and $[\xi] : \mathbb{R}^n \to \mathbb{R}^n$ are linear, we deduce that the mapping R_ξ is also linear. Moreover, we have

$$|R_\xi y| = |\xi x| = |[\xi]x| = |y|$$

and therefore R_ξ is an isometry. Since both $\mathcal{R}([\xi])$ and $\mathcal{R}(\xi)$ are finite dimensional, we have $\dim\big(\mathcal{R}([\xi])\big) = \dim\big(\mathcal{R}(\xi)\big)$ and hence $\operatorname{rank}([\xi]) = \operatorname{rank}(\xi)$.

Define $R_\xi y = 0$ if $y \in \big(\mathcal{R}([\xi])\big)^\perp$. Then R_ξ is a linear transform from $\mathbb{R}^n$ to $\mathbb{R}^N$ with $\operatorname{rank}(R_\xi) = \dim\big(\mathcal{R}([\xi])\big) = \dim\big(\mathcal{R}(\xi)\big) = \operatorname{rank}(\xi)$.

Lastly, we will prove that $R_\xi^T R_\xi = P_{\mathcal{R}([\xi])}$. By the parallelogram law, we see that

$$\begin{aligned}
(R_\xi y_1) \cdot (R_\xi z_1) &= \langle R_\xi y_1, R_\xi z_1 \rangle \\
&= \langle \xi x_1, \xi x_2 \rangle \\
&= \xi^T \xi \langle x_1, x_2 \rangle \\
&= [\xi]^2 \langle x_1, x_2 \rangle \\
&= \langle [\xi]x_1, [\xi]x_2 \rangle \\
&= y_1 \cdot z_1
\end{aligned}$$

for $y_1, z_1 \in \mathcal{R}([\xi])$. For $y, z \in \mathbb{R}^n$ with $y_1, z_1 \in \mathcal{R}([\xi])$ and $y_2, z_2 \in \big(\mathcal{R}([\xi])\big)^\perp$, we write the orthogonal decomposition $y = y_1 + y_2$ and $z = z_1 + z_2$. Thus, for all $z \in \mathbb{R}^n$, we have

$$(R_\xi^T R_\xi y) \cdot z = (R_\xi y) \cdot (R_\xi z) = (R_\xi y_1) \cdot (R_\xi z_1) = y_1 \cdot z_1 = (P_{\mathcal{R}([\xi])} y) \cdot z$$

and hence $R_\xi^T R_\xi y = P_{\mathcal{R}([\xi])} y$. □

We are now ready to prove for general case, when $\min\{N, n\} \geq 2$.

Theorem 3.1.11 (Zhang [133], Theorem 1.5). *Let F be defined by (3.1.14) with $\min\{N, n\} \geq 2$. Then*

$$F^{qc}(\xi) = \max_{|X|=1} F_X^{qc}(\xi),$$

and the maximum is achieved at $X = \frac{R_\xi}{|R_\xi|}$, *where* $F_X(\xi) = F\big(P_X(\xi)\big) + |P_{X^\perp}(\xi)|^2$ *and*

$$\begin{aligned} F_X^{qc}(\xi) &= g_X\left(\frac{|P_X(\xi)|}{x_1}\right) + |\xi|^2 - \frac{|P_X(\xi)|^2}{x_1^2} \\ &= \begin{cases} |\xi|^2 + \lambda & \text{if } \frac{|X\cdot\xi|}{x_1} \geq \sqrt{\lambda} \\ |\xi|^2 + 2\sqrt{\lambda}\left(\frac{|X\cdot\xi|}{x_1}\right) - \left(\frac{|X\cdot\xi|}{x_1}\right)^2 & \text{if } 0 \leq \frac{|X\cdot\xi|}{x_1} \leq \sqrt{\lambda} \end{cases} \end{aligned} \tag{3.1.21}$$

with

$$g_X(t) = \begin{cases} t^2 + \lambda & \text{if } t \geq \sqrt{\lambda} \\ 2t\sqrt{\lambda} & \text{if } 0 \leq t \leq \sqrt{\lambda} \end{cases}. \tag{3.1.22}$$

When $\min\{N, n\} = 2$, *then we have that*

$$F^{qc}(\xi) = F^{pc}(\xi),$$

the polyconvex envelope of $F(\xi)$.

Proof. We will prove the theorem in 2 steps.

Step 1. We first show that for $|X| = 1$, $F_X^{qc}(\cdot)$ is given by (3.1.21) with $g_X(\cdot)$ given by (3.1.22). We begin with the case $\operatorname{rank}(X) > 1$ and it follows from Proposition 3.1.8 that $\lambda_X = \frac{1}{x_1^2} - 1$ where x_1 is the largest eigenvalue of $[X]$. By the way we define λ_X, namely

$$\frac{1}{\lambda_X} = \max_{|a|=|b|=1} \frac{|P_X(a\otimes b)|^2}{|P_{X^\perp}(a\otimes b)|^2},$$

we can see that $|P_{X^\perp}(\cdot)|^2 - \lambda_X|P_X(\cdot)|^2$ is a rank one convex quadratic form and hence it is also quasiconvex (see Theorem 2.2.12). Moreover, for every rank one matrix $a \otimes b \in \mathbb{R}^{N\times n}$, we have $|P_{X^\perp}(a\otimes b)|^2 - \lambda_X|P_X(a\otimes b)|^2 \geq 0$. By letting

$$\Lambda_X = \left\{a\otimes b \colon |P_{X^\perp}(a\otimes b)|^2 - \lambda_X|P_X(a\otimes b)|^2 = 0\right\},$$

we assert that $P_X\colon \Lambda_X \to \operatorname{span}(X)$ is onto by taking $a = Xu$, $b = tu$ for unit eigenvector u and $t \in \mathbb{R}$. We will now prove the assertion. Using the formula of orthogonal projection gives us

$$\begin{aligned} P_X(a\otimes b) = \frac{\langle a\otimes b, X\rangle}{|X|^2}X &= \frac{\langle Xu\otimes tu, X\rangle}{|X|^2}X \\ &= \frac{t\langle Xuu^T, X\rangle}{|X|^2}X \\ &= \frac{t|Xu|^2}{|X|^2}X \\ &= sX \end{aligned}$$

where $s = \frac{t|Xu|^2}{|X|^2}$ is a constant and hence it follows that $P_X\colon \Lambda_X \to \operatorname{span}(X)$ is onto.

Turning back our focus on (3.1.14) and invoking the same procedure as in the proof of Theorem 3.1.6 gives us

$$F(\xi) = \Phi(|P_X(\xi)|) + |P_{X^\perp}(\xi)|^2 = F_X(\xi).$$

Let

$$F_X(\xi) = \Phi(|P_X(\xi)|) + \lambda_X |P_X(\xi)|^2 + |P_{X^\perp}(\xi)|^2 - \lambda_X |P_X(\xi)|^2,$$

we have to deal with $G = \Phi(|P_X(\xi)|) + \lambda_X |P_X(\xi)|^2$ since $H = |P_{X^\perp}(\xi)|^2 - \lambda_X |P_X(\xi)|^2$ is of rank one convex quadratic form. We use the definition of λ_X to get

$$G = \Phi(|P_X(\xi)|) + \lambda_X |P_X(\xi)|^2 = \begin{cases} \lambda + \frac{|P_X(\xi)|^2}{x_1^2} & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0 \end{cases}$$

and we finally have the convex relaxation along the one-dimensional space $\mathrm{span}(X)$

$$C_X\big(\Phi(|P_X(\xi)|) + \lambda_X |P_X(\xi)|^2\big) = g_X\left(\frac{|P_X(\xi)|}{x_1}\right).$$

Once again, by (3.1.1), we have

$$F_X^{rc}(\xi) \geq F_X^{qc}(\xi) \geq C_X\big(\Phi(|P_X(\xi)|) + \lambda_X |P_X(\xi)|^2\big) + |P_{X^\perp}(\xi)|^2 - \lambda_X |P_X(\xi)|^2.$$

We need the reverse inequality. Now, let $B_1, B_2 \in \mathrm{span}(X)$ be such that $tB_1 + (1-t)B_2 = P_X(\xi)$ where $t \in [0,1]$. Further let $B = B_1 - B_2$, then there exist some $a \otimes b \in \Lambda_X$ such that $P_X(a \otimes b) = B$. We set $\xi_1 = \xi + (1-t)a \otimes b$ and $\xi_2 = \xi - ta \otimes b$ to get

$$\begin{cases} t\xi_1 + (1-t)\xi_2 = \xi, \\ \xi_1 - \xi_2 = a \otimes b, \\ P_X(\xi_1 - \xi_2) = a \otimes b = B, \\ P_X(\xi_1) = B_1, \ P_X(\xi_2) = B_2. \end{cases}$$

We invoke the iteration method for rank one convexification to get

$$\begin{aligned} F_X^{rc}(\xi) &\leq tF_X(\xi_1) + (1-t)F_X(\xi_2) \\ &= t\left(\Phi(|P_X(\xi_1)|) + \lambda_X |P_X(\xi_1)|^2 + |P_{X^\perp}(\xi_1)|^2 - \lambda_X |P_X(\xi_1)|^2\right) \\ &\quad + (1-t)\left(\Phi(|P_X(\xi_2)|) + \lambda_X |P_X(\xi_2)|^2 + |P_{X^\perp}(\xi_2)|^2 - \lambda_X |P_X(\xi_2)|^2\right) \\ &= th\big(P_X(\xi_1)\big) + (1-t)h\big(P_X(\xi_2)\big) + t\big(|P_{X^\perp}(\xi_1)|^2 - \lambda_X |P_X(\xi_1)|^2\big) \\ &\quad + (1-t)\big(|P_{X^\perp}(\xi_2)|^2 - \lambda_X |P_X(\xi_2)|^2\big) \\ &= th(B_1) + (1-t)h(B_2) + t\omega(\xi_1) + (1-t)\omega(\xi_2) \end{aligned}$$

where for $i = 1, 2$,

$$h\Big(P_X(\xi_i)\Big) = \Phi(|P_X(\xi_i)|) + \lambda_X |P_X(\xi_i)|^2$$

and $\omega(\xi_i) = |P_{X^\perp}(\xi_i)|^2 - \lambda_X |P_X(\xi_i)|^2$. Taking the infimum on $B_1, B_2 \in \operatorname{span}(X)$ with $tB_1 + (1-t)B_2 = P_X(\xi)$ and since $\inf\Big\{t\omega(\xi_1) + (1-t)\omega(\xi_2)\Big\} = \omega\Big(t\xi_1 + (1-t)\xi_2\Big) = \omega(\xi)$, we obtain

$$\begin{aligned} F_X^{(1)}(\xi) &\leq C_1 h(tB_1 + (1-t)B_2 + \omega(\xi) \\ &= C_1 h\Big(P_X(\xi)\Big) + |P_{X^\perp}(\xi)|^2 - \lambda_X |P_X(\xi)|^2 \\ &= C_1\Big(\Phi(|P_X(\xi)|) + \lambda_X |P_X(\xi)|^2\Big) + |P_{X^\perp}(\xi)|^2 - \lambda_X |P_X(\xi)|^2. \end{aligned}$$

Repeating the process gives us

$$F_X^{(k)}(\xi) \leq C_k \left(\Phi(|P_X(\xi)|) + \lambda_X |P_X(\xi)|^2\right) + |P_{X^\perp}(\xi)|^2 - \lambda_X |P_X(\xi)|^2$$

for $k = 1, 2, \ldots$ and passing to the limit as $k \to \infty$, we obtain

$$F_X^{rc}(\xi) \leq C_X \left(\Phi(|P_X(\xi)|) + \lambda_X |P_X(\xi)|^2\right) + |P_{X^\perp}(\xi)|^2 - \lambda_X |P_X(\xi)|^2.$$

Hence we have the equality.

If $\operatorname{rank}(X) = 1$ and $X = a_0 \otimes b_0$, then $\lambda_X = 0$ and we have

$$F_X(\xi) = \Phi(|P_X(\xi)|) + |P_{X^\perp}(\xi)|^2.$$

By a similar method as of above, we can show that

$$F_X^{rc}(\xi) = F_X^{qc}(\xi) = C_X \Phi(|P_X(\xi)|) + |P_{X^\perp}(\xi)|^2. \tag{3.1.23}$$

The inequality

$$F_X^{rc}(\xi) \geq F_X^{qc}(\xi) \geq C_X \Phi(|P_X(\xi)|) + |P_{X^\perp}(\xi)|^2$$

is obvious.

To prove the reverse inequality, take $P_X(\xi) = t_0(X)$ and $X_1 = t_1 X, X_2 = t_2 X \in \operatorname{span}(X)$ with $tX_1 + (1-t)X_2 = t_0 X = P_X(\xi)$ for some $t \in (0, 1)$, then

$$X_1 - X_2 = (t_1 - t_2)X$$

which simply means that $\operatorname{rank}(X_1 - X_2) = 1$. Furthermore, by letting $\xi_1 = \xi + (1-t)X$ and $\xi_2 = \xi - tX$, we get

$$\begin{cases} \operatorname{rank}(\xi_1 - \xi_2) = 1, \\ t\xi_1 + (1-t)\xi_2 = \xi, \\ P_{X^\perp}(\xi_1) = P_{X^\perp}(\xi_2) = P_{X^\perp}(\xi), \\ P_X(\xi_1) = X_1,\ P_X(\xi_2) = X_2. \end{cases}$$

Therefore

$$F_X^{(1)}(\xi) \leq tF_X(\xi_1) + (1-t)F_X(\xi_2) \leq t\Phi(|X_1|) + (1-t)\Phi(|X_2|) + |P_{X^\perp}(\xi)|^2$$

and taking the infimum as before to obtain

$$F_X^{(1)} \leq C_1\Phi(|P_X(\xi)|) + |P_{X^\perp}(\xi)|^2.$$

Repeating the procedure as usual and passing to the limit as $k \to \infty$ gives us

$$F_X^{rc} \leq C_X\Phi(|P_X(\xi)|) + |P_{X^\perp}(\xi)|^2.$$

Hence, we have (3.1.23) and that concludes Step 1.

Step 2. We wish to prove that for a fixed $\xi \neq 0$, $\max_{|X|=1} F_X^{qc}(\xi)$ is achieved at $\hat{R}_\xi = \frac{R_\xi}{|R_\xi|}$. In particular,

$$F^{qc}(\xi) = F^{qc}_{\hat{R}_\xi}(\xi). \tag{3.1.24}$$

We need to consider two cases for this step.

We will start with the first one, in which the formula (3.1.15) is *known* since it implicitly establishes the fact that F_X^{qc} achieves its maximum at $X = \hat{R}_\xi$. From Proposition 3.1.10, we have $\operatorname{rank}(R_\xi) = \operatorname{rank}(\xi) = \operatorname{rank}([\xi]) := m \geq 1$. Therefore $|R_\xi| = |\xi| = \sqrt{m}$ and the all non-zero eigenvalues of $[\hat{R}_\xi]$ equal to $\frac{1}{\sqrt{m}}$. In fact,

$$[\hat{R}_\xi] = \sqrt{\hat{R}_\xi^T\hat{R}_\xi} = \sqrt{\frac{R_\xi^T}{|R_\xi^T|}\cdot\frac{R_\xi}{|R_\xi|}} = \frac{\sqrt{R_\xi^T R_\xi}}{\sqrt{m}} = \frac{\sqrt{P_{\mathcal{R}([\xi])}}}{\sqrt{m}}.$$

Substituting $X = \hat{R}_\xi$ in the formula above and writing $\xi = R_\xi[\xi]$ gives us

$$\lambda_{\hat{R}_\xi} = \frac{1}{(1/\sqrt{m})^2} - 1 = m - 1$$

and

$$|P_{\hat{R}_\xi}(\xi)|^2 = \left|\frac{\langle\xi, \hat{R}_\xi\rangle}{|\hat{R}_\xi|^2}\hat{R}_\xi\right|^2 = \left(\frac{\operatorname{tr}(\hat{R}_\xi^T\xi)}{\sqrt{m}}\right)^2 = \frac{\left(\operatorname{tr}(\hat{R}_\xi^T\hat{R}_\xi[\xi])\right)^2}{m} = \frac{(\operatorname{tr}[\xi])^2}{m}.$$

Hence, by (3.1.21), we have

$$F^{qc}_{\hat{R}_\xi}(\xi) = g_{\hat{R}_\xi}(\operatorname{tr}[\xi]) + |\xi|^2 - (\operatorname{tr}[\xi])^2 \tag{3.1.25}$$

with

$$g_{\hat{R}_\xi}(\operatorname{tr}[\xi]) = \begin{cases} \lambda + (\operatorname{tr}[\xi])^2 & \text{if } \operatorname{tr}[\xi] \geq \lambda \\ 2\sqrt{\lambda}(\operatorname{tr}[\xi]) & \text{if } 0 \leq \operatorname{tr}[\xi] \leq \sqrt{\lambda} \end{cases}. \tag{3.1.26}$$

If we compare (3.1.11) with (3.1.25) and (3.1.26), we get (3.1.24). Moreover, for each X with $|X| = 1$, $F_X(\cdot) \leq F(\cdot)$ and hence

$$\max_{|X|=1} F^{qc}_{\hat{R}_\xi}(\xi) \leq F^{qc}(\xi).$$

However, the upper bound $F^{qc}(\xi)$ is reached at $\hat{R}_\xi$ which proves that the maximum is reached at $F^{qc}_{\hat{R}_\xi}(\xi)$.

On the other hand, the second case is when the formula (3.1.15) is *not known*. We want to show that

$$\max_{|X|=1} F^{qc}_X(\xi) = F^{qc}_{\hat{R}_\xi}(\xi).$$

Let $t = \frac{|X \cdot \xi|}{x_1}$, where x_1 is the largest eigenvalue of $[X]$, and define

$$\phi(t) = \begin{cases} \lambda & \text{if } t \geq \sqrt{\lambda} \\ 2t\sqrt{\lambda} - t^2 & \text{if } 0 \leq t \leq \sqrt{\lambda} \end{cases}.$$

It is clear that $\phi(t)$ is increasing and so the maximum of $F^{qc}_X(\xi)$ can be attained if we can find $\max_{|X|=1}\left(\frac{|X \cdot \xi|}{x_1}\right)$. Applying Proposition 3.1.9 to X and ξ gives us

$$\frac{|X \cdot \xi|}{x_1} = \frac{|\operatorname{tr}(X^T \xi)|}{x_1} \leq \sum_{i=1}^{n} \frac{a_i x_i}{x_1} \leq \sum_{i=1}^{n} a_i$$

since $x_i \leq x_1$ for $i = 2, 3, \ldots, n$. Next, if we assume that $\operatorname{rank}(\xi) = m \leq n$ and take $X = \hat{R}_\xi$, we have that $|R_\xi| = \sqrt{m}$ and $|\hat{R}_\xi| = 1$, respectively. Once again, for each X with $|X| = 1$, $F_X(\cdot) \leq F(\cdot)$ which implies that $\max_{|X|=1} F^{qc}_X(\xi) \leq F^{qc}(\xi)$. Meanwhile,

$$\frac{|X \cdot \xi|}{x_1} = \frac{\hat{R}_\xi \cdot \xi}{x_1} = \frac{1}{\sqrt{m}} \frac{\operatorname{tr}(R^T_\xi R_\xi[\xi])}{1/\sqrt{m}} = |\operatorname{tr}[\xi]| = \sum_{i=1}^{n} a_i$$

and that proves that the maximum is achieved at $\hat{R}_\xi$.

Recall that the class of functions considered here are the sum of convexification along the one-dimensional space $\operatorname{span}(X)$ and rank one convex quadratic forms. When $\min\{N, n\} = 2$, it follows from Theorem 2.2.12 that the functions are in fact polyconvex and hence $F^{qc} = F^{pc}$ in these cases. □

3.2 Young measures and relaxation theorem

In the previous section, we gave a brief introduction on the history of the relaxation theorem in which a minimizer can still be obtained for non-quasiconvex variational problems under certain coercivity and growth assumptions on the integrand F. This section will draw our attention to the theory of Young measure and we will show that it can be used to prove the relaxation theorem for vector-valued variational problems but in a simpler and more abstract way that allows us to work with less restrictive assumptions on the integrand.

Young measures were originally introduced as *generalized curves* by Young [130] to treat problems in the calculus of variations and optimal control that could not be solved using classical methods. In particular, even non-convex problems could be shown to have solutions in the form of generalized curves (McShane [96, 97], Young [130]), and thus providing a positive answer to Hilbert's 20th problem.

The introduction of quasiconvexity by Morrey [102] and the work of in particular Tartar [118] generated much interest in the application of Young measures in vector-valued calculus of variations. Among other notable works that follow are by Balder [8], Ball [11], Berliocchi-Lasry [16], Kristensen [83] and Warga [127]. The main breakthrough was in 1991 when Kinderlehrer-Pedegral [70] gave the characterization of Young measures generated by gradients which also gave the existence result for non-quasiconvex problems. The advantage of Young measure in this aspect is that the integral functionals are always lower semicontinuous in the class of Young measures, even though by definition Young measure carries only minimal information about the weakly convergent sequences.

The theory of Young measure has undergone developments in several stages and it is now a standard tool in the calculus of variations. While there are many available references to Young measures, this thesis follows closely the work of Sychev [115], in which he showed that Young measures can be characterized as measurable functions with values in a special compact metric space. Instead of analyzing the behaviour of integral functionals along weakly convergent sequences, the broad spectrum of properties possessed by these measurable functions enable us to only analyze the operations with Young measures generated by these sequences. We will start by introducing notations, definitions and basic results.

Definitions and notations

Before we give the definition of Young measure, we should first recall the definitions of strong and weak measurable functions, duality results obtained by the Riesz representation theorem and some auxiliary results.

Definition 3.2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded measurable set. Let X be a Banach space and X^* the dual space of X.*

(i) A function $u\colon \Omega \to X$ is said to be strongly measurable if there exists a sequence of simple and measurable function $u_j : \Omega \to X$ such that $u_j \to u$ a.e. $x \in \Omega$.

(ii) A function $u\colon \Omega \to X$ is said to be weakly measurable if $x \mapsto T\big(u(x)\big)$ is measurable for all $T \in X^*$.

(iii) A function $u\colon \Omega \to X^*$ is said to be weakly measurable if $x \mapsto \langle u(x), y\rangle = u(x)y$ is measurable for all $y \in X$ where $\langle \cdot, \cdot \rangle\colon X^* \times X \to \mathbb{R}$ is the duality pairring.*

We write, for $p \in [1, \infty]$,

$$L^p(\Omega; X) = \left\{ u\colon \Omega \to X : u \text{ is strongly measurable and } \int_\Omega \|u\|_X^p \,\mathrm{d}x < \infty \right\},$$

$$L^p_w(\Omega; X) = \left\{ u\colon \Omega \to X : \begin{array}{l} u \text{ is weakly measurable, } x \longmapsto \|u(x)\|_X \text{ is} \\ \text{measurable and } \int_\Omega \|u\|_X^p \,\mathrm{d}x < \infty \end{array} \right\},$$

and

$$L^p_{w*}(\Omega; X^*) = \left\{ u\colon \Omega \to X^* : \begin{array}{l} u \text{ is weak* measurable, } x \longmapsto \|u(x)\|_{X^*} \\ \text{is measurable and } \int_\Omega \|u\|_{X^*}^p \,\mathrm{d}x < \infty \end{array} \right\}.$$

Then we have the following result as a consequence of the Riesz representation theorem (see Fonseca-Leoni [49], Theorem 2.112).

Theorem 3.2.2 (Riesz representation theorem for $L^p(\Omega; X)$). *Let X be a separable Banach space with dual X^*. Then for $p \in [1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$,*

$$L^p(\Omega; X)^* \cong L^q_{w^*}(\Omega; X^*)$$

under the duality

$$\langle u, v\rangle = \int_\Omega \langle u(x), v(x)\rangle \,\mathrm{d}x,$$

where $u \in L^p(\Omega; X)$ and $v \in L^q_{w^*}(\Omega; X^*)$.

We further denote the space $C_0(\mathbb{R}^d)$ as the closure of $C_0^\infty(\mathbb{R}^d)$ in the supremum norm:

$$C_0(\mathbb{R}^d) = \left\{ \Phi \in C(\mathbb{R}^d) \colon \lim_{|v|\to\infty} \Phi(v) = 0 \right\}.$$

The Riesz representation theorem asserts that there is an isometric isomorphism between the dual space of $C_0(\mathbb{R}^d)$ and the space $\mathcal{M}(\mathbb{R}^d)$ of all bounded Radon measures supported in R^d; i.e.

$$C_0(\mathbb{R}^d)^* \cong \mathcal{M}(\mathbb{R}^d).$$

We can identify this via the action of a measure μ on the function Φ, or simply the duality pairring,

$$\langle \Phi, \mu \rangle = \int_{\mathbb{R}^d} \Phi(v)\,\mathrm{d}\mu.$$

The space $\mathcal{M}(\mathbb{R}^d)$ is a Banach space and the norm is given by the total variation

$$\|\mu\|_{\mathcal{M}} := \sup_{\|\Phi\|_{C_0}\leq 1} \int_{R^d} \Phi(v)\,\mathrm{d}\mu = |\mu|(\mathbb{R}^d).$$

We further denote $\mathcal{M}_1^+(\mathbb{R}^d)$ to be the space of probability measures $(\mu_x)_{x\in\Omega} \in \mathcal{M}(\mathbb{R}^d)$.

Since $C_0(\mathbb{R}^d)$ is a separable Banach space, we now have

$$L^1\big(\Omega; C_0(\mathbb{R}^d)\big)^* \cong L^\infty_{w^*}\big(\Omega; \mathcal{M}(\mathbb{R}^d)\big)$$

under the duality

$$\langle \psi, \mu \rangle = \int_\Omega \langle \psi(x,v), \mu_x \rangle\,\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^d} \psi(x,v)\,d\mu_x(v)\,\mathrm{d}x \tag{3.2.1}$$

for $\psi \in L^1\big(\Omega; C_0(\mathbb{R}^d)\big)$ and $\mu \in L^\infty_{w^*}\big(\Omega; \mathcal{M}(\mathbb{R}^d)\big)$. The norm in $L^\infty_{w^*}\big(\Omega; \mathcal{M}(\mathbb{R}^d)\big)$ is given by

$$\|\mu\|_{\infty;\mathcal{M}} := \operatorname*{ess\,sup}_{x\in\Omega} \|\mu\|_{\mathcal{M}} < \infty$$

and it is clearly a Banach space. This form of duality is extensively used in Sychev [115] and hence in our discussion.

The following result is a characterization of measurable functions in view of Lusin's theorem (see Sychev [115], Theorem 3.2):

Theorem 3.2.3 (Lusin type characterization of measurable functions). *Let $\Omega \subset \mathbb{R}^n$ be bounded and measurable. Let (M,d) be a compact metric space. A function $g\colon \Omega \to (M,d)$ is measurable if and only if for any $\varepsilon > 0$, there exists a compact set $\Omega_\varepsilon \subset \Omega$ such that $\operatorname{meas}(\Omega \setminus \Omega_\varepsilon) \leq \varepsilon$ and the restriction of g to Ω_ε is continuous.*

For $A \in \mathbb{R}^{N\times n}$ and $x \in \mathbb{R}^n$, we denote by Ax the vector defined by matrix multiplication and $l_A \colon \mathbb{R}^n \to \mathbb{R}$ a linear function such that $l_A(x) = Ax$ everywhere. If we let Ω be a bounded open set with $\mathrm{meas}(\partial\Omega) = 0$ and $\tilde{\Omega}$ be an arbitrary bounded open set, then it follows from the exhaustion argument that for any $\varepsilon > 0$, there exists a decomposition of $\tilde{\Omega}$,

$$\tilde{\Omega} = \bigcup_i (a_i + \varepsilon_i \bar{\Omega}) \cup N$$

where $i \in \mathbb{N}$, $\varepsilon_i < \varepsilon$ and N is a set of measure zero. We have the following:

Lemma 3.2.4 (Sychev [115], Lemma 2.2)**.** *Let $u_0 \in l_A + W_0^{1,p}(\Omega;\mathbb{R}^N)$. For each $j \in \mathbb{N}$, consider the decomposition of $\tilde{\Omega}$ given by*

$$\tilde{\Omega} = \bigcup_i (a_i^j + \varepsilon_i^j \Omega) \cup N_j$$

where $i \in \mathbb{N}$, $\varepsilon_i^j \leq \frac{1}{j}$ and the set N_j is of measure zero. Define

$$u_j(x) = \begin{cases} Ax + \varepsilon_i^j u_0\left(\frac{x-a_i^j}{\varepsilon_i^j}\right) & \textit{if } x \in a_i^j + \varepsilon_i^j \Omega, \\ Ax & \textit{elsewehere.} \end{cases}$$

Then

$$u_j \rightharpoonup l_A \textit{ in } W_0^{1,p}(\Omega;\mathbb{R}^N)$$

and the sequence $|Du_j|^p$ is equiintegrable with the same modulus of equiintegrability as the function $|Du_0|^p$ multiplied by the factor $\frac{\mathrm{meas}\,\tilde{\Omega}}{\mathrm{meas}\,\Omega}$.

For a compact set with non-zero measure, we have the following result that we will use on several occasions.

Proposition 3.2.5 (Sychev [115], Proposition 2.3)**.** *Let $K \subset \mathbb{R}^n$ be a compact set of non-zero measure. Then, for each $\eta > 0$, there exists an open set O_η (consisting, possibly, of several domains) with smooth boundary and such that*

$$\sup_{x\in O_\eta} \mathrm{dist}(x,K) \leq \eta$$

and

$$\mathrm{meas}\big((K \setminus O_\eta) \cup (O_\eta \setminus K)\big) \to 0 \textit{ as } \eta \to 0.$$

Lastly, it is worth mentioning about the decomposition lemma (Kristensen [82], Theorem 3.10) which shows a property of Sobolev spaces:

Theorem 3.2.6 (Decomposition lemma). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $p \in (1, \infty)$. Let $\{u_j\}$ be a sequence bounded in $W^{1,p}(\Omega; \mathbb{R}^N)$. Then there exists a subsequence $\{u_j\}$ (not relabeled) and a sequence $\{v_j\} \in W^{1,p}(\Omega; \mathbb{R}^N)$ such that $|v_j - u_j| + |D(v_j - u_j)| \to 0$ in measure and $|Dv_j|^p$ is equiintegrable.*

Young measure theory and measurable functions

Here, we give the important definitions and results by Sychev [115] showing how sequences of Young measures can be characterized by means of measurable functions with values in a special compact metric space to give a new approach to Young measure theory. We note that the results will be stated without proof and interested readers can refer to Appendix B (or Sychev [115]) for proofs.

Definition 3.2.7. *A family $(\mu_x)_{x\in\Omega} \subset \mathcal{M}_1^+(\mathbb{R}^d)$ is called a Young measure if there exists a sequence of measurable functions $z_j \colon \Omega \to \mathbb{R}^d$ such that for each $\Phi \in C_0(\mathbb{R}^d)$*

$$\Phi(z_j) \overset{*}{\rightharpoonup} \langle \Phi, \mu_x \rangle \text{ in } L^\infty(\Omega).$$

For this, we write $z_j \xrightarrow{Y} (\mu_x)_{x\in\Omega}$.

Let $K_m = \left\{\mu \in \mathcal{M}(\mathbb{R}^d) \colon \|\mu\|_{\mathcal{M}} \le m\right\}$ and let $\{\Phi_i \colon i \in \mathbb{N}\}$ $(\Phi_i \neq 0)$ be a dense set in $C_0(\mathbb{R}^d)$. We define the metric

$$\rho(\mu, \nu) := \sum_{i=1}^{\infty} \frac{1}{2^i \|\Phi_i\|_C} \left| \langle \Phi_i, \mu \rangle - \langle \Phi_i, \nu \rangle \right| \tag{3.2.2}$$

for $\mu, \nu \in \mathcal{M}(\mathbb{R}^d)$. This provides K_m with weak* topology for each $m > 0$. It follows from the Riesz representation theorem that (K_m, ρ) is a compact metric space.

Definition 3.2.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded measurable set. A family of Radon measures $(\mu_x)_{x\in\Omega}$, where $\mu_x \in K_m$ for a.e. $x \in \Omega$, is called weak* measurable if the function $\langle \Phi, \mu_{(\cdot)} \rangle$ is measurable for any $\Phi \in C_0(\mathbb{R}^d)$ and we denote the set of these families as $L^\infty_{w*}(\Omega; K_m)$. A sequence $(\mu_x^j)_{x\in\Omega}$ of such families converges weakly* to a family $(\mu_x)_{x\in\Omega}$ in $L^\infty_{w*}(\Omega; K_m)$ if*

$$\langle \Phi, \mu^j_{(\cdot)} \rangle \overset{*}{\rightharpoonup} \langle \Phi, \mu_{(\cdot)} \rangle \text{ in } L^\infty(\Omega) \text{ as } j \to \infty,$$

for each $\Phi \in C_0(\mathbb{R}^d)$ and for this, we use the notation

$$(\mu_x^j)_{x\in\Omega} \overset{*}{\rightharpoonup} (\mu_x)_{x\in\Omega} \text{ in } L^\infty_{w*}(\Omega; K_m).$$

It is clear that

$$L^\infty_{w*}\big(\Omega;\mathcal{M}(\mathbb{R}^d)\big)=\bigcup_m L^\infty_{w*}(\Omega;K_m),$$

and hence

$$L^1\big(\Omega;C_0(\mathbb{R}^d)\big)^*\cong\bigcup_m L^\infty_{w*}(\Omega;K_m).$$

The following lemma shows the equivalence of measurable functions and weak* measurable families defined as above, and can be proved by the use of Lusin type characterization of measurable functions (Theorem 3.2.3).

Lemma 3.2.9 (Sychev [115], Lemma 3.3)**.** *Let $\Omega\subset\mathbb{R}^n$ be a bounded measurable set. Let $\mu_x\in K_m$ for a.e. $x\in\Omega$. The family $(\mu_x)_{x\in\Omega}$ is weak* measurable if and only if $\mu\colon\Omega\to(K_m,\rho)$ is a measurable function.*

We now define the average $\mathrm{Av}(\mu_x)_{x\in\Omega}$ of weak* measurable family of measures as

$$\langle\Phi,\mathrm{Av}(\mu_x)_{x\in\Omega}\rangle:=\fint_\Omega\langle\Phi,\mu_x\rangle\,\mathrm{d}x$$

where $\Phi\in C_0(\mathbb{R}^d)$. Note that if $\mu_x\in K_m$ for a.e. $x\in\Omega$, then $\mathrm{Av}(\mu_x)_{x\in\Omega}$ is a linear functional over $C_0(\mathbb{R}^d)$ bounded in norm by m and hence $\mathrm{Av}(\mu_x)_{x\in\Omega}\in K_m$. It turns out that, quantitatively, the convergence of the elements in the family $\mathrm{Av}(\mu_x)_{x\in\Omega}$ can be transformed into convergence of the families.

Lemma 3.2.10. *Let $\Omega\subset\mathbb{R}^n$ be a bounded and measurable set and let $(\mu^1_x)_{x\in\Omega},(\mu^2_x)_{x\in\Omega}\in L^\infty_{w*}(\Omega;K_m)$. Then the following assertions hold:*

(i) If $\Omega_\delta\subset\Omega$ is measurable such that $\mathrm{meas}(\Omega\setminus\Omega_\delta)\le\delta\,\mathrm{meas}\,\Omega$ and $\rho(\mu^1_x,\mu^2_x)\le\delta$ for all $x\in\Omega_\delta$, then

$$\rho\left(\mathrm{Av}(\mu^1_x)_{x\in\Omega},\mathrm{Av}(\mu^2_x)_{x\in\Omega}\right)\le(2m+1)\delta.$$

(ii) If $\rho\big(\mu^j(\cdot),\mu(\cdot)\big)\to0$ in measure, where $(\mu_x)_{x\in\Omega},(\mu^j_x)_{x\in\Omega}\in L^\infty_{w*}(\Omega;K_m)$, then

$$(\mu^j_x)_{x\in\Omega}\overset{*}{\rightharpoonup}(\mu_x)_{x\in\Omega}\text{ in }L^\infty_{w*}(\Omega;K_m)\text{ as }j\to\infty.$$

One theorem that is of general importance in Young measure theory is the fundamental theorem for Young measures which has been established by Balder [8], Ball [11], Evans [43], Tartar [118, 119] and Young [130, 131]. It shows that all bounded sequences of Radon measure generate some Young measures up to a subsequence. The following version of the fundamental theorem for Young measures is due to Sychev [115] (Theorem 3.6):

Theorem 3.2.11 (Fundamental theorem for Young measures). *Let $\Omega \subset \mathbb{R}^n$ be a bounded measurable set. Let $(\mu_x^j)_{x\in\Omega} \in L^\infty_{w^*}(\Omega; K_m)$. Then there exists a subsequence $(\mu_x^j)_{x\in\Omega}$ (not relabeled) and a $(\mu_x)_{x\in\Omega} \in L^\infty_{w^*}(\Omega; K_m)$ such that*

$$(\mu_x^j)_{x\in\Omega} \overset{*}{\rightharpoonup} (\mu_x)_{x\in\Omega} \text{ in } L^\infty_{w^*}(\Omega; K_m).$$

If $(\mu_x^j)_{x\in\Omega}$ is a family of probability measures, then $(\mu_x)_{x\in\Omega}$ also consists of probability measures, provided there exists a function $g\colon R^d \to \mathbb{R}^+$ such that

$$\lim_{|v|\to\infty} g(v) = \infty$$

and

$$\int_\Omega \int_{\mathbb{R}^d} g(v)\,\mathrm{d}\mu_x^j\,\mathrm{d}x \le c.$$

In particular, each sequence of measurable functions $z_j\colon \Omega \to \mathbb{R}^d$ such that $\int_\Omega g\big(z_j(x)\big)\,\mathrm{d}x \le c$ contains a subsequence generating a Young measure.

The first part of the theorem is in fact a compactness result and the standard version easily follows from the duality of $L^\infty_{w^*}\big(\Omega; \mathcal{M}(\mathbb{R}^d)\big)$ and $L^1\big(\Omega; C_0(\mathbb{R}^d)\big)$ and Banach-Alaoglu theorem (see Ball [11], Evans [43], Kristensen [82]). The proof in the context of Young measures as measurable functions is contained in Sychev [116, 117].

For the remaining of the discussion, we will need the tightness for $(\mu_x^j)_{x\in\Omega} \in L^\infty_{w^*}(\Omega; K_m)$, $j \in \mathbb{N}$. Let $F(x, v)\colon \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^+$ be a Carathéodory integrand. Then $(\mu_x^j)_{x\in\Omega}$ is said to satisfy the tightness condition with the integrand F on $\Omega \subset \mathbb{R}^n$ if

$$\lim_{M\to\infty} \sup_j \int_\Omega \left(\int_{\mathbb{R}^d} \Psi_M\big(F(x,\cdot)\big)\,\mathrm{d}\mu_x^j \right) \mathrm{d}x = 0 \tag{3.2.3}$$

where $\Psi_M\colon \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying:

(i) $0 \le \Psi_M(t) \le t$ everywhere, and

(ii) $\Psi_M(t) = \begin{cases} 0 & \text{if } t < M \\ t & \text{if } t \ge M \end{cases}$.

Note that if $\mu_{(\cdot)}^j = \delta_{z_j(\cdot)}$ where δ_a denotes the usual Dirac mass, then (3.2.3) coincides with equi-integrability of the sequence $F\big(\cdot, z_j(\cdot)\big)$. Sychev [115] showed that the tightness condition on sequences of Young measures gives a characterization of these sequences, apart from the usual lower semicontinuity result.

Theorem 3.2.12 (Sychev [115], Theorem 3.7)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded measurable set. Let $(\mu_x^j)_{x\in\Omega}$ be a sequence of families of probability measures with support in $\mathbb{R}^d$ and let $F(x,v)\colon \Omega\times\mathbb{R}^d \to \mathbb{R}$ be a Carathéodory integrand. Suppose that $(\mu_x^j)_{x\in\Omega}$ satisfies (3.2.3) with the negative part F^- of F and $(\mu_x^j)_{x\in\Omega}$ generates a family of probability measures $(\mu_x)_{x\in\Omega}$. Then*

$$\liminf_{j\to\infty} \int_\Omega \left(\int_{\mathbb{R}^d} F(x,v)\,\mathrm{d}\mu_x^j\right)\,\mathrm{d}x \geq \int_\Omega \left(\int_{\mathbb{R}^d} F(x,v)\,\mathrm{d}\mu_x\right)\,\mathrm{d}x. \tag{3.2.4}$$

Moreover

$$\lim_{j\to\infty} \int_\Omega \int_{\mathbb{R}^d} F(x,v)\,\mathrm{d}\mu_x^j\,\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^d} F(x,v)\,\mathrm{d}\mu_x\,\mathrm{d}x \tag{3.2.5}$$

if and only if $(\mu_x^j)_{x\in\Omega}$ satisfies (3.2.3) with $|F|$. In this case

$$\int_{\mathbb{R}^d} F(\cdot,v)\,\mathrm{d}\mu_{(\cdot)}^j \rightharpoonup \int_{\mathbb{R}^d} F(\cdot,v)\,\mathrm{d}\mu_{(\cdot)} \text{ in } L^1. \tag{3.2.6}$$

The following result sums up the properties of sequences measurable functions that generates Young measures.

Proposition 3.2.13 (Sychev [115], Proposition 3.8)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded measurable set. Let $z_j\colon \Omega \to \mathbb{R}^d$ be a sequence of measurable functions such that $z_j \xrightarrow{Y} (\mu_x)_{x\in\Omega}$. Then the following assertions hold:*

(i) The sequence z_j converges in measure if and only if μ_x is a Dirac mass for a.e. $x \in \Omega$.

(ii) If there exists a sequence $y_j\colon \Omega \to \mathbb{R}^d$ such that $z_j - y_j \to 0$ in measure as $j \to \infty$, then $y_j \xrightarrow{Y} (\mu_x)_{x\in\Omega}$.

Gradient p-Young measure theory

Next, we will discuss on gradient p-Young measure theory which is basically generated by the gradient of weakly convergent sequences. The relaxation theorem or precisely the quasiconvexification of the integrand F is given with respect to the gradient variable ξ and hence the importance of gradient p-Young measure theory. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $\operatorname{meas}(\partial\Omega) = 0$.

Definition 3.2.14. *A Young measure $(\mu_x)_{x\in\Omega}$ is a gradient p-Young measure, $p \in [1,\infty)$, if it is generated by gradients Du_j of a sequence $u_j \in W^{1,p}(\Omega;\mathbb{R}^N)$ such that $u_j \rightharpoonup u_0$ in $W^{1,p}(\Omega;\mathbb{R}^N)$ and $|Du_j|^p$ is equiintegrable. For this, we write $u_j \xrightarrow{GY^p} (\mu_x)_{x\in\Omega}$.*

If $u_j \rightharpoonup u_0$ in $W^{1,p}(\Omega;\mathbb{R}^N)$, then it follows from Theorem 3.2.12 that $Du_0(x) = \int_{\mathbb{R}^{N\times n}} (\cdot)\mathrm{d}\mu_x$ for a.e. $x \in \Omega$. This function u_0 is called the underlying deformation.

Definition 3.2.15. *A Young measure $(\mu_x)_{x\in\Omega}$ is called a homogeneous Young measure if μ_x does not depend on x. We denote $GM_p(A)$ as the set of all homogeneous gradient p-Young measures with the centre of mass at A and $GM_\infty(A)$ as the set of all homogeneous measures generated by gradients of sequences converging weakly* in $W^{1,\infty}(\Omega;\mathbb{R}^N)$.*

The following results give the conditions in which Young measure generated by gradients of minimizing sequences are actually gradient p-Young measures generated by sequences in the space of $C_0^\infty(\Omega;\mathbb{R}^N)$ (see also Kinderlehrer-Pedegral [72], Kristensen [82]).

Proposition 3.2.16 (Sychev [115], Proposition 4.4)**.**

(i) Let $p\in(1,\infty)$ and let $(\mu_x)_{x\in\Omega}$ be a Young measure generated by gradients of a sequence u_j bounded in $W^{1,p}(\Omega;\mathbb{R}^N)$. Let u_0 be the underlying deformation. Then $(\mu_x)_{x\in\Omega}$ is a gradient p-Young measure generated by a sequence $v_j\in u_0+C_0^\infty(\Omega;\mathbb{R}^N)$.

(ii) Let $p\in[1,\infty)$ and let $u_j\rightharpoonup u_0\in W^{1,p}(\Omega;\mathbb{R}^N)$ in $W^{1,p}_{\text{loc}}(\Omega;\mathbb{R}^N)$. Let $|Du_j|^p$ be equiintegrable and $Du_j\xrightarrow{Y}(\mu_x)_{x\in\Omega}$. Then there exists a sequence $v_j\in u_0+C_0^\infty(\Omega;\mathbb{R}^N)$ such that $v_j\xrightarrow{GY^p}(\mu_x)_{x\in\Omega}$.

It is easy to see that we can make use of the decomposition lemma (Theorem 3.2.6) in the proof of Proposition 3.2.16(i). The proposition below gives the conditions for sequences of gradient p-Young measures with underlying deformations to generate a gradient p-Young measure.

Proposition 3.2.17 (Sychev [115], Proposition 4.5)**.** *Let $p\in[1,\infty)$. Suppose that $(\mu_x^j)_{x\in\Omega}$ is a sequence of gradient p-Young measures such that $(\mu_x^j)_{x\in\Omega}\overset{*}{\rightharpoonup}(\mu_x)_{x\in\Omega}$ as $j\to\infty$ and the underlying deformations u_j are equibounded in $W^{1,p}(\Omega;\mathbb{R}^N)$.*

(i) Let $p>1$ and $\int_\Omega\int_{\mathbb{R}^{N\times n}}(|v|^p+1)\,\mathrm{d}\mu_x^j\,\mathrm{d}x<c$ for $v\in\mathbb{R}^{N\times n}$. Then $(\mu_x)_{x\in\Omega}$ is a gradient p-Young measure.

(ii) Assume that the sequence $(\mu_x^j)_{x\in\Omega}$ satisfies the tightness condition (3.2.3) with the integrand $(|\cdot|^p+1)$. Then $(\mu_x)_{x\in\Omega}$ is a gradient p-Young measure.

Young measures and relaxation theorem

Gradient p-Young measure provides a direct proof for the relaxation theorem. We first give results leading to the characterization of gradient p-Young measures which will then be followed by results leading to the relaxation theorem.

Theorem 3.2.18 (Kinderlehrer-Pedegral [70, 71, 72], Sychev [115], Theorem 4.2)**.** *Let $p \in [1, \infty)$ and let $(\mu_x)_{x\in\Omega}$ be a gradient p-Young measure with underlying deformation u_0. Then*

(i) ***(Averaging principle)*** *If there exists an $A \in \mathbb{R}^{N\times n}$ such that $u_0 - l_A \in W_0^{1,p}(\Omega; \mathbb{R}^N)$, then $\mathrm{Av}(\mu_x)_{x\in\Omega} \in GM_p(A)$. If $u_0 - l_A \in W_0^{1,\infty}(\Omega; \mathbb{R}^N)$, then $\mathrm{Av}(\delta_{Du_0(x)})_{x\in\Omega} \in GM_\infty(A)$.*

(ii) ***(Localization principle)*** *For a.e. $x \in \Omega$, the measure μ_x is a homogeneous gradient p- Young measure.*

The averaging and localization principles can be used to obtain necessary and sufficient conditions for quasiconvexity of an integrand F and also a representation formula for quasiconvex envelope, which is similar to that of Dacorogna's relaxation theorem (Theorem 3.1.3). We will state a series of results which will be eventually used to prove the relaxation theorem and the proofs of the results are given in the Appendix B.

Corollary 3.2.19 (Sychev [115], Corollary of Theorem 4.2)**.** *Let $p \in [1, \infty)$ and let $F \colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a continuous function satisfying*

$$|F(\xi)| \le a|v|^p + b$$

where constants $a, b > 0$. Then

(i) we have

$$\inf_{\mu\in GM_p(A)} \langle F, \mu\rangle = \inf_{\mu\in GM_\infty(A)} \langle F, \mu\rangle = \inf_{\varphi\in C_0^\infty(\Omega;\mathbb{R}^N)} \fint_\Omega F\big(A + D\varphi(x)\big)\,\mathrm{d}x$$

(ii) the function F is quasiconvex at A if and only if

$$\inf_{\mu\in GM_p(A)} \langle F, \mu\rangle \ge F(A).$$

Theorem 3.2.20 (Sychev [115], Theorem 4.6)**.** *Let $p \in [1, \infty)$ and let $F \colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a continuous function satisfying*

$$a_1|v|^p + b_1 \le F(v) \le a_2|v|^p + b_2, . \tag{3.2.7}$$

where constants $a_2 \ge a_1 > 0$ and $b_2 \ge b_1 > 0$. Then the quasiconvex envelope F^{qc} exists and is given by the formula

$$F^{qc}(A) = \inf_{\mu\in GM_p(A)} \int_{\mathbb{R}^{N\times n}} F(v)\,\mathrm{d}\mu. \tag{3.2.8}$$

Moreover, F^{qc} is continuous and has the same lower and upper bounds as in (3.2.7).

The following result on the characterization of gradient p-Young measure was first established by Kinderlehrer-Pedegral [70], Section 4–6. The version that we give here is due to Sychev [115], Theorem 4.3.

Theorem 3.2.21 (Characterization of gradient p-Young measures). *Let $p \in [1, \infty)$. A family $(\mu_x)_{x\in\Omega} \in L^\infty_{w*}(\Omega; K_1)$ of probability measures is a gradient p-Young measure if and only if*

(i) there exists $u_0 \in W^{1,p}(\Omega; \mathbb{R}^N)$ such that $Du_0(x) = \int_{\mathbb{R}^{N\times n}} (\cdot)\, \mathrm{d}\mu_x$ for a.e. $x \in \Omega$,

(ii) for a.e. $x \in \Omega$, $F\big(Du_0(x)\big) \le \int_{\mathbb{R}^{N\times n}} F(v)\, \mathrm{d}\mu_x$ holds for any quasicovex function F satisfying $c \le F(v) \le a|v|^p + b$ where constants $c < \infty$ and $a, b > 0$, and

(iii) $\int_\Omega \int_{\mathbb{R}^{N\times n}} (|v|^p + 1)\, \mathrm{d}\mu_x\, \mathrm{d}x < \infty$.

Recall that quasiconvexity is a necessary and sufficient conditions for lower semicontinuity. It turns out that quasiconvexity is intimately related to equiintegrability which plays a key role in Young measure theory.

Theorem 3.2.22 (Sychev [115], Theorem 5.1)**.** *Let $p \in [1, \infty)$. Let $u_0 \in W^{1,p}(\Omega; \mathbb{R}^N)$ and $F \colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ be a Carathéodory integrand such that*

$$|F(x, u, v)| \le a|v|^p + b$$

where constants $a, b > 0$.

(i) If $F\big(x, u_0(x), \cdot\big)$ is quasiconvex at $Du_0(x)$ for a.e. $x \in \Omega$, then

$$\liminf_{j\to\infty} \mathcal{F}(u_j, \Omega) \ge \mathcal{F}(u_0, \Omega) \text{ for any sequence } u_j \rightharpoonup u_0 \text{ in } W^{1,p}(\Omega; \mathbb{R}^N)$$

such that the negative parts of $F\big(x, u_j(x), Du_j(x)\big)$ are equiintegrable.

(ii) If $\liminf_{j\to\infty} \mathcal{F}(u_j, \Omega) \ge \mathcal{F}(u_0, \Omega)$ for any sequence $u_j \rightharpoonup u_0$ in $W^{1,p}(\Omega; \mathbb{R}^N)$ such that the sequence $|Du_j|^p$ is equiintegrable, then the function $F\big(x, u_0(x), \cdot\big)$ is quasiconvex at $Du_0(x)$ for a.e. $x \in \Omega$.

We are now ready to prove the relaxation theorem by the use of Young measures.

Theorem 3.2.23 (Sychev [115], Theorem 1.2)**.** *Let $p \in (1, \infty)$ and let $F \colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ be a Carathéodory integrand satisfying*

$$c_1|\xi|^p + a_1(x, u) \le F(x, u, \xi) \le c_2|\xi|^p + a_2(x, u), \tag{3.2.9}$$

where $a_1, a_2 \in L^1(\Omega \times \mathbb{R}^N)^+$ *and constants* $c_2 \geq c_1 > 0$. *Let*

$$F^{qc}(x, u, \xi_0) := \inf_{\varphi \in C_0^\infty(\Omega;\mathbb{R}^N)} \fint_\Omega F\big(x, u, \xi_0 + D\varphi(y)\big)\,\mathrm{d}y \tag{3.2.10}$$

and

$$\mathcal{F}^{qc}(u, \Omega) := \int_\Omega F^{qc}\big(x, u(x), Du(x)\big)\,\mathrm{d}x.$$

Then

(i) F^{qc} *is a Carathéodory integrand satisfying*

$$c_1|\xi|^p + a_1(x, u) \leq F^{qc}(x, \xi) \leq c_2|\xi|^p + a_2(x, u) \tag{3.2.11}$$

and the function $\xi \mapsto F^{qc}(x, u, \xi)$ *is quasiconvex for almost every* $x \in \Omega$ *and for all* $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N\times n}$,

(ii) $\liminf_{j\to\infty} \mathcal{F}(u_j, \Omega) \geq \mathcal{F}^{qc}(\bar{u}, \Omega)$ *if* $u_j \rightharpoonup \bar{u}$ *in* $W^{1,p}(\Omega; \mathbb{R}^N)$, *and*

(iii) the weak lower semicontinuous envelope of the functional $\mathcal{F}$ *is equal to* $\mathcal{F}^{qc}$*:*

$$\bar{\mathcal{F}}(u, \Omega) = \mathcal{F}^{qc}(u, \Omega).$$

In particular, there exists a sequence $u_j \in \bar{u} + C_0^\infty(\Omega; \mathbb{R}^N)$ *such that*

$$\begin{cases} |Du_j|^p \text{ is equiintegrable}, \\ u_j \rightharpoonup \bar{u} \text{ in } W^{1,p}(\Omega; \mathbb{R}^N), \\ \mathcal{F}(u_j, \Omega) \to \mathcal{F}^{qc}(\bar{u}, \Omega) \text{ as } j \to \infty. \end{cases}$$

Proof. Let $\Omega_j \subset \Omega$ be a sequence of compact sets such that $\operatorname{meas}(\Omega \setminus \Omega_j) \to 0$ as $j \to \infty$ and the restrictions of F to $\Omega_j \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$ are continuous. It follows from Theorem 3.2.20 that for each $(x, u) \in \Omega_j \times \mathbb{R}^N$,

$$F^{qc}(x, u, \cdot) = \inf_{\mu \in GM_p(\cdot)} \int_{\mathbb{R}^{N\times n}} F(x, u, \xi)\,\mathrm{d}\mu(\xi)$$

is a continuous and quasiconvex function satisfying (3.2.11). Let

$$V(x, u, A) = \left\{\mu \in GM_p(A)\colon\ \min \int_{\mathbb{R}^{N\times n}} F(x, u, \cdot)\mathrm{d}\mu(\cdot)\right\}.$$

It is clear from Theorem 3.2.12 and Proposition 3.2.17 that $V(x, u, A)$ is a non-empty compact set in the metric space (K_1, ρ).

We will now prove the continuity of F^{qc} restricted to $\Omega_j \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$. If we let $\mu^k \in V(x_k, u_k, \xi_k)$ and $(x_k, u_k, \xi_k) \to (x_0, u_0, \xi_0)$ as $k \to \infty$, then we obtain

$$\liminf_{k\to\infty} \langle F(x_k, u_k, \cdot), \mu^k \rangle = \liminf_{k\to\infty} F^{qc}(x_k, u_k, \xi_k).$$

For a subsequence μ^j of μ^k, we have

$$\liminf_{k\to\infty} \langle F(x_k, u_k, \cdot), \mu^k \rangle = \lim_{j\to\infty} \langle F(x_j, u_j, \cdot), \mu^j \rangle$$

and

$$\mu^j \overset{*}{\rightharpoonup} \mu_0 \text{ in } \mathcal{M}(\mathbb{R}^{N\times n}).$$

Proposition 3.2.17 implies that $\mu_0 \in GM_p(v_0)$ and subsequently by Theorem 3.2.12, we have

$$\langle F(x_0, u_0, \cdot), \mu_0 \rangle \le \liminf_{k\to\infty} \langle F(x_k, u_k, \cdot), \mu^k \rangle,$$

which proves the lower semicontinuity of F^{qc} restricted to $\Omega_j \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$ at (x_0, u_0, ξ_0).

To prove upper semicontinuity, if $\mu_0 \in V(x_0, u_0, \xi_0)$, then

$$\mu_0 \diamond \xi_k \in GM_p(\xi_k)$$

and

$$\langle |\cdot|^p + 1, \mu_0 \diamond \xi_k \rangle \to \langle |\cdot|^p + 1, \mu_0 \rangle \text{ as } k \to \infty,$$

which simply leads to

$$\lim_{k\to\infty} \langle F(x_k, u_k, \cdot), \mu_0 \diamond \xi_k \rangle = \langle F(x_0, u_0, \cdot), \mu_0 \rangle$$

by Theorem 3.2.12. It then follows that

$$\limsup_{k\to\infty} F^{qc}(x_k, u_k, \xi_k) \le \limsup_{k\to\infty} \langle F(x_k, u_k, \cdot), \mu_0 \diamond \xi_k \rangle = \langle F(x_0, u_0, \cdot), \mu_0 \rangle$$

and hence the upper semicontinuity of F^{qc} restricted to $\Omega_j \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$ at (x_0, u_0, ξ_0). Therefore we have proved the continuity of F^{qc} restricted to $\Omega_j \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$ at (x_0, u_0, ξ_0) and this holds for all $j \in \mathbb{N}$.

Next fix $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ and consider compact sets $\tilde{\Omega}_j \subset \Omega_j$ such that $\operatorname{meas}(\Omega_j \setminus \tilde{\Omega}_j) \le \frac{1}{j}$ and the restrictions of u and Du to $\tilde{\Omega}_j$ are continuous. For $x \in \Omega$, consider the multivalued mapping $P \colon \Omega \to V\big(x, u_0(x), Du_0(x)\big)$. Let x be such that the function $\xi \mapsto F\big(x, u_0(x), \xi\big)$ is continuous. Once again, by Theorem 3.2.12 and Proposition 3.2.17, $V\big(Du_0(x)\big)$ is a non-empty compact set in the metric ρ and $P(x)$ is closed for a.e. $x \in \Omega$.

Since the restriction of $F^{qc}\big(\cdot, u_0(\cdot), Du_0(\cdot)\big)$ to $\tilde{\Omega}_j$ is continuous, we have that the restriction of P to $\tilde{\Omega}_j$ is upper semicontinuous and hence P is measurable in Ω. By Theorem B.1.2, there exists a measurable selection of P and this selection is in fact a gradient p-Young measure by Theorem 3.2.21. For a sequence u_j associated with $(\mu_x)_{x\in\Omega}$, we have that

$$(u_j, Du_j) \xrightarrow{Y} (\delta_{u_0(x)} \otimes \mu_x)_{x\in\Omega}$$

and $|Du_j|^p$ is equiintegrable. Theorem 3.2.12 asserts that

$$\begin{aligned}\lim_{j\to\infty} \mathcal{F}(u_j, \Omega) &= \int_\Omega \int_{\mathbb{R}^{N\times n}} F\big(x, u_0(x), \cdot\big)\, \mathrm{d}\mu_x\, \mathrm{d}x \\ &= \int_\Omega F^{qc}\big(x, u_0(x), Du_0(x)\big)\, \mathrm{d}x = \mathcal{F}^{qc}(u_0, \Omega).\end{aligned}$$

By Corollary 3.2.19, the identity (3.2.10) holds for a.e. $x \in \Omega$ and for all $u \in \mathbb{R}^N$. Finally, it follows from Theorem 3.2.22 that F^{qc} is lower semicontinuous,

$$\lim_{j\to\infty} \mathcal{F}^{qc}(u_j, \Omega) \geq \mathcal{F}^{qc}(u_0, \Omega) \text{ whenever } u_j \rightharpoonup u_0 \text{ in } W^{1,p}(\Omega; \mathbb{R}^N)$$

and since $\mathcal{F}(u, \Omega) \geq \mathcal{F}^{qc}(u, \Omega)$ for $u \in W^{1,p}(\Omega; \mathbb{R}^N)$, we obtain the required result. □

Chapter 4

Quasiminimizers

In Chapter 2, we have discussed about the necessary and sufficient condition for lower semicontinuity and hence existence of minimizer for the variational problem (*P*). The fact that the functions considered are in a Sobolev space leads to the second issue in the direct method, namely regularity of the minimizers, since the functions (and hence minimizers) have derivatives only in a weak sense and therefore will not be continuous in general. The problem of proving that the minimizers of regular functionals are continuous, or differentiable, etc remained unsolved for a long time. The success of De Giorgi [35] and independently by Nash [108] in providing a positive response to Hilbert's 19th problem in the scalar case sparked more development in the study of regularity of minimizers. Briefly speaking, De Giorgi proved that if u is a solution to the differential equation in divergence form

$$D_i\Big(a_{ij}(x)D_jv\Big) = 0$$

where the coefficients a_{ij} are measurable and bounded (in particular may not be continuous) and satisfies the ellipticity condition

$$a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2, \quad \nu > 0,$$

then u is Hölder continuous. A function $u : \Omega \to \mathbb{R}$ is said to be Hölder continuous with exponent $\alpha \in (0, 1]$ in Ω if

$$[u]_{C^{0,\alpha}(\Omega)} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty.$$

The space of such functions is denoted by $C^{0,\alpha}$, and we may use the term $C^{0,\alpha}$-regularity for Hölder continuous functions. When $\alpha = 1$, we have Lipschitz continuity.

Between 1960 and 1961, Moser [105, 106] gave a new and elegant proof of regularity where he extended the classical Harnack inequality to general elliptic equations. This framework of Moser, together with that of De Giorgi and Nash, is known nowadays as De Giorgi-Nash-Moser theorem and has been extended in various directions, particularly by Ladyzenskaya-Uraltseva [86] to bounded solutions of non-linear elliptic equations.

However, the regularity theorem does not extend to vector-valued functions and this was verified by De Giorgi [37] who gave a counterexample for the linear elliptic system that was originally considered in his fundamental paper [35]. This was further extended by Giusti-Miranda [58] to non-linear systems with regular functionals. Around the same time, Maz'ya [94], independently, also found other counterexamples in this direction and in 1975 Nécas [109] showed for more general problems.

By adapting the ideas of De Giorgi [36] and Almgren's study of minimal surfaces [7], Morrey [104] proved partial regularity of weak solutions of non-linear elliptic system. This simply means that if the system is considered in Ω, then the regularity holds in an open set $\tilde{\Omega} \subset \Omega$ where $\operatorname{meas}(\Omega \setminus \tilde{\Omega}) = 0$. Among other authors who subsequently extended the result of partial regularity are Giusti [56], Giusti-Miranda [59] and Giaquinta-Giusti [53].

Proving regularity in the framework of De Giorgi-Nash-Moser has a drawback on its own - all the theorems only apply to $\mathcal{F}$-minimizers obtained by solving the associated Euler-Lagrange equation. It was not until 1982 that Giaquinta-Giusti [54] managed to prove Hölder continuity of minimizers of the functional $\mathcal{F}$ where there is no differentiability assumption on F. This leads to the use of the direct method for regularity problems. Regularity of $\mathcal{F}$-minimizers is an intriguing problem on its own and the present work is interested in the regularity properties of quasiminimizers that as we shall see in Chapter 7 are indeed also true minimizers. Though similar notions (for example almost minimizers) had been considered before in Geometric Measure Theory by Almgren, Bombieri and others, the formal introduction of quasiminimizers was made by Giaquinta-Giusti [55] as a tool for unified treatment of variational integrals, elliptic equations and quasiregular mappings in $\mathbb{R}^n$. They also showed that, with reference to their previous paper [54], De Giorgi's method can be extended to quasiminimizers and hence they are essentially Hölder continuous in the scalar case.

Shortly afterwards, DiBenedetto-Trudinger [38] proved Harnack inequality for quasiminimizers as well as weak Harnack inequalities for subquasiminimizers and superquasiminimizers. Ziemer [134] then gave a Wiener-type criterion sufficient for boundary regularity for quasiminimizers and also in the same year, Tolksdorf [121] obtained a Caccioppoli inequality and a convexity result for quasiminimizers.

Over the past decade or so, the study of quasiminimizer has experienced a rather steady growth, beginning with its extension to metric space, notably by Kinnunen-Shanmugalingam [74] and Björn [18]. Kinnunen-Martio [73] went on to establish potential theory for quasiminimizers, in which they showed the connection between superquasiharmonic functions and superquasiminimizers, in a similar way to that of superharmonic functions and superminimizers. The latest developement is probably in the sense of Björn-Björn [17] where they introduced radial (power-type) quasiminimizers. They presented the optimal quasiminimizing constant whenever the radial function and log-power function are quasiminimizers of the p-Dirichlet integral when $p \neq n$ and $p = n$, respectively. Similar results have been obtained when u is a super(quasi)minimizer and sub(quasi)minimizer. Björn-Björn concluded their paper with a brief discussion on local quasiminimizers and used the results of radial quasiminimizers to verify the non-locality of quasiminimizers as pointed out by Kinnunen-Martio [73].

The outline of this chapter is as follows: Section 4.1 will begin with the definition of quasiminimizers as introduced by Giaquinta-Giusti [55] and we make sense of the role played by quasiminimizers in unifying the treatment of different problems. As we have seen from above, quasiminimizers considered in different problems may be denoted in a particular way. The examples of vector-valued quasiminimizers including elliptic systems, quasiregular mappings and cubical and spherical quasiminimizers were given by Giaquinta-Giusti [55] for which we discuss in Section 4.2. We also mention briefly about regularity results that work already for scalar-valued quasiminimizers considered in the elliptic systems. Section 4.3 will see a series of more 'modern' quasiminimizers which are also scalar-valued quasiminimizers, namely one-dimensional, radial and local quasiminimizers.

4.1 Definitions

We begin by considering the functional

$$\mathcal{F}(u,\Omega) = \int_\Omega F\big(x, u(x), Du(x)\big)\,\mathrm{d}x$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set and $F\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is a Carathéodory integrand $F(x,u,\xi)$ satisfying the inequality

$$0 \le F(x,u,\xi) \le L|\xi|^p + b(x)|u|^\gamma + a(x) \tag{4.1.1}$$

with $1 < p \le \gamma < p^* = \frac{np}{n-p}$, so we always assume that $p < n$. Here $c > 0$ is a constant, $b \in L^{\frac{p^*}{p^*-\gamma}}(\Omega)^+$ and $a \in L^1(\Omega)^+$. The case $p \ge n$ is simpler since the Sobolev embedding theorem tells us that u belongs to every L^q when $p = n$ and it is Hölder continuous when $p > n$, and hence these cases can be incorporated by doing minor modifications in the proofs.

Definition 4.1.1. *Let $Q \ge 1$. We say that $u \in W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^N)$ is a quasiminimizer of the functional $\mathcal{F}$ if*

$$\mathcal{F}(u, K) \le Q\mathcal{F}(v, K) \tag{4.1.2}$$

for all $v \in W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^N)$ with $K =: \operatorname{supp}(v - u) \Subset \Omega$.

Sometimes, quasiminimizers are also called Q-quasiminimizers with reference to the constant Q in (4.1.2) and we shall use the latter for the rest of this thesis. The notion of Q-quasiminimizers can be viewed as perturbations of minimizers of variational integrals. Note that if $Q = 1$, then u is also a standard minimizer of the functional $\mathcal{F}$. When $N = 1$ so that the functions u and v above are real-valued, then u is a superquasiminimizer if inequality (4.1.2) holds for all $v - u \ge 0$. Likewise, u is a subquasiminimizer if inequality (4.1.2) holds for all $v - u \le 0$.

Though the definition above is a standard definition and holds for simple functionals such as p-Dirichlet integrals, it is in fact far too strong for general signed functionals. Therefore, we generalize the definition to signed local functionals. We first denote by $\mathcal{O}(\Omega)$ the family of all open subsets of Ω. We say that a functional $\mathcal{F}$ is local on $\mathcal{O}(\Omega)$ if for $u, v \in W^{1,p}(\Omega; \mathbb{R}^N)$

$$\mathcal{F}(u, \omega) = \mathcal{F}(v, \omega) \text{ whenever } u = v \text{ a.e. on } \omega \in \mathcal{O}(\Omega).$$

More details about local functionals can be found in Alberti [3].

Now let $\mathcal{F}\colon W^{1,p}(\Omega; \mathbb{R}^N) \times \mathcal{O}(\Omega) \to \mathbb{R}$ be a local functional satisfying the coercivity and growth conditions

$$L_1 \int_\omega |Du(x)|^p \,\mathrm{d}x - \mu(\omega) \le \mathcal{F}(u, \omega) \le L_2 \int_\omega |Du(x)|^p \,\mathrm{d}x + \mu(\omega) \tag{4.1.3}$$

for all $u \in W^{1,p}(\omega; \mathbb{R}^N)$ and for all $\omega \in \mathcal{O}(\Omega)$, where $L_2 \ge L_1 > 0$ are constants and μ is a bounded Radon measure on Ω. It is clear that instead of imposing some coercivity and growth conditions on the integrand, we considered the more relaxed conditions for local functionals, which is more general and we have taken the sign of $\mathcal{F}$ into consideration.

Definition 4.1.2. *Let $Q \ge 0$. We say that $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ is a Q-quasiminimizer of the functional $\mathcal{F}$ if*

$$\mathcal{F}(u, \omega) \le \mathcal{F}(v, \omega) + Q\left(\int_\omega |Dv(x)|^p \,\mathrm{d}x + \mu(\omega)\right) \tag{4.1.4}$$

for all $u, v \in W^{1,p}(\omega, \mathbb{R}^N)$ *with* $v \in W^{1,p}_u(\omega, \mathbb{R}^N)$ *and for all* $\omega \in \mathcal{O}(\Omega)$.

Similarly, u is a standard minimizer of the functional $\mathcal{F}$ if $Q = 0$. In reality, the functional $\mathcal{F}$ can be very complicated but the application of the notion of Q-quasiminimizers leads to a wide and flexible class of maps under general circumstances. Particularly, a map can be a minimizer for a very complicated and irregular integrand $F = F(x, u, Du)$. If the functional $\mathcal{F}(u, \Omega) = \int_\Omega F(x, u, Du)\,\mathrm{d}x$ satisfies the coercivity and growth conditions (4.1.3) with $\mu = \mathcal{L}^n$, then the minimizer is a Q-quasiminimizer of

$$\int_\Omega (|Du(x)|^p + 1)\,\mathrm{d}x$$

which appears much simpler. It is easy to see that the utility of Q-quasiminimizers unifies the treatment of different problems, thus providing a wide range of applications without resorting to the corresponding Euler equation. Apart from that, certain aspects of regularity theory works already for quasiminimizers including $C^{0,\alpha}$-regularity in the scalar case and higher integrability. A more recent result of Martio [92] shows that Q-quasiminimizers, when perturbed, are more flexible than solutions of differential equations.

Theorem 4.1.3 (Martio [92], Theorem 4.1)**.** *Let* $Q \geq 1$ *and let* u *be a* Q*-quasiminimizer of the* p*-Dirichlet integral* $\int_\Omega |Dv|^p\,\mathrm{d}x$. *Let* $g \in W^{1,p}_{\mathrm{loc}}(\Omega)$ *be such that* $|Dg| \leq c|Du|$ *a.e. in* Ω*, where* $c \in \left(0, Q^{-1/p}\right)$. *Then* $u + g$ *is a* Q'*-quasiminimizer with*

$$Q' = \frac{(1+c)^p}{(Q^{-1/p} - c)^p}.$$

4.2 Examples of vector-valued quasiminimizers

In this section, we will give some examples of vector-valued Q-quasiminimizers which were first considered in the paper of Giaquinta-Giusti [55]. We start with its relation to weak solutions of elliptic systems. We also briefly mention about regularity results for scalar-valued cases and leave the discussion on vector-valued functions to the next chapter.

Quasiminimizers and elliptic systems

The following discussion is partly taken from Giaquinta-Giusti [55] (see also Giusti [57]). In this section, we will explore the relation between weak solutions and Q-quasiminimizers. We first consider general systems in divergence form

$$\frac{\partial}{\partial x_i} A_i^\alpha\big(x, u(x), Du(x)\big) - B^\alpha\big(x, u(x), Du(x)\big) = 0. \tag{4.2.1}$$

where we use the usual summation convention of summing over repeated Latin indices from 1 to n and the Greek ones from 1 to N. If we multiply (4.2.1) with test functions $\varphi \in W_0^{1,p}(\Omega; \mathbb{R}^N)$ and followed by integration by parts, then we obtain the weak form

$$\int_\Omega \Big\{ A_i^\alpha\big(x, u(x), Du(x)\big) D_i\varphi^\alpha + B^\alpha\big(x, u(x), Du(x)\big)\varphi^\alpha \Big\} \, \mathrm{d}x = 0. \tag{4.2.2}$$

Note that if the coefficients A_i^α are differentiable, then we obtain the usual Euler-Lagrange equation.

Definition 4.2.1. *A function $u \in W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^N)$ is a weak solution if (4.2.2) holds for every $\varphi \in W_0^{1,p}(\Omega; \mathbb{R}^N)$.*

We begin by examining the simple case of linear systems,

$$\int_\Omega a_{ij}^{\alpha\beta}(x) D_j u^\beta D_i \varphi^\alpha \, \mathrm{d}x = 0 \tag{4.2.3}$$

for all $\varphi \in W_0^{1,2}(\Omega; \mathbb{R}^N)$ where the coefficients $a_{ij}^{\alpha\beta}(x)$ are bounded and measurable functions

$$\|a_{ij}^{\alpha\beta}\|_\infty \leq M \tag{4.2.4}$$

satisfying the strong ellipticity conditions

$$a_{ij}^{\alpha\beta}(x)\xi_i^\alpha \xi_j^\beta \geq \nu|\xi|^2, \quad \nu > 0. \tag{4.2.5}$$

We can easily show that if u is a weak solution, then it is also a Q-quasiminimizer of the Dirichlet integral. Let $v \in W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^N)$ such that $v = u$ outside a compact set $K \Subset \Omega$. Then, setting $\varphi = v - u$ and substituting into (4.2.3), we have

$$\int_K a_{ij}^{\alpha\beta}(x) D_j u^\beta D_i u^\alpha \, \mathrm{d}x = \int_K a_{ij}^{\alpha\beta}(x) D_j u^\beta D_i v^\alpha \, \mathrm{d}x.$$

By (4.2.4) and (4.2.5), we obtain

$$\nu \int_K |Du|^2 \, \mathrm{d}x \leq M \int_K |Du||Dv| \, \mathrm{d}x \leq M \left(\int_K |Du|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_K |Dv|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}$$

where the last inequality is obtained by the use of Hölder's inequality. It follows from simple algebra that u is indeed a Q-quasiminimizer of the Dirichlet integral with $Q = \frac{M^2}{\nu^2}$.

The following result is from [134], which can be easily elaborated to give a more precise version of Campanato's regularity results for minimizers and solutions to elliptic systems.

Theorem 4.2.2 (Ziemer [134]). *If $u \in W^{1,2}_{\text{loc}}(\Omega;\mathbb{R}^N)$ is a Q-quasiminimizer of the Dirichlet integral*

$$v \mapsto \int_\Omega |Dv|^2\, dx,$$

then

$$r \mapsto r^{-\frac{n-1}{Q}} \int_{B_r} |Du|^2\, dx$$

is increasing, where $B_r = B(x,r)$ is a ball with centre at $x \in \Omega$ and radius r.

Next, we suppose (4.2.1) is elliptic in the sense that

$$A_i^\alpha(x,u,\xi)\xi_i^\alpha \geq |\xi|^p - b_1(x)|u|^\gamma - a_1(x) \tag{4.2.6}$$

with $1 < p \leq \gamma < p^* = \frac{np}{n-p}$ and

$$|A(x,u,\xi)| \leq L|\xi|^{p-1} + b_2(x)|u|^\sigma + a_2(x) \tag{4.2.7}$$

with $\sigma = \gamma\frac{p-1}{p}$.

The term $B(x,u,\xi)$ can be distinguished into two cases. We first consider the simpler situation which is that of *controlled growth* conditions:

$$|B(x,u,\xi)| \leq H|\xi|^\tau + b_3(x)|u|^\delta + a_3(x) \tag{4.2.8}$$

where $\tau = p\frac{\gamma-1}{\gamma}$ and $\delta = \gamma\frac{p^*-1}{p^*}$. Assume that the functions a_i and b_i are positive.

Theorem 4.2.3 (Giaquinta-Giusti [55], Theorem 2.1). *Let $u \in W^{1,p}_{\text{loc}}(\Omega;\mathbb{R}^N)$ be a weak solution of (4.2.1) with coefficients A and B satisfying conditions (4.2.6), (4.2.7) and (4.2.8). Then u is a Q-quasiminimizer of the functional*

$$\mathcal{J}(u,\Omega) = \int_\Omega \Big(|Du|^p + b(x)|u|^\gamma + a(x)\Big)\,\mathrm{d}x \tag{4.2.9}$$

with

$$b(x) = b_1(x) + b_2(x)^{\frac{p}{p-1}} + b_3(x)^{\frac{p^*}{p^*-1}} \in L^{\frac{p^*}{p^*-\gamma}}$$

and

$$a(x) = a_1(x) + a_2(x)^{\frac{p}{p-1}} + a_3(x)^{\frac{p^*}{p^*-1}} + b(x)^{\frac{p^*}{p^*-\gamma}} \in L^1.$$

Proof. Let $v \in W^{1,p}_{\text{loc}}(\Omega;\mathbb{R}^N)$ with $K = \operatorname{supp}(v-u) \Subset \Omega$ and set $\varphi = v - u$. Since u is a weak solution of (4.2.1), we can substitute φ into (4.2.2) to obtain

$$\int_K A_i^\alpha(x,u,Du)D_iu^\alpha\,\mathrm{d}x = \int_K A_i^\alpha(x,u,Du)D_iv^\alpha\,\mathrm{d}x + \int_K B^\alpha(x,u,Du)(u^\alpha - v^\alpha)\,\mathrm{d}x$$

Substituting (4.2.6), (4.2.7) and (4.2.8) into the equation above gives us

$$\begin{aligned}\int_K (|Du|^p - b_1|u|^\gamma - a_1)\,\mathrm{d}x \quad &\leq \quad \int_K \left(L|Du|^{p-1} + b_2|u|^\sigma + a_2\right)|Dv|\,\mathrm{d}x \\ &\quad + \int_K \left(H|Du|^\tau + b_3|u|^\delta + a_3\right)|u-v|\,\mathrm{d}x\end{aligned}$$

which can be rewritten as

$$\begin{aligned}\int_K |Du|^p\,\mathrm{d}x \quad &\leq \quad \int_K b_1|u|^\gamma\,\mathrm{d}x + \int_K a_1\,\mathrm{d}x + L\int_K |Du|^{p-1}|Dv|\,\mathrm{d}x \\ &\quad + \int_K b_2|u|^\sigma|Dv|\,\mathrm{d}x + \int_K a_2|Dv|\,\mathrm{d}x + H\int_K |Du|^\tau|u-v|\,\mathrm{d}x \\ &\quad + \int_K b_3|u|^\delta|u-v|\,\mathrm{d}x + \int_K a_3|u-v|\,\mathrm{d}x. \qquad (4.2.10)\end{aligned}$$

Note that a_i and b_i, $i = 1, 2, 3$, are functions of x. We then use Young's inequality to find estimates for the last six terms on the right hand side, namely

$$\begin{aligned}|Du|^{p-1}|Dv| &\leq \varepsilon|Du|^p + c(\varepsilon)|Dv|^p; \\ b_2|u|^\sigma|Dv| &\leq \varepsilon|Dv|^p + c(\varepsilon)b_2^{\frac{p}{p-1}}|u|^\gamma \leq c\left(|Dv|^p + b_2^{\frac{p}{p-1}}|u|^\gamma\right); \\ a_2|Dv| &\leq \varepsilon|Dv|^p + c(\varepsilon)a_2^{\frac{p}{p-1}} \leq c\left(|Dv|^p + a_2^{\frac{p}{p-1}}\right); \\ |Du|^\tau|u-v| &\leq \varepsilon|Du|^p + c(\varepsilon)|u-v|^\gamma; \\ b_3|u|^\delta|u-v| &\leq \varepsilon|u-v|^{p^*} + c(\varepsilon)b_3^{\frac{p^*}{p^*-1}}|u|^\gamma; \\ a_3|u-v| &\leq \varepsilon|u-v|^{p^*} + c(\varepsilon)a_3^{\frac{p^*}{p^*-1}}.\end{aligned}$$

We substitute all the estimates into (4.2.10) to get

$$\begin{aligned}\int_K |Du|^p\,\mathrm{d}x \quad &\leq \quad c\int_K \left(b_1 + b_2^{\frac{p}{p-1}} + b_3^{\frac{p^*}{p^*-1}}\right)|u|^\gamma\,\mathrm{d}x + \int_K \left(a_1 + a_2^{\frac{p}{p-1}} + a_3^{\frac{p^*}{p^*-1}}\right)\mathrm{d}x \\ &\quad + c\int_K |Dv|^p\,\mathrm{d}x + \varepsilon(L+H)\int_K |Du|^p\,\mathrm{d}x \\ &\quad + \varepsilon\int_K |u-v|^{p^*}\,\mathrm{d}x + c(\varepsilon)\int_K |u-v|^\gamma\,\mathrm{d}x. \qquad (4.2.11)\end{aligned}$$

The second and third terms on the right hand side are fine, while the rests will need to be dealt with and more precisely we need to find the estimates for the last two terms. By letting

$$b = b_1 + b_2^{\frac{p}{p-1}} + b_3^{\frac{p^*}{p_*-1}},$$

one can use triangle inequality and Young's inequality to obtain

$$\begin{aligned} b|u|^\gamma &\le c(\gamma)\left(b|v|^\gamma + b|u-v|^\gamma\right) \\ &\le c(\gamma)b|v|^\gamma + c(\gamma)\left(\varepsilon|u-v|^{p^*} + c(\varepsilon)b^{\frac{p^*}{p^*-\gamma}}\right) \\ &= c(\gamma)b|v|^\gamma + \varepsilon c(\gamma)|u-v|^{p^*} + c(\gamma,\varepsilon)b^{\frac{p^*}{p^*-\gamma}}. \end{aligned} \tag{4.2.12}$$

Estimating $|u-v|^{p^*}$ by Sobolev's inequality and triangle inequality leads to

$$\begin{aligned} \int_K |u-v|^{p^*}\,\mathrm{d}x &\le c(n,p)\left(\int_K |Du-Dv|^p\,\mathrm{d}x\right)^{\frac{p^*}{p}} \\ &\le c\left(\int_K \big(|Du|^p+|Dv|^p\big)\,\mathrm{d}x\right)^{\frac{p^*}{p}} \\ &= c\left(\int_K \big(|Du|^p+|Dv|^p\big)\,\mathrm{d}x\right)^{\frac{p^*}{p}-1}\int_K \big(|Du|^p+|Dv|^p\big)\,\mathrm{d}x. \end{aligned}$$

If $\int_K |Dv|^p\,\mathrm{d}x \ge \int_K \big(|Du|^p + b|u|^\gamma\big)\,\mathrm{d}x$, then trivially $\mathcal{J}(u,K)\le\mathcal{J}(v,K)$ and there is nothing to be proved. Henceforth, we shall assume that

$$\int_K |Dv|^p\,\mathrm{d}x < \int_K \big(|Du|^p + b|u|^\gamma\big)\,\mathrm{d}x.$$

Applying this assumption into the previous inequality gives us

$$\begin{aligned} \int_K |u-v|^{p^*}\,\mathrm{d}x &\le c\left(\int_K \big(2|Du|^p+b|u|^\gamma\big)\,\mathrm{d}x\right)^{\frac{p^*}{p}-1}\int_K \big(|Du|^p+|Dv|^p\big)\,\mathrm{d}x \\ &\le c\left(\|Du\|_p, \|u\|_{p^*}, \|b\|_{\frac{p^*}{p^*-\gamma}}\right)\int_K \big(|Du|^p+|Dv|^p\big)\,\mathrm{d}x. \end{aligned}$$

Now, we have an estimate for $\int_K |u-v|^{p^*}\,\mathrm{d}x$ for which we substitute into (4.2.12) to obtain

$$\begin{aligned} \int_K b|u|^\gamma\,\mathrm{d}x \le\ & c(\gamma)\int_K b|v|^\gamma\,\mathrm{d}x + c(\gamma,\varepsilon)\int_K b^{\frac{p^*}{p^*-\gamma}}\,\mathrm{d}x \\ & + \varepsilon c(\gamma, \|Du\|_p, \|u\|_{p^*}, \|b\|_{\frac{p^*}{p^*-\gamma}})\int_K \big(|Du|^p+|Dv|^p\big)\,\mathrm{d}x. \end{aligned}$$

Lastly, we substitute all the estimates into (4.2.11), add $\int_K \big(b|u|^\gamma + a(x)\big)\,\mathrm{d}x$ on both sides and take $\varepsilon\to 0$ to conclude that u is a Q-quasiminimizer of the functional $\mathcal{J}$ where the value of Q also depends on u. □

When the general system (4.2.1) is an Euler-Lagrange equation, then we have $A_i^\alpha = \frac{\partial F}{\partial \xi_i^\alpha}$ and $B^\alpha = \frac{\partial F}{\partial u^\alpha}$. If the growth of $F(x,u,\cdot)$ is controlled by $|\cdot|^p$, then it is natural to expect $A(x,u,\cdot)$ to be controlled by $|\cdot|^{p-1}$. However, nothing changes the growth of B since it is the same as that of F. This leads to the second case in which $B(x,u,\xi)$ satisfies *natural growth* assumptions, namely

$$|B(x,u,\xi)| \le H|\xi|^p + a_3(x) \tag{4.2.13}$$

where $a_3 \in L^1(\Omega)^+$. One can clearly see that we have omitted the term $b_3|u|^\delta$ in (4.2.13). This is due to the fact that we consider only bounded solutions of (4.2.2), in which

$$\sup_\Omega u := M.$$

Therefore we can always assume that $b_1 = b_2 = 0$. Having considered that we should separate the case involving scalar-valued functions ($N = 1$) and vector-valued functions ($N > 1$), we will start with the former.

Theorem 4.2.4 (Giaquinta-Giusti [55], Theorem 2.2)**.** *Let $N = 1$ and let $u(x)$ be a bounded weak solution of (4.2.1) with coefficients A and B satisfying natural conditions (4.2.6), (4.2.7) and (4.2.13), where $b_1 = b_2 = 0$. Then u is a Q-quasiminimizer of the functional*

$$\mathcal{H}(u,\Omega) = \int_\Omega \big(|Du|^p + a(x)\big)\,\mathrm{d}x, \tag{4.2.14}$$

where

$$a(x) = a_1(x) + a_2(x)^{\frac{p}{p-1}} + a_3(x).$$

The idea of the proof is similar to that of Theorem 4.2.3. Since the functions $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ considered here are real-valued, we can even consider the obstacle problem: For a given function $\psi \in W^{1,p}(\Omega)$, we let $u \geq \psi$ be a weak supersolution of (4.2.1), meaning that

$$\int_\Omega \big\{A_i\big(x,u(x),Du(x)\big)D_i(v-u) + B\big(x,u(x),Du(x)\big)(v-u)\big\}\,\mathrm{d}x \geq 0$$

for every $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ with $v \geq \psi$ and $v \geq u$. Then, it can be shown that u is a superquasiminimizer of $\mathcal{H}$ as given in (4.2.14):

$$\mathcal{H}(u,K) \leq Q\mathcal{H}(v,K) \tag{4.2.15}$$

where $K = \operatorname{supp}(v-u) \Subset \Omega$. We can even drop the condition $v \geq \psi$ by first letting $w = \max\{v,\psi\}$ to get $w \geq \psi$ and $\operatorname{supp} w \Subset K$ so that (4.2.15) holds for w. Noting that $|Dw| \leq |Dv| + |D\psi|$, we deduce that u is indeed a Q-quasiminimizer of

$$\mathcal{H}(u,\Omega) + \int_\Omega |D\psi|^p\,\mathrm{d}x.$$

We now pass to a system of equations in which the boundedness of the solution is no longer sufficient to prove Q-quasiminimality of the functional $\mathcal{H}$ as in the case when $N = 1$. Giaquinta-Giusti [55] showed that a bounded weak solution u is still a Q-quasiminimizer of the same functional as long as we impose an additional smallness condition.

Theorem 4.2.5 (Giaquinta-Giusti [55], Theorem 2.3)**.** *Let $u \in W^{1,p}_{\text{loc}}(\Omega;\mathbb{R}^N)$ be a bounded weak solution of the system (4.2.1) with coefficients A and B satisfying conditions (4.2.6), (4.2.7) and (4.2.13), where $b_1 = b_2 = 0$. In addition, we assume the smallness condition*

$$2MH < 1.$$

Then u is a Q-quasiminimizer of the functional $\mathcal{H}$ given in (4.2.14).

Recall the functional $\mathcal{J}$ in Theorem 4.2.3:

$$\mathcal{J}(u,\Omega) = \int_\Omega \big(|Du(x)|^p + b(x)|u(x)|^\gamma + a(x)\big)\,\mathrm{d}x.$$

Note that it is possible to reduce the functional $\mathcal{F}(u,\Omega) = \int_\Omega F\big(x,u(x),Du(x)\big)\,\mathrm{d}x$ to $\mathcal{J}(u,\Omega)$ when the integrand $F(x,u,\xi)$ satisfies the inequalities

$$|\xi|^p - b(x)|u|^\gamma - a(x) \le F(x,u,\xi) \le L|\xi|^p + b(x)|u|^\gamma + a(x) \tag{4.2.16}$$

with $L \ge 1$, $a \in L^1(\Omega)^+$, $\gamma < p^*$ and $b \in L^{\frac{p^*}{p^*-\gamma}}(\Omega)$. In other words, a Q-quasiminimizer of $\mathcal{F}$ is a Q-quasiminimizer of $\mathcal{J}$ with a different constant Q. Let $v \in W^{1,p}_{\text{loc}}(\Omega)$ with $K = \operatorname{supp}(v-u) \Subset \Omega$. If $u \in W^{1,p}_{\text{loc}}(\Omega)$ is a Q-quasiminimizer of $\mathcal{F}$, then

$$\begin{aligned}\int_K \big(|Du|^p - b|u|^\gamma - a\big)\,\mathrm{d}x \le \int_K F(x,u,Du)\,\mathrm{d}x &\le Q\int_K F(x,v,Dv)\,\mathrm{d}x \\ &\le Q\int_K \big(L|Dv|^p + b|v|^\gamma + a\big)\,\mathrm{d}x\end{aligned}$$

which can be written as

$$\int_K \big(|Du|^p + b|u|^\gamma + a\big)\,\mathrm{d}x \le \tilde{Q}\int_K \big(|Dv|^p + b|v|^\gamma + b|u|^\gamma + a\big)\,\mathrm{d}x$$

to show that u is also a Q-quasiminimizer of $\mathcal{J}$ where the estimate for $b|u|^\gamma$ on the right hand side of the inequality follows from the proof of Theorem 4.2.3. Similarly, if u is a Q-quasiminimizer of $\mathcal{J}$, then it is also a Q-quasiminimizer of $\mathcal{F} + \int_\Omega (b|u|^\gamma + a)\,dx$.

It is worth noting that, in the scalar case, Q-quasiminimizers of the usual functional $\mathcal{F}(u,\Omega)$ with its integrand satisfying (4.2.16) does possess regularity in the sense of De Giorgi [35]. If $a \in L^q_{\text{loc}}(\Omega)$ where $q > \frac{n}{p}$, then a Q-quasiminimizer $u \in W^{1,p}_{\text{loc}}(\Omega)$ is locally bounded and one can use the local boundedness of u to pass to Hölder continuity. Interested reader may refer to the papers of Giaquinta-Giusti [54, 55] for the proofs. They also showed that the weak maximum principle applies to Q-quasiminimizers of the p-Dirichlet integral in [55].

Theorem 4.2.6 (**Weak maximum principle**)**.** *Let $u \in W^{1,p}(\Omega)$ be a Q-quasiminimizer of the Dirichlet integral*

$$\mathcal{F}(v, \Omega) = \int_\Omega |Dv|^p \,\mathrm{d}x.$$

Then

$$\sup_\Omega u = \sup_{\partial\Omega} u$$

and

$$\inf_\Omega u = \inf_{\partial\Omega} u.$$

Among other regularity properties that works for scalar-valued Q-quasiminimizers include Harnack's inequality (DiBenedetto-Trudinger [38]) and weak Liouville's property (Giaquinta-Giusti [55], Theorem 4.4).

Quasiminimizers and quasiregular mappings

We next examine Q-quasiminimizer that arises from quasiregular mappings and quasiregularity is a natural generalization of the notion of a holomorphic function. A map $u\colon \Omega \to \mathbb{R}^n$ is quasiregular if $u \in W^{1,n}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ and there exists a constant $\alpha \geq 1$ such that

$$|Du|^n \leq \alpha \det(Du),$$

where the norm used here, following convention, is the operator norm

$$|Du(x)| = \sup_{|h|=1} |Du(x)h|.$$

If, in addition, u is a homeomorphism, then u is called a quasiconformal mapping. When $n = 2$ and $\alpha = 1$, then we recover analytic functions of one complex variable or holomorphic functions (Lehto-Virtanen [87]), and hence a 1-quasiconformal mapping is a conformal mapping.

Theorem 4.2.7 (Giaquinta-Giusti [55], Theorem 2.4)**.** *A quasiregular map $u \in W^{1,n}(\Omega; \mathbb{R}^n)$ is a Q-quasiminimizer of the functional*

$$\int_\Omega |Du|^n \,dx.$$

Proof. Let $v \in W^{1,n}_u(\Omega; \mathbb{R}^n)$. For every $n \times n$ matrices A and B, we have

$$\det(A + B) \leq \det B + c \sum_{i=1}^{n-1} |A|^{n-i} |B|^i \tag{4.2.17}$$

where constant $c > 0$ and $A = (a_{ij})$ with $|A|^2 = \sum_{i,j} a_{ij}^2$. Take $A = Dv$ and $B = D(u - v)$ into (4.2.17) to obtain

$$\begin{aligned}
\det(Du) &= \det\big(Dv + D(u-v)\big) \\
&\leq \det\big(D(u-v)\big) + c\sum_{i=0}^{n-1} |Dv|^{n-i}|D(u-v)|^i \\
&\leq \det\big(D(u-v)\big) + c\sum_{i=0}^{n-1} |Dv|^{n-i}\big(|Du|^i + |Dv|^i\big) \\
&= \det\big(D(u-v)\big) + c\sum_{i=0}^{n-1} \big(|Dv|^{n-i}|Du|^i + |Dv|^n\big)
\end{aligned}$$

Set $K = \operatorname{supp}(u - v) \Subset \Omega$ and use the definition of quasiregularity to find

$$\begin{aligned}
\int_K |Du|^n \,\mathrm{d}x &\leq \alpha \int_K \det(Du)\,\mathrm{d}x \\
&\leq \alpha \int_K \det\big(D(u-v)\big)\,\mathrm{d}x + \alpha c \sum_{i=0}^{n-1} \int_K \big(|Dv|^{n-i}|Du|^i + |Dv|^n\big)\,\mathrm{d}x,
\end{aligned}$$

where constant $\alpha \geq 1$. Since $\int_\Omega \det(D\varphi)\,\mathrm{d}x = 0$ for all $\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^n)$, the first term on the right hand side becomes zero and we only need to estimate the second term. Letting $\tilde{c} = \alpha c$, the inequality becomes

$$\begin{aligned}
\int_K |Du|^n \,\mathrm{d}x &\leq \tilde{c}\int_K \sum_{i=0}^{n-1} |Dv|^{n-i}|Du|^i + \tilde{c}\int_K |Dv|^n\,\mathrm{d}x \\
&\leq \tilde{c}\int_K \big(\varepsilon|Du|^n + c(\varepsilon)|Dv|^n\big)\,\mathrm{d}x + \tilde{c}\int_K |Dv|^n\,\mathrm{d}x
\end{aligned}$$

where we have used Young's inequality. Thus with a suitable choice of $\varepsilon > 0$, we arrive at

$$\int_K |Du|^n \,\mathrm{d}x \leq Q\int_K |Dv|^n\,\mathrm{d}x$$

where $Q = Q(c, \alpha, \varepsilon, n)$. □

Cubical and spherical quasiminimizers

We will now consider cubical and spherical Q-quasiminimizers which involve only integrals on cubes and balls of $\mathbb{R}^n$, respectively. We once again consider a Carathéodory integrand F that satisfies (4.1.1).

Definition 4.2.8. *Let $Q \geq 1$. A function $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ is called a cubical Q-quasiminimizer of the functional $\mathcal{F}$ if*

$$\mathcal{F}(u, C) \leq Q\mathcal{F}(u + \varphi, C)$$

for every cube $C \subset \Omega$ and for every $\varphi \in W_0^{1,p}(C;\mathbb{R}^N)$.

Similarly, we can define spherical Q-quasiminimizer by considering balls $B \subset \Omega$ instead of cubes. By definition, we can see that a Q-quasiminimizer is also a cubical and a spherical Q-quasiminimizer. However, the converse is in general false when $n > 2$, which is shown in an example by Giaquinta-Giusti [55] (see also Giusti [57], Example 6.5).

Example 4.2.9 (Giaquinta-Giusti [55], pages 91–93)**.** *Let $n > 2$ and let $u(x)$ be a homogeneous function of degree β, $0 > \beta > 1 - \frac{n}{2}$, regular in $\mathbb{R}^n - 0$, without stationary points and non-constant on the boundary of any cube of $\mathbb{R}^n$. For this instance, we may take $u(x) = |x|^\beta$. Then u is a cubical Q-quasiminimizer of the Dirichlet integral $\int_\Omega |Du|^2\,\mathrm{d}x$ but it is not a Q-quasiminimizer.*

In the proof of the example above, we will need the following theorem which is similar to the Poincaré's inequality but with the exception that the average value of u is taken over the boundary of the domain Ω. We employ the concept of $(n-1)$-dimensional Hausdorff measure on $\partial\Omega$, denoted as H_{n-1}. Note that H_{n-1} coincides with the usual surface measure when $\partial\Omega$ is Lipschitz continuous.

Theorem 4.2.10 (Giusti [57], Theorem 3.20)**.** *Let Ω be a bounded open set, with a boundary $\partial\Omega$ connected and Lipschitz continuous. Then, there exists a constant $c = c(p,n,\Omega)$ such that for every $u \in W^{1,p}(\Omega;\mathbb{R}^N)$*

$$\int_{\partial\Omega} |u(x) - u_{\partial\Omega}|^p\,\mathrm{d}H_{n-1} \le c\int_\Omega |Du|^p\,\mathrm{d}x$$

where

$$u_{\partial\Omega} := \fint_{\partial\Omega} u\,\mathrm{d}x.$$

Moreover, if Ω is the cube C_R, we have

$$\int_{\partial C_R} |u(x) - u_{\partial C_R}|^p\,\mathrm{d}H_{n-1} \le c(p,n)R^{p-1}\int_{C_R} |Du|^p\,\mathrm{d}x. \tag{4.2.18}$$

If Ω is the ball B_R, the inequality (4.2.18) still holds with possibly a different constant c. We are now ready to show that u in Example 4.2.9 is in fact a cubical Q-quasiminimizer but not a Q-quasiminimizer and the proof follows from Giusti [57] (see Example 6.5).

Proof of Example 4.2.9. To prove that u is a cubical Q-quasiminimizer, we show that for any cube $C_R = C(x,R) \subset \Omega$, $Q \ge 1$ and noting that $v = u$ on $\partial\Omega$, we have

$$\int_{C_R} |Du(x)|^2\,\mathrm{d}x \le Q\int_{C_R} |Dv(x)|^2\mathrm{d}x.$$

It will suffice to prove that

$$R\int_{C_R}|Du(x)|^2\,\mathrm{d}x \leq c_1\int_{\partial C_R}|u(x)-u_{\partial C_R}|^2\,\mathrm{d}H_{n-1}, \tag{4.2.19}$$

since it follows from (4.2.18) that the right hand side of (4.2.19) satisfies

$$\int_{\partial C_R}|u-u_{\partial C_R}|^2\,\mathrm{d}H_{n-1} = \int_{\partial C_R}|v-v_{\partial C_R}|^2\,\mathrm{d}H_{n-1} \leq c(p,n)R\int_{C_R}|Dv|^2\,\mathrm{d}x. \tag{4.2.20}$$

The idea of the proof is to establish an upper bound when the integral on the left hand side of (4.2.19) is divided by the integral on the right hand side. To do this, we rewrite (4.2.19) by reducing to the cube $C = C(0,1)$ and set $x_0 = Ry_0$ and $x = R(y_0+y)$:

$$\begin{aligned} R\int_{C_R}|Du|^2\,\mathrm{d}x &= \int_C|Du(y_0+y)|^2\,\mathrm{d}y \\ &\leq c_1\int_{\partial C}|u-u_{\partial C}|^2\,\mathrm{d}H_{n-1}(y) \\ &\leq c_2\left(\fint_{\partial C}|u(y_0+y)|^2\,\mathrm{d}H_{n-1}(y) - \left(\fint_{\partial C}|u(y_0+y)|\,\mathrm{d}H_{n-1}(y)\right)^2\right). \end{aligned}$$

Let

$$F(y_0) = \int_C|Du(y_0+y)|^2\,\mathrm{d}y$$

and

$$G(y_0) = \fint_{\partial C}|u(y_0+y)|^2\,\mathrm{d}H_{n-1}(y) - \left(\fint_{\partial C}|u(y_0+y)|\,\mathrm{d}H_{n-1}(y)\right)^2.$$

We are now ready to show that there exists some constant Q that bounds $\frac{F}{G}$ from above. Note that, since u is not constant on the boundary of any cube, then both F and G are continuous positive functions. In fact, the ratio $\frac{F}{G}$ is bounded on compact sets, and therefore we only investigate its behaviour when $y_0 \to \infty$. By Taylor's expansion, we have for every $y \in C$

$$u(y_0+y) = u(y_0) + \langle Du(y_0), y\rangle + \frac{1}{2}\langle D^2u(y_0)y, y\rangle + O(|y_0|^{\beta-3})$$

and

$$Du(y_0+y) = Du(y_0) + O(|y_0|^{\beta-2}).$$

Hence, we have

$$F(y_0) = \operatorname{meas} C|Du(y_0)|^2 + O(|y_0|^{2\beta-3})$$

and

$$G(y_0) = \fint_{\partial C}\langle Du(y_0), y\rangle^2\,\mathrm{d}H_{n-1}(y) + O(|y_0|^{2\beta-3}) \geq c_4|Du(y_0)|^2 + O(|y_0|^{2\beta-3}).$$

Since Du is homogeneous of degree $\beta - 1$ and non-zero, then we have

$$|Du(y_0)| \geq c_5 |y_0|^{\beta-1}$$

which leads to the boundedness of $\frac{F}{G}$. Hence, u is a cubical Q-quasiminimizer of the Dirichlet integral $\int_\Omega |Du(x)|^2 \, \mathrm{d}x$.

On the other hand, u is not a Q-quasiminimizer of the Dirichlet integral. If we let $v = \min\{u, 1\}$, then it does not satisfy either local boundedness or weak maximum principle, and therefore u is not a Q-quasiminimizer. $\square$

4.3 Examples of scalar-valued quasiminimizers

One-dimensional quasiminimizers

We consider Q-quasiminimizers of one-dimensional p-Dirichlet integral

$$u \mapsto \int |u'|^p \, \mathrm{d}x, \quad p \geq 1. \tag{4.3.1}$$

The definition is in the sense of Definition 4.1.1 with $\Omega = I \subset \mathbb{R}$ where I is an open interval and the support of $(v - u)$ is essentially a closed subinterval of I.

Definition 4.3.1. *Let $Q \geq 1$ and $I \subset \mathbb{R}$ be an open interval. We say that $u \in W^{1,p}_{\mathrm{loc}}(I)$ is a Q-quasiminimizer of the p-Dirichlet integral, $p \geq 1$, if*

$$\int_a^b |u'|^p \, \mathrm{d}x \leq Q \int_a^b |v'|^p \, \mathrm{d}x$$

for all $v \in W^{1,p}_u(a, b)$ and for all closed intervals $[a, b] \subset I$.

It is easy to see that the p-Dirichlet integral $\int_a^b |v'|^p \, \mathrm{d}x$ attains its minimum when v is affine and hence u is a Q-quasiminimizer of the p-Dirichlet integral if and only if

$$\int_a^b |u'|^p \, \mathrm{d}x \leq Q \frac{\left|u(b) - u(a)\right|^p}{(b-a)^{p-1}}. \tag{4.3.2}$$

The inequality above implies that a Q-quasiminimizer u is absolutely continuous on every subinterval $[a, b] \subset I$ and that u is either constant or strictly monotone. Moreover, any Q-quasiminimizer in an open finite interval I has a continuous extension to the closed interval $\bar{I}$ such that (4.3.2) holds for all closed subintervals $[a, b] \subset \bar{I}$ (see Martio-Sbordone [93], Lemma 1), and we shall henceforth use this natural domain of definition for a Q-quasiminimizer. It is also worth mentioning that inequality (4.3.2) gives us the reverse Hölder's inequality

$$\fint_a^b |u'|^p \,\mathrm{d}x \leq Q \left(\fint_a^b u' \,\mathrm{d}x \right)^p ,$$

and even more, it is computed on the same set. A brief account about the importance of the reverse Hölder's inequality can be found in Section 5.2 of Chapter 5.

Originally discussed by Giaquinta-Giusti [55] alongside with the theory of Q-quasiminimizers and further investigated by Martio-Sbordone [93], one-dimensional Q-quasiminimizer exhibits an attractive feature – there are more available tools as compared to the multidimensional case. Martio-Sbordone [93] proved the higher integrability of one-dimensional Q-quasiminimizers, meaning that if u is a Q-quasiminimizer of the p-Dirichlet integral, then u is also a Q'-quasiminimizer of the q-Dirichlet integral, $q \in [1, p_1)$ where $p_1 = p_1(p, Q)$. Moreover, they even obtained the exact value for $p_1(p, Q)$. This was done by using higher integrability result of weights in the Gehring's class by Korenovskiĭ [80] where its optimal bound is known in the one-dimensional case.

Let $p \in (1, \infty)$ and $\omega \colon \bar{I} \to [0, \infty)$ be a weight in $L^1(I)$. Set

$$G_p(\omega) = \sup_J \left(\fint_J \omega^p \,\mathrm{d}x \right)^{\frac{1}{p}} \left(\fint_J \omega \,\mathrm{d}x \right)^{-1} ,$$

where the supremum is taken over all intervals $J \subset \bar{I}$. We say that ω belongs to G_p-class (or Gehring's class) if $G_p(\omega) < \infty$. The higher integrability result of Gehring [51] can now be stated as follows:

Theorem 4.3.2 (Martio-Sbordone [93], Theorem 3)**.** *For each $p > 1$ and $Q \in [1, \infty)$, there exists $p_0 = p_0(p, Q) > p$ such that if a weight ω satisfies $G_p(\omega) \leq Q$ in $\bar{I}$, then $G_s(\omega) < \infty$ for $s \in (1, p_0)$ and moreover $G_s(\omega)$ is bounded from above by some constant $c(p, s, Q)$.*

The proof of the theorem is essentially a direct consequence of Gehring's lemma [51] and the fact that one-dimensional Q-quasiminimizers satisfy the reverse Hölder's inequality computed on the same set. Likewise, a more detailed discussion on Gehring's lemma can be found in the next section which concerns interior regularity of Q-quasiminimizers in the vectorial case.

Furthermore set, for u a non-constant Q-quasiminimizer in $\bar{I}$,

$$Q_p = Q_p(u) = \sup_{[a,b]} \frac{(b-a)^{p-1}}{|u(b) - u(a)|^p} \int_a^b |u'|^p \,\mathrm{d}x$$

where the supremum is taken over all intervals $[a, b] \subset \bar{I}$. Note that Q_p is the least constant in (4.3.2). The following lemma states the connection between G_p and Q_p:

Lemma 4.3.3 (Martio-Sbordone [93], Lemma 2). *Let $u\colon \bar{I} \to \mathbb{R}$ be absolutely continuous and non-constant with $u' \geq 0$ a.e.. Then u is a Q_p-quasiminimizer with exponent $p > 1$ if and only if u' belong to the G_p-class with $G_p(u') = Q_p(u)^{1/p}$.*

We are now ready to state the higher integrability result for one-dimensional Q-quasiminimizers.

Theorem 4.3.4 (Martio-Sbordone [93], Theorem 4). *Let $u\colon \bar{I} \to \mathbb{R}$ be a Q_p-quasiminimizer and $p_0 = p_0(p, Q_p^{1/p}) > p$ where $p > 1$ and p_0 is as in Theorem 4.3.2. Then u is a Q'-quasiminimizer of the s-Dirichlet integral for each $s \in [1, p_0)$ where $Q' = Q'(p, s, Q_p)$.*

Proof. If $s = 1$, then the function u is either strictly monotone or constant and hence $Q_1 = 1$ is immediate. For $s \in (1, p_0)$, we will use Theorem 4.3.2 and Lemma 4.3.3. We first note that $G_p(u') = Q_p(u)^{1/p}$ implies that $G_s(u') < \infty$ for $s \in (1, p_0)$ and

$$G_s(u') \leq c(p, s, Q_p^{1/p}).$$

The definition of Gehring's class gives us

$$\sup_{[a,b]} \left(\fint_a^b |u'|^s \, \mathrm{d}x \right)^{\frac{1}{s}} \left(\fint_a^b u' \, \mathrm{d}x \right)^{-1} \leq c(p, s, Q_p^{1/p}),$$

and hence we have

$$\int_a^b |u'|^s \leq c(p, s, Q_p^{1/p}) \frac{|u(b) - u(a)|^s}{(b-a)^{s-1}},$$

which is exactly the Q-quasiminimality of u for the s-Dirichlet integral with $Q' = c(p, s, Q_p^{1/p})$. $\square$

In fact if $s \in (1, p)$, then we may use Hölder's inequality to obtain $Q_s = Q_p^{s/p}$:

$$\begin{aligned}
\int_a^b |u'|^s \, \mathrm{d}x &\leq \left(\int_a^b |u'|^p \, \mathrm{d}x \right)^{\frac{s}{p}} (b-a)^{\frac{p-s}{p}} \\
&\leq Q_p^{s/p} \frac{|u(b) - u(a)|^s}{(b-a)^{\frac{s(p-1)}{p}}} (b-a)^{\frac{p-s}{p}} \\
&= Q_p^{s/p} \frac{|u(b) - u(a)|^s}{(b-a)^{s-1}}.
\end{aligned}$$

Recall that Korenovskiĭ [80] (Theorem 2) determined the optimal higher integrability bound for a weight ω in the G_p-class (see also D'Apuzzo-Sbordone [34]) . Together with Lemma 4.3.3, we can even show that the higher integrability conditions of the derivative of Q-quasiminimizers are sharp. We first let $\gamma_{p,t}\colon [p, \infty) \to \mathbb{R}$, where $p > 1$ and $t > 1$, be the function

$$\gamma_{p,t}(x) = 1 - t^p \left(\frac{x-p}{x} \right) \left(\frac{x}{x-1} \right)^p. \tag{4.3.3}$$

Furthermore, let $p_1(p,t) \in (p,\infty)$ be the unique solution of the equation $\gamma_{p,t} = 0$. A detailed study in this aspect can be found in Section 3 of D'Apuzzo-Sbordone [34].

Theorem 4.3.5 (Martio-Sbordone [93], Theorem 6)**.** *Let $Q \geq 1$ and let $u \in W^{1,p}_{\mathrm{loc}}(I)$ is a Q-quasiminimizer of the p-Dirichlet integral (4.3.1), $p \geq 1$, in the closed interval $\bar{I} \subset \mathbb{R}$. Then we have the following integrability conditions:*

(i) If $p > 1$ and $Q > 1$, then $u' \in L^s(I)$ for $s \in \big[1, p_1(p, Q^{\frac{1}{p}})\big)$.

(ii) If $p = 1$, then $u' \in L^1(I)$.

(iii) If $p > 1$ and $Q = 1$, then $u' \in L^\infty(I)$.

Moreover, all the integrability conditions are sharp.

Proof.

(i) Since a Q-quasiminimizer u is strictly monotone, we may assume that u is increasing. Recall from Lemma 4.3.3 that $G_p(u') = Q_p(u)^{1/p}$. Using the optimal higher integrability bound result of Korenovskiĭ [80] (Theorem 2), we conclude that

$$u \in L^s(I) \quad \text{for} \quad 1 \leq s < p_1\big(p, G_p(u')\big) = p_1(p, Q_p^{1/p}).$$

To see that the integrability condition is sharp, let $\alpha = p_1(p, Q^{1/p})$ and consider for $x \in [0,1]$,

$$u(x) = \frac{\alpha}{\alpha - 1} x^{\frac{\alpha-1}{\alpha}}.$$

It follows from Proposition 2.3 of D'Apuzzo-Sbordone [34] that the derivative $u'(x) = x^{-\frac{1}{\alpha}}$ belongs to G_p-class with $G_p(u') = Q_p^{1/p}$ and hence u is a Q-quasiminimizer, as a direct consequence of Lemma 4.3.3. However, $u' \notin L^\alpha(0,1)$ and this proves that the open ended upper bound $p_1(p, Q^{1/p})$ is sharp.

(ii) It is trivial that $u' \in L^1(I)$. Recall that every increasing absolutely continuous function u is essentially a Q-quasiminimizer of the 1-Dirichlet integral and hence the integrability of u' cannot be improved.

(iii) Note that for $Q = 1$, a Q-quasiminimizer u of the p-Dirichlet integral is affine and therefore $u' \in L^\infty(I)$.

□

Example 4.3.6. *Solving (4.3.3) for $p = 2$ gives us*

$$p_1(2, t) = 1 + \frac{t}{(t^2 - 1)^{\frac{1}{2}}}$$

and therefore if $t = Q^{1/2}$ with $Q > 1$, we have

$$p_1(2, Q^{1/2}) = 1 + \left(\frac{Q}{Q-1}\right)^{\frac{1}{2}}.$$

Apart from that, Martio-Sbordone [93] also showed that the inverse function of a non-constant Q-quasiminimizer of the p-Dirichlet integral, $p > 1$, is also a Q'-quasiminimizer of some q-Dirichlet integral where $Q' = Q'\big(p, q, Q_p(u)\big) < \infty$. Similar to that of the previous result, the upper bound for q is sharp for $Q_p(u) > 1$ (see Theorem 14 and Corollary 15 of Martio-Sbordone [93]).

An interesting fact about Theorem 4.3.5 is that it can be extended to give higher integrability of the derivative of superminimizers.

Definition 4.3.7. *Let $u\colon \bar{I} \to \mathbb{R}$. We say that $u \in W^{1,p}(I)$ is a superminimizer of the p-Dirichlet integral, $p > 1$, if*

$$\int_I |u'|^p \,\mathrm{d}x \le \int_I |v'|^p \,\mathrm{d}x$$

for every $v \in W^{1,p}_u(I)$ and $v \ge u$.

The definition can be characterized in the sense of distribution, meaning that

$$\int_I |u'|^{p-2} u' \varphi' \,\mathrm{d}x \ge 0$$

for all non-negative $\varphi \in C^1_0(I)$. Recall that superminimizers are naturally connected to obstacle problem which is of general importance in non-linear potential theory. If we let $u \in W^{1,p}(I)$ and set

$$\mathcal{K}^+_u(I) = \left\{v \in W^{1,p}_u(I) \colon v \ge u\right\},$$

then one can easily see that there exists a unique function $u_+ \in \mathcal{K}^+_u(I)$ such that

$$\int_I |u'_+|^p \,\mathrm{d}x = \min_{v \in \mathcal{K}^+_u(I)} \int_I |v'|^p \,\mathrm{d}x.$$

This function u_+ is a superminimizer of the p-Dirichlet integral.

A detailed account about superminimizer and its connection with superharmonic functions can be found in Chapters 3, 5 and 7 of Heinonen-Kilpeläinen-Martio [62]. In 2003, Kinnunen-Martio

[73] established potential theory of quasiminimizers, in which they studied the connection between superharmonic functions and superquasiminimizers. Though it is an interesting theory on its own, we shall focus on just showing the higher integrability results.

Theorem 4.3.8 (Martio-Sbordone [93], Proposition 17)**.** *Let $u\colon \bar{I} \to \mathbb{R}$ be a superminimizer and a Q-quasiminimizer of the p-Dirichlet integral, $p \geq 1$. Then the higher integrability of u' is as in Theorem 4.3.5 and the bounds cannot be improved.*

The proof of the above simply follows from Theorem 4.3.5. We now move on to the obstacle problem:

Lemma 4.3.9 (Martio-Sbordone [93], Lemma 18)**.** *Let $u\colon \bar{I} \to \mathbb{R}$. If $u \in W^{1,p}(I)$ is a Q-quasiminimizer of the p-Dirichlet integral, then u_+ is also a Q-quasiminimizer of the p-Dirichlet integral.*

Proof. Let $[a,b] \subset \bar{I}$ and let $v \in W^{1,p}_{u_+}(a,b)$. We wish to show that

$$\int_a^b |u_+'|^p \,\mathrm{d}x \leq \int_a^b |v'|^p \,\mathrm{d}x. \tag{4.3.4}$$

We first let $v_1 = \min\{u_+, v\}$ and set

$$w = \begin{cases} \max\{v_1, u\} & \text{if } x \in [a,b] \\ u_+ & \text{if } x \in I \setminus [a,b] \end{cases}.$$

It is clear that $w \in W^{1,p}_u(I)$ and $w \geq u$ and therefore $w \in \mathcal{K}^+_u(I)$. Since u_+ is a superminimizer in $\mathcal{K}^+_u(I)$, then

$$\int_I |u_+'|^p \,\mathrm{d}x \leq \int_I |w'|^p \,\mathrm{d}x.$$

Since $u_+ = w$ outside $[a,b]$, we obtain

$$\int_a^b |u_+'|^p \,\mathrm{d}x \leq \int_a^b |w'|^p \,\mathrm{d}x = \int_{\{v_1<u\}} |u'|^p \,\mathrm{d}x + \int_{\{v_1 \geq u\}} |v_1'|^p \,\mathrm{d}x. \tag{4.3.5}$$

Note that the set $\{v_1 < u\}$ is open in $[a,b]$ and also $v_1 \in W^{1,p}_u(\{v_1 < u\})$. It then follows from Q-quasiminimality of u that

$$\int_{\{v_1<u\}} |u'|^p \,\mathrm{d}x \leq Q \int_{\{v_1<u\}} |v_1'|^p \,\mathrm{d}x$$

and substituting into (4.3.5), we have

$$\begin{aligned}
\int_a^b |u_+'|^p \,\mathrm{d}x &\le Q\int_{\{v_1<u\}} |v_1'|^p \,\mathrm{d}x + \int_{\{v_1\ge u\}} |v_1'|^p \,\mathrm{d}x \\
&\le Q\int_a^b |v_1'|^p \,\mathrm{d}x \\
&= Q\int_{\{u_+<v\}} |u_+'|^p \,\mathrm{d}x + Q\int_{\{u_+\ge v\}} |v'|^p \,\mathrm{d}x \\
&\le Q\int_{\{u_+<v\}} |v'|^p \,\mathrm{d}x + Q\int_{\{u_+\ge v\}} |v'|^p \,\mathrm{d}x
\end{aligned}$$

where the last inequality follows from the fact that u_+ is a superminimizer. The result then follows.

□

Lastly, if u is Q-quasiminimizer of the p-Dirichlet integral, $p > 1$, and u_+ is a solution to the $\mathcal{K}_u^+$-obstacle problem, then the derivative $u_+' \in L^s$ for $s \in \big[1, p_1(p, Q^{1/p})\big)$ and the upper bound is also sharp for $Q > 1$ (see Martio-Sbordone [93], Corollary 19).

Radial quasiminimizers

We consider the radial (power-type) functions

$$u(x) = |x|^\alpha$$

in $\mathbb{R}^n$. This type of function exhibits a good property – given the value of $p \neq n$ and for certain choice of α, it is indeed a Q-quasiminimizer of the p-Dirichlet integral in the unit ball $B = B(0,1)$ except at the origin:

$$\int_{B\setminus\{0\}} |Du|^p \,\mathrm{d}x \le Q\int_{B\setminus\{0\}} |Du + D\varphi|^p \,\mathrm{d}x$$

for all $\varphi \in W_0^{1,p}(\Omega)$. These results, together with that of optimal constant for Q, are clearly illustrated in the self-contained paper of Björn-Björn [17]. A thorough investigation on the conditions for the radial function to be super(quasi)harmonic, a super(quasi)minimizer or a sub(quasi)minimizer is also included. Similarly for log-power functions $u(x) = (-\log|x|)^\alpha$ when $p = n$. We shall be brief here and mention only the important results for Q-quasiminimizers and their corresponding best constant Q.

Theorem 4.3.10 (Björn-Björn [17]). *Let $u(x) = |x|^\alpha$.*

(i) ([17], Theorem 5.1) *Let $p \in (1, n)$. Then u is a Q-quasiminimizer in $B \setminus \{0\}$ if and only if $\alpha < 1 - \frac{n}{p}$ or $\alpha = 0$. Moreover, if $\alpha < 1 - \frac{n}{p}$, then*

$$Q = Q(\alpha, p, n) = \left(\frac{\alpha}{\beta}\right)^p \frac{\beta p - p + n}{\alpha p - p + n} \tag{4.3.6}$$

is the best Q-quasiminimizer constant where $\beta = \frac{p-n}{p-1}$.

(ii) ([17], Theorem 6.1) *Let $p > n \geq 1$. Then u is a Q-quasiminimizer in $B \setminus \{0\}$ if and only if $\alpha > 1 - \frac{n}{p}$ or $\alpha = 0$. Moreover, if $\alpha > 1 - \frac{n}{p}$, then (4.3.6) is the best Q-quasiminimizer constant.*

(iii) ([17], Theorem 7.1) *Let $p = n > 1$. Then u is a Q-quasiminimizer in $B \setminus \{0\}$ if and only if $\alpha = 0$.*

Remark 4.3.11. *Note that the positivity of the constant Q in (4.3.6) is guaranteed by the fact that $\beta p - p + n = \beta$.*

If one is to consider the log-power functions $u(x) = (-\log|x|)^\alpha$, then we can obtain the best Q-quasiminimizer constant for $p = n > 1$.

Theorem 4.3.12 (Björn-Björn [17], Theorem 7.3)**.** *Let $p = n > 1$ and $u(x) = (-\log|x|)^\alpha$. Then u is a Q-quasiminimizer in $B \setminus \{0\}$ if and only if $\alpha > 1 - \frac{1}{n}$ or $\alpha = 0$. Moreover, if $\alpha > 1 - \frac{1}{n}$, then*

$$Q = Q(\alpha, n) = \frac{\alpha^n}{\alpha n - n + 1}$$

is the best Q-quasiminimizer constant.

It is worth noting that in most cases, the radial or log-power function either fail to be Q-quasiminimizers of the p- Dirichlet integral or the best constant takes the value $Q = 1$.

Local quasiminimizers

It is well-known that the notion of p-harmonicity (and more generally minimality) is a local property: if u is p-harmonic in the open sets Ω_1 and Ω_2, then u is also p-harmonic in $\Omega_1 \cup \Omega_2$. One might wonder whether the locality is true for Q-quasiminimizers too. It was pointed out by Kinnunen-Martio [73] that the locality property exhibited by p-harmonic functions fails for Q-quasiminimizers. This was also verified by Björn-Björn [17] in their concluding section, using the examples of radial Q-quasiminimizers. We define local Q-quasiminimizers as follows:

Definition 4.3.13. *Let $Q \geq 1$. We say that u is a local Q-quasiminimizer in Ω if we can find finitely many open set $\Omega_1, \Omega_2, \ldots$, such that $\Omega = \bigcup_j \Omega_j$ and u is a Q-quasiminimizer in Ω_j for all j.*

Proposition 4.3.14 (Björn-Björn [17], Theorem 8.2)**.** *Let $p > 1$ and $u(x) = |x|^\alpha$ where $\alpha \in \mathbb{R}$. For every $\varepsilon > 0$, u is a local $(1+\varepsilon)$-quasiminimizer.*

Based on Björn-Björn's [17] earlier results where in most cases u fails to be a quasiminimizer or the best constant takes the value $Q = 1$, we can easily deduce that local $(1 + \varepsilon)$-quasiminimizers are not Q-quasiminimizers in general. The converse implication is also true.

Proposition 4.3.15 (Judin [68], Example 4.0.25 and Uppman [124], Section 2.2.3)**.** *Let $Q > 1$. There is a Q-quasiminimizer which is not a local Q'-quasiminimizer for any $Q' < Q$.*

Chapter 5

Regularity Results for Vector-Valued Quasiminimizers

In the previous chapter, we briefly mentioned about regularity results alongside the examples of scalar-valued Q-quasiminimizers. This chapter will focus on regularity results for vector-valued Q-quasiminimizers of an inhomogeneous p-Dirichlet integral.

We begin by first introducing another aspect on how the notion of Q-quasiminimizer can be extended to the variational problem (P). Defining our problem for local functionals that satisfies some coercivity and growth conditions (see Theorem 5.1.2), we show that if the corresponding variational problem has a minimizing sequence, we can always find another minimizing sequence for the variational problem and moreover they consist of Q-quasiminimizers of an inhomogeneous p-Dirichlet integral.

By focusing on the quasiminimizers of the inhomogeneous p-Dirichlet integral obtained as above, we give a series of known results leading to higher integrability in the interior. We begin by generalizing the Caccioppoli's inequality to a cubical Q-quasiminimizers of the inhomogeneous p-Dirichlet integral which then gives us an inhomogeneous reverse Poincaré's inequality on increasing support. This is particularly useful in showing the self-improving property of Q-quasiminimizers, by means of Gehring's lemma [51]. However, as expected this is only the regularity in the interior and we aim to show for global higher integrability. It turns out that global higher integrability is also achievable, proved by using the regularity in the interior and then extend to the whole set by the condition that we impose on the boundary of the set.

The outline of this chapter goes as such: We consider the existence of Q-quasiminimizers in Section 5.1. This establishes the motivation in considering Q-quasiminimizers of the inhomogeneous p-Dirichlet integral throughout the rest of the thesis. In Section 5.2, we focus on showing the higher integrability in the interior for Q-quasiminimizers. Lastly, we consider the issue of global higher integrability in Section 5.3.

5.1 Existence of quasiminimizers

Here, we would like to show the existence of Q-quasiminimizers within good minimizing sequences of a variational problem. The idea of finding good minimizing sequences in the sense below goes back to at least Marcellini-Sbordone [91]. In their paper, Yan-Zhou [128] (see Section 3) constructed such sequences using the Ekeland's variational principle [39].

Theorem 5.1.1 (Ekeland's variational principle). *Let (X, d) be a complete metric space where $d = d(u, v)$ denotes the distance between $u, v \in X$. Let $\mathcal{F}\colon X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, bounded from below and taking a finite value at some point. Assume that for some $u \in X$ and some $\varepsilon > 0$, we have*

$$\mathcal{F}(u) \leq \inf_X \mathcal{F} + \varepsilon. \tag{5.1.1}$$

Then there exists a point $v \in X$ such that

(i) $\mathcal{F}(v) \leq \mathcal{F}(u)$,

(ii) $d(u, v) \leq \varepsilon$, *and*

(iii) $\mathcal{F}(v) \leq \mathcal{F}(w) + \varepsilon d(v, w)$ *for all* $w \in X$.

Our result generalizes Yan-Zhou's construction [128] to local functionals of p-growth. Recall that a functional $\mathcal{F}\colon W^{1,p}(\Omega; \mathbb{R}^N) \times \mathcal{O}(\Omega) \to \mathbb{R}$ is local on $\mathcal{O}(\Omega)$ if for $u, v \in W^{1,p}(\Omega; \mathbb{R}^N)$

$$\mathcal{F}(u, \omega) = \mathcal{F}(v, \omega) \text{ whenever } u = v \text{ a.e. on } \omega \in \mathcal{O}(\Omega),$$

where $\mathcal{O}$ denotes the family of all open subsets of Ω. With an added mean coercivity condition (5.1.3) below as well as the usual growth condition, we prove that if the variational problem (*P*) has a minimizing sequence, then we can find another minimizing sequence consisting of Q-quasiminimizers of the inhomogeneous p-Dirichlet integral. This shows how Q-quasiminimizers appear naturally in this context, in addition to the examples discussed in Chapter 4, and more substantially they are higher integrable.

Theorem 5.1.2. *Let $\mathcal{F}\colon W^{1,p}(\Omega;\mathbb{R}^N) \times \mathcal{O}(\Omega) \to \mathbb{R}$ be a local functional that is $W^{1,1}$ lower semicontinuous on $\mathcal{O}(\Omega)$ and assume that*

$$|\mathcal{F}(u,\omega)| \leq c_1 \int_\omega (|Du|^p + 1)\ \mathrm{d}x \tag{5.1.2}$$

for all $u \in W^{1,p}(\omega;\mathbb{R}^N)$ and for all $\omega \in \mathcal{O}(\Omega)$ where $c_1 < \infty$ and $p > 1$ are constants. Assume furthermore that

$$\mathcal{F}(u,\omega) + c_2 \int_\omega (|Dg|^p + 1)\ \mathrm{d}x \geq c_3 \int_\omega |Du|^p\,\mathrm{d}x \tag{5.1.3}$$

for all $g \in W^{1,p}(\omega;\mathbb{R}^N)$, $u \in W^{1,p}_g(\omega;\mathbb{R}^N)$ and $\omega \in \mathcal{O}(\Omega)$ where constants $c_2 < \infty$ and $c_3 > 0$. Then there exists a constant $Q = Q(c_1,c_2,c_3,p) < \infty$ such that if $\{u_j\}$ is a minimizing sequence for the variational problem

$$\inf\left\{\mathcal{F}(u,\Omega):\ u \in W^{1,p}_g(\Omega;\mathbb{R}^N)\right\}, \tag{5.1.4}$$

then there exists another minimizing sequence $\{v_j\}$ for (5.1.4) consisting of Q-quasiminimizers of the inhomogeneous p-Dirichlet integral

$$\int_\Omega (|Du|^p + 1)\,\mathrm{d}x \tag{5.1.5}$$

and such that

$$\int_\Omega |Dv_j - Du_j|\,\mathrm{d}x \to 0. \tag{5.1.6}$$

Proof. The main essence of this proof is based on the proof of Theorem 6.3 in [55]. Let

$$m = \inf\left\{\mathcal{F}(u,\Omega):\ u \in W^{1,p}_g(\Omega;\mathbb{R}^N)\right\}.$$

It is clear that $m > -\infty$ by (5.1.3). Let $\varepsilon_j = \mathcal{F}(u_j,\Omega) - m$. For $u \in W^{1,p}(\Omega;\mathbb{R}^N)$, we define a complete metric space (X,d) by

$$X = W^{1,1}_g(\Omega;\mathbb{R}^N)$$
$$d = d(w,v) = \int_\Omega |Dw - Dv|\,\mathrm{d}x.$$

By assumption $\mathcal{F}(v,\Omega)$ is lower semicontinuous on (X,d). Hence Ekeland's variational principle yields $v_j \in W^{1,p}_g(\Omega;\mathbb{R}^N)$ such that

$$\begin{cases} \mathcal{F}(v_j,\Omega) \leq \mathcal{F}(u_j,\Omega), \\ \int_\Omega |Du_j - Dv_j|\,\mathrm{d}x \leq \varepsilon_j, \\ \mathcal{F}(v_j,\Omega) \leq \mathcal{F}(u,\Omega) + \varepsilon_j \int_\Omega |Du - Dv_j|\,\mathrm{d}x, \end{cases} \tag{5.1.7}$$

for all $u \in W_g^{1,p}(\Omega;\mathbb{R}^N)$. Since $\varepsilon_j > 0$ can be chosen to be arbitrarily small, we obtain (5.1.6).

To prove that v_j is quasiminimizing, fix j and let $\omega \in \mathcal{O}(\Omega)$. Then, by (5.1.3), we have for all $v_j \in W_u^{1,p}(\omega,\mathbb{R}^N)$

$$c_3 \int_\omega |Dv_j|^p \,\mathrm{d}x \le \mathcal{F}(v_j,\omega) + c_2 \int_\omega (|Du|^p + 1)\,\mathrm{d}x.$$

If we define

$$\tilde{u} = \begin{cases} u & \text{in} \quad \omega \\ v_j & \text{in} \quad \Omega \setminus \omega \end{cases}$$

with $u = v_j$ on $\partial\omega$, then $\tilde{u} \in W_g^{1,p}(\Omega;\mathbb{R}^N)$. Taking $\tilde{u}$ into the last inequality of (5.1.7), we have

$$\mathcal{F}(v_j,\omega) \le \mathcal{F}(u,\omega) + \varepsilon_j \int_\omega |Du - Dv_j|\,\mathrm{d}x$$

and subsequently

$$c_3 \int_\omega |Dv_j|^p \,\mathrm{d}x \le \mathcal{F}(u,\omega) + \varepsilon_j \int_\omega |Du - Dv_j|\,\mathrm{d}x + c_2 \int_\omega (|Du|^p + 1)\,\mathrm{d}x.$$

By growth condition (5.1.2) and triangle inequality, we have

$$c_3 \int_\omega |Dv_j|^p \,\mathrm{d}x \le (c_1 + c_2) \int_\omega (|Du|^p + 1)\,\mathrm{d}x + \varepsilon_j \int_\omega (|Du| + |Dv_j|)\,\mathrm{d}x.$$

Letting $c_3 > 2\varepsilon_j$ in the above inequality, we obtain

$$c_3 \int_\omega |Dv_j|^p \,\mathrm{d}x \le (c_1 + c_2) \int_\omega (|Du|^p + 1)\,\mathrm{d}x + \frac{c_3}{2} \int_\omega (|Du| + |Dv_j|)\,\mathrm{d}x.$$

Invoking Young's inequality on the last term of the right hand side gives us

$$\begin{aligned} c_3 \int_\omega |Dv_j|^p \,\mathrm{d}x \quad &\le \quad (c_1 + c_2) \int_\omega (|Du|^p + 1)\,\mathrm{d}x + \frac{c_3}{2} \int_\omega \Big(\frac{|Du|^p}{p} + \frac{p-1}{p} \Big)\,\mathrm{d}x \\ &\quad + \frac{c_3}{2} \int_\omega \Big(\frac{|Dv_j|^p}{p} + \frac{p-1}{p} \Big)\,\mathrm{d}x \\ &\le \quad \Big(c_1 + c_2 + \frac{c_3}{2} \Big) \int_\omega (|Du|^p + 1)\,\mathrm{d}x + \frac{c_3}{2} \int_\omega (|Dv_j|^p + 1)\,\mathrm{d}x. \end{aligned}$$

Performing simple algebra on the above inequality brings us to

$$c_3 \int_\omega (|Dv_j|^p + 1)\,\mathrm{d}x \le \Big(c_1 + c_2 + \frac{3c_3}{2} \Big) \int_\omega (|Du|^p + 1)\,\mathrm{d}x + \frac{c_3}{2} \int_\omega (|Dv_j|^p + 1)\,\mathrm{d}x$$

In the end, we arrive at

$$\int_\omega (|Dv_j|^p + 1)\,\mathrm{d}x \le Q \int_\omega (|Du|^p + 1)\,\mathrm{d}x.$$

where $Q = (2c_1 + 2c_2 + 3c_3)/c_3$. $\square$

We can clearly see from the result another extension of the notion of Q-quasiminimizer and more significantly in the variational problem: Whenever a functional satisfies the mean coercivity and growth conditions and the corresponding variational problem has a minimizing sequence, then the existence of Q-quasiminimizers of the inhomogeneous p-Dirichlet integral (5.1.5) within another minimizing sequence of the variational problem is guaranteed.

We will see in Chapter 6 that the conclusion of Theorem 5.1.2 will be applied to the proof of Theorem 6.1.1, together with the regularity properties of Q-quasiminimizers that we will discuss in the sections that follow. The versatility of Q-quasiminimizers for the variational problem will then be showcased in Chapter 7.

5.2 Higher integrability in the interior

Now we focus on showing regularity property of the gradient of the Q-quasiminimizer obtained in Theorem 5.1.2. We note that the results considered here are standard results, but we generalize them to our Q-quasiminimizers. We begin with Caccioppoli's inequality in the interior.

The history of the well-known Caccioppoli's inequality goes back to Boyarskiĭ [20, 21] where he proved the result for solutions of first order elliptic systems of Beltrami's type when $n = 2$. Meyers [98] showed that the same result still holds for solutions of second order elliptic equations with L^∞ coefficients and for any n. This then led to a much celebrated result of Gehring [51] in which he proved higher integrability of partial derivatives of quasiconformal mappings.

Our version of the Caccioppoli's inequality does not require any coercivity or growth condition and the proof is adapted from Theorem 6.5 of Giusti [57]. We state our results here for cubical Q-quasiminimizers. It is clear that similar results are valid for spherical Q-quasiminimizers, simply replaces all cubes by balls in the statements and proofs below.

Theorem 5.2.1 (Caccioppoli's inequality in interior). *Let $\Omega \subset \mathbb{R}^n$ be an open set. If $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ is a cubical Q-quasiminimizer of $\int_\Omega (|Dv|^p + 1)\,\mathrm{d}x$, then there exists $c = c(p,Q)$ such that*

$$\int_{C_r} |Du|^p\,\mathrm{d}x \le c\left\{\frac{1}{(R-r)^p}\int_{C_R}|u-u_R|^p\,\mathrm{d}x + |C_R|\right\} \tag{5.2.1}$$

and hence

$$\fint_{C_r} |Du|^p\,\mathrm{d}x \le c\left\{\frac{1}{(R-r)^p}\fint_{C_R}|u-u_R|^p\,\mathrm{d}x + 1\right\} \tag{5.2.2}$$

for all $x \in \Omega$, $0 < r < R$ and $C_R = C(x,R) \Subset \Omega$ where $|C_R| = \operatorname{meas} C_R$.

To prove the theorem, we need the following iteration lemma (Giaquinta-Giusti [54], Lemma 1.1).

Lemma 5.2.2 (Iteration lemma). *Let $Z(s)$ be a bounded non-negative function in the interval $[r, R]$. Suppose that for all numbers s, t such that $r \leq s < t \leq R$, we have*

$$Z(s) \leq \vartheta Z(t) + A + \frac{B}{(t-s)^\alpha},$$

where $A, B \geq 0$, $\alpha > 0$ and $\vartheta \in [0, 1)$ are constants. Then

$$Z(r) \leq c(\alpha, \vartheta)\left(A + \frac{B}{(R-r)^\alpha}\right).$$

Proof of Theorem 5.2.1. Let C_R be a cube strictly contained in Ω and let $\eta \in C_0^\infty(C_R; \mathbb{R}^N)$ be such that for $r \leq s < t \leq R$

$$\begin{cases} 0 \leq \eta \leq 1, \\ \eta = 1 \text{ in } C_s, \\ \eta = 0 \text{ in } C_R \setminus C_t, \\ |D\eta| \leq \frac{1}{t-s}. \end{cases}$$

Let

$$\varphi = \eta(u - u_R)$$

where $u_R = \fint_{C_R} u \, \mathrm{d}x$ and hence $\varphi \in W_0^{1,p}(C_t; \mathbb{R}^N)$. Let

$$v = u - \eta(u - u_R) = (1-\eta)(u - u_R) + u_R.$$

Then

$$Dv = (1-\eta)Du - D\eta \otimes (u - u_R)$$

and hence

$$\begin{aligned} |Dv|^p &\leq \Big|(1-\eta)Du - D\eta \otimes (u - u_R)\Big|^p \\ &\leq 2^{p-1}(1-\eta)^p|Du|^p + 2^{p-1}|D\eta|^p|u - u_R|^p \\ &\leq 2^{p-1}(1-\eta)^p|Du|^p + \frac{2^{p-1}}{(t-s)^p}|u - u_R|^p \end{aligned} \tag{5.2.3}$$

where the last inequality follows from the assumption on η. Since u is a cubical Q-quasiminimizer of the functional $\int_\Omega (|Dv|^p + 1)\, \mathrm{d}x$, we have

$$\int_{C_t} \Big(|Du|^p + 1\Big) \mathrm{d}x \leq Q \int_{C_t} \Big(|Dv|^p + 1\Big) \mathrm{d}x$$

which can be rearranged and estimated using (5.2.3) to obtain

$$\begin{aligned}
\int_{C_t} |Du|^p \, \mathrm{d}x &\leq Q \int_{C_t} |Dv|^p \, \mathrm{d}x + (Q-1)|C_t| \\
&\leq 2^{p-1} Q \int_{C_t} (1-\eta)^p |Du|^p \, \mathrm{d}x + \frac{2^{p-1}Q}{(t-s)^p} \int_{C_t} |u - u_R|^p \, \mathrm{d}x + (Q-1)|C_t| \\
&\leq 2^{p-1} Q \int_{C_t \setminus C_s} |Du|^p \, \mathrm{d}x + \frac{2^{p-1}Q}{(t-s)^p} \int_{C_t} |u - u_R|^p \, \mathrm{d}x + (Q-1)|C_t|.
\end{aligned}$$

We now add $2^{p-1}Q \int_{C_s} |Du|^p \, \mathrm{d}x$ to both sides to get

$$2^{p-1}Q \int_{C_s} |Du|^p \, \mathrm{d}x \leq (2^{p-1}Q - 1) \int_{C_t} |Du|^p \, \mathrm{d}x + \frac{2^{p-1}Q}{(t-s)^p} \int_{C_t} |u - u_R|^p \, \mathrm{d}x + (Q-1)|C_t|.$$

Divide both sides by $2^{p-1}Q$ and we have, for $c = c(p, Q) = \frac{2^{p-1}Q - 1}{2^{p-1}Q} \in (0, 1)$,

$$\begin{aligned}
\int_{C_s} |Du|^p \, \mathrm{d}x &\leq c \int_{C_t} |Du|^p \, \mathrm{d}x + \frac{1}{(t-s)^p} \int_{C_t} |u - u_R|^p \, \mathrm{d}x + \frac{Q-1}{2^{p-1}Q} |C_t| \\
&\leq c \int_{C_t} |Du|^p \, \mathrm{d}x + \frac{1}{(t-s)^p} \int_{C_R} |u - u_R|^p \, \mathrm{d}x + |C_R|,
\end{aligned}$$

where $\frac{Q-1}{2^{p-1}Q} \in (0, 1)$ and $|C_t| \leq |C_R|$. We now use the iteration lemma by setting

$$\begin{cases}
\vartheta = c, \\
\alpha = p, \\
A = |C_R|, \\
B = \int_{C_R} |u - u_R|^p \, \mathrm{d}x, \\
Z(t) = \int_{C_t} |Du|^p \, \mathrm{d}x,
\end{cases}$$

to conclude that

$$\int_{C_r} |Du|^p \, \mathrm{d}x \leq \tilde{c}(p, c) \left(\frac{1}{(R-r)^p} \int_{C_R} |u - u_R|^p \, \mathrm{d}x + |C_R| \right)$$

which then gives us (5.2.2) when both sides are divided by $|C_R|$ since clearly $\tilde{c}(p, c) = c(p, Q)$. $\square$

If our integral is set to be

$$\int_\Omega \left(|Dv|^2 + \lambda^2 \right)^{\frac{p}{2}} \, \mathrm{d}x$$

where $\lambda \geq 0$, we also have a similar inequality.

Theorem 5.2.3. *Let $\Omega \subset \mathbb{R}^n$ be an open set. If $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ is a cubical Q-quasiminimizer of $\int_\Omega (|Dv|^2 + \lambda^2)^{\frac{p}{2}} \, \mathrm{d}x$ where constant $\lambda \geq 0$, then there exists $c = c(p, Q)$ such that*

$$\int_{C_r} |Du|^p \, \mathrm{d}x \leq c \left\{ \frac{1}{(R-r)^p} \int_{C_R} |u - u_R|^p \, \mathrm{d}x + \lambda^p |C_R| \right\}$$

and hence

$$\fint_{C_r} |Du|^p \, \mathrm{d}x \leq c \left\{ \frac{1}{(R-r)^p} \fint_{C_R} |u - u_R|^p \, \mathrm{d}x + \lambda^p \right\}$$

for all $x \in \Omega$, $0 < r < R$ and $C_R = C(x, R) \Subset \Omega$.

Proof. The proof follows from that of Theorem 5.2.1 with some minor changes in the estimation. Let C_t, C_s, η to be the same as in the proof of Theorem 5.2.1. Since u is a cubical Q-quasiminimizer, then

$$\int_{C_t} \left(|Du|^2 + \lambda^2\right)^{\frac{p}{2}} \,\mathrm{d}x \le Q \int_{C_t} \left(|Dv|^2 + \lambda^2\right)^{\frac{p}{2}} \,\mathrm{d}x.$$

Note that

$$c_1\left(|Du|^p + \lambda^p\right) \le \left(|Du|^2 + \lambda^2\right)^{\frac{p}{2}} \le c_2\left(|Du|^p + \lambda^p\right)$$

for constants $c_2 \ge c_1 > 0$. Using the estimation above, we have

$$c_1 \int_{C_t} (|Du|^p + \lambda^p) \,\mathrm{d}x \le c_2 Q \int_{C_t} (|Dv|^p + \lambda^p) \,\mathrm{d}x$$

which we rearrange to obtain

$$\int_{C_t} |Du|^p \,\mathrm{d}x \le \frac{c_2 Q}{c_1} \int_{C_t} |Dv|^p \,\mathrm{d}x + \left(\frac{c_2 Q}{c_1} - 1\right) \lambda^p |C_t|$$

By letting $c_3 = \frac{c_2 Q}{c_1}$ and $c_4 = \frac{c_2 Q}{c_1} - 1$, we proceed with the same scheme as the previous proof to get

$$\int_{C_t} |Du|^p \,\mathrm{d}x \le 2^{p-1} c_3 \int_{C_t \setminus C_s} |Du|^p \,\mathrm{d}x + \frac{2^{p-1} c_3}{(t-s)^p} \int_{C_t} |u - u_R|^p \,\mathrm{d}x + c_4 \lambda^p |C_t|.$$

Adding $2^{p-1} c_3 \int_{C_s} |Du|^p \,\mathrm{d}x$ to both sides and subsequently dividing by $2^{p-1} c_3$, we have

$$\int_{C_s} |Du|^p \,\mathrm{d}x \le c \int_{C_t} |Du|^p \,\mathrm{d}x + \frac{1}{(t-s)^p} \int_{C_R} |u - u_R|^p \,\mathrm{d}x + \frac{c_4 \lambda^p}{2^{p-1} c_3} |C_R|,$$

where $c = c(p, Q) = \frac{2^{p-1} c_3 - 1}{2^{p-1} c_3} \in (0, 1)$ and $|C_t| \le |C_R|$. Finally, we obtain the conclusion by invoking the iteration lemma. $\square$

The Caccioppoli's inequality presented here is also known as inhomogeneous reverse Poincaré's inequality on increasing support. Note that the first term on the right hand side of (5.2.2) can be estimated by Poincaré-Sobolev's inequality to obtain

$$\fint_{C_R} |u - u_R|^p \,\mathrm{d}x \le c(p, Q) R^p \left(\fint_{C_R} |Du|^{\frac{np}{n+p}} \,\mathrm{d}x\right)^{\frac{n+p}{n}}$$

which we substitute into (5.2.2) to get

$$\fint_{C_r} |Du|^p \,\mathrm{d}x \le c \left(\fint_{C_R} |Du|^{\frac{np}{n+p}} \,\mathrm{d}x\right)^{\frac{n+p}{n}} + c$$

for all $C_R \subset \Omega$ where $c = c(n, p, Q, r, R)$.

Writing $F := |Du|^{\frac{np}{n+p}}$ and $q = \frac{n+p}{n} = 1 + \frac{p}{n} > 1$, we have the inhomogeneous reverse Hölder's inequality on increasing support

$$\fint_{C_r} F^q \,\mathrm{d}x \le c \left(\fint_{C_R} F \,\mathrm{d}x \right)^q + c$$

which has a self-improving property known as Gehring's lemma [51]. Gehring's lemma is useful for showing that higher integrability of the gradient under certain conditions.

Lemma 5.2.4 (Gehring's lemma). *Let C_0 be a fixed cube and let $F, G \in L^q(C)$ be non-negative functions. If for some $q > 1$*

$$\fint_{C_R} F^q \,\mathrm{d}x \le c \left(\fint_{C_{2R}} F \,\mathrm{d}x \right)^q + \fint_{C_{2R}} G^q \,\mathrm{d}x + c$$

for each cube C_R with $C_{2R} \subset C$, then there exists $\varepsilon = \varepsilon(c, q, n) > 0$ and $\tilde{c} = \tilde{c}(c, q, n) > 0$ such that

$$\left(\fint_{C_R} F^{\tilde{q}} \,\mathrm{d}x \right)^{\frac{1}{\tilde{q}}} \le \tilde{c} \left\{ \left(\fint_{C_{2R}} F^q \,\mathrm{d}x \right)^{\frac{1}{q}} + \left(\fint_{C_{2R}} G^{\tilde{q}} \,\mathrm{d}x \right)^{\frac{1}{\tilde{q}}} + 1 \right\}$$

for all $\tilde{q} \in [q, q + \varepsilon]$.

Note that if we only consider functions that are locally integrable, then the assumption on G is not included and hence the estimate will be free of the term G. We can easily see that if F is a function in the gradient of u, then we have obtained its higher integrability in the interior.

As a consequence of Gehring's lemma, we have the higher integrability of the gradient of a cubical Q-quasiminimizer of the inhomogeneous p-Dirichlet integral.

Theorem 5.2.5 (Giusti [57], Theorem 6.8)**.** *Let Ω be a bounded Lipschitz domain and let $g \in W^{1,\tilde{p}}(\Omega; \mathbb{R}^N)$ for some $\tilde{p} > p$. If $u \in W^{1,p}_g(\Omega; \mathbb{R}^N)$ is a cubical Q-quasiminimizer of the inhomogeneous p-Dirichlet integral $\int_\Omega (|Dv|^p + 1) \,\mathrm{d}x$, then there exists $\delta \in (0, \tilde{p} - p)$ such that $u \in W^{1,p+\delta}(\Omega; \mathbb{R}^N)$.*

In the next section, we will prove a similar result for Q-quasiminimizer but with a weaker condition, namely requiring only the complement of Ω to be uniformly p-thick, a concept that requires some understanding of variational p-capacity (see Appendix C). It is also worth mentioning the following theorem concerning the set of p-capacity zero (or removability of singularities).

Theorem 5.2.6 (Giaquinta-Giusti [55], Theorem 3.3)**.** *Let $E \subset \Omega$ be a closed set of p-capacity zero. Let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ be a Q-quasiminimizer of the functional $\mathcal{F}(u, \Omega)$ with the integrand F satisfying the coercivity and growth conditions (4.2.16) in $\Omega \setminus E$. Then u is a Q-quasiminimizer in Ω.*

5.3 Global higher integrability

In this section, we are concerned about the issue of global higher integrability where we prove a similar result to that of Theorem 5.2.5 with a weaker condition but yet sufficient, requiring only the complement of Ω to be uniformly p-thick. The definition and basic properties of variational p-capacity can be found in Appendix C.

The idea of proof follows from Kilpeläinen-Koskela [69] (pages 904–906) which employs the notion of variational p-capacity and a series of results such as Caccioppoli's inequality, Poincaré-Sobolev's inequality and Gehring's lemma. These results have been considered in the previous section and mainly focused on interior regularity for vector-valued Q-quasiminimizers, but they will be proven to be useful in this section.

Theorem 5.3.1. *Let $\Omega \subset \mathbb{R}^n$ be bounded open set such that Ω^c is uniformly p-thick (no restriction when $p > n$). Let $g \in W^{1,\tilde{p}}(\Omega;\mathbb{R}^N)$ for some $\tilde{p} > p$. If $u \in W^{1,p}_g(\Omega;\mathbb{R}^N)$ is a Q-quasiminimizer of the inhomogeneous p-Dirichlet integral*

$$\int_\Omega (|Dv|^p + 1)\,\mathrm{d}x,$$

then there exists a constant $\delta_0 = \delta_0(n, N, p, Q, c_0) \in (0, \tilde{p} - p]$ such that $u \in W^{1,p+\delta}(\Omega;\mathbb{R}^N)$ whenever $\delta \in (0, \delta_0)$. In particular, $|Du| \in L^{p+\delta}(\Omega)$ for all $\delta \in (0, \delta_0)$ and

$$\left(\int_\Omega |Du|^{p+\delta}\,\mathrm{d}x\right)^{\frac{1}{p+\delta}} \le c\left(\int_\Omega |Du|^p\,\mathrm{d}x\right)^{\frac{1}{p}} + c\left(\int_\Omega |Dg|^{p+\delta}\,\mathrm{d}x\right)^{\frac{1}{p+\delta}} + c$$

where constant $c = c(n, N, p, Q, c_0) \in (0, \infty)$.

Proof. Let B_0 be a ball with $\Omega \Subset \frac{1}{2}B_0$. Fix $r > 0$ and let B_r be a ball of radius r with $B_{2r} \subset B_0$. We will divide the proof into two cases.

Case 1: Let $B_{2r} \subset \Omega$. By Caccioppoli's estimate (see Theorem 5.2.1), we have

$$\fint_{B_r} |Du|^p\,\mathrm{d}x \le \frac{c}{r^p}\fint_{B_{2r}} |u - u_{2r}|^p\,\mathrm{d}x + c \tag{5.3.1}$$

where constant $c = c(p, Q) > 0$ and $u_{2r} = \fint_{B_{2r}} u\,\mathrm{d}x$. If we let $p_0 = \max\left\{\frac{np}{n+p}, 1\right\}$, then we have $p \in (p_0, p_0^*)$ where p_0^* is the Sobolev conjugate of p_0. By (5.3.1) and Poincaré-Sobolev's inequality, we obtain

$$\begin{aligned}
\fint_{B_r} |Du|^p \,\mathrm{d}x &\leq \frac{c}{r^p} \fint_{B_{2r}} |u - u_{2r}|^p \,\mathrm{d}x + c \\
&\leq c \left(\fint_{B_{2r}} |Du|^{p_0} \,\mathrm{d}x \right)^{\frac{p}{p_0}} + c \\
&\leq c \left(\fint_{B_{2r}} |Du|^{p(1-\varepsilon)} \,\mathrm{d}x \right)^{\frac{1}{1-\varepsilon}} + c
\end{aligned} \tag{5.3.2}$$

whenever $0 < \varepsilon \leq \min\left\{\frac{p}{n+p}, p-1\right\}$ and $c = c(n, N, p, Q) > 0$.

Case 2: Let $B_{2r} \setminus \Omega \neq \emptyset$. First, we choose a cut-off function $\eta \in C_0^\infty(B_{2r}; \mathbb{R}^N)$ such that for $r \leq s < t \leq 2r$

$$\begin{cases}
0 \leq \eta \leq 1, \\
\eta = 1 \text{ in } B_s, \\
\eta = 0 \text{ in } B_{2r} \setminus B_t, \\
|D\eta| \leq \frac{2}{t-s}.
\end{cases}$$

Then $\eta(u - g) \in W_0^{1,p}(B_t \cap \Omega; \mathbb{R}^N)$ and the Q-quasiminimality of u gives us

$$\int_{B_t \cap \Omega} (|Du|^p + 1) \,\mathrm{d}x \leq Q \int_{B_t \cap \Omega} (|Dv|^p + 1) \,\mathrm{d}x$$

where $v = u - \eta(u - g) = (1 - \eta)(u - g) + g$. Perform simple algebra to obtain

$$\int_{B_s \cap \Omega} |Du|^p \,\mathrm{d}x \leq Q \int_{B_t \cap \Omega} |Dv|^p \,\mathrm{d}x + Q|B_t \cap \Omega| - |B_s \cap \Omega|.$$

Here we note that $Q|B_t \cap \Omega| - |B_s \cap \Omega| < Q|B_t \cap \Omega|$ and

$$Dv = -D\eta \otimes (u - g) + (1 - \eta)(Du - Dg) + Dg.$$

We now have

$$\begin{aligned}
\int_{B_s \cap \Omega} |Du|^p \,\mathrm{d}x \leq\ & c \int_{B_t \cap \Omega} (1 - \eta)^p |Du|^p \,\mathrm{d}x + c \int_{B_t \cap \Omega} |u - g|^p |D\eta|^p \,\mathrm{d}x \\
&+ c \int_{B_t \cap \Omega} \big((1 - \eta)^p + 1\big) |Dg|^p \,\mathrm{d}x + c|B_t \cap \Omega|,
\end{aligned}$$

Applying the assumptions of the cut-off function η yield

$$\begin{aligned}
\int_{B_s \cap \Omega} |Du|^p \,\mathrm{d}x \leq\ & c \int_{(B_t \setminus B_s) \cap \Omega} |Du|^p \,\mathrm{d}x + \frac{c}{(t - s)^p} \int_{B_t \cap \Omega} |u - g|^p \,\mathrm{d}x \\
&+ c \int_{B_t \cap \Omega} |Dg|^p \,\mathrm{d}x + c|B_t \cap \Omega|
\end{aligned}$$

and add $c \int_{B_s \cap \Omega} |Du|^p \,\mathrm{d}x$ to both sides to obtain

$$\begin{aligned}(c+1)\int_{B_s\cap\Omega}|Du|^p\,\mathrm{d}x &\leq c\int_{B_t\cap\Omega}|Du|^p\,\mathrm{d}x+\frac{c}{(t-s)^p}\int_{B_t\cap\Omega}|u-g|^p\,\mathrm{d}x\\ &\quad +c\int_{B_t\cap\Omega}|Dg|^p\,\mathrm{d}x+c|B_t\cap\Omega|.\end{aligned}$$

We now divide both sides of the inequality by $c+1$ and observe that $B_t\subset B_{2r}$ to get

$$\int_{B_s\cap\Omega}|Du|^p\,\mathrm{d}x\leq\tilde{c}\int_{B_t\cap\Omega}|Du|^p\,\mathrm{d}x+\frac{\tilde{c}}{(t-s)^p}\int_{B_{2r}\cap\Omega}|u-g|^p\,\mathrm{d}x+\tilde{c}\int_{B_{2r}\cap\Omega}|Dg|^p\,\mathrm{d}x+\tilde{c}|B_{2r}\cap\Omega|,$$

where $\tilde{c}=\frac{c}{c+1}\in(0,1)$. Applying the iteration lemma gives us

$$\int_{B_r\cap\Omega}|Du|^p\,\mathrm{d}x\leq\frac{c}{r^p}\int_{B_{2r}\cap\Omega}|u-g|^p\,\mathrm{d}x+c\int_{B_{2r}\cap\Omega}|Dg|^p\,\mathrm{d}x+c|B_{2r}\cap\Omega|. \tag{5.3.3}$$

We define $u-g=0$ in $\mathbb{R}^n\setminus\overline{\Omega}$ and therefore $u-g=0$ in Ω^c except for a set of p-capacity zero. We may now apply Lemma C.0.7 to estimate $\frac{c}{r^p}\int_{B_{2r}\cap\Omega}|u-g|^p\,\mathrm{d}x$. We first let $q=p(1-\varepsilon)$, where

$$\begin{cases}0<\varepsilon<\min\left\{\frac{p}{n+p},p-1\right\} & \text{if } p\leq n\\ 0<\varepsilon<\min\left\{\frac{p-n}{p},\frac{1}{2}\right\} & \text{if } p>n\end{cases}.$$

If

$$\kappa=\begin{cases}\frac{n}{n-q} & \text{if } q<n\\ 2 & \text{if } q>n\end{cases},$$

then $\kappa q\geq p$. We may now apply Lemma C.0.7 to obtain

$$\begin{aligned}\left(\frac{c}{r^p}\fint_{B_{2r}}|u-g|^p\,\mathrm{d}x\right)^{\frac{1}{p}} &= \frac{c}{r}\left(\fint_{B_{2r}}|u-g|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\\ &\leq \frac{c}{r}\left(\fint_{B_{2r}}|u-g|^{\kappa q}\,\mathrm{d}x\right)^{\frac{1}{\kappa q}}\\ &\leq \frac{c}{r}\left(\frac{1}{\mathrm{cap}_q(N(u-g);B_{4r})}\fint_{B_{2r}}|Du-Dg|^q\,\mathrm{d}x\right)^{\frac{1}{q}}\\ &\leq c\left(\frac{r^{n-q}}{\mathrm{cap}_q(N(u-g);B_{4r})}\fint_{B_{2r}}|Du-Dg|^q\,\mathrm{d}x\right)^{\frac{1}{q}}\\ &\leq c\left(\frac{r^{n-q}}{\mathrm{cap}_q(N(u-g);B_{4r})}\fint_{B_{2r}\cap\Omega}|Du-Dg|^q\,\mathrm{d}x\right)^{\frac{1}{q}},\end{aligned}$$

where $N(u-g)=\{x\in B_{2r}\colon u(x)=g(x)\}$, the constants c derived for each inequality are actually different from the previous one and the last inequality is due to $u-g=0$ in Ω^c except for a set of p-capacity zero. If $p>n$, then $q>n$ and hence we use the uniform p-thickness of every non-empty set to obtain

$$\mathrm{cap}_q(N(u-g);B_{4r})\geq\mathrm{cap}_q(B_{2r}\setminus\Omega;B_{4r})\geq c_0\,\mathrm{cap}_q(B_{2r};B_{4r})\geq cr^{n-q}. \tag{5.3.4}$$

If $p \le n$, then it follows from Proposition C.0.4 that Ω^c is uniformly q-thick if $\varepsilon \in \big(0, \varepsilon_0(n,p,c_0)\big)$. Similarly, for ε small enough, we have (5.3.4). Putting the estimates together, we finally have

$$\begin{aligned}\left(\frac{1}{r^n}\int_{B_r\cap\Omega}|Du|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \quad &\le \quad c + c\left(\frac{1}{r^n}\int_{B_{2r}\cap\Omega}|Du|^q\,\mathrm{d}x\right)^{\frac{1}{q}} \\ &\quad + c\left(\frac{1}{r^n}\int_{B_{2r}\cap\Omega}|Dg|^p\,\mathrm{d}x\right)^{\frac{1}{p}}. \end{aligned} \tag{5.3.5}$$

Let

$$f(x) = \begin{cases} |Du(x)|^{p(1-\varepsilon)} & \text{if } x \in \Omega \\ 0 & \text{elsewhere} \end{cases}$$

and

$$h(x) = \begin{cases} |Dg(x)|^{p(1-\varepsilon)} & \text{if } x \in \Omega \\ 0 & \text{elsewhere} \end{cases}.$$

Further let $s = \frac{1}{1-\varepsilon} > 1$, where $\varepsilon \in (0,1)$ is so small that both (5.3.2) and (5.3.5) hold. Then $f, h \in L^s(B_0; \mathbb{R}^N)$ and we obtain the following inhomogeneous reverse Hölder's inequality on increasing supports

$$\fint_{B_r} f^s\,\mathrm{d}x \le c\left(\fint_{B_{2r}} f\,\mathrm{d}x\right)^s + c\fint_{B_{2r}} h^s\,\mathrm{d}x + c$$

whenever $B_{2r} \subset B_0$. It then follows from Gehring's lemma that there exist $\tilde{\varepsilon} = \tilde{\varepsilon}(c,s,n) > 0$ and $\tilde{c} = \tilde{c}(c,s,n) > 0$ such that

$$\left(\fint_{B_r} f^t\,\mathrm{d}x\right)^{\frac{1}{t}} \le \tilde{c}\left(\fint_{B_{2r}} f^s\,\mathrm{d}x\right)^{\frac{1}{s}} + \tilde{c}\left(\fint_{B_{2r}} h^t\,\mathrm{d}x\right)^{\frac{1}{t}} + \tilde{c} \tag{5.3.6}$$

for all $t \in [s, s+\tilde{\varepsilon}]$. Since Ω is bounded, we can choose a finite number of balls $B(x_j, r_j)$, $j = 1, \ldots, J$ such that

$$B(x_j, 2r_j) \subset B_0$$

and

$$\Omega \subset \bigcup_{j=1}^{J} B(x_j, r_j).$$

Multiplying (5.3.6) by $|B_{2r}|^{\frac{1}{t}}$ and summing over $B(x_j, r_j)$ gives us

$$\sum_{j=1}^{J}\left(\int_{B(x_j,r_j)} f^t\,\mathrm{d}x\right)^{\frac{1}{t}} \le cJ\left(\int_{B_0} f^s\,\mathrm{d}x\right)^{\frac{1}{s}} + cJ\left(\int_{B_0} h^t\,\mathrm{d}x\right)^{\frac{1}{t}} + c.$$

It is clear that

$$\sum_{j=1}^{J}\left(\int_{B(x_j,r_j)} f^t\,\mathrm{d}x\right)^{\frac{1}{t}} \ge \left(\int_{\bigcup_{j=1}^{J} B(x_j,r_j)} f^t\,\mathrm{d}x\right)^{\frac{1}{t}} \ge \left(\int_{\Omega} f^t\,\mathrm{d}x\right)^{\frac{1}{t}},$$

whereas the integrals on the right hand side are no greater than the integrals over Ω by the fact that $f = 0$ and $h = 0$ outside Ω. We now have

$$\Big(\int_\Omega f^t \,\mathrm{d}x\Big)^{\frac{1}{t}} \leq c\Big(\int_\Omega f^s \,\mathrm{d}x\Big)^{\frac{1}{s}} + c\Big(\int_\Omega h^t \,\mathrm{d}x\Big)^{\frac{1}{t}} + c \tag{5.3.7}$$

where the constant c has been adjusted and therefore

$$f \in L^{s+\delta'}(\Omega;\mathbb{R}^N)$$

for some $\delta' = \delta'(n, N, p, Q, c_0) \in [0, \tilde{\varepsilon}]$ provided that

$$g \in L^{s+\delta'}(\Omega;\mathbb{R}^N).$$

We substitute $|Du|$ and $|Dg|$ into (5.3.7) to get

$$\begin{aligned} &\Big(\int_\Omega |Du|^{p(1+\delta'(1-\varepsilon))} \,\mathrm{d}x\Big)^{\frac{1}{p(1+\delta'(1-\varepsilon))}} \\ \leq \;\; & c\Big(\int_\Omega |Du|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} + \Big(\int_\Omega |Dg|^{p(1+\delta'(1-\varepsilon))} \,\mathrm{d}x\Big)^{\frac{1}{p(1+\delta'(1-\varepsilon))}} + c \end{aligned}$$

Clearly $1 + \delta'(1-\varepsilon) > 1$, so we may write $p\big(1 + \delta'(1-\varepsilon)\big)$ as $p + \delta$ and therefore

$$\Big(\int_\Omega |Du|^{p+\delta} \,\mathrm{d}x\Big)^{\frac{1}{p+\delta}} \leq c\Big(\int_\Omega |Du|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} + \Big(\int_\Omega |Dg|^{p+\delta} \,\mathrm{d}x\Big)^{\frac{1}{p+\delta}} + c$$

provided that $\delta \in (0, \delta_0)$ where $\delta_0 = \delta_0(n, N, p, Q, c_0) \in (0, \tilde{p} - p]$ and $g \in W^{1,\tilde{p}}(\Omega;\mathbb{R}^N)$ for some $\tilde{p} > p$. Finally, it follows from Sobolev's inequality that $u \in W^{1,p+\delta}(\Omega;\mathbb{R}^N)$. $\square$

The higher integrability of our Q-quasiminimizers is a much needed property in the next chapter. Besides, we shall see in Chapter 7 that the higher integrability also extends to minimizers of a certain variational problem which is in line with the essence of the direct method as mentioned in the introduction – to prove more regularity of the minimizers.

Chapter 6

Characterization of Mean Coercivity

It is known that the existence of minimizer for the functional $\mathcal{F}$ is guaranteed when the integrand F is quasiconvex and satisfies some pointwise coercivity and growth conditions, say

$$c_1|\xi|^p - c_2 \leq F(\xi) \leq c_3(|\xi|^p + 1)$$

for all $\xi \in \mathbb{R}^{N\times n}$ where constants $c_3 \geq c_1 > 0$ and $c_2 < \infty$. The pointwise coercivity condition can be weakened to mean coercivity (Giaquinta [52], Giaquinta-Giusti [54], Iwaniec-Lutoborski [66], Yan-Zhou [129]): there exist $c > 0$ and $\tilde{c} < \infty$ such that

$$\int_\Omega F(Du)\,\mathrm{d}x \geq c\int_\Omega |Du|^p\,\mathrm{d}x - \tilde{c}\int_\Omega (|Dg|^p + 1)\,\mathrm{d}x$$

for all $u \in W_g^{1,p}(\Omega;\mathbb{R}^N)$. In this chapter, we would like to investigate the connection between mean coercivity and strong quasiconvexity (closely related to uniform strict quasiconvex as defined by Evans [42]).

Definition 6.0.2. *Let $p \in (1,\infty)$. A continuous integrand $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is said to be strongly p-quasiconvex if and only if there exists $\varepsilon > 0$ such that $F - \varepsilon|\cdot|^p$ is quasiconvex. More precisely, the inequality*

$$\int_\Omega \Big(F\big(\xi + D\varphi(x)\big) - F(\xi)\Big)\,\mathrm{d}x \geq \varepsilon\int_\Omega |D\varphi(x)|^p\,\mathrm{d}x$$

holds for all $\xi \in \mathbb{R}^{N\times n}$ and for all $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^N)$.

Strong quasiconvexity is closely related to one of the conditions for partial regularity of minimizers (see for example Section 4 of Evans [42]). It turns out that a continuous integrand F satisfying p-growth condition (2.0.2) is mean p-coercive on $W_g^{1,p}(\Omega;\mathbb{R}^N)$ if and only if F is strongly p-quasiconvex at some $\xi \in \mathbb{R}^{N\times n}$.

This characterization of mean coercivity result will be clearly elaborated in the following section and the proof requires results such as existence of Q-quasiminimizers within good minimizing sequences and higher integrability of these Q-quasiminimizers, which were established in the previous chapter. We also show the relation between strongly 2-quasiconvex and strongly p-quasiconvex integrands for $p \in (2, \infty)$ in Section 6.2. In addition, if for some $c > 0$, $F - c|\cdot|^2$ satisfies the strict Legendre-Hadamard condition and hence strictly rank one convex, then it is equal to a 2-quasiconvex integrand in a neighbourhood of each point.

6.1 Mean coercivity and strong quasiconvexity

Theorem 6.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set such that Ω^c, the complement of Ω, is uniformly p-thick. Suppose $F \colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is a continuous integrand satisfying for some $c_1 < \infty$ and $p \in (1, \infty)$ the growth condition*

$$|F(\xi)| \leq c_1(|\xi|^p + 1) \tag{6.1.1}$$

for all $\xi \in \mathbb{R}^{N\times n}$. Then there exists a $c_2 > 0$ such that

$$(F - c_2|\cdot|^p)^{qc}(0) > -\infty, \tag{6.1.2}$$

where for a function G we denote by G^{qc} its quasiconvex envelope if and only if there exist constants $c_3 > 0$ and $c_4 < \infty$, such that for all u, $g \in W^{1,p}(\Omega;\mathbb{R}^N)$ with $u \in W^{1,p}_g(\Omega;\mathbb{R}^N)$

$$\int_\Omega F(Du)\,\mathrm{d}x \geq c_3 \int_\Omega |Du|^p\,\mathrm{d}x - c_4 \int_\Omega (|Dg|^p + 1)\,\mathrm{d}x. \tag{6.1.3}$$

In turn, (6.1.2) and (6.1.3) are both equivalent to the existence of a $c_5 > 0$ such that $F - c_5|\cdot|^p$ is quasiconvex at some $\xi \in \mathbb{R}^{N\times n}$.

Before we give the proof, we first give an example showcasing that mean coercivity is very distinct from pointwise coercivity.

Example 6.1.2. *Let $F \colon \mathbb{R}^{2\times 2} \to \mathbb{R}$. Define $F(\xi) = c|\xi_+|^2 - |\xi_-|^2$ for all $\xi \in \mathbb{R}^{2\times 2}$ where constant $c > 1$. Let $p = 2$ and recall the properties of conformal and anti-conformal parts of ξ from Remark 3.1.5. We can see that F is not pointwise coercive, but it is mean coercive. Hence F is strongly 2-quasiconvex or $F - c|\cdot|^2$ is quasiconvex. The Beurling–Ahlfors' inequality (see Iwaniec-Martin[67]) states that for $p \in (1, \infty)$, there exists $c(p) < \infty$ such that*

$$\|(D\varphi)_+\|_{L^p} \leq c(p)\|(D\varphi)_-\|_{L^p}$$

and

$$\|(D\varphi)_-\|_{L^p} \le c(p)\|(D\varphi)_+\|_{L^p}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^2;\mathbb{R}^2)$. *If we identify* $\mathbb{R}^2 \cong \mathbb{C}$, *then* $(D\varphi)_+$ *corresponds to* $\frac{\partial \varphi}{\partial z}$ *and* $(D\varphi)_-$ *corresponds to* $\frac{\partial \varphi}{\partial \bar{z}}$, *where* $\bar{z}$ *is the complex conjugate of* z. *This can be generalized to show that* $|\xi_\pm|^p$ *is indeed strongly* p*-quasiconvex. In the case when* $F\colon \mathbb{R}^n \to \mathbb{R}^n$, *then we employ Korn's inequality (see Ciarlet [23] and Korn [81]) in a similar way.*

Proof of Theorem 6.1.1. Let $G = F - c_2|\cdot|^p$. By similar argument to that of Section 3.1,

$$G^{qc}(\xi) > -\infty$$

for all $\xi \in \mathbb{R}^{N\times n}$.

Next we show that G^{qc} is quasiconvex. Let $\xi \in \mathbb{R}^{N\times n}$ and given $\varepsilon > 0$, we can find quasiconvex f such that $f \le G^{qc}$ and

$$f(\xi) + \varepsilon > G^{qc}(\xi).$$

Since f is quasiconvex , we have by definition

$$f(\xi)|\Omega| \le \int_\Omega f(\xi + D\varphi)\,\mathrm{d}x \tag{6.1.4}$$

for every piecewise affine $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^N)$ where $|\Omega| =: \operatorname{meas}\Omega$. Hence,

$$\int_\Omega G^{qc}(\xi + D\varphi)\,\mathrm{d}x \ge \int_\Omega f(\xi + D\varphi)\,\mathrm{d}x \ge f(\xi)|\Omega| > \Big(G^{qc}(\xi) - \varepsilon\Big)|\Omega|.$$

Thus by taking $\varepsilon > 0$ to be arbitrarily small, we have

$$\int_\Omega G^{qc}(\xi + D\varphi)\,\mathrm{d}x \ge G^{qc}(\xi)|\Omega|, \tag{6.1.5}$$

and hence the quasiconvexity of G^{qc}.

We will now show that (6.1.1) and (6.1.2) imply (6.1.3). From $G^{qc} \le G = F - c_2|\cdot|^p$, we get that $G^{qc} + c_2|\cdot|^p \le F$ and clearly

$$G^{qc} + c_2|\cdot|^p \le F^{qc}. \tag{6.1.6}$$

Since both G^{qc} and F^{qc} are bounded above by $c_1(|\cdot|^p + 1)$, it follows from Lemma 2.2.21 that they are also bounded below by $-\tilde{c}c_1(|\cdot|^p + 1)$ for some constant $\tilde{c} = \tilde{c}(n, N)$. By Lemma 2.2.19, we therefore have that

$$\fint_\Omega G^{qc}(D\varphi)\,\mathrm{d}x \ge G^{qc}(0) \quad \text{and} \quad \fint_\Omega F^{qc}(D\varphi)\,\mathrm{d}x \ge F^{qc}(0)$$

for all $\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^N)$. Take $\varphi = u - g \in W_0^{1,p}(\Omega;\mathbb{R}^N)$ and let $m = G^{qc}(0)$. Then, from inequality (6.1.6), we see that

$$\begin{aligned} & m|\Omega| + c_2 \int_\Omega |Du - Dg|^p \,\mathrm{d}x \\ \leq \;& \int_\Omega F^{qc}(Du - Dg) \,\mathrm{d}x \\ = \;& \int_\Omega \big(F^{qc}(Du) + F^{qc}(Du - Dg) - F^{qc}(Du)\big) \,\mathrm{d}x \\ \leq \;& \int_\Omega F(Du) \,\mathrm{d}x + c_6 \int_\Omega \big(|Du|^{p-1} + |Dg|^{p-1} + 1\big)\, |Dg| \,\mathrm{d}x, \end{aligned} \tag{6.1.7}$$

where we used Lipschitz estimate for quasiconvex (and so in particular separately convex, see Proposition 2.2.18) functions in the last inequality. Invoking Young's inequality on (6.1.7) and choosing the right constant, we have

$$m|\Omega| + c_2 \int_\Omega |Du - Dg|^p \,\mathrm{d}x \leq \int_\Omega F(Du) \,\mathrm{d}x + c_7 \int_\Omega (|Dg|^p + 1) \,\mathrm{d}x + \frac{c_2}{4} \int_\Omega |Du|^p \,\mathrm{d}x. \tag{6.1.8}$$

Next, using Lemma 1.1 in [89] in which

$$|a + b|^p \leq (1+\varepsilon)^{p-1} |a|^p + \left(1 + \frac{1}{\varepsilon}\right)^{p-1} |b|^p$$

with $a = Du - Dg, b = Dg$ and $(1+\varepsilon)^{p-1} = 2$, we obtain

$$|Du|^p \leq 2|Du - Dg|^p + 2(2^{\frac{1}{p-1}} - 1)^{1-p} |Dg|^p$$

and hence substituting into (6.1.8) gives us

$$\frac{c_2}{4} \int_\Omega |Du|^p \,\mathrm{d}x \leq \int_\Omega F(Du) \,\mathrm{d}x + c_4 \int_\Omega (|Dg|^p + 1) \,\mathrm{d}x$$

for all $u \in W_g^{1,p}(\Omega;\mathbb{R}^N)$. Choosing $c_3 = \frac{c_2}{4}$, we finally have (6.1.3). The opposite implication is a consequence of Dacorogna's relaxation formula for the quasiconvex envelope (see Theorem 3.1.3).

We now prove the last statement. Assume that $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is a continuous integrand satisfying (6.1.1) and (6.1.2). Put $G = F - \frac{c_2}{2}|\cdot|^p$. By Dacorogna's relaxation formula, we have

$$G^{qc}(0) = \inf \left\{ \fint_\Omega G\big(D\varphi(x)\big) \,\mathrm{d}x \colon \varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^N) \right\}$$

and we assert that we may take $\varphi_j \to 0$ in $L^p(\Omega;\mathbb{R}^N)$ such that

$$\fint_\Omega G\big(D\varphi_j(x)\big) \,\mathrm{d}x \to G^{qc}(0)$$

which was first observed by Yan-Zhou (see [129], Lemma 3.1). To prove our assertion, we may choose $\Omega = C = (-\frac{1}{2}, \frac{1}{2})^n$ since Dacorogna's relaxation formula is independent of the choice of the set Ω. Let $\varepsilon > 0$ and take $\varphi = \varphi_\varepsilon \in W_0^{1,\infty}$ such that

$$G^{qc}(0) + \varepsilon > \fint_C G\big(D\varphi(x)\big)\,\mathrm{d}x.$$

We prove the assertion for open sets C_j which are constructed by dilation and translation of C with

$$|\partial C_j| = 0.$$

Let $m = \|\varphi\|_{L^\infty}$ and for $j \in J \subseteq \mathbb{N}$, choose x_j such that

$$C_j = x_j + \frac{1}{\lceil \frac{m}{\varepsilon} \rceil} C$$

are disjoint and so

$$C = \bigcup_{j \in J} C_j \cup N$$

where N denotes the null set. Define

$$\psi(x) = \begin{cases} \frac{1}{\lceil \frac{m}{\varepsilon} \rceil} \varphi\big(\lceil \frac{m}{\varepsilon} \rceil (x - x_j)\big) & \text{if } x \in C_j \text{ for some } j \\ 0 & \text{elsewhere} \end{cases}.$$

Hence $\psi \in W_0^{1,\infty}(C;\mathbb{R}^N)$. We now have

$$\begin{aligned} \int_C G\big(D\psi(x)\big)\,\mathrm{d}x &= \sum_{j\in J} \int_{C_j} G\left(D\varphi\left(\left\lceil \frac{m}{\varepsilon} \right\rceil (x - x_j)\right)\right)\mathrm{d}x \\ &= \frac{1}{\lceil \frac{m}{\varepsilon} \rceil^n} \sum_{j \in J} \int_{\lceil \frac{m}{\varepsilon} \rceil C_j} G\big(D\varphi(y)\big)\,\mathrm{d}y \\ &= \sum_{j\in J} \int_{C_j} G\big(D\varphi(y)\big)\,\mathrm{d}y \\ &= \int_C G\big(D\varphi(y)\big)\,\mathrm{d}y. \end{aligned}$$

It is clear that

$$\|\psi\|_{L^\infty(C)} = \|\psi\|_{L^\infty(C_j)} = \frac{m}{\lceil \frac{m}{\varepsilon} \rceil} < \varepsilon$$

and by taking $\varepsilon = \frac{1}{j}$, we put $\varphi_j = \psi$. Finally, we have $\varphi_j \to 0$ in $L^p(C;\mathbb{R}^N)$ as $j \to \infty$ and by Dacorogna's relaxation formula, it follows that

$$\fint_\Omega G\big(D\varphi_j(x)\big)\,\mathrm{d}x \to G^{qc}(0)$$

and hence the assertion.

Put $l := (F - c_2|\cdot|^p)^{qc}(0) > -\infty$, we have

$$\begin{aligned} l &\le \fint_\Omega \big(F(D\varphi_j) - c_2|D\varphi_j|^p\big)\,\mathrm{d}x \\ &= \fint_\Omega \Big(F(D\varphi_j) - \frac{c_2}{2}|D\varphi_j|^p - \frac{c_2}{2}|D\varphi_j|^p\Big)\,\mathrm{d}x \end{aligned}$$

and thus

$$\fint_\Omega |D\varphi_j|^p\,\mathrm{d}x \le \frac{2}{c_2}\left[\fint_\Omega \Big(F(D\varphi_j) - \frac{c_2}{2}|D\varphi_j|^p\Big)\,\mathrm{d}x - l\right]$$

showing that $\{D\varphi_j\}$ is bounded in $L^p(\Omega;\mathbb{R}^{N\times n})$. It follows that $\varphi_j \rightharpoonup 0$ in $W_0^{1,p}(\Omega;\mathbb{R}^N)$ and that up to a subsequence (for convenience not relabeled) $\{D\varphi_j\}$ generates a Young measure $(\mu_x)_{x\in\Omega}$. By definition, $(\mu_x)_{x\in\Omega}$ is then a gradient p-Young measure with underlying deformation 0, so by localization principle (see Theorem 3.2.18) follows that for a.e. $x \in \Omega$ each μ_x is a homogeneous gradient p-Young measure with underlying deformation 0. Invoking Theorem 5.1.2 gives us the existence of $\{\psi_j\} \subset W_0^{1,p}(\Omega;\mathbb{R}^N)$ such that each ψ_j is a Q-quasiminimizer of $\int_\Omega(|Du|^p + 1)\,\mathrm{d}x$ for some $Q \ge 1$ and $\{\psi_j\}$ is a minimizing sequence satisfying

$$\fint_\Omega G(D\psi_j)\,\mathrm{d}x \to G^{qc}(0)$$

and

$$\int_\Omega |D\psi_j - D\varphi_j|\,\mathrm{d}x \to 0.$$

Therefore, we have

$$D\psi_j \xrightarrow{Y} (\mu_x)_{x\in\Omega}.$$

The higher integrability of $D\psi_j$ is guaranteed by Theorem 5.3.1. It follows from De la Vallé-Poussin criterion for equiintegrability and the fundamental theorem for Young measures that

$$G^{qc}(0) = \fint_\Omega \int_{\mathbb{R}^{N\times n}} G\,d\mu_x\,\mathrm{d}x.$$

The averaging principle (see Theorem 3.2.18) tells us that the probability measure μ on $\mathbb{R}^{N\times n}$ defined by

$$\int_{\mathbb{R}^{N\times n}} \Phi\,d\mu := \fint_\Omega \int_{\mathbb{R}^{N\times n}} \Phi\,d\mu_x\,\mathrm{d}x, \qquad \Phi \in C_0(\mathbb{R}^{N\times n})$$

is a homogeneous gradient p-Young measure with centre of mass $\bar{\mu} = 0$. Now G^{qc} is quasiconvex and of p-growth and hence it follows from Corollary 3.2.19 that $\int_{\mathbb{R}^{N\times n}} G^{qc}\,d\mu \ge G^{qc}(0)$ (see also [83], Proposition 1.9).

We assert that $G^{qc} = G$ on the support of μ, $\operatorname{supp}\mu$. Clearly, $G \geq G^{qc}$ so $G - G^{qc} \geq 0$, and hence from

$$\begin{aligned} 0 \leq \int_{\mathbb{R}^{N\times n}} (G - G^{qc})\, d\mu &= \int_{\mathbb{R}^{N\times n}} G\, d\mu - \int_{\mathbb{R}^{N\times n}} G^{qc}\, d\mu \\ &= G^{qc}(0) - \int_{\mathbb{R}^{N\times n}} G^{qc}\, d\mu \leq 0 \end{aligned}$$

we get $G - G^{qc} = 0$ μ-a.e.. The continuity of $G - G^{qc}$ leads to $G = G^{qc}$ on $\operatorname{supp}\mu$ as asserted. Consequently, we have shown that from (6.1.1) and (6.1.2) follows that $F - \frac{c_2}{2}|\cdot|^p$ is quasiconvex at some $\xi \in \mathbb{R}^{N\times n}$ with $c_5 = \frac{c_2}{2}$. The converse is clear. □

Note how the conclusions of Theorem 5.1.2 and Theorem 5.3.1 have been used in the proof of the second part. By showing that there exists a minimizing sequence $\{\varphi_j\}$ generating a gradient p-Young measure, we deduce from Theorem 5.1.2 that there is another minimizing sequence $\{\psi_j\}$ such that it is Q-quasiminimizing $\int_\Omega (|Du|^p + 1)\, \mathrm{d}x$ and its gradient generates the same Young measure. The higher integrabilitity of this Q-quasiminimizer allows the gradient to pass to equiintegrability and hence $\{\psi_j\}$ generates the same gradient p-Young measure. We then use the results on Young measure as mentioned in Section 3.2 of Chapter 3 to conclude the proof.

6.2 Strong quasiconvex integrands

We have seen how strong quasiconvexity is closely related to mean coercivity in the previous section. The question that we would like to answer here is that if it possible for an integrand that is strongly 2-quasiconvex everywhere and strongly p-quasiconvex somewhere for $p \in (2, \infty)$ to be strongly p-quasiconvex everywhere for $p \in (2, \infty)$. We show that this stability property holds for $F \in C^2$. For this, we need the following lemma:

Lemma 6.2.1. *Assume that $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is twice differentiable and $F(0) = 0$. Let $m > 0$. Then for every $\xi_0, \xi \in \mathbb{R}^{N\times n}$ with $|\xi_0|, |\xi| \leq m$, we have*

$$F(\xi) - F(\xi + \xi_0) + F(\xi_0) \geq \big(F'(0) - F'(\xi_0)\big)[\xi] - c_m|\xi|^2 \tag{6.2.1}$$

where $c_m > 0$ is a constant depending on m.

Proof. The fact that $F \in C^2$ allows us to use fundamental theorem of calculus twice to obtain

$$\begin{aligned} F(\xi) - F(\xi+\xi_0) + F(\xi_0) &= \int_0^1 \big(F'(t\xi)[\xi] - F'(t\xi+\xi_0)[\xi]\big)\,\mathrm{d}t \\ &= F'(0)[\xi] + \int_0^1 \big(F'(t\xi) - F'(0)\big)[\xi]\,\mathrm{d}t \\ &\quad - \left\{F'(\xi_0)[\xi] + \int_0^1 \big(F'(t\xi+\xi_0) - F'(\xi_0)\big)[\xi]\,\mathrm{d}t\right\} \\ &= \big(F'(0) - F'(\xi_0)\big)[\xi] + \int_0^1\int_0^1 F''(st\xi)[\xi, t\xi]\,\mathrm{d}s\,\mathrm{d}t \\ &\quad - \int_0^1\int_0^1 F''(st\xi+\xi_0)[\xi, t\xi]\,\mathrm{d}s\,\mathrm{d}t. \end{aligned}$$

The compactness of the image is guaranteed by the fact that F'' is continuous, $|st\xi| \leq m$ and $|st\xi + \xi_0| \leq 2m$. Therefore, we have

$$\left|\int_0^1\int_0^1 F''(st\xi)[\xi, t\xi]\,\mathrm{d}s\,\mathrm{d}t\right| \leq c_m|\xi|^2$$

and

$$\left|\int_0^1\int_0^1 F''(st\xi+\xi_0)[\xi, t\xi]\,\mathrm{d}s\,\mathrm{d}t\right| \leq c_m|\xi|^2.$$

Substituting the estimates into the main equation gives us (6.2.1). □

Theorem 6.2.2. *Let $B \subset \mathbb{R}^n$ be a bounded open ball and $m > 0$. Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a C^2 integrand satisfying for some $c_1 < \infty$ and $p \in (2,\infty)$ the growth condition*

$$|F(\xi)| \leq c_1(|\xi|^p + 1)$$

for all $\xi \in \mathbb{R}^{N\times n}$. Suppose that there exists $c_2 > 0$ such that

$$F - c_2|\cdot|^2$$

is quasiconvex and there exists $c_3 > 0$ such that

$$F - c_3|\cdot|^p$$

is quasiconvex at ξ_0 where $|\xi_0| \leq m$. Then for each m, there exists a constant $c = c(m) > 0$ such that

$$F - c|\cdot|^p$$

is quasiconvex for all ξ satisfying $|\xi| \leq m$.

Proof. Without loss of generality, we may assume that $\xi = 0$ and $F(0) = 0$. We will prove the theorem by contradiction. Assume that $F - c|\cdot|^p$ is not quasiconvex for all $c < \infty$. Then for all $j \in \mathbb{N}$, there exists piecewise affine $\varphi_j \in W_0^{1,\infty}(B;\mathbb{R}^N)$ such that

$$\fint_B F(D\varphi_j)\,\mathrm{d}x < \frac{1}{j}\fint_B |D\varphi_j|^p \mathrm{d}x. \tag{6.2.2}$$

Note that $F - c_2|\cdot|^2$ is quasiconvex and hence

$$c_2\fint_B |D\varphi_j|^2\,\mathrm{d}x \le \fint_B F(D\varphi_j)\,\mathrm{d}x < \frac{1}{j}\fint_B |D\varphi_j|^p \mathrm{d}x. \tag{6.2.3}$$

We may write

$$\begin{aligned}
\int_B F(D\varphi_j)\,dx &= \int_B \big(F(D\varphi_j + \xi_0) - F(\xi_0)\big)\,\mathrm{d}x \\
&\quad + \int_B \big(F(D\varphi_j) - F(D\varphi_j + \xi_0) + F(\xi_0)\big)\,\mathrm{d}x \\
&\ge c_3\int_B |D\varphi_j|^p\,\mathrm{d}x + \int_{|D\varphi_j|\le m} \big(F(D\varphi_j) - F(D\varphi_j + \xi_0) + F(\xi_0)\big)\,\mathrm{d}x \\
&\quad + \int_{|D\varphi_j|>m} \big(F(D\varphi_j) - F(D\varphi_j + \xi_0) + F(\xi_0)\big)\,\mathrm{d}x
\end{aligned} \tag{6.2.4}$$

where we have made use of the fact that $F - c_3|\cdot|^p$ is quasiconvex at ξ_0. Estimating the second term of (6.2.4) by Lemma 6.2.1 gives us

$$\begin{aligned}
&\int_{|D\varphi_j|\le m} \big(F(D\varphi_j) - F(D\varphi_j + \xi_0) + F(\xi_0)\big)\,\mathrm{d}x \\
\ge\ &\int_{|D\varphi_j|\le m} \big(F'(0) - F'(\xi_0)\big)[D\varphi_j]\,\mathrm{d}x - c_m\int_{|D\varphi_j|\le m} |D\varphi_j|^2 \mathrm{d}x,
\end{aligned} \tag{6.2.5}$$

where $c_m > 0$ is a constant depending on m. Note that $\int_B \big(F'(0) - F'(\xi_0)\big)[D\varphi_j]\,\mathrm{d}x = 0$ and we may write

$$\begin{aligned}
0 &= \int_B \big(F'(0) - F'(\xi_0)\big)[D\varphi_j]\,\mathrm{d}x \\
&= \int_{|D\varphi_j|\le m} \big(F'(0) - F'(\xi_0)\big)[D\varphi_j]\,\mathrm{d}x + \int_{|D\varphi_j|>m} \big(F'(0) - F'(\xi_0)\big)[D\varphi_j]\,\mathrm{d}x.
\end{aligned}$$

The estimate for the second term on the right hand side goes as follows:

$$\begin{aligned}
\int_{|D\varphi_j|>m} \big(F'(0) - F'(\xi_0)\big)[D\varphi_j]\,\mathrm{d}x &\le c_m\int_{|D\varphi_j|>m} |D\varphi_j|\,\mathrm{d}x \\
&\le \frac{c_m}{m}\int_{|D\varphi_j|>m} |D\varphi_j|^2\,\mathrm{d}x \\
&\le \tilde{c}_m\int_B |D\varphi_j|^2\,\mathrm{d}x.
\end{aligned}$$

This gives us

$$\int_{|D\varphi_j|\le m}\big(F'(0)-F'(\xi_0)\big)[D\varphi_j]\,\mathrm{d}x \ge -\tilde{c}_m\int_B |D\varphi_j|^2\,\mathrm{d}x.$$

Substituting the inequality above together with (6.2.3) into (6.2.5), we have

$$\int_{|D\varphi_j|\le m}\big(F(D\varphi_j)-F(D\varphi_j+\xi_0)+F(\xi_0)\big)\,\mathrm{d}x \ge -\frac{c_m}{c_2 j}\int_B |D\varphi_j|^p\mathrm{d}x,$$

where c_m remains as a constant that depends on m. Finally, we will now estimate the last term of (6.2.4), namely

$$\int_{|D\varphi_j|>m}\big(F(D\varphi_j)-F(D\varphi_j+\xi_0)+F(\xi_0)\big)\,\mathrm{d}x.$$

Note that the quasiconvexity of $F-c_2|\cdot|^2$ allows us to use Proposition 2.2.18 to obtain

$$\begin{aligned}
&\int_{|D\varphi_j|>m}\big(F(D\varphi_j)-F(D\varphi_j+\xi_0)+F(\xi_0)\big)\,\mathrm{d}x\\
\ge\quad & -c_4\int_{|D\varphi_j|>m}\big(|D\varphi_j|^{p-1}+|\xi_0|^{p-1}+1\big)\,|\xi_0|\,\mathrm{d}x+\int_{|D\varphi_j|>m}F(\xi_0)\,\mathrm{d}x
\end{aligned}$$

for some constant $c_4>0$. Invoking Young's inequality and Chebychev's inequality give us

$$\int_{|D\varphi_j|>m}\big(F(D\varphi_j)-F(D\varphi_j+\xi_0)+F(\xi_0)\big)\,\mathrm{d}x \ge -\frac{c_5}{m^p}\int_B|D\varphi_j|^p\,\mathrm{d}x - c_6\int_B|D\varphi_j|^{p-1}\,\mathrm{d}x.$$

We substitute all the estimates into (6.2.4) to obtain

$$\int_B F(D\varphi_j)\,dx \ge \left(c_3-\frac{c_m}{c_2 j}-\frac{c_5}{m^p}\right)\int_B|D\varphi_j|^p\,\mathrm{d}x - c_6\int_B|D\varphi_j|^{p-1}\,\mathrm{d}x. \tag{6.2.6}$$

To find the estimate for the last term of (6.2.6), we need to consider two cases, namely $p>3$ and $p\in(2,3)$. We first let $p>3$ and we use interpolation inequality for L^p-norms to find

$$\begin{aligned}
\int_B|D\varphi_j|^{p-1}\,\mathrm{d}x &\le \left(\int_B|D\varphi_j|^2\,\mathrm{d}x\right)^{\frac{1}{p-2}}\left(\int_B|D\varphi_j|^p\,\mathrm{d}x\right)^{\frac{p-3}{p-2}}\\
&\le \left(\frac{1}{c_2 j}\int_B|D\varphi_j|^p\,\mathrm{d}x\right)^{\frac{1}{p-2}}\left(\int_B|D\varphi_j|^p\,\mathrm{d}x\right)^{\frac{p-3}{p-2}}\\
&= \left(\frac{1}{c_2 j}\right)^{\frac{1}{p-2}}\int_B|D\varphi_j|^p\,\mathrm{d}x.
\end{aligned}$$

Substituting this into (6.2.6) leads to

$$\int_B F(D\varphi_j)\,dx \ge c\int_B|D\varphi_j|^p\,dx,\quad c<\infty$$

which is a contradiction. Hence $F-c|\cdot|^p$ is quasiconvex for all $p>3$.

We will now prove for $p\in(2,3)$. Since $1<p-1<2$, we may apply Lemma 6.2.3 (see below) to obtain

$$\left(\int_B|D\varphi_j|^{p-1}\,\mathrm{d}x\right)^{\frac{1}{p-1}} \le |B|^{\frac{1}{p-1}-\frac{1}{2}}\left(\int_B|D\varphi_j|^2\,\mathrm{d}x\right)^{\frac{1}{2}}.$$

It then follows from (6.2.3) that

$$\int_B |D\varphi_j|^{p-1}\,\mathrm{d}x \le |B|^{\frac{3-p}{2}} \left(\frac{1}{c_2 j}\right)^{\frac{p-1}{2}} \left(\int_B |D\varphi_j|^p\,\mathrm{d}x\right)^{\frac{p-1}{2}}$$

and since $\frac{p-1}{2} < 1$, we have

$$\int_B |D\varphi_j|^{p-1}\,\mathrm{d}x \le c_7 \int_B |D\varphi_j|^p\,\mathrm{d}x.$$

We once again obtain a contradiction and hence the theorem. $\square$

In the proof above, we have used the following elementary result:

Lemma 6.2.3 (**Consequence of Hölder's inequality**). *Let B be a ball with Lebesgue measure $|B|$. Suppose that $0 < p < q < \infty$. If $f \in L^q(B)$, then $f \in L^p(B)$ and*

$$\|f\|_p \le |B|^{\frac{1}{p}-\frac{1}{q}} \|f\|_q.$$

If $f \in L^\infty(B)$, then $f \in L^p(B)$ and

$$\|f\|_p \le |B|^{\frac{1}{p}} \|f\|_q.$$

It is known that, in quadratic forms, quasiconvexity is equivalent to rank one convexity. If a function $F \in C^2(\mathbb{R}^{N\times n})$, then rank one convexity is equivalent to the Legendre-Hadamard condition:

$$D^2F(\xi)(a\otimes b, a\otimes b)\,\mathrm{d}x \ge 0 \tag{6.2.7}$$

for every $\xi \in \mathbb{R}^{N\times n}$, $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^n$. We obtain a strict Legendre-Hadamard condition if for some $\xi \in \mathbb{R}^{N\times n}$, (6.2.7) holds for all $a, b \ne 0$. In other words, there exists $c > 0$ such that

$$D^2F(\xi)(a\otimes b, a\otimes b)\,\mathrm{d}x \ge c|a|^2|b|^2 \tag{6.2.8}$$

for all $a \in \mathbb{R}^N$, $b \in \mathbb{R}^n$. It then follows from the applications of Fourier transform and Plancherel's theorem that (6.2.8) is equivalent to

$$\int_B D^2F(\xi)\big(D\varphi(x), D\varphi(x)\big)\,\mathrm{d}x \ge c\int_B |D\varphi(x)|^2\mathrm{d}x \tag{6.2.9}$$

for all $\varphi \in C_0^\infty(\Omega; \mathbb{R}^N)$ with $\operatorname{supp}\varphi \subset B$ where B is the open unit ball. The following result, which is related to Theorem 6.2.2, shows that an integrand satisfying the strict Legendre-Hadamard condition is equal to a strongly 2-quasiconvex integrand in a neighbourhood of each point.

Lemma 6.2.4 (Kristensen [84], Lemma 2)**.** *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a C^2 function and suppose there exist constants $c_1, s > 0$ such that $F - c_1|\cdot|^2$ is rank one convex for $|\xi_0| \leq s$. Then, for all $\xi_0 \in \mathbb{R}^{N\times n}$, there exists a function $G\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ such that $G - c_2|\cdot|^2$ is quasiconvex with at most quadratic growth where $0 < c_2 < c_1$ and*

$$G = F \quad \textit{whenever} \quad |\xi_0| \leq r \textit{ for some } r \in (0, s).$$

Proof. Note that since $F - c_1|\cdot|^2$ is rank one convex, then F is strongly rank one convex and $F - c_2|\cdot|^2$ is also rank one convex for every $c_2 \in (0, c_1)$. If we write

$$F - c_1|\cdot|^2 = (F - c_2|\cdot|^2) - (c_1 - c_2)|\cdot|^2,$$

we can see that $F - c_2|\cdot|^2$ is also strongly rank one convex. Thus we have the equivalent form of (6.2.9) for $F - c_2|\cdot|^2$. The rest of the proof follows as in Lemma 2 of [84]. □

Chapter 7

Quasiminimizers and Relaxation

Recall that the advantage of the notion of Q-quasiminimizer is its flexibility and that it incorporates many related notions. For example if $\bar{u}$ is a minimizer for a functional

$$\mathcal{F}(u,\Omega) = \int_\Omega F\big(x, u(x), Du(x)\big)\mathrm{d}x$$

that satisfies the coercivity and growth conditions (4.1.3), then it is also a Q-quasiminimizer of

$$\int_\Omega (|Du(x)|^p + 1)\ \mathrm{d}x.$$

One could wonder if the converse is true. In this chapter, we consider the inhomogeneous p-Dirichlet integral

$$\int_\Omega \big(|Du(x)|^p + a(x)\big)\,\mathrm{d}x, \tag{7.0.1}$$

where $a \in L^1(\Omega)^+$. Instead of the usual definition of Q-quasiminimizer, we consider strong Q-quasiminimizer.

Definition 7.0.5. *Let $p \in (1,\infty)$, $Q \in [1,\infty)$ and $\Omega \subset \mathbb{R}^n$ be open. We say that $u \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ is a strong Q-quasiminimizer of the inhomogeneous p-Dirichlet integral if the inequality*

$$\int_{\{D\varphi\neq 0\}} \big(|Du(x)|^p + a(x)\big)\,\mathrm{d}x \le Q \int_{\{D\varphi\neq 0\}} \big(|Du(x)+D\varphi(x)|^p + a(x)\big)\,\mathrm{d}x$$

holds for all $\varphi \in W^{1,p}_0(\Omega;\mathbb{R}^N)$ where $a \in L^1(\Omega)^+$.

Note that the definition of strong Q-quasiminimality is taken over the set when $\{D\varphi \neq 0\}$. Contrary to the usual definition of Q-quasiminimality, this stronger version satisfies the local property exhibited by p-harmonic functions (see Chapter 4).

It turns out that a strong Q-quasiminimizer of the functional in (7.0.1) is indeed an absolute minimizer of a quasiconvex x-dependent integral, namely $\int_\Omega F^{qc}\big(x, Du(x)\big)\,\mathrm{d}x$, where the integrand is the quasiconvex envelope of an explicit integrand (see (7.0.2) below). The result goes as follows:

Theorem 7.0.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $Q > 1$, $p \in (1, \infty)$ and $a \in L^1(\Omega)^+$. Then $\bar{u} \in W^{1,p}(\Omega; \mathbb{R}^N)$ is a strong Q-quasiminimizer for the integral $\int_\Omega \big(|Du(x)|^p + a(x)\big)\,\mathrm{d}x$ if and only if $\bar{u}$ is an absolute minimizer for the integral $\int_\Omega F^{qc}\big(x, Du(x)\big)\,\mathrm{d}x$, where for each $x \in \Omega$, $F^{qc}(x, \cdot\,)$ is the quasiconvex envelope of*

$$F(x, \xi) = \begin{cases} Q\big(|\xi|^p + a(x)\big) & \text{if } \xi \neq D\bar{u}(x) \\ |D\bar{u}(x)|^p + a(x) & \text{if } \xi = D\bar{u}(x) \end{cases}. \tag{7.0.2}$$

We have seen how Q-quasiminimizers appear naturally in various problems in Chapter 4 and also we showed the existence of Q-quasiminimizers for an inhomogeneous p-Dirichlet integral within minimizing sequences of a variational problem in Theorem 5.1.2. This result yields the existence of Q-quasiminimizers and can help in proving the existence of a minimizer for variational problems, since by Theorem 5.3.1 Q-quasiminimizers possess higher integrability. An example of the utility of this approach was the paper of Marcellini-Sbordone [91].

In fact, when $p = 2$, we can even write F^{qc} explicitly in terms of the Kohn-Strang relaxation formula and in addition, we consider the formulae for polyconvex and convex envelopes in lower dimensions. Finally, when $N = 1$ such that u is real-valued, we show that Theorem 7.0.6 holds for strong Q-superquasiminimizers and superminimizers, respectively.

This chapter is organized in the following way: In Section 7.1, we consider Theorem 7.0.6 in the general case and setting $p = 2$, we note that it is possible to write the relaxation formula explicitly and we extend Theorem 7.0.6 to polyconvex and convex x-dependent integrals in lower dimensions. The case when the function u is real-valued is considered in Section 7.2.

7.1 Quasiminimizers and absolute minimizers

We begin this section by proving the main result.

Proof of Theorem 7.0.6. Clearly, $F(x, \xi)$ is a normal integrand satisfying

$$|\xi|^p + a(x) \leq F(x, \xi) \leq Q\big(|\xi|^p + a(x)\big). \tag{7.1.1}$$

Since a similar relaxation for Carathéodory integrands works for normal integrands, we let

$$F^{qc}\big(x, D\bar{u}(x)\big) = \inf_{\varphi \in C_0^\infty(\Omega;\mathbb{R}^N)} \fint_\Omega F\big(x, D\bar{u}(x) + D\varphi(y)\big)\,\mathrm{d}y$$

and

$$\mathcal{F}^{qc}(u,\Omega) = \int_\Omega F^{qc}\big(x, Du(x)\big)\,\mathrm{d}x.$$

Then by Theorem 3.2.23, we find that F^{qc} is also a Carathéodory integrand satisfying

$$|\xi|^p + a(x) \le F^{qc}(x,\xi) \le Q\big(|\xi|^p + a(x)\big) \tag{7.1.2}$$

and the function $\xi \mapsto F^{qc}(x,\xi)$ is quasiconvex for a.e. $x \in \Omega$. Therefore we have

$$F^{qc}\big(x, D\bar{u}(x)\big) = |D\bar{u}(x)|^p + a(x) \text{ a.e..}$$

It follows from (7.1.2) that any absolute minimizer of the integral $\int_\Omega F^{qc}\big(x, Du(x)\big)\,\mathrm{d}x$ is a Q-quasiminimizer of $\int_\Omega \big(|Du(x)|^p + a(x)\big)\,\mathrm{d}x$.

We now prove the converse implication. Assume that $\bar{u}$ is a strong Q-quasiminimizer of the inhomogeneous p-Dirichlet integral, meaning that

$$\int_{\{D\varphi\neq 0\}} \big(|D\bar{u}(x)|^p + a(x)\big)\,\mathrm{d}x \le Q\int_{\{D\varphi\neq 0\}} \big(|D\bar{u}(x) + D\varphi(x)|^p + a(x)\big)\,\mathrm{d}x$$

holds for all $\varphi \in W_0^{1,p}(\omega;\mathbb{R}^N)$ and $\omega \in \mathcal{O}(\Omega)$, where $\mathcal{O}(\Omega)$ is the family of all open subsets in Ω. We now have

$$\begin{aligned}
&\int_\omega F\big(x, D\bar{u}(x) + D\varphi(x)\big)\,\mathrm{d}x \\
= \;& Q\int_{\{D\varphi\neq 0\}\cap\omega} \big(|D\bar{u}(x) + D\varphi(x)|^p + a(x)\big)\,\mathrm{d}x + \int_{\{D\varphi=0\}\cap\omega} \big(|D\bar{u}(x)|^p + a(x)\big)\,\mathrm{d}x \\
\ge \;& \int_{\{D\varphi\neq 0\}\cap\omega} \big(|D\bar{u}(x)|^p + a(x)\big)\,\mathrm{d}x + \int_{\{D\varphi=0\}\cap\omega} \big(|D\bar{u}(x)|^p + a(x)\big)\,\mathrm{d}x \\
= \;& \int_\omega F\big(x, D\bar{u}(x)\big)\,\mathrm{d}x,
\end{aligned}$$

and hence

$$\int_\omega F^{qc}\big(x, D\bar{u}(x) + D\varphi(x)\big)\,\mathrm{d}x \ge \int_\omega F^{qc}\big(x, D\bar{u}(x)\big)\,\mathrm{d}x.$$

Since quasiconvexity is independent of the choice of the set, we find that $\bar{u}$ is indeed an absolute minimizer of $\int_\Omega F^{qc}\big(x, Du(x)\big)\,\mathrm{d}x$. $\square$

Restricting our function $F(x,\xi)$ to the case $p=2$, we can write out the explicit formula for the convex, polyconvex, rank one convex and quasiconvex envelopes of $F(x,\xi)$ in terms of Kohn-Strang relaxation formula (see Chapter 3). In order to make use of Kohn-Strang relaxation formula, the function has to be in the form of Kohn-Strang function (3.1.14). This we will show in the following theorem. Note that for simplicity, we let $\xi_0 = D\bar{u}(x)$.

Theorem 7.1.1. *Let $F\colon \Omega\times\mathbb{R}^{N\times n}\to\mathbb{R}_+$ be a function given by*

$$F(x,\xi) = \begin{cases} Q\big(|\xi|^2 + a(x)\big) & \text{if } \xi \neq \xi_0 \\ |\xi_0|^2 + a(x) & \text{if } \xi = \xi_0 \end{cases} \tag{7.1.3}$$

where $Q>1$, $a\in L^1(\Omega)^+$ and $\xi,\xi_0\in\mathbb{R}^{N\times n}$. Then,

$$F^c(x,\xi) = b(x,\xi) + QG^c(x,\xi) \tag{7.1.4}$$

and

$$F^{rc}(x,\xi) = F^{qc}(x,\xi) = b(x,\xi) + QG^{qc}(x,\xi) \tag{7.1.5}$$

where

$$b(x,\xi) = 2Q\xi_0\cdot\xi + a(x) - (2Q-1)|\xi_0|^2$$

is an affine function of ξ and the function $G\colon\Omega\times\mathbb{R}^{N\times n}\to\mathbb{R}_+$ is given by

$$G(x,\xi) = \begin{cases} |\xi-\xi_0|^2 + \frac{Q-1}{Q}\big(|\xi_0|^2 + a(x)\big) & \text{if } \xi\neq\xi_0 \\ 0 & \text{if } \xi = \xi_0 \end{cases},$$

and its convexification G^c can be expressed as in (3.1.13) and quasiconvexification $G^{qc}(x,\xi)$ as in Theorem 3.1.11, with $\xi-\xi_0$ in place of ξ and $\lambda = \frac{Q-1}{Q}\big(|\xi_0|^2+a(x)\big)$. Moreover, when $\min\{N,n\}=2$, we have

$$F^{pc}(x,\xi) = F^{qc}(x,\xi) = F^{rc}(x,\xi). \tag{7.1.6}$$

Proof. First, we write

$$\begin{aligned} Q\big(|\xi|^2 + a(x)\big) &= Q\big(|\xi-\xi_0|^2 - |\xi_0|^2 + 2\xi\cdot\xi_0 + a(x)\big) \\ &= Q\big(|\xi-\xi_0|^2 + a(x)\big) + Q(2\xi_0\cdot\xi - |\xi_0|^2) \end{aligned}$$

and hence $F(\xi)$ can now be rewritten as

$$\begin{aligned} F(x,\xi) &= Q(2\xi_0\cdot\xi-|\xi_0|^2)+\begin{cases} Q\big(|\xi-\xi_0|^2+a(x)\big) & \text{if } \xi\neq\xi_0 \\ |\xi_0|^2-Q|\xi_0|^2+a(x) & \text{if } \xi=\xi_0 \end{cases} \\ &= 2Q\xi_0\cdot\xi+a(x)-(2Q-1)|\xi_0|^2 \\ &\quad +Q\begin{cases} |\xi-\xi_0|^2+\frac{Q-1}{Q}\big(|\xi_0|^2+a(x)\big) & \text{if } \xi\neq\xi_0 \\ 0 & \text{if } \xi=\xi_0 \end{cases} \\ &= b(x,\xi)+QG(x,\xi) \end{aligned}$$

where we have shifted the origin to $|\xi-\xi_0|$,

$$b(x,\xi)=2Q\xi_0\cdot\xi+a(x)-(2Q-1)|\xi_0|^2$$

and

$$G(x,\xi)=\begin{cases} |\xi-\xi_0|^2+\frac{Q-1}{Q}\big(|\xi_0|^2+a(x)\big) & \text{if } \xi\neq\xi_0 \\ 0 & \text{if } \xi=\xi_0 \end{cases}.$$

Note that $b(x,\xi)$ is an affine function of ξ and it follows that it is convex. Meanwhile $G(x,\xi)$ is now in the form of generalized Kohn-Strang formula (3.1.14). Applying the convexification and quasiconvexification on $F(x,\xi)=b(x,\xi)+QG(x,\xi)$ gives us

$$F^c(x,\xi)=b(x,\xi)+QG^c(x,\xi)$$

and

$$F^{qc}(x,\xi)=b(x,\xi)+QG^{qc}(x,\xi),$$

respectively. The explicit formula of convex envelope G^c follows immediately from (3.1.13), whereas the explicit form of quasiconvex envelope G^{qc} follows from Theorem 3.1.6 (when $N=n=2$) and Theorem 3.1.11 (when $\min\{N,n\}\geq 2$).

Note that in the proof of Theorem 3.1.11, we showed that the quasiconvex envelope is equal to the rank one convex envelope and therefore we have that $G^{qc}(x,\xi)=G^{rc}(x,\xi)$. Consequently, we also have $F^{qc}(x,\xi)=F^{rc}(x,\xi)$ which is precisely (7.1.5). Finally for the case of $\min\{N,n\}=2$, Theorem 3.1.11 yields the equality (7.1.6). □

By the above theorem, we conclude with the following result:

Corollary 7.1.2. *Let $\Omega\subset\mathbb{R}^n$ be a bounded open set, $Q>1$, $a\in L^1(\Omega)^+$ and $\min\{N,n\}=2$. Then $\bar{u}\in W^{1,2}(\Omega;\mathbb{R}^N)$ is a strong Q-quasiminimizer for the integral $\int_\Omega\big(|Du(x)|^2+a(x)\big)\,\mathrm{d}x$ if and only if $\bar{u}$ is an absolute minimizer for the integral $\int_\Omega F^{pc}\big(x,Du(x)\big)\,\mathrm{d}x$, where for each $x\in\Omega$, $F^{pc}(x,\cdot\,)$ is the polyconvex envelope of (7.1.3). If $\min\{N,n\}=1$, then the result holds for the convex envelope $F^c(x,\cdot)$.*

7.2 Superquasiminimizers and superminimizers

In this section, we wish to show that Theorem 7.0.6 remains valid for strong superquasiminimizers of the integral $\int_\Omega \big(|Du(x)|^p + a(x)\big)\,\mathrm{d}x$ and superminimizers of the given x-dependent convex integral.

In potential theory, Q-superquasiminimizer is related to superquasiharmonic functions in a similar way to that of superminimizer and superharmonic functions. Problems in this direction have been previously considered by Kinnunen-Martio [73] where they showed the interesting features of Q-quasiminimizers in potential theory and more substantially the regularity properties that follow. Indeed, Q-superquasiminimizers do enjoy some of the properties of superminimizers. We will not mention more about this here but instead interested readers may refer to the paper of Kinnunen-Martio [73].

Before we proceed with our result, we need to deal with the issue of non-triviality of one-sided relaxation and for this, we show that the integrand $F(x,\cdot)$ is convex at the gradient of the superminimizer for a.e. $x \in \Omega$. We can then conclude that the relaxation method also applies for real-valued superminimizers and subminimizers.

Theorem 7.2.1. *Let $F\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a Carathéodory integrand satisfying the coercivity and growth conditions*

$$\frac{1}{c}|\xi|^p - a(x) \le F(x,\xi) \le c|\xi|^p + a(x)$$

for all $\xi \in \mathbb{R}^n$ and a.e. $x \in \Omega$, where constant $c > 0$ and $a \in L^1(\Omega)^+$. Assume that $\bar{u} \in W^{1,p}(\Omega)$ is a superminimizer of $\mathcal{F}(u,\Omega) = \int_\Omega F\big(x, Du(x)\big)\,\mathrm{d}x$:

$$\mathcal{F}(\bar{u},\Omega) \le \mathcal{F}(\bar{u}+\varphi,\Omega)$$

for all non-negative $\varphi \in W_0^{1,p}(\Omega)$. Then the integrand $F(x,\cdot)$ is convex at $D\bar{u}(x)$ for a.e. $x \in \Omega$.

Proof. Let $\psi \in W_0^{1,\infty}\big((-1,1)^n\big)$ with $\|\psi\|_{L^\infty} \le 1$. For $k \in \mathbb{N}$, let $C = C(x_0, r)$ be the cube with centre $x_0 \in \Omega$ and side length $r > 0$. Write

$$C = \bigcup_{j=1}^{2^{kn}} C_j^{(k)}$$

in terms of the C-dyadic cubes obtained by subdividing C into 2^{kn} congruent subcubes $C_j^{(k)} = C(x_j^{(k)}, 2^{-k}r)$. Extend ψ to $\mathbb{R}^n$ by $\psi(x) = 0$ for $x \in \mathbb{R}^n \setminus (-1,1)^n$ and define

$$\varphi_k(x) = \sum_{j=1}^{2^{kn}} 2^{-k} r\, \psi\left(\frac{x - x_j^{(k)}}{2^{-k}r}\right) \in W_0^{1,\infty}(C).$$

Let $2C = C(x_0, 2r)$ and take $\rho \in W_0^{1,\infty}(2C)$ such that it satisfies $\mathbb{1}_C \le \rho \le \mathbb{1}_{2C}$ and $|D\rho| \le \frac{1}{r}$ a.e.. Put $\varphi = \varphi_k + 2^{-k} r\rho$ and hence $\varphi \in W_0^{1,\infty}(2C)^+$. From superminimimality, we have

$$\begin{aligned} & \int_{2C} F\big(x, D\bar{u}(x)\big)\,\mathrm{d}x \\ \le\ & \int_{2C} F\big(x, D\bar{u}(x) + D\varphi(x)\big)\,\mathrm{d}x \\ =\ & \int_C F\big(x, D\bar{u}(x) + D\varphi_k(x)\big)\,\mathrm{d}x + \int_{2C\setminus C} F\big(x, D\bar{u}(x) + 2^{-k} r\, D\rho\big)\,\mathrm{d}x \\ =\ & \sum_{j=1}^{2^{kn}} \int_{C_j^{(k)}} F\Big(x, D\bar{u}(x) + D\psi\big(\frac{x - x_j^{(k)}}{2^{-k} r}\big)\Big)\,\mathrm{d}x + \int_{2C\setminus C} F\big(x, D\bar{u}(x) + 2^{-k} r\, D\rho\big)\,\mathrm{d}x \\ \xrightarrow{k\to\infty}\ & \int_C 2^{-n} \int_{(-1,1)^n} F\big(x, D\bar{u}(x) + D\psi(y)\big)\,\mathrm{d}y\,\mathrm{d}x + \int_{2C\setminus C} F\big(x, D\bar{u}(x)\big)\,\mathrm{d}x \end{aligned}$$

and so it follows that

$$⨍_C F\big(x, D\bar{u}(x)\big)\,\mathrm{d}x \le ⨍_C 2^{-n} \int_{(-1,1)^n} F\big(x, D\bar{u}(x) + D\psi(y)\big)\,\mathrm{d}y\,\mathrm{d}x.$$

Since this inequality is true for all cubes $2C$ and the Lebesgue measure $\mathcal{L}^n$ is a regular Borel measure, it follows that there exists a null set $N_\psi \subset \Omega$ such that

$$F\big(x_0, D\bar{u}(x_0)\big) \le ⨍_{(-1,1)^n} F\big(x_0, D\bar{u}(x_0) + D\psi(y)\big)\,\mathrm{d}y \tag{7.2.1}$$

for all $x_0 \in \Omega \setminus N_\psi$. Since F is Carathéodory, we only need (7.2.1) to hold for a $W^{1,\infty}$-dense subset of ψ in $C_0^\infty(\Omega;\mathbb{R}^N)$. It is clear that $C_0^\infty(\Omega;\mathbb{R}^N)$ is separable in $W^{1,\infty}$-norm (for instance by Weierstrass approximation theorem), so we may find a $W^{1,\infty}$-dense countable subset $\{\psi_j : j \in \mathbb{N}\}$ of $C_0^\infty(\Omega;\mathbb{R}^N)$ for which (7.2.1) holds for all $x_0 \in \Omega \setminus \tilde{N}$, where $\tilde{N} = \bigcup_{j\in\mathbb{N}} N_\psi$ remains a null set. Thus $F(x_0, \cdot)$ is quasiconvex at $Du(x_0)$ for all $x_0 \in \Omega \setminus \tilde{N}$, and hence is convex there by Lemma 2.2.6 because $N = 1$. □

We can now easily deduce that Theorem 7.0.6 also works for strong Q-superquasiminimizers and superminimizers.

Theorem 7.2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $Q > 1$, $p \in (1,\infty)$ and $a \in L^1(\Omega)^+$. Then $\bar{u} \in W^{1,p}(\Omega)$ is a strong Q-superquasiminimizer of the integral $\int_\Omega \big(|Du(x)|^p + a(x)\big)\,\mathrm{d}x$ if and only if $\bar{u}$ is an superminimizer of the integral $\int_\Omega F^c\big(x, Du(x)\big)\,\mathrm{d}x$, where for each $x \in \Omega$, $F^c(x,\cdot)$ is the convex envelope of (7.0.2).*

Proof. The proof is similar to that of Theorem 7.0.6. Since $F(x,\xi)$ is a normal integrand satisfying

$$|\xi|^p + a(x) \le F(x,\xi) \le Q\big(|\xi|^p + a(x)\big)$$

and $\bar{u} \in W^{1,p}(\Omega)$ is a superminimizer of $\int_\Omega F\big(x, Du(x)\big)\,\mathrm{d}x$, it follows from Theorem 7.2.1 that $F(x,\cdot)$ is convex at $D\bar{u}(x)$ a.e. $x \in \Omega$. Hence, we obtain

$$F^c\big(x, D\bar{u}(x)\big) = F\big(x, D\bar{u}(x)\big) = |D\bar{u}(x)|^p + a(x) \text{ a.e. } x \in \Omega.$$

At the same time, we also have

$$|\xi|^p + a(x) \le F^c(x,\xi) \le Q\big(|\xi|^p + a(x)\big),$$

and therefore we conclude that $\bar{u}$ is indeed a Q-superquasiminimizer of the inhomogeneous p-Dirichlet integral.

The proof of the converse implication follows the proof of Theorem 7.0.6. Assume that $\bar{u}$ is a strong Q-superquasiminimizer of the inhomogeneous p-Dirichlet integral, meaning that

$$\int_{\{D\varphi\neq 0\}} \big(|D\bar{u}(x)|^p + a(x)\big)\,\mathrm{d}x \le Q \int_{\{D\varphi\neq 0\}} \big(|D\bar{u}(x) + D\varphi(x)|^p + a(x)\big)\,\mathrm{d}x$$

holds for all non-negative $\varphi \in W_0^{1,p}(\omega)$ and for all $\omega \in \mathcal{O}(\Omega)$. Therefore we now use the fact that $\bar{u}$ is a strong Q-superquasiminimizer to obtain

$$\begin{aligned}
& \int_\omega F\big(x, D\bar{u}(x) + D\varphi(x)\big)\,\mathrm{d}x \\
= \;& Q\int_{\{D\varphi\neq 0\}\cap\omega} \big(|D\bar{u}(x) + D\varphi(x)|^p + a(x)\big)\,\mathrm{d}x + \int_{\{D\varphi=0\}\cap\omega} \big(|D\bar{u}(x)|^p + a(x)\big)\,\mathrm{d}x \\
\ge \;& \int_{\{D\varphi\neq 0\}\cap\omega} \big(|D\bar{u}(x)|^p + a(x)\big)\,\mathrm{d}x + \int_{\{D\varphi=0\}\cap\omega} \big(|D\bar{u}(x)|^p + a(x)\big)\,\mathrm{d}x \\
= \;& \int_\omega F\big(x, D\bar{u}(x)\big)\,\mathrm{d}x,
\end{aligned}$$

and hence

$$\int_\omega F^c\big(x, D\bar{u}(x) + D\varphi(x)\big)\,\mathrm{d}x \ge \int_\omega F^c\big(x, D\bar{u}(x)\big)\,\mathrm{d}x$$

holds for all non-negative $\varphi \in W^{1,p}(\omega)$. It then follows that $\bar{u}$ is indeed a superminimizer of $\int_\Omega F^c\big(x, Du(x)\big)\,\mathrm{d}x$. $\square$

Appendix A

Approximation by Piecewise Affine Functions

On several occasions, we will have to construct functions whose gradients take only a finite number of values. This can be easily done with piecewise affine functions, which is a natural approximation for the Sobolev functions that we consider throughout the thesis. Here, we provide a self-contained section on how Sobolev functions can be approximated by piecewise affine functions. This requires some understanding of the multipoint Taylor formula which has a direct application to the finite element method for solving second-order variational problems over a space X.

The simplest form of the finite element method in this case is called the Galerkin's method where it consists of defining discrete problems over finite-dimensional subspaces U_j of U. A subspace U_j can be constructed by first establishing a triangulation T_j over the set $\bar{\Omega}$ where $\bar{\Omega}$ is a finite union of finite elements $K \in T_j$. Then the functions $u_j \in U_j$ are piecewise polynomials, in the sense that for each $K \in T_j$, the spaces

$$P_K = \{u_j \text{ restricted to } K \colon u_j \in U_j\}$$

consist of polynomials. Lastly, there exists a basis in the space U_j whose functions have small supports. For the application of multipoint Taylor formula to the finite element method, please refer to Section 6 of [24].

Note that for functions and maps in Lebesgue and Sobolev spaces, we always consider them in terms of their precise representatives:

Definition A.0.3. *Let $u \in L^p_{\text{loc}}(\Omega;\mathbb{R}^N)$, $p \in [1,\infty]$. The precise representative of u, denoted by u^*, is the map $u^* \colon \Omega \to \mathbb{R}^N$ defined by*

$$u^*(x) = \begin{cases} \lim_{r\to 0^+} \fint_{B(x,r)} u(y)\,\mathrm{d}y & \text{if the limit exists} \\ 0 & \text{otherwise} \end{cases}.$$

A.1 Multipoint Taylor formula

The studies of multipoint Taylor formula can be found for instance in the work of Ciarlet-Wagschal [24]. Let $E = \mathbb{R}^n$ be the n-dimensional Euclidean space. Let there be given $(n+1)$ points $A_i = a_{1i}, a_{2i}, \cdots, a_{ni}$, $1 \le i \le n+1$, such that the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,n+1} \\ a_{21} & a_{22} & \cdots & a_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,n+1} \\ 1 & 1 & \cdots & 1 \end{pmatrix} \tag{A.1.1}$$

is regular. The barycentric coordinates $\lambda_j = \lambda_j(P)$, $1 \le j \le n+1$ of any point $P \in E$, with respect to the $(n+1)$ points A_i, are the unique solutions of the linear system

$$\sum_{j=1}^{n+1} a_{ij}\lambda_j = x_i,\ 1 \le i \le n+1 \quad \text{and} \quad \sum_{j=1}^{n+1} \lambda_j = 1$$

since the matrix A is regular. Meanwhile, the barycentric coordinates are affine functions of the coordinates $x_1, x_2, \cdots, x_n$ of P since

$$\lambda_i = \sum_{j=1}^{n} b_{ij}x_j + b_{in+1}, \quad 1 \le i \le n+1 \tag{A.1.2}$$

where the matrix $B = (b_{ij})$ is the inverse matrix of A. The barycentric coordinates are independent of the basis in E and hence

$$\sum_{i=1}^{n+1} \lambda_i \overrightarrow{PA_i} = \overrightarrow{0}. \tag{A.1.3}$$

A n-simplex is the convex hull S of $(n+1)$ points A_i which are called the vertices of the n-simplex satisfying the conditions

$$0 \le \lambda_i \le 1, \quad 1 \le i \le n+1.$$

Thus, we denote

$$S = \left\{ x = \sum_{j=1}^{n+1} a_{ij}\lambda_j \colon 0 \le \lambda_j \le 1,\ 1 \le j \le n+1,\ \sum_{j=1}^{n+1} \lambda_j = 1 \right\}.$$

It is clear that a 2-simplex is a triangle whereas a 3-simplex is a tetrahedron. The barycentre of an n-simplex S is the point of S whose barycentric coordinates are all equal to $\frac{1}{n+1}$.

Given a C^{k+1}function $\phi\colon K \to \mathbb{R}$ and an open segment (A, P) whose closure $[A, P] \subset K$. The Taylor formula of the $(k+1)$-st order between P and A can be applied and we can write

$$\phi(A) = \phi(P) + \sum_{l=1}^{k} \frac{1}{l!} D^l \phi(P) \cdot (\overrightarrow{PA}^l) + \frac{1}{(k+1)!} D^{k+1} \phi(\Pi(P)) \cdot (\overrightarrow{PA}^{k+1}) \tag{A.1.4}$$

for some point $\Pi(P)$ of the open segment (A, P). A subset $K \in E$ is said to be S-admissible if and only if whenever K contains a point P, it also contains the closed segments $[P, A_i]$ for all $1 \leq i \leq n+1$.

Definition A.1.1. *A function ϕ is said to belong to the class J^{k+1} if and only if the domain of the definition of ϕ contains K, and between any point $P \in K$ and any point $A \in S$, the Taylor formula (A.1.4) holds.*

Observe that, from the fact that $\sum_{i=1}^{n+1} \lambda_i = 1$, we have

$$\sum_{i=1}^{n+1} D\lambda_i(P) = 0, \tag{A.1.5}$$

and by (A.1.2), we obtain $\frac{\partial \lambda_i}{\partial x_k} = b_{ik}$, $1 \leq i \leq n+1$, $1 \leq k \leq n$. Thus, applying (A.1.2) once again gives us

$$D\lambda_i(P) \cdot (\overrightarrow{PA_j}) = \sum_{k=1}^{n} b_{ik} a_{kj} - \sum_{k=1}^{n} b_{ik} x_k = \delta_{ij} - \lambda_i(P), \tag{A.1.6}$$

where δ_{ij} is the Kronecker's delta. Since λ_i are affine,

$$D\lambda_i(P)\colon E \to \mathbb{R}, \quad 1 \leq i \leq n+1$$

are independent of P and hence can be denoted by $D\lambda_i$. We shall derive an estimate for $\|D\lambda_i\|$ in the following lemma.

Lemma A.1.2. *Given an n-simplex S, let h' be the diameter of the inscribed sphere of S. Then*

$$\|D\lambda_i\| \leq \frac{1}{h'} \tag{A.1.7}$$

for all $1 \leq i \leq n+1$.

Proof. By definition, we have

$$\|D\lambda_i\| = \sup \left\{ |D\lambda_i \cdot X| \colon X \in E, \|X\|_E = 1 \right\}.$$

Given any vector $X \in E$ which satisfies $\|X\|_E = 1$, we can write X as

$$X = \frac{c}{h'}\overrightarrow{PQ} \tag{A.1.8}$$

for $P, Q \in S$ where $c \leq 1$ is a constant. Since $Q \in S$, there exist $(n+1)$ real numbers μ_j such that

$$\overrightarrow{PQ} = \sum_{j=1}^{n+1} \mu_j \overrightarrow{PA_j}, \text{ with } 0 \leq \mu_j \leq 1,\ 1 \leq j \leq n+1, \quad \text{and} \quad \sum_{j=1}^{n+1} \mu_j = 1. \tag{A.1.9}$$

It follows from (A.1.6), (A.1.8) and (A.1.9) that

$$D\lambda_i \cdot X = \frac{c}{h'} \sum_{j=1}^{n+1} \mu_j D\lambda_i \cdot (\overrightarrow{PA_j}) = \frac{c}{h'}(\mu_i - \lambda_i).$$

Since the quantity λ_i also satisfies $0 \leq \lambda_i \leq 1$ for $P \in S$, we have

$$|D\lambda_i \cdot X| \leq \frac{c}{h'} \leq \frac{1}{h'}$$

which finally proves (A.1.7). □

Definition A.1.3. *Let $S \subset E$ be an n-simplex with vertices A_i, $1 \leq i \leq n+1$. Given a real-valued function ϕ defined at the points A_i, we say that $\tilde{\phi}\colon E \to \mathbb{R}$ is an approximation of type 1 of ϕ if and only if $\tilde{\phi}$ is an affine function and $\tilde{\phi}(A_i) = \phi(A_i)$, $1 \leq i \leq n+1$.*

Theorem A.1.4. *The unique approximation of type 1 is given by*

$$\tilde{\phi}(P) = \sum_{i=1}^{n+1} \lambda_i(P)\phi(A_i) \tag{A.1.10}$$

for all $P \in E$.

Proof. We can see that the first condition of Definition A.1.3 is satisfied since λ_i's are affine functions of $x_1, x_2, \cdots, x_n$. The second condition is also satisfied since $\lambda_i(A_j) = \delta_{ij},\ 1 \leq i, j \leq n+1$. □

Theorem A.1.5. *Let $K \in E$ be S-admissible and let $\phi \in J^2(K)$. Then, for all $P \in K$, $P \neq A_i$, $1 \leq i \leq n+1$,*

$$\tilde{\phi}(P) - \phi(P) = \frac{1}{2}\sum_{i=1}^{n+1} \left\{ D^2\phi(\Pi_i(P)) \cdot [\overrightarrow{PA_i}, \overrightarrow{PA_i}] \right\} \lambda_i(P), \tag{A.1.11}$$

$$D\tilde{\phi}(P) - D\phi(P) = \frac{1}{2}\sum_{i=1}^{n+1} \left\{ D^2\phi(\Pi_i(P)) \cdot [\overrightarrow{PA_i}, \overrightarrow{PA_i}] \right\} D\lambda_i(P). \tag{A.1.12}$$

where the points $\Pi_i(P)$ are some points on the open segments (P, A_i), $1 \leq i \leq n+1$.

Proof. Since $\phi \in J^2(K)$, we can apply the Taylor formula (A.1.4) to get

$$\phi(A_i) = \phi(P) + D\phi(p) \cdot (\overrightarrow{PA_i}) + \frac{1}{2} D^2\phi(\Pi_i(P)) \cdot [\overrightarrow{PA_i}, \overrightarrow{PA_i}] \tag{A.1.13}$$

for $1 \leq i \leq n+1$. Substitute (A.1.13) into (A.1.10) and as a direct consequence of $\sum_{i=1}^{n+1} \lambda_i = 1$ and (A.1.3), we therefore obtain (A.1.11).

To prove (A.1.12), firstly we differentiate (A.1.10) with respect to P to get

$$D\tilde{\phi}(P) = \sum_{i=1}^{n+1} \phi(A_i) D\lambda_i.$$

Once again, substituting (A.1.13) into the above equation, we have

$$\begin{aligned} D\tilde{\phi}(P) &= \phi(P) \sum_{i=1}^{n+1} D\lambda_i + \sum_{i=1}^{n+1} \left\{ D\phi(P) \cdot (\overrightarrow{PA_i}) \right\} D\lambda_i \\ &\quad + \frac{1}{2} \sum_{i=1}^{n+1} \left\{ D^2\phi(\Pi_i(P)) \cdot [\overrightarrow{PA_i}, \overrightarrow{PA_i}] \right\} D\lambda_i. \end{aligned}$$

The first term on the right hand side is zero by (A.1.5) and hence

$$D\tilde{\phi}(P) = \sum_{i=1}^{n+1} \left\{ D\phi(P) \cdot (\overrightarrow{PA_i}) \right\} D\lambda_i + \frac{1}{2} \sum_{i=1}^{n+1} \left\{ D^2\phi(\Pi_i(P)) \cdot [\overrightarrow{PA_i}, \overrightarrow{PA_i}] \right\} D\lambda_i.$$

In order to prove (A.1.12), we need to show that

$$D\phi(P) = \sum_{i=1}^{n+1} \left\{ D\phi(P) \cdot (\overrightarrow{PA_i}) \right\} D\lambda_i.$$

Observe that we can apply the linear mapping to the $(n+1)$ vectors $\overrightarrow{PA_j}$, $1 \leq j \leq n+1$ and relation (A.1.6) to get

$$\begin{aligned} \sum_{i=1}^{n+1} \left\{ D\phi(P) \cdot (\overrightarrow{PA_i}) \right\} D\lambda_i \cdot (\overrightarrow{PA_j}) &= D\phi(P) \cdot (\overrightarrow{PA_j}) - D\phi(P) \cdot \left(\sum_{i=1}^{n+1} \lambda_i \overrightarrow{PA_i} \right) \\ &= D\phi(P) \cdot (\overrightarrow{PA_j}) \end{aligned}$$

since $\sum_{i=1}^{n+1} \lambda_i \overrightarrow{PA_i} = 0$ by (A.1.3). Therefore, we finally obtain the required (A.1.12). □

Corollary A.1.6. *Let $\phi \in J^2(S)$ be given and assume that*

$$M_2 = \sup \left\{ \left| \frac{\partial^2 \phi}{\partial x_i \partial x_j}(Q) \right| : 1 \leq i, j \leq n,\ Q \in S \right\} < +\infty. \tag{A.1.14}$$

Then, denoting by $\tilde{\phi}$ the unique approximation of type 1 of ϕ, we have

$$\sup \left\{ |\tilde{\phi}(P) - \phi(P)| \colon P \in S \right\} \leq C M_2 h^2, \tag{A.1.15}$$

and for any $1 \le i \le n$*,*

$$\sup \left\{ \left| \frac{\partial \tilde{\phi}}{\partial x_i}(P) - \frac{\partial \phi}{\partial x_i}(P) \right| : P \in S \right\} \le C' M_2 \frac{h^2}{h'}, \tag{A.1.16}$$

where

$$\begin{aligned} h &= \text{diameter of } S, \\ h' &= \text{diameter of the inscribed sphere of } S, \\ C, C' &= \text{numerical constants which are independent of the geometry of } S. \end{aligned}$$

Proof. It is clear that condition (A.1.14) implies that

$$M_2' = \sup \left\{ \|D^2\phi(Q)\| : Q \in S \right\} < +\infty.$$

By (A.1.11), we have

$$|\tilde{\phi}(P) - \phi(P)| \le \frac{1}{2} \sum_{i=1}^{n+1} \lambda_i(P) \left| D^2\phi\big(\Pi_i(P)\big) \cdot [\overrightarrow{PA_i}, \overrightarrow{PA_i}] \right|$$

We can see that

$$\left| D^2\phi(\Pi_i(P)) \cdot [\overrightarrow{PA_i}, \overrightarrow{PA_i}] \right| \le M_2' h^2$$

and since $0 \le \lambda_i(P) \le 1$ for all $0 \le i \le n+1$, we thus obtain (A.1.15). Similarly, equation (A.1.12) and the result of Lemma A.1.2 lead to

$$|D\tilde{\phi}(P) - D\phi(P)| \le \frac{1}{2} \sum_{i=1}^{n+1} D\lambda_i(P) \left| D^2\phi\big(\Pi_i(P)\big) \cdot [\overrightarrow{PA_i}, \overrightarrow{PA_i}] \right| \le C' M_2 \frac{h^2}{h'},$$

which is the statement (A.1.16). □

A.2 Approximations by piecewise affine functions

A function $\varphi \colon \Omega \to \mathbb{R}^N$ is said to be piecewise affine in Ω if it is continuous and if there exists a finite number of open sets $\omega_1, \omega_2, \ldots, \omega_n$ on which φ is affine and such that $\bigcup_{i=1}^n \overline{\omega}_i = \overline{\Omega}$. A nice property of piecewise affine functions is that they can be used to approximate Sobolev functions. We will employ the exhaustion argument and an approximation lemma [75], together with application of multipoint Taylor formula, in the proof.

Proposition A.2.1 (Exhaustion argument). *Let* $\Omega, \Theta \subset \mathbb{R}^n$ *be non-empty bounded open sets and suppose that* $\operatorname{meas}(\partial\Omega) = 0$*. Let* $\varepsilon > 0$*. Then there exists at most countably many points* $x_i \in \Theta$ *and numbers* $\lambda_i \in (0, \varepsilon)$ *for* $i \in I$ *such that* $x_i + \lambda_i\bar{\Omega} \subset \Theta$ *are pairwise disjoint, and*

$$\operatorname{meas}\Big(\Theta \setminus \bigcup_{i \in I} \big(x_i + \lambda_i \bar{\Omega}\big)\Big) = 0.$$

Proof. We first note that an open subset of $\mathbb{R}^n$ is the disjoint union of closed cubes. Let $C \supset \bar{\Omega}$ be a closed cube. Then there is an at most countable family $x_i \in \Theta$ and $\lambda_i \in (0, \varepsilon)$ for $i \in I$ such that $x_i + \lambda_i C \subset \Theta$ are pairwise disjoint and $\Theta = \bigcup_{i \in I} \big(x_i + \lambda_i C\big)$. In other word, we write Θ as a disjoint union of translations and dilations of C.

We next use translation invariance and n-homogeneity of Lebesgue measure to get

$$\begin{aligned}
\operatorname{meas}\Big(\Theta \setminus \bigcup_{i\in I}\big(x_i + \lambda_i \bar{\Omega}\big)\Big) &= \sum_{i \in I} \operatorname{meas}\Big(\big(x_i + \lambda_i C\big) \setminus \big(x_i + \lambda_i \bar{\Omega}\big)\Big) \\
&= \sum_{i \in I} {\lambda_i}^n \operatorname{meas}\big(C \setminus \bar{\Omega}\big) \\
&= \alpha \sum_{i \in I} {\lambda_i}^n \operatorname{meas} C \\
&= \alpha \sum_{i \in I} \operatorname{meas}\big(x_i + \lambda_i C\big) \\
&= \alpha \operatorname{meas} \Theta
\end{aligned}$$

where $\alpha = \dfrac{\operatorname{meas}\big(C \setminus \bar{\Omega}\big)}{\operatorname{meas} C} < 1$. For some finite subset $I_1 \subset I$, we have

$$\operatorname{meas}\Big(\Theta \setminus \bigcup_{i \in I_1} \big(x_i + \lambda_i \bar{\Omega}\big)\Big) \leq \sqrt{\alpha} \operatorname{meas} \Theta$$

since $\sqrt{\alpha} > \alpha$ for $\alpha < 1$. Observe that the set

$$\Theta_1 = \Theta \setminus \bigcup_{i \in I_1} \big(x_i + \lambda_i \bar{\Omega}\big)$$

is open and hence repeating the above procedure on Θ_1 and so on gives us the required result since $\sqrt{\alpha} < 1$. □

Lemma A.2.2 (Kirchheim [75], Lemma 3.2). *Let* $\min\{N, n\} \geq 1$ *and let* $\Omega \subset \mathbb{R}^n$ *be an open set with finite measure. Let* $A, B \in \mathbb{R}^{N \times n}$ *with* $\operatorname{rank}(A - B) = 1$ *and let* u_ξ *be such that*

$$Du_\xi = \xi = tA + (1-t)B$$

for all $x \in \bar{\Omega}$ *where* $t \in [0, 1]$. *Write* $A - B$ *as* $a \otimes b$ *and suppose we are given more vectors* $b_3, \cdots, b_k$ *such that*

$$0 \in \operatorname{int}_{\mathbb{R}^n}\big(\operatorname{conv}(\{b, -b, b_3, \cdots, b_k\})\big)$$

where conv *denotes the convex hull. Then, for every* $\varepsilon > 0$, *there is a* $u \in \operatorname{Aff}_{piec}(\bar{\Omega}, \mathbb{R}^N)$ *such that*

$$
\begin{cases}
u(x) = u_\xi(x) = \xi x \textit{ if } x \in \partial\Omega, \\
\|u - u_\xi\|_{L^\infty(\Omega)} < \varepsilon, \\
Du(x) \in \{A, B, \xi + a \otimes b_3, \cdots, \xi + a \otimes b_k\} \textit{ a.e. in } \Omega, \\
\operatorname{meas} \Omega_A > (1-\varepsilon)\, t \operatorname{meas} \Omega, \\
\operatorname{meas} \Omega_B > (1-\varepsilon)(1-t) \operatorname{meas} \Omega,
\end{cases}
\tag{A.2.1}
$$

where $\Omega_A = \{x \in \Omega \colon Du(x) = A\}$ *and* $\Omega_B = \{x \in \Omega \colon Du(x) = B\}$.

Proof. Without loss of generality, assume that $\xi = 0$ with

$$A = (1-t)\, a \otimes b \quad \text{and} \quad B = -t\, a \otimes b.$$

We now prove for the case when $n = 1$ and $a = 1$, the general case follows by multiplying the scalar solution with a vector a. Let

$$P = \{x \in \mathbb{R}^n \colon 1 + \langle x, b_i \rangle > 0 \text{ for all } i \leq k,\ b_1 = b,\ b_2 = -b\}.$$

It is clear that P is a convex polyhedron containing the origin in its interior. Since P is contained in the polar of the convex hull of the b_i's, which is a neighbourhood of the origin, then P is bounded. Now, we take the auxiliary function $v \colon \mathbb{R} \to \mathbb{R}$ which is 1-periodic and fulfills $v(0) = 0$ and

$$v'(y) = \begin{cases} 1-t & \text{if } y \in (0,t) \\ -t & \text{if } y \in (t,1) \end{cases}.$$

For $l \geq 1$, set

$$u_l(x) := \min\left\{ \min_{i=3,\cdots,k} \big\{1 + \langle x, b_i \rangle\big\}, \frac{1}{l} v\big(l \langle x, b \rangle\big) \right\}.$$

By the construction of v, we have $v \geq 0$ and hence we can see that $u_l \geq 0$ on P. Meanwhile, we have $\langle x, b_i \rangle = -1$ in all boundary points and hence $u_l = 0$ on ∂P. Since u is affine on a finite decomposition of P into convex pieces, it is clear that

$$Du_l \in \{(1-t)b, -tb, b_3, \cdots, b_k\} \text{ a.e. in } P.$$

By setting $y \in (1-s)P$, $s > 0$, we find that

$$1 + (1-s)\langle x, b_i \rangle = 1 + \langle x, b_i \rangle - s\langle x, b_i \rangle \geq s$$

and therefore on $(1-s)P$

$$\min_{i \geq 3} \{1 + \langle x, b_i \rangle\} \geq s.$$

By taking $l = l_0$ sufficiently large, we have obtained the last two results as stated in (A.2.1). Next, by Proposition A.2.1, we can find $x_i \in \Omega$, $\lambda_i \in (0,1)$ such that

$$\text{meas}\left(\Omega \setminus \bigcup_{i \in I} P_i\right) = 0$$

where $P_i = x_i + \lambda_i P$. Define for $x \in \Omega$, the function

$$u(x) := \begin{cases} \lambda_i u_\xi\left(\dfrac{x - x_i}{\lambda_i}\right) & \text{if } x \in P_i \text{ for some } i \in I \\ \xi x & \text{elsewhere} \end{cases}. \tag{A.2.2}$$

It is clear that

$$u(x) = \xi x \text{ if } x \in \partial P_i \text{ for some } i \text{ or } x \in \partial\Omega$$

and hence $\|u - u_\xi\|_{L^\infty(\Omega)} < \varepsilon$. □

The following proposition is a standard result in the studies of finite elements method of numerical anlysis [40]:

Proposition A.2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $u\colon \bar{\Omega} \to \mathbb{R}$ be continuous and C^∞ smooth on Ω. Then, there exists a sequence $\{u_i\}$ of piecewise affine functions on Ω such that when $i \to \infty$,*

$$Du_i \to Du \text{ uniformly over every compact subset of } \Omega \tag{A.2.3}$$

and

$$u_i \to u \text{ uniformly.} \tag{A.2.4}$$

If, in addition, the support of u is compact in Ω, we can impose on the u_i, $i \in \mathbb{N}$ the additional condition that they are zero outside a fixed compact $K \subset \Omega$. In particular,

$$u_i(x) = 0 \tag{A.2.5}$$

for all $x \in \partial\Omega$.

Proof. Let S be a net of n-simplexes with diameter no greater than 2^{-i} and further let S' be the cover of $\bar{\Omega}$. Let u_i be the piecewise affine function which coincides with u at the vertices of the net. Applying the multipoint Taylor formula stated in Corollary A.1.6, taking $h \le 2^{-i}$, we can see that we have (A.2.3) and (A.2.4). □

Summing up all the previous results, we are ready to prove the result of approximation by piecewise affine function. Of course, there are many variations of this result that one can easily find in Ekeland-Témam [40] (Chapter X), Dacorogna-Marcellini [33] (Chapter 10) and Dacorogna [29] (Theorem 12.14 and Theorem 12.15).

Theorem A.2.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $p \in [1, \infty]$ and $u \in W_0^{1,p}(\Omega)$. Then, for every $\varepsilon > 0$, there exists $u_\varepsilon \in \mathrm{Aff}_{piec}(\bar{\Omega})$ such that*

$$\begin{cases} u_\varepsilon \in W_0^{1,p}(\Omega), \\ \|u_\varepsilon\|_{W^{1,p}(\Omega)} \leq \|u\|_{W^{1,p}(\Omega)} + \varepsilon, \\ \|u_\varepsilon - u\|_{W^{1,p}(\Omega)} \leq \varepsilon. \end{cases}$$

(If $p = \infty$, we only have $\|u_\varepsilon - u\|_{W^{1,q}} \leq \varepsilon$ for any $q < \infty$.)

Proof. By definition of $W_0^{1,p}(\Omega)$, there exists $v \in C_0^\infty(\Omega)$ such that when $p \in [1, \infty)$

$$\|u - v\|_{W^{1,p}(\Omega)} \leq \frac{\varepsilon}{2},$$

and when $p = \infty$

$$\begin{cases} \|u - v\|_{W^{1,q}(\Omega)} \leq \dfrac{\varepsilon}{2} \text{ for all } q < \infty, \\ \|v\|_{W^{1,p}(\Omega)} \leq \|u\|_{W^{1,p}(\Omega)} + \dfrac{\varepsilon}{2}. \end{cases}$$

As a result of Proposition A.2.3, there exists $u_\varepsilon \in \mathrm{Aff}_{piec}(\bar{\Omega})$, null outside a compact $\Omega' \subset \Omega$, converging uniformly to v such that

$$Du_\varepsilon \to Dv \quad \text{uniformly}.$$

Thus, we have

$$\|v - u_\varepsilon\|_{W^{1,p}(\Omega)} \leq \frac{\varepsilon}{2}.$$

By triangle inequality, we obtain for $p \in [1, \infty)$,

$$\|u - u_\varepsilon\|_{W^{1,p}(\Omega)} \leq \|u - v\|_{W^{1,p}(\Omega)} + \|v - u_\varepsilon\|_{W^{1,p}(\Omega)} \leq \varepsilon.$$

Similarly, we obtain the result for $p = \infty$. Apart from that, for $p \in [1, \infty)$, we can see from Proposition A.2.3 that $u_\varepsilon \in W_0^{1,p}(\Omega)$ and as a consequence,

$$\|u_\varepsilon\|_{W^{1,p}(\Omega)} \leq \|u_\varepsilon - u\|_{W^{1,p}(\Omega)} + \|u\|_{W^{1,p}(\Omega)} \leq \varepsilon + \|u\|_{W^{1,p}(\Omega)}.$$

□

Appendix B

Young Measures

B.1 A theorem on measurable selections

Definition B.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded measurable set and let (M, d) be a compact metric space. A mapping $V: \Omega \to 2^M$ is a closed measurable multivalued mapping if*

(i) the set $V(x)$ is closed for a.e. $x \in \Omega$, and

(ii) the set $\left\{x \in \Omega\colon V(x) \cap \tilde{M} \neq \emptyset\right\}$ is measurable for any closed $\tilde{M} \subset M$.

Theorem B.1.2 (Kuratowski-Ryll-Nardzewski [85])**.** *If $V: \Omega \to 2^M$ is a closed measurable multivalued mapping, then there exists a measurable map $g\colon \Omega \to (M, d)$ such that $g(x) \in V(x)$ for a.e. $x \in \Omega$.*

B.2 Proofs of Young measure theory

In this section, we give the proofs of the results presented in Section 3.2 (see also Sychev [115]).

Proof of Lemma 3.2.9.

Assume that the family $(\mu_x)_{x\in\Omega}$ is weak* measurable. Then, by definition, the function $\langle \Phi, \mu_{(\cdot)} \rangle$ is measurable for any $\Phi \in C_0(\mathbb{R}^d)$. Note that (K_m, ρ) is a compact metric space. Let $\{\Phi_i\} \subset C_0(\mathbb{R}^d)$ be a sequence of functions which is dense in $C_0(\mathbb{R}^d)$. By Theorem 3.2.3, for $i \in \mathbb{N}$ and given $\varepsilon > 0$, there exists a compact set $\Omega_i \subset \Omega$ such that $\mathrm{meas}(\Omega \setminus \Omega_i) \leq \frac{\varepsilon}{2^i}$ and the restriction of $\langle \Phi_i, \mu_{(\cdot)} \rangle\colon \Omega \to \mathbb{R}$ to Ω_i is continuous. Clearly, $\mathrm{meas}(\Omega \setminus \bigcap \Omega_i) \leq \varepsilon$ and the restrictions of all functions $\langle \Phi_i, \mu_{(\cdot)} \rangle$ to $\bigcap \Omega_i$ are continuous. This implies that $\mu\colon \bigcap \Omega_i \to (K_m, \rho)$ is continuous and since $\varepsilon > 0$ is arbitrary, we have that $\mu\colon \Omega \to (K_m, \rho)$ is a measurable function.

Next, we prove the converse implication. Assume that $\mu\colon \Omega \to (K_m, \rho)$ is a measurable function. Then, by Theorem 3.2.3, for any $\varepsilon > 0$, there exists a compact set $\Omega_\varepsilon \subset \Omega$ such that meas $(\Omega \setminus \Omega_\varepsilon) \leq \varepsilon$ and the restriction of μ to Ω_ε is continuous. It follows that for each $\Phi \in C_0(\mathbb{R}^d)$, the restriction of $\langle \Phi, \mu_{(\cdot)} \rangle$ to Ω_ε is continuous and hence $\langle \Phi, \mu_{(\cdot)} \rangle$ is measurable for each $\Phi \in C_0(\mathbb{R}^d)$. Therefore $(\mu_x)_{x\in\Omega}$ is weak* measurable by Definition 3.2.8. □

Proof of Lemma 3.2.10.

(i) If we let $\Psi_i = \dfrac{\Phi_i}{\|\Phi_i\|_C}$, then

$$\begin{aligned}
&\rho(\mathrm{Av}(\mu_x^1)_{x\in\Omega}, \mathrm{Av}(\mu_x^2)_{x\in\Omega}) \\
= &\sum_{i=1}^{\infty} \frac{1}{2^i \|\Phi_i\|_C} \left| \langle \Phi_i, \mathrm{Av}(\mu_x^1)_{x\in\Omega} \rangle - \langle \Phi_i, \mathrm{Av}(\mu_x^2)_{x\in\Omega} \rangle \right| \\
= &\sum_{i=1}^{\infty} \frac{1}{2^i} \left| \langle \Psi_i, \mathrm{Av}(\mu_x^1)_{x\in\Omega} \rangle - \langle \Psi_i, \mathrm{Av}(\mu_x^2)_{x\in\Omega} \rangle \right| \\
= &\sum_{i=1}^{\infty} \frac{1}{2^i \mathrm{meas}\, \Omega} \left| \int_\Omega \langle \Psi_i, \mu_x^1 \rangle \,\mathrm{d}x - \int_\Omega \langle \Psi_i, \mu_x^2 \rangle \,\mathrm{d}x \right| \\
= &\sum_{i=1}^{\infty} \frac{1}{2^i \mathrm{meas}\, \Omega} \left| \int_\Omega \int_{\mathbb{R}^d} \Psi_i(v) \mathrm{d}\mu_x^1 \,\mathrm{d}x - \int_\Omega \int_{\mathbb{R}^d} \Psi_i(v) \mathrm{d}\mu_x^2 \,\mathrm{d}x \right| \\
\leq &\sum_{i=1}^{\infty} \frac{1}{2^i \mathrm{meas}\, \Omega} \int_\Omega \left| \int_{\mathbb{R}^d} \Psi_i(v) \mathrm{d}\mu_x^1 - \int_{\mathbb{R}^d} \Psi_i(v) \mathrm{d}\mu_x^2 \right| \mathrm{d}x \\
\leq &\ 2m\delta + \sum_{i=1}^{\infty} \frac{1}{2^i \mathrm{meas}\, \Omega} \int_{\Omega_\delta} \left| \int_{\mathbb{R}^d} \Psi_i(v) \mathrm{d}\mu_x^1 - \int_{\mathbb{R}^d} \Psi_i(v) \mathrm{d}\mu_x^2 \right| \mathrm{d}x \\
= &\ 2m\delta + \frac{1}{\mathrm{meas}\, \Omega} \int_{\Omega_\delta} \left(\sum_{i=1}^{\infty} \frac{1}{2^i} \left| \int_{\mathbb{R}^d} \Psi_i(v) \mathrm{d}\mu_x^1 - \int_{\mathbb{R}^d} \Psi_i(v) \mathrm{d}\mu_x^2 \right| \right) \mathrm{d}x \\
= &\ 2m\delta + \frac{1}{\mathrm{meas}\, \Omega} \int_{\Omega_\delta} \rho(\mu_x^1, \mu_x^2) \,\mathrm{d}x \\
\leq &\ 2m\delta + \delta = (2m+1)\delta.
\end{aligned}$$

(ii) By the first statement of the theorem, we have that, for measurable $\tilde{\Omega} \subset \Omega$,

$$\mathrm{Av}(\mu^j_{(x)})_{x\in\tilde{\Omega}} \overset{*}{\rightharpoonup} \mathrm{Av}(\mu_x)_{x\in\tilde{\Omega}} \text{ as } j \to \infty.$$

It follows that $\langle \Phi, \mu^j_{(\cdot)} \rangle \overset{*}{\rightharpoonup} \langle \Phi, \mu_{(\cdot)} \rangle$ in $L^\infty(\Omega)$ for any $C_0(\mathbb{R}^d)$ and by definition, we have

$$(\mu_x^j)_{x\in\Omega} \overset{*}{\rightharpoonup} (\mu_x)_{x\in\Omega} \text{ in } L^\infty_{w^*}(\Omega; K_m) \text{ as } j \to \infty.$$

□

Proof of Theorem 3.2.11. We will begin with the proof of the compactness result. Let $D = [-a, a]^n$ where $a > 0$ is large enough such that $\Omega \subset D$. For each $i \in \mathbb{N}$, decompose D into cubes of equal size D^i_k, $k = 1, \ldots, 2^{ni}$. Let $B^i_k = D^i_k \cap \Omega$ for $i \in \mathbb{N}$ and $k = 1, \ldots, 2^{ni}$. The diagonalization process allows us to select a subsequence $(\mu^j_x)_{x\in\Omega}$ (not relabeled) such that

$$\mathrm{Av}(\mu^j_x)_{x\in B^i_k} \overset{*}{\rightharpoonup} \mu^i_k \text{ in } \mathcal{M}(\mathbb{R}^d)$$

for every $i \in \mathbb{N}$ and $k = 1, \ldots, 2^{ni}$, where $\mu^i_k \in B^i_k$. Our aim is to find a function $\mu\colon B \to (K_m, \rho)$ for which $\mu^i_k = \mathrm{Av}(\mu_x)_{x\in B^i_k}$, where B will be defined in a later part of this proof.

We now consider $\nu^i\colon \Omega \to (K_m, \rho)$ of auxiliary functions such that

$$\nu^i(x) = \mu^i_k \text{ for } x \in B^i_k, k = 1, \ldots, 2^{ni}.$$

By construction, we find that if $i' \geq i$, then

$$\mathrm{Av}(\nu^{i'})_{x\in B^i_k} = \mu^i_k \text{ for every fixed } i \in \mathbb{N} \text{ and } k = 1, \ldots, 2^{ni}.$$

If we let

$$\tilde{\Omega} := \bigcap_{i=1}^{\infty} \bigcup_{k=1}^{2^{ni}} B^i_k,$$

then it is clear that $\mathrm{meas}(\Omega \setminus \tilde{\Omega}) = 0$. For any $x \in \tilde{\Omega}$, there exists a unique sequence $B^i_{k(i)}$ such that $x \in B^i_{k(i)}$ for $i \in \mathbb{N}$ and note that $B^{i+1}_{k(i+1)} \subset B^i_{k(i)}$. For such points x, define $V(x)$ as the set of all cluster points of the sequence $\mu^i_{k(i)}$. For the remaining x, we let $V(x) = K_m$. It is clear that $V(x)$ is closed for any $x \in \tilde{\Omega}$.

We aim to show that the second condition of Definition B.1.1 is also satisfied and hence by Theorem B.1.2, we may conclude that V admits a measurable selection $(\mu_x)_{x\in\Omega}$ and $\mathrm{Av}(\mu_x)_{x\in B^i_k} = \mu^i_k$ for any such selection. Note that if $O \subset K_m$ is open and

$$O_c = \left\{\mu \in O\colon \rho(\mu, K_m \setminus O) > \frac{1}{c}\right\},$$

then

$$V^{-1}(O) = (\Omega \setminus \tilde{\Omega}) \cup \bigcup_c \left(\bigcap_k \bigcup_{i\geq k} (\nu^i)^{-1}(O_c) \right),$$

where ν_k are piecewise constant functions. Thus $V^{-1}(O)$ is measurable and hence $V\colon \Omega \to 2^{K_m}$ is a closed measurable multivalued mapping. Then by Theorem B.1.2, there exists a measurable selection $\mu\colon \Omega \to (K_m, \rho)$ such that $\mu_x \in V(x)$ for a.e. $x \in \Omega$.

Now let $B = B^{i_0}_{k_0}$ and let $\nu = \mu^{i_0}_{k_0}$. Our aim is to prove that

$$\mathrm{Av}(\mu_x)_{x\in B} = \nu$$

and we will be done. By Theorem 3.2.3, for a fixed $\varepsilon > 0$, there exists a compact set $\Omega_\varepsilon \subset \tilde{\Omega} \cap B$ such that $\mathrm{meas}(B \setminus \Omega_\varepsilon) \le \varepsilon\, \mathrm{meas}\, B$ and the restriction of $\mu\colon \Omega \to (K_m, \rho)$ to Ω_ε is continuous. Let i' be sufficiently large such that $\mathrm{diam}(B^{i'}_{k'}) < \delta$, where $\rho\big(\mu(x), \mu(y)\big) \le \varepsilon$ for $x, y \in \Omega_\varepsilon$ such that $|x - y| \le \delta$.

Consider a compact set $\tilde{\Omega}_\varepsilon \subset \Omega_\varepsilon$ that consists only of Lebesgue points of the set Ω_ε and satisfies $\mathrm{meas}(B \setminus \tilde{\Omega}_\varepsilon) \le \varepsilon\, \mathrm{meas}\, B$. Then for any $x \in \tilde{\Omega}_\varepsilon$, there exists $i \ge i'$ and $k = 1, \dots, 2^{ni}$ such that

$$x \in B^i_k \subset B \quad \text{and} \quad \rho\Big(\mu^i_k, \mu(x)\Big) < \varepsilon.$$

This implies that there also exists a finite cover Ω_ε by the sets B^i_k, where $i \ge i'$ and $k = 1, \dots, 2^{ni}$, such that

$$\rho\Big(\mu^i_k, \mu(y)\Big) < 2\varepsilon \text{ for all } y \in B^i_k \cap \tilde{\Omega}_\varepsilon.$$

Note that we have either

$$B^i_k \subset B^{i'}_{k'} \quad \text{or} \quad B^i_k \cap B^{i'}_{k'} = \emptyset$$

for $i \ge i'$, $k = 1, \dots, 2^{ni}$ and $k' = 1, \dots, 2^{ni'}$. Thus there exists a similar cover consisting of disjoint sets $\tilde{B}_1, \dots, \tilde{B}_t$. Let $\tilde{\mu}^1, \dots, \tilde{\mu}^t$ be the averages μ^i_k associated with $\tilde{B}_1, \dots, \tilde{B}_t$. respectively. Let

$$\tilde{\mu}_x = \begin{cases} \tilde{\mu}^i & \text{on } \tilde{B}_i,\ i = 1, \dots, t \\ \mathrm{Av}(\mu_x)_{x \in (B \setminus \cup_{i=1}^t \tilde{B}_i)} & \text{on } B \setminus \cup_{i=1}^t \tilde{B}_i \end{cases}.$$

This gives us $\mathrm{Av}(\tilde{\mu}_x)_{x\in B} = \nu$.

Therefore $\rho\big(\mu(x), \tilde{\mu}(x)\big) \le 2\varepsilon$ holds for all $x \in \tilde{\Omega}_\varepsilon$ where $\mathrm{meas}(B \setminus \Omega_\varepsilon) \le \varepsilon\, \mathrm{meas}\, B$ and by Lemma 3.2.10, we have $\rho\Big(\mathrm{Av}(\tilde{\mu}_x)_{x\in B}, \mathrm{Av}(\mu_x)_{x\in B}\Big) \le (2m+1)2\varepsilon$. Since ε is arbitrary and since $\mathrm{Av}(\tilde{\mu}_x)_{x\in B} = \nu$, we finally have $\mathrm{Av}(\mu_x)_{x\in B} = \nu$.

We prove the second part of the theorem by contradiction. We first note that μ_x is non-negative for a.e. $x \in \Omega$ and $\|\mu_x\|_{\mathcal{M}} \le 1$. Therefore we have $m = 1$. We need to show that $\|\mu_x\|_{\mathcal{M}} = 1$ for a.e. $x \in \Omega$. For $k \in \mathbb{N}$, let $\Omega_k \subset \Omega$ be an increasing sequence compact set such that $\mathrm{meas}(\Omega \setminus \Omega_k) \to 0$ as $k \to \infty$ and the restirctions of $\mu\colon \Omega \to (K_1, \rho)$ to Ω_k are continuous. Let $i \in \mathbb{N}$ and

$$\Omega_{i,k} = \left\{ x \in \Omega_k \colon \|\mu_x\|_{\mathcal{M}} \le 1 - \frac{1}{i} \right\}.$$

It is clear that $\Omega_{i,k} \subset \Omega_k$ is closed.

We now suppose that $\mathrm{meas}(\Omega_{i,k}) > 0$. Consider $\mu^j = \mathrm{Av}(\mu_x^j)_{x\in\Omega_{i,k}}$ and $\mu = \mathrm{Av}(\mu_x)_{x\in\Omega_{i,k}}$. It follows that

$$\begin{cases} \mu^j \overset{*}{\rightharpoonup} \mu, \\ \int_{\mathbb{R}^d} g(v) \mathrm{d}\mu^j \le m, \\ \|\mu\|_{\mathcal{M}} \le 1 - \frac{1}{i}. \end{cases}$$

In particular, for all $r > 0$,

$$\begin{aligned} c \ge \int_{\mathbb{R}^d} g(v)\,\mathrm{d}\mu^j \ge \int_{\mathbb{R}^d \setminus B(0,r)} g(v)\,\mathrm{d}\mu^j \ge \int_{\mathbb{R}^d \setminus B(0,r)} \inf g(v)\,\mathrm{d}\mu^j \\ = \mu^j\big(\mathbb{R}^d \setminus B(0,r)\big) \inf\{g(v) \colon |v| \ge r\} \end{aligned}$$

holds. Thus, for any sufficiently large r, we have

$$\mu^j\big(B(0,r)\big) \ge 1 - \frac{1}{2^i}$$

for all $j \in \mathbb{N}$. If $\Phi \colon \mathbb{R}^d \to [0,1]$ is continuous and

$$\Phi(v) = \begin{cases} 1 & \text{if } |v| \le r \\ 0 & \text{if } |v| \ge 2r \end{cases},$$

then

$$\int_{\mathbb{R}^d} \Phi(v)\,\mathrm{d}\mu^j \ge 1 - \frac{1}{2i} > 1 - \frac{1}{i} \ge \int_{\mathbb{R}^d} \Phi(v)\,\mathrm{d}\mu$$

for all $j \in \mathbb{N}$ which leads to a contradiction that $\mathrm{meas}(\Omega_{i,k}) = 0$ since $\mu^j \overset{*}{\rightharpoonup} \mu$. Hence

$$\mathrm{meas}\Big(\bigcup_{i,k} \Omega_{i,k}\Big) = 0$$

which gives us $\|\mu_x\|_{\mathcal{M}} = 1$ for a.e. $x \in \Omega$. □

Proof of Theorem 3.2.12. Let Ω_k be a sequence of compact sets such that $\mathrm{meas}(\Omega \setminus \Omega_k) \to 0$ as $k \to \infty$ and the restrictions of F to $\Omega_k \times \mathbb{R}^d$ are continuous. Assume first that $F > -\infty$. Consider a sequence of functions $\Phi_i \colon \mathbb{R}^d \to [0,1]$ such that

$$\Phi_i(v) = \begin{cases} 1 & \text{if } v \in B(0,i) \\ 0 & \text{if } v \in \mathbb{R}^d \setminus B(0,2i) \end{cases}.$$

It is clear that $\Phi_i(v)$ is non-decreasing in i for any fixed $v \in \mathbb{R}^d$. For fixed $i,k \in \mathbb{N}$, let $\omega \colon [0,\infty] \to [0,\infty]$ be a modulus of continuity of F restricted to $\Omega_k \times [-2i,2i]^d$:

$$\omega(s) := \sup\{|F(x,v) - F(y,v)| \colon x,y \in \tilde{\Omega}_k \subset \Omega_k, \|x-y\| < s\}$$

where $s > 0$. Decompose Ω_k on the sets $K_1, \ldots, K_m$ where $\operatorname{diam}(K_p) \leq \delta, p = 1, \ldots, m$. Let $x_p, y_p \in K_p$. Then for any fixed p, for $v \in \mathbb{R}^d$ and since $(\mu_x^j)_{x\in\Omega}$ generates the family $(\mu_x)_{x\in\Omega} \subset \mathcal{M}_1^+(\mathbb{R}^d)$, we obtain

$$|\Phi_i(v)||F(x_p, v) - F(y_p, v)| \leq 1 \cdot \omega(|x_p - y_p|) \leq \omega(\delta)$$

and

$$\int_{\mathbb{R}^d} \Phi_i(v) F(x_p, v) \,\mathrm{d}\mu_{(\cdot)}^j \overset{*}{\rightharpoonup} \int_{\mathbb{R}^d} \Phi_i(v) F(x_p, v) \,\mathrm{d}\mu_{(\cdot)} \text{ in } L^\infty \text{ as } j \to \infty.$$

Letting $\delta \to 0$ gives us

$$\int_{\mathbb{R}^d} \Phi_i(v) F(\cdot, v) \,\mathrm{d}\mu_{(\cdot)}^j \overset{*}{\rightharpoonup} \int_{\mathbb{R}^d} \Phi_i(v) F(\cdot, v) \,\mathrm{d}\mu_{(\cdot)} \text{ in } L^\infty(\Omega_k) \text{ as } j \to \infty.$$

The condition $F > -\infty$ implies that

$$\lim_{i\to\infty} \int_{\Omega_k} \left(\int_{\mathbb{R}^d} \Phi_i(v) F(x, v) \,\mathrm{d}\mu_x^j \right) \mathrm{d}x = \int_{\Omega_k} \left(\int_{\mathbb{R}^d} F(x, v) \,\mathrm{d}\mu_x^j \right) \mathrm{d}x$$

and

$$\lim_{i\to\infty} \int_{\Omega_k} \left(\int_{\mathbb{R}^d} \Phi_i(v) F(x, v) \,\mathrm{d}\mu_x \right) \mathrm{d}x = \int_{\Omega_k} \left(\int_{\mathbb{R}^d} F(x, v) \,\mathrm{d}\mu_x \right) \mathrm{d}x.$$

The fact that $(\mu_x^j)_{x\in\Omega}$ generates the family $(\mu_x)_{x\in\Omega} \subset \mathcal{M}_1^+(\mathbb{R}^d)$ and applying Fatou's lemma leads us to

$$\liminf_{j\to\infty} \int_{\Omega_k} \left(\int_{\mathbb{R}^d} F(x, v) \,\mathrm{d}\mu_x^j \right) \mathrm{d}x \geq \int_{\Omega_k} \left(\int_{\mathbb{R}^d} F(x, v) \,\mathrm{d}\mu_x \right) \mathrm{d}x,$$

which gives us the first result of the theorem. Similarly, if $(\mu_x^j)_{x\in\Omega}$ satisfies the tightness condition with F^+, then we obtain equality (3.2.5) and hence (3.2.6). Since $\operatorname{meas}(\Omega \setminus \Omega_k) \to 0$ as $k \to \infty$, we obtain the required result. The converse is clear from Dunford-Pettis theorem.

We will now prove the theorem for general integrands. Consider the auxiliary integrands $F^n = \max\{F, -n\}$. By construction, we have

$$\liminf_{j\to\infty} \int_{\Omega} \left(\int_{\mathbb{R}^d} F^n(x, v) \,\mathrm{d}\mu_x^j \right) \mathrm{d}x \geq \int_{\Omega} \left(\int_{\mathbb{R}^d} F^n(x, v) \,\mathrm{d}\mu_x \right) \mathrm{d}x.$$

Since $(\mu_x^j)_{x\in\Omega}$ satisfies (3.2.3) with F^- and by Fatou's lemma, we have (3.2.4). The rest of the theorem follows by similar argument. $\square$

Proof of Proposition 3.2.13.

(i) Assume that $z_j \to z_0$ in measure. Then $\Phi(z_j) \to \Phi(z_0)$ in $L^1(\Omega)$ for each $\Phi \in C_0(\mathbb{R}^d)$. This implies that

$$\Phi(z_j) \overset{*}{\rightharpoonup} \langle \Phi, \delta_{z_0(\cdot)} \rangle = \Phi(z_0) \text{ in } L^\infty$$

for all $\Phi \in C_0(\mathbb{R}^d)$ and hence z_j generates the family $\delta_{z_0(\cdot)}$.

Next, assume that μ_x is a Dirac mass for a.e. $x \in \Omega$. Given $\varepsilon > 0$, there exists a bounded continuous integrand $F\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ and a set $\Omega_\varepsilon \subset \Omega$ such that $0 \leq F \leq 1$ everywhere and $\operatorname{meas}(\Omega \setminus \Omega_\varepsilon) < \varepsilon$. For each $x \in \Omega_\varepsilon$, we have

$$F(x,v) = \begin{cases} 1 & \text{if } v \in B\big(z_0(x), \frac{\varepsilon}{2}\big) \\ 0 & \text{if } v \notin B\big(z_0(x), \varepsilon\big) \end{cases} .$$

By Theorem 3.2.12, we obtain

$$F\big(\cdot, z_j(\cdot)\big) \overset{*}{\rightharpoonup} F\big(\cdot, z_0(\cdot)\big) \text{ in } L^\infty(\Omega).$$

Moreover, we have $\int_\Omega F\big(x, z_0(x)\big)\,\mathrm{d}x \geq \operatorname{meas}\Omega_\varepsilon$ which then leads to

$$\lim_{j\to\infty} \operatorname{meas}\big(\{x \in \Omega\colon z_j(x) \in B(z_0, \varepsilon\}\big) \geq \operatorname{meas}\Omega_\varepsilon \geq \operatorname{meas}\Omega_\varepsilon - \varepsilon$$

and hence $z_j \to z_0$ in measure.

(ii) Since $z_j - y_j \to 0$ in measure as $j \to \infty$, then for each $\Phi \in C_0(\mathbb{R}^d)$,

$$\Phi(z_j) - \Phi(y_j) \to 0 \text{ in } L^1(\Omega)$$

and the consequence is clear.

□

Proof of Proposition 3.2.16.

(i) By the decomposition lemma, there exists a subsequence u_j (not relabeled) and a sequence $w_j \in W^{1,p}(\Omega;\mathbb{R}^N)$ such that $|D(w_j - u_j)| \to 0$ in measure and $|Dw_j|^p$ is equiintegrable. By Proposition 3.2.13, it is clear that $Dw_j \xrightarrow{Y} (\mu_x)_{x\in\Omega}$. Without loss of generality, we assume that $w_j \rightharpoonup u_0$ in $W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$. Let $\Omega_k \Subset \Omega$ be an increasing sequence of sets with smooth boundary such that $\operatorname{meas}(\Omega \setminus \Omega_k) \to 0$ as $k \to \infty$. Let $\varphi_k \in C_0^\infty(\Omega_{k+1};\mathbb{R}^N)$ be a sequence such that

$$\begin{cases} 0 \leq \varphi_k \leq 1 & \text{on } \Omega_{k+1}, \\ \varphi = 1 & \text{on } \Omega_k. \end{cases}$$

We further consider a sequence

$$v_k = u_0 + (w_{j(k)} - u_0)\varphi_k. \tag{B.2.1}$$

Since $w_j - u_0 \to 0$ in $L^p_{\text{loc}}(\Omega)$ as $j \to \infty$, we obtain

$$\|v_k - u_0\|_{L^p(\Omega)} = \|(w_{j(k)} - u_0)\phi_k\|_{L^p(\Omega)} \leq \|(w_{j(k)} - u_0)\|_{L^p(\Omega_{k+1})} \to 0$$

as $j(k) \to \infty$. From (B.2.1), we have

$$Dv_k = Du_0 + (Dw_{j(k)} - Du_0)\varphi_k + (w_{j(k)} - u_0) \otimes D\varphi_k$$

and therefore

$$\begin{aligned} \|Dv_k - Dw_{j(k)}\|_{L^p(\Omega)} &\leq \|Du_0\|_{L^p(\Omega \setminus \Omega_k)} + \|Dw_{j(k)}\|_{L^p(\Omega\setminus\Omega_k)} \\ &\quad + \|w_{j(k)} - u_0\|_{L^p(\Omega_{k+1}\setminus\Omega_k)}\|D\varphi_k\|_C. \end{aligned}$$

The fact that $\operatorname{meas}(\Omega \setminus \Omega_k) \to 0$ as $k \to \infty$ gives us

$$\|Du_0\|_{L^p(\Omega\setminus\Omega_k)} \to 0$$

and

$$\|Dw_{j(k)}\|_{L^p(\Omega\setminus\Omega_k)} \to 0$$

as $j(k) \to \infty$. Lastly, $\|w_{j(k)} - u_0\|_{L^p(\Omega_{k+1}\setminus\Omega_k)} \to 0$ for the same reason as before. It is now clear that for this choice of $j(k)$,

$$\begin{cases} v_k \rightharpoonup u_0 \text{ in } W_0^{1,p}(\Omega;\mathbb{R}^N), \\ |Dv_k|^p \text{ is equiintegrable,} \\ Dv_k - Dw_{j(k)} \to 0 \text{ in measure.} \end{cases}$$

Once again, Proposition 3.2.13 implies that $Dv_k \xrightarrow{Y} (\mu_x)_{x\in\Omega}$. Finally, taking the mollifier η_ε with $\varepsilon > 0$ sufficiently small, we have $v_k \in u_0 + C_0^\infty(\Omega;\mathbb{R}^N)$.

(ii) This can be proved by the same arguments by taking $w_j = u_j$.

□

Proof of Proposition 3.2.17.

(i) Let u_j be the underlying deformation for $(\mu_x^j)_{x\in\Omega}$. Fix $j \in \mathbb{N}$. Then by Proposition 3.2.16, there exists a sequence $u_k^j \in u_j + C_0^\infty(\Omega;\mathbb{R}^N)$ such that for each fixed $j \in \mathbb{N}$

$$\begin{cases} u_k^j \rightharpoonup u_j \text{ in } W_0^{1,p}(\Omega;\mathbb{R}^N) \text{ as } k \to \infty, \\ (\delta_{Du_k^j(x)})_{x\in\Omega} \overset{*}{\rightharpoonup} (\mu_x^j)_{x\in\Omega} \text{ in } L^\infty_{w^*}(\Omega;K_1) \text{ as } k \to \infty, \\ \text{the functions } |Du_k^j|^p \text{ are equiintegrable.} \end{cases}$$

By Theorem 3.2.12, we have

$$\lim_{k\to\infty}\int_\Omega(|Du_k^j(x)|^p+1)\,\mathrm{d}x=\int_\Omega\int_{\mathbb{R}^{N\times n}}(|v|^p+1)\,\mathrm{d}\mu_x^j\,\mathrm{d}x$$

and

$$\lim_{j\to\infty}\int_\Omega\int_{\mathbb{R}^{N\times n}}(|v|^p+1)\,\mathrm{d}\mu_x^j\,\mathrm{d}x=\int_\Omega\int_{\mathbb{R}^{N\times n}}(|v|^p+1)\,\mathrm{d}\mu_x\,\mathrm{d}x. \tag{B.2.2}$$

By diagonalization argument, we may isolate a sequence $u_{k(j)}^j$ bounded in $W^{1,p}(\Omega;\mathbb{R}^N)$: For each fixed $j\in\mathbb{N}$ and given $\varepsilon>0$, there exists $k(j)$ such that

$$\left|\int_\Omega(|Du_{k(j)}^j(x)|^p+1)\,\mathrm{d}x-\int_\Omega\int_{\mathbb{R}^{N\times n}}(|v|^p+1)\,\mathrm{d}\mu_x^j\,\mathrm{d}x\right|<\varepsilon$$

and hence

$$\begin{aligned}&\left|\int_\Omega(|Du_{k(j)}^j(x)|^p+1)\,\mathrm{d}x\right|\\ \leq\ &\left|\int_\Omega(|Du_{k(j)}^j(x)|^p+1)\,\mathrm{d}x-\int_\Omega\int_{\mathbb{R}^{N\times n}}(|v|^p+1)\,\mathrm{d}\mu_x^j\,\mathrm{d}x\right|\\ &+\left|\int_\Omega\int_{\mathbb{R}^{N\times n}}(|v|^p+1)\,\mathrm{d}\mu_x^j\,\mathrm{d}x-\int_\Omega\int_{\mathbb{R}^{N\times n}}(|v|^p+1)\,\mathrm{d}\mu_x\,\mathrm{d}x\right|\\ \leq\ &\varepsilon+\tilde\varepsilon\end{aligned}$$

as $j\to\infty$ where the bound $\tilde\varepsilon$ can be identified through (B.2.2). We therefore obtain

$$(\delta_{Du_{k(j)}^j(x)})_{x\in\Omega}\overset{*}{\rightharpoonup}(\mu_x)_{x\in\Omega}\text{ in }L_{w^*}^\infty(\Omega;K_1)\text{ as }j\to\infty$$

and

$$\lim_{j\to\infty}\int_\Omega(|Du_{k(j)}^j(x)|^p+1)\,\mathrm{d}x-\int_\Omega\int_{\mathbb{R}^{N\times n}}(|v|^p+1)\,\mathrm{d}\mu_x^j\,\mathrm{d}x=0.$$

It follows from Proposition 3.2.16 that $(\mu_x)_{x\in\Omega}$ is a gradient p-Young measure.

(ii) By Theorem 3.2.12, the tightness condition implies that

$$\int_{\mathbb{R}^{N\times n}}(|v|^p+1)\,\mathrm{d}\mu_{(\cdot)}^j\rightharpoonup\int_{\mathbb{R}^{N\times n}}(|v|^p+1)\,\mathrm{d}\mu_{(\cdot)}\text{ in }L^1(\Omega)\text{ as }j\to\infty.$$

It follows from the first assertion that

$$\lim_{j\to\infty}\int_\Omega(|Du_{k(j)}^j(x)|^p+1)\,\mathrm{d}x=\int_\Omega\int_{\mathbb{R}^{N\times n}}(|v|^p+1)\,\mathrm{d}\mu_x\,\mathrm{d}x$$

and hence the functions $|Du_{k(j)}^j(x)|^p$ are equiintegrable. As a consequence of Proposition 3.2.16, we conclude that $(\mu_x)_{x\in\Omega}$ a gradient p-Young measure.

□

Proof of Theorem 3.2.18.

(i) Without loss of generality, assume that $0 \in \Omega$. Consider $(\mu_x)_{x\in\Omega} = (\delta_{Du_0(x)})_{x\in\Omega}$. For each $i \in \mathbb{N}$, we consider a cover of Ω by disjoint sets of the form

$$\Omega_j^i = (a_j^i + \varepsilon_j^i \Omega) \cup N_i, \quad j \in \mathbb{N}$$

where $\operatorname{diam}(\Omega_j^i) \leq \frac{1}{i}$ and meas $N_i = 0$. Suppose that either

$$\Omega_{j'}^{i'} \subset \Omega_j^i \quad \text{or} \quad \Omega_{j'}^{i'} \cap \Omega_j^i = \emptyset$$

for each $i' \geq i$ and $j' \in \mathbb{N}$. Then we have

$$\Omega = \bigcap_i \left(\bigcup_j (a_j^i + \varepsilon_j^i \Omega) \cup N_i \right).$$

If we let $\tilde{\Omega} = \bigcap_i (\bigcup_j \Omega_j^i)$, we obtain that $\operatorname{meas}(\Omega \setminus \tilde{\Omega}) = 0$. We first define that

$$v^i(x) = \begin{cases} \varepsilon_j^i u_0\left(\frac{x - a_j^i}{\varepsilon_j^i}\right) & \text{if } x \in \Omega_j^i, i, j \in \mathbb{N} \\ l_A(x) & \text{elsewhere} \end{cases}$$

and it is clear that

$$\operatorname{Av}(\delta_{Dv^i(x)})_{x\in\Omega_j^i} = \operatorname{Av}(\delta_{Du_0(x)})_{x\in\Omega}$$

for each $j \in \mathbb{N}$. Our aim is to prove that each subsequence $(\delta_{Dv^k(x)})_{x\in\Omega}$ of the original sequence $(\delta_{Dv^i(x)})_{x\in\Omega}$ satisfies the following:

$$(\delta_{Dv^k(x)})_{x\in\Omega} \overset{*}{\rightharpoonup} \operatorname{Av}(\delta_{Du_0(x)})_{x\in\Omega} \text{ in } L^\infty_{w*}(\Omega; K_1) \text{ for } k \in \mathbb{N}.$$

Let $(\delta_{Dv^k(x)})_{x\in\Omega}$ be a subsequence (not relabeled) such that

$$(\delta_{Dv^k(x)})_{x\in\Omega} \xrightarrow{Y} (\mu_x)_{x\in\Omega}. \tag{B.2.3}$$

Then for each $x_0 \in \tilde{\Omega}$, there exists a subsequence $\Omega_{j(i)}^i$ such that $x_0 \in \Omega_{j(i)}^i$ for each $i \in \mathbb{N}$. By Theorem 3.2.3 and since $\rho(\operatorname{Av}(\mu_x)_{x\in\Omega_{j(i)}^i}, \mu_{x_0}) \to 0$ in measure, we obtain from Lemma 3.2.10 that

$$\operatorname{Av}(\mu_x)_{x\in\Omega_{j(i)}^i} \overset{*}{\rightharpoonup} \mu_{x_0} \text{ in } C_0(\mathbb{R}^{N\times n})^*$$

as $i \to \infty$ for a.e. $x_0 \in \tilde{\Omega}$. By our assumption (B.2.3), it is easy to see that as $k \to \infty$

$$\operatorname{Av}(\delta_{Dv^k(x)})_{x\in\Omega_{j(i)}^i} \overset{*}{\rightharpoonup} \operatorname{Av}(\mu_x)_{x\in\Omega_{j(i)}^i} \text{ in } C_0(\mathbb{R}^{N\times n})^*$$

and for any $k \geq i$,

$$\mathrm{Av}(\delta_{Dv^k(x)})_{x\in\Omega^i_{j(i)}} = \mathrm{Av}(\delta_{Du_0(x)})_{x\in\Omega}$$

Hence, it follows that

$$\mathrm{Av}(\mu_x)_{x\in\Omega^i_{j(i)}} = \mathrm{Av}(\delta_{Du_0(x)})_{x\in\Omega}$$

for all $i \in \mathbb{N}$ and subsequently

$$\mu_{x_0} = \mathrm{Av}(\delta_{Du_0(x)})_{x\in\Omega}$$

for a.e. $x_0 \in \tilde{\Omega}$. It is now clear that each subsequence $(\delta_{Dv^k(x)})_{x\in\Omega^i_j}$ of the original sequence $(\delta_{Dv^i(x)})_{x\in\Omega}$ has a subsequence $(\delta_{Dv^k(x)})_{x\in\Omega^i_{j(i)}}$ converging weakly* in $L^\infty_{w^*}(\Omega; K_1)$ to the homogeneous Young measure $\mathrm{Av}(\delta_{Du_0(x)})_{x\in\Omega}$ and therefore so does the original sequence. The equiintegrability of $|Dv^i|^p$ follows from Lemma 3.2.4. Hence we obtain

$$\mathrm{Av}(\delta_{Du_0(x)})_{x\in\Omega} \in GM_p(A).$$

On the other hand, if $u_0 \in l_A + W^{1,\infty}_0(\Omega;\mathbb{R}^N)$, then $v^i - l_A \overset{*}{\rightharpoonup} 0$ in $W^{1,\infty}_0(\Omega;\mathbb{R}^N)$, which leads to $\mathrm{Av}(\delta_{Du_0(x)})_{x\in\Omega} \in GM_\infty(A)$.

Let $(\mu_x)_{x\in\Omega}$ be a gradient p-Young measure with the underlying deformation $u_0 \in l_A + W^{1,p}_0(\Omega;\mathbb{R}^N)$. Then by Proposition 3.2.16, there exists a sequence $u_i \in l_A + C^\infty_0(\Omega;\mathbb{R}^N)$ such that

$$u_i \xrightarrow{GY^p} (\mu_x)_{x\in\Omega}.$$

Note that the functions $|Du_i|^p$ are equiintegrable. Since

$$\mathrm{Av}(\delta_{Du_i(x)})_{x\in\Omega} \overset{*}{\rightharpoonup} \mathrm{Av}(\mu_x)_{x\in\Omega}$$

and the modulus of p-equiintegrability of gradients of sequences generating $\mathrm{Av}(\delta_{Du_i(x)})_{x\in\Omega}$ does not exceed modulus of equiintegrability of $|Du_i|^p$, we have that the generated sequence satisfies the requirements of Proposition 3.2.17 and hence $\mathrm{Av}(\mu_x)_{x\in\Omega} \in GM_p(A)$.

(ii) Let $(\mu_x)_{x\in\Omega}$ be a gradient p-Young measure. Then there exists a sequence of compact sets $\Omega_k \subset \Omega$ such that $\mathrm{meas}(\Omega \setminus \Omega_k) \to 0$ as $k \to \infty$, the restrictions of $\mu\colon \Omega \to (K_1, \rho)$ to Ω_k are continuous and all points of Ω_k are Lebesgue for the map $x \longmapsto \langle |\cdot|^p + 1, \mu_x\rangle$.

Let Du_i be a sequence such that $u_i \xrightarrow{GY^p} (\mu_x)_{x\in\Omega}$. Let $x_0 \in \Omega_k$ be a Lebesgue point and let $B(x_0,\varepsilon) \subset \Omega$ for an $\varepsilon > 0$. For $j \in \mathbb{N}$, let

$$u^j_i(x_0 + y) = j\left(u_i\big(x_0 + \frac{y}{j}\big) - u_i(x_0)\right) \text{ on } B(x_0,\varepsilon) \subset \Omega, i \in \mathbb{N}.$$

If $|y| \le \varepsilon$, then we have

$$Du_i^j(x_0 + y) = Du_i\left(x_0 + \frac{y}{j}\right)$$

and hence for $i \in \mathbb{N}$,

$$u_i^j \xrightarrow{GY^p} (\mu_x^j)_{x \in B(x_0,\varepsilon)}$$

with

$$\mu^j_{x_0+y} = \mu_{x_0+\frac{y}{j}} \text{ for } y \in B(x_0,\varepsilon).$$

Since the restrictions of $\mu\colon \Omega \to (K_1,\rho)$ to Ω_k are continuous and by using Lemma 3.2.10, we obtain that

$$(\mu_x^j)_{x\in B(x_0,\varepsilon)} \overset{*}{\rightharpoonup} \mu_{x_0} \text{ in } L^\infty_{w*}(B(x_0,\varepsilon);K_1),$$

where μ_{x_0} is a homogeneous gradient p-Young measure. Recall that x_0 is a Lebesgue point of the map $x \longmapsto \langle |v|^p + 1, \mu_x\rangle$ and thus

$$\lim_{j\to\infty} \int_{B(x_0,\varepsilon)} \langle |v|^p + 1, \mu_x^j\rangle \,\mathrm{d}x = \int_{B(x_0,\varepsilon)} \langle |v|^p + 1, \mu_{x_0}\rangle \,\mathrm{d}x.$$

By Theorem 3.2.12, we see that $(\mu_x^j)_{x\in\Omega}$ satisfies the tightness condition with the integrand $|\cdot|^p + 1$ and it follows from Proposition 3.2.17 that μ_{x_0} is a homogeneous gradient p-Young measure.

□

Proof of Corollary 3.2.19.

(i) It is clear that $\inf\limits_{\mu\in GM_p(A)} \langle F,\mu\rangle \le \inf\limits_{\mu\in GM_\infty(A)} \langle F,\mu\rangle$. Let $\varphi \in l_A + C_0^\infty(\Omega;\mathbb{R}^N)$. Then the averaging principle implies that $\mathrm{Av}(\delta_{D\varphi(x)})_{x\in\Omega} \in GM_\infty(A)$ and hence

$$\inf_{\mu\in GM_\infty(A)} \langle F,\mu\rangle \le \inf_{\varphi\in C_0^\infty(\Omega;\mathbb{R}^N)} \fint_\Omega F\big(A + D\varphi(x)\big)\,\mathrm{d}x.$$

To prove the converse, note that by Proposition 3.2.16, there exists a sequence $u_j \in l_A + C_0^\infty(\Omega;\mathbb{R}^N)$ such that $u_j \xrightarrow{GY^p} \mu$, $\mu \in GM_p(A)$. By Theorem 3.2.12,

$$\lim_{j\to\infty} \int_\Omega F\big(Du_j(x)\big)\,\mathrm{d}x = \langle F,\mu\rangle \operatorname{meas}\Omega.$$

It follows that

$$\inf_{\varphi\in C_0^\infty(\Omega;\mathbb{R}^N)} \fint_\Omega F\big(A + D\varphi(x)\big)\,\mathrm{d}x \le \inf_{\mu\in GM_p(A)} \langle F,\mu\rangle.$$

(ii) Since F is quasiconvex at A, then for any $j \in \mathbb{N}$ we have

$$\int_\Omega F\big(Du_j(x)\big)\,\mathrm{d}x \geq F(A)\,\mathrm{meas}\,\Omega.$$

From the first assertion, we obtain $\langle F, \mu\rangle \geq F(A)$.

Conversely, if $\varphi \in l_A + C_0^\infty(\Omega;\mathbb{R}^N)$ and since $\mathrm{Av}(\delta_{D\varphi(x)})_{x\in\Omega} \in GM_\infty(A)$, then

$$\langle F, \mathrm{Av}(\delta_{D\varphi(x)})_{x\in\Omega}\rangle \geq F(A)$$

which therefore implies

$$\fint_\Omega F\big(A + D\varphi(x)\big)\,\mathrm{d}x \geq F(A),$$

the quasiconvexity of F at A.

$\square$

Proof of Theorem 3.2.20. We first consider the case $p > 1$.

Let

$$I(\mu) := \int_{\mathbb{R}^{N\times n}} F(v)\,\mathrm{d}\mu.$$

Since F is bounded, by Theorem 3.2.12 and Proposition 3.2.17, the infimum of $I(\mu)$ over $GM_p(A)$ exists. Let

$$V(A) := \{\mu \in GM_p(A)\colon\ \min I(\mu)\}.$$

If $A_j \to A$ as $j \to \infty$ and $\mu^j \in GM_p(A_k)$, then for a subsequence μ^k we have

$$\lim_{k\to\infty} I(\mu^k) = \liminf_{j\to\infty} I(\mu^j) \quad \text{and} \quad \mu^k \overset{*}{\rightharpoonup} \mu.$$

By Proposition 3.2.17, μ is a gradient p-Young measure with the center of mass at A. It follows from Theorem 3.2.12 that $\lim\limits_{k\to\infty} I(\mu^k) \geq I(\mu)$ which implies

$$\liminf_{j\to\infty} F^{qc}(A_j) \geq F^{qc}(A)$$

and thus the lower semicontinuity of F^{qc} at A. On the other hand, note that if $\mu \in V(A)$, then the measures $\mu \diamond A_k$ are gradient p-Young measure centered at A_j, respectively, and

$$\mu^j \overset{*}{\rightharpoonup} \mu \quad \text{and} \quad \limsup_{j\to\infty} F^{qc}(A) \leq \lim_{j\to\infty} I(\mu^j) = I(\mu),$$

which proves the upper semicontinuity of F^{qc} at A. Therefore F^{qc} is continuous.

Next, we prove the quasiconvexity of F^{qc} by proving the inequality

$$\fint_{\Omega} F^{qc}\big(A + D\varphi(x)\big)\,\mathrm{d}x \geq F^{qc}(A)$$

for piecewise affine functions $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^N)$. For a fixed φ and for $k = 1,\ldots,j$, let Ω_k be a finite collection of subdomains of Ω on which $A + D\varphi$ has constant values $A_1,\ldots,A_j$, respectively. Let $\mu^k \in V(A_k), k = 1,\ldots,j$. Then by Proposition 3.2.16, there exists functions $u_k \in C_0^\infty(\Omega_k;\mathbb{R}^N)$ such that

$$\int_{\Omega_k} F^{qc}(A_k)\,\mathrm{d}x = \int_{\Omega_k}\int_{\mathbb{R}^{N\times n}} F(v)\,\mathrm{d}\mu^k\,\mathrm{d}x \geq \int_{\Omega_k} F\big(A_k + Du_k(x)\big)\,\mathrm{d}x - \frac{\varepsilon}{j}.$$

Define

$$\tilde{u}(x) = \begin{cases} \varphi(x) + u_k(x) & \text{if } x \in \Omega_k, k = 1,\ldots,j \\ \varphi(x) & \text{elsewhere} \end{cases}.$$

Then,

$$\begin{aligned}
&\int_{\Omega} F^{qc}\big(A + D\varphi(x)\big)\,\mathrm{d}x \\
=\;& \int_{\Omega\setminus\bigcup_{k=1}^{j}\Omega_k} F^{qc}\big(A + D\varphi(x)\big)\,\mathrm{d}x + \sum_{k=1}^{j}\int_{\Omega_k} F^{qc}(A_k)\,\mathrm{d}x \\
\geq\;& \int_{\Omega\setminus\bigcup_{k=1}^{j}\Omega_k} F^{qc}\big(A + D\varphi(x)\big)\,\mathrm{d}x + \sum_{k=1}^{j}\int_{\Omega_k} F\big(A_k + Du_k(x)\big)\,\mathrm{d}x - \varepsilon \\
\geq\;& \int_{\Omega\setminus\bigcup_{k=1}^{j}\Omega_k} \Big[F^{qc}\big(A + D\varphi(x)\big) - F\big(A + D\varphi(x)\big)\Big]\,\mathrm{d}x \\
&+ \int_{\Omega} F\big(A + D\tilde{u}(x)\big)\,\mathrm{d}x - \varepsilon.
\end{aligned}$$

By the averaging principle, we have $\mathrm{Av}(\delta_{D\tilde{u}(x)})_{x\in\Omega} \in GM_p(A)$ which implies that

$$\int_{\Omega} F\big(A + D\tilde{u}(x)\big)\,\mathrm{d}x = \langle F, \mathrm{Av}(\delta_{D\tilde{u}(x)})_{x\in\Omega}\rangle\,\mathrm{meas}\,\Omega.$$

Moreover,

$$\int_{\Omega\setminus\bigcup_{k=1}^{j}\Omega_k} \Big[F^{qc}\big(A + D\varphi(x)\big) - F\big(A + D\varphi(x)\big)\Big]\,\mathrm{d}x \to 0 \text{ as } j \to \infty$$

and since $\varepsilon > 0$ may be chosen arbitrarily small, we obtain

$$\int_{\Omega} F^{qc}\big(A + D\tilde{u}(x)\big)\,\mathrm{d}x \geq F^{qc}(A)\,\mathrm{meas}\,\Omega,$$

the quasiconvexity of F^{qc}. Recall the definition of quasiconvex envelope:

$$F^{qc}(A) = \sup\{G(A)\colon G \leq F \text{ and } G \text{ quasiconvex}\}.$$

If G is a quasiconvex function such that $G \leq F$, then for any $\mu \in V(A)$ and by Theorem 3.2.18, we obtain

$$F^{qc}(A) = \langle F, \mu \rangle \geq \langle G, \mu \rangle \geq G(A).$$

It is now clear that $F^{qc}(v) \geq a_1|v|^p + b_1$ and the upper bound simply follows from $F^{qc}(v) \leq F(v) \leq a_2|v|^p + b_2$.

We will now move to the case when $p = 1$. First consider a family of auxiliary integrands, with $\lambda > 0$,

$$F_\lambda(\cdot) := F(\cdot) + \lambda|\cdot|^2$$

and their quasiconvex envelopes F_λ^{qc}. For any fixed $v \in \mathbb{R}^{N\times n}$, we have

$$\tilde{F}(v) = \lim_{\lambda\to 0} F_\lambda^{qc}(v).$$

By construction, F_λ^{qc} are continuous and bounded below by b_1. Therefore $\tilde{F}$ is upper semincontinuous and bounded below by b_1. For any $\varphi \in C_0^\infty(\Omega;\mathbb{R}^N)$ and $A \in \mathbb{R}^{N\times n}$, we also have

$$\fint_\Omega F_\lambda^{qc}\big(A + D\varphi(x)\big)\,\mathrm{d}x \geq F_\lambda^{qc}(A)$$

and the same holds for $\tilde{F}$ by monotone convergence theorem which proves the quasiconvexity of $\tilde{F}$.

Our next aim is to show that $\tilde{F}$ is a continuous quasiconvex function and subsequently $\tilde{F} = F^{qc}$. If $A_j \to A$, then there exists $\varphi_j \in l_A + C_0^\infty(\Omega;\mathbb{R}^N)$ such that $\mathrm{meas}(\{x \in \Omega\colon \varphi_j \neq A_j\}) \to 0$ and $\|\varphi_j\|_{W^{1,\infty}} \leq c$ where constant $c < \infty$. Then

$$\tilde{F}(A_j) - \fint_\Omega \tilde{F}\big(A + D\varphi_j(x)\big)\,\mathrm{d}x \to 0$$

and hence

$$\liminf_{j\to\infty} \tilde{F}(A_j) \geq \tilde{F}(A).$$

Therefore, $\tilde{F}$ is continuous and quasiconvex. It then follows from Dini convergence theorem that

$$F_\lambda^{qc} \to \tilde{F} \text{ uniformly as } \lambda \to 0^+$$

and so $\tilde{F} = F^{qc}$.

Our last aim is to prove that we have

$$\tilde{F}(A) = \inf_{\mu\in GM_1(A)} \langle F, \mu \rangle.$$

Recall from Corollary 3.2.19:

$$\begin{aligned}
\tilde{F}(A) = \lim_{\lambda\to 0^+} F_\lambda^{qc}(A) &= \lim_{\lambda\to 0^+} \inf_{\mu\in GM_2(A)} \langle F(\cdot) + \lambda|\cdot|^2, \mu\rangle \\
&= \lim_{\lambda\to 0^+} \inf_{\mu\in GM_\infty(A)} \langle F(\cdot) + \lambda|\cdot|^2, \mu\rangle \\
&= \inf_{\mu\in GM_\infty(A)} \langle F, \mu\rangle \\
&= \inf_{\mu\in GM_1(A)} \langle F, \mu\rangle
\end{aligned}$$

This completes the proof of the theorem. □

Proof of Theorem 3.2.21. Necessity of condition (i) follows from Theorem 3.2.12.

As for (ii), observe that since F is quasiconvex, then we have

$$F(Du_0) \leq \fint_\Omega F\big(Du_j(x)\big)\,\mathrm{d}x$$

where $u_j \in u_0 + C_0^\infty(\Omega;\mathbb{R}^N)$. Since $(\mu_x)_{x\in\Omega}$ is a gradient p-Young measure, it follows from Theorem 3.2.12 that

$$\lim_{j\to\infty} \int_\Omega F\big(Du_j(x)\big)\,\mathrm{d}x = \int_\Omega \langle F, \mu_x\rangle\,\mathrm{d}x$$

and hence $F\big(Du_0(x)\big) \leq \int_{\mathbb{R}^{N\times n}} F(v)\,\mathrm{d}\mu_x$.

To obtain (iii), we obtain by Theorem 3.2.12 that

$$\lim_{j\to\infty} \int_\Omega (|Du_j(x)|^p + 1)\,\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^{N\times n}} (|v|^p + 1)\,\mathrm{d}\mu_x\,\mathrm{d}x$$

and this is finite by equiintegrability of $|Du_j|^p$.

We will divide the proof for sufficiency into a few cases.

(a) Homogeneous case: We first show that $GM_p(A)$ is a convex set. Let $\mu^1, \mu^2 \in GM_p(A)$ and $t \in (0,1)$. Let $\Omega_1, \Omega_2 \subset \Omega$ be disjoint such that

$$\begin{cases} \operatorname{meas}(\partial\Omega_1) = \operatorname{meas}(\partial\Omega_1) = 0, \\ \operatorname{meas}\Omega_1 = t\operatorname{meas}\Omega, \\ \operatorname{meas}\Omega_2 = (1-t)\operatorname{meas}\Omega. \end{cases}$$

Then by Proposition 3.2.16, there exists $u_j^1 \in l_A + C_0^\infty(\Omega_1;\mathbb{R}^N)$ and $u_j^2 \in l_A + C_0^\infty(\Omega_2;\mathbb{R}^N)$ such that $u_j^1 \overset{GY^p}{\longrightarrow} \mu^1$ and $u_j^2 \overset{GY^p}{\longrightarrow} \mu^2$, respectively. Therefore,

$$\mu = \begin{cases} \mu^1 & \text{on } \Omega_1 \\ \mu^2 & \text{on } \Omega_2 \end{cases}$$

is a gradient p-Young measure. It follows from the averaging principle that

$$\mathrm{Av}(\mu) = t\mu^1 + (1-t)\mu^2 \in GM_p(A)$$

and hence the convexity of $GM_p(A)$.

To prove that $\mu \in GM_p(A)$, we will prove that there exists a sequence $\mu^k \in GM_p(A)$ such that

$$\rho(\mu^k, \mu) + |\langle \Phi_0, \mu^k \rangle - \langle \Phi_0, \mu \rangle| \to 0 \text{ as } k \to \infty \tag{B.2.4}$$

where $\Phi_0 = |\cdot|^p + 1$. If $\rho(\mu^k, \mu) \to 0$, then $\mu^k \overset{*}{\rightharpoonup} \mu$ and it follows from Proposition 3.2.17 that $\mu \in GM_p(A)$ provided $|\langle \Phi_0, \mu^k \rangle - \langle \Phi_0, \mu \rangle| \to 0$. We will prove (B.2.4) by contradiction. Suppose that (B.2.4) does not hold. Then, there exists $L \in \mathbb{N}$ such that for all $l \geq L$ and $\varepsilon > 0$,

$$\inf_{\nu \in GM_p(A)} \left\{ |\langle \Phi_0, \nu \rangle - \langle \Phi_0, \mu \rangle| + \sum_{i=1}^{l} \frac{1}{2^i \|\Phi_i\|_C} |\langle \Phi_i, \nu \rangle - \langle \Phi_i, \mu \rangle| \right\} > \varepsilon$$

where the sequence $\{\Phi_i\}$ is dense in $C_0(\Omega; \mathbb{R}^{N\times n})$. Then

$$\mathcal{C} = \left(\langle \Phi_0, \nu \rangle, \frac{1}{2\|\Phi_1\|_C} \langle \Phi_1, \nu \rangle, \cdots, \frac{1}{2^l \|\Phi_l\|_C} \langle \Phi_l, \nu \rangle \right) \subset \mathbb{R}^{l+1}$$

is convex in term of convexity of $GM_p(A)$ and

$$\left(\langle \Phi_0, \mu \rangle, \frac{1}{2\|\Phi_1\|_C} \langle \Phi_1, \mu \rangle, \cdots, \frac{1}{2^l \|\Phi_l\|_C} \langle \Phi_l, \mu \rangle \right) \notin \bar{\mathcal{C}}$$

where $\bar{\mathcal{C}}$ denotes the closure of the convex set. By the Hahn-Banach separation theorem, there exists a vector $c \in \mathbb{R}^{l+1}$ and $\delta > 0$ such that

$$\inf_{\nu \in GM_p(A)} \sum_{i=0}^{l} c_i \langle \Phi_i, \nu \rangle > \sum_{i=0}^{l} c_i \langle \Phi_i, \mu \rangle + \delta.$$

Letting $F = \sum_{i=0}^{l} c_i \Phi_i$ gives us

$$\inf_{\nu \in GM_p(A)} \langle F, \nu \rangle > \langle F, \mu \rangle + \delta.$$

Since integrand F satisfies the conditions of Theorem 3.2.20, we have

$$F^{qc}(A) = \inf_{\nu \in GM_p(A)} \langle F, \nu \rangle > \langle F, \mu \rangle + \delta \geq \langle F^{qc}, \mu \rangle + \delta$$

which contradicts (ii) of the theorem. Therefore $\mu \in GM_p(A)$.

(b) Let $\mu \in GM_p(A)$ and let $\tilde{\Omega} \subset \Omega$ be an open set such that $\mathrm{meas}(\partial\tilde{\Omega}) = 0$. Then

$$\tilde{\mu}_x = \begin{cases} \mu \diamond Du_0(x) & \text{if } x \in \tilde{\Omega} \\ \delta_{Du_0(x)} & \text{if } x \in (\Omega \setminus \tilde{\Omega}) \end{cases}.$$

If $\Omega_1, \ldots, \Omega_l$ are disjoint measurable open subsets of Ω with $\mathrm{meas}(\partial\Omega_i) = 0$ and $\mu^1 \in GM_p(A_1), \ldots, \mu^l \in GM_p(A_l)$ respectively, then it is clear that

$$\tilde{\mu}_x = \begin{cases} \mu^i \diamond Du_0(x) & \text{if } x \in \Omega_i, i = 1, \ldots, l \\ \delta_{Du_0(x)} & \text{if } x \in (\Omega \setminus \cup_{i=1}^{l}\Omega_i) \end{cases} \tag{B.2.5}$$

is a gradient p-Young measure.

On the other hand, if Ω_i are compact sets, then it follows from Proposition 3.2.5 that there exists an open set O_{η_i} with smooth boundary such that

$$\sup_{x \in O_{\eta_i}} \mathrm{dist}(x, \Omega_i) \leq \eta_i$$

and

$$\mathrm{meas}\big((\Omega_i \setminus O_{\eta_i}) \cup (O_{\eta_i} \setminus \Omega_i)\big) \to 0 \text{ as } \eta_i \to 0$$

for each $i = 1, \ldots, l$. If we let $u_0 \in l_{A_i} + W_0^{1,p}(\Omega; \mathbb{R}^N)$, then we can use Lemma 3.2.4 to find that (B.2.5) is a gradient p-Young measure.

As for the general case , we can approximate Ω_i with compact subsets and the claim follows from the arguments as before.

(c) Lastly, we consider the general case of non-homogeneous measure $(\mu_x)_{x\in\Omega}$. For each $k \in \mathbb{N}$, there exists a compact set $\Omega_k \subset \Omega$ such that $\mathrm{meas}(\Omega \setminus \Omega_k) \leq \frac{1}{k}$, the restrictions of $u_0, Du_0, \int_{\mathbb{R}^{N\times n}}(|v|^p + 1)$ and $\mu\colon \Omega \to (K_1, \rho)$ to Ω_k are continuous and μ_x satisfies (ii) of the theorem for each $x \in \Omega_k$. Our aim is to prove that the measure

$$(\mu_x^k)_{x\in\Omega} = \begin{cases} \mu_x & \text{if } x \in \Omega_k \\ \delta_{Du_0(x)} & \text{if } x \in \Omega \setminus \Omega_k \end{cases}$$

is a gradient p-Young measure.

For a fixed $k \in \mathbb{N}$, suppose that $D = [-a, a]^n$ such that $\Omega_k \subset D$ and let $D_j^1, j = 1, \ldots, 2^n$, be quadrants of D. For each $i > 1$, decompose $D_j^i, j = 1, \ldots, 2^{ni}$, in 2^n cubes of equal size in the similar way. Let $B^i_j = D^i_j \cap \Omega_k$ for $i \in \mathbb{N}$ and $j = 1, \dots, 2^{ni}$. For a fixed $i \in \mathbb{N}$, let $x_j \in B^i_j$ and

$$\mu^{i,k}_x = \begin{cases} \mu(x_j) \diamond Du_0(x) & \text{if } x \in B^i_j, j = 1, \dots, 2^{ni} \\ \delta_{Du_0(x)} & \text{elsewhere} \end{cases}.$$

Then by the second case, it is clear that $(\mu^{i,k}_x)_{x\in\Omega}$ is a gradient p-Young measure. Since the restriction of $\mu^k \colon \Omega \to (K_1, \rho)$ to Ω_k is continuous, it follows from Lemma 3.2.10 that as $i \to \infty$

$$\rho(\mu^{i,k}_{(\cdot)}, \mu^k_{(\cdot)}) \to 0 \text{ in measure} \quad \Rightarrow \quad (\mu^{i,k}_x)_{x\in\Omega} \overset{*}{\rightharpoonup} (\mu^k_x)_{x\in\Omega}.$$

Recall that the restrictions of $\int_{\mathbb{R}^{N\times n}} (|v|^p + 1)\, \mathrm{d}\mu_{(\cdot)}$ to Ω_k are continuous. Therefore, we have

$$\lim_{i\to\infty} \int_{\Omega_k} \int_{\mathbb{R}^{N\times n}} (|v|^p + 1)\, \mathrm{d}\mu^{i,k}_x\, \mathrm{d}x = \int_{\Omega_k} \int_{\mathbb{R}^{N\times n}} (|v|^p + 1)\, \mathrm{d}\mu^k_x\, \mathrm{d}x$$

and it follows from Theorem 3.2.12 that $(\mu^{i,k}_x)_{x\in\Omega}$ satisfies the tightness condition with $|v|^p + 1$. By the second assertion of Proposition 3.2.17, we find that $(\mu^k_x)_{x\in\Omega}$ is a gradient p-Young measure for each $k \in \mathbb{N}$ and $(\mu_x)_{x\in\Omega}$ is a gradient p-Young measure .

$\square$

Proof of Theorem 3.2.22.

(i) Without loss of generality, assume that $u_j \xrightarrow{GY^p} (\mu_x)_{x\in\Omega}$ and $(u_j, Du_j) \xrightarrow{Y} (\delta_{u_0(x)} \otimes \mu_x)_{x\in\Omega}$. It follows from Theorem 3.2.12 that

$$\liminf_{j\to\infty} \mathcal{F}(u_j, \Omega) \geq \int_\Omega \int_{\mathbb{R}^{N\times n}} F\big(x, u_0(x), \cdot\big)\, \mathrm{d}\mu_x\, \mathrm{d}x.$$

By Proposition 3.2.16, $(\mu_x)_{x\in\Omega}$ is a gradient p-Young measure and the localization principle asserts that $\mu_x \in GM_p\big(Du_0(x)\big)$ for a.e. $x \in \Omega$. Since F is quasiconvex at $Du_0(x)$, we have

$$\begin{aligned} \liminf_{j\to\infty} \mathcal{F}(u_j, \Omega) &\geq \int_\Omega \int_{\mathbb{R}^{N\times n}} F\big(x, u_0(x), \cdot\big)\, \mathrm{d}\mu_x\, \mathrm{d}x \\ &\geq \int_\Omega F\big(x, u_0(x), Du_0(x)\big)\, \mathrm{d}x = \mathcal{F}(u_0, \Omega), \end{aligned}$$

where the second inequality follows from Corollary 3.2.19.

(ii) We will prove by contradiction. Let $\Omega_j \subset \Omega$ be an increasing sequence of compact sets such that $\operatorname{meas}(\Omega \backslash \Omega_j) \leq \frac{1}{j}$, the restrictions of u_0 and Du_0 to Ω_j are continuous, and the restriction of F to $\Omega_j \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$ is continuous. Suppose that for a Lebesgue point $x_0 \in \Omega_j$, the function $F\big(x_0, u_0(x_0), \cdot\big)$ is not quasiconvex at $Du_0(x_0)$. By Corollary 3.2.19, there exists $\varepsilon > 0$ and $\mu \in GM_p\big(Du_0(x_0)\big)$ such that

$$F\big(x_0, u_0(x_0), Du_0(x_0)\big) > \int_{\mathbb{R}^{N\times n}} F\big(x_0, u_0(x_0), \cdot\big)\, \mathrm{d}\mu + \varepsilon \tag{B.2.6}$$

This inequality also holds for all $x \in \Omega_j$ which are sufficiently close to x_0 and μ_x obtained from μ by shifting the centre of mass from $Du_0(x_0)$ to $Du_0(x)$. It then follows from Theorem 3.2.21 that

$$\tilde{\mu}_x = \begin{cases} \mu_x & \text{if } x \in \Omega_j, x \text{ sufficiently close to } x_0 \\ \delta_{Du_0(x)} & \text{elsewhere} \end{cases}$$

is a gradient p-Young measure. Since (B.2.6) also holds for all such x, we can see that lower semicontinuity fails and hence the result.

□

Appendix C

Variational Capacity

Variational p-capacity (or p-capacity) plays an important role in potential theory and in understanding the pointwise behaviour of functions in a Sobolev space.

Definition C.0.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. The p-capacity of a compact set $K \subset \Omega$ is defined as*

$$\operatorname{cap}_p(K;\Omega) := \inf\left\{\int_\Omega |Du|^p \mathrm{d}x \colon u \in C_0^\infty(\Omega),\ u = 1 \text{ on } K\right\}$$

We extend this definition to arbitrary subsets in two steps. First, if $E \subset \Omega$ is an open set, then

$$\operatorname{cap}_p(E;\Omega) := \sup_{\substack{K \subset E \\ K \text{ compact}}} \operatorname{cap}_p(K;\Omega)$$

and then for an arbitrary set $A \subset \Omega$,

$$\operatorname{cap}_p(A;\Omega) := \inf_{\substack{A \subset E \subset \Omega \\ E \text{ open}}} \sup_{\substack{K \subset E \\ K \text{ compact}}} \operatorname{cap}_p(K;\Omega).$$

The set function defined above is also known as the relative p-capacity (relative to the ambient open set Ω). When the underlying set Ω is dropped, we have a variant of p-capacity known as the p-Sobolev capacity, in which for a set $A \subset \mathbb{R}^n$,

$$\operatorname{cap}_p(A) = \inf\left\{\|u\|_{W^{1,p}(\mathbb{R}^n)} \colon u = 1 \text{ a.e. on the neighbourhood of } A\right\}.$$

This definition is suited for characterizing the fine properties of Sobolev functions [44]. It is not hard to show that p-capacity enjoys the basic properties of a positive outer measure (see for instance Section 4.7 of Evans-Gariepy [44] and Chapter 2 of Heinonen-Kilpeläinen-Martio [62]).

Theorem C.0.2. *Let $A \subset \Omega$. The set function $A \mapsto \operatorname{cap}_p(A;\Omega)$ has the following properties:*

(i) Monotonicity: If $A_1 \subset A_2$, then $\operatorname{cap}_p(A_1;\Omega) \leq \operatorname{cap}_p(A_2;\Omega)$.

(ii) Monotonicity in second argument: If Ω_1 and Ω_2 are open and $\Omega_1 \subset \Omega_2$, then for every $A \subset \Omega_1$,

$$\operatorname{cap}_p(A;\Omega_2) \leq \operatorname{cap}_p(A;\Omega_1).$$

(iii) Strong additivity: If K_1 and K_2 are compact subsets of Ω, then

$$\operatorname{cap}_p(K_1 \cup K_2;\Omega) + \operatorname{cap}_p(K_1 \cap K_2;\Omega) \leq \operatorname{cap}_p(K_1;\Omega) + \operatorname{cap}_p(K_2;\Omega).$$

(iv) Continuity from above on compact subsets: If $K_1 \supset K_2 \supset \cdots \supset K_i \supset \cdots$ are compact subsets of Ω with $K = \bigcap_i K_i$, then

$$\operatorname{cap}_p(K;\Omega) = \lim_{i\to\infty} \operatorname{cap}_p(K_i;\Omega).$$

(v) Continuity from below: If $A_1 \subset A_2 \subset \cdots \subset \bigcup_i A_i = A \subset \Omega$, then

$$\operatorname{cap}_p(A;\Omega) = \lim_{i\to\infty} \operatorname{cap}_p(A_i;\Omega)$$

(vi) σ-subadditivity: If $A = \bigcup_i A_i \subset \Omega$, then

$$\operatorname{cap}_p(A;\Omega) \leq \sum_i \operatorname{cap}_p(A_i;\Omega).$$

Note that, for all $p > 1$, we have by a routine calculation

$$\operatorname{cap}_p\big(\bar{B}(x,r);B(x,2r)\big) = cr^{n-p}$$

where constant $c = c(n,p) > 0$, $B(x,r)$ denotes the open ball of radius r centered at x and $\bar{B}(x,r)$ the corresponding closed ball. In particular,

$$\begin{cases} \operatorname{cap}_p\big(\bar{B}(x,r);\mathbb{R}^n\big) = c_1(n,p)r^{n-p} & \text{if } 1 < p < n, \\ \operatorname{cap}_p\big(\bar{B}(x,r);B(x,2r)\big) = c_2(\log 2)^{1-p} & \text{if } p = n, \\ \operatorname{cap}_p\big(\{x\};B(x,r)\big) = c_3(n,p)r^{n-p} & \text{if } p > n. \end{cases}$$

The computation can be found in Chapter 2 of [62].

An important concept that is defined in terms of p-capacity is uniform thickness. This gives us a weak form of regularity of the boundary of a set Ω which is much weaker than having a Lipschitz boundary, but which is nevertheless sufficient for proving global higher integrability results (see [62], Chapter 6). In fact, this notion is the main reason for this appendix.

Definition C.0.3. *Let $p \in (1, \infty)$. A set Ω is said to be uniformly p-thick if there exist positive constants c_0 and R such that*

$$\operatorname{cap}_p \left(\Omega \cap \bar{B}(x,r); B(x,2r)\right) \geq c_0 \operatorname{cap}_p \left(\bar{B}(x,r); B(x,2r)\right)$$

whenever $x \in \Omega$ and $r \in (0, R)$.

It follows immediately from the definition of p-capacity that every non-empty set is uniformly p-thick when $p > n$. Hence we need to only consider the case when $p \leq n$. Details about p-capacity and uniform p-thickness can be found in [44], [62] and [135]. The following result shows a self-improving property of a uniformly p-thick set.

Theorem C.0.4 (Lewis [88], Theorem 1)**.** *Let $p \in (1, n]$. If a set Ω is uniformly p-thick, then there exists $q \in (1, p)$ such that Ω is uniformly q-thick.*

It is worth mentioning that if we replace p-capacity by Lebesgue measure, we obtain a measure density condition: there exists a constant $c > 0$ such that

$$|\Omega \cap B(x,r)| \geq c r^n$$

for all $x \in \Omega$ and for all $r \in (0, 1]$. Any set that satisfies the measure density condition is uniformly p-thick for all $p > 1$. In fact, Granlund [60] has already considered global higher integrability in the elliptic case when the complement Ω^c satisfies a measure density condition.

Recall the importance of Lusin's type theorem in showing that functions are almost everywhere continuous in their respective spaces. Similar to sets of measure zero, a set $E \subset \mathbb{R}^n$ is said to be of p-capacity zero if

$$\operatorname{cap}_p(E \cap \Omega; \Omega) = 0$$

for every $\Omega \subset \mathbb{R}^n$. The p-capacity replaces the Lebesgue measure in Lusin's type theorems for $W^{1,p}$ Sobolev functions. The relevant notion here is highlighted in the next definition.

Definition C.0.5. *A function u is p-quasicontinuous if for each $\varepsilon > 0$, there exists an open set $\omega \subset \Omega$ such that $\operatorname{cap}_p(\omega; \Omega) \leq \varepsilon$ and the restriction of u to $\Omega \setminus \omega$ is continuous.*

The definition, in turn, gives us a precise representative of Sobolev functions and also a version of Sobolev inequality.

Theorem C.0.6 (Ziemer [135], Theorem 3.3.3)**.** *Let $\Omega \subset \mathbb{R}^n$ be open. For each $u \in W^{1,p}(\Omega)$, there is a p-quasicontinuous representative such that*

$$\lim_{r \to 0} \fint_{B(x,r)} u(y) \, \mathrm{d}y = u(x)$$

for all $x \in \Omega$ except on a set E of p-capacity zero.

Lemma C.0.7 (**Sobolev inequality**, Maz'ya [95], Chapter 10)**.** *Suppose that $q \in (1, \infty)$ and that u is a q-quasicontinuous function in $W^{1,q}(B)$, where B is a ball. Let $N(u) = \{x \in B \colon u(x) = 0\}$. Then*

$$\left(\fint_B |u|^{\kappa q} \, \mathrm{d}x \right)^{1/\kappa q} \leq c \left(\frac{1}{\mathrm{cap}_q(N(u); 2B)} \int_B |Du|^q \, \mathrm{d}x \right)^{1/q}$$

where $c = c(n,q) > 0$ and

$$\kappa = \begin{cases} \frac{n}{n-q} & \text{if } q \in (1,n) \\ 2 & \text{if } q \in [n, \infty) \end{cases} .$$

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