Roughness properties of paths and signals

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Declaration of Originality

I hereby declare that this thesis contains no material which has been accepted or is currently being submitted for any other degree, diploma, certificate or other qualifications at the University of Oxford or elsewhere. This thesis and the research to which it refers [25, 23, 22, 24], are the product of my own work and were carried out in collaboration with my supervisor Professor Rama Cont. Work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

Abstract

Functions and processes with irregular behaviour in time are ubiquitous in physics, engineering, and finance and have been the focus of various pathwise theories of integration in stochastic analysis, in which the degree of 'roughness' of the function plays an important role. This thesis focuses on various concepts of 'roughness' for continuous functions and processes and their interplay with pathwise integration. We first explore these issues using the concept of pathwise quadratic variation, then expand results to the more general setting of p-th order variation.

The first chapter discusses some motivations and background for the questions explored in the thesis and provides an overview of the results.

In the second chapter, we study quadratic variation along a sequence of partitions and its dependence with respect to the choice of the partition sequence. We introduce a property which we call *quadratic roughness*, and show that for Hölder-continuous paths satisfying this roughness condition, the quadratic variation along 'balanced' partitions is invariant with respect to the choice of the partition sequence. Typical paths of Brownian motion satisfy this quadratic roughness property almost-surely along partitions with fine enough mesh. Using these results we derive a formulation of the pathwise Föllmer-Itô calculus which is invariant with respect to the partition sequences. Furthermore, we provide an invariance result for local time under quadratic roughness.

In the third chapter, instead of balanced partition sequences (which is a key condition in Chapter 2) we consider (finitely) refining partition sequences, without any bound on mesh size. We construct a generalized Haar basis along any such finite refining sequence of partitions. We provide a closed-form representation of quadratic variation in terms of Faber-Schauder coefficients along this basis. Further, we construct a class of continuous processes with linear and prescribed quadratic variations along any given finitely refining partition sequence. We provide an example of a rough class of continuous processes with invariant quadratic variations along finitely refining sequences of partitions. Brownian motion belongs to this 'rough' class, but we also give examples of processes with $\frac{1}{2}$-Hölder continuity in this class. Finally, we extend these constructions to higher dimensions.

In the fourth chapter of the thesis, we consider a more general concept of roughness based on p-th variation and the associated notions of variation and roughness index of a continuous function. We define the *normalized p-th variation* of a path and use it to introduce a pathwise estimator to estimate the order of roughness of a signal. We investigate the finite sample performance of our estimator for measuring the roughness of sample paths of stochastic processes using detailed numerical experiments based on sample paths of fractional Brownian motion and Takagi-Landsberg functions.

In the final chapter we use our 'roughness' estimator (discussed in Chapter 4) to investigate the statistical evidence for the use of 'rough' fractional processes with Hurst exponent $H < 0.5$ for the modelling of volatility of financial assets, using a non-parametric, model-free approach. Detailed numerical experiments based on stochastic volatility models show that, even when the instantaneous volatility has diffusive dynamics with the same roughness as Brownian motion, the realized volatility exhibits rough behaviour corresponding to a Hurst exponent significantly smaller than 0.5, which suggests that the origin of the roughness observed in realized volatility time-series lies in the estimation error rather than the volatility process itself. Comparison of roughness estimates for realized and instantaneous volatility in fractional volatility models with different values of Hurst exponent shows that, irrespective of the value of H, realized volatility always exhibits 'rough' behaviour with an apparent Hurst index $\hat{H} < 0.5$ but this is not necessarily indicative of a similar rough behaviour of the spot volatility process which may have $H \geq 1/2$.

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Notations and Common Definitions

Abbreviations

- a.s. = almost surely
- a.e. = almost everywhere
- i.e. = id est = that is
- w.r.t. = with respect to
- i.i.d. = independent and identically distributed
- Def. = definition
- Thm. = theorem
- iff = if and only if
- càdlàg = right continuous with left limits
- QV = quadratic variation
- BM= Brownian motion
- fBM = fractional Brownian motion
- IV = instantaneous volatility
- RV = realized volatility
- OU process = Ornstein-Uhlenbeck process
- fOU = fractional Ornstein-Uhlenbeck process

Notations and definitions

- $\forall$ = For all
- $\exists$ = There exists
- $\implies$ = Implies
- $\mathbb{R}$ = Set of all real numbers
- $\mathbb{R}_+$ = Set of all positive real numbers
- S_d^+ = Set of symmetric semidefinite positive matrices
- $[.]_\pi$ = Quadratic variation along sequence of partition π
- $[.]_\pi^{(p)}$ = p-th variation along sequence of partition π
- $[.,.]_\pi$ = Quadratic covariation along sequence of partition π
- $D([0,T],\mathbb{R}^d)$ = The space of $\mathbb{R}^d$ valued càdlàg functions
- $C^0([0,T],\mathbb{R}^d)$ = The space of $\mathbb{R}^d$ valued continuous functions
- $C^\nu([0,T],\mathbb{R}^d)$ The space of Hölder continuous functions with exponent ν
- $\Pi([0,T])$ = Set of all finite partitions of $[0,T]$
- $|\pi^n| = \sup\{|t_i^n - t_{i-1}^n|, i = 1, \cdots, N(\pi^n)\}$
- $\underline{\pi^n} = \inf\{|t_i^n - t_{i-1}^n|, i = 1, \cdots, N(\pi^n)\}$
- $\mathbb{T}$ = Dyadic partition sequence of $[0,1]$.
- $\mathbb{B}([0,T])$ = Set of all balanced partition sequence of $[0,T]$.
- $Q_\pi([0,T],\mathbb{R}^d)$ = Class of all continuous function with finite quadratic variation along π.
- $R_\pi^\beta([0,T],\mathbb{R}^d)$ = Class of all function with quadratic roughness property along partition sequence π of order β.
- $\mathcal{Q}([0,T],\mathbb{R}^d) = C^{\frac{1}{2}-}([0,T],\mathbb{R}^d) \cap Q_{\mathbb{T}}([0,T],\mathbb{R}^d)$.
- L_t^σ = Local time along partition sequence σ up to time t.
- $\psi_{m,k}^\pi$ = Generalized Haar basis constructed along sequence of partition π of $[0,T]$.
- $e_{m,k}^\pi$ = Faber-Schauder function constructed along sequence of partition π of $[0,T]$.

- $\mathcal{B}^\pi$ = Class of processes with linear quadratic variation (Equation 3.10).
- $\mathcal{A}^\pi$ = Class of processes with linear quadratic variation across coarsening (Equation 3.28).
- $V_\pi^p([0,T],\mathbb{R}^d)$ = The class of all continuous function with finite p-th variation along π.
- $p^\pi(x)$ = Variation index of x along partition sequence π.
- $H^\pi(x)$ = Roughness index of x along partition sequence π.
- $N_\pi^p([0,T],\mathbb{R}^d)$ = The class of all continuous function with finite normalized p-th variation along π.
- $w(x,p,\pi)$ = Normalized p-th variation of x along partition sequence π.
- $\mathcal{X}^p$ = Class of Takagi-Landsberg functions constructed with Hurst index $h = 1/p$.
- $W(L,K,\pi,p,t,X)$ = Normalized p-th variation statistics.
- $\widehat{H}_{L,K}$ = Estimator for roughness index.
- $\widehat{p}_{L,K}$ = Estimator for variation index.
- S_t = Price at time t.
- σ_t = Instantaneous or spot volatility.
- B_t = Brownian motion.
- B_t^H = fractional Brownian motion.

Chapter 1

Overview

Functions and processes with irregular behaviour in time are ubiquitous in physics, engineering, and finance and have prompted the development of various families of stochastic models whose sample path properties mimic some of the irregularities observed in data. From a mathematical perspective, many models for such functions have been studied [72, 49]. Also, the irregularity of such functions leads to obstacles for differential and integral analysis. Starting with Itô [52] this has been the focus of various pathwise theories of integration in stochastic analysis, in which the degree of 'roughness' of the function plays an important role.

Beginning with Mandelbrot and VanNess [61], fractional Brownian motion and fractional Gaussian noise have been used as building blocks of stochastic models of various phenomena in physics, engineering [60] and finance [5, 13, 18, 20, 42, 68, 76]. This raises the question of determining and measuring "roughness". The most widely studied roughness measures are the Hurst parameter [50], Hölder continuity and Hausdorff dimensions, Besov regularity [67], ρ-irregularity [15], IR-roughness [6].

This thesis focuses on various concepts of 'roughness' for continuous functions and processes and their interplay with pathwise integration. We first explore these issues using the concept of pathwise quadratic variation, then expand results to the more general setting of p-th order variation.

Quadratic roughness The concept of quadratic variation of a path along a sequence of partitions was first introduced by Föllmer [35]. The quadratic variation of a (real-valued) random process $(X(t), t \in [0,T])$ with càdlàg sample paths is defined as the limit in the sense of (uniform) convergence in probability, of the sum of squared increments

$$\sum_{\pi_n} \left(X(t^n_{k+1} \wedge t) - X(t^n_k \wedge t)\right)^\top \left(X(t^n_{k+1} \wedge t) - X(t^n_k \wedge t)\right) \tag{1.1}$$

along a sequence of partitions $\pi^n = \left(0 = t^n_0 < t^n_1 < \cdots < t^n_{N(\pi^n)} = T\right)$ with vanishing step. Although quadratic variation for a stochastic process X is usually defined as a limit in probability of (1.1), it is essentially a pathwise property. In his seminal paper *Calcul d'Itô sans probabilités* [35], Hans Föllmer showed that for $f \in C^2(\mathbb{R})$ and $x \in C^0([0,T],\mathbb{R})$ one can define a pathwise integral $\int_0^{\cdot} (\nabla f \circ X) d^\pi X$ as a pointwise limit of left Riemann sums along (π^n) and this integral satisfies a change of variable formula. This 'pathwise Itô formula' could be potentially used as a starting point for a purely pathwise construction of the Itô calculus but, unlike the analogous theory for Riemann-Stieltjes or Young integrals, the construction in [35] seems to depend on the choice of the sequence of partitions (π^n): both the quadratic variation $[X]_\pi$ and the pathwise integral are defined as limits along this sequence of partitions π. So the dependence of $[x]_\pi$ on π leaves no hope for the uniqueness of the Itô integral.

On the other hand, as shown by Lévy [58, 59] and Dudley [33], for typical paths of Brownian motion the sums (1.1) converge to a unique limit along *any* sequence of partitions that are refining or whose mesh decreases to zero fast enough. Therefore, there exists a large set of paths - containing all typical Brownian paths - for which one should be able to define the stochastic integral independently of the choice of the partition sequence $(\pi^n)_{n\geq 1}$ for a large class of such sequences[1]. We clarify these issues by investigating in detail the dependence of quadratic variation with respect to the sequence of partitions and deriving sufficient conditions for the stability of quadratic variation with respect to this choice.

[1]For different sequence of partitions we might have a different null set outside where Equation (1.1) converge to a unique limit

Quadratic variation along refining partitions Examples of functions with (non-zero) finite quadratic variation are given by typical sample paths of Brownian motion and semi-martingales, but explicit constructions of such functions are given by Gantert [41], Schied [70] and Mishura and Schied [62], in the spirit of Takagi's construction [72]. These constructions are based on a Schauder representation associated with a dyadic partition sequences and exploit certain identities, which is a result of the dyadic nature of the constructions. The question therefore arises whether such constructions may be carried out for non-dyadic and, more generally, non-uniform partition sequences and whether the quadratic variation of the resulting functions is invariant with respect to the partition sequence. We investigate these questions by providing several constructions of paths and processes with finite quadratic variation along refining sequences of partitions, extending previous constructions to the case of non-uniform partitions. We also provide class of processes for which the pathwise quadratic variation is invariant under 'coarsening'.

Roughness and p-th variation Cont and Perkowski [28] have shown that Föllmer's pathwise Itô calculus may be extended to paths with arbitrary regularity using the concept of p-th variation along a sequence of time partitions for even integer $p > 0$. For paths with finite p-th variation along a sequence of time partitions, there is also a change of variable formula for p times continuously differentiable functions. In fact, these have been recently extended by Cont and Jin [26] for all $p > 1$.

For such paths and processes with non-zero p-th variation, there are many measures of roughness such as the Hurst exponent [50], Hölder exponent, Besov regularity [67], ρ-irregularity [15], IR-roughness [6].

We introduce a model-free pathwise roughness index based on the concept of *normalized* p-th variation of a signal, which identifies the correct order p and provides good finite sample performance for fractional processes.

Application to financial data: is volatility rough? A recent strand of literature, starting with Gatheral et al. [42], has suggested the use of fractional Brownian models with $H < 1/2$ for modelling volatility. However, it has not been lost on experts working in this area that the estimation results in the previous literature on long-range dependence in volatility, which pointed out towards Hurst exponents $H > 0.5$ (≈ 0.55) [5, 18, 57] seem to contradict the claims in the recent 'rough volatility' literature, which points to values of H much smaller than 0.5 (closer to 0.1). Together with the well-known statistical issues plaguing the estimation of Hurst exponents [9, 66], these conflicting results call for a critical examination of the empirical evidence for 'rough volatility'. We address these questions in detail by re-examining the statistical evidence from high-frequency financial data, in an attempt to clarify whether the assertion that 'volatility is rough' is supported by empirical evidence.

1.1 Outline of thesis

Chapter 2: Quadratic variation and quadratic roughness

In the second chapter of the thesis, we study the concept of quadratic variation of a continuous path along a sequence of partitions and its dependence with respect to the choice of the partition sequence. Section 2.1 recalls the definition of quadratic variation along a sequence of partitions, following [16, 35]. Section 2.2 defines the class of balanced sequences of partitions and discusses the asymptotic comparability of such partitions. Section 2.3 introduces the concept of *quadratic roughness* and explores some of its properties. In particular we show that Brownian paths satisfy this property almost-surely (Theorem 2.14). Section 2.4 shows that the quadratic roughness of a path is a sufficient condition for the invariance of quadratic variation with respect to the choice of partitions (Theorem 2.16). This result allows to give a definition of quadratic variation invariant with respect to the choice of the partition sequences (Proposition 2.21). Section 2.5.1 builds on these results to arrive at a robust formulation of the pathwise Föllmer-Itô calculus. Section 2.5.2 provides sufficient condition for invariant pathwise local time.

Chapter 3: Quadratic variation along refining partitions: constructions and examples

We present several constructions of paths and processes with finite quadratic variation along a refining sequence of partitions, extending previous constructions to the non-uniform case. In this chapter instead of balanced partition sequences we restrict ourselves on (finitely) refining partitions, which do not have any uniform bound on step sizes but do have some locally bounded branching properties. Section 3.1 develops a non-uniform orthogonal Haar basis (Extension of Haar basis [46]) along any finite refining sequence of partitions, which provides a unique Faber-Schauder expansion of any continuous function. Section 3.2 extends Gantert's [41] quadratic variation formula along dyadic partition to general quadratic variation (and covariation) formula along any finitely refining (non-uniform) partition sequence. In Section 3.3, we construct a class of processes with *linear* quadratic variation along an arbitrary finitely refining partition π of $[0, 1]$, extending the construction in [70] beyond the dyadic case. We also construct class of processes with *prescribed* quadratic variation, extending results of Mishura and Schied [62]. Section 3.4 discusses the dependence of quadratic variation with respect to the partition sequence. Theorem 3.24 provides a class of processes with finite quadratic variation along a finitely refining partition π whose quadratic variation is invariant under coarsening of the partitions (Definition 3.20). Typical Brownian paths are shown to belong to this class, but also process with $\frac{1}{2}$-Hölder exponent belongs to the class. This gives us an interesting class of processes that are 'smoother' than Brownian motion in the sense of Hölder continuity, but still 'rough' enough to have pathwise quadratic variation invariant across different finitely refining partitions. Finally, Section 3.5 discusses extensions of these constructions to higher dimensions.

Chapter 4: Roughness and variation index of a signal

We extend the concept of roughness from quadratic variation to general p-th variation. We introduced a pathwise concept of *normalized* p-th variation. In section 4.2 we introduce the variation and roughness index of a path and study their properties. Section 4.3 introduces normalized p-th variation and provides several examples of paths and processes with linear normalized p-th variation. Finally, Section 4.4 and Section 4.5 discuss the properties of the roughness estimator, its asymptotic properties and its finite sample performance.

Chapter 5: Is volatility rough?

In the final chapter, we use our ‘roughness’ estimator to investigate the statistical evidence whether volatility of a financial price process is rough or not. In section 5.1 we provide the history of fractional processes and its recent development in modelling volatility as rough processes. Section 5.2 recalls the spot and realized volatility of financial price process and includes theoretical convergence results. In Section 5.3 we compare various estimators of the roughness index for instantaneous volatility σ_t with those obtained from realized volatility RV using price trajectories simulated from stochastic volatility models with varying degrees of ‘roughness’. Our simulation study shows even if the volatility is coming from a Brownian diffusion model the estimated roughness of realized volatility exhibits apparent ’rough’ behaviour. Finally, in Section 5.4 we apply our roughness estimator based on the normalized p-th variation statistic to high-frequency financial time series.

Our results show that the properties of SP500 time series previously presented as ’evidence’ for rough volatility seem perfectly compatible with those observed in a classical Brownian stochastic volatility model. AAPL time series exhibit similar features. This suggests that, rather than an empirical fact, ’roughness of volatility’ may well be an artefact due to the estimation error.

Chapter 2

Pathwise quadratic variation and quadratic roughness

Chapter based on: Rama Cont, Purba Das. Quadratic variation and quadratic roughness [25].

In this chapter, we study the concept of *quadratic variation* plays a central role in stochastic analysis and in the modern theory of stochastic integration [32, 65]. The quadratic variation of a (real-valued) random process $(X(t), t \in [0,T])$ with càdlàg sample paths is defined as the limit in the sense of (uniform) convergence in probability, of the sum of squared increments

$$\sum_{\pi_n} \left(X(t^n_{k+1} \wedge t) - X(t^n_k \wedge t)\right)^{\top} \left(X(t^n_{k+1} \wedge t) - X(t^n_k \wedge t)\right) \tag{2.1}$$

computed along a sequence of partitions $\pi^n = (0 = t^n_0 < t^n_1 < \cdots < t^n_{N(\pi^n)} = T)$ with vanishing step size $|\pi^n| = \sup_{i=1,\cdots,N(\pi^n)} |t^n_i - t^n_{i-1}| \to 0$. The relevance of this notion, as opposed to $p-$variation, is underlined by the fact that large classes of random processes –such as Brownian motion and diffusion processes– have finite quadratic variation, while at the same time possessing infinite $2-$variation.

Although quadratic variation for a stochastic process X is usually defined as a limit in probability of (2.1), it is essentially a pathwise property. In his seminal paper *Calcul d'Itô sans probabilités* [35], Hans Föllmer introduced the class of càdlàg paths $X \in D([0,T],\mathbb{R})$ with finite quadratic variation along a sequence of partitions $\pi = (\pi^n)$, for which (2.1) has a limit with Lebesgue decomposition

$[X]_\pi(t) = [X]^c(t) + \sum_{0\leq s\leq t}(\Delta X_s)^2$ and showed that for $f \in C^2(\mathbb{R})$ one can define the integral $\int_0^{\cdot}(\nabla f \circ X)d^\pi X$ as a pointwise limit of left Riemann sums along (π^n):

$$\int_0^T (\nabla f \circ X)d^\pi X := \lim_{n\to\infty} \sum_{\pi^n} \nabla f(X(t)).(X(t_{i+1}^n \wedge T) - X(t_i^n \wedge T)), \tag{2.2}$$

and this integral satisfies a change of variable formula:

$$\begin{aligned} f(X(t)) = f(X(0)) + \int_0^t (\nabla f \circ X)d^\pi X + \frac{1}{2}\int_0^t \nabla^2 f(X(s)).d[X]_\pi^c \\ + \sum_{s\in[0,t]} \Big(f(X(s)) - f(X(s-)) - \nabla f(X(s))\Delta X(s)\Big). \end{aligned} \tag{2.3}$$

This 'pathwise Itô formula' may be used as a starting point for a purely pathwise construction of the Itô calculus [35, 17] but, unlike the analogous theory for Riemann-Stieltjes or Young integrals, the construction in [35] seems to depend on the choice of the sequence of partitions (π^n): both the quadratic variation $[X]_\pi$ and the pathwise integral (2.2) are defined as limits along this sequence of partitions. In fact, as shown by Freedman [38, p. 47], (following an idea of Campbell) for any continuous function x one can construct a sequence of partitions π such that $[x]_\pi = 0$. This result was extended by Davis et al. [30] where they have shown that given any continuous path x and any increasing function A with infinite variation, one can construct a sequence of partitions π such that $[x]_\pi = A$. These negative results seem to suggest that the dependence of $[x]_\pi$ on π leaves no hope for the uniqueness of the quantities in Equation (2.3).

On the other hand, as shown by Lévy [58, 59] and Dudley [33], for typical paths of Brownian motion the sums in (2.1) converge to a unique limit along *any* sequence of partitions which are refining or whose mesh decreases to zero fast enough. Therefore, there exists a large set of paths-containing all typical Brownian paths - for which one should be able to define the quantities in Equation (2.3) independent of the choice of the partition sequence $(\pi^n)_{n\geq 1}$ for a large class of sequences.

We clarify these issues by investigating in detail the dependence of quadratic variation with respect to the sequence of partitions and deriving sufficient conditions for the stability of quadratic variation with respect to the choice of partition sequence. These conditions are related to an irregularity property of the path, which we call *quadratic roughness* (Def. 2.10): this property requires cross-products of increments along the partition to average to zero at certain scales and is different from other notions of roughness such as Hölder roughness [39] or the concept of $\rho-$irregularity as put forth by Catellier and Gubinelli [15]. Hölder roughness, like Hölder regularity, involves the amplitude of increments of a function, whereas our definition crucially involves the sign of the increments (or 'phase' in the multidimensional case). The relation between quadratic roughness and ρ-irregularity [15] is less clear. The ρ-irregularity is based on the smoothness of the local time of a path; while our approach relies only on the existence of quadratic variation along certain partition sequences and does not require the existence of a local time, it is possible that such properties would be implied by the existence of a smooth local time.

2.1 Quadratic variation along a sequence of partitions

Let $T > 0$. We denote $D([0,T],\mathbb{R}^d)$ the space of $\mathbb{R}^d$-valued right-continuous functions with left limits (càdlàg functions), $C^0([0,T],\mathbb{R}^d)$ the subspace of continuous functions and, for $0 < \nu < 1$, $C^\nu([0,T],\mathbb{R}^d)$ the space of Hölder continuous functions with exponent ν:

$$C^\nu([0,T],\mathbb{R}^d) = \left\{ x \in C^0([0,T],\mathbb{R}^d) \quad \Big| \sup_{(t,s)\in[0,T]^2, t\neq s} \frac{\|x(t)-x(s)\|}{|t-s|^\nu} < +\infty \right\} \subset C^0([0,T],\mathbb{R}^d),$$

$$\text{and} \qquad C^{\nu-}([0,T],\mathbb{R}^d) = \bigcap_{0\leq\alpha<\nu} C^\alpha([0,T],\mathbb{R}^d).$$

We denote by $\Pi([0,T])$ the set of all finite partitions of $[0,T]$. A sequence of partitions of $[0,T]$ is a sequence $(\pi^n)_{n\geq 1}$ of elements of $\Pi([0,T])$:

$$\pi^n = \left(0 = t_0^n < t_1^n < \cdots < t_{N(\pi^n)}^n = T\right).$$

We denote $N(\pi^n)$ the number of intervals in the partition π^n and

$$|\pi^n| = \sup\{|t_i^n - t_{i-1}^n|, i = 1, \cdots, N(\pi^n)\}, \tag{2.4}$$

$$\underline{\pi^n} = \inf\{|t_i^n - t_{i-1}^n|, i = 1, \cdots, N(\pi^n)\}, \tag{2.5}$$

the size of the largest (resp. the smallest) interval of π^n.

Example 2.1. *Let $k \geq 2$ be an integer. The k-adic partition sequence of $[0,T]$ is defined by*

$$\pi^n = \left(t_j^n = \frac{j\,T}{k^n}, \qquad j = 0, \cdots, k^n \right).$$

We have $\underline{\pi^n} = |\pi^n| = T/k^n$.

Example 2.2 (Lebesgue partition). *Given $x \in D([0,T],\mathbb{R}^d)$ define*

$$\lambda_0^n(x) = 0, \text{ and} \forall k \geq 1; \quad \lambda_{k+1}^n(x) = \inf\{t \in (\lambda_k^n(x), T], \quad \|x(t) - x(\lambda_k^n(x))\| \geq 2^{-n}\}$$

and $N(\lambda^n(x)) = \inf\{k \geq 1, \,:\, \lambda_k^n(x) = T\}$. We call the sequence $\lambda^n(x) = (\lambda_k^n(x))$ the (dyadic) Lebesgue partition associated to x.

Furthermore, if the function $x \in C^\alpha([0,T],\mathbb{R}^d)$, then $\underline{\pi^n} \geq \frac{1}{2^{\frac{n}{\alpha}}}$

Definition 2.1 (Quadratic variation of a path along a sequence of partitions)**.** *Let $\pi^n = (0 = t_0^n < t_1^n < \cdots < t_{N(\pi^n)}^n = T)$ be a sequence of partitions of $[0,T]$ with vanishing mesh $|\pi^n| = \sup_{i=0,\cdots,N(\pi^n)-1} |t_{i+1}^n - t_i^n| \to 0$. A càdlàg function $x \in D([0,T],\mathbb{R})$ is said to have finite quadratic variation along the sequence of partitions $(\pi^n)_{n\geq 1}$ if the sequence of measures*

$$\sum_{t_j^n \in \pi^n} (x(t_{j+1}^n) - x(t_j^n))^2 \delta_{t_j^n}$$

where $\delta_{t_j^n}$ denotes a unit point mass at t_j^n, converges vaguely on $[0,T]$ to a Radon measure μ such that $t \mapsto [x]_\pi^c(t) = \mu([0,t]) - \sum_{0<s\leq t} |\Delta x(s)|^2$ is continuous and increasing. The increasing function $[x]_\pi : [0,T] \to \mathbb{R}_+$ defined by

$$[x]_\pi(t) = \mu([0,t]) = \lim_{n\to\infty} \sum_{\pi_n} (x(t_{k+1}^n \wedge t) - x(t_k^n \wedge t))^2 \tag{2.6}$$

is called the quadratic variation of x along the sequence of partitions π. We denote $Q_\pi([0,T],\mathbb{R})$ the set of càdlàg paths with these properties.

$Q_\pi([0,T],\mathbb{R})$ is not a vector space (see e.g [70]). The extension of pathwise quadratic variation to vector-valued paths requires some care [35]:

Definition 2.2 (Pathwise quadratic variation for a vector valued path)**.** *A càdlàg path $x = (x^1, ..., x^d) \in D([0,T], \mathbb{R}^d)$ is said to have finite quadratic variation along $\pi = (\pi^n)_{n\geq 1}$ if for all $i, j = 1, \cdots, d$ we have $x^i \in Q_\pi([0,T], \mathbb{R})$ and $x^i + x^j \in Q_\pi([0,T], \mathbb{R})$. We then denote $[x]_\pi \in D([0,T], S_d^+)$ the matrix-valued function defined by*

$$[x]_\pi^{i,j}(t) = \frac{[x^i + x^j]_\pi(t) - [x^i]_\pi(t) - [x^j]_\pi(t)}{2}$$

where S_d^+ is the set of symmetric semidefinite positive matrices. We denote by $Q_\pi([0,T], \mathbb{R}^d)$ the set of functions satisfying these properties.

For $x \in Q_\pi([0,T], \mathbb{R}^d)$, $[x]_\pi$ is a càdlàg function with values in the cone S_d^+ of semidefinite positive symmetric $d \times d$ matrices: $[x]_\pi \in D([0,T], S_d^+)$.
We say a function $f : [0,T] \to S_d^+$ is increasing if for $t \geq s \geq 0,\ f(t) - f(s) \in S_d^+$.

As shown in [16], the above definitions may be more simply expressed in terms of convergence of discrete approximations. For continuous paths, we have the following characterization [21, 16] for quadratic variation:

Proposition 2.3. *$x \in C^0([0,T], \mathbb{R}^d)$ has finite quadratic variation along partition sequence $\pi = (\pi^n, n \geq 1)$ if and only if the sequence of functions $([x]_{\pi^n},\ n \geq 1)$ defined by*

$$[x]_{\pi^n}(t) := \sum_{t_j^n \in \pi^n} \left(x(t_{j+1}^n \wedge t) - x(t_j^n \wedge t)\right)^\top \left(x(t_{j+1}^n \wedge t) - x(t_j^n \wedge t)\right),$$

converges uniformly on $[0,T]$ to a continuous (increasing) function $[x]_\pi \in C^0([0,T], S_d^+)$.

The notion of quadratic variation along a sequence of partitions is different from the p-variation for $p = 2$. The p-variation involves taking a supremum over *all* partitions, whereas quadratic variation is a limit taken along a specific partition sequence $(\pi^n)_{n\geq 1}$. In general $[x]_\pi$ given by (2.6) is smaller than the p-variation for $p = 2$. In fact, for diffusion processes, the typical situation is that p-variation is (almost-surely) infinite for $p = 2$ [34, 73] while the quadratic variation is finite for sequences satisfying some mesh size condition. For instance, typical paths of Brownian motion have finite quadratic variation along any sequence of partitions with mesh size $o(1/\log n)$ [33, 31] while simultaneously having infinite p-variation almost surely for $p \leq 2$ [59, p. 190]:

$$\inf_{\pi\in\Pi(0,T)} \sum_{\pi} |W(t_{k+1}) - W(t_k)|^2 = 0, \quad \text{while} \quad \sup_{\pi\in\Pi(0,T)} \sum_{\pi} |W(t_{k+1}) - W(t_k)|^2 = \infty$$

almost-surely.

The quadratic variation of a path along a sequence of partitions strongly depends on the chosen sequence. In fact, as shown by Freedman [38, p. 47], given any continuous function, one can always construct a sequence of partitions along which the quadratic variation is zero. This result was extended by Davis et al. [30] who show that, given any continuous path $x \in C^0([0,T],\mathbb{R})$ and any increasing function $A : [0,T] \to \mathbb{R}_+$ one can construct a partition sequence π such that $[x]_\pi = A$. Notwithstanding these negative results, we shall identify a class of paths x for which $[x]_\pi$ is uniquely defined across the class of *balanced* partition sequences, which we now define.

2.2 Balanced partition sequences

One difficulty in comparing quadratic variation along two different partition sequences is the lack of uniform bounds on the partition intervals and the lack of comparability between two partitions. In this section, we introduce the class of *balanced* partition sequences which allow such bounds.

We shall say two (real) sequences $a = (a_n)_{n\geq 1}$ and $b = (b_n)_{n\geq 1}$ are asymptotically comparable, denoted $a_n \asymp b_n$, if $|a_n| = O(|b_n|)$ and $|b_n| = O(|a_n|)$. If both sequences are strictly positive then

$$a_n \asymp b_n \quad \Longleftrightarrow \quad \limsup_{n\to\infty} \frac{|b_n|}{|a_n|} < \infty \quad \text{and} \quad \limsup_{n\to\infty} \frac{|a_n|}{|b_n|} < \infty.$$

$$\Longleftrightarrow \; \exists M_0 < \infty \;\text{ s.t. }\; \forall n \in \mathbb{N} : \quad \frac{|b_n|}{|a_n|} < M_0 \quad \text{and} \quad \frac{|a_n|}{|b_n|} < M_0.$$

2.2.1 Definition and properties

Definition 2.4 (Balanced partition sequence)**.** *Let* $\pi^n = \Big(0 = t_0^n < t_1^n < \cdots < t_{N(\pi^n)}^n = T\Big)$ *be a sequence of partitions of interval* $[0,T]$ *and*

$$\underline{\pi^n} = \inf_{i=0,\cdots,N(\pi^n)-1} |t_{i+1}^n - t_i^n|, \qquad |\pi^n| = \sup_{i=0,\cdots,N(\pi^n)-1} |t_{i+1}^n - t_i^n|.$$

We say $(\pi^n)_{n\geq 1}$ *is* balanced partition sequence *if*

$$\exists\, c > 0,\ \forall n \geq 1, \quad \frac{|\pi^n|}{\underline{\pi^n}} \leq c. \tag{2.7}$$

This condition means that all intervals in the partition sequence π^n are asymptotically comparable. Note that, since $\underline{\pi^n} N(\pi^n) \leq T$, any balanced sequence of partitions satisfies the following inequality.

$$\underline{\pi^n} \leq |\pi^n| \leq c\ \underline{\pi^n} \leq \frac{cT}{N(\pi^n)}. \tag{2.8}$$

We denote by $\mathbb{B}([0,T])$, the set of all balanced partition sequences of $[0,T]$.

Proposition 2.5 (Properties of balanced partition sequence)**.** *Let* $\pi = (\pi^n)_{n\geq 1}$ *be a sequence of partitions of* $[0,T]$ *with mesh* $|\pi^n| \to 0$*. Then:*

(i) $\pi \in \mathbb{B}([0,T]) \iff \liminf_{n\to\infty} N(\pi^n)\underline{\pi^n} > 0$ *and* $\limsup_{n\to\infty} N(\pi^n)|\pi^n| < \infty$.

(ii) let $N(\pi^n, t_1, t_2)$ *be the number of partition points of* π^n *in* $[t_1, t_2]$*. If* $\pi \in \mathbb{B}([0,T])$ *then for any* $h > 0$*,*

$$\limsup_{n\to\infty} \frac{\sup_{t\in[0,T-h]} N(\pi^n, t, t+h)}{\inf_{t\in[0,T-h]} N(\pi^n, t, t+h)} < \infty.$$

(iii) if $\pi = (\pi^n;\ n \geq 1) \in \mathbb{B}([0,T])$ *then*

$$\limsup_n \frac{N(\pi^{n+1})}{N(\pi^n)} < \infty \iff \limsup_n \frac{|\pi^n|}{|\pi^{n+1}|} < \infty \iff \limsup_n \frac{\underline{\pi^n}}{\underline{\pi^{n+1}}} < \infty. \tag{2.9}$$

(iv) if $g \in C^1([0,T],\mathbb{R})$ *is strictly increasing with* $\inf g' > 0$ *then the image under* g *of a balanced partition sequence of* $[0,T]$ *is also a balanced partition sequence of* $[g(0), g(T)]$.

Proof. (i) For any sequence of partitions π of $[0,T]$ and for any $n \geq 1$:

$$N(\pi^n)\underline{\pi^n} \leq T \leq N(\pi^n)|\pi^n|.$$

For proof of ($\Rightarrow$): Using the balanced property, $\liminf_{n\to\infty} N(\pi^n)\underline{\pi^n}$ $= \liminf_{n\to\infty} N(\pi^n)|\pi^n|\frac{\underline{\pi^n}}{|\pi^n|} \geq \liminf_{n\to\infty} \frac{1}{c}N(\pi^n)|\pi^n| \geq \frac{T}{c} > 0$.
Similarly, $\limsup_{n\to\infty} N(\pi^n)|\pi^n| = \limsup_{n\to\infty} N(\pi^n)\underline{\pi^n}\frac{|\pi^n|}{\underline{\pi^n}}$ $\leq \limsup_{n\to\infty} cN(\pi^n)\underline{\pi^n} \leq cT < \infty$.
For proof of ($\Leftarrow$): $\limsup_{n\to\infty} \frac{|\pi^n|}{\underline{\pi^n}} = \limsup_{n\to\infty} \frac{N(\pi^n)|\pi^n|}{N(\pi^n)\underline{\pi^n}} \leq \frac{\limsup_{n\to\infty} N(\pi^n)|\pi^n|}{\liminf_{n\to\infty} N(\pi^n)\underline{\pi^n}} < \infty$.
(ii) For any sequence of partitions π with vanishing mesh and for any fixed $h > 0$ there exists a N_0 such that for all $n \geq N_0$, $|\pi^n| < h$. So for all $n \geq N_0$ and for all $t \in [0, T-h]$, $N(\pi^n, t, t+h) \geq 1$. Hence:

$$\underline{\pi^n} \leq \frac{h}{N(\pi^n, t, t+h)} \leq |\pi^n|.$$

So

$$\limsup_{n\to\infty} \frac{\sup_{t\in[0,T-h]} N(\pi^n, t, t+h)}{\inf_{t\in[0,T-h]} N(\pi^n, t, t+h)} \leq \limsup_{n\to\infty} \frac{|\pi^n|}{h} \times \frac{h}{\underline{\pi^n}} < \infty.$$

(iii) For any balanced sequence of partitions π of $[0,T]$ and for any $n \geq 1$:

$$\frac{1}{c}N(\pi^n)|\pi^n| \leq N(\pi^n)\underline{\pi^n} \leq T \leq N(\pi^n)|\pi^n| \leq cN(\pi^n)\underline{\pi^n},$$

where, c is a constant $< \infty$. So the equivalence follows.
(iv) Let $\pi = (\pi^n)_{n\geq 1}$ be any balanced sequence of partitions of $[0,T]$:

$$\pi^n = (0 = t_1^n < t_2^n < \cdots < t_{N(\pi^n)}^n = T).$$

Now, define the new partition $g(\pi) = (g(\pi^n))_{n\geq 1}$ as follows:

$$g(\pi^n) = \left(g(0) = g(t_1^n) < g(t_2^n) < \cdots < g(t_{N(\pi^n)}^n = g(T))\right).$$

Now, from mean value theorem there exists $u_k^n, v_k^n \in [t_k^n, t_{k+1}^n]$ such that,

$$\left|\limsup_{n\to\infty} \frac{|g(\pi^n)|}{\underline{g(\pi^n)}}\right| = \left|\limsup_{n\to\infty} \frac{\sup_{\pi^n}(g(t_{k+1}^n) - g(t_k^n))}{\inf_{\pi^n}(g(t_{k+1}^n) - g(t_k^n))}\right| = \left|\limsup_{n\to\infty} \frac{\sup_{\pi^n} g'(u_k^n)(t_{k+1}^n - t_k^n)}{\inf_{\pi^n} g'(v_k^n)(t_{k+1}^n - t_k^n)}\right|$$

$$\leq \left|\limsup_{n\to\infty} \frac{\sup_{\pi^n} g'(u_k^n)}{\inf_{\pi^n} g'(v_k^n)}\right| \times \left|\limsup_{n\to\infty} \frac{\sup_{\pi^n}(t_{k+1}^n - t_k^n)}{\inf_{\pi^n}(t_{k+1}^n - t_k^n)}\right| \leq \frac{\max g'}{\inf g'}\, c < \infty.$$

$\blacksquare$

Definition 2.6 (Asymptotic comparability of balanced partitions)**.** *We will say that two balanced partition sequences $\tau = (\tau^n)_{n\geq 1}$ and $\sigma = (\sigma^n)_{n\geq 1}$ are (asymptotically) comparable if*

$$0 < \liminf_{n\to\infty} \frac{|\sigma^n|}{|\tau^n|} \leq \limsup_{n\to\infty} \frac{|\sigma^n|}{|\tau^n|} < \infty. \tag{2.10}$$

Since the partition sequences are balanced (not true for a general partition sequence), an equivalent condition will be:

$$0 < \liminf_{n\to\infty} \frac{N(\sigma^n)}{N(\tau^n)} \leq \limsup_{n\to\infty} \frac{N(\sigma^n)}{N(\tau^n)} < \infty. \tag{2.11}$$

We denote $\tau \asymp \sigma$ (or $\tau^n \asymp \sigma^n$).

Note that for general (not balanced) sequences of partitions Inequality (2.10) neither implies nor is implied by Inequality (2.11): this is purely a consequence of (2.8). If $\tau \asymp \sigma$ then the number of partition points of τ^n (respectively σ^n) in any consecutive partition points of σ^n (respectively τ^n) remains bounded as $n \to \infty$.

The following lemma shows how one can adjust the rate at which the mesh of a balanced sequence converges to zero.

Lemma 2.7. *Let $\tau = (\tau^n)_{n\geq 1}$ and $\sigma = (\sigma^n)_{n\geq 1}$ be two balanced partition sequences of $[0,T]$ with $\limsup_n \frac{|\sigma^n|}{|\tau^n|} < 1$ and mesh $|\tau^n| \xrightarrow{n\to\infty} 0$.*

(i) There exists a subsequence $(\tau^{k(n)})_{n\geq 1}$ of τ such that:

$$|\tau^{k(n)}| \to 0 \quad \textit{and,} \quad \limsup_n \frac{|\sigma^n|}{|\tau^{k(n)}|} \geq 1.$$

(ii) Furthermore if we also assume $\limsup_n \frac{|\tau^n|}{|\tau^{n+1}|} < \infty$, then there exists a subsequence $(\tau^{k(n)})_{n\geq 1}$ of τ which is asymptotically comparable to σ: $\tau^{k(n)} \asymp \sigma^n$.

(iii) There exists $r : \mathbb{N} \mapsto \mathbb{N}$ with $\lim_{n\to\infty} r(n) = \infty$ such that

$$\limsup_n \frac{|\sigma^{r(n)}|}{|\tau^n|} \geq 1.$$

Note that $r : \mathbb{N} \mapsto \mathbb{N}$ in Lemma 2.7 (iii) is not necessarily injective i.e. $(\sigma^{r(n)}, n \geq 1)$ is not necessarily a subsequence of $(\sigma^n, n \geq 1)$.

Proof. Denote the partition points of τ^n and σ^n respectively by $\big(t_k^n;\ k = 0, \cdots, N(\tau^n)\big)$ and $\big(s_l^n;\ l = 0, \cdots, N(\sigma^n)\big)$.
Proof of (i): From the assumption, $\limsup_n \frac{|\sigma^n|}{|\tau^n|} < 1$ which implies $\liminf_n \frac{|\tau^n|}{|\sigma^n|} > 1$. Then there exists $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, $\frac{|\tau^n|}{|\sigma^n|} > 1$. Since we are only concerned about the limiting behaviour as $n \to \infty$ we will only consider $n > N_0$. We define $k(n)$ as follows.

$$k(n) = \inf\{k \geq n; \quad |\tau^k| \leq |\sigma^n|\} < \infty \quad \text{since,} \quad |\tau^k| \overset{k\to\infty}{\to} 0. \tag{2.12}$$

We now consider the subsequence $(\tau^{k(n)})_{n\geq 1}$ of τ. From the definition of $k(n)$:

$$\limsup_n \frac{|\sigma^n|}{|\tau^{k(n)}|} \geq 1.$$

Proof of (ii): If $\limsup_n \frac{|\tau^n|}{|\sigma^n|} < \infty$ we set $k(n) = n$; otherwise if $\limsup_n \frac{|\tau^n|}{|\sigma^n|} = +\infty$, define $k(n)$ as in Equation (2.12). Now for $i = 1, \cdots, N(\sigma^n)$ define $j(i,n)$ as follows.

$$j(i,n) = \inf\{j \geq 1, \quad t_j^{k(n)} \in (s_i^n, s_{i+1}^n]\}.$$

Then we have:

$$t_{j(i,n)-1}^{k(n)} \leq s_k^n < t_{j(i,n)}^{k(n)} < \cdots < t_{j(i+1,n)-1}^{k(n)} \leq s_{i+1}^n < t_{j(i+1,n)}^{k(n)}.$$

If for some i, $|j(i+1,n) - j(i,n)| \to \infty$ as $n \to \infty$ then, from the above construction of $k(n)$ and using the well balanced property of σ^n and $\tau^{k(n)}$ we have: $\limsup_n \frac{|\sigma^n|}{|\tau^{k(n)}|} \to \infty$ and $\limsup_n \frac{|\sigma^n|}{|\tau^{k(n)-1}|} < 1$. Hence, $\limsup_n \left(\frac{|\sigma^n|}{|\tau^{k(n)}|} - \frac{|\sigma^n|}{|\tau^{k(n)-1}|}\right) = \limsup_n \frac{|\sigma^n|}{|\tau^{k(n)-1}|}\left[\frac{|\tau^{k(n)-1}|}{|\tau^{k(n)}|} - 1\right] \to \infty$ which is a contradiction because of our assumption. Hence the cluster size $j(i+1,n) - j(i,n)$ is uniformly bounded:

$$\forall i, n \geq 1, \ \exists M, \quad \text{such that,} \ |j(i+1,n) - j(i,n)| \leq M < \infty.$$

So there exists a constant c_0 such that

$$1 \leq \liminf_n \frac{|\sigma^n|}{|\tau^{k(n)}|} \leq \limsup_n \frac{|\sigma^n|}{|\tau^{k(n)}|} \leq c_0 < \infty. \tag{2.13}$$

Therefore, $(\tau^{k(n)})_{n\geq 1}$ and $(\sigma^n)_{n\geq 1}$ are (asymptotically) comparable.

Proof of (iii): Since $\limsup_n \frac{|\sigma^n|}{|\tau^n|} < 1$, the set $\{n \geq 1,\ \frac{|\sigma^n|}{|\tau^n|} \geq 1\}$ is finite and the set,

$$A = \{n \geq 1,\ \frac{|\sigma^n|}{|\tau^n|} < 1\}$$

is infinite. Now from the assumption there exists $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, $\frac{|\tau^n|}{|\sigma^n|} > 1$. Now for $n \leq N_0$ set $r(n) = 1$, for $n > N_0$ and $n \notin A$, set $r(n) = n$ and

$$\text{for} \quad n > N_0;\ n \in A \qquad r(n) = \sup\{r \leq n,\ |\sigma^r| > |\tau^n|\} < \infty.$$

Then,

$$r(n) \leq n \quad \textit{and}, \quad \limsup_{n\to\infty} \frac{|\sigma^{r(n)}|}{|\tau^n|} = \limsup_{n\in A} \frac{|\sigma^{r(n)}|}{|\tau^n|} \geq 1.$$

■

2.2.2 Quadratic variation along balanced partition sequences

If a path has quadratic variation along a sequence of partitions, then it also has (the same) quadratic variation along any sub-sequence. This simple remark has interesting implications when the partition sequences are balanced: comparing the sum of squared increments along the original sequence with the sum along a sub-sequence (with finer mesh), we obtain that, under some scaling conditions on the mesh, cross-products of increments along the finer partition average to zero across the coarser partition.

Lemma 2.8 (Averaging property of cross-products of increments)**.** *Let $x \in C^\alpha([0,T],\mathbb{R}^d)$ for some $\alpha > 0$ and $\sigma^n = \left(0 = s_0^n < s_1^n < \cdots < s_{N(\sigma^n)}^n = T\right)$ be a balanced sequence of partitions of $[0,T]$ such that $x \in Q_\sigma([0,T],\mathbb{R}^d)$. Let $\kappa > \frac{1}{2\alpha}$ and $(\sigma^{l_n})_{n\geq 1}$ a subsequence of σ^n with $|\sigma^{l_n}| = O\left(|\sigma^n|^\kappa\right)$. For $k = 1, \cdots, N(\sigma^n)$ define $p(k,n) = \inf\{m \geq 1 : \quad s_m^{l_n} \in (s_k^n, s_{k+1}^n]\}$. Then*

$$\sum_{k=1}^{N(\sigma^n)} \sum_{p(k,n)\leq i\neq j<p(k+1,n)-1} \left(x(s_{i+1}^{l_n}) - x(s_i^{l_n})\right)^\top \left(x(s_{j+1}^{l_n}) - x(s_j^{l_n})\right) \xrightarrow{n\to\infty} 0.$$

Proof. We provide the proof for $d = 1$. The extension to $d > 1$ is straightforward extension of 1-dimensional case. Let σ^{l_n} be a sub-sequence of σ^n satisfying $|\sigma^{l_n}| = O\left(|\sigma^n|^\kappa\right)$. Denote,

$$[x]_{\sigma^n}(t) = \sum_{k=1}^{N(\sigma^n)-1} \left(x(s^n_{k+1} \wedge t) - x(s^n_k \wedge t)\right)^\top \left(x(s^n_{k+1} \wedge t) - x(s^n_k \wedge t)\right), \quad \text{and,}$$

$$[x]_{\sigma^{l_n}}(t) = \sum_{s^{l_n}_k \in \sigma^{l_n}} \left(x(s^{l_n}_{k+1} \wedge t) - x(s^{l_n}_k \wedge t)\right)^\top \left(x(s^{l_n}_{k+1} \wedge t) - x(s^{l_n}_k \wedge t)\right).$$

Then $\left|[x]_{\sigma^{l_n}}(t) - [x]_{\sigma^n}(t)\right| \to 0$. Grouping the points of σ^{l_n} along partition points of σ^n, we obtain:

$$\left|[x]_{\sigma^n}(T) - [x]_{\sigma^{l_n}}(T)\right| = \left|\sum_{\sigma^n} \left(x(s^n_{i+1}) - x(s^n_i)\right)^2 - \sum_{\sigma^{l_n}} \left(x(s^{l_n}_{i+1}) - x(s^{l_n}_i)\right)^2\right|$$

$$= \Bigg|\sum_{\sigma^n} \left(\left(x(s^n_{i+1}) - x(s^n_i)\right)^2 - \sum_{j=p(i,n)}^{p(i+1,n)-2} \left(x(s^{l_n}_{j+1}) - x(s^{l_n}_j)\right)^2 \right)$$
$$+ \sum_{k=1}^{N(\sigma^n)} \left(x(s^{l_n}_{p(k,n)}) - x(s^{l_n}_{p(k,n)-1})\right)^2\Bigg|$$
$$\geq \left|\sum_{\sigma^n} \left(\left(x(s^n_{i+1}) - x(s^n_i)\right)^2 - \sum_{j=p(i,n)}^{p(i+1,n)-2} \left(x(s^{l_n}_{j+1}) - x(s^{l_n}_j)\right)^2 \right)\right|$$
$$- \sum_{k=1}^{N(\sigma^n)} \left(x(s^{l_n}_{p(k,n)}) - x(s^{l_n}_{p(k,n)-1})\right)^2.$$

Using the $\alpha-$Hölder continuity of x, the last term in the above equation is bounded above by $\sum_{k=1}^{N(\sigma^n)} C|\sigma^{l_n}|^{2\alpha} \leq CN(\sigma^n)|\sigma^{l_n}|^{2\alpha}$. Now using the balanced property of σ^{l_n} (subsequence of a balanced sequence of partitions is also balanced), we further get the above bounded as:

$$\sum_{i=1}^{N(\sigma^n)} C|\sigma^{l_n}|^{2\alpha} \leq \frac{C_1 N(\sigma^n)}{N(\sigma^{l_n})^{2\alpha}} \leq C_2 \times N(\sigma^n)^{1-2\alpha\kappa} \xrightarrow{n\to\infty} 0,$$

since $1 - 2\kappa\alpha < 0$. So writing the first term of the previous equation explicitly we finally obtain,

$$\lim_{n\to\infty}\left|\sum_{k=1}^{N(\sigma^n)}\sum_{p(k,n)\leq i\neq j<p(k+1,n)-2}\left(x(s_{i+1}^{l_n})-x(s_i^{l_n})\right)\left(x(s_{j+1}^{l_n})-x(s_j^{l_n})\right)\right|$$
$$\leq \lim_{n\to\infty}|[x]_{\sigma^n}-[x]_{\sigma^{l_n}}| = 0.$$

■

2.3 Quadratic roughness

2.3.1 Quadratic roughness along a sequence of partitions

Lemma 2.8 shows that if a function has finite quadratic variation along a balanced partition sequence, then the cross-products of the increments along any subsequence with *sufficiently small mesh* average to zero along the original (coarser) sequence. Intuitively, this means that there is enough cancellation across neighbouring increments such that their cross-products average to zero under coarse-graining. This can only occur if the increments over any scale have alternating signs, which is an indicator of the 'roughness' of the function itself. We will now introduce a slightly extended version of this property, which we call *quadratic roughness*, and show that this property plays a crucial role in the stability of quadratic variation with respect to the choice of partition.

Definition 2.9 (Super-sequence). *We call $d^n = \pi^{r(n)}$ a super-sequence of $\pi = (\pi^n)_{n\geq 1}$ if the map $r : \mathbb{N} \to \mathbb{N}$ is non-decreasing, $k \geq r(k)$ for all $k \in \mathbb{N}$ and $r(k) \xrightarrow{k\to\infty} \infty$.*

Definition 2.10 (Quadratic roughness). *Let $\mathbb{T} = (\mathbb{T}^n)_{n\geq 1}$ be the dyadic partition of $[0,T]$ and $\pi^n = \left(0 = s_0^n < s_1^n < \cdots < s_{N(\pi^n)}^n = T\right)$ be a balanced sequence of partitions of $[0,T]$ with vanishing mesh $|\pi^n| \to 0$. We say that $x \in C^0([0,T],\mathbb{R}^d) \cap Q_{\mathbb{T}}([0,T],\mathbb{R}^d)$ has the* quadratic roughness *property with coarsening index $0 < \beta < 1$ along π on $[0,T]$ if there exists a subsequence or super-sequence $d^n = \left(0 = t_1^n < t_2^n < \cdots < t_{N(d^n)}^n = T\right)$ of $\mathbb{T}$ with the following properties:*

(i) $|d^n|^\beta = O\left(|\pi^n|\right)$ *and,*

(ii) for all $t \in [0,T]$ *:*

$$\sum_{j=1}^{N(\pi^n)-1} \sum_{t_i^n \neq t_{i'}^n \in (s_j^n, s_{j+1}^n]} \Big(x(t_{i+1}^n \wedge t) - x(t_i^n \wedge t)\Big)^\top \Big(x(t_{i'+1}^n \wedge t) - x(t_{i'}^n \wedge t)\Big) \xrightarrow{n\to\infty} 0.$$

We denote by $R_\pi^\beta([0,T],\mathbb{R}^d)$ *the set of paths satisfying this quadratic roughness property.*

In other words, the quadratic roughness property states that cross-products of increments along the dyadic partition d^n average to zero when grouped along π^n. Note that, since $\beta < 1$, the number of terms in the inner sum in (ii) grows to infinity as n grows, so (ii) is the result of compensation across terms, reminiscent of the law of large numbers.

Remark 2.11 (**Choice of reference partition**). In the quadratic roughness definition (2.10), we have used the dyadic partition as a 'reference partition' to which other (balanced) partitions are compared. In fact, as it will be clear in the proof below, the dyadic partition may be replaced by any other balanced sequence of partitions σ with vanishing mesh $|\sigma^n| \to 0$ satisfying $\sup_n \frac{|\sigma^n|}{|\sigma^{n+1}|} < \infty$ without changing the statements of any of the results.

Remark 2.12. The second condition in the Quadratic roughness definition can be rewritten as difference of quadratic variations of x plus $O(N(\pi^n))$ many boundary terms i.e.,

$$\begin{aligned}
\textstyle\sum_{j=1}^{N(\pi^n)-1} \sum_{t_i^n \neq t_{i'}^n \in (s_j^n, s_{j+1}^n]} &\Big(x(t_{i+1}^n \wedge t) - x(t_i^n \wedge t)\Big)^\top \Big(x(t_{i'+1}^n \wedge t) - x(t_{i'}^n \wedge t)\Big) \\
&= [x]_{\pi^n} - [x]_{d^n} \\
+ \textstyle\sum_{\pi^n} \Big[x(t_k^n) - x(s_{p(n,k)}^n) + x(s_{p(n,k+1)}^n) - x(t_{k+1}^n)\Big]^\top &\Big[x(t_k^n) - x(s_{p(n,k)}^n) + x(s_{p(n,k+1)}^n) - x(t_{k+1}^n)\Big] \\
+ \textstyle\sum_{\pi^n} \Big(x(t_k^n) - x(s_{p(n,k)}^n) &+ x(s_{p(n,k+1)}^n) - x(t_{k+1}^n)\Big)^\top \Big(x(t_{k+1}^n) - x(t_k^n)\Big).
\end{aligned}$$

Now the last two sums in general do not go to zero, but under certain regularity conditions like Hölder continuity, one can show that the last two sums indeed go to zero. Though the new representation involves difference of quadratic variation across partitions, unlike the second condition of quadratic roughness, this representation do not involve only increments of x along partition sequence π.

As a consequence of the quadratic roughness (Definition 2.10) if $x \in R^{\beta}_{\pi}([0,T],\mathbb{R}^d)$ for some sequence of partitions π of $[0,T]$, then $x \in Q_{\mathbb{T}}([0,T],\mathbb{R}^d)$ (but not necessarily $x \in Q_{\pi}([0,T],\mathbb{R}^d)$).

Proposition 2.13 (Properties of quadratic roughness). *Let $\pi = (\pi^n)_{n\geq 1}$ be a balanced partition sequence of $[0,T]$ with vanishing mesh (i.e. $|\pi^n| \to 0$) and $x \in R^{\beta}_{\pi}([0,T],\mathbb{R}^d)$ with $0<\beta<1$. Then:*

1. *For any interval $I=[a,b]\subset[0,T]$, the path x also has the quadratic roughness property on I along stopped partition $\pi_I = (\pi_I^n)_{n\geq 1} = ((\pi^n \cap I)\cup\{a,b\})_{n\geq 1}$. i.e. $x \in R^{\beta}_{\pi_I}(I,\mathbb{R}^d)$.*
2. *For any subsequence/super-sequence $\tau^n = \pi^{k(n)}$ of π, we have $x \in R^{\beta}_{\tau}([0,T],\mathbb{R}^d)$.*
3. *For any $\lambda\in\mathbb{R}$, $\lambda x \in R^{\beta}_{\pi}([0,T],\mathbb{R}^d)$.*
4. *If y is a function of finite variation then, $x+y \in R^{\beta}_{\pi}([0,T],\mathbb{R}^d)$.*

   Furthermore, if $d=1$ and if y is a function with finite p-variation with $p<2$ then, $x+y\in R^{\beta}_{\pi}([0,T],\mathbb{R})$.
5. *For $\gamma\in[\beta,1)$, $R^{\beta}_{\pi}([0,T],\mathbb{R}^d)\subset R^{\gamma}_{\pi}([0,T],\mathbb{R}^d)$.*

Proof. The proof of $1-3$ are direct consequences of Definition 2.10.
Proof of 4: We will proof the statement for $d=1$ first, and then give general argument for $d>1$. Let $\pi^n = (0 = t_0^n < t_2^n < \cdots < t^n_{N(\pi^n)} = T)$. Since $x \in R^{\beta}_{\pi}([0,T],\mathbb{R}^d)$, there exists a dyadic sub/super-sequence $(d^n)_{n\geq 1}$ with the two following property:
(i) $|d^n|^{\beta} O\left(|\pi^n|\right)$ and,
(ii) For all $t\in[0,T]$:

$$\sum_{k=1}^{N(\pi^n)} \sum_{s_i^n \neq s_{i'}^n \in (t_k^n, t_{k+1}^n]} \left(x(s_{i+1}^n\wedge t) - x(s_i^n \wedge t)\right)^{\top}\left(x(s_{i'+1}^n\wedge t) - x(s_{i'}^n\wedge t)\right) \xrightarrow{n\to\infty} 0, \tag{2.14}$$

where, $d^n = \Big(0 = t_1^n < t_2^n < \cdots < t_{N(d^n)}^n = T\Big)$. Now the pathwise quadratic variation along π and d respectively, at level n as follows:

$$[x]_{\pi^n}(t) = \sum_{k=0}^{N(\pi^n)-1} \big(x(t_{k+1}^n \wedge t) - x(t_k^n \wedge t)\big)^\top \big(x(t_{k+1}^n \wedge t) - x(t_k^n \wedge t)\big), \quad \text{and,}$$

$$[x]_{d^n}(t) = \sum_{k=0}^{N(d^n)-1} \left(x(s_{k+1}^n \wedge t) - x(s_k^n \wedge t)\right)^\top \left(x(s_{k+1}^n \wedge t) - x(s_k^n \wedge t)\right).$$

Define for $k = 1, 2, \cdots N(\pi^n)$:

$$p(n,k) = \inf\{m \geq 1 : \quad s_m^n \in (t_k^n, t_{k+1}^n]\}.$$

Then we have the following inequality between partition points of π^n and d^n:

$$s_{p(n,k)-1}^n \leq t_k^n < s_{p(n,k)}^n < \cdots < s_{p(n,k+1)-1}^n \leq t_{k+1}^n < s_{p(n,k+1)}^n, \tag{2.15}$$

where, $p(n, N(\pi^n)) - 1 = N(d^n)$. We define an intermediate partition defined as $\sigma = (\sigma^n)$ and a corresponding intermediate limit $[x]_{\sigma^n}(t)$ to show that $[x]_{\pi^n}(t)$ and $[x]_{d^n}(t)$ has exact same limit:

$$\sigma^n = (0 = s_{p(n,1)}^n < s_{p(n,1)}^n < \cdots < s_{p(n,N(\pi^n))-1}^n = T) \qquad \text{and,}$$

$$[x]_{\sigma^n}(t) = \sum_{k=0}^{N(\pi^n)-1} \big(x(s_{p(n,k+1)}^n \wedge t) - x(s_{p(n,k)}^n \wedge t)\big)^\top \big(x(s_{p(n,k+1)}^n \wedge t) - x(s_{p(n,k)}^n \wedge t)\big).$$

The roughness assumption preciously tells us for all $t \in [0,T]$, we have $[x]_\sigma(t) = [x]_d(t) = [x]_{\mathbb{T}}(t)$. So $x \in Q_\sigma([0,T], \mathbb{R}^d)$. Hence to show that $x + y \in R_\pi^\beta([0,T], \mathbb{R}^d)$ we have to show that $[x+y]_\sigma(t) = [x+y]_d(t) = [x+y]_{\mathbb{T}}(t)$. Now,

$$\begin{aligned}
[x+y]_{\sigma^n}(t) = &\sum_{k=0}^{N(\pi^n)-1} \big((x+y)(s_{p(n,k+1)}^n \wedge t) - (x+y)(s_{p(n,k)}^n \wedge t)\big)^\top \\
&\big((x+y)(s_{p(n,k+1)}^n \wedge t) - (x+y)(s_{p(n,k)}^n \wedge t)\big) \\
= &\sum_{k=0}^{N(\pi^n)-1} \big(x(s_{p(n,k+1)}^n \wedge t) - x(s_{p(n,k)}^n \wedge t)\big)^\top \big(x(s_{p(n,k+1)}^n \wedge t) - x(s_{p(n,k)}^n \wedge t)\big) \\
+ &\sum_{k=0}^{N(\pi^n)-1} \big(y(s_{p(n,k+1)}^n \wedge t) - y(s_{p(n,k)}^n \wedge t)\big)^\top \big(y(s_{p(n,k+1)}^n \wedge t) - y(s_{p(n,k)}^n \wedge t)\big)
\end{aligned}$$

$$+2\sum_{k=0}^{N(\pi^n)-1}\big(x(s^n_{p(n,k+1)}\wedge t)-x(s^n_{p(n,k)}\wedge t)\big)^\top\big(y(s^n_{p(n,k+1)}\wedge t)-y(s^n_{p(n,k)}\wedge t)\big).$$

Replacing $[x]_{d^n}$ in the above equality we obtain the following inequality:

$$\begin{aligned}&|[x+y]_{\sigma^n}(T)-[x]_{\sigma^n}(T)|\\ &\le\left|\sum_{k=0}^{N(\pi^n)-1}\big(y(s^n_{p(n,k+1)})-y(s^n_{p(n,k)})\big)^\top\big(y(s^n_{p(n,k+1)})-y(s^n_{p(n,k)})\big)\right|\\ &+2\left|\sum_{k=0}^{N(\pi^n)-1}\big(x(s^n_{p(n,k+1)})-x(s^n_{p(n,k)})\big)^\top\big(y(s^n_{p(n,k+1)})-y(s^n_{p(n,k)})\big)\right|.\end{aligned}\tag{2.16}$$

We will now show that the first term of the above inequality goes to zero as $n\to\infty$.

$$\begin{aligned}&\left|\sum_{k=0}^{N(\pi^n)-1}\big(y(s^n_{p(n,k+1)})-y(s^n_{p(n,k)})\big)^\top\big(y(s^n_{p(n,k+1)})-y(s^n_{p(n,k)})\big)\right|\\ &\le\left|\sum_{k=0}^{N(\pi^n)-1}\big(y(s^n_{p(n,k+1)})-y(s^n_{p(n,k)})\big)^p\big(y(s^n_{p(n,k+1)})-y(s^n_{p(n,k)})\big)^{2-p}\right| \qquad (2.17)\\ &\le\sup_k\left|\big(y(s^n_{p(n,k+1)})-y(s^n_{p(n,k)})\big)^{2-p}\right|\left|\sum_{k=0}^{N(\pi^n)-1}\big(y(s^n_{p(n,k+1)})-y(s^n_{p(n,k)})\big)^p\right|\to 0.\end{aligned}$$

The above quantity goes to zero as y has a bounded p-variation and since $2-p>0$ the first term goes to zero.

Now using the Hölder inequality on the last term of the inequality (2.16) we get:

$$\begin{aligned}&\left|\sum_{k=0}^{N(\pi^n)-1}\big(x(s^n_{p(n,k+1)})-x(s^n_{p(n,k)})\big)^\top\big(y(s^n_{p(n,k+1)})-y(s^n_{p(n,k)})\big)\right|\\ &\le\left|\sum_{k=0}^{N(\pi^n)-1}\big(x(s^n_{p(n,k+1)})-x(s^n_{p(n,k)})\big)^\top\big(x(s^n_{p(n,k+1)})-x(s^n_{p(n,k)})\big)\right|^{\frac{1}{2}}\\ &\times\left|\sum_{k=0}^{N(\pi^n)-1}\big(y(s^n_{p(n,k+1)})-y(s^n_{p(n,k)})\big)^\top\big(y(s^n_{p(n,k+1)})-y(s^n_{p(n,k)})\big)\right|^{\frac{1}{2}}.\end{aligned}$$

From roughness assumption x has finite quadratic variation along σ and we already show that quadratic variation of y along σ is zero, so the quadratic cross-variation of x and y along σ is also zero. Hence we obtain $|[x+y]_{\sigma^n}(T) - [x]_{\sigma^n}(T)| \to 0$. Similarly we can obtain $|[x+y]_{\mathbb{T}^n}(T) - [x]_{\mathbb{T}^n}(T)| \to 0$. So, for the function $x+y$ we have for all $t \in [0,T]$: $[x+y]_{\mathbb{T}}(t) = [x+y]_{\sigma}(t)$ and hence $x+y \in R^{\beta}_{\pi}([0,T],\mathbb{R}^d)$.

Now for the general d-dimension the equation in 2.17 is not valid anymore. But we can proceed for $p=1$ in the following way:

$$\leq \left| \sum_{k=0}^{N(\pi^n)-1} \left(y(s^n_{p(n,k+1)-1}) - y(s^n_{p(n,k)-1}) \right)^{\top} \left(y(s^n_{p(n,k+1)-1}) - y(s^n_{p(n,k)-1}) \right)^{1} \right|$$

$$\leq \sup_k \left\| \left(y(s^n_{p(n,k+1)-1}) - y(s^n_{p(n,k)-1}) \right) \right\| \times \left| \sum_{k=0}^{N(\pi^n)-1} \left\| y(s^n_{p(n,k+1)-1}) - y(s^n_{p(n,k)-1}) \right\| \right| \to 0.$$

The above quantity goes to zero as y has a bounded variation so the second term is finite and the first term goes to zero as $y \in C^0$. The rest of the proof in higher dimensions follows exactly in the same way.

Proof of 5. Take $x \in R^{\beta}_{\pi}([0,T],\mathbb{R}^d)$, then there exists a dyadic subsequence or super-sequence $(d^n)_{n\geq 1}$ which satisfy Condition (i) and (ii) of Definition 2.10. Also since mesh size of dyadic partition goes to 0, there exists N_0 such that $|d^n| \leq 1$ for all $n \geq N_0$. Now if we fix $\gamma \geq \beta$ then for all $n \geq N_0$: $|d^n|^{\gamma} \leq |d^n|^{\beta} = O\left(|\pi^n|\right)$. So $x \in R^{\beta}_{\pi}([0,T],\mathbb{R}^d)$ which concludes the proof.

Denote the partition points of τ^n and σ^n respectively by $(t^n_k, k = 0..N(\tau^n))$ and $(s^n_l, l = 0..N(\sigma^n))$.

Proof of (i): From the assumption we have, $\limsup_n \frac{|\tau^n|}{|\sigma^n|} > 1$. Then there exists $N_0 \in \mathbb{N}$ such that for $n \geq N_0$, $\frac{|\tau^n|}{|\sigma^n|} \geq 1$. Since we are only concerned about the limiting behaviour when $n \to \infty$ we will only consider $n > N_0$ throughout the rest of the proof.

If $\limsup_n \frac{|\tau^n|}{|\sigma^n|} < \infty$ we set $k(n) = n$; otherwise if $\limsup_n \frac{|\tau^n|}{|\sigma^n|} = +\infty$ we define:

$$k(n) = \inf\{k \geq n, \quad |\tau^k| \leq |\sigma^n|\} < \infty \quad \text{since} \quad |\tau^k| \overset{k\to\infty}{\to} 0. \tag{2.18}$$

We now consider the subsequence $(\tau^{k(n)})_{n\geq 1}$ of τ. From the definition of $k(n)$:

$$\limsup_n \frac{|\sigma^n|}{|\tau^{k(n)}|} \geq 1.$$

Proof of (ii): Define $k(n)$ as in (2.18) for $i = 1..N(\sigma^n)$,

$$j(i,n) = \inf\{j \geq 1, \quad t_j^{k(n)} \in (s_i^n, s_{i+1}^n]\}.$$

Then we have

$$t_{j(i,n)-1}^{k(n)} \leq s_k^n < t_{j(i,n)}^{k(n)} < \cdots < t_{j(i+1,n)-1}^{k(n)} \leq s_{i+1}^n < t_{j(i+1,n)}^{k(n)}.$$

If for some i, j(i+1,n)- j(i,n)$\rightarrow \infty$ as $n \rightarrow \infty$ then, from the above construction of $k(n)$ and using the well balanced property of σ^n and $\tau^{k(n)}$ we have: $\limsup_n \frac{|\sigma^n|}{|\tau^{k(n)}|} \rightarrow \infty$ and $\limsup_n \frac{|\sigma^n|}{|\tau^{k(n)-1}|} < 1$. Hence, $\limsup_n \frac{|\sigma^n|}{|\tau^{k(n)-1}|} \left[\frac{|\tau^{k(n)-1}|}{|\tau^{k(n)}|} - 1\right] \rightarrow \infty$ which is a contradiction because of our assumption. Hence, the size $j(i+1,n) - j(i,n)$ of clusters is uniformly bounded:

$$\forall k, n \geq 1, j(i+1,n) - j(i,n) \leq M < \infty.$$

So there exists a constant c_0 such that

$$1 \leq \limsup_n \frac{|\sigma^n|}{|\tau^{k(n)}|} \leq c_0 < \infty. \tag{2.19}$$

Therefore $(\tau^{k(n)})_{n\geq 1}$ and $(\sigma^n)_{n\geq 1}$ are (asymptotically) comparable.
Proof of (iii): If $\limsup_n \frac{|\sigma^n|}{|\tau^n|} < 1$ then the set $\{n \geq 1, \ \frac{|\sigma^n|}{|\tau^n|} \geq 1\}$ is finite and the set,

$$A = \{n \geq 1, \ \frac{|\sigma^n|}{|\tau^n|} < 1\}$$

is infinite. Now define $r : \mathbb{N} \mapsto \mathbb{N}$ as follows: we set $r(n) = n$ for $n \notin A$ and

$$r(n) = \inf\{k \geq 1, \ |\sigma^k| > |\tau^n|\} < \infty \qquad \text{for} \quad n \in A.$$

Then

$$\limsup_{n\rightarrow\infty} \frac{|\sigma^{r(n)}|}{|\tau^n|} = \limsup_{n\in A} \frac{|\sigma^{r(n)}|}{|\tau^n|} \geq 1.$$

■

2.3.2 Quadratic roughness of Brownian paths

We will now show that the quadratic roughness property is satisfied almost-surely by typical sample paths of Brownian motion.

Theorem 2.14 (Quadratic roughness of Brownian paths)**.** *Let W be a Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $T > 0$ and $(\pi^n)_{n\geq 1}$ a balanced sequence of partitions of $[0, T]$ with*

$$(\log n)^2 |\pi^n| \overset{n\to\infty}{\to} 0. \tag{2.20}$$

Then the sample paths of W almost-surely satisfy the quadratic roughness property for any $0 < \beta < 1$:

$$\forall \beta \in (0,1), \quad \mathbb{P}\left(\ W \in R^{\beta}_{\pi}([0,T],\mathbb{R})\ \right) = 1.$$

Proof. Let W be a Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which we take to be the canonical Wiener space without loss of generality i.e. $\Omega = C^0([0,T],\mathbb{R})$, $W(t,\omega) = \omega(t)$.

Take $\beta \in (0,1)$. We know $\pi^n = (0 = t^n_0 < t^n_1 < \cdots < t^n_{N(\pi^n)} = T)$ be a balanced sequence of partitions of $[0, T]$ satisfying Equation (2.20). Now we will define a subsequence $\left(\mathbb{T}^{l_n}\right)$ of $(\mathbb{T}^n)$ such that $|\mathbb{T}^{l_n}|^\beta = O(|\pi^n|)$.

If $|\mathbb{T}^n|^\beta = O(|\pi^n|)$, then take $l_n = n$. Otherwise if $\limsup_n \frac{|\mathbb{T}^n|^\beta}{|\pi^n|} = \infty$, then we define l_n as follows,

$$l_n = \inf\{l \geq n : |\pi^n| \geq |\mathbb{T}^l|^\beta\} < \infty \qquad \text{since } |\mathbb{T}^l|^\beta = \frac{1}{2^{l\beta}} \xrightarrow{l\to\infty} 0.$$

So from the construction of l_n we then get:

$$|\mathbb{T}^{l_n}| \leq |\pi^n|^{1/\beta} < |\mathbb{T}^{l_n - 1}|.$$

Since the subsequence $(\mathbb{T}^{l_n})$ is also balanced, there exist constants c_1 and c_2 such that $c_1 N(\mathbb{T}^{l_n}) \geq N(\pi^n)^{1/\beta} > c_2 N(\mathbb{T}^{l_n})$. Hence the dyadic subsequence $\mathbb{T}^{l_n}$ satisfies

$$|\mathbb{T}^{l_n}|^\beta = O(|\pi^n|)$$

Now we will show that $\mathbb{T}^{l_n}$ satisfies Condition (ii) of Definition 2.10. Define $a^n_{ii'} = \sqrt{(t^{l_n}_{i+1} - t^{l_n}_i)(t^{l_n}_{i'+1} - t^{l_n}_{i'})}$ if $\exists\ j \in \{1, 2, \cdots, N(\pi^n)\}$ such that $p(n, j-1) \leq$

$i \neq i' < p(n,j)$. Otherwise, set $a^n_{ii'} = 0$. Let

$$S_\pi(\mathbb{T}^{l_n}, W) = \sum_{j=1}^{N(\pi^n)} \sum_{p(n,j-1) \le i \neq i' < p(n,j)} \left(W(t^{l_n}_{i+1}) - W(t^{l_n}_i)\right)^\top \left(W(t^{l_n}_{i'+1}) - W(t^{l_n}_{i'})\right)$$

$$= \sum_{i,i'=1}^{N(\mathbb{T}^{l_n})} a^n_{ii'} X^n_i X^n_{i'},$$

$$\text{where,} \quad X^n_i = \frac{W(t^{l_n}_{i+1}) - W(t^{l_n}_i)}{\sqrt{t^{l_n}_{i+1} - t^{l_n}_i}} \sim^{i.i.d} N(0,1) \quad \text{for} \;\; i = 0, \cdots, N(\mathbb{T}^{l_n}) - 1.$$

Now let,

$$\Lambda^2 = \sum_{1 \le i,i' \le N(\mathbb{T}^{l_n})} (a^n_{ii'})^2 = \sum_{j=1}^{N(\pi^n)} \sum_{p(n,j-1) \le i \neq i' < p(n,j)} (a^n_{ii'})^2$$

$$\le \sum_{j=1}^{N(\pi^n)} \sum_{p(n,j-1) \le i \neq i' < p(n,j)} \Delta t^{l_n}_i \Delta t^{l_n}_{i'} \le \sum_{j=1}^{N(\pi^n)} |\pi^n|^2 \le |\pi^n| \sum_{j=1}^{N(\pi^n)} |\pi^n| \le cT|\pi^n|.$$

The last inequality is due to the fact that π is a balanced sequence. The Hanson-Wright inequality [48] then implies that there exists constants C_1 and C_2 such that

$$\forall \delta > 0, \quad \forall n \ge 1, \qquad \mathbb{P}\left(\left|S_\pi(\mathbb{T}^{l_n}, W)\right| > \delta \right) \le 2\exp(-\min\{C_1 \frac{\delta}{\sqrt{|\pi^n|}}, C_2 \frac{\delta^2}{|\pi^n|}\}).$$

Since $|\pi^n|(\log n)^2 \to 0$ for large n, the upper bound is determined by the first term $\exp\left(-C_1 \frac{\delta}{\sqrt{|\pi^n|}}\right)$. If we denote $\varepsilon_n^2 = |\pi^n|(\log n)^2$ then $\varepsilon_n \to 0$ and we can rewrite this bound as

$$\mathbb{P}\left(\left|S_\pi(\mathbb{T}^{l_n}, W)\right| > \delta \right) \le 2\exp(-\min\{\frac{C_1\delta \;\log n}{\varepsilon_n}, \frac{C_2\delta^2 \;(\log n)^2}{\varepsilon_n^2}\}) \; \le \frac{2C}{n^{C_1\delta/\varepsilon_n}}. \tag{2.21}$$

The series $\sum_n \frac{1}{n^{C_1\delta/\varepsilon_n}} < \infty$ is absolutely convergent. So we can apply the Borel-Cantelli lemma to obtain for each $\delta > 0$ a set Ω_δ with $\mathbb{P}(\Omega_\delta) = 1$ and $N_\delta \in \mathbb{N}$ such that

$$\forall \omega \in \Omega_\delta, \quad \forall n \ge N_\delta, \quad \left|S_\pi(\mathbb{T}^{l_n}, W)(\omega)\right| \le \delta.$$

Now if we set

$$\Omega_\pi = \Omega_0 \cap \left(\bigcap_{m \ge 1} \Omega_{1/m} \right) \qquad \text{then} \quad \mathbb{P}(\Omega_\pi) = 1$$

and for paths in Ω_π we have $S_\pi(\mathbb{T}^{l_n}, \omega) \to 0$ simultaneously :

$$\forall \omega \in \Omega_\pi, \qquad S_\pi(\mathbb{T}^{l_n}, \omega) \overset{n\to\infty}{\to} 0,$$

Therefore, $\mathbb{P}\left(W \in R^\beta_\pi([0,T], \mathbb{R}) \right) = 1.$ ■

The proof above uses independence of increments, which then implies that the cross-products of increments averages to zero due to a concentration inequality. However, the quadratic roughness property is a pathwise property, and also holds for classes of stochastic processes with dependent increments, as the following example shows:

Example 2.3 (Quadratic roughness of mixed Brownian motion)**.** *Let $H > \frac{1}{2}$ and $\delta > 0$ and*

$$M^{H,\delta} = B + \delta B^H$$

where B is a Brownian motion and B^H is a fractional Brownian motion with Hurst parameter H on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then for any balanced sequence $(\pi^n)_{n\geq 1}$ of partitions of $[0,T]$ with

$$(\log n)^2 |\pi^n| \overset{n\to\infty}{\to} 0. \tag{2.22}$$

the sample paths of $M^{H,\delta}$ almost-surely satisfy the quadratic roughness property on $[0,T]$:

$$\forall \beta \in (0,1), \quad \mathbb{P}\left(M^{H,\delta} \in R^\beta_\pi([0,T], \mathbb{R}) \right) = 1.$$

This is an application of Proposition 2.13 (iv) and Theorem 2.14.

2.4 Uniqueness of quadratic variation along balanced partitions

The following lemma shows that the quadratic roughness property is a necessary condition for the stability of quadratic variation with respect to the choice of the partition sequences:

Lemma 2.15 (Necessity of quadratic roughness). *Let $x \in C^\alpha([0,T],\mathbb{R}^d) \cap Q_{\mathbb{T}}([0,T],\mathbb{R}^d)$. Let $\pi = (\pi^n)_{n\geq 1}$ be a balanced partition sequence of $[0,T]$ such that $x \in Q_\pi([0,T],\mathbb{R}^d)$. Then:*

$$\Big(\forall t \in [0,T]),\ [x]_\pi(t) = [x]_{\mathbb{T}}(t)\ \Big) \qquad \Rightarrow \qquad \forall \beta \in (0,2\alpha),\ x \in R^\beta_\pi([0,T],\mathbb{R}^d).$$

Proof. Take $\beta \in (0,2\alpha)$. Since π is a balanced sequence of partitions and since for dyadic partition $\frac{|\mathbb{T}^n|}{|\mathbb{T}^{n+1}|} = 2 < \infty$, we can construct a subsequence $\left(\mathbb{T}^{l_n}\right)$ of $(\mathbb{T}^n)$ such that $|\mathbb{T}^{l_n}| = O(|\pi^n|)$ as follows.

Firstly, if $|\mathbb{T}^n|^\beta = O(|\pi^n|)$, then take $l_n = n$. Otherwise, for $\limsup_n \frac{|\mathbb{T}^n|^\beta}{|\pi^n|} = \infty$, we construct l_n as follows,

$$l_n = \inf\{l \geq n : |\pi^n| \geq |\mathbb{T}^l|^\beta\} < \infty, \qquad \text{since } |\mathbb{T}^l|^\beta = \frac{1}{2^{l\beta}} \to 0.$$

Since $l_n \geq n$, we also have $l_n \to \infty$. So from the construction of l_n we get the following inequality:

$$\forall n \geq 1, \qquad |\mathbb{T}^{l_n}| \leq |\pi^n|^{1/\beta} < |\mathbb{T}^{l_n-1}|.$$

Since the subsequence $(\mathbb{T}^{l_n})$ is also balanced, there exists constants c_1 and c_2 such that,

$$\forall n \geq 1, \qquad c_1 N(\mathbb{T}^{l_n}) \geq N(\pi^n)^{1/\beta} > c_2 N(\mathbb{T}^{l_n-1}).$$

The points of the partition $\mathbb{T}^{l_n}$ are interspersed among those of π^n. Define for $k = 1,\cdots,N(\pi^n)$:

$$p(n,k) = \inf\{m \geq 1 : \quad s^n_m \in (t^n_k, t^n_{k+1}]\},$$

where, $\pi^n = (0 = t^n_1 < t^n_2 < \cdots < t^n_{N(\pi^n)} = T)$ and $\mathbb{T}^{l_n} = (0 = s^n_1 < s^n_2 < \cdots < s^n_{N(\mathbb{T}^{l_n})} = T)$. Then we have

$$s^n_{p(n,k)-1} \leq t^n_k < s^n_{p(n,k)} < \cdots < s^n_{p(n,k+1)-1} \leq t^n_{k+1} < s^n_{p(n,k+1)}, \tag{2.23}$$

where, $p(n,N(\pi^n)) - 1 = N(\mathbb{T}^{l_n})$. From the construction of l_n and the fact that $\limsup_n \frac{|\mathbb{T}^n|}{|\mathbb{T}^{n+1}|} = 2 < \infty$, we can conclude that $|\pi^n| \asymp |\mathbb{T}^{l_n}|^\beta$. To prove this, assume for contradiction $|\pi^n|$ and $|\mathbb{T}^{l_n}|^\beta$ are not asymptotically comparable. Then

$\sup_{i=1,\cdots,N(\pi^n)} |p(n,i+1)-p(n,i)| \to \infty$ as $n \to \infty$. Then, from the above definition of l_n and using the balanced property of π^n and $\mathbb{T}^{l_n}$ we have $\limsup_n \frac{|\pi^n|}{|\mathbb{T}^{l_n}|^\beta} \to \infty$. Since $\limsup_n \frac{|\pi^n|}{|\mathbb{T}^{l_n-1}|^\beta}$ is bounded by 1 for all $n \geq 1$, from the construction of l_n, we thus have

$$\infty = \limsup_n \left(\frac{|\pi^n|}{|\mathbb{T}^{l_n}|^\beta}\right) - \limsup_n \left(\frac{|\pi^n|}{|\mathbb{T}^{l_n-1}|^\beta}\right) \leq \limsup_n \left(\frac{|\pi^n|}{|\mathbb{T}^{l_n}|^\beta} - \frac{|\pi^n|}{|\mathbb{T}^{l_n-1}|^\beta}\right)$$
$$= \limsup_n \frac{|\pi^n|}{|\mathbb{T}^{l_n-1}|^\beta}\left(\frac{|\mathbb{T}^{l_n-1}|^\beta}{|\mathbb{T}^{l_n}|^\beta} - 1\right) < \infty$$

which is a contradiction, and the last inequality is form the fact that $\frac{|\mathbb{T}^{l_n-1}|^\beta}{|\mathbb{T}^{l_n}|^\beta} = 2^\beta < \infty$ and $\limsup_n \frac{|\pi^n|}{|\mathbb{T}^{l_n-1}|^\beta} \leq 1$. Hence the sequence $\sup_{i=1,\cdots,N(\pi^n)}\ p(n,i+1)-p(n,i)$ is bounded as $n \to \infty$:

$$\exists M > 0, \text{ such that } \forall i, n \geq 1, \qquad p(n,i+1) - p(n,i) \leq M < \infty.$$

Therefore, $(\mathbb{T}^{l_n})^\beta_{n\geq 1}$ and $(\pi^n)_{n\geq 1}$ are (asymptotically) comparable i.e. the sequences

$$\frac{N(\mathbb{T}^{l_n})^\beta}{N(\pi^n)} \qquad \text{and} \qquad \frac{|\mathbb{T}^{l_n}|^\beta}{|\pi^n|} \tag{2.24}$$

are uniformly bounded. Now since, $\forall t \in [0,T],\ [x]_\pi(t) = [x]_\mathbb{T}(t)$, we have:

$$\left|[x]_{\pi^n}(t) - [x]_{\mathbb{T}^{l_n}}(t)\right| \to 0.$$

For convenience denote $(\mathbb{T}^{l_n}) = (d^n)$. We will only give the proof for $t = T$, for $t < T$ we will get one additional boundary term which goes to zero.

Decomposing $\Delta^n_k = x(t^n_{k+1}) - x(t^n_k)$ along the partition points of d^n, we obtain,

$$\underbrace{x(t^n_{k+1}) - x(t^n_k)}_{\Delta^n_k} = \underbrace{(x(t^n_{k+1}) - x(s^n_{p(n,k+1)})}_{D_k} \qquad - \underbrace{(x(t^n_k) - x(s^n_{p(n,k)}))}_{B_k}$$
$$+ \underbrace{\sum_{i=p(n,k)}^{p(n,k+1)-1} (x(s^n_{i+1}) - x(s^n_i))}_{C_k}. \tag{2.25}$$

Grouping together the terms in $[x]_{d^n}$ according to Equation (2.23) yields

$$[x]_{\pi^n}(T) - [x]_{d^n}(T)$$

$$= \sum_{k=1}^{N(\pi^n)-1} \left[\Delta_k^{n\top}\Delta_k^n - \sum_{i=p(n,k)}^{p(n,k+1)-1} \left(x(s_{i+1}^n) - x(s_i^n)\right)^\top \left(x(s_{i+1}^n) - x(s_i^n)\right)\right]$$

$$= \sum_{k=1}^{N(\pi^n)-1} \left[\Delta_k^{n\top}\Delta_k^n - C_k^\top C_k\right]$$

$$+ \sum_{k=1}^{N(\pi^n)-1} \left[C_k^\top C_k - \sum_{i=p(n,k)}^{p(n,k+1)-1} \left(x(s_{i+1}^n) - x(s_i^n)\right)^\top \left(x(s_{i+1}^n) - x(s_i^n)\right)\right].$$

Now, the second term of the previous equation can also be represented as follows.

$$\sum_{k=1}^{N(\pi^n)-1} \left[C_k^\top C_k - \sum_{i=p(n,k)}^{p(n,k+1)-1} \left(x(s_{i+1}^n) - x(s_i^n)\right)^\top \left(x(s_{i+1}^n) - x(s_i^n)\right)\right]$$

$$= \sum_{k=1}^{N(\pi^n)-1} \Bigg[\left(x(s_{p(n,k+1)}^n) - x(s_{p(n,k)}^n)\right)^\top \left(x(s_{p(n,k+1)}^n) - x(s_{p(n,k)}^n)\right)$$

$$- \sum_{i=p(n,k)}^{p(n,k+1)-1} \left(x(s_{i+1}^n) - x(s_i^n)\right)^\top \left(x(s_{i+1}^n) - x(s_i^n)\right) \Bigg]$$

$$= \sum_{j=1}^{N(\pi^n)-1} \sum_{s_i^n \neq s_{i'}^n \in (t_j^n, t_{j+1}^n]} \left(x(s_{i+1}^n) - x(s_i^n)\right)^\top \left(x(s_{i'+1}^n) - x(s_{i'}^n)\right).$$

This is precisely the roughness term of x along partition π. Hence to show that

$$\sum_{j=1}^{N(\pi^n)-1} \sum_{t_i^n \neq t_{i'}^n \in (s_j^n, s_{j+1}^n]} \left(x(t_i^n \wedge t) - x(t_{i-1}^n \wedge t)\right)^\top \left(x(t_{i'}^n \wedge t) - x(t_{i'-1}^n \wedge t)\right) \xrightarrow{n\to\infty} 0,$$

we only have to show that: $$\left|[x]_{\pi^n}(T) - [x]_{d^n}(T) - \left(\sum_{k=1}^{N(\pi^n)-1} \Delta_k^{n\top}\Delta_k^n - C_k^\top C_k\right)\right| \to 0.$$

Now, From the assumption $|[x]_{\pi^n}(T) - [x]_{d^n}(T)| \to 0$. So we only need to show that
$\left|\sum_{k=1}^{N(\pi^n)-1} \Delta_k^{n\top}\Delta_k^n - C_k^\top C_k\right| \to 0$. This is a consequence from the fact that quadratic variation along π exists and $x \in C^\alpha([0,T],\mathbb{R}^d)$ with $\frac{\alpha}{\beta} > \frac{1}{2}$. A similar line of proof is adopted in more detail in Theorem 2.16. ■

We will now show that quadratic roughness is also a *sufficient* condition for the uniqueness of quadratic variation along balanced partition sequences.

Our main result is that quadratic roughness along such a sequence of partitions implies uniqueness of pathwise quadratic variation:

Theorem 2.16. *Let π be a balanced sequence of partitions of $[0,T]$ and $x \in C^{\alpha}([0,T],\mathbb{R}^d) \cap R^{\beta}_{\pi}([0,T],\mathbb{R}^d)$ for some $0 < \beta < 2\alpha$. Then*

$$x \in Q_{\pi}([0,T],\mathbb{R}^d), \qquad \text{and} \qquad \forall t \in [0,T], \quad [x]_{\pi}(t) = [x]_{\mathbb{T}}(t).$$

Proof. Let, $\pi^n = \left(0 = t_1^n < t_2^n < \cdots < t^n_{N(\pi^n)} = T\right)$. Since $x \in R^{\beta}_{\pi}([0,T],\mathbb{R}^d)$, from Definition 2.10 we know there exists a sub/super-sequence $d = (d^n)_{n\geq 1}$ of the dyadic partition $\mathbb{T} = (\mathbb{T}^n)$ with $d^n = \left(0 = s_1^n < s_2^n < \cdots < s^n_{N(d^n)} = T\right)$ such that $|d^n|^{\beta} = O(|\pi^n|)$ and for all $t \in [0,T]$:

$$\sum_{j=1}^{N(\pi^n)-1} \sum_{s_i^n \neq s_{i'}^n \in (t_j^n, t_{j+1}^n]} \left(x(s_{i+1}^n \wedge t) - x(s_i^n \wedge t)\right)^{\top} \left(x(s_{i'+1}^n \wedge t) - x(s_{i'}^n \wedge t)\right) \xrightarrow{n\to\infty} 0. \tag{2.26}$$

So there exists $C < \infty$ and $N_0 \in \mathbb{N}$ such that:

$$\forall n \geq N_0, \qquad |d^n|^{\beta} \leq C|\pi^n|$$

We will assume $n \geq N_0$ for the rest of the proof. Since both $d = (d^n)$ and $\pi = (\pi^n)$ are balanced sequence of partitions there exists $C_1 < \infty$ such that:

$$\forall n \geq 1, \qquad N(\pi^n) \leq C_1 N(d^n)^{\beta}.$$

$d = (d^n)_{n\geq 1}$ is also a balanced sequence of partitions of $[0,T]$ and the points of π^n are interspersed along those of d^n. Define for $k = 1, 2, \cdots, N(\pi^n)$:

$$p(n,k) = \inf\{m \geq 1 : \quad s_m^n \in (t_k^n, t_{k+1}^n]\}.$$

Then we get the following inequality regarding the partition points of π^n and d^n.

$$s^n_{p(n,k)-1} \leq t_k^n < s^n_{p(n,k)} < \cdots < s^n_{p(n,k+1)-1} \leq t^n_{k+1} < s^n_{p(n,k+1)}, \tag{2.27}$$

where $p(n, N(\pi^n)) - 1 = N(d^n)$ and $p(n,0) = 1$. For all $t \in [0,T]$ we will show that by grouping the points of d^n according to the intervals defined by π^n and use the roughness property of x along π:

$$[x]_{\pi^n}(t) = \sum_{k=1}^{N(\pi^n)-1} \left(x(t_{k+1}^n \wedge t) - x(t_k^n \wedge t)\right)^{\top} \left(x(t_{k+1}^n \wedge t) - x(t_k^n \wedge t)\right) \quad \text{and}$$

$$[x]_{d^n}(t) = \sum_{k=1}^{N(d^n)-1} \big(x(s^n_{k+1}\wedge t) - x(s^n_k \wedge t)\big)^\top \big(x(s^n_{k+1}\wedge t) - x(s^n_k \wedge t)\big)$$

have the same limits. We define an auxiliary partition

$$\sigma^n = (0 = s^n_{p(n,1)} < s^n_{p(n,2)} < \cdots < s^n_{p(n,N(\pi^n))-1} = T) \qquad \text{and,}$$

$$[x]_{\sigma^n}(t) = \sum_{k=1}^{N(\pi^n)-1} \big(x(s^n_{p(n,k+1)}\wedge t) - x(s^n_{p(n,k)} \wedge t)\big)^\top \big(x(s^n_{p(n,k+1)}\wedge t) - x(s^n_{p(n,k)} \wedge t)\big)$$

and we will show that $[x]_{\pi^n}(t)$ and $[x]_{d^n}(t)$ have the same limit as $[x]_{\sigma^n}(t)$. We shall give the proof for $t = T$; for $t < T$ we have an additional boundary term that goes to zero.

Decomposing $\Delta^n_k = x(t^n_{k+1}) - x(t^n_k)$ along the partition points of d^n, we obtain,

$$\underbrace{x(t^n_{k+1}) - x(t^n_k)}_{\Delta^n_k} = \underbrace{(x(t^n_{k+1}) - x(s^n_{p(n,k+1)})}_{D_k} - \underbrace{(x(t^n_k) - x(s^n_{p(n,k)}))}_{B_k}$$
$$+ \underbrace{\sum_{i=p(n,k)}^{p(n,k+1)-1} (x(s^n_{i+1}) - x(s^n_i))}_{C_k}.$$

Grouping together the terms in $[x]_{d^n}$ according to Equation (2.27) yields

$$[x]_{\pi^n}(T) - [x]_{d^n}(T)$$
$$= \sum_{k=1}^{N(\pi^n)-1} \left[\Delta^{n\top}_k \Delta^n_k - \sum_{i=p(n,k)}^{p(n,k+1)-1} \big(x(s^n_{i+1}) - x(s^n_i)\big)^\top \big(x(s^n_{i+1}) - x(s^n_i)\big) \right]$$
$$= \sum_{k=1}^{N(\pi^n)-1} \big[\Delta^{n\top}_k \Delta^n_k - C^\top_k C_k\big] + \sum_{k=1}^{N(\pi^n)-1} \left[C^\top_k C_k - \sum_{i=p(n,k)}^{p(n,k+1)-1} \big(x(s^n_{i+1}) - x(s^n_i)\big)^\top \big(x(s^n_{i+1}) - x(s^n_i)\big) \right].$$

Now, the second term of the previous equation:

$$\sum_{k=1}^{N(\pi^n)-1} \left[C^\top_k C_k - \sum_{i=p(n,k)}^{p(n,k+1)-1} \big(x(s^n_{i+1}) - x(s^n_i)\big)^\top \big(x(s^n_{i+1}) - x(s^n_i)\big) \right]$$
$$= \sum_{k=1}^{N(\pi^n)-1} \Big[\big(x(s^n_{p(n,k+1)}) - x(s^n_{p(n,k)})\big)^\top \big(x(s^n_{p(n,k+1)}) - x(s^n_{p(n,k)})\big)$$

$$- \sum_{i=p(n,k)}^{p(n,k+1)-1} \left(x(s_{i+1}^n) - x(s_i^n)\right)^\top \left(x(s_{i+1}^n) - x(s_i^n)\right) \Bigg]$$

$$= \sum_{j=1}^{N(\pi^n)-1} \sum_{s_i^n \neq s_{i'}^n \in (t_j^n, t_{j+1}^n]} \left(x(s_{i+1}^n) - x(s_i^n)\right)^\top \left(x(s_{i'+1}^n) - x(s_{i'}^n)\right) \xrightarrow{n\to\infty} 0.$$

The last limit is precisely (2.26) which arises from the roughness of x along π. Additionally note that

$$\forall n \geq 1,\ [x]_{\sigma^n} = \sum_{k=0}^{N(\pi^n)-1} C_k^\top C_k = [x]_{d^n} \quad \text{and,}$$

$$[x]_\sigma = \lim_{n\to\infty} [x]_{\sigma^n} = \lim_{n\to\infty} \sum_{k=0}^{N(\pi^n)-1} C_k^\top C_k = [x]_d = [x]_{\mathbb{T}} < \infty.$$

So $x \in Q_\sigma([0,T], \mathbb{R}^d)$. Now to show that $|[x]_{\pi^n}(T) - [x]_{d^n}(T)| \to 0$ we only need to show that $\left|\sum_{k=1}^{N(\pi^n)-1} \left[\Delta_k^{n\top} \Delta_k^n - C_k^\top C_k\right]\right| \to 0$. Since,

$$\sum_{k=1}^{N(\pi^n)-1} \Delta_k^{nt} \Delta_k^n = \sum_{k=1}^{N(\pi^n)} (C_k + D_k - B_k)^\top (C_k + D_k - B_k)$$

$$= \sum_{k=1}^{N(\pi^n)-1} {C_k}^\top C_k + \sum_{k=1}^{N(\pi^n)-1} (D_k - B_k)^\top (D_k - B_k) - 2 \sum_{k=1}^{N(\pi^n)-1} {C_k}^\top B_k + 2 \sum_{k=1}^{N(\pi^n)-1} {C_k}^\top D_k.$$

we finally obtain,

$$\left|\sum_{k=1}^{N(\pi^n)-1} \left[\Delta_k^{n\top} \Delta_k^n - C_k^\top C_k\right]\right|$$

$$\leq \left|\sum_{k=1}^{N(\pi^n)-1} (D_k - B_k)^\top (D_k - B_k)\right| + \left|2 \sum_{k=1}^{N(\pi^n)-1} {C_k}^\top B_k\right| + \left|2 \sum_{k=1}^{N(\pi^n)-1} {C_k}^\top D_k\right|.$$

Now we will show that as $n \to \infty$, $\left|\sum_{k=1}^{N(\pi^n)-1} (\Delta_k^{n\top} \Delta_k^n - C_k^\top C_k)\right| \to 0$. Since $x \in C^\alpha([0,T], \mathbb{R}^d)$ we have :

$$\forall t \in [0, T-h], \quad \forall h > 0, \qquad \|x(t+h) - x(t)\| \leq \|x\|_\alpha h^\alpha.$$

Now,

$$\left|\sum_{k=1}^{N(\pi^n)-1} D_k^\top D_k\right| \leq \sum_{k=1}^{N(\pi^n)-1} ||D_k||^2 \leq \sum_{k=1}^{N(\pi^n)-1} \|x\|_\alpha^2 |d^n|^{2\alpha}$$

$$\leq \|x\|_\alpha^2 N(\pi^n)|d^n|^{2\alpha} \leq cC^{\frac{1}{\beta}} N(\pi^n)|\pi^n|^{2\alpha/\beta} \overset{n\to\infty}{\to} 0$$

since $\frac{2\alpha}{\beta} > 1$. Similarly we have $\sum_{k=1}^{N(\pi^n)-1} B_k^\top B_k \to 0$. Therefore,

$$\sum_{k=1}^{N(\pi^n)-1} |(D_k - B_k)^\top (D_k - B_k)| \leq 2\left|\sum_{k=1}^{N(\pi^n)-1} D_k^\top D_k\right| + 2\left|\sum_{k=1}^{N(\pi^n)-1} B_k^\top B_k\right| \to 0.$$

Using Hölder's inequality,

$$\left|\sum_{k=1}^{N(\pi^n)-1} D_k^\top C_k\right| \leq \left(\sum_{k=1}^{N(\pi^n)-1} \|D_k\|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{N(\pi^n)-1} \|C_k\|^2\right)^{\frac{1}{2}}.$$

Since the quadratic variation of x along the sequence of partitions σ exists and finite; the sequence $\sum_{k=1}^{N(\pi^n)-1} \|C_k\|^2$ is bounded. Combining this with the estimate above we obtain

$$\left|\sum_{k=1}^{N(\pi^n)-1} D_k^\top C_k\right| \to 0.$$

Similarly, we have, $\left|\sum_{k=1}^{N(\pi^n)-1} B_k^\top C_k\right| \to 0$ as $n \to \infty$. Therefore, $\sum_{k=1}^{N(\pi^n)-1} \left[\Delta_k^{n\top} \Delta_k^n - C_k^\top C_k\right] \to 0$. Hence,

$$\Big|[x]_{\pi^n}(T) - [x]_{d^n}(T)\Big|$$

$$= \left|\sum_{k=1}^{N(\pi^n)-1} \left[\Delta_k^{n\top}\Delta_k^n - C_k^\top C_k\right] + \sum_{k=1}^{N(\pi^n)-1}\left[C_k^\top C_k - \sum_{i=p(n,k)}^{p(n,k+1)-1} \big(x(s_i^n) - x(s_{i-1}^n)\big)^\top \big(x(s_i^n) - x(s_{i-1}^n)\big)\right]\right|$$

$$\leq \left|\sum_{k=1}^{N(\pi^n)-1} \left[\Delta_k^{n\top}\Delta_k^n - C_k^\top C_k\right]\right|$$

$$+ \left|\sum_{k=1}^{N(\pi^n)-1}\left[C_k^\top C_k - \sum_{i=p(n,k)}^{p(n,k+1)-1} \big(x(s_i^n) - x(s_{i-1}^n)\big)^\top \big(x(s_i^n) - x(s_{i-1}^n)\big)\right]\right| \xrightarrow{n\to\infty} 0.$$

Since $d = (d^n)_{n\geq 1}$ is a sub/super-sequence of the dyadic partition, for all $t \in [0,T]$, $[x]_d = [x]_{\mathbb{T}}$. So, $x \in Q_\pi([0,T],\mathbb{R}^d)$ and also,

$$\forall t \in [0,T], \qquad [x]_\pi(t) = [x]_{\mathbb{T}}(t).$$

This concludes the proof. ■

For Hölder continuous paths, the quadratic roughness property along a balanced partition sequence implies existence of quadratic variation along the same sequence:

Corollary 2.17. *Let $\pi = (\pi^n)_{n\geq 1}$ be a balanced partition sequence of $[0,T]$ with $|\pi^n| \to 0$. Then*

$$\forall \beta \in (0, 2\alpha), \qquad R^\beta_\pi([0,T],\mathbb{R}^d) \cap C^\alpha([0,T],\mathbb{R}^d) \subset Q_\pi([0,T],\mathbb{R}^d) \cap Q_{\mathbb{T}}([0,T],\mathbb{R}^d).$$

Proof. Let $\pi \in \mathbb{B}([0,T])$. If $x \in R^\beta_\pi([0,T],\mathbb{R}^d) \cap C^\alpha([0,T],\mathbb{R}^d)$ then from Definition 2.10, $x \in Q_{\mathbb{T}}([0,T],\mathbb{R}^d)$. Since $\beta \in (0,2\alpha)$, from Theorem 2.16 the quadratic variation of x along π exists and is equal its quadratic variation along the dyadic partition. So $x \in Q_\pi([0,T],\mathbb{R}^d)$. ■

In general without the Hölder continuity assumption on $x \in C^0([0,T],\mathbb{R}^d)$, roughness along a partition sequence π does not imply the existence of quadratic variation along π. The following lemma is a simple application of Theorem 2.16:

Lemma 2.18. *Let $\pi = (\pi^n)_{n\geq 1}$ and $\sigma = (\sigma^n)_{n\geq 1}$ be balanced sequences of partitions of $[0,T]$. If $x \in C^\alpha([0,T],\mathbb{R}^d) \cap R^\beta_\pi([0,T],\mathbb{R}^d) \cap R^\gamma_\sigma([0,T],\mathbb{R}^d)$ for some $\beta, \gamma \in (0, 2\alpha)$ then:*

$$x \in Q_\pi([0,T],\mathbb{R}^d) \cap Q_\sigma([0,T],\mathbb{R}^d) \qquad \textit{and,} \qquad \forall t \in [0,T] \qquad [x]_\pi(t) = [x]_\sigma(t).$$

The quadratic roughness property is invariant under C^2 transformation.

Corollary 2.19. *Let $\pi = (\pi^n)_{n\geq 1}$ be a balanced sequence of partitions of $[0,T]$. If $x \in R^\beta_\pi([0,T],\mathbb{R}) \cap C^\alpha([0,T],\mathbb{R})$ for some $0 < \beta < 2\alpha \wedge 1$ then:*

$$\forall f \in C^2([0,T],\mathbb{R}), \forall t \in [0,T]: \quad [f \circ x]_\pi(t) = [f \circ x]_{\mathbb{T}}(t)$$

Proof. Since π and x satisfies the conditions of Theorem 2.16, we can conclude $\forall t \in [0,T]: [x]_\pi(t) = [x]_\mathbb{T}(t)$. So $[f \circ x]_\pi(t)$ can be expresses as:

$$[f \circ x]_\pi(t) = \lim_{n\to\infty} \sum_{\pi^n \cap [0,t]} \left(f \circ x(t_{i+1}^n) - f \circ x(t_i^n)\right)^2 = \int_0^t f' \circ x(u) d[x]_\pi(u)$$

$$= \int_0^t f' \circ x(u) d[x]_\mathbb{T}(u) = \lim_{n\to\infty} \sum_{\mathbb{T}^n \cap [0,t]} \left(f \circ x(s_{j+1}^n) - f \circ x(s_j^n)\right)^2 = [f \circ x]_\mathbb{T}(t).$$

■

2.4.1 Invariant definition of quadratic variation

Let $\mathbb{T} = (\mathbb{T}^n)_{n\geq 1}$ be the dyadic sequence of partitions of $[0,T]$. Define,

$$\mathcal{Q}([0,T],\mathbb{R}^d) = C^{\frac{1}{2}-}([0,T],\mathbb{R}^d) \cap Q_\mathbb{T}([0,T],\mathbb{R}^d). \tag{2.28}$$

Lemma 2.20. *The class $\mathcal{Q}([0,T],\mathbb{R}^d)$ is non-empty and contains all 'typical' Brownian paths.*

Proof. Let W be a Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which we take to be the canonical Wiener space without loss of generality. For dyadic partition $\mathbb{T}$, since we have $|\mathbb{T}^n| = \frac{1}{2^n}$, so $|\mathbb{T}^n| \log(n) \to 0$. So from Dudley [34] we can conclude:

$$\mathbb{P}\left[W \in Q_\mathbb{T}\left([0,T],\mathbb{R}^d\right)\right] = 1.$$

Brownian paths are almost-surely α-Hölder for $\alpha < \frac{1}{2}$, so

$$\mathbb{P}\left(W \in Q_\mathbb{T}([0,T],\mathbb{R}^d) \cap C^{\frac{1}{2}-}([0,T],\mathbb{R}^d)\right) = 1,$$

hence the result follows. ■

Based on the results above we can now give an 'intrinsic' definition of pathwise quadratic variation for paths in $\mathcal{Q}([0,T],\mathbb{R}^d)$ which does not rely on a particular partition sequence:

Proposition 2.21 (Quadratic variation map)**.** *There exists a unique map:*

$$[\,.\,] \quad : \mathcal{Q}\left([0,T],\mathbb{R}^d\right) \;\to\; C^0\left([0,T], S_d^+\right)$$

$$x \mapsto [x]$$

such that for all $t \in [0,T]$

$$\forall \pi \in \mathbb{B}([0,T]),\ \forall \beta \in (0,1),\ \forall x \in R^{\beta}_{\pi}([0,T],\mathbb{R}^d) \cap \mathcal{Q}([0,T],\mathbb{R}^d); \quad [x]_\pi(t) = [x](t).$$

We call $[x]$ *the quadratic variation of* x*.*

Proof. Let $\pi \in \mathbb{B}([0,T])$. Then for $\beta \in (0,1)$ and for any $x \in R^{\beta}_{\pi}([0,T],\mathbb{R}^d) \cap \mathcal{Q}([0,T],\mathbb{R}^d)$ take $\alpha = \frac{\beta+1}{4} < \frac{1}{2}$ as $\beta < 1$. So we have

$$x \in R^{\beta}_{\pi}([0,T],\mathbb{R}^d) \cap C^{\alpha}([0,T],\mathbb{R}^d) \cap Q_{\mathbb{T}}([0,T],\mathbb{R}^d).$$

Then Theorem 2.16 implies

$$x \in Q_\pi([0,T],\mathbb{R}^d) \qquad \text{and,} \qquad \forall t \in [0,T],\ [x]_\pi(t) = [x]_{\mathbb{T}}(t).$$

By the same argument the quadratic variation does not depend on the choice of $\pi \in \mathbb{B}([0,T])$, so the result follows. ■

Remark 2.22. If X is a continuous $\mathbb{P}$-semimartingale then its image $[X]$ under the map defined in Proposition 2.21 coincides almost-surely with the probabilistic definition of quadratic variation as a limit in probability [54, 65]. Building on [54], Karandikar and Rao [55] construct a (different) quadratic variation map which shares this property. In contrast to [55], our construction does not use any probabilistic tools, does not rely on specific path-dependent partitions and identifies explicitly the domain of definition of the map (rather than implicitly in terms of the support of a probability measure).

2.5 Pathwise Itô calculus

In this section we give a robust formulation of the pathwise Föllmer-Itô calculus and extend the results to local time.

2.5.1 Pathwise integration and the Föllmer-Itô formula

Using Theorem 2.16 and Proposition 2.21, we can give a formulation of Föllmer's pathwise Itô calculus which is invariant with respect to the choice of the sequence of partitions π.

Theorem 2.23 (Invariance of the Föllmer integral)**.** *There exists a unique map*

$$\begin{array}{rrcl} I: & C^2(\mathbb{R}^d)\times\mathcal{Q}([0,T],\mathbb{R}^d) & \to & \mathcal{Q}([0,T],\mathbb{R}) \\ & (f,x) & \to & I(f,x)=\int_0^{\cdot}(\nabla f\circ x).dx, \end{array}$$

such that: $\forall\pi\in\mathbb{B}([0,T])$, $\forall\beta\in(0,1)$, $\forall x\in R^{\beta}_{\pi}([0,T],\mathbb{R}^d)\cap\mathcal{Q}([0,T],\mathbb{R}^d)$, $\forall t\in[0,T]$,

$$I(f,x)(t)=\int_0^t(\nabla f\circ x).d^{\pi}x=\lim_{n\to\infty}\sum_{\pi^n}\nabla f(x(t_i^n)).(x(t_{i+1}^n\wedge t)-x(t_i^n\wedge t)).$$

We denote $I(f,x)=\int_0^{\cdot}(\nabla f\circ x)dx$*. Furthermore,*

$\forall f\in C^2(\mathbb{R}^d),\ \forall\pi\in\mathbb{B}([0,T]),\ \forall\beta\in(0,1),\ \forall x\in R^{\beta}_{\pi}([0,T],\mathbb{R}^d)\cap\mathcal{Q}([0,T],\mathbb{R}^d)$,

$$f(x(t))-f(x(0))=\int_0^t(\nabla f\circ x).dx+\frac{1}{2}\int_0^t<\nabla^2 f(x),d[x]> \tag{2.29}$$

$$\text{and}\qquad\left[\int_0^{\cdot}(\nabla f\circ x)\ dx)\right]_{\pi}(t)=\int_0^t<(\nabla f\circ x)^{\top}(\nabla f\circ x),d[x]>. \tag{2.30}$$

Proof. For any $\beta\in(0,1)$, take $\alpha=\frac{\beta+1}{4}<\frac{1}{2}$. So, if $x\in Q_{\pi}([0,T],\mathbb{R}^d)\cap\mathcal{Q}([0,T],\mathbb{R}^d)$ then $x\in R^{\beta}_{\pi}([0,T],\mathbb{R}^d)\cap C^{\alpha}([0,T],\mathbb{R}^d)\cap C_{\mathbb{T}}([0,T],\mathbb{R}^d)$. Then for any balanced partition sequence $\pi\in\mathbb{B}([0,T])$ the pathwise Ito formula [35] implies

$$\int_0^t(\nabla f\circ x).d^{\pi}x=f(x(t))-f(x(0))-\frac{1}{2}\int_0^t<\nabla^2 f(x),d[x]_{\pi}>,$$

$$\text{and}\qquad\int_0^t(\nabla f\circ x).d^{\mathbb{T}}x=f(x(t))-f(x(0))-\frac{1}{2}\int_0^t<\nabla^2 f(x),d[x]_{\mathbb{T}}>.$$

Since all assumptions of Theorem 2.16 are satisfied for the path x along the partition sequence π, we conclude $[x]_{\mathbb{T}}=[x]_{\pi}$. This argument is true for all $\beta\in(0,1)$ and for all $\pi\in\mathbb{B}([0,T])$. So:

$$\forall\beta\in(0,1),\ \forall\pi\in\mathbb{B}([0,T])\ \forall t\in[0,T]:\quad\int_0^t(\nabla f\circ x).d^{\pi}x=\int_0^t(\nabla f\circ x).d^{\mathbb{T}}x$$

i.e. the pathwise integral $\int_0^t (\nabla f \circ x).d^\pi x$ along a balanced sequence of partitions π does not depend on choice of π. To show $I(f,x) \in \mathcal{Q}([0,T],\mathbb{R})$ we first note that by [2, Lemma 4.11] we have $I(f,x) \in C^{\frac{1}{2}-}([0,T],\mathbb{R})$.

For all $\beta \in (0,1)$ and for all $\pi \in \mathbb{B}([0,T])$ using the Theorem 2.16 we can conclude $x \in \mathcal{Q}([0,T],\mathbb{R}) \cap \mathcal{R}^\beta_\pi([0,T],\mathbb{R})$ implies $x \in Q_\pi([0,T],\mathbb{R})$. Now applying the pathwise isometry formula [3, Theorem 2.1], to the integral $\int_0^\cdot (\nabla f \circ x).d^\pi x$ we obtain that $\int_0^\cdot (\nabla f \circ x).d^\pi x = \int_0^\cdot (\nabla f \circ x).d^{\mathbb{T}} x \in Q_\pi([0,T],\mathbb{R}) \cap Q_{\mathbb{T}}([0,T],\mathbb{R})$ and

$$\left[\int_0^\cdot \nabla f \circ x \; dx\right]_\pi (t) = \int_0^t < (\nabla f \circ x)^\top (\nabla f \circ x), d[x]_\pi > .$$

From Theorem 2.16 we have $[x]_{\mathbb{T}} = [x]_\pi$, so $\left[\int_0^\cdot \nabla f \circ x \; dx\right]_\pi (t)$ does not depend on choice of balanced partition π. As a consequence:

$$\left[\int_0^\cdot \nabla f \circ x \; dx\right]_\pi (t) = \left[\int_0^\cdot \nabla f \circ x \; dx\right]_{\mathbb{T}} (t) = \int_0^t < (\nabla f \circ x)^\top (\nabla f \circ x), d[x] > .$$

So finally $I(f,x) \in \mathcal{Q}([0,T],\mathbb{R})$. ■

2.5.2 Local time

Pathwise analogues of (semimartingale) local time have been considered in [10, 28, 29, 56, 63, 77] in the context of Tanaka-type formulas for convex functions or functions with Sobolev regularity. One such construction of such local times involves taking a limit of a sequence of discrete approximations of occupation densities along a sequence of time partitions [28, 30].

Given a sequence of partition $\sigma = (\sigma^n)_{n\geq 1}$ and a path $x \in C^0([0,T],\mathbb{R}) \cap Q_\sigma([0,T],\mathbb{R})$, define the function $L_t^{\sigma^n} : \mathbb{R} \to \mathbb{R}$ by

$$L_t^{\sigma^n}(u) := 2 \sum_{t_j^n \in \sigma^n \cap [0,t]} \mathbb{1}_{[[x(t_j^n), x(t_{j+1}^n))]}(u) \; |x(t_{j+1}^n \wedge t) - u|.$$

where $[[u,v)] := [u,v)$ if $u \leq v$ and $[[u,v)] := [v,u)$ if $u > v$. $L_t^{\sigma^n}$ is bounded and zero outside $[\min x, \max x]$.

Following [77, 10, 29, 63] we say that x has (L^2)- local time on $[0,T]$ along σ if the sequence $(L_t^{\sigma^n}, n \geq 1)$ converges weakly in $L^2(\mathbb{R})$ to a limit L_t^σ for all $t \in [0,T]$:

$$\forall t \in [0,T], \quad \forall h \in L^2(\mathbb{R}), \quad \int L_t^{\sigma^n}(u) h(u) du \xrightarrow{n\to\infty} \int L_t^\sigma(u) h(u) du.$$

The local time along π satisfies the occupation time formula [77, 10, 63]: for every Borel set $A \in \mathcal{B}(R)$,

$$\int_A L_t^\pi(u)du = \frac{1}{2}\int_0^t \mathbb{1}_A(x)d[x]_\pi$$

and the following extension of the pathwise Ito formula (2.29) to functions in the Sobolev space $W^{2,2}(\mathbb{R})$ (see e.g. [30, Thm 3.1]):

$$\forall f \in W^{2,2}(\mathbb{R}),\ f(x(t)) - f(x(0)) = \int_0^t (f' \circ x).d^\pi x + \frac{1}{2}\int_{\mathbb{R}} L_t^\pi(u) f''(u)du \quad (2.31)$$

where the first integral is a limit of left Riemann sums along π:

$$\int_0^t (f' \circ x).d^\pi x := \lim_{n\to\infty} \sum_{\pi^n} f'(x(t_i^n)).(x(t_{i+1}^n) - x(t_i^n)).$$

Unlike the intrinsic definition of occupation densities for real functions (see e.g. [43]), the above construction depends on the choice of the partition sequence π and a natural question is therefore to clarify the dependence of this local time on the choice of the partition sequence. Note that, differently from [43, 44], L_t^π is the density of a *weighted* occupation measure, weighted by quadratic variation $[x]_\pi$ so a necessary condition for the uniqueness of L_t^π is the uniqueness of $[x]_\pi$.

We now show that the quadratic roughness property implies an invariance property of the *local time* with respect to the sequence of partitions:

Theorem 2.24 (Invariance of local time under quadratic roughness)**.** *Let $x \in C^\alpha([0,T],\mathbb{R}) \cap R_\pi^\beta([0,T],\mathbb{R})$ with $0 < \beta \le 2\alpha \le 1$. Assume x has local time L_t^π on $[0,t]$ along π. Then if x has local time $L_t^{\mathbb{T}}$ on $[0,t]$ along the dyadic partition sequence $\mathbb{T} \in \mathbb{B}(0,T])$ we can conclude*

$$\forall t \in [0,T], \qquad L_t^\pi(u) = L_t^{\mathbb{T}}(u) \qquad du - a.e.$$

This defines a unique element $L_t \in L^2(\mathbb{R})$ which we call the local time of x on $[0,t]$.

This result shows that for paths satisfying the quadratic roughness property, the (L^2)- local time is an intrinsic object associated with the path x, independent of the (balanced) sequence of partitions used in the construction.

Proof. From [77, Satz 9] for any Borel set $A \in \mathcal{B}(R)$ we have the occupation density formula as:

$$\int_A L_t^\pi(u)du = \frac{1}{2}\int_0^t \mathbb{1}_A(x)d[x]_\pi.$$

If the local time along $\mathbb{T}$ exists, we also have:

$$\forall A \in \mathcal{B}(R), \quad \int_A L_t^{\mathbb{T}}(u)du = \frac{1}{2}\int_0^t \mathbb{1}_A(x)d[x]_{\mathbb{T}}.$$

Since π is balanced, Theorem 2.16 implies that $[x]_\pi = [x]_{\mathbb{T}}$. Hence,

$$\forall A \in \mathcal{B}(R), \quad \int_A L_t^\pi(u)du = \int_A L_t^{\mathbb{T}}(u)du,$$

which implies $L_t^\pi = L_t^{\mathbb{T}}$ almost everywhere. ∎

An important consequence of this result is the uniqueness of limits of left Riemann sums for integrands in the Sobolev space $W^{1,2}(\mathbb{R})$ and a robust version of the pathwise Tanaka formula [10, 30]:

Corollary 2.25 (Uniqueness of Föllmer integral on $W^{1,2}(\mathbb{R})$ and pathwise Tanaka formula)**.**
Under the assumptions of Theorem 2.24 we have:

$$\forall h \in W^{1,2}(\mathbb{R}), \forall t \in [0,T], \qquad \int_0^t (h \circ x)d^\pi x = \int_0^t (h \circ x)d^{\mathbb{T}} x.$$

Designating this common value by $\int_0^t (h \circ x)dx$, we obtain for all $f \in W^{2,2}(\mathbb{R})$

$$\forall t \in [0,T], \quad f(x(t)) - f(x(0)) = \int_0^t (f' \circ x).dx + \frac{1}{2}\int_{\mathbb{R}} L_t(u)f''(u)du, \quad (2.32)$$

where the pathwise integral and the local time may be computed with respect to any balanced partition sequence along which x has quadratic roughness and local time.

Chapter 3

Quadratic variation along refining partitions: constructions and examples

Chapter based on: Rama Cont, Purba Das. Quadratic variation along refining partitions: Constructions and examples [23].

Examples of functions with (non-zero) finite quadratic variation are given by typical sample paths of Brownian motion and semi-martingales, but explicit constructions of such functions have also been given by Gantert [41], Schied [70] and Mishura and Schied [62], in the spirit of Takagi's construction [72]. These constructions are based on a Faber-Schauder representation associated with a dyadic sequence of partitions and exploit certain identities which result from the dyadic nature of the construction. The question therefore arises whether such constructions may be carried out for non-dyadic and, more generally, non-uniform partitions sequences. In this chapter we first provide constructions of general Haar basis, and extended Garnert's [41] quadratic variation and co-vairation formula to the non-uniform case. We also present several constructions of paths and processes with finite quadratic variation along a refining sequence of partitions, extending previous constructions to the non-uniform case.

Though it is well-known that for semimartingales and, more generally, Dirichlet processes [36], quadratic variation, defined as a limit in probability, is invariant with respect to the choice of the partition sequence as long as it has vanishing step size. Lévy [58, 59], shows that, for Brownian motion almost surely for any refining sequence of partitions π the quadratic variation is linear and same across partitions, i.e. $\mathbb{P}([W]_\pi(t) = t) = 1$. Though conditions for such an invariance of quadratic variation have been studied in Chapter 2 but some of the aforementioned constructions, based on the dyadic partition, do not fulfil these conditions mentioned in Chapter 2. The question therefore arises whether such constructions may be carried out for non-dyadic and, more generally, non-uniform partitions sequences and whether the quadratic variation of the resulting functions is invariant with respect to the partition sequence. We identify a class of paths and processes whose quadratic variation along a partition sequence is invariant under *coarsening* of the partition sequence (non necessarily balanced or uniform). This 'rough' class is shown to include typical sample paths of Brownian motion, but also includes paths which are $\frac{1}{2}$-Hölder continuous (So smoother than BM in terms of Hölder continuity). Finally, we extend these constructions to higher dimensional. This class of 'Rough' processes are continuous but nowhere differentiable (Like Takagi functions [72]).

3.1 Schauder system associated with a finitely refining partition sequence

The constructions in [41, 70, 62] make use of the Haar basis [46] and Schauder system [69, 71] both of which are associated with a dyadic partition sequences. We extend these constructions to the class of finitely refining partition sequences. We construct an orthonormal non-uniform Haar basis and a corresponding Schauder system along any finitely refining sequence of partitions. Francois et al. [37], have introduced a generalized Haar basis as well, but their construction does not generate an orthonormal basis like our construction.

3.1.1 Sequences of interval partitions

Definition 3.1 (Refining sequence of partition). *A sequence of partitions* $\pi = (\pi^n)_{n\geq 1}$ *of* $[0,T]$ *with*

$$\pi^n = \left(0 = t_1^n < t_2^n < \cdots < t_{N(\pi^n)}^n = T\right),$$

is said to be a refining (or nested) sequence of partitions if

$$\textit{for all } m \geq 1, \quad t \in \pi^m \implies t \in \cap_{n\geq m}\pi^n.$$

In particular, $\pi^1 \subseteq \pi^2 \subseteq \cdots$. Now we introduce a subclass of refining partitions that have a 'finite branching' property at every level.

Definition 3.2 (Finitely refining sequence of partitions). *We call a sequence of partitions* π *of* $[0,T]$ *to be a finitely refining sequence of partitions if* π *is refining with mesh* $|\pi^n| \to 0$ *and there exists* $M < \infty$ *such that the number of partition points of* π^{n+1} *within any two consecutive partition points of* π^n *is always bounded above by* M*, irrespective of* $n \in \mathbb{N}$.

For a finitely refining sequence of partitions π, there exists $M < \infty$ such that $\sup_n \frac{N(\pi^n)}{M^n} \leq 1$. A subsequence of a finitely refining sequence may not be a finitely refining sequence but has to be a refining sequence. This property ensures the partition has locally finite branching at every step but do not ensure any global bound on partitions size. This is ensured by the balanced property (Chapter 2, Def. 2.4). If a sequence of partitions π of $[0,T]$ is finitely refining and balanced at the same time (for example dyadic/uniform partition) then

$$\limsup_n \frac{|\pi^n|}{\underline{\pi^{n+1}}} < \infty.$$

Definition 3.3 (Complete refining partition). *A refining sequence of partitions* $\pi = (\pi^n)_{n\geq 1}$ *of* $[0,T]$ *is said to be complete refining if there exist positive constants* ϵ *and* M *such that:*

$$\textit{for all } n \geq 1, \quad 1+\epsilon \leq \frac{|\pi^n|}{|\pi^{n+1}|} \leq M.$$

3.1.2 Haar basis associated with a finitely refining partition sequence

Let π be a finitely refining sequence of refining partition of $[0,1]$

$$\pi^n = \big(0 = t_0^n < t_1^n < \cdots < t_{N(\pi^n)}^n = 1\big)$$

with mesh $|\pi^n| \to 0$. Now define $p(n,k)$ as follows:

$$p(n,k) = \inf\{j \geq 0 \ : \ t_j^{n+1} \geq t_k^n\}.$$

Since π is refining the following inequality holds:

$$\text{for all } \ k = 0, \cdots, N(\pi^n)-1, \qquad 0 \leq t_k^n = t_{p(n,k)}^{n+1} < t_{p(n,k)+1}^{n+1} < \cdots < t_{p(n,k+1)}^{n+1} = t_{k+1}^n \leq 1. \tag{3.1}$$

We now define the Haar basis associated with such as partition sequence:

Definition 3.4 (Haar basis)**.** *The Haar basis associated with a finitely refining partition sequence $\pi = (\pi^n)_{n\geq 1}$ is a collection of piece-wise constant functions $\{\psi_{m,k,i}, m = 0,1,\cdots, k = 0,\cdots, N(\pi^m)-1, i = 1,\cdots, p(m,k+1)-p(m,k)\}$ defined as follows:*

$$\psi_{m,k,i}(t) = \begin{cases} 0 & \text{if } t \notin \left[t_{p(m,k)}^{m+1}, t_{p(m,k)+i}^{m+1}\right) \\ \left(\frac{t_{p(m,k)+i}^{m+1} - t_{p(m,k)+i-1}^{m+1}}{t_{p(m,k)+i-1}^{m+1} - t_{p(m,k)}^{m+1}} \times \frac{1}{t_{p(m,k)+i}^{m+1} - t_{p(m,k)}^{m+1}}\right)^{\frac{1}{2}} & \text{if } t \in \left[t_{p(m,k)}^{m+1}, t_{p(m,k)+i-1}^{m+1}\right) \\ -\left(\frac{t_{p(m,k)+i-1}^{m+1} - t_{p(m,k)}^{m+1}}{t_{p(m,k)+i}^{m+1} - t_{p(m,k)+i-1}^{m+1}} \times \frac{1}{t_{p(m,k)+i}^{m+1} - t_{p(m,k)}^{m+1}}\right)^{\frac{1}{2}} & \text{if } t \in \left[t_{p(m,k)+i-1}^{m+1}, t_{p(m,k)+i}^{m+1}\right). \end{cases} \tag{3.2}$$

Note, $t_{p(m,k)+i-1}^{m+1} \in \pi^{m+1} \backslash \pi^m$ for all i and $t_{p(m,k)}^{m+1} = t_k^m \in \pi^m \cap \pi^{m+1}$. Since π is a finitely refining sequence of partitions $p(m,k+1) - p(m,k) \leq M < \infty$, for all m,k.

For any finitely refining partition π, the family of functions $\{\psi_{m,k,i}\}_{m,k,i}$ can be reordered as $\{\psi_{m,k}\}_{m,k}$. For each level $m \in \{0,1,\cdots\}$, the values of k run from 0 to $N(\pi^{m+1}) - N(\pi^m) - 1$ (after reordering).

Example 3.1. *Figure 3.1 represents the generalized Haar basis $\psi_{m,k}^{\pi}$ for $m = 0,1,2,3$ constructed along triadic sequence.*

Figure 3.2 represents the generalized Haar basis $\psi^{\pi}_{m,k}$ for $m = 1, 2, 3, 4$ constructed along partition sequence π. Where the non-uniform (not balanced) doubly refining sequence of partitions $\pi^n = \left(0 = t^n_1 < \cdots < t^n_{N(\pi^n)}\right)$ as follows.

$$\forall k = 1, \cdots, 2^n, \qquad t^{n+1}_{2k} = t^n_k \ \text{ and, } \ t^{n+1}_{2k+1} = t^n_k + \frac{2(t^n_{k+1} - t^n_k)}{3}$$

The following properties are easily derived from the definition:

Proposition 3.5. *The non-uniform Haar basis along a finitely refining sequence of partitions $\pi = (\pi^n)_{n\geq 1}$ has the following properties:*
(i). For fixed $m \in \{0\}\cup\mathbb{N}$, the piece-wise constant functions $\psi_{m,k,i}(t)$ and $\psi_{m,k',i'}(t)$ have disjoint supports for all $k \neq k' \in \{0, 1, \cdots, N(\pi^m) - 1\}$ and for all i, i'.
(ii). For fixed $m \in \{0\} \cup \mathbb{N}$ and fixed k, the support of the piece-wise constant function $\psi_{m,k,i}(t)$ is contained in the support of $\psi_{m,k,i'}(t)$ as soon as $i \leq i'$.
(iii). For all $m \in \{0\} \cup \mathbb{N}$, for all $k \in \{0, 1, \cdots, N(\pi^m) - 1\}$ and for all i

$$\int_{\mathbb{R}} \psi_{m,k,i}(t)dt = \int_0^1 \psi_{m,k,i}(t)dt = 0.$$

(iv). Orthogonality:

$$\int_{\mathbb{R}} \psi_{m,k,i}(t)\psi_{m',k',i'}(t)dt = \int_0^1 \psi_{m,k,i}(t)\psi_{m',k',i'}(t)dt = \mathbb{1}_{m,m'}\mathbb{1}_{k,k'}\mathbb{1}_{i,i'},$$

where $\mathbb{1}_{a,b}$ is 1 if $a = b$ and 0 otherwise.

As a consequence of (iii) and (iv), the family $\{\psi_{m,k,i}; \quad \forall m, k, i\}$ is an orthonormal family.

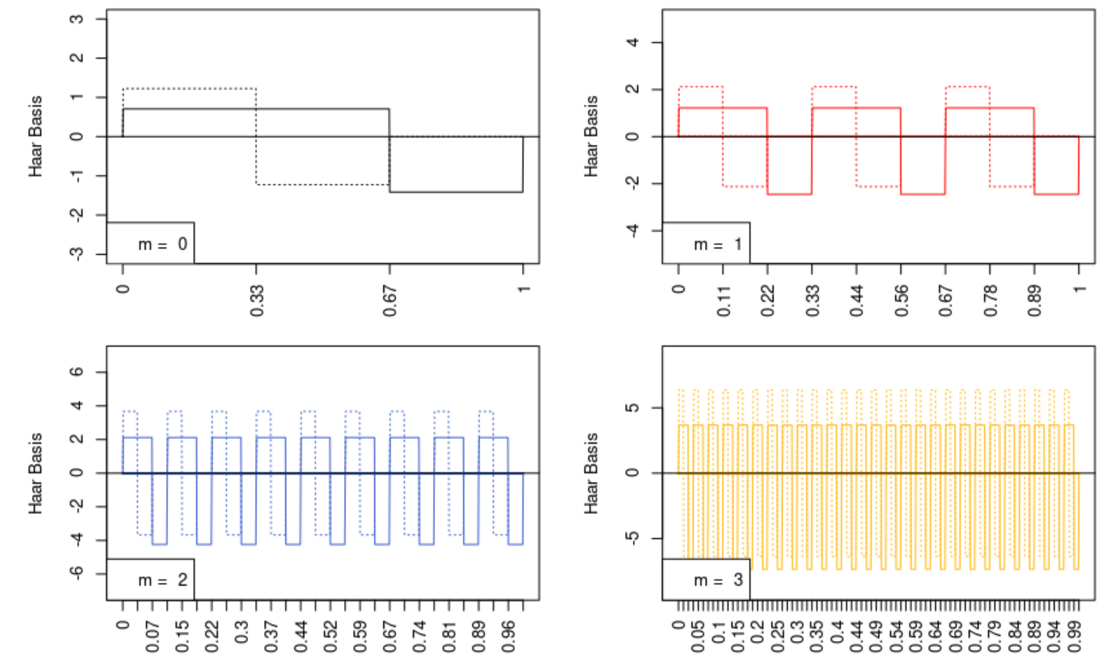

Figure 3.1: Haar basis along triadic partition

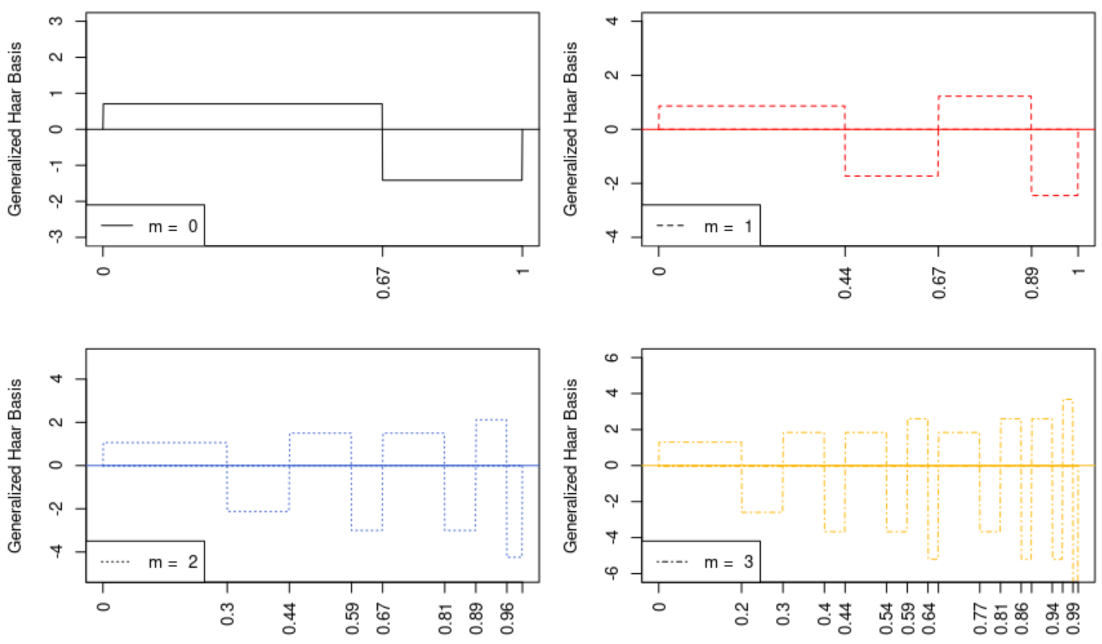

Figure 3.2: Haar basis along a non-uniform not balanced doubly refining partition π (Defined in Example 3.1).

3.1.3 Schauder representation of a continuous function

The Schauder basis functions $e^{\pi}_{m,k,i}$ are obtained by integrating the Haar basis functions:

$$e^{\pi}_{m,k,i} : [0,1] \to \mathbb{R} \text{ where,}$$

$$e^{\pi}_{m,k,i}(t) = \int_0^t \psi_{m,k,i}(s)ds = \left(\int_{t^{m+1}_{p(m,k)}}^{t \wedge t^{m+1}_{p(m,k)+i}} \psi_{m,k,i}(s)ds \right) \mathbb{1}_{[t^m_k, t^{m+1}_{p(m,k)+i}]}.$$

For all m, k, i the functions $e^{\pi}_{m,k,i} : [0,1] \to \mathbb{R}$ are continuous but not differentiable and

$$e^{\pi}_{m,k,i}(t) = \begin{cases} 0 & \text{if } t \notin \left[t^{m+1}_{p(m,k)}, t^{m+1}_{p(m,k)+i}\right) \\ \left(\frac{t^{m+1}_{p(m,k)+i} - t^{m+1}_{p(m,k)+i-1}}{t^{m+1}_{p(m,k)+i-1} - t^{m+1}_{p(m,k)}} \times \frac{1}{t^{m}_{p(m,k)+i} - t^{m}_{p(m,k)}} \right)^{\frac{1}{2}} \times (t - t^{m+1}_{p(m,k)}) & \text{if } t \in \left[t^{m+1}_{p(m,k)}, t^{m+1}_{p(m,k)+i-1}\right) \\ \left(\frac{t^{m+1}_{p(m,k)+i-1} - t^{m+1}_{p(m,k)}}{t^{m+1}_{p(m,k)+i} - t^{m+1}_{p(m,k)+i-1}} \times \frac{1}{t^{m+1}_{p(m,k)+i} - t^{m+1}_{p(m,k)}} \right)^{\frac{1}{2}} \times (t^{m+1}_{p(m,k)+i} - t) & \text{if } t \in \left[t^{m+1}_{p(m,k)+i-1}, t^{m+1}_{p(m,k)+i}\right) \end{cases} . \quad (3.3)$$

Example 3.2. *Figure 3.3 represents the Faber-Schauder basis $e^{\pi}_{m,k}$ for $m = 0, 1, 2$ constructed along triadic sequence.*

Figure 3.4 represents the Faber Schauder basis $e^{\pi}_{m,k}$ for $m = 0, 1, 2, 3$ constructed along partition sequence π. Where the non-uniform (not balanced) doubly refining sequence of partitions $\pi^n = \left(0 = t^n_1 < \cdots < t^n_{N(\pi^n)}\right)$ as follows.

$$\forall k = 1, \cdots, 2^n, \qquad t^{n+1}_{2k} = t^n_k \ \ \textit{and,} \ \ t^{n+1}_{2k+1} = t^n_k + \frac{2(t^n_{k+1} - t^n_k)}{3}$$

Assume that $x \in C^0([0,1], \mathbb{R})$ is a continuous function with the following Schauder representation along a finitely refining sequence of partitions π:

$$x(t) = a_0 + a_1 t + \sum_{m=0}^{\infty} \sum_{k=0}^{N(\pi^{m+1}) - N(\pi^m) - 1} \theta_{m,k} e^{\pi}_{m,k}(t),$$

where, for all m, k, the coefficients $a_0, a_1, \theta_{m,k} \in \mathbb{R}$; are constants. Denote by $x^N(t) : [0,1] \to \mathbb{R} \in C^0([0,1], \mathbb{R})$ the linear interpolation of x along partition points of π^N:

$$x^N(t) = a_0 + a_1 t + \sum_{m=0}^{N-1} \sum_{k=0}^{N(\pi^{m+1}) - N(\pi^m) - 1} \theta_{m,k} e^{\pi}_{m,k}(t).$$

Lemma 3.6. *For all $N \geq 2$, for all $t \in \pi^N$ one have, $x(t) = x^N(t)$.*

Proof. From the construction of $e^{\pi}_{m,k}$, for all $m \geq N$, and for all k, we have $e^{\pi}_{m,k}(t^N_i) = 0$. So for $t \in \pi^N$:

$$x(t) = \sum_{m=0}^{\infty} \sum_{k=0}^{N(\pi^{m+1})-N(\pi^m)-1} \theta_{m,k} e^{\pi}_{m,k}(t) = \sum_{m=0}^{N-1} \sum_{k=0}^{N(\pi^{m+1})-N(\pi^m)-1} \theta_{m,k} e^{\pi}_{m,k}(t) = x^N(t).$$

■

If the sequence of partitions π has vanishing mesh then as a limit the continuous function x^N converges to $x \in C^0([0,1],\mathbb{R})$ in uniform norm

$$\lim_{N\to\infty} \sup_{t\in[0,1]} \left| x^N(t) - x(t) \right| = 0.$$

Theorem 3.7. *Let π be a finitely refining sequence of partitions of $[0,T]$. Then any $x \in C^0([0,1],\mathbb{R})$ has a unique Schauder representation:*

$$x(t) = x(0) + (x(1) - x(0))t + \sum_{m=0}^{\infty} \sum_{k=0}^{N(\pi^{m+1})-N(\pi^m)-1} \theta_{m,k} e^{\pi}_{m,k}(t).$$

If the support of the function $e^{\pi}_{m,k}$ is $[t^{m,k}_1, t^{m,k}_3]$ and its maximum is attained at time $t^{m,k}_2$ then, the coefficient $\theta_{m,k}$ has a closed form representation as follows:

$$\theta_{m,k} = \frac{\left[\left(x(t^{m,k}_2) - x(t^{m,k}_1) \right) (t^{m,k}_3 - t^{m,k}_2) - \left(x(t^{m,k}_3) - x(t^{m,k}_2) \right) (t^{m,k}_2 - t^{m,k}_1) \right]}{\sqrt{(t^{m,k}_2 - t^{m,k}_1)(t^{m,k}_3 - t^{m,k}_2)(t^{m,k}_3 - t^{m,k}_1)}}. \tag{3.4}$$

Proof. Take the function y as $y(t) = x(t) - x(0) + (x(0) - x(1))t$. Since x is a continuous function, so does y. Also for the function y we have $y(0) = y(1) = 0$. So without loss of generality we will assume $x(0) = x(1) = 0$ for the rest of the proof.

Since $t^{m,k}_1, t^{m,k}_2, t^{m,k}_3 \in \pi^{m+1}$, using Proposition 3.6 we get:

$$x(t^{m,k}_1) = x^{m+1}(t^{m,k}_1), \quad x(t^{m,k}_2) = x^{m+1}(t^{m,k}_2) \quad \text{and} \quad x(t^{m,k}_3) = x^{m+1}(t^{m,k}_3).$$

Now we can write the increment $x(t^{m,k}_2) - x(t^{m,k}_1)$ as follows.

$$x(t^{m,k}_2) - x(t^{m,k}_1) = \left(x^{m+1}(t^{m,k}_2) - x^{m+1}(t^{m,k}_1) \right)$$

$$= \sum_{n=0}^{m} \sum_{\{k:\psi_{n,k}(t_1^{m,k})\neq 0\}} \theta_{n,k} \times \psi_{n,k}(t_1^{m,k}) \times (t_2^{m,k} - t_1^{m,k}),$$

where k is such that for which the function $\psi_{n,k}$ has strictly positive value in the interval $(t_1^{m,k}, t_2^{m,k})$. Now one can notice that for all $n < m$, $\psi_{n,k(.)}(t_1^{m,k}) = \psi_{n,k(.)}(t_2^{m,k})$. So for the expansion of weighted second difference $\left(x(t_2^{m,k}) - x(t_1^{m,k})\right)\left(t_3^{m,k} - t_2^{m,k}\right) - \left(x(t_3^{m,k}) - x(t_2^{m,k})\right)\left(t_2^{m,k} - t_1^{m,k}\right)$, all values cancel out except for the term involving $\theta_{m,k}$. So we get the following identity:

$$\left(x(t_2^{m,k}) - x(t_1^{m,k})\right)(t_3^{m,k} - t_2^{m,k}) - \left(x(t_3^{m,k}) - x(t_2^{m,k})\right)(t_2^{m,k} - t_1^{m,k}))$$

$$= \theta_{m,k}\left[\psi_{m,k}(t_1^{m,k}) \times (t_2^{m,k} - t_1^{m,k})(t_3^{m,k} - t_2^{m,k}) - \psi_{m,k}(t_2^{m,k}) \times (t_3^{m,k} - t_2^{m,k})(t_2^{m,k} - t_1^{m,k})\right]$$

$$= \theta_{m,k} \times (t_3^{m,k} - t_2^{m,k})(t_2^{m,k} - t_1^{m,k}) \left[\left(\frac{t_3^{m,k} - t_2^{m,k}}{t_2^{m,k} - t_1^{m,k}} \times \frac{1}{t_3^{m,k} - t_1^{m,k}}\right)^{\frac{1}{2}} + \left(\frac{t_2^{m,k} - t_1^{m,k}}{t_3^{m,k} - t_2^{m,k}} \times \frac{1}{t_3^{m,k} - t_1^{m,k}}\right)^{\frac{1}{2}}\right]$$

$$= \theta_{m,k} \times \sqrt{(t_3^{m,k} - t_2^{m,k})(t_2^{m,k} - t_1^{m,k})} \times \left[\sqrt{t_3^{m,k} - t_1^{m,k}}\right].$$

Note that the value of $\theta_{m,k}$ only depends on the function x and the partition π. So the result follows. ■

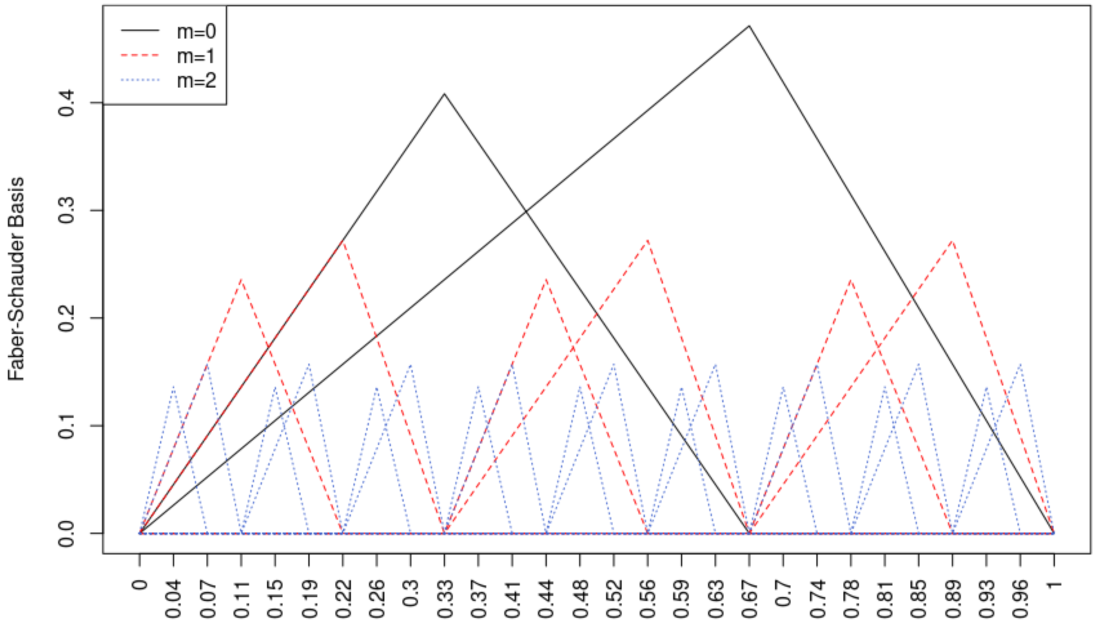

Figure 3.3: Plot of Faber-Schauder basis $e^{\pi}_{m,k}$ for $m = 0, 1, 2$ along triadic partition sequence.

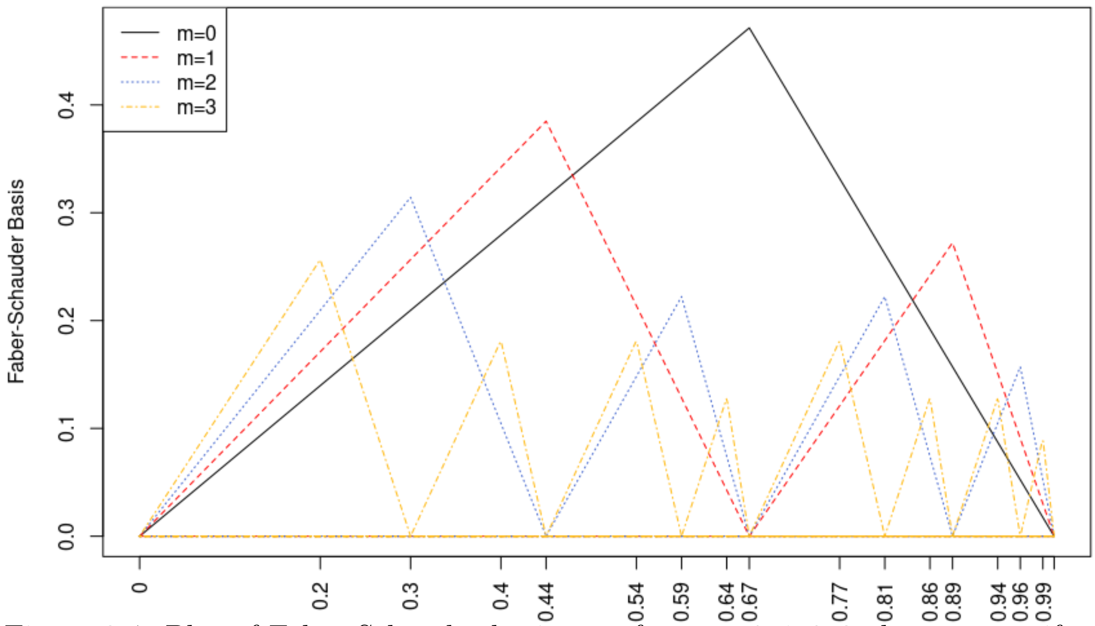

Figure 3.4: Plot of Faber-Schauder basis $e^{\pi}_{m,k}$ for $m = 0, 1, 2, 3$ along non-uniform non balanced doubly refining partition sequence π (Defined in Example 3.2).

3.2 Quadratic variation along finitely refining partitions

Gantert [41] derives a formula for the quadratic variation of a continuous function along the dyadic partition sequence, which only involves the dyadic Faber-Schauder basis coefficients. In this section, we generalize these results to any finitely refining sequence of partitions.

Notation: For a function $x \in C^0([0,1],\mathbb{R})$ and a sequence of partitions π of $[0,1]$, we denote

$$[x]_{\pi^n}(t) := \sum_{i=0}^{N(\pi^n)-1} \big(x(t_{i+1}^n \wedge t) - x(t_i^n \wedge t)\big)^2,$$

the quadratic variation of x along π at level n.

Proposition 3.8. *Let π be a finitely refining sequence of partitions of $[0,1]$ with vanishing mesh and $(e^{\pi}_{m,k})$ be the associated Schauder basis. Let $x \in C^0([0,1],\mathbb{R})$ given by*

$$x(t) = x(0) + \big(x(1) - x(0)\big)t + \sum_{m=0}^{\infty} \sum_{k=0}^{N(\pi^{m+1})-N(\pi^m)-1} \theta_{m,k} e^{\pi}_{m,k}(t).$$

Then the quadratic variation of x along π_n is given by:

$$[x]_{\pi^n}(t) = \sum_{m=0}^{n-1} \sum_{k=0}^{N(\pi^{m+1})-N(\pi^m)-1} a^n_{m,k}(t)\theta^2_{m,k} + \sum_{m,m'} \sum_{\substack{k,k' \\ (m,k)\neq(m',k')}} b^n_{m,k,m',k'}(t)\theta_{m,k}\theta_{m',k'}.$$

Denoting by $[t_1^{m,k}, t_3^{m,k}]$ the support of $e^{\pi}_{m,k}$, $t_2^{m,k}$ the point at which it reaches it maximum and

$$\Delta t_i^n = t_{i+1}^n \wedge t - t_i^n \wedge t,$$

we have the following closed form expression for $a^n_{m,k}(t)$ and $b^n_{m,k,m',k'}(t)$:

$$a^n_{m,k}(t) = \left\{ \left[\sum_{t_i^n \in [t_1^{m,k}, t_2^{m,k}]} (\Delta t_i^n)^2 \right] \times \frac{t_3^{m,k} - t_2^{m,k}}{t_2^{m,k} - t_1^{m,k}} \right.$$

$$
+\left[\sum_{t_i^n\in[t_2^{m,k},t_3^{m,k}]}(\Delta t_i^n)^2\right]\times\frac{t_2^{m,k}-t_1^{m,k}}{t_3^{m,k}-t_2^{m,k}}\Bigg\}\times\frac{1}{t_3^{m,k}-t_1^{m,k}},
$$

$$
b^n_{m,k,m',k'}(t)=\psi_{m',k'}(t_1^{m,k})\times\left\{\frac{\sum_{t_i^n\in[t_1^{m,k},t_2^{m,k}]}(\Delta t_i^n)^2}{t_2^{m,k}-t_1^{m,k}}-\frac{\sum_{t_i^n\in[t_2^{m,k},t_3^{m,k}]}(\Delta t_i^n)^2}{t_3^{m,k}-t_2^{m,k}}\right\}
\times\sqrt{\frac{(t_2^{m,k}-t_1^{m,k})(t_3^{m,k}-t_2^{m,k})}{t_3^{m,k}-t_1^{m,k}}}
$$

if $\operatorname{supp}(e^n_{m,k})\subset\operatorname{supp}(e^n_{m',k'})$ *and* $b^n_{m,k,m',k'}(t)=0$ *otherwise.*

Remark 3.9. Similar to the dyadic case [41], the coefficients $a^n_{m,k}$ and $b^n_{m,k,m',k'}$ only depend on the sequence of partitions π and not on the path $x\in C^0([0,1],\mathbb{R})$.

Proof. We compute $[x]_{\pi^n}(1)$. For $t\in[0,1]$, the calculations are analogously done with the stopped path $x(t\wedge .)$.

$$
[x]_{\pi^n}(1)=\sum_{i=0}^{N(\pi^n)-1}\left(x(t_{i+1}^n)-x(t_i^n)\right)^2
$$

$$
=\sum_{i=0}^{N(\pi^n)-1}\left(\sum_{m=0}^{n-1}\sum_{\{k:\psi_{m,k}(t_i^n)\neq0\}}\theta_{m,k}\left(e^\pi_{m,k}(t^n_{i+1})-e^\pi_{m,k}(t_i^n)\right)\right)^2
$$

$$
=\sum_{i=0}^{N(\pi^n)-1}\left(\sum_{m=0}^{n-1}\sum_{\{k:\psi_{m,k}(t_i^n)\neq0\}}\theta_{m,k}\int_{t_i^n}^{t_{i+1}^n}\psi_{m,k}(u)du\right)^2
$$

$$
=\sum_{i=0}^{N(\pi^n)-1}\left(\sum_{m=0}^{n-1}\sum_{\{k:\psi_{m,k}(t_i^n)\neq0\}}\theta_{m,k}\times\psi_{m,k}(t_i^n)(t^n_{i+1}-t^n_i)\right)^2.
$$

Since π is a finitely refining sequence of partitions, there exists an upper bound M on the cardinality of the set

$$
\{k\geq1,\quad \psi_{m,k}(t_i^n)\neq0\}
$$

for any $m\leq n$. So in the above expression of $[x]_{\pi^n}(1)$ if we look at the coefficient of $\theta^2_{m,k}$ for some pair (m,k) we get:

$$
\sum_{\{i:\psi_{m,k}(t_i^n)\neq0\}}\left[\psi_{m,k}(t_i^n)(t^n_{i+1}-t^n_i)\right]^2
$$

$$= \left\{ \left[\sum_{\Delta t_i^n \subset [t_1^{m,k}, t_2^{m,k}]} (\Delta t_i^n)^2 \right] \times \frac{t_3^{m,k} - t_2^{m,k}}{t_2^{m,k} - t_1^{m,k}} + \left[\sum_{\Delta t_i^n \subset [t_2^{m,k}, t_3^{m,k}]} (\Delta t_i^n)^2 \right] \times \frac{t_2^{m,k} - t_1^{m,k}}{t_3^{m,k} - t_2^{m,k}} \right\} \times \frac{1}{t_3^{m,k} - t_1^{m,k}}.$$

For two pairs (m,k) and (m',k') if $e^n_{m,k}$ and $e^n_{m',k'}$ have disjoint support then $\psi_{m,k}(t)\psi_{m',k'}(t) = 0$ for all t, hence coefficient of $\theta_{m,k}\theta_{m',k'}$ is always zero. For two pairs (m,k) and (m',k') with $\mathrm{supp}(e^n_{m,k}) \subset \mathrm{supp}(e^n_{m',k'})$; $\psi_{m',k'}(t)$ is a non-zero constant for all t. This is a consequence of the fact $\{\psi_{m,k}\}$ is orthonormal. Now if we look at the coefficient of $\theta_{m,k}\theta_{m',k'}$ for the case when $\mathrm{supp}(e^n_{m,k}) \subset \mathrm{supp}(e^n_{m',k'})$, we get:

$$\sum_{\{i:\psi_{m,k}(t_i^n)\neq 0\}} \left[\psi_{m,k}(t_i^n)(t_{i+1}^n - t_i^n)\right] \times \left[\psi_{m',k'}(t_i^n)(t_{i+1}^n - t_i^n)\right]$$

$$= \sum_{\{i:\psi_{m,k}(t_i^n)\neq 0\}} \left[\psi_{m,k}(t_i^n)\psi_{m',k'}(t_1^{m,k})\right] \times (t_{i+1}^n - t_i^n)^2$$

$$= \psi_{m',k'}(t_1^{m,k}) \times \left\{ \frac{\sum_{\Delta t_i^n \subset [t_1^{m,k}, t_2^{m,k}]} (\Delta t_i^n)^2}{t_2^{m,k} - t_1^{m,k}} - \frac{\sum_{\Delta t_i^n \subset [t_2^{m,k}, t_3^{m,k}]} (\Delta t_i^n)^2}{t_3^{m,k} - t_2^{m,k}} \right\}$$
$$\times \sqrt{\frac{(t_2^{m,k} - t_1^{m,k})(t_3^{m,k} - t_2^{m,k})}{t_3^{m,k} - t_1^{m,k}}}.$$

So the result follows. ■

The following example is an example of continuous function with bounded Schauder coefficients along dyadic partition, but quadratic variation along dyadic partition does not exist.

Example 3.3. *Consider the sequence $\{\mathbb{T}^n\}_n$ of dyadic partitions and the continuous function $x \in C^0([0,1],\mathbb{R})$ defined as follows:*

$$x(t) = \sum_{m=0}^{\infty} \sum_{k=0}^{2^m-1} \theta^{\mathbb{T}}_{m,k} e^{\mathbb{T}}_{m,k}(t), \qquad \text{where, } \theta^{\mathbb{T}}_{m,k} = 1 + (-1)^m.$$

For the function x defined above, in the similar line of argument provided in [70, Proposition 2.7.] we can show that

$$[x]_{\mathbb{T}^{2n}}(t) = \frac{4}{3}t \qquad \text{and, } [x]_{\mathbb{T}^{2n+1}}(t) = \frac{8}{3}t.$$

Note that: $\mathbb{T}$ is a finitely refining and balanced sequence of partitions with $\frac{|\mathbb{T}^n|}{|\mathbb{T}^{n+1}|} = \frac{\underline{\mathbb{T}^n}}{\underline{\mathbb{T}^{n+1}}} = 2$ and x had bounded Schauder coefficients along dyadic representation.

Theorem 3.10 (Quadratic covariation representation). *Let π be a finitely refining sequence of partitions of $[0,1]$ with vanishing mesh and $(e^{\pi}_{m,k})$ be the associated Schauder basis. Let $x, y \in C^0([0,1],\mathbb{R})$ with unique Faber-Schauder representations*

$$x(t) = x(0) + (x(1) - x(0))t + \sum_{m=0}^{\infty} \sum_{k=0}^{N(\pi^{m+1})-N(\pi^m)-1} \theta_{m,k} e^{\pi}_{m,k}(t), \qquad \text{and,}$$

$$y(t) = y(0) + (y(1) - y(0))t + \sum_{m=0}^{\infty} \sum_{k=0}^{N(\pi^{m+1})-N(\pi^m)-1} \eta_{m,k} e^{\pi}_{m,k}(t).$$

Then, the quadratic covariation of x and y at level n along the sequence of partitions π can be represented as:

$$[x,y]_{\pi^n}(t) := \sum_{\pi^n} \left(x(t^n_{i+1} \wedge t) - x(t^n_i \wedge t)\right) \left(y(t^n_{i+1} \wedge t) - y(t^n_i \wedge t)\right)$$

$$= \sum_{m=0}^{n-1} \sum_{k=0}^{N(\pi^{m+1})-N(\pi^m)-1} a^n_{m,k}(t) \theta_{m,k} \eta_{m,k} + \sum_{m,m'} \sum_{\substack{k,k' \\ (m,k) \neq (m',k')}} b^n_{m,k,m',k'}(t) \theta_{m,k} \eta_{m',k'}.$$

Denoting by $[t^{m,k}_1, t^{m,k}_3]$ the support of $e^{\pi}_{m,k}$, $t^{m,k}_2$ the point at which it reaches it maximum and

$$\Delta t^n_i = t^n_{i+1} \wedge t - t^n_i \wedge t,$$

we have the following closed form expression for $a^n_{m,k}$ and $b^n_{m,k,m',k'}$.

$$a^n_{m,k}(t) = \left\{ \left[\sum_{\Delta t^n_i \subset [t^{m,k}_1, t^{m,k}_2]} (\Delta t^n_i)^2 \right] \times \frac{t^{m,k}_3 - t^{m,k}_2}{t^{m,k}_2 - t^{m,k}_1} + \left[\sum_{\Delta t^n_i \subset [t^{m,k}_2, t^{m,k}_3]} (\Delta t^n_i)^2 \right] \times \frac{t^{m,k}_2 - t^{m,k}_1}{t^{m,k}_3 - t^{m,k}_2} \right\} \times \frac{1}{t^{m,k}_3 - t^{m,k}_1},$$

and,

$$b^n_{m,k,m',k'}(t) = \psi_{m',k'}(t^{m,k}_1) \times \left\{ \frac{\sum_{\Delta t^n_i \subset [t^{m,k}_1, t^{m,k}_2]} (\Delta t^n_i)^2}{t^{m,k}_2 - t^{m,k}_1} - \frac{\sum_{\Delta t^n_i \subset [t^{m,k}_2, t^{m,k}_3]} (\Delta t^n_i)^2}{t^{m,k}_3 - t^{m,k}_2} \right\} \times \sqrt{\frac{(t^{m,k}_2 - t^{m,k}_1)(t^{m,k}_3 - t^{m,k}_2)}{t^{m,k}_3 - t^{m,k}_1}},$$

if $\mathrm{supp}(e^n_{m,k}) \subset \mathrm{supp}(e^n_{m',k'})$ *and* $b^n_{m,k,m',k'}(t) = 0$ *otherwise.*

Proof. The proof is similar to that of Theorem 3.8. ∎

We will now derive some bounds on the coefficients $a^n_{m,k}$[1] and $b^n_{m,k,m',k'}$[2] which appear in the expression of quadratic variation in Theorem 3.8.

Proposition 3.11. *If π is a finitely refining sequence of partitions of $[0,1]$ then*

$$0 \leq \underline{\pi^n} \leq a^n_{m,k} \leq |\pi^n|.$$

If we also assume the sequence of partitions π is balanced, then there exists $C > 0$ such that

$$\mathrm{supp}(e^n_{m,k}) \subset \mathrm{supp}(e^n_{m',k'}) \quad \Rightarrow \; 0 \leq |b^n_{m,k,m',k'}| \leq C(|\pi^n| - \underline{\pi^n})\sqrt{\frac{|\pi^m|}{|\pi^{m'}|}}.$$

If $\mathrm{supp}(e^n_{m,k}) \cap \mathrm{supp}(e^n_{m',k'}) = \emptyset$ then $b^n_{m,k,m',k'} = 0$.

Proof. From Theorem 3.8 we have the expression of $a^n_{m,k}$ as follows:

$$a^n_{m,k} = \left\{ \left[\sum_{\Delta t^n_i \subset [t^{m,k}_1, t^{m,k}_2]} (\Delta t^n_i)^2 \right] \times \frac{t^{m,k}_3 - t^{m,k}_2}{t^{m,k}_2 - t^{m,k}_1} + \left[\sum_{\Delta t^n_i \subset [t^{m,k}_2, t^{m,k}_3]} (\Delta t^n_i)^2 \right] \times \frac{t^{m,k}_2 - t^{m,k}_1}{t^{m,k}_3 - t^{m,k}_2} \right\} \times \frac{1}{t^{m,k}_3 - t^{m,k}_1}.$$

Since $\underline{\pi^n} \times (\Delta t^n_i) \leq (\Delta t^n_i)^2 \leq |\pi^n| \times (\Delta t^n_i)$, for all m, k we can bound $a^n_{m,k}$ as follows.

$$a^n_{m,k} \leq |\pi^n| \left[\left\{ \left[\sum_{\Delta t^n_i \subset [t^{m,k}_1, t^{m,k}_2]} \Delta t^n_i \right] \frac{t^{m,k}_3 - t^{m,k}_2}{t^{m,k}_2 - t^{m,k}_1} + \left[\sum_{\Delta t^n_i \subset [t^{m,k}_2, t^{m,k}_3]} \Delta t^n_i \right] \frac{t^{m,k}_2 - t^{m,k}_1}{t^{m,k}_3 - t^{m,k}_2} \right\} \times \frac{1}{t^{m,k}_3 - t^{m,k}_1} \right]$$

$$= |\pi^n| \times \left[(t^{m,k}_3 - t^{m,k}_2) + (t^{m,k}_2 - t^{m,k}_1) \right] \frac{1}{t^{m,k}_3 - t^{m,k}_1} = |\pi^n|.$$

Similarly using the other side of the inequality, we get for all n, m, k: $\underline{\pi^n} \leq a^n_{m,k} \leq |\pi^n|$. So the first part of the result follows. For the second part of the proposition, for any (m, k), under the balanced assumption on π, we have $|\psi_{m,k}(t)| \leq C_1 \sqrt{\frac{1}{|\pi^m|}}$. So under balanced assumption:

$$|b^n_{m,k,m',k'}|$$

$$= \left| \psi_{m',k'}(t^{m,k}_1) \right| \times \left| \frac{\sum_{\Delta t^n_i \subset [t^{m,k}_1, t^{m,k}_2]} (\Delta t^n_i)^2}{t^{m,k}_2 - t^{m,k}_1} - \frac{\sum_{\Delta t^n_i \subset [t^{m,k}_2, t^{m,k}_3]} (\Delta t^n_i)^2}{t^{m,k}_3 - t^{m,k}_2} \right| \times \sqrt{\frac{(t^{m,k}_2 - t^{m,k}_1)(t^{m,k}_3 - t^{m,k}_2)}{t^{m,k}_3 - t^{m,k}_1}}$$

[1]throughout the rest of the chapter in some places we wrote $a^n_{m,k}$ for $a^n_{m,k}(1)$

[2]similarly we wrote $b^n_{m,k,m',k'}$ for $b^n_{m,k,m',k'}(1)$

$$\leq C_2\sqrt{\frac{1}{|\pi^{m'}|}}\left|\frac{\sum_{\Delta t_i^n\subset[t_1^{m,k},t_2^{m,k}]}(\Delta t_i^n)^2}{t_2^{m,k}-t_1^{m,k}}-\frac{\sum_{\Delta t_i^n\subset[t_2^{m,k},t_3^{m,k}]}(\Delta t_i^n)^2}{t_3^{m,k}-t_2^{m,k}}\right|\times\sqrt{|\pi^m|}.$$

Since $\underline{\pi^n}\times(\Delta t_i^n)\leq(\Delta t_i^n)^2\leq|\pi^n|\times(\Delta t_i^n)$, for all $(m,k)\neq(m',k')$ we can bound $|b^n_{m,k,m',k'}|$ as follows.

$$|b^n_{m,k,m',k'}|\leq C_3\sqrt{\frac{1}{|\pi^{m'}|}}\left|\frac{|\pi^n|\sum_{\Delta t_i^n\subset[t_1^{m,k},t_2^{m,k}]}(\Delta t_i^n)}{t_2^{m,k}-t_1^{m,k}}-\frac{\underline{\pi^n}\sum_{\Delta t_i^n\subset[t_2^{m,k},t_3^{m,k}]}(\Delta t_i^n)}{t_3^{m,k}-t_2^{m,k}}\right|\times\sqrt{|\pi^m|}$$

$$\leq C(|\pi^n|-\underline{\pi^n})\sqrt{\frac{|\pi^m|}{|\pi^{m'}|}}.$$

■

As a consequence of Proposition 3.11, for any uniform partition (such as dyadic partition), $b^n_{m,k,m',k'}=0$ for all $m,k,m',k',n\geq 0$.

Lemma 3.12. *Consider a balanced finitely refining sequence π of partitions satisfying $t^n_{i+1}-t^n_i=\frac{1}{N(\pi^n)}(1+\epsilon^n_i)$ with $\sup_i|\epsilon^n_i|=o(\frac{1}{n})$ for all $n\geq 1$. Then for any function $x\in C^0([0,1],\mathbb{R})$*

$$x(t)=x(0)+(x(1)-x(0))t+\sum_{m=0}^{\infty}\sum_{k=0}^{N(\pi^{m+1})-N(\pi^m)-1}\theta_{m,k}e^{\pi}_{m,k}(t)$$

with bounded Schauder coefficients we have

$$[x]_{\pi^n}(t)=\sum_{m=0}^{n-1}\sum_{k=0}^{N(\pi^{m+1})-N(\pi^m)-1}a^n_{m,k}(t)\theta^2_{m,k}.$$

If $[t_1^{m,k},t_3^{m,k}]$ the support of $e^\pi_{m,k}$, $t_2^{m,k}$ the point at which it reaches it maximum and

$$\Delta t^n_i=t^n_{i+1}\wedge t-t^n_i\wedge t,$$

then:

$$a^n_{m,k}(t)=\left\{\left[\sum_{\Delta t_i^n\subset[t_1^{m,k},t_2^{m,k}]}(\Delta t_i^n)^2\right]\times\frac{t_3^{m,k}-t_2^{m,k}}{t_2^{m,k}-t_1^{m,k}}+\left[\sum_{\Delta t_i^n\subset[t_2^{m,k},t_3^{m,k}]}(\Delta t_i^n)^2\right]\times\frac{t_2^{m,k}-t_1^{m,k}}{t_3^{m,k}-t_2^{m,k}}\right\}\times\frac{1}{t_3^{m,k}-t_1^{m,k}}.$$

Note: The above assumption is true for any uniform partition π, say dyadic or triadic partition as in this case $b^n_{m,k,m',k'}=0$ for all m,m',k,k'. But Lemma 3.12 does not require having $b^n_{m,k,m',k'}=0$.

Proof. We compute $[x]_{\pi^n}(1)$. For $t \in [0,1]$, the calculations are analogously done with the stopped path $x(t \wedge .)$.

For any pair (m,k), under the balanced assumption on π we have; $|\psi_{m,k}(t)| \leq C_1\sqrt{\frac{1}{|\pi^m|}}$, where constant C_1 is independent of m and k. We will show that the second term on the quadratic variation formula in Theorem 3.8:
$\sum_{m,m'} \sum_{\substack{k,k' \\ (m,k)\neq(m',k')}} b^n_{m,k,m',k'}\theta_{m,k}\theta_{m',k'}$ goes to 0 as $n \to \infty$. From the construction of $b^n_{m,k,m',k'}$ we know that if support of $e^n_{m,k}$ and support of $e^n_{m',k'}$ are disjoint, then: $b^n_{m,k,m',k'} = 0$. So,

$$\sum_{m,m'} \sum_{\substack{k,k' \\ (m,k)\neq(m',k')}} b^n_{m,k,m',k'}\theta_{m,k}\theta_{m',k'}$$

$$= \sum_{m=0}^{n-1} \sum_{k=0}^{N(\pi^{m+1})-N(\pi^m)-1} \sum_{m'=0}^{m} \sum_{\substack{k':\text{ Support of } e^n_{m,k}\subset e^n_{m',k'} \\ (m,k)\neq(m',k')}} b^n_{m,k,m',k'}\theta_{m,k}\theta_{m',k'}$$

$$\leq M \sum_{m=0}^{n-1} \sum_{k=0}^{N(\pi^{m+1})-N(\pi^m)-1} \sum_{m'=0}^{m} |\theta_{m,k}\theta_{m',k'(.)}| \times \psi_{m',k'(.)}(t_1^{m,k}) \times \sqrt{\frac{(t_2^{m,k}-t_1^{m,k})(t_3^{m,k}-t_2^{m,k})}{t_3^{m,k}-t_1^{m,k}}}$$
$$\times \left| \frac{\sum_{\Delta t_i^n \subset [t_1^{m,k}, t_2^{m,k}]} (\Delta t_i^n)^2}{t_2^{m,k}-t_1^{m,k}} - \frac{\sum_{\Delta t_i^n \subset [t_2^{m,k}, t_3^{m,k}]} (\Delta t_i^n)^2}{t_3^{m,k}-t_2^{m,k}} \right|$$

under the balanced assumption on π:

$$\leq C_2 \sum_{m=0}^{n-1} \sum_{k=0}^{N(\pi^{m+1})-N(\pi^m)-1} \sum_{m'=0}^{m} |\theta_{m,k}\theta_{m'k'(.)}| \times \sqrt{\frac{1}{|\pi^{m'}|}} \times \sqrt{|\pi^m|}$$
$$\times \left| \frac{\sum_{\Delta t_i^n \subset [t_1^{m,k}, t_2^{m,k}]} (\Delta t_i^n)^2}{t_2^{m,k}-t_1^{m,k}} - \frac{\sum_{\Delta t_i^n \subset [t_2^{m,k}, t_3^{m,k}]} (\Delta t_i^n)^2}{t_3^{m,k}-t_2^{m,k}} \right|$$

$$\leq C_3 \sum_{m=0}^{n-1} \sum_{k=0}^{N(\pi^{m+1})-N(\pi^m)-1} \sum_{m'=0}^{m} \left| \frac{\sum_{\Delta t_i^n \subset [t_1^{m,k}, t_2^{m,k}]} (\Delta t_i^n)^2}{t_2^{m,k}-t_1^{m,k}} - \frac{\sum_{\Delta t_i^n \subset [t_2^{m,k}, t_3^{m,k}]} (\Delta t_i^n)^2}{t_3^{m,k}-t_2^{m,k}} \right|.$$

The last inequality follows from the fact that x has a bounded Schauder coefficients along a refining sequence of partitions π and $\sqrt{\frac{|\pi^m|}{|\pi^{m'}|}} \leq 1$ for all $m' \leq m$. So the above inequality will reduce as follows:

$$\leq C_3 \sum_{m=0}^{n-1} \sum_{k=0}^{N(\pi^{m+1})-N(\pi^m)-1} \sum_{m'=0}^{m} \left| \frac{|\pi^n| \sum_{\Delta t_i^n \subset [t_1^{m,k}, t_2^{m,k}]} (\Delta t_i^n)}{t_2^{m,k}-t_1^{m,k}} - \frac{\underline{\pi}^n \sum_{\Delta t_i^n \subset [t_2^{m,k}, t_3^{m,k}]} (\Delta t_i^n)}{t_3^{m,k}-t_2^{m,k}} \right|$$

$$= C_3 \sum_{m=0}^{n-1} \sum_{k=0}^{N(\pi^{m+1})-N(\pi^m)-1} \sum_{m'=0}^{m} ||\pi^n| - \underline{\pi^n}| \leq C_4 \sum_{m=0}^{n-1} \sum_{k=0}^{N(\pi^{m+1})-N(\pi^m)-1} \sum_{m'=0}^{m} \left| \frac{1}{N(\pi^n)} \sup_i |\epsilon_i^n| \right|$$

$$\leq C_4 \left| \frac{1}{N(\pi^n)} \sup_i |\epsilon_i^n| \right| \sum_{m=0}^{n-1} (m+1) \left[N(\pi^{m+1}) - N(\pi^m) \right] \leq C_5 \times n \sup_i |\epsilon_i^n| \to 0.$$

So the lemma follows. ■

3.3 Processes with prescribed quadratic variation along a finitely refining partition sequence

3.3.1 Processes with linear quadratic variation

A well-known example of process with linear quadratic variation i.e. constant quadratic variation per unit time is Brownian motion, which satisfies this property almost surely along any refining partition. Schied [70] provided a subclass $\mathcal{X}$ of $Q_{\mathbb{T}}([0,1], \mathbb{R})$, such that for all $x \in \mathcal{X}$, the quadratic variation along the dyadic partition is $[x]_{\mathbb{T}}(t) = t$. However Brownian motion is not included in the class $\mathcal{X}$ given in [70].

In this subsection, for any fixed finitely refining sequence of partitions π, we construct a class $\mathcal{B}^\pi$ of processes with linear quadratic variation along π and we show that Brownian motion belongs to $\mathcal{B}^\pi$. With some additional conditions on the sequence of partitions, we also provide an almost sure convergence result. The class $\mathcal{X}$ defined in [70] has a non-empty intersection with $\mathcal{B}^{\mathbb{T}}$.

Let W be a Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which we take to be the canonical Wiener space without loss of generality i.e. $\Omega = C^0([0,T], \mathbb{R}), W(t,\omega) = \omega(t)$. For finitely refining sequence of partitions π of $[0,1]$, the quadratic variation of W along π is linear almost surely, i.e. $\forall t \in [0,1],\ \mathbb{P}([W]_\pi(t) = t) = 1$ [58, 59]. On the other hand, W can also be represented in terms of its Schauder expansion along π.

Lemma 3.13. *Let π be a finitely refining sequence of partitions and W be a Brownian motion. Then W has the following Schauder expansion along the partition* *sequence π:*

$$W(t) = W(0) + (W(1) - W(0))t + \sum_{m=0}^{\infty} \sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} \eta_{mk} e^{\pi}_{m,k}(t),$$

where $\eta_{m,k} \sim^{IID} N(0,1)$ are independent and identically distributed.

Proof. $(\eta_{m,k})$ is a Gaussian family. If the support of the function $e^{\pi}_{m,k}$ is $[t_1^{m,k}, t_3^{m,k}]$ and the maximum is attained at time $t_2^{m,k}$ then, applying Theorem 3.7 the coefficient $\eta_{m,k}$ has a closed-form representation as follows.

$$\eta_{m,k} = \frac{\left[(W(t_2^{m,k}) - W(t_1^{m,k}))(t_3^{m,k} - t_2^{m,k}) - (W(t_3^{m,k}) - W(t_2^{m,k}))(t_2^{m,k} - t_1^{m,k})\right]}{\sqrt{(t_2^{m,k} - t_1^{m,k})(t_3^{m,k} - t_2^{m,k})(t_3^{m,k} - t_1^{m,k})}}. \tag{3.5}$$

Since W is a Brownian motion,

$$\mathbb{E}(\eta_{m,k}) = \frac{\left[\mathbb{E}(W(t_2^{m,k}) - W(t_1^{m,k}))(t_3^{m,k} - t_2^{m,k}) - \mathbb{E}(W(t_3^{m,k}) - W(t_2^{m,k}))(t_2^{m,k} - t_1^{m,k})\right]}{\sqrt{(t_2^{m,k} - t_1^{m,k})(t_3^{m,k} - t_2^{m,k})(t_3^{m,k} - t_1^{m,k})}} = 0 \quad \text{and,}$$

$$\begin{aligned} Var(\eta_{m,k}) &= \frac{Var\left[(W(t_2^{m,k}) - W(t_1^{m,k}))(t_3^{m,k} - t_2^{m,k})\right] + Var\left[(W(t_3^{m,k}) - W(t_2^{m,k}))(t_2^{m,k} - t_1^{m,k})\right]}{(t_2^{m,k} - t_1^{m,k})(t_3^{m,k} - t_2^{m,k})(t_3^{m,k} - t_1^{m,k})} \\ &+ \frac{Cov\left((W(t_2^{m,k}) - W(t_1^{m,k}))(t_3^{m,k} - t_2^{m,k}), (W(t_3^{m,k}) - W(t_2^{m,k}))(t_2^{m,k} - t_1^{m,k})\right)}{(t_2^{m,k} - t_1^{m,k})(t_3^{m,k} - t_2^{m,k})(t_3^{m,k} - t_1^{m,k})} \\ &= \frac{(t_2^{m,k} - t_1^{m,k})(t_3^{m,k} - t_2^{m,k})^2 + (t_2^{m,k} - t_1^{m,k})^2(t_3^{m,k} - t_2^{m,k})}{(t_2^{m,k} - t_1^{m,k})(t_3^{m,k} - t_2^{m,k})(t_3^{m,k} - t_1^{m,k})} = 1. \end{aligned}$$

Using the orthogonality of increments of Brownian motion we can show that $Cov(\eta_{m,k}, \eta_{m'k'}) = \mathbb{1}_{m=m'}\mathbb{1}_{k=k'}$. Along with the fact that $\eta_{m,k}$ is Gaussian, we can conclude $\eta_{m,k} \sim^{IID} N(0,1)$. ■

For Brownian motion W the quadratic variation along π can be represented using the explicit representation of quadratic variation (Theorem 3.8) as follows:

$$[W]_\pi(t) = \lim_{n\to\infty} [W]_{\pi^n}(t), \quad \text{with:}$$

$$[W]_{\pi^n}(t) = \sum_{m=0}^{n-1}\sum_{k} a^n_{m,k}(t)\eta^2_{m,k} + \sum_{m,m'=0}^{n-1}\sum_{\substack{k,k'\\(m,k)\neq(m',k')}} b^n_{m,k,m',k'}(t)\eta_{m,k}\eta_{m',k'}.$$

Now we know that for Brownian motion $\mathbb{E}[W]_{\pi}(t) = \lim_{n\to\infty}\mathbb{E}[W]_{\pi^n}(t) = t$. So,

$$\lim_{n\to\infty}\left[\mathbb{E}\sum_{m=0}^{n-1}\sum_{k} a^n_{m,k}(t)\eta^2_{m,k} + \mathbb{E}\sum_{m,m'=0}^{n-1}\sum_{\substack{k,k'\\(m,k)\neq(m',k')}} b^n_{m,k,m',k'}(t)\eta_{m,k}\eta_{m',k'}\right] = t$$

$$\implies \lim_{n\to\infty}\left[\sum_{m=0}^{n-1}\sum_{k} a^n_{m,k}(t)\mathbb{E}\eta^2_{m,k} + \sum_{m,m'=0}^{n-1}\sum_{\substack{k,k'\\(m,k)\neq(m',k')}} b^n_{m,k,m',k'}(t)\mathbb{E}[\eta_{m,k}\eta_{m',k'}]\right] = t$$

$$\implies \lim_{n\to\infty}\left[\sum_{m=0}^{n-1}\sum_{k} a^n_{m,k}(t)\right] = t. \tag{3.6}$$

Since $a^n_{m,k}(t)$ only depends on the refining partition π, and *not* on the path of Brownian motion, the above invariant is true for any finitely refining sequence of partitions π. For Brownian motion we also know that $\lim_{n\to\infty}\mathbb{E}([W]_{\pi^n}(t)-t)^2 = 0$. This implies, $\lim_{n\to\infty}\mathbb{E}([W]_{\pi^n}(t))^2 = t^2$. So,

$$\lim_{n\to\infty}\mathbb{E}\left[\sum_{m=0}^{n-1}\sum_{k} a^n_{m,k}(t)\eta^2_{m,k} + \sum_{m,m'=0}^{n-1}\sum_{\substack{k,k'\\(m,k)\neq(m',k')}} b^n_{m,k,m',k'}(t)\eta_{m,k}\eta_{m',k'}\right]^2 = t^2$$

$$\implies \lim_{n\to\infty}\mathbb{E}\Bigg[\sum_{m=0}^{n-1}\sum_{k}(a^n_{m,k}(t))^2\eta^4_{m,k} + \sum_{m,m'=0}^{n-1}\sum_{\substack{k,k'\\(m,k)\neq(m',k')}} a^n_{m,k}(t)a^n_{m',k'}(t)\eta^2_{m,k}\eta^2_{m',k'}$$

$$+ \sum_{m,m'=0}^{n-1}\sum_{\substack{k,k'\\(m,k)\neq(m',k')}} (b^n_{m,k,m',k'}(t))^2\eta^2_{m,k}\eta^2_{m',k'}\Bigg] = t^2$$

$$\implies \lim_{n\to\infty}\Bigg[3\sum_{m=0}^{n-1}\sum_{k}(a^n_{m,k}(t))^2 + \sum_{m,m'=0}^{n-1}\sum_{\substack{k,k'\\(m,k)\neq(m',k')}} a^n_{m,k}(t)a^n_{m',k'}(t)$$

$$+\sum_{m,m'=0}^{n-1}\sum_{\substack{k,k'\\(m,k)\neq(m',k')}}(b^n_{m,k,m',k'}(t))^2\Bigg] = t^2$$

$$\implies \lim_{n\to\infty}\left(\sum_m^{n-1}\sum_k a^n_{m,k}(t)\right)\left(\sum_{m'}^{n-1}\sum_{k'} a^n_{m',k'}(t)\right)$$

$$+\lim_{n\to\infty}\left[2\sum_{m=0}^{n-1}\sum_k (a^n_{m,k}(t))^2+\sum_{m,m'=0}^{n-1}\sum_{\substack{k,k'\\(m,k)\neq(m',k')}}(b^n_{m,k,m',k'}(t))^2\right] = t^2.$$

From Equation (3.6) we know that the first sum converges to t^2. So the above equality reduces to:

$$\lim_{n\to\infty}\left[2\sum_{m=0}^{n-1}\sum_k (a^n_{m,k}(t))^2+\sum_{m,m'=0}^{n-1}\sum_{\substack{k,k'\\(m,k)\neq(m',k')}}(b^n_{m,k,m',k'}(t))^2\right] = 0.$$

Since both the two summations in the limit are positive we get the following two identities:

$$\lim_{n\to\infty}\left[\sum_{m=0}^{n-1}\sum_k (a^n_{m,k}(t))^2\right] = 0 \text{ and,} \tag{3.7}$$

$$\lim_{n\to\infty}\left[\sum_{m,m'=0}^{n-1}\sum_{\substack{k,k'\\(m,k)\neq(m',k')}}(b^n_{m,k,m',k'}(t))^2\right] = 0. \tag{3.8}$$

Since both $a^n_{m,k}(t)$ and $b^n_{m,k,m',k'}(t)$ are only dependent of the sequence of partitions π and *not* dependents on the Brownian path W, Equation (3.7) and Equation (3.8) are true for any finitely refining sequence of partitions π of $[0,1]$.

In the following theorem, we provide a class of processes with linear quadratic variation along a finitely refining partition sequence π.

Theorem 3.14. *Let π be a finitely refining sequence of partitions with vanishing mesh $|\pi^n| \to 0$. Define, for $t \in [0,1]$,*

$$X(t) = X(0) + (X(1) - X(0))t + \sum_{m=0}^{\infty}\sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} \eta_{mk} e^{\pi}_{m,k}(t),$$

where $(\eta_{m,k}, m \in \mathbb{N}, k = 1, \cdots, N(\pi^{m+1}) - N(\pi^m))$ *is a family of random variables with*

$$\mathbb{E}\eta_{m,k} = 0, \qquad \mathbb{E}\eta_{m,k}\eta_{m',k'} = \mathbb{1}_{m,m'}\mathbb{1}_{k,k'}, \qquad \mathbb{E}\eta_{m,k}^4 < \infty \quad \textit{and,}$$

$$\mathbb{E}(\eta_{m,k}^{\alpha}\eta_{m_1,k_1}^{\beta}\eta_{m_2,k_2}^{\gamma}\eta_{m_3,k_3}^{\delta}) = \mathbb{E}(\eta_{m,k}^{\alpha})\mathbb{E}(\eta_{m_1,k_1}^{\beta})\mathbb{E}(\eta_{m_2,k_2}^{\gamma})\mathbb{E}(\eta_{m_3,k_3}^{\delta})$$

for all integers $\alpha, \beta, \gamma, \delta$ *such that* $\alpha + \beta + \gamma + \delta = 4$*. Then:*

$$\forall\, \epsilon > 0, \qquad \lim_{n\to\infty} \mathbb{P}(|[X]_{\pi^n}(t) - t| > \epsilon) = 0.$$

Furthermore, if the sequence of partitions π *is complete refining and balanced then:*

$$X \in Q_\pi([0,1], \mathbb{R}) \textit{ almost surely, and } \mathbb{P}([X]_\pi(t) = t) = 1.$$

Note that the coefficients are neither assumed independent nor Gaussian, so this class of processes contains examples of processes other than Brownian motion.

Proof. Using Theorem 3.8 the quadratic variation of X along π at level n can be represented as:

$$[X]_{\pi^n}(t) = \sum_{m=0}^{n-1}\sum_k a_{m,k}^n(t)\eta_{m,k}^2 + \sum_{m,m'=0}^{n-1} \sum_{\substack{k,k' \\ (m,k)\neq(m',k')}} b_{m,k,m',k'}^n(t)\eta_{m,k}\eta_{m',k'}.$$

Now using the assumptions on the coefficient $\eta_{m,k}$, we will show that

$$\forall t \in [0,1],\ \lim_{n\to\infty} \mathbb{E}[X]_{\pi^n}(t) = t.$$

So

$$\lim_{n\to\infty} \mathbb{E}[X]_{\pi^n}(t) = \lim_{n\to\infty} \mathbb{E}\left[\sum_{m=0}^{n-1}\sum_k a_{m,k}^n(t)\eta_{m,k}^2 + \sum_{m,m'=0}^{n-1} \sum_{\substack{k,k' \\ (m,k)\neq(m',k')}} b_{m,k,m',k'}^n(t)\eta_{m,k}\eta_{m',k'}\right]$$

$$= \lim_{n\to\infty}\left[\sum_{m=0}^{n-1}\sum_k a_{m,k}^n(t)\mathbb{E}\eta_{m,k}^2 + \sum_{m,m'=0}^{n-1} \sum_{\substack{k,k' \\ (m,k)\neq(m',k')}} b_{m,k,m',k'}^n(t)\mathbb{E}\left(\eta_{m,k}\eta_{m',k'}\right)\right]$$

$$= \lim_{n\to\infty}\left[\sum_{m=0}^{n-1}\sum_k a_{m,k}^n(t)\right] = t.$$

The last equality follows from Equation (3.6). Now to prove $[X]_{\pi^n}(t) \to t$ in probability, we only need to show that $\lim_{n\to\infty} \mathbb{E}([X]_{\pi^n}(t))^2 = t^2$. So:

$$\lim_{n\to\infty} \mathbb{E}\left([X]_{\pi^n}(t)\right)^2 = \lim_{n\to\infty} \mathbb{E}\left[\sum_{m=0}^{n-1}\sum_k a^n_{m,k}(t)\eta^2_{m,k} + \sum_{m,m'=0}^{n-1} \sum_{\substack{k,k' \\ (m,k)\neq(m',k')}} b^n_{m,k,m',k'}(t)\eta_{m,k}\eta_{m',k'}\right]^2$$

$$\begin{aligned}
= \lim_{n\to\infty} \mathbb{E}\Bigg[&\sum_{m=0}^{n-1}\sum_k (a^n_{m,k}(t))^2\eta^4_{m,k} + \sum_{m,m'=0}^{n-1} \sum_{\substack{k,k' \\ (m,k)\neq(m',k')}} a^n_{m,k}(t)a^n_{m',k'}(t)\eta^2_{m,k}\eta^2_{m',k'} \\
&+ \sum_{m,m'=0}^{n-1} \sum_{\substack{k,k' \\ (m,k)\neq(m',k')}} (b^n_{m,k,m',k'}(t))^2\eta^2_{m,k}\eta^2_{m',k'} \\
&+ \sum_{m=0}^{n-1}\sum_k \sum_{m',m''=0}^{n-1} \sum_{\substack{k',k'' \\ (m',k')\neq(m'',k'')}} a^n_{m,k}(t)\eta^2_{m,k} b^n_{m',k',m'',k''}(t)\eta_{m',k'}\eta_{m'',k''}\Bigg]
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
= \lim_{n\to\infty} \Bigg[&\sum_{m=0}^{n-1}\sum_k (a^n_{m,k}(t))^2\mathbb{E}\eta^4_{m,k} + \sum_{m,m'=0}^{n-1} \sum_{\substack{k,k' \\ (m,k)\neq(m',k')}} a^n_{m,k}(t)a^n_{m',k'}(t) \\
&+ \sum_{m,m'=0}^{n-1} \sum_{\substack{k,k' \\ (m,k)\neq(m',k')}} (b^n_{m,k,m',k'}(t))^2\Bigg] \\
= \lim_{n\to\infty} &\left(\sum_{m}^{n-1}\sum_k a^n_{m,k}(t)\right)\left(\sum_{m'}^{n-1}\sum_{k'} a^n_{m',k'}(t)\right) \\
+ \lim_{n\to\infty} &\left[\sum_{m=0}^{n-1}\sum_k (a^n_{m,k}(t))^2(\mathbb{E}\eta^4_{m,k} - 1) + \sum_{m,m'=0}^{n-1} \sum_{\substack{k,k' \\ (m,k)\neq(m',k')}} (b^n_{m,k,m',k'}(t))^2\right].
\end{aligned}$$

Using Equation (3.6) we know that the first sum converges to t^2. The last two sum can be bounded above as follows.:

$$\left|\lim_{n\to\infty}\left[\sum_{m=0}^{n-1}\sum_k (a^n_{m,k}(t))^2(\mathbb{E}\eta^4_{m,k}-1)+\sum_{m,m'=0}^{n-1}\sum_{\substack{k,k'\\(m,k)\neq(m',k')}}(b^n_{m,k,m',k'}(t))^2\right]\right|$$

$$\leq \lim_{n\to\infty}\left[C\sum_{m=0}^{n-1}\sum_k (a^n_{m,k}(t))^2+\sum_{m,m'=0}^{n-1}\sum_{\substack{k,k'\\(m,k)\neq(m',k')}}(b^n_{m,k,m',k'}(t))^2\right]=0.$$

The last equality follows using the Equality (3.7) and Equality (3.8). So we have $\lim_{n\to\infty}\mathbb{E}[X]_{\pi^n}(t)=t$, and correspondingly $\lim_{n\to\infty}\mathbb{E}\left([X]_{\pi^n}(t)-t\right)^2=0$. So $[X]_{\pi^n}(t)\to t$ in probability.

Now we will prove the almost sure convergence. Since for this part we have already assumed π is balanced, from the previous calculations and using the bounds from Proposition 3.11 we get the bound on $Var([X]_{\pi^n}(t))$ as follows:

$$Var([X]_{\pi^n}(t))\leq\left|C\sum_{m=0}^{n-1}\sum_k (a^n_{m,k}(t))^2+\sum_{m,m'=0}^{n-1}\sum_{\substack{k,k'\\(m,k)\neq(m',k')}}(b^n_{m,k,m',k'}(t))^2\right|$$

$$\leq\left||\pi^n|^2N(\pi^n)+C(|\pi^n|-\underline{\pi^n})^2\sum_{m=0}^{n-1}\sum_{k=0}^{N(\pi^{m+1})-N(\pi^m)-1}\sum_{m'=0}^{m-1}\frac{|\pi^m|}{|\pi^{m'}|}\right|.$$

Since, π is also complete refining there exists $C_0<\infty$ such that $\sum_{m'=0}^{m-1}\frac{|\pi^m|}{|\pi^{m'}|}\leq C_0$. So we get the bound on variance as follows.

$$Var([X]_{\pi^n}(t))\leq C_1|\pi^n|.$$

Now take $\epsilon_n=|\pi^n|^{\frac{1}{4}}$, then from Markov inequality we have:

$$\mathbb{P}(|[X]_{\pi^n}(t)-t|\geq\epsilon_n)\leq\frac{Var([X]_{\pi^n}(t))}{\epsilon_n^2}\leq C\sqrt{|\pi^n|}.$$

Since π is a complete refining sequence of partitions of $[0,1]$, $\sum_{n=0}^{\infty}\sqrt{|\pi^n|}<\infty$. So using Borel-Cantelli Lemma we can show, $\mathbb{P}(|[X]_{\pi^n}(t)-t|\geq\epsilon_n,\ \text{infinitely often})=0$, where $\epsilon_n=|\pi^n|^{\frac{1}{4}}\to 0$. Hence, we have $[X]_{\pi^n}(t)\to t$ almost-surely. So as a consequence $[X]_\pi(t)=\lim_{n\to\infty}[X]_{\pi^n}(t)$ exists almost surely and $[X]_\pi(t)=t$ almost surely. ■

To summarise, for any finitely refining sequence of partitions π we define

$$\mathcal{B}^\pi = \Bigg\{ X : \Omega \times [0,1] \mapsto \mathbb{R}, \quad X(t) = X(0) + (X(1) - X(0))\, t + \sum_{m=0}^{\infty} \sum_{k=0}^{N(\pi^{m+1})-N(\pi^m)} \eta_{m,k} e^\pi_{m,k}(t)$$
$$\text{where, } \mathbb{E}(\eta_{m,k}) = 0,\ \mathbb{E}(\eta_{m,k}\eta_{m',k'}) = \delta_{m,m'}\delta_{k,k'},\ \mathbb{E}(\eta^4_{m,k}) \le M < \infty, \text{ and,} \tag{3.10}$$
$$\text{for integers } \alpha + \beta + \gamma + \delta = 4,$$
$$\mathbb{E}(\eta^\alpha_{m,k}\eta^\beta_{m_1,k_1}\eta^\gamma_{m_2,k_2}\eta^\delta_{m_3,k_3}) = \mathbb{E}(\eta^\alpha_{m,k})\mathbb{E}(\eta^\beta_{m_1,k_1})\mathbb{E}(\eta^\gamma_{m_2,k_2})\mathbb{E}(\eta^\delta_{m_3,k_3}) \Bigg\}.$$

Then for any $X \in \mathcal{B}^\pi$, we have $[X]_{\pi^n}(t) \to t$ in probability. Furthermore, if π is also balanced and complete refining partition sequence, then the convergence is almost surely.

Corollary 3.15. *For any balanced complete refining sequence of partitions π, we have $\mathcal{B}^\pi \subset Q_\pi([0,1],\mathbb{R})$ almost surely.*

3.3.2 Processes with prescribed quadratic variation

A well known method for constructing a process with prescribed quadratic variation is via time-changed Brownian motion. Let W be a Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\phi : [0,\infty) \to [0,\infty)$ a continuous increasing function with $\phi(0) = 0$ and define $Y(t) = W(\phi(t))$. Then for any refining partition π we have

$$[Y]_\pi(t) = \phi(t)$$

almost surely.

In this subsection, we will construct a class of processes with prescribed quadratic varation, using a different construction based on the Schauder expansion. We will show that our class contains time-changed Brownian motion, but also other processes which may not be semimartingales.

Without loss of generality for the rest of this section we will also assume $\phi(1) = 1$. We first study the Schauder expansion of a time-changed Brownian motion: the proof of the following is based on straightforward calculations.

Lemma 3.16 (Schauder expansion of a time-changed Brownian motion)**.** *Let π to be a finitely refining sequence of partitions and $Y(t) = W(\phi(t))$, where W is* *a Brownian motion and $\phi : [0,1] \to [0,1]$ an increasing function with $\phi(0) = 0$. Then Y has the following Schauder expansion:*

$$Y(t) = Y(0) + (Y(1) - Y(0))t + \sum_{m=0}^{\infty} \sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} \eta_{mk}(Y) e^{\pi}_{m,k}(t),$$

where $\eta_{m,k}(Y) \sim \mathcal{N}(0, w^{\pi,\phi}_{m,k})$ are independent and

$$w^{\pi,\phi}_{m,k} = \frac{(\phi(t_2^{m,k}) - \phi(t_1^{m,k}))(t_3^{m,k} - t_2^{m,k})^2 + (t_2^{m,k} - t_1^{m,k})^2(\phi(t_3^{m,k}) - \phi(t_2^{m,k}))}{(t_2^{m,k} - t_1^{m,k})(t_3^{m,k} - t_2^{m,k})(t_3^{m,k} - t_1^{m,k})}, \tag{3.11}$$

where $[t_1^{m,k}, t_3^{m,k}] = \mathrm{supp}(e^{\pi}_{m,k})$ and $e^{\pi}_{m,k}$ attains its maximum at $t_2^{m,k}$.

We note that $w^{\pi,\phi}_{m,k}$ are non-random and only depend on the partition sequence and the function ϕ.

For any finitely refining sequence of partitions π, and for any continuous increasing function ϕ with $\phi(0) = 0$, similar to Equation (3.6),3.7,3.8 we have the corresponding identities (which are only dependent on π and ϕ but not on the path).

$$\lim_{n\to\infty} \left[\sum_{m=0}^{n-1} \sum_{k} a^{n,\pi}_{m,k}(t) w^{\pi,\phi}_{m,k} \right] = \phi(t), \tag{3.12}$$

$$\lim_{n\to\infty} \left[\sum_{m=0}^{n-1} \sum_{k} (a^{n,\pi}_{m,k}(t))^2 (w^{\pi,\phi}_{m,k})^2 \right] = 0, \tag{3.13}$$

$$\lim_{n\to\infty} \left[\sum_{m,m'=0}^{n-1} \sum_{\substack{k,k' \\ (m,k)\neq(m',k')}} (b^{n,\pi}_{m,k,m',k'}(t))^2 w^{\pi,\phi}_{m,k} w^{\pi,\phi}_{m',k'} \right] = 0. \tag{3.14}$$

The following theorem provides us with a broader class of processes with prescribed quadratic variation:

Theorem 3.17. *Let π be a finitely refining sequence of partitions with vanishing mesh $|\pi^n| \to 0$ and $\phi : [0,\infty) \to [0,\infty)$ an increasing function with $\phi(0) = 0$. Define X :*

$$X(t) = X(0) + (X(1) - X(0))t + \sum_{m=0}^{\infty} \sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} \eta_{mk} e^{\pi}_{m,k}(t).$$

where $(\eta_{m,k}, m \in \mathbb{N}, k = 1..N(\pi^{m+1}) - N(\pi^m))$ *is a family of random variables with*

$$\mathbb{E}\eta_{m,k} = 0, \qquad \mathbb{E}\eta_{m,k}\eta_{m',k'} = \mathbb{1}_{m,m'}\mathbb{1}_{k,k'}w^{\pi,\phi}_{m,k}, \qquad \mathbb{E}\eta^4_{m,k} < \infty$$

where $w^{\pi,\phi}_{m,k}$ *is given by* (3.11) *and*

$$\mathbb{E}(\eta^{\alpha}_{m,k}\eta^{\beta}_{m_1,k_1}\eta^{\gamma}_{m_2,k_2}\eta^{\delta}_{m_3,k_3}) = \mathbb{E}(\eta^{\alpha}_{m,k})\mathbb{E}(\eta^{\beta}_{m_1,k_1})\mathbb{E}(\eta^{\gamma}_{m_2,k_2})\mathbb{E}(\eta^{\delta}_{m_3,k_3})$$

for all integers $\alpha, \beta, \gamma, \delta$ *such that* $\alpha + \beta + \gamma + \delta = 4$. *Then*

$$\forall\, \epsilon > 0; \qquad \lim_{n\to\infty} \mathbb{P}(|[X]_{\pi^n}(t) - \phi(t)| > \epsilon) = 0.$$

Furthermore, if the sequence of partitions π *is complete refining and balanced and* ϕ *has a bounded derivative then*

$$X \in Q_\pi([0,1], \mathbb{R}) \text{ almost surely} \qquad \text{and} \qquad \mathbb{P}\left([X]_\pi(t) = \phi(t)\right) = 1.$$

Proof. We skip the proof of the above theorem as the proof is very similar to the proof of Theorem 3.14. The proof particularly uses Identity (3.12), (3.13), (3.14). For the proof of almost sure convergence, we use the fact that if ϕ has bounded derivatives and π is balanced, then the weights $w^{\pi,\phi}_{m,k}$ are almost surely bounded as well. ■

The assumptions of π and ϕ for almost sure convergence in Theorem 3.17 are sufficient conditions but not necessary. To summarise, for any finitely refining sequence of partitions π and for any continuous increasing function ϕ with $\phi(0) = 0$, define the class of processes $\mathcal{B}^{\pi,\phi}_0$ as follows.

$$\mathcal{B}^{\pi,\phi}_0 = \Bigg\{ X : \Omega \times [0,1] \mapsto \mathbb{R} : \; X(t) = X(0) + (X(1) - X(0))\,t + \sum_{m=0}^{\infty} \sum_{k=0}^{N(\pi^{m+1})-N(\pi^m)} \eta_{m,k} e^{\pi}_{m,k}(t) \tag{3.15}$$

$$\text{with, } \mathbb{E}(\eta_{m,k}) = 0, \; \mathbb{E}(\eta_{m,k}\eta_{m',k'}) = \delta_{m,m'}\delta_{k,k'}w_{m,k}, \; \mathbb{E}(\eta^4_{m,k}) \le M < \infty,$$

$$\text{and for, } (\alpha, \beta, \gamma, \delta) \in (0,1,3,4)$$

$$\mathbb{E}(\eta^{\alpha}_{m,k}\eta^{\beta}_{m_1,k_1}\eta^{\gamma}_{m_2,k_2}\eta^{\delta}_{m_3,k_3}) = \mathbb{E}(\eta^{\alpha}_{m,k})\mathbb{E}(\eta^{\beta}_{m_1,k_1})\mathbb{E}(\eta^{\gamma}_{m_2,k_2})\mathbb{E}(\eta^{\delta}_{m_3,k_3}) \text{ if } \alpha + \beta + \gamma + \delta = 4 \Bigg\}.$$

Then for any $X \in \mathcal{B}^{\pi,\phi}_0$, we have $[X]_{\pi^n}(t) \to \phi(t)$ in probability. If π is also balanced, complete refining and the continuous increasing function ϕ has $\phi(0) = 0$ and bounded derivatives then the convergence is in an almost sure sense.

Corollary 3.18. *Let π be any finitely refining sequence of partition and $\phi \in C^0([0,1],\mathbb{R})$ be an increasing function with $\phi(0) = 0$. Then the time changed Brownian motion defined as $Y(t) = W(\phi(t))$ belongs to the class $\mathcal{B}_0^{\pi,\phi}$*

Corollary 3.19. *For any balanced complete refining sequence of partitions π and for any increasing $\phi \in C^0([0,1],\mathbb{R})$ with bounded derivatives, we have $\mathcal{B}_0^{\pi,\phi} \subset Q_\pi([0,1],\mathbb{R})$ almost surely.*

3.4 A class of processes with quadratic variation invariant under coarsening

The quadratic variation of a path along a sequence of partitions strongly depends on the chosen sequence of partitions. As shown by Freedman [38, p. 47], given any continuous function, one can always construct a sequence of partitions along which the quadratic variation is zero. This result has been extended by Davis et al. [30] where they have shown that, given any continuous path $x \in C^0([0,T],\mathbb{R})$ and any increasing function $A : [0,T] \to \mathbb{R}_+$ (not necessarily continuous) one can construct a partition sequence π such that $[x]_\pi = A$. Another result by Mishura and Schied [62] provides a way to construct a vector space of functions with a prescribed quadratic variation. Notwithstanding these negative results, the quadratic variation of a function along a sequence of partitions π is always the same as that along any subsequences of π and the Chapter 2 also identifies a class of partitions and a class of d-dimensional paths where quadratic variation is partition invariant. In this section, we shall identify a class of processes x for which $[x]_\pi$ is uniquely defined across any coarsening of the initial finitely refining partition π.

One main difficulty in comparing the quadratic variation along two different partition sequences is the lack of structural similarity between the two sequences of partitions and/or lack of local bounds on the number of partition intervals.

For Brownian motion almost surely for any refining sequence of partitions π the quadratic variation is linear and same across partitions, i.e. $\mathbb{P}([W]_\pi(t) = t) = 1$. Now from Lemma 3.13 we can see along any finitely refining partition sequence π

the coefficients $\eta^{\pi}_{m,k}$ for Brownian motion are i.i.d. $\mathcal{N}(0,1)$. So for two finitely refining partition sequences π and σ, if we compare the corresponding Schauder basis coefficients $\eta^{\pi}_{m,k}$ and $\eta^{\sigma}_{m,k}$ for Brownian motions, both of them have the same distribution. This uniformity of coefficients of Brownian motion contributes towards partition sequence independent quadratic variation of Brownian motions.

In this section, we provide a class of 'rough' continuous processes for which the Schauder expansion has similar properties across certain 'related' sequences of refining partitions. As expected, our 'rough' class contains Brownian motion but also contains processes that are smoother than Brownian motion in terms of Hölder continuity.

3.4.1 Invariance of quadratic variation

Coarsening A partition may be refined by adding points to it. The inverse operation, which we call coarsening, corresponds to removing points i.e. subsampling or grouping of partition points. We will be specifically interested in coarsening that preserve the finitely refining property but may modify the asymptotic rate of decrease of the mesh size:

Definition 3.20 (Coarsening of a partition sequence)**.** *Let $\pi^n = (0 = t^n_0 < t^n_1 < \cdots < t^n_{N(\pi^n)} = T)$ be a finitely refining sequence of partitions of $[0,T]$ with vanishing mesh $|\pi^n| \to 0$. A coarsening of π is a sequence of subpartitions of π^n:*

$$A^n = (0 = t^n_{p(n,0)} < t^n_{p(n,1)} < \cdots < t^n_{p(n,N(A^n))} = T),$$

such that $(A^n)_{n\geq 1}$ is a finitely refining partition sequence of $[0,T]$.

Remark 3.21. $t \in A^n$ implies $t \in \pi^n$. Also if $\sigma = (\sigma^n)_{n\geq 1}$ is a coarsening of $\pi = (\pi^n)_{n\geq 1}$, then for any subsequence $\tau = (\pi^{K(n)})_{n\geq 1}$ of π; $\sigma^{K(n)}$ is also a coarsening of τ.

Take π be a finitely refining sequence of partitions of $[0,1]$ and take $\sigma = (\sigma^n)_{n\geq 1}$ to be a coarsening of π. Let $x \in C^0([0,1],\mathbb{R})$. Then the x can be expanded along the non-uniform Schauder system corresponding to partition sequences π and σ respectively i.e.

$$x(t) = x(0) + (x(1) - x(0))t + \sum_{m=0}^{\infty} \sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} \eta_{m,k} e^{\pi}_{m,k}(t)$$

$$= x(0) + (x(1) - x(0))t + \sum_{j=0}^{\infty} \sum_{l=1}^{N(\sigma^{j+1})-N(\sigma^j)} \theta_{j,l} e^{\sigma}_{j,l}(t),$$

where, $\{\eta_{m,k}\}$ and $\{\theta_{j,l}\}$ are corresponding coefficients of the Schauder system expansion along sequence of partition π and σ respectively. If the support of the function $e^{\sigma}_{j,l}$ is $[s_1^{j,l}, s_3^{j,l}]$ and its maximum is attained at time $s_2^{j,l}$ then, the coefficient $\theta_{j,l}$ has a closed form representation as follows (Proposition 3.7):

$$\theta_{j,l} = \frac{\left[\left(x(s_2^{j,l}) - x(s_1^{j,l})\right)(s_3^{j,l} - s_2^{j,l}) - \left(x(s_3^{j,l}) - x(s_2^{j,l})\right)(s_2^{j,l} - s_1^{j,l})\right]}{\sqrt{(s_2^{j,l} - s_1^{j,l})(s_3^{j,l} - s_2^{j,l})(s_3^{j,l} - s_1^{j,l})}}$$

$$= \left[\frac{(s_3^{j,l} - s_2^{j,l})\left(\sum_{m=0}^{\infty}\sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} \eta_{m,k}\left(e^{\pi}_{m,k}(s_2^{j,l}) - e^{\pi}_{m,k}(s_1^{j,l})\right)\right)}{\sqrt{(s_2^{j,l} - s_1^{j,l})(s_3^{j,l} - s_2^{j,l})(s_3^{j,l} - s_1^{j,l})}}\right.$$

$$\left. - \frac{(s_2^{j,l} - s_1^{j,l})\left(\sum_{m=0}^{\infty}\sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} \eta_{m,k}\left(e^{\pi}_{m,k}(s_3^{j,l}) - e^{\pi}_{m,k}(s_2^{j,l})\right)\right)}{\sqrt{(s_2^{j,l} - s_1^{j,l})(s_3^{j,l} - s_2^{j,l})(s_3^{j,l} - s_1^{j,l})}}\right]$$

$$= \sum_{m=0}^{\infty} \sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} \left[\frac{(s_3^{j,l} - s_2^{j,l})\left(e^{\pi}_{m,k}(s_2^{j,l}) - e^{\pi}_{m,k}(s_1^{j,l})\right) - (s_2^{j,l} - s_1^{j,l})\left(e^{\pi}_{m,k}(s_3^{j,l}) - e^{\pi}_{m,k}(s_2^{j,l})\right)}{\sqrt{(s_2^{j,l} - s_1^{j,l})(s_3^{j,l} - s_2^{j,l})(s_3^{j,l} - s_1^{j,l})}}\right] \eta_{m,k}.$$

Denote

$$A^{m,k}_{j,l} = \frac{(s_3^{j,l} - s_2^{j,l})\left(e^{\pi}_{m,k}(s_2^{j,l}) - e^{\pi}_{m,k}(s_1^{j,l})\right) - (s_2^{j,l} - s_1^{j,l})\left(e^{\pi}_{m,k}(s_3^{j,l}) - e^{\pi}_{m,k}(s_2^{j,l})\right)}{\sqrt{(s_2^{j,l} - s_1^{j,l})(s_3^{j,l} - s_2^{j,l})(s_3^{j,l} - s_1^{j,l})}}. \tag{3.16}$$

Since the function $e^{\pi}_{m,k}$ only depends on π not on the path $x \in C^0([0,1],\mathbb{R})$, the coefficient $A^{m,k}_{j,l}$ only depends on the refining partitions σ and π but not on the continuous path x. So the expression for $\theta_{j,l}$ can be represented as an infinite expansion of η's.

$$\theta_{j,l} = \sum_{m=0}^{\infty} \sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} A^{m,k}_{j,l} \eta_{m,k}. \tag{3.17}$$

The above equation holds for any two finitely refining partitions, but since σ is a coarsening of π, $A^{m,k}_{j,l} = 0$ for all $m > j+1$, $\forall l, k$. So the Equation (3.17) reduces to:

$$\theta_{j,l} = \sum_{m=0}^{j+1} \sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} A_{j,l}^{m,k} \eta_{m,k}. \tag{3.18}$$

Now if we take the path x to be a typical path of Brownian motion, then $\eta_{m,k} \sim^{IID} \mathcal{N}(0,1)$ and $\theta_{j,l} \sim^{IID} \mathcal{N}(0,1)$. So,

$$\mathbb{E}\theta_{j,l}^2 = 1$$

$$\implies \mathbb{E}\left[\sum_{m=0}^{j+1} \sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} A_{j,l}^{m,k} \eta_{m,k}\right]^2 = 1.$$

For Brownian motion $\mathbb{E}\eta_{m,k}\eta_{m',k'} = \mathbb{E}\delta_{m,m'}\delta_{k,k'} = \mathbb{1}_{m=m'}\mathbb{1}_{k=k'}$ and for any fixed pair (j,l), the above sum is a finite sum. So the above equality reduces to:

$$\left[\sum_{m=0}^{j+1} \sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} \sum_{m'=0}^{j+1} \sum_{k'=1}^{N(\pi^{m'+1})-N(\pi^{m'})} (A_{j,l}^{m,k})(A_{j,l}^{m',k'})\mathbb{E}\left(\eta_{m,k}\eta_{m',k'}\right)\right] = 1$$

$$\implies \sum_{m=0}^{j+1} \sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} (A_{j,l}^{m,k})^2 \mathbb{E}\eta_{m,k}^2 = 1$$

$$\implies \sum_{m=0}^{j+1} \sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} (A_{j,l}^{m,k})^2 = 1. \tag{3.19}$$

Similarly, for Brownian motion the cross-correlations of the coefficients are 0. So for pairs $(j,l) \neq (j',l')$:

$$\mathbb{E}(\theta_{j,l}\theta_{j',l'}) = 0$$

$$\implies E\left[\left(\sum_{m=0}^{j+1} \sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} A_{j,l}^{m,k} \eta_{m,k}\right)\left(\sum_{m'=0}^{j'+1} \sum_{k'=1}^{N(\pi^{m'+1})-N(\pi^{m'})} A_{j',l'}^{m',k'} \eta_{m',k'}\right)\right] = 0$$

$$\implies \left[\sum_{m=0}^{(j\wedge j')+1} \sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} A_{j,l}^{m,k} A_{j',l'}^{m,k}\left(\mathbb{E}\eta_{m,k}^2\right)\right] = 0$$

$$\implies \sum_{m=0}^{(j\wedge j')+1} \sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} A_{j,l}^{m,k} A_{j',l'}^{m,k} = 0. \tag{3.20}$$

Comparing the fourth moment of the coefficient $\theta_{j,l}$ for Brownian paths we get:

$$\mathbb{E}\theta_{j,l}^4 = 3\left(\mathbb{E}\theta_{j,l}^2\right)^2$$

$$\implies \mathbb{E}\left(\sum_{m=0}^{j+1}\sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} A_{j,l}^{m,k}\eta_{m,k}\right)^4 = 3$$

$$\implies \mathbb{E}\left(\sum_{m=0}^{j+1}\sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} (A_{j,l}^{m,k})^4\eta_{m,k}^4 + \sum_{m,k}\sum_{\substack{m',' k \\ (m,k)\neq(m',k')}} (A_{j,l}^{m,k})^2(A_{j,l}^{m',k'})^2\eta_{m,k}^2\eta_{m',k'}^2\right) = 3$$

$$\implies \left(\sum_{m=0}^{j+1}\sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} 3(A_{j,l}^{m,k})^4 + \sum_{m,k}\sum_{\substack{m',' k \\ (m,k)\neq(m',k')}} (A_{j,l}^{m,k})^2(A_{j,l}^{m',k'})^2\right) = 3$$

$$\implies \left[\sum_{m=0}^{j+1}\sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} 2(A_{j,l}^{m,k})^4 + \left(\sum_{m,k}(A_{j,l}^{m,k})^2\right)\left(\sum_{m',' k}(A_{j,l}^{m',k'})^2\right)\right] = 3.$$

Substituting Equation (3.19) twice in the second sum we get the following identity:

$$\sum_{m=0}^{j+1}\sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} (A_{j,l}^{m,k})^4 = 1. \tag{3.21}$$

Similarly, exploring the uncorrelated property of the coefficients θ for Brownian motion leads to the following equalities:

$$\sum_{m=0}^{(j\wedge j')+1}\sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} \left((A_{j,l}^{m,k})^2(A_{j',l'}^{m,k})^2\right) = 0 \quad \text{and,} \tag{3.22}$$

$$\sum_{m=0}^{(j\wedge j')+1}\sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} \left((A_{j,l}^{m,k})^3(A_{j',l'}^{m,k})\right) = 0 \quad \text{and,} \tag{3.23}$$

$$\sum_{m=0}^{j\wedge j'\wedge j_1+1}\sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} \left((A_{j,l}^{m,k})^2(A_{j',l'}^{m,k})(A_{j_1,l_1}^{m,k})\right) = 0 \quad \text{and,} \tag{3.24}$$

$$\sum_{m=0}^{j\wedge j'\wedge j_1\wedge j_2+1}\sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} \left((A_{j,l}^{m,k})(A_{j',l'}^{m,k})(A_{j_1,l_1}^{m,k})(A_{j_2,l_2}^{m,k})\right) = 0. \tag{3.25}$$

The following theorem provides properties of Schauder coefficients under coarsening.

Theorem 3.22. *Let π be a finitely refining sequence of partitions of* $[0,1]$ *and* $\sigma = (\sigma^n)_{n\geq 1}$ *be a coarsening of* π. *Define for* $t \in [0,1]$

$$x(t) = x(0) + (x(1) - x(0))t + \sum_{m=0}^{\infty} \sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} \eta_{m,k} e^{\pi}_{m,k}(t)$$

where

$$\mathbb{E}\eta_{m,k} = 0,\ \mathbb{E}\eta_{m,k}\eta_{m',k'} = \mathbb{1}_{m,m'}\mathbb{1}_{k,k'}, \qquad \mathbb{E}\eta^4_{m,k} = M < \infty \qquad \text{and} \tag{3.26}$$

$$\mathbb{E}(\eta^{\alpha}_{m,k}\eta^{\beta}_{m_1,k_1}\eta^{\gamma}_{m_2,k_2}\eta^{\delta}_{m_3,k_3}) = \mathbb{E}(\eta^{\alpha}_{m,k})\mathbb{E}(\eta^{\beta}_{m_1,k_1})\mathbb{E}(\eta^{\gamma}_{m_2,k_2})\mathbb{E}(\eta^{\delta}_{m_3,k_3}) \tag{3.27}$$

for all integer exponents $\alpha, \beta, \gamma, \delta$ *satisfying* $\alpha + \beta + \gamma + \delta = 4$. *Then* $(\theta_{j,l}, j \in \mathbb{N}, 1 \leq l \leq N(\sigma^{l+1}) - N(\sigma^l))$ *defined by Equations* (3.16)-(3.18) *also satisfies the properties* (3.26)-(3.27).

Proof. $\mathbb{E}\theta_{j,l}$ and $\mathbb{E}\theta^2_{j,l}$ can be expanded as follows.

$$\mathbb{E}\theta_{j,l} = \mathbb{E}\left[\sum_{m=0}^{j+1} \sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} A^{m,k}_{j,l}\eta_{m,k}\right] = \sum_{m=0}^{j+1} \sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} A^{m,k}_{j,l}\mathbb{E}\eta_{m,k} = 0 \ \text{ and,}$$

$$\mathbb{E}\theta^2_{j,l} = \mathbb{E}\left(\sum_{m=0}^{j+1} \sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} A^{m,k}_{j,l}\eta_{m,k}\right)^2 = \sum_{m=0}^{j+1} \sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} (A^{m,k}_{j,l})^2 = 1.$$

The last identity follows from the Equation (3.19). For the covariation the following identity can be obtained.

$$\mathbb{E}\theta_{j,l}\theta_{j',l'} = \mathbb{E}\left[\left(\sum_{m=0}^{j+1} \sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} A^{m,k}_{j,l}\eta_{m,k}\right)\left(\sum_{m'=0}^{j'+1} \sum_{k'=1}^{N(\pi^{m'+1})-N(\pi^{m'})} A^{m',k'}_{j',l'}\eta_{m',k'}\right)\right]$$

$$= \sum_{m=0}^{j\wedge j'+1} \sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} (A^{m,k}_{j,l})(A^{m,k}_{j',l'}) = 0.$$

The last equality follows from Equation 3.20. Now the fourth moment of $\theta_{j,l}$ can be represented as follows.

$$\mathbb{E}\theta^4_{j,l} = \mathbb{E}\left(\sum_{m=0}^{j+1} \sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} A^{m,k}_{j,l}\eta_{m,k}\right)^4$$

$$= \sum_{m=0}^{j+1} \sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} (A_{j,l}^{m,k})^4 \mathbb{E}\eta_{m,k}^4 + \sum_{m,k} \sum_{\substack{m',k' \\ (m,k)\neq(m',k')}} \left(A_{j,l}^{m,k}\right)^2 \left(A_{j,l}^{m',k'}\right)^2 \mathbb{E}(\eta_{m,k}^2 \eta_{m',k'}^2)$$

$$\leq M \sum_{m=0}^{j+1} \sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} \left(A_{j,l}^{m,k}\right)^4 + \sum_{m,k} \sum_{\substack{m',k' \\ (m,k)\neq(m',k')}} \left(A_{j,l}^{m,k}\right)^2 \left(A_{j,l}^{m',k'}\right)^2$$

$$= (M-1) \sum_{m=0}^{j+1} \sum_{k=1}^{N(\pi^{m+1})-N(\pi^m)} \left(A_{j,l}^{m,k}\right)^4 + \left(\sum_{m,k} \left(A_{j,l}^{m,k}\right)^2\right) \left(\sum_{m',k'} \left(A_{j,l}^{m',k'}\right)^2\right) < \infty.$$

The last inequality follows from the fact $\mathbb{E}\eta_{m,k}^4 = M$ and using Equation (3.19) and (3.21). The uncorrelated property of θ is a consequence of Equation (3.22, 3.23, 3.24, 3.25) and the fact that $\mathbb{E}\eta_{m,k}^4 = M < \infty$. So the result follows. ■

Remark 3.23. The assumptions of the above theorem are sufficient but may not be necessary. Note that, unlike the Brownian motion case, the coefficients in the non-uniform Schauder basis expansion of typical paths satisfying the assumption of Theorem 3.22 only have uncorrelated properties and do not necessarily have i.i.d. properties.

For any finitely refining sequence of partitions π of $[0,1]$ we can define the following class of processes:

$$\mathcal{A}^\pi = \Bigg\{ x : \Omega \times [0,1] \mapsto \mathbb{R},\ x(t) = x(0) + (x(1)-x(0))\,t + \sum_{m=0}^{\infty} \sum_{k=0}^{N(\pi^{m+1})-N(\pi^m)} \eta_{m,k} e_{m,k}^\pi(t)$$
$$\text{where } \mathbb{E}(\eta_{m,k}) = 0,\ \mathbb{E}(\eta_{m,k}\eta_{m',k'}) = \delta_{m,m'}\delta_{k,k'},\ \mathbb{E}(\eta_{m,k}^4) = M < \infty, \text{ and}$$
$$\text{for all integers } 0 \leq \alpha,\beta,\gamma,\delta \leq 4 \quad \text{with } \alpha+\beta+\gamma+\delta = 4$$
$$\mathbb{E}(\eta_{m,k}^\alpha \eta_{m_1,k_1}^\beta \eta_{m_2,k_2}^\gamma \eta_{m_3,k_3}^\delta) = \mathbb{E}(\eta_{m,k}^\alpha)\mathbb{E}(\eta_{m_1,k_1}^\beta)\mathbb{E}(\eta_{m_2,k_2}^\gamma)\mathbb{E}(\eta_{m_3,k_3}^\delta) \Bigg\}. \quad (3.28)$$

Then $\mathcal{A}^\pi \subset \mathcal{B}^\pi$ and we have the following result:

Theorem 3.24 (Invariance of Quadratic variation). *For any finitely refining sequence of partitions π, take a process $x \in \mathcal{A}^\pi$. Then for any coarsening σ of π we have:*

$$\forall t \in [0,1],\ [x]_{\sigma^n}(t) \to t \text{ and, } [x]_{\pi^n}(t) \to t \text{ in probability.}$$

Furthermore, if both π and σ are complete refining and balanced then:

$$\mathbb{P}\left(x \in Q_\pi([0,1],\mathbb{R}) \cap Q_\sigma([0,1],\mathbb{R})\right) = 1 \textit{ and, } [x]_\pi(t) = [x]_\sigma(t) \textit{ almost surely.}$$

Proof. Since $x \in \mathcal{A}^\pi$ for a finitely refining sequence of partitions π of $[0,1]$, $x \in C^0([0,1],\mathbb{R})$. Now for any coarsening σ of π, Theorem 3.22 concludes the corresponding Schauder coefficients $\theta^\sigma_{j,l}$ and $\eta^\pi_{m,k}$ have same uncorrelated properties. So the result follows from Theorem 3.14. ■

The following is an example of a path that does not satisfy the assumptions of Theorem 3.22 and whose quadratic variation (unlike Theorem 3.24) is not invariant under coarsening.

Example 3.4 (Example of continuous function with different quadratic variation along two different balanced finitely refining sequence of partition)**.** *Define (See Fig. 3.6)*

$$x(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{2^m-1} e^{\mathbb{T}}_{m,k}(t),$$

Then the quadratic variation of x along $\mathbb{T}$ is different from the quadratic variation of x along π, where $\pi^n = \left(0, \frac{1}{2^n}, \frac{2}{2^n}, \frac{4}{2^n}, \cdots, \frac{3i+1}{2^n}, \frac{3i+2}{2^n}, \cdots, 1\right)$. Note that the function x belongs to the class of functions defined in [70] and both the partition sequences π and $\mathbb{T}$ are finitely refining and π is coarsening of $\mathbb{T}$. Also, x has linear quadratic variation along both sequence of partitions π and $\mathbb{T}$, but they are not same for all $t \in (0,1]$.

Not surprisingly, Brownian motion belongs to the class $\mathcal{A}^\pi$ for any finitely refining sequence of partitions π, as for Brownian motion the coefficient of Schauder system expansion follows i.i.d. $\mathcal{N}(0,1)$. But the class of paths in $\mathcal{A}^\pi$ is not just a typical path of Brownian motion.

Example 3.5 (Process with $\frac{1}{2}$-Hölder continuous paths in the class $\mathcal{A}^{\mathbb{T}}$)**.** *Define (See Fig. 3.5)*

$$X(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{2^m-1} \theta_{m,k} e^{\mathbb{T}}_{m,k}(t),$$

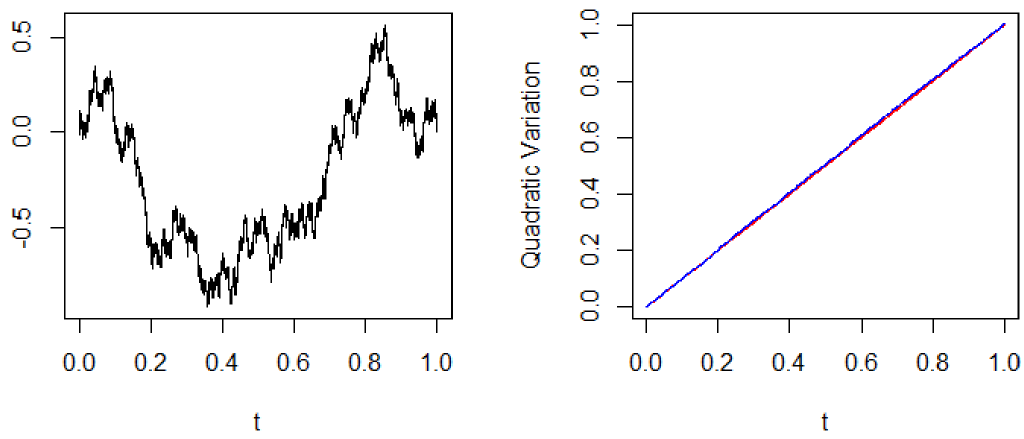

Figure 3.5: Left: Plot of the function x defined in Example 3.5 truncated at $n = 12$. Right: The black line represented the quadratic variation of the function x at level n=12 with respect to dyadic partition. The blue line represents the quadratic variation of the function x at level n=12 with respect to the partition π.

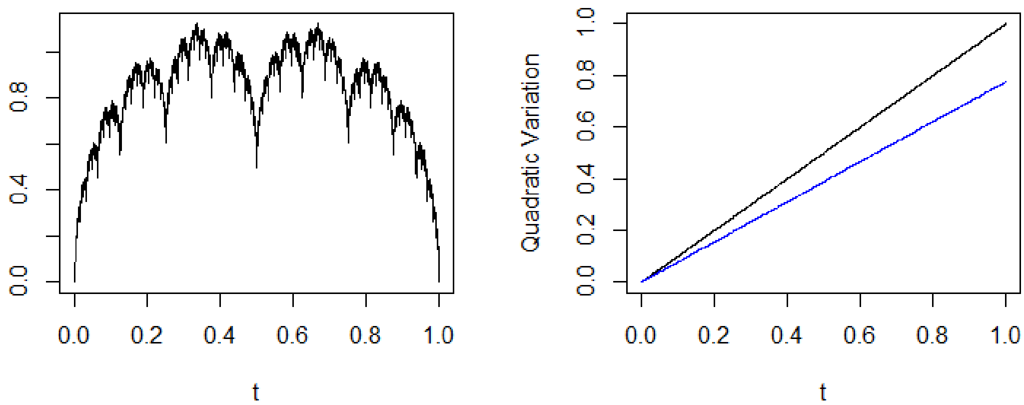

Figure 3.6: Left: Sample path of X defined in Example 3.4 truncated at $n = 12$. Right: The red line represented the quadratic variation of X at level n=12 along the dyadic partition. The blue line represents the quadratic variation of X at level n=12 along π.

where $\theta_{m,k}$ are i.i.d. random variables with

$$\theta_{m,k} = \begin{cases} 1, & \text{with probability } \frac{1}{2} \\ -1, & \text{with probability } \frac{1}{2} \end{cases}.$$

From the results of Mishura and Schied [62] we know that $X \in Q_{\mathbb{T}}([0,1], \mathbb{R})$ and from the construction $X \in \mathcal{A}^{\mathbb{T}}$. The process X belongs to the class $\mathcal{X}$ defined in [62], which is a class of function with $\frac{1}{2}$-Hölder continuity. So our 'rough' class $\mathcal{A}^{\mathbb{T}}$ contains $\frac{1}{2}$-Hölder continuous paths.

So $\mathcal{A}^{\pi}$ is an interesting class of processes and contains a processes smoother than Brownian motion in the sense of Hölder continuity, but still rough enough to have quadratic variation invariant across different finitely refining partitions.The following examples extend the constructions in Example 3.5 and 3.6 to a non-uniform and non-balanced partition.

Example 3.6. *Let $\pi^n = \left(0 = t_1^n < \cdots < t_{N(\pi^n)}^n\right)$ where*

$$\forall k = 1, \cdots, 2^n, \qquad t_{2k}^{n+1} = t_k^n \;\; \text{and,} \;\; t_{2k+1}^{n+1} = t_k^n + \frac{t_{k+1}^n - t_k^n}{2.5}$$

and define $x : [0,1] \to \mathbb{R}$ as follows (See Fig 3.7).

$$\forall t \in [0,1], \qquad x(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{2^m - 1} e_{m,k}^{\pi}(t).$$

From Figure 3.7 we can see the quadratic variation along a random coarsening partition differs form the quadratic variation along the given partition π.

Example 3.7. *Define the sequence of partition $\pi = (\pi^n)_{n \geq 1}$ with $\pi^n = \left(0 = t_1^n < \cdots < t_{N(\pi^n)}^n\right)$ as follows.*

$$\forall k = 1, \cdots, 2^n, \qquad t_{2k}^{n+1} = t_k^n \;\; \text{and,} \;\; t_{2k}^{n+1} = t_k^n + \frac{t_{k+1}^n - t_k^n}{2.5}$$

Define the process $X : [0,1] \times \Omega \to \mathbb{R}$ as (See Fig 3.8)

$$X(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{2^m - 1} \theta_{m,k} e_{m,k}^{\pi}(t),$$

where $\theta_{m,k}$ are i.i.d. random variables with

$$\theta_{m,k} = \begin{cases} 1, & \textit{with probability } \frac{1}{2} \\ -1, & \textit{with probability } \frac{1}{2} \end{cases}.$$

From Figure 3.8 we can see the quadratic variation along a random coarsening partition mathches with the quadratic variation along the given partition π.

3.4.2 Properties and lemmas

In this subsection, we will discuss some general properties of a process that contains $\mathcal{A}^\pi$, for any finitely refining sequence of partitions π.

For convenience of the next section let us reorder the complete orthonormal basis $\{\psi_{m,k}\}_{m,k}$ as $\{\psi_i\}_i$. Since $\{\psi_i\}_i$ is a complete orthonormal basis, for all $x \in \mathcal{A}^\pi$ we can express x in the Schauder basis expansion along π as follows.

$$x(t) = \sum_{i=0}^{\infty} \eta_i \int_0^t \psi_i^\pi(u)du,$$

where, $\mathbb{E}(\eta_i) = 0,\ \mathbb{E}(\eta_i \eta_{i'}) = \delta_{i,i'},\ \mathbb{E}(\eta_i^4) < \infty$. Now define,

$$I_t(s) := \begin{cases} 1 & s < t \\ 0 & s \geq t \end{cases} \quad \text{and} \quad < f, g >:= \int_0^1 f(t)g(t)dt$$

Then,

$$\int_0^t \psi_i^\pi(u)du =< I_t, \psi_i^\pi > .$$

Since $\{\psi_i\}_i$ is a complete orthonormal basis we have,

$$I_t = \sum_{i=0}^{\infty} < I_t, \psi_i^\pi > \psi_i^\pi \ \text{ and } \ t = ||I_t||^2 = \sum_{i=0}^{\infty} < I_t, \psi_i^\pi >^2 . \tag{3.29}$$

Lemma 3.25. *For any finitely refining sequence of partitions π take $x \in \mathcal{A}^\pi$. For any two times t and $s \in [0,T]$: $\mathbb{E}[x(t)x(s)] = t \wedge s$, where $t \wedge s = \min(t,s)$.*

Proof. Corresponding to π we have a complete orthonormal set of basis $\{\psi_i^\pi\}_i$ (as an example non-uniform Haar basis defined in Section 3.1). So:

$$\mathbb{E}[x(t)x(s)] = \mathbb{E}\left[\left(\sum_{i=0}^{\infty} \eta_i \int_0^t \psi_i^\pi(u)du\right)\left(\sum_{i=0}^{\infty} \eta_i \int_0^s \psi_i^\pi(u)du\right)\right]$$

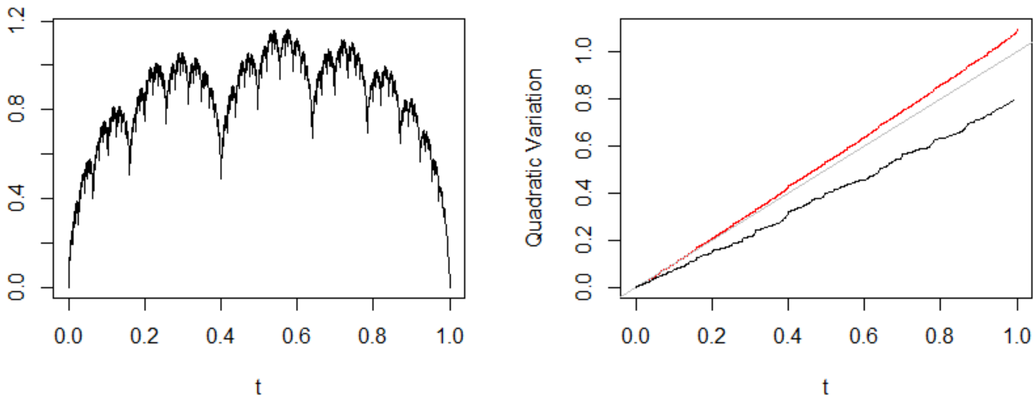

Figure 3.7: Left: Plot of the function x defined in Example 3.6 truncated at level $n = 12$. Right: The red line represents the quadratic variation of the function x at level n=12 with respect to partition π. The black line represents the quadratic variation of the function x at level n=12 for a random partition and the gray line represents for $y = x$ line.

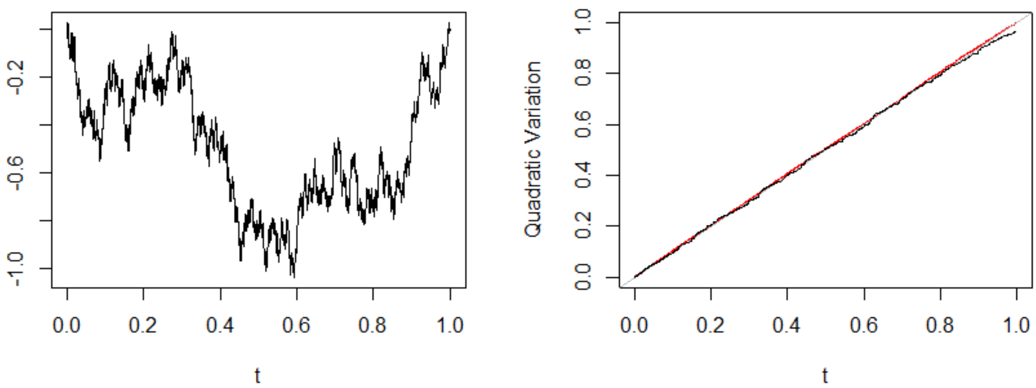

Figure 3.8: Left: Sample path of X defined in Example 3.7 truncated at $n = 12$. Right: The red line represented the quadratic variation of X at level n=12 with respect to partition π. The black line represents the quadratic variation of X at level n=12 for a random partition and the gray line represents for $y = x$ line.

$$= \sum_{i=0}^{\infty} \mathbb{E}\eta_i^2 \left(\int_0^t \psi_i^\pi(u)du \right) \left(\int_0^s \psi_i^\pi(u)du \right)$$

$$= \sum_{i=0}^{\infty} < I_t, \psi_i^\pi >< I_s, \psi_i^\pi > = < I_t, I_s > = t \wedge s.$$

■

As a consequence of the above for any finitely refining sequence of partition π and for any $x \in \mathcal{A}$, we have uncorrelated property of disjoint increments of x, i.e. if we have two disjoint interval $[t_1, t_2], [s_1, s_2] \subset [0, T]$ then for all $x \in \mathcal{A}^\pi$, we have $\mathbb{E}\left[(x(t_2) - x(t_1))(x(s_2) - x(s_1))\right] = 0$.

Theorem 3.26. *Let $\{\phi_i\}$ be an arbitrary complete orthonormal basis and let $\eta_1, \eta_2, \eta_3 \cdots$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{E}\eta_i = 0, \mathbb{E}\eta_i^2 = 1$ and $\mathbb{E}\eta_i\eta_j = \delta_{ij}$ for $i, j = 1, 2, \cdots,$ define*

$$X_t^n = \sum_{i=1}^{n} \eta_i \int_0^t \phi_i(s)ds. \tag{3.30}$$

Then for each t, X_t^n is a Cauchy sequence in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ whose limit X_t is a random variable with mean zero and variance t.

For any finitely refining sequence of partition π the assumption of the above theorem is satisfied for all $x \in \mathcal{A}^\pi$.

Proof. Since $\{\phi_i\}_i$ is a complete orthonormal basis we have

$$I_t = \sum_{i=0}^{\infty} < I_t, \phi_i > \phi_i \quad \text{and} \quad t = |t|^2 = \sum_{i=0}^{\infty} < I_t, \phi_i >^2 .$$

So we can have the following expression for $\mathbb{E}\left(X_t^n - X_t^m\right)^2$ where $n > m$ as follows.

$$\mathbb{E}\left(X_t^n - X_t^m\right)^2 = \mathbb{E}\left(\sum_{i=m+1}^{n} \eta_i \int_0^t \phi_i(s)ds \right)^2 = \sum_{i=m+1}^{n} \mathbb{E}\eta_i^2 \left(\int_0^t \phi_i(s)ds \right)^2$$

$$= \sum_{i=m+1}^{n} < I_t, \phi_i >^2 \xrightarrow{m,n\to\infty} 0.$$

Thus X_t^n is a Cauchy sequence in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. The mean and the variance of the limiting random variable X_t can be represented as:

$$\mathbb{E}X_t = \lim_{n\to\infty} \mathbb{E}X_t^n = \lim_{n\to\infty} \mathbb{E}\left(\sum_{i=0}^{n} \eta_i \int_0^t \phi_i(s)ds\right) = \lim_{n\to\infty} \sum_{i=0}^{n} \mathbb{E}\left(\eta_i\right) \int_0^t \phi_i(s)ds = 0 \quad \text{and,}$$

$$Var(X_t) = \lim_{n\to\infty} Var(X_t^n) = \lim_{n\to\infty} \left[\mathbb{E}\left(\sum_{i=0}^{n} \eta_i \int_0^t \phi_i(s)ds\right)^2 - \left(\mathbb{E}\sum_{i=0}^{n} \eta_i \int_0^t \phi_i(s)ds\right)^2\right]$$

$$= \lim_{n\to\infty} \sum_{i=0}^{n} \mathbb{E}\left(\eta_i\right)^2 \left(\int_0^t \phi_i(s)ds\right)^2 = \lim_{n\to\infty} \sum_{i=0}^{n} < I_t, \phi_i >^2 = t.$$

So the lemma follows. ■

The above result is valid for any orthonormal basis (non just for non-uniform Haar basis). For the following continuity result, let consider a non-uniform Haar basis. So Equation (3.30) is as follows.

$$X^n(t) = \sum_{m=0}^{n-1} \sum_{k=0}^{N(\pi^{m+1})-N(\pi^m)-1} \theta_{m,k} e^{\pi}_{m,k}(t), \quad \text{and,} \tag{3.31}$$

$$X(t) = \lim_{n\to\infty} x^n(t) = \sum_{m=0}^{\infty} \sum_{k=0}^{N(\pi^{m+1})-N(\pi^m)-1} \theta_{m,k} e^{\pi}_{m,k}(t)$$

Theorem 3.27 (Continuity of path). *Take a balanced, finitely refining sequence of partitions π of $[0,1]$. Then under the assumption $\mathbb{E}(\theta^4_{m,k}) < M$, for all m, k the sequence $X^n(t)$ defined in Equation (3.31), converges uniformly in t, almost surely to $X(t)$. Thus the process $X(t) = \lim_{n\to\infty} X^n(t)$ is a stochastic process with continuous sample paths.*

Proof. Let define $y^n(t) = X^{n+1}(t) - X^n(t) = \sum_{k=0}^{N(\pi^{n+1})-N(\pi^n)-1} \theta_{n,k} e^{\pi}_{n,k}(t)$, then if we can show that the function y^n is continuous and converges to 0 uniformly so the result follows. Now since $e_{n,k}$ is a continuous function over t for all n, k, so for every $n \in \mathbb{N} \cap \{0\}$: y^n is a continuous function over t. Since π is finitely refining the number of elements in

$$\{t \in [0,1], \quad e_{n,k}(t) \neq 0\}$$

is bounded by some M independent of n. Now define:

$$H_n = \sup_{t\in[0,1]} |y^n(t)| = \sup_{t\in[0,1]} |X^{n+1}(t) - X^n(t)|$$

$$= \sup_{t\in[0,1]} \left| \sum_{k=0}^{N(\pi^{n+1})-N(\pi^n)-1} \theta_{n,k} e^{\pi}_{n,k}(t) \right| \leq C_1 |\pi^n|^{\frac{1}{2}} \times \sup_k |\theta_{n,k}|.$$

For the last inequality we use the fact that for a balanced sequence of partitions π, $\sup_{t\in[0,1]} |e_{n,k}(t)| \leq C|\pi^n|^{\frac{1}{2}}$. Thus for any constant c_n,

$$\mathbb{P}\left(H_n > C_1|\pi^n|^{\frac{1}{2}} c_n\right) \leq \mathbb{P}\left(\sup_k |\theta_{n,k}| > c_n\right) = \mathbb{P}\left(\cup_k \{|\theta_{n,k}| > c_n\}\right)$$

$$\leq \sum_k \mathbb{P}\left(|\theta_{n,k}| > c_n\right) \leq C_2 N(\pi^n) \times \frac{\mathbb{E}|\theta_{n,k}|^4}{c_n^4} \leq C_0 \times \frac{N(\pi^n)}{c_n^4} \leq C\frac{M^n}{c_n^4}, \tag{3.32}$$

where, C, M are finite constants independent of n. The last inequality is a consequence of Markov inequality. We now choose $c_n = |\pi^n|^{\epsilon-\frac{1}{2}}$ for some $\epsilon > 0$ with $8\epsilon < 1$. Then the right-hand side of Inequality (3.32) is $C_0\frac{N(\pi^n)}{c_n^4} = C_0\frac{N(\pi^n)}{|\pi^n|^{4\epsilon-2}} \leq M_1|\pi^n|^{1-4\epsilon} \leq M_2 M^{n(4\epsilon-1)}$ (The two inequality follows as π is balanced). Now we know that $M^{n(4\epsilon-1)}$ is a general team of in a convergent series. Also b_n defined as $b_n = C_1|\pi^n|^{\frac{1}{2}}c_n = C_1|\pi^n|^{\frac{1}{2}}|\pi^n|^{\epsilon-\frac{1}{2}} = C_1|\pi^n|^{\epsilon} \to 0$ as $n \to \infty$. So using Borel-Cantelli Lemma, Inequality (3.32) deduces to,

$$\mathbb{P}[H_n > b_n \text{ infinitely often }] = 0$$

Since $b_n \to 0$, this shows that H_n is a convergent series and completes the proof. ■

In Theorem 3.27, the assumption of the reference partition π to be balanced is sufficient but *not* necessary.

3.5 Extension to the multidimensional case

In this section, we extend the previous results discussed to a multidimensional setting.

Non-uniform multidimensional Haar basis. Fix a finitely refining sequence of partitions π of $[0,1]$. The one dimensional non-uniform Haar basis can be represented as $\{h_{m,k,j}\}$, where $m = 0, 1, \cdots$ and $k = 0, \cdots, N(\pi^n)$ and there exists $M < \infty$ such that $j < M$. Then the function $h_{m,k,j} : [0,1] \to \mathbb{R}$ for all m, k and j can be expressed as:

$$h_{m,k,j}(t) = \begin{cases} 0 & \text{if } t \notin \left[t^{m+1}_{p(m,k)}, t^{m+1}_{p(m,k)+j}\right) \\ \left(\frac{t^{m+1}_{p(m,k)+j} - t^{m+1}_{p(m,k)+j-1}}{t^{m+1}_{p(m,k)+j-1} - t^{m+1}_{p(m,k)}} \times \frac{1}{t^{m+1}_{p(m,k)+j} - t^{m+1}_{p(m,k)}}\right)^{\frac{1}{2}} & \text{if } t \in \left[t^{m+1}_{p(m,k)}, t^{m+1}_{p(m,k)+j}\right) \\ -\left(\frac{t^{m+1}_{p(m,k)+j-1} - t^{m+1}_{p(m,k)}}{t^{m+1}_{p(m,k)+j} - t^{m+1}_{p(m,k)+j-1}} \times \frac{1}{t^{m+1}_{p(m,k)+j} - t^{m+1}_{p(m,k)}}\right)^{\frac{1}{2}} & \text{if } t \in \left[t^{m+1}_{p(m,k)+j-1}, t^{m+1}_{p(m,k)+j}\right) \end{cases}, \tag{3.33}$$

where, $p(m,k)$ is defined in Equation (3.1). The non-uniform Haar basis $\{h_{m,k,j}\}$ is an orthogonal basis in one dimension. For convenience, reorder the non-uniform Haar basis to $\{h_{m,k}\}$, where $m = 0, 1, \cdots$ and $k = 0, 1, \cdots, N(\pi^{m+1}) - N(\pi^m) - 1$. Now we will define d-dimensional non-uniform Haar basis in the canonical way. Define $\{h^i_{m,k}\}$ for all $m = 0, 1, \cdots$, $k = 0, 1, \cdots, N(\pi^{m+1}) - N(\pi^m) - 1$ and $i = 1, 2, \cdots, d$ as follows.

$$h^i_{m,k}(t) : [0,1] \to \mathbb{R}^d \quad \text{such that,} \quad h^i_{m,k}(t) = h_{m,k}(t) \times \mathbf{e}_i, \tag{3.34}$$

where, $\mathbf{e}_i$ is a d-dimensional column vector with 1 at i^{th} entry and 0 elsewhere. Clearly, $\{\mathbf{e}_i\}_{i=1,\cdots,d}$ is an orthogonal basis of $\mathbb{R}^d$. Denote $\mathbf{0}$ to be a d-dimensional column vector with all entry as 0. Now from the definition of $h^i_{m,k}$ we get

$$\int_0^1 h^i_{m,k} = \mathbf{0}; \quad \int_0^1 < h^i_{m,k}, h^i_{m,k} >= \mathbf{e}_i \quad \text{and,} \int_0^1 < h^i_{m,k}, h^j_{m',k'} >= \mathbb{1}_{i=j}\mathbb{1}_{m=m'}\mathbb{1}_{k=k'}\mathbf{e}_i.$$

So $\{h^i_{m,k}\}$, where $m = 0, 1, \cdots$, $k = 0, 1, \cdots, N(\pi^{m+1}) - N(\pi^m) - 1$ and $i = 1, \cdots, d$ form an orthonormal basis in $\mathbb{R}^d$. The Schauder basis $e^{\pi,i}_{m,k} : [0,1] \to \mathbb{R}^d$ is defined as $e^i_{m,k}(t) = \left(\int_0^t h_{m,k}(u)du\right)\mathbf{e}_i$ for $m \in \mathbb{N}$, $k = 0, 1, \cdots, N(\pi^{m+1} - N(\pi^m) - 1)$ and $i = 1, \cdots, d$.

The following theorem shows that any d-dimensional continuous function can be represented uniquely w.r.t. the d-dimensional non-uniform Schauder system associated with a finitely refining partition sequence.

Theorem 3.28. *Let π be a finitely refining sequence of partitions of $[0,1]$. Then any continuous function $x = (x_1, x_2, \cdots, x_d) \in C^0([0,1], \mathbb{R}^d)$ has a unique Schauder representation associated with π:*

$$x_i(t) = x_i(0) + (x_i(1) - x_i(0))t + \sum_{m=0}^{\infty} \sum_{k=0}^{N(\pi^{m+1})-N(\pi^m)-1} \theta^{i}_{m,k} e^{\pi}_{m,k}(t).$$

If the support of the function $e^{\pi}_{m,k}$ is $[t_1^{m,k}, t_3^{m,k}]$ and its maximum is attained at time $t_2^{m,k}$ then

$$\forall i = 1, \cdots, d \qquad \theta^{i}_{m,k} = \frac{\left[(x_i(t_2^{m,k}) - x_i(t_1^{m,k}))(t_3^{m,k} - t_2^{m,k}) - (x_i(t_3^{m,k}) - x_i(t_2^{m,k}))(t_2^{m,k} - t_1^{m,k})\right]}{\sqrt{(t_2^{m,k} - t_1^{m,k})(t_3^{m,k} - t_2^{m,k})(t_3^{m,k} - t_1^{m,k})}}.$$

Proof. The proof is a straightforward extension of the one-dimensional case in Theorem 3.7. ■

We now give a multi-dimensional version of Theorem 3.8.

Theorem 3.29. *Let π be a finitely refining sequence of partitions of $[0,1]$ with vanishing mesh and*

$$x_i(t) = x_i(0) + (x_i(1) - x_i(0))t + \sum_{m=0}^{\infty} \sum_{k=0}^{N(\pi^{m+1})-N(\pi^m)-1} \theta^{i}_{m,k} e^{\pi}_{m,k}(t).$$

Then

$$[x_i, x_j]_{\pi^n}(t) = \sum_{m=0}^{n-1} \sum_{k=0}^{N(\pi^{m+1})-N(\pi^m)-1} a^n_{m,k}(t) \theta^{i}_{m,k} \theta^{j}_{m,k} + \sum_{m,m'} \sum_{\substack{k,k' \\ (m,k) \neq (m',k')}} b^n_{m,k,m',k'}(t) \theta^{i}_{m,k} \theta^{j}_{m',k'}. \quad (3.35)$$

If $[t_1^{m,k}, t_3^{m,k}]$ is the support of the function $e^{\pi}_{m,k}$, $t_2^{m,k}$ is maximum and $\Delta t_i^n = t_{i+1}^n \wedge t - t_i^n \wedge t$, then:

$$a^n_{m,k}(t) = \left\{ \left[\sum_{\Delta t_i^n \subset [t_1^{m,k}, t_2^{m,k}]} (\Delta t_i^n)^2 \right] \times \frac{t_3^{m,k} - t_2^{m,k}}{t_2^{m,k} - t_1^{m,k}} + \left[\sum_{\Delta t_i^n \subset [t_2^{m,k}, t_3^{m,k}]} (\Delta t_i^n)^2 \right] \times \frac{t_2^{m,k} - t_1^{m,k}}{t_3^{m,k} - t_2^{m,k}} \right\} \times \frac{1}{t_3^{m,k} - t_1^{m,k}},$$

and,

$$b^n_{m,k,m',k'}(t) = \psi_{m',k'}(t_1^{m,k}) \times \left\{ \frac{\sum_{\Delta t_i^n \subset [t_1^{m,k}, t_2^{m,k}]} (\Delta t_i^n)^2}{t_2^{m,k} - t_1^{m,k}} - \frac{\sum_{\Delta t_i^n \subset [t_2^{m,k}, t_3^{m,k}]} (\Delta t_i^n)^2}{t_3^{m,k} - t_2^{m,k}} \right\} \times \sqrt{\frac{(t_2^{m,k} - t_1^{m,k})(t_3^{m,k} - t_2^{m,k})}{t_3^{m,k} - t_1^{m,k}}},$$

if $\mathrm{Supp}(e^n_{m,k}) \subset \mathrm{Supp}(e^n_{m',k'})$, *and* $b^n_{m,k,m',k'}(t) = 0$ *if* $\mathrm{Supp}(e^n_{m,k}) \cap \mathrm{Supp}(e^n_{m',k'}) = \emptyset$.

The following example is a 2-dimensional extension of the construction given in Section 3.4.

Example 3.8. *[Example of process in 2 dimension with linear quadratic variation] Define the class of processes $x \in C^0([0,T],\mathbb{R}^2)$ as follows. For all $t \in [0,T]$:*

$$x(t) = \left(x_1(0) + (x_1(1) - x_1(0))\, t + \sum_{m=0}^{\infty} \sum_{k=0}^{2^m-1} \theta^{①}_{m,k} e_{m,k}(t), \qquad x_2(0) + (x_2(1) - x_2(0))\, t + \sum_{m=0}^{\infty} \sum_{k=0}^{2^m-1} \theta^{②}_{m,k} e_{m,k}(t) \right),$$

where, $\theta^{①}_{m,k}, \theta^{②}_{m,k} \in \{-1,1\}$. This is the two dimensional extension of [70]. Then the quadratic variation of x can be think of a 2×2 matrix:

$$[x]_{T^n}(t) = \begin{pmatrix} \frac{1}{2^n}\sum_{m=0}^{n-1}\sum_{k=0}^{2^m-1}(\theta^{①}_{m,k})^2 \mathbb{1}_{[0,t]} & \frac{1}{2^n}\sum_{m=0}^{n-1}\sum_{k=0}^{2^m-1}(\theta^{①}_{m,k})(\theta^{②}_{m,k}) \mathbb{1}_{[0,t]} \\ \frac{1}{2^n}\sum_{m=0}^{n-1}\sum_{k=0}^{2^m-1}(\theta^{①}_{m,k})(\theta^{②}_{m,k}) \mathbb{1}_{[0,t]} & \frac{1}{2^n}\sum_{m=0}^{n-1}\sum_{k=0}^{2^m-1}(\theta^{②}_{m,k})^2 \mathbb{1}_{[0,t]} \end{pmatrix}.$$

Since $\theta^{①}_{m,k}, \theta^{②}_{m,k} \in \{-1,1\}$ we get $\frac{1}{2^n}\sum_{m=0}^{n-1}\sum_{k=0}^{2^m-1}(\theta^{①}_{m,k})^2 = \frac{2^n-1}{2^n} \xrightarrow{n\to\infty} 1$, similarly, $\frac{1}{2^n}\sum_{m=0}^{n-1}\sum_{k=0}^{2^m-1}(\theta^{②}_{m,k})^2 \xrightarrow{n\to\infty} 1$.

If we further assume $\theta^{①}_{m,k}, \theta^{②}_{m,k}$ are independent (not just uncorrelated) with $\mathbb{E}(\theta^{①}_{m,k}) = 0$, then we get:

$$\mathbb{E}\left(\frac{1}{2^n}\sum_{m=0}^{n-1}\sum_{k=0}^{2^m-1}(\theta^{①}_{m,k})(\theta^{②}_{m,k})\right) = \frac{1}{2^n}\sum_{m=0}^{n-1}\sum_{k=0}^{2^m-1}\mathbb{E}\left(\theta^{①}_{m,k}\right)\mathbb{E}\left(\theta^{②}_{m,k}\right) = 0 \qquad \textit{and,}$$

$$Var\left(\frac{1}{2^n}\sum_{m=0}^{n-1}\sum_{k=0}^{2^m-1}(\theta^{①}_{m,k})(\theta^{②}_{m,k})\right) = \mathbb{E}\left(\frac{1}{2^n}\sum_{m=0}^{n-1}\sum_{k=0}^{2^m-1}(\theta^{①}_{m,k})(\theta^{②}_{m,k})\right)^2$$

$$= \left(\frac{1}{2^n}\right)^2 \mathbb{E}\left[\sum_{m=0}^{n-1}\sum_{k=0}^{2^m-1}\left((\theta^{①}_{m,k})(\theta^{②}_{m,k})\right)^2 + \sum_{m=0}^{n-1}\sum_{k=0}^{2^m-1}\sum_{\substack{m'=0 \\ (m,k)\neq(m',k')}}^{n-1}\sum_{k'=0}^{2^{m'}-1}\left((\theta^{①}_{m,k})(\theta^{②}_{m,k})\right)\left((\theta^{①}_{m',k'})(\theta^{②}_{m',k'})\right)\right]$$

$$= \left(\frac{1}{2^n}\right)^2 \left[\sum_{m=0}^{n-1}\sum_{k=0}^{2^m-1} 1 + \sum_{m=0}^{n-1}\sum_{k=0}^{2^m-1}\sum_{\substack{m'=0 \\ (m,k)\neq(m',k')}}^{n-1}\sum_{k'=0}^{2^{m'}-1}\mathbb{E}\left((\theta^{①}_{m,k})(\theta^{②}_{m,k})\right)\mathbb{E}\left((\theta^{①}_{m',k'})(\theta^{②}_{m',k'})\right)\right]$$

$$= \left(\frac{1}{2^n}\right)^2 \left[\sum_{m=0}^{n-1}\sum_{k=0}^{2^m-1} 1\right] = \frac{1}{2^n} - \left(\frac{1}{2^n}\right)^2 \xrightarrow{n\to\infty} 0.$$

We can see, $Var\left(\frac{1}{2^n}\sum_{m=0}^{n-1}\sum_{k=0}^{2^m-1}(\theta_{m,k}^{①})(\theta_{m,k}^{②})\right)$ *has an upper-bound of* $\left(\frac{1}{2^n}-\frac{1}{2^{2n}}\right)$ *which is the general term of a summable series. So using Borel-Cantelli lemma we can conclude*

$$\left(\frac{1}{2^n}\sum_{m=0}^{n-1}\sum_{k=0}^{2^m-1}(\theta_{m,k}^{①})(\theta_{m,k}^{②})\right)\to 0 \quad \textit{almost surely.}$$

So $[x]_{T^n}(t)\to t\boldsymbol{Id}_{2\times 2}$ *almost surely.*

Remark 3.30. The process described in Example 3.8 has the same quadratic variation as two-dimensional Brownian Motion. But in contrast with Brownian paths it has $\frac{1}{2}$-Hölder continuous paths.

If we take $\theta_{m,k}^{②}=1$ in Example 3.8, then the corresponding process x has different quadratic variation along Triadic partition than that of 2-dimensional Brownian motion.

If we take $\theta_{m,k}^{②}$ and $\theta_{m,k}^{②}$ are independent and $+1$ and -1 both with probability $\frac{1}{2}$ in Example 3.8, then the process x has the same quadratic variation along any finitely refining sequence of partitions which is coarsening of dyadic partition. This is a higher-dimensional extension of the process we discussed in Section 3.4. We have skipped the proof of this argument, as it follows in the similar line of the proofs discussed in Section 3.4.

Chapter 4

Roughness and variation index of a signal

Chapter based on: Rama Cont, Purba Das. Measuring the roughness of a signal [22].

Cont and Perkowski [28] showed that Föllmer's pathwise Itô calculus may be extended to paths with arbitrary regularity using the concept of p-th variation along a sequence of time partitions for even integers $p > 0$. For paths with finite and strictly positive p-th variation along a sequence of time partitions, Cont and Perkowski [28] obtained the following change of variable formula for $p \in 2\mathbb{N}$, $S \in V_\pi^p$ and $f \in C^p(\mathbb{R}, \mathbb{R})$ (the p-th variation is defined in Def. 4.1):

$$f(S(t)) - f(S(0)) = \int_0^t f'(S(s))\, dS(s) + \frac{1}{p!} \int_0^t f^{(p)}(S(s))\, d[S]^{(p)}(s),$$

where the integral

$$\int_0^t f'(S(s))\, dS(s) := \lim_{n\to\infty} \sum_{[t_j, t_{j+1}] \in \pi_n} \sum_{k=1}^{p-1} \frac{f^{(k)}(S(t_j))}{k!} (S(t_{j+1} \wedge t) - S(t_j \wedge t))^k$$

is defined as a (pointwise) limit of compensated Riemann sums. These results were subsequently extended by Cont and Jin [26] to the case of all $p > 1$. These results show that the class of paths with finite p-th order variation is an interesting class for which a calculus is available. In order to apply such results one needs to identify the correct roughness order p.

For such paths and processes with non-zero p-th variation, there are many measures of roughness such as the Hurst exponent [50], Hölder exponent, Besov regularity [67], ρ-irregularity [15], IR-roughness [6].

We introduce a model-free pathwise roughness index based on the concept of *normalized* p-th variation of a signal, which identifies the correct order p and provides good finite sample performance for fractional processes.

We then introduce a method for estimating the roughness of a function based on a discrete sample, using the concept of normalized p-th variation along a sequence of partitions. We investigate its finite sample performance for measuring the roughness of sample paths of stochastic processes using detailed numerical experiments based on sample paths of Fractional Brownian motion and Takagi-Landsberg functions.

4.1 p-th variation along a sequence of partitions

The concept of p-th variation along a sequence of partitions $\pi = (\pi^n)_{n\geq 1}$ with $0 = t_0^n < \cdots < t_k^n < \cdots < t_{N(\pi^n)}^n = T$ is defined in the line of [28], which is an extension of Föllmer's pathwise quadratic variation [35].

Let δ_t denote a unit point mass at t.

Definition 4.1 (p-th variation along a sequence of partitions)**.** *Let $p > 0$. A continuous path $x \in C^0([0,T],\mathbb{R})$ is said to have p-th variation along a sequence of partitions $\pi = (\pi^n)_{n\geq 1}$ of $[0,T]$ if the sequence of measures,*

$$\mu^n := \sum_{t_j^n \in \pi^n} \delta_{t_j^n} |x(t_{j+1}^n) - x(t_j^n)|^p,$$

converges weakly to a Radon measure μ without atoms. In that case, we write $x \in V_\pi^p([0,T],\mathbb{R})$ and $[x]^{(p)}(t) := \mu([0,t])$ for $t \in [0,T]$, and we call $[x]^{(p)}$ the p-th variation *of a path x along the sequence of partitions π.*

Remark 4.2. The following are some remarks regarding p-th variation along a sequence of partitions π of $[0,T]$.

1. For $p = 2$, for any sequence of partitions π with mesh $|\pi| \to 0$, the two classes $Q_\pi([0,T],\mathbb{R})$ and $V_\pi^p([0,T],\mathbb{R})$ are the same. So for a path x, we use the sort notation $[x]_\pi$, for $[x]_\pi^{(p)}$ when $p = 2$.

2. Functions in $V^p_\pi([0,T],\mathbb{R})$ do not necessarily have finite p-variation in the usual sense. Recall that the p-variation of a function $f \in C([0,T],\mathbb{R})$ is defined (see Dudley [34]) as follows:

$$\|f\|_{\text{p-var}} := \Big(\sup_{\pi\in\Pi([0,T])} \lim_n \sum_{t_j^n\in\pi} |f(t_{j+1}^n) - f(t_j^n)|^p \Big)^{1/p},$$

where the supremum is taken over the set $\Pi([0,T])$ of all partitions π of $[0,T]$. A typical example where p-th variation and p variation are different is provided for the case of $p=2$ in Chapter 2. The p-variation involves taking a supremum over *all* partitions, whereas p-th variation is a limit taken along a given sequence $(\pi^n)_{n\geq 1}$. So in general $[x]^{(p)}_\pi$ is smaller than the p-variation.

3. If $S \in V^p(\pi)$ and $q > p$, then $S \in V^q(\pi)$ with $[S]^{(q)}_\pi \equiv 0$.

Definition 4.1 is a general definition for any càdlàg paths, but for continuous functions the following lemma gives a simple characterization of p-th variation:

Lemma 4.3. *Let $x \in C^0([0,T],\mathbb{R})$. Then $x \in V^p_\pi([0,T],\mathbb{R})$ if and only if there exists a continuous function $[x]^{(p)}_\pi$ such that*

$$\forall t \in [0,T], \qquad \sum_{\substack{t_j^n\in\pi^n:\\ t_j^n\leq t}} \left| x(t_{j+1}^n) - x(t_j^n) \right|^p \overset{n\to\infty}{\longrightarrow} [x]^{(p)}_\pi(t). \tag{4.1}$$

If this property holds, then the convergence in (4.1) *is uniform.*

Indeed, the weak convergence of measures on $[0,T]$ is equivalent to the pointwise convergence of their cumulative distribution functions at all continuity points of the limiting cumulative distribution function, and if the limiting cumulative distribution function is continuous, the convergence is uniform.

Remark 4.4. For $x \in V^p_\pi([0,T],\mathbb{R})$ we have $[x]^{(p)}_\pi(T) < \infty$. If

$$\sum_{\pi^n\cap(a,b)} \left| x(t_{i+1}^n) - x(t_i^n) \right|^p \to \infty$$

for all $(a,b) \subset [0,T]$ then we will write $[x]^{(p)}_\pi(t) = \infty$.

4.2 Variation index and roughness index

For a given continuous path, there are many known estimates which are associated with roughness of paths. Some of the common estimates for roughness are Hölder exponents (irregularity), Hurst exponents, ρ-irregularity and so on. Even though for fractional processes like fractional Brownian motions these different roughness estimators are linked through self-similarity and hence the same, for a general continuous function there is no reason for these different estimators to be the same. To formalize the concept of roughness, we introduce the notion of variation index and roughness index of a path:

Definition 4.5 (Variation index)**.** *The variation index of a path $x \in C^0([0,T],\mathbb{R})$ along a partition sequence π is defined as follows:*

$$p^\pi(x) := \inf\left\{p \geq 1 \ : \ \limsup_{n\uparrow\infty} \sum_{\pi^n} \left|x(t^n_{i+1}) - x(t^n_i)\right|^p = 0\right\}.$$

Note that for $x \in C^0([0,T],\mathbb{R})$ we do *not* necessarily have $\limsup_{n\uparrow\infty} \sum_{\pi^n} \left|x(t^n_{i+1}) - x(t^n_i)\right|^{p^\pi(x)} = 0$.

Definition 4.6 (Roughness index)**.** *The roughness index of a path $x \in C^0([0,T],\mathbb{R})$ (along π) is defined as a reciprocal of the variation index:*

$$H^\pi(x) = \frac{1}{p^\pi(x)}.$$

Remark 4.7. The above definition of Roughness index is closely linked with Hurst roughness exponent defined by Han and Schied [47]. The advantage of using the above definition is under Hölder continuity assumption, the roughness index defined above always exists.

Remark 4.8. The roughness and the variation index is defined along a particular sequence of partitions π. It is possible to construct a continuous function x, and two partition sequences π and σ such that $p^\pi(x) \neq p^\sigma(x)$.

We now provide some properties of variation index.

Lemma 4.9. *Take a continuous function x and a partition sequence π with vanishing mesh. If there exists $q^* \in [1,\infty)$ such that* $\limsup_{n\uparrow\infty}\sum_{\pi^n}\left|x(t^n_{i+1})-x(t^n_i)\right|^{q^*} \in (0,\infty)$ *then:*

$$p^\pi(x) = q^*.$$

Proof. Take any $p > q^*$ and define $\epsilon = p - q^* > 0$. Also define $\limsup_{n\uparrow\infty}\sum_{\pi^n}\left|x(t^n_{i+1})-x(t^n_i)\right|^{q^*} = C$. Then:

$$\limsup_{n\uparrow\infty}\sum_{\pi^n}\left|x(t^n_{i+1})-x(t^n_i)\right|^{p} = \limsup_{n\uparrow\infty}\sum_{\pi^n}\left|x(t^n_{i+1})-x(t^n_i)\right|^{q^*+\epsilon}$$
$$= \limsup_{n\uparrow\infty}\left[\sup_i\left|x(t^n_{i+1})-x(t^n_i)\right|^{\epsilon}\sum_{\pi^n}\left|x(t^n_{i+1})-x(t^n_i)\right|^{q^*}\right]$$
$$\le C\limsup_{n\uparrow\infty}\left[\sup_i\left|x(t^n_{i+1})-x(t^n_i)\right|^{\epsilon}\right] = 0$$

where the last equality follows from the fact that $x \in C^0([0,T],\mathbb{R})$, $\epsilon > 0$ and $|\pi^n| \to 0$. The above argument is true for any $p > q^*$. So for all $p > q^*$, $\limsup_{n\uparrow\infty}\sum_{\pi^n}\left|x(t^n_{i+1})-x(t^n_i)\right|^{p} = 0$ and hence $p^\pi(x) \le q^*$. On the other hand $\limsup_{n\uparrow\infty}\sum_{\pi^n}\left|x(t^n_{i+1})-x(t^n_i)\right|^{q^*} > 0$, so we also get $p^\pi(x) \ge q^*$. This concludes the proof. ■

The following lemma shows that for Hölder continuous paths, the roughness index is always smaller than the Hölder exponent.

Lemma 4.10. *Take π be a sequence of partitions of vanishing mesh. Then $x \in C^\alpha([0,T],\mathbb{R})$ implies $p^\pi(x) \le \frac{1}{\alpha}$.*

Proof. Take $q^* \in (\frac{1}{\alpha},\infty)$. Then:

$$\limsup_{n\uparrow\infty}\sum_{\pi^n}\left|x(t^n_{i+1})-x(t^n_i)\right|^{q^*} \le \limsup_{n\uparrow\infty}\sum_{\pi^n}\left|t^n_{i+1}-t^n_i\right|^{\alpha q^*}$$
$$= \limsup_{n\uparrow\infty}\sum_{\pi^n}\left|t^n_{i+1}-t^n_i\right|^{1+(\alpha q^*-1)} \le \limsup_{n\uparrow\infty}\left[\sup_i\left|t^n_{i+1}-t^n_i\right|^{(\alpha q^*-1)}\sum_{\pi^n}\left|t^n_{i+1}-t^n_i\right|\right]$$
$$\le T\limsup_{n\uparrow\infty}|\pi^n|^{\alpha q^*-1} = 0$$

where the first inequality is due to the fact that x is α-Hölder and the last equality holds true as partition π has vanishing mesh $|\pi|^n \to 0$. Since the above argument is true for all $q^* > \frac{1}{\alpha}$, the variation index $p^\pi(x) \le \frac{1}{\alpha}$. ■

Though for a continuous function x, variation index always exists (it can be ∞) it is not clear if the function $x \in V_\pi^{p^\pi(x)}([0,T],\mathbb{R})$. In fact in general it is not even obvious if the variation index $p^\pi(x)$ is the same with

$$r^\pi(x) := \inf\left\{p \geq 1 \; : \; \liminf_{n\uparrow\infty} \sum_{\pi^n} \left|x(t_{i+1}^n) - x(t_i^n)\right|^p = 0\right\}.$$

We can only say $p^\pi(x) \geq r^\pi(x)$.

In a very recent paper, Han and Schied [47] have defined a notion of (pathwise) Hurst roughness exponent. The following lemma shows that the Hurst roughness exponent (whenever exists) coincides with the roughness index of a function.

Lemma 4.11. *Take a continuous function x and a partition sequence π with vanishing mesh. Then:*

$$p^\pi(x) = r^\pi(x) \qquad \Longrightarrow \qquad [x]_\pi^{(q)}(t) = \begin{cases} 0 & \text{if } q > p^\pi(x) \\ \infty & \text{if } q < p^\pi(x) \end{cases}.$$

Proof. From the definition of the variation index we can conclude for all $q > p^\pi(x)$, $\limsup_{n\uparrow\infty} \sum_{\pi^n} \left|x(t_{i+1}^n) - x(t_i^n)\right|^q = 0$. Similarly for all $q > r^\pi(x) = p^\pi(x)$ we have $\liminf_{n\uparrow\infty} \sum_{\pi^n} \left|x(t_{i+1}^n) - x(t_i^n)\right|^q = 0$. So

$$\forall q > p^\pi(x) : \qquad x \in V^q([0,T],\mathbb{R}) \quad \text{ and, } \quad [x]_\pi^{(q)}(t) = 0.$$

For $q < p^\pi(x)$ one has, $\limsup_{n\uparrow\infty} \sum_{\pi^n} \left|x(t_{i+1}^n) - x(t_i^n)\right|^q > 0$. Now fix $q < p^\pi(x)$ and (if possible) assume
$\limsup_{n\uparrow\infty} \sum_{\pi^n} \left|x(t_{i+1}^n) - x(t_i^n)\right|^q < \infty$. Define $\bar{q} = \frac{q+p^\pi(x)}{2} < p^\pi(x)$ and $\epsilon = \bar{q} - q > 0$. So

$$0 < \limsup_{n\uparrow\infty} \sum_{\pi^n} \left|x(t_{i+1}^n) - x(t_i^n)\right|^{\bar{q}} = \limsup_{n\uparrow\infty} \sum_{\pi^n} \left|x(t_{i+1}^n) - x(t_i^n)\right|^{q+\epsilon}$$

$$\leq \limsup_{n\uparrow\infty} \left[\sup_i \left|x(t_{i+1}^n) - x(t_i^n)\right|^\epsilon \times \sum_{\pi^n} \left|x(t_{i+1}^n) - x(t_i^n)\right|^q\right]$$

$$\leq \limsup_{n\uparrow\infty} \left[\sup_i \left|x(t_{i+1}^n) - x(t_i^n)\right|^\epsilon\right] \times \limsup_{n\uparrow\infty} \left[\sum_{\pi^n} \left|x(t_{i+1}^n) - x(t_i^n)\right|^q\right] = 0$$

where the last equality follows from the fact that second term is finite and first term $\sup_i \left|x(t_{i+1}^n) - x(t_i^n)\right|^\epsilon \to 0$ as x is continuous, $\epsilon > 0$ and π has vanishing mesh. This is a contradiction, so

$$\forall q < p^\pi(x): \quad \limsup_{n\uparrow\infty} \sum_{\pi^n} \left|x(t_{i+1}^n) - x(t_i^n)\right|^q = \infty.$$

Similarly we can also show that

$$\forall q < r^\pi(x): \quad \liminf_{n\uparrow\infty} \sum_{\pi^n} \left|x(t_{i+1}^n) - x(t_i^n)\right|^q < \infty.$$

So combining the above two equation we get

$$\forall q < p^\pi(x) = r^\pi(x): \quad [x]_\pi^{(q)} = \lim_{n\uparrow\infty} \sum_{\pi^n} \left|x(t_{i+1}^n) - x(t_i^n)\right|^q = \infty.$$

■

Remark 4.12. In Lemma 4.11 the assumption $p^\pi(x) = r^\pi(x)$ is crucial. In fact this assumption ensures that Hurst roughness exponent (defined by Han and Schied [47]) exists and is equal to the roughness index $H^\pi(x)$.

We now provide a sufficient condition under which $p^\pi(x) = r^\pi(x)$.

Proposition 4.13 (Sufficient condition)**.** *Let $x \in C^0([0,T],\mathbb{R})$ and π be a sequence of partitions with vanishing mesh. If there exists $p \geq 1$ such that $x \in V^p([0,T],\mathbb{R})$ and $0 < [x]_\pi^{(p)} < \infty$, then:*

$$p^\pi(x) = r^\pi(x) = p.$$

Proof. First, we will show that $p^\pi(x) = r^\pi(x) \leq p$. Fix any $q > p$ and define $\epsilon = q - p > 0$. So

$$[x]_\pi^{(p)} = \lim_{n\uparrow\infty} \sum_{\pi^n} \left|x(t_{i+1}^n - x(t_i^n))\right|^q = \lim_{n\uparrow\infty} \sum_{\pi^n} \left|x(t_{i+1}^n - x(t_i^n))\right|^{p+\epsilon}$$

$$\leq \lim_{n\uparrow\infty} \left[\sup_i \left|x(t_{i+1}^n - x(t_i^n))\right|^\epsilon \times \sum_{\pi^n} \left|x(t_{i+1}^n - x(t_i^n))\right|^p\right]$$

$$\leq [x]_\pi^{(p)} \times \lim_{n\uparrow\infty} \sup_i \left|x(t_{i+1}^n - x(t_i^n))\right|^\epsilon = 0$$

where the last inequality follows from the fact that $x \in C^0([0,T],\mathbb{R})$, $\epsilon > 0$ and $|\pi^n| \downarrow 0$. Since $[x]_\pi^{(p)}$ exists and equal to 0, both the limsup and the lininf are also equal to zero for all $q > p$. So we have $p^\pi(x) = r^\pi(x) \leq p$.

Now we will show that $p^\pi(x) = r^\pi(x) \geq p$ as well. This argument follows from the definition of $p^\pi(x)$ and $r^\pi(x)$, as $[x]^{(p)}(x) > 0$ implies,
$\limsup_{n\uparrow\infty} \sum_{\pi^n} \left|x(t_{i+1}^n - x(t_i^n))\right|^p = \liminf_{n\uparrow\infty} \sum_{\pi^n} \left|x(t_{i+1}^n - x(t_i^n))\right|^p > 0$. Now we only need to show that $x \in \cap_{q\geq 1} V_\pi^q([0,T],\mathbb{R})$. From the above proof it is clear that $x \in \cap_{q\geq p} V_\pi^q([0,T],\mathbb{R})$, and for all $q < p$ we have $[x]^{(q)}(T) = \infty$. So we conclude $x \in \cap_{q\geq 1} V_\pi^q([0,T],\mathbb{R})$. ■

In general, the converse of the statement is not true. i.e. $p^\pi(x) = r^\pi(x)$ does not grantee that $x \in V^p([0,T],\mathbb{R})$.

Example 4.1. *Consider the sequence $\{\mathbb{T}^n\}_n$ of dyadic partitions and the continuous function $x \in C^0([0,1],\mathbb{R})$ defined as follows:*

$$x(t) = \sum_{m=0}^{\infty} \sum_{k=0}^{2^m-1} \theta_{m,k}^{\mathbb{T}} e_{m,k}^{\mathbb{T}}(t), \qquad \text{with, } \theta_{m,k}^{\mathbb{T}} = 1 + (-1)^m,$$

where the graph of $e_{m,k}^{\mathbb{T}}$ is a wedge with height $2^{-\frac{m+2}{2}}$, width 2^{-m}, centred at $c = 2^{\frac{k-\frac{1}{2}}{2^m}}$. In particular, for any fixed level $m \geq 0$, the functions $e_{m,k}^{\mathbb{T}}$ have disjoint supports for distinct k. For the function x, similar to [62, Proposition 2.7] we can show that:

$$[x]_{\mathbb{T}^{2n}}(t) = \frac{4}{3}t \qquad \text{and } \ [x]_{\mathbb{T}^{2n+1}}(t) = \frac{8}{3}t.$$

So in particular $x \notin V^2([0,T],\mathbb{R})$, but we have $p^{\mathbb{T}}(x) = r^{\mathbb{T}}(x) = 2$.

Proposition 4.14. *For any sequence of partitions π with $|\pi| \downarrow 0$ and for any $x \in \cup_{p\geq 1} V_\pi^p([0,T],\mathbb{R})$, if $p^\pi(x) = r^\pi(x)$ then:*

$$p^\pi(x) = \inf\left\{p \geq 1 \ : \ x \in V_\pi^p([0,T],\mathbb{R})\right\}.$$

Proof. This is a simple consequence of Lemma 4.11. ■

4.2.1 Roughness index and p-th variation

Notation: For a function $x \in C^0([0,1],\mathbb{R})$ and a sequence of partitions π of $[0,1]$, we denote

$$[x]^{(p)}_{\pi^n}(t) := \sum_{i=0}^{N(\pi^n)-1} \left| x(t^n_{i+1} \wedge t) - x(t^n_i \wedge t) \right|^p ,$$

the p-th variation of x along π at level n.

For a continuous path, the pathwise p-th variation plays an important role in determining the 'roughness' of a function. In the following lemma, we show the interplay of variation index with its corresponding p-th variation.

Lemma 4.15. *For any sequence of partitions π with $|\pi| \downarrow 0$ and for any $x \in \cup_{p\geq 1} V^p_\pi([0,T],\mathbb{R})$, if the variation index $p^\pi(x)$ exists and $x \in V^{p^\pi(x)}_\pi([0,T],\mathbb{R})$ then:*

$$[x]^{(q)}_\pi(t) = \begin{cases} 0 & \textit{if } q > p^\pi(x) \\ 0 \leq [x]^{(p)}_\pi < \infty & \textit{if } q = p^\pi(x) \\ \infty & \textit{if } q < p^\pi(x) \end{cases} , \tag{4.2}$$

where $p^\pi(x)$ is the variation index along sequence of partitions π.

Proof. This lemma is a simple consequence of Lemma 4.9, Lemma 4.11 and Proposition 4.13. ■

We summarize this property by saying that a function $x \in \cup_{p\geq 1} V^p_\pi([0,T],\mathbb{R})$ is smoother (along the partition sequence π) than a function $y \in \cup_{p\geq 1} V^p_\pi([0,T],\mathbb{R})$ if and only if $p^\pi(x) < p^\pi(y)$.

Practical challenges in using p-th variation as a measure of roughness:

Though Lemma 4.15 gives a very interesting characteristic of p-th variation, from a practical prospective even if for high-frequency data, the limiting quantity $[x]^{(q)}_\pi$ is never observed and we only observe a discrete signal of the continuous path. So, the observed quantity $[x]^{(q)}_{\pi^n}(T) = \sum_{\pi^n} \left| x(t^n_{i+1}) - x(t^n_i) \right|^q$ is always a finite number and neither equal to ∞ nor 0. Lemma 4.2 is based on asymptotic behaviour and does not have an obvious finite sample analogue, so may not be an efficient method for determining the roughness of a process.

In the following section, we introduce a model-free roughness estimator which has good finite sample behaviour, using the concept of *normalized p-th variation* along a partition sequence π.

4.3 Normalized p-th variation

We introduce a normalized version of p-th variation which has better asymptotic properties. The normalized p -th variation is inspired from the probabilistic definition of weighted quadratic variation for fractional Brownian motion [74, Chapter 5] but our construction is purely pathwise.

Definition 4.16 (Normalized p-th variation along a sequence of partitions)**.** *Let π be a sequence of partitions of $[0,T]$ with mesh $|\pi^n| \to 0$.*

$$\pi^n = \left(0 = t_1^n < t_2^n < \cdots < t_{N(\pi^n)}^n = T\right).$$

Take $x \in V_\pi^p([0,T],\mathbb{R})$. Then x is said to have normalized p-th variation along π *if there exists a continuous function $w(x,p,\pi) : [0,T] \to \mathbb{R}$ such that:*

$$\forall t \in [0,T], \qquad \sum_{\pi^n \cap [0,t]} \frac{\left|x\left(t_{i+1}^n\right) - x\left(t_i^n\right)\right|^p}{[x]_\pi^{(p)}(t_{i+1}^n) - [x]_\pi^{(p)}(t_i^n)} \times \left(t_{i+1}^n - t_i^n\right) \xrightarrow{n\to\infty} w(x,p,\pi)(t). \quad (4.3)$$

We call $w(x,p,\pi)$ the normalized p-th variation *of x along partition sequence π, and $N_\pi^p([0,T],\mathbb{R})$ the class of all functions for which normalized p-th variation*[1] *exists and finite.*

Remark 4.17. For $x \in N_\pi^p([0,T],\mathbb{R})$ we have $w(x,p,\pi)(T) < \infty$. If

$$\sum_{\pi^n \cap (a,b)} \frac{\left|x\left(t_{i+1}^n\right) - x\left(t_i^n\right)\right|^p}{[x]_\pi^{(p)}(t_{i+1}^n) - [x]_\pi^{(p)}(t_i^n)} \times \left(t_{i+1}^n - t_i^n\right) \xrightarrow{n\to\infty} \infty$$

for all $(a,b) \subset [0,T]$ then we will write $w(x,p,\pi)(t) = \infty$.

Notation: For a function $x \in C^0([0,1],\mathbb{R})$ and a sequence of partitions π of $[0,T]$, we denote the 'unfeasible estimator'

$$w^n(x,p,\pi)(t) := \sum_{\pi^n \cap [0,t]} \frac{\left|x\left(t_{i+1}^n\right) - x\left(t_i^n\right)\right|^p}{[x]_\pi^{(p)}(t_{i+1}^n) - [x]_\pi^{(p)}(t_i^n)} \times \left(t_{i+1}^n - t_i^n\right)$$

the normalized p-th variation of x along π at level n.

[1] For $p = 2$ we will call this quantity as 'normalized quadratic variation'.

4.3.1 Properties of normalized p-th variations

We will now show that the normalized p-th variation has sharper bounds than p-th variation.

Theorem 4.18. *Take π be a sequence of partitions of $[0,T]$ with mesh $|\pi^n| \to 0$. Let $x \in V_\pi^p([0,T],\mathbb{R})$ with $[x]_\pi^{(p)} \in (0,\infty)$ for some $p > 1$. Then for all $t \in (0,T]$ and for all $q > p$; $w(x,q,\pi)(t) = \infty$.*

Proof. We will prove the result for $t = T$, for general t the proof generalizes without further complication. Since $q > p$ and the function x has finite p-variation along π, the pathwise q-th variation $[x]_\pi^{(q)}(t) = 0$ for all $t \in [0,T]$. Hence, given any fix $h > 0$ we have $[x]_\pi^{(q)}(t+h) - [x]_\pi^{(q)}(t) = 0$. Now, given any $M > 0$ and any $n > 1$, we will show that $w(x,q,\pi)(t) > M$. For fix $n > 1$ we have:

$$\text{for all } t_i^n \in \pi^n, \qquad [x]_\pi^{(q)}(t_{i+1}^n) - [x]_\pi^{(q)}(t_i^n) = 0.$$

Since $x \in V_\pi^p([0,T],\mathbb{R})$ for some $p > 0$, for fix large enough n, we have $\sum_{\pi^n} \left|x(t_{i+1}^n) - x(t_i^n)\right|^q \times (t_{i+1}^n - t_i^n) > 0$. So we get the lower bound of $w^n(x,q,\pi)$ as follows.

$$w^n(x,q,\pi)(T) = \sum_{\pi^n} \frac{\left|x(t_{i+1}^n) - x(t_i^n)\right|^q}{[x]_\pi^{(q)}(t_{i+1}^n) - [x]_\pi^{(q)}(t_i^n)} \times (t_{i+1}^n - t_i^n)$$

$$\geq \min_{t_i^n \in \pi^n} \left[\frac{1}{[x]_\pi^{(q)}(t_{i+1}^n) - [x]_\pi^{(q)}(t_i^n)}\right] \sum_{\pi^n} \left|x(t_{i+1}^n) - x(t_i^n)\right|^q \times (t_{i+1}^n - t_i^n)$$

$$\geq C \left[\frac{1}{\max_{t_i^n \in \pi^n}([x]_\pi^{(q)}(t_{i+1}^n) - [x]_\pi^{(q)}(t_i^n))}\right] = \infty > M.$$

Since for all $n > 1$, $w^n(x,q,\pi)(T) = \infty$ we can conclude the following:

$$w(x,q,\pi)(T) = \lim_{n\to\infty} w^n(x,q,\pi)(T) = \infty.$$

■

Theorem 4.19. *Let π be a sequence of partitions of $[0,T]$ with mesh $|\pi^n| \to 0$. Let $x \in V_\pi^p([0,T],\mathbb{R})$ with $[x]_\pi^{(p)} \in (0,\infty)$ for some $p > 1$. Then for all $t \in [0,T]$ and $\forall q < p$; $w(x,q,\pi)(t) = 0$.*

Proof. We will prove the result for $t = T$, for general t the proof follows exactly the same way. Since the function x has finite p-variation along π and $q < p$ from the assumption, the pathwise q -th variation $[x]_\pi^{(q)}(t) = \infty$ for all $t \in [0,T]$. Hence, given any fixed $h > 0$ we have $[x]_\pi^{(q)}(t+h) - [x]_\pi^{(q)}(t) = \infty$. Now, given any $\epsilon > 0$ and any $n > 1$, we will show that $w(x,q,\pi)(t) < \epsilon$ by showing that $w^n(x,q,\pi)(t) < \epsilon$ for all large n. For fix $n > 1$ we have:

$$\text{for all } t_i^n \in \pi^n, \qquad [x]_\pi^{(q)}(t_{i+1}^n) - [x]_\pi^{(q)}(t_i^n) = \infty.$$

So,

$$\begin{aligned}
w^n(x,q,\pi)(T) &= \sum_{\pi^n} \frac{\left|x(t_{i+1}^n) - x(t_i^n)\right|^q}{[x]_\pi^{(q)}(t_{i+1}^n) - [x]_\pi^{(q)}(t_i^n)} \times (t_{i+1}^n - t_i^n) \\
&\leq \max_{t_i^n \in \pi^n} \left[\frac{\left|x(t_{i+1}^n) - x(t_i^n)\right|^q}{[x]_\pi^{(q)}(t_{i+1}^n) - [x]_\pi^{(q)}(t_i^n)} \right] \times \sum_{\pi^n} (t_{i+1}^n - t_i^n) \\
&\leq T \times \max_{t_i^n \in \pi^n} \left[\frac{\left|x(t_{i+1}^n) - x(t_i^n)\right|^q}{[x]_\pi^{(q)}(t_{i+1}^n) - [x]_\pi^{(q)}(t_i^n)} \right] \\
&\leq T \times \max_{t_i^n \in \pi^n} \left|x(t_{i+1}^n) - x(t_i^n)\right|^q \times \left[\frac{1}{\min_{t_i^n \in \pi^n} \left([x]_\pi^{(q)}(t_{i+1}^n) - [x]_\pi^{(q)}(t_i^n)\right)} \right] = 0 < \epsilon.
\end{aligned}$$

Since for all $n > 1$, $w^n(x,q,\pi)(T) = 0$;

$$w(x,q,\pi)(T) = \lim_{n \to \infty} w^n(x,q,\pi)(T) = 0.$$

■

Theorems 4.18 and Theorem 4.19 show that alike p-th variation, if a function has finite non-zero p-th variation along a sequence of partitions π, then for all $q \neq p$ the normalized p-th variation is either infinite or zero. But for functions with finite p-th variation, it is not clear in general whether the function also has finite normalized p-th variation or not. The following Theorem provides a sufficient condition when the normalized p-th variation exists and is linear.

Theorem 4.20 (Linear normalized p-th variation). *Let $x \in V_\pi^p([0,T],\mathbb{R})$ for some $p > 1$ where π be a sequence of partitions of $[0,T]$ with mesh $|\pi| \to 0$. If we further assume, the p-th variation is strictly increasing and the derivative $\frac{d}{du}[x]_\pi^{(p)}(u)$ exists and continuous then:*

$$x \in N_\pi^p([0,T],\mathbb{R}) \qquad \textit{and} \qquad \forall t \in [0,T],\ w(x,p,\pi)(t) = t.$$

Proof. For convenience assume $g(u) = \frac{d}{du}[x]^{(p)}_\pi(u)$. Since the p-th variation is strictly increasing we have $\sup_{t\in[0,T]} \frac{1}{g(u)} < \infty$. Since $g(u)$ is continuous in a compact interval $[0,T]$, it is also bounded. So for all $t \in [0,T]$, we have $\frac{1}{g(u)} \in (0,\infty)$. So as a consequence of mean value theorem,

$$w^n(x,p,\pi)(t) = \sum_{\pi^n\cap[0,t]} \frac{\left|x(t^n_{i+1}) - x(t^n_i)\right|^p}{[x]^{(p)}_\pi(t^n_{i+1}) - [x]^{(p)}_\pi(t^n_i)} \times (t^n_{i+1} - t^n_i) = \sum_{\pi^n\cap[0,t]} \frac{\left|x(t^n_{i+1}) - x(t^n_i)\right|^p}{g(u^n_i)},$$

where $u^n_i \in [t^n_i, t^n_{i+1}]$ for all $n \geq 1$ and for all $i = 0, \cdots, N(\pi^n) - 1$. Finally using properties of Riemann integral we can conclude:

$$\sum_{\pi^n\cap[0,t]} \frac{\left|x(t^n_{i+1}) - x(t^n_i)\right|^p}{g(u^n_i)} \xrightarrow{n\to\infty} \int_0^t \frac{1}{g(u)} d[x]^{(p)}_\pi(u) = \int_0^t \frac{d[x]^{(p)}_\pi(u)/du}{g(u)} du = \int_0^t du = t.$$

Since the limit always exists we conclude the proof. ∎

Remark 4.21. If a function $x \in C^0([0,T],\mathbb{R})$ satisfies the assumptions of Theorem 4.20, then $x \cup_{p\geq 1} V^p_\pi([0,T],\mathbb{R})$ and $p^\pi(x) = r^\pi(x)$ (using Lemma 4.13). So Proposition 4.10 concludes: $p^\pi(x) = \inf\left\{p \geq 1 \ : \ [x]^{(p)}_\pi(T) < \infty\right\}$. So throughout the rest of the chapter we use the above proposition as a parallel definition of variation index.

4.3.2 Examples of linear normalized p-th variation

As expected, Brownian motion almost surely has linear normalized quadratic variation.

Example 4.2 (Normalized quadratic variation for Brownian motion)**.** *Let B be a Wiener process on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, and $(\pi^n)_{n\geq 1}$ be a sequence of partitions of $[0,T]$ with $|\pi^n|\log n \to 0$. Then:*

$$\mathbb{P}\left(w(B,2,\pi)(t) = t\right) = 1.$$

Proof. Let B be a Wiener process on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, which we take to be the canonical Wiener space without loss of generality, i.e. $\Omega = C^0([0,T],\mathbb{R})$, $B(t,\omega) = \omega(t)$ and $\mathbb{P}$ is the Wiener measure. Let $\pi^n = (0 = t^n_0 < t^n_1 < \cdots <$

$t^n_{N(\pi^n)} = T$) be a sequence of partitions of $[0,T]$ satisfying $|\pi^n| \log n \to 0$. Then the results of Dudley [33] imply that

$$\mathbb{P}\left(\sum_{\pi^n} |B(t^n_{i+1} \wedge t) - B(t^n_i \wedge t)|^2 \stackrel{n\to\infty}{\to} t \right) = 1.$$

So if we set $\Omega_0 = \Omega \cap Q_\pi([0,T],\mathbb{R})$ then $\mathbb{P}(\Omega_0) = 1$ and any $\omega \in \Omega_0$ satisfies $[\omega]_\pi(t) = t$. Now for any $\omega \in \Omega_0$ we also have the following:

$$w(\omega,2,\pi)(t) = \lim_{n\to\infty} \sum_{\pi^n\cap[0,t]} \frac{\left(\omega(t^n_{i+1}) - \omega(t^n_i)\right)^2}{[\omega]_\pi(t^n_{i+1}) - [\omega]_\pi(t^n_i)} \times (t^n_{i+1} - t^n_i)$$

$$= \lim_{n\to\infty} \sum_{\pi^n\cap[0,t]} \frac{(\omega(t^n_{i+1}) - \omega(t^n_i))^2}{(t^n_{i+1} - t^n_i)} \times (t^n_{i+1} - t^n_i) = \lim_{n\to\infty} \sum_{\pi^n\cap[0,t]} (\omega(t^n_{i+1}) - \omega(t^n_i))^2 = t.$$

So for Brownian motion, normalized quadratic variation up to time t is almost surely equal to t. ■

Remark 4.22. From Lemma 4.2 we know that, for any partition sequence π with $|\pi^n|\log(n) \to 0$, there exists $\Omega_\pi \subset \Omega$ with $\mathbb{P}(\Omega_\pi) = 1$ such that:

$$\forall \omega \in \Omega_\pi,\ \forall t \in [0,T], \qquad w(\omega,2,\pi)(t) = t.$$

We also have the following relation between quadratic variation and normalized-QV for Brownian paths in the class Ω_π.

$$\forall \omega \in \Omega_\pi,\ \forall t \in [0,T], \qquad w(\omega,2,\pi)(t) = [\omega]_\pi(t).$$

So the measure zero set for quadratic variation and normalized-quadratic variation of Brownian motion are the same.

Remark 4.23. In the above example, instead of taking any partition sequence with $|\pi^n|\log n \to 0$, we can also take any refining sequence of partitions.

In general, the stochastic integral w.r.t. Brownian motion does not have linear quadratic variation, but does have linear normalized quadratic variation.

Example 4.3 (Stochastic integrals)**.** *Let $X(t) = \int_0^t \sigma(u)dB_u$ where σ is an adapted process with $\int_0^T \sigma^2(u)du < \infty$. Then, for any refining partition sequence π with vanishing mesh,*

$$\mathbb{P}\left(w(X,2,\pi)(t) = t\right) = 1.$$

Proof. This is an immediate consequence of Theorem 4.20. ■

Remark 4.24. In the statement of Example 4.3, we can replace the assumption of refining partitions with partitions satisfying $|\pi^n|\log(n) \to 0$.

Example 4.4 (Normalized p-th variation for fractional Brownian motion)**.** *Let B^H be a fractional Brownian motion with Hurst index H, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then for any sequence of partitions π of $[0,T]$ with mesh $|\pi| \to 0$, we have*

$$w^n\left(B^H, \frac{1}{H}, \pi\right)(t) \xrightarrow{n\to\infty} t \quad \textit{in probability.}$$

Furthermore, for dyadic partitions $\mathbb{T} = (\mathbb{T}^n)_{n\geq 1}$ the convergence is in almost sure sense, i.e.

$$\mathbb{P}\left(\lim_{n\to\infty} w^n\left(B^H, \frac{1}{H}, \mathbb{T}\right)(t) = t\right) = 1.$$

Proof. The proof is a consequence of [75, Proposition 4.1]. ■

Schied [70] and Schied and Mishura [62] provide several examples of functions with prescribed p-th variation. For all $p > 1$ the class of functions $\mathcal{X}^p$ defined in [62] is described as follows:

$$\mathcal{X}^p = \left\{ x \in C^0([0,1],\mathbb{R}) \;\middle|\; x(t) = \sum_{m=0}^{\infty} 2^{m(\frac{1}{2}-\frac{1}{p})} \sum_{k=0}^{2^m-1} \theta_{m,k} e_{m,k}(t); \right.$$
$$\left. \text{where coefficients } \; \theta_{m,k} \in \{-1,+1\} \right\}. \tag{4.4}$$

The graph of $e_{m,k}$ is a wedge with height $2^{-\frac{m+2}{2}}$, width 2^{-m}, centred at $c = 2^{\frac{k-\frac{1}{2}}{2^m}}$. In particular, the functions $e_{m,k}$ have disjoint support for distinct k and fixed level m.

Example 4.5. *For dyadic partition sequence $\mathbb{T}$ and for any function $x \in \mathcal{X}^p$, the variation index $p^{\mathbb{T}}(x) = p$. Furthermore, the normalized p-th variation exists and is linear:*

$$\forall x \in \mathcal{X}^p, \; \forall t \in [0,1], \qquad w(x,p,\mathbb{T})(t) = t.$$

Proof. This is an immediate consequence of Theorem 4.20. But we give a direct proof below.

Schied and Mishura [62] have shown that for $x \in \mathcal{X}$, the p-th variation along the dyadic partition is positive and linear w.r.t. time: $[x]^{(p)}_{\mathbb{T}}(t) = t2^{1-p}\mathbb{E}[|Z_p|^p]$. Where $Z_p = \sum_{m=0}^{\infty} 2^{m(\frac{1}{p}-1)Y_m}$ for an i.i.d. sequence $Y_0, Y_1, \cdots$ of $\{-1, 1\}$-valued random variable with $\mathbb{P}[Y_n = +1] = \frac{1}{2}$. Since the p-variation $[x]^{(p)}_{\mathbb{T}}(t) \in (0, \infty)$ Lemma 4.13 concludes $p^{\mathbb{T}}(x) = p$. For the second part we have,

$$w^n(x, p, \mathbb{T})(t) = \sum_{\mathbb{T}^n \cap [0,t]} \frac{\left(x(\frac{i+1}{2^n}) - x(\frac{i}{2^n})\right)^2}{[x]_{\mathbb{T}}(\frac{i+1}{2^n}) - [x]_{\mathbb{T}}(\frac{i}{2^n})} \times \left(\frac{i+1}{2^n} - \frac{i}{2^n}\right)$$

$$= \frac{1}{2^{1-p}\mathbb{E}[|Z_p|^p]} \sum_{\mathbb{T}^n \cap [0,t]} \left(x\left(\frac{i+1}{2^n}\right) - x\left(\frac{i}{2^n}\right)\right)^2 \xrightarrow{n\to\infty} t.$$

This concludes the proof. ■

4.4 Estimating roughness from discrete observations

We now explain how to use the concept of normalized p-th variation to estimate the roughness of a path.

Given observations on a refining time partition π^L, we define the 'normalized p-th variation statistic' which is the discrete counterpart of the normalized p-th variation:

$$W(L, K, \pi, p, t, X) := \sum_{\pi^K \cap [0,t]} \frac{\left|X(t^K_{i+1}) - X(t^K_i)\right|^p}{\sum_{\pi^L \cap [t^K_i, t^K_{i+1}]} \left|X(t^L_{j+1}) - X(t^L_j)\right|^p} \times \left(t^K_{i+1} - t^K_i\right). \quad (4.5)$$

The definition of the statistic (4.5) involves two frequencies: a 'block' frequency K and a sampling frequency $L \gg K$. As the partition is refining, π^K is a subpartition of π^L. The denominator is estimated by grouping the sample of size L into K many groups, where each group contains $\frac{L}{K}$ consecutive data points.

The statistic (4.5) converges to the normalized p-th variation (4.3) as the sampling frequency L and block frequency increase:

$$\lim_{K\to\infty} \lim_{L\to\infty} W(L, K, \pi, p, t, x) = w(x, p, \pi)(t). \quad (4.6)$$

It is thus natural to define roughness estimators for a discretely sampled signal in terms of (4.5).

The *variation index estimator* $\widehat{p}_{L,K}(X)$ associated with the signal sampled on π^L is then obtained by computing $W(L,K,\pi,p,t,X)$ for different values of p and solving the following equation for $p^{\pi}_{L,K}(X)$,

$$W(L,K,\pi,\widehat{p}^{\pi}_{L,K}(X),T,X) = T. \tag{4.7}$$

One can either fix a window length T or solve (4.7) in a least squares sense across several values of T.

An estimator for the roughness index is subsequently defined as:

$$\widehat{H}^{\pi}_{L,K}(X) = \frac{1}{\widehat{p}^{\pi}_{L,K}(X)}. \tag{4.8}$$

We will denote the roughness estimator (4.8) as $\widehat{H}_{L,K}$ when the dataset and the corresponding partition sequence are clear.

Lemma 4.25 (Monotonicity). *For fix sampling frequency L and block frequency K with $L \gg K$, the function $W(L,K,\pi,p,t,x)$ is a (strictly) increasing function on p.*

Proof. The proof follows from the fact that $[x]^{(p')}_{\pi^n}(t+\Delta t) - [x]^{(p')}_{\pi^n}(t) > [x]^{(p)}_{\pi^n}(t+\Delta t) - [x]^{(p)}_{\pi^n}(t)$ for all $1 < p' < p < \infty$ and the fact that for $L \gg K$ the denominator of $W(L,K,\pi,p,t,x)$ is dominant term in the sum. ■

Note: From Figure 4.1 we can see that for fractional Brownian motions $W(L,K,\pi,p,t,x)$ is monotonically decreasing in $1/p \in (0,1)$ and hence monotonically increasing as a function of p.

The following proposition gives conditions for the convergence of the normalized p-th variation statistic under high-frequency asymptotics in a pathwise setting:

Proposition 4.26 (Convergence of the normalized p-th variation statistic). *Let $X \in V^{\bar{p}}_{\pi}([0,T],\mathbb{R}) \cap C^{\alpha}([0,T],\mathbb{R})$ where, $\bar{p} = p^{\pi}(X)$ and π be a balanced partition sequence of $[0,T]$ with mesh $|\pi| \to 0$. If we further assume, the $\bar{p}$-th variation is* *strictly increasing and the derivative* $\frac{d}{du}[X]^{(\bar{p})}_{\pi}(u)$ *exists and continuous, then there exists sequence* (L_n, K_n) *such that* $L_n > K_n$ *and*

$$\lim_{n\to\infty} W(L_n, K_n, \pi, p, t, X) = w(X, p, \pi)(t)$$

uniform over p *on the interval* $[a, b] \subset (1, p^{\pi}(X))$.

Proof. For $p < p^{\pi}(X)$, Lemma 4.19 shows that $w(X, p, \pi)(t) = 0$. So need to show that given any $\epsilon > 0$, there exists a $N_0(\epsilon), N_1(K, \epsilon) < \infty$ such that for all $K > N_0(\epsilon)$ and for all $L > N_1(K, \epsilon)$ one has $W(L, K, \pi, p, t, X) \leq \epsilon$. Now since $X \in C^{\alpha}([0, T], \mathbb{R})$ we have for all $t, t' \in [0, T]$, $|X(t) - X(t')| \leq ||X||_{\alpha}|t - t'|^{\alpha}$. Since the partition sequence π balanced, there exists constant $C < \infty$ such that $\sup_n N(\pi^n)|\pi^n| \leq CT$. We now define $N_0, N_1(K)$ as follows.

$$N_0(\epsilon) = 1 + \inf\left\{n \in \mathbb{N},\ ||X||_{\alpha}T|\pi^n|^{a\alpha} \leq \sqrt{\epsilon}\right\}$$

$$N_1(K, \epsilon) = \inf\left\{n \in \mathbb{N} :\ \forall h > \underline{\pi^K}, \forall t \in [0, T - h), \left|[X]^{(b)}_{\pi^n}(t + h) - [X]^{(b)}_{\pi^n}(t)\right| \geq \frac{2}{\sqrt{\epsilon}}\right\}$$

As $\underline{\pi}^n \to 0$ and $[X]^{(p)}_{\pi^L} \to \infty$ for $p < p^{\pi}(x)$, $N_0(\epsilon)$ and $N_1(K, \epsilon)$ exists and finite. Now take any $K > N_0(\epsilon)$ and $L > N_1(K, \epsilon)$. So,

$$\begin{aligned}
|W(L, K, \pi, p, t, X)| &= \sum_{\pi^K\cap[0,t]} \frac{\left|X(t^K_{i+1}) - X(t^K_i)\right|^p}{\sum_{\pi^L\cap[t^K_i, t^K_{i+1}]}\left|X(t^L_{j+1}) - X(t^L_j)\right|^p} \times \left(t^K_{i+1} - t^K_i\right) \\
&\leq ||X||^a_{\alpha}|\pi^K|^{a\alpha} \sum_{\pi^K\cap[0,t]} \frac{1}{\sum_{\pi^L\cap[t^K_i, t^K_{i+1}]}\left|X(t^L_{j+1}) - X(t^L_j)\right|^p} \times \left(t^K_{i+1} - t^K_i\right) \\
&\leq T||X||^a_{\alpha}|\pi^K|^{a\alpha} \frac{1}{\min_{\pi^K\cap[0,t]}\left|X(t^L_{j+1}) - X(t^L_j)\right|^p} \\
&\leq \sqrt{\epsilon} \times \frac{1}{\min_{\pi^K\cap[0,t]}\left|X(t^L_{j+1}) - X(t^L_j)\right|^p} \leq \sqrt{\epsilon} \times \frac{\sqrt{\epsilon}}{2} < \epsilon.
\end{aligned} \tag{4.9}$$

In the above proof we use the fact that for any $t \neq t' \in [0, T]$, for all $n \leq \infty$ and for all $p' < p \in (1, \infty)$, $\left|[X]^{(p)}(t') - [X]^{(p)}(t)\right| < \left|[X]^{(p')}(t') - [X]^{(p')}(t)\right|$. ■

Since for $p > p^{\pi}(X)$, one has $w(X, p, \pi)(t) = \infty$, we fix a $C > 2$ and regularize the normalized p-th variation statistics $w(X, p, \pi)$ by taking $\max\{w(X, p, \pi), C\}$ which is always finite.

Proposition 4.27. *Let* $X \in V^{\bar{p}}_{\pi}([0, T], \mathbb{R})$ *where,* $\bar{p} = p^{\pi}(X)$ *and* π *be a partition sequence of* $[0, T]$ *with mesh* $|\pi| \to 0$. *If we further assume, the* $\bar{p}$*-th variation is* *strictly increasing and the derivative* $\frac{d}{du}[X]^{(\bar{p})}_{\pi}(u)$ *exists and continuous, then there exists sequence* (L_n, K_n) *such that* $L_n > K_n$ *and*

$$\lim_{n\to\infty} \min\{W(L_n, K_n, p, \pi, t, X), C\} = \min\{w(X, p, \pi)(t), C\}$$

uniformly over p *on the interval* $[a, b] \subset (p^{\pi}(X), \infty)$.

Proof. Fix $C > 2$. For $p > p^{\pi}(X)$, Lemma 4.18 shows that $w(X, p, \pi)(t) = \infty$. So, on the interval $[a, b] \subset (p^{\pi}(X), \infty)$, we have $\min\{w(X, p, \pi)(t), C\} = C$. To show the convergence of the regularized normalized p-th variation along a subsequence, it is enough to show that there exists N_0 and $N_1(K)$ such that for all $K > N_0$ and for all $L > N_1(K)$ and for all $p \in [a, b]$, $W(L, K, \pi, p, T, X) > C$. Now define N_0 and $N_1(K)$ as follows.

$$N_0 = \inf\{k \geq 1 : \max_{t_i^n \in \pi^n} |X(t_{i+1}^n) - X(t_i^n)|\} < \infty \ \ (\text{ as } x \in C^0([0, T], \mathbb{R}) \text{ and, } \pi \downarrow 0).$$

$$N_1(K) = \inf\left\{L : \forall L' > L; [X]^{(a)}_{\pi^{L'}}(T) \leq \frac{\epsilon_K}{C}\right\} < \infty \ \ (\text{ as } [X]^{(a)}_{\pi^{L'}} \xrightarrow{L'\to\infty} 0)$$

where, $\epsilon_K := \sum_{\pi^K} \left|X(t_{i+1}^K) - X(t_i^K)\right|^b (t_{i+1}^n - t_i^n)$. So for all $p \in [a, b]$, for all $K > N_0$ and, for all $L > N_1(K)$

$$\begin{aligned}
W(L, K, \pi, p, T, x) &= \sum_{\pi^K} \frac{|X(t_{i+1}^K) - X(t_i^K)|^p}{\sum_{\pi^L \cap [t_i^K, t_{i+1}^K]} |X(t_{j+1}^L) - X(t_j^L)|^p} (t_{i+1}^K - t_i^K) \\
&\geq \sum_{\pi^K} \frac{|X(t_{i+1}^K) - X(t_i^K)|^b}{\sum_{\pi^L \cap [t_i^K, t_{i+1}^K]} |X(t_{j+1}^L) - X(t_j^L)|^a} (t_{i+1}^K - t_i^K) \\
&\geq \sum_{\pi^K} \frac{|X(t_{i+1}^K) - X(t_i^K)|^b}{\frac{\epsilon_K}{C+1}} (t_{i+1}^K - t_i^K) = C.
\end{aligned}$$

So $\min\{W(L, K, p, \pi, t, X), C\} = C$, which concludes the proof. ∎

This leads us to the consistency of the roughness estimator:

Theorem 4.28 (Pathwise consistency of estimator)**.** *Under conditions of Proposition 4.26 and Proposition 4.27 there exists sequence* (L_n, K_n) *such that* $L_n > K_n$ *and*

$$\lim_{n\to\infty} \widehat{H}^{\pi}_{L_n, K_n}(X) = H^{\pi}(X) \ \ \textit{and,}$$
$$\lim_{n\to\infty} \widehat{p}^{\pi}_{L_n, K_n}(X) = p^{\pi}(X)$$

Proof. Let us assume, there exists $\epsilon > 0$ such that, $A := \lim_{n\to\infty} \widehat{p}^{\pi}_{L_n,K_n}(X) \in (max\{p^{\pi}(X)-\epsilon, T\}, (p^{\pi}(X)+\epsilon))^c$. This implies there exists a subsequence of pairs (L^n, K^n) with $L^n >> K^n \to \infty$ such that one of the following holds,
1. $\widehat{p}^{\pi}_{L^n,K^n}(X) \in [1, max\{p^{\pi}(X)-\epsilon, T\}]$ for all n.
2. $\widehat{p}^{\pi}_{L^n,K^n}(X) \in [p^{\pi}(X)+\epsilon, \infty)$ for all n. But using Proposition 4.26 and Proposition 4.27, $W(L^n, K^n, \pi, \widehat{p}^{\pi}, T, X)$ converges to either 0 or to a constant $> C$. This is a contradiction, hence the theorem follows. ■

Lemma 4.29 (Representing L as a function of K in the Estimator $W(L, K, \pi, p, T, X)$)**.** *Let $X \in C^{\alpha}([0,T], \mathbb{R})$ for some $\alpha \in (0,1)$, π be a balanced partition sequence and $p \in (1, p^{\pi}(X)) \cup (p^{\pi}(X), \infty)$. Further, assume the $p^{\pi}(X)$-th variation is strictly increasing and the derivative $\frac{d}{du}[X]^{(p^{\pi}(X))}_{\pi}(u)$ exists and continuous. Then,*

$$\lim_{K\to\infty} W(L(K), K, \pi, p, T, X) = w(X, p, \pi)(T)$$

where, $L(K)$ is defined as following.

$$\begin{aligned} L(K) &:= \inf\left\{n \geq K \ : \ [X]^{(p)}_{\pi^n}(T) < [X]^{(p)}_{\pi^K}(T) \times (\underline{\pi^K})^2\right\} \mathbb{1}_{p>p^{\pi}(X)} \\ &+ \inf\left\{n \geq K : \ \forall h > \underline{\pi^K}, \forall t \in [0, T-h), \left|[X]^{(p)}_{\pi^n}(t+h) - [X]^{(p)}_{\pi^n}(t)\right| \right. \\ &\qquad \left. \geq T||X||^p_{\alpha} N(\pi^K)|\pi^K|^p \right\} \mathbb{1}_{p<p^{\pi}(X)}. \end{aligned}$$

Proof. For $p > p^{\pi}(X)$, since under the assumptions of the lemma, $[X]^{(p)}_{\pi} \to 0$, $\inf\left\{n \geq K \ : \ [X]^{(p)}_{\pi^n}(T) < [X]^{(p)}_{\pi^K}(T) \times (\underline{\pi^K})^2\right\}$ exists and finite. So the estimator can be lower bounded as:

$$\begin{aligned} W(L(K), K, \pi, p, T, X) &= \sum_{\pi^K} \frac{|X(t^K_{i+1}) - X(t^K_i)|^p}{\sum_{\pi^L \cap [t^K_i, t^K_{i+1}]} |X(t^{L(K)}_{j+1}) - X(t^{L(K)}_j)|^p} (t^K_{i+1} - t^K_i) \\ &\geq \sum_{\pi^K} \frac{|X(t^K_{i+1}) - X(t^K_i)|^p}{\sum_{\pi^{L(K)} \cap [0,T]} |X(t^{L(K)}_{j+1}) - X(t^{L(K)}_j)|^p} (t^K_{i+1} - t^K_i) \\ &\geq \frac{1}{[X]^{(p)}_{\pi^{L(K)}}(T)} \sum_{\pi^K} |X(t^K_{i+1}) - X(t^K_i)|^p \underline{\pi^K} \\ &\geq \frac{1}{\underline{\pi^K}} \xrightarrow{K\to\infty} \infty = w(X, p, \pi)(T). \end{aligned}$$

On the other hand for $1 < p < p^{\pi}(X)$, under the assumption of lemma we know that, $\forall i, |X(t^K_{i+1}) - X(t^K_i)|^p \leq ||X||^p_{\alpha} |\pi^K|^p$. Since $[X]^{(p)}_{\pi^n} \to \infty$ and π is a balanced partition sequence, $\inf \Big\{ n \geq K : \ \forall h > \underline{\pi^K}, \forall t \in [0, T-h), \Big| [X]^{(b)}_{\pi^n}(t+h) - [X]^{(b)}_{\pi^n}(t) \Big|$ $\geq T||X||^p_\alpha N(\pi^n) \left|\pi^K\right|^p \Big\}$ exists and finite. So the estimator can be upper bounded as:

$$
\begin{aligned}
W(L(K), K, \pi, p, T, X) &= \textstyle\sum_{\pi^K} \frac{|X(t^K_{i+1}) - X(t^K_i)|^p}{\sum_{\pi^L \cap [t^K_i, t^K_{i+1}]} |X(t^{L(K)}_{j+1}) - X(t^{L(K)}_j)|^p} (t^K_{i+1} - t^K_i) \\
&\leq \textstyle\sum_{\pi^K} \frac{|X(t^K_{i+1}) - X(t^K_i)|^p}{T||X||^p_\alpha N(\pi^K) |\pi^K|^p} (t^K_{i+1} - t^K_i) \\
&\leq \textstyle\sum_{\pi^K} \frac{1}{TN(\pi^K)} (t^K_{i+1} - t^K_i) = \frac{1}{N(\pi^K)} \xrightarrow{K \to \infty} 0 = w(X, p, \pi)(T)
\end{aligned}
$$

■

4.5 Finite sample behaviour of the roughness estimator

We will now study the finite sample behaviour of the roughness estimator $\widehat{H}^\pi_{L,K}(X)$ using high-frequency simulations of fractional Brownian motions and Takagi-Landsberg functions. In the simulation examples unless mentioned otherwise we will use a uniform partition sequence of $[0, 1]$ with:

$$\pi^n = \left(0 < \frac{1}{n} < \frac{2}{n} < \cdots < 1\right).$$

4.5.1 Simulation experiments for diffusion models

To assess the finite sample accuracy of the estimator we compare the roughness index estimator $\widehat{H}^\pi_{L,K}$ with the underlying Hurst exponent $H \in \{0.1, 0.3, 0.5, 0.8\}$. For every simulated fractional brownian paths, we compute $W(L = 300 \times 300, K = 300, \pi, q, t = 1, X = B^H)$ for different values of q, in order to estimate $\widehat{H}_{L,K}$. In figure 4.1, the black line is the value of $\log\big(W(L = 300 \times 300, K = 300, \pi, q = p, t = 1, X = B^H)\big)$ plotted against roughness index $1/p$ in log-scale. The blue horizontal line represents the estimated roughness index $\widehat{H}_{L,K}$ whereas the dotted horizontal line represents the Hurst parameter. Figure 4.2 shows the histograms of the estimator $\widehat{H}^\pi_{L,K}$ generated from 150 independent paths. We observe that for datasets with length $L = 300 \times 300$ our roughness estimator $\widehat{H}_{L=300\times300,K=300}$ has satisfactory accuracy. Table 4.1 provides summary statistics for roughness index $\widehat{H}$ of simulated fractional Brownian motions. Figure 4.3 represents a similar plot for simulated fractional Brownian motion with Hurst parameter $H = 0.1$. In Figure 4.3, in left, similar to Figure 4.1, $\log(W(L = 2000 \times 2000, K = 2000, \pi, p, t = 1, X = B^H))$ is plotted against $H = 1/p$ and the right plot represents the histograms of the estimator $\widehat{H}_{L=2000\times2000,K=2000}$. The summary statistics for the estimator are provided in Table 4.2. To compute the estimator $\widehat{H}_{L,K}$ we have different possible choices of $K \ll L$. Figure 4.4 shows how the estimator $\widehat{H}^{\pi}_{L,K}$ varies with K for fractional Brownian motion with Hurst parameter $H = 0.1$. The black line represents the $\widehat{H}_{L,K}$ plotted against different values of K whereas the blue vertical line represents the value for $L = 300 \times 300, K = 300$. We observe that when we vary K in the neighbourhood $K \approx \sqrt{L}$ the estimator performs reasonably well and is not very sensitive to the choice of K in this vicinity.

In summary, we observe that for realistic sample sizes and frequencies encountered in high-frequency financial data, the estimator is quite accurate and not sensitive to the block size K in the range $K \approx \sqrt{L}$.

H	Min.	Lower quartile	Median	Mean	Upper quartile	Max.
0.1	0.0650	0.0920	0.1030	0.1009	0.1100	0.1440
0.3	0.2730	0.2940	0.2980	0.2976	0.3020	0.3180
0.5	0.4820	0.4940	0.4980	0.4978	0.5020	0.5140
0.8	0.7570	0.7820	0.7900	0.7891	0.7940	0.8220

Table 4.1: Summary statistics for estimated roughness index $\widehat{H}_{L,K}$ for fractional Brownian motion B^H with $L = 300 \times 300, K = 300$.

Hurst idx H	Min.	Lower quartile	Median	Mean	Upper quartile	Max.
0.1	0.086	0.096	0.099	0.099	0.103	0.117

Table 4.2: Summary statistics for estimated roughness index $\widehat{H}_{L,K}$ for fractional Brownian motion B^H with $H = 0.1, L = 2000 \times 2000, K = 2000$.

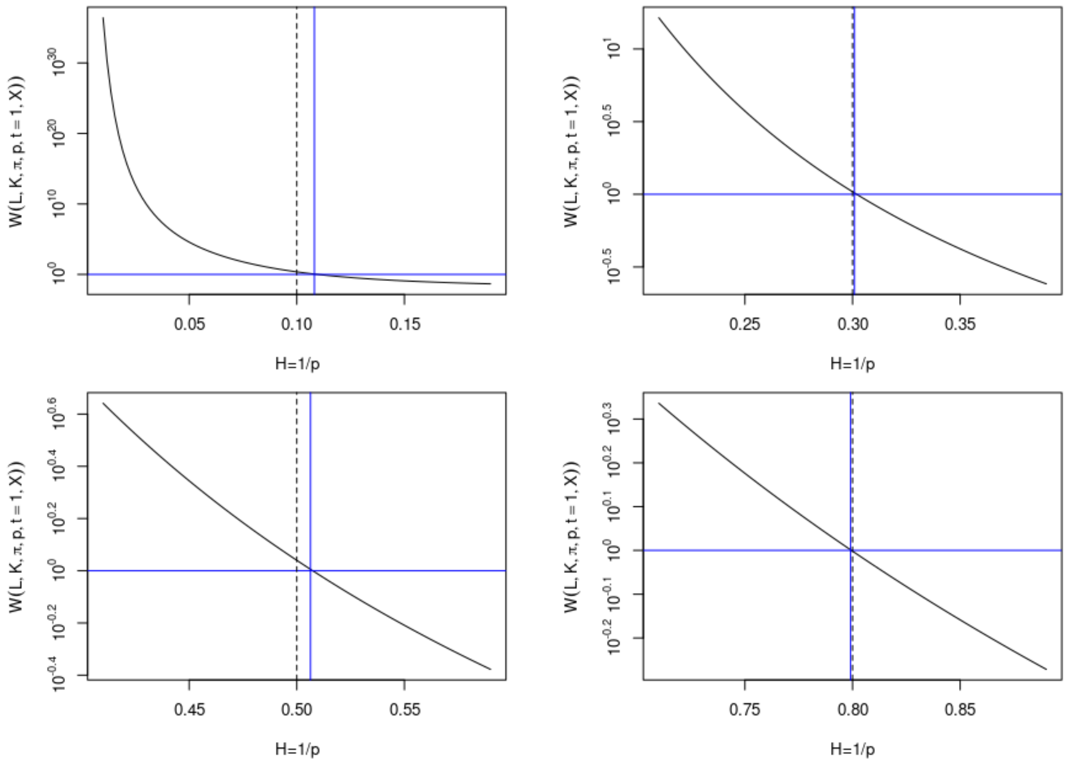

Figure 4.1: We simulate a fractional Brownian motion with Hurst parameter H for $H \in \{0.1, 0.3, 0.5, 0.8\}$. The black line represents the log of normalized p-th variation statistics plotted against $H = 1/p$. The blue vertical line represents $\hat{H}_{L,K}$ using the normalized p-th variation statistics (with $L = 300 \times 300, K = 300$), whereas the dotted line represents the true Hurst parameter H for the corresponding fractional Brownian motions.

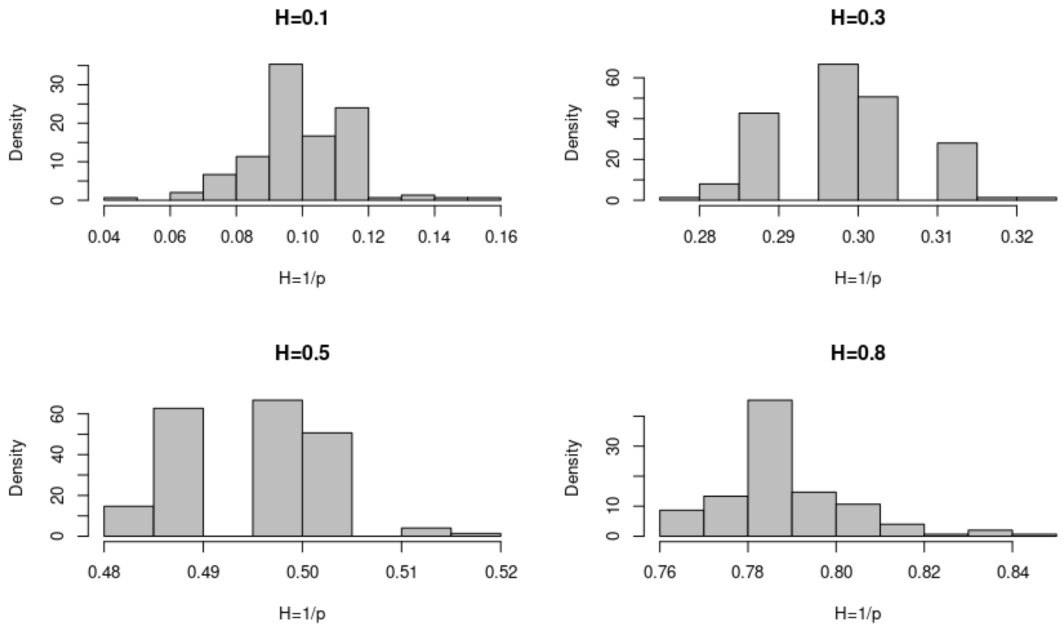

Figure 4.2: Histogram of estimated roughness index $\hat{H}_{L,K}$ via normalized p-th variation statistics (with $L = 300 \times 300, K = 300$) generated by 150 independent simulations. Data simulated from fractional Brownian motion with Hurst parameter $H \in \{0.1, 0.3, 0.5, 0.8\}$ resp..

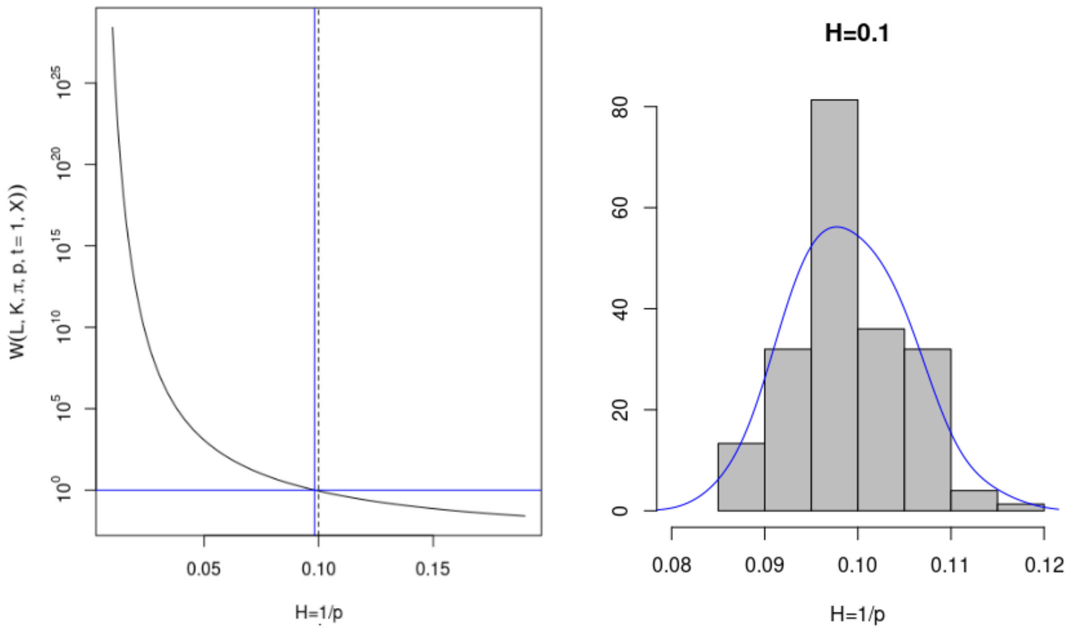

Figure 4.3: Data simulated from a fractional Brownian motion with Hurst parameter $H = 0.1$. **Left:** The log of normalized p-th variation statistic is plotted against $H = 1/p$ in black. The blue vertical line represents the estimated roughness index $\hat{H}_{L,K}$ (with $L = 2000 \times 2000, K = 2000$), whereas the green line represents for true Hurst index $H = 0.1$. **Right:** Histogram of estimated roughness index $\hat{H}_{L,K}$ generated by simulating $n = 150$ independent fractional Brownian motion with Hurst parameter 0.1. The blue line represents the corresponding kernel plot generated by Gaussian density.

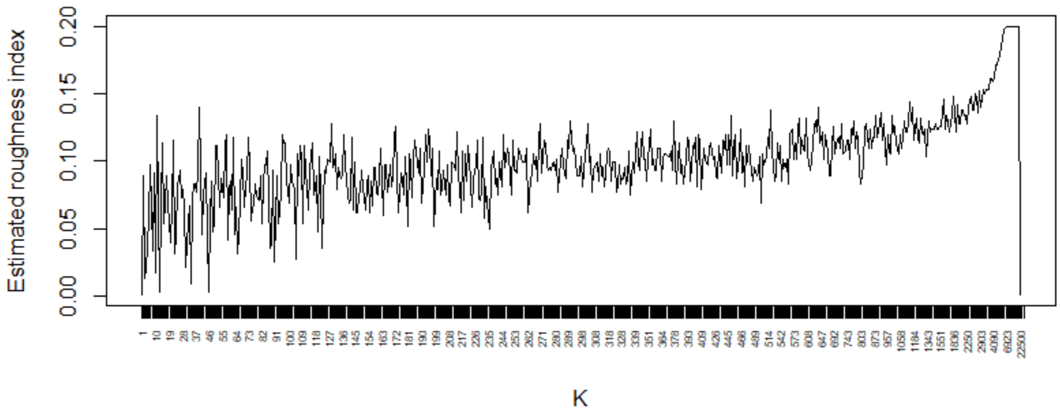

Figure 4.4: The solid black line represents the estimated roughness index $\hat{H}_{L=300\times300,K}$ plotted against different values of K for a simulated fractional Brownian motion with Hurst parameter 0.1. The blue vertical line represents for $K = 300, L = 300 \times 300$ whereas the blue horizontal line represents the true Hurst index value $H = 0.1$.

4.5.2 Simulation experiments with Takagi-Landsberg functions

We will now study the finite sample behaviour of the roughness estimator $\hat{H}^{\pi}_{L,K}(X)$ using high-frequency simulations of Takagi-Landsberg functions. Misure and Schied [62] have introduced a large class $\mathcal{X}^p$ of (signed) Takagi-Landsberg functions with Hurst parameter $H = 1/p \in (0,1)$.

$$\mathcal{X}^p = \left\{ x \in C^0([0,1],\mathbb{R}) \;\middle|\; x(t) = \sum_{m=0}^{\infty} 2^{m(\frac{1}{2}-\frac{1}{p})} \sum_{k=0}^{2^m-1} \theta_{m,k} e_{m,k}(t); \right.$$
$$\left. \text{where coefficients } \theta_{m,k} \in \{-1,+1\} \right\}. \tag{4.10}$$

The graph of $e_{m,k}$ is a wedge with height $2^{-\frac{m+2}{2}}$, width 2^{-m}, centred at $c = 2^{\frac{k-\frac{1}{2}}{2^m}}$. In particular, the functions $e_{m,k}$ have disjoint support for distinct k and fixed level m. The class $\mathcal{X}^p$ is very interesting as it is a class of deterministic continuous functions with known variation/ roughness index. So unlike fractional Brownian motions, given a choice of coefficients $\theta_{m,k} \in \{+1,-1\}$, the Takagi-Landsberg functions x is deterministic and does not involve any probabilistic notion.

Example 4.5 shows that for any $p > 1$ and for any function $x \in \mathcal{X}^p$, the variation index is equal to p. For paths in $\mathcal{X}^p$, we now compare the roughness (resp. variation) index estimator $\hat{\mathbf{H}}^{\pi}_{L,K}$ (resp. $\hat{p}^{\pi}_{L,K}$) with the true roughness (resp. variation) index $H = 1/p$.

Notation: For a function $x \in \mathcal{X}^{\pi}$, we denote x^n to be the truncated Takagi-Landsberg function at level $n \in \mathbb{N}$. i.e.,

$$x(t) = \sum_{m=0}^{n-1} 2^{m(\frac{1}{2}-\frac{1}{p})} \sum_{k=0}^{2^m-1} \theta_{m,k} e_{m,k}(t) \tag{4.11}$$

Example 4.6. *For fix $p > 1$ define the function* $x(t) = \sum_{m=0}^{\infty} 2^{m(\frac{1}{2}-\frac{1}{p})} \sum_{k=0}^{2^m-1} \theta_{m,k} e_{m,k}(t) \in \mathcal{X}^p$ *for the following choice of coefficients $\theta_{m,k}$.*

(i) For all m,k, the corresponding Faber-Schauder coefficients $\theta_{m,k} = +1$.

(ii) For all m,k, the corresponding Faber-Schauder coefficients $\theta_{m,k} = (-1)^k$.

(iii) For all m, k, the corresponding Faber-Schauder coefficients $\theta_{m,k} = (-1)^m$.

(iv) For all m, k, the corresponding Faber-Schauder coefficients $\theta_{m,k} = (-1)^{m+k}$.

From the class $\mathcal{X}^p$ we simulate the corresponding truncated Takagi-Landsberg functions x^n for $n = 18$ for the above example. In Figure 4.5: left, we plot the truncated Takagi-Landsberg functions $x^{n=18}(t) = \sum_{m=0}^{17} 2^{m(\frac{1}{2}-\frac{1}{p})} \sum_{k=0}^{2^m-1} e_{m,k}(t)$ for Hurst index $H = 1/p \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$. In Figure 4.5: right, we plot the estimated roughness index $\widehat{H}_{L,K}$ for $K = \sqrt{2^{18}} = 512, L = 512 \times 512$ plotted against the true Hurst index of the corresponding function x^n. Similarly, Figure 4.6, Figure 4.7 and Figure 4.8 respectively represent the plot of truncated function $x^{n=18}(t)$ defined in Example 4.6(ii), (iii) and (iv) and it's corresponding estimated roughness index $\hat{H}_{L,K}$. In the following table we provide the estimated roughness index for the functions considered in Example 4.6.

H	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
Ex. 4.6 (i)	0.0991	0.1992	0.2991	0.3991	0.4973	0.5973	0.6936	0.7845
Ex. 4.6 (ii)	0.0991	0.1991	0.2991	0.3991	0.4992	0.5973	0.6936	0.7845
Ex. 4.6 (iii)	0.0991	0.1991	0.2991	0.3991	0.4991	0.5971	0.6965	0.7831
Ex. 4.6 (iv)	0.0991	0.1991	0.2991	0.3991	0.4991	0.5973	0.6955	0.7864

Table 4.3: Estimated roughness index for Example 4.6

Given $p > 1$ we will now consider some example of processes X which almost surely belongs to the class $\mathcal{X}^p$ and these processes also have interesting roughness property (See [23]).

Example 4.7. *For fix $p > 1$ define the function $X(t) = \sum_{m=0}^{\infty} 2^{m(\frac{1}{2}-\frac{1}{p})} \sum_{k=0}^{2^m-1} \theta_{m,k} e_{m,k}(t) \in \mathcal{X}^p$ for the following choice of coefficients $\theta_{m,k}$.*

(i) For all m, k, the corresponding Faber Schauder coefficients $\theta_{m,k}$ are i.i.d. random variable with $\mathbb{P}(\theta_{m,k} = +1) = \mathbb{P}(\theta_{m,k} = -1) = 0.5$.

(ii) For all m, k, the corresponding Faber Schauder coefficients $\theta_{m,k}$ are i.i.d. random variable with $\mathbb{P}(\theta_{m,k} = +1) = 0.7$ and $\mathbb{P}(\theta_{m,k} = -1) = 0.3$.

(iii) For all m, k, the corresponding Faber Schauder coefficients $\theta_{m,k}$ are i.i.d. random variable with $\mathbb{P}(\theta_{m,k} = +1) = 0.8$ and $\mathbb{P}(\theta_{m,k} = -1) = 0.2$.

Similar to Example 4.6, from the class $\mathcal{X}^p$ we simulate sample paths of corresponding truncated Takagi-Landsberg process X^n for $n = 18$. In Figure 4.5: left, we plot the truncated Takagi-Landsberg functions $x^{n=18}(t) = \sum_{m=0}^{17} 2^{m(\frac{1}{2}-\frac{1}{p})} \sum_{k=0}^{2^m-1} e_{m,k}(t)$ for Hurst index $H = 1/p \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$. In Figure 4.9: right, we plot the estimated roughness index $\hat{\mathbf{H}}_{L,K}$ for $K = \sqrt{2^{18}} = 512, L = 512 \times 512$ plotted against the true Hurst index of the simulated path X^n. Similarly, Figure 4.10 and Figure 4.11 resp. represents the plot of a simulated path from the truncated process $X^{n=18}(t)$ defined in Example 4.7(ii) and (iii), and it's corresponding estimated roughness index $\hat{H}_{L,K}$. In the following table we provide the Estimated roughness index for the functions concidered in Example 4.7.

Hurst index H	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
Ex.4.7(i)	0.0992	0.1991	0.2991	0.3991	0.4981	0.5973	0.6936	0.7848
Ex.4.7(ii)	0.0993	0.1991	0.2994	0.3971	0.4991	0.5978	0.6936	0.7844
Ex.4.7(iii)	0.0991	0.1994	0.2993	0.3991	0.4973	0.5972	0.6939	0.7845

Table 4.4: Estimated roughness index for Example 4.7

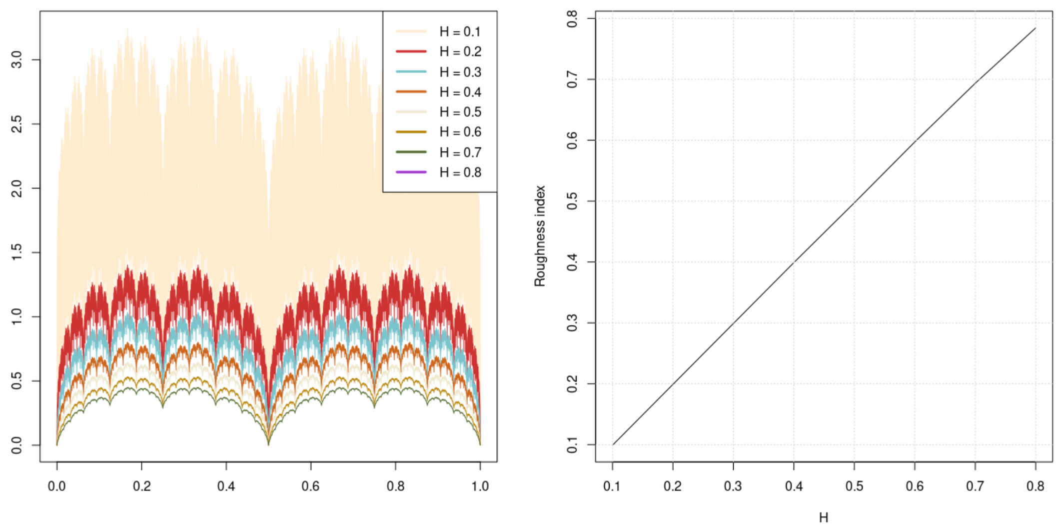

Figure 4.5: **Left:** For $H = 1/p \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$, plot of function x^n defined in Example 4.6(i) where $n = 18$. **Right:** Plot of estimated roughness index $\hat{H}_{L,K}$ ploted against the true Hurst exponent of the path.

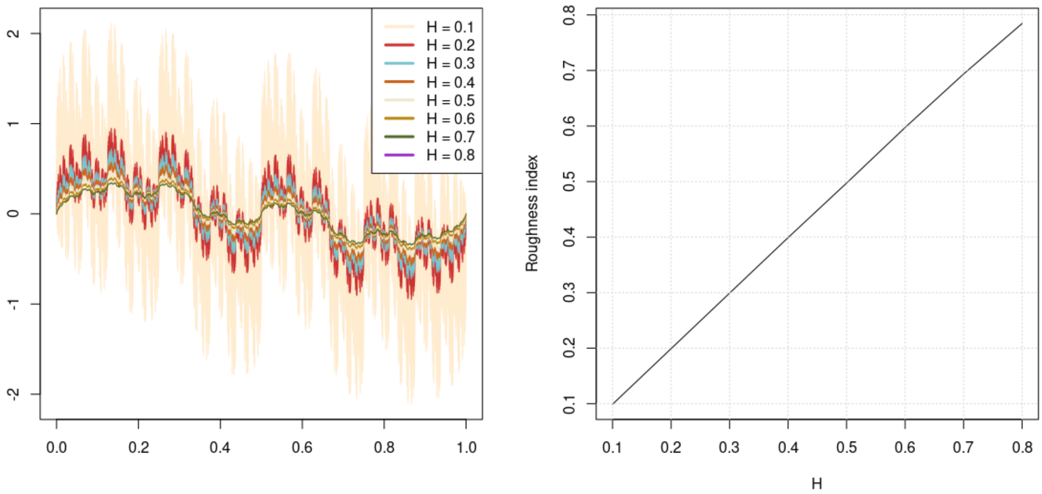

Figure 4.6: **Left:** For $H = 1/p \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$, plot of function x^n defined in Example 4.6(ii) where $n = 18$. **Right:** Plot of estimated roughness index $\hat{H}_{L,K}$ ploted against the true Hurst exponent of the path.

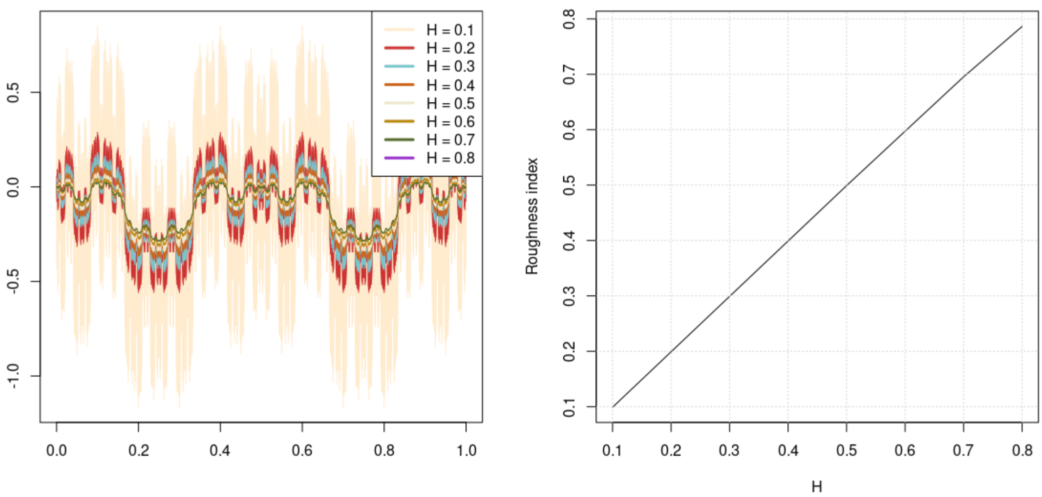

Figure 4.7: **Left:** For $H = 1/p \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$, plot of function x^n defined in Example 4.6(iii) where $n = 18$. **Right:** Plot of estimated roughness index $\hat{H}_{L,K}$ ploted against the true Hurst exponent of the path.

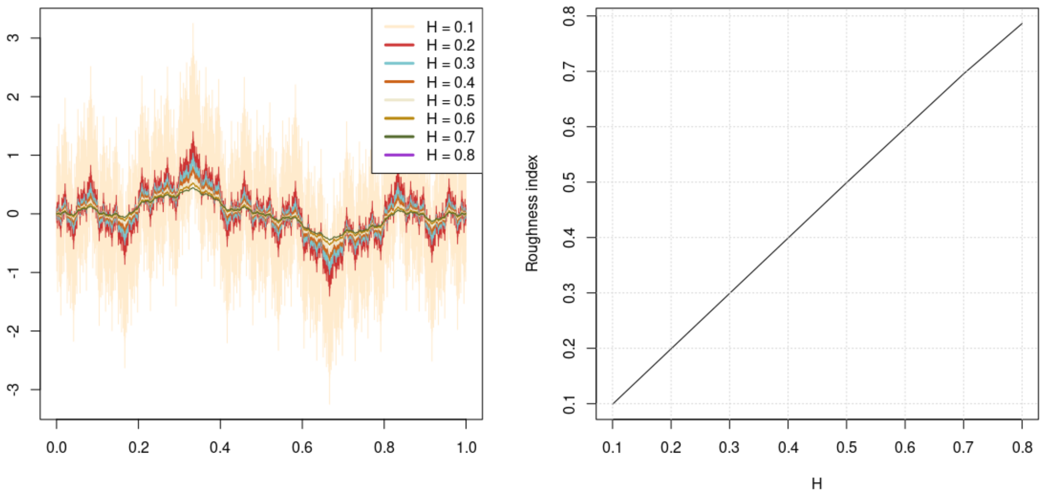

Figure 4.8: **Left:** For $H = 1/p \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$, plot of function x^n defined in Example 4.6(iv) where $n = 18$. **Right:** Plot of estimated roughness index $\hat{H}_{L,K}$ ploted against the true Hurst exponent of the path.

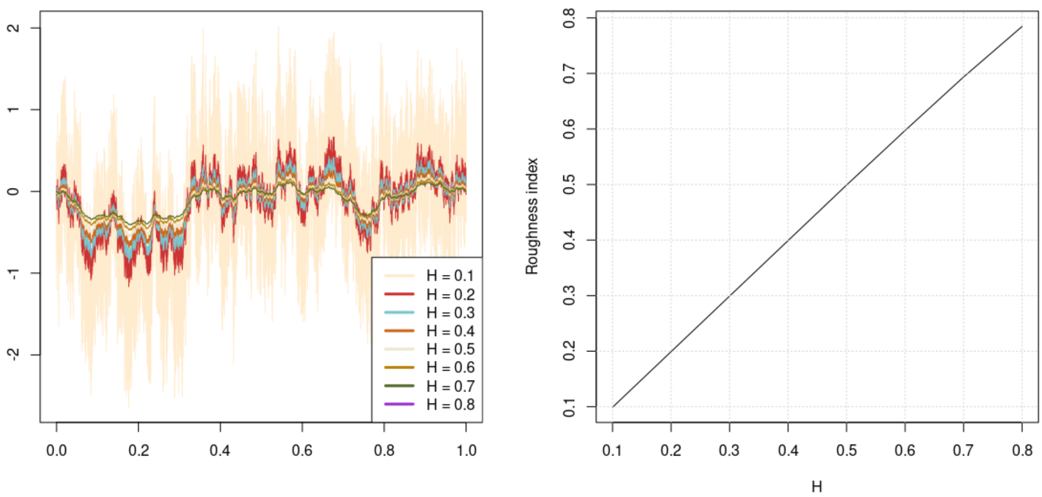

Figure 4.9: **Left:** For $H = 1/p \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$, a realization of process X^n defined in Example 4.7(i) where $n = 18$. **Right:** Plot of estimated roughness index $\hat{H}_{L,K}$ ploted against the true Hurst exponent of the path.

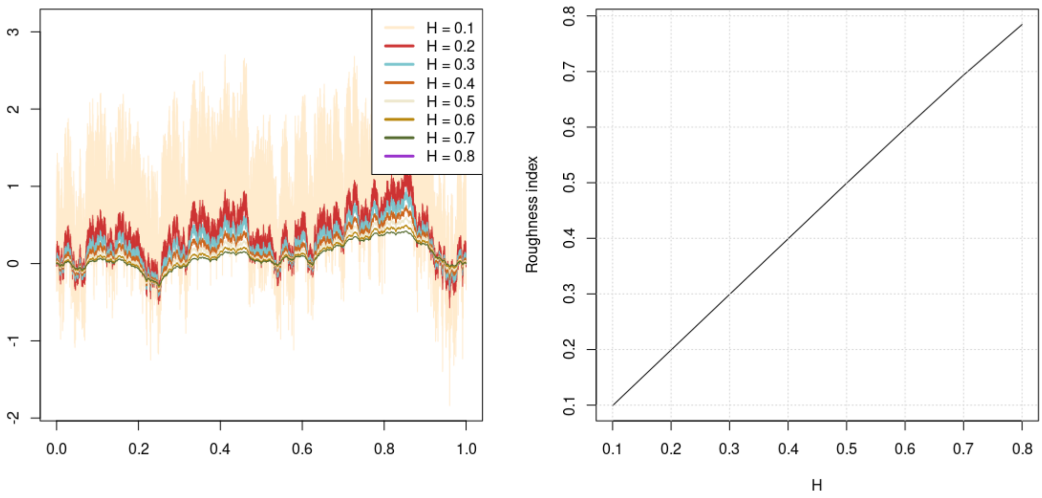

Figure 4.10: **Left:** For $H = 1/p \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$, a realization of process X^n defined in Example 4.7(ii) where $n = 18$. **Right:** Plot of estimated roughness index $\hat{H}_{L,K}$ ploted against the true Hurst exponent of the path.

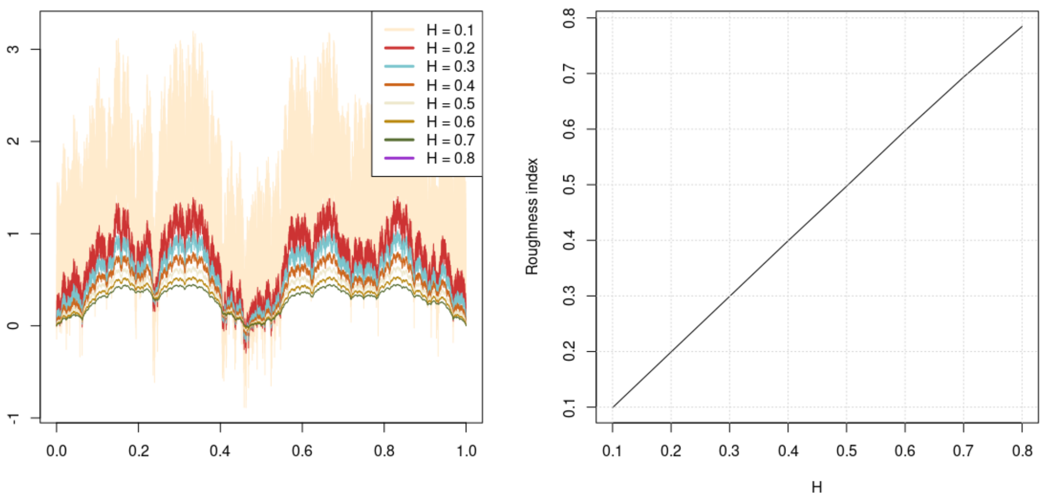

Figure 4.11: **Left:** For $H = 1/p \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$, a realization of process X^n defined in Example 4.7 (iii) where $n = 18$. **Right:** Plot of estimated roughness index $\hat{H}_{L,K}$ ploted against the true Hurst exponent of the path.

4.6 Discussion

We have introduced a pathwise estimator for the roughness of a signal based on discrete observations, and studied its asymptotic behaviour as the frequency of observations is increased, as well as its finite sample performance for some examples. Our estimator is based on the normalized p-th variation statistic, introduced in Definition 4.5.

The two main advantages of using the normalized p-th variation statistic for measuring the 'roughness' of a process are as follows. On contrary to most current estimators [45, 1, 11] on the estimation of Hurst index for specific classes of processes our method of estimating roughness/ variation index is purely pathwise: we do not assume a particular probabilistic structure or distributional property for the underlying process. Secondly, for the normalized p-th variation statistics the dataset does not have to be observed over a fixed (uniform) time grid, which is often the case for financial data as in most situations the data is usually observed when there is a market movement, rather than over uniform time intervals.

Chapter 5

Application to financial data: Is volatility rough?

Chapter based on: Rama Cont, Purba Das (2022) Rough volatility - fact or artefact? [24].

5.1 Fractional processes in finance: from long-range dependence to 'rough volatility'

Beginning with Mandelbrot and VanNess [61], fractional Brownian motion and fractional Gaussian noise have been used as building blocks of stochastic models of various phenomena in physics, engineering [60] and finance [5, 13, 18, 20, 42, 68, 76]. Fractional Brownian motion has two remarkable properties which have contributed towards its adoption as a building block in stochastic models: first, its ability to model long-range dependence, as measured by the slow decay $\sim T^{2H-2}$ of auto-correlation functions of increments, where $0 < H < 1$ is the Hurst exponent; second, its ability to generate trajectories which have varying levels of Hölder regularity ('roughness'). The former is a property that manifests itself over long time scales while the latter manifests itself over short time scales and, in general, these two properties are unrelated. But in the case of fractional Brownian motion, the two properties are linked through self-similarity and governed by the Hurst exponent $0 < H < 1$: for $H > 1/2$ one obtains long-range dependence in increments and trajectories smoother than Brownian motion while for $H < 1/2$

one obtains 'anti-correlated' increments and trajectories rougher than Brownian motion[1]. Processes driven by fractional Gaussian noise with $H < 1/2$ are thus sometimes referred to as 'rough processes'.

In early applications to financial data [5, 13, 18, 76], fractional processes were adopted in order to model *long range dependence* effects in financial time series [19]. More specifically, statistical evidence of *volatility clustering* [20] - positive dependence of the amplitude of returns over long time scales - led to the development of stochastic volatility models driven by fractional Brownian motion. A well-known example of such a fractional stochastic volatility model is the one proposed by Comte and Renault [18] who modelled the dynamics of the (instantaneous) volatility $\sigma(t)$ of an asset as:

$$Y(t) = \ln \sigma(t) \qquad dY(t) = -\gamma Y(t)dt + \theta dB^H(t) \tag{5.1}$$

where B^H is a fractional Brownian motion (fBM) with Hurst exponent H. This long-range dependence in volatility is modelled by choosing $1 > H > 1/2$ [18, 13, 14, 51, 57].

A recent strand of literature, starting with Gatheral et al. [42], has suggested the use of fractional Brownian models with $H < 1/2$ for modelling volatility. The motivation of this approach, beginning with [42], is that empirical data on volatility estimators suggest that volatility is 'rough' i.e., has a (Hölder) roughness which is strictly less than 1/2. Unlike previous studies based on auto-correlations of various volatility estimators over long time scales [5, 13, 19], rough volatility models [42] rely on the analysis of the behaviour of volatility estimators over very short intraday time scales in order to assess the 'roughness' of these signals.

However, it has not been lost on experts working in this area that the estimation results in the previous literature on long-range dependence in volatility, which pointed out towards Hurst exponents $H > 0.5$ (and around 0.55) [5, 18, 57] seem to contradict the claims in the recent 'rough volatility' literature, which points to values of H much smaller than 0.5 (closer to 0.1). Together with the well-known statistical issues plaguing the estimation of Hurst exponents [9, 66], these conflicting results call for a critical examination of the empirical evidence for 'rough volatility'.

[1]Incidentally, Hurst and Hölder happen to have the same initials, adding to the confusion...

Compounding this issue is the fact that (spot) volatility is not directly observed but estimated from data with an inherent estimation error, known in the context of high-frequency data as 'microstructure noise', which has been the subject of many studies [7, 53, 57]. This estimation error is far from i.i.d.: it is known to possess path-dependent features [53]. As a result, measures of roughness for realized volatility indicators may be quite different from those of the underlying 'spot volatility'. This is simply because the convergence of high-frequency volatility estimators in L^p norms does not imply in any way their functional convergence in Hölder norms or other norms related to roughness.

As already pointed out by Rogers [68], these two properties, namely the short-range behaviour which determines the roughness of the path, and the long-range dependence property, can (and should) be modeled through different mechanisms (see also [8]). Bennedsen et al [8] discuss several such approaches.

Unfortunately the focus of the literature on parametric models based on fractional Brownian motion or fractional Gaussian noise concentrates these two, very different, properties in a single parameter: the Hurst exponent H. Such parametric approaches [12, 40, 42] proceed as follows: one estimates a parametric model for volatility dynamics based on some fractional Gaussian driving noise with Hurst exponent $0 < H < 1$ using a MLE [40] or method of moments [12]. Then, based on the estimated value of this parameter H, one concludes that "volatility is rough" if $\hat{H} < 0.5$.

The validity of such approaches hinges of course on the assumption that the class of models used is well-specified. As pointed out by Bennedsen et al [8], this is unlikely to be the case for SDEs driven by fractional Gaussian noise if one wants to accommodate both (long-range) dependence properties and (short-range) roughness properties.

To avoid this caveat, we propose a model-free *nonparametric* method which focuses solely on the roughness properties of sample paths. Although less ambitious in its scope (we only focus on roughness properties rather than developing a full model for volatility dynamics), our approach is robust to the specification errors and estimation biases which plague parametric methods.

5.1.1 Contribution

In the chapter, we address these questions in detail by re-examining the statistical evidence from high-frequency financial data, in an attempt to clarify whether the assertion that 'volatility is rough' (i.e. rougher than typical paths of Brownian motion) is supported by empirical evidence. We investigate the statistical evidence for the use of 'rough' fractional processes with Hurst exponent $H < 0.5$ for the modelling of volatility of financial assets, using a non-parametric, model-free approach.

A non-parametric method for estimating the roughness of a function/path based on a (high-frequency) discrete sample, is described in Chapter 4. In Chapter 4 we also provide the finite sample behaviour of the roughness estimator. In this chapter, we apply this method to estimate the roughness of realized volatility signals based on high-frequency observations. Through a detailed numerical experiment based on a stochastic volatility model, we show that even when the instantaneous (spot) volatility has diffusive dynamics with the same roughness as Brownian motion, the realized volatility exhibits rough behaviour corresponding to a Hurst exponent significantly smaller than 0.5. Similar behavior is observed in financial data as well, which suggests that the origin of the roughness observed in realized volatility time-series may lie in the estimation error rather than the volatility process itself. Comparison of roughness estimates for realized and instantaneous volatility in fractional volatility models for different values of Hurst parameter H shows that whatever the value of H for the (spot) volatility process, realized volatility always exhibits 'rough' behaviour.

Our results are broadly consistent with the points raised by Rogers [66], but we pinpoint more precisely the origin of the apparent 'rough' behaviour of volatility as being the estimation error inherent in the estimation of realized volatility. In particular, our results question whether the empirical evidence presented from high-frequency volatility estimates supports the 'rough volatility' hypothesis.

5.2 Spot volatility and realized volatility

Contrary to prices of an asset which may be observed and sampled directly from market data, *(spot) volatility* is not directly observable and as a consequence must be estimated from prices. Stochastic volatility models represent the price of a financial asset as the solution of a stochastic differential equation driven by a Brownian motion:

$$dS_t = \sigma_t S_t dB_t + \mu_t S_t dt \tag{5.2}$$

where the coefficient σ_t represents the instantaneous or 'spot' volatility. In general, σ_t is represented as a random process itself driven by fractional processes.

In a practical situation, the price S_t at time t, is usually observed over a non-uniform time grids of $[0, T]$:

$$\pi^n = \left(0 = t_0^n < t_1^n < \cdots < t_{N(\pi^n)}^n = T\right). \tag{5.3}$$

In order to study high-frequency asymptotics of roughness estimators, we assume $|\pi^n| \to 0$ as n increases; here the index n may be thought of as a 'sampling frequency'. An example to keep in mind is the dyadic partition sequence: $\pi^n = \left(t_i^n = \frac{iT}{2^n}, i = 0, \cdots, 2^n\right)$ but none of the results below requires a uniform grid.

The spot volatility process σ_t may then be recovered from the *quadratic variation* of the log-price $X = \log S$ along this particular grid:

$$\sigma_t^2 = \frac{d}{dt}[\log S]_\pi(t) \tag{5.4}$$

where the quadratic variation $[\log S]_\pi$ of the log-price

$$[\log S]_\pi(t) = \lim_{n\to\infty} \sum_{\pi^n \cap [0,t]} \left(\log \frac{S(t_{i+1}^n)}{S(t_i^n)}\right)^2 = \lim_{n\to\infty} RV_t(\pi^n)^2$$

is computed as a high-frequency limit of the *realized variance* [7, 4] along the sampling grid π^n, defined as

$$RV_t(\pi^n)^2 = \sum_{\pi^n \cap [0,t]} \left(\log \frac{S(t_{i+1}^n)}{S(t_i^n)}\right)^2 = \sum_{\pi^n \cap [0,t]} \left(X(t_{i+1}^n) - X(t_i^n)\right)^2. \tag{5.5}$$

The realized volatility is defined as the square root of the realized variance.

Definition 5.1 (Realized volatility)**.** *The realized volatility of a price process S over time interval $[t, t+\delta]$ sampled along the time partition π^n is defined as:*

$$RV_{t,t+\Delta}(\pi^n) = \sqrt{\sum_{\pi^n \cap [t,t+\Delta]} \left(X(t^n_{i+1}) - X(t^n_i)\right)^2} = \sqrt{[X]_{\pi^n}(t+\Delta) - [X]_{\pi^n}(t)} \quad (5.6)$$

where $X = \log S$.

If the price S_t follows a stochastic volatility model (5.2) with **instantaneous volatility** σ_t then realized variance converges to the quadratic variation of $\log S$ (also called '*integrated variance*') as sampling frequency increases [7, 53]:

$$RV_t(\pi^n)^2 \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} IV_t := \int_0^t \sigma_u^2 du, \qquad RV_{t,t+\Delta}(\pi^n) \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} \sqrt{IV_{t,t+\Delta}} = \sqrt{\int_t^{t+\Delta} \sigma_u^2 du} \quad (5.7)$$

Along a single price path observed at high-frequency, we can compute the realized variance (5.6) and the realized volatility $RV_{t,t+\delta}(\pi^n)$ in (5.7) may be used as a practical indicator of volatility:

$$RV_{t,t+\Delta}(\pi^n) \simeq \sqrt{\Delta}\ \sigma_t.$$

Several empirical studies have attempted to estimate the roughness of 'realized volatility' signals using high-frequency data [4, 7, 27, 53, 64]. A well-known reference is the study of Gatheral et al. [42] where the authors estimate the roughness index of S&P500 realized volatility by performing the following logarithmic regression to :

$$m(q,\Delta) = \frac{1}{n}[\log RV]^{(q)}_{\pi^n} = \frac{1}{n}\sum_{t=1}^{n} |\log(RV_{t+\Delta}) - \log(RV_t)|^q \approx C_q \Delta^{\xi_q}. \quad (5.8)$$

The coefficients ξ_q are also shown to behave linearly in q:

$$\xi_q \approx q\ \widehat{H_R}.$$

Regression of ξ_q on q yields an estimate $\widehat{H_R}$ of Hurst/Hölder index, for which Gatheral et al. [42] report the value $\widehat{H_R} = 0.13$. Based on these observations, they propose a fractional SDE for (spot) volatility:

$$d\log\sigma_t^2 = \mu_t dt + \eta dB_t^H.$$

As we see from Equation 5.8, the method used in [41] actually uses p-th variation of the $\log(RV)$ to calculate the roughness of the underlying volatility process. Figure 5.1 is a replication of the log-regression model described above to estimate the roughness index of the volatility of 5-min S&P 500. However, in an interesting simulation study using paths simulated from a Brownian OU volatility process, Rogers [66] showed that the scaling behavior claimed as evidence for 'rough volatility' is also observed in a Brownian OU model over a range of time scales, and that estimators of the roughness index based on log-regression of empirical $p-$th variation have poor accuracy.

Similar evidence for the lack of accuracy of such estimators based on linear-regression of p-th variation is shown by Fukasawa et al. [40].

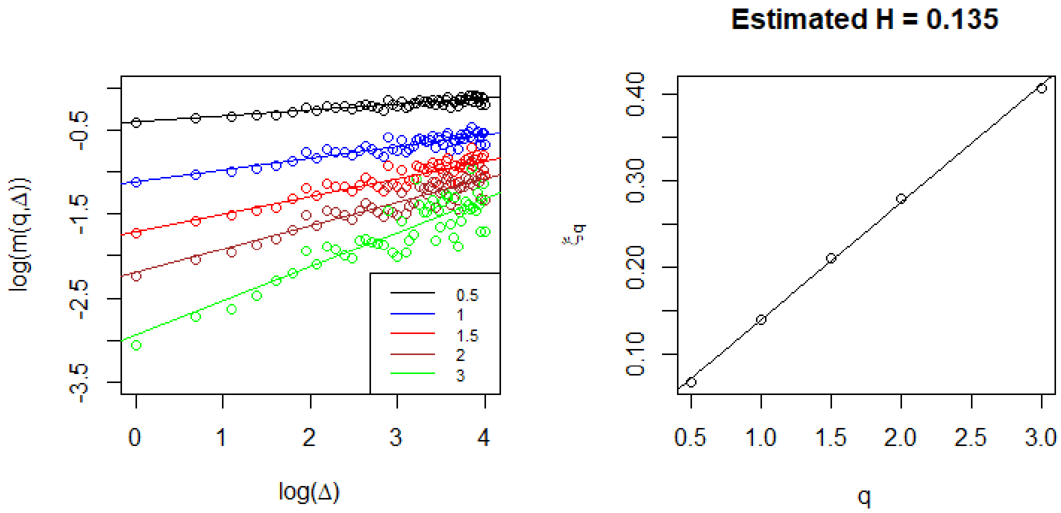

Figure 5.1: **Left:** A reproduction of the linear-regression method introduced by Gatheral et al. [42] using SPX 5-minute realized volatility from the Oxford-Man Institutes Realized Library. **Right:** Regression of ξ_q vs q: the estimated slope is $\widehat{H}_R = 0.135$.

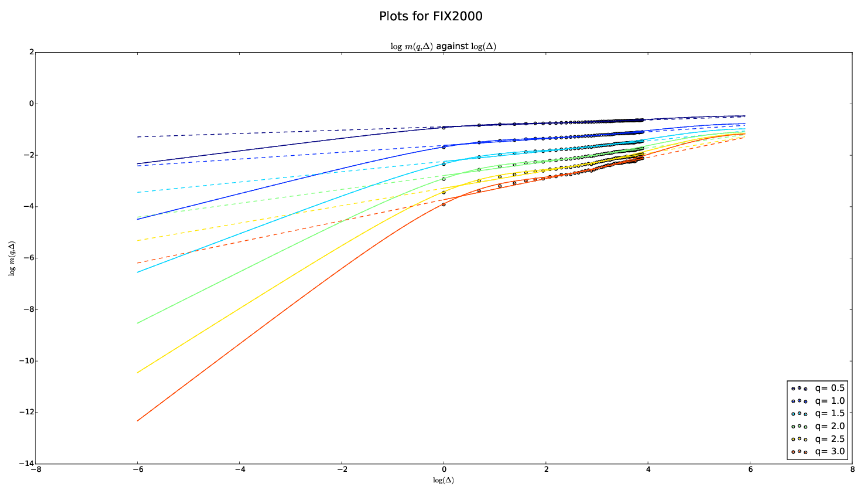

Figure 5.2: As shown by Rogers (2019), the scaling behavior claimed as evidence for 'rough volatility' is also observed in a Brownian OU model over a range of time scales.

5.3 Simulation experiments

We now compare various estimators of the roughness index for instantaneous volatility σ_t with those obtained from realized volatility $RV_{t,t+\Delta}(\pi^n)$ using price trajectories simulated from stochastic volatility models with varying degrees of "roughness".

5.3.1 Stochastic volatility diffusion models

Let us first consider the following stochastic volatility where volatility is simply (the modulus of) a Brownian motion:

Example 5.1.

$$dS_t = \sigma_t S_t dB_t, \quad \textit{with} \quad \sigma_t = |W_t|, \tag{5.9}$$

where B, W are independent Brownian motions.

Figure 5.3 represents a path of the instantaneous volatility σ_t and the realized volatility $RV_t(\pi^L)$ computed as in Equation 5.6 by taking 300 consecutive data-points. The right plot of Figure 5.3 represents the estimation error, which is defined as the difference of instantaneous and realized volatility. The ACF of the estimation error shows a complex time-dependent pattern which rules out i.i.d. behavior and indicates a complex dependence structure.

The estimated roughness index of instantaneous and realized volatility are observed to be very different. In the left graph of Figure 5.4 we plot $\log(W(K = 500, L = 500 \times 500, \pi, p, t = 1, X = RV))$ against $H = 1/p$ for the realized volatility. On the other hand, the right graph is the same plot with the same set of parameters but for instantaneous volatility. The estimated roughness index for realized volatility ($\widehat{H}_{L=500\times500,K=500}(RV) = 0.27$) is significantly smaller than the roughness index of the instantaneous volatility ($\widehat{H}_{L=500\times500,K=500}(\sigma) = 0.49$) suggesting rougher behaviour of realized volatility. As in our simulation study we do not have any measurement errors, this roughness behaviour purely comes from estimation error. In some studies it is assumed that the estimation error or the log-estimation error is i.i.d. (see e.g. [40]) but as we can see from this diffusion example, the estimation error is far from i.i.d.

The solid black lines in Figure 5.5 and Figure 5.6 respectively represent the estimated roughness index $\widehat{H}_{L=300\times300,K}$ plotted against different values of K for the realized and instantaneous volatility (model 5.9). The blue vertical line represents for $K = 500, L = 500 \times 500$. From the figures, we can observe that irrespective of the choice of K for the finite sample dataset of length $L = 500 \times 500$, the realized volatility is significantly rougher than the instantaneous volatility.

We now compare our roughness estimator with the log-regression method used in Gatheral et al. [42] for the model 5.9. It turns out that even with the linear-regression model, similar 'rougher' realized volatility is observed even if the instantaneous volatility has Brownian diffusive behaviour. Figure 5.7 and Figure 5.8 show that the realized volatility has a significant smaller roughness index than the instantaneous volatility even with respect to the linear regression method. In this example it is clear that the roughness observed in realized volatility is attributable to the discretization error ('estimation error') and not the roughness of the spot volatility process, which is Brownian.

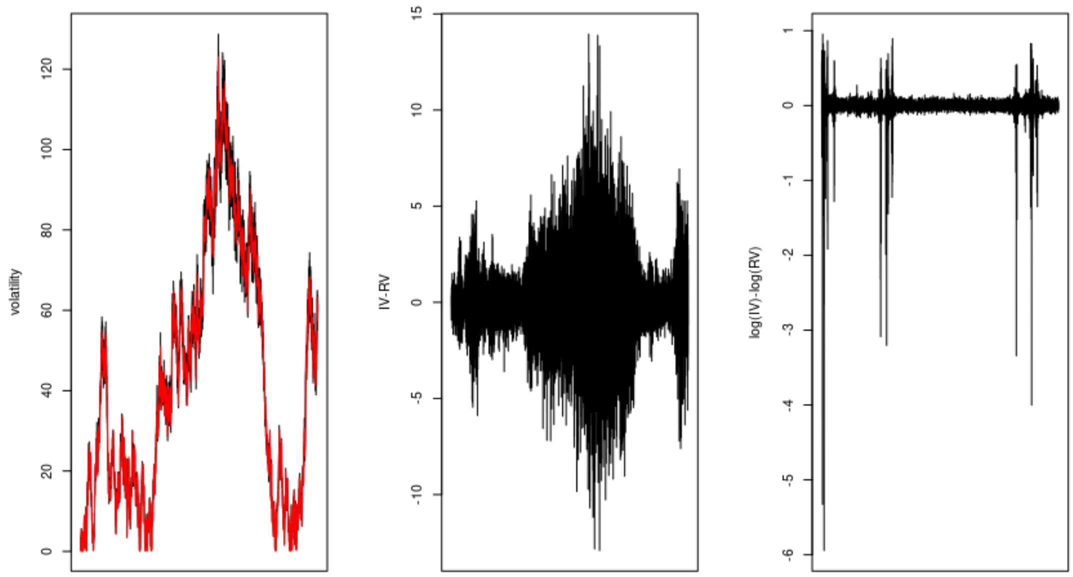

Figure 5.3: Simulation model: $\sigma_t = |B'_t|, dS_t = S_t\sigma_t dB_t$, where B_t and B'_t are Brownian motions independent of each other. **Left:** The red line represents instantaneous volatility σ_t whereas the black line represents realized volatility RV_t. **Right:** Corresponding estimation error for the left simulated path.

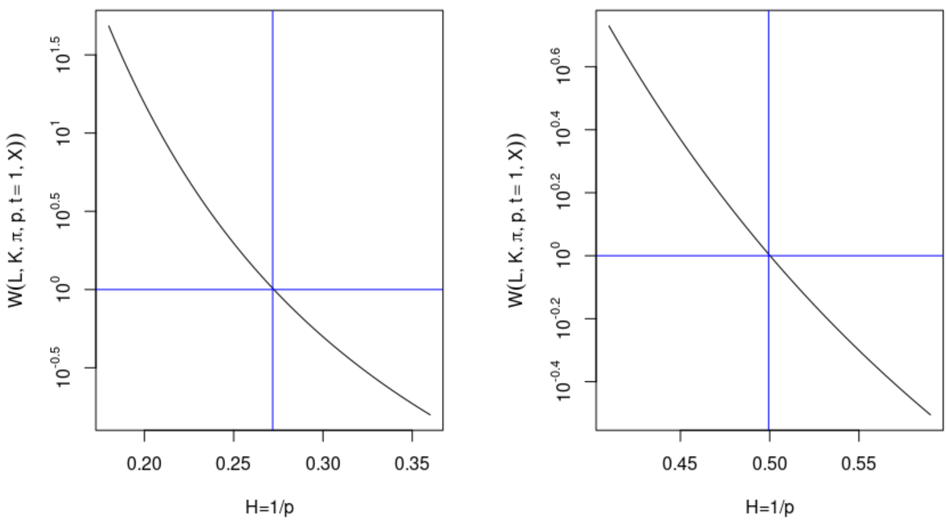

Figure 5.4: **Left:** Estimated roughness index $\widehat{H}_{L,K}$ (via normalized p-variation statistic with $L = 500 \times 500, K = 500$), for realized volatility derived from a Brownian diffusion model, shown in Figure 5.3. **Right:** Estimated roughness index $\widehat{H}_{L,K}$ for instantaneous volatility of the same price path.

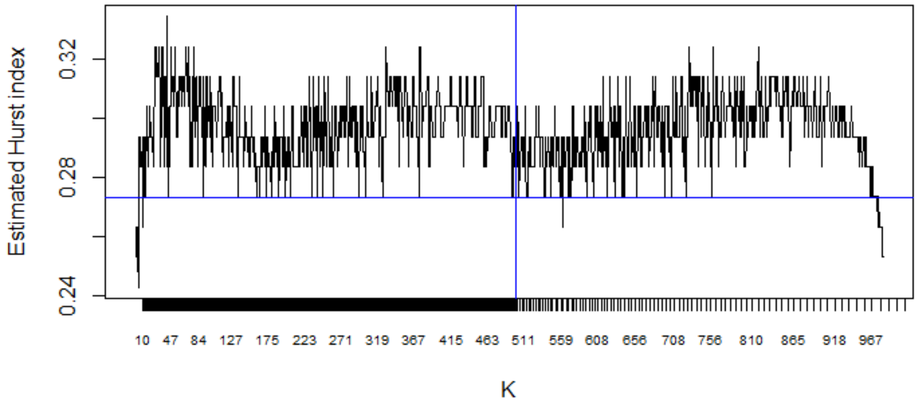

Figure 5.5: The solid black line represents the estimated roughness index $\widehat{H}_{L,K}$ via normalized p-th variation statistic $W(L = 500 \times 500, K, \pi, q, t = 1, X = RV)$ plotted against different values of K for the realized volatility shown in Figure 5.3. The blue vertical line represents $L = 500 \times 500, K = 500$ whereas the blue horizontal line represents $\widehat{H} = 0.273$.

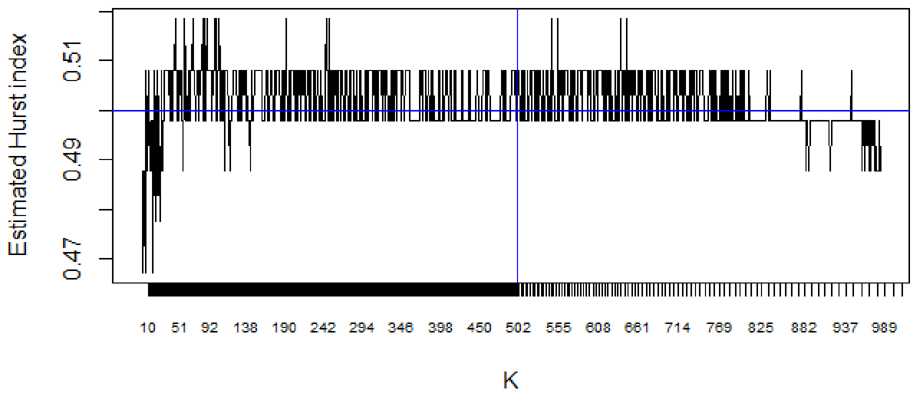

Figure 5.6: The solid black line represents the estimated roughness index $\widehat{H}_{L,K}$ via normalized p-th variation statistic $W(L = 500 \times 500, K, \pi, q, t = 1, X = IV)$ plotted against different values of K for the instantaneous volatility shown in Figure 5.3. The blue vertical line represents $L = 500 \times 500, K = 500$ whereas the blue horizontal line represents true Hurst parameter $H = 0.5$.

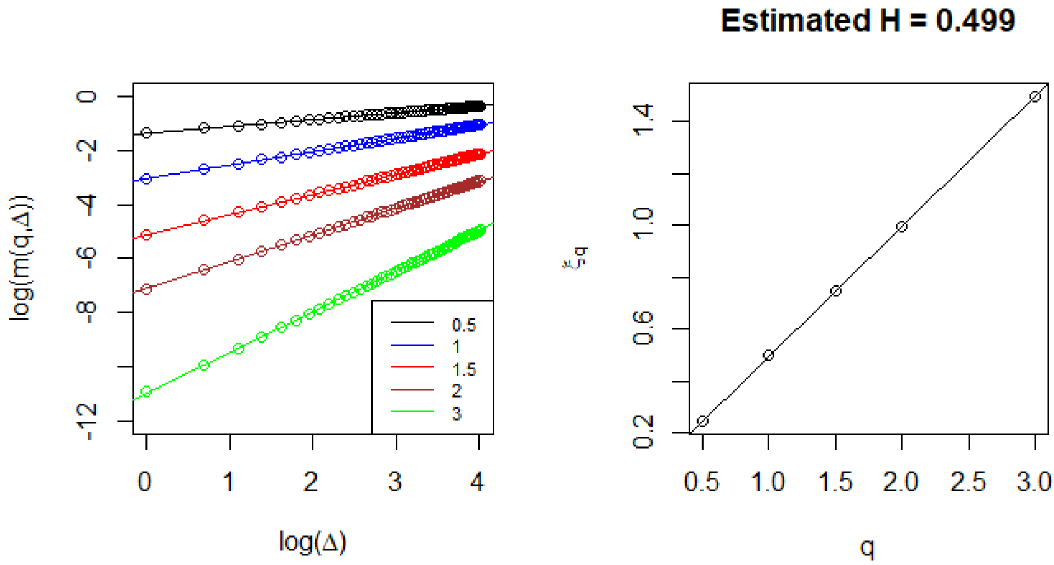

Figure 5.7: **Left:** Scaling analysis of instantaneous volatility of a simulated Brownian stochastic volatility model (Example 5.1) using the same log-regression method used by Gatheral et al. [42]. **Right:** regression coefficients ξ_q as a function of q. The estimated roughness index is $\widehat{H_R} = 0.499$.

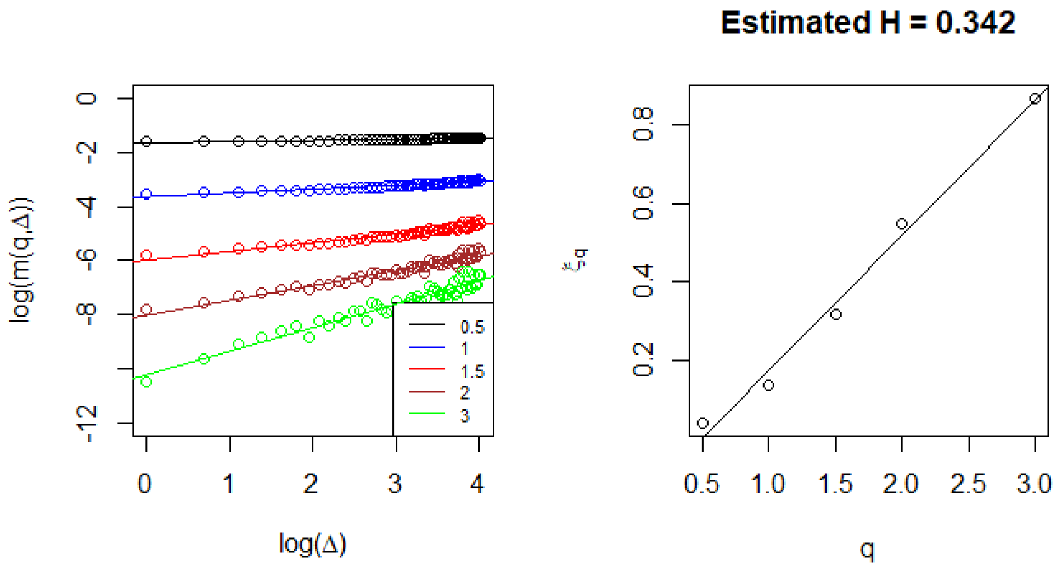

Figure 5.8: **Left:** Scaling analysis of realized volatility of a simulated Brownian stochastic volatility model (Example 5.1) using the same log-regression method used by Gatheral et al. [42]. **Right:** regression coefficients ξ_q as a function of q. The estimated roughness index is $\widehat{H_R} = 0.342$.

Next we consider a more realistic mean-reverting volatility model in which the volatility follows a Brownian OrnsteinUhlenbeck process:

Example 5.2 (OU-SV model)**.**

$$dS_t = S_t \sigma_t dB_t, \qquad \sigma_t = \sigma_0 e^{Y_t}, \quad dY_t = -\gamma Y_t dt + \theta dB_t. \tag{5.10}$$

In the simulation, we use $\sigma_0 = 1, Y_0 = 0$ and $\gamma = \theta = 1$. The left plot of Figure 5.9 represents the realized volatility (respectively instantaneous volatility) of the price process in black (respectively red) simulated from the above stochastic volatility model 5.10. The right plot of Figure 5.9 represents the corresponding estimation error, which is the difference between the realized and the instantaneous volatility. Visually the plot suggests the estimation error has a complicated dependence structure, and the i.i.d. assumption for estimation error, as put forth for example in [40], is not very realistic.

Now we compare the distribution of the estimator $\widehat{H}_{L,K}$ with ($L = 300 \times 300, K = 300$) for realized and instantaneous volatility across 2500 independent paths drawn from (5.10). The left plot in Figure 5.10 is the distribution of $\widehat{H}_{L,K}$ for the realized volatility while the right plot corresponds to the same for instantaneous volatility. The following table provides summary statistics for the estimator $\widehat{H}_{L,K}$ with $L = 300 \times 300, K = 300$ across 2500 independent sample paths for realized volatility and instantaneous volatility respectively.

	Realized volatility	Instantaneous volatility
Min.	0.087	0.528
1st Quantile	0.128	0.552
Median	0.136	0.556
Mean	0.137	0.557
3rd Quantile	0.148	0.563
Max.	0.181	0.581

Table 5.1: Estimated roughness index $\widehat{H}_{L,K}, L = 300 \times 300, K = 300$ for realized volatility and instantaneous volatility for the diffusion model (5.9) with $H = 0.5$.

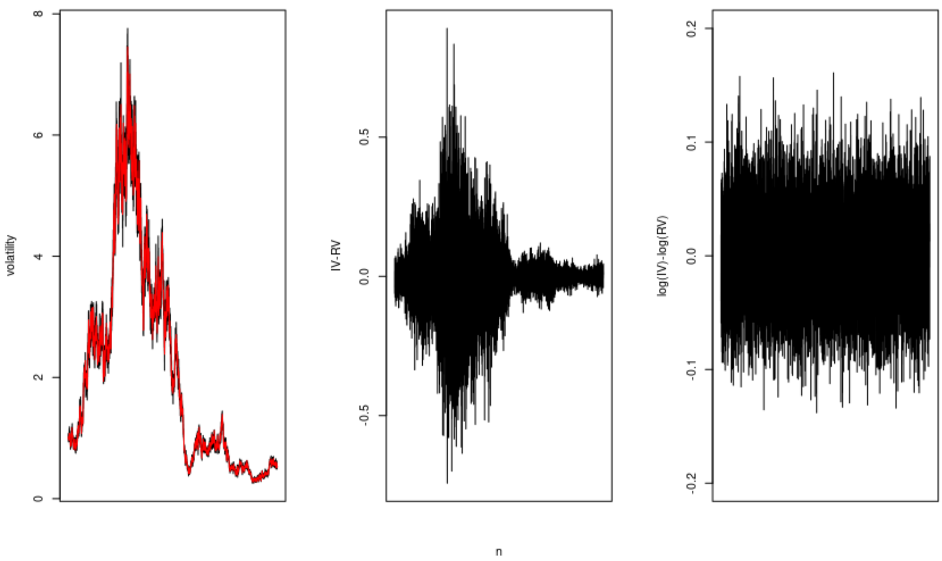

Figure 5.9: **Left:** Realized volatility (in black) vs instantaneous volatility (red) for the OU price model (5.10). **Right:** estimation error.

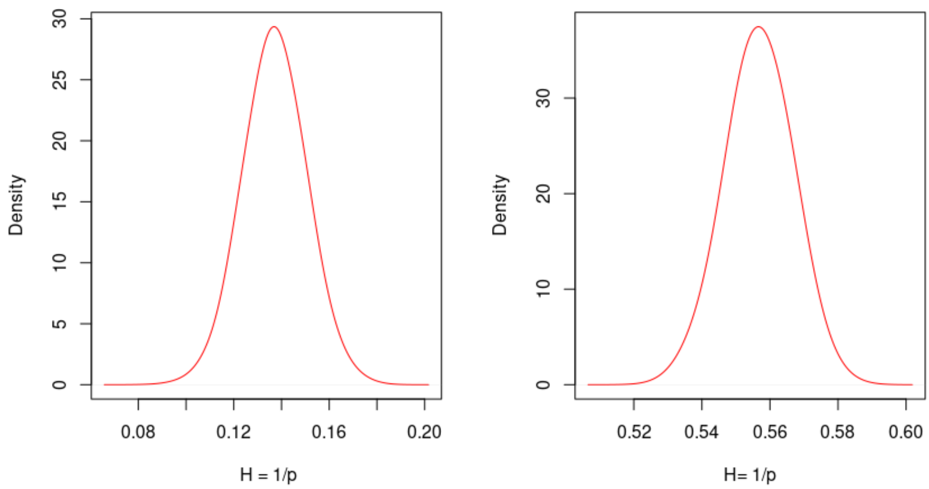

Figure 5.10: Distribution of the estimated roughness index $\widehat{H}_{L,K}$ for $(L = 300 \times 300, K = 300)$ across 2500 independent simulations for the OU-SV model (5.10) with roughness index $H = 0.5$. **Left:** Realized volatility. **Right:** Instantaneous volatility.

5.3.2 A fractional Ornstein-Uhlenbeck model

In both previous examples, instantaneous volatility follow a diffusive behaviour similar to Brownian motion with $H = \frac{1}{2}$, yet the realized volatility exhibits rough behaviour with an estimated roughness index significantly smaller than 0.5.

We now consider the more general case of a price process generated by a stochastic volatility model of the type (5.1) where instantaneous volatility has a general roughness index $H \in (0, 1)$ and explore how the roughness index of the instantaneous volatility reflects on the roughness index of realized volatility and the corresponding estimation error.

Example 5.3. *[Fractional OU process] Consider the following price process, where the volatility is described by a fractional OrnsteinUhlenbeck process:*

$$dS_t = \sigma_t S_t dB_t, \quad \sigma_t = \sigma_0 e^{Y_t}; \qquad dY_t = -\gamma Y_t dt + \theta dB_t^H, \tag{5.11}$$

where $\gamma = \theta = \sigma_0 = 1$, B is a Brownian motion and B^H fractional Brownian motion with Hurst index $H \in (0, 1)$.

We compute the realized volatility and compare the estimated roughness index $\widehat{H}_{L,K}$ (with, $L = 300 \times 300, K = 300$) of instantaneous and realized volatility in the following table.

H	Instantaneous volatility	Realized volatility
0.10	0.130	0.190
0.20	0.215	0.250
0.30	0.310	0.258
0.40	0.413	0.207
0.50	0.507	0.130
0.60	0.601	0.087
0.70	0.678	0.061
0.80	0.756	0.052

Table 5.2: Comparison of estimated roughness index for realized and instantaneous volatility

The corresponding pictures for price process, realized volatility and the instantaneous volatility from Model 5.11 with Hurst index $H = \{0.05, 0.1, 0.2, 0.3, 0.4, 0.5$

$, 0.6, 0.7, 0.8\}$ respectively are presented in Figure 5.11. Visually we can observe that for smaller H, the instantaneous volatility is rougher than realized volatility but as we increase H the realized volatility shows significantly rougher behaviour than the instantaneous volatility. In Figure 5.12 for the simulated models in Figure 5.11, we plot the estimated roughness index $\widehat{H}_{L,K}(RV)$ and $\widehat{H}_{L,K}(\sigma)$ respectively in the red and blue line. Though the estimated roughness index of instantaneous volatility (represented in blue line) gives an accurate estimate of Hurst index H, the roughness index for realized volatility always stays below 0.3. In particular, when the instantaneous volatility exhibits smoother behaviour (corresponding to $H \geq 0.5$) the estimated roughness index of realized volatility turns out to be a poor estimate for the Hurst index.

Figure 5.13 shows the corresponding estimators $\widehat{H}_{L,K}(RV)$ and $\widehat{H}_{L,K}(\sigma)$ for 100 independent simulated price paths from (5.11). The bold black lines represent the mean across 100 independent simulations whereas the dotted lines represent the corresponding 25% and 75% confidence intervals. For the price process (5.11), no matter what the value of the Hurst exponent for instantaneous volatility, the roughness index of realized volatility $\widehat{H}_{L,K}(RV)$ is always estimated to be between 0 to 0.3.

These examples illustrate our point: one *cannot draw the conclusion* that (spot) 'volatility is rough' i.e. reject the null hypothesis $H(\sigma) = 1/2$ just because realized volatility exhibits 'rough' behaviour with $\widehat{H}_{L,K}(RV) < \frac{1}{2}$ or $\widehat{H_R} < 1/2$, *even* when these estimators exhibit values well below 1/2.

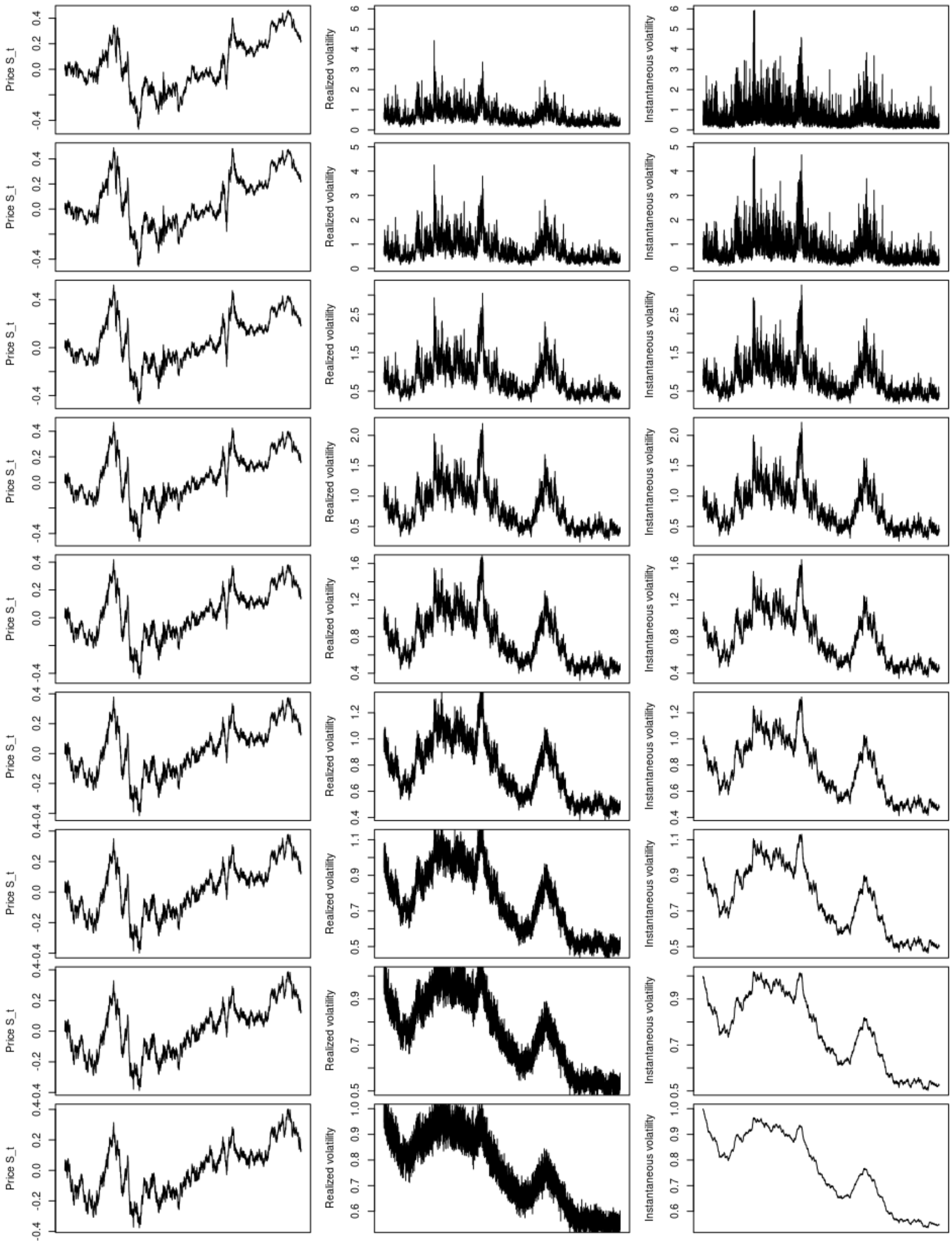

Figure 5.11: **Left:** Simulated price path S_t of fractional OU model (Equation 5.11) with H={0.05,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8} respectively, **Center:** Realized volatility, **Right:** Instantaneous volatility.

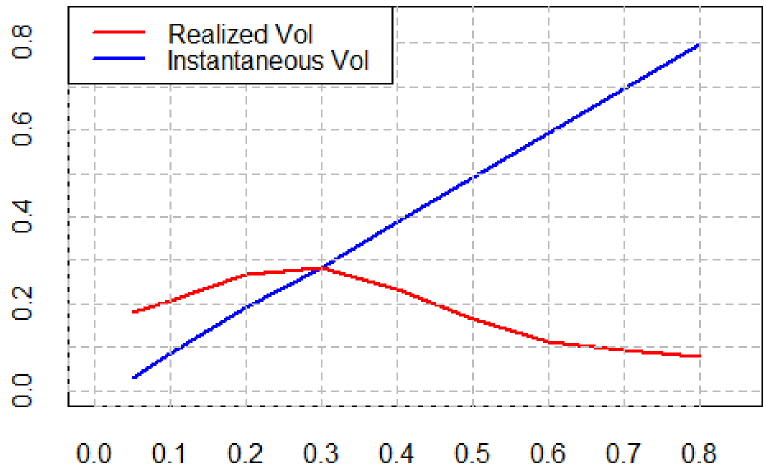

Figure 5.12: Estimated roughness index $\widehat{H}_{L=300\times300,K=300}$ for realized volatility and instantaneous volatility from a high-frequency fractional-OU stochastic volatility model (Equation (5.11)), plotted for price path generated with different values of H.

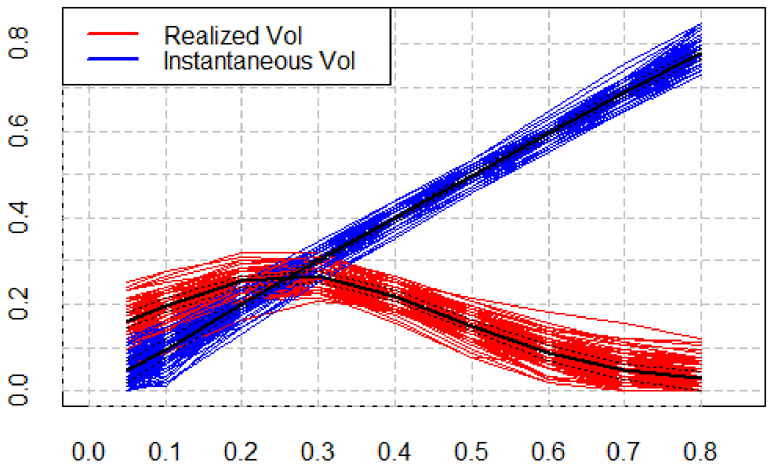

Figure 5.13: Estimated roughness index $\widehat{H}_{L=300\times300,K=300}$ for realized volatility and instantaneous volatility for a high-frequency simulation of a fractional-OU stochastic volatility model (5.11), plotted against different Hurst index H for 100 independent price paths.

5.4 Application to high-frequency financial data

Having extensively explored the performance of our roughness estimator $\widehat{H}_{L,K}$ based on the normalized p-th variation statistic for spot and realized volatility on simulated price paths, we now apply it to high-frequency financial time series.

5.4.1 AAPL

Figure 5.14 (left) shows the second-by-second record of AAPL stock price. The right graph of Figure 5.14 is $\log(W(L = 1900 \times 1900, K = 1900, \pi, p, t = 1, X))$ plotted against Hurst parameter $H = 1/p$ for 'AAPL' price.

The left plot of Figure 5.15 represents 1-minute realized volatility of 'AAPL' in 2016. The right graph of Figure 5.15 is $\log(W(L = 310 \times 310, K = 310, \pi, p, t = 1, X = RV))$ plotted against $H = 1/p$ for the 1-min AAPL realized volatility. Fixing the value of $L = 310 \times 310$, if we deviate the value of K a little, then the estimated roughness index varies between 0.08 to 0.22. This is consistent with the results of Gatheral et al. [42] regarding realized volatility. But as our simulation study suggests, the roughness index of realized volatility may be very different from that of spot volatility which is the quantity modelled in continuous-time stochastic volatility models.

5.4.2 SP500

Several studies on rough volatility, including the original study [42], are based on the Oxford-Man institute Realized Volatility dataset [2]. Figure 5.16 represents the plot of 5-minute realized volatility of SP500. The X-axis represents date. The right graph of Figure 5.16 is $\log(W(L = 70 \times 70, K = 70, \pi, p, t = 1, X = RV))$ plotted against Hurst parameter $H = 1/p$ for the 5-min Oxford-Man institute realized volatility data. Fixing the value of $L = 70 \times 70$, if we deviate the value of K a little, the estimated roughness index $\widehat{H}_{L,K}$ varies between 0.05 to 0.25. This finding is consistent with Gatheral's findings [42]. Because of the limited available data in the Oxford-Man Institute Realized Volatility data set, the value of L in the estimator can not be taken any higher.

[2] `https://realized.oxford-man.ox.ac.uk/data`

Overall, the picture that emerges from S&P500 and AAPL data is quite similar to the one observed in simulations of diffusion-type stochastic volatility models discussed in Section 5.3.1. As observed in Section 4.5, these observations are fully compatible with a diffusion-type stochastic volatility model such as (5.10) and one cannot reject the null hypothesis $H(\sigma) = 1/2$ just because realized volatility exhibits 'rough' behaviour with $\widehat{H}_{L,K}(RV) < \frac{1}{2}$ or $\widehat{H_R} < 1/2$, *even* though these estimators exhibit values around 0.1.

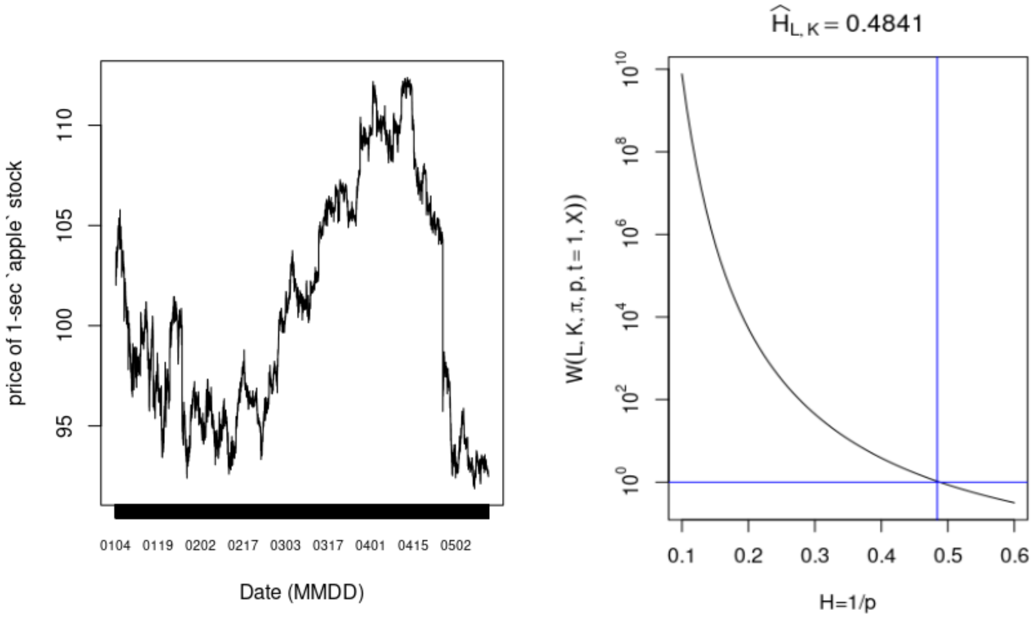

Figure 5.14: **Left:** price of AAPL 04/Jan/2016 - 11/May/2016 (90 days). **Right:** Roughness index estimator $\widehat{H}_{L,K}$(with $L = 1400 \times 1400$, $K = 1400$) for AAPL stock price data.

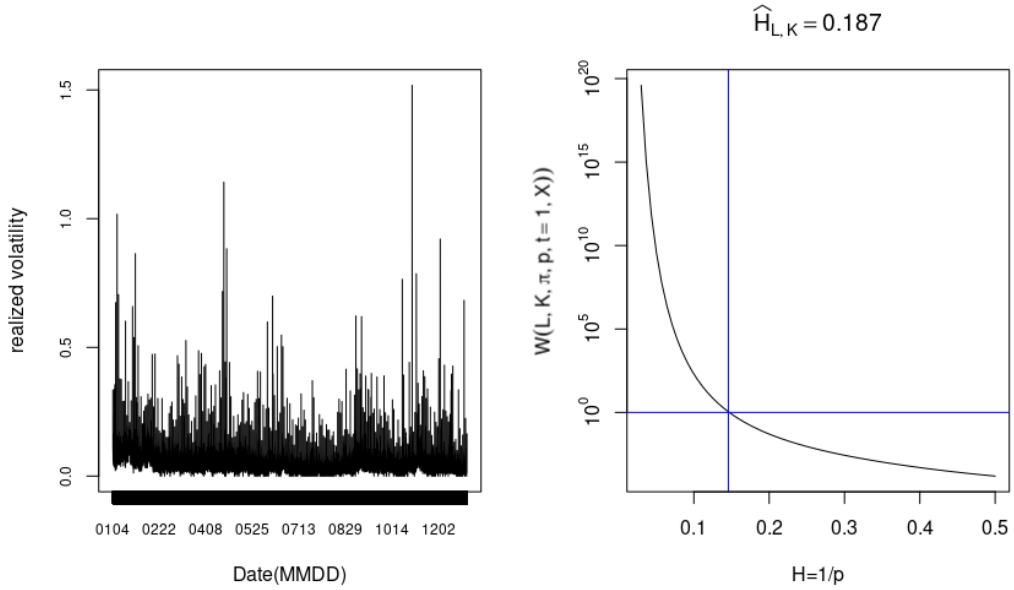

Figure 5.15: **Left:** Plot of 1-min realized volatility of 'AAPL' (year 2016). **Right:** Estimated roughness index $\widehat{H}_{L,K}$ (with $L = 310 \times 310, K = 310$) for the 1-min realized volatility (estimated roughness index $\widehat{H}_{L,K} \in [.08 - .22]$).

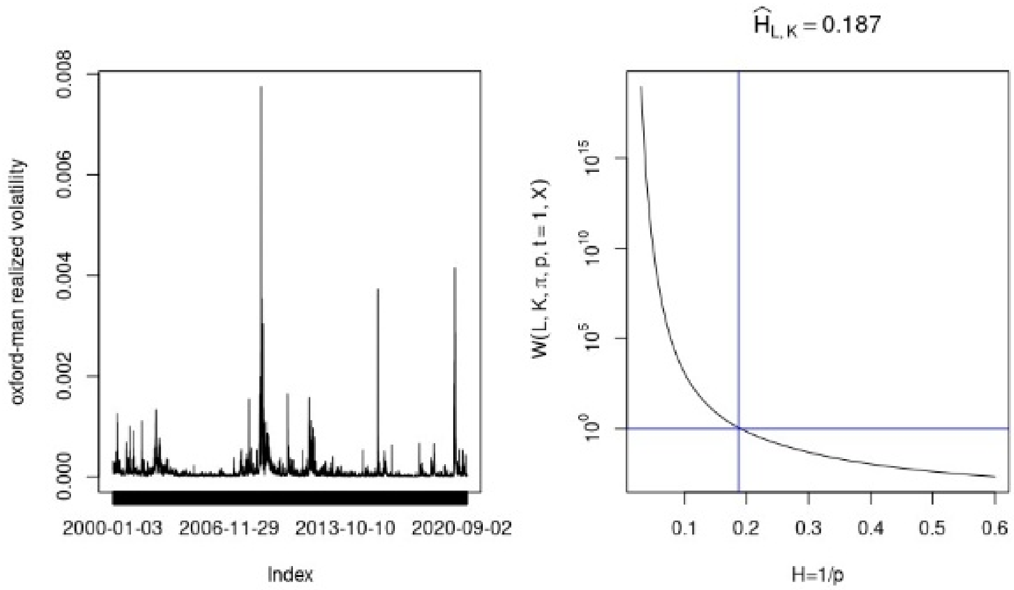

Figure 5.16: **Left:** S&P500 5-min realized volatility. **Right:** Estimated roughness index $\widehat{H}_{L,K} \in [.05 - .25]$ with $L = 70 \times 70, K = 70$.

5.5 Rough volatility ... or estimation error?

Given the large literature on 'rough volatility' in quantitative finance, it is somewhat surprising that the initial claim [42] that one needs to model the spot volatility process using a 'rough' fractional noise with Hurst exponent $H < 1/2$ has not been examined more closely, especially given that several follow-up studies [40, 66] point to the fact that the observations in [42] may well be compatible with a Brownian diffusion model for volatility.

Our detailed examples illustrate that, for stochastic-volatility diffusion models driven by Brownian motion as described in Examples 5.1 and 5.2, the realized volatility has a roughness index $\widehat{H_{L,K}} \approx 0.3$ so exhibits an 'apparent roughness' which instantaneous volatility does not have, both in terms of normalized p-th variation statistics and also in terms of the linear- regression method used by Gatheral et al. [42]. Clearly in these simulation examples this is entirely due to the discretization error or 'estimation error'.

These results suggest that one cannot hastily conclude that the roughness observed in realized volatility is an indicator of similar behaviour in spot volatility, as implicitly assumed in the 'rough volatility' literature; the observations in high-frequency financial data are in fact compatible with a stochastic volatility model drive by Brownian motion and the origin of this apparent roughness may very well lie in estimation error rather than the noise process driving spot volatility.

Also, as shown in Example 5.3, the rough behaviour of realized volatility does not lead us to reject the hypothesis that the underlying spot volatility may be modelled with a Brownian diffusion model or even a smoother model with long-range dependence and $H > 1/2$. This observation, together with Occam's razor", pleads for the use of Markovian stochastic volatility models which seem compatible with the empirical evidence but are far more tractable.

We are thus drawn to concur with Rogers [66] that "the notion that volatility is rough, that is, governed by a fractional Brownian motion (with $H < 1/2$), is not an incontrovertible established fact; simpler models explain the observations just as well."

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