A priori regularity results for discrete solutions to elliptic problems

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Abstract

This thesis is concerned with the development and analysis of a discrete counterpart of the well-known De-Giorgi-type regularity theory for solutions of elliptic partial differential equations in the setting of finite element approximations. We consider a finite element space consisting of piecewise affine functions on shape-regular triangulations of polyhedral Lipschitz domains $\Omega \subset \mathbb{R}^n$. We identify conditions for the mesh and the data to prove a global *a priori* Hölder-norm bound for finite element approximations of solutions to linear elliptic equations of the form $-\text{div}(A\nabla u) = f - \text{div}F$, where $A \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ is a uniformly elliptic matrix-valued function and $F \in L^p(\Omega; \mathbb{R}^n)$ and $f \in L^q(\Omega)$ are given functions for $p > n$ and $q > \frac{n}{2}$. After proving a Caccioppoli-type inequality for discrete subsolutions, we use iteration techniques to establish *a priori* L^∞- and Hölder-norm bounds. In particular, all estimates are proved for adaptively refined, highly graded meshes.

Next, we establish a local L^∞-norm bound for finite element approximations of solutions to p-Laplacian systems on non-obtuse meshes. Again, we do not require our mesh to be quasi-uniform but we do allow highly graded meshes. As there is no natural notion of a positive part for vector functions, we use different techniques than in the scalar, linear uniformly elliptic case.

We then apply the results to the finite element approximation of solutions to a system that describes the steady flow of a chemically reacting incompressible fluid. The convergence of a series of approximations was already established in the literature in two space dimensions, but the lack of a De-Giorgi-type regularity theory in the finite element setting prevented a direct generalisation to the physically relevant case of three space dimensions. We show that the theory that was developed in this thesis is under certain restrictions strong enough to overcome this limitation, thus enabling us to extend the convergence theory for finite element approximations of the problem to three space dimensions.

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Statement of Originality

This thesis is entirely my own work if not otherwise indicated. It has never been submitted for another degree of this university or any other institution. Chapter 3 and parts of Chapter 2 are based a joint paper with Endre Süli and Lars Diening that has been submitted to the IMA Journal of Numerical Analysis on 9 April 2020.

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Chapter 1

Introduction

In 1957, the nineteenth problem (*"Are all solutions of regular problems in the calculus of variations analytic?"*) from David Hilbert's famous list of 23 problems [41] presented at the Second International Congress of Mathematicians in Paris at the turn of the century was finally solved independently by Ennio de Giorgi [18] and John Nash [52]. As an intermediate result, De Giorgi proved that functions $u \in W^{1,2}$, which satisfy an inequality that estimates the norm of ∇u in terms of the norm of u locally on a level set are Hölder continuous. He then proved that solutions to elliptic equations satisfy this inequality, whereby he deduced local Hölder-continuity of weak solutions to linear equations. Hence, by using existing results of Hopf [42] and Morrey [51] to deduce higher regularity of solutions to elliptic partial differential equations and respective problems form the Calculus of Variations he thereby solved Hilbert's nineteenth problem.

Since then, the De-Giorgi-Nash-Moser estimate has been a well-known fact: Given a bounded domain $\Omega \subset \mathbb{R}^n$, a uniformly elliptic matrix-valued function $A \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ and functions $f \in L^q(\Omega; \mathbb{R})$ and $F \in L^p(\Omega; \mathbb{R}^n)$ with $q > \frac{n}{2}$ and $p > n$, weak solutions $u \in W_0^{1,2}(\Omega; \mathbb{R})$ to

$$\begin{aligned} -\text{div}(A\nabla u) &= f - \text{div}F \text{ on } \Omega, \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{1.1}$$

are in fact Hölder continuous. De Giorgi's iteration technique turned out to be flexible enough to be applied to a variety of non-linear or parabolic problems. For further references, we refer for example to [21] and [28].

While the finite element method is one of the most general and powerful techniques for the numerical approximation of solutions to partial differential equations, there is currently no discrete counterpart of the De Giorgi theory for elliptic boundary-value problems of the form (1.1) under the regularity hypotheses on the functions A, f and F stated above. Our first aim is to fill this gap by identifying conditions under which De-Giorgi-type regularity results hold in the discrete setting, for continuous piecewise affine finite element approximations of the problem (1.1).

First, we will survey the available literature. As a model problem, most authors study Poisson's equation:

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{1.2}$$

The first step on the way to proving Hölder regularity is usually a local L^∞-bound on the solution. The earliest result of this kind in the finite element literature was given by Nitsche in [54], where the author proved an $O(h)$ error bound in the L^∞-norm for equations of the form $-\text{div}(A\nabla u)+cu = f$ in two space dimensions with a uniformly elliptic $A \in W^{1,\infty}(\Omega, \mathbb{R}^{2\times 2})$, a nonnegative function $c \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$ on a convex domain Ω, subject to a homogeneous Dirichlet boundary condition, and a continuous, piecewise affine approximation on α–κ-regular triangulations of granularity $h > 0$. (i.e. the size of any angle is bounded below by α and the ratio of the side-lengths of any two triangles is bounded above by κ.) Then, Helfrich proved in [40] that this error bound was optimal in the case of $f \in L^2(\Omega)$ and piecewise affine approximations, and the order of convergence could not be improved. In [17], Ciarlet and Raviart extended Nitsche's result to n space dimensions with $f \in L^q(\Omega)$ and $q > \frac{n}{2}$ on regular simplicial subdivisions of *nonnegative type*. (In the case of Poisson's equation in two space dimensions, a triangulation is guaranteed to be of nonnegative type if all angles are non-obtuse.)

On a polygonal domain in $\mathbb{R}^2$ with a quasi-uniform mesh, with $f \in L^\infty(\Omega)$ and a finite element space consisting of piecewise affine functions, Natterer [53] proved a uniform regularity result in weighted Sobolev spaces to deduce an error bound of order $O(h^{2-\varepsilon})$ in the L^∞-norm. Soon thereafter, Schatz and Walbin [56] proved a local L^∞-best-approximation property. They analysed approximate solutions to

$$-\text{div}(a\nabla u) + b \cdot \nabla u + du = f$$

for smooth uniformly elliptic matrix-valued functions a, vector-valued functions b and scalar-valued functions c on a quasi-uniform and shape-regular mesh with a few additional technical assumptions on the finite element space that are, for example, satisfied by Lagrange or Hermite elements. Their final result is given by the following estimate between the analytical solution u and the finite element approximation u_h on a triangulation of Ω of granularity h:

$$\|u - u_h\|_{L^\infty(\Omega_1)} \le C \left(\left(\log \frac{1}{h} \right)^{\overline{r}} \|u - \chi\|_{L^\infty(\Omega)} + \|u - u_h\|_{W^{-p,q}(\Omega)} \right).$$

Here, $W^{-s,p}(\Omega)$ is the dual of $W_0^{s,p'}(\Omega)$ with $s > 0$, $1 < p < \infty$, and $\frac{1}{p} + \frac{1}{p'} = 1$, $\overline{r}$ is 1 if the optimal order in the mesh size h in which finite element functions can approximate L^q

functions is 2 (as is the case for a continuous, piecewise affine finite element approximation) or higher and 0 if this order is 3 or higher, χ is an arbitrary finite element function and $\Omega_1 \Subset \Omega$ and C is a positive constant independent of h and χ. Choosing $\chi \equiv 0$ yields a uniform L^∞-estimate, given L^∞-regularity of the solution to the continuous problem u. The methods of proof of these results are based on pointwise estimates for discrete Green's functions. This also implies that a generalization of those results to non-linear equations using the same technique is impossible.

An alternative way to prove uniform Hölder-norm bounds is obtaining $W^{1,p}$-norm bounds for $p > n$ first and then using the continuous embedding $W^{1,p}(\Omega) \hookrightarrow C^\alpha(\overline{\Omega})$ with $\alpha = 1 - \frac{n}{p}$. In two space dimensions and with piecewise affine elements, Rannacher and Scott [55] showed the $W^{1,p}$-stability of the Ritz-projection for $p \geq 2$ on triangulations which satisfy the condition that each triangle contains a circle of radius $c_1 h$ and is contained in a circle of radius $c_2 h$ with positive constants c_1 and c_2 (i.e. quasi-uniformity and shape-regularity). Therein, they used this result to deduce that the $W^{1,p}$-norm of the approximation error behaves like $O(h)$ and the L^p-norm behaves like $O(h^2)$, thereby eliminating the logarithmic factor that appeared in all previous results for piecewise affine elements.

If one considers the more general class of elliptic problems of the form $-\mathrm{div}(A\nabla u) = f - \mathrm{div}F$ with $A \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ a uniformly elliptic, matrix-valued function, $f \in L^{\frac{pn}{p+n}}(\Omega)$ and $F \in L^p(\Omega; \mathbb{R}^n)$, $p \in (1, \infty)$, subject to a homogeneous Dirichlet boundary condition, on a bounded open convex polyhedral domain in $n = 2$ or $n = 3$ space dimensions, one obtains a uniform $W^{1,p}$-norm bound on a sequence of finite element approximations on quasi-uniform triangulations for all $p \in (2, 2 + \varepsilon)$ with a possibly small ε as a direct consequence of Proposition 8.6.2 in [11] and Theorem 5.1 of [37]. However, this indirect approach does not yield a uniform Hölder-norm bound for a sequence of finite element approximations in more than $n = 2$ space dimensions.

A De-Giorgi-type iteration is used in the paper by Aguilera and Caffarelli [3] to prove an h-uniform C^α-estimate at least for continuous piecewise affine approximations of solutions at least to Laplace's equation $\Delta u = 0$. The authors assert without proof that their results generalize to more difficult equations. In this paper, a quasi-uniform, shape-regular and uniformly acute mesh is assumed.

As before, another approach to establish a uniform Hölder-norm bound is to first prove a uniform $W^{1,\infty}(\Omega)$-norm bound. In [38], the authors established a $W^{1,\infty}$-best-approximation property for finite element approximations of solutions to Poisson's equation (1.2). They assumed quasi-uniform meshes and a conforming finite element space S_h of piecewise polynomials of arbitrary degree $k \geq 1$ and a smooth right-hand side f to prove that

$$\|\nabla(u - u_h)\|_{L^\infty(\Omega)} \leq C \inf_{\chi \in S_h} \|\nabla(u - \chi)\|_{L^\infty(\Omega)}, \tag{1.3}$$

where $u_h \in S_h$ is the finite element approximation of the solution of equation (1.2) and u is the solution in the continuous case. By taking $\chi \equiv 0$ followed by the application of the triangle inequality, this implies that

$$\|\nabla u_h\|_{L^\infty(\Omega)} \leq C\|\nabla u\|_{L^\infty(\Omega)}.$$

Together with standard regularity theory for solutions to equation (1.2) with a smooth right-hand side f and the well known embedding $W^{1,\infty} \hookrightarrow C^\alpha$, this implies a uniform Hölder-estimate for u_h. However, a bound on $\|\nabla u\|_{L^\infty(\Omega)}$ does not always exist for all domains Ω and the results only hold for Poisson's equation.

If one assumes C^α-regularity of A in equation (1.1), an *a priori* C^α-bound and optimal error estimates for finite element approximations of solutions to elliptic systems are proved in [31]. The author uses local comparison arguments to prove estimates in Camapanato spaces to extend the results of [55] to elliptic systems. Again, the finite element space consists of piecewise affine functions. However, there is no C^α-regularity of solutions to equation (1.1) even in the continuous case for systems. This suggests that to derive estimates that only require $A \in L^\infty(\Omega; \mathbb{R}^{n\times n})$, we have to use a different technique.

All of the results that we cited above assumed a quasi-uniform triangulation of the domain Ω and exclude highly graded meshes. This excludes finite element schemes that are based on an adaptive refinement of the mesh. The strategy of the schemes is the following. One solves the problem on a coarse grid and then estimates the local errors. After that, one marks simplices based on those estimates and locally refines the mesh. In short, the scheme is given by the flow

$$\text{Solve} \to \text{Estimate} \to \text{Mark} \to \text{Refine}.$$

The first proof of linear error reduction for the Poisson problem $-\Delta u = f$ using this scheme was given in [32] by Dörfler. Therein, the author used piecewise affine elements and newest vertex bisection. The proposed marking strategy ensures that that the marked edges contribute at least to a fixed proportion of the global error estimator. However, the initial mesh had to be fine enough, dependent of the oscillations of the right-hand side f. In [50], the latter restriction was avoided by extending the marking strategy to not only ensure that sufficiently many simplices are marked such that they contribute to a fixed proportion of the error estimator, but also to a fixed proportion of the data oscillation error. The authors also include finite element approximations of solutions to the equation $-\text{div}(A\nabla u) = f$ where A

is assumed to be a piecewise constant, uniformly elliptic matrix-valued function. This was extended in [49] to include not only Lipschitz-continuous, uniformly elliptic matrix-valued functions A, but also lower order terms, i.e. finite element approximations to solutions to the equation $-\text{div}(A\nabla u) + b \cdot \nabla u + cu = f$.

To the best of our knowledge, there is no *a priori* regularity theory for finite element approximations to elliptic equations on highly graded meshes at the time this thesis was written. Here, we use the term *a priori* in the sense that we want to have knowledge about regularity properties of the sequence of approximate solutions, uniformly with respect to the mesh size, and uniform bounds on the respective norms of the sequence of approximate solutions without the need to compute said approximations. In [19], the authors obtain a best approximation property as in inequality (1.3) for solutions to Poisson's equation (1.2) for $f \in L^\infty(\Omega)$ on slowly varying (a technical condition that is stronger than shape-regularity but weaker than quasi-uniformity) meshes in 2 and 3 space dimensions. The approach is again based on Galerkin orthogonality and point-wise estimates for discrete Green's functions.

A well studied nonlinear generalisation of the Poisson problem is the p-Poisson problem that is given by

$$-\text{div}(|\nabla u|^{p-2}\nabla u) = f$$

for $p \in (1,\infty)$. Obviously, $|\nabla u|^{p-2}$ is neither uniformly elliptic nor bounded for $p \neq 2$ and the natural space for weak solutions is $W^{1,p}$. In [36], the authors use piecewise affine functions on triangulations of the domain $\Omega \subset \mathbb{R}^2$ where the diameter of the triangles is bounded above by h. They prove the strong convergence $u_h \to u$ as $h \to 0$ and the error estimate

$$\|u_h - u\|_{W^{1,p}(\Omega)} \leq C_1 \|u\|_{W^{2,p}(\Omega)}^{\frac{1}{p-1}} \|u\|_{W^{1,p}(\Omega)}^{\frac{p-2}{p-1}} h^{\frac{1}{p-1}}$$

for a constant C_1 and $p \geq 2$ and

$$\|u_h - u\|_{W^{1,p}(\Omega)} \leq C_2 \|u\|_{W^{2,p}(\Omega)}^{\frac{1}{3-p}} \|u\|_{W^{1,p}(\Omega)}^{\frac{2-p}{3-p}} h^{\frac{1}{3-p}}$$

a constant C_2 and $1 < p < 2$. Note that this requires $u \in W^{2,p}(\Omega)$, which is not guaranteed for general data f. However, it is satisfied in a wide range of cases. Those estimates were improved in [16]. Therein, the authors assumed a slightly more general class of degenerate or singular elliptic equations that includes the p-Laplacian and a conforming finite element space of piecewise polynomials. They proved the estimates

$$\|u - u_h\|_{W^{1,p}(\Omega)} \leq C\|u - v_h\|_{W^{1,p,}(\Omega)}^{s}$$

for all v_h in the respective finite element space with $s = \frac{p}{2}$ for $1 < p < 2$ and $s = \frac{2}{p}$ for $p > 2$. If we assume piecewise affine functions and $u \in W^{2,p}(\Omega)$, this implies that there is a constant C such that

$$\|u - u_h\|_{W^{1,p}(\Omega)} \le Ch^s$$

by the Bramble-Hilbert Lemma. (See [11][Lemma 4.3.8].)

Barrett and Liu achieved a major breakthrough in the analysis of the finite element approximation of solutions to the p-Poisson problem in [7]. Therein, the authors found that the natural quantity to estimate the error is given by the quasi-norm

$$|v|_{(p,\sigma)} = \left(\int_\Omega (|\nabla u| + |\nabla v|)^{p-\sigma} |v|^\sigma \, \mathrm{d}x \right)^{\frac{1}{p}}.$$

The authors use a finite element space of piecewise affine functions V_h in two space dimensions. The main error estimate follows from the approximation property given by

$$|u - u_h|_{(p,2+\delta_2)} \le C|u - v_h|_{(p,2-\delta_2)}$$

for any $\delta_1 \in [0, 2)$, $\delta_2 \ge 0$ and $v_h \in V_h$, where u is the solution to the continuous problem and $u_h \in V_h$ its finite element approximation, and v_h is any element of the finite element space. For $1 < p < 2$ this leads to optimal $W^{1,p}(\Omega)$-error bounds

$$\|u - u_h\|_{W^{1,p}(\Omega)} \le Ch$$

if $u \in W^{3,1}(\Omega) \cap C^{2,\frac{2-p}{p}}(\overline{\Omega})$. In the degenerate case $p > 2$, the authors obtain the optimal convergence order for the error in the $W^{1,\frac{4}{3}}(\Omega)$-norm if $u \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$. Building on works of Babuška and others (see [6] and [5]), Ainsworth and Kay analysed the convergence of the p-method for the approximation of solutions to the p-Poisson problem in $n = 2$ space dimensions in [4]. (Here, the p in p-method means the polynomial degree of the elements and is not to be confused with the exponent p in the p-Laplacian.) The p-method refers to increasing the polynomial degree of the elements instead of decreasing the granularity of the partition (referred to as the h-method). The rate of convergence in terms of the number of degrees of freedom is shown to be at least equal to that of the h-method while achieving exponential convergence in the case of smooth solutions. This was primarily achieved by extending the approximation theory to L^p-spaces. The first convergence result for an adaptive algorithm for the p-Poisson problem was given in [60]. The author used finite element spaces consisting of piecewise affine functions in n space dimensions on shape-regular meshes. The only discrete regularity result that was used therein is the stability estimate $\|\nabla u_h\|_{L^p(\Omega)} \le C\|f\|_{W^{-1,p'}(\Omega)}$ that follows directly from the variational formulation of the p-Poisson problem. First steps towards establishing results on adaptive methods for approximations of solutions of the p-Poisson problem have been made in [46], where the authors proved local *a posteriori* error estimators from above and below that may be used for the *mark* step of an adaptive scheme. In [24], a sharper semi-norm *a posteriori* error estimate is proved to establish a linear error reduction result for an adaptive algorithm to approximate solutions to the φ-Laplacian equation $-\mathrm{div}\left(\frac{\varphi(|\nabla u|)}{|\nabla u|}\nabla u\right) = f$, where φ is an N-function that satisfies the Δ_2-condition and $\varphi'(t) \sim t\varphi''(t)$. (For precise definitions, see Definition 4.12.) In [10], this was further improved by showing quasi-optimality and avoiding an *interior node condition*, which demanded that every marked triangle and all of its faces contain a new node. Little seems to be known about *a priori* regularity results to finite element approximations of solutions to p- or φ-Laplacian equations or systems. In [25], a discrete maximum principle for piecewise affine approximation on non-obtuse meshes was given.

Let us now briefly discuss our own results. In Theorem 3.25, we will prove a uniform *a priori* Hölder estimate on discrete solutions to $-\mathrm{div}(a\nabla u_h) = f - \mathrm{div}F$. On the one hand, we allow adaptively refined, highly graded meshes and only require shape-regularity, which is an improvement over [19]. We also only require $A \in L^\infty(\Omega; \mathbb{R}^{n\times n})$; in particular, we will not assume Hölder continuity of A as in [31]. Furthermore, the De-Giorgi-type iteration technique is flexible enough to include scalar, uniformly elliptic nonlinearities, see Theorem 3.30. Therefore, our results are a first step towards similar estimates for more complex, non-linear problems. On the other hand, we have a few restrictions. The theorem is only valid for finite element spaces of piecewise affine functions. Additionally, we shall assume that there is a function $G \in L^p(\Omega; \mathbb{R}^n)$, such that $-\mathrm{div}G \geq |\mathrm{div}F|$ in the sense of distributions (cf. Assumption $\star$ in Definition 3.10). This assumption is satisfied in a number of non-trivial cases, but it is restrictive, in particular because it implies that $\mathrm{div}F$ has to be a signed measure. We also require the mesh to be A-non-obtuse. Even with A being the identity matrix, which means that A-non-obtuseness becomes the regular non-obtuseness, this condition is still restrictive. In general, the common algorithms (for example the one proposed in [57]) for local mesh refinement produce obtuse angles. However, avoiding the assumption of uniform acuteness as in [3] allows us to include important special cases such as the n-dimensional hypercube with a Kuhn-simplex triangulation that gets locally refined.

We will prove a local L^∞-estimate for discrete solutions to systems of the form

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f - \mathrm{div}F$$

in Theorem 4.25. We are able to allow fully adaptive refinement of the mesh, but still need the triangulation to be non-obtuse. If we assume a uniformly acute triangulation of the domain, we also find local L^∞-estimates to φ-Laplacian systems of the form $-\mathrm{div}\left(\frac{\varphi'(|\nabla u|)}{|\nabla u|}\nabla u\right) =$

0. This is not only an interesting result in itself, but the locality of the estimates suggests that it may be a first step towards local L^∞-error estimates for nonlinear elliptic problems. However, the derivation of those error estimates is beyond of the scope of this project.

In Chapter 5, we will apply a generalized version of Theorem 3.30 to the finite element scheme that was proposed in [43] to approximate solutions to a system that models the flow of a chemically reacting, non-Newtonian fluid, that is given by

$$\mathrm{div} u = 0, \tag{1.4}$$

$$\mathrm{div}(u \otimes u) - \mathrm{div}\,(S(c, Du)) = \nabla p + f, \tag{1.5}$$

$$\mathrm{div}(cu) - \mathrm{div}(k(Du, c)\nabla c) = 0. \tag{1.6}$$

where $u : \Omega \to \mathbb{R}^n$ is the velocity field, $p : \Omega \to \mathbb{R}$ is the pressure field and $c : \Omega \to \mathbb{R}$ is the concentration of hyluronan. For more details and precise definitions, we refer to Section 5.1. In Theorem 5.12, we will extend Theorem 3.25 to include equations with lower order terms on uniformly acute meshes. This will allow us to prove a De-Giorgi-type estimate for c as a finite element approximation to a solution to equation (1.6). This allows us to generalise the convergence result of [43] to include $n = 3$ space dimensions without the regularisation that was necessary in [44].

Chapter 2

Preliminaries

In this chapter, we will introduce function spaces, mesh conditions and some other definitions which will be needed in the thesis. The first section contains a brief overview of classical properties of Lebesgue and Sobolev spaces. In the second section, we will introduce the finite element spaces that we will be working with and prove some important properties of these spaces.

2.1 Lebesgue and Sobolev spaces

We will now review some important elementary definitions and properties of Lebesgue and Sobolev spaces. First, we will introduce a few notational conventions.

For two non-negative expressions a and b, we will write $a \lesssim b$ if there is a positive constant C such that $a \leq Cb$. We will write $a \gtrsim b$ if there is a constant $c > 0$ such that $a \geq cb$. If we have $a \lesssim b$ and $a \gtrsim b$, we will write $a \sim b$. If this notation is used within a proof, the implicit constants are allowed to depend on the quantities that the constants in the respective theorem or lemma depend on. If this notation is used outside of proofs, it will be either clear from the context or stated explicitly, what the implicit constants depend on. The maximum of two real numbers x and y will be denoted either by $\max\{a, b\}$ or in short by $a \vee b$. The positive part of an expression u will be denoted by $u_{\dotplus} := u \vee 0$. (Note that $\dotplus$ is not the standard notation for the positive part. In Definition 3.9, we will introduce the nodal positive part of a piecewise affine function and denote it with the standard $+$ because this notion will be used more frequently than the point-wise expression that we shall denote by $\dotplus$.) For a measurable set $A \subset \mathbb{R}^n$, we will denote its Lebesgue-measure by $|A|$ and its characteristic function by χ_A. We will start by providing a precise definition of a Lipschitz domain.

Definition 2.1 (cf. [2] 4.9)**.** *A bounded domain $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain, if for all $x \in \partial\Omega$, there is a neighbourhood U_x such that $U_x \cap \partial\Omega$ is the graph of a Lipschitz continuous* *function.*

Next, we need the spaces of p-integrable functions

Definition 2.2 (cf. [2] Chapter 2)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain. For $p \geq 1$, a measurable function $f : \Omega \to \mathbb{R}^m$ is p-integrable, if*

$$\int_\Omega |f|^p \,\mathrm{d}x < \infty.$$

The space of equivalence classes of almost everywhere equal p-integrable functions is denoted by $L^p(\Omega; \mathbb{R}^m)$. If the target space or the domain are clear, we will omit those. We will use equivalence classes and representatives of those classes interchangeably. Equipped with the norm

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p \,\mathrm{d}x \right)^{\frac{1}{p}},$$

the space L^p is a Banach space. The space of essentially bounded functions is denoted by L^∞, its norm is given by

$$\|f\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x \in \Omega} |f(x)|.$$

We will use the terms sup *and* ess sup *interchangeably.*

We will also need the notions of weak derivative and Sobolev spaces.

Definition 2.3 (cf. [2], Chapter 3)**.** *Given $p \geq 1$ and an open set Ω, a function $f \in L^p(\Omega; \mathbb{R}^m)$ is weakly partially differentiable in the direction x_i, if there is are measurable functions $\partial_{x_i} f : \Omega \to \mathbb{R}^m$ for all $i \in \{1, \dots, n\}$, such that*

$$\int_\Omega f \cdot \partial_{x_i} \varphi \,\mathrm{d}x = - \int_\Omega (\partial_{x_i} f) \cdot \varphi \,\mathrm{d}x$$

for all $\varphi \in C_0^\infty(\Omega, \mathbb{R}^m)$. A function $f : \Omega \to \mathbb{R}^m$ belongs to the Sobolev space $W^{k,p}$ if f and all its weak partial derivatives are in $L^p(\Omega)$. The norm is given by

$$\|f\|^p_{W^{k,p}(\Omega)} = \sum_{|\alpha| \leq k} \|\partial^\alpha f\|^p_{L^p(\Omega)}.$$

Furthermore, we will denote the completion of $C_0^\infty(\Omega)$ under the $W^{k,p}$-norm by $W_0^{k,p}(\Omega)$.

We will now collect a few useful facts and basic properties of those spaces.

- Young's inequality: $ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$ for $a, b \geq 0$, $1 < p < \infty$ and $1 = \frac{1}{p} + \frac{1}{p'}$.
- Hölder's inequality: $\|fg\|_{L^p(\Omega)} \leq \|f\|_{L^{p_1}(\Omega)} \|g\|_{L^{p_2}(\Omega)}$ for $p, p_1, p_2 \in [1, \infty]$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

- Density (Meyers–Serrin): For $1 \le p < \infty$ and $k \in \mathbb{N}$, we have
$W^{k,p}(\Omega) = \overline{C^\infty(\Omega) \cap W^{k,p}(\Omega)}^{\|\cdot\|_{W^{k,p}(\Omega)}}$.

Next, we will give the two most important versions of Poincaré's inequality. For a proof, see for example [2][Theorem 6.30].

Theorem 2.4 (Poincaré's inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz-domain that is bounded in at least one direction. Then, there exists a constant $C > 0$ that depends on the domain and $p \in [1, \infty)$, such that*

$$\int_\Omega |u|^p \, \mathrm{d}x \le C \int_\Omega |\nabla u|^p \, \mathrm{d}x$$

for all $u \in W_0^{1,p}(\Omega)$.

One of the most general type of domains where a Poincaré-type inequality holds is a so called α-John domain. We will cite this definition from [27]. First we shall define α-cigars and α-carrots.

Definition 2.5. *Given a rectifiable path γ that is parametrised by arch-length and an angle $\alpha > 0$, we define the α-cigar along γ as*

$$\operatorname{cig}(\gamma, \alpha) := \bigcup_{t \in [0, |\gamma|]} \left\{ B(\gamma(t), \frac{1}{\alpha} \min\{t, |\gamma| - t\}) \right\}$$

and the α-carrot along γ as

$$\operatorname{car}(\gamma, \alpha) := \bigcup_{t \in [0, |\gamma|]} \{B(\gamma(t), \alpha t)\}.$$

An example of an α-cigar is shown later in Figure 2.5 and an example of an α-carrot is shown later in Figure 2.4.

Definition 2.6. *A domain $\Omega \subset \mathbb{R}^n$ is called an α-John domain, $\alpha > 0$, if every pair of distinct points $a, b \in \Omega$ can be joined by a rectifiable path γ such that*

$$\operatorname{cig}(\gamma, \alpha) \subset \Omega.$$

Remark 2.7. *Obviously, any bounded convex set is an α-John domain.*

Theorems 3.8 and 5.1 of [27] then prove the following proposition.

Proposition 2.8. *Let Ω be a bounded α-John domain for some $\alpha > 0$. Then, for all $f \in W^{1,q}(\Omega)$ holds*

$$\|f - \langle f \rangle_\Omega\|_{L^q(\Omega)} \le C \operatorname{diam}(\Omega) \|\nabla f\|_{L^q(\Omega)}.$$

The constant C only depends on α, n and q.

Proof. Theorem 3.12 of [27] guarantees that Ω satisfies an emanating chain condition with constants only depending on Ω. Then Theorem 5.1 of [27] guarantees that Poincaré's inequality is true in weighted Sobolev spaces. As we have chosen the weight $w \equiv 1$, the dependence of the constant on the weight can be dropped. □

Fore more details on the Poincaré constant of star shaped sets that arise in the context of finite element methods, see [61]. Following arguments from [23][Lemma 8.2.3], we will now show that it is not necessary to subtract the average over the entire domain. In fact, it is enough to subtract the average over a subset $\omega \subset \Omega$ as long as $\frac{|\omega|}{|\Omega|}$ is bounded below.

Corollary 2.9. *Let Ω be a bounded α-John domain. Furthermore, let $\omega \subset \Omega$ be a measurable set with $\frac{|\omega|}{|\Omega|} \geq \beta > 0$. Then, there is a $C > 0$, such that for all $f \in W^{1,q}(\Omega)$, we have*

$$\|f - \langle f\rangle_\omega\|_{L^q(\Omega)} \leq C\left(1 + \frac{1}{\beta}\right)\operatorname{diam}(\Omega)\|\nabla f\|_{L^q(\Omega)}.$$

C depends only on α, n, q.

Proof. By the triangle inequality, we have

$$\|f - \langle f\rangle_\omega\|_{L^q(\Omega)} \leq \|f - \langle f\rangle_\Omega\|_{L^q(\Omega)} + \|\langle f\rangle_\omega - \langle f\rangle_\Omega\|_{L^q(r)}. \tag{2.1}$$

From Proposition 2.8, we know that

$$\|f - \langle f\rangle_\Omega\|_{L^q(\Omega)} \leq C\operatorname{diam}(\Omega)\|\nabla f\|_{L^q(\Omega)}, \tag{2.2}$$

where C is the constant from Proposition 2.8 that only depends on α, n and q. On the other hand, we find

$$\|\langle f\rangle_\omega - \langle f\rangle_\Omega\|_{L^q(r)} = |\langle f\rangle_\omega - \langle f\rangle_\Omega||\Omega|^{\frac{1}{q}} \tag{2.3}$$

We estimate

$$\begin{aligned} |\langle f\rangle_\omega - \langle f\rangle_\Omega| &= \left|\fint_\omega f\,\mathrm{d}x - \fint_\omega \langle f\rangle_\Omega\,\mathrm{d}x\right| \\ &= \frac{|\Omega|}{|\omega|}\frac{1}{|\Omega|}\left|\int_\omega f - \langle f\rangle_\Omega\,\mathrm{d}x\right| \\ &\leq \frac{1}{\beta|\Omega|}\int_\Omega |f - \langle f\rangle|\,\mathrm{d}x \\ &\leq \frac{1}{\beta|\Omega|^{\frac{1}{q}}}\|f - \langle f\rangle_\Omega\|_{L^q(\Omega)} \end{aligned} \tag{2.4}$$

where we have used Hölder's inequality in the last step. Combining inequalities (2.2), (2.3) and (2.4) yields

$$\|\langle f\rangle_\omega - \langle f\rangle_\Omega\|_{L^q(r)} \leq \frac{C}{\beta}\operatorname{diam}(\Omega)\|\nabla f\|_{L^q(\Omega)}. \tag{2.5}$$

Now combining inequalities (2.1), (2.2) and (2.5) proves the Corollary. □

The next function space that we will need is the space of Hölder continuous functions. Let us write

$$\operatorname*{osc}_A f := \sup_{x \in A,\, y \in A} |f(x) - f(y)|$$

for the oscillations of a function $f : \Omega \to \mathbb{R}^m$ on a set $A \subset \Omega$. For $m = 1$, this simplifies to

$$\operatorname*{osc}_A f = \sup_A f - \inf_A f.$$

Definition 2.10. *We will say a function $f : \Omega \to \mathbb{R}^m$ is Hölder continuous with the exponent α, if*

$$\sup_{x,y \in \Omega} \frac{|f(x) - f(y)|}{|x-y|^\alpha} \le C \tag{2.6}$$

for some constant C. The smallest constant C that satisfies this inequality for a function f will be denoted by $|f|_{C^\alpha(\overline{\Omega})}$. Note that $|\cdot|_{C^\alpha(\overline{\Omega})}$ is a seminorm on the space of Hölder-continuous functions $C^\alpha\left(\overline{\Omega}\right)$. Inequality (2.6) can be simplified to

$$\operatorname*{osc}_{B(x_0,R)} f \le C R^\alpha$$

for all $x_0 \in \Omega$ and $R > 0$ with $B(x_0, R) \subset \Omega$.

Finally, we will give the most important embedding theorems.

Theorem 2.11. *Suppose that Ω is a bounded open Lipschitz domain. The following embeddings are continuous:*

- $W^{1,p}(\Omega) \hookrightarrow L^{p^\star}(\Omega)$ *with* $\frac{1}{p} = \frac{1}{p^\star} + \frac{1}{n}$ *and* $1 \le p < n$,
- $W^{1,p}(\Omega) \hookrightarrow C^\alpha\left(\overline{\Omega}\right)$ *for* $n < p \le \infty$ *and* $0 \le \alpha \le 1 - \frac{n}{p}$.

The first embedding will be referred to as the Sobolev embedding, the second one as Morrey's embedding. Furthermore, the following embeddings are compact:

- $W^{1,p}(\Omega) \hookrightarrow L^{p^\star}(\Omega)$ *with* $\frac{1}{p} < \frac{1}{p^\star} + \frac{1}{n}$ *and* $1 \le p < n$,
- $W^{1,n}(\Omega) \hookrightarrow L^p(\Omega)$ *for all* $p \in [1, \infty)$,
- $C^\alpha\left(\overline{\Omega}\right) \hookrightarrow C^\beta\left(\overline{\Omega}\right)$ *for* $\alpha > \beta$.

For more details see [2], Chapters 4 and 6.

2.2 Finite element spaces, triangulations and mesh conditions

This section is based on the third the section of a joint paper with Lars Diening and Endre Süli, entitled *Uniform Hölder-norm bounds for finite element approximations of second-order elliptic equations*, submitted to the IMA Journal of Numerical Analysis on 9 April 2020, and available from arXiv:2004.09341 [math.NA]. In this section, we will introduce common mesh conditions that will be used throughout and prove properties of the corresponding triangulations. We will always assume that $\Omega \subset \mathbb{R}^n$ is a polyhedral Lipschitz domain and $\mathcal{T}_h$ is a triangulation of that domain. Here, by a triangulation of Ω we mean a subdivision of $\overline{\Omega}$ into closed n-dimensional simplices with pairwise disjoint interiors, whose union is $\overline{\Omega}$ and the intersection between two simplices is either empty or a complete sub-simplex. The (0-dimensional) vertices of the simplices will be referred to as nodes, the 1-dimensional intersections between two simplices will be referred to as edges and the $n-1$-dimensional ($n \geq 3$) intersections between two simplices will be referred to as faces. First, we will define the Lagrange basis for a triangulation.

Definition 2.12. *Let $\mathcal{T}_h$ be a triangulation of the polyhedral Lipschitz domain $\Omega \subset \mathbb{R}^n$. For a node x_i of the triangulation $\mathcal{T}_h$, we define the associated Lagrange basis function ψ_i via $\psi_i(x_j) = \delta_{i,j}$ and ψ_i is an affine function of n variables on any $T \in \mathcal{T}_h$.*

Now, we will define different mesh conditions.

Definition 2.13. *We call a triangulation $\mathcal{T}_h$ shape-regular with shape-regularity parameter $\Gamma > 1$, if one has, for each $T \in \mathcal{T}_h$,*

$$h_T \leq R_{i,T}\Gamma, \tag{2.7}$$

where $R_{i,T}$ is the radius of the largest n-dimensional ball contained in T (which we shall refer to as the inscribed ball of T) and $h_T := \operatorname{diam} T$.

Next, we will introduce two important notions: *A-nonobtuseness* and *uniform A-acuteness.*

Definition 2.14. *Let $A \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ be a uniformly elliptic matrix-valued function. We call a triangulation $\mathcal{T}_h$ of Ω A-non-obtuse, if*

$$\int_T A\nabla\psi_i \cdot \nabla\psi_j \,\mathrm{d}x \leq 0 \tag{2.8}$$

for any $T \in \mathcal{T}_h$ where ψ_i and ψ_j are the Lagrange basis functions of V_h that are associated with the nodes x_i and x_j (see Definition 2.12) and for any $i \neq j$.

We will call a triangulation $\mathcal{T}_h$ uniformly A-acute with the acuteness constant γ if the estimate

$$\int_T A\nabla\psi_i \cdot \nabla\psi_j \,\mathrm{d}x \leq -\gamma \|\nabla\psi_i\|_{L^2(T)} \|\nabla\psi_j\|_{L^2(T)} \tag{2.9}$$

holds for any $T \in \mathcal{T}_h$ and any $i \neq j$ with $T \subset \operatorname{supp}\psi_i \cap \operatorname{supp}\psi_j$.

Note that if A is the product of a scalar function and the identity matrix, this definition coincides with the geometric idea of a non-obtuse triangulation. The existence and construction of uniformly A-acute and A-non-obtuse triangulations is discussed in detail in [15].

We will now recall a few properties of shape-regular families of triangulations $\mathcal{T}_h$. For any simplex $T \in \mathcal{T}_h$, there is an invertible affine transformation B_T with bounded inverse that maps T onto the simplex in $\mathbb{R}^n$ with the nodes $(0, \dots, 0)$, $(h_T, 0, \dots, 0)$, $\dots$, $(0, \dots, 0, h_T)$. The norms of the gradients of B_T and its inverse are uniformly bounded for all $T \in \mathcal{T}_h$ and the bound only depends on the shape-regularity parameter Γ. Furthermore, for any node x_i of a simplex $T \in \mathcal{T}_h$ we have that

$$|\nabla\psi_i| \sim h_T^{-1}, \tag{2.10}$$

where the implicit constant only depends on the shape-regularity constant Γ and the dimension n. Furthermore, if $T \in \mathcal{T}_h$ and $S \in \mathcal{T}_h$ have a nonempty intersection, we have

$$h_T \sim h_S. \tag{2.11}$$

Again, the implicit constant in equation (2.11) only depends on the shape-regularity constant and the dimension. For simplicity, for any set A contained in $\overline{\Omega}$, we will write

$$\overline{\Omega(A)} := \bigcup_{T \in \mathcal{T}_h \,:\, T \cap A \neq \emptyset} T,$$

and the interior of that set is then

$$\Omega(A) := \overline{\Omega(A)} \setminus \partial\overline{\Omega(A)}.$$

In particular, we will write $P_i := \Omega(\{x_i\})$ for any patch around a node x_i. We will also write

$$\Omega'(A) := \bigcup_{i \,:\, x_i \in A} P_i. \tag{2.12}$$

In short, $\Omega(A)$ is the union of all simplices that touch A whereas $\Omega'(A)$ is the union of all simplices that have a node in A.

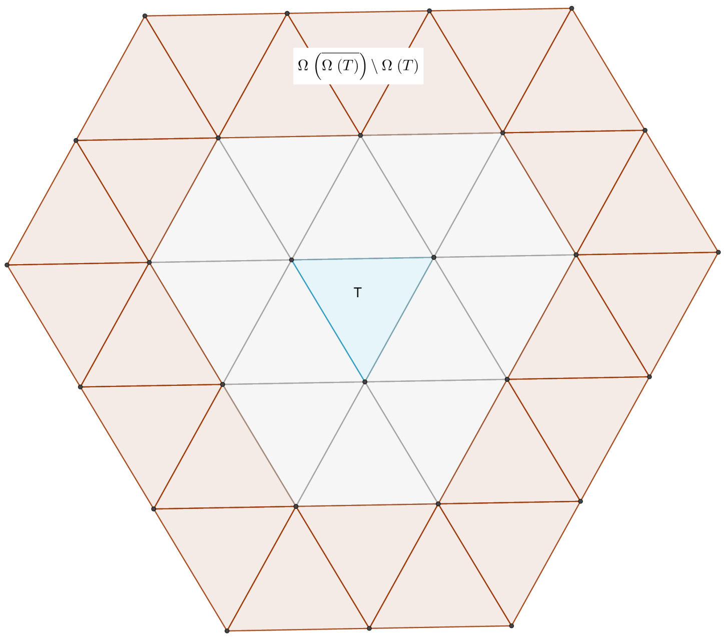

Figure 2.1: T and $\Omega(\overline{\Omega(T)})$

Lemma 2.15. *Let $\mathcal{T}_h$ be a shape-regular triangulation of Ω. Let $T \in \mathcal{T}_h$ be a simplex and $h_T = \operatorname{diam} T$. Then, there exists a $\sigma > 0$ that only depends on the shape-regularity constant Γ and the dimension, such that*

$$\operatorname{dist}\left(T, \Omega(\overline{\Omega(T)}) \setminus \Omega\left(T\right)\right) \geq \sigma h_T.$$

Note that $\operatorname{dist}\left(T, \Omega(\overline{\Omega(T)}) \setminus \Omega\left(T\right)\right) = \operatorname{dist}\left(T, \Omega \setminus \Omega\left(T\right)\right)$ *if* $\overline{\Omega(T)} \cap \partial\Omega = \emptyset$.

Proof. Given $T \in \mathcal{T}_h$, let $S_j \in \mathcal{T}_h$, $j = 1, \ldots, N$, be the collection of all those simplices that have non-empty intersection with T. Thanks to the shape-regularity of $\mathcal{T}_h$ the integer N only depends on the shape-regularity parameter Γ. As $\operatorname{dist}\left(T, \Omega(\overline{\Omega(T)}) \setminus \Omega\left(T\right)\right) \geq \min_{j=1,\ldots,N} R_{i,S_j}$, where R_{i,S_j} is the radius of the maximal ball inscribed in S_j, the assertion follows from shape-regularity, which implies that $R_{i,S_j} \sim h_{S_j}$, and the equation (2.11), which guarantees that $h_{S_j} \sim h_T$ for all $j = 1, \ldots, N$. See also Figure 2.1. □

We want to be able to deal with domains that are not convex. For example in Pacman-domains we could run into situations, where $\Omega(x_0, B)$ contains two different connected components. We will only want to look at the connected component that contains the centre x_0.

Definition 2.16. *For $x_0 \in \Omega$ and $R > 0$, we write $\mathcal{B}(x_0, R)$ for the connected component of $B(x_0, R) \cap \Omega$ that contains x_0. Note that for $B(x_0, R) \subset \Omega$, we have $B(x_0, R) = \mathcal{B}(x_0, R)$.*

Lemma 2.15 is essential to prove the following statement.

Lemma 2.17. *Let $\mathcal{T}_h$ be a shape-regular triangulation of the polyhedral domain Ω. Suppose that $x_0 \in T$ for some $T \in \mathcal{T}_h$ and $R \geq h_T$. There exists a number $Q > 1$ that only depends on the shape-regularity parameter Γ, the Lipschitz constant of the boundary of Ω and the dimension, such that*

$$\Omega(\mathcal{B}(x_0, R)) \subset \mathcal{B}(x_0, QR). \tag{2.13}$$

Proof. Each simplex $S \in \mathcal{T}_h$ is a bounded set; there is therefore an $\tilde{R} > 0$ such that $\Omega(\mathcal{B}(x_0, R)) \subset \mathcal{B}(x_0, \tilde{R})$. If we denote the radius of the circumscribed ball of $S \in \mathcal{T}_h$ by $R_{c,S}$ and use shape-regularity as defined in Definition 2.13, we find that

$$\tilde{R} \leq R + \max_{S \in \mathcal{T}_h : S \cap T \neq \emptyset} R_{c,S} \lesssim R + \max_{S \in \mathcal{T}_h : S \cap T \neq \emptyset} h_S. \tag{2.14}$$

If $S \subset \overline{\Omega\,(T)}$, we have $S \cap T \neq \emptyset$ by definition, and equation (2.11) and the assumption $R \geq h_T$ therefore yield

$$h_S \leq \tilde{\sigma} h_T \leq \tilde{\sigma} R \tag{2.15}$$

for some $\tilde{\sigma}$ that only depends on the shape-regularity constant and the dimension.

We denote the length of the shortest path between S and x_0 in $\mathcal{B}(x_0)$ by $d_\Omega(x_0, S)$. If $S \subset \overline{\Omega \setminus \Omega(T)}$, we also have $x_0 \in \Omega \setminus \Omega\,(S)$ and $\mathcal{B}(x_0, R) \cap S \neq \emptyset$, and therefore $d_\Omega(S, x_0) \geq \operatorname{dist}\left(S, \Omega(\overline{\Omega(S)}) \setminus \Omega(S))\right)$. Together with Lemma 2.15 this implies that $d_\Omega(x_0, S) \geq \sigma h_S$. As Ω is Lipschitz and therefore satisfies a uniform exterior cone condition, there is a constant $\sigma_\Omega > 1$ that depends on the Lipschitz constant of the boundary, such that $d_\Omega(x_0, S) \leq \sigma_\Omega R$. In total, we find

$$\sigma R \geq d_\Omega(x_0, S) \geq \operatorname{dist}\left(S, \Omega\left(\overline{\Omega(S)}\right)\right) \geq \sigma h_s \tag{2.16}$$

for $S \subset \overline{\Omega \setminus \Omega(T)}$. Inserting inequalities (2.15) and (2.16) into inequality (2.14) now yields $\tilde{R} \leq (1 + \max\{\frac{\sigma_\Omega}{\sigma}, \frac{1}{\tilde{\sigma}}\})R$, which proves the lemma for $Q = 1 + \max\{\frac{\sigma_\Omega}{\sigma}, \frac{1}{\tilde{\sigma}}\}$. □

Lemma 2.18. *Let $\mathcal{T}_h$ be a shape-regular triangulation of the polyhedral domain Ω. There exists a constant $\kappa > 0$ that only depends on the shape-regularity constant Γ, such that for any $T \in \mathcal{T}_h$, $x_0 \in T$ and $R \geq h_T$, we have*

$$\mathcal{B}(x_0, \kappa R) \subset \Omega'(\mathcal{B}(x_0, R)). \tag{2.17}$$

Proof. Recall that $h_T = \operatorname{diam} T$. Thus, all nodes of T belong to $\mathcal{B}(x_0, R)$. In particular, this means that $\Omega(T) \subset \Omega'(\mathcal{B}(x_0, R))$. Now assume that there is an $S \in \mathcal{T}_h$ such that $\mathcal{B}(x_0, R) \cap S \neq \emptyset$, but none of the nodes of S belong to $\mathcal{B}(x_0, R)$. (If that were not the case, we would have $\mathcal{B}(x_0, R) \subset \Omega'(\mathcal{B}(x_0, R))$ and there would be nothing to show.) We connect a point on $\partial\mathcal{B}(x_0, R) \cap S$ with x_0 by a line-segment and denote the length of the part of the segment within S by a and the part outside of S by b. (See Figure 2.2.) Then, we have

$$R = a + b. \tag{2.18}$$

We know that

$$a \leq \operatorname{diam} S = h_S. \tag{2.19}$$

On the other hand $x_0 \notin \Omega(S)$. Denote the point that a and b share by x_1 and as before, denote the length of the shortest path between x_1 and x_0 by $d_\Omega(x_1, x_0)$. Then, we have

$$d_\Omega(x_1, x_0) \geq \operatorname{dist}(S, \Omega(\overline{\Omega(S)}) \setminus \Omega(S)) \geq \sigma h_S \tag{2.20}$$

by Lemma 2.15. Again, because Ω is Lipschitz and it therefore satisfies an exterior cone condition, there is a constant σ_Ω that only depends on the Lipschitz constant of Ω, such that

$$d_\Omega(x_0, x_1) \leq \sigma_\Omega b. \tag{2.21}$$

Inequalities (2.19), (2.20) and (2.21) yield $a \leq \sigma\sigma_\Omega b$. Using this in the equality (2.18) gives

$$R = a + b \leq \left(1 + \frac{\sigma_\Omega}{\sigma}\right) b. \tag{2.22}$$

As this is true for all simplices $S \in \mathcal{T}_h$ that intersect $\mathcal{B}(x_0, R)$ and do not have a node in $\mathcal{B}(x_0, R)$, this implies inclusion (2.17) for $\kappa = \frac{\sigma}{\sigma + \sigma_\Omega}$. $\square$

Remark 2.19. *Our choice of $\mathcal{B}(x_0, R)$ instead of only $B(x_0, R)$ near the boundary is necessary to exclude cases where two different connected components of $\Omega \cap B(x_0, R)$ might be connected in Ω via an arbitrarily long path and therefore the simplices in this other component might be very big compared to the ones in the component we were starting in. We will see in Section 3.4, where we will discuss regularity at the boundary, that this approach still leads to a C^α-estimate.*

Now, we will define the finite element space that we will be working on.

Definition 2.20. *Given a triangulation $\mathcal{T}_h$ of the domain Ω, we denote the space of continuous functions that are affine on every $T \in \mathcal{T}_h$ by V_h. Note that $V_h \subset W^{1,\infty}(\Omega)$ and we can write*

$$\begin{aligned} u_h &= \sum_i u_h(x_i)\psi_i, \\ \nabla u_h &= \sum_i u_h(x_i)\nabla\psi_i \end{aligned} \tag{2.23}$$

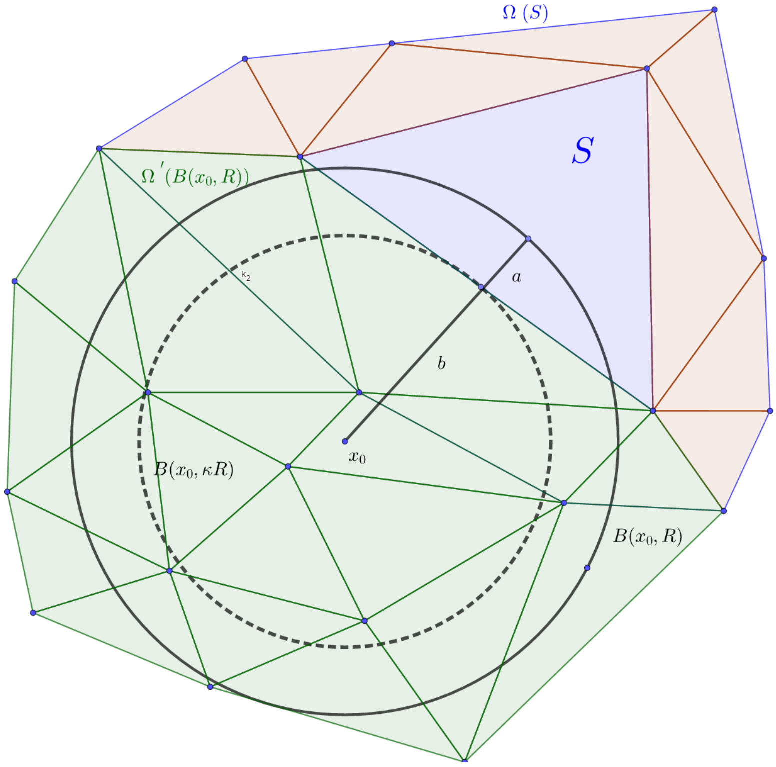

Figure 2.2: S and $\mathcal{B}(x_0, R)$

for any $u_h \in V_h$ with the Lagrange basis functions ψ_i introduced in Definition 2.12. This also shows that we can write a projection Π_h onto V_h for every continuous function f as

$$\Pi_h f = \sum_i f(x_i)\psi_i. \tag{2.24}$$

$\Pi_h f$ is also called the Lagrange interpolant of f. Furthermore, we will denote the space of continuous functions that are affine on every $T \in \mathcal{T}_h$ and zero on $\partial\Omega$ by $V_{h,0} := W_0^{1,1}(\Omega) \cap V_h$.

We will collect a few lemmas about functions defined on triangulations. In particular, this will lead to a stronger version of Poincaré's inequality. We will always assume that V_h and $V_{h,0}$ are finite element spaces associated with a shape-regular triangulation $\mathcal{T}_h$ of a polyhedral domain Ω. In particular, this means that (2.10) is true. Π_h will always denote the Lagrange projection operator onto V_h as defined in equation (2.24).

Lemma 2.21. *Let $\mathcal{T}_h$ be a shape-regular triangulation of the polyhedral Lipschitz domain Ω. For all $f \in C^0\left(\overline{\Omega}\right)$, we have*

$$\sup_T |\Pi_h f| \leq \sup_T |f|. \tag{2.25}$$

Furthermore, there is constant C that depends on the shape-regularity constant of the mesh and the dimension, such that for any $T \in \mathcal{T}_h$ and $f \in C^1\left(\overline{T}\right)$, we have

$$\sup_T |f - \Pi_h f| + \sup_T h_T |\nabla \Pi_h f| \leq C \sup_T h_T |\nabla f|. \tag{2.26}$$

on all $T \in \mathcal{T}_h$.

Proof. As an affine function on a simplex T, $\Pi_h f$ attains its maximum and minimum on T on one of the nodes of T. This means that we have

$$\sup_T |\Pi_h f| \leq \max_{x_i \in T} |\Pi_h f| = \max_{x_i \in T} |f| \leq \sup_T |f|,$$

which proves the inequality (2.25). The bound $\sup_T |f - \Pi_h f| \leq \sup_T h_T |\nabla f|$ is standard from approximation theory. To prove that $\sup_T h_T |\nabla \Pi_h f| \lesssim \sup_T h_T |\nabla f|$, we can assume that $f(x) = 0$ for some $x \in T$ because $\nabla f = \nabla(f - c)$ for all $f \in C^1\left(\overline{T}\right)$ and all constants c. Write $\mathcal{I}_T$ for the index set of the $n+1$ indices of the nodes of T. We also use the relation (2.10) to get

$$\sup_T |\nabla \Pi_h f| = \sup_T \left| \sum_i f(x_i) \nabla \psi_i \right| \leq (n+1) \max_T |f(x_i)| h_T^{-1} \lesssim \max_T |\nabla f|.$$

This concludes the proof of inequality (2.26). □

The following important Lemma 2.23 is true for polynomials defined on simplices $T \in \mathcal{T}_h$. We will denote the space of polynomials of degree k or lower by $\mathcal{P}_k$.

Remark 2.22. *Let $a_1, \ldots, a_m \in \mathcal{P}_k$. Then, $a_1 \cdots a_m = 0$ if and only if at least one of the functions $a_1, \ldots, a_m$ is identically zero.*

Lemma 2.23. *Let $a_1, \ldots, a_m \in \mathcal{P}_k$. Then,*

$$\prod_{j=1}^m \max_T |a_j| \sim \fint_T \left| \prod_{j=1}^m a_j \right| \, \mathrm{d}x. \tag{2.27}$$

The constant hidden in the $\sim$ depends on k, n, m and the shape-regularity parameter of T.

Proof. Since the expressions on both sides of (2.27) are homogeneous in $a_1, \ldots, a_m$, it suffices to show that there are constants c and C such that $0 < c \leq \fint_T \left| \prod_{j=1}^m a_j \right| \, \mathrm{d}x \leq C$ for any $a_1, \ldots, a_m \in \mathcal{P}_k$ with $\max_T |a_1| = \cdots = \max_T |a_m| = 1$ and any shape-regular simplex T. Obviously, we can choose $C = 1$. We will prove the existence of c with a compactness argument. Equation (2.27) is invariant under linear scaling. Thus, we can assume that $h_T = 1$ in the sense of Definition 2.13. First, we fix a shape-regular simplex T and note that the mapping $g : (a_1, \ldots, a_m) \mapsto \fint_T |a_1 \cdots a_m| \, \mathrm{d}x$ is continuous on $\mathcal{P}_k^{\otimes m}$, as this space is finite-dimensional and all norms are equivalent. Furthermore, the set $\{(a_1,\dots,a_m) : \max_T |a_1| = \cdots = \max_T |a_m| = 1\} \subset \mathcal{P}_k^{\otimes m}$ is closed and bounded and therefore compact. Thus, it suffices to show that $g(a_1,\dots,a_m) > 0$ for any $(a_1,\dots,a_m)$ with $\max_T |a_1| = \cdots = \max_T |a_m| = 1$ to get

$$\inf_{\max_T |a_1| = \cdots = \max_T |a_m| = 1} \fint_T |a_1 \cdots a_m| \,\mathrm{d}x > 0. \tag{2.28}$$

Assume the contrary, i.e. $\fint_T |a_1 \cdots a_m| \,\mathrm{d}x = 0$. Then, $a_1 \cdots a_m = 0$ on T. But, because of Remark 2.22, this implies that at least one of the $a_1,\dots,a_m$ has to be identically zero, which contradicts $\max_T |a_1| = \cdots = \max_T |a_m| = 1$.

If we now fix a node x_0 of T, we define the set $\mathcal{A} \subset \mathbb{R}^{n\times n}$ as the set of n-tuples $(x_1,\dots,x_n)$ that, together with x_0, form a shape-regular simplex with diameter $h_T = 1$ and see that $\mathcal{A}$ is bounded and closed and therefore compact. Furthermore, the mapping $f : \mathcal{A} \to \mathbb{R}$ defined via

$$(x_1,\dots,x_n) \mapsto \inf_{\max_T |a_1| = \cdots = \max_T |a_m| = 1} \fint_{\operatorname{convhull}(x_0,\dots,x_n)} |a_1 \cdots a_m| \,\mathrm{d}x$$

is continuous. This means that it suffices to show that $f(a_1,\dots,a_n) > 0$ to prove (2.27). But this is obviously true due to the inequality (2.28). □

Furthermore, we have the following identity.

Lemma 2.24. *For all $f_h, g_h \in V_h$ and $a, b \in \mathbb{R}$, we have*

$$\Pi_h(f_h g_h) - f_h g_h = \Pi_h\big((f_h - a)(g_h - b)\big) - (f_h - a)\,(g_h - b)\,.$$

Proof. Π_h is linear and constant functions are contained in V_h. This means that we have $\Pi_h(a g_h) = a g_h$, $\Pi_h(b f_h) = b f_h$ and $\Pi_h(ab) = ab$. This leaves us with

$$\begin{aligned}
&\Pi_h\big((f_h - a)(g_h - b)\big) - (f_h - a)\,(g_h - b) \\
&\quad = \Pi_h(f_h g_h) - \Pi_h(a g_h) - \Pi_h(b f_h) + \Pi_h(ab) - f_h g_h + a g_h + b f_h - ab \\
&\quad = \Pi_h(f_h g_h) - f_h g_h.
\end{aligned}$$

This concludes the proof. □

This leads to the following estimate.

Lemma 2.25. *There are constants C_1 and C_2 that depend on the shape-regularity constant and the dimension, such that for all $v_h, w_h \in V_h$, we have*

$$\begin{aligned}
&\max_T |v_h w_h - \Pi_h(v_h w_h)| + \max_T h_T \,|\nabla(v_h w_h - \Pi_h(v_h w_h))| \\
&\quad \le C_1 \fint_T |v_h - \langle v_h\rangle_T| \,\mathrm{d}y \fint_T |w_h - \langle w_h\rangle_T| \,\mathrm{d}y
\end{aligned} \tag{2.29}$$

and

$$\max_T |v_h w_h - \Pi_h(v_h w_h)| + \max_T h_T |\nabla(v_h w_h - \Pi_h(v_h w_h))| \\ \leq C_2 h_T \fint_T |\nabla v_h| \, \mathrm{d}y \fint_T |w_h - \langle w_h \rangle_T| \, \mathrm{d}y \tag{2.30}$$

for any $T \in \mathcal{T}_h$.

Proof. Because v_h, w_h are affine functions on every $T \in \mathcal{T}_h$, we deduce by an inverse estimate that

$$\max_T h_T |\nabla(v_h w_h - \Pi_h(v_h w_h))| \lesssim \max_T |v_h w_h - \Pi_h(v_h w_h)|. \tag{2.31}$$

Now, Lemma 2.24 yields

$$\max_T |v_h w_h - \Pi_h(v_h w_h)| = \max_T |(v_h - \langle v_h \rangle_T)(w_h - \langle w_h \rangle_T) \\ - \Pi_h((v_h - \langle v_h \rangle_T)(w_h - \langle w_h \rangle_T))|. \tag{2.32}$$

Combining this with the triangle inequality and inequality (2.25) gives

$$\max_T |v_h w_h - \Pi_h(v_h w_h)| \lesssim \max_T |(v_h - \langle v_h \rangle_T)(w_h - \langle w_h \rangle_T)| \\ \leq \max_T |(v_h - \langle v_h \rangle_T)| \max_T |(w_h - \langle w_h \rangle_T)|. \tag{2.33}$$

We can now apply Lemma 2.23 to inequality (2.33) to get

$$\max_T |v_h w_h - \Pi_h(v_h w_h)| \lesssim \fint_T |v_h - \langle v_h \rangle_T| \, \mathrm{d}x \fint_T |w_h - \langle w_h \rangle_T| \, \mathrm{d}x, \tag{2.34}$$

which proves the bound (2.29). The bound (2.30) follows directly from equation (2.29) by Poincaré's inequality on T. □

Remark 2.26. *On the right-hand side of equations* (2.29) *and* (2.30), *the integral means can be dropped, because we have*

$$\fint_T |v_h - \langle v_h \rangle| \, \mathrm{d}x \leq \fint_T |v_h| \, \mathrm{d}x + \fint_T |\langle v_h \rangle| \, \mathrm{d}x \leq 2 \fint_T |v_h| \, \mathrm{d}x.$$

Next, we will prove a Jensen type inequality.

Lemma 2.27. *For every non-negative* $\eta_h \in V_h$ *and* $q \geq 1$ *we have*

$$\eta_h^q \leq \Pi_h(\eta_h^q).$$

Proof. Recall that we denote the Lagrange basis functions of V_h by ψ_i. (See Definition 2.12.) We know that

$$\sum_i \psi_i = \Pi_h(1) = 1.$$

This allows us to use Jensen's inequality to get

$$\eta_h^q(x) = \Big(\sum_j \psi_j(x) \eta_h(x_j) \Big)^q \leq \sum_j \psi_j(x) \eta_h^q(x_j) = \big(\Pi_h(\eta_h^q)\big)(x).$$

□

Finally, we will prove two versions of Poincaré's inequality for functions in V_h.

Lemma 2.28 (Poincaré's inequality on patches)**.** *There is a constant $C > 0$ that depends on the dimension and the shape-regularity constant, such that the following holds: Let x_i be a node of the triangulation $\mathcal{T}_h$ and $P_i = \Omega(x_i)$ the respective patch and let $v_h \in V_h$ be a function with $v_h(x_0) = 0$ for some node $x_0 \in \overline{P_i}$. Then, we have Poincaré's inequality on P_i, that is*

$$\int_{P_i} |v_h| \,\mathrm{d}x \le C h_i \int_{P_i} |\nabla v_h| \,\mathrm{d}x,$$

where $h_i = h_T$ for some $T \in \mathcal{T}_h$ with $T \subset \overline{P_i}$ in the sense of Definition 2.13. Note, that any $S, T \in \mathcal{T}_h$ with $S, T \subset \overline{P_i}$ share the node x_i, which means that $h_T \sim h_S$ by equation 2.11 and the notion of h_i makes sense.

Proof. Let $T_0 \subset P_i$ a simplex that has x_0 as a node. Note that $\operatorname{diam} P_i \sim h_i$ and $|T_0| \sim |P_i|$ by shape-regularity. Then, Poincaré's inequality yields

$$\int_{P_i} \left| v_h - \langle v_h \rangle_{T_0} \right| \mathrm{d}x \lesssim h_i \int_{P_i} |\nabla v_h| \,\mathrm{d}x. \tag{2.35}$$

On the other hand, we have Poincaré's inequality on T_0 because v_h is affine on any $T \in \mathcal{T}_h$ and we get

$$\int_{P_i} \left| \langle v_h \rangle_{T_0} \right| \mathrm{d}x \le \frac{|P_i|}{|T_0|} \int_{T_0} |v_h| \,\mathrm{d}x \lesssim \int_{T_0} |v_h| \,\mathrm{d}x \lesssim h_i \int_{T_0} |\nabla v_h| \,\mathrm{d}x \le h_i \int_{P_i} |\nabla v_h| \,\mathrm{d}x. \tag{2.36}$$

Combining equations (2.35) and (2.36) yields

$$\int_{P_i} |v_h| \,\mathrm{d}x \le \int_{P_i} \left| v_h - \langle v_h \rangle_{T_0} \right| \mathrm{d}x + \int_{P_i} \left| \langle v_h \rangle_{T_0} \right| \mathrm{d}x \lesssim h_i \int_{P_i} |\nabla v_h| \,\mathrm{d}x.$$

This concludes the proof. ◻

Theorem 2.29 (Poincaré's inequality for V_h)**.** *Let $\mathcal{T}_h$ be a shape-regular triangulation of the polyhedral Lipschitz domain Ω with respective finite element space V_h. Let $v_h \in V_h$ nonnegative function and let $\Theta = \bigcup_i T_i \setminus \partial \left(\bigcup_i T_i \right)$ be a connected set with diameter R for a set of simplices $\{T_i\} \subset \mathcal{T}_h$. Suppose*

$$\left| \Theta \cap \left(\bigcup_{v_h(x_i)=0} P_i \right) \right| \ge \gamma |\Theta| \tag{2.37}$$

for some $\gamma > 0$. Then, there is a constant $C > 0$ that only depends on γ, the Poincaré constant of Θ, the shape-regularity constant Γ of the mesh and the dimension, such that

$$\int_{\Theta} |v_h| \,\mathrm{d}x \le C R \int_{\Theta} |\nabla v_h| \,\mathrm{d}x.$$

Proof. First, note that Θ is a bounded Lipschitz domain and therefore, its Poincaré constant is finite. Let $\mathcal{N}$ be the index set of nodes that are either nodes x_i with $v_h(x_i) = 0$ or that are connected to one of those nodes in $\overline{\Theta}$ by an edge. We write $w_h := \sum_{i\in\mathcal{N}} v_h(x_i)\psi_i$ and $\tilde{w}_h := \sum_{i\notin\mathcal{N}} v_h(x_i)\psi_i$. Note that $v_h = w_h + \tilde{w}_h$ and $\tilde{w}_h = 0$ on $\Theta \cap \left(\bigcup_{v_h(x_i)=0} P_i\right)$ and thus, $⨍_{\Theta\cap\left(\bigcup_{v_h(x_i)=0} P_i\right)} \tilde{w}\,\mathrm{d}x = 0$. Therefore, assumption (2.37) implies that we can use Poincaré's inequality from Corollary 2.9 on $\tilde{w}_h$. This leaves us with

$$\int_\Theta |\tilde{w}_h|\,\mathrm{d}x \lesssim R \int_\Theta |\nabla \tilde{w}_h|\,\mathrm{d}x. \tag{2.38}$$

On the other hand, we know that w_h has a zero on every patch $P_i \cap \Theta = \Omega(x_i) \cap \Theta$ for any node $x_i \in \overline{\Theta}$ because every node x_i with $w_h(x_i) \neq 0$ has $i \in \mathcal{N}$ and is therefore connected to a node with $v(x_j) = 0$ by an edge. This means that we can use Lemma 2.28 on any of those patches. This gives

$$\int_\Theta |w_h|\,\mathrm{d}x \leq \sum_{i\in\mathcal{N}} \int_{P_i\cap\Theta} |w_h|\,\mathrm{d}x \lesssim \sum_{i\in\mathcal{N}} h_i \int_{P_i\cap\Theta} |\nabla w_h|\,\mathrm{d}x.$$

We have $h_i \lesssim R$. Furthermore, the shape-regularity of $\mathcal{T}_h$ guarantees that each simplex can only be part of a uniformly bounded number of patches. Thus, we get

$$\int_\Theta |w_h|\,\mathrm{d}x \lesssim R \int_\Theta |\nabla w_h|\,\mathrm{d}x. \tag{2.39}$$

Adding the inequalities (2.38) and (2.39) yields

$$\int_\Theta |v_h|\,\mathrm{d}x \leq \int_\Theta |w_h|\,\mathrm{d}x + \int_\Theta |\tilde{w}_h|\,\mathrm{d}x \lesssim R\left(\int_\Theta |\nabla w_h|\,\mathrm{d}x + \int_\Theta |\nabla \tilde{w}_h|\,\mathrm{d}x\right). \tag{2.40}$$

We define

$$O := \left(\bigcup_{i\in\mathcal{N}} \overline{P_i}\right) \cap \Theta.$$

Note that by definition, if $i \in \mathcal{N}$, then P_i has a node where v_h has a zero. Therefore, we get

$$\int_O |\nabla v_h|\,\mathrm{d}x \gtrsim \sum_{i\in\mathcal{N}} \int_{P_i\cap\Theta} |\nabla v_h|\,\mathrm{d}x \gtrsim \sum_{i\in\mathcal{N}} h_i^{-1} \int_{P_i\cap\Theta} |v_h|\,\mathrm{d}x, \tag{2.41}$$

where h_i is the diameter of one of the simplices $T \subset P_i$ and where we have used the fact that any $T \in \mathcal{T}_h$ is only part of finitely many patches and Lemma 2.28.

For simplicity, let us write $\Psi_\mathcal{N} = \sum_{i\in\mathcal{N}} \psi_i$. On each $T \in \mathcal{T}_h$, we therefore have $|\Psi_\mathcal{N}| \leq 1$ and $|\nabla\Psi_\mathcal{N}| \lesssim h_T^{-1}$. Furthermore, we can write

$$w_h = \Pi_h\left(v_h \Psi_\mathcal{N}\right).$$

Thus, we can use equation (2.26) and the product rule to find

$$|\nabla w_h| \lesssim |\nabla v_h| + h_T^{-1}|v_h| \tag{2.42}$$

on every $T \in \mathcal{T}_h$. Additionally, we have

$$\int_O |\nabla w_h| \,\mathrm{d}x \leq \sum_{i \in \mathcal{N}} \int_{P_i \cap \Theta} |\nabla w_h| \,\mathrm{d}x. \tag{2.43}$$

Combining inequalities (2.42), (2.43) and equation (2.11) yields

$$\int_O |\nabla w_h| \lesssim \sum_{i \in \mathcal{N}} \int_{P_i \cap \Theta} |\nabla v_h| + h_i^{-1} |v_h| \,\mathrm{d}x. \tag{2.44}$$

Therefore, we can combine inequalities (2.41) and (2.44) to get

$$\int_O |\nabla w_h| \lesssim \int_O |\nabla v_h| \,\mathrm{d}x. \tag{2.45}$$

Analogously, we find

$$\int_O |\nabla \tilde{w}_h| \,\mathrm{d}x \lesssim \int_O |\nabla v_h| \,\mathrm{d}x. \tag{2.46}$$

Now note that $w_h = 0$ on $\Theta \setminus O$. As $v_h = w_h + \tilde{w}_h$, we therefore have

$$|\nabla v_h| = |\nabla w_h| + |\nabla \tilde{w}_h| \tag{2.47}$$

on $\Theta \setminus O$. This means that we can finally combine inequalities (2.45), (2.46) and (2.47) to find

$$\int_\Theta |\nabla w_h| + |\nabla \tilde{w}_h| \,\mathrm{d}x \lesssim \int_\Theta |\nabla v_h| \,\mathrm{d}x. \tag{2.48}$$

Together with inequality (2.40), that completes the proof. □

We will want to apply the Poincaré inequality to sets of the form $\Omega(B(x_0, R))$. It remains to be shown that the Poincaré constant of those sets stays uniformly bounded. As we might have a highly refined mesh, $\Omega(B(x_0, R))$ might have a lot of small spikes, which makes it impossible to use the Lipschitz condition for the bound on the Poincaré constant. However, the fact that all the spikes have an angle that is uniformly bounded below because of the shape-regularity of the mesh, it is possible to show that $\Omega(B(x_0, R))$ is always α-John with α bounded below. By Corollary 2.9 it is enough to show that a set of the form $\Omega(\mathcal{B}(x_0, R)) \cup B(x_0, R)$ is indeed an α-John domain. Note that if $B(x_0, R) \subset \Omega$, we have $\Omega(\mathcal{B}(x_0, R)) \cup B(x_0, R) = \Omega(B(x_0, R))$.

Lemma 2.30. *Let $\mathcal{T}_h$ be a shape-regular triangulation of the polyhedral Lipschitz-domain Ω. Then, for any $x_0 \in \Omega$ and $R > 0$, the set*

$$\Omega(\mathcal{B}(x_0, R)) \cup B(x_0, R)$$

is an α-John domain, where α only depends on the shape-regularity constant Γ, the Lipschitz constant of $\partial\Omega$ and the dimension n.

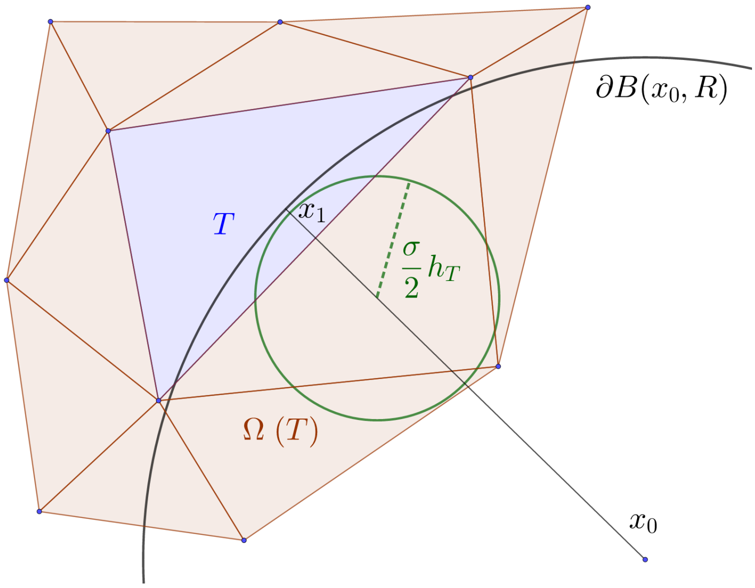

Figure 2.3: $B(x_2, \frac{\sigma}{2} h_T)$ in a patch

Proof. We will prove the theorem in three steps. We will first show that if a simplex $T \in \mathcal{T}_h$ touches $B(x_0, R)$, but is not completely in $B(x_0, R)$, there is indeed a ball $B(y_0, r) \subset B(x_0, R) \cap \Omega(T)$ with $r \gtrsim h_T$. In a second step, we will show that that given a set Θ that is a union of m simplices that are connected via faces (i.e. $(n-1)$-dimensional subsimplices), there is a constant σ_m that depends only on the shape-regularity constant of $\mathcal{T}_h$ and m, such that given $T \subset \Theta$, there is an α-carrot in Θ that ends in $B(y, r) \subset \Theta$ if $r \leq \sigma_m h_T$ and starts in $y_0 \in \Theta \setminus B(y, r) \cap T$. In a third step, we will conclude that $\Omega(\mathcal{B}(x_0, R)$ is indeed α-John.

Let us first assume that we are away from the boundary of Ω, i.e. we have $B(x_0, R) = \mathcal{B}(x_0, R)$.

STEP 1: For a $T \in \mathcal{T}_h$, assume $T \cap B(x_0, R) \neq \emptyset$ and $T \not\subset B(x_0, R)$, thus $\partial B(x_0, R) \cap T \neq \emptyset$. Take a point $x_1 \in B(x_0, R) \cap T$. By Lemma 2.15 we know that $\mathrm{dist}(T, \Omega(\overline{\Omega(T)}) \setminus \Omega(T)) \geq \sigma h_T$ (σ only depends on shape-regularity.) Denote the point that is $\frac{\sigma}{2}$ away from x_1 on the radius between x_0 and x_1 by x_2. Thus, $B(x_2, \frac{\sigma}{2} h_T) \subset B(x_0, R) \cap \Omega(T)$ (See Figure 2.3.), which concludes Step 1.

STEP 2. Consider a collection of m simplices $T_i \in \mathcal{T}_h$, $i \in \mathcal{I}$, for some index set $\mathcal{I}$ with $m \in \mathbb{N}$ elements, such that for any $i \in \mathcal{I}$, there is a $j \in \mathcal{I}$, such that $T_i \cap T_j$ is an $(n-1)$-dimensional subsimplex. We write $\Theta = \bigcup_{i \in \mathcal{I}} T_i$. We take two arbitrary points x_1 and x_2 in Θ. Recall the definition of the α-carrot $\mathrm{car}(\gamma, \alpha)$ from Definition 2.5:

$$\mathrm{car}(\gamma, \alpha) := \bigcup_{t \in [0, |\gamma|]} \{B(\gamma(t), \alpha t\}\,.$$

We want to show that there is a $\zeta > 0$ that only depends on the shape-regularity parameter Γ and m, such that if $x_1 \in T$ for some $T \in \mathcal{T}_h$ and $B(x_2, r) \subset \Theta$ for $r = \zeta h_T$, there exists a rectifiable path γ that connects x_1 and x_2 and an $\alpha > 0$ such that $\mathrm{car}(\gamma, \alpha) \subset \Theta$ and α is bounded below by a constant that only depends on ζ, m and the shape-regularity constant Γ. Let us first construct γ. Denote the $T \in \mathcal{T}_h$ with $x_1 \in T$ by T_1 and the one with $x_2 \in T$ by T_2. Let us now construct the shortest sequence $\{S_i\}_{i=1}^{m_{x_1,x_2}}$ of simplices with $S_1 = T_1$, $S_{m_{x_1,x_2}} = T_2$ and $S_i \cap S_{i+1}$ is an $(n-1)$-dimensional subsimplex. Obviously, $m_{x_1,x_2} \leq m$. Denote the centre of the largest inscribed ball of S_i by y_i. Now, we connect x_1 with y_1 by a line, then y_1 with y_2, and so on, and finally $y_{m_{x_1,x_2}}$ with x_2. (See Figure 2.4.) From equation (2.11) we know that there is an $\sigma_0 > 1$ such that $h_{S_i} \leq \sigma_0^m h_T$ and the radii of the largest inscribed balls of S_i (denoted by r_i are bounded below by $\frac{\sigma_1}{\sigma_0^m}$ for some positive $\sigma_1 < 1$ that only depends on shape-regularity. Therefore $m\frac{\sigma_1}{\sigma_0^m} h_T \leq |\gamma| \leq (m+2)\sigma_0^m h_T$. (Those constants could be improved by noting that going from T to the next simplex, and so on, the diameter of the next simplex is only smaller or larger by a factor σ and one could work with a geometric series instead of just taking the minimum/maximum there.) There is a maximum angle α_1 that is only bounded by shape-regularity such that the beginning of the carrot $\mathrm{car}(\gamma, \alpha)$ is still in T_1. Furthermore, we need to have $\alpha \leq \alpha_2 := \frac{\zeta}{(m+2)\sigma_0^m}$ to ensure that the end of the carrot is still in Θ. Therefore, $\mathrm{car}(\gamma, \min\{\alpha_1, \alpha_2\}) \subset \Theta$. This completes Step 2.

STEP 3: If the two points that we want to connect via an α-cigar are in $B(x_0, R)$, the existence of the respective cigar is trivial. If one point x_1 lays outside of $B(x_0, R)$, it has to be in a simplex $T \in \mathcal{T}_h$ that intersects the circle. By Step 1, we find a ball $B_1(\tilde{x}_1, \frac{\sigma}{2} h_T) \subset B(x_0, R) \cap \Omega(T)$ that has a radius of $\frac{\sigma}{2} h_T$. As it is also in the patch around T, we can find a rectifiable path γ_1 from x_1 to $\tilde{x}_1$ and an angle $\alpha_1 \leq 1$, such that $\mathrm{car}(\gamma_1, \alpha_1) \subset \Omega(B(x_0, R))$ by Step 2. ($\zeta = \frac{\sigma}{2}$, m is the number of simplices in $\Omega(T)$ that is bounded above by a constant that only depends on shape-regularity and the dimension.) The angle α is bounded below by a constant that only depends on shape-regularity and the dimension as the number of simplices in a patch is bounded above by those two quantities. If the second point x_2 is in $B(x_0, R)$, we can just connect $\tilde{x}_1$ and x_0 by a line and then x_0 and x_2 by another line and denote the path by γ_0. Then, the α_1-cigar $\mathrm{cig}(\gamma_1 \cup \gamma_2, \alpha_1)$ lies completely in $\Omega(B(x_0, R))$. If x_2 is also not in $B(x_0, R)$ and in $S \in \mathcal{T}_h$, we construct $B(\tilde{x}_2, \frac{\sigma}{2} h_S)$ as above in Step 2 and the respective carrot $\mathrm{car}(\gamma_2, \alpha_2)$ by step 1. We then connect the $\tilde{x}_1$ and x_0 by a line and then we connect x_0 and $\tilde{x}_2$ by a line and denote the respective path by γ_0. Then, the cigar $\mathrm{cig}(\gamma_1 \cup \gamma_0 \cup \gamma_2, \min\{\alpha_1, \alpha_2\})$ lies completely in $\Omega(B(x_0, R))$. (See Figure 2.5.) This completes the proof away from the boundary of Ω.

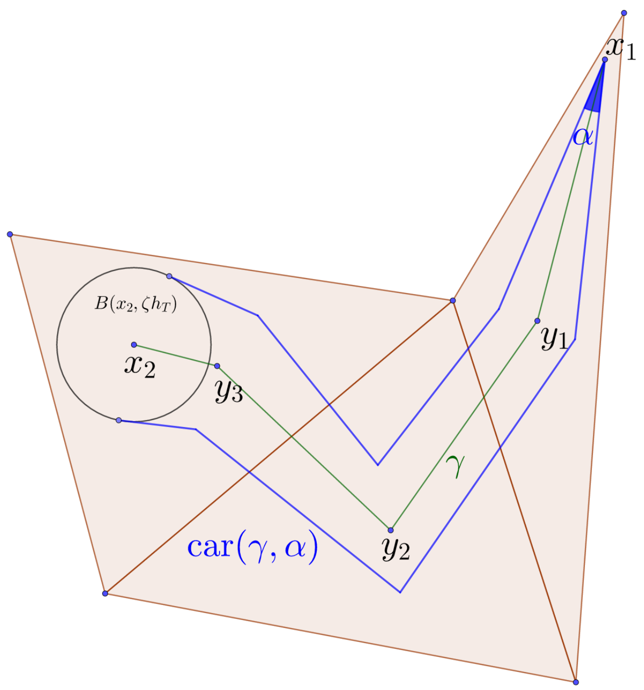

Figure 2.4: α-carrot in a set of three simplices.

If we are near the boundary of Ω, we note that we are indeed looking at the set $\Omega(\mathcal{B}(x_0, R)) \cup B(x_0, R)$. Thus, by definition this contains $B(x_0, R)$ as before and is connected. To find the balls in Step 2, we are looking at $\Omega(T) \cup (B(x_0, R) \setminus \Omega(\mathcal{B}(x_0, R))$. For the parts outside of Ω the role of shape-regularity is played by the outer cone condition that is satisfied by a Lipschitz domain. Then, the constants depend also on the Lipschitz constant of the boundary $\partial\Omega$. □

We will also need a lemma that concerns connecting pairs of nodes in an A-non-obtuse mesh in a way that allows us to work as if we were on a uniformly A-acute mesh. This seems to be a small change but allows us, for example, in the case where A is the identity matrix to use meshes that are generated by newest vertex bisection on a square, which are important cases, especially if we consider that it is one of the novelties of our approach that we do not demand quasi-uniformity of the mesh, i.e. we do not need h_T from equation (2.10) to be uniform on the entire domain.

Lemma 2.31. *Let $\mathcal{T}_h$ be an A-nonobtuse, shape-regular triangulation of a polyhedral domain $\Omega \subset \mathbb{R}^n$. Denote the nodes of a simplex $T \in \mathcal{T}_h$ by $x_0, \ldots, x_n$. Then, there is a constant $\tau \in (0, 1)$ that only depends on the dimension n and the shape-regularity parameter Γ of $\mathcal{T}_h$, such that any pair of nodes x_j and x_k of T can be connected by a sequence of distinct*

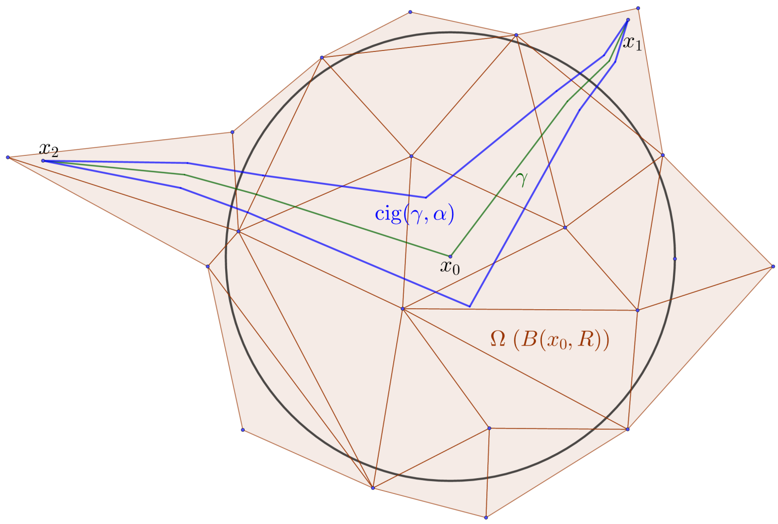

Figure 2.5: α-cigar in $\Omega(B(x_0,R))$.

nodes $\{x_j = y_0, y_1, \ldots, y_N = x_k\}$ *of* T*, with* $N \le n$ *and without loops, belonging to the same simplex* T *and such that*

$$-\int_\Omega A\nabla\psi_i \cdot \nabla\psi_{i+1}\,\mathrm{d}x \ge \tau \int_\Omega A\nabla\psi_i \cdot \nabla\psi_i\,\mathrm{d}x, \qquad i = 0,\ldots,N-1, \tag{2.49}$$

where ψ_i *is the basis function associated with the node* y_i *in the sense of Definition 2.12*

Proof. The proof proceeds in two steps. For two $\mathbb{R}^n$-valued functions ζ and ξ defined on Ω we shall write

$$\langle\zeta,\xi\rangle_{T,A} := \int_T A\zeta\cdot\xi\,\mathrm{d}x \quad \text{and} \quad \langle\zeta,\xi\rangle_A := \int_\Omega A\zeta\cdot\xi\,\mathrm{d}x.$$

STEP 1. Let $L \subset \{0,\ldots,n\}$ and $M := \{0,\ldots,n\}\setminus L$ with $L, M \neq \emptyset$. We will show that

$$\max_{l\in L,\, m\in M}(-\langle\nabla\psi_l,\nabla\psi_m\rangle_{T,A}) \ge \tau \max_{r\in\{0,\ldots,n\}}\langle\nabla\psi_r,\nabla\psi_r\rangle_A \tag{2.50}$$

for some $\tau > 0$ that depends on the shape regularity constant Γ. Since the $\langle\nabla\psi_l,\nabla\psi_m\rangle_{T,A} \le 0$ for $l \neq m$, it suffices to prove that

$$(I) := \sum_{l\in L,\, m\in M}(-\langle\nabla\psi_l,\nabla\psi_m\rangle_{T,A}) \gtrsim \max_{r\in\{0,\ldots,n\}}\langle\nabla\psi_r,\nabla\psi_r\rangle_A.$$

As $\sum_{r=0}^n \psi_r = 1$ on T and $L\cup M = \{0,\ldots,n\}$, we obtain

$$(I) = \sum_{l\in L}\Big(-\langle\nabla\psi_l,\nabla 1\rangle_{T,A} + \sum_{m\in L}\langle\nabla\psi_l,\nabla\psi_m\rangle_{T,A}\Big) = \sum_{l,m\in L}\langle\nabla\psi_l,\nabla\psi_m\rangle_{T,A} =: (II).$$

Now define $\eta := \sum_{l\in L}\psi_l$. Since $\eta(x_m) = 0$ for some $m \in M$ and A is uniformly elliptic, we obtain by Poincaré's inequality, Lemma 2.23 and inverse estimates that

$$\begin{aligned}(II) = \langle\nabla\eta,\nabla\eta\rangle_{T,A} \geq c\|\nabla\eta\|^2_{L^2(T)} \sim h_T^{-2}\|\eta\|^2_{L^2(T)} \sim h_T^{-2}|T|\,\|\eta\|^2_{L^\infty(T)}\\ \sim h_T^{-2}|T| \sim \max_{r\in\{0,\dots,n\}}\langle\nabla\psi_r,\nabla\psi_r\rangle_{T,A} \sim \max_{r\in\{0,\dots,n\}}\langle\nabla\psi_r,\nabla\psi_r\rangle_A,\end{aligned}$$

where we have used the shape-regularity of the triangulation. That completes Step 1.

STEP 2. Let us now assume that we were not able to find a sequence without loops that connects x_j and x_k so that inequality (2.49) holds for the τ from step 1. (Note that if a sequence that satisfies inequality (2.49) had loops, we could simply remove the loop from the sequence and it would still satisfy inequality (2.49).) This would imply that we can find disjoint sets $L \subset \{1,\dots,n\}$ and $M \subset \{1,\dots,n\}$ with $j \in L$ and $k \in M$ such that inequality (2.50) does not hold, but this is obviously a contradiction. □

We will also need the following formula for the scalar product of the gradients of two functions in V_h on simplices.

Lemma 2.32. *Let $\mathcal{T}_h$ be a triangulation of the polyhedral domain Ω. Let $v_h, w_h \in V_h$ be arbitrary finite element functions. We write $v_h(x_i) =: v_i$ and $w_h(x_i) =: w_i$ for all nodes x_i and $I_T = \{i : x_i \in T\}$. Then, on each $T \in \mathcal{T}_h$ we have that*

$$\nabla v_h \cdot \nabla w_h = -\frac{1}{2}\sum_{\substack{i,j\in I_T\\ i\neq j}}(v_i - v_j)\cdot(w_i - w_j)\nabla\psi_i\cdot\nabla\psi_j. \tag{2.51}$$

Proof. We know that $\sum_i \psi_i \equiv 1$ and therefore $\sum_i \nabla\psi_i \equiv 0$. This gives

$$\begin{aligned}\sum_{\substack{i,j\in I_T\\ i\neq j}}(v_i - v_j)\cdot(w_i - w_j)(-\nabla\psi_i\cdot\nabla\psi_j) &= \sum_{i,j\in I_T}(v_i - v_j)\cdot(w_i - w_j)(-\nabla\psi_i\cdot\nabla\psi_j)\\ &= -\sum_{i\in I_T} v_i\cdot w_i\,\nabla\psi_i\cdot\Big(\sum_{j\in I_T}\nabla\psi_j\Big) - \sum_{j\in I_T} v_j\cdot w_j\,\nabla\psi_j\cdot\Big(\sum_{i\in I_T}\nabla\psi_i\Big)\\ &\quad + \sum_{i,j\in I_T} v_i\cdot w_j(\nabla\psi_i\cdot\nabla\psi_j) + \sum_{i,j\in I_T} v_j\cdot w_i(\nabla\psi_i\cdot\nabla\psi_j)\\ &= 2\nabla v_h\cdot\nabla w_h,\end{aligned}$$

which proves the claim. □

For the modulus of the gradient of a function from V_h, we can find an even simpler formula.

Lemma 2.33. *Let $\mathcal{T}_h$ be a shape-regular triangulation of the polyhedral domain $\Omega \subset \mathbb{R}^n$ with respective finite element space V_h. For any simplex $T \in \mathcal{T}_h$, we write $\mathcal{I}_T := \{i : x_i \in T\}$. For any $v_h \in V_h$, we have*

$$|\nabla v_h| \sim \frac{1}{h_T} \sum_{i \in \mathcal{I}_T} |v_h(x_j) - v_h(x_i)| \tag{2.52}$$

on any $T \in \mathcal{T}_h$ with diameter h_T. The implicit constant only depends on the shape-regularity constant and the dimension.

Proof. The shape-regularity of the triangulation $\mathcal{T}_h$ implies that for every $T \in \mathcal{T}_h$, there is an affine transformation $\Gamma_T : \mathbb{R}^n \to \mathbb{R}^n$ with $|\nabla \Gamma_T| \leq C$ and $|\nabla \Gamma_T^{-1}| \leq C$, where C does only on the shape-regularity constant and the dimension, that maps T to the simplex spanned by the points $(0, \dots, 0)$, $(h_T, 0, \dots, 0)$,$\cdots$, $(0, \dots, 0, h_T)$. Without loss of generality, assume that x_j gets mapped on $(0, \dots, 0)$. On this simplex, we have

$$|\nabla (v_h \circ \Gamma_T^{-1})| = \frac{1}{h_T} \left(\sum_{i \in \mathcal{I}_T} |v_h(x_j) - v_h(x_i)|^2 \right)^{\frac{1}{2}} \sim \frac{1}{h_T} \sum_{i \in \mathcal{I}_T} |v_h(x_j) - v_h(x_i)|, \tag{2.53}$$

where we have used the equivalence of norms in $\mathbb{R}^n$ in the last step. This means that we can write $v_h = v_h \circ \Gamma_T^{-1} \circ \Gamma_T$ and use equation (2.53) and the boundedness of the gradients of Γ_T and Γ_T^{-1} to conclude the proof of the lemma. $\square$

Chapter 3

C^α-regularity of discrete solutions to linear elliptic equations

In this chapter, we will show *a priori* regularity results for discrete solutions to linear elliptic partial differential equations. We will give a brief overview of the De Giorgi theory for this type of equations. The subsequent sections will focus on developing a discrete De Giorgi theory. The chapter is based on the joint paper with Lars Diening and Endre Süli, entitled *Uniform Hölder-norm bounds for finite element approximations of second-order elliptic equations*, submitted to the IMA Journal of Numerical Analysis on 9 April 2020, and available from arXiv:2004.09341 [math.NA].

3.1 Local Hölder continuity in the continuous case

First, we will present a brief overview of the proof of the local Hölder continuity of weak solutions to elliptic equations. The result was first established by De Giorgi and independently by Nash and Moser. This is, by now, a classical result in the PDE analysis literature, however since our proof of the discrete counterpart of this result proceeds along similar, but much more technical lines, readers may find this short overview helpful, regardless. The main ideas of the proofs in this section have been used in [14]. For simplicity, we will restrict ourselves to the homogeneous case. We begin by proving the Caccioppoli inequality stated in the next theorem.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain and $A \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ a uniformly elliptic matrix-valued function, i.e., there is a $d > 0$ such that*

$$A(x)v \cdot v \geq d|v|^2 \tag{3.1}$$

for any $v \in \mathbb{R}^n$ and almost all $x \in \Omega$. Let $u \in W^{1,2}(\Omega)$ be a weak solution to $\operatorname{div}(A\nabla u) = 0$*, i.e.,*

$$\int_\Omega A\nabla u \cdot \nabla\varphi \,\mathrm{d}x = 0 \tag{3.2}$$

for all $\varphi \in W_0^{1,2}(\Omega)$*. Then, we have that*

$$\int_\Omega |\nabla(u-c)_+|^2|\eta|^2 \,\mathrm{d}x \le C \int_\Omega |(u-c)_+|^2 |\nabla \eta|^2 \,\mathrm{d}x \tag{3.3}$$

for any function $\eta \in C_0^\infty(\Omega)$ *and any* $c > 0$*, where* $(u-c)_+ = (u-c) \vee 0$ *and a constant* C *that only depends on* $\|A\|_{L^\infty(\Omega)}$ *and* d*.*

Proof. We test equation (3.2) against $\varphi = (u-c)_+\eta^2$. Note that $\nabla u = \nabla(u-c) = \nabla(u-c)_+$ on $\operatorname{supp}(u-c)_+$. This gives

$$\begin{aligned} 0 &= \int_\Omega A\nabla u \cdot \nabla\left(\eta^2(u-c)_+\right) \,\mathrm{d}x \\ &= \int_\Omega A\nabla(u-c)_+ \cdot \nabla((u-c)_+)\eta^2 \,\mathrm{d}x + \int_\Omega A\nabla(u-c)_+ \cdot 2(\nabla\eta)\eta(u-c)_+ \,\mathrm{d}x \\ &=: I + II. \end{aligned} \tag{3.4}$$

We use the uniform ellipticity of A to deduce that

$$I \gtrsim \int_\Omega |\nabla(u-c)_+|^2 \eta^2 \,\mathrm{d}x. \tag{3.5}$$

The boundedness of A, Hölder's inequality, and Young's inequality yield

$$\begin{aligned} |II| &\lesssim \int_\Omega |\nabla(u-c)_+||\eta||\nabla\eta||(u-c)_+| \,\mathrm{d}x \\ &\le \varepsilon \int_\Omega |\nabla(u-c)_+|^2|\eta|^2 \,\mathrm{d}x + C_\varepsilon \int_\Omega |(u-c)_+|^2|\nabla\eta|^2 \,\mathrm{d}x. \end{aligned} \tag{3.6}$$

Inserting the inequalities (3.5) and (3.6) into equation (3.4) and absorbing the first term of II into I proves the claim. □

Theorem 3.2. *Let* $\Omega \subset \mathbb{R}^n$ *be a domain and* $A \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ *a uniformly elliptic matrix-valued function. There is a constant* $C > 0$*, that depends only on* $\|A\|_{L^\infty(\Omega)}$*,* d *and* n*, such that if* $u \in W^{1,2}(\Omega)$ *is a weak solution to* $\operatorname{div}(A\nabla u) = 0$*, we have, for every* $c > 0$*,*

$$\sup_{B(x_0,R)} |(u-c)_+|^2 \le C \fint_{B(x_0,2R)} |(u-c)_+|^2 \,\mathrm{d}x \tag{3.7}$$

for any ball $B(x_0,R)$ *with* $B(x_0,2R) \subset \Omega$*.*

Proof. We define $\gamma_k = \gamma_\infty\left(1 - 2^{-k}\right)$ and $B_k := B\left(x_0, \left(1+2^{-k}\right)R\right)$ where $\gamma_\infty > 0$ is is to be chosen later. Clearly, $\gamma_0 = 0$, $\lim_{k\to\infty}\gamma_k = \gamma_\infty$, and $B(x_0,2R) = B_0 \supset B_1 \supset \cdots \supset B_k \supset B_{k+1} \supset \cdots \supset B(x_0,R)$. We then define compactly supported C^∞-functions φ_k such that

$$\begin{gathered} \operatorname{supp}\varphi_k \subset B_k, \quad 0 \le \varphi_k \le 1, \\ \varphi_k \equiv 1 \text{ on } B_{k+1}, \\ |\nabla\varphi_k| \lesssim R^{-1}2^k, \end{gathered} \tag{3.8}$$

and the sequence

$$U_k := \fint_{B(x_0,2R)} \big|(u-c-\gamma_k)_+\big|^2 |\varphi_k|^2 \,\mathrm{d}x, \qquad k = 0,1,\dots. \tag{3.9}$$

We use scaling-invariant norms $|||\cdot|||_p$ with $p \in [1,\infty)$ defined by $|||f|||_p^p := \fint_{B(x_0,2R)} |f|^p \,\mathrm{d}x$ and apply Hölder's inequality, the Sobolev embedding theorem, and equation (3.3) to get (assume that $n \geq 3$ for simplicity, and let $2^\star := 2n/(n-2)$ denote the critical Sobolev index; the bounds below are easily adjusted in the case of $n = 2$ to reach the desired conclusion by choosing $2^\star$ as a large positive integer):

$$\begin{aligned} U_k &\leq |||(u-c-\gamma_k)_+\varphi_k|||_{2^\star}^2 |||\chi_{\{u-c>\gamma_k\}\cap \operatorname{supp}\varphi_k}|||_n^2 \\ &\lesssim R^2 |||\nabla((u-c-\gamma_k)_+\varphi_k)|||_2^2 |||\chi_{\{u-c>\gamma_k\}\cap \operatorname{supp}\varphi_k}|||_n^2 \\ &\lesssim R^2 \Big(|||\nabla((u-c-\gamma_k)_+)\varphi_k|||_2^2 + |||(u-c-\gamma_k)_+\nabla\varphi_k|||_2^2 \Big) |||\chi_{\{u-c>\gamma_k\}\cap \operatorname{supp}\varphi_k}|||_n^2 \\ &\lesssim R^2 |||(u-c-\gamma_k)_+\nabla\varphi_k|||_2^2 \, |||\chi_{\{u-c>\gamma_k\}\cap \operatorname{supp}\varphi_k}|||_n^2, \end{aligned} \tag{3.10}$$

where in the transition to the last line we applied Theorem 3.1. Note that the constant that appears from the Sobolev embedding theorem only depends on the dimension because the cutoff function φ_k ensures that $(u-c-\gamma_k)_+\varphi_k = 0$ outside of $B(x_0, 2R)$. For the first factor on the right-hand side in the final line of inequality (3.10) we use that $|\nabla\varphi_k| \lesssim R^{-1}2^k$ and $\varphi_{k-1} \equiv 1$ on $\operatorname{supp}\varphi_k$ (see (3.8)), and that (γ_k) is monotonic increasing sequence, to get

$$|||(u-c-\gamma_k)_+\nabla\varphi_k|||_2^2 \lesssim R^{-2}2^{2k} |||(u-c-\gamma_{k-1})_+\varphi_{k-1}|||_2^2. \tag{3.11}$$

On $\{u-c>\gamma_k\}$, we have

$$u-c-\gamma_{k-1} > \gamma_k - \gamma_{k-1} = \gamma_\infty \left(2^{-(k-1)} - 2^{-k}\right) = \gamma_\infty 2^{-k}. \tag{3.12}$$

Together with inequality (3.12) this yields

$$\begin{aligned} \int_{B(x_0,2R)} [(u-c-\gamma_{k-1})_+]^2 \varphi_{k-1}^2 \,\mathrm{d}x &\geq \int_{\operatorname{supp}\varphi_k} \chi_{\{u-c>\gamma_k\}} [(u-c-\gamma_{k-1})_+]^2 \,\mathrm{d}x \\ &\geq \gamma_\infty^2 \, 2^{-2k} |\operatorname{supp}\varphi_k \cap \{u-c>\gamma_k\}|. \end{aligned} \tag{3.13}$$

We can use inequality (3.13) to obtain the following weak-type estimate:

$$|||\chi_{\{u-c>\gamma_k\}\cap \operatorname{supp}\varphi_k}|||_n^2 = \left(\frac{|\operatorname{supp}\varphi_k \cap \{u-c>\gamma_k\}|}{|B(x_0,2R)|} \right)^{\frac{2}{n}} \lesssim 2^{2k} \left(\frac{|||(u-c-\gamma_{k-1})_+\varphi_{k-1}|||_2^2}{\gamma_\infty^2} \right)^{\frac{2}{n}}. \tag{3.14}$$

Inserting the inequalities (3.11) and (3.14) in inequality (3.10) yields, with a positive constant C, independent of γ_∞ and k,

$$U_k \leq C 2^{3k} U_{k-1} \left(\frac{U_{k-1}}{\gamma_\infty^2} \right)^{\frac{2}{n}}, \qquad k = 1,2,\dots.$$

This then allows us to apply Corollary 3.32 from the Appendix with $b = 2^3$ and $\alpha = 2/n$ and $\gamma = \gamma_\infty^2 := C^{n/2} 2^{3n^2/4} U_0$, to deduce that $U_k \to 0$ as $k \to \infty$, and therefore $(u - c - \gamma_\infty)_+ = 0$ a.e. on $B(x_0, R)$, regardless of the sign of $u - c$. Hence, $|(u-c)_+|^2 \leq \gamma_\infty^2$ on $B(x_0, R)$. On the ball $B(x_0, R) \subseteq \bigcap_{k \in \mathbb{N}} \operatorname{supp} \eta_k$ this means that

$$|(u-c)_+|^2 \leq \gamma_\infty^2 \sim U_0 \leq \fint_{B(x_0,2R)} |(u-c)_+|^2 \, \mathrm{d}x,$$

because $\gamma_0 = 0$ and $0 \leq \varphi_0 \leq 1$, which then implies (3.7). $\square$

From this result one can deduce the local C^α-continuity of weak solutions.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain and $A \in L^\infty(\Omega; \mathbb{R}^{n \times n})$ a uniformly elliptic matrix-valued function, i.e. there is a $d > 0$, such that $A(x) v \cdot v \geq d|v|^2$ almost everywhere for all $v \in \mathbb{R}^n$. Let $u \in W^{1,2}(\Omega)$ be a weak solution to $\operatorname{div}(A \nabla u) = 0$. Then, there are constants $C > 0$ and $\alpha > 0$ that only depend on $\|A\|_{L^\infty(\Omega)}$, d and n, such that*

$$\operatorname*{osc}_{B(x,r)} u \leq C r^\alpha$$

for all $r \in (0, R]$ such that $B(x, 4R) \subset \Omega$.

To prove Theorem 3.3 we require the following intermediate result.

Lemma 3.4. *Under the assumptions of Theorem 3.3, for each $\gamma \in (0,1)$ there exists a $\tau \in (0,1)$ that depends on γ, $\|A\|_{L^\infty(\Omega)}$, d and n such that for every ball $B(x_0, R)$ such that $B(x_0, 4R) \subset \Omega$, with $u \leq 1$ on $B(x_0, 4R)$ and $|\{u \leq 0\} \cap B(x_0, 2R)| \geq \gamma |B(x_0, 2R)|$, we have that*

$$\sup_{B(x_0,R)} u \leq 1 - \tau. \tag{3.15}$$

Proof. We define

$$\lambda_k := 1 - 2^{-k}, \qquad u_k := \frac{1}{1 - \lambda_k}(u - \lambda_k)_+, \qquad A_k := B(x_0, 2R) \cap \{u_k > 0\}.$$

As, by hypothesis, $u \leq 1$ on $B(x_0, 4R)$, it follows that $u_k \leq 1$ on A_k. Furthermore, we have $u_k > 0$ on A_k by definition of A_k. By applying Theorem 3.2 to u with $c = \lambda_k$ we deduce that

$$\sup_{B(x_0,R)} u_k \lesssim \left(\fint_{B(x_0,2R)} |u_k|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} = \left(\frac{1}{|B(x_0,2R)|} \int_{A_k} |u_k|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \lesssim \left(\frac{|A_k|}{R^n} \right)^{\frac{1}{2}}. \tag{3.16}$$

If we can find a $\bar{k}$ such that $|A_{\bar{k}}| \leq \beta R^n$ for a sufficiently small β, we will have that $u_{\bar{k}} \leq \frac{1}{2}$ on $B(x_0, R)$.

Assume that $u_{k+1} > 0$ on some set. There, we then have that

$$\begin{aligned} 0 < \frac{u - (1 - \frac{1}{2^{k+1}})}{\frac{1}{2^{k+1}}} &= 2^{k+1}\left(u - 1 + \frac{1}{2^k} - \frac{1}{2^k} + \frac{1}{2^{k+1}}\right) \\ &\le 2^{k+1}\left(u - \lambda_k - \frac{1}{2^{k+1}}\right) = 2\left(u_k - \frac{1}{2}\right). \end{aligned}$$

This means that

$$A_{k+1} \cap \left\{0 < u_k < \frac{1}{2}\right\} = \emptyset. \tag{3.17}$$

We have $\left\{0 < u_k < \frac{1}{2}\right\} \subset A_k$ and $A_{k+1} \subset A_k$. This yields

$$|A_k| \ge \left|\left(\left\{0 < u_k < \frac{1}{2}\right\} \cap B(x_0, 2R)\right) \cup A_{k+1}\right| = \left|\left(\left\{0 < u_k < \frac{1}{2}\right\} \cap B(x_0, 2R)\right)\right| + |A_{k+1}|. \tag{3.18}$$

Note that if $x \in A_{k+1}$ then $u_k(x) \ge \frac{1}{2}$, and if $x \in B(x_0, 2R) \setminus A_{k+1}$ then $0 < u_k(x) < \frac{1}{2}$. Note also that we have $|B(x_0, 2R) \cap \{u_k = 0\}| \ge \gamma |B(x_0, 2R)|$. Therefore, we can use Poincaré's inequality to get

$$\begin{aligned} |A_{k+1}| &= 2\int_{A_{k+1}} \frac{1}{2}\,\mathrm{d}x \le 2\int_{A_{k+1}} \frac{1}{2}\,\mathrm{d}x + 2\int_{B(x_0,2R)\setminus A_{k+1}} u_k\,\mathrm{d}x \\ &= 2\int_{B(x_0,2R)} \min\left\{u_k, \frac{1}{2}\right\}\mathrm{d}x \\ &\lesssim R\int_{B(x_0,2R)} \left|\nabla\left(\min\left\{u_k, \frac{1}{2}\right\}\right)\right|\mathrm{d}x \\ &= R\int_{\{0<u_k<\frac{1}{2}\}\cap B(x_0,2R)} |\nabla u_k|\,\mathrm{d}x \\ &\le R\left(\int_{B(x_0,2R)} |\nabla u_k|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \left|\left\{0 < u_k < \frac{1}{2}\right\} \cap B(x_0, 2R)\right|^{\frac{1}{2}}. \end{aligned} \tag{3.19}$$

By inequality (3.3), with $\eta \in C_0^\infty(B(x_0, 4R))$ such that $\eta \equiv 1$ on $B(x_0, 2R)$ and $|\nabla\eta| \lesssim R^{-1}$, we find that

$$R^2\int_{B(x_0,2R)} |\nabla u_k|^2\,\mathrm{d}x \lesssim \int_{B(x_0,4R)} |u_k|^2\,\mathrm{d}x \lesssim R^n. \tag{3.20}$$

Now, by inserting inequalities (3.18) and (3.20) into inequality (3.19) yields

$$|A_{k+1}| \le R^{\frac{n}{2}}\left(|A_k| - |A_{k+1}|\right)^{\frac{1}{2}}.$$

Consequently, we can use the iteration from Lemma 3.33 to deduce the existence of a $\bar{k}$ such that $u_{\bar{k}} \le \frac{1}{2}$ on $B(x_0, R)$. This gives

$$2^{\bar{k}}\left(u - 1 + \frac{1}{2^{\bar{k}}}\right) \le \frac{1}{2}$$

and therefore

$$u \le 1 - \frac{1}{2^{\bar{k}+1}},$$

which proves the lemma for $\tau = \frac{1}{2^{\bar{k}+1}}$. □

Proof of Theorem 3.3. First, note that $\operatorname{osc}(c_1 u + c_2) = |c_1| \operatorname{osc} u$ for constants $c_1, c_2 \in \mathbb{R}$. We define $\tilde{u} := u - \frac{1}{2}\left(\inf_{B(x_0,2R)} u + \sup_{B(x_0,2R)} u\right)$ and $\tilde{\tilde{u}} = \frac{1}{\|\tilde{u}\|_{L^\infty(B(x_0,2R))}}\tilde{u}$. This gives $\operatorname{osc}_{B(x_0,2R)} \tilde{\tilde{u}} = 2$. As u is a solution to $-\operatorname{div}(A\nabla u) = 0$, $-u$ is a solution as well. Note that we have either $|\{\tilde{\tilde{u}} \le 0\} \cap B(x_0, 2R)| \ge \frac{1}{2}|B(x_0, 2R)|$ or $|\{-\tilde{\tilde{u}} \ge 0\} \cap B(x_0, 2R)| \ge \frac{1}{2}|B(x_0, 2R)|$. Thus we can assume that $|\{\tilde{\tilde{u}} \le 0\} \cap B(x_0, 2R)| \ge \frac{1}{2}|B(x_0, 2R)|$ without loss of generality. Of course, this means that $|\{\tilde{\tilde{u}}_+ = 0\} \cap B(x_0, 2R)| \ge \frac{1}{2}|B(x_0, 2R)|$; also, clearly, $-1 \le \tilde{\tilde{u}} \le 1$, and we can therefore apply Lemma 3.4 with $\gamma = \frac{1}{2}$ to deduce the existence of a $\tau \in (0, 1)$ such that

$$\begin{aligned} \operatorname*{osc}_{B(x_0,R)} \tilde{\tilde{u}} &\le 1 + \sup_{B(x_0,R)} \tilde{\tilde{u}} \le 2 - \tau = \operatorname*{osc}_{B(x_0,2R)} \tilde{\tilde{u}} - \tau \\ &= \operatorname*{osc}_{B(x_0,2R)} \tilde{\tilde{u}} - \frac{\tau}{2} \operatorname*{osc}_{B(x_0,2R)} \tilde{\tilde{u}} = \left(1 - \frac{\tau}{2}\right) \operatorname*{osc}_{B(x_0,2R)} \tilde{\tilde{u}} = 2^{-\alpha_1} \operatorname*{osc}_{B(x_0,2R)} \tilde{\tilde{u}}, \end{aligned} \tag{3.21}$$

with $\alpha_1 := -\log_2(1 - \frac{\tau}{2}) \in (0, 1)$ (because $\tau \in (0, 1)$). Hence, upon rescaling (3.21),

$$\operatorname*{osc}_{B(x_0,R)} \tilde{u} \le 2^{-\alpha_1} \operatorname*{osc}_{B(x_0,2R)} \tilde{u}.$$

Now, Lemma 3.34 with $\kappa = 0$, $C = 0$, $\sigma = \frac{1}{2}$ and $\varphi(\rho) := \operatorname{osc}_{B(x_0,2\rho)} \tilde{u}$ gives

$$\operatorname*{osc}_{B(x_0,r)} \tilde{u} \le \operatorname*{osc}_{B(x_0,2r)} \tilde{u} = \varphi(r) \lesssim \left(\frac{r}{R}\right)^\alpha \varphi(R),$$

for some $\alpha \in (0, \alpha_1)$ and $0 \le r \le R$. This then implies, with R held fixed, the assertion of the theorem by noting that $\operatorname{osc}_{B(x_0,r)} u = \operatorname{osc}_{B(x_0,r)} \tilde{u}$. □

3.2 A discrete Caccioppoli-type inequality

In this section, we prove a discrete version of inequality (3.3). First, we will define what we mean by an approximate solution to an elliptic partial differential equation. In our case, the existence is guaranteed by the Lax-Milgram Theorem.

Definition 3.5. *Let $A \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ be a uniformly elliptic matrix-valued function. Furthermore, let $F \in L^p(\Omega; \mathbb{R}^n)$ and $f \in L^q(\Omega)$ be given functions with $p > n$ and $q > n/2$. Furthermore, let $\mathcal{T}_h$ be a triangulation of the polyhedral domain Ω and V_h the corresponding finite element space. We call $u_h \in V_{h,0}$ an approximate solution to*

$$-\operatorname{div}(A\nabla u) = f - \operatorname{div} F$$

with zero Dirichlet boundary condition provided that we have

$$\int_\Omega A\nabla u_h \cdot \nabla\varphi_h \,\mathrm{d}x = \int_\Omega f\varphi_h \,\mathrm{d}x + \int_\Omega F \cdot \nabla\varphi_h \,\mathrm{d}x \tag{3.22}$$

for every $\varphi_h \in V_{h,0}$.

To prove the Caccioppoli inequality (3.3) we had to test equation (3.2) against $(u - c)_+|\eta|^2$. In the discrete case however, $(u_h - c)_+|\eta|^2$ is not an admissible test function in equation (3.22) because it is not in $V_{h,0}$. In particular, this leads to the conclusion that we will have to use a nodal version $(u_h - c)_+$ of $(u_h - c)_+$. This unfortunately means that $\nabla(u_h - c)_+$ and ∇u_h will not coincide on $\operatorname{supp}(u_h - c)_+$ any more. To overcome this problem, we will introduce the notion of *discrete subsolution*, prove a Caccioppoli-type inequality for discrete subsolutions, and prove that the nodal version $(u_h - c)_+$ is indeed a discrete subsolution under an additional assumption on the right-hand side, which we shall state.

Definition 3.6. *Let $A \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ be a uniformly elliptic matrix-valued function and let $F \in L^p(\Omega; \mathbb{R}^n)$ and $f \in L^q(\Omega)$ be given functions with with $p > n$ and $q > n/2$. Furthermore, let $\mathcal{T}_h$ be a triangulation of the polyhedral domain Ω and V_h the corresponding finite element space. We call $u_h \in V_{h,0}$ a discrete subsolution to*

$$-\mathrm{div}(A\nabla u) = f - \mathrm{div}F$$

provided that we have

$$\int_\Omega A\nabla u_h \cdot \nabla\varphi_h \,\mathrm{d}x \le \int_\Omega f\varphi_h \,\mathrm{d}x + \int_\Omega F \cdot \nabla\varphi_h \,\mathrm{d}x \tag{3.23}$$

for every $\varphi_h \in V_{h,0}$ with $\varphi_h \ge 0$ on $\overline{\Omega}$. Of course, every solution in the sense of equation (3.22) is automatically a subsolution in the sense of inequality (3.23).

Henceforth we shall concentrate on the case of $n \ge 3$ and recall that $2^\star := 2n/(n-2)$. The statements and proofs of the results below are easily adjusted in the case of $n = 2$.

Theorem 3.7 (Caccioppoli inequality for discrete subsolutions)**.** *Let $A \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ be a uniformly elliptic matrix-valued function (i.e. $A(x)v \cdot v \ge d|v|^2$ almost everywhere for all $v \in \mathbb{R}^n$) and let $F \in L^p(\Omega; \mathbb{R}^n)$ and $f \in L^q(\Omega)$ be given functions with $\frac{1}{p} = \frac{1}{n} - \frac{\delta}{n}$ and $\frac{1}{q} = \frac{2}{n} - \frac{\delta}{n}$, where $\delta > 0$. Furthermore, let $\mathcal{T}_h$ be a shape-regular triangulation of the polyhedral domain Ω and V_h the corresponding finite element space. There is a constant $C > 0$ that depends on n, $\|A\|_{L^\infty(\Omega)}$, d, and the shape-regularity constant Γ such that the following holds: Let $u_h \in V_h$ be a discrete subsolution to $-\mathrm{div}(A\nabla u) = f - \mathrm{div}F$ in the sense of inequality (3.23), let $\eta \in C_0^\infty(\Omega)$ be nonnegative, and define $\eta_h = \Pi_h\eta$. Then, we have*

$$\begin{aligned}\int_\Omega |\nabla u_h|^2|\eta_h|^2 \,\mathrm{d}x &\le C\int_\Omega |u_h|^2|\nabla\eta_h|^2 \,\mathrm{d}x \\ &+ C\left(\|F\|^2_{L^p(\Omega)} + \|f\|^2_{L^q(\Omega)}\right)\left(\|\eta_h\|^2_{L^{2^\star}(\operatorname{supp} u_h)} + \|\nabla\eta_h\|^2_{L^2(\operatorname{supp} u_h)}\right)|\operatorname{supp}\eta_h \cap \operatorname{supp} u_h|^{\frac{2\delta}{n}}.\end{aligned} \tag{3.24}$$

Proof. We write $\rho_h = \Pi_h\left(\eta_h^2\right)$ and test equation (3.23) against $\varphi_h = \Pi_h\left(\rho_h u_h\right)$ to get

$$L := \int_\Omega A\nabla u_h \cdot \nabla\Pi_h\left(\rho_h u_h\right)\,\mathrm{d}x \le \int_\Omega f\Pi_h\left(\rho_h u_h\right)\,\mathrm{d}x + \int_\Omega F\cdot\nabla\Pi_h\left(\rho_h u_h\right)\,\mathrm{d}x. \tag{3.25}$$

First we will consider the left-hand side of equation (3.25):

$$\begin{aligned} L &= \int_\Omega A\nabla u_h \cdot \nabla\Pi_h\left(\rho_h u_h\right)\,\mathrm{d}x \\ &= \int_\Omega A\nabla u_h \cdot \nabla\left(\rho_h u_h\right)\,\mathrm{d}x - \int_\Omega A\nabla u_h \cdot \nabla\left(\rho_h u_h - \Pi_h\left(\rho_h u_h\right)\right)\,\mathrm{d}x \\ &=: L_I - L_{II}. \end{aligned} \tag{3.26}$$

Next we decompose L_I in equation (3.26) once more to get

$$L_I = \int_\Omega \rho_h A\nabla u_h \cdot \nabla u_h\,\mathrm{d}x + \int_\Omega \left(A\nabla u_h\cdot\nabla\rho_h\right)u_h\,\mathrm{d}x =: L_{I_1} + L_{I_2}. \tag{3.27}$$

For L_{I_1} we use that A is uniformly elliptic (cf. inequality (3.1)) to get

$$\int_\Omega \rho_h A\nabla u_h\cdot\nabla u_h\,\mathrm{d}x \ge d\int_\Omega \rho_h|\nabla u_h|^2\,\mathrm{d}x =: \tilde{L}_{I_1}. \tag{3.28}$$

For L_{I_2} in equation (3.26) we use that $A \in L^\infty(\Omega;\mathbb{R}^{n\times n})$ to get

$$\begin{aligned} \left|L_{I_2}\right| &\le \|A\|_{L^\infty(\Omega)}\int_\Omega |\nabla u_h||\nabla\rho_h||u_h|\,\mathrm{d}x \\ &\lesssim \sum_{T\in\mathcal{T}_h}|T|\max_T|\nabla u_h|\max_T|\nabla\rho_h|\max_T|u_h| =: R_I. \end{aligned} \tag{3.29}$$

We will now estimate the term L_{II} in inequality (3.26). By Lemma 2.25 (cf. inequality (2.30)), $A\in L^\infty(\Omega;\mathbb{R}^{n\times n})$ and Remark 2.26 we get

$$\begin{aligned} \left|L_{II}\right| &\le \sum_{T\in\mathcal{T}_h}|T|\max_T|\nabla u_h|\,\max_T\left|\nabla(\rho_h u_h - \Pi_h(\rho_h u_h))\right| \\ &\lesssim \sum_{T\in\mathcal{T}_h}|T|\max_T|\nabla u_h|\max_T|\nabla\rho_h|\max_T|u_h| = R_I. \end{aligned} \tag{3.30}$$

Substituting inequalities (3.30) and (3.29) into equation (3.27), and inserting equation (3.27) and inequality (3.29) into (3.26), and then equation (3.26) back into (3.25) yields

$$\begin{aligned} \tilde{L}_{I_1} = \int_\Omega \rho_h|\nabla u_h|^2\,\mathrm{d}x &\lesssim \sum_{T\in\mathcal{T}_h}|T|\max_T|\nabla u_h|\max_T|\nabla\rho_h|\max_T|u_h| \\ &+ \int_\Omega f\Pi_h\left(\rho_h u_h\right)\,\mathrm{d}x + \int_\Omega F\cdot\nabla\Pi_h\left(\rho_h u_h\right)\,\mathrm{d}x =: R_I + R_{II} + R_{III}. \end{aligned} \tag{3.31}$$

Next, note that for any $T\in\mathcal{T}_h$ we have

$$\max_T|\nabla\rho_h| = \max_T\left|\nabla\Pi_h(\eta_h^2)\right| \lesssim \max_T|\nabla(\eta_h^2)| \lesssim \max_T|\eta_h|\max_T|\nabla\eta_h|, \tag{3.32}$$

where we have used the stability of the Lagrange projection Π_h in the second step. This allows us to estimate R_I from inequality (3.31):

$$R_I \lesssim \sum_{T\in\mathcal{T}_h} |T| \max_T |\nabla u_h| \max_T |\eta_h| \max_T |\nabla \eta_h| \max_T |u_h|.$$

Now we use Lemma 2.23 to get

$$R_I \lesssim \sum_T |T| \fint_T |\nabla u_h||\eta_h|\,|\nabla \eta_h||u_h|\;\mathrm{d}x.$$

By applying Young's inequality this yields

$$R_I \leq \sum_{T\in\mathcal{T}_h} |T| \left(\varepsilon \fint_T |\nabla u_h|^2 |\eta_h|^2\;\mathrm{d}x + C_\varepsilon \fint_T |\nabla \eta_h|^2 |u_h|^2\;\mathrm{d}x \right)$$

for any $\varepsilon > 0$. Finally, we note that we know from Lemma 2.27 that $\eta_h^2 \leq \rho_h$ and get

$$R_I \leq \varepsilon \tilde{L}_{I_1} + C_\varepsilon \int_\Omega |\nabla \eta_h|^2 |u_h|^2\;\mathrm{d}x \tag{3.33}$$

after performing the sums. We can now insert inequality (3.33) into (3.31) and absorb $\varepsilon\tilde{L}_{I_1}$ into the left-hand side to get

$$\begin{aligned}\tilde{L}_{I_1} = \int_\Omega \rho_h |\nabla u_h|^2\,\mathrm{d}x &\lesssim \int_\Omega |\nabla \eta_h|^2 |u_h|^2\;\mathrm{d}x \\ &+ \int_\Omega f \Pi_h\left(\rho_h u_h\right)\;\mathrm{d}x + \int_\Omega F\cdot\nabla\Pi_h\left(\rho_h u_h\right)\;\mathrm{d}x =: \tilde{R}_I + R_{II} + R_{III}.\end{aligned} \tag{3.34}$$

We will now consider R_{II} in inequality (3.34). Let $\mathcal{S} := \{T \in \mathcal{T}_h : T \subset \operatorname{supp}\eta_h \cap \operatorname{supp} u_h\}$; we then have that

$$|R_{II}| = \left|\int_\Omega f\Pi_h\left(\rho_h u_h\right)\;\mathrm{d}x\right| \leq \sum_{T\in\mathcal{S}} |T| \left(\fint_T |f|\,\mathrm{d}x\right) \max_T |u_h \rho_h|\,.$$

We can now use inequality (2.25) to deduce that

$$|R_{II}| \leq \sum_{T\in\mathcal{S}} |T| \left(\fint_T |f|\,\mathrm{d}x\right) \max_T |u_h| \left(\max_T |\eta_h|\right)^2.$$

Using Lemma 2.23, we get

$$\begin{aligned}|R_{II}| &\lesssim \sum_{T\in\mathcal{S}} |T| \left(\fint_T |f|\,\mathrm{d}x\right)\left(\fint_T |u_h\eta_h|\,\mathrm{d}x\right)\left(\fint_T |\eta_h|\,\mathrm{d}x\right) \\ &= \sum_{T\in\mathcal{S}} |T|^{-2} \|f\|_{L^1(T)} \|u_h\eta_h\|_{L^1(T)} \|\eta_h\|_{L^1(T)}.\end{aligned}$$

Recall that $f \in L^q(\Omega)$ with $\frac{1}{q} = \frac{2}{n} - \frac{\delta}{n}$ and $\delta > 0$. Hölder's inequality yields

$$
\begin{aligned}
\left|R_{II}\right| &\lesssim \sum_{T \in \mathcal{S}} |T|^{-2} \|f\|_{L^q(T)} |T|^{1-\frac{2}{n}+\frac{\delta}{n}} \|u_h \eta_h\|_{L^{2^\star}(T)} |T|^{\frac{1}{2}+\frac{1}{n}} \|\eta_h\|_{L^{2^\star}(T)} |T|^{\frac{1}{2}+\frac{1}{n}} \\
&= \sum_{T \in \mathcal{S}} |T|^{\frac{\delta}{n}} \|f\|_{L^q(T)} \|u_h \eta_h\|_{L^{2^\star}(T)} \|\eta_h\|_{L^{2^\star}(T)}.
\end{aligned}
$$

Now note that $\frac{1}{q} + \frac{1}{2^\star} + \frac{1}{2^\star} + \frac{\delta}{n} = 1$. We can use Hölder's inequality for sums to get

$$
\left|R_{II}\right| \lesssim \left(\sum_{T \in \mathcal{S}} |T|\right)^{\frac{\delta}{n}} \left(\sum_{T \in \mathcal{S}} \|f\|_{L^q(T)}^q\right)^{\frac{1}{q}} \left(\sum_{T \in \mathcal{S}} \|u_h \eta_h\|_{L^{2^\star}(T)}^{2^\star}\right)^{\frac{1}{2^\star}} \left(\sum_{T \in \mathcal{S}} \|\eta_h\|_{L^{2^\star}(T)}^{2^\star}\right)^{\frac{1}{2^\star}}. \tag{3.35}
$$

As the interiors of the elements T involved in the sums above are disjoint, we know that

$$
\sum_{T \in \mathcal{S}} \|g\|_{L^r(T)}^r = \|g\|_{L^r\left(\bigcup_{T \in \mathcal{S}} T\right)}^r \tag{3.36}
$$

for any $1 \leq r < \infty$ and any $g \in L^r\left(\bigcup_{T \in \mathcal{S}} T\right)$. Using this in inequality (3.35) and noting that $\mathcal{S} \subset \operatorname{supp} u_h$ gives

$$
\left|R_{II}\right| \lesssim \left|\operatorname{supp} \eta_h \cap \operatorname{supp} u_h\right|^{\frac{\delta}{n}} \|f\|_{L^q(\Omega)} \|u_h \eta_h\|_{L^{2^\star}(\Omega)} \|\eta_h\|_{L^{2^\star}(\operatorname{supp} u_h)}.
$$

Young's inequality and the Sobolev embedding theorem then yield

$$
\left|R_{II}\right| \lesssim \varepsilon \|\nabla (u_h \eta_h)\|_{L^2(\Omega)}^2 + C_\varepsilon \left|\operatorname{supp} \eta_h \cap \operatorname{supp} u_h\right|^{\frac{2\delta}{n}} \|f\|_{L^q(\Omega)}^2 \|\eta_h\|_{L^{2^\star}(\operatorname{supp} u_h)}^2. \tag{3.37}
$$

Note that

$$
\begin{aligned}
\|\nabla (u_h \eta_h)\|_{L^2(\Omega)}^2 &\lesssim \|u_h \nabla \eta_h\|_{L^2(\Omega)}^2 + \|\eta_h \nabla u_h\|_{L^2(\Omega)}^2 \\
&\leq \int_\Omega \rho_h |\nabla u_h|^2 \, \mathrm{d}x + \int_\Omega |u_h|^2 |\nabla \eta_h|^2 \, \mathrm{d}x,
\end{aligned} \tag{3.38}
$$

where we used Lemma 2.27 in the second step. Finally, substituting this into inequality (3.37) gives

$$
\begin{aligned}
\left|R_{II}\right| \lesssim \varepsilon \int_\Omega \rho_h |\nabla u_h|^2 \, \mathrm{d}x &+ \varepsilon \int_\Omega |u_h|^2 |\nabla \eta_h|^2 \, \mathrm{d}x \\
&+ C_\varepsilon \left|\operatorname{supp} \eta_h \cap \operatorname{supp} u_h\right|^{\frac{2\delta}{n}} \|f\|_{L^q(\Omega)}^2 \|\eta_h\|_{L^{2^\star}(\operatorname{supp} u_h)}^2.
\end{aligned} \tag{3.39}
$$

We will now focus on R_{III} in inequality (3.34). Splitting it up into a sum over simplices and using inequality (2.25) yields

$$
\begin{aligned}
|R_{III}| &= \left| \int_\Omega F \cdot \nabla\big(\Pi_h\left(u_h\rho_h\right)\big)\,\mathrm{d}x \right| \\
&\le \sum_{T\in\mathcal{S}} |T| \fint_T |F|\,\mathrm{d}x\, \max_T |\nabla \Pi_h\left(u_h\rho_h\right)| \\
&\lesssim \sum_{T\in\mathcal{S}} |T| \fint_T |F|\,\mathrm{d}x\, \max_T |\nabla\left(u_h\rho_h\right)| \\
&\le \sum_{T\in\mathcal{S}} |T| \fint_T |F|\,\mathrm{d}x\, \max_T |(\nabla u_h)\,\rho_h| + \sum_{T\in\mathcal{S}} |T| \fint_T |F|\,\mathrm{d}x\, \max_T |(\nabla \rho_h)\,u_h| \\
&=: R_{III_1} + R_{III_2}.
\end{aligned}
\tag{3.40}
$$

We rearrange R_{III_1} from inequality (3.40) and use Lemma 2.23:

$$
\begin{aligned}
R_{III_1} &\lesssim \sum_{T\in\mathcal{S}} |T| \fint_T |F|\,\mathrm{d}x \fint_T |(\nabla u_h)\eta_h|\,\mathrm{d}x \fint_T |\eta_h|\,\mathrm{d}x \\
&= \sum_{T\in\mathcal{S}} |T|^{-2} \int_T |F|\,\mathrm{d}x \int_T |(\nabla u_h)\eta_h|\,\mathrm{d}x \int_T |\eta_h|\,\mathrm{d}x.
\end{aligned}
$$

Recall that $F \in L^p(\Omega)$ with $\frac{1}{p} = \frac{1}{n} - \frac{\delta}{n}$. Hölder's inequality gives

$$
\begin{aligned}
R_{III_1} &\lesssim \sum_{T\in\mathcal{S}} |T|^{-2} \|F\|_{L^p(T)} |T|^{1-\frac{1}{n}+\frac{\delta}{n}} \|(\nabla u_h)\eta_h\|_{L^2(T)} |T|^{\frac{1}{2}} \|\eta_h\|_{L^{2^\star}(T)} |T|^{\frac{1}{2}+\frac{1}{n}} \\
&= \sum_{T\in\mathcal{S}} |T|^{\frac{\delta}{n}} \|F\|_{L^p(T)} \|(\nabla u_h)\eta_h\|_{L^2(T)} \|\eta_h\|_{L^{2^\star}(T)}.
\end{aligned}
$$

Now Hölder's inequality for sums and the identity (3.36) give

$$
R_{III_1} \lesssim |\operatorname{supp}\eta_h \cap \operatorname{supp} u_h|^{\frac{\delta}{n}} \|F\|_{L^p(\Omega)} \|(\nabla u_h)\eta_h\|_{L^2(\Omega)} \|\eta_h\|_{L^{2^\star}(\operatorname{supp} u_h)}.
$$

Finally, we can use $\eta_h^2 \le \rho_h$ (by Lemma 2.27) and Young's inequality to get

$$
R_{III_1} \lesssim \varepsilon \int_\Omega |\nabla u_h|^2 \rho_h\,\mathrm{d}x + C_\varepsilon \|F\|^2_{L^p(\Omega)} |\operatorname{supp}\eta_h \cap \operatorname{supp} u_h|^{\frac{2\delta}{n}} \|\eta_h\|^2_{L^{2^\star}(\operatorname{supp} u_h)}. \tag{3.41}
$$

Next we consider R_{III_2} from inequality (3.40). Recall $\max_T |\nabla\rho_h| \lesssim \max_T |\eta_h| \max_T |\nabla\eta_h|$ as in inequality (3.32). We get

$$
R_{III_2} \lesssim \sum_{T\in\mathcal{S}} |T| \fint_T |F|\,\mathrm{d}x\, \max_T |u_h| \max_T |\eta_h| \max_T |\nabla\eta_h|.
$$

With Lemma 2.23, this becomes

$$
R_{III_2} \lesssim \sum_{T\in\mathcal{S}} |T|^{-2} \int_T |F|\,\mathrm{d}x \int_T |u_h\eta_h|\,\mathrm{d}x \int_T |\nabla\eta_h|\,\mathrm{d}x.
$$

Completely analogously to R_{III_1}, we get

$$R_{III_2} \leq \varepsilon \|\nabla(u_h \eta_h)\|^2_{L^2(\Omega)} + C_\varepsilon |\operatorname{supp} \eta_h \cap \operatorname{supp} u_h|^{\frac{2\delta}{n}} \|F\|^2_{L^p(\Omega)} \|\nabla \eta_h\|^2_{L^2(\operatorname{supp} u_h)}.$$

We use inequality (3.38) to deduce that

$$\begin{aligned} R_{III_2} \leq \varepsilon \int_\Omega |\nabla u_h|^2 \rho_h \,\mathrm{d}x + \varepsilon \int_\Omega |u_h|^2 |\nabla \eta_h|^2 \,\mathrm{d}x \\ + C_\varepsilon |\operatorname{supp} \eta_h \cap \operatorname{supp} u_h|^{\frac{2\delta}{n}} \|F\|^2_{L^p(\Omega)} \|\nabla \eta_h\|^2_{L^2(\operatorname{supp} u_h)}. \end{aligned} \tag{3.42}$$

Inserting inequalities (3.41) and (3.42) into inequality (3.40) and using Poincaré's inequality on the last factor in the second summand in (3.41) gives, after renaming constants,

$$\begin{aligned} |R_{III}| \leq \varepsilon \int_\Omega |\nabla u_h|^2 \rho_h \,\mathrm{d}x + \varepsilon \int_\Omega |u_h|^2 |\nabla \eta_h|^2 \,\mathrm{d}x \\ + C_\varepsilon |\operatorname{supp} \eta_h \cap \operatorname{supp} u_h|^{\frac{2\delta}{n}} \|F\|^2_{L^p(\Omega)} \|\nabla \eta_h\|^2_{L^2(\operatorname{supp} u_h)}. \end{aligned} \tag{3.43}$$

We can finally substitute inequalities (3.39) and (3.43) into inequality (3.34) and absorb the

$$\varepsilon \int_\Omega |\nabla u_h|^2 \rho_h \,\mathrm{d}x$$

term into the left-hand side. This yields

$$\begin{aligned} \int_\Omega \rho_h |\nabla u_h|^2 \,\mathrm{d}x \lesssim \int_\Omega |\nabla \eta_h|^2 |u_h|^2 \,\mathrm{d}x \\ + |\operatorname{supp} \eta_h \cap \operatorname{supp} u_h|^{\frac{2\delta}{n}} \left(\|\eta_h\|^2_{L^{2^\star}(\operatorname{supp} u_h)} + \|\nabla \eta_h\|^2_{L^2(\operatorname{supp} u_h)} \right) \left(\|F\|^2_{L^p(\Omega)} + \|f\|^2_{L^q(\Omega)} \right). \end{aligned} \tag{3.44}$$

Using $\eta_h^2 \leq \rho_h$ from Lemma 2.27 on the left-hand side proves inequality (3.23) and the theorem. □

This discrete Caccioppoli inequality has the following direct corollary.

Corollary 3.8. *Let $A \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ be a uniformly elliptic matrix-valued function (i.e. $A(x)v \cdot v \geq d|v|^2$ for all $v \in \mathbb{R}^n$) and let $F \in L^p(\Omega; \mathbb{R}^n)$ and $f \in L^q(\Omega)$ be given functions with $\frac{1}{p} = \frac{1}{n} - \frac{\delta}{n}$ and $\frac{1}{q} = \frac{2}{n} - \frac{\delta}{n}$, where $\delta > 0$. Furthermore, let $\mathcal{T}_h$ be a shape-regular triangulation of the polyhedral domain Ω and V_h the corresponding finite element space. Then, there is a constant $C > 0$ that only depends on $\|A\|_{L^\infty(\Omega)}$, d, n and the shape-regularity constant Γ, such that the following holds: Let $u_h \in V_h$ be a discrete subsolution to $-\operatorname{div}(A\nabla u) = f - \operatorname{div} F$ in the sense of inequality (3.23), let $\eta \in C_0^\infty(\Omega)$ be nonnegative and define $\eta_h := \Pi_h \eta$. Then, we have*

$$\begin{aligned} \int_\Omega |\nabla (u_h \eta_h)|^2 \,\mathrm{d}x \leq C \int_\Omega |u_h|^2 |\nabla \eta_h|^2 \,\mathrm{d}x \\ + \left(\|F\|^2_{L^p} + \|f\|^2_{L^q} \right) \left(\|\nabla \eta_h\|^2_{L^2(\operatorname{supp} u_h)} + \|\eta_h\|^2_{L^{2^\star}(\operatorname{supp} u_h)} \right) |\operatorname{supp} \eta_h \cap \operatorname{supp} u_h|^{\frac{2\delta}{n}}. \end{aligned} \tag{3.45}$$

Proof. We use that

$$\int_\Omega |\nabla (u_h\eta_h)|^2 \lesssim \int_\Omega |\nabla u_h|^2 |\eta_h|^2 \,\mathrm{d}x + \int_\Omega |u_h|^2 |\nabla \eta_h|^2 \,\mathrm{d}x$$

and apply Theorem 3.7. □

We will now introduce the *nodal maximum* of two functions in V_h. This will also lead to a suitable notion of the *positive part* of a continuous piecewise affine function.

Definition 3.9. *Let $\mathcal{T}_h$ be a triangulation of the polyhedral domain Ω and V_h the corresponding finite element space of continuous piecewise affine functions. We define the nodal maximum of two functions $v_h \in V_h$ and $w_h \in V_h$ as*

$$v_h \curlyvee w_h := \sum_i (v_h(x_i) \vee w_h(x_i))\, \psi_i.$$

Furthermore, we define the nodal positive part of a function $v_h \in V_h$ as

$$(v_h)_+ := v_h \curlyvee 0 = \sum_i (v_h(x_i))_+\, \psi_i,$$

where we have denoted the point-wise positive part by $t_+ = t \vee 0$.

We will also need the following technical assumption of F, which will be referred to as assumption $(\star)$, to be able to prove a subsolution property for the nodal maximum of two subsolutions to equation (3.23).

Definition 3.10. *For $p > n$, let $F \in L^p(\Omega; \mathbb{R}^n)$. We will say that F satisfies assumption $(\star)$ if there exists a $G \in L^p(\Omega; \mathbb{R}^n)$ such that*

$$\int_\Omega G \cdot \nabla\varphi \,\mathrm{d}x \geq \left| \int_\Omega F \cdot \nabla\varphi \,\mathrm{d}x \right| \geq 0 \tag{3.46}$$

for any nonnegative $\varphi \in C_0^\infty(\Omega)$. By density, this inequality extends to all nonnegative $\varphi \in W_0^{1,p'}(\Omega)$.

Remark 3.11. *The condition (3.46) is restrictive, but still admits several nontrivial cases. First and foremost, it includes every $F \in L^p(\Omega; \mathbb{R}^n)$ where* $\mathrm{div} F$ *has a fixed sign as a distribution, with $G := -(sgn\, \mathrm{div} F)\, F$. It also includes cases such as the following, where* $\mathrm{div} F$ *changes sign in the sense of distributions: suppose that $n = 2$, $\Omega = (-2, 2)^2$ and, for $x = (x_1, x_2)^\mathrm{T} \in \Omega$, let*

$$F(x) := \begin{cases} -e_1 & \text{for } |x_1| \geq 1, \\ e_1 & \text{for } |x_1| < 1, \end{cases}$$

where $e_1 = (1,0)^{\mathrm{T}}$*. Obviously* $F \in L^\infty(\Omega;\mathbb{R}^2)$ *and* $\mathrm{div}F = 2\delta_{-1}\otimes\chi_{(-2,2)} - 2\delta_1\otimes\chi_{(-2,2)}$*, where* δ_t *denotes the Dirac-distribution concentrated on the point* $t\in\mathbb{R}$*. Consider the function* G *defined by*

$$G(x) = \begin{cases} 3e_1 & \text{for } x_1 \le -1,\\ e_1 & \text{for } -1 < x_1 < 1,\\ -e_1 & \text{for } x_1 \ge 1,\end{cases}$$

which gives $\mathrm{div}G = -2\delta_{-1}\otimes\chi_{(-2,2)} - 2\delta_1\otimes\chi_{(-2,2)}$*, whereby* $\mathrm{div}(G\pm F) = -4\delta_{\pm1}\otimes\chi_{(-2,2)} \le 0$ *in the sense of distributions, and therefore assumption* $(\star)$ *is satisfied in this case. In general,* $(\star)$ *is fulfilled if there is an* L^p*-function* G *with* $-\mathrm{div}G \ge |\mathrm{div}F|$ *in the sense of distributions. The restrictive feature that excludes a variety of functions* $F\in L^p(\Omega;\mathbb{R}^n)$ *where* $\mathrm{div}F$ *is a signed measure is the fact that we have required* G *to be in* $L^p(\Omega,\mathbb{R}^n)$*, as well.*

This leads to the following theorem, which connects the notions of *positive part* and *subsolution.*

Theorem 3.12. *Let* $\mathcal{T}_h$ *be an* A*-nonobtuse triangulation of the polyhedral domain* Ω*. Let* $u_h\in V_h$ *be a discrete subsolution to* $-\mathrm{div}(A\nabla v_h) = f_1 - \mathrm{div}F_1$ *and suppose that* $v_h \in V_h$ *is a discrete subsolution to* $-\mathrm{div}(A\nabla w_h) = f_2 - \mathrm{div}F_2$ *for* L^p*-functions* F_1 *and* F_2 *(for* $p>n$*) and* L^q*-functions* f_1 *and* f_2*. Suppose further that* F_1 *satisfies assumption* $(\star)$ *from Definition 3.10 with dominating function* $G_1\in L^p(\Omega;\mathbb{R}^n)$ *and* F_2 *satisfies assumption* $(\star)$ *with dominating function* $G_2\in L^p(\Omega,\mathbb{R}^n)$*. Then,* $u_h \curlyvee v_h$ *is a discrete subsolution to*

$$-\mathrm{div}(A\nabla(v_h\curlyvee w_h)) \le f_1\vee f_2 - \mathrm{div}(G_1+G_2).$$

Proof. As a first step, we will show that

$$\int_\Omega A\nabla(u_h\curlyvee v_h)\cdot\nabla\psi_j\,\mathrm{d}x \le \int_\Omega (f_1\vee f_2)\psi_j\,\mathrm{d}x + \left(\int_\Omega F_1\cdot\nabla\psi_j\,\mathrm{d}x\right)\vee\left(\int_\Omega F_2\cdot\nabla\psi_j\,\mathrm{d}x\right) \quad (3.47)$$

for all j. So we fix an arbitrary j and assume w.l.o.g. that $u_h(x_j)\ge v_h(x_j)$ and therefore $u_h(x_j)\vee v_h(x_j) = u_h(x_j)$. As $\mathcal{T}_h$ is A-nonobtuse, we have $\int_\Omega A\nabla\psi_i\cdot\nabla\psi_j\,\mathrm{d}x\le 0$ for all $i\ne j$ from equation (2.8) and find

$$\begin{aligned}&\int_\Omega A\nabla u_h\cdot\nabla\psi_j\,\mathrm{d}x = \sum_i u_h(x_i)\int_\Omega A\nabla\psi_i\cdot\nabla\psi_j\,\mathrm{d}x\\ &= u_h(x_j)\int_\Omega A\nabla\psi_j\cdot\nabla\psi_j\,\mathrm{d}x + \sum_{i\ne j}u_h(x_i)\int_\Omega A\nabla\psi_i\cdot\nabla\psi_j\,\mathrm{d}x\\ &\ge (u_h(x_j)\vee v_h(x_j))\int_\Omega A\nabla\psi_j\cdot\nabla\psi_j\,\mathrm{d}x + \sum_{i\ne j}(u_h(x_i)\vee v_h(x_i))\int_\Omega A\nabla\psi_i\cdot\nabla\psi_j\,\mathrm{d}x\\ &= \int_\Omega A\nabla(u_h\curlyvee v_h)\cdot\nabla\psi_j\,\mathrm{d}x.\end{aligned} \quad (3.48)$$

On the other hand, we have

$$\int_\Omega (f_1 \vee f_2)\psi_j \,\mathrm{d}x \geq \int_\Omega f_1 \psi_j \,\mathrm{d}x \tag{3.49}$$

and

$$\left(\int_\Omega F_1 \cdot \nabla\psi_j \,\mathrm{d}x\right) \vee \left(\int_\Omega F_2 \cdot \nabla\psi_j \,\mathrm{d}x\right) \geq \int_\Omega F_1 \cdot \nabla\psi_j \,\mathrm{d}x. \tag{3.50}$$

Combining inequalities (3.48), (3.49) and (3.50) with

$$\int_\Omega A\nabla u_h \cdot \nabla\psi_j \,\mathrm{d}x \leq \int_\Omega F_1 \cdot \nabla\psi_j + f_1\psi_j \,\mathrm{d}x,$$

which follows from the fact that u_h is a discrete subsolution to $-\mathrm{div}(A\nabla u_1) = f_1 - \mathrm{div}F_1$, yields inequality (3.47). We will now use the fact that F_1 and F_2 satisfy assumption $(\star)$ and therefore inequality (3.46) with dominating functions G_1 and G_2 respectively. We thus have that

$$\begin{aligned}\left(\int_\Omega F_1 \cdot \nabla\psi_j \,\mathrm{d}x\right) \vee \left(\int_\Omega F_2 \cdot \nabla\psi_j \,\mathrm{d}x\right) &\leq \left(\int_\Omega G_1 \cdot \nabla\psi_j \,\mathrm{d}x\right) \vee \left(\int_\Omega G_2 \cdot \nabla\psi_j \,\mathrm{d}x\right) \\ &\leq \int_\Omega (G_1 + G_2) \cdot \nabla\psi_j \,\mathrm{d}x.\end{aligned} \tag{3.51}$$

Together with inequality (3.47) this gives

$$\int_\Omega A\nabla(u_h \curlyvee v_h) \cdot \nabla\psi_j \,\mathrm{d}x \leq \int_\Omega (f_1 \vee f_2)\psi_j \,\mathrm{d}x + \int_\Omega (G_1 + G_2) \cdot \nabla\psi_j \,\mathrm{d}x. \tag{3.52}$$

In general, for a nonnegative $\varphi_h \in V_{h,0}$ we write $\varphi_h = \sum_j \varphi_h(x_j)\psi_j$ and use the fact that both sides of inequality (3.52) are linear in ψ_j:

$$\begin{aligned}\int_\Omega A\nabla(u_h \curlyvee v_h) \cdot \nabla\varphi_h \,\mathrm{d}x &= \sum_j \varphi_h(x_j) \int_\Omega A\nabla(u_h \vee v_h) \cdot \nabla\psi_j \,\mathrm{d}x \\ &\leq \sum_j \varphi_h(x_j) \int_\Omega (f_1 \vee f_2)\psi_j \,\mathrm{d}x + \int_\Omega (G_1 + G_2) \cdot \nabla\psi_j \,\mathrm{d}x \\ &= \int_\Omega (f_1 \vee f_2)\varphi_h \,\mathrm{d}x + \int_\Omega (G_1 + G_2) \cdot \nabla\varphi_h \,\mathrm{d}x.\end{aligned}$$

This implies that $u_h \curlyvee v_h$ is a discrete subsolution for the right-hand side $f_1 \vee f_2 - \mathrm{div}(G_1 + G_2)$ and proves the theorem. □

By setting $f_1 = f$, $F_1 = F$, $f_2 = 0$ and $F_2 = 0$ we deduce the following immediate corollary.

Corollary 3.13. *Let $\mathcal{T}_h$ be an A-nonobtuse triangulation of the polyhedral domain Ω. Let u_h be a discrete subsolution to $-\mathrm{div}(A\nabla u) = f - \mathrm{div}F$ for $F \in L^p(\Omega)$ and $f \in L^q(\Omega)$, with p and q as previously. Furthermore, suppose that F satisfies assumption $(\star)$ from Definition 3.10 with dominating function $G \in L^p(\Omega, \mathbb{R}^n)$ and let $c \in \mathbb{R}$ be a constant. Then the nodal positive part $(u_h - c)_+$ is a discrete subsolution to $-\mathrm{div}(A\nabla w) = |f| - \mathrm{div}G$.*

Proof. We have

$$\int_\Omega A\nabla(u_h - c)_+ \cdot \nabla\varphi_h \,\mathrm{d}x \leq \int_\Omega F \cdot \nabla\varphi_h + |f|\varphi_h \,\mathrm{d}x.$$

for any nonnegative $\varphi_h \in V_{h,0}$. Recall that we defined the nodal positive part as $(u_h - c)_+ = (u_h - c) \curlyvee 0$ in Definition 3.9. We have $\pm f \vee 0 \leq |f|$ and $\pm \int_\Omega F \cdot \nabla\varphi \,\mathrm{d}x \leq \int_\Omega G \cdot \nabla\varphi \,\mathrm{d}x$. Because $v_h \equiv 0$ solves $-\mathrm{div}(A\nabla v_h) = 0$, we can apply Theorem 3.12 with $f_1 = f$, $F_1 = F$, $f_2 \equiv 0$ and $F_2 \equiv 0$ to deduce the assertion of the corollary. ☐

Finally, Corollary 3.13 yields the desired discrete Caccioppoli-type inequalities for the truncated functions $(u_h - c)_+$ with $c \in \mathbb{R}$, which we now state and prove.

Corollary 3.14. *Let $\mathcal{T}_h$ be an A-nonobtuse triangulation of the polyhedral domain Ω. Let u_h be a discrete subsolution to $-\mathrm{div}(A\nabla u) = f - \mathrm{div}F$ for $F \in L^p(\Omega;\mathbb{R}^n)$ and $f \in L^q(\Omega)$, with p and q as previously defined. Furthermore, suppose that F satisfies assumption ($\star$) from Definition 3.10 with dominating function $G \in L^p(\Omega,\mathbb{R}^n)$. Let η_h be as in Theorem 3.7. Then, there are constants C_1 and C_2 that only depend on $\|A\|_{L^\infty(\Omega)}$, d, n and the shape-regularity constant Γ, such that, for any $c \in \mathbb{R}$ we have the following bounds:*

$$\begin{aligned} \int_\Omega |\nabla(u_h - c)_+\eta_h|^2 \,\mathrm{d}x \leq C_1 \int_\Omega (u_h - c)_+^2 |\nabla\eta_h|^2 \,\mathrm{d}x + \left(\|G\|_{L^p(\Omega)}^2 + \|f\|_{L^q(\Omega)}^2\right) \\ \cdot \left(\|\nabla\eta_h\|_{L^2(\mathrm{supp}\,(u_h - c)_+)} + \|\eta_h\|_{L^{2^\star}(\mathrm{supp}\,(u_h - c)_+)}^2\right) |\mathrm{supp}\,\eta_h \cap \mathrm{supp}\,(u_h - c)_+|^{\frac{2\delta}{n}}, \end{aligned} \tag{3.53}$$

and

$$\begin{aligned} \int_\Omega |\nabla\left((u_h - c)_+\eta_h\right)|^2 \,\mathrm{d}x \leq C_2 \int_\Omega (u_h - c)_+^2 |\nabla\eta_h|^2 \,\mathrm{d}x + \left(\|G\|_{L^p(\Omega)}^2 + \|f\|_{L^q(\Omega)}^2\right) \\ \cdot \left(\|\nabla\eta_h\|_{L^2(\mathrm{supp}\,(u_h - c)_+)} + \|\eta_h\|_{L^{2^\star}(\mathrm{supp}\,(u_h - c)_+)}^2\right) |\mathrm{supp}\,\eta_h \cap \mathrm{supp}\,(u_h - c)_+|^{\frac{2\delta}{n}}. \end{aligned} \tag{3.54}$$

Proof. Corollary 3.13 guarantees that $(u_h - c)_+$ is a discrete subsolutions to $-\mathrm{div}(A\nabla w) = |f| - \mathrm{div}G$. Furthermore $(u_h - c)_+$ is obviously nonnegative. Thus we can apply Theorem 3.7 and Corollary 3.8 to deduce the stated claims. ☐

3.3 Interior C^α-estimates for approximate solutions

We will now prove the desired uniform a priori C^α-bound for sequences of continuous piecewise affine finite element approximations in the interior of Ω. To this end, we will first prove an L^∞-bound, and will then deduce the discrete C^α-bound. We emphasize that we have not, so far, assumed any kind of (quasi-)uniformity of the triangulation, nor shall we do so. Our results therefore apply on graded and adaptively refined triangulations. The main result of this subsection is encapsulated in the following theorem.

Theorem 3.15. *Let $\frac{1}{p} = \frac{1}{n} - \frac{\delta}{n}$ and $\frac{1}{q} = \frac{2}{n} - \frac{\delta}{n}$ for some $\delta > 0$, let $f \in L^q(\Omega)$, suppose that the function $F \in L^p(\Omega;\mathbb{R}^n)$ satisfies assumption ($\star$) from Definition 3.10 with dominating function $G \in L^p(\Omega;\mathbb{R}^n)$ and let $A \in L^\infty(\Omega;\mathbb{R}^{n\times n})$ be a uniformly elliptic matrix-valued function (i.e. $A(x)v \cdot v \geq d|v|^2$ with $d > 0$ almost everywhere for all $v \in \mathbb{R}^n$). Furthermore, let $u_h \in V_h$ be the finite element approximation to the solution u of $-\mathrm{div}(A\nabla u) = f - \mathrm{div}F$ in the sense of Definition 3.5 on a shape-regular, A-nonobtuse triangulation $\mathcal{T}_h$ of the polyhedral domain Ω with respective finite element space V_h, i.e.,*

$$\int_\Omega A\nabla u_h \cdot \nabla\varphi_h \,\mathrm{d}x = \int_\Omega f\varphi_h \,\mathrm{d}x + \int_\Omega F \cdot \nabla\varphi_h \,\mathrm{d}x \tag{3.55}$$

for any function $\varphi_h \in V_{h,0}$. Furthermore, assume that $u_h|_{\partial\Omega} \in C^\beta(\partial\Omega)$. Let κ and Q be the constants from equation (2.13) and inclusion (2.17). Assume that $x_0 \in T$ for some $T \in \mathcal{T}_h$ and $B(x_0, 4\kappa^{-1}QR') \subset \Omega$ for some $R' \geq h_T$. Then, there is an $\alpha > 0$ and a $C > 0$ that only depend only on $\|A\|_{L^\infty(\Omega)}$, d, δ, the shape-regularity constant Γ, the dimension n, $\|f\|_{L^q(\Omega)}$, $\|G\|_{L^p(\Omega)}$ and $\|u_h\|_{L^\infty(\partial\Omega)}$, such that, for every ball $B(x_0,R) \subset \Omega$,

$$\operatorname*{osc}_{B(x_0,R)} u_h \leq CR^\alpha. \tag{3.56}$$

Remark 3.16. *The assumption $B(x_0, 4\kappa^{-1}QR') \subset \Omega$ for some $R' \geq h_T$ requires a certain fineness of the mesh. If there is no ball that satisfies this condition, we refer to the discussion of regularity near the boundary in Section 3.4 of this Chapter.*

The hypotheses of Theorem 3.15 will be assumed to hold throughout this section. We will need a few additional technical lemmas concerning the projections of the functions φ_k, which we have defined in equation (3.8). Before stating these, we introduce the following notation.

Definition 3.17. *For a constant $\lambda_\infty > 0$ that is to be chosen later, a radius $R > 0$, $x_0 \in \Omega$, such that $B(x_0, 2R) \subset \Omega$, and $k = 0, 1, 2...$, let*

$$\lambda_k := \left(1 - 2^{-k}\right)\lambda_\infty \qquad \text{and} \qquad B_k := B\left(x_0, \left(1 + 2^{-k}\right)R\right),$$

and consider a sequence of nonnegative C^∞-functions $\tilde\eta_k$, such that

$$\chi_{B_k} \leq \tilde\eta_k \leq \chi_{B_{k+1}} \qquad \text{and} \qquad |\nabla\tilde\eta_k| \leq 2^{k+1}R^{-1}.$$

We then define, for any triangulation $\mathcal{T}_h$,

$$\eta_k := \Pi_h\tilde\eta_k.$$

Lemma 3.18. *Let B_k and η_k be defined as in Definition 3.17. Then, we have*

(a) $\max_\Omega |\nabla \eta_k| \lesssim 2^k R^{-1}$;

(b) *If* $\max_T |\eta_{k+1}| > 0$ *for some* $T \in \mathcal{T}_h$*, then we have* $\max_T |\eta_k| = 1$;

(c) $\max_T |\nabla \eta_{k+1}| \lesssim 2^{k+1} R^{-1} \max_T |\eta_k|$;

(d) *If* a *is a polynomial, we have* $⨍_T |a|^2 |\nabla \eta_{k+1}|^2 \, \mathrm{d}x \lesssim R^{-2} 2^{2(k+1)} ⨍_T |a|^2 \eta_k^2 < dx$.

The implicit constants only depend on the shape-regularity constant Γ *and the dimension* n.

Proof. Assertion (a) is clear since

$$\max_\Omega |\nabla \eta_k| = \max_\Omega |\nabla (\Pi_h \tilde{\eta}_k)| \leq \max_\Omega |\nabla \tilde{\eta}_k| \lesssim 2^k R^{-1}$$

by equation (2.26). For assertion (b), we note that if $\max_T |\eta_{k+1}| > 0$, there is at least one node x_0 with $\eta_{k+1}(x_0) > 0$ and therefore $x_0 \in B_k$ and $\eta_k(x_0) = \tilde{\eta}_k(x_0) = 1$. Assertion (c) is a direct consequence of assertions (a) and (b). The inequality stated in assertion (d) follows by using Lemma 2.23:

$$⨍_T |a|^2 |\nabla \eta_{k+1}|^2 \, \mathrm{d}x \sim \max_T |a|^2 \max_T |\nabla \eta_{k+1}|^2$$
$$\lesssim 2^{2(k+1)} R^{-2} \max_T |a|^2 \max_T |\eta_k|^2 \sim 2^{2(k+1)} R^{-2} ⨍_T |a|^2 \eta_k^2 \, \mathrm{d}x.$$

That completes the proof of the lemma. $\square$

We will also need a discrete counterpart of the weak-type estimate (3.14), which we shall now state.

Lemma 3.19. *Consider* η_k *as in Definition 3.17 and let* $A_k := \{\eta_k^2 |(v_h - \lambda_k - c_0)_+|^2 > 0\}$ *for some* $c_0 \in \mathbb{R}$ *and* $v_h \in V_h$*. Then, we have*

$$|A_{k+1}| \leq C \frac{2^{2k}}{\lambda_\infty^2} \int_\Omega |\eta_k (u_h - \lambda_k - c_0)_+|^2 \, \mathrm{d}x \tag{3.57}$$

where the constant $C > 0$ *only depends on* n *and the shape-regularity constant* Γ.

Proof. First observe that A_{k+1} is a union of finitely many $T_i \in \mathcal{T}_h$. On every such T_i, we have $\max_{T_i} \eta_k = 1$ by Lemma 3.18, assertion (b), as $\max_{T_i} \eta_{k+1} > 0$. For at least one node x_0 of T_i we have $v_h(x_0) > \lambda_{k+1} + c_0$, and therefore

$$(v_h(x_0) - \lambda_k - c_0)_+ = v_h(x_0) - \lambda_k - c_0 > \lambda_{k+1} - \lambda_k = 2^{-k} \lambda_\infty.$$

Using this and Lemma 2.23 we get

$$
\begin{aligned}
|T_i| &\le \frac{2^{2k}}{\lambda_\infty^2} |T_i| \max_{T_i} (v_h - \lambda_k - c_0)_+^2 \max_{T_i} \eta_k^2 \\
&\sim \frac{2^{2k}}{\lambda_\infty^2} |T_i| \fint_{T_i} (v_h - \lambda_k - c_0)_+^2 \eta_k^2 \, \mathrm{d}x = \frac{2^{2k}}{\lambda_\infty^2} \int_{T_i} (v_h - \lambda_k - c_0)_+^2 \eta_k^2 \, \mathrm{d}x.
\end{aligned}
$$

As the T_i are disjoint and $\operatorname{supp} \eta_k \subset A_{k+1}$, summing over all i yields the assertion of the lemma. □

This result allows us to prove a discrete version of the L^∞-estimate from Theorem 3.2. We shall again confine ourselves to the case when $n \ge 3$ and write $2^\star := 2n/(n-2)$; the proofs below are easily adjusted when $n = 2$.

Theorem 3.20. *Assume that all the assumptions of Theorem 3.15 are satisfied. Furthermore, let $x_0 \in T$ for some $T \in \mathcal{T}_h$ be a point such that $B(x_0, 2R) \subset \Omega$ for some $R \ge h_T$. There is a constant $C > 0$ that only depends on n, $\|A\|_{L^\infty(\Omega)}$, d, δ and the shape-regularity constant Γ, such that*

$$
\max_{\Omega'(B(x_0,R))} (u_h - c)_+^2 \le C \fint_{\Omega(B(x_0,2R))} (u_h - c)_+^2 \, \mathrm{d}x + C \left(\|G\|_{L^p}^2 + \|f\|_{L^q}^2 \right) R^{2\delta} \tag{3.58}
$$

for all $c \in \mathbb{R}$, with $\Omega'(B(x_0,R))$ as defined in (2.12).

Proof. For $B = B(x_0, R)$ we use the notation $2B := B(x_0, 2R)$. Using the notions from Definition 3.17, we define the sequence

$$
a_k := \fint_{\Omega(2B)} (u_h - \lambda_k - c)_+^2 \eta_k^2 \, \mathrm{d}x \qquad \text{for } k = 0, 1, 2, \ldots .
$$

We write $A_k := \{(u_h - \lambda_k - c)_+^2 \eta_k^2 > 0\}$ as in Lemma 3.19 and use Hölder's inequality to deduce that

$$
\begin{aligned}
a_{k+1} &= \fint_{\Omega(2B)} (u_h - \lambda_{k+1} - c)_+^2 \eta_{k+1}^2 \, \mathrm{d}x \\
&\lesssim \left(\fint_{\Omega(2B)} \left| (u_h - \lambda_{k+1} - c)_+ \eta_{k+1} \right|^{2^\star} \mathrm{d}x \right)^{\frac{2}{2^\star}} \left(\frac{|A_{k+1}|}{|\Omega(2B)|} \right)^{\frac{2}{n}} .
\end{aligned} \tag{3.59}
$$

We have assumed that $R \ge h_T$. Recall that there is a $Q > 2$ such that $\Omega(2B) \subset B(x_0, QR)$ by inequality (2.13). This means that we have $|\Omega(2B)| \sim R^n$. Using this and the Sobolev embedding theorem in inequality (3.59) gives

$$
a_{k+1} \lesssim R^2 \fint_{\Omega(2B)} |\nabla((u_h - \lambda_{k+1} - c)_+ \eta_{k+1})|^2 \, \mathrm{d}x \left(\frac{|A_{k+1}|}{R^n} \right)^{\frac{2}{n}} . \tag{3.60}
$$

Note that this version of the Sobolev embedding also requires Poincaré's inequality. However, the cutoff function η_{k+1} guarantees that $(u_h - \lambda_{k+1} - c)_+\eta_{k+1}$ vanishes outside of $\Omega(2B) \subset B(x_0, 2QR)$. Therefore, the Poincaré constant is the one of a ball. We will consider the factors separately. For the first one we can apply the discrete Caccioppoli inequality (3.54) to get

$$\begin{aligned}
&\fint_{\Omega(2B)} |\nabla((u_h - \lambda_{k+1} - c)_+\eta_{k+1})|^2 \, \mathrm{d}x \\
&\lesssim \fint_{\Omega(2B)} (u_h - \lambda_{k+1} - c)_+^2 |\nabla \eta_{k+1}|^2 \, \mathrm{d}x \\
&\quad + \frac{1}{|\Omega(2B)|} \left(\|G\|^2_{L^p(\Omega)} + \|f\|^2_{L^q(\Omega)} \right) |\operatorname{supp} \eta_{k+1} \cap \operatorname{supp} (u_h - \lambda_{k+1} - c)_+|^{\frac{2\delta}{n}} \\
&\qquad \cdot \left(\|\eta_{k+1}\|^2_{L^{2^\star}(\operatorname{supp} (u - \lambda_{k+1} - c)_+)} + \|\nabla \eta_{k+1}\|^2_{L^2(\operatorname{supp} (u - \lambda_{k+1} - c)_+)} \right).
\end{aligned}$$

Now, by applying Lemma 3.18, assertion (d), to the first summand $\fint_{\Omega(2B)} (u_h - \lambda_{k+1} - c)_+^2 |\nabla \eta_{k+1}|^2 \, \mathrm{d}x$ (with $a = (u_h - \lambda_{k+1} - c)_+$) on every simplex $T \in \mathcal{T}$ and then adding up, and by using the bound $\max_\Omega |\nabla \eta_k| \le 2^k R^{-1}$ from Lemma 3.18, assertion (a), together with $\eta_h \le 1$, $|\Omega(2B)| \gtrsim |B(x_0, 2R)| \ge R^n$ and $|A_{k+1}| \le |\operatorname{supp} \eta_{k+1}| \lesssim R^n$ (and therefore $\left(\frac{|A_{k+1}|}{R^n}\right) \lesssim 1$) to estimate the second summand, we deduce that

$$\begin{aligned}
&\fint_{\Omega(2B)} |\nabla((u_h - \lambda_{k+1} - c)_+\eta_{k+1})|^2 \, \mathrm{d}x \\
&\lesssim \frac{2^{4k}}{R^2} \fint_{\Omega(2B)} |(u_h - c - \lambda_k)_+|^2 |\eta_k|^2 \, \mathrm{d}x \\
&\quad + R^{2\delta - 2} \left(\|G\|^2_{L^p(\Omega)} + \|f\|^2_{L^q(\Omega)} \right) \left(\left(\frac{|A_{k+1}|}{R^n} \right)^{1 - \frac{2}{n} + \frac{2\delta}{n}} + 2^{2k} \frac{|A_{k+1}|}{R^n} \right).
\end{aligned}$$

With the weak-type estimate (3.57) and because $\lambda_{k+1} > \lambda_k$, this becomes

$$\begin{aligned}
&\fint_{\Omega(2B)} |\nabla((u_h - \lambda_{k+1} - c)_+\eta_{k+1})|^2 \, \mathrm{d}x \\
&\lesssim \frac{2^{4k}}{R^2} \fint_{\Omega(2B)} |(u_h - c - \lambda_k)_+|^2 |\eta_k|^2 \, \mathrm{d}x + R^{2\delta - 2} \left(\|F\|^2_{L^p(\Omega)} + \|f\|^2_{L^q(\Omega)} \right) \\
&\quad \cdot \frac{2^{4k}}{\lambda_\infty^2} \left(\fint_{\Omega(2B)} |\eta_k (u_h - \lambda_k - c)_+|^2 \, \mathrm{d}x + \left(\fint_{\Omega(2B)} |\eta_k (u_h - \lambda_k - c)_+|^2 \, \mathrm{d}x \right)^{1 - \frac{2}{n} + \frac{2\delta}{n}} (\lambda_\infty^2)^{\frac{2}{n} - \frac{2\delta}{n}} \right) \\
&= \frac{2^{4k}}{R^2} \left(a_k + \left(a_k + a_k^{1 - \frac{2}{n} + \frac{2\delta}{n}} (\lambda_\infty^2)^{\frac{2}{n} - \frac{2\delta}{n}} \right) \frac{R^{2\delta} \left(\|G\|^2_{L^p(\Omega)} + \|f\|^2_{L^q(\Omega)} \right)}{\lambda_\infty^2} \right).
\end{aligned} \tag{3.61}$$

For the second factor of inequality (3.60) we use the weak-type estimate (3.57) to get

$$\begin{aligned}\left(\frac{|A_{k+1}|}{R^n}\right)^{\frac{2}{n}} &\leq \left(\frac{2^{2k}}{\lambda_\infty^2} \fint_\Omega |\eta_k(u_h - \lambda_k - c)_+|^2 \,\mathrm{d}x\right)^{\frac{2}{n}} \\ &= 2^{\frac{4k}{n}} \left(\frac{a_k}{\lambda_\infty}\right)^{\frac{2}{n}}.\end{aligned} \tag{3.62}$$

Substituting inequalities (3.61) and (3.62) into inequality (3.60) yields

$$a_{k+1} \lesssim 2^{4\left(1+\frac{1}{n}\right)k} a_k \left(\left(\frac{a_k}{\lambda_\infty^2}\right)^{\frac{2}{n}} + \left(\left(\frac{a_k}{\lambda_\infty^2}\right)^{\frac{2}{n}} + \left(\frac{a_k}{\lambda_\infty^2}\right)^{\frac{2\delta}{n}}\right) \frac{R^{2\delta}\left(\|G\|_{L^p(\Omega)}^2 + \|f\|_{L^q(\Omega)}^2\right)}{\lambda_\infty^2}\right).$$

If we now choose $\lambda_\infty^2 \sim \max\left\{R^{2\delta}\left(\|F\|_{L^p(\Omega)}^2 + \|f\|_{L^q(\Omega)}^2\right), \fint_{\Omega(2B)} (u_h - c)_+^2 \,\mathrm{d}x\right\}$, we obtain

$$a_{k+1} \lesssim 2^{4k\left(1+\frac{1}{n}\right)} a_k \left(\left(\frac{a_k}{\lambda_\infty^2}\right)^{\frac{2}{n}} + \left(\frac{a_k}{\lambda_\infty^2}\right)^{\frac{2\delta}{n}}\right).$$

With the help of Corollary 3.32 we get $a_k \to 0$, as $k \to \infty$, if

$$\lambda_\infty^2 \sim a_0 \leq \fint_{\Omega(2B)} (u_h - c)_+^2 \,\mathrm{d}x.$$

Because $\lim_{k\to\infty} a_k = 0$, passing to the limit $k \to \infty$ in the definition of a_k we deduce that $(u_h - c)_+^2 \leq \lambda_\infty^2$ on $\Omega'(B(x_0, R)) \subset \bigcap_k \operatorname{supp} \eta_k$. By recalling our choice of λ_∞^2 we thus have on $\Omega'(B(x_0, R)) \subset \bigcap_k \operatorname{supp} \eta_k$ the following bound:

$$\begin{aligned}(u_h - c)_+^2 \leq \lambda_\infty^2 &\lesssim \max\left\{\left(\|G\|_{L^p(\Omega)}^2 + \|f\|_{L^q(\Omega)}^2\right) R^{2\delta}, \fint_{\Omega(2B)} (u_h - c)_+^2 \,\mathrm{d}x\right\} \\ &\sim \left(\|G\|_{L^p(\Omega)}^2 + \|f\|_{L^q(\Omega)}^2\right) R^{2\delta} + \fint_{\Omega(2B)} (u_h - c)_+^2 \,\mathrm{d}x,\end{aligned}$$

which proves the stated claim. □

Next we will prove an estimate showing that if a value of u_h is small at a node, then it cannot be too large at neighbouring nodes. By enabling us to control not only the maximum of a function on a simplex, but also its minimum, this will help us to recover a property analogous to the one in inequality (3.17) that will be a discrete version of Lemma 3.4.

Lemma 3.21. *Under the assumptions of Theorem 3.15, let v_h be a discrete subsolution to $-\operatorname{div}(A\nabla u) = f - \operatorname{div} F$ with $0 \leq v_h \leq 1$. Then, there exist constants $C > 0$, $\tau \in (0, 1)$ and $N > 0$ that depend only on d, the shape-regularity parameter Γ and the dimension n such that if x_i and x_j are nodes of the same simplex $T \in \mathcal{T}_h$ with $\partial T \cap \partial\Omega = \emptyset$, then we have*

$$v_h(x_j) \leq 1 - \tau^N + \tau^N v_h(x_i) + C\left(\|F\|_{L^p(\Omega)} + \|f\|_{L^q(\Omega)}\right) h_T^\delta. \tag{3.63}$$

Proof. We follow the ideas from [3], Lemma 1.7. By Lemma 2.31, we can connect the nodes x_i and x_j by a sequence of nodes $\{x_i = y_0, y_1, \dots, y_N = x_j\}$ belonging to the same simplex T with

$$-\int_\Omega A\nabla\psi_i \cdot \nabla\psi_{i+1}\,\mathrm{d}x \geq \tau \int_\Omega A\nabla\psi_i \cdot \nabla\psi_i\,\mathrm{d}x, \qquad i = 0, \dots, N-1, \tag{3.64}$$

where we denote by ψ_i the basis function associated with the node y_i in the sense of Definition 2.12 and $\tau \in (0,1)$ only depends on the shape-regularity parameter Γ of the triangulation. Also note that all of the y_i are nodes of the same simplex T, which means that $N \leq n$. We denote the remaining nodes of $\mathcal{T}_h$ by y_i with $i \geq N+1$. We shall prove the claim by induction over the sequence of nodes connecting x_i and x_j. The base step (corresponding to $N = 0$ when $x_j = x_i$) is clear. Suppose therefore that $N \geq 1$.

Let v_h be a discrete subsolution to $-\mathrm{div}(A\nabla u) = f - \mathrm{div}F$, such that $0 \leq v_h \leq 1$. Fix an integer $k \in \{0, \dots, N-1\}$, test equation (3.23) against $\varphi_h = \psi_{k+1}$ and write $v_h = \sum_l v_h(x_l)\psi_l$ and $\langle f, g\rangle_A := \int_\Omega Af \cdot g\,\mathrm{d}x$ for all $f, g \in L^2(\Omega;\mathbb{R}^n)$ to get

$$\sum_l v_h(y_l)\langle \nabla\psi_l, \nabla\psi_{k+1}\rangle_A = \int_\Omega A\nabla v_h \cdot \nabla\psi_{k+1}\,\mathrm{d}x \leq \int_\Omega F \cdot \nabla\psi_{k+1} + f\psi_{k+1}\,\mathrm{d}x.$$

This leads to

$$\begin{aligned} v_h(y_{k+1}) &\leq -\sum_{l\neq k+1} \frac{\langle \nabla\psi_l, \nabla\psi_{k+1}\rangle_A}{\langle \nabla\psi_{k+1}, \nabla\psi_{k+1}\rangle_A} v_h(y_l) \\ &\quad + \int_\Omega \frac{F \cdot \nabla\psi_{k+1}}{\langle \nabla\psi_{k+1}, \nabla\psi_{k+1}\rangle_A}\,\mathrm{d}x + \int \frac{f\psi_{k+1}}{\langle \nabla\psi_{k+1}, \nabla\psi_{k+1}\rangle_A}\,\mathrm{d}x \\ &=: I + II + III. \end{aligned} \tag{3.65}$$

First, we consider term I. Since $0 \leq v_h \leq 1$, and because $\mathcal{T}_h$ is A-nonobtuse in the sense of inequality (2.8), we find that

$$\begin{aligned} I &= -\sum_{l\neq k+1} \frac{\langle \nabla\psi_l, \nabla\psi_{k+1}\rangle_A}{\langle \nabla\psi_{k+1}, \nabla\psi_{k+1}\rangle_A} v_h(y_l) \\ &= -\sum_{l\neq k,k+1} \frac{\langle \nabla\psi_l, \nabla\psi_{k+1}\rangle_A}{\langle \nabla\psi_{k+1}, \nabla\psi_{k+1}\rangle_A} v_h(y_l) - \frac{\langle \nabla\psi_k, \nabla\psi_{k+1}\rangle_A}{\langle \nabla\psi_{k+1}, \nabla\psi_{k+1}\rangle_A} v_h(y_k) \\ &\leq \sum_{l\neq k+1} -\frac{\langle \nabla\psi_l, \nabla\psi_{k+1}\rangle_A}{\langle \nabla\psi_{k+1}, \nabla\psi_{k+1}\rangle_A} + \frac{\langle \nabla\psi_k, \nabla\psi_{k+1}\rangle_A}{\langle \nabla\psi_{k+1}, \nabla\psi_{k+1}\rangle_A} - \frac{\langle \nabla\psi_k, \nabla\psi_{k+1}\rangle_A}{\langle \nabla\psi_{k+1}, \nabla\psi_{k+1}\rangle_A} v_h(y_k). \end{aligned}$$

Since $0 = \nabla 1 = \sum_l \nabla\psi_l$ and therefore $-\sum_{l\neq k+1} \frac{\langle \nabla\psi_l, \nabla\psi_{k+1}\rangle_A}{\langle \nabla\psi_{k+1}, \nabla\psi_{k+1}\rangle_A} = \frac{\langle \nabla\psi_{k+1}, \nabla\psi_{k+1}\rangle_A}{\langle \nabla\psi_{k+1}, \nabla\psi_{k+1}\rangle_A} = 1$, we get

$$\begin{aligned} I &\leq 1 + \frac{\langle \nabla\psi_k, \nabla\psi_{k+1}\rangle_A}{\langle \nabla\psi_{k+1}, \nabla\psi_{k+1}\rangle_A} - \frac{\langle \nabla\psi_k, \nabla\psi_{k+1}\rangle_A}{\langle \nabla\psi_{k+1}, \nabla\psi_{k+1}\rangle_A} v_h(y_k) \\ &= 1 + \frac{\langle \nabla\psi_k, \nabla\psi_{k+1}\rangle_A}{\langle \nabla\psi_{k+1}, \nabla\psi_{k+1}\rangle_A}(1 - v_h(y_k)). \end{aligned}$$

Because the sequence y_i satisfies inequality (3.64), this becomes

$$I \leq 1 - \tau + \tau v_h(y_k). \tag{3.66}$$

Next we consider term II from inequality (3.65). We know that $\operatorname{supp} \psi_{k+1} = \overline{\Omega(y_{k+1})}$ and $0 \leq \psi_l \leq 1$. Hence, $\|\psi_{k+1}\|_{L^r(\Omega)} \sim h_T^{\frac{n}{r}}$ as $y_{k+1} \in T$. By equation (2.10) we find that $\|\nabla\psi_{k+1}\|_{L^r(\Omega)} \sim h_T^{\frac{n}{r}-1}$. By the uniform ellipticity of A we have $\langle \nabla\psi_{k+1}, \nabla\psi_{k+1}\rangle_A \geq d\|\nabla\psi_{k+1}\|^2_{L^2(\Omega)}$. This gives

$$II \lesssim \frac{\|F\|_{L^p(\Omega)} \|\nabla\psi_{k+1}\|_{L^{\left(1-\frac{1}{n}+\frac{\delta}{n}\right)^{-1}}(\Omega)}}{\|\nabla\psi_{k+1}\|^2_{L^2(\Omega)}} \sim \|F\|_{L^p(\Omega)} \frac{h_T^{n-2+\delta}}{h_T^{n-2}} \sim \|F\|_{L^p(\Omega)} h_T^{\delta}. \tag{3.67}$$

Analogously, we find for term III, using Hölder's inequality, that

$$III \lesssim \frac{\|f\|_{L^q(\Omega)} \|\psi_{k+1}\|_{L^{\left(1-\frac{2}{n}+\frac{\delta}{n}\right)^{-1}}(\Omega)}}{\|\nabla\psi_{k+1}\|^2_{L^2(\Omega)}} \sim \|f\|_{L^q(\Omega)} \frac{h_T^{n-2+\delta}}{h_T^{n-2}} \sim \|f\|_{L^q(\Omega)} h_T^{\delta}. \tag{3.68}$$

Inserting the inequalities (3.66), (3.67) and (3.68) into inequality (3.65) gives

$$v_h(y_{k+1}) \leq 1 - \tau + \tau v_h(y_k) + C\left(\|F\|_{L^p(\Omega)} + \|f\|_{L^q(\Omega)}\right) h_T^{\delta}. \tag{3.69}$$

Now assume that

$$v_h(y_k) \leq 1 - \tau^k + \tau^k v_h(x_i) + C\left(\|F\|_{L^p(\Omega)} + \|f\|_{L^q(\Omega)}\right) h_T^{\delta} \sum_{m=0}^{k-1} \tau^m.$$

We recall that $y_0 := x_i$ and calculate

$$\begin{aligned}
v_h(y_{k+1}) &\leq 1 - \tau + \tau v_h(y_k) + C\left(\|F\|_{L^p(\Omega)} + \|f\|_{L^q(\Omega)}\right) h_T^{\delta} \\
&\leq 1 - \tau + \tau\left(1 - \tau^k + \tau^k v_h(y_0) + C\left(\|F\|_{L^p(\Omega)} + \|f\|_{L^q(\Omega)}\right) h_T^{\delta} \sum_{m=0}^{k-1} \tau^m\right) \\
&\quad + C\left(\|f\|_{L^q(\Omega)} + \|F\|_{L^p(\Omega)}\right) h_T^{\delta} \\
&= 1 - \tau^{k+1} + \tau^{k+1} v_h(y_0) + C\left(\|F\|_{L^p(\Omega)} + \|f\|_{L^q(\Omega)}\right) h_T^{\delta} \sum_{m=1}^{k} \tau^m \\
&\quad + C\left(\|F\|_{L^p(\Omega)} + \|f\|_{L^q(\Omega)}\right) h^{\delta} \\
&= 1 - \tau^{k+1} + \tau^{k+1} v_h(y_0) + C\left(\|F\|_{L^p(\Omega)} + \|f\|_{L^q(\Omega)}\right) h_T^{\delta} \sum_{m=0}^{k} \tau^m.
\end{aligned}$$

By induction, this proves by setting $k = N-1$ and recalling again that $y_0 := x_i$ and $y_N = x_j$ the bound

$$v_h(x_j) \leq 1 - \tau^N + \tau^N v_h(x_i) + C\left(\|F\|_{L^p(\Omega)} + \|f\|_{L^q(\Omega)}\right) h_T^{\delta} \sum_{m=0}^{N-1} \tau^m.$$

Because $\sum_{m=0}^{N-1} \tau^m$ is bounded and depends only on $N \leq n$ and τ, which depends only on the shape-regularity parameter, we deduce the assertion of the lemma. □

This leads to the following lemma, which is a discrete version of Lemma 3.4. It is a generalisation of ideas from [3], where the discussion was restricted to Laplace's equation on a quasi-uniform mesh.

Lemma 3.22. *Under the assumptions of Theorem 3.15, let $B(x_0, R)$ be a ball with $R \geq h_T$ and $B(x_0, QR) \subset \Omega$ where Q is the constant from inequality (2.13) and $x_0 \in T$ for some $T \in \mathcal{T}_h$. Furthermore, let v_h be a discrete subsolution to $-\mathrm{div}(A\nabla u) = f - \mathrm{div}F$ with $0 \leq v_h \leq 1$ on $B(x_0, QR)$. Assume further, that we have*

$$\Big| \bigcup_{i : v_h(x_i)=0} P_i \cap \Omega(B(x_0, 2R)) \Big| \geq \zeta |\Omega(B(x_0, 2R))| \tag{3.70}$$

for this ball and some $\zeta > 0$. Then, there exist constants R_0, $\theta \in (0,1)$ and $C > 0$ that only depend on ζ, n, $\|A\|_{L^\infty(\Omega)}$, d, δ and the shape-regularity parameter Γ, such that

$$\sup_{\Omega'(B(x_0,R))} v_h \leq 1 - \theta + C \left(\|G\|_{L^p(\Omega)} + \|f\|_{L^q(\Omega)} \right) R^\delta \tag{3.71}$$

for any $R \in (h_T, R_0)$.

Proof. We recall our notational convention $2B := B(x_0, 2R)$ and define, for any $R \geq h_T$,

$$\mu := 1 - \frac{\tau^N}{4}, \quad \lambda_k := 1 - (1-\mu)^k, \quad v_k := \frac{1}{1-\lambda_k}(v_h - \lambda_k)_+, \quad A_k := \Omega(2B) \cap \{u_k > 0\}.$$

Again, the operation $(...)_+$ is meant in a nodal sense and N and τ are the constants from Lemma 3.21.

As we have $v_h \leq 1$, we have $0 \leq v_k \leq 1$ as well. Note that v_k is a subsolution to $-\mathrm{div}(A\nabla v) = \frac{|f|}{(1-\mu)^k} - \frac{\mathrm{div}G}{(1-\mu)^k}$ because of Corollary 3.13, so the local L^∞-estimates from Theorem 3.20 hold. This means that we get

$$\begin{aligned} \sup_{\Omega'(B)} v_k &\lesssim \left(\fint_{\Omega(2B)} v_k^2 \,\mathrm{d}x \right)^{\frac{1}{2}} + C \left(\frac{\|G\|_{L^p} + \|f\|_{L^q}}{(1-\mu)^k} \right) R^\delta \\ &\lesssim \left(\frac{|A_k|}{R^n} \right)^{\frac{1}{2}} + C \left(\frac{\|G\|_{L^p} + \|f\|_{L^q}}{(1-\mu)^k} \right) R^\delta. \end{aligned} \tag{3.72}$$

If we can find a $\bar{k}$ such that $|A_{\bar{k}}| \leq \beta R^n + C_\beta R^{n+2\delta}$ for a small β, we will have $v_{\bar{k}} \leq \frac{1}{2} + \tilde{C} R^\delta$. Assume that $v_{k+1}(x_i) > 0$ for some $x_i \in \Omega(2B)$. Because the sequence $(\lambda_k)_{k\geq 0}$ is strictly monotonic increasing, if $v_{k+1}(x_i) > 0$ for some $x_i \in \Omega(2B)$, as has been assumed, then

$v_h(x_i) - \lambda_k > v_h(x_i) - \lambda_{k+1} > 0$, whereby $(v_h(x_i) - \lambda_k)_+ = v_h(x_i) - \lambda_k$. Hence, we have

$$
\begin{aligned}
0 < v_{k+1}(x_i) = \frac{v_h(x_i) - \lambda_{k+1}}{1 - \lambda_{k+1}} &= \frac{1 - \lambda_k}{1 - \lambda_{k+1}} \frac{v_h(x_i) - \lambda_k}{1 - \lambda_k} + \frac{\lambda_k - \lambda_{k+1}}{1 - \lambda_{k+1}} \\
&= \frac{1 - \lambda_k}{1 - \lambda_{k+1}} v_k(x_i) + \frac{\lambda_k - \lambda_{k+1}}{1 - \lambda_{k+1}} \\
&= \frac{v_k(x_i)}{1 - \mu} + \frac{(1-\mu)^{k+1} - (1-\mu)^k}{(1-\mu)^{k+1}} \\
&= \frac{v_k(x_i)}{1 - \mu} + \frac{1}{1 - \mu}(1 - \mu - 1) \\
&= \frac{v_k(x_i) - \mu}{1 - \mu},
\end{aligned}
$$

and therefore

$$\mu < v_k(x_i). \tag{3.73}$$

We also define a sequence R_k, such that

$$\frac{C\left(\|G\|_{L^p} + \|f\|_{L^q}\right)}{(1-\mu)^k} (2QR_k)^\delta = \frac{\tau^N}{4}. \tag{3.74}$$

By inequality (2.7) and inclusion (2.13), we know that

$$2QR \geq \operatorname{diam}\left(\Omega\left(B(x_0, 2R)\right)\right) \geq h_S \tag{3.75}$$

for any $S \in \mathcal{T}_h$ with $S \subset \Omega(B(x_0, 2R))$, as long as $R \geq h_T$. By combining equality (3.74) and inequality (3.75) we have that

$$\frac{C\left(\|G\|_{L^p} + \|f\|_{L^q}\right)}{(1-\mu)^k} h_S^\delta \leq \frac{\tau^N}{4} \tag{3.76}$$

for any $R \in [h_T, R_k]$ and any $S \subset \Omega(2B)$.

Assume that $R \leq R_k$ for some k. Again, note that v_k is a subsolution to $-\operatorname{div}(A\nabla v) = \frac{|f|}{(1-\mu)^k} - \frac{\operatorname{div} G}{(1-\mu)^k}$. This means, by Lemma 3.21, that for any node x_j of the same simplex $S \subset \Omega(B(x_0, 2R))$ as x_i, we get

$$
\begin{aligned}
1 - \frac{\tau^N}{4} = \mu < v_k(x_i) &\leq 1 - \tau^N + \tau^N v_k(x_j) + C\left(\|\tilde{G}\|_p + \|\tilde{f}\|_q\right) h_S^\delta \\
&\leq 1 - \tau^N + \tau^N v_k(x_j) + \frac{C\left(\|G\|_p + \|f\|_q\right)}{(1-\mu)^k} h_S^\delta.
\end{aligned}
$$

Now recall that $R \leq R_k$ and note the equality (3.76) to get

$$1 - \frac{\tau^N}{4} \leq 1 - \tau^N + \tau^N v_k(x_j) + \frac{\tau^N}{4}, \tag{3.77}$$

which implies that $v_k(x_j) > \frac{1}{2}$. Together with the already established bound $\frac{1}{2} \leq \mu < v_k(x_i)$ this gives

$$A_{k+1} \subset \left\{ v_k > \frac{1}{2} \right\} \cap \Omega(B(2R)),$$

as $v_k(x_k) > \frac{1}{2}$ at all nodes of A_{k+1}, because if an $S \in \mathcal{T}_h$ satisfies $S \subset A_k$, there is a node x_0 of S such that $v_k(x_0) > \lambda_{k+1}$, and this means that we have shown that $v_k(x_j) \geq \frac{1}{2}$ for any other node of that simplex S by inequality (3.77). Furthermore, as we are using piecewise affine basis functions, we know that $\max_S v_h$ and $\min_S v_h$ are attained at a node on any simplex $S \in \mathcal{T}_h$. Therefore, we get

$$\left\{0 < v_k < \frac{1}{2}\right\} \cap A_{k+1} = \emptyset.$$

However, as we have $\left\{0 < v_k < \frac{1}{2}\right\} \cap \Omega(2B) \subset A_k$ and $A_{k+1} \subset A_k$, it follows that

$$|A_k| \geq \left|\left(\left\{0 < v_k < \frac{1}{2}\right\} \cap \Omega(2B)\right) \cup A_{k+1}\right| = \left|\left\{0 < v_k < \frac{1}{2}\right\} \cap \Omega(B(2R))\right| + |A_{k+1}|. \tag{3.78}$$

Now note that if $v_h(x_i) = 0$, we also have $v_k(x_i) = 0$. This means that inequality (3.70) holds for all v_k with the same constant ζ. This allows us to use the Poincaré-type inequality from Theorem 2.29. By Corollary 2.9 and Lemma 2.30, we conclude that the Poincaré constant of $\Omega(B(x_0, 2R))$ only depends on the shape-regularity constant Γ. (Note that we are far away from the boundary, so the dependence on the Lipschitz constant of the boundary can be dropped.)

$$\begin{aligned}
|A_{k+1}| &\leq 2 \int_{\Omega(2B)} \min\left\{v_k, \frac{1}{2}\right\} \,\mathrm{d}x \\
&\lesssim R \int_{\Omega(2B)} \left|\nabla\left(\min\left\{v_k, \frac{1}{2}\right\}\right)\right| \,\mathrm{d}x \\
&\leq R \int_{\Omega(2B) \cap \{0 < v_k < \frac{1}{2}\}} |\nabla v_k| \,\mathrm{d}x \\
&\leq R \left(\int_{\Omega(2B)} |\nabla v_k|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \left|\left\{0 < v_k < \frac{1}{2}\right\} \cap \Omega(B(2R))\right|^{\frac{1}{2}}.
\end{aligned}$$

Here, the min is meant in the pointwise rather than in the nodal sense. As we have $0 \leq v_k \leq 1$, we can apply the discrete Caccioppoli-type inequality (3.54) to deduce that

$$\begin{aligned}
R^2 \int_{\Omega(2B)} |\nabla v_k|^2 \,\mathrm{d}x &\lesssim \int_{B(2Q^2R)} |v_k|^2 + \frac{C\left(\|G\|_{L^p}^2 + \|f\|_{L^q}^2\right)}{(1-\mu)^k} R^{n+2\delta} \\
&\lesssim R^n + C_k R^{n+2\delta}.
\end{aligned}$$

This gives

$$|A_{k+1}| \lesssim \left(R^n + \frac{C\left(\|G\|_{L^p}^2 + \|f\|_{L^q}^2\right)}{(1-\mu)^k} R^{n+2\delta}\right)^{\frac{1}{2}} \left|\left\{0 < v_k < \frac{1}{2}\right\} \cap \Omega(2B)\right|^{\frac{1}{2}}.$$

Combining this with (3.78) yields

$$|A_{k+1}| \lesssim \left(R^n + \frac{C\left(\|G\|_{L^p}^2 + \|f\|_{L^q}^2\right)}{(1-\mu)^k} R^{n+2\delta} \right)^{\frac{1}{2}} (|A_k| - |A_{k+1}|)^{\frac{1}{2}}.$$

Note that $|A_0| \lesssim R^n$. With the help of Lemma 3.33 we deduce that, for any $R \le R_k$,

$$\begin{aligned} |A_k| &\le \frac{\sqrt{\max_{i\le k}\left(CR^n + \frac{C\left(\|G\|_{L^p}^2+\|f\|_{L^q}^2\right)}{(1-\mu)^i} R^{n+2\delta}\right)}\sqrt{|A_0|}}{\sqrt{k}} \\ &\lesssim \frac{\sqrt{C}}{\sqrt{k}} R^n + \sqrt{\frac{C\left(\|G\|_{L^p}^2 + \|f\|_{L^q}^2\right)}{k(1-\mu)^k}} R^{n+\delta}, \end{aligned}$$

where we have used that that all norms on $\mathbb{R}^2$ are equivalent, so in particular $\sqrt{x^2+y^2} \sim |x| + |y|$. This means that for every $\beta > 0$, there exists a $\bar{k}$ such that, for any $R \le R_{\bar{k}}$,

$$|A_{\bar{k}}| \le \beta R^n + C_\beta \left(\|G\|_{L^p} + \|f\|_{L^q}\right) R^{n+\delta}.$$

Now, choose β small enough, such that inequality (3.72) gives, for some $C > 0$,

$$\sup_{\Omega'(B(R))} v_{\bar{k}} \le \frac{1}{2} + C\left(\|G\|_{L^p} + \|f\|_{L^q}\right) R^\delta.$$

Recall that $v_k = \frac{(v_h - \lambda_k)_+}{1-\lambda_k}$. Then, we get on $\Omega'(B(R))$ that

$$\begin{aligned} v_h(x) &\le (1-\lambda_{\bar{k}}) v_{\bar{k}}(x) + \lambda_{\bar{k}} \\ &\le (1-\lambda_{\bar{k}}) \left(\frac{1}{2} + C\left(\|G\|_{L^p} + \|f\|_{L^q}\right) R^\delta \right) + \lambda_{\bar{k}} \\ &= \frac{1}{2}(\lambda_{\bar{k}} + 1) + (1-\lambda_{\bar{k}}) C\left(\|G\|_{L^p} + \|f\|_{L^q}\right) R^\delta \\ &= \frac{1}{2}\left(2 - (1-\mu)^{\bar{k}}\right) + (1-\lambda_{\bar{k}}) C\left(\|G\|_{L^p} + \|f\|_{L^q}\right) R^\delta \\ &\le 1 - \frac{1}{2}(1-\mu)^{\bar{k}} + C\left(\|G\|_{L^p} + \|f\|_{L^q}\right) R^\delta \\ &= 1 - \frac{1}{2}\left(\frac{\tau^N}{4}\right)^{\bar{k}} + C\left(\|G\|_{L^p} + \|f\|_{L^q}\right) R^\delta. \end{aligned}$$

This proves the lemma with $\theta = \frac{1}{2}\left(\frac{\tau}{4}\right)^{\bar{k}}$ and $R_0 = R_{\bar{k}}$. □

Lemma 3.22 allows us to prove a bound on the oscillation of u_h, stated in the following lemma. Again, it is a generalisation of a result from [3].

Lemma 3.23. *Under the assumptions of Theorem 3.15, let R_0 be the constant from Lemma 3.22, let $x_0 \in T$ for some $T \in \mathcal{T}_h$ and $R \in (h_T, R_0)$, and suppose that $B(x_0, 2QR) \subset \Omega$. Then, there exist constants $\theta \in (0,1)$ and $C > 0$ that only depend on n, $\|A\|_{L^\infty(\Omega)}$, d, δ and the shape-regularity parameter Γ, such that*

$$\operatorname*{osc}_{\Omega'(B(x_0,R))} u_h \le (1-\theta) \operatorname*{osc}_{B(x_0,2QR)} u_h + C\left(\|G\|_{L^p} + \|f\|_{L^q}\right) R^\delta. \tag{3.79}$$

Proof. We begin by noting that u_h is by construction continuous on $\overline{\Omega}$. As $\operatorname{osc}(cu_h + d) = |c| \operatorname{osc} u_h$, we first set $\tilde{u}_h = u_h - \frac{1}{2}\left(\max_{\Omega(2B)} u_h + \min_{\Omega(2B)} u_h\right)$. We then rescale to $\tilde{\tilde{u}}_h = \tilde{u}_h / \|\tilde{u}_h\|_{L^\infty(\Omega(2B))}$ to get $-1 \leq \tilde{\tilde{u}}_h \leq 1$ on $\Omega(2B)$. Then, $\tilde{\tilde{u}}_h$ is still an approximate solution to $-\operatorname{div}(A\nabla u) = \tilde{f} - \operatorname{div}\tilde{F}$ with a rescaled right-hand side, i.e., for $\tilde{f} = f/\|\tilde{u}_h\|_{L^\infty(\Omega(2B))}$ and $\tilde{F} = F/\|\tilde{u}_h\|_{L^\infty(\Omega(2B))}$ and

$$2 = \operatorname*{osc}_{\Omega(2B)} \tilde{\tilde{u}}_h \leq \operatorname*{osc}_{B(x_0,2QR)} \tilde{\tilde{u}}_h.$$

Note that

$$\bigcup_{i\,:\,u_+(x_i)=0} P_i = \bigcup_{i\,:\,u_h(x_i)\leq 0} P_i$$

and

$$\left(\bigcup_{i\,:\,u_h(x_i)\leq 0} P_i\right) \cup \left(\bigcup_{i\,:\,-u_h(x_i)\leq 0} P_i\right) = \Omega.$$

This means that the inequality (3.70) is satisfied for at least one of the functions $(\tilde{\tilde{u}}_h)_+$ and $(-\tilde{\tilde{u}}_h)_+$. Note that u_h is a finite element solution and $\tilde{F}$ and $-\tilde{F}$ both satisfy assumption $\star$ with the dominating function $\tilde{G} = G/\|\tilde{u}_h\|_{L^\infty(\Omega(2B))}$. Thus, both $(\tilde{\tilde{u}}_h)_+$ and $(-\tilde{\tilde{u}}_h)_+$ are subsolutions, with right-hand sides $|\tilde{f}|$, $\tilde{G}$. As $\operatorname{osc} \tilde{\tilde{u}}_h = \operatorname{osc}(-\tilde{\tilde{u}}_h)$, we are free to choose the one for which inequality (3.70) holds.

We can therefore apply Lemma 3.22 to either $(\tilde{\tilde{u}}_h)_+$ or $(-\tilde{\tilde{u}}_h)_+$ and get

$$\begin{aligned}
\operatorname*{osc}_{\Omega'(B(R))} \tilde{\tilde{u}}_h &\leq \sup_{B(R)} (\tilde{\tilde{u}}_h)_+ + 1 \leq 2 - \theta + D\left(\|\tilde{G}\|_{L^p} + \|\tilde{f}\|_{L^q}\right) R^\delta \\
&= \left(1 - \frac{\theta}{2}\right) \operatorname*{osc}_{B(x_0,2QR)} \tilde{\tilde{u}}_h + \frac{D\left(\|G\|_{L^p} + \|f\|_{L^q}\right)}{\|\tilde{u}_h\|_{L^\infty(\Omega(2B))}} R^\delta.
\end{aligned}$$

Hence, after multiplying through by $\|\tilde{u}_h\|_{L^\infty(\Omega(2B))}$, using that $\operatorname{osc} u_h = \|\tilde{u}_h\|_{L^\infty(\Omega(2B))} \operatorname{osc} \tilde{\tilde{u}}_h$, and redefining $\theta \rightsquigarrow \frac{1}{2}\theta$ we have that

$$\operatorname*{osc}_{\Omega'(B(x_0,R))} u_h \leq (1-\theta) \operatorname*{osc}_{B(x_0,2QR)} u_h + D\left(\|G\|_{L^p} + \|f\|_{L^q}\right) R^\delta,$$

which completes the proof. □

We are now ready to prove the interior Hölder-regularity result from Theorem 3.15.

Proof of Theorem 3.15. We will prove inequality (3.56) for different values of R. If

$$R \in [h_T, R_0]$$

(see the conditions of Lemma 3.23 and equation (2.7)), we can follow arguments that were used in the proof of [8] for the non-discrete case and use Lemma 3.23 to get

$$\operatorname*{osc}_{\Omega'(B(x_0,R))} u_h \leq (1-\theta) \operatorname*{osc}_{B(x_0,2QR)} u_h + D\left(\|G\|_{L^p(\Omega)} + \|f\|_{L^q(\Omega)}\right) R^\delta. \tag{3.80}$$

Now, the inclusion (2.17) guarantees that there is a $\kappa > 0$ depending on shape-regularity, such that $B(\kappa R) \subset \Omega'(B(R))$. This gives, together with inequality (3.80), that

$$\operatorname*{osc}_{B(x_0,\kappa R)} u_h \leq (1-\theta) \operatorname*{osc}_{B(x_0,2QR)} u_h + D\left(\|G\|_{L^p(\Omega)} + \|f\|_{L^q(\Omega)}\right) R^\delta, \tag{3.81}$$

which allows us to apply Lemma 3.34 with $\varphi(r) = \operatorname{osc}_{B(2Qr)} u_h$, $\beta = 2Q\kappa^{-1}$, $\tau = \frac{\kappa}{2Q}$, $1-\theta = (2Q\kappa^{-1})^{-\alpha_1}$ and $\alpha_2 = \delta$. If it turns out that $\delta \geq \alpha_1$, we can use a weaker norm for f and F in our Caccioppoli estimates that result in a smaller δ. This leaves us with

$$\operatorname*{osc}_{B(x_0,R)} u_h \leq \left(\frac{C}{R_0^\delta} + C\right) R^\delta. \tag{3.82}$$

For $R \leq h_T$, assume first that $B(x_0, R) \subset \Omega(T)$. We note that u_h is piecewise affine on every $S \in \mathcal{T}_h$. Denote by $L_{u_h,S}$ the Lipschitz constant of u_h on a simplex $S \in \mathcal{T}_h$. For a ball $B(x_0, R)$ we have that

$$\operatorname*{osc}_{B(x_0,R)} u_h \leq \left(\max_{\substack{S\in\mathcal{T}_h \\ S\cap B(x_0,R)\neq\emptyset}} L_{u_h,S}\right) R. \tag{3.83}$$

On the other hand, the $L_{u_h,S}$ are given by

$$L_{u_h,S} = \frac{\operatorname{osc}_{B_{i,S}} u_h}{2R_{i,S}}, \tag{3.84}$$

where $B_{i,S}$ is the inscribed ball of S and $R_{i,S}$ its radius. Denote by y_0 the centre of $B_{i,S}$. We then find using inequality (3.82) that

$$\operatorname*{osc}_{B_{i,S}} u_h \leq \operatorname*{osc}_{B(y_0,h_S)} u_h \lesssim h_S^\delta. \tag{3.85}$$

By shape-regularity, we have $R_{i,S} \gtrsim h_S$. Substituting this and inequality (3.85) into (3.84) yields

$$L_{u_h,S} \sim h_S^{\delta-1}. \tag{3.86}$$

Because $B(x_0, R) \subset \Omega(T)$, we get $h_S \sim h_T$ for all $S \in \mathcal{T}_h$ with $S \cap B(x_0, R) \neq \emptyset$ from equation (2.11). Inserting this, together with the relation (3.86), into inequality (3.83) yields

$$\operatorname*{osc}_{B(x_0,R)} u_h \lesssim h_T^{\delta-1} R \leq R^\delta. \tag{3.87}$$

By Lemma (2.15) there is a $\sigma > 0$, depending only on the shape-regularity constant, such that if $B(x_0, R) \cap (\Omega \setminus \Omega(T)) \neq \emptyset$, we necessarily have $R \geq \sigma h_T$. However, we can then use inequality (3.82) to deduce that

$$\operatorname*{osc}_{B(x_0,R)} u_h \leq \operatorname*{osc}_{B(x_0,h_T)} u_h \lesssim h_T^\delta \leq \frac{1}{\sigma^\delta} R^\delta. \tag{3.88}$$

Taking C as the maximum constant from inequalities (3.82), (3.90), (3.87) and (3.88) finally gives

$$\operatorname*{osc}_{B(x_0,R)} u_h \leq CR^\delta$$

and proves inequality (3.56) in the case of $R < h_T$. Finally, we will focus on the case $R > R_0$. First, note that

$$\operatorname*{osc}_{B(x_0,R)} u_h \leq \operatorname*{osc}_{B(x_0,R)} (u_h)_+ + \operatorname*{osc}_{B(x_0,R)} (-u_h)_+. \tag{3.89}$$

Furthermore, we have $2CQ\left(\frac{4}{\tau^N}\right)^{\bar{k}+1} R^\delta \geq 1$ and get

$$\begin{aligned}\operatorname*{osc}_{B(x_0,R)} (u_h)_+ &= \operatorname*{osc}_{B(x_0,R)} ((u_h)_+ - \zeta) \leq 2\|(u_h)_+ - \zeta\|_{L^\infty(B(R))} \\ &\leq 4CQ\|(u_h)_+ - \zeta\|_{L^\infty(B(x_0,R))} \left(\frac{4}{\tau^N}\right)^{\bar{k}+1} R^\delta\end{aligned} \tag{3.90}$$

for every constant function ζ. We will now show that $\|(u_h)_+ - \zeta\|_{L^\infty(B(x_0,R))}$ is uniformly bounded. For simplicity, we write $v_h = ((u_h)_+ - \|u_h\|_{L^\infty(\partial\Omega)})_+$. We set $\zeta = \langle v_h\rangle_{\Omega(B(x_0,2\kappa^{-1}QR))}$ and note that $(u_h)_+ \geq 0$ to find

$$\begin{aligned}&\|(u_h)_+ - \langle v_h\rangle_{\Omega(B(x_0,2\kappa^{-1}QR))}\|_{L^\infty(B(x_0,R))} \\ \leq& \left\|\Big((u_h)_+ - \|u_h\|_{L^\infty(\partial\Omega)}\Big)_+ - \langle v_h\rangle_{\Omega(B(x_0,2\kappa^{-1}QR))}\right\|_{L^\infty(B(x_0,R))} + \|u_h\|_{L^\infty(\partial\Omega)} \\ =& \|v_h - \langle v_h\rangle_{\Omega(B(x_0,2\kappa^{-1}QR))}\|_{L^\infty(B(x_0,R))} + \|u_h\|_{L^\infty(\partial\Omega)}.\end{aligned} \tag{3.91}$$

Applying inequality (3.58) from Theorem 3.20 yields

$$\begin{aligned}&\left\|\left(v_h - \langle v_h\rangle_{\Omega(B(x_0,2\kappa^{-1}QR))}\right)_+\right\|^2_{L^\infty(B(x_0,R))} \\ \leq& C_1 \fint_{\Omega(B(x_0,2\kappa^{-1}QR))} |v_h - \langle v_h\rangle_{\Omega(B(x_0 2\kappa^{-1}QR))}|^2 \,\mathrm{d}x + C_2\left(\|f\|^2_{L^p(\Omega)} + \|G\|^2_{L^q(\Omega)}\right) R^{2\delta}.\end{aligned} \tag{3.92}$$

Recall from Lemma 2.30, that the Poincaré constant of $\Omega(B(2\kappa^{-1}QR)))$ only depends on the shape-regularity constant Γ. Thus, we can use Poincaré's inequality to find

$$\begin{aligned}&R^{-n}\|v_h - \langle v_h\rangle_{\Omega(B(2\kappa^{-1}QR))}\|^2_{L^2(\Omega(B(2\kappa^{-1}QR)))} \\ \lesssim& R^{2-n}\|\nabla v_h\|^2_{L^2(\Omega(B(2\kappa^{-1}QR)))} \leq R_0^{2-n}\|v_h\|^2_{L^2(\Omega)}.\end{aligned}$$

where we have used $R > R_0$ in the last step. Note that $v_h = (u_h - \|u_h\|_{L^\infty(\partial\Omega)})_+ = 0$ on $\partial\Omega$. Furthermore, v_h is a non-negative discrete subsolution to $-\mathrm{div}(A\nabla v_h) \leq |f| - \mathrm{div}G$. Thus, we can test against v_h and find

$$\|\nabla v_h\|^2_{L^2(\Omega)} \lesssim \int_\Omega A\nabla v_h \cdot \nabla v_h \,\mathrm{d}x \leq \int_\Omega F \cdot \nabla v_h + f v_h \,\mathrm{d}x.$$

We can now use Hölder's inequality and the Sobolev embedding theorem. Also note that $p > n \geq 2$ and $q > \frac{n}{2} \geq \frac{2n}{n+2}$; hence,

$$\begin{aligned}\|\nabla v_h\|_{L^2(\Omega)}^2 &\leq \|\nabla v_h\|_{L^2(\Omega)}\|F\|_{L^2(\Omega)} + \|f\|_{L^{\frac{2n}{n+2}}(\Omega)}\|u_h\|_{L^{2^\star}(\Omega)} \\ &\lesssim \|\nabla v_h\|_{L^2(\Omega)}\|F\|_{L^p(\Omega)} + \|f\|_{L^q(\Omega)}\|\nabla v_h\|_{L^2(\Omega)}.\end{aligned}$$

Using Young's inequality gives

$$\|\nabla v_h\|_{L^2(\Omega)}^2 \lesssim \left(\|f\|_{L^p(\Omega)}^2 + \|G\|_{L^q(\Omega)}^2\right).$$

Together with inequalities (3.91), (3.92) and (3.90) this gives a uniform bound on $\operatorname{osc}_{B(x_0,R)_+}(u_h)_+$. The same estimates are true for $(-u_h)_+$. Therefore, inequality (3.89) proves inequality (3.56) for $R \geq R_0$ and thus, completes the proof of Theorem 3.15. □

3.4 C^α-Regularity at the boundary

So far we have always assumed that we are far away from the boundary $\partial\Omega$ of Ω. In order to prove uniform Hölder regularity up to the boundary we will use suitable truncations of u_h to be able to use $u_h\eta_h^2$ as a test function in equation (3.23) even if η_h is not necessarily compactly supported in Ω. For convenience we will write $\Omega^c := \mathbb{R}^n \setminus \Omega$.

Definition 3.24. *A domain $\Omega \subset \mathbb{R}^n$ satisfies a uniform outer cone condition if, for any $x \in \partial\Omega$, there is a cone C with $C \cap \Omega = \emptyset$, with its tip at x and angle greater than some $\alpha_0 > 0$. In particular, this means that the complement Ω^c is fat in the sense that there is a $\mu_0 > 0$, such that for any $x \in \partial\Omega$ and $R > 0$, we have*

$$|B(x,R) \cap \Omega^c| \geq \mu_0|B(x,R)|. \tag{3.93}$$

We note that a Lipschitz polyhedral domain automatically satisfies a uniform cone condition and μ_0 depends only on the Lipschitz constant of the boundary. This allows us to state the main theorem of the section.

Theorem 3.25. *Let $\Omega \subset \mathbb{R}^n$ be a polyhedral domain (which thereby satisfies a uniform outer cone condition in the sense of Definition 3.24). Furthermore, let p, q be defined via $\frac{1}{p} = \frac{1}{n} - \frac{\delta}{n}$ and $\frac{1}{q} = \frac{2}{n} - \frac{\delta}{n}$, let $f \in L^q(\Omega)$ and let $F \in L^p(\Omega;\mathbb{R}^n)$ satisfy assumption $(\star)$ with dominating function $G \in L^p(\Omega;\mathbb{R}^n)$, and let $A \in L^\infty(\Omega;\mathbb{R}^{n\times n})$ be a uniformly elliptic matrix-valued function (i.e. $A(x)v \cdot v \geq d|v|^2$ for all $v \in \mathbb{R}^n$). Let $\mathcal{T}_h$ be an A-nonobtuse, shape-regular triangulation of the polyhedral domain Ω with respective finite element spaces V_h and $V_{h,0}$. Let $u_h \in V_h$ be a continuous piecewise affine finite element approximation to the solution of the equation $-\operatorname{div}(A\nabla u) = f - \operatorname{div}F$. Furthermore, assume that $u_h|_{\partial\Omega} \in C^\beta(\partial\Omega)$. There* *are constants* $\alpha \in (0,1)$ *and* C*, that only depend on* $|u_h|_{\partial\Omega}|_{C^\beta}$, $\|A\|_{L^\infty(\Omega)}$, d, δ, $\|f\|_{L^q(\Omega)}$, $\|G\|_{L^p(\Omega)}$, β*, the Lipschitz constant of* $\partial\Omega$, $\operatorname{diam}\Omega$ *and the shape-regularity parameter* Γ *of* $\mathcal{T}_h$ *such that*

$$u_h \in C^\alpha(\overline{\Omega})$$

and

$$|u_h|_{C^\alpha(\overline{\Omega})} \leq C.$$

Again, we will break down the proof into several lemmas. Recall from Definition 2.16 that we write $\mathcal{B}(x_0,R)$ for the connected component of $B(x_0,R) \cap \Omega$ that contains x_0. Taking only the connected component is important that the estimates from Lemma 2.17 and Lemma 2.18 hold near the boundary.

Henceforth we shall assume that $n \geq 3$ and write $2^\star := 2n/(n-2)$. The proofs are easily adjusted in the case of $n = 2$.

Theorem 3.26. *Let* $\mathcal{T}_h$ *be an* A*-nonobtuse, shape-regular triangulation of the polyhedral domain* Ω*. Let* u_h *be a nonnegative subsolution to* $-\operatorname{div}(A\nabla u_h) = f - \operatorname{div}F$ *for* $F \in L^p$ *and* $f \in L^q$*. Furthermore, suppose that* F *satisfies assumption* $(\star)$ *from Definition 3.10 with dominating function* G*. For any* $\eta \in C^\infty(\mathbb{R}^n)$ *define* $\eta_h := \Pi_h\eta$*. Suppose furthermore that* $u_h = 0$ *on* $\partial\Omega \cap \operatorname{supp}\eta_h$*. Then, there is a* $C > 0$ *that only depends on* n, δ, $\|A\|_{L^\infty(\Omega)}$, d *and the shape-regularity parameter* Γ *of* $\mathcal{T}_h$*, such that*

$$\begin{aligned}\int_\Omega |\nabla u_h|^2 |\eta_h|^2 \,\mathrm{d}x &\leq C\int_\Omega u_h^2 |\nabla\eta_h|^2 \,\mathrm{d}x \\ &+ C\left(\|F\|^2_{L^p(\Omega)} + \|f\|^2_{L^q(\Omega)}\right)\left(\|\nabla\eta_h\|_{L^2(\operatorname{supp} u_h)} + \|\eta_h\|^2_{L^{2^\star}(\operatorname{supp} u_h)}\right)|\operatorname{supp}\eta_h|^{\frac{2\delta}{n}}.\end{aligned} \tag{3.94}$$

Proof. Note that $\Pi_h(u_h\Pi_h(\eta_h^2)) \in V_{h,0}$ because we assumed that $u_h = 0$ on $\partial\Omega \cap \operatorname{supp}\eta_h$. This means that we can test inequality (3.23) against $\varphi_h = \Pi_h(u_h\Pi_h(\eta_h^2))$ and follow the steps of the proof of Theorem 3.7. □

Thus we arrive at the following L^∞-norm bound that is valid near the boundary.

Theorem 3.27. *Under the assumptions of Theorem 3.25, suppose that* v_h *is a discrete subsolution to* $-\operatorname{div}(A\nabla u) = f - \operatorname{div}F$*. Suppose furthermore that* $v_h = 0$ *on* $\partial\Omega \cap \Omega(B(x_0,2R))$ *for some ball* $B(x_0,R)$*. For every* $c \geq 0$*, we then have*

$$\sup_{\Omega'(\mathcal{B}(x_0,R))} (v_h - c)_+^2 \leq CR^{-n}\int_{\Omega(\mathcal{B}(x_0,2R))} (v_h - c)_+^2 \,\mathrm{d}x + C\left(\|f\|^2_{L^q(\Omega)} + \|G\|^2_{L^p(\Omega)}\right)R^{2\delta} \tag{3.95}$$

for a constant $C > 0$ *that only depends on* n, $\|A\|_{L^\infty(\Omega)}$, d, δ *and the shape-regularity parameter* Γ *of* $\mathcal{T}_h$*.*

Proof. By Theorem 3.12, $(v_h - c - \lambda_k)_+$ (in the notation of the proof of Theorem 3.20) is a non-negative subsolution. We also have $(v_h - c - \lambda_k)_+ = 0$ on $\partial\Omega \cap \Omega(B(x_0, 2R))$ because $c \geq 0$, $\lambda_k > 0$ and $v_h = 0$ on $\partial\Omega \cap \Omega(B(x_0, 2R))$. Therefore, the result follows using inequality (3.94) and proceeding as in the proof of Theorem 3.20. □

Lemma 3.28. *Under the assumptions of Theorem 3.25, let $v_h \in V_h$ be a discrete subsolution to the equation $-\mathrm{div}(A\nabla u) = f - \mathrm{div}F$ with $0 \leq v_h \leq 1$. Then, there exist constants $\tau \in (0,1)$, $N > 0$ and $C > 0$ that only depend on n, δ, $\|A\|_{L^\infty(\Omega)}$, d and the shape-regularity parameter Γ of $\mathcal{T}_h$ such that if x_i and x_j are nodes of the same simplex $T \in \mathcal{T}_h$ with $v_h = 0$ on $\overline{\Omega(T)} \cap \partial\Omega$ then we have*

$$v_h(x_i) \leq 1 - \tau^N + \tau^N v_h(x_j) + C\left(\|F\|_{L^p(\Omega)} + \|f\|_{L^q(\Omega)}\right) h_T^\delta. \tag{3.96}$$

Proof. We would like to follow the steps in the proof of Lemma 3.21, which requires being able to test equation (3.23) against Lagrange basis functions ψ_i. If $x_i \in \Omega$, we have $\psi_i \in V_{h,0}$ and inequality (3.65) holds. Let us therefore assume that $x_i \in \partial\Omega$. We then have that $v_h(x_i) = 0$ because of the assumption $v_h = 0$ on $\overline{\Omega(T)} \cap \partial\Omega$. This yields the following inequality:

$$\begin{aligned}\int_\Omega A\nabla v_h \cdot \nabla\psi_i \,\mathrm{d}x &= \sum_j v_h(x_j) \int_\Omega A\nabla\psi_j \cdot \nabla\psi_i \,\mathrm{d}x \\ &= \sum_{j\neq i} v_h(x_j) \int_\Omega A\nabla\psi_j \cdot \nabla\psi_i \,\mathrm{d}x \leq 0,\end{aligned} \tag{3.97}$$

where we have used in the last step that the mesh is A-nonobtuse. Inequality (3.65) thereby simplifies to

$$v_h(y_{k+1}) \leq -\sum_{l\neq k+1} \frac{\langle\nabla\psi_l, \nabla\psi_{k+1}\rangle_A}{\langle\nabla\psi_{k+1}, \nabla\psi_{k+1}\rangle_A} v_h(y_l). \tag{3.98}$$

This means that we can proceed as in the proof of Lemma 3.21 from here on. □

This leads to the following variant of Lemma 3.22.

Lemma 3.29. *Under the assumptions of Theorem 3.25, let $B(x_0, R)$ be a ball with $R \geq h_T$ where $x_0 \in T$ for some $T \in \mathcal{T}_h$. Furthermore, let v_h be a discrete subsolution to $-\mathrm{div}(A\nabla u) = f - \mathrm{div}F$ with $0 \leq v_h \leq 1$ on $\mathcal{B}(x_0, QR)$. Assume further that*

$$|B(x_0, QR) \cap \Omega^c| \geq \mu_0 |\Omega(\mathcal{B}(x_0, QR))| \tag{3.99}$$

for some $\mu_0 > 0$, where Q is the constant from inclusion (2.13) and $v_h = 0$ on $\partial\Omega \cap \Omega\left(\overline{\Omega(B(2R))}\right)$.

Then, there exist constants R_0, $\theta \in (0,1)$ and $C > 0$ that only depend on n, $\|A\|_{L^\infty(\Omega)}$, d, δ, the Lipschitz constant of $\partial\Omega$ and the shape-regularity parameter Γ of $\mathcal{T}_h$ such that

$$\sup_{\Omega'(\mathcal{B}(x_0,R))} v_h \leq 1 - \theta + C\left(\|G\|_p + \|f\|_q\right) R^\delta \tag{3.100}$$

for any $R \in (h_T, R_0)$.

Proof. We can follow the proof of Lemma 3.22 step by step, because the condition $v_h = 0$ on $\partial\Omega \cap \Omega\left(\overline{\Omega(B(2R))}\right)$ ensures that inequalities (3.94), (3.95) and (3.96) hold. Furthermore, assumption (3.99) guarantees, together with $v_h = 0$ on $\partial\Omega\cap\Omega\left(\overline{\Omega(B(2R))}\right)$, that Poincaré's inequality can be applied to v_h on $\Omega(\mathcal{B}(x_0, QR) \cup B(x_0, QR)$ where v_h is extended by zero outside of $\Omega(\mathcal{B}(x_0, QR))$. By Corollary 2.9 and Lemma 2.30, the Poincaré constant of $\Omega(\mathcal{B}(x_0, QR) \cup B(x_0, QR)$ only depends on shape-regularity and the Lipschitz constant of the boundary. □

This now allows us to prove Theorem 3.25.

Proof of Theorem 3.25. If we have $B(x_0, 4\kappa^{-1}QR') \subset \Omega$ for an $x_0 \in T$ for some $T \in \mathcal{T}_h$ and $R' \geq h_T$, Theorem 3.15 yields

$$\operatorname*{osc}_{B(x_0,R)} u_h \lesssim R^\alpha.$$

On the other hand, if $B(x_0, 4\kappa^{-1}QR') \cap \partial\Omega \neq \emptyset$ for every $R' \geq h_T$, we write $u_h = (u_h)_+ - (-u_h)_+$ and consider $(u_h)_+$ and $(-u_h)_+$ separately; both are nonnegative subsolutions in V_h. We write $\xi = \min\{\beta, \delta\}$ (See assumptions of Theorem 3.25). Let us consider $(u_h)_+$. We can assume that $u_h(x) = 0$ for some point of $\partial\Omega$ because we are only interested in oscillations and could consider $u_h - u_h(x)$. Recall that $u_h|_{\partial\Omega} \in C^\beta(\partial\Omega) \subset C^\xi(\partial\Omega)$ and write $|u_h|_{\partial\Omega}|_{C^\xi} = D$. Then, we have $((u_h)_+ - D\,\mathrm{diam}(\Omega)^\xi)_+ \in V_{h,0}$. By Theorem 3.27, this gives that $((u_h)_+ - D\,\mathrm{diam}(\Omega)^\xi)_+$ is uniformly bounded, which means that $(u_h)_+$ is uniformly bounded by $\sup_\Omega\left(((u_h)_+ - D\,\mathrm{diam}(\Omega)^\xi)_+\right) + D\mathrm{diam}(\Omega)^\xi$.

If $R \geq h_T$ and $B(x_0, 4\kappa^{-1}QR) \cap \Omega^c \neq \emptyset$, we know that $B(y, \kappa^{-1}QR) \subset B(x_0, 5\kappa^{-1}QR)$ for some $y \in \partial\Omega$ and together with inequality (3.93) this gives assumption (3.99). We can also assume that $u_h(y) = 0$. We write

$$v_h = \frac{((u_h)_+ - DR^\xi)_+}{\|((u_h)_+ - DR^\xi)_+\|_{L^\infty(\mathcal{B}(x_0,5\kappa^{-1}QR))}}$$

with $D = c5\kappa^{-1}QR$. This means that

$$\operatorname*{osc}_{\mathcal{B}(x_0,5\kappa^{-1}QR)} v_h = \|v_h\|_{L^\infty(\mathcal{B}(x_0,5\kappa^{-1}QR))} = 1$$

and

$$v_h|_{\partial\Omega\cap B(x_0,5\kappa^{-1}QR)} = 0.$$

Note that v_h is a discrete subsolution to

$$-\text{div}(A\nabla v) = \frac{1}{\|((u_h)_+ - DR^\xi)_+\|_{L^\infty(\mathcal{B}(x_0,5\kappa^{-1}QR))}}(-\text{div}G + f).$$

Hence we can apply Lemma 3.29 for $R \in [h_T, \frac{1}{5}\kappa R_0]$ to get

$$\sup_{\mathcal{B}(x_0,\kappa R)} v_h \leq 1 - \theta + C\frac{\|G\|_{L^p(\Omega)} + \|f\|_{L^q(\Omega)}}{\|((u)_+ - DR^\delta)_+\|_{L^\infty(\mathcal{B}(x_0,5\kappa^{-1}QR))}}R^\xi.$$

This leads to

$$\begin{aligned}
\operatorname*{osc}_{\mathcal{B}(x_0,\kappa R)}(u_h)_+ &= \sup_{\mathcal{B}(x_0,\kappa R)}(u_h)_+ \leq \sup_{\mathcal{B}(x_0,\kappa R)}((u_h)_+ - DR^\xi)_+ + DR^\xi \\
&= \|((u_h)_+ - DR^\xi)_+\|_{L^\infty(\mathcal{B}(x_0,5\kappa^{-1}QR))}\sup_{\mathcal{B}(x_0,\kappa R)} v_h + DR^\xi \\
&\leq (1-\theta)\|((u_h)_+ - DR^\xi)_+\|_{L^\infty(\mathcal{B}(x_0,5\kappa^{-1}QR))} \\
&\qquad + \left(C\|G\|_{L^p(\Omega)} + C\|f\|_{L^q(\Omega)} + D\right)R^\xi \\
&\leq (1-\theta)\|(u_h)_+\|_{L^\infty(\mathcal{B}(x_0,5\kappa^{-1}QR))} \\
&\qquad + \left(C\|G\|_{L^p(\Omega)} + C\|f\|_{L^q(\Omega)} + D\right)R^\xi \\
&\leq (1-\theta)\operatorname*{osc}_{\mathcal{B}(x_0,5\kappa^{-1}QR)}(u_h)_+ + \left(C\|G\|_{L^p(\Omega)} + C\|f\|_{L^q(\Omega)} + D\right)R^\xi.
\end{aligned} \tag{3.101}$$

This means that we can apply Lemma 3.34 to deduce that

$$\operatorname*{osc}_{\mathcal{B}(x_0,R)}(u_h)_+ \lesssim \left(\|G\|_{L^p(\Omega)} + \|f\|_{L^q(\Omega)} + D\right)R^\xi.$$

The proofs for $R \leq h_T$ and $R \geq \frac{1}{5}\kappa R_0$ are completely analogous to the respective parts in the proof of Theorem 3.15. We then repeat this argument for $(-u_h)_+$. By combining the resulting bound on $\text{osc}_{\mathcal{B}(x_0,R)}(-u_h)_+$ with the above bound on $\text{osc}_{\mathcal{B}(x_0,R)}(u_h)_+$ we deduce that

$$\operatorname*{osc}_{\mathcal{B}(x_0,R)}(u_h) \leq C\left(\|G\|_{L^p(\Omega)} + \|f\|_{L^q(\Omega)} + D\right)R^\xi. \tag{3.102}$$

To finish the proof, we finally have to look at cases where $B(x_0,R)$ has multiple connected components. On a single connected component we have

$$|u_h(x) - u_h(y)| \leq C\left(\|G\|_{L^p(\Omega)} + \|f\|_{L^q(\Omega)} + D\right)|x-y|^\xi$$

by inequality (3.102). Now, let x and y be two points in $\Omega(B(x_0,R))$ that are in different connected components. We find points $a, b \in \partial\Omega$ with $|a-b| \leq |x-y|$, such that $|x-a| \leq |x-y|$ and $|y-b| \leq |x-y|$ and a is in the same connected component of $\Omega(B(x_0,R))$ as

x and b is in the same connected component as y (for example by just connecting x and y by a line and taking a and b as the intersections between that line and $\partial\Omega$). Recall that $u_h|_{\partial\Omega} = D$ and therefore, we have $|u(a) - u(b)| \leq D|a-b|^\xi$. Therefore, we can write

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - u(a)| + |u(a) - u(b)| + |u(b) - u(y)| \\ &\leq C \left(\|G\|_{L^p(\Omega)} + \|f\|_{L^q(\Omega)} + D\right) \left(|x-a|^\xi + |a-b|^\xi + |b-y|^\xi\right). \end{aligned} \tag{3.103}$$

Note that $0 < \xi < 1$. From Jensen's inequality we deduce that $(a_1^\xi + a_2^\xi + a_3^\xi) \leq 3\left(\frac{1}{3}(a+b+c)\right)^\xi$. This and $|x-a| + |a-b| + |b-y| \leq 3|x-y|$ yield together with inequality (3.103) that

$$\begin{aligned} |u(x) - u(y)| &\leq C \left(\|G\|_{L^p(\Omega)} + \|f\|_{L^q(\Omega)} + D\right) 3\Big(\frac{1}{3}(|x-a| + |a-b| + |b-y|)\Big)^\xi \\ &\leq C \left(\|G\|_{L^p(\Omega)} + \|f\|_{L^q(\Omega)} + D\right) 3|x-y|^\xi. \end{aligned}$$

Finally, this proves

$$|u_h|_{C^\alpha(\overline{\Omega})} \lesssim \|G\|_{L^p(\Omega)} + \|f\|_{L^q(\Omega)} + D,$$

which completes the proof. □

We conclude with the application of the discrete De Giorgi theory to a nonlinear elliptic problem.

Theorem 3.30 (Uniformly elliptic nonlinear equations)**.** *Let $\mathcal{T}_h$ be a shape-regular, non-obtuse triangulation of the polyhedral Lipschitz domain Ω with associated finite element space V_h. Furthermore, let $F \in L^p(\Omega; \mathbb{R}^n)$ satisfy assumption ($\star$) from Definition 3.10 with dominating function $G \in L^p(\Omega)$. and let $f \in L^q(\Omega)$ with q and p defined as in Theorem 3.25. Let $a : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfy $0 < d \leq a(\cdot,\cdot,\cdot) \leq D < \infty$. Furthermore, let $u_h \in V_{h,0}$ be a discrete solution to $-\mathrm{div}(a(x,u_h,\nabla u_h)\nabla u_h) = f - \mathrm{div}F$, i.e., a function $u_h \in V_h$, that satisfies*

$$\int_\Omega a(x,u_h,\nabla u_h)\nabla u_h \cdot \nabla\varphi_h \,\mathrm{d}x = \int_\Omega f\varphi_h \,\mathrm{d}x + \int_\Omega F \cdot \nabla\varphi_h \,\mathrm{d}x \tag{3.104}$$

for all $\varphi_h \in V_{h,0}$. Furthermore, assume that

$$\big|u_h|_{\partial\Omega}\big|_{C^\beta(\overline{\Omega})} \leq D'$$

for constants $\beta > 0$ and $D' > 0$. Then, there are exist $\alpha \in (0,1)$ and $C > 0$ depending only on n, d, D, $\|f\|_{L^q(\Omega)}$, δ, $\|G\|_{L^p(\Omega,\mathbb{R}^n)}$, $\mathrm{diam}\,\Omega$,D', β, the Lipschitz constant of $\partial\Omega$ and the shape-regularity parameter Γ of $\mathcal{T}_h$, such that

$$|u_h|_{C^\alpha(\overline{\Omega})} \leq C.$$

Proof. First note that any nonobtuse triangulation is A-nonobtuse if A has the form $\mathbb{1}_{n\times n}$. The same steps as in the proof to Theorem 3.12 (with $f_2 = 0$ and $F_2 = 0$) show that

$$\int_\Omega a(x, u_h, \nabla u_h)\nabla(u_h - c)_+ \cdot \nabla\varphi_h \,\mathrm{d}x \le \int_\Omega f\varphi_h \,\mathrm{d}x + \int_\Omega G \cdot \nabla\varphi_h \,\mathrm{d}x \tag{3.105}$$

for any non-negative $\varphi_h \in V_{h,0}$. Because $a(x, u_h, \nabla u_h)\mathbb{1}_{n\times n}$ is a uniformly elliptic matrix for any $x \in \Omega$, $u_h(x) \in \mathbb{R}$ and $\nabla u_h(x) \in \mathbb{R}^n$, we can follow the proofs of Theorem 3.25 and Theorem 3.15. □

In this chapter, we have shown that under the condition of shape-regularity and A-non-obtuseness, discrete solutions to $-\operatorname{div}(a\nabla u) = f - \operatorname{div} F$ are uniformly Hölder-continuous up to the boundary of the domain Ω if F satisfies condition $\star$ from Definition 3.10 and if a is bounded and uniformly elliptic. In the next chapter, we will analyse discrete solutions to p- and φ-Laplacian systems where a is given by $|\nabla u|^{p-2}$ or $\varphi(|\nabla u|)|\nabla u|^{-1}$, respectively, and therefore fails to be uniformly elliptic or bounded. We will derive uniform L^∞-estimates.

3.5 Appendix

We will now give the proofs of the technical iteration Lemmas that were used in this Chapter.

Lemma 3.31 (Fast geometric convergence, cf. [21] Lemma 4.1)**.** *Let $\alpha > 0$, $C > 0$ and $b > 1$ be real numbers and (a_k) a sequence of nonnegative real numbers with the properties*

$$0 \le a_{k+1} \le Cb^k a_k^{1+\alpha},$$

$$0 \le a_0 \le C^{-\frac{1}{\alpha}} b^{-\frac{1}{\alpha^2}}.$$

Then we have $a_k \le C^{-\frac{1}{\alpha}} b^{-\frac{1+k\alpha}{\alpha^2}} \to 0$ as $k \to \infty$.

Proof. We use induction:
The base case $k = 0$ follows directly from the second property.
The induction step is straightforward: let $a_k \le C^{-\frac{1}{\alpha}} b^{-\frac{1+k\alpha}{\alpha^2}}$ for some k then we get

$$\begin{aligned} a_{k+1} &\le Cb^k a_k^{1+\alpha} \le Cb^k \left(C^{-\frac{1}{\alpha}} b^{-\frac{1+k\alpha}{\alpha^2}}\right)^{1+\alpha} \\ &\le Cb^k C^{-1-\frac{1}{\alpha}} b^{-\frac{1+(k+1)\alpha}{\alpha^2} - k} = C^{-\frac{1}{\alpha}} b^{-\frac{1+(k+1)\alpha}{\alpha^2}}. \end{aligned}$$

□

From this we easily deduce the following result.

Corollary 3.32. *Let $\alpha > 0$, $C > 0$, $b > 1$ and $\gamma > 0$ be real numbers and a_k a sequence of nonnegative real numbers, such that*

$$0 \le a_{k+1} \le C b^k a_k \left(\frac{a_k}{\gamma} \right)^\alpha .$$

Then we have $a_k \to 0$ as $k \to \infty$ if $\gamma = a_0 C^{\frac{1}{\alpha}} b^{\frac{1}{\alpha^2}}$.

Proof. Apply Lemma 3.31 to the sequence $\left(\frac{a_k}{\gamma}\right)$. □

Lemma 3.33 (cf. [3] A2)**.** *Suppose that a sequence (a_k) with $a_k > 0$ satisfies*

$$a_{k+1}^2 \le c_k (a_k - a_{k+1})$$

for some bounded, nonnegative sequence c_k. Then, we have that

$$a_{k+1} \le \frac{\sqrt{\max_{i \le k} c_i}\sqrt{a_0}}{\sqrt{k}}, \; k \ge 1.$$

Proof. Since we necessarily have $a_k \ge a_{k+1} \ge 0$, we get via a telescoping sum

$$\begin{aligned} k\, a_{k+1}^2 &\le \sum_{i=1}^k a_{i+1}^2 \le \sum_{i=1}^k c_k (a_i - a_{i+1}) \\ &\le \left(\max_{i \le k} c_i \right) (a_1 - a_{k+1}) \le \left(\max_{i \le k} c_i \right) a_1 . \end{aligned}$$

Dividing by k and taking the square root concludes the proof. □

Lemma 3.34 (cf. [8], Lemma B.3)**.** *Assume that $\varphi(\rho)$ is a real-valued, nonnegative, non-decreasing function defined on the interval $[0, R_1]$, and that we have $C, \alpha_1, \alpha_2 > 0$ with $\alpha_2 \le \alpha_1$ and $0 < \beta < 1$, such that*

$$\varphi(\beta R) \le \beta^{\alpha_1} \varphi(R) + A R^{\alpha_2}$$

for all $R \le R_1$ such that $\beta R \ge R_0$. Then, there is a $c > 0$ such that for all $R_0 \le r \le R \le R_1$ we have

$$\varphi(r) \le c \left(\left(\frac{r}{R} \right)^{\alpha_2} \varphi(R) + C r^{\alpha_2} \right).$$

Chapter 4

Local Boundedness of discrete solutions to p-Laplacian systems

We will now focus on p-Laplacian systems. Unfortunately, there is no natural notion of a positive part for vector-valued functions. This means that we have to find a different truncation to apply the De-Giorgi-iteration method. This only leads to an L^∞-estimate on the solution itself, not on truncated solutions. However, the latter would be needed to obtain C^α-estimates the way we did in the case of uniformly elliptic equations, which means that we have to restrict ourselves to L^∞-estimates. On the other hand, we will be able to drop the subsolution method which means that we are able to work with right-hand sides that do not satisfy assumption ($\star$). For uniformly acute meshes, we will show an L^∞-estimate for homogeneous φ-Laplacian systems, a common generalisation of the p-Laplace equation. Before that, we will give a brief overview of the proof of a local L^∞-estimate in the continuous case. We note that in the non-discrete case a vast range of regularity results is available. Interior $C^{1,\alpha}$-estimates for minimisers for variational problems exhibiting p-growth are available since the 60s, see for example [59, 58, 45, 33, 20, 1, 30] and the references therein.

4.1 The continuous case

We will first define a weak solution to a p-Laplacian system. For simplicity, we will restrict ourselves to the homogeneous case. However, we will have a non-zero right-hand side in the discrete case. The techniques are again inspired by [14] (where it was used for linear scalar equations).

Definition 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $p \in (1, \infty)$ be a fixed real number. We call a function $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ a weak solution to the equation $-\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0$ if* *we have*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \,\mathrm{d}x = 0 \tag{4.1}$$

for every $\varphi \in C_0^\infty(\Omega;\mathbb{R}^m)$*. By density, this extends to all* $\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$*. Note that* $\cdot$ *means the Frobenius matrix scalar product or the usual vector scalar product depending on the context. However, it will always be clear which scalar product is meant.*

We will now give the main theorem of this section. For the sake of clarity, we will restrict ourselves to the case $p \in (1, n)$. However, the proofs are easily adjusted for $p \geq n$ by taking an arbitrary large q instead of $p^\star$ when using the Sobolev embedding theorem.

Theorem 4.2. *For* $p > 1$*, let* $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ *be a weak solution to* $-\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0$*. Let* $x_0 \in \Omega$ *and* $R > 0$ *be such that* $B(x_0, 2R) \subset \Omega$*. Then, we have that*

$$\sup_{B(x_0,R)} |u|^p \leq C \fint_{B(x_0,2R)} |u|^p \,\mathrm{d}x \tag{4.2}$$

for a constant $C > 0$ *that only depends on* p *and* n*.*

To be able to prove this theorem, we need a new truncation.

Definition 4.3. *For any* $\lambda > 0$ *and* $v \in \mathbb{R}^m$*, we define*

$$\mathcal{S}_\lambda v = (|v| - \lambda)_+ \frac{v}{|v|} \tag{4.3}$$

for $|v| > 0$ *and* $\mathcal{S}_\lambda v = 0$ *for* $|v| = 0$*.*

Remark 4.4. *We will later show that* $|\mathcal{S}_\lambda x - \mathcal{S}_\lambda y| \leq |x - y|$ *in inequality (4.29). Let us write* $(\delta_{\varepsilon,i} v)(x) = \frac{1}{\varepsilon}(v(x + \varepsilon e_i) - v(x))$ *almost everywhere for the difference quotient in* i*-direction, where* e_i *is the respective unit vector. We find*

$$|\delta_{\varepsilon,i}(\mathcal{S}_\lambda v)| \leq |\delta_{\varepsilon,i} v|$$

almost everywhere. Therefore, we get

$$\|\delta_{\varepsilon,i}(\mathcal{S}_\lambda v)\|_{L^p(\Omega')} \leq \|\delta_{\varepsilon,i} v\|_{L^p(\Omega')}$$

for all $\varepsilon > 0$ *and all* $\Omega' \subset \Omega$ *with* $\mathrm{dist}(\Omega', \partial\Omega) > \varepsilon$*. By [48][Theorem 1.49], we know that* $\|\delta_{\varepsilon,i} v\|_{L^p(\Omega')}$ *stays bounded if* $v \in W^{1,p}(\Omega)$*. Therefore,* $\|\delta_{\varepsilon,i}(\mathcal{S}_\lambda v)\|_{L^p(\Omega')}$ *also stays bounded and we can conclude with [48][Theorem 1.49] that* $\mathcal{S}_\lambda v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ *if* $v \in W^{1,p}(\Omega)$ *and* $\mathcal{S}_\lambda$ *is applied point-wise almost everywhere.*

To be able to prove a Caccioppoli-type inequality for $\mathcal{S}_\lambda u$, we need the following proposition. Note that this section is meant to guide the reader through the non-discrete case to help understand the discrete case. We will use some more formal calculations that have to be justified.

Proposition 4.5. *Let $\lambda > 0$ and $p \in (1, \infty)$ and let $v \in W^{1,p}(\Omega; \mathbb{R}^m)$. Almost everywhere, we get*

$$\nabla v \cdot \nabla \mathcal{S}_\lambda v \geq \frac{(|v| - \lambda)_+}{|v|} |\nabla v|^2. \tag{4.4}$$

Proof. First, we recall that for any function $f \in W^{1,p}(\Omega)$, we have $\nabla(f_+) = \chi_{\{f>0\}} \nabla f$ and $\nabla|f| = (\text{sign} f)\,\nabla f$ (see [47][Remark 1.2, Lemma 1.7]). By approximating $\mathcal{S}_\lambda v$ by $\mathcal{S}_{\lambda,\varepsilon} v = \frac{(\sqrt{|v|^2+\varepsilon}-\lambda)_+}{\sqrt{|v|^2+\varepsilon}} v$ and then taking the limit $\varepsilon \to 0$, we find

$$\nabla \mathcal{S}_\lambda v = \frac{|v| - \lambda}{|v|} \nabla v + \frac{v}{|v|} \nabla |v| - \frac{|v| - \lambda}{|v|} \frac{v}{|v|} \nabla |v|$$

on $\{|v| > \lambda\}$. Together with $\nabla |v| = \frac{v}{|v|} \nabla v$ and by taking the scalar product with ∇v, this gives

$$\nabla \mathcal{S}_\lambda v \cdot \nabla v = \frac{|v| - \lambda}{|v|} |\nabla v|^2 + \left(1 - \frac{|v| - \lambda}{|v|}\right) \big|\nabla |v|\big|^2. \tag{4.5}$$

Using $0 \leq \frac{|v|-\lambda}{|v|} \leq 1$ proves the claim. $\square$

Furthermore, applying the Cauchy–Schwarz inequality to inequality (4.4) and dividing by $|\nabla v|$ (for $|\nabla v| \neq 0$) yields

$$|\nabla \mathcal{S}_\lambda v| \geq \frac{(|v| - \lambda)_+}{|v|} |\nabla v| \tag{4.6}$$

almost everywhere. For $|\nabla v| = 0$, inequality (4.6) simplifies to $|\nabla S_\lambda v| \geq 0$ which is obviously true.

We are now able to prove a Caccioppoli-type inequality.

Lemma 4.6. *For $p \in (1, \infty)$, let $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ be a weak solution to $-\text{div}(|\nabla u|^{p-2} \nabla u) = 0$. Then, there exists a constant $C > 0$ that only depends on p and n, such that*

$$\int_\Omega \frac{(|u| - \lambda)_+}{|u|} |\nabla u|^p |\eta|^p \, \mathrm{d}x \leq C \int_\Omega \frac{(|u| - \lambda)_+}{|u|} |u|^p |\nabla \eta|^p \, \mathrm{d}x \tag{4.7}$$

for every nonnegative $\eta \in C_0^\infty(\Omega; \mathbb{R})$ and every $\lambda > 0$.

Proof. We know that $\mathcal{S}_\lambda u \in W^{1,p}_{\text{loc}}$ by Remark 4.4. Therefore, $(\mathcal{S}_\lambda u)\eta^p \in W_0^{1,p}(\Omega)$ is an admissible test function. We test equation (4.1) against $\varphi = \mathcal{S}_\lambda u \eta^p$ to get

$$\begin{aligned} 0 &= \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \left(\mathcal{S}_\lambda u \eta^p\right) \mathrm{d}x \\ &= \int_\Omega \eta^p |\nabla u|^{p-2} \nabla u \cdot \nabla \mathcal{S}_\lambda u \, \mathrm{d}x + \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \eta \, \mathcal{S}_\lambda u \, p \eta^{p-1} \, \mathrm{d}x := I + II. \end{aligned} \tag{4.8}$$

First, we use inequality (4.4) on I from equation (4.8) to get

$$I \geq \int_\Omega \frac{(|u|-\lambda)_+}{|u|} |\nabla u|^p |\eta|^p \, \mathrm{d}x. \tag{4.9}$$

Recall $|\mathcal{S}_\lambda u| = (|u|-\lambda)_+$. Then, we estimate II from equation (4.8) with Young's inequality to get

$$\begin{aligned} II &\lesssim \int_\Omega |\nabla u|^{p-1} |\nabla \eta| |\eta|^{p-1} |u| \frac{(|u|-\lambda)_+}{|u|} \, \mathrm{d}x \\ &\leq \varepsilon \int_\Omega \frac{(|u|-\lambda)_+}{|u|} |\nabla u|^p |\eta|^p \, \mathrm{d}x + C_\varepsilon \int_\Omega \frac{(|u|-\lambda)_+}{|u|} |u|^p |\nabla \eta|^p \, \mathrm{d}x. \end{aligned} \tag{4.10}$$

Now, inserting inequalities (4.9) and (4.10) into equation (4.8) and absorbing the term

$$\varepsilon \int_\Omega \frac{(|u|-\lambda)_+}{|u|} |\nabla u|^p |\eta|^p \, \mathrm{d}x$$

into the left-hand side proves inequality (4.7) and thus, the lemma.

□

Lemma 4.6 allows us to prove the local L^∞-estimate stated in Theorem 4.2.

Proof of Theorem 4.2. We define the sequence

$$\lambda_k = \lambda_\infty \left(1 - 2^{-k}\right) \tag{4.11}$$

where λ_∞ is to be chosen later. Furthermore, we define the nested balls $B_k = B(x_0, R(1+2^{-k}))$ and C^∞-functions η_k with

$$\chi_{B_k} \leq \eta_k \leq \chi_{B_{k+1}}, \tag{4.12}$$

$$|\nabla \eta_k| \lesssim R^{-1} 2^k. \tag{4.13}$$

We define the sequence $a_k := \fint_{B(x_0,2R)} |\mathcal{S}_{\lambda_k} u|^p \eta_k^p \, \mathrm{d}x$. Using the scaling-invariant norms $|||f|||_q^q := \fint_{B(x_0,2R)} |f|^q \, \mathrm{d}x$, Hölder's inequality and the Sobolev embedding theorem, we find that

$$\begin{aligned} a_{k+1} = |||\mathcal{S}_{\lambda_{k+1}} u \eta_{k+1}|||_p^p &\leq |||\mathcal{S}_{\lambda_{k+1}} u \eta_{k+1}|||_{p^\star}^p \left(\frac{|\operatorname{supp} \mathcal{S}_{\lambda_{k+1}} u \cap B_k|}{|B(x_0,2R)|} \right)^{\frac{p}{n}} \\ &\lesssim R^p |||\nabla \left(\mathcal{S}_{\lambda_{k+1}} u \eta_{k+1} \right)|||_p^p \left(\frac{|\operatorname{supp} \mathcal{S}_{\lambda_{k+1}} u \cap B_k|}{|B(x_0,2R)|} \right)^{\frac{p}{n}} \end{aligned} \tag{4.14}$$

for $p \in (1,n)$. For $p \geq n$ the proof is easily adjusted by taking a large enough (depending only on n and p) positive number instead of $p^\star$ as $L^q(\Omega) \hookrightarrow W^{1,p}(\Omega)$ for all $q \in (1,\infty)$ in this case. We know that $|u| \geq \lambda_{k+1}$ on $\operatorname{supp} \mathcal{S}_{\lambda_{k+1}} u$. On this set, we therefore have

$$|\mathcal{S}_{\lambda_k} u| \geq \lambda_{k+1} - \lambda_k = \lambda_\infty \left(2^{-k} - 2^{-(k+1)} \right) = \lambda_\infty 2^{-(k+1)}. \tag{4.15}$$

Together with $\eta_k \equiv 1$ on $\operatorname{supp}\eta_{k+1} = B_k$, inequality (4.15) yields

$$a_k = \fint_{B(x_0,2R)} |\mathcal{S}_{\lambda_k} u|^p \eta_k^p \,\mathrm{d}x \geq |B(x_0,2R)|^{-1} \frac{2^{-p(k+1)}}{\lambda_\infty^p} |\operatorname{supp} \mathcal{S}_{\lambda_{k+1}} u \cap B_k|. \tag{4.16}$$

On the other hand, we use the product rule and the convexity of $t \mapsto t^p$ to find

$$|||\nabla \left((\mathcal{S}_{\lambda_{k+1}} u)\eta_{k+1}\right)|||_p^p \lesssim |||\left(\nabla \mathcal{S}_{\lambda_{k+1}} u\right) \eta_{k+1}|||_p^p + |||\mathcal{S}_{\lambda_{k+1}} u \,\nabla \eta_{k+1}|||_p^p. \tag{4.17}$$

Define $\lambda = \frac{1}{2}(\lambda_{k+1} + \lambda_k)$ and assume that $|u| > \lambda_{k+1}$. Then, we have

$$\frac{(|u| - \lambda)_+}{|u|} \geq \frac{\lambda_{k+1} - \lambda}{\lambda_{k+1}} = \frac{2^{-k} - 2^{-(k+1)}}{2\left(1 - 2^{-(k+1)}\right)} \gtrsim 2^{-k}. \tag{4.18}$$

Using this and $|\nabla \mathcal{S}_{\lambda_{k+1}} u| \leq |\nabla u|$, we find that

$$\fint_{B(x_0,2R)} |\nabla \mathcal{S}_{\lambda_{k+1}} u|^p \eta_{k+1}^p \,\mathrm{d}x \lesssim 2^k \fint_{B(x_0,2R)} \frac{(|u| - \lambda)_+}{|u|} |\nabla u|^p \eta_{k+1}^p \,\mathrm{d}x. \tag{4.19}$$

This means that we can apply inequality (4.7) to inequality (4.19) to get

$$\fint_{B(x_0,2R)} |\nabla \mathcal{S}_{\lambda_{k+1}} u|^p \eta_{k+1}^p \,\mathrm{d}x \lesssim 2^k \fint_{B(x_0,2R)} \frac{(|u| - \lambda)_+}{|u|} |u|^p |\nabla \eta_{k+1}|^p \,\mathrm{d}x. \tag{4.20}$$

Now, assume that $|u| > \lambda$. Analogously to inequality (4.18), we then find that

$$\frac{|\mathcal{S}_{\lambda_k} u|}{|u|} > \frac{\lambda - \lambda_k}{\lambda_k} \geq 2^{-(k+2)}. \tag{4.21}$$

Inserting this into inequality (4.20) yields

$$\fint_{B(x_0,2R)} |\nabla \mathcal{S}_{\lambda_{k+1}} u|^p \eta_{k+1}^p \,\mathrm{d}x \lesssim 2^{(p+1)k} \fint_{B(x_0,2R)} |\mathcal{S}_{\lambda_k} u|^p |\nabla \eta_{k+1}|^p \,\mathrm{d}x, \tag{4.22}$$

where we have also used that $\frac{|u|-\lambda}{|u|} \leq 1$. Furthermore, we have $\eta_k \equiv 1$ on $\operatorname{supp}\eta_{k+1} = B_k$ from inequality (4.12) and $|\nabla \eta_{k+1}| \lesssim R^{-1} 2^k$ by definition in equation (4.13). Using this in inequality (4.22) gives

$$\fint_{B(x_0,2R)} |\nabla \mathcal{S}_{\lambda_{k+1}} u|^p \eta_{k+1}^p \,\mathrm{d}x \lesssim 2^{(2p+1)k} R^{-p} \fint_{B(x_0,2R)} |\mathcal{S}_{\lambda_k} u|^p |\eta_k|^p \,\mathrm{d}x. \tag{4.23}$$

We can now use inequality (4.23) and $|\mathcal{S}_{\lambda_{k+1}} u| \leq |\mathcal{S}_{\lambda_k} u|$ in inequality (4.9) to get

$$|||\nabla \left(\mathcal{S}_{\lambda_{k+1}} u \eta_{k+1}\right)|||_p^p \lesssim R^{-p} 2^{(2p+1)k} |||\mathcal{S}_{\lambda_k} u \eta_k|||_p^p. \tag{4.24}$$

Finally, we can insert inequalities (4.16) and (4.24) into inequality (4.14) to get

$$a_{k+1} \lesssim 2^{\left(2p+1+\frac{p}{n}\right)k} a_k \left(\frac{a_k}{\lambda_\infty^p}\right)^{\frac{p}{n}}. \tag{4.25}$$

If we choose $\lambda_\infty^p \sim a_0$, we can apply Corollary 3.32 to conclude that $a_k \to 0$ and therefore

$$\sup_{B(x_0,R)} |u|^p \leq \lambda_\infty^p \sim a_0 \leq \fint_{B(x_0,2R)} |u|^p \,\mathrm{d}x.$$

This proves Theorem 4.2. □

4.2 The discrete truncation S_λ

We want to prove a similar L^∞-estimate in the discrete setting. If we apply the truncation $\mathcal{S}_\lambda$ pointwise on a function $u_h \in V_h$, the result would in general no longer be in V_h. This means that we have to use a nodal version of the truncation.

Definition 4.7. *Let $\mathcal{T}_h$ be a triangulation of the polyhedral domain $\Omega \subset \mathbb{R}^n$ and V_h the respective finite element space with values in $\mathbb{R}^m$ and Lagrange basis functions ψ_i. For any $u_h \in V_h$ and $\lambda > 0$, we define*

$$S_\lambda u_h := \sum_i \mathcal{S}_\lambda u_h(x_i)\psi_i = \sum_i \frac{(|u_h(x_i)| - \lambda)_+}{|u_h(x_i)|} u_h(x_i)\psi_i = \Pi_h(\mathcal{S}_\lambda u_h). \tag{4.26}$$

The crucial estimate in the continuous case is $\nabla \mathcal{S}_\lambda u \cdot \nabla u \gtrsim \frac{(|u|-\lambda)_+}{|u|}|\nabla u|^2$. In order to be able to prove similar estimates in the discrete setting, we need estimates on the difference of two truncated vectors. We will give the most important ones in the following lemma.

Lemma 4.8. *Let $\mathcal{S}_\lambda$ be defined as in equation (4.3). Then, we have*

$$(\mathcal{S}_\lambda x - \mathcal{S}_\lambda y) \cdot (x - y) \geq \frac{1}{2}\left(\frac{(|x| - \lambda)_+}{|x|} + \frac{(|y| - \lambda)_+}{|y|}\right)|x - y|^2, \tag{4.27}$$

$$(\mathcal{S}_\lambda x - \mathcal{S}_\lambda y) \cdot (x - y) \geq |\mathcal{S}_\lambda x - \mathcal{S}_\lambda y|^2, \tag{4.28}$$

$$|x - y| \geq |\mathcal{S}_\lambda x - \mathcal{S}_\lambda y| \tag{4.29}$$

for every $\lambda \in \mathbb{R}$ and every $x, y \in \mathbb{R}^m$.

Proof. It is easier to work with

$$T_\lambda x := x - \mathcal{S}_\lambda x = \begin{cases} x, \text{ if } |x| \leq \lambda, \\ \lambda \frac{x}{|x|} \text{ if } |x| > \lambda. \end{cases} \tag{4.30}$$

As a first step, we will show that

$$(x - T_\lambda x) \cdot (z - T_\lambda x) \leq 0 \tag{4.31}$$

for every $x \in \mathbb{R}^m$ and every $z \in \overline{B(0,\lambda)} \subset \mathbb{R}^m$. For the more general case of orthogonal projections onto convex sets in $\mathbb{R}^m$ instead of the specific projection T_λ, this inequality is shown in [26, Lemma 10]. Inequality (4.31) is obviously true for $|x| \leq \lambda$ because $x - T_\lambda x = 0$ in that case. Therefore, let us assume that $|x| > \lambda$. The $(m-1)$-dimensional hyperplane L_x through $T_\lambda x \in \partial B(0,\lambda)$ that is tangential to the ball $B(0,\lambda) \subset \mathbb{R}^m$ is characterised by the equation $L_x = \{y \subset \mathbb{R}^m : (y - T_\lambda x) \cdot (x - T_\lambda x) = 0\}$. $B(0,\lambda)$ is convex. This implies that L_x separates x and $B(0,\lambda)$. This means that we can deduce $(z - T_\lambda x) \cdot (x - T_\lambda x) \leq 0$.

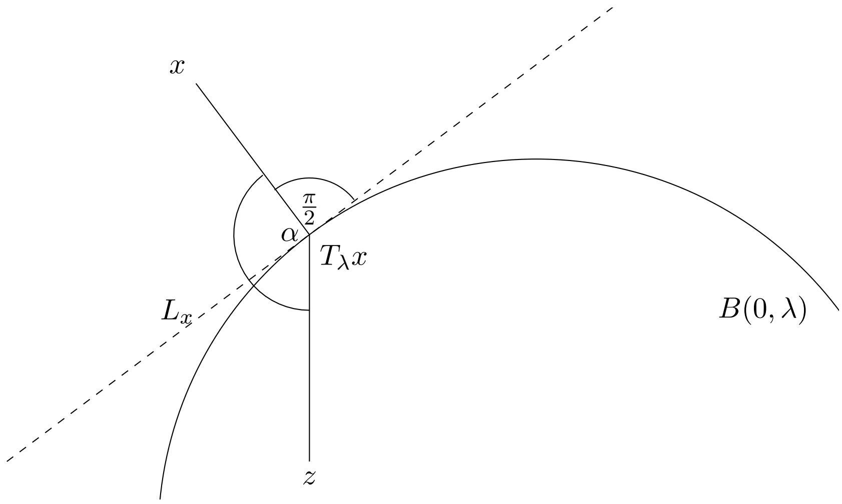

Figure 4.1: The projection T_λ

For reference, see that the angle α in Figure 4.1 is necessarily larger than or equal to $\frac{\pi}{2}$. This proves inequality (4.31). We will now show that

$$0 \leq (T_\lambda x - T_\lambda y) \cdot (x - y) \leq \frac{1}{2}\left(\frac{\min\{|x|, \lambda\}}{|x|} + \frac{\min\{|y|, \lambda\}}{|y|}\right)|x - y|^2. \tag{4.32}$$

Obviously, we have $T_\lambda y \in \overline{B(0,\lambda)}$ for all $y \in \mathbb{R}^m$. This allows us to apply inequality (4.31) to get

$$\begin{aligned}
&(T_\lambda x - T_\lambda y) \cdot (x - y) \\
=&\underbrace{(T_\lambda x - T_\lambda y) \cdot (x - T_\lambda x)}_{\geq 0} + \underbrace{(T_\lambda x - T_\lambda y) \cdot (T_\lambda y - y)}_{\geq 0} + |T_\lambda x - T_\lambda y|^2 \\
\geq&0.
\end{aligned}$$

This proves the lower bound in inequality (4.32). For the upper bound, we distinguish three cases.

(a) Assume that $|x|, |y| \leq \lambda$. In this case, inequality (4.32) simplifies to $0 \leq (x - y) \cdot (x - y) = |x - y|^2$, which is obviously true.

(b) Assume that $|x|, |y| \geq \lambda$.

We calculate

$$
\begin{aligned}
(T_\lambda x - T_\lambda y)\cdot(x-y) &= \left(\frac{\lambda}{|x|}x - \frac{\lambda}{|y|}y\right)\cdot(x-y)\\
&= \lambda\left(|x|+|y| - \left(\frac{1}{|x|}+\frac{1}{|y|}\right)x\cdot y\right)\\
&= \lambda\left(\frac{1}{|x|}+\frac{1}{|y|}\right)(|x||y| - x\cdot y)\\
&= \lambda\left(\frac{1}{|x|}+\frac{1}{|y|}\right)\tfrac{1}{2}\left(|x-y|^2 - \big||x|-|y|\big|^2\right)\\
&\leq \tfrac{1}{2}\left(\frac{\lambda}{|x|}+\frac{\lambda}{|y|}\right)|x-y|^2,
\end{aligned}
$$

which is exactly the upper bound in equality (4.32).

(c) Assume that $|y| \leq \lambda \leq |x|$.

Let $\alpha := \frac{\lambda}{|x|}$; then $0 \leq \alpha \leq 1$ and we calculate

$$
\begin{aligned}
(T_\lambda x - T_\lambda y)\cdot(x-y) &= \big(\alpha x - y\big)\cdot(x-y)\\
&= \alpha|x|^2 + |y|^2 - (\alpha+1)x\cdot y\\
&= \frac{\alpha+1}{2}|x-y|^2 + \underbrace{\frac{\alpha-1}{2}}_{\leq 0}\underbrace{(|x|^2-|y|^2)}_{\geq 0}\\
&\leq \frac{\alpha+1}{2}|x-y|^2.
\end{aligned}
$$

This is also exactly the upper bound in inequality (4.32) and therefore concludes the proof of inequality (4.32).

Then, we use the definition of T_λ in equation (4.30) and find

$$
(\mathcal{S}_\lambda x - \mathcal{S}_\lambda y)\cdot(x-y) = |x-y|^2 - (T_\lambda x - T_\lambda y)\cdot(x-y). \tag{4.33}
$$

This means that we can use inequality (4.32) in equation (4.33) to find

$$
\begin{aligned}
(\mathcal{S}_\lambda x - \mathcal{S}_\lambda y)\cdot(x-y) &\geq |x-y|^2 - \frac{1}{2}\left(\frac{\min\{|x|,\lambda\}}{|x|} + \frac{\min\{|y|,\lambda\}}{|y|}\right)|x-y|^2\\
&\geq \frac{1}{2}\left(\frac{|x|-\min\{|x|,\lambda\}}{|x|} + \frac{|y|-\min\{|y|,\lambda\}}{|y|}\right)|x-y|^2.
\end{aligned}
$$

Now, note that $|x| - \min\{|x|,\lambda\} = (|x|-\lambda)_+$. This proves inequality (4.27).

For the proof of inequality (4.28) we use equation (4.31) to find that

$$
(x - T_\lambda x)\cdot(T_\lambda y - T_\lambda x) \leq 0. \tag{4.34}
$$

With inequality (4.34), we estimate

$$\begin{aligned}&|T_\lambda x - T_\lambda y|^2 - (T_\lambda x - T_\lambda y)\cdot(x-y)\\ =&(T_\lambda x - x)\cdot(T_\lambda x - T_\lambda y) + (y - T_\lambda y)\cdot(T_\lambda x - T_\lambda y) \le 0.\end{aligned} \tag{4.35}$$

Now, combining inequality (4.35) with the identity

$$|\mathcal{S}_\lambda x - \mathcal{S}_\lambda y|^2 - (\mathcal{S}_\lambda x - \mathcal{S}_\lambda y)\cdot(x-y) = |T_\lambda x - T_\lambda y|^2 - (T_\lambda x - T_\lambda y)\cdot(x-y)$$

proves inequality (4.28). Inequality (4.29) is obvious for $\mathcal{S}_\lambda x - \mathcal{S}_\lambda y = 0$. If $\mathcal{S}_\lambda x - \mathcal{S}_\lambda y \neq 0$, inequality (4.29) follows directly from inequality (4.29) by the Cauchy-Schwarz inequality and dividing by $|\mathcal{S}_\lambda x - \mathcal{S}_\lambda y|$. □

In the case of a uniformly elliptic triangulation $\mathcal{T}_h$, the previous lemma allows us to get a similar estimate to inequality (4.4) in the discrete case.

Lemma 4.9. *Let $\mathcal{T}_h$ be a shape-regular, uniformly acute triangulation of the polyhedral domain Ω and let $V_h \subset C(\Omega;\mathbb{R}^m)$ be the respective finite element space. There exists a constant $C > 0$ that only depends on n, the shape-regularity parameter Γ and the uniform acuteness of the mesh, such that for any $u_h \in V_h$ and $\lambda > 0$, we have, on every $T \in \mathcal{T}_h$,*

$$\nabla u_h \cdot \nabla S_\lambda u_h \ge C \max_T \left(\frac{(|u_h| - \lambda)_+}{|u_h|}\right) |\nabla u_h|^2. \tag{4.36}$$

Proof. Given a $T \in \mathcal{T}_h$, we write $I_T := \{i : x_i \in T\}$ and denote the node with $|u_h(x_0)| = \max_T |u_h|$ by x_0. For simplicity, we write $u_i := u_h(x_i)$. On T, we use equation (2.51) from Lemma 2.32 and inequality (4.27) from Lemma 4.8 to get

$$\begin{aligned}\nabla u_h \cdot \nabla S_\lambda u_h &= \frac{1}{2}\sum_{i,j\in I_T} (u_i - u_j)\cdot(\mathcal{S}_\lambda u_i - \mathcal{S}_\lambda u_j)(-\nabla\psi_i\cdot\nabla\psi_j)\\ &\ge \frac{1}{2}\sum_{i,j\in I_T}\left(\frac{(|u_i|-\lambda)_+}{|u_i|} + \frac{(|u_j|-\lambda)_+}{|u_j|}\right)|u_i - u_j|^2(-\nabla\psi_i\cdot\nabla\psi_j)\\ &\ge \frac{(|u_0|-\lambda)_+}{|u_0|}\sum_{i\in I_T}|u_0 - u_i|^2(-\nabla\psi_i\cdot\nabla\psi_0).\end{aligned} \tag{4.37}$$

On the other hand we know that $\mathcal{T}_h$ is uniformly acute. This means that inequality (2.9) implies that $-\nabla\psi_i\cdot\nabla\psi_j > \varepsilon_1|\nabla\psi_i|\,|\nabla\psi_j|$. Furthermore, the shape-regularity of $\mathcal{T}_h$ implies inequality (2.10) which gives $\frac{|\nabla\psi_i|}{|\nabla\psi_j|} > \varepsilon_2$. Together with the triangle inequality, this yields

$$\begin{aligned}|\nabla u_h|^2 &= \frac{1}{2}\sum_{i,j\in I_T}|u_i - u_j|^2(-\nabla\psi_i\cdot\nabla\psi_j)\\ &\le 2\sum_{i,j\in I_T}|u_i - u_0|^2(-\nabla\psi_i\cdot\nabla\psi_j)\\ &\le \frac{2(n+1)}{\varepsilon_1\varepsilon_2}\sum_{i\in I_T}|u_i - u_0|^2(-\nabla\psi_0\cdot\nabla\psi_i).\end{aligned} \tag{4.38}$$

Combining inequalities (4.37) and (4.38) concludes the proof of inequality (4.36). □

Unfortunately, the condition of uniform acuteness is necessary, even in the case of a scalar function u_h. Take for example a triangle with vertices $x_0 = (0, 0)$, $x_1 = (1, 0)$ and $x_2 = (0, 1)$ and let $u_0 = \lambda$, $u_1 = \lambda(1 + \varepsilon)$ and $u_2 = 0$ with $\varepsilon > 0$. Then,

$$(\nabla\psi_j \cdot \nabla\psi_k)_{j,k=0,\dots,2} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

and $(S_\lambda u_h)(x_0) = 0$, $(S_\lambda u_h)(x_1) = \varepsilon\lambda$, $(S_\lambda u_h)(x_0) = 0$. Thus,

$$\nabla u_h \cdot \nabla S_\lambda u_h = \varepsilon^2\lambda^2,$$
$$\max_T \frac{(|u_h| - \lambda)_+}{|u_h|} |\nabla u_h|^2 = \frac{\varepsilon}{1+\varepsilon}(1 + \varepsilon^2)\lambda^2 \geq \varepsilon\lambda^2.$$

This implies that inequality (4.36) cannot hold if we allow right angles. In the case of a triangulation that is only non-obtuse we therefore have to rely on a weaker estimate, which we state in the following lemma.

Lemma 4.10. *Let $\mathcal{T}_h$ be a non-obtuse triangulation of the polyhedral domain Ω and let $V_h \subset C(\Omega; \mathbb{R}^m)$ be the respective finite element space. Then, we have*

$$\nabla u_h \cdot \nabla S_\lambda u_h \geq |\nabla S_\lambda u_h|^2. \tag{4.39}$$

Proof. This follows directly from equation (2.51) and inequality (4.28). □

We will also need a second interesting property of S_λ that is given in the following lemma.

Lemma 4.11. *Let $\mathcal{T}_h$ be a shape-regular triangulation of the polyhedral domain $\Omega \subset \mathbb{R}^n$ and let $V_h \subset C(\Omega; \mathbb{R}^m)$ be the respective finite element space. There is a constant $C > 0$ that only depends on n and the shape-regularity constant Γ, such that, for any $\lambda > 0$, $u_h \in V_h$, and $T \in \mathcal{T}_h$, we have*

$$\max_T |S_\lambda u_h||\nabla u_h| \leq C \max_T |u_h||\nabla S_\lambda u_h|. \tag{4.40}$$

Proof. We fix an arbitrary $T \in \mathcal{T}_h$ and write $\mathcal{I}_T := \{i : x_i \in T\}$. Without loss of generality, we assume that x_0 is the node with $\max_T |u_h| = |u(x_0)|$. On T, Lemma 2.33 implies that

$$|\nabla S_\lambda u_h| \sim h_T^{-1} \sum_{i\in\mathcal{I}_T} |S_\lambda u_h(x_i) - S_\lambda u_h(x_0)|. \tag{4.41}$$

This means that we can combine inequality (4.27) from Lemma 4.8 and inequality (4.41) to get

$$\begin{aligned} |\nabla S_\lambda u_h| &\gtrsim h_T^{-1} \sum_{i\in\mathcal{I}_T} \left(\frac{(|u_h(x_0)| - \lambda)_+}{|u_h(x_0)|} + \frac{(|u_h(x_i)| - \lambda)_+}{|u_h(x_i)|} \right) |u_h(x_0) - u_h(x_i)| \\ &\geq h_T^{-1} \frac{(|u_h(x_0)| - \lambda)_+}{|u_h(x_0)|} \sum_{i\in\mathcal{I}_T} |u_h(x_0) - u_h(x_i)| \end{aligned} \tag{4.42}$$

on T. Now, Lemma 2.33 implies $|\nabla u_h| \sim h_T^{-1} \sum_{i \in \mathcal{I}_T} |u_0 - u_i|$ on T, which gives

$$|\nabla S_\lambda u_h| \gtrsim \frac{(|u_h(x_0)| - \lambda)_+}{|u_h(x_0)|} |\nabla u_h| \tag{4.43}$$

on T. Together with the fact that $(|u_h(x_0)| - \lambda)_+ = \max_T |S_\lambda u_h|$, this concludes the proof of inequality (4.40). □

4.3 L^∞-estimates for discrete solutions to φ-Laplacian systems

Inequality (4.36) is strong enough to give a local L^∞-bound not only for solutions to p-Laplacian systems, but also for the φ-Laplacian system. In order to do that, we have to introduce N-functions and their basic properties. Those are standard in the analysis of Orlicz spaces; see for example [22] or [39]. Because we rely on inequality (4.36), we need to restrict ourselves to uniformly acute meshes.

Definition 4.12. *We call a function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ an N-Function, if it satisfies*

$$\varphi(t) = \int_0^t \varphi'(s) \mathrm{d}s \tag{4.44}$$

with a left-continuous function $\varphi' : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ that satisfies

$$\varphi'(0) = 0, \tag{4.45}$$

$$\varphi'(t) > 0 \text{ for } t > 0, \tag{4.46}$$

$$\varphi'(s) \leq \varphi'(t) \text{ for } s \leq t. \tag{4.47}$$

If there is a constant $c \in \mathbb{R}$ that is independent of t, such that φ satisfies

$$\varphi(2t) \leq c\varphi(t) \tag{4.48}$$

for every $t \geq 0$, we say that φ fulfils the Δ_2-condition and denote the smallest constant c by $\Delta_2(\varphi)$. We say $f : \Omega \to \mathbb{R}^m$ belongs to the Orlicz-class $\mathcal{L}^\varphi(\Omega)$ if and only if we have

$$\int_\Omega \varphi(|f|) \, \mathrm{d}x < \infty.$$

Note that $\mathcal{L}^\varphi(\Omega)$ is a convex set but in general not a vector space. For $\varphi(t) = t^p$, we have $\mathcal{L}^\varphi(\Omega) = L^p(\Omega)$. We can deduce the following properties of N-functions from Definition 4.12.

Proposition 4.13. *Let $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be an N-function. Then, φ is a strictly monotonically increasing, convex function. If $\Delta_2(\varphi) < \infty$, we have*

$$\varphi(t) \sim \varphi'(t)t, \tag{4.49}$$

where the implicit constant depends on $\Delta_2(\varphi)$. Furthermore, there is a number $\Theta > 1$ that depends on $\Delta_2(\varphi)$, such that for every $t \in (0,1)$ and every $s \in \mathbb{R}_0^+$, we have

$$\varphi(ts) \leq t^{\Theta}\varphi(s). \tag{4.50}$$

Proof. Convexity and monotonicity of φ are standard. Equation (4.49) can be found in [22][Equation 2.4], inequality (4.50) is an easy consequence of [39][Lemma 2.2.7]. □

We will also have to define the conjugate N-function.

Definition 4.14. *Let φ be an N-function. We denote the right inverse of φ' by $(\varphi')^{-1}$ and define the conjugate N-function via*

$$\varphi^*(t) = \int_0^t (\varphi')^{-1} \mathrm{d}s.$$

This function is gains its importance from the following properties.

Proposition 4.15. *Let φ, φ^* be a pair of conjugate N-functions that both satisfy the Δ_2-condition. Then, Young's inequality is true. This means that there is a constant C_ε for every $\varepsilon > 0$, such that*

$$st \leq \varepsilon\varphi(s) + C_\varepsilon\varphi^*(t) \tag{4.51}$$

for every $s, t > 0$, where C_ε depends on ε, $\Delta_2(\varphi)$ and $\Delta_2(\varphi^*)$.

Furthermore, we have

$$\varphi^*(\varphi'(t)) \sim \varphi(t), \tag{4.52}$$

where the implicit constant only depends on $\Delta_2(\varphi)$ and $\Delta_2(\varphi^*)$.

We are now able to define φ-Laplacian systems.

Definition 4.16. *Let $\Omega \subset \mathbb{R}^n$ be a domain and φ an N-function. We call a function $u \in W^{1,1}(\Omega;\mathbb{R}^m)$ with $\nabla u \in \mathcal{L}^{\varphi}(\Omega)$ weakly φ-harmonic, if it is a weak solution to $\Delta_\varphi u := \operatorname{div}\left(\frac{\varphi'(|\nabla u|)}{|\nabla u|}\nabla u\right) = 0$, i.e.*

$$\int_\Omega \frac{\varphi'(|\nabla u|)}{|\nabla u|}\nabla u \cdot \nabla\psi \,\mathrm{d}x = 0$$

for every $\psi \in C_0^\infty(\Omega;\mathbb{R}^m)$. This extends to every function $\psi \in W^{1,1}(\Omega;\mathbb{R}^m)$ with $\nabla\psi \in \mathcal{L}^{\varphi}(\Omega)$.

We will now give the definition of a discrete solution to this equation.

Definition 4.17. *Let $\Omega \subset \mathbb{R}^n$ be a polyhedral domain and let φ be an N-function. Let $\mathcal{T}_h$ be a triangulation of Ω with respective finite element spaces V_h and $V_{h,0}$. We call a function $u_h \in V_h$ with values in $\mathbb{R}^m$ discretely φ-harmonic, if it is a discrete solution to $\Delta_\varphi u := \operatorname{div}\left(\frac{\varphi'(|\nabla u|)}{|\nabla u|}\nabla u\right) = 0$, i.e.*

$$\int_\Omega \frac{\varphi'(|\nabla u_h|)}{|\nabla u_h|}\nabla u_h \cdot \nabla \rho_h \,\mathrm{d}x = 0 \tag{4.53}$$

for every $\rho_h \in V_{h,0}$.

To prove a local L^∞-estimate for discrete solutions to the φ-Laplacian system, we will have to use inequality (4.36). This means that we have to assume a uniformly acute mesh. The main theorem of this section reads as follows.

Theorem 4.18. *Let $\mathcal{T}_h$ be a shape-regular, uniformly acute triangulation of the polyhedral domain Ω with the respective finite element space $V_h \subset C(\Omega;\mathbb{R}^m)$. Let φ be an N-function with $\Delta_2(\varphi) < \infty$ and $\Delta_2(\varphi^*) < \infty$ and let $u_h \in V_h$ be a discrete solution to $-\Delta_\varphi u = 0$ in the sense of Definition 4.17. Furthermore, let $B(x_0, R)$ be a ball with $B(x_0, 2R) \subset \Omega$. Then, there exists a constant $C > 0$ that only depends on n, $\Delta_2(\varphi)$, $\Delta_2(\varphi^*)$, the uniform acuteness of the mesh and the shape-regularity parameter Γ, such that*

$$\sup_{\Omega'(B(x_0,R))} \varphi\left(\frac{|u_h|}{R}\right) \le C \fint_{\Omega(B(x_0,2R))} \varphi\left(\frac{|u_h|}{R}\right) \mathrm{d}x.$$

First, we will prove a Caccioppoli-type inequality for discrete solutions to φ-Laplacian systems. Before we start, we note that, given an N-function φ with $\Delta_2(\varphi) < \infty$, there holds $\varphi(\eta^{q-1}t) \le \eta^q \varphi(t)$ for any $0 \le \eta \le 1$ and any $q \ge q' = \frac{\Theta}{\Theta-1}$ because of inequality (4.50).

Lemma 4.19 (Caccioppoli inequality for the φ-Laplace system)**.** *Under the assumptions of Theorem 4.18, let $q \in \mathbb{N}$ be such that we have*

$$\varphi^*(s^{q-1}t) \le s^q \varphi^*(t) \tag{4.54}$$

for every $t \in \mathbb{R}_0^+$ and every $s \in (0,1)$. Suppose also that $\eta \in C_0^\infty(\Omega)$ with $0 \le \eta \le 1$ and write $\eta_h := \Pi_h \eta$. Then, we have

$$\sum_{T\in\mathcal{T}_h} \max_T \frac{(|u_h|-\lambda)_+}{|u_h|} \int_T \varphi(|\nabla u_h|)\,\eta_h^q \,\mathrm{d}x \le C \sum_{T\in\mathcal{T}_h} \max_T \frac{(|u_h|-\lambda)_+}{|u_h|} \int_T \varphi(|u_h||\nabla \eta_h|)\,\mathrm{d}x \tag{4.55}$$

for every $\lambda > 0$, where the constant $C > 0$ only depends on n, $\Delta_2(\varphi)$, $\Delta_2(\varphi^*)$, the uniform acuteness of the mesh and the shape-regularity parameter Γ.

Proof. We test equation (4.53) against $\rho_h = \Pi_h\left(S_\lambda u_h \eta^q\right)$. This gives

$$\begin{aligned}
0 &= \int_\Omega \frac{\varphi'(|\nabla u_h|)}{|\nabla u_h|}\nabla u_h \cdot \nabla \Pi_h\left(S_\lambda u_h \eta_h^q\right)\,\mathrm{d}x = \int_\Omega \frac{\varphi'(|\nabla u_h|)}{|\nabla u_h|}\nabla u_h \cdot \nabla\left(S_\lambda u_h \eta_h^q\right)\,\mathrm{d}x \\
&+ \int_\Omega \frac{\varphi'(|\nabla u_h|)}{|\nabla u_h|}\nabla u_h \cdot \nabla\left(\Pi_h\left(S_\lambda u_h \eta_h^q\right) - S_\lambda u_h \eta_h^q\right)\,\mathrm{d}x \\
&=: I + II.
\end{aligned} \tag{4.56}$$

We will first look at I in equation (4.56). We get

$$\begin{aligned}
I &= \int_\Omega \frac{\varphi'(|\nabla u_h|)}{|\nabla u_h|}\nabla u_h \cdot \left(\nabla S_\lambda u_h\right)\eta_h^q\,\mathrm{d}x \\
&\quad + q\int_\Omega \frac{\varphi'(|\nabla u_h|)}{|\nabla u_h|}\nabla u_h \cdot \left(\nabla \eta_h\right) S_\lambda u_h \eta_h^{q-1}\,\mathrm{d}x =: I_1 + I_2.
\end{aligned} \tag{4.57}$$

In our notation from equations (4.56) and (4.57), these two equations leave us with

$$I_1 \leq |I_2| + |II|. \tag{4.58}$$

Note that u_h is affine on every $T \in \mathcal{T}_h$, which means that ∇u_h and $\nabla S_\lambda u_h$ are constant on every $T \in \mathcal{T}_h$. This means that the notation $\nabla v_h(T)$ makes sense for every $v_h \in V_h$. With equations (4.36) and (4.49), we find

$$\begin{aligned}
I_1 &= \sum_{T\in\mathcal{T}_h} \frac{\varphi'(|\nabla u_h(T)|)}{|\nabla u_h(T)|}\nabla u_h(T)\cdot\nabla S_\lambda u_h(T)\int_T \eta_h^q\,\mathrm{d}x \\
&\gtrsim \sum_{T\in\mathcal{T}_h} \max_T \frac{(|u_h|-\lambda)_+}{|u_h|}\varphi'(|\nabla u_h(T)|)|\nabla u_h(T)|\int_T \eta_h^q\,\mathrm{d}x \\
&\gtrsim \sum_{T\in\mathcal{T}_h} \max_T \frac{(|u_h|-\lambda)_+}{|u_h|}\int_T \varphi(|\nabla u_h|)\eta_h^q\,\mathrm{d}x.
\end{aligned} \tag{4.59}$$

For I_2 from equation (4.57), we note that $|(S_\lambda u_h)(x_i)| = (|u_h|-\lambda)_+(x_i)$ at every node x_i and that any function in V_h takes its maximum over a $T \in \mathcal{T}_h$ at a node. This gives

$$\begin{aligned}
I_2 &\lesssim \sum_{T\in\mathcal{T}_h} |T|\varphi'(|\nabla u_h(T)|)|\nabla\eta_h(T)|\max_T|\eta_h|^{q-1}\max_T|S_\lambda u_h| \\
&= \sum_{T\in\mathcal{T}_h} |T|\max_T\frac{(|u_h|-\lambda)_+}{|u_h|}\varphi'(|\nabla u_h(T)|)\max_T \eta_h^{q-1}|\nabla\eta_h(T)|\max_T|u_h|.
\end{aligned} \tag{4.60}$$

We use Young's inequality (4.51) in inequality (4.60) to get

$$\begin{aligned}
I_2 &\lesssim \varepsilon\sum_{T\in\mathcal{T}_h}|T|\max_T\frac{(|u_h|-\lambda)_+}{|u_h|}\varphi^*\left(\max_T \eta^{q-1}\varphi'(|\nabla u_h(T)|)\right) \\
&\quad + C_\varepsilon\sum_{T\in\mathcal{T}_h}|T|\max_T\frac{(|u_h|-\lambda)_+}{|u_h|}\varphi\left(\max_T|u_h||\nabla\eta_h(T)|\right).
\end{aligned}$$

We now use equation (4.54), $\varphi^*(\varphi'(t)) \sim \varphi(t)$ from equation (4.52), and inequality (2.27) to get

$$\begin{aligned} I_2 \le \varepsilon \sum_{T\in\mathcal{T}_h} |T| \max_T \frac{(|u_h|-\lambda)_+}{|u_h|} \varphi(|\nabla u_h(T)|) \max_T \eta_h^q \\ + C_\varepsilon \sum_{T\in\mathcal{T}_h} |T| \max_T \frac{(|u_h|-\lambda)_+}{|u_h|} \varphi\left(C \fint_T |u_h||\nabla \eta_h| \,\mathrm{d}x \right). \end{aligned} \tag{4.61}$$

We use inequality (2.27), $\Delta_2(\varphi) < \infty$ and Jensen's inequality in inequality (4.61) to get

$$\begin{aligned} I_2 \lesssim \varepsilon \sum_{T\in\mathcal{T}_h} \max_T \frac{(|u_h|-\lambda)_+}{|u_h|} \int_T \varphi(|\nabla u_h|)\eta_h^q \,\mathrm{d}x \\ + C_\varepsilon \sum_{T\in\mathcal{T}_h} \max_T \frac{(|u_h|-\lambda)_+}{|u_h|} \int_T \varphi(|u_h||\nabla \eta_h|) \,\mathrm{d}x. \end{aligned} \tag{4.62}$$

Now, we will focus on II from equation (4.56). We get

$$\begin{aligned} II &\le \int_\Omega \varphi'(|\nabla u_h|) \left|\nabla \left(\Pi_h \left(S_\lambda u_h \eta_h^q\right) - S_\lambda u_h \eta_h^q\right)\right| \mathrm{d}x \\ &\le \sum_{T\in\mathcal{T}_h} |T| \varphi'(|\nabla u_h(T)|) \max_T \left|\nabla \left(\Pi_h \left(S_\lambda u_h \eta_h^q\right) - S_\lambda u_h \eta_h^q\right)\right|. \end{aligned}$$

We use equation (2.30) from Lemma 2.25 in this inequality to deduce that

$$\begin{aligned} II &\lesssim \sum_{T\in\mathcal{T}_h} |T| \varphi'(|\nabla u_h(T)|) \max_T |S_\lambda u_h| \max_T |\nabla \eta_h^q| \\ &\lesssim \sum_{T\in\mathcal{T}_h} |T| \varphi'(|\nabla u_h(T|) \max_T |S_\lambda u_h| |\nabla \eta_h(T)| \max_T \eta_h^{q-1}. \end{aligned}$$

This is the same term as the right-hand side of equation (4.60). We can therefore follow the exact same steps as for I_2 to get

$$\begin{aligned} II \le \varepsilon \sum_{T\in\mathcal{T}_h} \max_T \frac{(|u_h|-\lambda)_+}{|u_h|} \int_T \varphi(|\nabla u_h|)\eta_h^q \,\mathrm{d}x \\ + C_\varepsilon \sum_{T\in\mathcal{T}_h} \max_T \frac{(|u_h|-\lambda)_+}{|u_h|} \int_T \varphi(|u_h||\nabla \eta_h|) \,\mathrm{d}x. \end{aligned} \tag{4.63}$$

This means that we can insert the inequalities (4.59), (4.62) and (4.63) into equation (4.58) and absorb into the left-hand side to get

$$\sum_{T\in\mathcal{T}_h} \max_T \frac{(|u_h|-\lambda)_+}{|u_h|} \int_T \varphi\left(|\nabla u_h|\right) \eta_h^q \,\mathrm{d}x \lesssim \sum_{T\in\mathcal{T}_h} \max_T \frac{(|u_h|-\lambda)_+}{|u_h|} \int_T \varphi\left(|u_h||\nabla \eta_h|\right) \mathrm{d}x.$$

This proves the Lemma. □

For our De-Giorgi-type iteration, we will again need sequences of cut-off functions and nested balls.

Definition 4.20. *For $\lambda_\infty > 0$, we define*

$$\lambda_k := \lambda_\infty \left(1 - 2^{-k}\right).$$

Furthermore, let $B(x_0, R) \subset \Omega$ be a ball with $B(x_0, 2R) \subset \Omega$. We define the sequence of balls $B_k = B\left(x_0, \left(1 + 2^{-k}\right) R\right)$ and the sequence of functions $\tilde{\eta}_k \in C_0^\infty(\Omega)$ such that

$$\chi_{B_{k+1}} \le \tilde{\eta}_k \le \chi_{B_k}, \tag{4.64}$$

$$|\nabla \tilde{\eta}_k| \lesssim R^{-1} 2^k. \tag{4.65}$$

Define $\eta_k := \Pi_h(\tilde{\eta}_k)$.

We will state a corollary analogous to inequality (4.24). Note that the natural scaling for solutions to φ-Laplacian systems is $u(x) \to Ru\left(\frac{x}{R}\right)$. Therefore, the natural quantity to estimate is $\varphi\left(\frac{|u|}{R}\right)$.

Corollary 4.21. *We define B_k and λ_k as in Definition 4.20. Under the assumptions of Lemma 4.19, there are constants $\alpha > 0$ and $C > 0$ that only depend on n, $\Delta_2(\varphi)$, $\Delta_2(\varphi^*)$, the uniform acuteness of the mesh and the shape-regularity parameter Γ, such that*

$$R \fint_{\Omega(B(x_0,2R))} \left|\nabla \left(\varphi\left(\frac{|S_{\lambda_{k+1}} u_h|}{R}\right) \eta_{k+1}^q\right)\right| \mathrm{d}x \le C 2^{\alpha k} \fint_{\Omega(B(x_0,2R))} \varphi\left(\left|\frac{S_{\lambda_k} u_h}{R}\right|\right) \eta_k^q \,\mathrm{d}x$$

for every $k \in \mathbb{N}$.

Proof. For simplicity, we write $B := \Omega(B(x_0, 2R))$. We note that $\left|\nabla |S_{\lambda_{k+1}} u_h|\right| \le |\nabla S_{\lambda_{k+1}} u_h|$ and calculate

$$\begin{aligned}
&R \fint_B \left|\nabla \left(\varphi\left(\frac{|S_{\lambda_{k+1}} u_h|}{R}\right) \eta_{k+1}^q\right)\right| \mathrm{d}x \\
&\lesssim \fint_B \varphi'\left(\frac{|S_{\lambda_{k+1}} u_h|}{R}\right) |\nabla S_{\lambda_{k+1}} u_h| \eta_{k+1}^q \,\mathrm{d}x + \fint_B \varphi\left(\frac{|S_{\lambda_{k+1}} u_h|}{R}\right) R |\nabla \eta_{k+1}| \eta_{k+1}^{q-1} \,\mathrm{d}x \\
&:= I + II.
\end{aligned} \tag{4.66}$$

For I in equation (4.66), we use Young's inequality (4.51) and equation (4.52) to find

$$\begin{aligned}
I &\lesssim \fint_B \varphi^*\left(\varphi'\left(\frac{|S_{\lambda_{k+1}} u_h|}{R}\right)\right) \eta_{k+1}^q \,\mathrm{d}x + \fint_B \varphi(|\nabla S_{\lambda_{k+1}} u_h|) \eta_{k+1}^q \,\mathrm{d}x \\
&\sim \fint_B \varphi\left(\frac{|S_{\lambda_{k+1}} u_h|}{R}\right) \eta_{k+1}^q \,\mathrm{d}x + \fint_B \varphi(|\nabla S_{\lambda_{k+1}} u_h|) \eta_{k+1}^q \,\mathrm{d}x =: I_1 + I_2.
\end{aligned} \tag{4.67}$$

First, we will look at I_1. This is easy because φ is monotonic increasing, $|S_{\lambda_{k+1}} u_h| \le |S_{\lambda_k} u_h|$ and $\eta_{k+1} \le \eta_k$. We get

$$I_1 \le \fint_B \varphi\left(\frac{|S_{\lambda_k} u_h|}{R}\right) \eta_k^q \,\mathrm{d}x. \tag{4.68}$$

For I_2 in inequality (4.67), we want to use Lemma 4.19. Let us define

$$\lambda = \frac{\lambda_{k+1} + \lambda_k}{2}.$$

On simplices $T \in \mathcal{T}_h$ where $\nabla S_{\lambda_{k+1}} u_h \neq 0$, we have $\max_T |u_h| > \lambda_{k+1}$ and, analogously to inequality (4.21), we therefore get

$$\max_T \frac{(|u_h| - \lambda)_+}{|u_h|} \geq \frac{\lambda_{k+1} - \lambda}{\lambda_{k+1}} = \frac{2^{-k} - 2^{-(k+1)}}{2\left(1 - 2^{-(k+1)}\right)} \geq 2^{-(k+2)}$$

on those simplices. For I_2 from equation (4.67), this leads to

$$I_2 \leq 2^{k+2} \frac{1}{|B|} \sum_{T \in \mathcal{T}_h} \max_T \frac{(|u_h| - \lambda)_+}{|u_h|} \int_T \varphi\left(|\nabla S_{\lambda_{k+1}} u_h|\right) \eta_{k+1}^q \, dx.$$

Now, we deduce that $|\nabla S_{\lambda_{k+1}} u_h| \leq |\nabla u_h|$ from inequality (4.39) via the Cauchy–Schwarz inequality to get

$$I_2 \leq 2^{k+2} \frac{1}{|B|} \sum_{T \in \mathcal{T}_h} \max_T \frac{(|u_h| - \lambda)_+}{|u_h|} \int_T \varphi(|\nabla u_h|) \eta_{k+1}^q \, dx.$$

This enables us to use Lemma 4.19 to establish

$$\begin{aligned} I_2 &\lesssim \frac{2^k}{|B|} \sum_{T \in \mathcal{T}_h} \max_T \frac{(|u_h| - \lambda)_+}{|u_h|} \int_T \varphi\left(|u_h||\nabla \eta_{k+1}|\right) dx \\ &\leq \frac{2^k}{|B|} \sum_{T \in \mathcal{T}_h} |T| \max_T \frac{(|u_h| - \lambda)_+}{|u_h|} \varphi\left(\max_T |u_h||\nabla \eta_{k+1}(T)|\right), \end{aligned} \tag{4.69}$$

where we have used the fact that φ is monotonic increasing in the last step. Note that $t \mapsto \frac{t-\lambda}{t}$ is increasing for $t > \lambda$. Furthermore, on simplices $T \in \mathcal{T}_h$ with $\max_T(|u_h| - \lambda)_+ \neq 0$, we obviously have $\max_T |u_h| > \lambda$. On those simplices, we therefore get

$$\frac{\max_T\left(|u_h| - \lambda_k\right)_+}{\max_T |u_h|} = \max_T \frac{|u_h| - \lambda_k}{|u_h|} \geq \frac{\lambda - \lambda_k}{\lambda} = \frac{\frac{1}{2}\left(2^{-(k+1)} - 2^{-k}\right)}{1 - 2^{k+1} - 2^{k+2}} \geq 2^{-(k+2)}. \tag{4.70}$$

Furthermore, we know from equations (4.65) and (2.26) that

$$|\nabla \eta_{k+1}(T)| \lesssim \frac{2^k}{R}. \tag{4.71}$$

We can use inequalities (4.70) and (4.71) to find

$$\varphi\left(\max_T |u_h||\nabla \eta_k(T)|\right) \leq \varphi\left(C 2^{2k} \max_T \frac{|S_{\lambda_k} u_h|}{R}\right) \chi_{\operatorname{supp} \eta_{k+1}}. \tag{4.72}$$

Later, we will want to use inequality (4.50). We write l for the lowest natural number with $l \geq \frac{q}{\Theta}$. Lemma 3.18 (b) ensures that for simplices $T \subset \operatorname{supp} \eta_{k+1}$, we have $\max_T \eta_k = 1$. This allows us to insert $\max_T \eta_k^l$ into inequality (4.72) to get

$$\varphi\left(\max_T |u_h||\nabla \eta_k(T)|\right) \leq \varphi\left(C 2^{2k} \max_T \frac{|S_{\lambda_k} u_h|}{R} \max_T \eta_k^l\right). \tag{4.73}$$

$S_{\lambda_k} u_h$ is affine on T and η_k^l a polynomial of degree l. This means that we can use equation (2.27) and Jensen's inequality in inequality (4.73) to find

$$\begin{aligned} \varphi\left(\max_T |u_h||\nabla \eta_k(T)|\right) &\leq \varphi\left(\fint_T C' 2^{2k} \frac{|S_{\lambda_k} u_h|}{R} \eta_k^l \, dx\right) \\ &\leq \fint_T \varphi\left(C' 2^{2k} \frac{|S_{\lambda_k} u_h|}{R} \eta_k^l\right) dx. \end{aligned} \tag{4.74}$$

Recall that φ satisfies the Δ_2-condition. Using inequalities (4.48) and (4.50), we can deduce from inequality (4.74) that

$$\varphi\left(\max_T |u_h||\nabla \eta_k(T)|\right) \leq \Delta_2(\varphi)^{\log_2(C')+2k} \fint_T \varphi\left(\frac{|S_{\lambda_k} u_h|}{R}\right) \eta_k^{l\Theta} \, dx. \tag{4.75}$$

By the definition of l, we have $l\Theta \geq q$. We also write $\beta := 2\log_2(\Delta_2(\varphi))$. This leaves us with

$$\varphi\left(\max_T |u_h||\nabla \eta_k(T)|\right) \lesssim 2^{\beta k} \fint_T \varphi\left(\frac{|S_{\lambda_k} u_h|}{R}\right) \eta_k^q \, dx. \tag{4.76}$$

Now, we can insert inequality (4.76) into inequality (4.69) to get

$$I_2 \lesssim \frac{2^{(1+\beta)k}}{|B|} \sum_{T \in \mathcal{T}_h} |T| \max_T \frac{(|u_h| - \lambda)_+}{|u_h|} \fint_T \varphi\left(\frac{|S_{\lambda_k} u_h|}{R}\right) \eta_k^q \, dx.$$

Finally, we use that $\max_T \frac{(|u_h|-\lambda)_+}{|u_h|} \leq 1$ to get

$$\begin{aligned} I_2 &\lesssim \frac{2^{(1+\beta)k}}{|B|} \sum_{T \in \mathcal{T}_h} |T| \fint_T \varphi\left(\frac{|S_{\lambda_k} u_h|}{R}\right) \eta_k^q \, dx \\ &= 2^{(1+\beta)k} \fint_B \varphi\left(\frac{|S_{\lambda_k} u_h|}{R}\right) \eta_k^q \, dx. \end{aligned} \tag{4.77}$$

We will consider at II from inequality (4.66). We use Young's inequality (4.51) to get

$$\begin{aligned} II &= \fint_B \varphi\left(\frac{|S_{\lambda_{k+1}} u_h|}{R}\right) R |\nabla \eta_{k+1}| \eta_{k+1}^{q-1} \, dx \\ &\lesssim \fint_B \varphi\left(\frac{|S_{\lambda_{k+1}} u_h|}{R}\right) R^q |\nabla \eta_{k+1}|^q \, dx + \fint_B \varphi\left(\frac{|S_{\lambda_{k+1}} u_h|}{R}\right) \eta_{k+1}^q \, dx \\ &=: II_1 + II_2. \end{aligned} \tag{4.78}$$

For II_2, we use $|S_{\lambda_{k+1}} u_h| \leq |S_{\lambda_k} u_h|$, the monotonicity of φ, and $\eta_{k+1} \leq \eta_k$ to get

$$II_2 = \fint_B \varphi\left(\frac{|S_{\lambda_{k+1}} u_h|}{R}\right) \eta_{k+1}^q \, dx \leq \fint_B \varphi\left(\frac{|S_{\lambda_k} u_h|}{R}\right) \eta_k^q \, dx. \tag{4.79}$$

To deal with II_1, we once more define l as the lowest natural number with $l \geq \frac{q}{\Theta}$ where Θ is the constant from inequality (4.50). From Lemma 3.18 (b), we know that we have $\max_T \eta_k = 1$ on all relevant simplices $T \subset \operatorname{supp} \eta_{k+1}$ and inequalities (4.65) and (2.26) imply $|\nabla \eta_{k+1}| \lesssim 2^{k+1} R^{-1}$. This gives

$$\begin{aligned} II_1 &\leq \frac{1}{|B|} \sum_{T \in \mathcal{T}_h} |T| \varphi \left(\frac{\max_T |S_{\lambda_{k+1}} u_h|}{R} \right) R^q |\nabla \eta_{k+1}|(T)^q \\ &\lesssim \frac{1}{|B|} \sum_{\substack{T \in \mathcal{T}_h \\ T \subset \operatorname{supp} \eta_{k+1}}} |T| \varphi \left(\frac{\max_T |S_{\lambda_{k+1}} u_h|}{R} \max_T \eta_k^l \right) 2^{qk}. \end{aligned} \tag{4.80}$$

Completely analogously to inequalities (4.73) to (4.76), we have

$$\varphi \left(\frac{\max_T |S_{\lambda_{k+1}} u_h|}{R} \max_T \eta_k^l \right) \lesssim \fint_T \varphi \left(\frac{|S_{\lambda_{k+1}} u_h|}{R} \right) \eta_k^q \, \mathrm{d}x. \tag{4.81}$$

Inserting inequality (4.81) into inequality (4.80) and using $|S_{\lambda_{k+1}} u_h| \leq |S_{\lambda_k} u_h|$ finally yields

$$II_1 \lesssim \frac{2^{qk}}{|B|} \sum_{T \in \mathcal{T}_h} |T| \fint_T \varphi \left(\frac{|S_{\lambda_{k+1}} u_h|}{R} \right) \eta_k^q \, \mathrm{d}x \leq 2^{qk} \fint_B \varphi \left(\frac{|S_{\lambda_k} u_h|}{R} \right) \eta_k^q \, \mathrm{d}x. \tag{4.82}$$

Recall from inequalities (4.66), (4.67), and (4.78) that

$$R \fint_B \left| \nabla \left(\varphi \left(\frac{|S_{\lambda_{k+1}} u_h|}{R} \right) \eta_{k+1}^q \right) \right| \mathrm{d}x \lesssim I_1 + I_2 + II_1 + II_2. \tag{4.83}$$

Inserting inequalities (4.68), (4.77), (4.79), and (4.82) into inequality (4.83) and defining $\alpha = \max\{1 + \beta, q\}$ yields

$$R \fint_B \left| \nabla \left(\varphi \left(\frac{|S_{\lambda_{k+1}} u_h|}{R} \right) \eta_{k+1}^q \right) \right| \mathrm{d}x \lesssim 2^{\alpha k} \fint_B \varphi \left(\left| \frac{S_{\lambda_k} u_h}{R} \right| \right) \eta_k^q \, \mathrm{d}x.$$

□

We also need a new weak-type inequality in the spirit of equation (3.57).

Lemma 4.22. *Under the assumptions of Theorem 4.18, we define η_k and λ_k as in Definition 4.20. Then, there are constants $\alpha > 0$ and $C > 0$ that only depend on n, $\Delta_2(\varphi)$, $\Delta_2(\varphi^*)$ and the shape-regularity parameter Γ, such that*

$$\left| \operatorname{supp} \left(\eta_{k+1} S_{\lambda_{k+1}} u_h \right) \right| \leq C \frac{2^{\alpha k}}{\varphi \left(\frac{\lambda_\infty}{R} \right)} \int_\Omega \varphi \left(\frac{|S_{\lambda_k} u_h|}{R} \right) \eta_k^q \, \mathrm{d}x.$$

The uniform acuteness of the triangulation $\mathcal{T}_h$ is not necessary.

Proof. Note that $\operatorname{supp}\left(\eta_{k+1} S_{\lambda_{k+1}} u_h\right)$ is the closure of the disjoint union of the interiors of those simplices $T \in \mathcal{T}_h$ with $\max_T |u_h| > \lambda_{k+1}$ and $\max_T \eta_{k+1} > 0$. The former implies that

$$2^{k+1}\left(\max_T |u_h| - \lambda_k\right) \geq 2^{k+1}(\lambda_{k+1} - \lambda_k) = \lambda_\infty \tag{4.84}$$

on the relevant $T \in \mathcal{T}_h$. On the other hand, $\max_T \eta_{k+1} > 0$ implies $\max_T \eta_k = 1$ by Lemma 3.18 (b). Again, we define $l \in \mathbb{N}$ as the lowest natural number with $l \geq \frac{q}{\Theta}$, where Θ is the constant from equation (4.50). Together with equation (4.84) and the monotonicity of φ, we find that

$$|T| \leq \frac{1}{\varphi\left(\frac{\lambda_\infty}{R}\right)} |T| \varphi\left(2^{k+1} \max_T \frac{|S_{\lambda_k} u_h|}{R} \max_T \eta_k^l\right). \tag{4.85}$$

Now, we can apply Lemma 2.23, Jensen's inequality, and inequalities (4.50) and (4.48) to inequality (4.85) to get

$$\begin{aligned}
|T| &\leq \frac{1}{\varphi\left(\frac{\lambda_\infty}{R}\right)} |T| \varphi\left(C 2^{k+1} \fint_T \frac{|S_{\lambda_k} u_h|}{R} \eta_k^l \, \mathrm{d}x\right) \\
&\leq \frac{1}{\varphi\left(\frac{\lambda_\infty}{R}\right)} |T| \fint_T \varphi\left(C 2^{k+1} \frac{|S_{\lambda_k} u_h|}{R} \eta_k^l\right) \mathrm{d}x \\
&\lesssim \frac{2^{\log_2(\Delta_2(\varphi))k}}{\varphi\left(\frac{\lambda_\infty}{R}\right)} \int_T \varphi\left(\frac{|S_{\lambda_k} u_h|}{R}\right) \eta_k^q \, \mathrm{d}x.
\end{aligned}$$

Finally, summing over all T with $\max_T |u_h| > \lambda_{k+1}$ and $\max_T \eta_{k+1} > 0$ concludes the proof of the corollary. □

This allows us to prove the local L^∞-estimate for φ-harmonic functions.

Proof of Theorem 4.18. We define B_k and η_k as in Definition 4.20 and q as in Corollary 4.21. Now, we write

$$a_k := \fint_{\Omega(B(x_0,2R))} \varphi\left(\frac{|S_{\lambda_k} u_h|}{R}\right) \eta_k^q \, \mathrm{d}x.$$

Using scaling-invariant norms $|||f|||_p := \left(\fint_{\Omega(B(x_0,2R))} |f|^p \, \mathrm{d}x\right)^{\frac{1}{p}}$, Hölder's inequality, and the Sobolev embedding theorem, we estimate

$$\begin{aligned}
a_{k+1} &= \left|\left|\left| \varphi\left(\frac{|S_{\lambda_{k+1}} u_h|}{R}\right) \eta_{k+1} \right|\right|\right|_1 \\
&\leq \left|\left|\left| \varphi\left(\frac{|S_{\lambda_{k+1}} u_h|}{R}\right) \eta_{k+1} \right|\right|\right|_{1^*} \left|\left|\left| \chi_{\operatorname{supp}\left(\varphi\left(\frac{|S_{\lambda_{k+1}} u_h|}{R}\right)\eta_{k+1}^q\right)} \right|\right|\right|_n \\
&\leq R \left|\left|\left| \nabla\left(\varphi\left(\frac{|S_{\lambda_{k+1}} u_h|}{R}\right) \eta_{k+1}^q\right) \right|\right|\right|_1 \left(\frac{\left|\operatorname{supp} |S_{\lambda_{k+1}} u_h| \eta_{k+1}^q\right|}{|\Omega(B(x_0,2R))|}\right)^{\frac{1}{n}},
\end{aligned} \tag{4.86}$$

where we have used that $\varphi(t) = 0$ if and only if $t = 0$. We use Corollary 4.21 to find that

$$R \left\| \nabla \left(\varphi \left(\frac{|S_{\lambda_{k+1}} u_h|}{R} \right) \eta_{k+1}^q \right) \right\|_1 \lesssim 2^{\alpha k} \fint_{\Omega(B(x_0,2R))} \varphi \left(\frac{|S_{\lambda_k} u_h|}{R} \right) \eta_k^q \,\mathrm{d}x = 2^{\alpha k} a_k. \tag{4.87}$$

Lemma 4.22 implies that

$$\frac{|\mathrm{supp}\,(|S_{\lambda_{k+1}} u_h| \eta_{k+1})|}{|\Omega(B(x_0, 2R))|} \lesssim \frac{2^{\alpha' k}}{\varphi\left(\frac{\lambda_\infty}{R}\right)} \fint_{\Omega(B(x_0,2R))} \varphi \left(\frac{|S_{\lambda_k} u_h|}{R} \right) \eta_k^q \,\mathrm{d}x = 2^{\alpha' k} \frac{a_k}{\varphi\left(\frac{\lambda_\infty}{R}\right)}. \tag{4.88}$$

Finally, inserting inequalities (4.87) and (4.88) into inequality (4.86) yields

$$a_{k+1} \lesssim 2^{\left(\alpha + \frac{\alpha'}{n}\right)k} a_k \left(\frac{a_k}{\varphi\left(\frac{\lambda_\infty}{R}\right)} \right)^{\frac{1}{n}}. \tag{4.89}$$

We choose λ_∞ such that $\varphi\left(\frac{\lambda_\infty}{R}\right) \sim a_0$. By Corollary 3.32, inequality (4.89) therefore implies that $a_k \to 0$. Using the monotonicity of φ, this therefore gives

$$\max_{\Omega'(B(x_0,R))} \varphi \left(\frac{|u_h|}{R} \right) \leq \varphi \left(\frac{\lambda_\infty}{R} \right) \sim a_0 \leq \fint_{\Omega(B(x_0,2R))} \varphi \left(\frac{|u_h|}{R} \right) \mathrm{d}x.$$

This proves the theorem. □

4.4 L^∞-estimates for discrete solutions to p-Laplacian systems

The restriction to uniformly acute meshes in Lemma 4.9 is quite restrictive, because many standard examples of local refinement algorithms produce right angles. See for example a square domain with newest vertex bisection. This is why we want to use only Lemmas 4.10 and 4.11. Unfortunately, this means that we have to choose different approaches for the singular $1 < p < 2$ and degenerate $2 < p < n$ cases of the p-Laplacian system. This implies that we cannot use this technique for the φ-Laplacian case. We begin with a definition of a discrete solution to a p-Laplacian system.

Definition 4.23. *Let $\mathcal{T}_h$ be a triangulation of the polyhedral domain $\Omega \subset \mathbb{R}^n$ with respective finite element spaces $V_h \subset C(\Omega; \mathbb{R}^m)$ and $V_{h,0} \subset C(\Omega; \mathbb{R}^m)$. For $1 < p < n$, we assume $F \in L^{\frac{n}{p-1}}(\Omega; \mathbb{R}^{m \times n})$ and $f \in L^{\frac{n}{p}}(\Omega; \mathbb{R}^m)$. We say that u_h is a discrete solution to $-\mathrm{div}\left(|\nabla u|^{p-2} \nabla u\right) = -\mathrm{div} F + f$, if we have*

$$\int_\Omega |\nabla u_h|^{p-2} \nabla u_h \cdot \nabla \varphi_h \,\mathrm{d}x = \int_\Omega F \cdot \nabla \varphi_h \,\mathrm{d}x + \int_\Omega f \cdot \varphi_h \,\mathrm{d}x \tag{4.90}$$

for all $\varphi_h \in V_{h,0}$.

Remark 4.24. *If $p \geq n$, we would have to assume $F \in L^{p'+\delta}(\Omega, \mathbb{R}^n)$ and $f \in L^{1+\delta}(\Omega)$. The following theorems and lemmas are stated and proved in the case $p < n$. The proofs are easily adjusted by using the fact that for $p \geq n$, we have $W^{1,p} \hookrightarrow L^s$ for all $s \in (1, \infty)$ and using a very large exponent instead of $p^\star$. How large the exponent has to be only depends on p, n, δ and the shape-regularity parameter Γ.*

We will now state the main theorem of the section. We will assume $1 < p < n$.

Theorem 4.25. *Let $\mathcal{T}_h$ be a shape-regular, non-obtuse triangulation of the polyhedral domain $\Omega \subset \mathbb{R}^n$ with respective finite element spaces $V_h \subset C(\Omega; \mathbb{R}^m)$ and $V_{h,0} \subset C(\Omega; \mathbb{R}^m)$. Given $1 < p < n$ and $0 < \delta < p - 1$, define q and r via $\frac{1}{q} = \frac{p-1}{n} - \frac{\delta}{n}$ and $\frac{1}{r} = \frac{p}{n} - \frac{\delta}{n}$. Let $F \in L^q(\Omega; \mathbb{R}^{m\times n})$ and $f \in L^r(\Omega; \mathbb{R}^m)$ be given functions, and let $u_h \in V_h$ be a discrete solution to $-\mathrm{div}(|\nabla u_h|^{p-2}\nabla u_h) = -\mathrm{div}F + f$ in the sense of Definition 4.23. Furthermore, let $B(x_0, R)$ be a ball with $B(x_0, 2R) \subset \Omega$ and $R \geq h_T$, where $x_0 \in T$ for some $T \in \mathcal{T}_h$. Then, we have*

$$\sup_{\Omega'(B(x_0,R))} |u_h|^p \leq C \fint_{\Omega(B(x_0,2R))} |u_h|^p \,\mathrm{d}x + C\left(\|f\|^{p'}_{L^r(\Omega(B(x_0,2R)))} + \|F\|^{p'}_{L^q(\Omega(B(x_0,2R)))}\right) R^{p'\delta},$$

where the constant C only depends on p, n, δ and the shape-regularity parameter Γ.

First, we need a Caccioppoli-type inequality for the function u_h.

Lemma 4.26. *Under the assumptions of Theorem 4.25, there is a constant $C > 0$ that only depends on p, n, δ and the shape-regularity constant Γ, such that*

$$\begin{aligned}\int_\Omega |\nabla u_h|^p |\eta_h|^p \,\mathrm{d}x \leq& C \int_\Omega |u_h|^p |\nabla \eta_h|^p \,\mathrm{d}x \\ &+ C\left(\|F\|^{p'}_{L^q(\operatorname{supp}\eta_h)} + \|f\|^{p'}_{L^r(\operatorname{supp}\eta_h)}\right) \left|\operatorname{supp}\eta_h \cap \operatorname{supp} u_h\right|^{1-\frac{p}{n}+p'\frac{\delta}{n}}\end{aligned} \tag{4.91}$$

for any $\eta_h \in V_{h,0}$ with $0 \leq \eta_h \leq 1$.

Proof. We write $\rho_h := \Pi_h\left(\eta_h^p\right)$ and test equation (4.90) against $\varphi_h = \Pi_h(u_h\rho_h)$. This leaves us with

$$\begin{aligned} L_I + L_{II} :=& \int_\Omega |\nabla u_h|^{p-2}\nabla u_h \cdot \nabla\left(u_h\rho_h\right) \mathrm{d}x \\ &+ \int_\Omega |\nabla u_h|^{p-2}\nabla u_h \cdot \nabla\left(\Pi_h\left(u_h\rho_h\right) - u_h\rho_h\right) \mathrm{d}x \\ =& \int_\Omega F \cdot \nabla\Pi_h\left(u_h\rho_h\right) \mathrm{d}x \\ &+ \int_\Omega f \cdot \Pi_h\left(u_h\rho_h\right) \mathrm{d}x =: R_I + R_{II}. \end{aligned} \tag{4.92}$$

First, we look at L_I.

$$\begin{aligned} L_I &= \int_\Omega |\nabla u_h|^{p-2}\nabla u_h \cdot \nabla\left(u_h\rho_h\right)\,\mathrm{d}x \\ &= \int_\Omega |\nabla u_h|^p \rho_h\,\mathrm{d}x + \int_\Omega |\nabla u_h|^{p-2}\nabla u_h\cdot\nabla\rho_h u_h\,\mathrm{d}x \\ &=: L_{I_1} + L_{I_2}. \end{aligned} \tag{4.93}$$

For L_{I_2}, we find

$$\begin{aligned} |L_{I_2}| &\le \int_\Omega |\nabla u_h|^{p-1}|\nabla\rho_h||u_h|\,\mathrm{d}x \\ &\le \sum_{T\in\mathcal{T}_h} |T||\nabla u_h(T)|^{p-1}|\nabla\rho_h(T)|\max_T |u_h| =: L_{III}. \end{aligned} \tag{4.94}$$

For L_{II} from equation (4.92), we use Lemmas 2.23 and 2.25 to find

$$\begin{aligned} L_{II} &= \sum_{T\in\mathcal{T}_h} |T| \fint_T |\nabla u_h|^{p-2}\nabla u_h\cdot\nabla\left(\Pi_h\left(u_h\rho_h\right) - u_h\rho_h\right)\,\mathrm{d}x \\ &\le \sum_{T\in\mathcal{T}_h} |T||\nabla u_h(T)|^{p-1}\max_T\left|\nabla\left(\Pi_h\left(u_h\rho_h\right) - u_h\rho_h\right)\right| \\ &\lesssim \sum_{T\in\mathcal{T}_h} |T||\nabla u_h(T)|^{p-1}\max_T |u_h||\nabla\rho_h(T)| = L_{III}. \end{aligned} \tag{4.95}$$

So far, we have shown that

$$\int_\Omega |\nabla u_h|^p\rho_h\,\mathrm{d}x = L_{I_1} \lesssim L_{III} + R_I + R_{II}. \tag{4.96}$$

We also use inequality (2.26) to find

$$|\nabla\rho_h(T)| = \max_T\left|\nabla\Pi_h(\eta_h^p)\right| \lesssim \max_T|\nabla(\eta_h^p)| \lesssim \max_T|\eta_h^{p-1}|\max_T|\nabla\eta_h|. \tag{4.97}$$

We use equation (4.97), Young's inequality, equation (2.27) and Jensen's inequality to conclude that

$$\begin{aligned} L_{III} &\lesssim \sum_{T\in\mathcal{T}_h} |T||\nabla u_h(T)|^{p-1}\max_T|u_h||\nabla\eta_h(T)|\max_T|\eta_h|^{p-1} \\ &\lesssim \varepsilon\sum_{T\in\mathcal{T}_h}|T||\nabla u_h(T)|^p\max_T|\eta_h|^p + C_\varepsilon\sum_{T\in\mathcal{T}_h}|T|\max_T|u_h|^p|\nabla\eta_h(T)|^p \\ &\lesssim \varepsilon\sum_{T\in\mathcal{T}_h}|T||\nabla u_h(T)|^p\max_T|\rho_h| + C_\varepsilon\sum_{T\in\mathcal{T}_h}|T|\left(\fint_T|u_h|\,\mathrm{d}x\right)^p|\nabla\eta_h(T)|^p \\ &\lesssim \varepsilon\sum_{T\in\mathcal{T}_h}|T||\nabla u_h(T)|^p\fint_T|\rho_h|\,\mathrm{d}x + C_\varepsilon\sum_{T\in\mathcal{T}_h}|T|\fint_T|u_h|^p\,\mathrm{d}x|\nabla\eta_h(T)|^p \\ &= \varepsilon\int_\Omega|\nabla u_h|^p\rho_h\,\mathrm{d}x + C_\varepsilon\int_\Omega|u_h|^p|\nabla\eta_h|^p\,\mathrm{d}x. \end{aligned} \tag{4.98}$$

Inserting inequality (4.98) into inequality (4.96) and absorbing the ε-term into the left-hand side then yields

$$\int_\Omega |\nabla u_h|^p \rho_h \,\mathrm{d}x \lesssim \int_\Omega |u_h|^p |\nabla \eta_h|^p \,\mathrm{d}x + R_I + R_{II}. \tag{4.99}$$

Now, we can focus on R_I from equation (4.92). With the help of inequality (2.26), we find

$$\begin{aligned}
R_I &= \int_\Omega F \cdot \nabla \Pi_h \left(u_h \rho_h\right) \mathrm{d}x \\
&\leq \sum_{T\in\mathcal{T}_h} |T| \fint_T |F| \,\mathrm{d}x \max_T \left|\nabla \Pi_h \left(u_h \rho_h\right)\right| \\
&\leq \sum_{T\in\mathcal{T}_h} |T| \fint_T |F| \,\mathrm{d}x \max_T \left|\nabla \left(u_h \rho_h\right)\right| \\
&\leq \sum_{T\in\mathcal{T}_h} |T| \fint_T |F| \,\mathrm{d}x \max_T |u_h| |\nabla \rho_h(T)| \\
&\quad + \sum_{T\in\mathcal{T}_h} |T| \fint_T |F| \,\mathrm{d}x \max_T |\eta_h|^p |\nabla u_h(T)| =: R_{I_1} + R_{I_2}.
\end{aligned} \tag{4.100}$$

First, we will look at R_{I_1} in inequality (4.100). Let us define $\mathcal{S} := \{T \in \mathcal{T}_h : T \subset \operatorname{supp} \eta_h \cap \operatorname{supp} u_h\}$. We use inequality (4.97) and equation (2.27) to get

$$\begin{aligned}
R_{I_1} &= \sum_{T\in\mathcal{T}_h} |T| \fint_T |F| \,\mathrm{d}x \max_T |u_h| |\nabla \rho_h(T)| \\
&\lesssim \sum_{T\in\mathcal{S}} |T| \fint_T |F| \,\mathrm{d}x \max_T |u_h| |\nabla \eta_h(T)| \max_T |\eta_h|^{p-1} \\
&\lesssim \sum_{T\in\mathcal{S}} |T| \fint_T |F| \,\mathrm{d}x \fint_T |u_h| |\nabla \eta_h| \,\mathrm{d}x.
\end{aligned} \tag{4.101}$$

Now, applying Hölder's inequality for sums and Jensen's inequality to inequality (4.101) yields

$$\begin{aligned}
R_{I_1} &\lesssim \left(\sum_{T\in\mathcal{S}} |T| \left(\fint_T |F| \,\mathrm{d}x\right)^q\right)^{\frac{1}{q}} \cdot \left(\sum_{T\in\mathcal{S}} |T|\right)^{1-\frac{p-1}{n}-\frac{1}{p}+\frac{\delta}{n}} \\
&\quad \cdot \left(\sum_{T\in\mathcal{S}} |T| \left(\fint_T |u_h| |\nabla \eta_h| \,\mathrm{d}x\right)^p\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{T\in\mathcal{S}} |T| \fint_T |F|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \cdot \left|\operatorname{supp} \eta_h \cap \operatorname{supp} u_h\right|^{1-\frac{p-1}{n}-\frac{1}{p}+\frac{\delta}{n}} \\
&\quad \cdot \left(\sum_{T\in\mathcal{S}} |T| \fint_T |u_h|^p |\nabla \eta_h|^p \,\mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned} \tag{4.102}$$

This means that we can carry out the sums and use Young's inequality to get

$$\begin{aligned} R_{I_1} &\lesssim \|F\|_{L^q(\operatorname{supp}\eta_h)} \left|\operatorname{supp}\eta_h \cap \operatorname{supp} u_h\right|^{1-\frac{p-1}{n}-\frac{1}{p}+\frac{\delta}{n}} \|u_h \nabla \eta_h\|_{L^p(\Omega)} \\ &\lesssim \|F\|^{p'}_{L^q(\operatorname{supp}\eta_h)} \left|\operatorname{supp}\eta_h \cap \operatorname{supp} u_h\right|^{1-\frac{p}{n}+p'\frac{\delta}{n}} + \|u_h \nabla \eta_h\|^p_{L^p(\Omega)}. \end{aligned} \tag{4.103}$$

For R_{I_2} in inequality (4.100), we can analogously write

$$\begin{aligned} R_{I_2} &= \sum_{T\in\mathcal{T}_h} |T| \fint_T |F| \,\mathrm{d}x \max_T |\eta_h|^p |\nabla u_h(T)| \\ &\lesssim \sum_{T\in\mathcal{S}} |T| \fint_T |F| \,\mathrm{d}x \fint_T \eta_h |\nabla u_h| \,\mathrm{d}x \max_T |\eta_h|^{p-1} \\ &\leq \sum_{T\in\mathcal{S}} |T| \fint_T |F| \,\mathrm{d}x \fint_T \eta_h |\nabla u_h| \,\mathrm{d}x. \end{aligned} \tag{4.104}$$

Using Hölder's inequality for sums and Jensen's inequality in inequality (4.104) then yields

$$\begin{aligned} R_{I_2} &\lesssim \left(\sum_{T\in\mathcal{S}} |T| \left(\fint_T |F| \,\mathrm{d}x\right)^q\right)^{\frac{1}{q}} \left(\sum_{T\in\mathcal{S}} |T|\right)^{1-\frac{p-1}{n}-\frac{1}{p}+\frac{\delta}{n}} \\ &\quad \cdot \left(\sum_{T\in\mathcal{S}} |T| \left(\fint_T \eta_h |\nabla u_h| \,\mathrm{d}x\right)^p\right)^{\frac{1}{p}} \\ &\lesssim \left(\sum_{T\in\mathcal{S}} |T| \fint_T |F|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \left|\operatorname{supp}\eta_h \cap \operatorname{supp} S_\lambda u_h\right|^{1-\frac{p-1}{n}-\frac{1}{p}+\frac{\delta}{n}} \\ &\quad \cdot \left(\sum_{T\in\mathcal{S}} |T| \fint_T \eta_h^p |\nabla u_h|^p \,\mathrm{d}x\right)^{\frac{1}{p}}. \end{aligned} \tag{4.105}$$

Then, we can carry out the remaining sums in inequality (4.105) and use Lemma 2.27 to get

$$R_{I_2} \lesssim \left(\int_\Omega \rho_h |\nabla u_h|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \|F\|_{L^q(\operatorname{supp}\eta_h)} \left|\operatorname{supp}\eta_h \cap \operatorname{supp} u_h\right|^{1-\frac{p-1}{n}-\frac{1}{p}+\frac{\delta}{n}}. \tag{4.106}$$

This means that we can use Young's inequality in inequality (4.106) to get

$$R_{I_2} \leq \varepsilon \int_\Omega \rho_h |\nabla u_h|^p \,\mathrm{d}x + C_\varepsilon \|F\|^{p'}_{L^q(\operatorname{supp}\eta_h)} \left|\operatorname{supp}\eta_h \cap \operatorname{supp} u_h\right|^{1-\frac{p}{n}+p'\frac{\delta}{n}}. \tag{4.107}$$

For R_{II} from inequality (4.96) as defined in equation (4.92), we estimate

$$R_{II} = \int_\Omega f \cdot \Pi_h \left(u_h \rho_h\right) \,\mathrm{d}x \leq \sum_{T\in\mathcal{T}_h} |T| \fint_T |f| \,\mathrm{d}x \max_T |u_h| \max_T |\eta_h|^p. \tag{4.108}$$

Now, recall the assumption $0 \leq \eta_h \leq 1$ and estimate inequality (4.108) with equation (2.27) to get

$$R_{II} \lesssim \sum_{T \in \mathcal{S}} |T| \fint_T |f| \,\mathrm{d}x \fint_T |u_h| \eta_h \,\mathrm{d}x. \tag{4.109}$$

Again, we use Hölder's inequality for sums and Jensen's inequality in inequality (4.109):

$$\begin{aligned} R_{II} &\lesssim \left(\sum_{T \in \mathcal{S}} |T| \left(\fint_T |f| \,\mathrm{d}x \right)^r \right)^{\frac{1}{r}} \cdot \left(\sum_{T \in \mathcal{S}} |T| \right)^{1 - \frac{p-1}{n} - \frac{1}{p} + \frac{\delta}{n}} \cdot \left(\sum_{T \in \mathcal{S}} |T| \left(\fint_T |u_h| \eta_h \,\mathrm{d}x \right)^{p^\star} \right)^{\frac{1}{p^\star}} \\ &\lesssim \left(\sum_{T \in \mathcal{S}} |T| \fint_T |f|^r \,\mathrm{d}x \right)^{\frac{1}{r}} \cdot \left(\sum_{T \in \mathcal{S}} |T| \right)^{1 - \frac{p-1}{n} - \frac{1}{p} + \frac{\delta}{n}} \cdot \left(\sum_{T \in \mathcal{S}} |T| \fint_T |u_h|^{p^\star} \eta_h^{p^\star} \,\mathrm{d}x \right)^{\frac{1}{p^\star}}. \end{aligned} \tag{4.110}$$

Carrying out the sums in inequality (4.110) yields

$$R_{II} \lesssim \|f\|_{L^r(\operatorname{supp} \eta_h)} \|u_h \eta_h\|_{L^{p^\star}(\Omega)} \left|\operatorname{supp} \eta_h \cap \operatorname{supp} u_h\right|^{1 - \frac{p-1}{n} - \frac{1}{p} + \frac{\delta}{n}}. \tag{4.111}$$

The Sobolev–Poincaré inequality allows us to estimate

$$\|u_h \eta_h\|_{L^{p^\star}(\Omega)} \lesssim \|\nabla (u_h \eta_h)\|_{L^p(\Omega)} \leq \|u_h \nabla \eta_h\|_{L^p(\Omega)} + \|\eta_h \nabla u_h\|_{L^p(\Omega)}. \tag{4.112}$$

Then, inserting inequality (4.112) into inequality (4.111) gives

$$\begin{aligned} R_{II} &\lesssim \|f\|_{L^r(\operatorname{supp} \eta_h)} \|u_h \nabla \eta_h\|_{L^p(\Omega)} \left|\operatorname{supp} \eta_h \cap \operatorname{supp} u_h\right|^{1 - \frac{p-1}{n} - \frac{1}{p} + \frac{\delta}{n}} \\ &\quad + \|f\|_{L^r(\operatorname{supp} \eta_h)} \|\eta_h \nabla u_h\|_{L^p(\Omega)} \left|\operatorname{supp} \eta_h \cap \operatorname{supp} u_h\right|^{1 - \frac{p-1}{n} - \frac{1}{p} + \frac{\delta}{n}}, \end{aligned} \tag{4.113}$$

which means that we can apply Young's inequality to get

$$R_{II} \lesssim \|u_h \nabla \eta_h\|_{L^p(\Omega)}^p + \varepsilon \int_\Omega |\nabla u_h|^p \rho_h \,\mathrm{d}x + C_\varepsilon \|f\|_{L^r(\operatorname{supp} \eta_h)}^{p'} \left|\operatorname{supp} \eta_h \cap \operatorname{supp} u_h\right|^{1 - \frac{p}{n} + p' \frac{\delta}{n}}. \tag{4.114}$$

We are now able to insert inequalities (4.102) and (4.107) into inequality (4.100) and then inequalities (4.100) and (4.114) into inequality (4.127) and absorb the ε-terms into the left-hand side to get

$$\begin{aligned} \int_\Omega |u_h|^p \eta_h^p \,\mathrm{d}x &\lesssim 2^{(2+p)k} \int_\Omega |u_h|^p |\nabla \eta_h|^p \,\mathrm{d}x \\ &\quad + \left(\|f\|_{L^r(\operatorname{supp} \eta)}^{p'} + \|F\|_{L^q(\operatorname{supp} \eta_h)}^{p'} \right) \left| \operatorname{supp} \eta_h \cap \operatorname{supp} u_h \right|^{1 - \frac{p}{n} + p' \frac{\delta}{n}} \end{aligned}$$

and thus, conclude the proof of the lemma. □

First, we will look at the singular case $1 < p < 2$.

Lemma 4.27. *Under the assumptions of Theorem 4.25, let $p \in (1,2)$. Let $\eta_h \in V_{h,0}$ be a cut-off function satisfying $0 \leq \eta_h \leq 1$ and define λ_k as in Definition 4.20. Then, we have*

$$\begin{aligned}
\int_\Omega |\nabla S_{\lambda_{k+1}} u_h|^p \eta_h^p \,\mathrm{d}x &\leq C 2^{(2+p)k} \int_\Omega |S_{\lambda_k} u_h|^p |\nabla \eta_h|^p \,\mathrm{d}x \\
&+ C \left(\|f\|_{L^r(\operatorname{supp}\eta_h)}^{p'} + \|F\|_{L^q(\operatorname{supp}\eta_h)}^{p'} \right) \left| \operatorname{supp}\eta_h \cap \operatorname{supp}\left(S_{\frac{1}{2}(\lambda_k+\lambda_{k+1})} u_h \right) \right|^{1-\frac{p}{n}+p'\frac{\delta}{n}} \\
&+ C \left(\|f\|_{L^r(\operatorname{supp}\eta_h)} + \|F\|_{L^q(\operatorname{supp}\eta_h)} \right) \|(\nabla S_{\lambda_k} u_h)\,\eta_h\|_{L^p(\Omega)} \\
&\quad \cdot \left| \operatorname{supp}\eta_h \cap \operatorname{supp}\left(S_{\frac{1}{2}(\lambda_k+\lambda_{k+1})} u_h \right) \right|^{1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}},
\end{aligned} \tag{4.115}$$

where the constant $C > 0$ depends only on p, n and the shape-regularity constant Γ.

Proof. Fix $k \in \mathbb{N}$ and define $\lambda = \frac{1}{2}(\lambda_k + \lambda_{k+1})$ and write $\rho_h = \Pi_h\left(\eta_h^p\right)$. We test equation (4.90) against $\varphi_h = \Pi_h(S_\lambda u_h \rho_h)$. This gives

$$\begin{aligned}
L_I + L_{II} &:= \int_\Omega |\nabla u_h|^{p-2} \nabla u_h \cdot \nabla \left(S_\lambda u_h \rho_h \right) \mathrm{d}x \\
&\quad + \int_\Omega |\nabla u_h|^{p-2} \nabla u_h \cdot \nabla \left(\Pi_h\left(S_\lambda u_h \rho_h \right) - S_\lambda u_h \rho_h \right) \mathrm{d}x \\
&= \int_\Omega F \cdot \nabla \Pi_h \left(S_\lambda u_h \rho_h \right) \mathrm{d}x \\
&\quad + \int_\Omega f \cdot \Pi_h \left(S_\lambda u_h \rho_h \right) \mathrm{d}x =: R_I + R_{II}.
\end{aligned} \tag{4.116}$$

First, we look at L_I.

$$\begin{aligned}
L_I &= \int_\Omega |\nabla u_h|^{p-2} \nabla u_h \cdot \nabla \left(S_\lambda u_h \rho_h \right) \mathrm{d}x \\
&= \int_\Omega |\nabla u_h|^{p-2} \nabla u_h \cdot \left(\nabla S_\lambda u_h \right) \rho_h \,\mathrm{d}x + \int_\Omega |\nabla u_h|^{p-2} \nabla u_h \cdot \left(\nabla \rho_h \right) S_\lambda u_h \,\mathrm{d}x \\
&=: L_{I_1} + L_{I_2}.
\end{aligned} \tag{4.117}$$

For L_{I_1}, we recall that Lemma 4.10 implies $\nabla S_\lambda u_h \cdot \nabla u_h \geq |\nabla S_\lambda u_h|^2$ to find

$$L_{I_1} = \int_\Omega |\nabla u_h|^{p-2} \nabla u_h \cdot \left(\nabla S_\lambda u_h \right) \rho_h \,\mathrm{d}x \geq \int_\Omega |\nabla u_h|^{p-2} |\nabla S_\lambda u_h|^2 \rho_h \,\mathrm{d}x. \tag{4.118}$$

Now, note that ∇v_h is constant on any $T \in \mathcal{T}_h$ for every $v_h \in V_h$ and write $\nabla v_h(T)$ for that matrix. Furthermore, Lemma 4.11 implies that $|\nabla S_\lambda u_h(T)| \max_T |u_h| \gtrsim |\nabla u_h(T)| \max_T |S_\lambda u_h|$. We also know that $\max_T \rho_h = \max_T \eta_h^p$. Using this in equation (4.125) yields

$$\begin{aligned}
L_{I_1} &\geq \sum_{T \in \mathcal{T}_h} |T| |\nabla u_h(T)|^{p-2} |\nabla S_\lambda u_h(T)|^2 \max_T |\rho_h| \\
&\gtrsim \sum_{T \in \mathcal{T}_h} |T| |\nabla u_h(T)|^p \frac{\max_T |S_\lambda u_h|^2}{\max_T |u_h|^2} \max_T \eta_h^p.
\end{aligned} \tag{4.119}$$

We will now focus on L_{I_2} from equation (4.117). We find

$$\begin{aligned} |L_{I_2}| &\leq \int_\Omega |\nabla u_h|^{p-1} |\nabla \rho_h| |S_\lambda u_h| \,\mathrm{d}x \\ &\leq \sum_{T\in\mathcal{T}_h} |T| |\nabla u_h(T)|^{p-1} |\nabla \rho_h(T)| \max_T |S_\lambda u_h| =: L_{III}. \end{aligned} \tag{4.120}$$

For L_{II} from equation (4.116), we proceed as in inequality (4.95) to find

$$L_{II} \lesssim \sum_{T\in\mathcal{T}_h} |T| |\nabla u_h(T)|^{p-1} \max_T |S_\lambda u_h| |\nabla \rho_h(T)| = L_{III}. \tag{4.121}$$

Inserting equation (4.117) and inequalities (4.119), (4.120) and (4.121) into equation (4.116) yields

$$\sum_{T\in\mathcal{T}_h} |T| |\nabla u_h(T)|^p \frac{\max_T |S_\lambda u_h|^2}{\max_T |u_h|^2} \max_T \eta_h^p \lesssim L_{III} + R_1 + R_2. \tag{4.122}$$

For simplicity, we write $\mathcal{S} := \{T \in \mathcal{T}_h : \max_T |u_h| > \lambda, T \subset \operatorname{supp} \eta_h\}$. We use equation (4.97) to get

$$L_{III} \lesssim \sum_{T\in\mathcal{S}} |T| |\nabla u_h(T)|^{p-1} \frac{\max_T |S_\lambda u_h|}{\max_T |u_h|} \max_T |u_h| \max_T |\eta_h|^{p-1} |\nabla \eta_h(T)|. \tag{4.123}$$

Now, we use Young's inequality in equation (4.123) and note that $p' > 2$ because $1 < p < 2$ and $\max_T \rho_h = \max_T \eta_h^p$. This gives

$$\begin{aligned} L_{III} &\leq \varepsilon \sum_{T\in\mathcal{S}} |T| \left(\frac{\max_T |S_\lambda u_h|}{\max_T |u_h|} \right)^{p'} |\nabla u_h(T)|^p \max_T |\eta_h|^p \\ &\quad + C_\varepsilon \sum_{T\in\mathcal{S}} |T| \max_T |u_h|^p |\nabla \eta_h(T)|^p \\ &\leq \varepsilon \sum_{T\in\mathcal{S}} |T| \left(\frac{\max_T |S_\lambda u_h|}{\max_T |u_h|} \right)^{2} |\nabla u_h(T)|^p \max_T |\rho_h| \\ &\quad + C_\varepsilon \sum_{T\in\mathcal{S}} |T| \max_T |u_h|^p |\nabla \eta_h(T)|^p. \end{aligned} \tag{4.124}$$

Putting this in inequality (4.122) and absorbing into the left-hand side gives

$$\sum_{T\in\mathcal{T}_h} |T| |\nabla u_h(T)|^p \frac{\max_T |S_\lambda u_h|^2}{\max_T |u_h|^2} \max_T \eta_h^p \lesssim \sum_{T\in\mathcal{S}} |T| \max_T |u_h|^p |\nabla \eta_h(T)|^p + R_1 + R_2. \tag{4.125}$$

For $T \in \mathcal{S}$, we have $\max_T |u_h| > \lambda$ by definition. This allows us to estimate

$$\frac{\max_T |S_{\lambda_k} u_h|}{\max_T |u_h|} = \max_T \frac{(|u_h| - \lambda)_+}{|u_h|} \geq \frac{\lambda_{k+1} - \lambda}{\lambda_{k+1}} \geq \frac{\lambda_{k+1} - \lambda_k}{2\,(\lambda_{k+1})} = \frac{2^{-k} - 2^{-(k+1)}}{2\left(1 - 2^{-(k+1)}\right)} \geq 2^{-(k+2)},$$

which leads to

$$\sum_{T\in\mathcal{S}} |T| \max_T |u_h|^p |\nabla \eta_h(T)| \lesssim 2^{pk} \sum_{T\in\mathcal{S}} |T| \left(\max_T |S_{\lambda_k} u_h| \right)^p |\nabla \eta_h(T)|^p. \tag{4.126}$$

Therefore, we can apply Lemma 2.23 and Jensen's inequality to inequality (4.124) to conclude that

$$\begin{aligned}\sum_{T\in\mathcal{S}} |T| \max_T |u_h|^p |\nabla\eta_h(T)| &\lesssim 2^{pk} \sum_{T\in\mathcal{S}} |T| \left(\fint_T |S_{\lambda_k} u_h| \,\mathrm{d}x \right)^p |\nabla\eta_h(T)|^p \\ &\lesssim 2^{pk} \sum_{T\in\mathcal{S}} |T| \fint_T |S_{\lambda_k} u_h|^p \,\mathrm{d}x |\nabla\eta_h(T)|^p \\ &\leq 2^{pk} \int_\Omega |S_{\lambda_k} u_h|^p |\nabla\eta_h|^p \,\mathrm{d}x.\end{aligned}$$

Putting this into inequality (4.125) gives

$$\sum_{T\in\mathcal{T}_h} |T| |\nabla u_h(T)|^p \frac{\max_T |S_\lambda u_h|^2}{\max_T |u_h|^2} \max_T \eta_h^p \lesssim 2^{pk} \int_\Omega |S_{\lambda_k} u_h|^p |\nabla\eta_h|^p \,\mathrm{d}x + R_1 + R_2. \tag{4.127}$$

Now, we look at R_I from equation (4.116). In direct analogy to inequality (4.100), we get

$$\begin{aligned}R_I \leq &\sum_{T\in\mathcal{S}} |T| \fint_T |F| \,\mathrm{d}x \max_T |S_\lambda u_h| |\nabla\rho_h(T)| \\ &+ \sum_{T\in\mathcal{S}} |T| \fint_T |F| \,\mathrm{d}x \max_T |\eta_h|^p |\nabla S_\lambda u_h(T)| =: R_{I_1} + R_{I_2}.\end{aligned} \tag{4.128}$$

For R_{I_1}, we proceed as in inequalities (4.101) and (4.102) to find

$$\begin{aligned}R_{I_1} \leq &\left(\sum_{T\in\mathcal{S}} |T| \fint_T |F|^q \,\mathrm{d}x \right)^{\frac{1}{q}} \cdot \left| \operatorname{supp} \eta_h \cap \operatorname{supp} S_\lambda u_h \right|^{1-\frac{p-1}{n}-\frac{1}{p}+\frac{\delta}{n}} \\ &\cdot \left(\sum_{T\in\mathcal{S}} |T| \fint_T |S_\lambda u_h|^p |\nabla\eta_h|^p \,\mathrm{d}x \right)^{\frac{1}{p}}.\end{aligned} \tag{4.129}$$

In analogy to inequality (4.103), we carry out the sums and use Young's inequality to get

$$\begin{aligned}R_{I_1} &\lesssim \|F\|_{L^q(\operatorname{supp}\eta_h)} \left| \operatorname{supp} \eta_h \cap \operatorname{supp} S_\lambda u_h \right|^{1-\frac{p-1}{n}-\frac{1}{p}+\frac{\delta}{n}} \|S_\lambda u_h \nabla\eta_h\|_{L^p(\Omega)} \\ &\lesssim \|F\|_{L^q(\operatorname{supp}\eta_h)}^{p'} \left| \operatorname{supp} \eta_h \cap \operatorname{supp} S_\lambda u_h \right|^{1-\frac{p}{n}+p'\frac{\delta}{n}} + \|S_{\lambda_k} u_h \nabla\eta_h\|_{L^p(\Omega)}^p,\end{aligned} \tag{4.130}$$

where we have also used that $|S_{\lambda_k} u_h| \geq |S_\lambda u_h|$.

For R_{I_2} in inequality (4.128), we apply the same steps as in inequalities (4.104) and (4.105) to get

$$\begin{aligned}R_{I_2} \lesssim &\left(\sum_{T\in\mathcal{S}} |T| \fint_T |F|^q \,\mathrm{d}x \right)^{\frac{1}{q}} \left| \operatorname{supp} \eta_h \cap \operatorname{supp} S_\lambda u_h \right|^{1-\frac{p-1}{n}-\frac{1}{p}+\frac{\delta}{n}} \\ &\cdot \left(\sum_{T\in\mathcal{S}} |T| \fint_T \eta_h^p |\nabla S_\lambda u_h|^p \,\mathrm{d}x \right)^{\frac{1}{p}}.\end{aligned} \tag{4.131}$$

After that, we can carry out the remaining sums in inequality (4.131) and use $|S_\lambda u_h| \le |S_{\lambda_k} u_h|$ to get

$$R_{I_2} \lesssim \|\eta_h \nabla S_{\lambda_k} u_h\|_{L^p(\Omega)} \|F\|_{L^q(\operatorname{supp}\eta_h)} \left|\operatorname{supp}\eta_h \cap \operatorname{supp} S_\lambda u_h\right|^{1-\frac{p-1}{n}-\frac{1}{p}+\frac{\delta}{n}}. \tag{4.132}$$

For R_{II} from inequality (4.122) as defined in equation (4.116), we proceed as in inequalities (4.109) to (4.113) to get

$$\begin{aligned} R_{II} &\lesssim \|f\|_{L^r(\operatorname{supp}\eta_h)} \|S_\lambda u_h \nabla \eta_h\|_{L^p(\Omega)} \left|\operatorname{supp}\eta_h \cap \operatorname{supp} S_\lambda u_h\right|^{1-\frac{p-1}{n}-\frac{1}{p}+\frac{\delta}{n}} \\ &\quad + \|f\|_{L^r(\operatorname{supp}\eta_h)} \|\eta_h \nabla S_\lambda u_h\|_{L^p(\Omega)} \left|\operatorname{supp}\eta_h \cap \operatorname{supp} S_\lambda u_h\right|^{1-\frac{p-1}{n}-\frac{1}{p}+\frac{\delta}{n}}. \end{aligned}$$

Using Young's inequality on the first summand finally yields

$$\begin{aligned} R_{II} &\lesssim \|S_{\lambda_k} u_h \nabla \eta_h\|_{L^p(\Omega)}^p + \|f\|_{L^r(\operatorname{supp}\eta_h)}^{p'} \left|\operatorname{supp}\eta_h \cap \operatorname{supp} S_\lambda u_h\right|^{1-\frac{p}{n}+p'\frac{\delta}{n}} \\ &\quad + \|f\|_{L^r(\operatorname{supp}\eta_h)} \|\eta_h \nabla S_\lambda u_h\|_{L^p(\Omega)} \left|\operatorname{supp}\eta_h \cap \operatorname{supp} S_\lambda u_h\right|^{1-\frac{p-1}{n}-\frac{1}{p}+\frac{\delta}{n}}. \end{aligned} \tag{4.133}$$

Finally, inserting inequalities (4.130) and (4.132) into inequality (4.128) and then inequalities (4.128) and (4.133) into inequality (4.127) yields

$$\begin{aligned} &\int_\Omega |\nabla S_{\lambda_{k+1}} u_h|^p \eta_h^p \,\mathrm{d}x \lesssim 2^{(2+p)k} \int_\Omega |S_{\lambda_k} u_h|^p |\nabla \eta_h|^p \,\mathrm{d}x \\ &\quad + \left(\|f\|_{L^r(\operatorname{supp}\eta_h)}^{p'} + \|F\|_{L^q(\operatorname{supp}\eta_h)}^{p'} \right) \left|\operatorname{supp}\eta_h \cap \operatorname{supp} S_\lambda u_h\right|^{1-\frac{p}{n}+p'\frac{\delta}{n}} \\ &\quad + \left(\|f\|_{L^r(\operatorname{supp}\eta_h)} + \|F\|_{L^q(\operatorname{supp}\eta_h)} \right) \|\nabla S_{\lambda_k} u_h \eta_h\|_{L^p} \left|\operatorname{supp}\eta_h \cap \operatorname{supp} S_\lambda u_h\right|^{1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}}, \end{aligned}$$

and thus, proves the lemma. □

Unfortunately, this bound is not strong enough to apply the De Giorgi iteration technique directly. The idea is now to iterate the application of the previous lemma until the exponent on the measure of the small set in the last summand is large enough.

Lemma 4.28. *Under the assumptions of Theorem 4.25, define λ_k, η_h and B_k as in Definition 4.20. Furthermore, assume $p \in (1,2)$ and*

$$\lambda_\infty^p \ge \max \left\{ \left(\|F\|_{L^q(\Omega(B(x_0,2R)))}^{p'} + \|f\|_{L^r(\Omega(B(x_0,2R)))}^{p'} \right) R^{p'\delta}, \fint_{\Omega(B(x_0,2R))} |u_h|^p \eta_0^p \,\mathrm{d}x \right\}. \tag{4.134}$$

Then, there are constants C, α_1, α_2, α_3 for every $m \in \mathbb{N}$, that do not depend on k, but only *on m, p, n, and the shape-regularity constant Γ, such that*

$$
\begin{aligned}
&\fint_{\Omega(B(x_0,2R))} |\nabla S_{\lambda_{k+m}} u_h|^p \eta_{k+m}^p \,\mathrm{d}x \le C 2^{\alpha_1 k} \fint_{\Omega(B(x_0,2R))} R^{-p} |S_{\lambda_k} u_h|^p \eta_k^p \,\mathrm{d}x \\
&+ C 2^{\alpha_2 k} \left(R^{-p} \fint_{\Omega(B(x_0,2R))} |S_{\lambda_k} u_h|^p \eta_k^p \,\mathrm{d}x \right)^{1-\frac{p}{n}+p'\frac{\delta}{n}} \left(\frac{\lambda_\infty}{R}\right)^{p\left(\frac{p}{n}-p'\frac{\delta}{n}\right)} \\
&+ C 2^{\alpha_3 k} \left(R^{-p} \fint_{\Omega(B(x_0,2R))} |S_{\lambda_k} u_h|^p \eta_k^p \,\mathrm{d}x \right)^{\left(1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}\right)\left(\sum_{i=0}^{m-1}\frac{1}{p^i}\right)} \\
&\quad \cdot \left(\frac{\lambda_\infty}{R}\right)^{p\left(1-\left(1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}\right)\left(\sum_{i=0}^{m-1}\frac{1}{p^i}\right)\right)}
\end{aligned}
\tag{4.135}
$$

for every $k \in \mathbb{N}$.

Proof. We define

$$
\begin{aligned}
a_k &:= R^{-p} \fint_{\Omega(B(x_0,2R))} |S_{\lambda_k} u_h|^p \eta_k^p \,\mathrm{d}x, \\
b_k &:= \fint_{\Omega(B(x_0,2R))} |\nabla S_{\lambda_{k+1}} u_h|^p \eta_k^p \,\mathrm{d}x.
\end{aligned}
$$

Note that both sequences are decreasing. To simplify our notation, we write $\Omega(B(x_0,2R)) =: B$. First, we will show that $b_k \lesssim \frac{\lambda_\infty^p}{R^p}$. Note that $\operatorname{supp}\eta_k \subset B$ for all $k \in \mathbb{N}$. We use the Caccioppoli inequality (4.91) for u_h from Lemma 4.26 to find

$$
\begin{aligned}
b_k \le b_1 \le \fint_B |\nabla u_h|^p |\eta_1| \,\mathrm{d}x &\lesssim \fint_B |u_h|^p |\nabla \eta_1|^p \,\mathrm{d}x \\
&+ R^{-n}\left(\|F\|_{L^r(B)}^{p'} + \|f\|_{L^q(B)}^{p'} \right) |\operatorname{supp}\eta_1 \cap \operatorname{supp} u_h|^{1-\frac{p}{n}+\delta\frac{p'}{n}}.
\end{aligned}
\tag{4.136}
$$

Recall $\max_T |\nabla \eta_{k+1}| \lesssim R^{-1} 2^k \max_T \eta_k$ from Definition 4.20 and Lemma 3.18 (c). We find

$$
\begin{aligned}
\fint_B |u_h|^p |\nabla \eta_1|^p \,\mathrm{d}x &\le |B|^{-1} \sum_{T \in \mathcal{T}_h} |T| \max_T |u_h|^p \max_T |\nabla \eta_1|^p \\
&\le |B|^{-1} R^{-p} \sum_{T \in \mathcal{T}_h} |T| \left(\max_T |u_h| \max_T |\eta_0| \right)^p
\end{aligned}
\tag{4.137}
$$

and apply Lemma 2.23 and Jensen's inequality to inequality (4.137) to get

$$
\begin{aligned}
\fint_B |u_h|^p |\nabla \eta_1|^p \,\mathrm{d}x &\lesssim |B|^{-1} R^{-p} \sum_{T \in \mathcal{T}_h} |T| \left(\fint_T |u_h||\eta_0| \,\mathrm{d}x \right)^p \\
&\lesssim |B|^{-1} R^{-p} \sum_{T \in \mathcal{T}_h} |T| \fint_T |u_h|^p |\eta_0|^p \,\mathrm{d}x \\
&= a_0.
\end{aligned}
\tag{4.138}
$$

Furthermore, we know from equation 2.13 that $|B| \sim R^n$ and as $\operatorname{supp}\eta_0 = B$, this implies that $|\operatorname{supp}\eta_1 \cap \operatorname{supp} u_h| \lesssim R^n$. Inserting this and inequality (4.138) into inequality (4.136) and using the assumptions $\lambda_\infty^p \geq \left(\|F\|_{L^q(B)}^{p'} + \|f\|_{L^r(B)}^{p'}\right) R^{p'\delta}$ and $\lambda_\infty \geq \fint_B |u_h|^p \eta_0^p \,\mathrm{d}x = a_0$ (from inequality (4.134)) yields that

$$b_k \lesssim \frac{\lambda_\infty^p}{R^p}. \tag{4.139}$$

Now, we want to start an induction over m. We use inequality (4.115) to find

$$\begin{aligned}
&b_{k+1} \lesssim 2^{(2+p)k} \fint_B |S_{\lambda_k} u_h|^p |\nabla \eta_{k+1}|^p \,\mathrm{d}x \\
&+ |B|^{-1} \left(\|f\|_{L^r(B)}^{p'} + \|F\|_{L^q(B)}^{p'}\right) \left|\operatorname{supp}\eta_{k+1} \cap \operatorname{supp}\left(S_{\frac{1}{2}(\lambda_k+\lambda_{k+1})} u_h\right)\right|^{1-\frac{p}{n}+p'\frac{\delta}{n}} \\
&+ |B|^{-1} \left(\|f\|_{L^r(B)} + \|F\|_{L^q(B)}\right) \|\nabla S_{\lambda_k} u_h \eta_h\|_{L^p(\Omega)} \\
&\quad \cdot \left|\operatorname{supp}\eta_h \cap \operatorname{supp}\left(S_{\frac{1}{2}(\lambda_{k+1}+\lambda_{k+1})} u_h\right)\right|^{1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}}.
\end{aligned} \tag{4.140}$$

Completely analogously to the calculations in inequalities (4.137) and (4.138), we find

$$\fint_B |S_{\lambda_k} u_h|^p |\nabla \eta_{k+1}|^p \,\mathrm{d}x \lesssim 2^k a_k. \tag{4.141}$$

Next, we will use the weak-type estimate from Lemma 4.22. Furthermore, recall from equation 2.13 that $|B| \sim R^n$. Additionally, recall that we assumed that

$$\lambda_\infty^p \geq \left(\|f\|_{L^r(B)}^{p'} + \|F\|_{L^q(B)}^{p'}\right) R^{\delta p'}.$$

We find that

$$\begin{aligned}
&R^{-n} \left(\|f\|_{L^r(B)}^{p'} + \|F\|_{L^q(B)}^{p'}\right) \left|\operatorname{supp}\eta_{k+1} \cap \operatorname{supp}\left(S_{\frac{1}{2}(\lambda_k+\lambda_{k+1})} u_h\right)\right|^{1-\frac{p}{n}+p'\frac{\delta}{n}} \\
&\lesssim R^{-n} 2^{\alpha_2 k} \left(\|f\|_{L^r(B)}^{p'} + \|F\|_{L^q(B)}^{p'}\right) \left(R^{-p} R^{-n} \int_B |S_\lambda u_h|^p \eta_k^p \,\mathrm{d}x\right)^{1-\frac{p}{n}+p'\frac{\delta}{n}} \\
&\quad \cdot \lambda_\infty^{-p\left(1-\frac{p}{n}+p'\frac{\delta}{n}\right)} R^{(p+n)\left(1-\frac{p}{n}+p'\frac{\delta}{n}\right)} \\
&\lesssim 2^{\alpha_2 k} \left(\|f\|_{L^r(B)}^{p'} + \|F\|_{L^q(B)}^{p'}\right) \left(R^{-p} \fint_B |S_\lambda u_h|^p \eta_k^p \,\mathrm{d}x\right)^{1-\frac{p}{n}+p'\frac{\delta}{n}} \\
&\quad \cdot \lambda_\infty^{-p} \left(\frac{\lambda_\infty}{R}\right)^{p\left(\frac{p}{n}-p'\frac{\delta}{n}\right)} R^{p'\delta} \\
&\lesssim 2^{\alpha_2 k} a_k^{1-\frac{p}{n}+p'\frac{\delta}{n}} \left(\frac{\lambda_\infty}{R}\right)^{p\left(\frac{p}{n}-p'\frac{\delta}{n}\right)}.
\end{aligned} \tag{4.142}$$

Analogously, we get

$$
\begin{aligned}
&|B|^{-1}\left(\|f\|_{L^r(B)}+\|F\|_{L^q(B)}\right)\|(\nabla S_{\lambda_k}u_h)\,\eta_h\|_{L^p(\Omega)} \\
&\qquad\cdot\left|\operatorname{supp}\eta_h\cap\operatorname{supp}\left(S_{\frac{1}{2}(\lambda_{k+1}+\lambda_{k+1})}u_h\right)\right|^{1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}} \\
&\lesssim R^{-n}2^{\alpha_3 k}\left(\|f\|_{L^r(B)}+\|F\|_{L^q(B)}\right)\left(R^{-p}\fint_B |S_{\lambda_k}u_h|^p\eta_h^p\,\mathrm{d}x\right)^{1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}} \\
&\qquad\cdot\|(\nabla S_{\lambda_k}u_h)\,\eta_h\|_{L^p(\Omega)}R^{-\frac{n}{p}}R^{\frac{n}{p}}\lambda_\infty^{-p\left(1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}\right)}R^{n\left(1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}\right)}R^{p\left(1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}\right)} \\
&=2^{\alpha_3 k}a_k^{1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}}b_k^{\frac{1}{p}}\left(\frac{\lambda_\infty}{R}\right)^{p\left(\frac{p-1}{n}-\frac{\delta}{n}\right)},
\end{aligned}
\tag{4.143}
$$

where we have also used that $\left(\|f\|_{L^r(B)}+\|F\|_{L^q(B)}\right)R^{\delta}\leq\lambda_\infty^{\frac{p}{p'}}$. From now on, we will use the α_i as generic positive constants that are allowed to change from line to line. We insert the inequalities (4.141), (4.142), and (4.143) into inequality (4.140) to get

$$
b_{k+1}\lesssim 2^{\alpha_1 k}a_k+2^{\alpha_2 k}a_k^{1-\frac{p}{n}+p'\frac{\delta}{n}}\left(\frac{\lambda_\infty}{R}\right)^{p\left(\frac{p}{n}-p'\frac{\delta}{n}\right)}+2^{\alpha_3 k}a_k^{1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}}b_k^{\frac{1}{p}}\left(\frac{\lambda_\infty}{R}\right)^{p\left(\frac{p-1}{n}-\frac{\delta}{n}\right)}.
\tag{4.144}
$$

Recall that $b_k\lesssim\frac{\lambda_\infty^p}{R^p}$ from inequality (4.139). Then, inequality (4.144) is exactly inequality (4.135) for $m=1$ and serves as a base case for an induction proof. Therefore, let us assume that inequality (4.135) is true for some $m\in\mathbb{N}$. Then, we find

$$
\begin{aligned}
b_{k+m+1}&\lesssim 2^{\alpha_1 k}a_{k+m}+2^{\alpha_2 k}a_{k+m}^{1-\frac{p}{n}+p'\frac{\delta}{n}}\left(\frac{\lambda_\infty}{R}\right)^{p\left(\frac{p}{n}-p'\frac{\delta}{n}\right)} \\
&\quad+2^{\alpha_3 k}a_{k+m}^{1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}}b_{k+m}^{\frac{1}{p}}\left(\frac{\lambda_\infty}{R}\right)^{p\left(\frac{p-1}{n}-\frac{\delta}{n}\right)} \\
&\lesssim 2^{\alpha_1 k}a_k+2^{\alpha_2 k}a_k^{1-\frac{p}{n}+p'\frac{\delta}{n}}\left(\frac{\lambda_\infty}{R}\right)^{p\left(\frac{p}{n}-p'\frac{\delta}{n}\right)} \\
&\quad+2^{\alpha_3 k}a_k^{1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}}b_{k+m}^{\frac{1}{p}}\left(\frac{\lambda_\infty}{R}\right)^{p\left(\frac{p-1}{n}-\frac{\delta}{n}\right)}.
\end{aligned}
\tag{4.145}
$$

Using inequality (4.135) on b_{k+m} leaves us with

$$
\begin{aligned}
&b_{k+m}^{\frac{1}{p}} a_k^{1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}} \left(\frac{\lambda_\infty}{R}\right)^{p\left(\frac{p-1}{n}-\frac{\delta}{n}\right)} \\
&\lesssim a_k^{1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}} \left(\frac{\lambda_\infty}{R}\right)^{p\left(\frac{p-1}{n}-\frac{\delta}{n}\right)} \\
&\qquad \cdot \Bigg(2^{\alpha_1 k} a_k + 2^{\alpha_2 k} a_k^{1-\frac{p}{n}+p'\frac{\delta}{n}} \left(\frac{\lambda_\infty}{R}\right)^{p\left(\frac{p}{n}-p'\frac{\delta}{n}\right)} \\
&\qquad\quad + 2^{\alpha_3 k} a_k^{\left(1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}\right)\left(\sum_{i=0}^{m-1}\frac{1}{p^i}\right)} \left(\frac{\lambda_\infty}{R}\right)^{p\left(1-\left(1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}\right)\left(\sum_{i=0}^{m-1}\frac{1}{p^i}\right)\right)} \Bigg)^{\frac{1}{p}}.
\end{aligned}
$$

The equivalence of norms in $\mathbb{R}^3$ then yields

$$
\begin{aligned}
&b_{k+m}^{\frac{1}{p}} a_k^{1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}} \left(\frac{\lambda_\infty}{R}\right)^{p\left(\frac{p-1}{n}-\frac{\delta}{n}\right)} \\
&\lesssim a_k^{1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}} \left(\frac{\lambda_\infty}{R}\right)^{p\left(\frac{p-1}{n}-\frac{\delta}{n}\right)} \\
&\qquad \cdot \Bigg(2^{\alpha_1 k} a_k^{\frac{1}{p}} + 2^{\alpha_2 k} a_k^{\frac{1}{p}\left(1-\frac{p}{n}+p'\frac{\delta}{n}\right)} \left(\frac{\lambda_\infty}{R}\right)^{\frac{p}{n}-p'\frac{\delta}{n}} \\
&\qquad\quad + 2^{\alpha_3 k} a_k^{\left(1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}\right)\left(\sum_{i=0}^{m-1}\frac{1}{p^i}\right)\frac{1}{p}} \left(\frac{\lambda_\infty}{R}\right)^{p\left(1-\left(1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}\right)\left(\sum_{i=0}^{m-1}\frac{1}{p^i}\right)\right)\frac{1}{p}} \Bigg).
\end{aligned}
\tag{4.146}
$$

We will look at the three terms separately. For the first one, we use Young's inequality to get

$$
\begin{aligned}
&a_k^{1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}} \left(\frac{\lambda_\infty}{R}\right)^{p\left(\frac{p-1}{n}-\frac{\delta}{n}\right)} a_k^{\frac{1}{p}} \\
&\lesssim a_k^{1-\frac{p}{n}+p'\frac{\delta}{n}} \left(\frac{\lambda_\infty}{R}\right)^{p\left(\frac{p}{n}-p'\frac{\delta}{n}\right)} + a_k.
\end{aligned}
\tag{4.147}
$$

For the second one, we recall $\frac{p'}{p} = p' - 1$ and calculate

$$
\begin{aligned}
&a_k^{1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}} \left(\frac{\lambda_\infty}{R}\right)^{p\left(\frac{p-1}{n}-\frac{\delta}{n}\right)} a_k^{\frac{1}{p}\left(1-\frac{p}{n}+p'\frac{\delta}{n}\right)} \left(\frac{\lambda_\infty}{R}\right)^{\frac{p}{n}-p'\frac{\delta}{n}} \\
&= a_k^{1-\frac{p}{n}+p'\frac{\delta}{n}} \left(\frac{\lambda_\infty}{R}\right)^{p\left(\frac{p}{n}-p'\frac{\delta}{n}\right)}.
\end{aligned}
\tag{4.148}
$$

For the third one, we finally get

$$
\begin{aligned}
&a_k^{1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}} \left(\frac{\lambda_\infty}{R}\right)^{p\left(\frac{p-1}{n}-\frac{\delta}{n}\right)} \\
&\quad \cdot a_k^{\left(1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}\right)\left(\sum_{i=0}^{m-1}\frac{1}{p^i}\right)\frac{1}{p}} \left(\frac{\lambda_\infty}{R}\right)^{p\left(1-\left(1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}\right)\left(\sum_{i=0}^{m-1}\frac{1}{p^i}\right)\right)\frac{1}{p}} \\
&= a_k^{\left(1-\frac{1}{p}+\frac{p-1}{n}+\frac{\delta}{n}\right)\left(1+\sum_{i=1}^{m}\frac{1}{p^i}\right)} \left(\frac{\lambda_\infty}{R}\right)^{p\left(1-\left(1-\frac{p-1}{n}-\frac{1}{p}+\frac{\delta}{n}\right)-\left(1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}\right)\left(\sum_{i=1}^{m}\frac{1}{p^i}\right)\right)} \\
&= a_k^{\left(1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}\right)\left(\sum_{i=0}^{m}\frac{1}{p^i}\right)} \left(\frac{\lambda_\infty}{R}\right)^{p\left(1-\left(1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}\right)\left(\sum_{i=0}^{m}\frac{1}{p^i}\right)\right)}.
\end{aligned}
\tag{4.149}
$$

Inserting inequalities (4.147), (4.148) and (4.149) into inequality (4.146) completes the induction step and concludes the proof. □

Proof of Theorem 4.25 in the singular case $1 < p < 2$. We know that $\sum_{i=0}^{\infty}\frac{1}{p^i} = \frac{p}{p-1} = p'$. We choose an $m_0 \in \mathbb{N}$, such that

$$
p' - \sum_{i=0}^{m_0-1}\frac{1}{p^i} = p'\frac{\delta-\varepsilon}{n}\left(1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}\right)^{-1}
\tag{4.150}
$$

for some $\varepsilon \in (0,\delta)$.

Furthermore, choose $\lambda_\infty^p = d_1 \max\{\left(\|F\|_{L^r(\Omega(B(x_0,2R)))}^{p'} + \|f\|_{L^q(\Omega(B(x_0,2R)))}^{p'}\right) R^{p'\delta}, \int_\Omega |u_h|^p \eta_0^p \,\mathrm{d}x\}$, where the constant $d_1 > 1$ is to be chosen later. For simplicity, we define the scaling-invariant norms $|||g|||_s^s := \fint_{\Omega(B(x_0,2R))} |g|^s \,\mathrm{d}x$ and define the sequence

$$
a_k = |||R^{-1} S_{\lambda_{km_0}} u_h \eta_{km_0}|||_p^p.
\tag{4.151}
$$

Hölder's inequality and the Sobolev embedding theorem yield

$$
\begin{aligned}
a_{k+1} &= |||R^{-1} S_{\lambda_{(k+1)m_0}} u_h \eta_{(k+1)m_0}|||_p^p \\
&\lesssim R^{-p} |||S_{\lambda_{(k+1)m_0}} u_h \eta_{(k+1)m_0}|||_{p^\star}^p \left(\frac{|\operatorname{supp} S_{\lambda_{(k+1)m_0}} u_h \cap \operatorname{supp} \eta_{(k+1)m_0}|}{R^n}\right)^{\frac{p}{n}} \\
&\lesssim |||\nabla\left(S_{\lambda_{(k+1)m_0}} u_h \eta_{(k+1)m_0}\right)|||_p^p \left(\frac{|\operatorname{supp} S_{\lambda_{(k+1)m_0}} u_h \cap \operatorname{supp} \eta_{(k+1)m_0}|}{R^n}\right)^{\frac{p}{n}} \\
&\lesssim |||\nabla(S_{\lambda_{(k+1)m_0}} u_h) \eta_{(k+1)m_0}|||_p^p \left(\frac{|\operatorname{supp} S_{\lambda_{(k+1)m_0}} u_h \cap \operatorname{supp} \eta_{(k+1)m_0}|}{R^n}\right)^{\frac{p}{n}} \\
&\quad + |||S_{\lambda_{(k+1)m_0}} u_h \nabla(\eta_{(k+1)m_0})|||_p^p \left(\frac{|\operatorname{supp} S_{\lambda_{(k+1)m_0}} u_h \cap \operatorname{supp} \eta_{(k+1)m_0}|}{R^n}\right)^{\frac{p}{n}}.
\end{aligned}
\tag{4.152}
$$

The weak-type estimate from Lemma 4.22 yields

$$
\frac{|\operatorname{supp} S_{\lambda_{(k+1)m_0}} u_h \cap \operatorname{supp} \eta_{(k+1)m_0}|}{R^n} \lesssim 2^{pk} |||S_{\lambda_{km_0}} u_h \eta_{km_0}|||_p^p R^{-p} \frac{R^p}{\lambda_\infty^p} = 2^{pk} a_k \frac{R^p}{\lambda_\infty^p}.
\tag{4.153}
$$

Analogously to inequalities (4.137) and (4.138), we find

$$\|\|S_{\lambda_{(k+1)m_0}} u_h \nabla(\eta_{(k+1)m_0})\|\|_p^p \lesssim 2^{pk} a_k. \tag{4.154}$$

Finally, inequality (4.135) gives that

$$\begin{aligned}\|\|\nabla(S_{\lambda_{(k+1)m_0}} u_h)\eta_{(k+1)m_0}\|\|_p^p &\lesssim 2^{\alpha_1 k} a_k + 2^{\alpha_2 k} a_k^{1-\frac{p}{n}+p'\frac{\delta}{n}} \left(\frac{\lambda_\infty^p}{R^p}\right)^{\frac{p}{n}-p'\frac{\delta}{n}} \\ &+ 2^{\alpha_3 k} a_k^{\left(1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}\right)\left(\sum_{i=0}^{m_0-1}\frac{1}{p^i}\right)} \left(\frac{\lambda_\infty}{R}\right)^{p\left(1-\left(1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}\right)\left(\sum_{i=0}^{m-1}\frac{1}{p^i}\right)\right)}.\end{aligned} \tag{4.155}$$

Now note that our choice of m_0 in equation (4.150) implies that

$$\left(1-\frac{1}{p}-\frac{p-1}{n}+\frac{\delta}{n}\right)\left(\sum_{i=0}^{m_0-1}\frac{1}{p^i}\right) = 1-\frac{p}{n}+p'\frac{\varepsilon}{n}.$$

Inserting this into inequality (4.155) yields the bound

$$\begin{aligned}&\|\|\nabla(S_{\lambda_{(k+1)m_0}} u_h)\eta_{(k+1)m_0}\|\|_p^p \\ &\lesssim 2^{\alpha_1 k} a_k + 2^{\alpha_2 k} a_k^{1-\frac{p}{n}+p'\frac{\delta}{n}} \left(\frac{\lambda_\infty^p}{R^p}\right)^{\frac{p}{n}-p'\frac{\delta}{n}} + 2^{\alpha_3 k} a_k^{1-\frac{p}{n}+p'\frac{\varepsilon}{n}} \left(\frac{\lambda_\infty^p}{R^p}\right)^{\frac{p}{n}-p'\frac{\varepsilon}{n}}.\end{aligned} \tag{4.156}$$

This means that we can insert inequalities (4.153), (4.154) and (4.156) into inequality (4.152) to get

$$a_{k+1} \lesssim 2^{\beta_1 k} a_k \left(\frac{R^p a_k}{\lambda_\infty^p}\right)^{\frac{p}{n}} + 2^{\beta_2 k} a_k \left(\frac{R^p a_k}{\lambda_\infty^p}\right)^{p'\frac{\delta}{n}} + 2^{\beta_2 k} a_k \left(\frac{R^p a_k}{\lambda_\infty^p}\right)^{p'\frac{\varepsilon}{n}}.$$

Corollary 3.32 implies that $a_k \to 0$ if $\lambda_\infty^p \gtrsim R^p a_0$. The implicit constant also determines d_1. This means that $|S_{\lambda_\infty} u_h| = 0$ on $\Omega'(B(x_0,R))$. Together with

$$\lambda_\infty^p = d_1 \max\{(\|F\|_{L^r(\Omega(B(x_0,2R)))}^{p'} + \|f\|_{L^q(\Omega(B(x_0,2R)))}^{p'})R^{p'\delta}, \fint_{\Omega(B(x_0,2R))} |u_h|^p \,\mathrm{d}x\},$$

this leaves us with

$$\begin{aligned}\sup_{\Omega'(B(x_0,R))} |u_h|^p \le \lambda_\infty^p \sim &\fint_{\Omega(B(x_0,2R))} |u_h|^p \,\mathrm{d}x \\ &+ \left(\|F\|_{L^r(\Omega(B(x_0,2R)))}^{p'} + \|f\|_{L^q(\Omega(B(x_0,2R)))}^{p'}\right) R^{p'\delta}\end{aligned}$$

and thus, proves Theorem 4.25 for $1 < p < 2$. □

Now, we will focus on the degenerate case $2 < p < n$. We will find a similar Caccioppoli-type inequality that we can iterate. In this case, we have $p' < 2$. This means the problematic term is not the one that comes from F and f, but the one that comes from the p-Laplacian operator itself.

Lemma 4.29. *Under the assumptions of Theorem 4.25, fix $2 < p < n$ and let $\eta_h \in V_{h,0}$ be a cut-off function with $0 \le \eta_h \le 1$. Furthermore, define λ_k as in Definition 4.20. Then, there is a constant $C > 0$ that only depends on p, n and the shape-regularity constant Γ, such that*

$$\begin{aligned}\int_\Omega |\nabla S_{\lambda_{k+1}} u_h|^p \eta_h^p \,\mathrm{d}x &\le C 2^{pk} \left(\int_\Omega |S_{\lambda_k} u_h|^p |\nabla \eta_h|^p \,\mathrm{d}x \right)^{\frac{1}{p}} \left(\int_\Omega |\nabla S_{\lambda_k} u_h|^p \eta_h^p \,\mathrm{d}x \right)^{1-\frac{1}{p}} \\ &\quad + C \int_\Omega |S_{\lambda_k} u_h|^p |\nabla \eta_h|^p \,\mathrm{d}x \\ &\quad + C \left(\|f\|^{p'}_{L^q(\operatorname{supp} \eta_h)} + \|F\|^{p'}_{L^r(\operatorname{supp} \eta_h)} \right) \left|\operatorname{supp} \eta_h \cap \operatorname{supp} S_{\lambda_{k+1}} u_h\right|^{1-\frac{p}{n}+p'\frac{\delta}{n}}.\end{aligned} \tag{4.157}$$

Proof. We write $\rho_h = \Pi_h(\eta_h^p)$ and test equation (4.90) against $\varphi_h = \Pi_h\left(\rho_h S_{\lambda_{k+1}} u_h\right)$ to get

$$\begin{aligned}&\int_\Omega |\nabla u_h|^{p-2} \nabla u_h \cdot \nabla \left(S_{\lambda_{k+1}} u_h \rho_h\right) \,\mathrm{d}x \\ &\qquad + \int_\Omega |\nabla u_h|^{p-2} \nabla u_h \cdot \nabla \left(\Pi_h \left(S_{\lambda_{k+1}} u_h \rho_h\right) - S_\lambda u_h \rho_h\right) \,\mathrm{d}x \\ &= \int_\Omega F \cdot \nabla \Pi_h \left(S_{\lambda_{k+1}} u_h \rho_h\right) \,\mathrm{d}x + \int_\Omega f \cdot \Pi_h \left(S_{\lambda_{k+1}} u_h \rho_h\right) \,\mathrm{d}x.\end{aligned} \tag{4.158}$$

We write $\mathcal{S} := \{T \in \mathcal{T}_h : T \subset \operatorname{supp} S_{\lambda_{k+1}} u_h \cap \operatorname{supp} \eta_h\}$. Completely analogously to the singular case (see equations (4.116) to (4.122)), we find

$$\begin{aligned}\int_\Omega |\nabla u_h|^{p-2} |\nabla S_{\lambda_{k+1}} u_h|^2 \rho_h \,\mathrm{d}x &\le \sum_{T \in \mathcal{S}} |T| |\nabla u_h(T)|^{p-1} \max_T |S_{\lambda_{k+1}} u_h| |\nabla \rho_h(T)| \\ + \int_\Omega F \cdot \nabla \Pi_h \left(S_{\lambda_{k+1}} u_h \rho_h\right) \,\mathrm{d}x &+ \int_\Omega f \Pi_h \left(S_{\lambda_{k+1}} u_h \rho_h\right) \,\mathrm{d}x =: L + R_I + R_{II}.\end{aligned} \tag{4.159}$$

We have assumed $p > 2$; this allows us to use $|\nabla S_{\lambda_{k+1}} u_h| \le |\nabla u_h|$ (from inequality (4.39)) to get

$$\begin{aligned}\int_\Omega |\nabla S_{\lambda_{k+1}} u_h|^p \rho_h &\le \sum_{T \in \mathcal{S}} |T| |\nabla u_h(T)|^{p-1} \max_T |S_{\lambda_{k+1}} u_h| |\nabla \rho_h(T)| \\ + \int_\Omega F \cdot \nabla \Pi_h \left(S_{\lambda_{k+1}} u_h \rho_h\right) \,\mathrm{d}x &+ \int_\Omega f \Pi_h \left(S_{\lambda_{k+1}} u_h \rho_h\right) \,\mathrm{d}x =: L + R_I + R_{II}.\end{aligned} \tag{4.160}$$

We will look at L from inequality (4.160). Note that we have $\max_T |u_h| > \lambda_{k+1}$ on the relevant simplices $T \in \mathcal{S}$. This means that $\frac{\max_T |u_h|}{\max_T |S_{\lambda_k} u_h|} < 2^k$. Therefore, we can use inequality (4.40) from Lemma 4.11 to get

$$|\nabla u_h(T)| = \frac{\max_T |u_h|}{\max_T |S_{\lambda_k} u_h|} |\nabla u_h(T)| \frac{\max_T |S_{\lambda_k} u_h|}{\max_T |u_h|} \lesssim 2^k |\nabla S_{\lambda_k} u_h| \tag{4.161}$$

on every $T \in \mathcal{S}$. This enables us to use inequality (4.161) and Lemma 2.27 to estimate L from inequality (4.159):

$$\begin{aligned}L &= \sum_{T \in \mathcal{S}} |T| |\nabla u_h(T)|^{p-1} \max_T |S_{\lambda_{k+1}} u_h| |\nabla \rho_h(T)| \\ &\lesssim 2^{(p-1)k} \sum_{T \in \mathcal{S}} |T| |\nabla S_{\lambda_k} u_h(T)|^{p-1} \max_T |S_{\lambda_k} u_h| \max_T |\eta_h|^{p-1} |\nabla \eta_h(T)|.\end{aligned} \tag{4.162}$$

Now, we use Hölder's inequality for sums and Lemma 2.23 in inequality (4.162).

$$
\begin{aligned}
L &\lesssim 2^{pk}\left(\sum_{T\in\mathcal{T}_h}|T||\nabla S_{\lambda_k}u_h|^p\max_T|\eta_h|^p\right)^{1-\frac{1}{p}}\left(\sum_{T\in\mathcal{T}_h}|T|\max_T|S_{\lambda_k}u_h|^p|\nabla\eta_h|^p\right)^{\frac{1}{p}}\\
&\lesssim 2^{pk}\left(\sum_{T\in\mathcal{T}_h}|T||\nabla S_{\lambda_k}u_h(T)|^p\left(\fint_T|\eta_h|\,\mathrm{d}x\right)^p\right)^{1-\frac{1}{p}}\\
&\quad\cdot\left(\sum_{T\in\mathcal{T}_h}|T|\left(\fint_T|S_{\lambda_k}u_h|\,\mathrm{d}x\right)^p|\nabla\eta_h(T)|^p\right)^{\frac{1}{p}}.
\end{aligned}
\tag{4.163}
$$

Then, we apply Jensen's inequality to inequality (4.163) and carry out the sums to get

$$
\begin{aligned}
L &\lesssim 2^{pk}\left(\sum_{T\in\mathcal{T}_h}|T||\nabla S_{\lambda_k}u_h(T)|^p\fint_T|\eta_h|^p\,\mathrm{d}x\right)^{1-\frac{1}{p}}\left(\sum_{T\in\mathcal{T}_h}|T|\fint_T|S_{\lambda_k}u_h|^p\,\mathrm{d}x|\nabla\eta_h(T)|^p\right)^{\frac{1}{p}}\\
&= 2^{pk}\left(\int_\Omega|\nabla S_{\lambda_k}u_h|^p|\eta_h|^p\,\mathrm{d}x\right)^{1-\frac{1}{p}}\left(\int_\Omega|S_{\lambda_k}u_h|^p|\nabla\eta_h|^p\,\mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned}
\tag{4.164}
$$

For R_I and R_{II} from equation (4.160), we proceed exactly as in inequalities (4.100) to (4.114) to get

$$
\begin{aligned}
R_I+R_{II} &\lesssim \varepsilon\int_\Omega\rho_h|\nabla S_{\lambda_{k+1}}u_h|^p\,\mathrm{d}x+\int_\Omega|S_{\lambda_{k+1}}u_h|^p|\nabla\eta_h|^p\,\mathrm{d}x\\
&\quad+C_\varepsilon\left(\|F\|^{p'}_{L^r(\operatorname{supp}\eta_h)}+\|f\|^{p'}_{L^q(\operatorname{supp}\eta_h)}\right)\left|\operatorname{supp}\eta_h\cap\operatorname{supp}S_{\lambda_{k+1}}u_h\right|.
\end{aligned}
\tag{4.165}
$$

We can now insert inequalities (4.164) and (4.165) into inequality (4.160) and absorb the ε-term into the left-hand side, which gives inequality (4.157) and therefore concludes the proof of the Lemma. □

Again, we will iterate the application of this inequality.

Lemma 4.30. *Let $2<p<n$. Under the assumptions of Theorem 4.25, we define η_k and λ_k as in Definition 4.20. Furthermore, assume that*

$$
\lambda_\infty^p \geq \max\left\{\left(\|f\|^{p'}_{L^q(\Omega(B(x_0,2R)))}+\|F\|^{p'}_{L^r(\Omega(B(x_0,2R)))}\right)R^{p'\delta},\ \fint_{\Omega(B(x_0,2R))}|u_h|^p\eta_0^p\,\mathrm{d}x\right\}.
\tag{4.166}
$$

Then, for every $m\in\mathbb{N}$, there are constants C, α_1, α_2, α_3, that only depend on m, p, n and *the shape-regularity constant* Γ*, but not on* k*, such that*

$$\begin{aligned}\fint_{\Omega(B(x_0,2R))} |\nabla S_{\lambda_{k+m}} u_h|^p |\eta_{k+m}|^p \,\mathrm{d}x &\leq C 2^{\alpha_1 k} \fint_{\Omega(B(x_0,2R))} R^{-p} |S_{\lambda_k} u_h|^p |\eta_k|^p \,\mathrm{d}x \\ &+ C 2^{\alpha_2 k} \left(\fint_{\Omega(B(x_0,2R))} R^{-p} |S_{\lambda_k} u_h|^p |\eta_k|^p \,\mathrm{d}x \right)^{1-\frac{p}{n}+p'\frac{\delta}{n}} \left(\frac{\lambda_\infty^p}{R^p} \right)^{\frac{p}{n}-p'\frac{\delta}{n}} \\ &+ C 2^{\alpha_3 k} \left(\fint_{\Omega(B(x_0,2R))} R^{-p} |S_{\lambda_k} u_h|^p |\eta_k|^p \,\mathrm{d}x \right)^{1-\frac{1}{p'^m}} \left(\frac{\lambda_\infty^p}{R^p} \right)^{\frac{1}{p'^m}} .\end{aligned} \tag{4.167}$$

Proof. We write $B := \Omega(B(x_0, 2R))$ and define the decreasing sequences

$$\begin{aligned} a_k &:= \fint_B R^{-p} |S_{\lambda_k} u_h|^p |\eta_k|^p \,\mathrm{d}x, \\ b_k &:= \fint_B |\nabla S_{\lambda_k} u_h|^p |\eta_k|^p \,\mathrm{d}x. \end{aligned} \tag{4.168}$$

With inequality (4.157), we find that

$$\begin{aligned} b_{k+1} &\lesssim 2^{pk} \left(\fint_B |S_{\lambda_k} u_h|^p |\nabla \eta_{k+1}|^p \,\mathrm{d}x \right)^{\frac{1}{p}} \left(\fint_B |\nabla S_{\lambda_k} u_h|^p \eta_{k+1}^p \,\mathrm{d}x \right)^{1-\frac{1}{p}} \\ &+ \fint_B |S_{\lambda_k} u_h|^p |\nabla \eta_{k+1}|^p \,\mathrm{d}x \\ &+ R^{-n} \left(\|f\|_{L^q(B)}^{p'} + \|F\|_{L^r(B)}^{p'} \right) \left| \operatorname{supp} \eta_{k+1} \cap \operatorname{supp} S_{\lambda_{k+1}} u_h \right|^{1-\frac{p}{n}+p'\frac{\delta}{n}} . \end{aligned} \tag{4.169}$$

The derivation of inequalities (4.141) and (4.142) does not depend on p. Therefore, we can insert both into inequality (4.169) to get

$$b_{k+1} \lesssim 2^{\alpha_1 k} a_k + 2^{\alpha_2 k} a_k^{1-\frac{p}{n}+p'\frac{\delta}{n}} \left(\frac{\lambda_\infty^p}{R^p} \right)^{\frac{p}{n}-p'\frac{\delta}{n}} + 2^{\alpha_3 k} a_k^{\frac{1}{p}} b_k^{\frac{1}{p'}}, \tag{4.170}$$

where α_1, α_2 and α_3 are generic positive constants that can change during the proof. From inequalities (4.136) to (4.139) we know that $b_k \lesssim \frac{\lambda_\infty^p}{R^p}$. Inserting this into inequality (4.170), we get the bound

$$b_{k+1} \lesssim 2^{\alpha_1 k} a_k + 2^{\alpha_2 k} a_k^{1-\frac{p}{n}+p'\frac{\delta}{n}} \left(\frac{\lambda_\infty^p}{R^p} \right)^{\frac{p}{n}-p'\frac{\delta}{n}} + 2^{\alpha_3 k} a_k^{\frac{1}{p}} \left(\frac{\lambda_\infty^p}{R^p} \right)^{\frac{1}{p'}}, \tag{4.171}$$

which serves as a base case to prove inequality (4.167) via induction. For the induction step, we assume that inequality (4.167) is true for some $m \in \mathbb{N}$. We then use inequality

(4.171) to get that

$$
\begin{aligned}
b_{k+m+1} &\lesssim 2^{\alpha_1 k} a_{k+m} + 2^{\alpha_2 k} a_{k+m}^{1-\frac{p}{n}+p'\frac{\delta}{n}} \left(\frac{\lambda_\infty^p}{R^p}\right)^{\frac{p}{n}-p'\frac{\delta}{n}} + 2^{\alpha_3 k} a_{k+m}^{\frac{1}{p}} b_{k+m}^{1-\frac{1}{p}} \\
&\lesssim 2^{\alpha_1 k} a_k + 2^{\alpha_2 k} a_k^{1-\frac{p}{n}+p'\frac{\delta}{n}} \left(\frac{\lambda_\infty^p}{R^p}\right)^{\frac{p}{n}-p'\frac{\delta}{n}} \\
&\quad + 2^{\alpha_3 k} a_k^{\frac{1}{p}} \left(a_k + a_k^{1-\frac{p}{n}+p'\frac{\delta}{n}} \left(\frac{\lambda_\infty^p}{R^p}\right)^{\frac{p}{n}-p'\frac{\delta}{n}} + a_k^{1-\frac{1}{p'^m}} \left(\frac{\lambda_\infty^p}{R^p}\right)^{\frac{1}{p'^m}} \right)^{\frac{1}{p'}}.
\end{aligned}
$$

From this, the equivalence of norms in $\mathbb{R}^3$ gives the bound

$$
\begin{aligned}
b_{k+m+1} &\lesssim 2^{\alpha_1 k} a_k + 2^{\alpha_2 k} a_k^{1-\frac{p}{n}+p'\frac{\delta}{n}} \left(\frac{\lambda_\infty^p}{R^p}\right)^{\frac{p}{n}-p'\frac{\delta}{n}} \\
&\quad + 2^{\alpha_3 k} \left(a_k + a_k^{\frac{1}{p}} a_k^{\left(1-\frac{n}{n}+p'\frac{\delta}{n}\right)\frac{1}{p'}} \left(\frac{\lambda_\infty^p}{R^p}\right)^{\left(\frac{p}{n}-p'\frac{\delta}{n}\right)\frac{1}{p'}} + a_k^{\frac{1}{p'}-\frac{1}{p'^{m+1}}+\frac{1}{p}} \left(\frac{\lambda_\infty^p}{R^p}\right)^{\frac{1}{p'^{m+1}}} \right).
\end{aligned}
\tag{4.172}
$$

Young's inequality yields

$$
a_k^{\frac{1}{p}} a_k^{\left(1-\frac{n}{n}+p'\frac{\delta}{n}\right)\frac{1}{p'}} \left(\frac{\lambda_\infty^p}{R^p}\right)^{\left(\frac{p}{n}-p'\frac{\delta}{n}\right)\frac{1}{p'}} \lesssim a_k + a_k^{1-\frac{n}{n}+p'\frac{\delta}{n}} \left(\frac{\lambda_\infty^p}{R^p}\right)^{\frac{p}{n}-p'\frac{\delta}{n}}. \tag{4.173}
$$

Furthermore, we have

$$
\frac{1}{p'} - \frac{1}{p'^{m+1}} + \frac{1}{p} = 1 - \frac{1}{p'^{m+1}}. \tag{4.174}
$$

Now, inserting inequality (4.173) and equation (4.174) into inequality (4.172) completes the induction and therefore proves the Lemma. □

This allows us to prove Theorem 4.25 for the degenerate case $2 < p < n$.

Proof of Thoerem 4.25 for $2 < p < n$. For simplicity, we write $B := \Omega(B(x_0, 2R))$ and use the scaling-invariant norms $|||g|||_s^s := \fint_B |g|^s \, \mathrm{d}x$. Furthermore assume that

$$
\lambda_\infty^p = d_1 \max \left\{ \left(\|f\|_{L^q(B)}^{p'} + \|F\|_{L^r(B)}^{p'} \right) R^{p'\delta}, \fint_B |u_h|^p \eta_0^p \, \mathrm{d}x \right\}
$$

for a constant $d_1 > 1$ that is to be chosen later. Define m_0, such that

$$
\frac{1}{p'^{m_0}} = \frac{p}{n} - \varepsilon \tag{4.175}
$$

for some $\varepsilon > 0$. We write

$$
a_k := \fint_B R^{-p} |S_{\lambda_{m_0 k}} u_h|^p \eta_{k m_0}^p \, \mathrm{d}x
$$

and estimate with Hölder's inequality and the Sobolev-embedding:

$$\begin{aligned}
a_{k+1} &= R^{-p}|||S_{\lambda_{(k+1)m_0}}u_h\eta_{(k+1)m_0}|||_p^p \\
&\lesssim R^{-p}|||S_{\lambda_{(k+1)m_0}}u_h\eta_{(k+1)m_0}|||_{p^\star}^p\left(\frac{|\operatorname{supp} S_{\lambda_{m_0(k+1)}}u_h \cap \operatorname{supp}\eta_{m_0(k+1)}|}{|B|}\right)^{\frac{p}{n}} \\
&\lesssim \left|\left|\left|\nabla\left(S_{\lambda_{(k+1)m_0}}u_h\eta_{(k+1)m_0}\right)\right|\right|\right|_p^p\left(\frac{|\operatorname{supp} S_{\lambda_{m_0(k+1)}}u_h \cap \operatorname{supp}\eta_{m_0(k+1)}|}{|B|}\right)^{\frac{p}{n}} \qquad (4.176)\\
&\lesssim \left|\left|\left|\left(\nabla S_{\lambda_{(k+1)m_0}}u_h\right)\eta_{(k+1)m_0}\right|\right|\right|_p^p\left(\frac{|\operatorname{supp} S_{\lambda_{m_0(k+1)}}u_h \cap \operatorname{supp}\eta_{m_0(k+1)}|}{|B|}\right)^{\frac{p}{n}} \\
&\quad + |||S_{\lambda_{(k+1)m_0}}u_h\nabla\left(\eta_{(k+1)m_0}\right)|||_p^p\left(\frac{|\operatorname{supp} S_{\lambda_{m_0(k+1)}}u_h \cap \operatorname{supp}\eta_{m_0(k+1)}|}{|B|}\right)^{\frac{p}{n}}.
\end{aligned}$$

The weak-type estimate from Lemma 4.22 yields analogously to inequality (4.153) that

$$\frac{|\operatorname{supp} S_{\lambda_{(k+1)m_0}}u_h \cap \operatorname{supp}\eta_{(k+1)m_0}|}{R^n} \lesssim 2^{pk}|||S_{\lambda_{km_0}}u_h\eta_{km_0}|||_p^p R^{-p}\frac{R^p}{\lambda_\infty^p} = 2^{pk}a_k\frac{R^p}{\lambda_\infty^p}. \qquad (4.177)$$

Analogously to inequality (4.154), we find that

$$|||S_{\lambda_{(k+1)m_0}}u_h\nabla\left(\eta_{(k+1)m_0}\right)|||_p^p \lesssim 2^{pk}a_k. \qquad (4.178)$$

Then, we can use inequality (4.167) from Lemma 4.30 and our choice of m_0 in equation (4.175) to get

$$\begin{aligned}
&|||\nabla\left(S_{\lambda_{(k+1)m_0}}u_h\right)\eta_{(k+1)m_0}|||_p^p \\
&\lesssim 2^{\alpha_1 k}a_k + 2^{\alpha_2 k}a_k^{1-\frac{p}{n}+p'\frac{\delta}{n}}\left(\frac{\lambda_\infty^p}{R^p}\right)^{\frac{p}{n}-p'\frac{\delta}{n}} + 2^{\alpha_3 k}a_k^{1-\frac{p}{n}+\varepsilon}\left(\frac{\lambda_\infty^p}{R^p}\right)^{\frac{p}{n}-\varepsilon}.
\end{aligned} \qquad (4.179)$$

This means that we can insert inequalities (4.177), (4.178) and (4.179) into inequality (4.176) to get

$$a_{k+1} \lesssim 2^{\beta_1}a_k\left(\frac{R^p}{\lambda_\infty^p}a_k\right)^{\frac{p}{n}} + 2^{\beta_2 k}a_k\left(\frac{R^p}{\lambda_\infty^p}a_k\right)^{p'\frac{\delta}{n}} + 2^{\beta_3 k}a_k\left(\frac{R^p}{\lambda_\infty^p}a_k\right)^{\varepsilon}. \qquad (4.180)$$

Furthermore, we recall that we also assumed

$$\lambda_\infty^p = d_1\max\left\{\left(\|f\|_{L^q(B)}^{p'} + \|F\|_{L^r(B)}^{p'}\right)R^{p'\delta}, \fint_B |u_h|^p\eta_0^p\,\mathrm{d}x\right\}.$$

Thus, we can use Corollary 3.32 to fix d_1 large enough to deduce $a_k \to 0$ because $\frac{\lambda_\infty^p}{R^p}$ is large enough in comparison to a_0. This implies that

$$\begin{aligned}
\sup_{\Omega'(B(x_0,R))}|u_h|^p \lesssim \lambda_\infty^p \lesssim &\fint_{\Omega(B(x_0,2R))}|u_h|^p\,\mathrm{d}x \\
&+ \left(\|f\|_{L^q(\Omega(B(x_0,2R)))0}^{p'} + \|F\|_{L^r(\Omega(B(x_0,2R)))}^{p'}\right)R^{p'\delta}.
\end{aligned}$$

and therefore proves Theorem 4.25 in the degenerate case $2 < p < n$. □

We have completed the proof of Theorem 4.25. We have shown a uniform L^∞-estimate for discrete solutions to $-\text{div}(|\nabla u|^{p-2}\nabla u) = f - \text{div}F$ for $1 < p < \infty$ on non-obtuse meshes. In the next chapter, we will briefly discuss future plans and a possible application of Theorem 3.25.

Chapter 5

Application to incompressible chemically reacting fluids

In this chapter, we will discuss the application of the theory that was developed in Chapter 3 to a more complex problem. We will look at a finite element approximation of generalized Navier–Stokes type system that describes steady flows of a chemically reacting incompressible non-Newtonian fluid. For example, this model is used to describe synovial fluids where there is a small concentration of hyaluronan. This concentration is small enough to describe the fluid as a single substance, but it significantly influences the properties of that fluid. The system was analysed in [12] and [13], while its finite element approximation was introduced in [43]. However, the authors had to restrict themselves to two dimensions in this paper. The case of three dimensions was tackled in [44]. Because of the lack of a discrete De Giorgi theory, the system was regularized in three dimensions. Then, the authors proved convergence of the finite element approximation of the regularized system and finally proved that the regularized system converges to the real system using a continuous De Giorgi estimate. If not indicated otherwise, we will follow the argumentation of [43]. Our main contribution here is that, by using the discrete De Giorgi theory we are able to avoid the two-stage passage to the limit from [44], and extend the results of [43] to three space dimensions.

5.1 The system of equations

First, we will introduce the system of equations. We will follow [43]. On a domain $\Omega \subset \mathbb{R}^n$, it is given by

$$\operatorname{div} u = 0, \tag{5.1}$$

$$\operatorname{div}(u \otimes u) - \operatorname{div}\left(S(c, Du)\right) = \nabla p + f, \tag{5.2}$$

$$\operatorname{div}(cu) - \operatorname{div}(q_c(c, \nabla c, Du)) = 0. \tag{5.3}$$

Here, $u : \Omega \to \mathbb{R}^n$ is the velocity field, $p : \Omega \to \mathbb{R}$ is the pressure field and $c : \Omega \to \mathbb{R}$ is the concentration of hyaluronan. We also define the symmetric part of the gradient $Du = \frac{1}{2}(\nabla u + \nabla u^\dagger)$. $S : \mathbb{R} \times \mathbb{R}^{n\times n}_{sym} \to \mathbb{R}^{n\times n}_{sym}$ denotes the stress tensor and $q_c : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}_{sym} \to \mathbb{R}^n$ denotes the diffusive flux. Equation (5.1) describes mass conservation in incompressible fluids, equation (5.2) is the momentum equation and the concentration equation (5.3) describes the diffusion of hyaluronan. Note that the dependence of the stress tensor on the concentration of hyluronan c and the dependence of the flux q_c on the velocity u couples the equations in a very complex way. Indeed, we will see that the space that u belongs to depends on c. Let $c_d \in W^{1,p}(\Omega)$ be a given positive function for $p > n$. We prescribe the following Dirichlet boundary conditions:

$$u|_{\partial\Omega} = 0, \tag{5.4}$$

$$c|_{\partial\Omega} = c_d|_{\partial\Omega}. \tag{5.5}$$

We assume that the matrix-valued function S satisfies the following conditions:

$$|S(s, A)| \lesssim |A|^{r(s)-1} + 1, \tag{5.6}$$

$$(S(s, A_1) - S(s, A_2)) \cdot (A_1 - A_2) \geq 0, \tag{5.7}$$

$$S(s, A) \cdot A \gtrsim |A|^{r(s)} + |S|^{r'(s)} - \gamma. \tag{5.8}$$

Here, r is a given Hölder-continuous function with $1 < r^- < r(s) < r^+ < \infty$ and r' is its Hölder conjugate that is given by $1 = \frac{1}{r(s)} + \frac{1}{r'(s)}$. Note that $\cdot$ indicates the (Frobenius) matrix scalar product or the usual vector scalar product depending on the context, but it will always be clear which scalar product is meant. Here, we can see that the main difficulty in analysing those equations is the coupling of the momentum equation and the diffusion equation via the variable exponent $r(s)$. If r was a constant, the estimates (not necessarily the equations, though) would decouple and the analysis would be rather straightforward. (See for example [9] for the analysis of a finite elements approximation of solutions to the p-Stokes system.) For the concentration flux, we assume that $q_c(s, a, A)$ is linear with respect to the second variable and satisfies

$$|q_c(s, a, A)| \lesssim |a|, \tag{5.9}$$

$$q_c(s, a, A) \cdot a \gtrsim |a|^2. \tag{5.10}$$

In [43], the authors give

$$S(s, B) = \nu(s, B)B, \qquad q_c(s, a, A) = K(s, A)a$$

as prototypical examples with $\nu(s,B) \sim \left(\kappa_1 + \kappa_2 |B|^2\right)^{\frac{r(s)-2}{2}}$, where κ_1 and κ_2 are positive constants and K is uniformly elliptic and bounded. Because of the assumptions of Theorem 3.30, we will assume here that $K(x,A) = k(x,A)$ is a scalar function.

Inequality (5.6) suggests that we have to use variable-exponent spaces for the velocity field u. We will now give the precise definition of those spaces. Following [43], we will introduce these spaces spaces and their properties without proof.

Definition 5.1. *Let $r : \Omega \to (1,\infty)$ be a measurable function. We define space $L^{r(\cdot)}$ by*

$$L^{r(\cdot)}(\Omega) = \left\{ u \in L^1_{loc}(\Omega) : \int_\Omega |u(x)|^{r(x)} \,\mathrm{d}x \right\} \tag{5.11}$$

and equip it with the standard Luxemburg-norm

$$\|u\|_{L^{r(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{r(x)} \mathrm{d}x \leq 1 \right\}. \tag{5.12}$$

This is a Banach space. Its dual space is given by $L^{r'(\cdot)}(\Omega)$ with $1 = \frac{1}{r(x)} + \frac{1}{r'(x)}$. We also define the respective Sobolev spaces via

$$W^{1,r(\cdot)}(\Omega) = \left\{ u \in L^{r(\cdot)}(\Omega) \cap W^{1,1}_{loc}(\Omega) : |\nabla u| \in L^{r(\cdot)} \right\} \tag{5.13}$$

equipped with the norm

$$\|u\|_{W^{1,r(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{\nabla u}{\lambda} \right|^{r(x)} + \left| \frac{u}{\lambda} \right|^{r(x)} \mathrm{d}x \leq 1 \right\}. \tag{5.14}$$

For $r \equiv p$, those spaces are equal to $L^p(\Omega)$ and $W^{1,p}(\Omega)$. In analogy with classical spaces, we will also need

$$W^{1,r(\cdot)}_0(\Omega) := \left\{ u \in W^{1,r(\cdot)}(\Omega) : u = 0 \text{ on } \partial\Omega \right\}, \tag{5.15}$$

$$W^{1,r(\cdot)}_{0,\mathrm{div}}(\Omega) := \left\{ u \in W^{1,r(\cdot)}_0(\Omega) : \mathrm{div} u = 0 \right\}, \tag{5.16}$$

$$L^{r(\cdot)}_0(\Omega) := \left\{ u \in L^{r(\cdot)}(\Omega) : \int_\Omega u(x) \,\mathrm{d}x = 0 \right\}. \tag{5.17}$$

We will also need the log-Hölder space $C_{\log}(\overline{\Omega})$ of functions $r : \overline{\Omega} \to \mathbb{R}$ that satisfy

$$|r(x) - r(x)| \leq \frac{C_{\log}(r)}{-\log|x-y|} \tag{5.18}$$

for some constant $C_{\log}(r)$. Obviously, $C_{\log}(\overline{\Omega}) \subset C^\alpha(\overline{\Omega})$ for all $\alpha > 0$.

We will collect a few properties of those spaces in the following proposition from [43][Proposition 5.2]. Note that the assumption that Ω is a bounded Lipschitz domain is sufficient, but not in all cases necessary. For example, property (a) only requires Ω to be an *extension domain* (see [23][Theorem 9.1.7]).

Proposition 5.2. *Let Ω be a bounded Lipschitz domain. For $r \in C_{\log}(\overline{\Omega})$ with $1 < r^- \leq r \leq r^+ < \infty$, we have the following properties:*

(a) Density: $\overline{C^\infty\left(\overline{\Omega}\right)}^{\|\cdot\|_{W^{1,r(\cdot)}(\Omega)}} = W^{1,r(\cdot)}(\Omega)$.

(b) Embedding: If $r^+ < n$, we have $W^{1,r(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ for $q(x) \leq r^\star(x)$ with $\frac{1}{r(x)} - \frac{1}{n} = \frac{1}{r^\star(x)}$ for all $x \in \overline{\Omega}$. The embedding is compact, if $q(x) < r^\star(x)$ for all $x \in \overline{\Omega}$.

(c) Hölder-type inequality:

$$\|uv\|_{L^{r(\cdot)}(\Omega)} \leq 2\|u\|_{L^{r_1(\cdot)}(\Omega)}\|v\|_{L^{r_2(\cdot)}(\Omega)}$$

if $\frac{1}{r(x)} = \frac{1}{r_1(x)} + \frac{1}{r_2(x)}$ for all $x \in \Omega$.

(d) Poincaré's inequality:

$$\|u\|_{L^{r(\cdot)}(\Omega)} \leq C(\Omega, C_{\log}(r))\mathrm{diam}(\Omega)\|\nabla u\|_{L^{r(\cdot)}(\Omega)}$$

for all $u \in W_0^{1,r(\cdot)}(\Omega)$.

(e) Korn's inequality:

$$\|\nabla u\|_{L^{r(\cdot)}(\Omega)} \leq C(\Omega, C_{\log}(r))\|Du\|_{L^{r(\cdot)}(\Omega)}$$

for all $u \in W^{1,r(\cdot)}(\Omega;\mathbb{R}^n)$.

With those definitions in place, we can define the problem that we want to solve. Note again that we use the symbol $\cdot$ for the scalar product of vectors or matrices, depending on the context.

Definition 5.3. *Let $r : \mathbb{R} \to [r^-, r^+]$ be a Hölder-continuous function with $r^- > 1$ and $r^+ < \infty$ and let $\Omega \subset \mathbb{R}^n$ be a domain. Furthermore, let $f \in \left(W_0^{1,r^-}(\Omega;\mathbb{R}^n)\right)'$ and $c_d \in W^{1,p}(\Omega)$ be given functions for $p > n$. We want to find $c \in W^{1,2}(\Omega) \cap C^\alpha(\overline{\Omega})$ for some $\alpha > 0$ with $c|_{\partial\Omega} = c_d|_{\partial\Omega}$, $u \in W_0^{1,r(c)}(\Omega;\mathbb{R}^n)$, and $p \in L_0^{(r(c))'}(\Omega)$, such that*

$$\int_\Omega S(c, Du) \cdot \nabla\psi - (u \otimes u) \cdot \nabla\psi \,\mathrm{d}x - \langle \mathrm{div}\psi, p\rangle = \langle f, \psi\rangle \qquad \forall \psi \in W_0^{1,\infty}(\Omega;\mathbb{R}^n), \quad (5.19)$$

$$\int_\Omega \nu \,\mathrm{div}u \,\mathrm{d}x = 0 \qquad \forall \nu \in L_0^{r(c)'}(\Omega), \quad (5.20)$$

$$\int_\Omega q_c(c, \nabla c, Du) \cdot \nabla\varphi - cu \cdot \nabla\varphi = 0 \qquad \forall \varphi \in W_0^{1,2}(\Omega). \quad (5.21)$$

With the help of the inf-sup condition

$$\sup_{0 \neq v \in W_0^{1,r(c)}(\Omega,\mathbb{R}^n), \|v\|_{W^{1,r(c)}(\Omega)} \leq 1} \langle \mathrm{div}v, q\rangle \gtrsim \|q\|_{L^{r(c)'}(\Omega)} \qquad \forall q \in L_0^{r(c)}(\Omega)$$

that is proved in [43][Proposition 2.5] we can guarantee the existence of a pressure and restate the problem in an equivalent way where we only allow divergence-free test functions for the momentum equation.

Definition 5.4. *Let $r : \mathbb{R} \to [r^-, r^+]$ be a Hölder-continuous function with $r^- > 1$ and $r^+ < \infty$ and let $\Omega \subset \mathbb{R}^n$ be a domain. Furthermore, let $f \in \left(W_0^{1,r^-}(\Omega;\mathbb{R}^n)\right)'$ and $c_d \in W^{1,p}(\Omega)$ be given functions for $p > n$. We want to find $c \in W^{1,2}(\Omega) \cap C^\alpha(\overline{\Omega})$ for some $\alpha > 0$ with $c|_{\partial\Omega} = c_d|_{\partial\Omega}$, and $u \in W^{1,r(c)}_{0,\mathrm{div}}(\Omega;\mathbb{R}^n)$ such that*

$$\int_\Omega S(c, Du) \cdot \nabla\psi - (u \otimes u) \cdot \nabla\psi \,\mathrm{d}x = \langle f, \psi\rangle \qquad \forall \psi \in W^{1,\infty}_{0,\mathrm{div}}(\Omega;\mathbb{R}^n), \tag{5.22}$$

$$\int_\Omega q_c(c, \nabla c, Du) \cdot \nabla\varphi - cu \cdot \nabla\varphi = 0 \qquad \forall \varphi \in W_0^{1,2}(\Omega;\mathbb{R}). \tag{5.23}$$

The existence of solutions to this problem was shown in [12] for $r^- > \max\left\{\frac{2n}{n-2}, \frac{n}{2}\right\}$. This threshold was improved in [13] to $r^- > \frac{n}{2}$. Note that $r^- > \frac{n}{2}$ is exactly the bound that guarantees $c \in C^\alpha(\overline{\Omega})$ via the classical De Giorgi theory. In the following section, we will define finite element spaces to formulate the finite element approximation of the problem stated in equations (5.22) and (5.23) and prove the existence of solutions to the discrete problem.

5.2 The finite element approximation

In [43] the authors are very general in their definition of finite element spaces. For the space for the approximation of the concentration, we will have to restrict ourselves to piecewise affine functions. By $\mathcal{T}_h^m$, we will denote a sequence of shape-regular triangulations of the polyhedral Lipschitz domain $\Omega \subset \mathbb{R}^n$.

We define the spaces $X_h^m \subset C(\overline{\Omega};\mathbb{R}^n)$ and $Q_h^m \subset L^\infty(\Omega;\mathbb{R})$ as finite-dimensional subspaces. We also assume that on each simplex $T \in \mathcal{T}_h^m$, functions in X_h^m are in $W^{1,\infty}(T)$. Furthermore, we assume that X_h^m has a locally supported basis, which means that if $\operatorname{supp}\psi \cap T \neq \emptyset$ for some basis function ψ and $T \in \mathcal{T}_h^m$, we have $\operatorname{supp}\psi \subset \overline{\Omega(T)}$. For P_h^m we assume that the basis consists of piecewise polynomials. For each basis function p, we require $\operatorname{supp} p = T$ for some $T \in \mathcal{T}_h^m$. As before, we will write $X_{h,0}^m$ for the space of all X_h^m-functions that vanish at the boundary of Ω. Furthermore, we need the discretely divergence-free space $X_{h,\mathrm{div}}^m$ that is defined as

$$X_{h,\mathrm{div}}^m := \{v_h \in X_{h,0}^m : \langle \mathrm{div} v_h, q\rangle = 0 \quad \forall Q_h^m\}. \tag{5.24}$$

We will also need the subspace of Q_h^m of functions that vanish in the mean, i.e.

$$Q_{h,0}^m := \left\{q_h \in Q_h^m : \int_\Omega q_h \,\mathrm{d}x = 0\right\}. \tag{5.25}$$

Furthermore, we will need the following approximability assumptions:

$$\inf_{v_h \in X_{h,0}^m} \|v - v_h\|_{W^{1,s}(\Omega)} \to 0 \quad \forall v_h \in W_0^{1,s} \text{ as } m \to \infty, \tag{5.26}$$

$$\inf_{q_h \in Q_h^m} \|q - q_h\|_{L^s(\Omega)} \to 0 \quad \forall q_h \in L_0^s \text{ as } m \to \infty. \tag{5.27}$$

For $X_{h,0}^m$, we need a projection $\Pi_{h,\mathrm{div}}^m : W_0^{1,1} \to X_h^m$ that preserves the discrete divergence, i.e.,

$$\langle \mathrm{div} v, q_h \rangle = \langle \mathrm{div} \Pi_{h,\mathrm{div}}^m v, q_h \rangle \quad \forall v \in W_0^{1,1}, q_h \in Q_h^m, \tag{5.28}$$

and is locally $W^{1,1}$-stable, i.e.,

$$\fint_T |\Pi_{h,\mathrm{div}}^m v| + h_T |\nabla \Pi_{h,\mathrm{div}}^m v| \,\mathrm{d}x \lesssim \fint_{\Omega(T)} |v| + h_T |\nabla v| \,\mathrm{d}x \tag{5.29}$$

for all $T \in \mathcal{T}_h^m$ and $v \in W^{1,1}$. Note that this also implies $W^{1,s}$-stability for all $s \in [1, \infty]$. Using this, we also find

$$\|v - \Pi_{h,\mathrm{div}}^m v\|_{W^{1,s}(\Omega)} \leq \|v - z\|_{W^{1,s}(\Omega)} + \|\Pi_{h,\mathrm{div}}^m v - z\|_{W^{1,s}(\Omega)} \lesssim \|v - z\|_{W^{1,s}(\Omega)}$$

for all $z \in X_h^m$ and $s \in [1, \infty]$, which implies that

$$\|v - \Pi_{h\,\mathrm{div}}^m v\|_{W^{1,s}(\Omega)} \lesssim \inf_{z \in X_h^m} \|v - z\|_{W^{1,s}(\Omega)} \to 0 \tag{5.30}$$

as $m \to \infty$ by the approximability assumption from equation (5.26). With the help of the key-estimate (see [29] for the proof and more details)

$$\left(\fint_Q |f(y)| \,\mathrm{d}y \right)^{r(x)} \lesssim \fint_Q |f(y)|^{r(y)} \,\mathrm{d}y + |Q|^m \tag{5.31}$$

for every cube or ball $Q \subset \mathbb{R}^n$ with $|Q| \leq 1$ and

$$\fint_Q |f| \,\mathrm{d}x \leq |Q|^{-m},$$

we also find a stability result in the variable exponent space. (The implicit constant depends on $|r|_{C_{\log}(\overline{\Omega})}$). To this end use the equivalence of norms in finite-dimensional spaces, a scaling argument, the $W^{1,1}$-stability of $\Pi^m_{h,\mathrm{div}}$ and the key estimate (5.31) to find

$$\begin{aligned}\int_T |\nabla\Pi^m_{h,\mathrm{div}} f(x)|^{r(x)}\,\mathrm{d}x &\lesssim \int_T \left(\fint_T |\Pi^m_{h,\mathrm{div}}\nabla f(y)|\,\mathrm{d}y\right)^{r(x)}\mathrm{d}x\\ &\lesssim \int_T \left(\fint_{\Omega(T)} |\nabla f(y)|\,\mathrm{d}y\right)^{r(x)}\mathrm{d}x\\ &\lesssim \int_T \left(\fint_{\Omega(T)} |\nabla f(y)|^{r(y)}\,\mathrm{d}y + h_T^n\right)\mathrm{d}x\\ &\lesssim \int_{\Omega(T)} |\nabla f(x)|^{r(x)}\,\mathrm{d}x + |T|\max_{T\in\mathcal{T}_h} h_T^n.\end{aligned} \tag{5.32}$$

Now, we can just add up inequality (5.32) over all $T \in \mathcal{T}_h^m$ and note the finite overlap of the patches $\Omega(T)$ to find

$$\int_\Omega \left|\nabla\Pi^m_{h,\mathrm{div}} f(x)\right|^{r(x)}\mathrm{d}x \lesssim \int_\Omega |\nabla f(x)|^{r(x)}\,\mathrm{d}x + |\Omega|\max_{T\in\mathcal{T}_h} h_T^n. \tag{5.33}$$

Analogously, we need a projection $\Pi^m_{h,Q} : L^1(\Omega) \to Q_h^m$ that satisfies a stability condition:

$$\fint_T |\Pi^m_{h,Q} q|\,\mathrm{d}x \lesssim \fint_{\Omega(T)} |q|\,\mathrm{d}x. \tag{5.34}$$

Similarly to $\Pi^m_{h,\mathrm{div}}$, this implies $L^s(\Omega)$-stability and $\|\Pi^m_{h,Q} v - v\|_{L^s(\Omega)} \to 0$. The existence of the projection $\Pi^m_{h,\mathrm{div}}$ of course depends on the choice of the discrete velocity and pressure spaces. For example, one can use continuous piecewise quadratic functions for the discrete velocity space X_h^m and discontinuous piecewise constants for the discrete pressure space Q_h^m. For more details, see for example [9].

In [43], a similar property is assumed for the discrete concentration space. We will have to use the space of piecewise affine functions to apply the theory from Chapter 3. In this chapter, we will again write V_h^m and $V_{h,0}^m$ for this space. Because of inequalities (2.25) and (2.26), the same stability properties hold for the projections onto this space.

We are now able to state the finite element approximation of the problem given in equations (5.19), (5.20) and (5.21). Note that we enforce the skew-symmetry of the convective term in the momentum equation because u_h is no longer point-wise divergence-free.

Definition 5.5. *Given a sequence of triangulations $\mathcal{T}_h^m$ with respective finite element spaces X_h^m, P_h^m and V_h^m and $c_d \in W^{1,p}(\Omega)$ for $p > n$, then $(u_h^m, p_h^m, c_h^m) \in X_{h,0} \times Q_{h,0}^m \times V_h^m$ is a* *finite element solution to equations* (5.19) *to* (5.21), *if*

$$\frac{1}{2}\int_\Omega ((u_h^m \otimes v_h)\cdot \nabla u_h^m - (u_h^m \otimes u_h^m)\cdot \nabla v_h)\,\mathrm{d}x \\ + \int_\Omega S(c_h^m, Du_h^m)\cdot Dv_h \,\mathrm{d}x - \langle \mathrm{div} v_h, p_h^m\rangle = \langle f, v_h\rangle \quad \forall v_h \in X_{h,0}^m, \tag{5.35}$$

$$\int_\Omega (\mathrm{div} u_h^m)\, q_h = 0 \qquad \forall q_h \in P_h^m, \tag{5.36}$$

$$\int_\Omega q_c(c_h^m, \nabla c_h^m, Du_h^m)\cdot \nabla\varphi_h - \frac{1}{2}(c_h^m u_h^m \cdot \nabla \varphi_h - u_h^m \cdot \nabla c_h^m \varphi_h)\,\mathrm{d}x = 0 \qquad \forall \varphi_h \in V_{h,0}^m \tag{5.37}$$

with $c_h^m = \Pi_{h,V}^m c_d$ *on* $\partial\Omega$.

Analogously to the continuous case, we want to simplify this problem by allowing only discretely divergence-free test functions. The discrete inf-sup condition

$$\sup_{0\neq v_h \in X_{h,0}^m, \|v_h\|_{W^{1,r^m}(\Omega)}\leq 1} \langle \mathrm{div} v_h, q_h\rangle \gtrsim \|q_h\|_{L^{(r^m)'}(\Omega)} \qquad \forall q_h \in Q_{h,0}^m \tag{5.38}$$

if $r^m \to r$ in $C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$ that was proved in Proposition 3.8 of [43] guarantees the existence of a discrete pressure p_h^m in this case. Furthermore, we have $S(c, Du)\cdot \nabla v = S(c, Du)\cdot Dv$.

We get the following set of equations.

Definition 5.6. *Given a sequence of triangulations* $\mathcal{T}_h^m$ *with corresponding finite element spaces* X_h^m, Q_h^m *and* V_h^m *and* $c_d \in W^{1,p}(\Omega)$ *for* $p > n$, *we say that* $(u_h^m, c_h^m) \in X_{h,0,\mathrm{div}}^m \times V_h^m$ *is a finite element solution to equations* (5.22) *and* (5.23), *if*

$$\frac{1}{2}\int_\Omega ((u_h^m \otimes v_h)\cdot \nabla u_h^m - (u_h^m \otimes u_h^m)\cdot \nabla v_h)\,\mathrm{d}x \\ + \int_\Omega S(c_h^m, Du_h^m)\cdot Dv_h \,\mathrm{d}x = \langle f, v_h\rangle \quad \forall v_h \in X_{h,0,\mathrm{div}}^m, \tag{5.39}$$

$$\int_\Omega q_c(c_h^m, \nabla c_h^m, Du_h^m)\cdot \nabla\varphi_h - \frac{1}{2}(c_h^m u_h^m \cdot \nabla \varphi_h - u_h^m \cdot \nabla c_h^m \varphi_h)\,\mathrm{d}x = 0 \qquad \forall \varphi_h \in V_{h,0}^m \tag{5.40}$$

with $c_h^m = \Pi_{h,V}^m c_d$ *on* $\partial\Omega$.

In [43], the existence of solutions to equations (5.39) and (5.40) is proved with the help of the following lemma that is a consequence of Brouwer's fixed point theorem and given in [34][Chapter 9].

Lemma 5.7. *Given a continuous function* $f : \mathbb{R}^M \to \mathbb{R}^M$ *that satisfies*

$$\exists R > 0 : \forall \left(x \in \mathbb{R}^M : |x| = r \right) : f(x) \cdot x \geq 0,$$

there exists an $x \in B(0, r)$ *such that* $f(x) = 0$.

Following the arguments of [43], prove the existence of solutions to equations (5.19) to (5.21) and to equations (5.39) and (5.40).

Lemma 5.8. *Given a sequence of triangulations* $\mathcal{T}_h^m$ *with respective finite element spaces* X_h^m, P_h^m *and* V_h^m *and* $c_d \in W^{1,p}(\Omega)$ *for* $p > n$, *then there exist sequences* $(u_h^m, p_h^m, c_h^m) \in X_{h,0} \times Q_{h,0}^m \times V_h^m$ *and* $(u_h^m, c_h^m) \in X_{h,0,\mathrm{div}}^m \times V_h^m$ *that are finite element solutions to equations (5.19) to (5.21) in the sense of Definition 5.5 or to equations (5.22) and (5.23) in the sense of Definition 5.6, respectively.*

Proof. We follow the arguments of [43]. First, we show the existence of solutions to equations (5.22) and (5.23). Let us use L^2-orthonormal bases $\{x_i\}_{i=1}^N$ and $\{v_i\}_{i=1}^M$ of $X_{h,0,\mathrm{div}}^m$ and $V_{h,0}^m$, respectively. For simplicity, we write $c_d^m = \Pi_{h,V}^m c_d$. The solutions u_h^m and c_h^m will have the form

$$u_h^m = \sum_{i=1}^N \alpha_i x_i \qquad c_h^m = \sum_{i=1}^M \beta_i v_i + c_d^m.$$

It is enough that equations (5.39) and (5.40) are satisfied for any basis function as test function, i.e.,

$$\frac{1}{2} \int_\Omega \left((u_h^m \otimes u_h^m) \cdot \nabla x_i - (u_h^m \otimes x_i) \cdot \nabla u_h^m \right) \mathrm{d}x + \int_\Omega S(c_h^m, Du_h^m) \cdot \nabla x_i \, \mathrm{d}x = \langle f, x_i \rangle \quad \forall i \leq N, \tag{5.41}$$

$$\int_\Omega q_c(c_h^m, \nabla c_h^m, Du_h^m) \cdot \nabla v_i - \frac{1}{2}(c_h^m u_h^m \cdot \nabla v_j - u_h^m \cdot \nabla c_h^m v_j) \, \mathrm{d}x = 0 \qquad \forall j \leq M. \tag{5.42}$$

The strategy is the following: we will start with $c_1^m = c_d^m$ and prove the existence of a solution u_1^m to equation (5.41) for $c_h^m = c_1^m$. We will then prove the existence of a solution c_2^m to equation (5.42) for $u_h^m = u_1^m$. This process can then be iterated and gives rise to the sequences u_k^m and c_k^m. Finally, we will prove that those sequences converge to solutions to equations (5.39) and (5.40).

For simplicity, we write $\alpha = (\alpha_1, \ldots, \alpha_N) \subset \mathbb{R}^N$ and for a given α, we write $\tilde{u}(\alpha) = \sum_{i=1}^N \alpha_i x_i$ and define the function $A : \mathbb{R}^N \to \mathbb{R}^N$ via

$$A(\alpha) := (A_1(\alpha), \ldots, A_N(\alpha))$$

where we define

$$A_j(\alpha) := \int_\Omega S_c(c_1^m, D\tilde{u}(\alpha)) \cdot \nabla x_j \\ + \frac{1}{2}\left((\tilde{u}(\alpha) \otimes \tilde{u}(\alpha)) \cdot \nabla x_j - (\tilde{u}(\alpha) \otimes x_j) \cdot \nabla \tilde{u}(\alpha)\right) \mathrm{d}x - \langle f, x_j \rangle.$$

Note that this expression is linear in the x_j entries. This means that if we take the scalar product $A(\alpha)\cdot\alpha$, the skew-symmetric part vanishes. Using equation (5.8) and the embedding from Proposition 5.2 and $r > r^-$, we get

$$\begin{aligned} A(\alpha) \cdot \alpha &= \int_\Omega S_c(D\tilde{u}(\alpha), c_1^m) \cdot \nabla \tilde{u}(\alpha) \, \mathrm{d}x - \langle f, \tilde{u}(\alpha) \rangle \\ &\geq C_1 \int_\Omega |D\tilde{u}(\alpha)|^{r \circ c_1^m} \, \mathrm{d}x - C_2 - |\langle f, \tilde{u}(\alpha) \rangle| \\ &\geq C_1 \int_\Omega |D\tilde{u}(\alpha)|^{r^-} \, \mathrm{d}x - C_2 - \|f\|_{\left(W_0^{1,r^-}(\Omega)\right)'} \|\tilde{u}(\alpha)\|_{W_0^{1,r^-}(\Omega)} \\ &\geq C_1 \|\tilde{u}(\alpha)\|_{W_0^{1,r^-}(\Omega)}^{r^-} - C_2 - C(\varepsilon)\|f\|_{\left(W_0^{1,r^-}(\Omega)\right)'}^{(r^-)'} - \varepsilon \|\tilde{u}(\alpha)\|_{W_0^{1,r^-}(\Omega)}^{r^-} \\ &\geq (C_1 - \varepsilon)\|\tilde{u}(\alpha)\|_{L^2(\Omega)}^{r^-} - C_3, \end{aligned}$$

where we have also used the Sobolev embedding theorem and $r^- > \frac{n}{2}$. Recall that we have chosen the basis $\{x_i\}_{i=1}^M$ to be L^2-orthonormal. Thus, we have $\|\tilde{u}(\alpha)\|_{L^2(\Omega)} = |\alpha|$. Therefore, we get $A(\alpha) \cdot \alpha \geq 0$ for $|\alpha| \geq \frac{C_3}{C_1 - \varepsilon}$. This means that we can apply Lemma 5.7 and deduce that there exists an $\bar{\alpha} \in \mathbb{R}^M$ such that $A(\bar{\alpha}) = 0$. Therefore, $\tilde{u}(\bar{\alpha}) =: u_1^m$ satisfies equation (5.39) for $c_h^m = c_d^m$.

Before we can proceed with the iteration, we have to derive a bound on the $L^{r^-}(\Omega)$-norm of Du_1^m and therefore, by Korn's inequality, the $W^{1,r^-}(\Omega)$-norm of u_1^m. We test equation (5.39) against $v_h = u_1^m$ and note that the skew-symmetric part vanishes again to get

$$\int_\Omega S(c_1^m, Du_1^m) \cdot \nabla u_1^m \, \mathrm{d}x = \langle f, u_1^m \rangle.$$

Then, the coercivity of S that was given in equation (5.8) yields

$$\int_\Omega |Du_1^m|^{r(c_1^m)} \, \mathrm{d}x - C \leq \|f\|_{\left(W_0^{1,r^-}(\Omega)\right)'} \|u_1^m\|_{W^{1,r^-}(\Omega)}.$$

Now, we can apply Young's inequality to the right-hand side. Furthermore, note that the Hölder-type inequality from Proposition 5.2 implies that $L^{r(\cdot)}(\Omega) \hookrightarrow L^{r^-}(\Omega)$. This gives

$$\|Du_1^m\|_{L^{r^-}(\Omega)}^{r^-} \leq C(\varepsilon)\|f\|_{\left(W_0^{1,r^-}(\Omega)\right)'}^{(r^-)'} + \varepsilon \|u_1^m\|_{W^{1,r^-}(\Omega)}^{r^-} + C_2.$$

This means that we can use Korn's inequality and absorb $\varepsilon\|u_1^m\|_{W^{1,r^-}(\Omega)}^{r^-}$ into the left-hand side to get

$$\|u_1^m\|_{W^{1,r^-}(\Omega)} \leq C. \tag{5.43}$$

Note that C is independent of m.

For the iteration scheme, we will now take u_1^m and will prove the existence of a c_2^m that satisfies equation (5.40) for $u_h^m = u_1^m$. Recall that we defined $\{v_i\}_{i=1}^M$ to be an orthonormal basis for $V_{h,0}^m$. For any $\beta \in \mathbb{R}^M$, we write $\beta = (\beta_1, \dots, \beta_M)$ and $\tilde{c}(\beta) = c_d^m + \sum_{i=1}^M \beta_i v_i$. We define the continuous function $B : \mathbb{R}^M \to \mathbb{R}^M$ via

$$B(\beta) := (B_1(\beta), \dots, B_M(\beta)) \tag{5.44}$$

with

$$B_i(\beta) := \int_\Omega q_c(\tilde{c}(\beta), \nabla\tilde{c}(\beta), Du_1^m) \cdot \nabla v_i - \frac{1}{2}(\tilde{c}(\beta)u_1^m \cdot \nabla v_i - \nabla\tilde{c}(\beta) \cdot u_1^m v_i)\,\mathrm{d}x. \tag{5.45}$$

For the scalar product $B(\beta) \cdot \beta$, we write

$$\begin{aligned} B(\beta) \cdot \beta = &\int_\Omega q_c(\tilde{c}(\beta), \nabla\tilde{c}(\beta), Du_1^m) \cdot \nabla(\tilde{c}(\beta) - c_d^m)\,\mathrm{d}x \\ &- \frac{1}{2}\int_\Omega \tilde{c}(\beta)u_1^m \cdot \nabla(\tilde{c}(\beta) - c_d^m)\,\mathrm{d}x \\ &+ \frac{1}{2}\int_\Omega \nabla\tilde{c}(\beta) \cdot u_1^m(\tilde{c}(\beta) - c_d^m)\,\mathrm{d}x. \end{aligned} \tag{5.46}$$

The terms in the skew-symmetric convective term that contain $\tilde{c}(\beta)$ twice will cancel out, which means that equation (5.46) simplifies to

$$\begin{aligned} B(\beta) \cdot \beta = &\int_\Omega q_c(\tilde{c}(\beta), \nabla\tilde{c}(\beta), Du_1^m) \cdot \nabla(\tilde{c}(\beta) - c_d^m)\,\mathrm{d}x \\ &+ \frac{1}{2}\int_\Omega \tilde{c}(\beta)u_1^m \cdot \nabla c_d^m - \nabla\tilde{c}(\beta) \cdot u_1^m c_d^m\,\mathrm{d}x. \end{aligned} \tag{5.47}$$

We can add a zero in the second term of the right-hand side of equation (5.47) to get

$$\begin{aligned} B(\beta) \cdot \beta = &\int_\Omega q_c(\tilde{c}(\beta), \nabla\tilde{c}(\beta), Du_1^m) \cdot \nabla(\tilde{c}(\beta) - c_d^m)\,\mathrm{d}x \\ &+ \frac{1}{2}\int_\Omega (\tilde{c}(\beta) - c_d^m)u_1^m \cdot \nabla c_d^m - \nabla(\tilde{c}(\beta) - c_d) \cdot u_1^m c_d^m\,\mathrm{d}x \\ =\!: &\; I + II + III. \end{aligned} \tag{5.48}$$

Note that q_c is linear in its second variable. We estimate I from equation (5.48) with the help of the coercivity (5.10) and boundedness (5.9) of q_c. Hence,

$$
\begin{aligned}
I &= \int_\Omega q_c(\tilde{c}(\beta), \nabla(\tilde{c}(\beta) - c_d^m), Du_1^m) \cdot \nabla(\tilde{c}(\beta) - c_d^m)\,\mathrm{d}x \\
&\quad + \int_\Omega q_c(\tilde{c}(\beta), \nabla c_d^m, Du_1^m) \cdot \nabla(\tilde{c}(\beta) - c_d^m)\,\mathrm{d}x \\
&\geq C_1 \|\nabla(\tilde{c}(\beta) - c_d^m)\|_{L^2(\Omega)}^2 - C_2 \int_\Omega |\nabla c_d||\nabla(\tilde{c}(\beta) - c_d^m)|\,\mathrm{d}x \\
&\geq C_1 \|\nabla(\tilde{c}(\beta) - c_d^m)\|_{L^2(\Omega)}^2 - \varepsilon \|\nabla(\tilde{c}(\beta) - c_d^m)\|_{L^2(\Omega)}^2 - C(\varepsilon)\|\nabla c_d^m\|_{L^2(\Omega)}^2 \\
&\geq (C_1 - \varepsilon)\|\tilde{c}(\beta) - c_d^m\|_{L^2(\Omega)}^2 - C,
\end{aligned}
\tag{5.49}
$$

where we have also used Young's inequality and the uniform boundedness of $\|\nabla c_d\|_{L^2(\Omega)}$.

For II from equation (5.48), we choose $t > 2$ such that $\frac{1}{2} - \frac{1}{t} \leq \frac{1}{(r^-)^\star}$. This means that $\frac{1}{t} + \frac{1}{(r^-)^\star} + \frac{1}{2} \geq 1$ and $t \geq 2^\star$. Therefore, we can use the Sobolev embedding theorem and Young's inequality to estimate II:

$$
\begin{aligned}
2|II| &= \left| \int_\Omega (\tilde{c}(\beta) - c_d^m) u_1^m \cdot \nabla c_d^m \,\mathrm{d}x \right| \\
&\lesssim \|\tilde{c}(\beta) - c_d^m\|_{L^t(\Omega)} \|u_1^m\|_{L^{(r^-)^\star}(\Omega)} \|\nabla c_d^m\|_{L^2(\Omega)} \\
&\lesssim \|\nabla(\tilde{c}(\beta) - c_d^m)\|_{L^2(\Omega)} \|u_1^m\|_{W^{1,r^-}(\Omega)} \\
&\leq \varepsilon \|\nabla(\tilde{c}(\beta) - c_d^m)\|_{L^2(\Omega)}^2 + C(\varepsilon)\|u_1^m\|_{W^{1,r^-}(\Omega)}.
\end{aligned}
\tag{5.50}
$$

For III from equation (5.48), we can use that c_d^m is uniformly bounded to find

$$
\begin{aligned}
2|III| &= \left| \int_\Omega \nabla(\tilde{c}(\beta) - c_d^m) \cdot u_1^m c_d^m \,\mathrm{d}x \right| \\
&\leq \|c_d^m\|_{L^\infty(\Omega)} \|u_1^m\|_{L^2(\Omega)} \|\nabla(\tilde{c}(\beta) - c_d^m)\|_{L^2(\Omega)} \\
&\leq \varepsilon \|\nabla(\tilde{c}(\beta) - c_d^m)\|_{L^2(\Omega)}^2 + C(\varepsilon)\|u_1^m\|_{W^{1,r^-}(\Omega)}^2,
\end{aligned}
\tag{5.51}
$$

where we have also used the embedding $W^{1,r^-}(\Omega) \hookrightarrow L^2(\Omega)$ and Young's inequality. Note that the L^2-orthonormality of the v_i and Poincaré's inequality yield

$$
\|\nabla(\tilde{c}(\beta) - c_d^m)\|_{L^2(\Omega)}^2 \gtrsim \|\tilde{c}(\beta) - c_d\|_{L^2(\Omega)} = |\beta|^2.
$$

Furthermore, recall from inequality (5.43) that $\|u_1^m\|_{W^{1,r^-}(\Omega)}$ is uniformly bounded. Now, we can insert equations (5.49), (5.50) and (5.51) into equation (5.48) to get

$$
B(\beta) \cdot \beta \gtrsim C \|\nabla(\tilde{c}(\beta) - c_d^m)\|_{L^2(\Omega)}^2 - C' \gtrsim |\beta|^2 - C'. \tag{5.52}
$$

Thus, we can apply Lemma 5.7 to deduce the existence of a $\bar{\beta} \in \mathbb{R}^M$ with a uniformly bounded norm, such that $B(\bar{\beta}) = 0$. This implies the existence of a $c_2^m \in V_h^m$ that satisfies equation (5.39) for $u^m = u_1^m$.

We will now prove a uniform L^2-bound on ∇c_1^m. We test equation (5.40) against $\varphi_h = c_2^m - c_d^m \in V_{h,0}^m$ to get

$$\int_\Omega q_c(c_2^m, \nabla c_2^m, Du_1^m) \cdot \nabla(c_2^m - c_d^m)\,\mathrm{d}x \\ - \int_\Omega \frac{1}{2}(c_2^m u_1^m \cdot \nabla(c_2^m - c_d^m) - u_h^m \cdot \nabla c_h^m (c_2^m - c_d^m))\,\mathrm{d}x = 0.$$

After cancelling out the terms that contain c_2^m twice in the skew-symmetric term, boundedness (5.9) and coercivity (5.10) of q_c then yield

$$\begin{aligned}\|\nabla c_2^m\|_{L^2(\Omega)}^2 &\lesssim \int_\Omega |\nabla c_2^m||\nabla c_d^m| + \frac{1}{2}(c_2^m u_1^m \cdot \nabla c_d^m - u_h^m \cdot \nabla c_h^m c_d^m)\,\mathrm{d}x \\ &\leq \varepsilon\|\nabla c_2^m\|_{L^2(\Omega)}^2 + C(\varepsilon)\|\nabla c_d^m\|_{L^2(\Omega)}^2 + \frac{1}{2}\int_\Omega (c_2^m u_1^m \cdot \nabla c_d^m - u_1^m \cdot \nabla c_2^m c_d^m)\,\mathrm{d}x,\end{aligned} \tag{5.53}$$

where we have also used Young's inequality in the last step. Now, we recall that $\|c_d^m\|_{L^\infty(\Omega)}$ is uniformly bounded by $\|c_d\|_{L^\infty(\Omega)}$ and that $r^- > \frac{n}{2}$ and therefore $W^{1,r^-}(\Omega) \hookrightarrow L^2(\Omega)$ for all $n \geq 2$. We use the Sobolev embedding theorem and Young's inequality to get

$$\begin{aligned}\left|\int_\Omega u_1^m \cdot (\nabla c_2^m) c_d^m\,\mathrm{d}x\right| &\lesssim \|c_d^m\|_{L^\infty(\Omega)} \int_\Omega |\nabla c_2^m||u_1^m|\,\mathrm{d}x \\ &\leq \varepsilon\|\nabla c_2^m\|_{L^2(\Omega)}^2 + C(\varepsilon)\|u_1^m\|_{L^2(\Omega)}^2 \\ &\leq \varepsilon\|\nabla c_2^m\|_{L^2(\Omega)}^2 + C(\varepsilon)\|u_1^m\|_{W^{1,r^-}(\Omega)}^2.\end{aligned} \tag{5.54}$$

Now, we use integration by parts and the triangle inequality to find that

$$\left|\int_\Omega c_2^m u_1^m \cdot \nabla c_d^m\,\mathrm{d}x\right| \leq \left|\int_\Omega u_1^m \cdot (\nabla c_2^m) c_d^m\,\mathrm{d}x\right| + \left|\int_\Omega (\mathrm{div} u_1^m) c_d^m c_2^m\,\mathrm{d}x\right|. \tag{5.55}$$

Note that $(r^-)' = \left(1 - \frac{1}{r^-}\right)^{-1} < \left(1 - \frac{2}{n}\right)^{-1} \leq \left(\frac{1}{2} - \frac{1}{n}\right)^{-1}$. This means that we can use the Sobolev embedding theorem and Young's inequality to find that

$$\begin{aligned}\left|\int_\Omega (\mathrm{div} u_1^m) c_d^m c_2^m\,\mathrm{d}x\right| &\leq \|c_d^m\|_{L^\infty(\Omega)}^2 \|\mathrm{div} u_1^m\|_{L^{r^-}(\Omega)} \|c_2^m\|_{L^{(r^-)'}(\Omega)} \\ &\lesssim \|u_1^m\|_{W^{1,r^-}(\Omega)} \|\nabla c_2^m\|_{L^2(\Omega)} \\ &\leq \varepsilon\|\nabla c_2^m\|_{L^2(\Omega)}^2 + C(\varepsilon)\|u_1^m\|_{W^{1,r^-}(\Omega)}^2.\end{aligned} \tag{5.56}$$

This enables us to insert inequalities (5.54), (5.55) and (5.55) into inequality (5.53) and absorb the ε-terms into the left hand side to find

$$\|\nabla c_n^2\|_{L^2(\Omega)} \lesssim \|\nabla c_d^m\|_{L^2(\Omega)}^2 + \|u_1^m\|_{W^{1,r^-}(\Omega)}^2 \leq C. \tag{5.57}$$

where we have used the m-uniform boundedness of $\|\nabla c_d^m\|_{L^2(\Omega)}$ and $\|u_1^m\|_{W^{1,r^-}(\Omega)}$ from inequality (5.43).

Now, we can iterate this procedure. We find u_2^m that solves equation (5.39) for $c_h^m = c_2^m$. Then we take c_3^m as the solution to equation (5.40) with $u_h^m = u_2^m$. For $m \in \mathbb{N}$, the iteration gives sequences $\{c_k^m\}_{k=0}^\infty$ and $\{u_k^m\}_{k=1}^\infty$ where c_k^m solves equation (5.40) for $u_h^m = u_k^m$ and u_k^m solves equation (5.39) for $c_h^m = c_{k-1}^m$. By induction, the estimate (5.43) holds for all u_k^m and the bound (5.57) holds for all c_k^m.

Now, we equip the finite-dimensional spaces $V_{h,0}^m$ and $X_{h,0,\text{div}}^m$ with the norms $\|\cdot\|_{W^{1,2}(\Omega)}$ and $\|\cdot\|_{W^{1,r^-}(\Omega)}$, respectively. Then, the Bolzano–Weierstraß theorem guarantees the existence of (strongly) convergent subsequences of $\{c_k^m\}_{k=0}^\infty$ and $\{u_k^m\}_{k=1}^\infty$. Without relabelling the subsequences, this means that

$$u_k^m \to \tilde{u}_h^m, \tag{5.58}$$

$$c_k^m - c_d^m \to \tilde{c}_h^m - c_d^m \tag{5.59}$$

for some $\tilde{u}_h^m \in X_{h,\text{div}}^m$ and $\tilde{c}_h^m$. It remains to show that the pair $(\tilde{u}_h^m, \tilde{c}_h^m)$ solves equations (5.39) and (5.40).

Because we are working on finite dimensional spaces, all norms are equivalent, so we also have strong convergence in $W^{1,\infty}(\Omega)$ for both, $\{c_k^m\}_{k=1}^\infty$ and $\{u_k^m\}_{k=1}^\infty$. This implies uniform convergence. Therefore, we have

$$u_k^m \to \tilde{u}_h^m, \qquad Du_k^m \to D\tilde{u}_h^m, \qquad c_k^m \to \tilde{c}_h^m, \qquad \nabla u_n^k \to \nabla \tilde{u}_h^m$$

almost everywhere uniformly. Furthermore, S and q_c are continuous, which leads to

$$S(c_k^m, DU_k^m) \to S(\tilde{c}_h^m, D\tilde{u}_h^m), \qquad q_c(c_k^m, \nabla c_k^m, Du_k^m) \to q_c(\tilde{c}_h^m, \nabla \tilde{c}_h^m, D\tilde{u}_h^m)$$

almost everywhere uniformly. This allows us to pass to the limit $k \to \infty$ in equations (5.39) and (5.40) to get

$$\begin{aligned} \frac{1}{2}\int_\Omega ((\tilde{u}_h^m \otimes \tilde{u}_h^m) \cdot \nabla v_h - (\tilde{u}_h^m \otimes v_h) \cdot \nabla \tilde{u}_h^m)\, \mathrm{d}x \\ + \int_\Omega S(\tilde{c}_h^m, D\tilde{u}_h^m) \cdot \nabla v_h \, \mathrm{d}x = \langle f, v_h \rangle \quad & \forall v_h \in X_{h,0,\text{div}}^m, \\ \int_\Omega q_c(\tilde{c}_h^m, \nabla \tilde{c}_h^m, D\tilde{u}_h^m) \cdot \nabla \varphi_h - \frac{1}{2}(\tilde{c}_h^m \tilde{u}_h^m \cdot \nabla \varphi_h - \tilde{u}_h^m \cdot \nabla \tilde{c}_h^m \varphi_h)\, \mathrm{d}x = 0 \quad & \forall \varphi_h \in V_{h,0}. \end{aligned}$$

The boundary condition $\tilde{c}_h^m = c_d^m$ on $\partial\Omega$ is fulfilled by construction. This establishes the existence of a discrete solution to equations (5.39) and (5.40) for any $m \in \mathbb{N}$. If we go back to Problem Q and equations (5.19) to (5.21), the discrete inf-sup condition (5.38) guarantees the existence of a discrete pressure p_h^m. □

Before we proceed, we will verify a few estimates of the form of inequalities (5.43) and (5.57).

Lemma 5.9. *Let $u_h^m \in X_{h,0,\mathrm{div}}^m$ and $c_h^m \in V_h^m$ satisfy equations (5.39) and (5.40). Define $\mu = \max\{r^+, \frac{n}{n-2} - \varepsilon\}$ for some $\varepsilon > 0$. Then, we have*

$$\int_\Omega |\nabla u_h^m|^{r(c_h^m)} \,\mathrm{d}x + \int_\Omega |S(c_h^m, Du_h^m)|^{r'(c_h^m)} \,\mathrm{d}x \le C_1, \tag{5.60}$$

$$\int_\Omega |\nabla c_h^m|^2 \,\mathrm{d}x + \int_\Omega |q_c(c_h^m, \nabla c_h^m, Du_h^m)|^2 \,\mathrm{d}x \le C_2, \tag{5.61}$$

$$\|p_h^m\|_{L^{\mu'}(\Omega)} \le C_3. \tag{5.62}$$

The constants C_1, C_2 and C_3 do not depend on m.

Proof. We test equation (5.39) against $\varphi_h = u_h^m$. The skew-symmetric term vanishes, so we get

$$\int_\Omega S(c_h^m, Du_h^m) \cdot Du_h^m \,\mathrm{d}x = \langle f, u_h^m \rangle. \tag{5.63}$$

We can apply inequality (5.8) to the left-hand side of equation (5.63) and the standard duality estimate on the right-hand side of equation (5.63) to get

$$\int_\Omega |Du_h^m|^{r(c_h^m)} + |S(c_h^m, Du_h^m|^{r'(c_h^m)} - C_1 \le \|f\|_{\left(W_0^{1,r^-}(\Omega)\right)'} \|u_h^m\|_{W^{1,r^-}(\Omega)}.$$

Then, we can use Korn's inequality and Young's inequality to prove inequality (5.60).

Now, we test equation (5.40) against $c_h^m - c_n^d$. This leaves us with

$$\int_\Omega q_c(c_h^m, \nabla c_h^m, Du_h^m) \cdot \nabla(c_h^h - c_d^m) - \frac{1}{2}(c_h^m u_h^m \cdot \nabla(c_h^m - c_d^m) - u_h^m \cdot \nabla c_h^m (c_h^m - c_h^m)) \,\mathrm{d}x = 0.$$

We proceed as in inequalities (5.53) to (5.56) to deduce

$$|\nabla c_h^m|^2 \lesssim 1 + \|u_h^m\|^2_{W^{1,r^-}(\Omega)}. \tag{5.64}$$

Note that inequality (5.9) implies $|q_c(c_h^m, \nabla c_h^m, Du_h^m)|^2 \lesssim |\nabla c_h^m|^2$. Then, inequality (5.60) and (5.64) imply inequality (5.61).

Finally, the discrete inf-sup condition (5.38) yields

$$\|p_h^m\|_{L^{s'}(\Omega)} \lesssim \sup_{0 \ne v_h \in X_{h,0}^m, \|v_h\|_{W^{1,s}(\Omega)} \le 1} \langle \mathrm{div} v_h, p_h^m \rangle \tag{5.65}$$

for all $s > 1$. For an arbitrary $v_h \in X_{h,0}^m$, we can use the PDE (5.19) to get

$$\begin{aligned} \langle \mathrm{div} v, p_h^m \rangle &\le \left| \int_\Omega S(c_h^m, Du_h^m) \cdot Dv_h + \frac{1}{2}\left(u_h^m \otimes v_h \cdot \nabla u_h^m - u_h^m \otimes u_h^m \cdot \nabla v_h\right) \mathrm{d}x - \langle f, v_h \rangle \right| \\ &\lesssim \|S(c_h^m, u_h^m)\|_{L^{\left(r(c_h^m)\right)'}(\Omega)} \|Dv_h\|_{L^{r(c_h^m)}(\Omega)} + \|v_h\|_{W^{1,r^-}(\Omega)} \|f\|_{W^{-1,r^-}(\Omega)} \\ &\quad + \|u_h^m\|_{L^{(r^-)^*}(\Omega)} \|v_h\|_{L^{s^*}(\Omega)} \|u_h^m\|_{W^{1,r^-}(\Omega)} + \|u_h^m\|^2_{L^{(r^-)^*}(\Omega)} \|v_h\|_{W^{1,s}(\Omega)}, \end{aligned} \tag{5.66}$$

where $\frac{1}{s} = 1 - \frac{2}{(r^-)^\star}$. For $r^- > \frac{n}{2}$, this means that $s < \frac{n}{n-2}$. Now, we deduce from inequality (5.6) and $(r-1)r' = r$ that

$$\|S(c_h^m, u_h^m)\|_{L^{(r(c_h^m))'}(\Omega)} \lesssim \|u_h^m\|_{W^{1,r(c_h^m)}(\Omega)}. \tag{5.67}$$

By the Sobolev embedding theorem, we also have

$$\|u_h^m\|_{L^{(r^-)^\star}(\Omega)} \|v_h\|_{L^{s^\star}(\Omega)} \|u_h^m\|_{W^{1,r^-}(\Omega)} \lesssim \|u_h^m\|^2_{L^{(r^-)^\star}(\Omega)} \|v_h\|_{W^{1,s}(\Omega)}. \tag{5.68}$$

Let us now define $\mu = \max\{s, r^+\}$. Then, $W^{1,\mu}(\Omega) \hookrightarrow L^{r(c_h^m)}(\Omega)$ and $W^{1,\mu}(\Omega) \hookrightarrow L^s(\Omega)$. This and inequalities (5.65), (5.66), (5.67) and (5.68) yield inequality (5.62) which completes the proof of the lemma. □

5.3 Elliptic equations with lower order terms

In the convergence analysis of the sequences u_h^m and c_h^m that satisfy equations (5.39) and (5.40) which were defined in the previous section, we want to apply the theory from Chapter 3 to equation (5.23). Before we can proceed, we have to generalise Theorem 3.25 to include lower order terms. Obviously, we cannot guarantee that $u_h^m c_h^m$ satisfies Assumption $(\star)$ from Definition 3.10. Thus, any attempt to see the lower order terms as part of the source term is bound to fail. One could integrate by parts, but then we would need a bound on ∇c_h^m and in general, it is not possible to bound it in a stronger norm than L^2. In $n = 3$ dimensions this would limit us to the case $r^- > 2$, which would not include Newtonian fluids.

After an integration by parts, equation (5.40) will have the following form. Let $A \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ be a uniformly elliptic matrix-valued function and let $b \in L^p(\Omega; \mathbb{R}^n)$, $c \in L^q(\Omega)$ and $g \in C^\alpha \cap V_h$ be given functions with $\frac{1}{p} = \frac{1}{n} - \frac{\delta}{n}$ and $\frac{1}{q} = \frac{2}{n} - \frac{\delta}{n}$ for some $\delta > 0$ and $\alpha > 0$. We look at functions $u_h \in V_h$ that satisfy

$$\begin{aligned} \int_\Omega A\nabla u_h \cdot \nabla \varphi_h + b u_h \cdot \nabla \varphi_h + c u_h \varphi_h \,\mathrm{d}x = 0, \\ u_h|_{\partial\Omega} = g|_{\partial\Omega} \end{aligned} \tag{5.69}$$

for all $\varphi_h \in V_{h,0}$. This means that we have to deal with lower order terms. The naïve approach of seeing bu_h as part of the source term and using the theorems in Chapter 3 is bound to fail because we cannot guarantee that bu_h satisfies Assumption $(\star)$, that in is core is a regularity assumption. However, the only part in Chapter 3 where we used assumption $(\star)$ was Theorem 3.12, where we proved the subsolution property of the nodal maximum of two discrete solutions. Under additional assumptions on the mesh, we can prove a subsolution property even in the case where bu_h does not satisfy Assumption $(\star)$.

Lemma 5.10. *For given functions $A \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ and $b \in L^p(\Omega; \mathbb{R}^n)$, $f, g \in L^q(\Omega; \mathbb{R})$ where A is uniformly elliptic, $\frac{1}{p} = \frac{1}{n} - \frac{\delta}{n}$, and $\frac{1}{q} = \frac{2}{n} - \frac{\delta}{n}$, let $\mathcal{T}_h$ be a shape-regular, uniformly A-acute triangulation of the polyhedral Lipschitz domain $\Omega \subset \mathbb{R}^n$. Furthermore, let $u_h \in V_h$ and $v_h \in V_h$ satisfy the inequalities*

$$\int_\Omega A\nabla u_h \cdot \nabla\varphi_h + bu_h \cdot \nabla\varphi_h \,\mathrm{d}x \leq \int_\Omega f\varphi_h \,\mathrm{d}x \qquad \forall \varphi_h \in V_{h,0} \text{ with } \varphi_h \geq 0, \tag{5.70}$$

$$\int_\Omega A\nabla u_h \cdot \nabla\varphi_h + bu_h \cdot \nabla\varphi_h \,\mathrm{d}x \leq \int_\Omega g\varphi_h \,\mathrm{d}x \qquad \forall \varphi_h \in V_{h,0} \text{ with } \varphi_h \geq 0 \tag{5.71}$$

with

$$u_h|_{\partial\Omega} = w_1|_{\partial\Omega},$$
$$v_h|_{\partial\Omega} = w_2|_{\partial\Omega}.$$

for some given functions $w_1, w_2 \in C^\alpha(\partial\Omega)$.

Then, the nodal maximum $u_h \curlyvee v_h$ satisfies the inequality

$$\int_\Omega A\nabla(u_h \curlyvee v_h) \cdot \nabla\varphi_h \,\mathrm{d}x + \int_\Omega (u_h \curlyvee v_h) b \cdot \nabla\varphi_h \,\mathrm{d}x \leq \int_\Omega (f \vee g)\varphi_h \,\mathrm{d}x \tag{5.72}$$

for all nonnegative $\varphi_h \in V_{h,0}$ if $\max_{T\in\mathcal{T}_h} h_T$ is small enough.

Proof. For simplicity, we write

$$\mathcal{A}(u, v) = \int_\Omega A\nabla u \cdot \nabla v \,\mathrm{d}x + \int_\Omega ub \cdot \nabla v \,\mathrm{d}x.$$

Obviously, this form is bilinear in u and v. Using the uniform A-acuteness of the triangulation $\mathcal{T}_h$ (i.e. inequality (2.9)), we find

$$\begin{aligned} -\mathcal{A}(\psi_i, \psi_j) &= -\int_\Omega A\nabla\psi_i \cdot \nabla\psi_j \,\mathrm{d}x - \int_\Omega \psi_i b \cdot \nabla\psi_j \,\mathrm{d}x \\ &\geq c\varepsilon \|\nabla\psi_i\|_{L^2(\Omega)} \|\nabla\psi_j\|_{L^2(\Omega)} - \|b\|_{L^p(\Omega)} \|\psi_i \nabla\psi_j\|_{L^{p'}(\Omega)} \end{aligned} \tag{5.73}$$

for $i \neq j$ and $\frac{1}{p} + \frac{1}{p'} = 1$. If the respective supports of ψ_i and ψ_j have a non-empty overlap, we have $h_i \sim h_j \sim h_T$. Inequality (2.10) yields

$$\|\nabla\psi_i\|_{L^2(\Omega)} \sim \|\nabla\psi_j\|_{L^2(\Omega)} \sim h_T^{\frac{n}{2}-1} \tag{5.74}$$

and

$$\|\psi_i \nabla\psi_j\|_{L^{p'}(\Omega)} \sim h_T^{\frac{n}{p'}-1} = h_T^{n-2+\delta}. \tag{5.75}$$

Now, we insert equations (5.74) and (5.75) into inequality (5.73). For some constants $c_1 > 0$ and $c_2 > 0$, we get

$$-\mathcal{A}(\psi_i, \psi_j) \geq c_1 h_T^{n-2} - c_2 \|b\|_{L^p} h_T^{n-2+\delta} = h_T^{n-2} \left(c_1 - c_2 \|b\|_{L^p} h_T^{\delta} \right). \tag{5.76}$$

If we now choose $\max_{T\in\mathcal{T}} h_T \leq \frac{c_2}{c_1}\|b\|_{L^p}(\Omega)$, we conclude that

$$-\mathcal{A}(\psi_i,\psi_j) \geq 0. \tag{5.77}$$

This implies that we can proceed as in equation (3.48). First, we fix some j and assume w.l.o.g. that $u(x_j) \geq v(x_j)$ and note that equation (5.72) becomes $\mathcal{A}(u_h,\varphi_h) \leq \int_\Omega f\varphi_h \,\mathrm{d}x$. Using inequality (5.77), we estimate

$$\begin{aligned}
\mathcal{A}(u_h \curlyvee v_h, \psi_j) &= \sum_i (u_h(x_i) \vee v_h(x_i))\mathcal{A}(\psi_i,\psi_j)\\
&= (u_h(x_j) \vee v_h(x_j))\mathcal{A}(\psi_j,\psi_j) + \sum_{i\neq j}(u_h(x_i) \vee v_h(x_i))\mathcal{A}(\psi_i,\psi_j)\\
&\leq u_h(x_j)\mathcal{A}(\psi_j,\psi_j) + \sum_{i\neq j} u_h(x_i)\mathcal{A}(\psi_i,\psi_j)\\
&= \mathcal{A}(u_h,\psi_j) \leq \int_\Omega f\psi_j \,\mathrm{d}x \leq \int_\Omega (f\vee g)\psi_j \,\mathrm{d}x.
\end{aligned} \tag{5.78}$$

Now, we write $\varphi_h = \sum_j \varphi_h(x_j)\psi_j$ with $\varphi_h(x_j) \geq 0$ and use inequality (5.78) find

$$\mathcal{A}(u_h \curlyvee v_h) = \sum_j \varphi_h(x_j)\mathcal{A}(u_h \curlyvee v_h,\psi_j) \leq \sum_j \varphi_h(x_j)\int_\Omega (f\vee g)\psi_j \,\mathrm{d}x = \int_\Omega (f\vee g)\varphi_h \,\mathrm{d}x.$$

This proves the lemma. □

A similar property to inequality (5.75) has been shown with a similar technique in [62] for $b \in L^\infty(\Omega)$ in order to prove a discrete maximum principle for linear elliptic equations.

Note that this is the only point where we needed assumption ($\star$) in our proofs in Chapter 3. Recall that $u_h \in V_h$ implies $u_h \in L^\infty(\Omega)$. (However, the L^∞-norm of u_h is not necessarily uniformly bounded, a priori.) Therefore, we can apply Theorems 3.20 and 3.27 to equation (5.69) with $F = u_h b$ and $f = u_h c$ because $\|u_h b\|_{L^p(\Omega)} \leq \|u_h\|_{L^\infty(\Omega)}\|b\|_{L^p(\Omega)}$ and $\|u_h c\|_{L^q(\Omega)} \leq \|u_h\|_{L^\infty(\Omega)}\|c\|_{L^q(\Omega)}$. We will now show that it is indeed possible to deduce a uniform L^∞-bound on u_h in this case.

Lemma 5.11. *For a given uniformly elliptic function $A \in L^\infty(\Omega;\mathbb{R}^{n\times n})$ and a function $b \in L^p(\Omega;\mathbb{R}^n)$ with $\frac{1}{p} = \frac{1}{n} - \frac{\delta}{n}$ for some $\delta > 0$, let $\mathcal{T}_h$ be a shape-regular, uniformly A-acute triangulation of the polyhedral Lipschitz domain $\Omega \subset \mathbb{R}^n$. Let $u_h \in V_h$ be a discrete solution to equation (5.69). Then, there is a uniform global bound on $\|u_h\|_{L^\infty(\Omega)}$ that is independent of the mesh size, provided that $\max_{T\in\mathcal{T}} h_T \leq C$ for some constant C that depends only on the shape-regularity constant Γ, the uniform A-acuteness of $\mathcal{T}_h$, $\|b\|_{L^p(\Omega)}$, and δ.*

Proof. For interiour balls away from the boundary, Theorem 3.20 implies

$$\sup_{\Omega'(B(x_0,R))} |u_h|^2 \leq C_1 \fint_{\Omega(B(x_0,2R))} |u_h|^2 \,\mathrm{d}x + C_2\|u_h\|^2_{L^\infty(\Omega)}\left(\|b\|^2_{L^p(\Omega)} + \|c\|^2_{L^q(\Omega)}\right)R^{2\delta}. \tag{5.79}$$

For balls near the boundary, we define $G = \max_{\partial\Omega} g$. Then, Theorem 3.27 implies

$$\begin{aligned}\sup_{\Omega'(\mathcal{B}(x_0,R))} |u_h|^2 \leq C_3 R^{-n} \int_{\Omega(2\mathcal{B}(x_0,R))} |u_h|^2 \,\mathrm{d}x \\ + C_4 \|u_h\|^2_{L^\infty(\Omega)} \left(\|b\|^2_{L^p(\Omega)} + \|c\|^2_{L^q(\Omega)} \right) R^{2\delta} + G^2.\end{aligned} \tag{5.80}$$

Let us now choose $R = R_0 := (2\max\{C_2, C_4\}(\|b\|^2_{L^p(\Omega)} + \|c\|^2_{L^q(\Omega)})^{-\frac{1}{2\delta}}$ in inequalities (5.79) and (5.80). Also recall from equation 2.13 that $|\Omega(B(x_0,R))| \sim R^n$. This gives

$$\sup_{\Omega'(\mathcal{B}(x_0,R_0))} |u_h|^2 \leq C R_0^{-n} \int_{\Omega(B(x_0,R_0))} |u_h|^2 \,\mathrm{d}x + \frac{1}{2}\|u_h\|^2_{L^\infty(\Omega)} + G^2. \tag{5.81}$$

Note that $\int_{\Omega(B(x_0,R))} |u_h|^2 \,\mathrm{d}x \leq \int_\Omega |u_h|^2 \,\mathrm{d}x$. This means that we can take the supremum over all $x_0 \in \Omega$ to find

$$\begin{aligned}\|u_h\|^2_{L^\infty(\Omega)} &= \sup_{x_0\in\Omega} \sup_{\Omega'(\mathcal{B}(x_0,R_0))} |u_h|^2 \\ &\leq C R^{\frac{n}{2\delta}} \int_\Omega |u_h|^2 \,\mathrm{d}x + \frac{1}{2}\|u_h\|^2_{L^\infty(\Omega)} + G^2.\end{aligned}$$

Therefore, we can absorb the term $\frac{1}{2}\|u_h\|^2_{L^\infty(\Omega)}$ into the left-hand side. This proves the uniform boundedness of $\|u_h\|_{L^\infty}$. The same result can be attained by using the discrete maximum principle as stated in [62] after generalizing it to include $b \in L^n(\Omega;\mathbb{R}^n)$. □

This allows us to apply Theorem 3.25 to discrete solutions to equation (5.69) with $F = u_h b$ and $f = u_h c$ to find a uniform global bound on $|u_h|_{C^\alpha(\overline{\Omega})}$ for some $\alpha > 0$. Let us summarize the result in a theorem.

Theorem 5.12. *Let $\Omega \subset \mathbb{R}^n$ be a polyhedral Lipschitz domain. For a given uniformly elliptic $A \in L^\infty(\Omega;\mathbb{R}^{n\times n})$, let $\mathcal{T}_h$ be a shape-regular and uniformly A-acute triangulation of Ω with respective finite element spaces V_h and $V_{h,0}$. Let $g \in V_h$, $b \in L^p(\Omega)$ and $f \in L^q(\Omega)$ be given functions with $|g|_{C^\beta(\overline{\Omega})} \leq C$ for some C and $\beta > 0$, $\frac{1}{p} = \frac{1}{n} - \frac{\delta}{n}$ and $\frac{1}{q} = \frac{2}{n} - \frac{\delta}{n}$ for some $\delta > 0$. Let $u_h \in V_h$ satisfy*

$$\begin{aligned}\int_\Omega A\nabla u_h \cdot \nabla\varphi_h + b u_h \cdot \nabla\varphi_h + c u_h \varphi_h \,\mathrm{d}x = 0, \\ u_h|_{\partial\Omega} = g|_{\partial\Omega}\end{aligned} \tag{5.82}$$

for all $\varphi_h \in V_{h,0}$. Then, there are constants $C_1 > 0$, $\alpha > 0$ that that depend only on n, d, $\|A\|_{L^\infty(\Omega)}$, $\|b\|_{L^p(\Omega)}$, $\|c\|_{L^q(\Omega)}$, β, $|g|_{C^\beta(\overline{\Omega})}$, $\operatorname{diam}(\Omega)$ and the shape-regularity parameter Γ, such that

$$|u_h|_{C^\alpha(\overline{\Omega})} \leq C_1$$

if $\max_{T\in\mathcal{T}_h} h_T \leq C_2$ where C_2 is a constant that depends only on the shape-regularity parameter Γ of $\mathcal{T}_h$, the uniform A-acuteness of $\mathcal{T}_h$, $\|b\|_{L^p(\Omega)}$, and δ.

Proof. Lemma 5.11 gives a uniform bound on $\|u_h\|_{L^\infty(\Omega)}$. Hence, we have $\|ub\|_{L^p(\Omega)} \lesssim \|b\|_{L^p(\Omega)}$ and $\|uc\|_{L^q(\Omega)} \lesssim \|c\|_{L^q(\Omega)}$. Lemma 5.10 guarantees that the nodal positive part$(u_h - c)_+$ satisfies a subsolution property. This means that we can apply Theorem 3.25 on u_h with $F = ub$ and $f = uc$ without having to require ub to satisfy assumption $(\star)$ because the only part in the proof of Theorem 3.25 where those conditions were needed, were the proofs of the subsolution properties. This gives a uniform Hölder-norm bound and completes the proof. □

Finally, we can apply the theory to the discrete concentration equation (5.37).

Theorem 5.13. *Let $\mathcal{T}_h^m$ be a sequence of uniformly acute triangulations of the polyhedral Lipschitz domain $\Omega \subset \mathbb{R}^n$ with a uniformly bounded shape-regularity parameter Γ with respective finite element spaces as defined in Section 5.2. Furthermore, assume $q_c(x, v, A) = k(x, A)v$ for a scalar function $k \in L^\infty(\mathbb{R} \times \mathbb{R}^{n\times n}; \mathbb{R})$ with $k(x, A) \geq k^- > 0$. Let $u_h^m \in X_{h,\mathrm{div}}^m$ and $c_h^m \in V_h^m$ solve equations (5.39) and (5.40). Then, there is an $\alpha > 0$, such that $|c_h^m|_{C^\alpha(\overline{\Omega})}$ is bounded uniformly in m, provided that $\max_{T\in\mathcal{T}_h^m} h_T$ is small enough.*

Proof. The function c_h^m satisfies equation (5.40), i.e.

$$\int_\Omega k(c_h^m, Du_h^m)\nabla c_h^m \cdot \nabla\varphi_h - \frac{1}{2}(c_h^m u_h^m \cdot \nabla\varphi_h - u_h^m \cdot \nabla c_h^m \varphi_h)\,\mathrm{d}x = 0 \tag{5.83}$$

for all $\varphi_h \in V_{h,0}$. After an integration by parts in the last summand of the left-hand side, this becomes

$$\int_\Omega k(c_h^m, Du_h^m)\nabla c_h^m \cdot \nabla\varphi_h - c_h^m u_h^m \cdot \nabla\varphi_h - \frac{1}{2}c_h^m(\mathrm{div} u_h^m)\varphi_h\,\mathrm{d}x = 0 \tag{5.84}$$

for all $\varphi_h \in V_{h,0}$. As k is a scalar function, a uniformly acute triangulation $\mathcal{T}_h^m$ is always uniformly k-acute. We have defined the boundary condition for c_h^m as $c_d^m = \Pi_{h,V} c_d$. The Lagrange interpolation projection is obviously stable under the Hölder-seminorm, i.e. $|c_d^m|_{C^\beta(\overline{\Omega})} \leq |c_d|_{C^\beta(\overline{\Omega})}$.

Recall from equation (5.60) from Lemma 5.9 that

$$\int_\Omega |\nabla u_h^m|^{r(c_h^m)}\,\mathrm{d}x \leq C.$$

Because $r \geq r^-$, this implies that $\|u_h^m\|_{W^{1,r^-}(\Omega)}^{r^-}$ is uniformly bounded. Furthermore, recall that $r^- > \frac{n}{2}$ which means that there is a $\delta > 0$, such that $\frac{1}{r^-} = \frac{2}{n} - \frac{\delta}{n}$. Therefore, we have $\|\mathrm{div} u_h^m\|_{L^{\left(\frac{2}{n}-\frac{\delta}{n}\right)^{-1}}(\Omega)} \leq \|u_h^m\|_{W^{1,r^-}(\Omega)}$. Furthermore, the Sobolev embedding theorem implies that $\|u_h^m\|_{L^{\left(\frac{1}{n}-\frac{\delta}{n}\right)^{-1}}(\Omega)} \leq \|u_h\|_{W^{1,r^-}(\Omega)}$.

This means that we can apply Theorem 5.12 with $a = k(c_h^m, Du_h^m)$, $b = -u_h^m$ and $c = -\frac{1}{2}\mathrm{div} u_h^m$ to conclude the proof. □

5.4 Convergence Analysis

We are now able to prove the convergence of the sequences that were defined in Section 5.2. We will follow the arguments of [43]. However, because of Theorem 5.13, these arguments also apply to the physically relevant case of $n = 3$ space dimensions.

First, let us now collect the convergence results that we can extract from the bounds which we have proved in Lemma 5.9 and Theorem 5.13. We will use the reflexivity of the spaces in question and the compact Sobolev embedding. From equation (5.60), we see that we can extract subsequences (not relabelled), such that

$$u_h^m \rightharpoonup u \qquad \text{weakly in } W_0^{1,r^-}(\Omega;\mathbb{R}^n), \tag{5.85}$$

$$u_h^m \to u \qquad \text{strongly in } L^{n+\varepsilon}(\Omega;\mathbb{R}^n), \varepsilon > 0 \tag{5.86}$$

$$S(c_h^m, u_h^m) \rightharpoonup \tilde{S} \qquad \text{weakly in } L^{(r^+)'}(\Omega;\mathbb{R}^{n\times n}). \tag{5.87}$$

Equation (5.62) yields

$$p_h^m \rightharpoonup p \qquad \text{weakly in } L^{\mu'}(\Omega) \tag{5.88}$$

for $\mu = \max\{r^+, \frac{n}{n-2} + \varepsilon\}$. Furthermore, equation (5.61) gives

$$c_h^m \rightharpoonup c \qquad \text{weakly in } W^{1,2}(\Omega), \tag{5.89}$$

$$q_c(c_h^m, \nabla c_h^m, Du_h^m) \rightharpoonup \tilde{q} \qquad \text{weakly in } L^2(\Omega;\mathbb{R}^n). \tag{5.90}$$

Finally, we use Theorem 5.13 and the fact that $C^\beta(\overline{\Omega}) \hookrightarrow C^\alpha(\overline{\Omega})$ compactly for $\beta < \alpha$. This gives

$$c_h^m \to c \qquad \text{strongly in } C^\beta(\overline{\Omega}). \tag{5.91}$$

First, we have to establish the fact that the limit u is indeed in $W_{0,\text{div}}^{1,r(c)}(\Omega;\mathbb{R}^n)$. We have $c_h^m \to c$ in $C^\beta(\overline{\Omega})$ from equation (5.91). This means that for a given $\varepsilon > 0$, we can choose m large enough to have $|r(c_h^m) - r(c)| < \frac{\varepsilon}{\Theta}$ with a $\Theta > 1$ that is large enough such that $r(c) - \frac{\Theta+1}{\Theta}\varepsilon > 1$. From the estimate in inequality (5.60), we deduce that

$$\begin{aligned} C_1 &\geq \int_\Omega |\nabla u_h^m|^{r(c_h^m)}\,\mathrm{d}x \geq \int_{\Omega\cap\{|\nabla u_h^m|>1\}} |\nabla u_h^m|^{r(c_h^m)}\,\mathrm{d}x \\ &\geq \int_{\Omega\cap\{|\nabla u_h^m|>1\}} |\nabla u_h^m|^{r(c_h^m)-r(c)+r(c)-\varepsilon}\,\mathrm{d}x \geq \int_{\Omega\cap\{|\nabla u_h^m|>1\}} |\nabla u_h^m|^{r(c)-\frac{\Theta+1}{\Theta}\varepsilon}\,\mathrm{d}x. \end{aligned} \tag{5.92}$$

Furthermore, it is obvious that

$$\int_{\Omega\cap\{|\nabla u_h^m|\leq 1\}} |\nabla u_h^m|^{r(c)-\frac{\Theta+1}{\Theta}\varepsilon}\,\mathrm{d}x \leq \tilde{C}_1. \tag{5.93}$$

Adding up the inequalities (5.92) and (5.93) yields

$$\int_\Omega |\nabla u_h^m|^{r(c)-\frac{\Theta+1}{\Theta}\varepsilon}\,\mathrm{d}x \lesssim 1. \tag{5.94}$$

By reflexivity of the respective spaces, inequality (5.94) allows us to find a subsequence (not relabelled), such that

$$u_h^m \rightharpoonup u \qquad \text{weakly in } W^{1,r(c)-\frac{\Theta+1}{\Theta}\varepsilon}(\Omega). \tag{5.95}$$

Thus, we have

$$\int_\Omega |\nabla u_h^m|^{r(c)-\frac{\Theta+1}{\Theta}\varepsilon} \,\mathrm{d}x \lesssim 1. \tag{5.96}$$

Therefore, we can use Fatou's Lemma for $\varepsilon \to 0$ to deduce the desired estimate

$$\int_\Omega |\nabla u|^{r(c)} \,\mathrm{d}x \lesssim 1. \tag{5.97}$$

Analogously to inequalities (5.92) to (5.97), we can find

$$\int_\Omega \left|\tilde{S}\right|^{r'(c)} \,\mathrm{d}x \lesssim 1. \tag{5.98}$$

Now, we need to check that we indeed have $\operatorname{div} u = 0$. We take an arbitrary $v \in C_0^\infty(\overline{\Omega})$ and test $\operatorname{div} u_h^m$ against $\Pi_{h,Q}^m v$. Then, $u_h^m \in X_{h,\mathrm{div}}^m$ implies that

$$\begin{aligned} 0 &= \int_\Omega \left(\Pi_{h,Q}^m v\right) \operatorname{div} u_h^m \,\mathrm{d}x \\ &= \int_\Omega \left(\Pi_{h,Q}^m v - v\right) \operatorname{div} u_h^m \,\mathrm{d}x + \int_\Omega v \left(\operatorname{div} u_h^m - \operatorname{div} u\right) \,\mathrm{d}x + \int_\Omega v \operatorname{div} u \,\mathrm{d}x. \end{aligned} \tag{5.99}$$

Note that $\Pi_{h,Q}^m v \to v$ strongly in $L^s(\Omega)$ for all $s \in (1,\infty)$ by inequality (5.34). Together with the bound (5.60), this means that

$$\int_\Omega \left(\Pi_{h,Q}^m v - v\right) \operatorname{div} u_h^m \,\mathrm{d}x \to 0 \tag{5.100}$$

as $m \to \infty$. Furthermore, the weak convergence (5.85) implies that

$$\int_\Omega v \left(\operatorname{div} u_h^m - \operatorname{div} u\right) \,\mathrm{d}x \to 0 \tag{5.101}$$

as $m \to \infty$. Therefore, letting $m \to \infty$ in equation (5.99) implies together with equations (5.100) and (5.101) that

$$\int_\Omega v \operatorname{div} u \,\mathrm{d}x = 0 \qquad \forall v \in C_0^\infty(\Omega). \tag{5.102}$$

Of course, this implies $\operatorname{div} u = 0$ almost everywhere in Ω.

With a similar technique, we can also verify the limit of the convective term in the momentum equation. For an arbitrary $v \in W^{1,\infty}(\Omega;\mathbb{R}^n)$, we have $\Pi_{h,X,\mathrm{div}}^m v \to v$ strongly in $W^{1,s}(\Omega)$ for all $s \in [1,\infty)$ by equation (5.30). Furthermore, note that $W_0^{1,r(c^m)}(\Omega) \hookrightarrow L^{2(1+\varepsilon)}(\Omega)$ for some $\varepsilon > 0$ because $n \geq 2$ and $r^- > \frac{n}{2}$. Therefore, we can deduce

$$\int_\Omega (u_h^m \otimes u_h^m) \cdot \nabla \Pi_{h,X,\mathrm{div}}^m v \,\mathrm{d}x \to \int_\Omega u \otimes u \cdot \nabla v \,\mathrm{d}x \tag{5.103}$$

as $m \to \infty$. Now, we are assuming $r^- > \frac{2n}{n+1}$. This is a restriction only for $n = 2$ where it translates to $r^- > \frac{4}{3}$. For $n \geq 3$, we have $\frac{n}{2} \geq \frac{2n}{n+1}$. This leads to $(r^-)^\star > (r^-)'$ and we therefore have the compact embedding $W_0^{1,r^-}(\Omega) \hookrightarrow L^{(r^-)'+\varepsilon}(\Omega)$ for some $\varepsilon > 0$. This means that we can find some $s \in (1, \infty)$, such that

$$\begin{aligned}
&\|u_h^m \otimes \Pi_{h,X,\mathrm{div}}^m v - u \otimes v\|_{L^{(r^-)'}(\Omega)} \\
\leq &\|u_h^m \otimes \left(\Pi_{h,X,\mathrm{div}}^m v - v\right)\|_{L^{(r^-)'}(\Omega)} + \|(u_h^m - u) \otimes v\|_{L^{(r^-)'}(\Omega)} \\
\leq &\|u_h^m\|_{L^{(r^-)'+\varepsilon}(\Omega)} \|\Pi_{h,X,\mathrm{div}}^m v - v\|_{L^s(\Omega)} + \|u_h^m - u\|_{L^{(r^-)'+\varepsilon}(\Omega)} \|v\|_{L^s(\Omega)}.
\end{aligned} \tag{5.104}$$

The first summand tends to zero as $m \to \infty$ because of property (5.30) of the projection operator, while $\|u_h^m\|_{L^{(r^-)'+\varepsilon}(\Omega)}$ stays bounded because of the continuous embedding $W_0^{1,r^-}(\Omega) \hookrightarrow L^{(r^-)'+\varepsilon}(\Omega)$ and the estimate from inequality (5.60). The second summand of inequality (5.104) also tends to zero as $m \to \infty$ because of the weak convergence of u_h^m in $W_0^{1,r^-}(\Omega)$ and the compact embedding $L^{(r^-)'+\varepsilon}(\Omega) \hookrightarrow W_0^{1,r^-}(\Omega)$. Therefore, we have $u_h^m \otimes \Pi_{h,X,\mathrm{div}}^m v \to u \otimes v$ strongly in $L^{(r^-)'}(\Omega)$ and thus

$$\int_\Omega (u_h^m \otimes \Pi_{h,X,\mathrm{div}}^m v) \cdot \nabla u_h^m \,\mathrm{d}x \to \int_\Omega (u \otimes v) \cdot \nabla u \,\mathrm{d}x = -\int_\Omega (u \otimes u) \cdot \nabla v \,\mathrm{d}x \tag{5.105}$$

as $m \to \infty$, where we have also used that $\mathrm{div} u = 0$ a.e. in Ω from equation (5.102).

Additionally, we note that $\mathrm{div}\Pi_{h,X,\mathrm{div}}^m v \to \mathrm{div} v$ strongly in $L^s(\Omega)$ for all $v \in W^{1,\infty}(\Omega)$ because of equation (5.30). Together with the weak convergence $p_h^m \rightharpoonup p$ in $L^{\mu'}(\Omega)$ for some $\mu' > 1$ from equation (5.88), this yields

$$\langle p_h^m, \mathrm{div}\Pi_{h,X,\mathrm{div}}^m v\rangle \to \langle p, \mathrm{div} v\rangle \tag{5.106}$$

as $m \to \infty$. Similarly, we get

$$\int_\Omega S(c_h^m, Du_h^m) \cdot D\Pi_{h,X,\mathrm{div}}^m v \,\mathrm{d}x \to \int_\Omega \tilde{S} \cdot Dv \,\mathrm{d}x \tag{5.107}$$

as $m \to \infty$ because of the weak convergence from equation (5.87). Finally, we test the velocity equations (5.35) and (5.39) against $\Pi_{h,X,\mathrm{div}}^m v$ for an arbitrary $v \in W_0^{1,\infty}(\Omega; \mathbb{R}^n)$ and $v \in W_{0,\mathrm{div}}^{1,\infty}(\Omega; \mathbb{R}^n)$ respectively and use the limits (5.103), (5.105), (5.106), and (5.107) to pass to the limit $m \to \infty$. We get

$$\int_\Omega \tilde{S} \cdot Dv - (u \otimes u) \cdot \nabla v \,\mathrm{d}x - \langle \mathrm{div} v, p\rangle = \langle f, v\rangle \quad \forall v \in W_0^{1,\infty}(\Omega; \mathbb{R}^n), \tag{5.108}$$

$$\int_\Omega \tilde{S} \cdot Dv - (u \otimes u) \cdot \nabla v \,\mathrm{d}x = \langle f, v\rangle \quad \forall v \in W_{0,\mathrm{div}}^{1,\infty}(\Omega; \mathbb{R}^n). \tag{5.109}$$

Next, we will look at the concentration equation (5.40). From the strong $C^\beta(\overline{\Omega})$-convergence of c_h^m (5.91) follows the strong convergence $c_h^m \to c$ in $L^\infty(\Omega)$ as $m \to \infty$. Together with the strong $L^2(\Omega)$-convergence of u_h^m from equation (5.86), this yields

$$\|u_h^m c_h^m - uc\|_{L^2(\Omega)} \leq \|u_h^m - u\|_{L^2(\Omega)} \|c\|_{L^\infty(\Omega)} + \|u_h^m\|_{L^2(\Omega)} \|c_h^m - c\|_{L^\infty(\Omega)} \to 0. \tag{5.110}$$

Analogously, we have $\Pi_{h,V}^m v \to v$ strongly in $W^{1,2}(\Omega)$ for all $v \in W_0^{1,2}(\Omega)$ by the stability and approximability properties of V_h^m. Furthermore, the weak convergence $u_h^m \rightharpoonup u$ in $W_0^{1,r^-}(\Omega)$ from equation (5.85) implies strong convergence in $L^{n+\varepsilon}(\Omega)$ for some $\varepsilon > 0$ because $r^- > \frac{n}{2}$ and the compact Sobolev embedding theorem. Additionally, note that $\frac{1}{2^\star} > \left(\frac{1}{2} - \frac{1}{n(1+\varepsilon)}\right)^{-1} = \frac{2n(1+\varepsilon)}{n-2+n\varepsilon}$. This gives

$$\begin{aligned} &\|u_h^m \Pi_{h,V}^m v - uv\|_{L^2(\Omega)} \\ \le &\|u_h^m\|_{L^{n+\varepsilon}(\Omega)} \|\Pi_{h,V}^m v - v\|_{L^{\frac{2n(1+\varepsilon)}{n-2+n\varepsilon}}(\Omega)} + \|u_h^m - u\|_{L^{n+\varepsilon}(\Omega)} \|v\|_{L^{\frac{2n(1+\varepsilon)}{n-2+n\varepsilon}}(\Omega)} \\ \le &\|u_h^m\|_{L^{n+\varepsilon}(\Omega)} \|\Pi_{h,V}^m v - v\|_{W^{1,2}(\Omega)} + \|u_h^m - u\|_{L^{n+\varepsilon}(\Omega)} \|v\|_{W^{1,2}(\Omega)} \to 0 \end{aligned} \tag{5.111}$$

as $m \to \infty$. Therefore, we can combine the strong $L^2(\Omega)$-convergence of $u_h^m \Pi_{h,V}^m v$ and the weak $W^{1,2}(\Omega)$-convergence of c_h^m from equation (5.89) to find

$$\int_\Omega \Pi_{h,V}^m v u_h^m \cdot \nabla c_h^m \, \mathrm{d}x \to \int_\Omega v u \cdot \nabla c \, \mathrm{d}x \tag{5.112}$$

as $m \to \infty$ for every $v \in W_0^{1,2}(\Omega)$. Recall $\operatorname{div} u = 0$ a.e. in Ω from equation (5.102). Thus, an integration by parts in equation (5.112) yields

$$\int_\Omega \Pi_{h,V}^m v u_h^m \cdot \nabla c_h^m \, \mathrm{d}x \to -\int_\Omega c u \cdot \nabla v \, \mathrm{d}x \tag{5.113}$$

as $m \to \infty$ for every $v \in W_0^{1,2}(\Omega)$. Furthermore, recall the strong $L^2(\Omega)$-convergence $u_h^m c_h^m \to uc$ from equation (5.110). Together with $\Pi_{h,V}^m v \to v$ strongly in $W^{1,2}(\Omega)$, this gives

$$\int_\Omega c_h^m u_h^m \cdot \nabla \Pi_{h,V}^m v \, \mathrm{d}x \to \int_\Omega c u \cdot \nabla v \, \mathrm{d}x \tag{5.114}$$

as $m \to \infty$ for every $v \in W_0^{1,2}(\Omega)$. Finally, the weak convergence $q(c_h^m, \nabla c_h^m, D u_h^m) \rightharpoonup \tilde{q}$ from equation (5.90) yields

$$\int_\Omega q_c(c_h^m, \nabla c_h^m, D u_h^m) \cdot \nabla \Pi_{h,V}^m v \, \mathrm{d}x \to \int_\Omega \tilde{q} \cdot \nabla v \, \mathrm{d}x \tag{5.115}$$

as $m \to \infty$ for every $v \in W_0^{1,2}(\Omega)$.

After testing the concentration equation (5.37) against $v_h = \Pi_{h,V}^m v$ for an arbitrary $v \in W_0^{1,2}(\Omega)$, we can let $m \to \infty$ and with the help of equations (5.113), (5.114) and (5.115), we get

$$\int_\Omega \tilde{q} \cdot \nabla v - c u \cdot \nabla v \, \mathrm{d}x = 0. \tag{5.116}$$

Comparing equations (5.109) and (5.116) with equations (5.22) and (5.23), we see that the only thing that remains to be shown is to verify that u and c satisfy equations (5.22) and (5.23) is the identification of the limits $\tilde{S} = S(c, Du)$ and $\tilde{q} = q_c(c, \nabla c, Du)$.

Let us first make the technical assumption

$$\lim_{m\to\infty}\int_{\Omega}((S(c_h^m,Du_h^m)-S(c_h,Du))\cdot(Du_h^m-Du))^{\frac{1}{4}}\,\mathrm{d}x=0 \tag{5.117}$$

that we shall prove in the Appendix. Recall from the monotonicity assumption on S in inequality (5.7) that the integrand of equation (5.117) is nonnegative. Therefore, equation (5.117) is also true when integrating over a set $\Omega_\lambda\subset\Omega$ defined by

$$\Omega_\lambda:=\{x\in\Omega:|Du|\leq\lambda\}$$

for a given arbitrary constant $\lambda>0$. Furthermore, we can find a not relabelled subsequence of the integrands in equation (5.117) that converges to zero pointwise almost everywhere in Ω and therefore also in $\Omega_\lambda\subset\Omega$. For an arbitrary $\varepsilon>0$, we now can use Egoroff's Theorem (See Theorem 2.22 in [35]) to find a set $\Omega_\lambda^\varepsilon\subset\Omega_\lambda$ such that the sequence of integrands converges uniformly on $\Omega_\lambda^\varepsilon$ and $|\Omega_\lambda\setminus\Omega_\lambda^\varepsilon|<\varepsilon$. Obviously, this also implies

$$(S(c_h^m,Du_h^m)-S(c_h^m,Du))\cdot(Du_h^m-Du)\to 0 \tag{5.118}$$

almost everywhere uniformly on $\Omega_\lambda^\varepsilon$. Therefore, we get

$$\lim_{m\to\infty}\int_{\Omega_\lambda^\varepsilon}(S(c_h^m,Du_h^m)-S(c_h^m,Du))\cdot(Du_h^m-Du)\,\mathrm{d}x=0. \tag{5.119}$$

Additionally, $S(c_h^m,Du)\to S(c,Du)$ in $L^s(\Omega_\lambda^\varepsilon;\mathbb{R}^{n\times n})$ for all $s\in[1,\infty)$ because of the boundedness of Du on $\Omega_\lambda^\varepsilon$, the boundedness of S from inequality (5.6) and $c_h^m\to c$ from equation (5.91) and the uniform convergence theorem. Thus, we get

$$\int_{\Omega_\lambda^\varepsilon}S(c_h^m,Du)\cdot(Du_h^m-Du)\,\mathrm{d}x\to 0 \tag{5.120}$$

as $m\to\infty$, where we also have used the weak convergence $Du_h^m\rightharpoonup Du$ from equation (5.85). Furthermore, the weak convergence $S(c_h^m,Du_h^m)\rightharpoonup\tilde{S}$ from equation (5.87) yields

$$\int_{\Omega_\lambda^\varepsilon}S(c_h^m,Du_h^m)\cdot Du\,\mathrm{d}x\to\int_{\Omega_\lambda^\varepsilon}\tilde{S}\cdot Du\,\mathrm{d}x \tag{5.121}$$

as $m\to\infty$. Combining equations (5.119), (5.120) and (5.121) yields

$$\int_{\Omega_\lambda^\varepsilon}S(c_h^m,Du_h^m)\cdot Du_h^m\,\mathrm{d}x\to\int_{\Omega_\lambda^\varepsilon}\tilde{S}\cdot Du\,\mathrm{d}x \tag{5.122}$$

as $m\to\infty$.

Let us now fix an arbitrary $A\in L^\infty(\Omega;\mathbb{R}^{n\times n})$. The monotonicity of S from equation (5.7) gives

$$\int_{\Omega_\lambda^\varepsilon}(S(c_h^m,Du_h^m)-S(c_h^m,A))\cdot(Du_h^m-A)\,\mathrm{d}x\geq 0. \tag{5.123}$$

We will now use equation (5.122), the strong $L^q(\Omega_\lambda^\varepsilon;\mathbb{R}^{n\times n})$-convergence $S(c_h^m, A) \to S(c, A)$ and the weak $L^{r^-}(\Omega;\mathbb{R}^{n\times n})$-convergence $Du_h^m \rightharpoonup Du$ to take the limit $m \to \infty$ in equation (5.123). This gives

$$\begin{aligned} 0 &\le \int_{\Omega_\lambda^\varepsilon} \tilde{S} \cdot (Du - A) - S(c, A) \cdot (Du - A) \, \mathrm{d}x \\ &= \int_{\Omega_\lambda^\varepsilon} \Big(\tilde{S} - S(c, A)\Big) \cdot (Du - A) \, \mathrm{d}x. \end{aligned} \tag{5.124}$$

We now want to use Minty's trick. Recall that Du is bounded on $\Omega_\lambda^\varepsilon$. This means that inequality (5.124) is true for $A = Du \pm \eta B$ for all arbitrary $B \in L^\infty(\Omega;\mathbb{R}^{n\times n})$ and $\eta > 0$. Therefore, we get

$$\pm \int_{\Omega_\lambda^\varepsilon} \Big(\tilde{S} - S(c, Du \pm \eta B)\Big) \cdot B \, \mathrm{d}x \ge 0. \tag{5.125}$$

S is continuous by definition. We can take $\eta \to 0$ thanks to the dominated convergence theorem and get

$$\int_{\Omega_\lambda^\varepsilon} \Big(\tilde{S} - S(c, Du)\Big) \cdot B \, \mathrm{d}x = 0. \tag{5.126}$$

We conclude

$$\tilde{S} = S(c, Du) \text{ a.e. on } \Omega_\lambda^\varepsilon. \tag{5.127}$$

By construction, we have $\lim_{\varepsilon\to 0} \lim_{\lambda\to 0} |\Omega \setminus \Omega_\lambda^\varepsilon| = 0$. Therefore, we can let $\varepsilon \to 0$ and $\lambda \to 0$ in equation (5.127) to finally find

$$\tilde{S} = S(c, Du) \text{ a.e. on } \Omega. \tag{5.128}$$

Now, recall that from equation (5.117), we can extract a subsequence of integrands that converges almost everywhere in Ω. Furthermore, recall from equation (5.91) that $c_h^m \to c$ in $C^\beta(\overline{\Omega})$ and that S is strictly monotonic and continuous from equation (5.7). Therefore, we have

$$Du_h^m \to Du \text{ a.e. on } \Omega. \tag{5.129}$$

Additionally, recall that $q_c(c, \nabla c, Du)$ was assumed to be continuous and linear with respect to the second variable. Thus, we find

$$q_c(c_h^m, \nabla c_h^m, Du_h^m) \rightharpoonup q_c(c, \nabla c, Du) \text{ weakly in } L^2(\Omega;\mathbb{R}^n) \tag{5.130}$$

with the help of equations (5.89), (5.91) and (5.129) and the dominated convergence theorem. On the other hand, we know from equation (5.90) that $q_c(c_h^m, \nabla c_h^m, Du_h^m) \rightharpoonup \tilde{q}$ weakly in $L^2(\Omega;\mathbb{R}^n)$. Combining this with the weak limit in equation (5.130), we get

$$\tilde{q} = q_c(c, \nabla c, Du). \tag{5.131}$$

Finally, combining equations (5.128) and (5.131) with the limits (5.108), (5.109) and (5.116) proves that the limits u, c, p solve the systems (5.19)–(5.21) or (5.22)–(5.23).

In this chapter, we have extended the result of [43] to $n = 3$ or more space dimensions. We do not rely on regularization of the equations like in [44] any more, but have to restrict ourselves to uniformly acute triangulations, piecewise affine functions for the discrete concentration space and a concentration flux of the form $q_c(c, \nabla c, Du) = k(c, Du)\nabla c$ with a scalar function k.

5.5 Appendix

We will need a discrete version of the Lipschitz truncation in the variable exponent setting. This was first analysed in [25] and then generalised in [43, Theorem 3.15].

Theorem 5.14. *Let $\Omega \subset \mathbb{R}^n$ be a polyhedral Lipschitz domain and let $v_h^m \in X_{h,\mathrm{div}}^m$, $r^m \in C^\alpha(\overline{\Omega})$ be sequences that satisfy $0 < r^- \leq r^m \leq r^+ < \infty$ and*

$$\int_\Omega |\nabla v_h^m|^{r^m(x)} \, \mathrm{d}x \leq C, \tag{5.132}$$

$$v_h^m \rightharpoonup v \qquad \text{weakly in } W_0^{1,r^-}(\Omega; \mathbb{R}^n), \tag{5.133}$$

$$r^m \to r \qquad \text{strongly in } C^\alpha\left(\overline{\Omega}; \mathbb{R}\right) \tag{5.134}$$

for some $C > 0$, $\alpha > 0$.

Then, for every $j \in \mathbb{N}$, there exists a sequence λ_j^m with $(2^j)^{2^j} \leq \lambda_j^m \leq (2^{j+1})^{2^{j+1}}$ and a sequence $v_j^m \in X_h^m \subset W^{1,\infty}(\Omega, \mathbb{R}^n)$ with

$$\|\nabla v_j^m\|_{L^\infty(\Omega)} \lesssim \lambda_j^m \leq \left(2^{j+1}\right)^{j+1}. \tag{5.135}$$

Furthermore, we can extract a subsequence $\{v_j^m\}_{m=1}^\infty$ (not relabelled) for all $j \in \mathbb{N}$, such that

$$v_j^m \to v_j \qquad \text{strongly in } L^s(\Omega; \mathbb{R}^n) \qquad \forall s \in (1, \infty), \tag{5.136}$$

$$v_j^m \rightharpoonup v_j \qquad \text{weakly in } W^{1,s}(\Omega; \mathbb{R}^n) \qquad \forall s \in (1, \infty), \tag{5.137}$$

$$\nabla v_j^m \rightharpoonup^* \nabla v_j \quad \text{weakly-* in } L^\infty(\Omega; \mathbb{R}^{n \times n}) \tag{5.138}$$

for some $v_j \in W^{1,\infty}(\Omega; \mathbb{R}^n)$. Additionally, we have

$$\|v_j\|_{L^r(\Omega)} \leq C \tag{5.139}$$

and can extract a (not relabelled) subsequence, such that

$$v^j \rightharpoonup v \tag{5.140}$$

weakly in $W^{1,s}(\Omega)$ *for any* $s \in (1,\infty)$.

Moreover, if we extend v_h^m *by zero outside of* $\overline{\Omega}$, *we have*

$$\{x \in \Omega : v_j^m \neq v_h^m\} \subset \{x \in \Omega : M(\nabla v_h^m) > \kappa\lambda_j^m\} \tag{5.141}$$

where M *is the Hardy–Littlewood maximal function and* κ *is a constant that depends on the shape-regularity constant* Γ *of the mesh, that is defined in the following way:*

$$U_\lambda(v) \cap \Omega \subset \Omega(U_\lambda(v)) \subset U_{\kappa\lambda}(v) \cap \Omega \tag{5.142}$$

where $U_\lambda(v) := \{x \in \mathbb{R}^n : M(v) > \lambda\}$. *Furthermore, the following estimate is true for all* n, j:

$$\int_\Omega \left|\nabla v_j^m \chi_{\{v_j^m \neq v_h^m\}}\right|^{r^m(x)} \mathrm{d}x \lesssim \int_\Omega \left|\lambda_j^m \chi_{\{v_j^m \neq v_h^m\}}\right|^{r^m(x)} \mathrm{d}x \lesssim \frac{1}{2^j}. \tag{5.143}$$

Proof. [43, Theorem 3.15]. □

Proposition 5.15 (Discrete Bogovskiĭ Operator)**.** *Let* r^m *be a sequence in* $C^\alpha(\overline{\Omega})$ *with* $r^m \to r$ *for* $m \to \infty$ *and* $0 < r^- \leq r^m \leq r^+ < \infty$. *Furthermore, let* $\mathcal{B} : L_0^{r^m(\cdot)}(\Omega) \to W_0^{1,r^m(\cdot)}(\Omega,\mathbb{R}^n)$ *be the classical Bogovskiĭ operator in varying exponent spaces that is given by*

$$\mathrm{div}\mathcal{B}f = f \tag{5.144}$$

with

$$\|\mathcal{B}f\|_{W^{1,r(\cdot)}(\Omega;\mathbb{R}^n)} \leq \|f\|_{L^{r(\cdot)}(\Omega)}. \tag{5.145}$$

For more details see [23][Theorem 14.3.15.]. Then, the operators $\mathcal{K} : L^{r^m(\cdot)}(\Omega) \to L^{r^m(\cdot)}(\Omega)$ *and* $\mathcal{B}_h^m : \mathrm{div}X_h^m \to X_h^m$ *that are defined by*

$$\int_\Omega f\Pi_{h,Q}^m g \,\mathrm{d}x = \int_\Omega g\mathcal{K}f \,\mathrm{d}x \qquad \text{for all } g \in L^{(r^m)'(\cdot)}(\Omega) \tag{5.146}$$

$$\mathcal{B}_h^m(f) = \Pi_{h,\mathrm{div}}^m \mathcal{B}\mathcal{K}f \tag{5.147}$$

have the following properties:

(a) For all $f \in L^{r^m(\cdot)}(\Omega)$, *we have*

$$\|\mathcal{K}f\|_{L^{r^m(\cdot)}(\Omega)} \lesssim \sup_{q \in Q_h^m, \|q\|_{L^{(r^m)'(\cdot)}(\Omega)} \leq 1} \int_\Omega fq \,\mathrm{d}x.$$

(b) For all $f_h \in \mathrm{div}X_h^m$ *and* $q_h \in Q_h^m$, *we have*

$$\int_\Omega \mathrm{div}(\mathcal{B}_h^m f_h) q_h \,\mathrm{d}x = \int_\Omega f_h q_h \,\mathrm{d}x.$$

(c) If f_h^m is a sequence with $f_h^m \rightharpoonup f$ weakly in $W_0^{1,s}(\Omega;\mathbb{R}^n)$ as $m \to \infty$, we have

$$\mathcal{B}_h^m \operatorname{div} f_h^m \rightharpoonup \mathcal{B} \operatorname{div} f \text{ weakly in } W_0^{1,s}(\Omega;\mathbb{R}^n).$$

Proof. See [43], Section 3.4. □

Lemma 5.16. *Under the assumptions of Chapter 5, sections 5.1 and 5.2, we have*

$$\int_\Omega \left((S(c_h^m, Du_h^m) - S(c_h^m, Du)) \cdot (Du_h^m - Du) \right)^{\frac{1}{4}} \, \mathrm{d}x = 0.$$

Proof. We follow the argumentation in [43]. First note that we can use the monotonicity assumption on S from inequality (5.7). Furthermore, we use the boundedness of S from inequality (5.6) and the (m-uniform) boundedness of $\|Du_h^m\|_{L^{r(c_h^m)}(\Omega)}$ and $\|Du\|_{L^{r(c)}(\Omega)}$ from inequalities (5.60) and (5.97), respectively. We find

$$0 \le \limsup_{m\to\infty} \int_\Omega \left((S(c_h^m, Du_h^m) - S(c_h^m, Du)) \cdot (Du_h^m - Du) \right)^{\frac{1}{4}} \, \mathrm{d}x \le L < \infty. \tag{5.148}$$

Thus, it suffices to show $L = 0$. For an arbitrary (but fixed) $\gamma > 0$, let us define

$$A_\gamma = \{x \in \Omega : |Du| > \gamma\}. \tag{5.149}$$

We decompose the integral in inequality (5.150):

$$\begin{aligned} &\int_\Omega \left((S(c_h^m, Du_h^m) - S(c_h^m, Du)) \cdot (Du_h^m - Du) \right)^{\frac{1}{4}} \, \mathrm{d}x \\ \le &\int_{A_\gamma} \left((S(c_h^m, Du_h^m) - S(c_h^m, Du)) \cdot (Du_h^m - Du) \right)^{\frac{1}{4}} \, \mathrm{d}x \\ &+ \int_{\Omega \setminus A_\gamma} \left((S(c_h^m, Du_h^m) - S(c_h^m, Du)) \cdot (Du_h^m - Du) \right)^{\frac{1}{4}} \, \mathrm{d}x \\ =: &A^m(\gamma) + B^m(\gamma). \end{aligned} \tag{5.150}$$

To be able to estimate $A^m(\gamma)$, we use a weak-type estimate and $r^- > 1$ together with Hölder's inequality and the embedding $L^{r(c)}(\Omega) \hookrightarrow L^1(\Omega)$ from Proposition 5.2 (b) to find that

$$|A_\gamma| \le \int_\Omega \frac{|Du|}{\gamma} \, \mathrm{d}x \lesssim \frac{1}{\gamma}. \tag{5.151}$$

Again, we use the boundedness of S from inequality (5.6) and the (m-uniform) boundedness of $\|Du_h^m\|_{L^{r(c_h^m)}}$ and $\|Du\|_{L^{r(c)}}$ from inequalities (5.60) and (5.97), respectively. Together with Hölder's inequality and inequality (5.151), this yields

$$A^m(\gamma) \lesssim |A_\gamma|^{\frac{1}{2}} \lesssim \frac{1}{\sqrt{\gamma}}. \tag{5.152}$$

Now, recall the truncation T_γ from equation (4.30):

$$T_\gamma X = \begin{cases} X \text{ if } |X| \le \gamma, \\ \gamma \frac{X}{|X|} \text{ if } |X| > \gamma \end{cases} \tag{5.153}$$

for $X \in \mathbb{R}^{n\times n}$. Obviously, we have $T_\gamma Du = Du$ on $\Omega \setminus A_\gamma$. Thus, we can find for $B^m(\gamma)$ from equation (5.150):

$$\begin{aligned} B^m(\gamma) &= \int_{\Omega\setminus A_\gamma} ((S(c_h^m, Du_h^m) - S(c_h^m, T_\gamma Du)) \cdot (Du_h^m - T_\gamma Du))^{\frac{1}{4}} \,\mathrm{d}x \\ &\le \int_\Omega ((S(c_h^m, Du_h^m) - S(c_h^m, T_\gamma Du)) \cdot (Du_h^m - T_\gamma Du))^{\frac{1}{4}} \,\mathrm{d}x, \end{aligned} \tag{5.154}$$

where we have used that the integrand is nonnegative thanks to the monotonicity of S from inequality (5.7). We want to use the discrete Lipschitz truncation. Because $c_h^m \to c$ in $C^\beta\left(\overline{\Omega}\right)$, the assumptions of Theorem 5.14 are satisfied and therefore, we can find a sequence $u_h^{m,j}$ that satisfies (5.135) to (5.143). We decompose the integral once more and use Hölder's inequality to find

$$\begin{aligned} B^m(\gamma) \le &\left(\int_{\{u_h^m = u_h^{m,j}\}} ((S(c_h^m, Du_h^m) - S(c_h^m, T_\gamma Du)) \cdot (Du_h^m - T_\gamma Du)) \,\mathrm{d}x\right)^{\frac{1}{4}} |\Omega|^{\frac{3}{4}} \\ &+ \left(\int_{\{u_h^m \ne u_h^{m,j}\}} ((S(c_h^m, Du_h^m) - S(c_h^m, T_\gamma Du)) \cdot (Du_h^m - T_\gamma Du))^{\frac{1}{2}} \,\mathrm{d}x\right)^{\frac{1}{2}} \\ &\cdot |\{u_h^{m,j} \ne u_h^m\}|^{\frac{1}{2}} \\ =:\ & B^{m,j}(\gamma)|\Omega|^{\frac{3}{4}} + \tilde{B}^{m,j}(\gamma)|\{u_h^{m,j} \ne u_h^m\}|^{\frac{1}{2}}. \end{aligned} \tag{5.155}$$

We will first look at $\tilde{B}^{m,j}(\gamma)$. We use the estimate on $M(\nabla u)$ on $\{u_h^{m,j} \ne u_h^m\}$ from equation (5.141) with a weak-type estimate, the $L^1(\Omega)$-stability of the Hardy–Littlewood maximal function together with $\lambda_j^m \ge \left(2^j\right)^{\left(2^j\right)}$ from inequality (5.135) and the uniform $L^{r^-}(\Omega)$-bound on ∇u_h^m from inequality (5.60). We have that

$$|\{u_h^{m,j} \ne u_h^m\}| \le |\{M(Du_h^{m,j}) > \lambda_j^m\}| \lesssim \int_{\mathbb{R}^n} \frac{M(Du_h^{m,j})}{\lambda_j^m} \lesssim \frac{1}{(2^j)^{(2^j))}}. \tag{5.156}$$

Hence, we find

$$\tilde{B}^{m,j}(\gamma)|\{u_h^{m,j} \ne u_h^m\}|^{\frac{1}{2}} \lesssim \frac{1}{2^j}. \tag{5.157}$$

Now, we will look at $B^{m,j}(\gamma)$ from inequality (5.155). Note that we are working on the set $\{u_h^{mj} = u_h^m\}$. We rewrite the integral:

$$\begin{aligned} \left(B^{m,j}(\gamma)\right)^4 = &\int_\Omega (S(c_h^m, Du_h^m) - S(c_h^m, T_\gamma Du)) \cdot \left(Du_h^{m,j} - T_\gamma Du\right) \,\mathrm{d}x \\ &- \int_{\{u_h^m \ne u_h^{m,j}\}} (S(c_h^m, Du_h^m) - S(c_h^m, T_\gamma Du)) \cdot \left(Du_h^{m,j} - T_\gamma Du\right) \,\mathrm{d}x. \end{aligned} \tag{5.158}$$

First, we will look at the second summand of equation (5.158). Recall that $|T_\gamma Du| \leq \gamma$ and the bound of $Du_h^{m,j}$ from inequalities (5.135) and (5.143). We also use Young's inequality, the weak-type estimate (5.157) and the bound on the growth of S from inequality (5.6) to get

$$\begin{aligned}
&\left| \int_{\{u_h^m \neq u_h^{m,j}\}} (S(c_h^m, Du_h^m) - S(c_h^m, T_\gamma Du)) \cdot \left(Du_h^{m,j} - T_\gamma Du \right) \mathrm{d}x \right| \\
&\leq \int_{\{u_h^m \neq u_h^{m,j}\}} S(c_h^m, Du_h^m) \cdot Du_h^{m,j} \,\mathrm{d}x \\
&\quad + C(\gamma) \int_{\{u_h^m \neq u_h^m\}} \left(|S(c_h^m, Du_h^{m,j})| + |Du_h^{m,j}| + 1 \right) \mathrm{d}x \\
&\lesssim C(\gamma) \left(\int_{\{u_h^m \neq u_h^m\}} \left| \nabla u_h^{m,j} \right|^{r(c_h^m)} \mathrm{d}x + \left| \{u_h^m \neq u_h^{m,j}\} \right| \right) \\
&\lesssim \frac{C(\gamma)}{2^j}.
\end{aligned} \tag{5.159}$$

This leaves us with the first summand of the right-hand side of equation (5.158). We will have to use the weak formulation of the problem. However, $u_h^{m,j}$ is not necessarily discretely divergence-free and therefore not an applicable test-function. However, with the help of the discrete Bogovskiĭ operator in the variable exponent context that is given by Proposition 5.15 we can approximate $u_h^{m,j}$ by a discretely divergence-free function with zero trace. We define

$$w_h^{m,j} := \mathcal{B}^m \left(\mathrm{div} u_h^{m,j} \right), \tag{5.160}$$

$$v_h^{m,j} := u_h^{m,j} - w_h^{m,j}. \tag{5.161}$$

The convergence statement for the discrete Lipschitz truncation in equation (5.137) and for the discrete Bogovskiĭ operator in Proposition 5.15 (c) and the compact embedding $L^s(\Omega) \hookrightarrow W_0^{1,s}(\Omega)$ yield

$$v_h^{m,j} \rightharpoonup u^j - \mathcal{B}(\mathrm{div} u^j) =: v^j \quad \text{weakly in } W_0^{1,s}(\Omega;\mathbb{R}^n), \tag{5.162}$$

$$v_h^{m,j} \to v^j \quad \text{strongly in } L^s(\Omega;\mathbb{R}^n) \tag{5.163}$$

for any $s \in (1,\infty)$ as $m \to \infty$. Obviously, we have $u_h^{m,j} = v_h^{m,j} + w_h^{m,j}$. Hence, we can write

$$\begin{aligned}
&\int_\Omega \left(S\left(c_h^m, Du_h^m\right) - S\left(c_h^m, T_\gamma Du\right) \right) \cdot \left(Du_h^{m,j} - T_\gamma Du \right) \mathrm{d}x \\
&= \int_\Omega S(c_h^m, Du_h^m) \cdot Dv_h^{m,j} \,\mathrm{d}x + \int_\Omega S(c_h^m, Du_h^m) \cdot Dw_h^{m,j} \,\mathrm{d}x \\
&\quad - \int_\Omega S(c_h^m, Du_h^m) \cdot T_\gamma Du \,\mathrm{d}x - \int_\Omega S(c_h^m, T_\gamma Du) \cdot \left(Du_h^{m,j} - T_\gamma Du \right) \mathrm{d}x \\
&=: B_1^{m,j} + B_2^{m,j} - B_3^m(\gamma) - B_4^{m,j}(\gamma).
\end{aligned} \tag{5.164}$$

Let us first look at $B_2^{m,j}$. Recall the definition $w_h^{m,j} = \mathcal{B}^m \mathrm{div} u_h^{m,j}$. We use the uniform boundedness of $\|S(c_h^m, Du_h^m)\|_{L^{r(c_h^m)}(\Omega)}$ from inequality (5.60) and Hölder's inequality in variable exponent spaces (see Proposition 5.2 (c)) to find

$$\int_\Omega S(c_h^m, Du_h^m) \cdot Dw_h^{m,j} \,\mathrm{d}x \lesssim \|Dw_h^{m,j}\|_{L^{r(c_h^m)}(\Omega)} \lesssim \|\Pi_{h,\mathrm{div}}^m \mathcal{B}\mathcal{K}(\mathrm{div} u_h^{m,j})\|_{W^{1,r(c_h^m)}(\Omega)}. \tag{5.165}$$

Recall the stability of $\Pi_{h,\mathrm{div}}^m$ in the variable exponent setting from inequality (5.33). We find

$$\|\Pi_{h,\mathrm{div}}^m \mathcal{B}\mathcal{K}(\mathrm{div} u_h^{m,j})\|_{W^{1,r(c_h^m)}(\Omega)} \lesssim \left(\|\mathcal{B}\mathcal{K}(\mathrm{div} u_h^{m,j})\|_{W^{1,r(c_h^m)}(\Omega)} + \max_{T \in \mathcal{T}_h^n} h_T^m \right)^\beta \tag{5.166}$$

for some $\beta > 0$ that depends on r^+ and r^-. The Bogovskiĭ operator is bounded as an operator from $L_0^{r(\cdot)}(\Omega)$ to $W^{1,r(\cdot)}(\Omega; \mathbb{R}^n)$ (see inequality (5.145)). Together with Proposition 5.15 (a), this yields

$$\begin{aligned} \|\mathcal{B}\mathcal{K}(\mathrm{div} u_h^{m,j})\|_{W^{1,r(c_h^m)}(\Omega)} &\lesssim \|\mathcal{K}(\mathrm{div} u_h^{m,j})\|_{L^{r(c_h^m)}(\Omega)} \\ &\lesssim \sup_{q \in Q_h^m, \|q_h\|_{L^{(r(c^m))'}(\Omega)} \leq 1} \int_\Omega (\mathrm{div} u_h^{m,j}) q_h \,\mathrm{d}x. \end{aligned} \tag{5.167}$$

Now, we use the fact that u_h^m is discretely divergence free and Hölder's inequality to find

$$\begin{aligned} \int_\Omega (\mathrm{div} u_h^{m,j}) q_h \,\mathrm{d}x &= \sum_{\substack{T \in \mathcal{T}_h^m \\ T \subset \{u_h^{m,j} = u_h^m\}}} \int_\Omega (\mathrm{div} u_h^m) q_h \chi_T \,\mathrm{d}x \\ &\quad + \sum_{\substack{T \in \mathcal{T}_h^m \\ T \subset \Omega(\{u_h^{m,j} \neq u_h^m\})}} \int_\Omega (\mathrm{div} u_h^{m,j}) q_h \chi_T \,\mathrm{d}x \\ &\leq \left\| (\mathrm{div} u_h^{m,j}) \chi_{\Omega(\{u_h^{m,j} \neq u_h^m\})} \right\|_{L^{r(c_h^m)}(\Omega)} \left\| q_h \chi_{\Omega(\{u_h^{m,j} \neq u_h^m\})} \right\|_{L^{r'(c_h^m)}(\Omega)} \\ &\leq \left\| (\nabla u_h^{m,j}) \chi_{\Omega(\{u_h^{m,j} \neq u_h^m\})} \right\|_{L^{r(c_h^m)}(\Omega)} \|q_h\|_{L^{r'(c_h^m)}(\Omega)}. \end{aligned} \tag{5.168}$$

Then, the inclusion (5.142) and inequality (5.143) yield

$$\left\| (\nabla u_h^{m,j}) \chi_{\Omega(\{u_h^{m,j} \neq u_h^m\})} \right\|_{L^{r(c_h^m)}(\Omega)} \lesssim 2^{-\frac{j}{r^+}}. \tag{5.169}$$

Therefore, we can combine inequalities (5.166), (5.167), (5.168) and (5.169) to find

$$\|\Pi_{h,\mathrm{div}}^m \mathcal{B}\mathcal{K}(\mathrm{div} u_h^{m,j})\|_{W^{1,r(c_h^m)}} \lesssim \left(2^{-\frac{j}{r^+}} + \max_{T \in \mathcal{T}_h^m} h_T^n \right)^\beta \tag{5.170}$$

Together with inequality (5.165), this yields

$$\int_\Omega S(c_h^m, Du_h^m) \cdot Dw_h^{m,j} \,\mathrm{d}x \lesssim \left(2^{-\frac{j}{r^+}} + \max_{T \in \mathcal{T}_h^m} h_T^n \right)^\beta \tag{5.171}$$

and therefore

$$\limsup_{m\to\infty} B_2^{m,j} \lesssim 2^{-j\frac{\beta}{r^+}}. \tag{5.172}$$

We will now look at $B_1^{m,j}$ from equation (5.164). By testing equation (5.39) against $v_h^{m,j}$, we obtain

$$\int_\Omega S(c_h^m, Du_h^m)\cdot Dv_h^{m,j}\,\mathrm{d}x = -\frac{1}{2}\int_\Omega \Big((u_h^m\otimes v_h^{m,j})\cdot\nabla u_h^m - (u_h^m\otimes u_h^m)\cdot\nabla v_h^{m,j}\Big)\,\mathrm{d}x + \langle f, v_h^{m,j}\rangle.$$

With the help of equations (5.86), (5.109), (5.102) and (5.162), we can pass to the limit and get

$$\lim_{m\to\infty}\int_\Omega S(c_h^m, Du_h^m)\cdot Dv_h^{m,j}\,\mathrm{d}x = \int_\Omega (u\otimes u)\cdot\nabla v^j\,\mathrm{d}x + \langle f, v^j\rangle = \int_\Omega \tilde{S}\cdot Dv^j\,\mathrm{d}x. \tag{5.173}$$

Then, recall $v^j = u^j - \mathcal{B}(\mathrm{div}u^j)$. We use the boundedness of $\|\tilde{S}\|_{L^{r'(c)}(\Omega)}$ and weak lower-semicontinuity to find

$$\int_\Omega \tilde{S}\cdot D\mathcal{B}(\mathrm{div}u^j)\,\mathrm{d}x \lesssim \|\mathcal{B}(\mathrm{div}u^j)\|_{L^{r(c)}(\Omega)} \lesssim \limsup_{m\to\infty}\left\|\Pi_{h,\mathrm{div}}^m\mathcal{B}\mathcal{K}(\mathrm{div}u_h^{m,j})\right\|_{W^{1,r(c_h^m)}(\Omega)}. \tag{5.174}$$

Hence, we can use inequality (5.170) to find

$$\int_\Omega \tilde{S}\cdot D\mathcal{B}(\mathrm{div}u^j)\,\mathrm{d}x \lesssim 2^{-j\frac{\beta}{r^+}} \tag{5.175}$$

and therefore

$$\lim_{m\to\infty} B_1^{m,j} \lesssim \int_\Omega \tilde{S}\cdot Du^j\,\mathrm{d}x + 2^{-j\frac{\beta}{r^+}}. \tag{5.176}$$

For the analysis of $B_3^m(\gamma)$ from equation (5.164), we use the weak convergence of $S(c_h^m, Du_h^m)$ from equation (5.87) to find

$$\lim_{m\to\infty} B_3^m(\gamma) = \lim_{m\to\infty}\int_\Omega S(c_h^m, Du_h^m)\cdot T_\gamma(Du)\,\mathrm{d}x = \int_\Omega \tilde{S}\cdot T_\gamma(Du)\,\mathrm{d}x. \tag{5.177}$$

Now, we will look at $B_4^{m,j}$ from equation (5.164). We use the strong $C^\alpha\left(\overline{\Omega}\right)$-convergence $c_h^m \to c$ from equation (5.91) and the weak $L^s(\Omega)$-convergence $Du_h^{j,m} \rightharpoonup Du^j$ from equation (5.137) to find

$$\begin{aligned}\lim_{m\to\infty} B_4^{m,j}(\gamma) &= \lim_{m\to\infty}\int_\Omega S(c_h^m, T_\gamma Du)\cdot\Big(Du_h^{m,j} - T_\gamma Du\Big)\,\mathrm{d}x\\ &= \int_\Omega S(c, T_\gamma Du)\cdot\big(Du^j - T_\gamma Du\big)\,\mathrm{d}x.\end{aligned} \tag{5.178}$$

Finally, we insert the estimates (5.172), (5.176), (5.177) and (5.178) into equation (5.164) and together with inequalities (5.155), (5.157) (5.158) and (5.159) and the equivalence of norms in finite dimensions, this yields

$$\begin{aligned}\lim_{m\to\infty} B^m(\gamma) \le C&\left(\int_\Omega\Big(\tilde{S} - S(c, T_\gamma Du)\Big)\cdot\big(Du^j - T_\gamma Du\big)\,\mathrm{d}x\right)^{\frac{1}{4}}\\ &+ C(\gamma)\left(2^{-j} + 2^{-j\frac{\beta}{4r^+}}\right)\end{aligned} \tag{5.179}$$

for any $j \in \mathbb{N}$. Thus, we can combine the inequalities (5.150), (5.152) and (5.179) to find

$$\begin{aligned} &\lim_{m\to\infty} \int_\Omega ((S(c_h^m, Du_h^m) - S(c_h^m, Du)) \cdot (Du_h^m - Du))^{\frac{1}{4}} \, \mathrm{d}x \\ \leq &C \left(\int_\Omega \left(\tilde{S} - S(c, T_\gamma Du) \right) \cdot \left(Du^j - T_\gamma Du \right) \, \mathrm{d}x \right)^{\frac{1}{4}} \\ &+ C(\gamma) \left(2^{-j} + 2^{-j\frac{\beta}{r^+}} \right) + \frac{C}{\sqrt{\gamma}}. \end{aligned} \tag{5.180}$$

Then, we can take the limit $j \to \infty$ in inequality (5.180) and use the weak convergence $Du^j \rightharpoonup Du$ from equation (5.140) to find

$$\begin{aligned} &\lim_{j\to\infty} \lim_{m\to\infty} \int_\Omega ((S(c_h^m, Du_h^m) - S(c_h^m, Du)) \cdot (Du_h^m - Du))^{\frac{1}{4}} \, \mathrm{d}x \\ \leq &C \left(\int_\Omega \left(\tilde{S} - S(c, T_\gamma Du) \right) \cdot (Du - T_\gamma Du) \, \mathrm{d}x \right)^{\frac{1}{4}} + \frac{C}{\sqrt{\gamma}}. \end{aligned} \tag{5.181}$$

Finally, we can use the pointwise convergence $T_\gamma Du \to Du$ and the dominated convergence theorem to take the limit $\gamma \to \infty$ in equation (5.181) and have therefore proven that

$$\int_\Omega ((S(c_h^m, Du_h^m) - S(c_h^m, Du)) \cdot (Du_h^m - Du))^{\frac{1}{4}} \, \mathrm{d}x = 0.$$

□

Chapter 6

Conclusions

In this thesis, we have investigated a finite element counterpart of the regularity theory of De Giorgi for elliptic partial differential equations. Chapter 3 was concerned with linear equations of the form

$$-\mathrm{div}(A\nabla u) = f - \mathrm{div}F \text{ in } \Omega,$$
$$u = g \text{ on } \partial\Omega$$

where $A \in L^\infty(\Omega;\mathbb{R}^{n\times n})$ is a given matrix-valued, uniformly elliptic function, and $f \in L^q(\Omega)$, $F \in L^p(\Omega;\mathbb{R}^n)$ and $g \in C^\alpha(\partial\Omega)$ are given functions for $q > \frac{n}{2}$, $p > n$ and $\alpha > 0$. We have analysed finite element approximations in spaces of piecewise affine functions V_h. The proof of a uniform Hölder-norm bound was based on the notion of a *discrete subsolution*, i.e. functions $u_h \in V_h$ that satisfy

$$\int_\Omega A\nabla u_h \cdot \nabla\varphi_h \,\mathrm{d}x \le \int_\Omega F\cdot\nabla\varphi_h \,\mathrm{d}x + \int_\Omega f\varphi_h \,\mathrm{d}x$$

for every non-negative function $\varphi_h \in V_{h,0}$. The first main ingredient is a Caccioppoli-type inequality for discrete subsolutions that locally estimates ∇u_h in terms of u_h, which was given in Theorem 3.7 for discrete subsolutions on shape-regular meshes:

$$\int_\Omega |\nabla u_h|^2|\eta_h|^2 \,\mathrm{d}x \lesssim \int_\Omega |u_h|^2|\nabla\eta_h|^2 \,\mathrm{d}x + \left(\|F\|^2_{L^p(\Omega)} + \|f\|^2_{L^q(\Omega)}\right)\left(\|\eta_h\|^2_{L^{2^\star}(\operatorname{supp} u_h)} + \|\nabla\eta_h\|^2_{L^2(\operatorname{supp} u_h)}\right)|\operatorname{supp}\eta_h \cap \operatorname{supp} u_h|^{\frac{2\delta}{n}}$$

for every cutoff-function $\eta \in V_{h,0}$. For a De-Giorgi-type strategy, this estimate has to be true for truncations of the solution. In the continuous case, one of the most commonly used truncations is given by the positive part $(u-c)_+$. The point-wise truncation of a discrete subsolution would not belong to V_h, which is why we had to use a *nodal positive part* $(u_h - c)_+ = \sum_i (u_h(x_i) - c)_+\psi_i$, where the ψ_i are the Lagrange basis functions of V_h. Then, the second main ingredient is the proof that for a discrete subsolution u_h, the truncation $(u_h - c)_+$ is indeed a discrete subsolution, as well. This step required the following two conditions: The mesh has to be *A-non-obtuse*, i.e. $A\nabla\psi_i \cdot \psi_j \leq 0$ for all $i \neq j$ and F had to satisfy assumption $(\star)$, i.e. there has to be a $G \in L^p(\Omega;\mathbb{R}^n)$ such that $\int_\Omega G \cdot \nabla\varphi \, \mathrm{d}x \geq \left|\int_\Omega F \cdot \nabla\varphi \, \mathrm{d}x\right|$ for all non-negative functions $\varphi \in C_0^\infty(\Omega)$. With those results in place, we were able to prove a local L^∞-estimate

$$\max_{\Omega'(B(x_0,R))} (u_h - c)_+^2 \lesssim \fint_{\Omega(B(x_0,2R))} (u_h - c)_+^2 \, \mathrm{d}x + \left(\|G\|_{L^p}^2 + \|f\|_{L^q}^2\right) R^{2\delta}$$

for all $c \in \mathbb{R}$. Then, we focussed on establishing the desired *a priori* $C^\alpha\left(\overline{\Omega}\right)$-norm estimate. In Lemma 3.21, we established an estimate that guarantees that if u_h is small at some node x_i, it cannot be too large on neighbouring nodes. Then, the most important step was the oscillation decay given in Lemma 3.23:

$$\operatorname*{osc}_{\Omega'(B(x_0,R))} u_h \leq (1-\theta) \operatorname*{osc}_{B(x_0,2QR)} u_h + C\left(\|G\|_{L^p} + \|f\|_{L^q}\right) R^\delta.$$

This allowed us to prove a uniform *a priori* estimate of the C^α-norm of solutions far from the boundary of the domain Ω. With analogous techniques, we were also able to prove similar estimates up to the boundary of Ω in the subsequent section.

The main restrictions of the theorems that were proved in this chapter were the assumption that the mesh is A-non-obtuse and that F satisfies assumption $(\star)$. If one were able to find truncations that don't rely on the discrete subsolution technique, it might be possible to relax those conditions and make the theory applicable to a wider class of problems.

Chapter 4 was devoted to local estimates for finite element approximations of solutions to p- and φ-Laplacian systems. Here, the approach of using subsolutions no longer works as for vector-functions there is no natural notion of a positive part. This meant that we had to establish properties of the alternative truncation

$$\mathcal{S}_\lambda x = (|x| - \lambda)_+ \frac{x}{|x|}$$

for $x \in \mathbb{R}^m$. Again, this truncation had to be applied in a nodal way to guarantee that $S_\lambda v_h \in V_h$ for any $v_h \in V_h$. We then assumed a uniformly acute mesh and looked at finite element approximations of the φ-Laplacian system

$$-\operatorname{div}\left(\frac{\varphi'(|\nabla u|)}{|\nabla u|}\nabla u\right) = 0 \text{ in } \Omega,$$
$$u = g \text{ on } \partial\Omega$$

for an N-Function φ with $\max\{\Delta_2(\varphi), \Delta_2(\varphi^*)\} < \infty$. We first proved the Caccioppoli-type inequality

$$R \fint_{\Omega(B(x_0,2R))} \left|\nabla\left(\varphi\left(\frac{|S_{\lambda_{k+1}} u_h|}{R}\right)\eta_{k+1}^q\right)\right| \mathrm{d}x \lesssim 2^{\alpha k} \fint_{\Omega(B(x_0,2R))} \varphi\left(\left|\frac{S_{\lambda_k} u_h}{R}\right|\right)\eta_k^q \, \mathrm{d}x$$

for $\lambda_k = \lambda_\infty \left(1 - 2^{-k}\right)$ and cut-off functions η_k with nested balls as support that were precisely defined in Definition 4.20. Using a De-Giorgi-type iteration technique, this then allowed us to prove a local L^∞-estimate for finite element approximations to solutions to φ-Laplacian systems:

$$\max_{\Omega'(B(x_0,R))} \varphi\left(\frac{|u_h|}{R}\right) \lesssim \fint_{\Omega(B(x_0,2R))} \varphi\left(\frac{|u_h|}{R}\right) \mathrm{d}x.$$

For p-Laplacian systems (i.e. $\varphi(t) = p^{-1}t^p$), we were able to prove a similar local L^∞-estimate. The specific structure of the p-Laplacian allowed us to use slightly different approaches for the degenerate case $p > 2$ and the singular case $1 < p < 2$ which enabled us to also allow meshes that are only non-obtuse.

The main factor that prevents us from proving a Hölder-norm-estimate as in Chapter 3 is the fact that this L^∞-norm bound is only valid for u_h itself and not for a truncated function. However, the regularity theory for the continuous problem even yields a $C^{1,\alpha}$-norm bound. (See [30].) It is possible that finding a truncation that allows us to prove an L^∞-norm bound for truncated discrete solutions would enable us to prove a uniform Hölder-norm bound. A different practical extension would be to extend the local L^∞-norm estimates of the discrete solutions to local L^∞-estimates of the approximation error that would help to extend the analysis of adaptive schemes for p- and φ-Laplacian equations.

In Chapter 5, we showed an application of the discrete De Giorgi theory to prove convergence of a finite element approximation of solutions to a system of equations that models the steady flow of a chemically reacting non-Newtonian fluid, that is given by

$$\begin{aligned}
\int_\Omega S(c, Du) \cdot \nabla\psi - (u \otimes u) \cdot \nabla\psi \,\mathrm{d}x - \langle \mathrm{div}\psi, p\rangle = \langle f, \psi\rangle \qquad & \forall \psi \in W_0^{1,\infty}(\Omega; \mathbb{R}^n),\\
\int_\Omega \nu \,\mathrm{div} u \,\mathrm{d}x = 0 \qquad & \forall \nu \in L_0^{r(c)'}(\Omega),\\
\int_\Omega q_c(c, \nabla c, Du) \cdot \nabla\varphi - cu \cdot \nabla\varphi = 0 \qquad & \forall \varphi \in W_0^{1,2}(\Omega).
\end{aligned}$$

To be able to apply the discrete De Giorgi theory to the concentration equation, we have extended the results of Chapter 3 to equations which include lower order terms under the condition of that the mesh is uniformly acute. Then, we were able to prove an h-uniform Hölder-norm bound for the concentration c that lead to the strong C^β-convergence of a subsequence. Hence, we were able to extend the result of [43] to three space dimensions.

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