



Thesis submitted for the degree of
DOCTOR OF PHILOSOPHY

The critical mean-field random cluster model

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*For Luke (c. 1998–2018), the cat
who played fetch. Even though you
only did it once.*

Contents

Abstract	v
Acknowledgements	v
Conventions	vi
Declaration	vi
1 Introduction	1
I Review and preliminaries	9
2 Review: scaling limits of random graphs	11
2.1 Definitions	11
2.2 The Brownian continuum random tree	13
2.3 Tilted trees and a scaling limit	15
2.4 A point process of component sizes	26
3 The random cluster model: introduction	29
3.1 Changing measure	34
3.2 Heuristics for a suggested proof	37
4 Preliminary results	41
4.1 Glauber dynamics and coupling	41
4.2 Nesting of exploration processes	45
4.3 The Erdős–Rényi exploration process	52
4.3.1 Approaching zero	53
4.3.2 Convergence of expectations	63
4.4 Small additions and deletions	70
5 Interlude: A scaling limit for $\mathcal{G}(n, m)$	81
5.1 Convergence of the exploration process	81
5.2 Component sizes and a scaling limit	87
II Results for the random cluster model	91
6 $q < 1$: Handling the discrete exploration process	93
6.1 Approaching zero, revisited	93
6.2 A limit for the exploration process	95
6.3 Problems for $q > 1$	103
6.3.1 Approximate independence	103
6.3.2 Integrability and the limit in n	108
7 $q > 1$: Convergence and integrability	111
7.1 Results in terms of local time	112
7.1.1 Towards a limit	114
7.1.2 Uniform integrability	125

7.2	Converting to “real” time	133
7.3	$q < 1$	143
8	Other approaches	145
8.1	A limit for component sizes	145
8.1.1	Janson and Spencer, revisited	149
8.1.2	An alternative proof strategy	155
8.1.3	Proving Conjecture 8.10	159
8.1.4	Adapting for $\mathcal{G}(n, m, q)$	165
8.2	A conditional scaling limit for the random cluster model	166
9	Outlook	171
9.1	Completing the proof of a scaling limit for $q \in (0, 2)$	171
9.2	$q \geq 2$	174
9.3	Universality	176
A	Additional proofs	177
	Bibliography	185
	Further acknowledgements	191

Abstract

In this thesis we study the mean-field random cluster model, a random graph model in which, relative to the Erdős–Rényi model, graphs are biased to have a greater or lesser number of components, such that, in essence, large-deviations behaviour of the Erdős–Rényi model is seen. This model was independently proven to have an Erdős–Rényi-like phase transition by Stepanov [58, 59] and Bollobás, Grimmett and Janson [16], and was proven to have a critical window by Łuczak and Łuczak [43]. We make progress towards demonstrating that the mean-field random cluster model also possesses a metric space scaling limit in its critical window, of the kind shown to exist for the Erdős–Rényi random graph by Addario-Berry, Broutin and Goldschmidt [3, 2].

We begin by reviewing results from the literature on scaling limits, and on limits of sequences of component sizes, for the Erdős–Rényi model; we also review existing results for the mean-field random cluster model. We go on to offer several results that form partial proofs of the existence of a scaling limit for the latter model, for differing values of the clustering parameter q , as well as some results which, while not proofs, are suggestive that the missing pieces can be filled. Along the way, we prove a number of results of independent interest about both models, including that the fixed-edge-count (as opposed to independent-edge) version of the Erdős–Rényi model has a scaling limit. We conclude by reviewing what would be necessary to complete the partial proofs, and make additional suggestions for future research.

Acknowledgements

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I also thank my supervisor, Christina Goldschmidt, and everyone who, on a personal level, has supported me during this project. There are too many of these people to list here, but a fuller list is given at the very end of this thesis (p. 191).

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Conventions

- We shall use \mathbb{N} to denote the set $\{1, 2, 3, \dots\}$, and let $\mathbb{Z}^{\geq 0} := \mathbb{N} \cup \{0\}$.
- We will let $[n] = \{1, \dots, n\}$.
- For a set A and $k \in \mathbb{Z}^{\geq 0}$, let $A^{(k)} = \{B \subseteq A : |B| = k\}$.
- For functions $f, g : [0, \infty) \rightarrow \mathbb{R}$, we say that $f(x)$ is “asymptotically equivalent” to $g(x)$, denoted $f(x) \sim g(x)$, as $x \rightarrow a \in [0, \infty)$ (resp. as $x \rightarrow \infty$) if $f(x)/g(x) \rightarrow 1$ as $x \rightarrow a$ (resp. as $x \rightarrow \infty$). Similarly, for sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$, we say that $a_n \sim b_n$ as $n \rightarrow \infty$ if $a_n/b_n \rightarrow 1$ as $n \rightarrow \infty$.
- For a graph $G = (V(G), E(G))$, where $V(G)$ denotes its vertex set and $E(G)$ its edge set, we let $|G|$ be its number of vertices, and $e(G)$ its number of edges. In addition, we let $c(G)$ be its number of connected components.
- For a sequence of events $(A_n)_{n \in \mathbb{N}}$, “with high probability” (abbreviated to “w.h.p.”) will mean that A_n occurs with probability tending to 1 as $n \rightarrow \infty$.
- We shall sometimes refer to a connected component of a graph simply as a “component”.

Declaration

I, Alexander Homer, declare that this thesis is entirely my own work under the guidance of my supervisor, Prof. Christina Goldschmidt, with the exceptions of those occasions in which I have directly cited and referenced others' work within the text.

Chapter 1

Introduction

We begin by considering the Erdős–Rényi random graph $\mathcal{G}(n, p)$, in which every one of the $\binom{n}{2}$ possible edges appears in the graph independently with probability p . This model undergoes a phase transition, where, for $p = \kappa/n$ with κ constant, the large- n behaviour is very different either side of $\kappa = 1$. This phase transition was first discussed by Erdős and Rényi [27]—although the authors in fact consider the related model $\mathcal{G}(n, m)$, in which, of the $\binom{n}{2}$ possible edges, m are chosen without replacement, and these are inserted into the graph. (Erdős and Rényi had first discussed this model in [26]; the first discussion of $\mathcal{G}(n, p)$, without mention of the phase transition, was by Gilbert [30].)

To state the phase transition result, we first introduce some notation for stochastic bounding, which is adapted from [13, Defns. 14.4-3 and 14.4-4].

Definition 1.1. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables and $(a_n)_{n \in \mathbb{N}}$ a sequence of positive constants. Then we say that $X_n = O_p(a_n)$ if, for every $\varepsilon > 0$, there exist $K > 0$, $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\mathbb{P}\left(\left|\frac{X_n}{a_n}\right| \leq K\right) \geq 1 - \varepsilon.$$

We say that $X_n = \Omega_p(a_n)$ if $X_n^{-1} = O_p(a_n^{-1})$, and we say that $X_n = \Theta_p(a_n)$ if $X_n = O_p(a_n)$ and $X_n = \Omega_p(a_n)$.

In the same setting, we say that $X_n = o_p(a_n)$ if $X_n/a_n \xrightarrow{p} 0$, and we say that $X_n = \omega_p(a_n)$ if $X_n^{-1} = o_p(a_n^{-1})$.

Remark. It is easy to show that, if $X_n = o_p(a_n)$, then $X_n = O_p(a_n)$; this in turn implies that, if $X_n = \omega_p(a_n)$, then $X_n = \Omega_p(a_n)$. It is clearly impossible to have both that $X_n = o_p(a_n)$ and $X_n = \omega_p(a_n)$.

We now describe the behaviour of $\mathcal{G}(n, p)$ on either side of its phase transition (see, for

example, [34, Thms. 4.4 and 4.8 and Cor. 4.13] for a statement with proof):

Theorem 1.2. *Let $G_n \sim \mathcal{G}(n, \kappa/n)$, for some constant $\kappa \in (0, \infty) \setminus \{1\}$. For $i \in \mathbb{N}$, let $C_i^{(n)}$ denote the i -th largest component of G_n . Then:*

- *If $\kappa < 1$, then $|C_1^{(n)}| = O_p(\log n)$.*
- *If $\kappa > 1$, then there exists a constant $\zeta_\kappa > 0$ such that*

$$\frac{|C_1^{(n)}|}{n} \xrightarrow{p} \zeta_\kappa;$$

additionally, $|C_2^{(n)}| = O_p(\log n)$.

So we see that, below the phase transition, we only see small components of order $\log n$; meanwhile, above the phase transition, there is a unique order- n “giant component”, with other components being also of order $\log n$. The critical threshold occurs at $\kappa = 1$.

The discussion in this thesis starts with a result of Aldous [6]. This gives the limiting behaviour of the connected component sizes in an Erdős–Rényi random graph $G \sim \mathcal{G}(n, p)$ for values of p in the so-called “critical window” around $\kappa = 1$. Moreover, it gives the limiting behaviour of the number of *surplus edges* in each connected component, which we now define.

Definition 1.3. For a connected component C , let $s(C)$ denote its number of *surplus edges*, i.e.

$$s(C) = e(C) - |C| + 1.$$

For a general graph G , let $s(G)$ denote its number of *surplus edges*, which is the sum of the number of surplus edges in each of its components—or, alternatively, let

$$s(G) = e(G) - |G| + c(G) \tag{1.1}$$

where $c(G)$ is the number of components in G .

Remark. So a component has s surplus edges if it has s edges more than would be necessary to form a tree.

We need one more technical definition: that of an excursion of a function (adapted from [55, p. 311]).

Definition 1.4. For a continuous function $f : A \rightarrow \mathbb{R}$, where $A \subseteq \mathbb{R}$ is an interval, an *excursion* of f is a function $g : [0, \sigma] \rightarrow \mathbb{R}$, such that, for some $[a, b] \subseteq A$ with $\sigma = b - a$:

- for all $x \in [0, \sigma]$, $g(x) = f(a + x)$;
- $g(0) = g(\sigma) = 0$; and
- $g(x) \neq 0$ for $x \in (0, \sigma)$.

The corresponding *excursion interval* is (a, b) , and the *length* of the excursion g is $\sigma = b - a$.

We now set up Aldous's theorem. Fix some constant $\lambda \in \mathbb{R}$. For sufficiently large n that this is well-defined, let $C_1^{(n)}, C_2^{(n)}$ be the connected components of $G \sim \mathcal{G}(n, p)$ in descending order of size, where $p = 1/n + \lambda n^{-4/3} + o(n^{-4/3})$. (We break ties arbitrarily.)

Let $B(t)$ be a one-dimensional Brownian motion, let $X(t) = B(t) - t^2/2 + \lambda t$, and let $Y(t) = X(t) - \inf_{0 \leq s \leq t} X(s)$. Let $\mathbf{C} = (\gamma_1, \gamma_2, \dots)$ be the vector of excursion lengths of $Y(t)$ above, sorted into decreasing order. (It is a non-trivial fact that such an ordering is possible almost surely, which is implied by the result, Theorem 1.5, that follows.) Let W_i , an open interval in $[0, \infty)$, be the excursion interval of length γ_i .

Conditionally on $(Y(t), t \geq 0)$, let $(N(t), t \geq 0)$ be an inhomogeneous Poisson process, with rate $Y(t)$ at time t . Let ν_i be the number of points occurring in the interval W_i . (Equivalently, draw the graph of Y , and place a Poisson process of intensity 1 on the plane; let ν_i be the number of points lying in the plane during the interval W_i which lie strictly between the graph and the horizontal axis.) Let $\mathbf{S} = (\nu_1, \nu_2, \dots)$.

Define the vector

$$\mathbf{C}_n = (|C_1^{(n)}|, |C_2^{(n)}|, \dots),$$

where, if $C_k^{(n)}$ does not exist, we say that $|C_k^{(n)}| = 0$. Define

$$\mathbf{S}_n = (s(C_1^{(n)}), s(C_2^{(n)}), \dots)$$

similarly.

Theorem 1.5 ([6, Cor. 2]).

$$(n^{-2/3} \mathbf{C}_n, \mathbf{S}_n) \xrightarrow{d} (\mathbf{C}, \mathbf{S}), \tag{1.2}$$

with convergence in the l^2 norm for the first coordinate, and the product topology for the second.

Remark. Aldous states this result only for $p = 1/n + \lambda n^{-4/3}$ exactly, but his proof extends to allowing the extra $o(n^{-4/3})$ error term.

This is a beautiful result connecting two seemingly unrelated mathematical objects—a random graph, and a Brownian motion. The connection is given by the fact that a certain

exploration process for the graph, the *depth-first walk*, converges in distribution to X ; Aldous proves this as a step towards the proof of Theorem 1.6. We give the algorithm for the depth-first search, and then the result.

Algorithm A (Depth-first search). We assume we are carrying out the algorithm on a graph on n vertices, whose labels form a totally ordered set.

1. Initialise $X_n(0) = I_n(0) = N_n(0) = 0$, and set all vertices to be sleeping, and the stack to be empty.
2. For each $i = 1, \dots, n$:
 - (a) If the stack is non-empty, set the new current vertex, v_i , to be the vertex at the top of the stack, kill it, and remove it from the stack. Otherwise, set v_i to be the lowest-labelled sleeping vertex, and immediately kill it.
 - (b) If any sleeping vertices are adjacent to the current vertex, v_i , wake them up and add them to the top of the stack in order of label, with the lowest-labelled at the top. Let ζ_i be the number of such vertices discovered.
 - (c) Set $X_n(i) = X_n(i-1) + \zeta_i - 1$.
 - (d) Set $I_n(i) = \min \{ I_n(i-1), X_n(i) \}$.
 - (e) Let η_i be the number of awake (but not dead) vertices adjacent to the current vertex, and let $N_n(i) = \eta_i + N_n(i-1)$.
3. The algorithm terminates after n iterations—or, equivalently, it terminates at the conclusion of the current iteration once all vertices are dead.

Remark. In relation to step 2(e), it is not possible for any vertices currently dead to have any edges joining them to the current vertex, except for any edge already discovered at a previous step. This is discussed in Section 2.3.

In order to illustrate this process, we provide a worked example in Figure 1.1. It can be checked by carefully inspecting the algorithm that the number of vertices on the stack on the i -th iteration after step 2(a) (i.e. just before we start querying edges) is exactly $X_n(i-1) - I_n(i-1)$, and the number of components whose exploration is completed by the end of the i -th iteration is exactly $-I_n(i)$.

Theorem 1.6 ([6, § 2.2]). Let $G_n \sim \mathcal{G}(n, 1/n + \lambda n^{-4/3} + o(n^{-4/3}))$. Then

$$(n^{-1/3}X_n(\lfloor tn^{2/3} \rfloor), N_n(\lfloor tn^{2/3} \rfloor))_{t \geq 0} \xrightarrow{d} (X, N)$$

as $n \rightarrow \infty$ with respect to the Skorokhod metric on the space of càdlàg functions, where, for B a Brownian motion, we take

$$X(t) := B(t) + \lambda t - \frac{t^2}{2},$$

and where, conditional on X , we take N to be an inhomogeneous Poisson process of rate $X(t) - \inf_{0 \leq s \leq t} X(s)$ at time t .

Remark. Note that, as I_n is a continuous functional of X_n , we also have, by the Continuous Mapping Theorem, joint convergence of the correspondingly rescaled I_n to the running infimum of X . Further note that, again, Aldous only states this for $p = 1/n + \lambda n^{-4/3}$, but the proof can be appropriately modified.

Remark. Aldous actually proved this result using a subtly different algorithm, the *breadth-first search*: the only difference is the order in which vertices are added to or removed from the stack, and, as the vertices of an Erdős–Rényi random graph are exchangeable, the distributional statement is unchanged.

Aldous’s result does not give us complete information, however, about the metric properties of the limit object. This question was answered by Addario-Berry, Broutin, and Goldschmidt in a pair of papers [2, 3], which showed that the Erdős–Rényi model, in its critical window, possesses a *scaling limit*. We will discuss their work, and related results, in Chapter 2, as it is a fundamental starting point for the work that follows.

The aim of this thesis is to present work leading to a similar result for a model known as the random cluster model, in its mean-field form; we denote this $\mathcal{G}(n, p, q)$ (where $n \in \mathbb{N}$, $p \in (0, 1)$, and $q > 0$), and define it by saying that, if $G_n \sim \mathcal{G}(n, p, q)$, then, for any graph G on n vertices,

$$\mathbb{P}(G_n = G) = \frac{p^{e(G)}(1-p)^{\binom{n}{2}-e(G)}q^{c(G)}}{\sum_H p^{e(H)}(1-p)^{\binom{n}{2}-e(H)}q^{c(H)'}}$$

taking the sum over all graphs H on n vertices; here, $e(G)$ (resp. $c(G)$) denotes the number of edges (resp. connected components) of G . (In their paper introducing the model, Fortuin and Kasteleyn [28] use the word “cluster” as synonymous with “connected component”.) Note that, when $q = 1$, this corresponds exactly to the Erdős–Rényi model $\mathcal{G}(n, p)$.

This model is known also to have an Erdős–Rényi-like phase transition, though now at

$f(q)/n$ for some function of q [16, 58, 59]; when $q < 2$, for which $f(q) = q$, it is known to have a form of critical window, when $p = q/n + \lambda n^{-4/3}$ for $\lambda \in \mathbb{R}$ [43].

This thesis provides work towards a proof of the existence and form of the scaling limit of this model in its critical regime, that is, we conjecture, similar to that of the Erdős–Rényi model. Although we do not succeed in finding a complete result, we present a number of different partial proofs; as part of this, we prove a collection of results about the Erdős–Rényi and random cluster models, some of which are of independent interest.

Following the literature review on scaling limits and related concepts in Chapter 2, we will discuss more fully the existing results in the literature about the random cluster model, in Chapter 3. In this latter chapter, we also give an idea of the intended proof strategy that guides much of the work in this thesis. Then, in Chapter 4, we provide a selection of results that allow us to compare the exploration process for that model to the one for the Erdős–Rényi model that is relatively well-understood. That chapter also contains some results about the Erdős–Rényi exploration process (e.g. Proposition 4.6, Proposition 4.13) that, as far as the author is aware, are not contained within the existing literature. As a brief interlude, in Chapter 5 we prove that $\mathcal{G}(n, m)$, the model Erdős and Rényi first studied in [26], has the same scaling limit as $\mathcal{G}(n, p)$, throughout its own critical window; that concludes Part I of this thesis.

Part II provides the bulk of the results about the random cluster model. The aforementioned proof strategy (which appears in Section 3.2), which is intended to give a result similar to Theorem 1.5, has two major issues which prevent it from being a completed proof; we were able to solve one of them for $q < 1$, and the other for $q > 1$. To each of these solutions we devote a chapter (Chapters 6 and 7, respectively). Subsequently, in Chapter 8, we discuss an alternate potential proof strategy; we also show in that chapter that, conditionally on convergence of the rescaled sequence of component sizes, a metric space scaling limit must follow. Finally, in Chapter 9, we review the unresolved problems from this thesis, as well as suggesting some other avenues for future research.

Part I

Review and preliminaries

Chapter 2

Review: scaling limits of random graphs

As discussed towards the end of the introduction, this thesis considers an extension of the work of Addario-Berry, Broutin and Goldschmidt [3, 2], which extends Theorem 1.5 proved by Aldous [6] to provide a limiting structure for critical Erdős–Rényi random graphs. We discuss that work in this chapter—particularly the content of [3], the first of their pair of papers.

Towards the end of this chapter, we also describe the work of Janson and Spencer [40] on limiting component sizes of the critical Erdős–Rényi random graph. They give an explicit description of these sizes as a point process on $[0, \infty)$, which will prove useful in the chapters that follow.

2.1 Definitions

Definition 2.1. Let $M = (X, d)$ be a metric space. For $\alpha > 0$, we can define the metric space $\alpha M = (X, \alpha d)$, where αd is the usual definition for scalar multiples of functions; that is, αM is M with all distances rescaled by α .

Given two metric spaces, we may define a notion of distance between them; here, we use the Gromov–Hausdorff distance (for a reference, see, for instance [19, Defn. 7.3.1 and Thm. 7.3.25]).

Definition 2.2. Let (M, d) be a metric space. Then the *Hausdorff distance* between two compact subsets X, Y of M is given by

$$d_{\text{H}}(X, Y) = \inf \{ \varepsilon > 0 : X \subseteq F_{\varepsilon}(Y) \text{ and } Y \subseteq F_{\varepsilon}(X) \}, \quad (2.1)$$

where, for $X \subseteq M$, $F_\varepsilon(X) := \{x \in M : \inf_{y \in X} d(x, y) \leq \varepsilon\}$.

Definition 2.3. The triple (X, d, ρ) is a *rooted metric space* if (X, d) is a metric space and ρ is a point in X .

Definition 2.4. Let \mathcal{M}_{cmp} be the class of rooted compact metric spaces. Then the *Gromov–Hausdorff distance* is a pseudometric on \mathcal{M}_{cmp} given by, for $(X, d, \rho), (X', d', \rho') \in \mathcal{M}_{\text{cmp}}$,

$$d_{\text{GH}}(X, X') = \inf \left\{ \max \left\{ d_{\text{H}}(\phi(X), \phi'(X')), \delta(\phi(\rho), \phi'(\rho')) \right\} \right\}, \quad (2.2)$$

where the infimum is taken over all choices of metric space (M, δ) and all isometric embeddings $\phi : X \rightarrow M$ and $\phi' : X' \rightarrow M$.

Remark. Note that, for two rooted compact metric spaces X, X' (we suppress the specification of the metric and the root), $d_{\text{GH}}(X, X') = 0$ if and only if there is a root-preserving isometry between X and X' . (For unrooted metric spaces, the analogous result is stated, e.g., in [33, Prop. 3.6], with the rooted case being an easy consequence of this.)

Definition 2.5. We define \mathcal{M} to be the set of equivalence classes of rooted compact metric spaces under d_{GH} , with metric induced by d_{GH} . (By the remark above, this is also the set of rooted isometry classes of such spaces.) Abusing notation slightly, we likewise denote the metric on equivalence classes by d_{GH} .

We define \mathcal{M}_{Pol} to be the subset of \mathcal{M} in which the metric spaces are *Polish spaces*: that is, complete, separable metric spaces.

Remark. We will often refer to convergence of metric spaces to other metric spaces, and similar, since it is more concise and unlikely to cause too much confusion; formally, we are referring to the convergence of the equivalence classes discussed in the preceding definition.

Definition 2.6. Let M and M_n , for $n \in \mathbb{N}$, be random objects in \mathcal{M}_{Pol} . We say that M is the *metric space scaling limit* of the sequence (M_n) if there exists a sequence (α_n) such that, in the metric d_{GH} , it holds that $\alpha_n M_n \xrightarrow{d} M$ as $n \rightarrow \infty$. That is, if, for all bounded continuous functions $\phi : \mathcal{M}_{\text{Pol}} \rightarrow \mathbb{R}$ (where \mathbb{R} is endowed with the standard Euclidean metric),

$$\mathbb{E}[\phi(\alpha_n M_n)] \rightarrow \mathbb{E}[\phi(M)]$$

as $n \rightarrow \infty$.

When it is clear what kind of scaling limit we mean, we will sometimes just write *scaling limit*.

Remark. Given a definition for a distance between sequences of (equivalence classes of) metric spaces, we can define an analogous notion of the scaling limit of such objects.

We can consider graphs as rooted metric spaces by considering the space of their vertices, with distance given by the graph distance and root given by the lowest-labelled vertex.

Addario-Berry, Broutin and Goldschmidt [3] found the scaling limit of an Erdős–Rényi random graph, considered as a sequence of metric spaces—each being a metric space, endowed with the graph distance. An important part of this paper (and its companion, [2]) is finding the scaling limit of an Erdős–Rényi graph conditioned to be connected—which, informally, can be identified with a component of a critical Erdős–Rényi graph. We describe this below, but we first need to define another structure, the Brownian continuum random tree.

2.2 The Brownian continuum random tree

The Brownian continuum random tree was first defined by Aldous in a series of three papers [7, 8, 9]; since it is a necessary ingredient in the proof of the Erdős–Rényi scaling limit that we discuss in the sequel, we introduce it here.

The Brownian continuum random tree is an example of a real tree, a type of object whose definition we quote from [3, p. 371].

Definition 2.7. A metric space $M = (X, d)$ is a *real tree* if, for all $x, y \in X$:

- (a) there is a unique isometry $f_{x,y} : [0, d(x, y)] \rightarrow M$ such that $f_{x,y}(0) = x$ and such that $f_{x,y}(d(x, y)) = y$; and
- (b) for any continuous, injective map $g : [0, 1] \rightarrow M$ with $g(0) = x$ and $g(1) = y$, we have that $g([0, 1]) = f_{x,y}([0, d(x, y)])$.

Remark. That is, in a real tree, there is a unique path between any two points x and y , and this path is of length $d(x, y)$.

We now set up the machinery we will need to define the Brownian continuum random tree. Fix $\sigma > 0$. Given a continuous function $f : [0, \sigma] \rightarrow [0, \infty)$, with $f(0) = 0 = f(\sigma)$, we can define a pseudometric d_f^* on $[0, \sigma]$ by

$$d_f^*(x, y) = \begin{cases} f(x) + f(y) - 2 \min_{u \in [x, y]} f(u) & \text{for } x \leq y; \\ d_f^*(y, x) & \text{for } x > y. \end{cases} \quad (2.3)$$

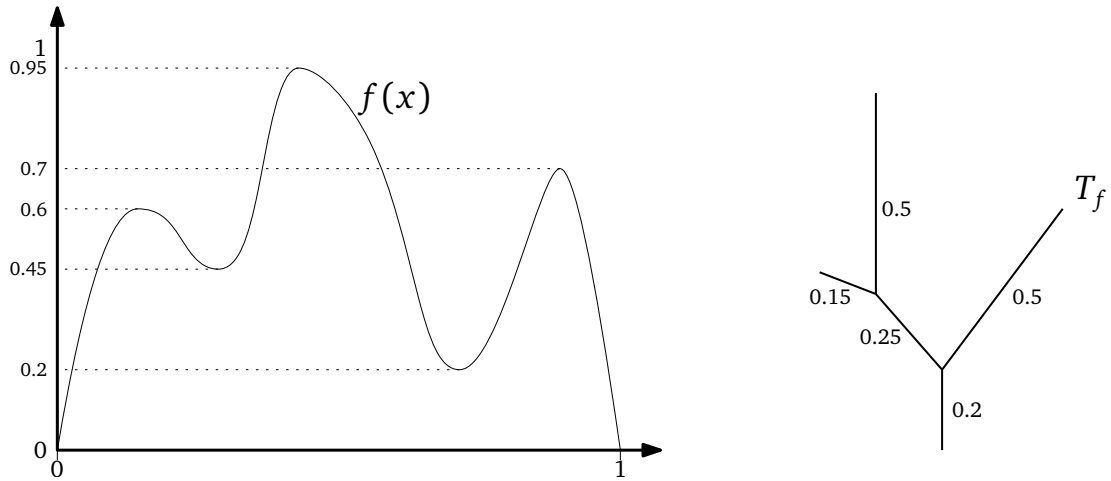


Figure 2.1: The diagram above illustrates, on the left, a function f , and, on the right, the real tree T_f corresponding to it. Along the segments of T_f are indicated their lengths.

This means that we can obtain a metric space T_f from $[0, \sigma]$ by identifying all points x, y where $d_f^*(x, y) = 0$: given a function f satisfying the hypotheses just stated, let \sim_f be the equivalence relation on $[0, \sigma]$ where we say that $x \sim_f y$ if and only if $d_f^*(x, y) = 0$. Then, given f again, we can define the metric space (T_f, d_f) by

$$T_f = [0, \sigma] / \sim_f \tag{2.4}$$

with metric d_f being the induced metric provided by d_f^* .

The object (T_f, d_f) (which we abbreviate to T_f , taking the d_f as implicit) is a real tree [3, p. 372]. The collapsing of $[0, 1]$ into T_f , for a particular function f , is illustrated in Figure 2.1. Heuristically, we can think of it as being formed when we apply glue to the underside of the function f , and then squeeze it together from both ends.

The Brownian continuum random tree is a special case of this construction.

Definition 2.8 ([42]). Let $(B(t) : t \geq 0)$ be a standard Brownian motion. Define random variables $L = \sup \{x \leq 1 : B(x) = 0\}$ and $R = \inf \{x \geq 1 : B(x) = 0\}$. For $0 \leq t \leq 1$, let

$$\mathbf{e}(t) = \frac{|B(L + (R - L)t)|}{\sqrt{R - L}}. \tag{2.5}$$

Then $\mathbf{e} = (\mathbf{e}(t) : 0 \leq t \leq 1)$ is a *Brownian excursion of length 1*, which may alternatively be called a *standard Brownian excursion*, or simply a *Brownian excursion*. Let the *Brownian excursion*

sion of length σ , $\mathbf{e}^{(\sigma)} = (\mathbf{e}^{(\sigma)}(t) : 0 \leq t \leq \sigma)$, have distribution given by:

$$(\mathbf{e}^{(\sigma)}(t) : 0 \leq t \leq \sigma) \stackrel{d}{=} \left(\sqrt{\sigma} \mathbf{e} \left(\frac{t}{\sigma} \right) : 0 \leq t \leq \sigma \right). \quad (2.6)$$

Definition 2.9. Let \mathbf{e} be a Brownian excursion. Then the random object $T_{2\mathbf{e}}$ is a *Brownian continuum random tree* (or *Brownian continuum random tree*).

The Brownian CRT is the scaling limit of uniform random labelled trees:

Theorem 2.10 ([7], Thm. 8). *Let T_n be a uniformly chosen random tree on $[n] = \{1, \dots, n\}$, considered as a metric space endowed with the graph distance. Then $n^{-1/2}T_n \xrightarrow{d} T$, where T is the Brownian continuum random tree.*

It is the scaling limit of many other random trees, as well; a list of some of these is given by Albenque and Goldschmidt [5, p. 1].

2.3 Tilted trees and a scaling limit

In this section and the next, we discuss the work of Addario-Berry, Broutin and Goldschmidt [2] on the scaling limit of an Erdős–Rényi random graph in the critical window. Although we will state the results formally, we mostly aim to give a heuristic description of their work; we will return to discuss some of the technicalities later, in Chapter 8, when we apply their methods to the model of interest for this thesis—as well as in Chapter 5, where we prove an analogous result for $\mathcal{G}(n, m)$, rather than $\mathcal{G}(n, p)$.

We first consider the limiting behaviour of a component of such a graph—or more formally, the behaviour of an Erdős–Rényi random graph with an appropriately chosen probability and number of edges, conditioned to be connected.

We can consider exploring a component through depth-first search. We defined this for general graphs as Algorithm A (stated on p. 4), and so we may apply it to connected graphs as a particular case. Note that we differ slightly from some other authors in our convention for what we consider the stack, in that we consider the current vertex *not* to be part of the stack. This will make notation simpler later on, though it adds some slight complications in the connected-graph case.

Definition 2.11. When applying Algorithm A, if vertex w is discovered on step i , at which point v_i is the current vertex, we say that w is a *child* of v_i , and v_i is the *parent* of w .

Remark. Note that every vertex but one in a connected graph has exactly one parent; that exception is the lowest-labelled vertex, which has no parent.

It can be checked that, when the algorithm is applied to a connected graph of size n , the function $X_n : \{0, \dots, n\} \rightarrow \mathbb{Z}$ is defined so that $X_n(i)$ is the number of vertices on the stack after step (a) of iteration $i + 1$ —unless there is no iteration $i + 1$ because the exploration is complete, in which case it takes the value -1 . Then it can be seen that $X_n(i) \geq 0$ for $i = 0, \dots, n - 1$ and $X_n(n) = -1$.

Recall that the “surplus” of a connected graph is the number of edges contained within it additionally to those required to form a tree on those vertices. We cannot, from just this, identify *which* are the surplus edges, since, in general, there is more than one spanning tree of the graph. However, we can use Algorithm A to form a particular spanning tree:

Definition 2.12. Let the *depth-first-search tree* of G be the subgraph of G with vertex set $[n]$ in which the only edges retained are those joining each vertex (but 1) to its parent.

We can then take as a convention that the surplus edges in the component are those not contained within the depth-first-search tree. They must therefore be edges that *could not* have been discovered by step (c) of the algorithm; at iteration i they are edges between the current vertex, v_i , and vertices already on the stack when the children of v_i are explored. The number of such vertices is $X_n(i)$: the number of vertices on the stack at the start of the iteration (that is, at the end of iteration $i - 1$). So the number of possible surplus edges is $\sum_{i=0}^{n-1} X_n(i)$; note that, if we interpolate between the points of the process $X_n(i)$ by considering $X_n(\lfloor t \rfloor)$, it is the number of points in $\mathbb{Z}^{\geq 0} \times \mathbb{Z}^{\geq 0}$ that lie strictly below the function, as shown in Figure 2.2. This motivates the following definition, of which we will make use later.

Definition 2.13. For a connected graph G on n vertices whose labels form a totally-ordered set, its *area* is defined as

$$a(G) := \sum_{i=0}^{n-1} X_n(i),$$

where $X_n(i)$ is defined via Algorithm A.

There is an even more direct correspondence between these points and the possible surplus edges. Consider the $X_n(i - 1)$ vertices on the stack during iteration i , once the current vertex has been removed. The stack structure means that the vertex k -th in the stack during iteration i (for $1 \leq k \leq X_n(i - 1)$), counting up from the bottom, will (since vertices are only removed from the stack at the start of an iteration) be v_j , for the first value of $j \geq i$ such that $X_n(j - 1) = k - 1$ (so that the stack, during iteration j , has $k - 1$ vertices on it).

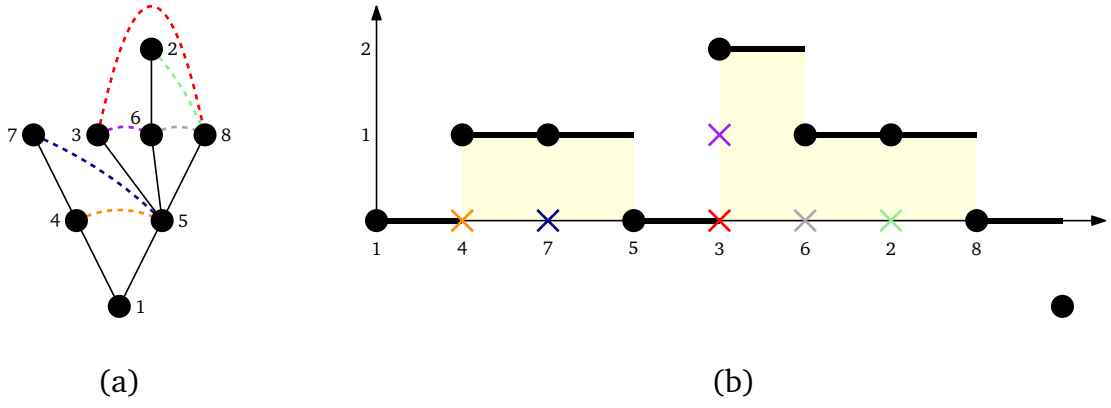


Figure 2.2: On the left is shown a depth-first-search tree for a graph. This corresponds to graphs with these solid black edges and some number (possibly none) of the coloured dashed edges, which are drawn between each vertex v and vertices on the stack at the time v is explored. No other edges can be present, as they would change the depth-first-search tree. On the right, meanwhile, is the corresponding set of values of $X_n(i)$, with the horizontal axis labelled not with values of i , but with the labels of the vertices in the order they are explored—shown at point i on the axis is the label of v_{i+1} . Note that there are six entries of $\mathbb{Z}^{\geq 0} \times \mathbb{Z}^{\geq 0}$ below the points in the sequence. Each corresponds to a possible surplus edge, namely that between vertices with given labels—the label directly below the cross, and the label at the next circle directly to the right of the point. This correspondence is indicated by the edge and cross colours.

The black “curve” shown is $X_n(\lfloor t \rfloor)$, for $t \in [0, 8]$, and the number of possible surplus edges can also be thought of as the area under this curve.

Thus, each point in $\mathbb{Z}^{\geq 0} \times \mathbb{Z}^{\geq 0}$ under the curve can be mapped to a unique possible surplus edge: if the point is at $(i - 1, k)$, for $i \in [n - 1]$ and $0 \leq k < X_n(i - 1)$, it represents the edge joining v_i with v_j for $j = \sup \{ p > i : X_n(p - 1) = k \}$, which vertex will be the $(k + 1)$ -th in the stack when v_i is the current vertex. This is illustrated in Figure 2.2.

This means that we can construct a component of an Erdős–Rényi random graph by a two-stage procedure: choose a depth-first-search tree, and then select surplus edges that correspond to points in the area under the depth-first walk on the tree.

Borrowing notation from [3], let $G_n^p \sim \mathcal{G}(n, p)$, conditioned on being connected. Meanwhile, define a graph \tilde{G}_n^p as follows: define \tilde{T}_n^p to be a tree on $[n]$ chosen such that, for each possible tree T ,

$$\mathbb{P}(\tilde{T}_n^p = T) \propto (1 - p)^{-a(T)},$$

with $a(T)$ being the area under the tree, as in Definition 2.13. Then, for each surplus edge allowed by Algorithm A, insert it independently with probability p . This graph is \tilde{G}_n^p .

Then the following holds.

Proposition 2.14 ([3, Propn. 8]). *For $n \in \mathbb{N}$ and $p \in (0, 1)$, $\tilde{G}_n^p \stackrel{d}{=} G_n^p$.*

Remark. This is an exact distributional statement, so p can depend on n .

To see how this relates to the Brownian CRT, we first consider a “tilted” Brownian excursion, defined by the following change of measure from Addario-Berry, Broutin and Goldschmidt [3, (3)].

Definition 2.15. For $\sigma > 0$, let

$$\mathcal{E}^{(\sigma)} = \{f \in C([0, \sigma]) : f(0) = f(\sigma) = 0, f(s) > 0 \text{ for } s \in (0, \sigma)\}$$

be the *set of excursions on $[0, \sigma]$* , where $C([0, \sigma])$ denotes the set of all continuous functions on $[0, \sigma]$. We consider it the *space of excursions on $[0, \sigma]$* when we endow it with the supremum norm.

We set $\mathcal{E} := \mathcal{E}^{(1)}$.

Recall from (2.6) that $\mathbf{e}^{(\sigma)}$ is a Brownian excursion of length $\sigma > 0$, which is almost surely an element of $\mathcal{E}^{(\sigma)}$.

Definition 2.16. We define the law of an *tilted excursion* of length $\sigma > 0$, $\tilde{\mathbf{e}}^{(\sigma)}$, by saying that, for any Borel subset $\mathcal{B} \subseteq \mathcal{E}^{(\sigma)}$ of the space of excursions on $[0, \sigma]$,

$$\mathbb{P}(\tilde{\mathbf{e}}^{(\sigma)} \in \mathcal{B}) = \frac{\mathbb{E}\left[\mathbb{1}_{\{\mathbf{e}^{(\sigma)} \in \mathcal{B}\}} \exp\left\{\int_0^\sigma \mathbf{e}^{(\sigma)}(u) du\right\}\right]}{\mathbb{E}\left[\exp\left\{\int_0^\sigma \mathbf{e}^{(\sigma)}(u) du\right\}\right]}. \quad (2.7)$$

If we don’t specify a length, we will mean the tilted excursion of length 1, which we denote $\tilde{\mathbf{e}}$.

Remark. This is well-defined, as the moment generating function of the area under a Brownian excursion of length one is finite for all values of the argument [36, § 13]; the fact that it is finite for all other lengths follows by Brownian scaling.

We return to the discrete setting.

Definition 2.17. Let T be a tree on $[n]$. For a vertex $v \in T$, let the *height* of v be $h_T(v) := d_T(1, v)$, where d_T is the graph distance on T .

Remark. If the vertex set is not $[n]$ but has a (partial) order with respect to which there is a minimal element, we can still make this definition work by considering distances from the minimal element instead. We will need to do so, for instance, when we come to consider components of a larger graph.

Definition 2.18. Let G be a connected graph on $[n]$ with depth-first-search tree T . Then the *DFS-height* of v in G , $h_G(v)$, is given by $h_G(v) = h_T(v)$.

Definition 2.19. Let $H_G : \{0, \dots, n\} \rightarrow \mathbb{Z}^{\geq 0}$ be the *height function* of the depth-first-search tree of G , defined by

$$H_G(i) = \begin{cases} h_G(v_{i+1}) & i = 0, \dots, n-1; \\ 0 & i = n. \end{cases}$$

When it is clear to which graph G we are referring, we will sometimes suppress G in the above, and write simply $h(v)$ and $H(i)$.

Now, let X_n be the function described above, applied to G_n^p (which we recall is distributed as $\mathcal{G}(n, p)$, conditioned on being connected). Let $H_n = H_{G_n^p}$. Fix $\sigma > 0$, and say that, for $t \in [0, \sigma]$, $\tilde{X}_n^{(\sigma)}(t) = (\sigma/n)^{1/2} X_n(\lfloor nt/\sigma \rfloor)$; define $\tilde{H}_n^{(\sigma)}(t)$ analogously. Then, in fact, the following holds:

Theorem 2.20 ([3]). *Suppose $p = p(n)$ is chosen so that $np^{2/3} \rightarrow \sigma$ as $n \rightarrow \infty$. Then, as $n \rightarrow \infty$,*

$$(\tilde{X}_n^{(\sigma)}, \tilde{H}_n^{(\sigma)}) \xrightarrow{d} (\tilde{\mathbf{e}}^{(\sigma)}, 2\tilde{\mathbf{e}}^{(\sigma)}),$$

with the convergence in distribution in the uniform norm.

Remark. Note that the two copies of $\tilde{\mathbf{e}}^{(\sigma)}$ on the right-hand side are *the same* realisation of the excursion. Note also that, in the full critical Erdős–Rényi random graph, in which $p \sim 1/n$, we expect to have components of size $\Theta_p(n^{2/3})$. So, when we are considering such a graph *conditioned to be connected*, taking $p = \Theta(n^{-3/2})$ is the correct scaling to see similar behaviour. (We have $p = \Theta(n^{-3/2})$ here, rather than $p \sim cn^{-3/2}$ for any particular c , to allow for the randomness in the sizes of the components in the full graph on that $n^{2/3}$ scaling.)

The individual convergence of the terms in the ordered pair in Theorem 2.20 is given by Theorems 12 and 15 of [3]; that they converge to the same realisation is implied by the proof of the latter, specifically using Lemma 16 of the same paper.

It is worth taking a minute to work out why we need to consider both the depth-first walk and the height function. The depth-first walk is easier to analyse, with a distribution

that can be characterised more readily by the distribution of the random graph. Moreover, the surplus edges truly are generated by the points lying beneath the depth-first walk, not the height function. However, it is the height function that more directly characterises the depth-first-search tree, which we put together with the surplus edges to obtain the graph.

Moving on, we now discuss why we obtain a *tilted* excursion. As noted by Aldous [9, Thm. 23] (see also [8, Thm. 2]), the height function of a uniform tree converges (when appropriately rescaled) to twice an un-tilted Brownian excursion. So, considering Proposition 2.14, we might expect the height function of an appropriately biased choice of tree to converge to an appropriately biased excursion—and in fact Definition 2.16 gives the continuous analogue of the bias applied to the discrete trees. (In fact, in the case of the uniform tree, it is again true that the depth-first walk and the height function converge to scalar multiples of the same realisation of the excursion—see [46, Thm. 2], applied to a Galton–Watson tree with $\text{Poi}(1)$ offspring distribution.)

There is, in fact, a close relationship between the height function and distances between arbitrary pairs of vertices in the tree. Recall that, for $k \in [n]$, the vertex v_k is the k -th vertex explored in depth-first order.

Lemma 2.21. *If d_T is the graph distance on the depth-first tree, then, for $i \leq j$,*

$$H_n(i) + H_n(j) - 2 \min_{k \in \{i, \dots, j\}} H_n(k) \leq d_T(v_i, v_j) \leq H_n(i) + H_n(j) - 2 \min_{k \in \{i, \dots, j\}} H_n(k) + 2. \quad (2.8)$$

Remark. Compare [3, Lma. 17].

Proof. This is obvious if $i = j$, so suppose not.

There are two cases: either v_j is a descendant of v_i , or the most recent common ancestor of v_i and v_j is some third vertex. (The vertex v_i cannot be a descendant of v_j because of the depth-first ordering.) In the first case, the distance between the two vertices is simply the difference in their heights, which is given by the left-hand side, since every vertex explored between the i -th and j -th is a descendant of v_i and so has height greater than $H_n(i)$.

In the second case, there is some most recent common ancestor v , which has some child w that is an ancestor of v_j (possibly v_j itself). Let $k(w)$ satisfy $w = v_{k(w)}$. Since $j > i$, we must have that w is a higher-labelled child of v than the child that is an ancestor of the i -th vertex, so $k(w) > i$, and meanwhile, as w is an ancestor of v_j , $k(w) \leq j$. Moreover, no vertex closer to the root than w is explored between i and j , since we are still exploring the descendants of v .

So

$$h(w) = 2 \min_{k \in \{i, \dots, j\}} H_n(k).$$

Now $h(w) = h(v) + 1$, and $d_T(v_i, v_j)$ is the distance down the tree from v_i to v , and then back up to v_j , which is given by

$$\begin{aligned} (h(v_i) - h(v)) + (h(v_j) - h(v)) &= h(v_i) + h(v_j) - 2h(v) = h(v_i) + h(v_j) - 2h(w) + 2 \\ &= H_n(i) + H_n(j) - 2 \min_{k \in \{i, \dots, j\}} H_n(k) + 2, \end{aligned}$$

the right-hand side. □

Note the similarity between (2.8) and (2.3), the latter expression being the one given above for the pseudometric applied to $[0, 1]$ given a function. Also, note that, when rescaled, the discrepancy vanishes as $n \rightarrow \infty$. Hence this is suggestive that in fact the discrete depth-first tree converges to the tree encoded by the appropriately sized tilted excursion, and indeed that is the case [3, pp. 385–386].

We have not yet addressed what happens in the limit to the surplus edges. Recall we have identified a correspondence between surplus edges and points in a grid under the curve, where we select each point independently with probability p . As we rescale with increased n , this grid of points moves closer together, and meanwhile the probability of choosing each decreases. This is the situation in which we might expect a Poisson point process to appear in the limit, provided that the number of available points and the probability of selecting each compensate each other appropriately.

But note that, in Theorem 2.20, to achieve convergence, we rescale the lengths of the excursions by a multiple of n , and the heights by a multiple of $n^{1/2}$ —so that, overall, the excursions are “growing like” $n^{3/2}$. Since we have chosen p so that $np^{2/3} \rightarrow \sigma$, this implies that $n^{3/2}p$ converges, and so we have the right scaling for a Poisson process to emerge.

So, in the limit, we imagine a Poisson process in the area under the curve $\check{e}^{(\sigma)}$, which generates the surplus edges. In the discrete picture, each added point corresponds to joining the vertex represented by the x -coordinate with that represented by the next “time”, to the right, at which the process’s value equals the y -coordinate. But as the depth-first walk X_n and the height process H_n converge to the *same* excursion (up to a scalar constant), this corresponds in the limit to generating a Poisson process under the curve used to form the tree, and then joining the corresponding points in the tree. (As the limit process of the height function is twice that of the depth-first search, this process will have half the rate.) By “corresponding points”

we mean the points that are the continuous-time analogues of the discrete notion we just discussed: so, for a point (x, y) under the excursion, we'd want to join the point represented by time x by the point represented by the next time after x that the process takes the value y . (See Figure 2.3 for an illustration of this.)

Of course, this is a purely heuristic argument, but it turns out (though we omit the technical details here) to give us the right idea to construct the limit object of appropriately-rescaled connected components of a critical Erdős–Rényi random graph—we consider, for each $\sigma > 0$, a random object $M^{(\sigma)}$, which we will shortly define.

Before we do, we define the following function g , whose definition we borrow from [3, § 4].

Definition 2.22. Fix $\sigma > 0$. Let $h \in \mathcal{E}^{(\sigma)}$, and let $\mathcal{Q} \subseteq [0, \sigma] \times [0, \infty)$ be a countable set. Then we define $g(h, \mathcal{Q})$ as follows. First take $T_h = [0, \sigma] / \sim_h$ (as given by (2.4)), and for $x \in [0, \sigma]$, let $t(x)$ denote the point in T_h corresponding to x (formally, when T_h is considered as a space of equivalence classes on $[0, \sigma]$, we let $t(x)$ be the class to which x belongs). Further, for each point $(x, y) \in \mathcal{Q}$ such that $2y < h(x)$, we identify the points $t(x)$ and $t(z)$, where

$$z = \inf \{ u > x : h(u) = 2y \}.$$

(Formally, we create a new equivalence relation \sim on T_h which is generated by the set of relations $t(x) \sim t(z)$ for pairs above, and create the metric space T_h / \sim , with the metric being the induced one.) The resulting metric space is $g(h, \mathcal{Q})$.

Remark. The factors of 2 that appear in this construction give it a slightly awkward character. They derive from the fact that our tree is constructed from the excursion corresponding to the height process, whereas our point set \mathcal{Q} is related to the excursion corresponding to the depth-first search—and, though these are based on the same tilted Brownian excursion, there is a factor of 2 difference between them, as given in Theorem 2.20. This means that it is natural, when giving a definition of the function g , to rescale one or the other.

Figure 2.3 gives an illustration of the process of identifying the points.

The object $g(h, \mathcal{Q})$ is not a real tree, because it contains cycles. However, its local structure is akin to that of a real tree. We call such an object an \mathbb{R} -graph (see [4, Defn. 2.2] for a formal definition).

Remark. Defining this formally relies on a definition of a real tree, which relies on a number of technical metric space results that are beyond the scope of this thesis.

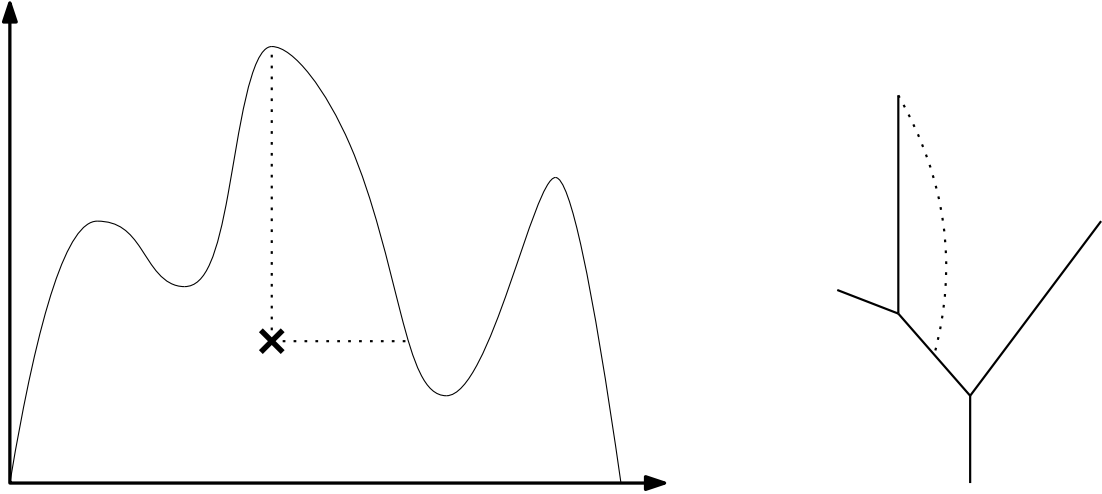


Figure 2.3: The diagram above illustrates the identification forming the structure described in Theorem 2.24. The tree on the right is the tree T_f corresponding to the function f shown on the left. The point selected from under its area (shown after the y -coordinate is scaled by 2, as required by the definition of g) then corresponds to two points on the tree, which are identified with each other. (Note that in this example only points directly underneath one of the three local maxima of the function correspond to leaves. In a tree formed from a Brownian excursion, this is true of almost all points, so the x -coordinate of a point selected from a Poisson process applied to the area under the curve will almost always correspond to a leaf.)

Definition 2.23. Let $\mathcal{M}^{(\sigma)} = g(2\tilde{\mathbf{e}}^{(\sigma)}, \mathcal{P})$, where \mathcal{P} is a Poisson point process on $[0, \sigma] \times [0, \infty)$ of rate 1.

$\mathcal{M}^{(\sigma)}$ will be our limit object, and thus we can now state the theorem.

Theorem 2.24 ([3, Thm. 22]). *Let $\sigma > 0$, and let $m = m(n)$ be a sequence of integers such that $n^{-2/3}m \rightarrow \sigma$ as $n \rightarrow \infty$. Let $p = p(n) \in (0, 1)$ satisfy $np \rightarrow 1$. Then, as $n \rightarrow \infty$,*

$$n^{-1/3}G_m^p \xrightarrow{d} \mathcal{M}^{(\sigma)},$$

in the Gromov–Hausdorff distance.

To extend this to the whole graph, as opposed to just a single connected component, we need to define a metric, or at least a topology, on a sequence of (isometry classes of) metric

spaces. We therefore define the metric

$$d_{\text{seq}}(\mathbf{A}, \mathbf{B}) = \left(\sum_{i=1}^{\infty} d_{\text{GH}}(A_i, B_i)^4 \right)^{\frac{1}{4}}, \quad (2.9)$$

where $\mathbf{A} = (A_1, A_2, \dots)$ and $\mathbf{B} = (B_1, B_2, \dots)$ are sequences of metric spaces. (Of course, this may take the value ∞ if the sequence on the right-hand side does not converge.)

We can now state the theorem.

Theorem 2.25 ([3], Thm. 24). *Let $\mathbf{M}^{(n)} = (M_1^{(n)}, M_2^{(n)}, \dots)$ be a sequence of metric spaces such that $M_i^{(n)}$ is the i -th largest component of $G \sim \mathcal{G}(n, p)$, treated as a metric space with the graph distance. Let $Z_i^{(n)} = |M_i^{(n)}|$, and let $\mathbf{Z}^{(n)} = (Z_1^{(n)}, Z_2^{(n)}, \dots)$. Then there exist \mathbf{Z}, \mathbf{M} such that*

$$(n^{-2/3}\mathbf{Z}^{(n)}, n^{-1/3}\mathbf{M}^{(n)}) \xrightarrow{d} (\mathbf{Z}, \mathbf{M}),$$

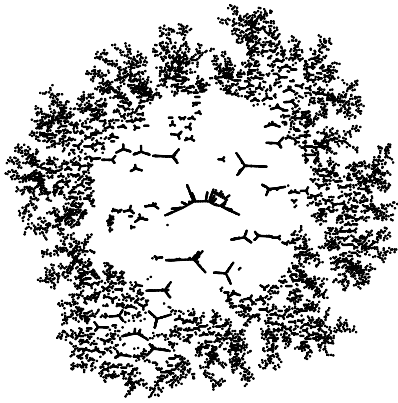
with convergence in the first co-ordinate in l^2 , and in the second co-ordinate in the metric d_{seq} . Moreover, the law of \mathbf{Z} is given by Theorem 1.5, while $\mathbf{M} = (M_1, M_2, \dots)$ is such that, conditionally on \mathbf{Z} , the random metric spaces M_1, M_2, \dots are independent and $M_i \stackrel{d}{=} \mathcal{M}^{(Z_i)}$.

Let us briefly summarise how this result is obtained. Theorem 2.24 shows that the exploration of a component of $G \sim \mathcal{G}(n, p)$, conditionally on its size, behaves as a tilted excursion of that size, with surplus edges given by a binomial point process under the excursion. It is also obvious that all components, conditionally on their sizes and numbers of surplus edges, are independent, so the M_i must be conditionally independent too.

The only thing then left is to obtain convergence in the metric d_{seq} rather than in the product topology. To do this, the authors prove a bound on the height of the discrete depth-first-search trees (biased in a corresponding discrete way to $\tilde{\mathbf{e}}^{(\sigma)}$) to show that large trees do not occur arbitrarily far down the sequence.

This is, again, a heuristic description. We will return to some of the details of their proof when we come to adapt it later.

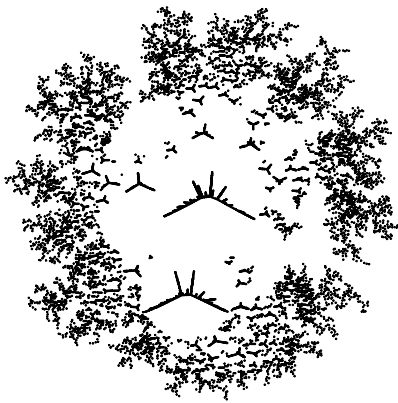
By way of illustration, in Figure 2.4, we provide some random graphs simulated for finite n at various points in the critical window, where we can begin to observe the limiting structure—including the $n^{2/3}$ scaling and shape of the largest component.



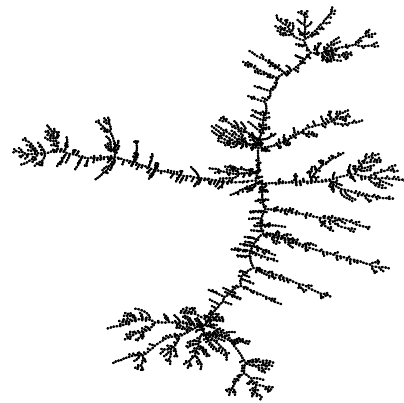
(a) $n = 10,000$, $\lambda = -1/2$; whole graph. In this realisation $|C_1^{(10,000)}| = 277$.



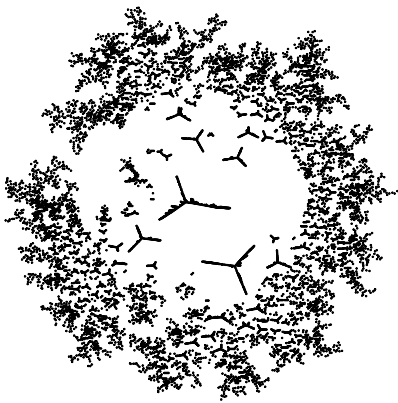
(b) $n = 100,000$, $\lambda = -1/2$; largest component. In this realisation $|C_1^{(100,000)}| = 881$.



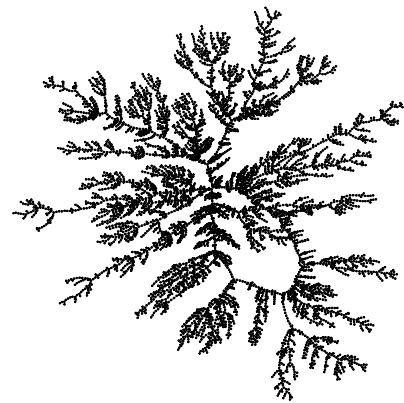
(c) $n = 10,000$, $\lambda = 0$; whole graph. In this realisation $|C_1^{(10,000)}| = 442$.



(d) $n = 100,000$, $\lambda = 0$; largest component. In this realisation $|C_1^{(100,000)}| = 2,668$.



(e) $n = 10,000$, $\lambda = 1/2$; whole graph. In this realisation $|C_1^{(10,000)}| = 317$.



(f) $n = 100,000$, $\lambda = 0$; largest component. In this realisation $|C_1^{(100,000)}| = 6,737$.

Figure 2.4: Simulations of various Erdős–Rényi random graphs for finite n , which correspond to points in the critical window. For $n = 10,000$, note that $n^{2/3} \approx 460$, and for $n = 100,000$, we have $n^{2/3} \approx 2,200$, giving us some idea of typical scales for the graphs. Made with R [54], using the igraph package [21].

2.4 A point process of component sizes

To conclude our discussion of the Erdős–Rényi random graph, we now discuss results of Janson and Spencer [40]. In this paper, they consider, for $G_n \sim \mathcal{G}(n, p)$ with $p = 1/n + \lambda n^{-4/3}$, rescaled component sizes given by

$$\xi_i^{(n)} = \frac{|C_i^{(n)}|}{n^{2/3}}.$$

They note that these may be placed into a random (multi)set

$$\Xi_n = \left\{ \xi_i^{(n)} : i \in [c(G_n)] \right\}$$

(where, as ever, $c(G)$ denotes the number of components of the graph G); they may then be considered as a point process on $(0, \infty]$ —or, alternatively, as a random measure on $(0, \infty]$, with point masses at the points of the set, which allows us to write, say, $\Xi_n(A)$ to denote $|\Xi_n \cap A|$. They note that it follows immediately from the results of Aldous [6] that this set converges to some Ξ in the vague topology, defined as follows:

Definition 2.26. Let $\mathfrak{R} = \mathfrak{R}((0, \infty))$ (resp. $\mathfrak{R} = \mathfrak{R}((0, \infty])$) be the class of Borel measures μ on $(0, \infty)$ (resp. $(0, \infty]$) such that $\mu(A) \in \mathbb{Z}^{\geq 0}$ (and in particular is finite) for every relatively compact Borel set $A \subseteq (0, \infty)$ (resp. $(0, \infty]$). Then the *vague topology* on \mathfrak{R} is defined such that, given $\mu_1, \mu_2, \dots \in \mathfrak{R}$, and any $\mu \in \mathfrak{R}$, we have $\mu_n \rightarrow \mu$ as $n \rightarrow \infty$ if and only if, for every real-valued continuous function f on $(0, \infty)$ (resp. $(0, \infty]$) with compact support,

$$\int f \, d\mu_n \rightarrow \int f \, d\mu$$

as $n \rightarrow \infty$.

Remark. Kallenberg [41, p. 169] provides a reference for the vague topology.

Further, Aldous [6] noted that Ξ could be considered as the set of lengths of excursions of reflected Brownian motion with parabolic drift, as we discussed in Chapter 1. (We will sometimes write $\Xi = \Xi^{(\lambda)}$ to emphasise the dependence on λ .) The contribution of Janson and Spencer in [40] is, among other things, to give an alternative description of Ξ in terms of its intensity and its Palm distributions—concepts which we now define.

Definition 2.27 ([60, 61]). For a point process on $(0, \infty)$, an *intensity function*, if one exists,

is a function f such that, for all Borel sets $A \subseteq (0, \infty)$

$$\mathbb{E}[\Xi(A)] = \int_A f(x) dx.$$

Remark. It is possible to define this in other ways: for instance, by letting intensity correspond to the probability of seeing a point in an interval, rather than the expected number of points (as in [22]). For a *simple* point process, in which a.s. no two points of the process coincide, these definitions coincide.

Definition 2.28 ([40, §8]). For a point process Ξ on a suitable space \mathfrak{S} , the *Palm distributions* are the conditional distributions of Ξ given the presence of a point $s \in \mathfrak{S}$, denoted $\mathcal{L}(\Xi | s \in \Xi)$.

Remark. The event on which we are conditioning in the definition above has probability 0 in many cases; a justification of this is given in Kallenberg [41, Ch. 12].

We now state the results of Janson and Spencer that we will use. Recall that e is a standard Brownian excursion, and define $A = \int_0^1 e(u) du$.

Theorem 2.29.

(a) [40, Thm. 4.1] The intensity $\Lambda = \Lambda^{(\lambda)}$ of $\Xi^{(\lambda)}$ is given by

$$\Lambda^{(\lambda)}(x) = \frac{1}{\sqrt{2\pi}} x^{-5/2} \Psi(x^{3/2}) \exp\left\{-\frac{1}{6}x^3 + \frac{\lambda}{2}x^2 - \frac{\lambda^2}{2}x\right\},$$

where

$$\Psi(t) := \mathbb{E}[e^{tA}].$$

(b) [40, Thm. 8.1] The Palm distribution $\mathcal{L}(\Xi^{(\lambda)} | s \in \Xi^{(\lambda)})$ equals, for every $s > 0$, the distribution of $\Xi^{(\lambda-s)} + \delta_s$, where δ_s is a point-mass at s .

Remark. In Corollary 8.4, the authors note that Ξ is actually a.s. simple, resolving the ambiguity about the meaning of “intensity”.

This formulation is particularly useful in calculations. For instance, as in the authors’ Corollary 4.2, we can see that, if $Z_\varepsilon = \sum_{\xi \in \Xi} \xi \mathbf{1}_{\{\xi > \varepsilon\}}$,

$$\mathbb{E}[Z_\varepsilon] = \frac{1}{\sqrt{2\pi}} \int_\varepsilon^\infty x^{-3/2} \Psi(x^{3/2}) \exp\left\{-\frac{1}{6}x^3 + \frac{\lambda}{2}x^2 - \frac{\lambda^2}{2}x\right\} dx. \quad (2.10)$$

We will use Theorem 2.29 to prove similar results in Chapter 7. However, we note that to convert from the integral representation to more-explicit asymptotic representations, we will

need asymptotics for the function Ψ . We have these: quoting [40, (3.4)], as $t \rightarrow \infty$,

$$\Psi(t) \sim \frac{1}{2}t^2 e^{t^2/24}, \quad (2.11)$$

which, for instance, tells us that the integral in (2.10) is finite. Meanwhile, we can exploit the fact that Ψ is a moment-generating function to find asymptotics for $t \downarrow 0$; for instance, that

$$\Psi(t) = 1 + t\mathbb{E}[A] + O(t^2),$$

when A (as in the theorem) is the area under a standard Brownian excursion. We discuss this in more detail later.

Chapter 3

The random cluster model:

introduction

In this thesis, we will consider the *mean-field random cluster model*, or the *random cluster model on the complete graph*. We start by motivating the definition; we then define the random cluster model in general, before specifying the setting of interest to this work.

We start by mentioning the Potts model, first described in the literature by Potts in 1951 [51]; this is a model used in statistical mechanics, which describes the behaviour of a collection of particles where each can have one of $q \in \mathbb{N}$ spin values. As it is not particularly relevant to our discussions, we do not discuss it in detail here, except to note that, in the $q = 2$ case, it is equivalent to the better-known Ising model (see Wu [63], who also gives an overview of the model and of some of its applications in statistical mechanics).

The random cluster model, devised by Fortuin and Kasteleyn [28], was introduced as a tool by which to analyse the Potts model, though it has since become the subject of research interest in its own right (see Grimmett [32] for an overview). It is defined on a connected graph G , and is the distribution in which each (not necessarily connected) subgraph H of G is chosen at random with probability proportional to

$$p^{e(H)}(1-p)^{e(G)-e(H)}q^{c(H)}$$

where $e(H)$ (resp. $e(G)$) denotes the number of edges of H (resp. G), and $c(H)$ denotes the number of (connected) components in H . Clearly if $q = 1$ this corresponds to Bernoulli percolation on the graph G , but the value of q biases the graph to have more or fewer components.

When $q \in \mathbb{N}$, the connection to the Potts model can be described as follows. Suppose we

consider the random cluster model on G ; call the random graph thus obtained H . Assign, uniformly at random, a spin state from the set $[q]$ to each component of H ; then assign each vertex the spin state of the component to which it belongs, before removing the structure of H , leaving only the vertices (which we may still consider as vertices of G). Then the spin states of the vertices behave according to the Potts model for particles on G . (See Grimmett [32, § 1.4] for a reference.)

Physicists are usually interested in taking the base graph G to have some particular geometry, such as taking G to be a lattice (potentially an infinite one, adapting the definition of the model appropriately). This covers many of the settings discussed in [32]. However, in this thesis, we will be interested in the *mean-field* random cluster model, or the random cluster model on the complete graph, in which we take G to be the complete graph K_n (for some $n \in \mathbb{N}$). To state the model more explicitly, we have that every possible graph H on $[n]$ occurs with probability

$$\frac{p^{e(H)}(1-p)^{\binom{n}{2}-e(H)}q^{c(H)}}{Z(n,p,q)}, \quad (3.1)$$

where $c(H)$ denotes the number of (connected) components of H ; here,

$$Z(n,p,q) := \sum_H p^{e(H)}(1-p)^{\binom{n}{2}-e(H)}q^{c(H)}, \quad (3.2)$$

where the sum is taken over all graphs H on $[n]$. We denote this by $\mathcal{G}(n,p,q)$. Since from now on we will only deal with the mean-field setting, we will often use the unqualified phrase “random cluster model” to refer to this setting.

If $q = 1$, this would be, as described above, Bernoulli percolation on the complete graph—that is to say, the Erdős–Rényi model, $\mathcal{G}(n,p)$. Consequently, taking $q < 1$ in $\mathcal{G}(n,p,q)$ biases towards having fewer components than $\mathcal{G}(n,p)$; taking $q > 1$ biases towards having more components.

Before we go on, we note that it is possible to define a model $\mathcal{G}(n,m,q)$, analogously to $\mathcal{G}(n,m)$: if $G_n \sim \mathcal{G}(n,m,q)$, we take G_n to be a random graph on n vertices and m edges, choosing each possibility H with probability proportional to $q^{c(H)}$. We will not discuss this version of the random cluster model in great detail, but will make some comments about it in Section 8.1.

Similarly to the Erdős–Rényi model, if we consider $\mathcal{G}(n,\kappa/n,q)$, with q constant, it can be shown that, for $n \rightarrow \infty$, this model exhibits a phase transition in the size of the largest component as κ passes through a critical value. (This was proved independently by Stepanov

[58, 59] and Bollobás, Grimmett and Janson [16].) The critical value of κ , which we denote as κ_c , depends on q as follows [16, Eqn. (2.1)]:

$$\kappa_c(q) = \begin{cases} q & \text{if } q \in (0, 2]; \\ 2^{\frac{q-1}{q-2}} \log(q-1) & \text{if } q > 2. \end{cases} \quad (3.3)$$

(For $q > 2$, Stepanov [58, p. 195] gives the value of κ_c implicitly in terms of the expressions

$$(1 - \alpha) \frac{e^{\alpha \kappa_c} - 1}{\alpha} = q \quad (3.4)$$

and

$$\frac{\alpha^2 \kappa_c}{2} \left(\frac{1}{q} - 1 \right) - \frac{\alpha \kappa_c}{q} - \log(1 - \alpha) = 0, \quad (3.5)$$

which he asserts can be solved to attain a function $\kappa_c(q)$ for $q > 2$; Bollobás, Grimmett and Janson [16, p. 299] implicitly show that the correct value of α is $(q-2)/(q-1)$, which does indeed solve both equations.)

As in the Erdős–Rényi model, we see a giant component for large κ , with other components being microscopic components of order $\log n$; for small κ , we only see the small components.

Theorem 3.1.

- (a) [58, 16] For $\kappa > \kappa_c$, the number of vertices in the largest component of the graph is $C_1 n + O_p(n^{3/4} \sqrt{\log n})$, for some constant C_1 depending on κ and q , with the second largest component being w.h.p. a tree, of order $C_2 \log n + o_p(\log n)$, for a constant C_2 depending on κ and q .
- (b) [16, Thm. 2.1] For $\kappa < \kappa_c$, the largest component of the graph is of size $D \log n + O_p(\log \log n)$, for a constant D depending on κ and q .

Remark. The part of Theorem 3.1(a) concerning the largest component is from [58, pp. 201–202]; that concerning the second-largest component is from [16, Thm. 2.2]. Each paper contains (weaker) statements about the other component. Meanwhile Theorem 3.1(b) is also given in slightly weaker form by [58, pp. 200–201].

Note that we have much more detailed information about the component sizes, in both regimes, for the Erdős–Rényi model, but the above theorems tell us that the basic behaviour is the same.

We move on to the case $\kappa = \kappa_c$: here, if $q > 2$, then we see something that looks like either the supercritical regime or the subcritical regime described above [16, Thm. 2.3] (or

[58, pp. 202–203]). For $q < 2$, Stepanov [58, p. 202] concludes that the largest component contains $O_p(n^{3/4}\sqrt{\log n})$ vertices; he obtains a weaker bound for $q = 2$.

For $q < 2$, Łuczak and Łuczak [43] prove a stronger result about this critical regime, and indeed show the existence of a *critical window* around $\kappa = q$ ($= \kappa_c$).

Theorem 3.2 ([43, Thm. 17]). *Let $G \sim \mathcal{G}(n, p, q)$, where $q \in (0, 2)$ and $p = q/n + \lambda n^{-4/3}$, for some $\lambda \in \mathbb{R}$. Then, if $|C_i^{(n)}|$ is the size of the i -th largest component of G ,*

$$|C_1^{(n)}|, |C_2^{(n)}| = \Theta_p(n^{2/3}).$$

Łuczak and Łuczak [43] also show that, for $q = 2$, this only holds for $\lambda < 0$, and that the behaviour is different for $\lambda \geq 0$.

Theorem 3.3 ([43, Thm. 18, (ii) and (v)]). *Let $G \sim \mathcal{G}(n, p, 2)$, where $p = 2/n + \lambda n^{-4/3}$, for some $\lambda \in \mathbb{R}$. Then, if $\lambda < 0$, letting $|C_i^{(n)}|$ denote the size of the i -th largest component of G ,*

$$|C_1^{(n)}|, |C_2^{(n)}| = \Theta_p(n^{2/3}).$$

If $\lambda > 0$, instead we have that there exists a function $l = l(n, \lambda) = (1 + o(1))\sqrt{3\lambda}n^{5/6}$ such that, for all $a \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \frac{n^{2/3}}{\sqrt{2\lambda}} \mathbb{P}\left(|C_1^{(n)}| = \left\lfloor l + \frac{an^{2/3}}{\sqrt{2\lambda}} \right\rfloor\right) = \frac{1}{\sqrt{2\pi}} e^{-a^2/2}.$$

Remark. So, for $\lambda > 0$, the size of the largest component is approximately normally distributed, with mean on the order of $n^{5/6}$ and standard deviation on the order of $n^{2/3}$. Meanwhile, if we take $\lambda = \lambda(n)$ and assume $\lambda \rightarrow 0$ as $n \rightarrow \infty$, we get different behaviour depending on the rate of this latter convergence to 0. The precise details are not germane to the discussions in this thesis; we will give an overview in Section 9.2, when discussing avenues for future research.

We work towards a proof of the existence of a scaling limit for all $q \in (0, 2)$. Specifically, we aim to prove the following.

Conjecture 3.4. *Let $G^{(n)} \sim \mathcal{G}(n, p, q)$, where $q \in (0, 2)$ and $p = q/n + \lambda n^{-4/3}$, for some $\lambda \in \mathbb{R}$. Let $C_i^{(n)}$ be the i -th largest component of $G^{(n)}$, which we treat as a metric space using the graph distance; likewise, treat $G^{(n)}$ as a sequence of its components, treated as metric spaces and listed in decreasing order of size.*

Then there exists $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2, \dots)$, a sequence of random metric spaces, such that

$$n^{-1/3} G^{(n)} \xrightarrow{d} \mathcal{C}$$

where convergence is in the metric d_{seq} defined in (2.9).

The difficulty is in proving the convergence of the component sizes and numbers of surplus edges, as evidenced by the following lemma, whose proof is immediate from the form of the measure (3.1).

Lemma 3.5. *Let $\tilde{G} \sim \mathcal{G}(n, p)$, and let $G \sim \mathcal{G}(n, p, q)$, for any $n \in \mathbb{N}$, $p \in [0, 1]$ and $q \in (0, \infty)$. Let $\ell \in \mathbb{N}$, and let $\mathbf{a} = (a_1, \dots, a_\ell) \in \mathbb{N}^\ell$ satisfy*

- $a_1 \geq a_2 \geq \dots$;
- $\sum_{i=1}^{\ell} a_i = n$.

Let C_i (resp. \tilde{C}_i) be the i -th largest component of G (resp. \tilde{G}), and let A (resp. \tilde{A}) be the event that, for all i , $|C_i| = a_i$ (resp. $|\tilde{C}_i| = a_i$). Then the distribution of \tilde{G} conditional on \tilde{A} is identical to the distribution of G conditional on A .

So, conditionally on the component structure, we just have an Erdős–Rényi random graph. But note that, for $q \neq 1$, if G is at the critical point then \tilde{G} will not be, and vice versa.

This strongly suggests that the bulk of the necessary work in proving Conjecture 3.4 is in showing that the component sizes converge. Indeed, we will show in Theorem 8.19 that this truly is all that is necessary to prove the conjecture.

We can also conjecture something about the form of the limit sequence of component sizes. Specifically, we conjecture the following.

Conjecture 3.6. *Let $\mathbf{C}_n = \left(|C_1^{(n)}|, |C_2^{(n)}|, \dots \right)$ be the decreasing sequence of component sizes in $G_n \sim \mathcal{G}(n, p, q)$, where $p = q/n + \lambda n^{-4/3}$. Then*

$$n^{-2/3} \mathbf{C}_n \xrightarrow{d} \mathbf{C}, \tag{3.6}$$

in l^2 , where \mathbf{C} is under a measure $\tilde{\mathbb{P}}$ that we characterise as follows.

Under the measure \mathbb{P} , let \mathbf{C} be the sequence of excursion lengths, in decreasing order, of $X - I$, where, for a Brownian motion $B(t)$, we have $X(t) = B(t) + \lambda t/q - t^2/2$, and $I(t) = \inf_{0 \leq u \leq t} X(u)$. Let L_λ be the limit in probability, as $t \rightarrow \infty$, of $\int_0^t (X(u) - I(u)) du - (\log t)/2$. Then, in (3.6), \mathbf{C} has the law that it does under $\tilde{\mathbb{P}}$, where

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \frac{\exp\{(q-1)L_\lambda\}}{\mathbb{E}[\exp\{(q-1)L_\lambda\}]}, \tag{3.7}$$

in which \mathbb{E} denotes expectation under \mathbb{P} .

There are several implicit assertions in this conjecture, such as that L_λ exists and is almost surely finite, and that $\mathbb{E}[\exp\{(q-1)L_\lambda\}] < \infty$. In fact, we can prove that L_λ exists (Proposition 7.1) and that, for $q > 1$, the expectation is finite (implied by Proposition 7.2), though we could not prove this for $q < 1$. We also have good evidence for the rest of this conjecture, as we will discuss in Section 3.2. Before that, though, we provide a fundamental lemma for the results that will follow.

3.1 Changing measure

We note the following result which underpins our approach. Henceforth, let $\mathbb{P}_{n,p,q}$ be a probability measure under which $G_n \sim \mathcal{G}(n, p, q)$. Let $\mathbb{P}_{n,p} = \mathbb{P}_{n,p,1}$, so that, under $\mathbb{P}_{n,p}$, the graph G_n follows the Erdős–Rényi model. Let $\mathbb{E}_{n,p,q}$ (resp. $\mathbb{E}_{n,p}$) denote expectation with respect to $\mathbb{P}_{n,p,q}$ (resp. $\mathbb{P}_{n,p}$). For an interval $A \subseteq \mathbb{R}$, let $D(A)$ be the space of càdlàg functions from A to \mathbb{R} , equipped with the Skorokhod metric.

Proposition 3.7. *For any bounded measurable functional $f : D([0, n])^2 \rightarrow \mathbb{R}$, and for any $n \in \mathbb{N}$, $p \in (0, 1)$ and $q > 0$,*

$$\begin{aligned} \mathbb{E}_{n,p,q}[f(X_n(\cdot), I_n(\cdot))] &= \frac{\mathbb{E}_{n,p}[f(X_n(\cdot), I_n(\cdot)) q^{-X_n(n)}]}{\mathbb{E}_{n,p}[q^{-X_n(n)}]} \\ &= \frac{\mathbb{E}_{n,\tilde{p}}[f(X_n(\cdot), I_n(\cdot)) q^{N_n(n)}]}{\mathbb{E}_{n,\tilde{p}}[q^{N_n(n)}]}, \end{aligned}$$

where

$$\tilde{p} = \frac{p}{p + q(1-p)}. \quad (3.8)$$

Remark. Note that, by definition, if $p \in (0, 1)$, then $\tilde{p} \in (0, 1)$ too, so this is a well-defined measure change.

Note that, if $p = q/n + \lambda n^{-4/3}$, then

$$\tilde{p} = \frac{1}{n} + \frac{\lambda}{qn^{4/3}} + O\left(\frac{1}{n^2}\right) \quad (3.9)$$

as $n \rightarrow \infty$. So we have transformed our problem from one about the random cluster model in what we know, from Theorem 3.2, to be its critical window, to one about the Erdős–Rényi model in its own critical window. Moreover, we note that the $\lambda n^{-4/3}$ term in the random cluster model setting translates to a $\lambda n^{-4/3}/q$ term in the Erdős–Rényi setting, which is indicative of why, in (3.7), we defined the limit distribution by changing measure with respect to the limit

of the “ λ/q ” point in the Erdős–Rényi critical window.

Before we go on to outline our approach, we must give the proof of Proposition 3.7, which is due to Christina Goldschmidt (personal communication).

Proof of Proposition 3.7. For the first equality, we simply need to note that $-I_n(i)$ records the number of components whose exploration was completed by the end of step i , and that therefore $-I_n(n)$ is the total number of components of the graph; this is equal to $-X_n(n)$, since at the end of the exploration we are trivially at the end of a component. The first equality then follows immediately from the distribution (3.1) of the random cluster model.

For the second part, we need to rewrite (3.1). For a graph G , let $s(G)$ denote the number of surplus edges it has, as in Definition 1.3, so that, as in (1.1),

$$s(G) = e(G) - |G| + c(G).$$

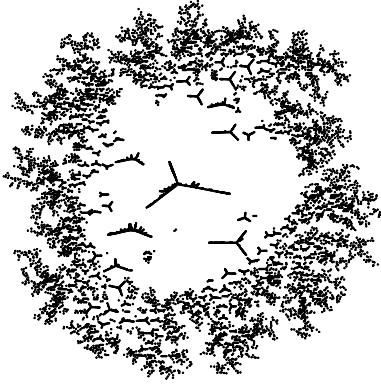
From this, we have that, for graphs H on n vertices, if $G_n \sim \mathcal{G}(n, p, q)$,

$$\begin{aligned} \mathbb{P}_{n,p,q}(G_n = H) &\propto \left(\frac{p}{1-p}\right)^{e(H)} q^{c(H)} \\ &= \left(\frac{p}{1-p}\right)^{e(H)} q^{s(H)-e(H)+n} \\ &\propto \left(\frac{p}{q(1-p)}\right)^{e(H)} q^{s(H)} \\ &= \left(\frac{\tilde{p}}{1-\tilde{p}}\right)^{e(H)} q^{s(H)}, \end{aligned} \tag{3.10}$$

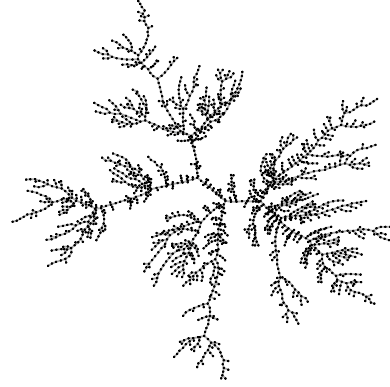
where the last line is obtained by multiplying the top and bottom of the fraction in (3.10) by $(p + q(1-p))^{-1}$. Since $N_n(n)$ records the total number of surplus edges in the graph, this gives the second part of the claim. \square

As well as letting us deal with a critical Erdős–Rényi graph rather than an off-critical one, this helps us in another way. We expect that, in the off-critical case, we are dealing with large-deviations behaviour of the graph by considering $q^{c(G_n)}$, since $c(G_n) = \Theta_p(n)$. (For example, Puhalskii [53, Thm. 2.2] shows this, in particular, for $p = \kappa/n$ for all $\kappa > 0$, from which the general case of $p \sim \kappa/n$ follows by an appropriate coupling.) By contrast, as we will see in Theorem 8.14, in the case of a critical Erdős–Rényi graph, $N_n = \Theta_p(\log n)$; since $q^{\log n} = n^{\log q}$, this suggests the measure change has a much less drastic effect.

Before we go on, we provide some illustrations of graphs drawn from the random cluster model for finite n (along with an Erdős–Rényi graph, for reference), in Figure 3.1; these are



(a) $n = 10,000$, $q = 1/2$; whole graph. In this realisation $|C_1^{(10,000)}| = 364$.



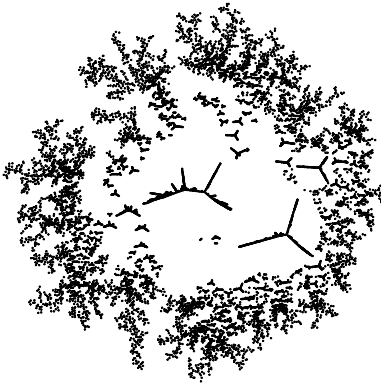
(b) $n = 100,000$, $q = 1/2$; largest component. In this realisation $|C_1^{(100,000)}| = 1,842$.



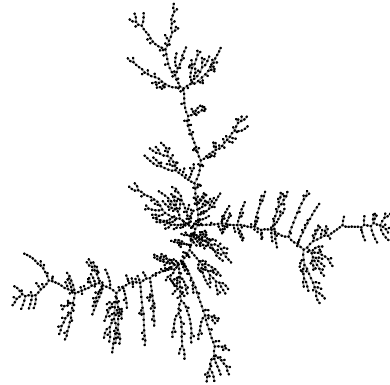
(c) $n = 10,000$, $q = 1$; whole graph. In this realisation $|C_1^{(10,000)}| = 716$.



(d) $n = 100,000$, $q = 1$; largest component. In this realisation $|C_1^{(100,000)}| = 2,760$.



(e) $n = 10,000$, $q = 3/2$; whole graph. In this realisation $|C_1^{(10,000)}| = 516$.



(f) $n = 100,000$, $q = 3/2$; largest component. In this realisation $|C_1^{(100,000)}| = 1,410$.

Figure 3.1: Simulations of graphs drawn from $\mathcal{G}(n, p, q)$, taking in each case $p = q/n$ exactly. The $q = 1$ (Erdős–Rényi) case was simulated using a standard function in the package used; for $q = 1/2$ rejection sampling was used, exploiting Proposition 3.7, and for $q = 3/2$ the simulation is an approximate one using the Metropolis–Hastings algorithm. Made with R [54], using the igraph [21] and progress [20] packages.

chosen to correspond to the $\lambda = 0$ point in the critical window.

3.2 Heuristics for a suggested proof

The rest of this thesis will be devoted to results towards a proof of Conjecture 3.4. More specifically, we will mostly work towards the following conjecture, which is an analogue to Theorem 1.6 (Aldous’s result from [6] on convergence of the exploration process). Consider exploring G_n using the exploration process we defined above for a component of a graph, Algorithm A, defined on p. 4, which we recall is denoted $(X_n, I_n, N_n) = (X_n(i), I_n(i), N_n(i))_{i=0}^n$.

Conjecture 3.8. *There exists a process X on $[0, \infty)$ with running infimum I under some measure \mathbb{P} such that, for all $t > 0$ and all bounded continuous functionals $f : D([0, t])^2 \rightarrow \mathbb{R}$,*

$$\mathbb{E}_{n,p,q}[f(n^{-1/3}X_n(\lfloor sn^{2/3} \rfloor), n^{-1/3}I_n(\lfloor sn^{2/3} \rfloor) : 0 \leq s \leq t)] \rightarrow \mathbb{E}[f(X(s), I(s) : 0 \leq s \leq t)]$$

as $n \rightarrow \infty$.

As in the case of the Erdős–Rényi graph, the conjecture would only give us that the exploration process converges, not that the sequence of component sizes does—towards the end of Section 6.2, we will also discuss how to carry out the conversion. We include I_n in the conjecture to emphasise that it too converges; it is also useful to consider them separately as, for the Erdős–Rényi model (and, as we shall see in Proposition 4.3, for the random cluster model) the pair (X_n, I_n) and the difference $X_n - I_n$ are Markov processes, but the individual processes X_n and I_n are not, in general. However, the convergence of I_n is implicit from the convergence of X_n .

Remark. We only include X_n and I_n in Conjecture 3.8, and not N_n , the surplus edge process. This is for two reasons. Firstly, an N_n term will appear in the heuristic calculations that follow (as well as in our formal proofs later on), and it will make these calculations simpler if this is the only term involving this process. Secondly, it is not necessary: as Proposition 4.3 shows, N_n has a conditional distribution (given X_n and I_n) that can be written down explicitly, and worked with separately should X_n and I_n converge; moreover, in Section 8.2 we will show that we only need convergence of the rescaled component sizes to obtain a full metric space scaling limit, and the component sizes are not affected by the number of surplus edges.

We now give heuristics for a proof of the above conjecture. By Proposition 3.7, we see that,

for \tilde{p} as above, and a bounded continuous functional f ,

$$\begin{aligned} & \mathbb{E}_{n,p,q} [f(n^{-1/3}X_n(\lfloor sn^{2/3} \rfloor), n^{-1/3}I_n(\lfloor sn^{2/3} \rfloor) : 0 \leq s \leq t)] \\ &= \frac{\mathbb{E}_{n,\tilde{p}} [f(n^{-1/3}X_n(\lfloor sn^{2/3} \rfloor), n^{-1/3}I_n(\lfloor sn^{2/3} \rfloor) : 0 \leq s \leq t) q^{N_n(n)}]}{\mathbb{E}_{n,\tilde{p}} [q^{N_n(n)}]}. \end{aligned}$$

We cannot simply take the limit of the right-hand side using Aldous's result from [6] (our Theorem 1.6), as we are no longer just looking at a functional of the process up to time $\lfloor tn^{2/3} \rfloor$ —that is, of $(n^{-1/3}X_n(\lfloor sn^{2/3} \rfloor), n^{-1/3}I_n(\lfloor sn^{2/3} \rfloor) : 0 \leq s \leq t)$ —but rather one which extends to times in the exploration outside the $n^{2/3}$ timescale. However, we expect that, for large $T \geq t$, the number of points arriving in the interval $[\lfloor Tn^{2/3} \rfloor + 1, n]$ should be approximately independent, for large n , of what happens up to time $\lfloor Tn^{2/3} \rfloor$. This would be exactly the case if we knew that $X_n(\lfloor Tn^{2/3} \rfloor) = I_n(\lfloor Tn^{2/3} \rfloor - 1) - 1$ (i.e., if we were at the end of a component), and, since the components appear in the exploration in size-biased random order, we expect to see smaller and smaller components, and for this therefore increasingly to be the case. So, informally, we would expect the approximate independence to hold.

Using this, we would have

$$\begin{aligned} & \mathbb{E}_{n,p,q} [f(n^{-1/3}X_n(\lfloor sn^{2/3} \rfloor), n^{-1/3}I_n(\lfloor sn^{2/3} \rfloor) : 0 \leq s \leq t)] \\ & \approx \frac{\mathbb{E}_{n,\tilde{p}} [f(n^{-1/3}X_n(\lfloor sn^{2/3} \rfloor), n^{-1/3}I_n(\lfloor sn^{2/3} \rfloor) : 0 \leq s \leq t) q^{N_n(\lfloor Tn^{2/3} \rfloor)}]}{\mathbb{E}_{n,\tilde{p}} [q^{N_n(\lfloor Tn^{2/3} \rfloor)}]} \\ & \quad \times \frac{\mathbb{E}_{n,\tilde{p}} [q^{N_n(n) - N_n(\lfloor Tn^{2/3} \rfloor)}]}{\mathbb{E}_{n,\tilde{p}} [q^{N_n(n) - N_n(\lfloor Tn^{2/3} \rfloor)}]} \\ & = \frac{\mathbb{E}_{n,\tilde{p}} [f(n^{-1/3}X_n(\lfloor sn^{2/3} \rfloor), n^{-1/3}I_n(\lfloor sn^{2/3} \rfloor) : 0 \leq s \leq t) q^{N_n(\lfloor Tn^{2/3} \rfloor)}]}{\mathbb{E}_{n,\tilde{p}} [q^{N_n(\lfloor Tn^{2/3} \rfloor)}]}, \end{aligned} \tag{3.11}$$

with the approximation improving as T grows. Since we are now approximately working on the $n^{2/3}$ timescale, we would hope to be able to use Aldous's result—Theorem 1.6—to deduce using that

$$\begin{aligned} & \frac{\mathbb{E}_{n,\tilde{p}} [f(n^{-1/3}X_n(\lfloor sn^{2/3} \rfloor), n^{-1/3}I_n(\lfloor sn^{2/3} \rfloor) : 0 \leq s \leq t) q^{N_n(\lfloor Tn^{2/3} \rfloor)}]}{\mathbb{E}_{n,\tilde{p}} [q^{N_n(\lfloor Tn^{2/3} \rfloor)}]} \\ & \rightarrow \frac{\mathbb{E} [f(X(s), I(s) : 0 \leq s \leq t) q^{N(T)}]}{\mathbb{E} [q^{N(T)}]}, \end{aligned} \tag{3.12}$$

as $n \rightarrow \infty$, where X , I and N are Aldous's limit processes. This can be done using the bounded convergence theorem for $q < 1$, though it would require some additional machinery for $q > 1$.

Taking the limit in T is not going to be immediately possible, since $N(T) \rightarrow \infty$ almost surely as $T \rightarrow \infty$ (a consequence, for instance, of Proposition 4.1(ii) of [31]), and thus the expectations in both numerator and denominator tend to 0 for $q < 1$, and to ∞ for $q > 1$. However, we can hope that an appropriately compensated process might converge. Conditionally on X , we have that $N(T) \sim \text{Poi}\left(\int_0^T (X(u) - I(u)) du\right)$, and so, applying the tower law we have

$$\begin{aligned} & \mathbb{E}\left[f(X(s), I(s) : 0 \leq s \leq t) q^{N(T)}\right] \\ &= \mathbb{E}\left[f(X(s), I(s) : 0 \leq s \leq t) \exp\left\{(q-1) \int_0^T (X(u) - I(u)) du\right\}\right], \end{aligned}$$

and similarly for the denominator. Then, if we can find some deterministic function $\mathfrak{Z}(T)$ such that

$$\frac{\exp\left\{(q-1) \int_0^T (X(u) - I(u)) du\right\}}{\mathfrak{Z}(T)} \xrightarrow{d} L \quad (3.13)$$

for some almost-surely non-zero random variable L (whose distribution implicitly depends on q), and such that the sequence $q^{N(T)}/\mathfrak{Z}(T)$ is uniformly integrable, then we should have, dividing through the top and bottom of the fraction by $\mathfrak{Z}(T)$, that

$$\frac{\mathbb{E}\left[f(X(s), I(s) : 0 \leq s \leq t) q^{N(T)}\right]}{\mathbb{E}\left[q^{N(T)}\right]} \rightarrow \frac{\mathbb{E}\left[f(X(s), I(s) : 0 \leq s \leq t) L\right]}{\mathbb{E}[L]}$$

as $T \rightarrow \infty$. Then we would have that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{E}_{n,p,q}\left[f\left(n^{-1/3} X_n(\lfloor sn^{2/3} \rfloor), n^{-1/3} I_n(\lfloor sn^{2/3} \rfloor) : 0 \leq s \leq t\right)\right] \\ &= \frac{\mathbb{E}\left[f(X(s), I(s) : 0 \leq s \leq t) L\right]}{\mathbb{E}[L]}, \end{aligned}$$

which would yield Conjecture 3.8. (We may also be able to say something explicit about the distribution of L .)

When it comes to considering the limiting behaviour of $q^{N(T)}$, we will show later (Corollary 7.3) that $N(T) = \Theta_p(\log T)$, so we would hope to be dealing with a polynomial-order measure-change term—that is, something comparatively small. (Proposition 7.2 makes this more precise.) We also note that this proof method is consistent with Conjecture 3.6 above about the distribution of the limit component sizes, in terms of a measure change from the Erdős–Rényi case, as long as we take the right choice of $\mathfrak{Z}(T)$. So evidence for this proof method is evidence towards that conjecture.

There are two major problems with the above ideas that prevent them from immediately providing a proof:

Problem 1: We need to make rigorous the notion of the independence of the far future from the early part of the process, as in (3.11).

Problem 2: We need to find an appropriate function \mathfrak{J} , as in (3.13).

Eventually, we will consider how to address these problems, but first, in Chapter 4, we discuss preliminary results allowing us to compare the Erdős–Rényi and random-cluster exploration processes. Chapter 5 then takes a brief detour concerning the $\mathcal{G}(n, m)$ form of the Erdős–Rényi model.

In Chapter 6, we use the results from Chapter 4 to resolve Problem 1 in the $q < 1$ case. For $q > 1$, we do not have a resolution to this problem. Furthermore, there is an additional issue: for the case $q < 1$, the convergence in (3.12) followed by the bounded convergence theorem, but an integrability result would be needed in the larger- q case. We discuss this in more detail towards the end of Chapter 6.

However, we can resolve Problem 2 for $q > 1$: we do so in Chapter 7, using an approach originating with Janson and Spencer [40], which suggests that, in the calculation above, the correct function $\mathfrak{J}(T)$ to take is $\mathfrak{J}(T) \propto \exp\{(q-1)(\log T)/2\} = T^{(q-1)/2}$, as required for Conjecture 3.6 to hold. Problem 2 remains unsolved in the case $q < 1$; we discuss towards the end of that chapter why the method fails.

Of course, there's no reason to think that the heuristic proof above is the only valid method. Going forward, then, in Chapter 8, we outline an alternative (though still incomplete) method that might be used to show that the sequence of component sizes converges. Later in the same chapter, we finally discuss components as metric spaces, showing that, were a proof of convergence of the component sizes to be found, this is sufficient to show the existence of a scaling limit.

Chapter 4

Preliminary results

This chapter will be devoted to proving preliminary results, predominantly concerning the Erdős–Rényi model. These results will be used in the proofs relating to the random cluster model in Chapter 6.

In Section 4.1 we establish that, for $q < 1$, we may couple a critical random cluster graph to a critical Erdős–Rényi graph; we establish in Section 4.2 that this means that we can nest the corresponding exploration processes too, as well as noting that, for any p and q , the exploration process for the random cluster model is a Markov process. Having established the possibility of comparison, in Section 4.3 we establish some useful results about the Erdős–Rényi exploration process. Finally, in Section 4.4, we consider adding a small number of vertices to, or deleting a small number of vertices from, an Erdős–Rényi-type graph, demonstrating that the effect on the exploration is negligible, which will be necessary in the proofs in Chapter 6.

All of these results are related to the problems discussed at the end of the last chapter; we will discuss the connections as we go along.

Some of these results are of independent interest: for example, in Proposition 4.13, we prove that the reflected limit process described by Aldous (in the theorem we quoted as Theorem 1.6) converges to 0 as the time $t \rightarrow \infty$.

4.1 Glauber dynamics and coupling

Our aim in this section will be to prove the following proposition, which allows us to compare the random cluster model (for $q < 1$) and the Erdős–Rényi model.

Proposition 4.1. *Let $q \in (0, 1)$; let $\lambda^{(1)} \in \mathbb{R}$, and let $\lambda^{(2)} = \max\{0, \lambda^{(1)}/q\}$. Define*

$$\begin{aligned} p^{(1)} &= \frac{q}{n} + \frac{\lambda^{(1)}}{n^{4/3}}; \\ p^{(2)} &= \frac{1}{n} + \frac{\lambda^{(2)}}{n^{4/3}}. \end{aligned}$$

Let $G^{(1)} \sim \mathcal{G}(n, p^{(1)}, q)$, and let $G^{(2)} \sim \mathcal{G}(n, p^{(2)})$. Then, for all sufficiently large n that $p^{(1)} \in (0, 1)$ and $p^{(2)} \in (0, 1)$, we have that $G^{(1)} \leq_{\text{st}} G^{(2)}$, in the sense that we may couple $G^{(1)}$ and $G^{(2)}$ so that $G^{(1)}$ is a subgraph of $G^{(2)}$.

We will prove this by considering *Glauber dynamics*; that is, a Markov chain (in our case, a continuous-time one) whose stationary distribution is the random graph in question.

Let \mathfrak{G}_n denote the set of all graphs of size n , and let $B = (b_{GH})$ be a square matrix indexed by \mathfrak{G}_n , where:

- for all $G, H \in \mathfrak{G}_n$ with $G \neq H$, we have $b_{GH} \geq 0$; and
- for all $G \in \mathfrak{G}_n$, we have

$$b_{GG} = - \sum_{\substack{H \in \mathfrak{G}_n \\ H \neq G}} b_{GH}.$$

(That is, B is a plausible generator matrix.) As noted by Grimmett [32, §8.2], if B satisfies the condition that, whenever $b_{GH} > 0$,

$$\frac{b_{HG}}{b_{GH}} = \frac{p^{e(G)}(1-p)^{-e(G)}q^{c(G)}}{p^{e(H)}(1-p)^{-e(H)}q^{c(H)}}, \quad (4.1)$$

then a continuous-time Markov chain on \mathfrak{G}_n with generator B , if it is irreducible, has as its stationary distribution $\mathcal{G}(n, p, q)$. (Indeed, it is also reversible.)

Grimmett in fact gives an example of such a chain [32, (8.1)]; we define a slightly different one. However, we will use the same idea $b_{GH} = 0$ whenever G and H differ by more than one edge. Specifically, for any graph G and any $e \in [n]^{(2)}$, let G_e^- denote the graph G with e removed (if indeed it was present in G in the first place), and G_e^+ denote G with e added (if it was not already present). Let $[n]^{(2)}$ denote the set of size-2 subsets of $[n]$. Then we take, for all $G \in \mathfrak{G}_n$, and for all $e \in [n]^{(2)}$,

$$\begin{aligned} b_{G_e^-, G_e^+} &= (1-p^{(2)}) \left(\frac{p^{(1)}}{1-p^{(1)}} \right) q^{-I_{e,G}}; \\ b_{G_e^+, G_e^-} &= 1-p^{(2)}, \end{aligned} \quad (4.2)$$

where

$$I_{e,G} = \mathbb{1}_{\{e \text{ is a bridge in } G_e^+\}},$$

an edge e of a graph G being a bridge if and only if there is no path between the endpoints of e in $G - e$; note that it is exactly when $I_{e,G} = 1$ that $c(G_e^-)$ and $c(G_e^+)$ differ, with G_e^- having one more component than G_e^+ .

It can then be checked that these transition rates satisfy (4.1) for the case $p = p^{(1)}$, and that a chain with generator B , where

$$b_{GG} = - \sum_{\substack{H \in \mathfrak{G}_n \\ H \neq G}} b_{GH}$$

and

$$B = (b_{GH})_{G,H \in \mathfrak{G}_n}$$

is irreducible. (The latter holds because, in the corresponding jump chain, there is always a positive probability that the next graph is any graph that differs from the current one by one edge; in this way we can construct a sequence of moves to pass from any graph on $[n]$ to any other.) Thus its invariant distribution is $\mathcal{G}(n, p^{(1)}, q)$.

Note in particular that, taking $q = 1$ and replacing $p^{(1)}$ with $p^{(2)}$ in (4.2), we obtain a chain with invariant distribution $\mathcal{G}(n, p^{(2)})$ that has transition rates

$$\begin{aligned} b_{G_e^-, G_e^+} &= p^{(2)} \\ b_{G_e^+, G_e^-} &= 1 - p^{(2)}. \end{aligned} \tag{4.3}$$

This is equivalent to setting up a separate, independent chain for each edge, with two states “on” and “off”, in which we switch from “off” to “on” at rate $p^{(2)}$ and make the reverse transition at rate $1 - p^{(2)}$. It is from this that we derive the chains used in the proof of the following lemma.

Lemma 4.2. *Let $q \in (0, 1)$, and $\lambda^{(1)} \in \mathbb{R}$; define $\lambda^{(2)}$, $p^{(1)}$ and $p^{(2)}$ as in Proposition 4.1. Then, for sufficiently large n that $p^{(1)}, p^{(2)} \in (0, 1)$, there exists a pair of continuous-time Markov chains $(G_s^{(1)})_{s \geq 0}$ and $(G_s^{(2)})_{s \geq 0}$, whose state space is the set of all graphs of size n , such that:*

$$(I) \quad G_0^{(1)} = G_0^{(2)} \sim \mathcal{G}(n, p^{(1)}, q);$$

$$(II) \quad (G_s^{(1)}) \text{ is stationary};$$

(III) $(G_s^{(2)})$ has as its stationary distribution $\mathcal{G}(n, p^{(2)})$, and $G_s^{(2)} \xrightarrow{d} \mathcal{G}(n, p^{(2)})$ as $s \rightarrow \infty$;

(IV) For all $s \geq 0$, we have $G_s^{(1)} \leq G_s^{(2)}$.

Proof. Start by taking $G_0^{(1)} = G_0^{(2)} \sim \mathcal{G}(n, p^{(1)}, q)$ as an initial distribution, so that (I) is immediately satisfied. We proceed by constructing a means to update the graphs such that the transition rates in $G^{(1)}$ satisfy (4.2), and the rates in $G^{(2)}$ satisfy (4.3).

For each $e \in [n]^{(2)}$, define two Poisson processes $U^{(e)}$ and $S^{(e)}$, independent of each other and across choices of e ; let $U^{(e)}$ have rate $p^{(2)}$ and $S^{(e)}$ have rate $1 - p^{(2)}$. Then, for each e , at each event time t of $U^{(e)}$, add e to $G_{t-}^{(2)}$, if it is not already present, to form $G_t^{(2)}$; likewise, if t is an event time of $S^{(e)}$, remove e from $G_{t-}^{(2)}$ to form $G_t^{(2)}$. This forms a chain whose transition rates satisfy (4.3), noting that, with probability 1, all the event times are distinct, which gives (III).

Now, to form $G^{(1)}$, we still (for each e , and if possible) remove e from $G^{(1)}$ at time t if it is an event time of $S^{(e)}$. However, we will thin each chain $U^{(e)}$. If t is an event time of $U^{(e)}$, and e is not present in $G_{t-}^{(1)}$, then with probability

$$\left(\frac{p^{(1)}(1 - p^{(2)})}{p^{(2)}(1 - p^{(1)})} \right) q^{-I_{e,G}}, \quad (4.4)$$

with $G = G_{t-}^{(1)}$, we add e to $G_{t-}^{(1)}$ to form $G_t^{(1)}$, and otherwise we do nothing. These rates likewise satisfy (4.2); however, it remains to show that the quantity in (4.4) is a valid probability—that is, that it lies between 0 and 1.

Since $q^{-I_{e,G}} \in \{1, q^{-1}\}$ where $q^{-1} > 1$, it is sufficient to show that, for all sufficiently large n ,

$$\frac{p^{(1)}(1 - p^{(2)})}{qp^{(2)}(1 - p^{(1)})} \in [0, 1]$$

It is clear that this is non-negative, as long as n is sufficiently large. Then note that it is less than or equal to 1 if and only if

$$p^{(2)} \geq \frac{p^{(1)}}{q + (1 - q)p^{(1)}},$$

which in turn holds if and only if

$$\begin{aligned} \lambda^{(2)} &\geq n^{4/3} \left[\frac{p^{(1)}}{q + (1 - q)p^{(1)}} - \frac{1}{n} \right] \\ &= \frac{\lambda^{(1)} - (1 - q)q/n^{2/3} - (1 - q)\lambda^{(1)}/n}{q + (1 - q)q/n + (1 - q)\lambda^{(1)}/n^{4/3}} \end{aligned}$$

This holds for all n with our choice of $\lambda^{(2)}$, showing (II).

Now suppose that t is an event time of one of the Poisson processes $U^{(e)}$ or $S^{(e)}$, for some e , and suppose that $G_{t-}^{(1)} \leq G_{t-}^{(2)}$. Then one of three things happens at time t : either e is set to be missing in both graphs $G_t^{(1)}$ and $G_t^{(2)}$, or e is set to be present in both graphs, or (if t was an addition time, e was not present in $G_{t-}^{(1)}$, and the event gets “thinned out”) e is set to be present in $G_t^{(2)}$ but not in $G_t^{(1)}$. Thus $G_t^{(1)} \leq G_t^{(2)}$. We note that nothing changes between event times (and that there are countably many of them which tend to infinity—i.e. the chain does not explode, which holds because the state space is finite). Then, by induction on the countably many event times, $G_t^{(1)} \leq G_t^{(2)}$ for all t , and so (IV) is satisfied also. \square

Proof of Proposition 4.1. Take $G^{(1)}$ and $G^{(2)}$ as in Lemma 4.2; then $G_t^{(1)} \xrightarrow{d} \mathcal{G}(n, p_G, q)$ trivially, and (non-trivially, but rather as a consequence of Markov chain theory on finite state spaces) $G_t^{(2)} \xrightarrow{d} \mathcal{G}(n, p_H)$, both as $t \rightarrow \infty$. Indeed, they converge jointly, so that the sequence

$$\mathbf{P}_k = \left(G_k^{(1)}, G_k^{(2)} \right)_{k \in \mathbb{N}}$$

converges in distribution. Thus, by Skorokhod’s representation theorem, we may consider a probability space on which the sequence \mathbf{P}_k converges almost surely, so that there is a random limit object $\mathbf{P} = (G^{(1)}, G^{(2)})$ in $(\mathcal{G}_n)^2$ where $\mathbf{P}_k \rightarrow \mathbf{P}$ almost surely. However, $G_k^{(1)} \leq G_k^{(2)}$ a.s. for each k , so, since $(\mathcal{G}_n)^2$ is a discrete and finite space, this relation must hold for the limit object too, so $G^{(1)} \leq G^{(2)}$. But we know that $G^{(1)} \sim \mathcal{G}(n, p^{(1)}, q)$ and $G^{(2)} \sim \mathcal{G}(n, p^{(2)})$. \square

4.2 Nesting of exploration processes

We will now start to consider the exploration process for the random cluster model. Recall that $\mathbb{P}_{n,p}$ denotes the measure on which our random graph, G , is distributed according to $\mathcal{G}(n, p)$, and that $\mathbb{P}_{n,p,q}$ is the same for $\mathcal{G}(n, p, q)$.

Consider exploring the random cluster model using Algorithm A (which we stated on p. 4). Recall that, for any graph, the number of vertices on the stack on the i -th iteration after step 2(a) (i.e. just before we start querying edges) is exactly $X_n(i-1) - I_n(i-1)$, and the number of components whose exploration is completed by the end of the i -th iteration is exactly $-I_n(i)$.

If $q = 1$ (i.e. under $\mathbb{P}_{n,p}$), the processes (X_n, I_n) , (X_n, I_n, N_n) and $X_n - I_n$ are all time-inhomogeneous Markov chains (by the independence of edges). Moreover, we have that the following distributional statements hold: for $i = 1, \dots, n$, conditionally on the triple

$(X_n(i-1), I_n(i-1), N_n(i-1))$:

$$\zeta_i \sim \text{Bin}(n-i-X_n(i-1)+I_n(i-1), p);$$

$$\eta_i \sim \text{Bin}(X_n(i-1)-I_n(i-1), p).$$

For $\mathcal{G}(n, p, q)$ with general q , we will not have such neat distributional statements—if we did, it would likely make it much easier to find a limit for the exploration process. However, we can still make some observations about the exploration process.

Proposition 4.3. *The processes $(X_n(k), I_n(k))_{0 \leq k \leq n}$ and $(X_n(k), I_n(k), N_n(k))_{0 \leq k \leq n}$ are both Markovian under the measure $\mathbb{P}_{n,p,q}$. Further, conditionally on $(X_n(k), I_n(k), N_n(k))$, the pair $(X_n(k+1), I_n(k+1))$ is independent of $N_n(k+1)$, with*

$$\eta_{k+1} = N_n(k+1) - N_n(k) \sim \text{Bin}(X_n(k) - I_n(k), p).$$

Moreover, if $Y_n(k) = X_n(k) - I_n(k)$, then $(Y_n)_{0 \leq k \leq n}$ is a Markov chain.

Remark. The explicit statement of the distribution of N_n is further evidence that it really is the component size distribution wherein lies the fundamental difference between $\mathcal{G}(n, p)$ and $\mathcal{G}(n, p, q)$.

Proof. We show only the claims about (X_n, I_n, N_n) ; the claims about (X_n, I_n) and Y_n may be shown similarly.

We set up notation. For each $k \in \mathbb{Z}^{\geq 0}$, let $\mathbf{X}_n^{(k)} = (X_n(0), \dots, X_n(k)) \in \mathbb{Z}^{k+1}$; define $\mathbf{I}_n^{(k)}$ and $\mathbf{N}_n^{(k)}$ analogously for each k . Suppose we have a sequence $(x_k, i_k, n_k)_{k \in \mathbb{Z}^{\geq 0}}$ in \mathbb{Z}^3 . For each $k \in \mathbb{N}$, let $\mathbf{x}^{(k)} = (x_0, \dots, x_k) \in \mathbb{Z}^{k+1}$; let $\mathbf{i}^{(k)} \in \mathbb{Z}^{k+1}$ and $\mathbf{n}^{(k)} \in \mathbb{Z}^{k+1}$ be defined analogously. Finally, denote, for each k , the events

$$E^{(k)} = \{ \mathbf{X}_n^{(k)} = \mathbf{x}^{(k)}, \mathbf{I}_n^{(k)} = \mathbf{i}^{(k)}, \mathbf{N}_n^{(k)} = \mathbf{n}^{(k)} \},$$

and

$$E_k = \{ X_n(k) = x_k, I_n(k) = i_k, N_n(k) = n_k \};$$

note that these events implicitly depend on our choice of the sequence $(x_k, i_k, n_k)_{k \in \mathbb{Z}^{\geq 0}}$.

Then we can express the Markov property as the statement that, for any $k \in \mathbb{N}$ and for any

choice of the sequence such that $\mathbb{P}_{n,p,q}(E^{(k)}) > 0$,

$$\mathbb{P}_{n,p,q}(E_{k+1} \mid E^{(k)}) = \mathbb{P}_{n,p,q}(E_{k+1} \mid E_k). \quad (4.5)$$

To show that this holds, note that, since the number of components of the graph is $-I_n(n)$, we have

$$\mathbb{P}_{n,p,q}(E_{k+1} \mid E^{(k)}) = \frac{\mathbb{P}_{n,p,q}(E^{(k+1)})}{\mathbb{P}_{n,p,q}(E^{(k)})} = \frac{\mathbb{E}_{n,p}[\mathbb{1}_{E^{(k+1)}} q^{-I_n(n)}]}{\mathbb{E}_{n,p}[\mathbb{1}_{E^{(k)}} q^{-I_n(n)}]} = \frac{\mathbb{E}_{n,p}[\mathbb{1}_{E_{k+1}} q^{-I_n(n)} \mid E^{(k)}]}{\mathbb{E}_{n,p}[q^{-I_n(n)} \mid E^{(k)}]}, \quad (4.6)$$

where we apply Proposition 3.7 twice for the second equality, and divide the numerator and denominator by $\mathbb{P}_{n,p}(E^{(k)})$ for the third. But, under $\mathbb{P}_{n,p}$, the process (X_n, I_n, N_n) is a Markov process, since under this measure $G_n \sim \mathcal{G}(n, p)$. So it holds that

$$\mathbb{P}_{n,p,q}(E_{k+1} \mid E^{(k)}) = \frac{\mathbb{E}_{n,p}[\mathbb{1}_{E_{k+1}} q^{-I_n(n)} \mid E_k]}{\mathbb{E}_{n,p}[q^{-I_n(n)} \mid E_k]},$$

and reversing the series of steps in (4.6) gives us (4.5).

For the distributional statements, note that, under $\mathbb{P}_{n,p}$, in fact, given E_k , the entire sequence $(X_n(k+1), \dots, X_n(n))$ is conditionally independent of $N_n(k+1)$, since they can be determined by inspection of disjoint collections of independent edges—and hence, by the definition of I_n , so is the entire collection $(X_n(j), I_n(j) : j = k+1, \dots, n)$. So in fact

$$\begin{aligned} & \mathbb{E}_{n,p}[\mathbb{1}_{E_{k+1}} q^{-I_n(n)} \mid E_k] \\ &= \mathbb{E}_{n,p}[\mathbb{1}_{\{X_n(k+1)=x_{k+1}, I_n(k+1)=i_{k+1}\}} q^{-I_n(n)} \mid E_k] \mathbb{P}_{n,p}(N_n(k+1) = n_{k+1} \mid E_k), \end{aligned}$$

and reversing the steps in (4.6) again on the left-hand term gives us that in fact

$$\mathbb{P}_{n,p,q}(E_{k+1} \mid E_k) = \mathbb{P}_{n,p,q}(X_n(k+1) = x_{k+1}, I_n(k+1) = i_{k+1} \mid E_k) \mathbb{P}_{n,p}(N_n(k+1) = n_{k+1} \mid E_k),$$

showing (since x_{k+1} , i_{k+1} and n_{k+1} were arbitrary) the claimed conditional independence. Moreover, under $\mathbb{P}_{n,p}$, and conditionally on E_k , we have that $N_n(k+1) - N_n(k)$ has the claimed binomial distribution. \square

Remark. Although the process I_n is measurable with respect to X_n , it does *not* hold that the processes X_n and (X_n, N_n) are Markovian in general, because the step distributions depend on the value of I_n , which in turn depends on the *history* of X_n .

When the stack size reaches 0, we can say something stronger about the residual graph.

Proposition 4.4. *Conditionally on $X_n(k) - I_n(k) = 0$, we have $G_n \setminus \{v_1, \dots, v_k\} \sim \mathcal{G}(n - k, p, q)$, up to relabelling of vertices.*

Proof. We prove this conditionally on $X_n(k) = I_n(k) = x$ and $N_n(k) = n$ for some $x, n \in \mathbb{Z}$, such that the probability of that event is non-zero. The claimed result follows by averaging.

When we inspect Algorithm A, the fact that $X_n(k) - I_n(k) = 0$ —i.e. that the stack is empty—tells us that there are no edges between $\{v_1, \dots, v_k\}$ and $\{v_{k+2}, \dots, v_n\}$, because there can never be any edges between pairs of vertices such that, at the end of any step of the exploration, one is dead and one is sleeping. This may mean that v_k completed the exploration of a component (which would be the case if $I_n(k) = I_n(k-1) - 1$), but this is not necessarily true: it may be that some vertices in $\{v_1, \dots, v_k\}$ are connected to vertices in the rest of the graph, but if so they can only be connected through the vertex v_{k+1} . The effect of this is that, in either case, the number of components of $G_n \setminus \{v_1, \dots, v_k\}$ is equal to the number of components in G_n spanned by $\{v_{k+1}, \dots, v_n\}$, which in turn is equal to $-I_n(n) + I_n(k)$. That is, if $X_n(k) = I_n(k)$, then

$$c(G_n \setminus \{v_1, \dots, v_k\}) = -I_n(n) + I_n(k). \quad (4.7)$$

We also note that, under $\mathbb{P}_{n,p}$, then the distribution of $G_n \setminus \{v_1, \dots, v_k\}$ is independent of $(X_n(k), I_n(k), N_n(k))$, since they are determined by inspecting disjoint collections of the independent edges in G_n .

Now, redefining notation from the proof of Proposition 4.3, let the event

$$E_k = \{X_n(k) = I_n(k) = x, N_n(k) = m, \{v_1, \dots, v_k\} = W\},$$

where $W \in \binom{[n]}{k}$ is some (non-random) size- k subset of $[n]$ such that $\mathbb{P}_{n,p,q}(E_k) > 0$. (We have already chosen x and m such that the probability that $X_n(k) = I_n(k) = x$ and $N_n(k) = m$ is non-zero.) Then, for any graph H on $[n] \setminus W$, it holds, similarly to (4.6), that

$$\begin{aligned} \mathbb{P}_{n,p,q}(G_n \setminus \{v_1, \dots, v_k\} = H \mid E_k) &= \frac{\mathbb{E}_{n,p}[\mathbb{1}_{\{G_n \setminus \{v_1, \dots, v_k\} = H\}} q^{-I_n(n)} \mid E_k]}{\mathbb{E}_{n,p}[q^{-I_n(n)} \mid E_k]} \\ &= \frac{q^{-x} \mathbb{E}_{n,p}[\mathbb{1}_{\{G_n \setminus \{v_1, \dots, v_k\} = H\}} q^{-I_n(n) + I_n(k)} \mid E_k]}{q^{-x} \mathbb{E}_{n,p}[q^{-I_n(n) + I_n(k)} \mid E_k]} \\ &= \frac{\mathbb{E}_{n,p}[\mathbb{1}_{\{G_n \setminus \{v_1, \dots, v_k\} = H\}} q^{c(G_n \setminus \{v_1, \dots, v_k\})} \mid E_k]}{\mathbb{E}_{n,p}[q^{c(G_n \setminus \{v_1, \dots, v_k\})} \mid E_k]}, \end{aligned}$$

where the last equality follows by (4.7). But, by the independence of $G_n \setminus \{v_1, \dots, v_k\}$ from

$(X_n(k), I_n(k), N_n(k))$ under $\mathbb{P}_{n,p}$, we can remove part of the conditioning, giving

$$\mathbb{P}_{n,p,q}(G_n \setminus \{v_1, \dots, v_k\} = H \mid E_k) = \frac{\mathbb{E}_{n,p}[\mathbb{1}_{\{G_n \setminus \{v_1, \dots, v_k\} = H\}} q^{c(G_n \setminus \{v_1, \dots, v_k\})} \mid \{v_1, \dots, v_k\} = W]}{\mathbb{E}_{n,p}[q^{c(G_n \setminus \{v_1, \dots, v_k\})} \mid \{v_1, \dots, v_k\} = W]}.$$

Then note that, under $\mathbb{P}_{n,p}$ and conditionally on its vertex labels, the random graph $G_n \setminus \{v_1, \dots, v_k\}$ is an Erdős–Rényi graph with on those vertices with probability p . Applying Proposition 3.7 again gives the result we need. \square

Remark. An important point to note is that nowhere in the above proof do we use the specific choices of v_i made in step 2(a) of Algorithm A (although we do also need to choose a vertex from the stack, if the stack is non-empty, so that it still holds that $X_n(i) - I_n(i)$ is the number of vertices on the stack when we start exploring). This means, for instance, that we would obtain the same results (Propositions 4.3 and 4.4) if we made a uniform random choice of vertex from the stack.

This observation is important because we will need to change our choice of “current vertex” to prove the following proposition, which is the main result of this section.

Proposition 4.5. *Let $q \in (0, 1)$, and $\lambda_G \in \mathbb{R}$. Define:*

$$\begin{aligned} \lambda_H &= \max \left\{ \frac{\lambda_G}{q}, 0 \right\}; \\ p_G &= \frac{q}{n} + \frac{\lambda_G}{n^{4/3}}; \\ p_H &= \frac{1}{n} + \frac{\lambda_H}{n^{4/3}}. \end{aligned}$$

Let (X_n, I_n, N_n) (resp. (X'_n, I'_n, N'_n)) be an exploration using Algorithm A of $\mathcal{G}(n, p_G, q)$ (resp. $\mathcal{G}(n, p_H)$), except that we choose our new v_i after each iteration by making a uniform random pick from the stack. Then it is possible to couple (X_n, I_n, N_n) and (X'_n, I'_n, N'_n) such that $X_n(i) - I_n(i) \leq X'_n(i) - I'_n(i)$ for all i , and such that $N_n(i) - N_n(i-1) \leq N'_n(i) - N'_n(i-1)$ for all i .

To address why this change is necessary, we note that we are aiming to exploit the fact that we have a nested coupling of the two random graphs—say, $G \sim \mathcal{G}(n, q/n, q)$ ($q < 1$) and $H \sim \mathcal{G}(n, 1/n)$ —such that $G \leq H$. To obtain the nesting of the exploration processes, we need to explore the vertices of the two graphs simultaneously and in the same order, while still exploring, in each, one component at a time. In general, this will not be a criterion satisfied by the depth-first order for either.

This motivates the following algorithm.

Algorithm B.

We explore graphs G and H simultaneously, where G is a subgraph of H . We will maintain separate stacks, called the G - and H -stacks, and have separate labels (“ G -sleeping”, “ H -awake” etc.), for each graph.

1. Initialise $X_G(0) = I_G(0) = N_G(0) = 0$, and likewise for H . Set all vertices to be sleeping, and both stacks to be empty.
2. For each $i = 1, \dots, n$:
 - (a) If the G -stack is non-empty, set the new current vertex, v_i , to be a uniform random vertex from the G -stack, which is immediately killed and removed from both stacks. Otherwise, if the H -stack is non-empty, set v_i to be a uniform random vertex from the H -stack, which is immediately killed and removed from that stack; if both stacks are empty, set v_i to be a uniform random sleeping vertex, and immediately set it to be both G -dead and H -dead.
 - (b) If any G -sleeping vertices are adjacent to v_i in G , make them G -awake and add them to the G -stack. Let $\zeta_{i,G}$ be the number of such vertices discovered. Repeat for H .
 - (c) Set $X_G(i) = X_G(i-1) + \zeta_{i,G} - 1$, and similarly for H .
 - (d) Set $I_G(i) = \min\{I_G(i-1), X_G(i)\}$, and similarly for H .
 - (e) Let $\eta_{i,G}$ be the number of G -awake vertices adjacent to v_i , and similarly for H . Let $N_G(i) = N_G(i-1) + \eta_{i,G}$, and similarly for H .

We note that Algorithm B is well-defined—that is, v_i is a vertex from the G -stack when the G -stack is non-empty, and from the H -stack when the H -stack is non-empty. We note further by induction that, for all $i = 0, \dots, n$, the G -stack, after iteration i , is a subset of the H -stack.

We now use this algorithm to couple the exploration processes.

Proof of Proposition 4.5. Take \mathbb{P} to be a probability measure where

$$G \sim \mathcal{G}(n, p_G, q);$$

$$H \sim \mathcal{G}(n, p_H);$$

$$G \leq H,$$

the existence of which is implied by Proposition 4.1.

We use Algorithm B to explore G and H simultaneously. We claim that the equalities in distribution $(X_G, I_G, N_G) \stackrel{d}{=} (X_n, I_n, N_n)$ and $(X_H, I_H, N_H) \stackrel{d}{=} (X'_n, I'_n, N'_n)$ hold. If so, then, as noted above, the stack size in the exploration of G is always less than or equal to that of H . But the stack size (after Step 2(a)) is $X_G(i) - I_G(i)$ for G , and similarly for H , so that, for $i = 0, \dots, i-1$

$$X_G(i) - I_G(i) \leq X_H(i) - I_H(i).$$

Finally, we have that $X_G(n) - I_G(n) = 0 = X_H(n) - I_H(n)$, since we are at the end of a component (the last one), which means that both processes hit a new infimum. So, for all i , we have $X_G(i) - I_G(i) \leq X_H(i) - I_H(i)$.

Correspondingly, $\eta_{i,G}$ (resp. $\eta_{i,H}$) counts the number of edges discovered from v_i to other vertices on the stack in the exploration of G (resp. H) at iteration i . But the stack for G is a subset of that for H , so we consider a subset of the possible edges as surplus edges; moreover, if any possible surplus edge exists in G , then, since $G \leq H$, that edge is present (and a surplus edge) in H . So $N_G(i) - N_G(i-1) \leq N_H(i) - N_H(i-1)$, and the algorithm gives a coupling of the kind required.

It remains to show that the equalities in distribution hold. Define the following:

$$A_{e,G} = \{e \text{ is an edge in } G\};$$

$$I_{e,G} = \mathbb{1}_{A_{e,G}};$$

$$\mathcal{F}_{i,G} = \sigma(v_j \text{ for } j \leq i, I_{e,G} \text{ for } e \in [n]^{\binom{2}{2}} \text{ with } v_j \in e \text{ for some } j \leq i),$$

and similarly for H . In addition, define

$$\mathcal{G}_i = \sigma(\mathcal{F}_{i,G}, \mathcal{F}_{i,H}).$$

Then we just need that, for the marginal explorations, the choices of v_i are appropriately distributed. That is, for $i = 1, \dots, n$:

- (A) conditionally on $\mathcal{F}_{i-1,G}$, we have that v_i is a uniform pick from the G -stack if the G -stack is non-empty, and otherwise a uniform pick from the G -sleeping vertices; and
- (B) conditionally on $\mathcal{F}_{i-1,H}$, we have that v_i is a uniform pick from the H -stack if the H -stack is non-empty, and otherwise a uniform pick from the H -sleeping vertices.

This follows by induction, and from the exchangeability of vertex labels in both the Erdős–

Rényi model and the random cluster model.

Specifically, even though we only consider the H -stack if the G -stack is empty, it holds by the exchangeability of vertex labels and the uniform random choices we are making that the G -stack is always, conditionally on its size and the vertices on the H -stack, a uniform random subset (of that size) of the H -stack. So, when considering only the marginal distribution of the exploration of H , choosing a uniform random vertex from the G -stack amounts to choosing a uniform random vertex from the H -stack. Similarly, when the G -stack is empty but the H -stack isn't, the H -stack is (again conditionally on its size and the G -exploration) a uniform subset (of that size) of the G -sleeping vertices, so a uniform pick from the H -stack is, in the marginal distribution for G , a uniform pick from the G -sleeping vertices. \square

Before we move on, we note briefly that essentially the same proof may be used to show the following proposition (whose proof we therefore omit).

Proposition 4.6. *Let (X_n, I_n, N_n) be an exploration of $\mathcal{G}(n, p_1)$, and let (X'_n, I'_n, N'_n) be an exploration of $\mathcal{G}(n, p_2)$, both using Algorithm A, except that we choose our new v_i after each iteration by making a uniform random pick from the relevant stack. Suppose $p_1 \leq p_2$. Then it is possible to couple (X_n, I_n, N_n) and (X'_n, I'_n, N'_n) such that $X_n(i) - I_n(i) \leq X'_n(i) - I'_n(i)$ for all i , and such that $N_n(i) - N_n(i-1) \leq N'_n(i) - N'_n(i-1)$ for all i .*

Remark. This does not immediately follow from setting $q = 1$ in Proposition 4.5, because we didn't consider arbitrary changes of the probability parameter there. However, as suggested above, the proof method transfers over almost identically, using the standard subgraph coupling of Erdős–Rényi graphs with different edge probabilities.

4.3 The Erdős–Rényi exploration process

Now that we have established a comparison with the critical Erdős–Rényi random graph, we prove some preliminary results about the Erdős–Rényi random graph scaling limit. We consider the exploration process given by Algorithm A for $\mathcal{G}(n, 1/n + \lambda n^{-4/3})$, with $\lambda \in \mathbb{R}$. In this section, to avoid unnecessarily cluttering notation, we will drop references to the measure under which we are working, since this will usually be obvious, and we will just use \mathbb{P} and \mathbb{E} throughout. Specificity will return in the following section.

We let X , I and N be the limiting processes defined by Aldous in [6], introduced above in

Theorem 1.5; for reference, these are defined, where B is a Brownian motion on $[0, \infty)$, by

$$X(t) = B(t) + \lambda t - \frac{t^2}{2} \quad (4.8)$$

$$I(t) = \inf_{s \leq t} X(s), \quad (4.9)$$

with, conditionally on $(X(t) : t \geq 0)$, the process N being an inhomogeneous Poisson process on $[0, \infty)$ with intensity $X(t) - I(t)$ at t .

4.3.1 Approaching zero

We wish to prove the following.

Proposition 4.7. *For $T > 0, \varepsilon > 0$, let the event $E_n(T, \varepsilon)$ be defined by*

$$E_n(T, \varepsilon) = \left\{ X_n(k) = I_n(k) \text{ for some } k \in \left[\lfloor Tn^{2/3} \rfloor, \lfloor (T + \varepsilon)n^{2/3} \rfloor \right] \right\} \\ \cap \left\{ N_n(\lfloor (T + \varepsilon)n^{2/3} \rfloor) = N_n(\lfloor Tn^{2/3} \rfloor) \right\}.$$

Then

$$\lim_{T \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{P}(E_n(T, 1/T)^c) = 0. \quad (4.10)$$

That is, we want to show that, for sufficiently large times T , the reflected process $X - I$ will hit 0 very soon after T , and no further Poisson points will be detected in the time between T and the time where the process hits 0.

To explain why this is useful, imagine we had a similar result for the random cluster model. If we knew exactly that the exploration hit 0 at $\lfloor Tn^{2/3} \rfloor$, then we could state precisely the behaviour of the exploration for the random cluster model, by Proposition 4.4, and indeed we'd have (by the Markov property) independence of the residuary from the explored portion—addressing Problem 1 in Section 3.2, which discussed showing “approximate independence” between these parts. (In fact, in Section 6.1, we will be able to use the comparison established in the preceding two sections to establish the analogous result for the random cluster model in the case $q < 1$.)

Such a result doesn't completely address the problem, because we only know that we hit 0 in a short space of time, not the exact point. We might hope this gives us the *approximate* independence we seek, though; in fact, it does, aided by the fact that we can also show that it is unlikely we see any more surplus edges in that short space of time. The results that will eventually allow us to make this rigorous are discussed in Section 4.4.

Note that we know that the limit in (4.10) in n exists, because we know that the scaling limit for $\mathcal{G}(n, 1/n + \lambda n^{-4/3})$ exists. It will therefore be helpful to have some bounds on the probability of certain events in the limit process.

Lemma 4.8. *For all $\varepsilon > 0$ and $T \geq 0$,*

$$\mathbb{P}(X(T) - I(T) > \varepsilon) \leq e^{-(T-2\lambda)\varepsilon}, \quad (4.11)$$

and so, in particular, $X(T) - I(T) \xrightarrow{P} 0$ as $T \rightarrow \infty$.

Proof. This is trivial for $T \leq 2\lambda$, so assume $T > 2\lambda$.

Note that $(\tilde{B}^{(T)}(s))_{0 \leq s \leq T}$ defined by

$$\tilde{B}^{(T)}(s) = B(T-s) - B(T)$$

is a Brownian motion run for time T . Extend it to a BM on $[0, \infty)$. Then define, for $s \geq 0$,

$$\tilde{X}^{(T)}(s) = \tilde{B}^{(T)}(s) + \frac{s(2T-s)}{2} - \lambda s,$$

and, for $0 \leq s \leq T$,

$$\tilde{J}^{(T)}(s) = \inf_{s \leq u \leq T} \tilde{X}^{(T)}(u).$$

Then it holds that, for $0 \leq s \leq T$:

$$\tilde{X}^{(T)}(s) = X(T-s) - X(T);$$

$$\tilde{J}^{(T)}(s) = I(T-s) - X(T).$$

We are interested in

$$\mathbb{P}(X(T) - I(T) > \varepsilon) = \mathbb{P}(\tilde{J}^{(T)}(0) < -\varepsilon) = \mathbb{P}\left(\inf_{0 \leq u \leq T} \tilde{X}^{(T)}(u) < -\varepsilon\right). \quad (4.12)$$

Now define $\tilde{Y}^{(T)}$ by

$$\tilde{Y}^{(T)}(s) = \tilde{B}^{(T)}(s) + s\left(\frac{T}{2} - \lambda\right),$$

noting that, for $0 \leq s \leq T$, $\tilde{Y}^{(T)}(s) \leq \tilde{X}^{(T)}(s)$. Thus

$$\begin{aligned} \mathbb{P}\left(\inf_{0 \leq u \leq T} \tilde{X}^{(T)}(u) < -\varepsilon\right) &\leq \mathbb{P}\left(\inf_{0 \leq u \leq T} \tilde{Y}^{(T)}(u) < -\varepsilon\right) \\ &\leq \mathbb{P}\left(\inf_{s \geq 0} \tilde{Y}^{(T)}(s) < -\varepsilon\right) \\ &\leq \mathbb{P}\left(\inf_{s \geq 0} \tilde{Y}^{(T)}(s) \leq -\varepsilon\right). \end{aligned} \tag{4.13}$$

Now $\tilde{Y}^{(T)}$ is a Brownian motion with linear drift term $T/2 - \lambda$. For $b > 0$, let τ_{-b} be defined by

$$\tau_{-b} = \inf \{s \geq 0 : \tilde{Y}_s^{(T)} = -b\},$$

which may take the value ∞ if $\tilde{Y}^{(T)}$ never hits $-b$. Then it can be shown (see, for example, [17, Eqn. 2.2.0.2(1)]) that, for $T \geq \max\{0, 2\lambda\}$,

$$\mathbb{P}\left(\inf_{s \geq 0} \tilde{Y}^{(T)}(s) \leq -\varepsilon\right) = \mathbb{P}(\tau_{-\varepsilon} < \infty) = e^{-(T-2\lambda)\varepsilon}.$$

Hence, from (4.13), for such T ,

$$\mathbb{P}\left(\inf_{0 \leq u \leq T} \tilde{X}^{(T)}(u) < -\varepsilon\right) \leq e^{-(T-2\lambda)\varepsilon},$$

which, coupled with (4.12), gives us (4.11). The final assertion follows since, by definition, $X(T) - I(T) \geq 0$. \square

We next give a deterministic technical lemma, which allows us to compare between the Brownian motion with quadratic drift (i.e., X), and a Brownian motion with (appropriately chosen non-zero starting point and) linear drift—crucially, even when their respective infima are subtracted. Indeed, it is more general than that.

Lemma 4.9. *Let $f, \tilde{f} : [0, \infty) \rightarrow \mathbb{R}$ be continuous functions, and $d, \tilde{d} \in \mathbb{R}$, such that, for all s , $f(s) + d \leq \tilde{f}(s) + \tilde{d}$, and such that $\tilde{f} - f$ is a (weakly) increasing function on $[0, \infty)$. Define:*

$$\begin{aligned} m(s) &= \min \left\{ \inf_{0 \leq u \leq s} f(u), -d \right\}; \\ \tilde{m}(s) &= \min \left\{ \inf_{0 \leq u \leq s} \tilde{f}(u), -\tilde{d} \right\}. \end{aligned}$$

Let $y(s) = f(s) - m(s)$ and $\tilde{y}(s) = \tilde{f}(s) - \tilde{m}(s)$. Then $y(s) \leq \tilde{y}(s)$ for all $s \geq 0$.

We defer the proof to Appendix A (p. 179).

So that we can apply this, we show a useful result about a Brownian motion with linear

drift which shows that, if it starts sufficiently close to its past infimum it is unlikely to get very far away again in unit time. Once we use the lemma above to convert back to the quadratic drift case, we will be able to use this to show that the area under the process across a unit time interval is small, reducing the “space” in which surplus edges can appear—and thus allowing us to show that it is unlikely we will have any.

Lemma 4.10. *Let $(W(s))_{s \geq 0}$ be a Brownian motion, and let $T > 0$ be non-random. Let $Q(s) = W(s) - Ms + \varepsilon/2$. Let*

$$P(s) = \min \left\{ \inf_{0 \leq u \leq s} Q(u), 0 \right\},$$

and let $R(s) = Q(s) - P(s)$. Then, for $M \rightarrow \infty$ and $\varepsilon = \varepsilon(M) \rightarrow 0$, with $\varepsilon > 0$ for all M and $M^2 e^{-\varepsilon M} \rightarrow 0$ as $M \rightarrow \infty$,

$$\mathbb{P}(\exists u \in [0, 1] \text{ s.t. } R(u) = \varepsilon) = O(e^{-\varepsilon M}).$$

Remark. Note that $M^2 e^{-\varepsilon M} \rightarrow 0$ (as $M \rightarrow \infty$) implies that $\varepsilon M \rightarrow \infty$ as $M \rightarrow \infty$, which will be required in the proof.

As Peskir [50] notes, R is a reflected Brownian motion with (linear) drift $-M$, started at $\varepsilon/2$. This is therefore a question about a hitting time of such a process, which has been analysed in several sources (e.g. [62, 29, 35]). We will use the following result, taken from [29].

Lemma 4.11 ([29, (3.7)]). *Let $B^{(\mu)}$ be a Brownian motion with linear drift $\mu > 0$, started at $x \geq 0$. Fix $b \geq x$, and define*

$$D(t) = \max \left\{ \sup_{0 \leq u \leq t} B^{(\mu)}(u) - b, 0 \right\}.$$

Let

$$\tau = \inf \{ t \geq 0 : B^{(\mu)}(t) - D(t) = 0 \}.$$

Then, for $\delta > 0$,

$$\mathbb{E}[e^{-\delta\tau}] = \frac{r e^{-s(b-x)} - s e^{-r(b-x)}}{r e^{-sb} - s e^{-rb}}, \quad (4.14)$$

where $r = r(\delta, \mu)$ and $s = s(\delta, \mu)$ are given by

$$r := -\mu + \sqrt{\mu^2 + 2\delta}$$

and

$$s := -\mu - \sqrt{\mu^2 + 2\delta}.$$

Proof of Lemma 4.10. We note that τ , as defined in Lemma 4.11, can be expressed as

$$\tau = \inf \left\{ t \geq 0 : B^{(\mu)}(t) = 0 \text{ or } B^{(\mu)}(t) - \sup_{0 \leq u \leq t} X(u) + b = 0 \right\} = \inf \{ t \geq 0 : R'(t) = 0 \},$$

where R' is a Brownian motion with drift μ , started at x , reflected at b . Taking $x = \varepsilon/2$, $b = \varepsilon$ and $\mu = M$, we have that $R' \stackrel{d}{=} \varepsilon - R$. So τ is identically distributed to the first hitting time of R at ε .

Hence, our result is equivalent to showing that $\mathbb{P}(\tau \leq 1) = O(e^{-\varepsilon M})$. We apply Lemma 4.11 by noting that

$$\mathbb{P}(\tau \leq 1) = \mathbb{P}(e^{-\tau} \geq e^{-1}) \leq e \mathbb{E}[e^{-\tau}], \quad (4.15)$$

by Markov's inequality. Thus, in turn, it is sufficient to show that

$$\mathbb{E}[e^{-\tau}] = O(e^{-\varepsilon M}).$$

It can be straightforwardly verified that, as $M \rightarrow \infty$,

$$r = r(1, M) = \Theta(M^{-1}) \quad (4.16)$$

and

$$s = s(1, M) = -2M + \Theta(M^{-1}). \quad (4.17)$$

Moreover,

$$r - s = 2\sqrt{\mu^2 + 2\delta} = 2M + \Theta(M^{-1}). \quad (4.18)$$

Manipulating (4.14) (with the appropriate substitutions for b and x) and using (4.16)–(4.18) gives us that

$$\begin{aligned} \mathbb{E}[e^{-\tau}] &= e^{s\varepsilon/2} \left(1 - \frac{se^{(s-r)\varepsilon/2}}{r} \right) \left(1 - \frac{se^{(s-r)\varepsilon}}{r} \right)^{-1} \\ &= O(e^{-\varepsilon M}) (1 + O(M^2 e^{-\varepsilon M})) (1 + O(M^2 e^{-2\varepsilon M})) = O(e^{-\varepsilon M}), \end{aligned}$$

provided that $M^2 e^{-\varepsilon M} \rightarrow 0$ as $M \rightarrow \infty$. With (4.15), this completes the result. \square

We now shift back to the quadratic drift case.

Lemma 4.12. *As $T \rightarrow \infty$, for $\alpha = \alpha(T) \rightarrow 0$ with $T^2 e^{-\alpha T} \rightarrow 0$,*

$$\mathbb{P}\left(X(T+s) - I(T+s) > \alpha \text{ for some } s \in [0, 1) \mid X(T) - I(T) \leq \frac{\alpha}{2}\right) = O(e^{-T\alpha}).$$

Proof. As notational setup, we take $R(s)$ as in Lemma 4.10, i.e. a reflected Brownian motion with linear drift $-M$, started at $\alpha/2$. We assume $T > \lambda$, and set $M = T - \lambda$. (Note that, as $T \rightarrow \infty$, $M \rightarrow \infty$ and $\alpha(T) \rightarrow 0$ with $\alpha M = \alpha T - \alpha \lambda \rightarrow \infty$.)

Let $A_{T,\alpha} = \{X(T+s) - I(T+s) \geq \alpha \text{ for some } s \in [0, 1)\}$. Then

$$\begin{aligned} & \mathbb{P}\left(X(T) - I(T) \leq \frac{\alpha}{2}\right) \mathbb{P}\left(X(T+s) - I(T+s) \geq \alpha \text{ for some } s \in [0, 1) \mid X(T) - I(T) \leq \frac{\alpha}{2}\right) \\ &= \mathbb{E}\left[\mathbb{1}_{A_{T,\alpha}} \mathbb{1}_{\{X(T) - I(T) \leq \frac{\alpha}{2}\}}\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{A_{T,\alpha}} \mathbb{1}_{\{X(T) - I(T) \leq \frac{\alpha}{2}\}} \mid X(T), I(T)\right]\right], \end{aligned}$$

by the tower law.

Now we apply Lemma 4.9, so that we can consider the computationally-easier linear drift case instead. If $F(s) = X(T+s)$ and $d = -I(T)$, then, taking $f = F$ in the statement of Lemma 4.9, then the $y(s)$ from that lemma equals $X(T+s) - I(T+s)$ exactly. Now take $\tilde{F}(s) = F(s) + \alpha/2 - X(T) + s^2/2$ and $\tilde{d} = 0$, so that \tilde{y} has identical distribution to R . In Lemma 4.9, let $\tilde{f} = \tilde{F}$. Then, on the event

$$\left\{X(T) - I(T) \leq \frac{\alpha}{2}\right\},$$

the condition that $f + d \leq \tilde{f} + \tilde{d}$ is satisfied, as is the condition that $\tilde{f} - f$ is an increasing function; thus, by Lemma 4.9, $y(s) \leq \tilde{y}(s)$. As such, on this event we have found a coupling of $X(T+s) - I(T+s)$ and $R(s)$ such that $X(T+s) - I(T+s) \leq R(s)$.

Then, if there exists a (random) point $s \in [0, 1)$ such that $X(T+s) - I(T+s) \geq \alpha$, certainly there exists $s \in [0, 1)$ such that $R(s) \geq \alpha$. Let $C_{T,\alpha} = \{R(s) \geq \alpha \text{ for some } s \in [0, 1)\}$, noting that the distribution of R implicitly depends on T . Then

$$\mathbb{E}\left[\mathbb{1}_{A_{T,\alpha}} \mathbb{1}_{\{X(T) - I(T) \leq \frac{\alpha}{2}\}} \mid X(T), I(T)\right] \leq \mathbb{E}\left[\mathbb{1}_{C_{T,\alpha}} \mathbb{1}_{\{X(T) - I(T) \leq \frac{\alpha}{2}\}} \mid X(T), I(T)\right];$$

applying the tower law again thus gives us that

$$\begin{aligned}
\mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{A_{T,\alpha}}\mathbb{1}_{\{X(T)-I(T)\leq\frac{\alpha}{2}\}}\mid X(T),I(T)\right]\right] &\leq \mathbb{E}\left[\mathbb{1}_{C_{T,\alpha}}\mathbb{1}_{\{X(T)-I(T)\leq\frac{\alpha}{2}\}}\right] \\
&= \mathbb{P}\left(C_{T,\alpha}\mid X(T)-I(T)\leq\frac{\alpha}{2}\right)\mathbb{P}\left(X(T)-I(T)\leq\frac{\alpha}{2}\right) \\
&= \mathbb{P}(C_{T,\alpha})\mathbb{P}\left(X(T)-I(T)\leq\frac{\alpha}{2}\right),
\end{aligned}$$

since $C_{T,\alpha}$ depends only on \tilde{F} , which is independent of $(X(T), I(T))$ (as these depend only on the process $(B_s)_{0\leq s\leq T}$). Therefore,

$$\begin{aligned}
\mathbb{P}\left(X(T+s)-I(T+s)\geq\alpha\text{ for some }s\in[0,1]\mid X(T)-I(T)\leq\frac{\alpha}{2}\right) \\
\leq \mathbb{P}(\exists u\in[0,1]\text{ s.t. }R(u)=\alpha) \\
= O(e^{-\alpha M}) = O(e^{-T\alpha}),
\end{aligned}$$

by Lemma 4.9, as required. □

We are now in a position to prove the main result about the Erdős–Rényi exploration limit.

Proof of Proposition 4.7. Let

$$E(T, \varepsilon) = \{X(u) = I(u) \text{ for some } u \in [T, T + \varepsilon] \text{ and } N(T + \varepsilon) = N(T)\}.$$

We will take $\varepsilon = 1/T$. We then seek to prove that, as $T \rightarrow \infty$,

$$\mathbb{P}\left(X(u) > I(u) \text{ for all } u \in \left[T, T + \frac{1}{T}\right]\right) \rightarrow 0 \tag{4.19}$$

and

$$\mathbb{P}\left(N\left(T + \frac{1}{T}\right) > N(T)\right) \rightarrow 0; \tag{4.20}$$

this is sufficient, by the union bound, to prove the claim, since, by Theorem 1.6,

$$\limsup_{n\rightarrow\infty}\mathbb{P}(E_n(T, \varepsilon)^c) = \lim_{n\rightarrow\infty}\mathbb{P}(E_n(T, \varepsilon)^c) = \mathbb{P}(E(T, \varepsilon)^c).$$

We take (4.19) and (4.20) in turn. For (4.19), we can write, for any $\delta > 0$,

$$\begin{aligned} & \mathbb{P}\left(X(u) > I(u) \text{ for all } u \in \left[T, T + \frac{1}{T}\right]\right) \\ & \leq \mathbb{P}(X(T) - I(T) > \delta) + \mathbb{P}\left(X(u) > I(u) \text{ for all } u \in \left[T, T + \frac{1}{T}\right] \mid X(T) - I(T) \leq \delta\right). \end{aligned} \quad (4.21)$$

Now by Lemma 4.8, for sufficiently large T , we have $\mathbb{P}(X(T) - I(T) > \delta) \leq e^{-(T-2\lambda)\delta}$.

For the second term, let

$$\tau = \inf\{u \geq 0 : X(T+u) = I(T+u)\},$$

and, for $0 \leq u \leq \tau$, define

$$L(u) = X(T+u) - X(T) + \frac{u^2}{2} + \delta.$$

L is a Brownian motion with linear drift $-(T-\lambda)$ on $[0, \tau]$, started at δ . For an independent Brownian motion B' , define that for $u > 0$,

$$L(u+\tau) = L(\tau) + B'(u) - (T-\lambda)u,$$

so that we have a Brownian motion with linear drift on $[0, \infty)$. It can be verified that, for $0 \leq u \leq \tau$ and conditional on $X(T) - I(T) \leq \delta$, we have $X(T+u) - I(T+u) \leq L(u)$; moreover, L is independent of $(X(T), I(T))$ by the Markov property. In particular, this means that, if $\tau_L = \inf\{u \geq 0 : L(u) = 0\}$, then $\tau_L \geq \tau$, and so

$$\begin{aligned} \mathbb{P}\left(X(T) > I(T) \text{ for all } u \in \left[T, T + \frac{1}{T}\right] \mid X(T) - I(T) \leq \delta\right) &= \mathbb{P}\left(\tau > \frac{1}{T} \mid X(T) - I(T) \leq \delta\right) \\ &\leq \mathbb{P}\left(\tau_L > \frac{1}{T}\right). \end{aligned}$$

Now, the density function of τ_L is given, according to [17, Eqn. 2.2.0.2] and for $t \geq 0$, by

$$f_{\tau_L}(t) = \frac{\delta}{\sqrt{2\pi}t^{3/2}} \exp\left\{-\frac{(\delta - (T-\lambda)t)^2}{2t}\right\}$$

as $t \rightarrow \infty$, with $\mathbb{P}(\tau_L < \infty) = 1$ [17, Eqn. 2.2.0.2(1)]; that is to say, τ_L has an inverse Gaussian distribution with mean $\delta/(T-\lambda)$ and shape parameter δ^2 .

Therefore

$$\mathbb{E}[\tau_L] = \frac{\delta}{T - \lambda}$$

and

$$\text{Var}(\tau_L) = \frac{1}{\delta^2} \left(\frac{\delta}{T - \lambda} \right)^3 = \frac{\delta}{(T - \lambda)^3}.$$

Thus, by Chebyshev's inequality,

$$\mathbb{P}\left(\tau_L > \frac{1}{T - \lambda}\right) \leq \frac{\delta}{(T - \lambda)(1 - \delta)^2},$$

as long as $\delta < 1$. So take $\delta = 1/2$, say. Then, putting everything into Equation (4.21), we have that

$$\mathbb{P}\left(X(u) > I(u) \text{ for all } u \in \left[T, T + \frac{1}{T}\right]\right) \leq e^{-(T-2\lambda)/2} + \frac{2}{T - \lambda} \rightarrow 0,$$

as required. (It is possible to optimise over δ to obtain a tighter bound, but we do not need to.)

We move on to (4.20). For $T > 1$, note that $0 \leq N(T + 1/T) - N(T) \leq N(T + 1) - N(T)$, so it suffices to show that $\mathbb{P}(N(T + 1) - N(T) > 0) \rightarrow 0$ as $T \rightarrow \infty$. Moreover, note that, conditional on the processes (X, I) , if we define

$$A(T) = \int_T^{T+1} (X(u) - I(u)) du$$

then $N(T + 1) - N(T) \sim \text{Poi}(A(T))$.

Now we split: for any $\alpha > 0$,

$$\begin{aligned} & \mathbb{P}(N(T + 1) - N(T) > 0) \\ & \leq \mathbb{P}(X(T) - I(T) > \alpha) + \mathbb{P}(\exists u \in [0, 1] \text{ s.t. } X(T + u) - I(T + u) = 2\alpha \mid X(T) - I(T) \leq \alpha) \\ & \quad + \mathbb{P}(N(T + 1) - N(T) > 0 \mid X(T) - I(T) \leq \alpha; \forall u \in [0, 1], X(T + u) - I(T + u) < 2\alpha). \end{aligned} \tag{4.22}$$

We choose $\alpha = T^{-1/2}$. Again, we can use the fact that $\mathbb{P}(X(T) - I(T) > \alpha) \leq e^{-(T-2\lambda)\alpha}$ for sufficiently large T . Meanwhile, by Lemma 4.12, we have that, if $T^2 e^{-\alpha T} \rightarrow 0$ as $T \rightarrow \infty$

(which is the case here),

$$\mathbb{P}(\exists u \in [0, 1] \text{ s.t. } X(T+u) - I(T+u) = 2\alpha \mid X(T) - I(T) \leq \alpha) = O(e^{-2T\alpha}).$$

Finally, by the Poisson distribution, the last term is easy to deal with:

$$\begin{aligned} & \mathbb{P}(N(T+1) - N(T) > 0 \mid X(T) - I(T) \leq \alpha; \forall u \in [0, 1], X(T+u) - I(T+u) < 2\alpha) \\ &= \mathbb{E}[1 - e^{-A(T)} \mid X(T) - I(T) \leq \alpha; \forall u \in [0, 1], X(T+u) - I(T+u) < 2\alpha] \\ &\leq 1 - e^{-2\alpha} = O(\alpha), \end{aligned}$$

since $f(x) = 1 - e^{-x}$ is an increasing function of x , and the conditioning implies $A(T) \leq 2\alpha$.

Then we have, by (4.22),

$$\mathbb{P}(N(T+1) - N(T) > 0) \leq e^{-T^{1/2} + 2\lambda T^{-1/2}} + O(e^{-2T^{1/2}}) + O(T^{-1/2}) \rightarrow 0,$$

as required. □

As an aside, we note that we can put together the “ingredients” above in a different way to strengthen the final assertion of Lemma 4.8.

Proposition 4.13. $X(T) - I(T) \rightarrow 0$ almost surely as $T \rightarrow \infty$.

Proof. Let $A_{T,\alpha} = \{X(T+s) - I(T+s) \geq \alpha \text{ for some } s \in [0, 1]\}$, as in the proof of Lemma 4.12.

Then

$$\begin{aligned} \mathbb{P}(A_{T,\alpha}) &\leq \mathbb{P}\left(X(T) - I(T) > \frac{\alpha}{2}\right) \\ &\quad + \mathbb{P}\left(X(T+s) - I(T+s) > \alpha \text{ for some } s \in [0, 1] \mid X(T) - I(T) \leq \frac{\alpha}{2}\right). \end{aligned}$$

By Lemma 4.8, we have that

$$\mathbb{P}\left(X(T) - I(T) > \frac{\alpha}{2}\right) \leq e^{-(T-2\lambda)\alpha/2}.$$

Meanwhile, we note that we may adapt the proofs of Lemmas 4.10 and 4.12 to show that, for fixed $\alpha > 0$,

$$\mathbb{P}\left(X(T+s) - I(T+s) \geq \alpha \text{ for some } s \in [0, 1] \mid X(T) - I(T) \leq \frac{\alpha}{2}\right) = O(e^{-T\alpha}).$$

Thus, we have that

$$\mathbb{P}(A_{T,\alpha}) \leq e^{-(T-2\lambda)\alpha/2} + O(Te^{-T\alpha}) = O(e^{-T\alpha/2}).$$

From this, we see that

$$\sum_{k=1}^{\infty} \mathbb{P}(A_{k,\alpha}) < \infty,$$

and thus, by the first Borel–Cantelli lemma,

$$\mathbb{P}(A_{k,\alpha} \text{ infinitely often}) = 0$$

for all $\alpha > 0$. The event $\{A_{k,\alpha} \text{ infinitely often}\}$ is equivalent to the event that there exist infinitely large values t such that $X(t) - I(t) \geq \alpha$. So

$$\begin{aligned} & \mathbb{P}\left(\bigcup_{\alpha>0} \{\forall T > 0 \exists t \geq T \text{ s.t. } X(t) - I(t) > \alpha\}\right) \\ &= \mathbb{P}\left(\bigcup_{j=1}^{\infty} \left\{\forall T > 0 \exists t \geq T \text{ s.t. } X(t) - I(t) > \frac{1}{j}\right\}\right) \\ &= \lim_{j \rightarrow \infty} \mathbb{P}\left(\forall T > 0 \exists t \geq T \text{ s.t. } X(t) - I(t) > \frac{1}{j}\right) = 0. \end{aligned}$$

Hence $\mathbb{P}(X(t) - I(t) \rightarrow 0) = 1$. □

4.3.2 Convergence of expectations

We have, by Aldous’s Theorem 1.6 that, when (X_n, I_n, N_n) is the exploration process (via the depth-first search, Algorithm A) for $\mathcal{G}(n, p)$,

$$(n^{-1/3}X_n(\lfloor n^{2/3}s \rfloor), n^{-1/3}I_n(\lfloor n^{2/3}s \rfloor), N_n(\lfloor n^{2/3}s \rfloor))_{s \geq 0} \xrightarrow{d} (X(s), I(s), N(s))_{s \geq 0}.$$

This, of course, doesn’t show that the corresponding expectations converge. Our aim in this section is to prove the following results, which will be useful technical lemmas later. Throughout this subsection, we slightly generalise to take $p = 1/n + \lambda n^{-4/3} + o(n^{-4/3})$.

Lemma 4.14. *Fix $s > 0$, and, for each $n \in \mathbb{N}$, define some $\mathfrak{R}_n(s)$, such that $n^{-2/3}\mathfrak{R}_n(s) \rightarrow s$. Then, as $n \rightarrow \infty$,*

$$\mathbb{E}[n^{-2/3}(X_n(\mathfrak{R}_n(s)) - I_n(\mathfrak{R}_n(s)))^2] \rightarrow \mathbb{E}[(X(s) - I(s))^2]; \quad (4.23)$$

moreover,

$$\mathbb{E}[N_n(\mathfrak{K}_n(s))] \rightarrow \mathbb{E}[N(s)]. \quad (4.24)$$

Lemma 4.15. *As $s \downarrow 0$,*

$$\mathbb{E}[(X(s) - I(s))^2] \rightarrow 0 \quad \text{and} \quad \mathbb{E}[N(s)] \rightarrow 0.$$

For Lemma 4.14, we will consider a discrete-time Markov chain defined as follows: under $\mathbb{P}_{n,p}$, define the process \tilde{X}_n , for $k \in \mathbb{Z}^{\geq 0}$, by

$$\tilde{X}_n(k) = \sum_{i=1}^k (\tilde{\zeta}_i - 1),$$

where

$$\tilde{\zeta} \stackrel{\text{i.i.d.}}{\sim} \text{Bin}(n, p).$$

Define

$$\tilde{I}_n(k) = \inf_{0 \leq i \leq k} \tilde{X}_n(i).$$

The utility of studying this process comes from the following lemma.

Lemma 4.16. *We may couple X_n and \tilde{X}_n such that, for all k , we have $X_n(k) \leq \tilde{X}_n(k)$, with $\tilde{X}_n(k) - X_n(k)$ an increasing function of k .*

Proof. We note that, conditional on \mathcal{F}_{i-1} , we have $\zeta_i \sim \text{Bin}(n - k - X_n(i) + I_n(i), p)$, so we may couple, for all i , the random variables ζ_i and $\tilde{\zeta}_i$ so that $\zeta_i \leq \tilde{\zeta}_i$. The result follows. \square

The process \tilde{X}_n is easier to analyse than X_n because it is a random walk. In particular, it is *skip-free to the left*: that is, $\mathbb{P}(\tilde{X}_n(k+1) - \tilde{X}_n(k) < -1) = 0$. (This latter property it shares with X_n). A collection of results about such processes are given by Brown, Peköz and Ross [18]; we cite the one that will be of use below, which comes from the proof of their Proposition 7.

Proposition 4.17 ([18]). *Let $S_k = \sum_{i=1}^k Y_i$, where $Y_i \in \mathbb{Z} \cap [-1, \infty)$ a.s., and $\mathbb{E}[Y_i] > 0$. For $r \in \mathbb{N}$, let $T_r \in \mathbb{N} \cup \{\infty\}$ be the first hitting time of $-r$. Then $\mathbb{P}(T_{-r} < \infty) = c^r$, where c is the only solution in $(0, 1)$ of*

$$\mathbb{E}[c^{Y_i}] = 1.$$

In our case, $Y_i = \tilde{\zeta}_i - 1$, $S_k = \tilde{X}_n(k)$, and

$$\mathbb{E}[c^{Y_i}] = c^{-1}(1 - p + pc)^n.$$

Lemma 4.18. Fix $\lambda > 0$ and $u > 0$, and recall that $p = 1/n + \lambda n^{-4/3} + a_n$, where $a_n = o(n^{-4/3})$ is some sequence. Then there exist constants $\beta = \beta(\lambda, u) \in (0, 1)$, $C = C(\lambda, u) > 0$ and $D = D(\lambda, u, (a_n)_{n \in \mathbb{N}})$, such that, for all $x > 0$, for all $k \in \{0, \dots, \lceil un^{2/3} \rceil\}$, and for all $n \in \mathbb{N}$ such that $n \geq D$,

$$\mathbb{P}(\tilde{X}_n(k) - \tilde{I}_n(k) \geq xn^{1/3}) \leq C\beta^x.$$

Proof. We will show this first for $p = 1/n + \lambda n^{-4/3}$ exactly (i.e. $a_n = 0$ for all n), before moving on to the general case. We note that

$$\mathbb{P}(\tilde{X}_n(k) - \tilde{I}_n(k) \geq xn^{1/3}) \leq \mathbb{P}\left(\tilde{X}_n(k) \geq \frac{x}{2}n^{1/3}\right) + \mathbb{P}\left(\tilde{I}_n(k) \leq -\frac{x}{2}n^{1/3}\right).$$

We have that $\tilde{X}_n(k) + k \sim \text{Bin}(nk, p)$. We can use a Chernoff bound for this: we have (for instance, by [34, Thm. 2.21]) that, for all $t > 0$,

$$\mathbb{P}(\tilde{X}_n(k) + k \geq nkp + t) \leq \exp\left\{-\frac{t^2}{2(nkp + t/3)}\right\}.$$

Here we want

$$t = \frac{x}{2}n^{1/3} + k(1 - np) = \frac{x}{2}n^{1/3} - \frac{k\lambda}{n^{1/3}},$$

which we have to check is greater than 0. Since $k \leq \lceil un^{2/3} \rceil$, this holds for $x > 2\lambda(u + 1)$, so fix some $\varepsilon \in (0, 1)$, which does not depend on x , k and n , and suppose $x \geq 2\lambda(u + 1)/(1 - \varepsilon)$.

(We will return to the case of smaller x later.)

We then get that, for all $k \leq \lceil un^{2/3} \rceil$ ($\leq (u + 1)n^{2/3}$),

$$\begin{aligned} \mathbb{P}\left(\tilde{X}_n(k) \geq \frac{x}{2}n^{1/3}\right) &\leq \exp\left\{-\frac{(xn^{1/3}/2 - k\lambda n^{-1/3})^2}{2k(1 + \lambda n^{-1/3}) + xn^{1/3}/3 - 2k\lambda n^{-1/3}/3}\right\} \\ &\leq \exp\left\{-x \frac{(1/2 - (u + 1)\lambda/x)^2}{n^{-1/3}/3 + 2(u + 1)/x + 4(u + 1)\lambda n^{-1/3}/(3x)}\right\}. \end{aligned}$$

From the fact that $n \geq 1$ and $x \geq 2\lambda(u + 1)/(1 - \varepsilon)$ (and $\varepsilon < 1$), we have that

$$\mathbb{P}\left(\tilde{X}_n(k) \geq \frac{x}{2}n^{1/3}\right) \leq \exp\left\{-x \frac{\varepsilon^2}{4(1 - 2\varepsilon/3 + (1 - \varepsilon)/\lambda)}\right\}, \quad (4.25)$$

where ε and λ do not depend on k , x or n . Thus we can find C_1 and β_1 such that

$$\mathbb{P}\left(\tilde{X}_n(k) \geq \frac{x}{2}n^{1/3}\right) \leq C_1\beta_1^x, \quad (4.26)$$

for all n and for all $x > 0$, taking C_1 large enough to cover the case of $x < 2\lambda(u+1)/(1-\varepsilon)$ as well.

We move on to \tilde{I}_n . Let $\tilde{I}_n(\infty) = \inf_{k \geq 0} \tilde{X}_n(k)$, where $-\tilde{I}_n(\infty) \in \mathbb{Z}^{\geq 0} \cup \{\infty\}$. Then, for all $k \in \mathbb{Z}^{\geq 0}$,

$$\begin{aligned} \mathbb{P}\left(\tilde{I}_n(k) \leq -\frac{x}{2}n^{1/3}\right) &\leq \mathbb{P}\left(\tilde{I}_n(\infty) \leq -\frac{x}{2}n^{1/3}\right) = \mathbb{P}\left(\tilde{I}_n(\infty) \leq -\left\lceil \frac{x}{2}n^{1/3} \right\rceil\right) \\ &= \mathbb{P}\left(T_{-\lceil xn^{1/3}/2 \rceil} < \infty\right) \end{aligned}$$

(with the notation T_{-r} taken from Proposition 4.17), where the last inequality holds by the skip-free property. Applying Proposition 4.17, we have that

$$\mathbb{P}\left(T_{-\lceil xn^{1/3}/2 \rceil} < \infty\right) = c^{\lceil xn^{1/3}/2 \rceil} \leq c^{xn^{1/3}/2},$$

for a constant $c \in (0, 1)$ that solves $c = (1-p+pc)^n$. (As an aside, note that this immediately gives us that $\mathbb{P}(\tilde{I}_n(\infty) = -\infty) = 0$.) Let $d = 1-c$, so we need to solve

$$1-d = (1-pd)^n$$

for $d \in (0, 1)$. Note that, since x is not present in its definition, d (and hence c) are independent of x . Using the bound that $(1-y)^n \leq 1-ny + n^2y^2/2$, we can rearrange to get

$$d \geq \frac{2(np-1)}{n^2p^2} = \frac{2\lambda n^{-1/3}}{(1+\lambda n^{-1/3})^2} \geq \frac{2\lambda n^{-1/3}}{(1+\lambda)^2}.$$

Since d is independent of x , we can therefore find a constant $D_2 = 2\lambda/(1+\lambda)^2 > 0$ such that, for all $x > 0$ and for all $n \in \mathbb{N}$, we have $d \geq D_2n^{-1/3}$, and thus $c \leq 1 - D_2n^{-1/3}$.

Take $x > 0$, and $n > D_2^3$. Then we have, combining the above, that

$$\mathbb{P}\left(\tilde{I}_n(k) \leq -\frac{x}{2}n^{1/3}\right) \leq (1 - D_2n^{-1/3})^{xn^{1/3}/2} \leq \exp\left(-\frac{x D_2}{2}\right),$$

which means that certainly in these cases we can find $\beta_2 \in (0, 1)$ and $C_2 > 0$ such that

$$\mathbb{P}\left(\tilde{I}_n(k) \leq -\frac{x}{2}n^{1/3}\right) \leq C_2\beta_2^x. \quad (4.27)$$

Thus, combining (4.26) and (4.27), there exist $C_1, C_2 > 0$ and $\beta_1, \beta_2 \in (0, 1)$ such that, for all $x > 0$, for all $k \leq \lceil n^{2/3}u \rceil$ and for all all $n > D_2^3$,

$$\mathbb{P}(\tilde{X}_n(k) - \tilde{I}_n(k) \geq xn^{1/3}) \leq C_1\beta_1^x + C_2\beta_2^x.$$

Taking $\beta = \max\{\beta_1, \beta_2\}$ and $C = C_1 + C_2$, we see that the claim of the lemma holds on the restriction that $a_n = 0$ for all n .

For the general case, where $(a_n)_{n \in \mathbb{N}}$ is arbitrary, suppose we take \tilde{X}'_n to be the same process but with $p = 1/n + (\lambda + 1)n^{-4/3}$, defining \tilde{I}'_n to be the running minimum accordingly, and assume n to be sufficiently large that $a_n \leq n^{-4/3}$. Then we may use a discrete-time analogue of Lemma 4.9 to couple our processes such that

$$\tilde{X}_n(k) - \tilde{I}_n(k) \leq \tilde{X}'_n(k) - \tilde{I}'_n(k).$$

We may then apply the result above to \tilde{X}'_n to obtain the claimed result. \square

Proof of Lemma 4.14. We start by showing (4.23), noting that, via Vitali's convergence theorem, it is sufficient to show that the sequence $(P_n)_{n > D_3}$ is uniformly integrable, where

$$P_n := n^{-2/3}(X_n(\mathfrak{R}_n(s)) - I_n(\mathfrak{R}_n(s)))^2,$$

and D_3 is a constant chosen to be at least the D from Lemma 4.18, and is also sufficiently large that, for all $n \geq D_3$, we have $\mathfrak{R}_n(s) \leq (s + 1)n^{2/3}$ (which is possible since $n^{-2/3}\mathfrak{R}_n(s) \rightarrow s$ as $n \rightarrow \infty$). That is, since P_n is non-negative, we wish to show that

$$\lim_{K \rightarrow \infty} \sup_{n > D_3} \mathbb{E}[P_n \mathbb{1}_{\{P_n \geq K\}}] = 0.$$

Note firstly that

$$\mathbb{E}[P_n \mathbb{1}_{\{P_n \geq K\}}] = K\mathbb{P}(P_n \geq K) + \int_K^\infty \mathbb{P}(P_n \geq x) dx.$$

For $\lambda > 0$, for all $n > D_3$ and for all $x > 0$, we use Lemma 4.18 (with $u = s + 1$) in conjunction

with Lemma 4.16 and a discrete analogue of Lemma 4.9 to show that, for $C > 0$ and $\beta \in (0, 1)$ depending only on s and λ ,

$$\mathbb{P}(P_n \geq x) = \mathbb{P}(X_n(\mathfrak{R}_n(s)) - I_n(\mathfrak{R}_n(s)) \geq n^{1/3} \sqrt{x}) \leq C \beta^{\sqrt{x}},$$

since $X_n - I_n$ is non-negative. For $\lambda \leq 0$, we obtain a similar uniform bound in n and x by using Proposition 4.6 to couple $X_n - I_n$, coming from the exploration of $\mathcal{G}(n, p)$, to the analogous process for $\mathcal{G}(n, 1/n + n^{-4/3})$. Thus, for all n and K ,

$$\mathbb{E}[P_n \mathbb{1}_{\{P_n \geq K\}}] \leq CK \beta^{\sqrt{K}} + \int_K^\infty C \beta^{\sqrt{x}} dx \leq C \left(K + \frac{2}{(\log \beta)^2} - \frac{2\sqrt{K}}{\log \beta} \right) \beta^{\sqrt{K}},$$

so that

$$0 \leq \limsup_{K \rightarrow \infty} \sup_{n > D_3} \mathbb{E}[P_n \mathbb{1}_{\{P_n \geq K\}}] \leq \limsup_{K \rightarrow \infty} \sup_{n > D_3} C \left(K + \frac{2}{(\log \beta)^2} - \frac{2\sqrt{K}}{\log \beta} \right) \beta^{\sqrt{K}} = 0,$$

since $\beta \in (0, 1)$. Thus (4.23) holds.

As it will be useful in the sequel, we note further that an easy consequence of the above is that

$$\mathbb{E}[n^{-1/3}(X_n(\mathfrak{R}_n(s)) - I_n(\mathfrak{R}_n(s)))] \rightarrow \mathbb{E}[X(s) - I(s)]. \quad (4.28)$$

Indeed, we may make a further observation. By Lemma 4.8, the values of C and β above also imply that, for $n > D_3$ and $k \leq \mathfrak{R}_n(s)$ ($\leq (s+1)n^{2/3}$),

$$\mathbb{E}[n^{-1/3}(X_n(k) - I_n(k))] = \int_0^\infty \mathbb{P}(X_n(k) - I_n(k) \geq xn^{1/3}) dx \leq \int_0^\infty C \beta^x dx = \frac{C}{(-\log \beta)}, \quad (4.29)$$

where the constants C and β depend on s and λ , but not on k or n .

We now consider (4.24). Fix $\delta > 0$, and note that $n^{-2/3}\mathfrak{R}_n(s) \rightarrow s$ as $n \rightarrow \infty$. Then, by the tower law, we firstly have that, for all sufficiently large n ,

$$\begin{aligned} \mathbb{E}[N_n(\mathfrak{R}_n(s))] &= \mathbb{E} \left[p \sum_{k=0}^{\mathfrak{R}_n(s)-1} (X_n(k) - I_n(k)) \right] = pn^{1/3} \sum_{k=0}^{\mathfrak{R}_n(s)-1} n^{-1/3} \mathbb{E}[X_n(k) - I_n(k)] \\ &= np \int_0^{s+\delta} \mu_n(u) \mathbb{1}_{\{n^{2/3}u < \mathfrak{R}_n(s)\}} du, \end{aligned}$$

where we define

$$\mu_n(u) = \mathbb{E}[n^{-1/3}(X_n(\lfloor un^{2/3} \rfloor) - I_n(\lfloor un^{2/3} \rfloor))].$$

By (4.28), we have that, for all $u > 0$, the function $\mu_n(u) \rightarrow \mu(u) := \mathbb{E}[X(u) - I(u)]$. In fact, we have that, on $[0, s + \delta]$,

$$\mu_n(u) \mathbb{1}_{\{n^{2/3}u < \hat{\kappa}_n(s)\}} \rightarrow \mu(u) \mathbb{1}_{\{u \leq s\}}.$$

Thus, by the bounded convergence theorem applied with the bound in (4.29), it holds that, as $n \rightarrow \infty$,

$$\int_0^{s+\delta} \mu_n(u) \mathbb{1}_{\{n^{2/3}u < \hat{\kappa}_n(s)\}} du \rightarrow \int_0^{s+\delta} \mu(u) \mathbb{1}_{\{u \leq s\}} du = \int_0^s \mu(u) du,$$

and of course $np \rightarrow 1$. Hence

$$\mathbb{E}[N_n(\hat{\kappa}_n(s))] \rightarrow \int_0^s \mu(u) du = \mathbb{E}\left[\int_0^s (X(u) - I(u)) du\right] = \mathbb{E}[N(s)],$$

(applying Tonelli's Theorem for the first equality, and the tower law for the second), giving (4.24). \square

Before we prove Lemma 4.15, we define some technical notation: we set

$$\text{Erf}(x) := \begin{cases} \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt & \text{for } x \geq 0; \\ -\text{Erf}(-x) & \text{for } x < 0, \end{cases}$$

and we define $\text{Erfc}(x) := 1 - \text{Erf}(x)$.

Proof of Lemma 4.15. We first note that, by Lemma 4.9, we may couple $X - I$ to $\hat{X} - \hat{I}$, where \hat{X} is a Brownian motion with constant linear drift λ (starting at 0) and \hat{I} its running infimum, such that $X - I \leq \hat{X} - \hat{I}$. (This is similar to what we did in the proof of Lemma 4.12.) Hence it must hold that, for all $s > 0$,

$$0 \leq \mathbb{E}[X(s) - I(s)] \leq \mathbb{E}[\hat{X}(s) - \hat{I}(s)] = \mathbb{E}[\hat{X}(s)] - \mathbb{E}[\hat{I}(s)]. \quad (4.30)$$

It is easy to obtain that $\mathbb{E}[\hat{X}(s)] = \lambda s = o(1)$ as $s \downarrow 0$. Now, we may use [17, Eqn. 2.1.1.4], which tells us that, for $y \geq 0$,

$$\mathbb{P}(-\hat{I}(s) \geq y) = \frac{1}{2} \text{Erfc}\left(\frac{y}{\sqrt{2s}} + \lambda\sqrt{\frac{s}{2}}\right) + \frac{1}{2} e^{-2\lambda y} \text{Erfc}\left(\frac{y}{\sqrt{2s}} - \lambda\sqrt{\frac{s}{2}}\right). \quad (4.31)$$

Using the fact that $\mathbb{E}[Y] = \int_0^\infty \mathbb{P}(Y \geq y) dy$ for a non-negative random variable Y , we can

use the previous equation to obtain that

$$\mathbb{E}[\hat{I}(s)] = \frac{1}{2}\lambda s - \left(\frac{1}{2\lambda} + \frac{1}{2}\lambda s\right) \operatorname{Erf}\left(\lambda\sqrt{\frac{s}{2}}\right) - \sqrt{\frac{s}{2\pi}}e^{-\lambda^2 s/2} = o(1) \quad (4.32)$$

as $s \downarrow 0$. Thus $\mathbb{E}[\hat{X}(s)] - \mathbb{E}[\hat{I}(s)] \rightarrow 0$ as $s \downarrow 0$, giving us from (4.30) that $\mathbb{E}[X(s) - I(s)] \rightarrow 0$.

Then we have that

$$\mathbb{E}[N(s)] = \mathbb{E}\left[\int_0^s (X(u) - I(u)) du\right] = \int_0^s \mathbb{E}[X(u) - I(u)] du$$

by Tonelli's theorem. Since $\mathbb{E}[X(u) - I(u)]$ is continuous at 0, we have by the bounded convergence theorem that the integral converges to 0 as $s \downarrow 0$, completing the proof for N .

For the result for $(X - I)^2$, we note that

$$0 \leq \mathbb{E}[(X(s) - I(s))^2] \leq \mathbb{E}[(\hat{X}(s) - \hat{I}(s))^2] \leq 2\mathbb{E}[\hat{X}(s)^2] + 2\mathbb{E}[\hat{I}(s)^2],$$

with the last bound following somewhat crudely by the parallelogram law. It is easy to calculate that $\mathbb{E}[\hat{X}(s)^2] = \lambda^2 s^2 + s = o(1)$ as $s \downarrow 0$. We can then use (4.31), and the fact that $\mathbb{E}[Y^2] = \int_0^\infty \mathbb{P}(Y^2 \geq z) dz = 2 \int_0^\infty y \mathbb{P}(Y \geq y) dy$ for a non-negative random variable Y , to deal with $\mathbb{E}[\hat{I}(s)^2]$. Via somewhat long-winded calculations, we obtain that

$$\mathbb{E}[\hat{I}(s)^2] = \frac{1}{2}\lambda^2 s^2 + s + \left(\frac{1}{2\lambda^2} - \frac{1}{2}\lambda^2 s^2 - s\right) \operatorname{Erf}\left(\lambda\sqrt{\frac{s}{2}}\right) - \left(\lambda s + \frac{1}{\lambda}\right) \sqrt{\frac{s}{2\pi}}e^{-\lambda^2 s/2} = o(1) \quad (4.33)$$

as $s \downarrow 0$. This completes the proof. \square

4.4 Small additions and deletions

We have established that, in the exploration process, once we are at a sufficiently large time (on an $n^{2/3}$ timescale), we will, after any such time, rapidly hit a point k where $X_n(k) = I_n(k)$.

We know that we hit such a point rapidly, but we don't know precisely when in the small interval this occurs. It will be important to show that this doesn't make much difference, in order to make rigorous the notion of "approximate independence" indicated by Problem 1.

We will prove two propositions motivated by the above. Recall that, for any graph H , $s(H)$

denotes the number of surplus edges it has, given by

$$s(H) = e(H) - |H| + c(H).$$

We make the following technical definition.

Definition 4.19. Define:

$$\bar{k}_n(T) = \left\lfloor \left(T + \frac{1}{T} \right) n^{2/3} \right\rfloor;$$

$$\underline{k}_n(T) = \lfloor T n^{2/3} \rfloor; \text{ and}$$

$$\delta_n(T) = \bar{k}_n(T) - \underline{k}_n(T).$$

Remark. Note that $\delta_n(T)$ is equal to either $\lfloor n^{2/3}/T \rfloor$ or $\lceil n^{2/3}/T \rceil$, depending on the values of T and n .

Proposition 4.20. Let $n \in \mathbb{N}, T > 0$, and suppose that, for some $c \in \mathbb{R}$, $p = p(n)$ satisfies $(np - 1)n^{1/3} \rightarrow c$ as $n \rightarrow \infty$. Work under $\mathbb{P}_{n-\underline{k}_n(T), p}$, and form $G_{n-\bar{k}_n(T)}$ from $G_{n-\underline{k}_n(T)}$ by deleting all vertices with labels strictly larger than $n - \bar{k}_n(T)$ (and any edges incident to such vertices). Then

$$\lim_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}_{n-\underline{k}_n(T), p} \left[s(G_{n-\underline{k}_n(T)}) - s(G_{n-\bar{k}_n(T)}) \right] = 0.$$

Proposition 4.21. Let $n \in \mathbb{N}, T > 0, q \in (0, 1)$ and $c \in \mathbb{R}$. Let $p = q/n + \lambda n^{-4/3}$, and let $p' = p'(n)$ satisfy $(np' - 1)n^{1/3} \rightarrow c$ as $n \rightarrow \infty$. We let $\mathbb{P}_{n, T}^*$ be a probability measure with respect to which $G^{(1)} \sim \mathcal{G}(n - \bar{k}_n(T), p, q)$, and where G is formed from $G^{(1)}$ by

- adding in vertices $n - \bar{k}_n(T) + 1, \dots, n - \underline{k}_n(T)$, and,
- for each possible edge incident to a new vertex, inserting it with probability p' , independently for each possible edge and independently of $G^{(1)}$.

Then

$$\lim_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}_{n, T}^* \left[s(G) - s(G^{(1)}) \right] = 0.$$

The construction in Proposition 4.21 may seem an artificial construction, but it will turn out to be the one we need in the $q < 1$ setting later. The one in Proposition 4.20 is more natural—we prove it both because it is a useful lemma for the other result, and because it is itself the result we *would* need if we were able to adapt this method for the $q > 1$ case. We will see this in Lemma 4.24.

Note that the quantities inside the expectations in both of the propositions above are non-negative, as shown by this lemma.

Lemma 4.22. *If $G \leq H$, then $s(G) \leq s(H)$.*

Moreover, suppose G_1, G_2, H_1 and H_2 are graphs such that G_1 and G_2 are on the same vertex set with $G_1 \leq G_2$, and H_1 and H_2 are on the same vertex set with $H_1 \leq H_2$. Moreover suppose $G_1 \leq H_1$, and $G_2 \leq H_2$, not necessarily on the same vertex set. Then

$$s(H_1) - s(G_1) \leq s(H_2) - s(G_2).$$

Proof. The first part of the lemma is obvious by considering the effect on the formula $e(G) - |G| + c(G)$ for the number of surplus edges of first adding the new vertices, and then the new edges. The second part may be argued similarly, by starting with H_1 (with subgraph G_1), and considering adding in first the edges in $E(G_2) \setminus E(G_1)$, and then the remaining edges necessary to form H_2 . Note that, in the first stage, when we add an edge that is in G_2 but not in H_2 , if it increases the number of G -surplus edges, it must also increase the number of H -surplus edges, since at any stage the vertex sets of the components of G are contained within the vertex sets of the components of H . \square

Before we begin the proofs of our main propositions, we need one more lemma.

Lemma 4.23. *Fix $s \in \mathbb{Z}^{\geq 0}$. Choose s vertices uniformly at random without replacement from a given graph G on m vertices, with components C_1, C_2, \dots (listed in descending order of size, with ties broken arbitrarily). Let N denote the number of components to which the s vertices belong. Then*

$$s \geq \mathbb{E}[N] \geq s - \frac{s^2}{2m(m-1)} \sum_i |C_i|^2.$$

Remark. In applying this, we will take the stack at a particular time to be the random set of vertices in the hypotheses of the lemma—so this will give a bound on the number of components of $\mathcal{G} \setminus \{v_1, \dots, v_k\}$ at time k , across which the stack vertices are distributed. Although the stack vertices are *not* a uniform random subset of the unexplored vertices, the exchangeability of the vertices, and the independence of edges, mean that this does not affect the distribution. (Alternatively, we may consider exploring the graph using the modified algorithm in which, when given a choice of awake or of sleeping vertices, we choose one uniformly at random rather than choosing the most-recently awoken, or the lowest-labelled; this gives an exploration process with identical distribution, and a set of stack vertices which truly are a uniform

random subset of the unexplored vertices.)

Proof. The first inequality is obvious, as is the second when $s = 0$. So suppose $s > 0$.

Consider choosing the s vertices one at a time, and let v_k ($k = 1, \dots, s$) be the k -th vertex chosen. We compare to the case where all vertices “land” in different components. Then, each time a vertex lands in a component already seen, the maximal number of components we can hope to cover is reduced by 1.

In other words, we have that

$$N = s - \sum_{k=1}^s \mathbb{1}_{A_k},$$

where

$$A_k = \{v_k \text{ is in the same component as } v_j \text{ for some } j < k\}.$$

Now let

$$B_{j,k} = \{v_j \text{ and } v_k \text{ are in the same component}\};$$

then

$$N \geq s - \sum_{1 \leq j < k \leq s} \mathbb{1}_{B_{j,k}},$$

so

$$\mathbb{E}[N] \geq s - \sum_{1 \leq j < k \leq s} \mathbb{P}(B_{j,k}) = s - \binom{s}{2} \mathbb{P}(B_{1,2}),$$

since the v_k s are exchangeable.

Now note that $\binom{s}{2} \leq s^2/2$, and that

$$\mathbb{P}(B_{1,2}) = \sum_i \mathbb{P}(v_1, v_2 \in C_i) = \sum_i \frac{|C_i|(|C_i| - 1)}{m(m-1)} \leq \frac{1}{m(m-1)} \sum_i |C_i|^2. \quad \square$$

We will start with the proof of Proposition 4.20, and then use it to prove Proposition 4.21.

Proof of Proposition 4.20. Clearly $G_{n-\bar{k}_n(T)} \sim \mathcal{G}(n - \bar{k}_n(T), p)$, so we have that

$$\mathbb{E}_{n-\underline{k}_n(T), p} [s(G) - s(G_{n-\bar{k}_n(T)})] = \mathbb{E}_{n-\underline{k}_n(T), p} [s(G_{n-\underline{k}_n(T)})] - \mathbb{E}_{n-\bar{k}_n(T), p} [s(G_{n-\bar{k}_n(T)})].$$

We now couple these two graphs in a different way. Consider exploring $G_{n-\underline{k}_n(T)}$ using Algorithm A. Then we let $G_n^*(T) = G_{n-\underline{k}_n(T)} \setminus \{v_1, \dots, v_{\delta_n(T)}\}$, recalling that v_i denotes the i -th vertex explored—so $G_n^*(T)$ is the residual graph when we remove the first $\delta_n(T)$ vertices explored in the depth-first search. (Recall that $\delta_n(T) = \bar{k}_n(T) - \underline{k}_n(T)$.) Under $\mathbb{P}_{n-\underline{k}_n(T), p}$, it

holds that $G_n^*(T) \sim \mathcal{G}(n - \bar{k}_n(T), p)$, up to relabelling of vertices. Hence,

$$\mathbb{E}_{n-\underline{k}_n(T), p} \left[s(G_{n-\underline{k}_n(T)}) \right] - \mathbb{E}_{n-\bar{k}_n(T), p} \left[s(G_{n-\bar{k}_n(T)}) \right] = \mathbb{E}_{n-\underline{k}_n(T), p} \left[s(G_{n-\underline{k}_n(T)}) - s(G_n^*(T)) \right],$$

and it is sufficient to show the right-hand side tends to 0. Here, the right-hand side is the expected number of surplus edges created by adding the early vertices $v_1, \dots, v_{\delta_n(T)}$ back into $G_n^*(T)$. Let $C(v)$ denote the component to which the vertex v belongs. Let S be the number of vertices on the stack after iteration $\delta_n(T)$ (noting that, if $v_{\delta_n(T)+1}$ is on the stack at this point, we have not yet removed it—as we will shortly do at the start of iteration $\delta_n(T) + 1$). Let M be the number of components of $G_n^*(T)$ to which the stack vertices belong. Note that, as all edges of $G_{n-\underline{k}_n(T)}$ are independent, $G_n^*(T)$ is independent of S , though the distribution of M is influenced by both.

Then it is straightforward that

$$s(G_{n-\underline{k}_n(T)}) - s(G_n^*(T)) = N_{n-\underline{k}_n(T)}(\delta_n(T)) + S - M. \quad (4.34)$$

Now, as the vertices of $G_n^*(T)$ are exchangeable, it holds that (under $\mathbb{P}_{n-\underline{k}_n(T), p}$), conditionally on S , the random variable M is distributed like the number of components spanned by S vertices chosen at random from $G_{n-\bar{k}_n(T)} \sim \mathcal{G}(n - \bar{k}_n(T), p)$.

By Lemma 4.23, it then holds that,

$$\mathbb{E}_{n-\underline{k}_n(T), p} [M \mid G_n^*(T), S] \geq S - \frac{S^2}{2(n - \bar{k}_n(T))(n - \bar{k}_n(T) - 1)} \sum_i |C_i^*|^2,$$

where we write C_i^* for the i -th-largest component of $G_n^*(T)$. By the tower law, and using the distribution asserted above for $G_n^*(T)$, we have

$$\begin{aligned} \mathbb{E}_{n-\underline{k}_n(T), p} [M' \mid S] &\geq S - \frac{S^2}{2(n - \bar{k}_n(T))(n - \bar{k}_n(T) - 1)} \mathbb{E}_{n-\bar{k}_n(T), p} \left[\sum_i |C_i|^2 \right] \\ &\geq S - \frac{S^2}{2(n - \bar{k}_n(T))(n - \bar{k}_n(T) - 1)} \mathbb{E}_{n, p} \left[\sum_i |C_i|^2 \right], \end{aligned} \quad (4.35)$$

where, under $\mathbb{P}_{m, r}$ for any m and r , we take C_i to be the i -th-largest component of G_m . (The last inequality in (4.35) holds because, if $G_i \sim \mathcal{G}(m_i, p)$ ($i = 1, 2$) with $m_1 \leq m_2$, we may couple the graphs such that $G_1 \leq G_2$). Now, as a consequence of Theorem 1.5, since $n^{1/3}(np - 1) \rightarrow c \in \mathbb{R}$, it holds that there exists a constant A (which clearly can be chosen not to depend on T) such

that

$$\mathbb{E}_{n,p} \left[\sum_i |C_i|^2 \right] \leq An^{4/3},$$

and thus

$$\mathbb{E}_{n-\underline{k}_n(T),p} [M | S] \geq S - \frac{AS^2 n^{4/3}}{2(n-\bar{k}_n(T))(n-\bar{k}_n(T)-1)}. \quad (4.36)$$

Hence, using the tower law and (4.34),

$$\begin{aligned} & \mathbb{E}_{n-\underline{k}_n(T),p} [s(G_{n-\underline{k}_n(T)}) - s(G_n^*(T))] \\ & \leq \mathbb{E}_{n-\underline{k}_n(T),p} [N_{n-\underline{k}_n(T)}(\delta_n(T))] + \frac{An^{4/3} \mathbb{E}_{n-\underline{k}_n(T),p} [S^2]}{2(n-\bar{k}_n(T))(n-\bar{k}_n(T)-1)}. \end{aligned}$$

But $S \leq X_{n-\underline{k}_n(T)}(\delta_n(T)) - I_{n-\underline{k}_n(T)}(\delta_n(T)) + 1$. So we have that

$$\begin{aligned} & \mathbb{E}_{n-\underline{k}_n(T),p} [s(G_{n-\underline{k}_n(T)}) - s(G_n^*(T))] \\ & \leq \mathbb{E}_{n-\underline{k}_n(T),p} [N_{n-\underline{k}_n(T)}(\delta_n(T))] \\ & \quad + \frac{An^2}{2(n-\bar{k}_n(T))(n-\bar{k}_n(T)-1)} \mathbb{E}_{n-\underline{k}_n(T),p} \left[\frac{(X_{n-\underline{k}_n(T)}(\delta_n(T)) - I_{n-\underline{k}_n(T)}(\delta_n(T)) + 1)^2}{n^{2/3}} \right]. \end{aligned}$$

Then, by Lemma 4.14, as $n \rightarrow \infty$, the right-hand side converges to

$$\mathbb{E} \left[N^{(\lambda-T)} \left(\frac{1}{T} \right) \right] + \frac{A}{2} \mathbb{E} \left[\left(X^{(\lambda-T)} \left(\frac{1}{T} \right) - I^{(\lambda-T)} \left(\frac{1}{T} \right) \right)^2 \right], \quad (4.37)$$

where (for this proof only) we say that $(X^{(\rho)}, I^{(\rho)}, N^{(\rho)})$ is the limiting process in Theorem 1.6 for $\mathcal{G}(n, 1/n + \rho n^{-4/3})$.

Assume without loss of generality that $T > \lambda$. Then the expression in (4.37) can easily be shown to be bounded above by

$$\mathbb{E} \left[N^{(0)} \left(\frac{1}{T} \right) \right] + \frac{A}{2} \mathbb{E} \left[\left(X^{(0)} \left(\frac{1}{T} \right) - I^{(0)} \left(\frac{1}{T} \right) \right)^2 \right].$$

By Lemma 4.15, this tends to 0 as $T \rightarrow \infty$, thus proving the claim. \square

Proof of Proposition 4.21. Let $\lambda_+ = \max\{\lambda, 0\}$, and define

$$p'' = \max \left\{ p', \frac{1}{n} + \frac{\lambda_+}{n^{4/3}q} \right\},$$

so that $n^{1/3}(np'' - 1) \rightarrow \max\{c, \lambda_+\}$ as $n \rightarrow \infty$. Then, applying Proposition 4.1 (which

couples Erdős–Rényi and random cluster graphs) and the standard coupling of Erdős–Rényi graphs across different values of the probability parameter, we may add edges to the graph to produce a graph $G' \sim \mathcal{G}(n - \underline{k}_n(T), p'')$ —that is, we can couple G to this G' such that $G \leq G'$. Let $G^{(1)'} \sim \mathcal{G}(n - \bar{k}_n(T), p'')$ be the subgraph of G' induced by $(1, \dots, n - \bar{k}_n(T))$. Then, applying Lemma 4.22, it holds that

$$0 \leq s(G) - s(G^{(1)}) \leq s(G') - s(G^{(1)'}),$$

and hence

$$0 \leq \mathbb{E}_{n,T}^* [s(G) - s(G^{(1)})] \leq \mathbb{E}_{n-\underline{k}_n(T), p''} [s(G_{n-\underline{k}_n(T)})] - \mathbb{E}_{n-\bar{k}_n(T), p''} [s(G_{n-\bar{k}_n(T)})].$$

We now apply Proposition 4.20 to the right-hand side to obtain the result. \square

To see how Proposition 4.21 and Proposition 4.20 might be useful, we fix some $q \in (0, 2) \setminus \{1\}$, and define the event

$$Z_n(T, k) = \{ \inf \{ i \geq T : X_n(i) - I_n(i) = 0 \} = k \},$$

as well as the quantities:

$$s_n\left(T, \frac{1}{T}\right) = \max_{\underline{k}_n(T) \leq k \leq \bar{k}_n(T)} \mathbb{E}_{n,p} [q^{N_n(n) - N_n(k)} \mid Z_n(T, k)]; \quad (4.38)$$

$$m_n\left(T, \frac{1}{T}\right) = \min_{\underline{k}_n(T) \leq k \leq \bar{k}_n(T)} \mathbb{E}_{n,p} [q^{N_n(n) - N_n(k)} \mid Z_n(T, k)]; \text{ and} \quad (4.39)$$

$$r_n\left(T, \frac{1}{T}\right) = \frac{m_n(T, 1/T)}{s_n(T, 1/T)}. \quad (4.40)$$

Thus $r_n(T, 1/T)$ quantifies, if k is the first time the reflected process hits zero in the interval in question—i.e. $[\lfloor Tn^{2/3} \rfloor, \lfloor (T + 1/T)n^{2/3} \rfloor]$ —how much the expectation of q , raised to the power of the number of surplus edges not yet discovered by k , can change depending on the value of k , assuming such a value exists. We prove that this variability is negligible in the limit, in a certain sense.

Lemma 4.24. *For all $q \in (0, 2)$ and $\lambda \in \mathbb{R}$, when, for*

$$\hat{p} = \frac{q}{n} + \frac{\lambda}{n^{-4/3}},$$

we take

$$p = \frac{\hat{p}}{\hat{p} + q(1-p)},$$

then

$$\lim_{T \rightarrow \infty} \liminf_{n \rightarrow \infty} r_n \left(T, \frac{1}{T} \right) = 1. \quad (4.41)$$

Remark. We will use this lemma when we provide our partial proof of the convergence of the exploration process in Section 6.2.

For this we will use the following technical lemma.

Lemma 4.25. *For $m_1, m_2 \in \mathbb{N}$, and for any $r \in (0, 1)$, we may couple $G_1 \sim \mathcal{G}(m_1, r)$ and $G_2 \sim \mathcal{G}(m_2, r)$ such that $G_1 \leq G_2$.*

We omit the proof, which follows by considering adding additional vertices and carrying out independent ‘‘coin flips’’ for the potential new edges.

Proof of Lemma 4.24. We note that, under $\mathbb{P}_{n,p}$ and conditionally on the event $Z_n(T, k)$, the graph on the unexplored vertices v_{k+1}, \dots, v_n is simply distributed as an Erdős–Rényi graph on those unexplored vertices (with the same value of the parameter p). Moreover, we note that the process $(X_n(k+i), I_n(k+i), N_n(k+i))_{i \geq 0}$ has conditional distribution identical to the unconditional distribution of the process $(X_{n-k}(i), I_{n-k}(i), N_{n-k}(i))_{i \geq 0}$. Thus

$$\mathbb{E}_{n,p} [q^{N_n(n) - N_n(k)} \mid Z_n(T, k)] = \mathbb{E}_{n-k,p} [q^{N_{n-k}(n-k)}].$$

Now note that, by Lemma 4.25, for $m_1 \leq m_2$, we can (for any r) couple $G_1 \sim \mathcal{G}(m_1, r)$ and $G_2 \sim \mathcal{G}(m_2, r)$ so that $G_1 \leq G_2$. Thus, by Lemma 4.22, $s(G_1)$ is stochastically dominated by $s(G_2)$. This means that we can see that the extremal values of k are going to provide the extrema of $\mathbb{E}_{n-k,p} [q^{N_{n-k}(n-k)}]$, but that the order varies with the sign of $q - 1$. Hence, we will split into cases.

We start with $q < 1$. Then we have, by the above, that

$$s_n \left(T, \frac{1}{T} \right) = \mathbb{E}_{n-\bar{k}_n(T), p} [q^{N_{n-\bar{k}_n(T)}(n-\bar{k}_n(T))}]$$

and

$$m_n \left(T, \frac{1}{T} \right) = \mathbb{E}_{n-\underline{k}_n(T), p} [q^{N_{n-\underline{k}_n(T)}(n-\underline{k}_n(T))}].$$

Now, for a fixed graph G on $[m]$, consider adding $x \in \mathbb{N}$ extra vertices, which we label $m + 1, \dots, m + x$. Let the probability measure \mathbb{Q}_r be such that, under it, we insert each edge between two vertices in $\{m + 1, \dots, m + x\}$, and each edge from one of these vertices into $[m]$, independently with probability r . Call the new graph H . Then we may define

$$\Psi(G, x, r) = \mathbb{E}_{\mathbb{Q}_r} [q^{s(H)-s(G)}],$$

where the expectation is taken with respect to the measure \mathbb{Q}_r . Note that Lemma 4.22 tells us that $s(H) - s(G) \geq 0$; note that $\Psi(G, x, r)$ only depends on the component sizes of G , which will allow us to apply Proposition 3.7.

Then it holds that

$$\mathbb{E}_{n-\underline{k}_n(T), p} [q^{N_{n-\underline{k}_n(T)}(n-\underline{k}_n(T))}] = \mathbb{E}_{n-\bar{k}_n(T), p} [q^{N_{n-\bar{k}_n(T)}(n-\bar{k}_n(T))} \Psi(G_{n-\bar{k}_n(T)}, \delta_n(T), p)],$$

and so, recalling that under $\mathbb{P}_{n,p,q}$ we have $G_n \sim \mathcal{G}(n, p, q)$, we have that

$$\begin{aligned} r_n \left(T, \frac{1}{T} \right) &= \frac{\mathbb{E}_{n-\bar{k}_n(T), p} [q^{N_{n-\bar{k}_n(T)}(n-\bar{k}_n(T))} \Psi(G_{n-\bar{k}_n(T)}, \delta_n(T), p)]}{\mathbb{E}_{n-\bar{k}_n(T), p} [q^{N_{n-\bar{k}_n(T)}(n-\bar{k}_n(T))}]} \\ &= \mathbb{E}_{n-\bar{k}_n(T), \hat{p}, q} [\Psi(G_{n-\bar{k}_n(T)}, \delta_n(T), p)] && \text{(by Proposition 3.7)} \\ &= \mathbb{E}_A [q^{s(H)-s(G)}], && (4.42) \end{aligned}$$

where this last expectation is taken under a probability measure \mathbb{P}_A for which:

- $G \sim \mathcal{G}(n - \bar{k}_n(T), \hat{p}, q)$, in which $\hat{p} = q/n + \lambda n^{-4/3}$;
- H is formed from G by:
 - adding vertices labelled with $n - \bar{k}_n(T) + 1, \dots, n - \underline{k}_n(T)$, and,
 - for each possible edge incident to one of the new vertices, including it with probability p , all new edges being independent and independent of G .

Now note that, by Jensen's inequality,

$$\mathbb{E}_A [q^{s(H)-s(G)}] \geq q^{\mathbb{E}_A [s(H)-s(G)]}.$$

However, by Proposition 4.21,

$$\lim_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}_A[s(H) - s(G)] = 0$$

(note that \mathbb{P}_A implicitly depends on T and n). This, and the fact that $q^{s(H)-s(G)} \leq 1$, implies that

$$\lim_{T \rightarrow \infty} \liminf_{n \rightarrow \infty} \mathbb{E}_A[q^{s(H)-s(G)}] = 1,$$

so that, by Equation (4.42),

$$\lim_{T \rightarrow \infty} \liminf_{n \rightarrow \infty} r_n\left(T, \frac{1}{T}\right) = 1. \quad (4.43)$$

Observing that the case $q = 1$ is trivial, we now consider the case where $q > 1$. In this case,

$$s_n\left(T, \frac{1}{T}\right) = \mathbb{E}_{n-\underline{k}_n(T), p}\left[q^{N_{n-\underline{k}_n(T)}(n-\underline{k}_n(T))}\right]$$

and

$$m_n\left(T, \frac{1}{T}\right) = \mathbb{E}_{n-\bar{k}_n(T), p}\left[q^{N_{n-\bar{k}_n(T)}(n-\bar{k}_n(T))}\right].$$

For a fixed graph G on $[m]$, consider deleting the x highest-labelled vertices to form a graph H . Then let

$$\Phi(G, x) = q^{s(H)-s(G)}$$

(where we may note that, by Lemma 4.8, $s(H) - s(G) \leq 0$, so $\Phi(G, x) \leq 1$). Thus it holds that

$$\mathbb{E}_{n-\bar{k}_n(T), p}\left[q^{N_{n-\bar{k}_n(T)}(n-\bar{k}_n(T))}\right] = \mathbb{E}_{n-\underline{k}_n(T), p}\left[q^{N_{n-\underline{k}_n(T)}(n-\underline{k}_n(T))}\Phi(G_{n-\underline{k}_n(T)}, \delta_n(T))\right],$$

giving, via Proposition 3.7 (and similarly to above),

$$r_n\left(T, \frac{1}{T}\right) = \mathbb{E}_{n-\underline{k}_n(T), p}\left[q^{s(H)-s(G)}\right], \quad (4.44)$$

in which $G = G_{n-\underline{k}_n(T)}$ and H is formed from G by deleting the $\delta_n(T)$ highest-labelled vertices.

Again, we may apply Jensen's inequality, to conclude that

$$\mathbb{E}_{n-\underline{k}_n(T), p}\left[q^{s(H)-s(G)}\right] \geq q^{-\mathbb{E}_{n-\underline{k}_n(T), p}[s(G)-s(H)]},$$

and, by Proposition 4.20 this time, we have that

$$\lim_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}_{n-k_n(T), p} [s(G) - s(H)] = 0.$$

As $q^{s(H)-s(G)} \leq 1$, we can use (4.44) to conclude, similarly to above, that

$$\lim_{T \rightarrow \infty} \liminf_{n \rightarrow \infty} r_n \left(T, \frac{1}{T} \right) = 1.$$

□

Chapter 5

Interlude: A scaling limit for $\mathcal{G}(n, m)$

Moving away from $\mathcal{G}(n, p)$ for a while—and from results of direct relevance to our later proofs connected to the random cluster model—we recall that the model $\mathcal{G}(n, m)$, where $n \in \mathbb{N}$ and $m \in \mathbb{Z} \cap [0, \binom{n}{2}]$, is the model in which a graph is selected uniformly at random from all graphs with n vertices and m edges. Here, we extend Theorem 1.5 and Theorem 2.25, which were results about $\mathcal{G}(n, p)$, by showing that near-identical results hold for $\mathcal{G}(n, m)$ also. To do this, we use some of the machinery established in the previous chapter.

5.1 Convergence of the exploration process

We first prove an analogue of Theorem 1.6. Analogously to the notation we defined in Section 3.1, we suppose that, under the measure $\mathbb{P}_{n,m}$, the graph $G_n \sim \mathcal{G}(n, m)$. Consider exploring G_n using Algorithm A (as defined on page 4).

We begin with a technical lemma.

Lemma 5.1. *Fix $t > 0$, and let $p = 1/n + \lambda n^{-4/3}$ for some $\lambda \in \mathbb{R}$. As $n \rightarrow \infty$,*

$$\mathbb{P}_{n,p}(|X_n(\lfloor n^{2/3}t \rfloor) + N_n(\lfloor n^{2/3}t \rfloor)| > n^{1/3} \log n) \rightarrow 0.$$

Proof. We start by observing that

$$\begin{aligned} & \mathbb{P}_{n,p}(|X_n(\lfloor n^{2/3}t \rfloor) + N_n(\lfloor n^{2/3}t \rfloor)| > n^{1/3} \log n) \\ & \leq \mathbb{P}_{n,p}\left(|n^{-1/3}X_n(\lfloor n^{2/3}t \rfloor)| > \frac{1}{2} \log n\right) + \mathbb{P}_{n,p}\left(N_n(\lfloor n^{2/3}t \rfloor) > \frac{1}{2}n^{1/3} \log n\right). \end{aligned} \quad (5.1)$$

For the first term, fix some $C > 0$. Then by Theorem 1.6,

$$\mathbb{P}_{n,p}\left(|n^{-1/3}X_n(\lfloor n^{2/3}t \rfloor)| > C\right) \rightarrow \mathbb{P}(|X(t)| > C).$$

But, for sufficiently large n ,

$$\mathbb{P}_{n,p}\left(\left|n^{-1/3}X_n(\lfloor n^{2/3}t \rfloor)\right| > \frac{1}{2}\log n\right) < \mathbb{P}_{n,p}\left(\left|n^{-1/3}X_n(\lfloor n^{2/3}t \rfloor)\right| > C\right).$$

Hence, for all C ,

$$0 \leq \limsup_{n \rightarrow \infty} \mathbb{P}_{n,p}\left(\left|n^{-1/3}X_n(\lfloor n^{2/3}t \rfloor)\right| > \frac{1}{2}\log n\right) < \mathbb{P}(|X(t)| > C).$$

But $X(t)$ is a.s. finite, so $\mathbb{P}(|X(t)| > C) \downarrow 0$ as $C \rightarrow \infty$. So the first term in (5.1) converges to 0.

For the second term, note by Markov's inequality that

$$\mathbb{P}_{n,p}\left(N_n(\lfloor n^{2/3}t \rfloor) > \frac{1}{2}n^{1/3}\log n\right) \leq \frac{\mathbb{E}[N_n(\lfloor n^{2/3}t \rfloor)]}{n^{1/3}(\log n)/2},$$

and use Lemma 4.14 to show that the right-hand side tends to 0. \square

Recall that, in Theorem 1.6, we had a pair of processes (X, N) such that $X(t) \stackrel{d}{=} B(t) + \lambda t - t^2/2$ where B is a Brownian motion; then, conditionally on X , $N(t)$ is an inhomogeneous Poisson process of rate $X(t) - \inf_{0 \leq s \leq t} X(s)$.

Theorem 5.2. *If the process (X_n, N_n) is the exploration process from $G_n \sim \mathcal{G}(n, m)$ in which*

$$m = \frac{n-1}{2}(1 + \lambda n^{-1/3} + O(n^{-2/3})),$$

then

$$\left(n^{-1/3}X_n(\lfloor tn^{2/3} \rfloor), N_n(\lfloor tn^{2/3} \rfloor)\right)_{t \geq 0} \xrightarrow{d} (X, N)$$

in the Skorokhod metric.

Remark. The $O(n^{-2/3})$ in the statement of the theorem above is not optimal; with more careful calculations, it is potentially possible to extend the result to $o(n^{-1/3})$, as in Theorems 1.5 and 1.6.

The proof of this theorem is partly due to Christina Goldschmidt (personal communication). In the proof we will need to invoke the following local limit theorem with an explicit error.

Lemma 5.3. *Let $m = (1 + \lambda n^{-1/3} + O(n^{-2/3}))(n-1)/2$. There is a universal constant C for all m'*

satisfying

$$|m' - m + \lfloor n^{2/3}t \rfloor| \leq n^{1/3} \log n,$$

and for all sufficiently large n , giving that, if

$$\tilde{M}_n(t) \sim \text{Bin}\left(\binom{n - \lfloor n^{2/3}t \rfloor}{2}, p\right),$$

then

$$\left| \mathbb{P}(\tilde{M}_n(t) = m') - \frac{1}{\sqrt{2\pi m}} \right| \leq C \frac{(\log n)^2}{n^{5/6}}. \quad (5.2)$$

The proof involves careful calculations similar to those we will use to prove (5.7) below; we defer it to Appendix A (p. 177).

Proof. We show that, as $n \rightarrow \infty$, for all $t > 0$ and for all continuous bounded real-valued functions f ,

$$\mathbb{E}_{n,m}[f(n^{-1/3}X_n(\lfloor n^{2/3}s \rfloor), N_n(\lfloor n^{2/3}s \rfloor)) : 0 \leq s \leq t] \rightarrow \mathbb{E}[f(X(s), N(s)) : 0 \leq s \leq t]. \quad (5.3)$$

Let $p = 1/n + \lambda n^{-4/3}$, so that $m = \binom{n}{2}p + O(n^{1/3})$. We know from Theorem 1.6 that the equivalent expression to (5.3) converges, namely that

$$\mathbb{E}_{n,p}[f(n^{-1/3}X_n(\lfloor n^{2/3}s \rfloor), N_n(\lfloor n^{2/3}s \rfloor)) : 0 \leq s \leq t] \rightarrow \mathbb{E}[f(X(s), N(s)) : 0 \leq s \leq t].$$

Pick a function f , and suppose that $|f| \leq C$ for some constant $C > 0$. Let M_n be the number of edges of G_n . This, of course, is identically equal to m under $\mathbb{P}_{n,m}$, but is random under $\mathbb{P}_{n,p}$ —specifically, $M_n \sim \text{Bin}\left(\binom{n}{2}, p\right)$. Then we have that

$$\begin{aligned} & \mathbb{E}_{n,m}[f(n^{-1/3}X_n(\lfloor n^{2/3}s \rfloor), N_n(\lfloor n^{2/3}s \rfloor)) : 0 \leq s \leq t] \\ &= \mathbb{E}_{n,p}[f(n^{-1/3}X_n(\lfloor n^{2/3}s \rfloor), N_n(\lfloor n^{2/3}s \rfloor)) : 0 \leq s \leq t \mid M_n = m] \\ &= \frac{\mathbb{E}_{n,p}[f(n^{-1/3}X_n(\lfloor n^{2/3}s \rfloor), N_n(\lfloor n^{2/3}s \rfloor)) : 0 \leq s \leq t] \mathbb{1}_{\{M_n = m\}}}{\mathbb{P}_{n,p}(M_n = m)}. \end{aligned}$$

Let

$$E_n = \{ |X_n(\lfloor n^{2/3}t \rfloor) + N_n(\lfloor n^{2/3}t \rfloor)| \leq n^{1/3} \log n \},$$

so that, by Lemma 5.1, E_n occurs with high probability. It is sufficient to show that

$$\frac{\mathbb{E}_{n,p}[f(n^{-1/3}X_n(\lfloor n^{2/3}s \rfloor), N_n(\lfloor n^{2/3}s \rfloor)) : 0 \leq s \leq t] \mathbb{1}_{\{M_n=m\}} \mathbb{1}_{E_n^c}]}{\mathbb{P}_{n,p}(M_n = m)} \rightarrow 0, \quad (5.4)$$

and that

$$\begin{aligned} \frac{\mathbb{E}_{n,p}[f(n^{-1/3}X_n(\lfloor n^{2/3}s \rfloor), N_n(\lfloor n^{2/3}s \rfloor)) : 0 \leq s \leq t] \mathbb{1}_{\{M_n=m\}} \mathbb{1}_{E_n}]}{\mathbb{P}_{n,p}(M_n = m)} \\ \rightarrow \mathbb{E}[f(X(s), N(s)) : 0 \leq s \leq t]. \end{aligned} \quad (5.5)$$

We will make use of a local limit theorem for M_n . We start with (5.4), and specifically its denominator: we note

$$d_{\text{TV}}(\text{Bin}(N, p), \text{Poi}(Np)) = p \left[\frac{1}{\sqrt{2\pi e}} + O\left(\min\left\{1, p + \frac{1}{\sqrt{Np}}\right\}\right) \right] \quad (5.6)$$

(see, e.g., [48, (39)], which cites Prokhorov [52]). We have $N = \binom{n}{2}$, and so the right-hand side is $O(n^{-1})$. Hence,

$$\left| \mathbb{P}\left(\text{Bin}\left(\binom{n}{2}, p\right) = m\right) - \mathbb{P}\left(\text{Poi}\left(\binom{n}{2}p\right) = m\right) \right| = O\left(\frac{1}{n}\right).$$

Meanwhile, let $a_n = \binom{n}{2}p - m = O(n^{1/3})$, so that, carrying out asymptotic calculations on the probability mass function for the Poisson distribution,

$$\mathbb{P}\left(\text{Poi}\left(\binom{n}{2}p\right) = m\right) = \frac{1}{\sqrt{2\pi m}} \left(1 + O\left(\frac{a_n^2}{m}\right)\right) \left(1 + O\left(\frac{1}{m}\right)\right) = \frac{1}{\sqrt{2\pi m}} + O(n^{-5/6}),$$

since $m = \Theta(n)$. Hence

$$\mathbb{P}_{n,p}(M_n = m) = \frac{1}{\sqrt{2\pi m}} + O(n^{-5/6}). \quad (5.7)$$

Having estimated the denominator, we move on to the numerator. Since f is bounded, it is then sufficient to show that

$$\mathbb{P}_{n,p}(\{M_n = m\} \cap E_n^c) = o(n^{-1/2})$$

in order to establish (5.4). By analysing Algorithm A, we see that, after step $\lfloor n^{2/3}t \rfloor$ of the process, we have discovered that there are exactly $X_n(\lfloor n^{2/3}t \rfloor) + \lfloor n^{2/3}t \rfloor$ tree edges, and exactly $N_n(\lfloor n^{2/3}t \rfloor)$ surplus edges, which are incident to at least one of the vertices $v_1, \dots, v_{\lfloor n^{2/3}t \rfloor}$. We know nothing about edges both of whose ends are later vertices in the exploration.

Let $(\mathcal{F}_k)_{k=0,\dots,n}$ be the natural filtration of $(X_n(k), N_n(k))_{k=0,\dots,n}$. Then we can write

$$\{M_n = m\} = \{\tilde{M}_n(t) = m - \lfloor n^{2/3}t \rfloor - X_n(\lfloor n^{2/3}t \rfloor) - N_n(\lfloor n^{2/3}t \rfloor)\},$$

where $\tilde{M}_n(t)$ is distributed as the number of edges in a copy of $\mathcal{G}(n - \lfloor n^{2/3}t \rfloor, p)$ that is independent of $\mathcal{F}_{\lfloor n^{2/3}t \rfloor}$, and so

$$\tilde{M}_n(t) \sim \text{Bin}\left(\binom{n - \lfloor n^{2/3}t \rfloor}{2}, p\right).$$

Thus,

$$\begin{aligned} & \mathbb{P}_{n,p}(\{M_n = m\} \cap E_n^c) \\ &= \mathbb{E}_{n,p} \left[\mathbb{P}(\tilde{M}_n(t) = m - \lfloor n^{2/3}t \rfloor - X_n(\lfloor n^{2/3}t \rfloor) - N_n(\lfloor n^{2/3}t \rfloor) \mid \mathcal{F}_{\lfloor n^{2/3}t \rfloor}) \mathbf{1}_{E_n^c} \right] \\ &\leq \mathbb{P}_{n,p}(E_n^c) \sup_k \mathbb{P}\left(\text{Bin}\left(\binom{n - \lfloor n^{2/3}t \rfloor}{2}, p\right) = k\right). \end{aligned}$$

The supremum here is attained by the probability that the binomial random variable is equal to its mode. So

$$\begin{aligned} \sup_k \mathbb{P}\left(\text{Bin}\left(\binom{n - \lfloor n^{2/3}t \rfloor}{2}, p\right) = k\right) &= \mathbb{P}\left(\text{Bin}\left(\binom{n - \lfloor n^{2/3}t \rfloor}{2}, p\right) = \left\lfloor \binom{n - \lfloor n^{2/3}t \rfloor}{2} p \right\rfloor\right), \\ &= O(n^{-1/2}), \end{aligned}$$

by a similar argument to that used above to prove (5.7). Meanwhile, by Lemma 5.1, $\mathbb{P}(E_n^c) \rightarrow 0$ as $n \rightarrow \infty$. So

$$\mathbb{P}_{n,p}(\{M_n = m\} \cap E_n^c) = o\left(\frac{1}{n^{1/2}}\right),$$

and thus we have established (5.4).

We move on to (5.5). For brevity, write

$$F_{n,t} = f(n^{-1/3}X_n(\lfloor n^{2/3}s \rfloor), N_n(\lfloor n^{2/3}s \rfloor)) : 0 \leq s \leq t.$$

Then, we can write

$$\begin{aligned}
& \mathbb{E}_{n,p} \left[f \left(n^{-1/3} X_n(\lfloor n^{2/3} s \rfloor), N_n(\lfloor n^{2/3} s \rfloor) : 0 \leq s \leq t \right) \mathbb{1}_{\{M_n=m\}} \mathbb{1}_{E_n} \right] \\
&= \mathbb{E}_{n,p} \left[F_{n,t} \mathbb{1}_{\{\tilde{M}_n(t)=m-\lfloor n^{2/3} t \rfloor - X_n(\lfloor n^{2/3} t \rfloor) - N_n(\lfloor n^{2/3} t \rfloor)\}} \mathbb{1}_{E_n} \right] \\
&= \mathbb{E}_{n,p} \left[F_{n,t} \mathbb{P} \left(\tilde{M}_n(t) = m - \lfloor n^{2/3} t \rfloor - X_n(\lfloor n^{2/3} t \rfloor) - N_n(\lfloor n^{2/3} t \rfloor) \mid \mathcal{F}_{\lfloor n^{2/3} t \rfloor} \right) \mathbb{1}_{E_n} \right],
\end{aligned}$$

where again $\tilde{M}_n(t)$ is distributed as the number of edges in a copy of $\mathcal{G}(n - \lfloor n^{2/3} t \rfloor, p)$ that is independent of the pre- $\lfloor n^{2/3} t \rfloor$ process, and so

$$\tilde{M}_n(t) \sim \text{Bin} \left(\binom{n - \lfloor n^{2/3} t \rfloor}{2}, p \right).$$

Now let $n' = n - \lfloor n^{2/3} t \rfloor$. Fix some $m' \in \mathbb{Z}^{\geq 0}$, where

$$|m' - m + \lfloor n^{2/3} t \rfloor| \leq n^{1/3} \log n. \quad (5.8)$$

(We note that the lower bound on m' tends to infinity as n does, and so in particular, if n is sufficiently large, (5.8) will crudely constrain $m' \geq 1$.) By Lemma 5.3 above, there is a universal constant C for all such m' , and for all sufficiently large n , giving that

$$\left| \mathbb{P}(\tilde{M}_n(t) = m') - \frac{1}{\sqrt{2\pi m}} \right| \leq C \frac{(\log n)^2}{n^{5/6}}.$$

(We allow C to depend on t .) From this, we can write that

$$\begin{aligned}
& \mathbb{E}_{n,p} \left[F_{n,t} \mathbb{P} \left(\tilde{M}_n(t) = m - \lfloor n^{2/3} t \rfloor - X_n(\lfloor n^{2/3} t \rfloor) - N_n(\lfloor n^{2/3} t \rfloor) \mid \mathcal{F}_{\lfloor n^{2/3} t \rfloor} \right) \mathbb{1}_{E_n} \right] \\
&= \mathbb{E}_{n,p} \left[F_{n,t} \left(\frac{1}{\sqrt{2\pi m}} + O \left(\frac{(\log n)^2}{n^{5/6}} \right) \right) \mathbb{1}_{E_n} \right] \\
&= \frac{1}{\sqrt{2\pi m}} \mathbb{E}_{n,p} [F_{n,t} \mathbb{1}_{E_n}] + O \left(\frac{(\log n)^2}{n^{5/6}} \right)
\end{aligned}$$

using the fact that f is bounded. From this and from (5.7), we have that

$$\begin{aligned}
& \frac{\mathbb{E}_{n,p} \left[f \left(n^{-1/3} X_n(\lfloor n^{2/3} s \rfloor), N_n(\lfloor n^{2/3} s \rfloor) : 0 \leq s \leq t \right) \mathbb{1}_{\{M_n=m\}} \mathbb{1}_{E_n} \right]}{\mathbb{P}_{n,p}(M_n = m)} \\
&= \frac{(2\pi m)^{-1/2} \mathbb{E}_{n,p} [F_{n,t} \mathbb{1}_{E_n}] + O(n^{-5/6} (\log n)^2)}{(2\pi m)^{-1/2} + O(n^{-5/6})} \\
&= \frac{\mathbb{E}_{n,p} [F_{n,t} \mathbb{1}_{E_n}] + o(1)}{1 + o(1)} \rightarrow \mathbb{E}[f(X(s), N(s) : 0 \leq s \leq t)]
\end{aligned}$$

by Theorem 1.6, using the fact that E_n is $\mathcal{F}_{\lfloor n^{2/3}t \rfloor}$ -measurable, and (by Lemma 5.1) has probability tending to 1 as $n \rightarrow \infty$ (and also, again, the fact that f is bounded and continuous). This establishes (5.5), and completes the proof. \square

5.2 Component sizes and a scaling limit

We want to establish an analogue of Theorem 1.5 for $\mathcal{G}(n, m)$. Let \mathbf{C}_n be the sequence of component sizes, and \mathbf{S}_n the number of surplus edges, in $G_n \sim \mathcal{G}(n, m)$, and let \mathbf{C} and \mathbf{S} be the objects defined in Chapter 1—that is, \mathbf{C} is the decreasing sequence of excursion lengths of $X - I$, and the i -th entry of \mathbf{S} is the number of points of a rate 1 Poisson process that occur under the excursion whose length was the i -th entry of \mathbf{C} .

Theorem 5.4. *Fix $\lambda \in \mathbb{R}$, and, for each $n \in \mathbb{N}$, set*

$$m = m(n) = \frac{n-1}{2} (1 + \lambda n^{-1/3} + O(n^{-2/3})).$$

As $n \rightarrow \infty$,

$$(n^{-2/3} \mathbf{C}_n, \mathbf{S}_n) \xrightarrow{d} (\mathbf{C}, \mathbf{S}), \quad (5.9)$$

with convergence in the l^2 norm for the first coordinate, and the product topology for the second.

So we need to convert from convergence of the exploration process to l^2 convergence of the component sizes, and to product convergence of the number of surplus edges.

We first show that the component sizes converge when treated as a point process, analogously to [6, Lma. 8] of Aldous. For the i -th largest component $C_i^{(n)}$ of $G_n \sim \mathcal{G}(n, m)$, let $L(C_i^{(n)}) = k - 1$ if the first vertex of $C_i^{(n)}$ to be explored is chosen as v_k . Let

$$\Theta_n = \left\{ \left(n^{-2/3} |C_i^{(n)}|, n^{-1/3} L(C_i^{(n)}) \right) : i \geq 1 \right\}. \quad (5.10)$$

Similarly, define the set of pairs

$$\Theta = \{ (|\mathbf{e}_i|, L(\mathbf{e}_i)) : i \geq 1 \},$$

where \mathbf{e}_i is the i -th-longest excursion of $X - I$, where $|\mathbf{e}_i|$ is its length, and where $L(\mathbf{e}_i)$ is its left end-point. (We know this is well-defined—i.e. that there is a largest excursion, a second-largest, etc.—because it is the same limit object as for $\mathcal{G}(n, p)$.) Recall the definition of the vague topology from Definition 2.26.

Lemma 5.5. As $n \rightarrow \infty$,

$$\Theta_n \xrightarrow{d} \Theta$$

with convergence in the vague topology.

We omit the proof; it follows from Theorem 5.2 and [6, Lma. 7], in much the same way that Aldous proves [6, Lma. 8].

To obtain the l^2 convergence part of Theorem 5.4, we will need to use [6, Propn. 15], which, for reference, we quote here. To set up some notation (borrowed from [6], with some minor tweaks), for a countable index set I , let $l_+^2(I)$ be the set of sequences $\mathbf{x} = (x_\gamma : \gamma \in I)$ such that each $x_\gamma \geq 0$ and $\sum_\gamma x_\gamma^2 < \infty$. We also need to define the concept of a *size-biased random ordering*, which we do now.

Definition 5.6 ([6, § 3.3]). Let $\mathbf{Y} = \{Y_\gamma : \gamma \in I\}$ be a set indexed by some countable index set I , with $\mathbf{Y} \in l_+^2(I)$. For each $\gamma \in I$, define E_γ to be a random variable, such that, conditional on \mathbf{Y} , we have $E_\gamma \sim \text{Exp}(Y_\gamma)$, with $(E_\gamma)_{\gamma \in I}$ being (conditionally) independent.

Then, for each $a \in [0, \infty)$, define

$$S(a) = \sum_{\gamma: E_\gamma < a} Y_\gamma < \infty$$

a.s. (by the assumption that $\mathbf{Y} \in l_+^2(I)$). Then define $S_\gamma = S(E_\gamma)$. Then the *size-biased point process* or *SBPP* associated with \mathbf{Y} is the set $\Theta = \{(Y_\gamma, S_\gamma) : \gamma \in I\}$.

Given a size-biased point process $\Theta = \{(y_\gamma, s_\gamma) : \gamma \in I\}$, write

$$\pi(\Theta) = \{y_\gamma : \gamma \in I\}.$$

Finally (as far as our notation is concerned), if \mathbf{Y} is a sequence indexed by I such that, for all k , there is a k -th-largest entry in \mathbf{Y} , let $\text{ord } \mathbf{Y}$ be that sequence rearranged into descending order; otherwise, it is not well-defined.

Proposition 5.7 ([6, Propn. 15]). Let I_1, I_2, \dots and I_∞ be index sets. Let $\mathbf{Y}_n \in l_+^2(I_n)$ for each $n \in \mathbb{N} \cup \{\infty\}$, and let Θ_n be the associated size-biased point process. Suppose $\Theta_n \xrightarrow{d} \Theta_\infty$ as $n \rightarrow \infty$, where Θ_∞ is a point process satisfying the following three conditions:

(1) If $(y, s) \in \Theta_\infty$, then

$$\sum_{\substack{y' : (y', s') \in \Theta_\infty, \\ s' < s}} y' = s;$$

(2) As $s_0 \rightarrow \infty$,

$$\max \{ y : (y, s) \in \Theta_\infty \text{ for some } s > s_0 \} \xrightarrow{p} 0;$$

(3) Almost surely,

$$\sup \{ s : (y, s) \in \Theta_\infty \text{ for some } y \} = \infty.$$

Then $\mathbf{Y}_\infty = \text{ord } \pi(\Theta_\infty)$ is well-defined and in l^2 , and $\text{ord } \mathbf{Y}_n \xrightarrow{d} \text{ord } \mathbf{Y}_\infty$ in l^2 .

Note that this result only imposes conditions on the candidate limit object. But the candidate limit in the $\mathcal{G}(n, m)$ case is identical to that for $\mathcal{G}(n, p)$, so conditions (1)–(3) hold for it, and the required vague convergence holds by Lemma 5.5. Thus the proof of Theorem 5.4 follows in an almost identical way to the proof by Aldous [6, § 3.3 and § 3.4] of the analogous result for $\mathcal{G}(n, p)$, and we consequently omit it.

Now we just need to show that a metric space scaling limit exists. We show this in the product topology, rather than convergence in the metric d_{seq} defined by (2.9), though it is potentially possible to extend the result.

Theorem 5.8. *Let $\mathbf{M}^{(n)} = (M_1^{(n)}, M_2^{(n)}, \dots)$ be a sequence of metric spaces such that $M_i^{(n)}$ is the i -th largest component of $G_n \sim \mathcal{G}(n, m)$, treated as a metric space with the graph distance, where*

$$m = m(n) = \frac{n-1}{2} (1 + \lambda n^{-1/3} + O(n^{-2/3})).$$

Let $Z_i^{(n)} = |M_i^{(n)}|$, and let $\mathbf{Z}^{(n)} = (Z_1^{(n)}, Z_2^{(n)}, \dots)$. Then there exist \mathbf{Z}, \mathbf{M} such that

$$\left(n^{-\frac{2}{3}} \mathbf{Z}^{(n)}, n^{-\frac{1}{3}} \mathbf{M}^{(n)} \right) \xrightarrow{d} (\mathbf{Z}, \mathbf{M}),$$

with convergence in the product topology, using d_{GH} , the Gromov–Hausdorff distance, to measure the distance between (isometry classes of) metric spaces. Moreover, the law of \mathbf{Z} is given by Theorem 5.4, while $\mathbf{M} = (M_1, M_2, \dots)$ is such that, conditional on \mathbf{Z} , M_1, M_2, \dots are independent and $M_i \stackrel{d}{=} \mathcal{M}^{(Z_i)}$.

Remark. So the scaling limit is the same as that for $\mathcal{G}(n, p)$ with appropriately chosen p .

Proof. We note firstly that, conditionally on being connected and on having a given number of surplus edges, a copy of $\mathcal{G}(n, m)$ is identically distributed to a copy of $\mathcal{G}(n, p)$ (they are both uniform picks from all the possible graphs). Informally, this means that, conditional on a sequence of component sizes and surplus edge counts, the two graphs are identically distributed.

Let $\mathbf{S}^{(n)}$ be the sequence of surplus edge counts for $G_n \sim \mathcal{G}(n, m)$ (so that $S_i^{(n)}$ is the number of surplus edges in $M_i^{(n)}$), and let \mathbf{S} be the corresponding limit sequence implied by Theorem 5.4 (with i -th entry S_i). Then that theorem tells us that

$$(n^{-2/3}\mathbf{Z}^{(n)}, \mathbf{S}^{(n)}) \xrightarrow{d} (\mathbf{Z}, \mathbf{S})$$

in the product topology. Exploiting the conditional equality in distribution above, it is then implicit from [2, p. 748] that the corresponding sequences of metric spaces $\mathbf{M}^{(n)}$ converge in the product topology (and jointly with $\mathbf{Z}^{(n)}$ and $\mathbf{S}^{(n)}$); the components are formed using the same function g given by Definition 2.22, but with different arguments to the function. Specifically, conditionally on \mathbf{Z} and \mathbf{S} , the i -th entry in \mathbf{M} , labelled M_i , is distributed as $g(2\mathbf{e}', \mathcal{S})$, where

- \mathbf{e}' is distributed such that, for any appropriate function f ,

$$\mathbb{E}[f(\mathbf{e}')] = \frac{\mathbb{E}\left[f(\mathbf{e}^{(Z_i)})\left(\int_0^{Z_i} e^{(Z_i)}\right)^{S_i}\right]}{\mathbb{E}\left[\left(\int_0^{Z_i} e^{(Z_i)}\right)^{S_i}\right]},$$

where $\mathbf{e}^{(\sigma)}$ is a Brownian excursion of length σ ;

- conditionally on \mathbf{e}' , we take \mathcal{S} to be a set of S_i points uniformly chosen from the area under \mathbf{e}' .

Thus we have that, in the product topology,

$$(n^{-2/3}\mathbf{Z}^{(n)}, \mathbf{S}^{(n)}, n^{-1/3}\mathbf{M}^{(n)}) \xrightarrow{d} (\mathbf{Z}, \mathbf{S}, \mathbf{M}). \quad (5.11)$$

But since the object \mathbf{M} described above was simply the scaling limit for $\mathcal{G}(n, p)$ conditional on \mathbf{Z} and \mathbf{S} , and since \mathbf{Z} and \mathbf{S} are the same objects that appear in the limit of $\mathcal{G}(n, p)$ (in both cases for $p = 1/n + \lambda n^{-4/3}$), it follows that \mathbf{M} must be the scaling limit of $\mathcal{G}(n, p)$ —and so, conditioned only on \mathbf{Z} , it must have the distribution asserted. So, dropping $\mathbf{S}^{(n)}$ and \mathbf{S} from (5.11), we have the result claimed. \square

Remark. To convert this to use d_{seq} we would need a bound on the heights of the depth-first search trees of $\mathcal{G}(n, m)$, along the lines of [3, Lma. 25].

Part II

Results for the random cluster model

Chapter 6

$q < 1$: Handling the discrete exploration process

In this chapter, we make progress towards showing that the exploration process for the random cluster model converges, at least in the case where $q < 1$.

We return to the heuristic proof given in Section 3.2, noting that we gave two problems with it that remained to be resolved. In Section 6.1, we show that, using the comparison established in Section 4.2, we may show that the exploration process for the critical random cluster model with $q < 1$ is well-behaved for large times, in that it rapidly reaches 0 without any surplus edges being added in the intervening time; effectively, this resolves Problem 1 from Section 3.2 for $q < 1$. We use this, in Section 6.2, to give a proof that the exploration process converges, conditionally on also resolving Problem 2. At the end of that section, we show that this is sufficient in order to show that the rescaled sequence of component sizes converges in l^2 .

In Section 6.3, we discuss possible approaches to resolving Problem 1 in the case where $q > 1$. In the next chapter, we will see that, for $q > 1$, it is Problem 2 that is easier to resolve.

6.1 Approaching zero, revisited

We now give an extension of Proposition 4.7 to the critical random cluster model for $q < 1$. We let (X, I, N) be an exploration of G according to Algorithm A under $\mathbb{P}_{n,p,q}$.

Proposition 6.1. *Fix $q \in (0, 1)$ and $\lambda \in \mathbb{R}$, and define $p = p(n, \lambda, q) = q/n + \lambda n^{-4/3}$. As before,*

let the event $E_n(T, \varepsilon)$ be defined by

$$E_n(T, \varepsilon) = \{X_n(k) = I_n(k) \text{ for some } k \in [\lfloor Tn^{2/3} \rfloor, \lfloor (T + \varepsilon)n^{2/3} \rfloor]\} \\ \cap \{N_n(\lfloor (T + \varepsilon)n^{2/3} \rfloor) = N_n(\lfloor Tn^{2/3} \rfloor)\}.$$

Then

$$\lim_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}_{n,p,q} \left(E_n \left(T, \frac{1}{T} \right)^c \right) = 0.$$

Remark. This time we do—at this point—require the lim sup in n rather than a limit, since we have not yet established the existence of any sort of limit for $\mathcal{G}(n, p, q)$.

Proof. We prove the claim for (X_n, I_n, N_n) where this is an exploration of $G_n \sim \mathcal{G}(n, p, q)$ according to Algorithm A, but modified to select the new vertex uniformly at random from the stack; as there is an equality in distribution, this establishes the result for the unmodified Algorithm A too.

Define

$$p' = \frac{1}{n} + \frac{\max\{\lambda, 0\}}{qn^{4/3}};$$

then, take the measure \mathbb{P} whose existence was established by Proposition 4.5, in which a copy of the exploration for $\mathcal{G}(n, p, q)$ —which we label (X_n, I_n, N_n) —is coupled to an exploration (X'_n, I'_n, N'_n) for $\mathcal{G}(n, p')$, with, for all i ,

$$X_n(i) - I_n(i) \leq X'_n(i) - I'_n(i)$$

and

$$N_n(i) - N_n(i-1) \leq N'_n(i) - N'_n(i-1).$$

It is then clear that, if $X'_n(k) = I'_n(k)$ for some $k \in [\lfloor Tn^{2/3} \rfloor, \lfloor (T + \varepsilon)n^{2/3} \rfloor]$, then, for that k , we have $X_n(k) = I_n(k)$. Similarly,

$$0 \leq N_n(\lfloor (T + \varepsilon)n^{2/3} \rfloor) - N_n(\lfloor Tn^{2/3} \rfloor) \leq N'_n(\lfloor (T + \varepsilon)n^{2/3} \rfloor) - N'_n(\lfloor Tn^{2/3} \rfloor),$$

so if $N'_n(\lfloor (T + \varepsilon)n^{2/3} \rfloor) = N'_n(\lfloor Tn^{2/3} \rfloor)$ then $N_n(\lfloor (T + \varepsilon)n^{2/3} \rfloor) = N_n(\lfloor Tn^{2/3} \rfloor)$. Thus, applying

this coupling we have that

$$\mathbb{P}_{n,p,q}(E_n(T, \varepsilon)^c) \leq \mathbb{P}_{n,p'}(E_n(T, \varepsilon)^c).$$

This means that

$$0 \leq \limsup_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}_{n,p,q}(E_n(T, 1/T)^c) \leq \limsup_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}_{n,1/n}(E_n(T, 1/T)^c).$$

By Proposition 4.7, the right-hand side is equal to 0. Hence

$$\limsup_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}_{n,q/n,q}(E_n(T, 1/T)^c) = 0,$$

establishing the claim. □

This, combined with the results in Section 4.4, is essentially enough to address Problem 1 in Section 3.2 in the case where $q < 1$. We show how in the next section.

6.2 A limit for the exploration process

We now put together a partial proof of the existence, and form, of the limit of the exploration process, in the case where $q < 1$. This proof follows the strategy outlined in Section 3.2, but this time we make sure our arguments are rigorous. We first state a conjecture for the missing part.

Conjecture 6.2. Fix $\lambda \in \mathbb{R}$, and define (X, I, N) to be real-valued processes on $[0, \infty)$ with distributions given by:

$$X(t) = B(t) + \lambda t - \frac{t^2}{2};$$

$$I(t) = \inf_{0 \leq s \leq t} X(s);$$

$(N(t))_{t \geq 0}$ is a Poisson process of rate $X(t) - I(t)$

(where $B(t)$ is a Brownian motion). Then, for $q \in (0, 1)$, there exists a function $\mathfrak{D}_q(T) > 0$ such that the random process $(R_q(T))_{T \geq 0}$, where

$$R_q(T) := \frac{1}{\mathfrak{D}_q(T)} \exp \left\{ (q-1) \int_0^T (X(u) - I(u)) du \right\}, \quad (6.1)$$

is uniformly integrable, and converges in probability as $T \rightarrow \infty$ to a random variable that is almost surely positive.

Remark. For $q \in (1, 2)$ we can prove this, as we prove in Chapter 7. Indeed, our proofs will imply that we can find $\mathfrak{D}_q(T)$ for $q \in (0, 1)$ such that the asserted convergence in probability to a non-zero random variable takes place, but the uniform integrability in this setting has proven elusive. Nevertheless, this conjecture seems plausible.

Theorem 6.3. Fix $q \in (0, 1)$. Let f be a bounded function, and let λ and (X, I, N) be defined as in Conjecture 6.2. Then, if that conjecture holds,

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}_{n,p,q} [f(n^{-1/3}X_n(\lfloor sn^{2/3} \rfloor), n^{-1/3}I_n(\lfloor sn^{2/3} \rfloor) : 0 \leq s \leq t)] \\ = \frac{\mathbb{E}[R_q f(X(s), I(s) : 0 \leq s \leq t)]}{\mathbb{E}[R_q]}, \end{aligned}$$

for some almost surely positive random variable R_q .

Proof. First, for $T, \varepsilon > 0$, define the event $E'_n(T, \varepsilon)$ by

$$\begin{aligned} E'_n(T, \varepsilon) \\ = \{ \text{for some } k \in [\lfloor Tn^{2/3} \rfloor, \lfloor (T + \varepsilon)n^{2/3} \rfloor], X_n(k) = I_n(k) \text{ and } N_n(k) = N_n(\lfloor Tn^{2/3} \rfloor) \}. \end{aligned}$$

Then, immediately, $E'_n(T, \varepsilon) \supseteq E_n(T, \varepsilon)$ as defined in Proposition 6.1. So, by that result, it holds that

$$\lim_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}_{n,p,q} \left(E'_n \left(T, \frac{1}{T} \right)^c \right) = 0.$$

In turn, we have that

$$\begin{aligned} \mathbb{E}_{n,p,q} [f(n^{-1/3}X_n(\lfloor sn^{2/3} \rfloor), n^{-1/3}I_n(\lfloor sn^{2/3} \rfloor) : 0 \leq s \leq t)] \\ = \mathbb{E}_{n,p,q} [f(n^{-1/3}X_n(\lfloor sn^{2/3} \rfloor), n^{-1/3}I_n(\lfloor sn^{2/3} \rfloor) : 0 \leq s \leq t) \mathbb{1}_{E'_n(T, 1/T)}] \\ + \mathbb{E}_{n,p,q} [f(n^{-1/3}X_n(\lfloor sn^{2/3} \rfloor), n^{-1/3}I_n(\lfloor sn^{2/3} \rfloor) : 0 \leq s \leq t) \mathbb{1}_{E'_n(T, 1/T)^c}]. \end{aligned} \quad (6.2)$$

Since f is a bounded function, we have by Proposition 6.1 that

$$\lim_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}_{n,p,q} [f(n^{-1/3}X_n(\lfloor sn^{2/3} \rfloor), n^{-1/3}I_n(\lfloor sn^{2/3} \rfloor) : 0 \leq s \leq t) \mathbb{1}_{E'_n(T, 1/T)^c}] = 0. \quad (6.3)$$

We now consider the other term. By Proposition 3.7, we have that

$$\begin{aligned}
& \mathbb{E}_{n,p,q} \left[f \left(n^{-1/3} X_n(\lfloor sn^{2/3} \rfloor), n^{-1/3} I_n(\lfloor sn^{2/3} \rfloor) : 0 \leq s \leq t \right) \mathbb{1}_{E'_n(T, 1/T)} \right] \\
&= \frac{\mathbb{E}_{n,\bar{p}} \left[q^{N_n(n)} f \left(n^{-1/3} X_n(\lfloor sn^{2/3} \rfloor), n^{-1/3} I_n(\lfloor sn^{2/3} \rfloor) : 0 \leq s \leq t \right) \mathbb{1}_{E'_n(T, 1/T)} \right]}{\mathbb{E}_{n,\bar{p}} \left[q^{N_n(n)} \right]} \\
&= \frac{\mathbb{E}_{n,\bar{p}} \left[q^{N_n(n)} f \left(n^{-1/3} X_n(\lfloor sn^{2/3} \rfloor), n^{-1/3} I_n(\lfloor sn^{2/3} \rfloor) : 0 \leq s \leq t \right) \mathbb{1}_{E'_n(T, 1/T)} \right]}{\mathbb{E}_{n,\bar{p}} \left[q^{N_n(n)} \mathbb{1}_{E'_n(T, 1/T)} \right]} p_n \left(T, \frac{1}{T} \right),
\end{aligned} \tag{6.4}$$

where

$$\begin{aligned}
p_n \left(T, \frac{1}{T} \right) &= \frac{\mathbb{E}_{n,\bar{p}} \left[q^{N_n(n)} \mathbb{1}_{E'_n(T, 1/T)} \right]}{\mathbb{E}_{n,\bar{p}} \left[q^{N_n(n)} \right]} \\
&= \mathbb{P}_{n,p,q} \left(E'_n(T, 1/T) \right)
\end{aligned}$$

by another application of Proposition 3.7. By Proposition 6.1, we then have that

$$\lim_{T \rightarrow \infty} \liminf_{n \rightarrow \infty} p_n \left(T, \frac{1}{T} \right) = 1, \tag{6.5}$$

and clearly $p_n(T, 1/T) \leq 1$.

Now consider the numerator of the fraction in (6.4). For brevity, we will write

$$F_{n,t} = f \left(n^{-1/3} X_n(\lfloor sn^{2/3} \rfloor), n^{-1/3} I_n(\lfloor sn^{2/3} \rfloor) : 0 \leq s \leq t \right).$$

We will partition the event $E'_n(T, 1/T)$ as follows. Define the events

$$Z_n(T, k) = \left\{ \inf \{ i \geq \lfloor Tn^{2/3} \rfloor : X_n(i) - I_n(i) = 0 \} = k \right\} \tag{6.6}$$

and

$$M_n(T, k) = \left\{ N_n(k) - N_n(\lfloor Tn^{2/3} \rfloor) = 0 \right\}. \tag{6.7}$$

Then

$$E'_n \left(T, \frac{1}{T} \right) = \bigsqcup_{k=\lfloor Tn^{2/3} \rfloor}^{\lfloor (T+1/T)n^{2/3} \rfloor} (Z_n(T, k) \cap M_n(T, k)) = \bigsqcup_{k=\underline{k}_n(T)}^{\bar{k}_n(T)} (Z_n(T, k) \cap M_n(T, k)), \tag{6.8}$$

letting \sqcup denote a disjoint union, and recalling the meaning of $\underline{k}_n(T)$ and $\bar{k}_n(T)$ from Definition 4.19; thus, we may write

$$\begin{aligned}\mathbb{E}_{n,\bar{p}}\left[q^{N_n(n)}F_{n,t}\mathbb{1}_{E'_n(T,1/T)}\right] &= \sum_{k=\underline{k}_n(T)}^{\bar{k}_n(T)} \mathbb{E}_{n,\bar{p}}\left[q^{N_n(n)}F_{n,t}\mathbb{1}_{Z_n(T,k)}\mathbb{1}_{M_n(T,k)}\right] \\ &= \sum_{k=\underline{k}_n(T)}^{\bar{k}_n(T)} \mathbb{E}_{n,\bar{p}}\left[q^{N_n(n)-N_n(k)}q^{N_n(\lfloor Tn^{2/3} \rfloor)}F_{n,t}\mathbb{1}_{Z_n(T,k)}\mathbb{1}_{M_n(T,k)}\right],\end{aligned}\tag{6.9}$$

where we have used the fact that, for each k , we are working on the event $M_n(T, k)$.

Now, by the Markov property for $Y_n = X_n - I_n$ (Proposition 4.3), for each k in the sum, conditionally on $Z_n(T, k)$, we have that

$$q^{N_n(\lfloor Tn^{2/3} \rfloor)}F_{n,t}\mathbb{1}_{M_n(T,k)}$$

is independent of

$$q^{N_n(n)-N_n(k)};$$

thus, for each k , we have that

$$\begin{aligned}\mathbb{E}_{n,\bar{p}}\left[q^{N_n(n)-N_n(k)}q^{N_n(\lfloor Tn^{2/3} \rfloor)}F_{n,t}\mathbb{1}_{Z_n(T,k)}\mathbb{1}_{M_n(T,k)}\right] \\ = \mathbb{E}_{n,\bar{p}}\left[q^{N_n(\lfloor Tn^{2/3} \rfloor)}F_{n,t}\mathbb{1}_{Z_n(T,k)}\mathbb{1}_{M_n(T,k)}\right]\mathbb{E}_{n,\bar{p}}\left[q^{N_n(n)-N_n(k)} \mid Z_n(T, k)\right].\end{aligned}$$

Recall that, in (4.38)–(4.40), we defined:

$$\begin{aligned}s_n\left(T, \frac{1}{T}\right) &= \sup_{\underline{k}_n(T) \leq k \leq \bar{k}_n(T)} \mathbb{E}_{n,\bar{p}}\left[q^{N_n(n)-N_n(k)} \mid Z_n(T, k)\right]; \\ m_n\left(T, \frac{1}{T}\right) &= \inf_{\underline{k}_n(T) \leq k \leq \bar{k}_n(T)} \mathbb{E}_{n,\bar{p}}\left[q^{N_n(n)-N_n(k)} \mid Z_n(T, k)\right]; \\ r_n\left(T, \frac{1}{T}\right) &= \frac{m_n(T, 1/T)}{s_n(T, 1/T)}.\end{aligned}$$

With these definitions, for each k ,

$$\begin{aligned}\mathbb{E}_{n,\bar{p}}\left[q^{N_n(\lfloor Tn^{2/3} \rfloor)}F_{n,t}\mathbb{1}_{Z_n(T,k)}\mathbb{1}_{M_n(T,k)}\right]m_n\left(T, \frac{1}{T}\right) \\ \leq \mathbb{E}_{n,\bar{p}}\left[q^{N_n(n)-N_n(k)}q^{N_n(\lfloor Tn^{2/3} \rfloor)}F_{n,t}\mathbb{1}_{Z_n(T,k)}\mathbb{1}_{M_n(T,k)}\right] \\ \leq \mathbb{E}_{n,\bar{p}}\left[q^{N_n(\lfloor Tn^{2/3} \rfloor)}F_{n,t}\mathbb{1}_{Z_n(T,k)}\mathbb{1}_{M_n(T,k)}\right]s_n\left(T, \frac{1}{T}\right),\end{aligned}$$

which means, summing back over k and using (6.8) and (6.9), that

$$\begin{aligned} & \mathbb{E}_{n,\bar{p}}\left[q^{N_n(\lfloor Tn^{2/3}\rfloor)}F_{n,t}\mathbb{1}_{E'_n(T,1/T)}\right]m_n\left(T,\frac{1}{T}\right) \\ & \leq \mathbb{E}_{n,\bar{p}}\left[q^{N_n(n)}F_{n,t}\mathbb{1}_{E'_n(T,1/T)}\right] \\ & \leq \mathbb{E}_{n,\bar{p}}\left[q^{N_n(\lfloor Tn^{2/3}\rfloor)}F_{n,t}\mathbb{1}_{E'_n(T,1/T)}\right]s_n\left(T,\frac{1}{T}\right). \end{aligned}$$

For the denominator of the fraction in (6.4), we can obtain analogous bounds by simply setting $f \equiv 1$. Substituting both sets of bounds into (6.4), we obtain

$$\begin{aligned} & \frac{\mathbb{E}_{n,\bar{p}}\left[q^{N_n(\lfloor Tn^{2/3}\rfloor)}F_{n,t}\mathbb{1}_{E'_n(T,1/T)}\right]}{\mathbb{E}_{n,\bar{p}}\left[q^{N_n(\lfloor Tn^{2/3}\rfloor)}\mathbb{1}_{E'_n(T,1/T)}\right]}P_n\left(T,\frac{1}{T}\right)r_n\left(T,\frac{1}{T}\right) \\ & \leq \mathbb{E}_{n,p,q}\left[F_{n,t}\mathbb{1}_{E'_n(T,1/T)}\right] \tag{6.10} \\ & \leq \frac{\mathbb{E}_{n,\bar{p}}\left[q^{N_n(\lfloor Tn^{2/3}\rfloor)}F_{n,t}\mathbb{1}_{E'_n(T,1/T)}\right]P_n(T,1/T)}{\mathbb{E}_{n,\bar{p}}\left[q^{N_n(\lfloor Tn^{2/3}\rfloor)}\mathbb{1}_{E'_n(T,1/T)}\right]r_n(T,1/T)}. \end{aligned}$$

By Lemma 4.24, we have that

$$\lim_{T \rightarrow \infty} \liminf_{n \rightarrow \infty} r_n\left(T,\frac{1}{T}\right) = 1,$$

and clearly $r_n(T, 1/T) \leq 1$. It remains to show that the other term, which expands out to

$$\frac{\mathbb{E}_{n,\bar{p}}\left[q^{N_n(\lfloor Tn^{2/3}\rfloor)}f\left(n^{-1/3}X_n(\lfloor sn^{2/3}\rfloor), n^{-1/3}I_n(\lfloor sn^{2/3}\rfloor) : 0 \leq s \leq t\right)\mathbb{1}_{E'_n(T,1/T)}\right]}{\mathbb{E}_{n,\bar{p}}\left[q^{N_n(\lfloor Tn^{2/3}\rfloor)}\mathbb{1}_{E'_n(T,1/T)}\right]}, \tag{6.11}$$

behaves appropriately. Since $q < 1$, we have a bounded function of

$$\left(n^{-1/3}X_n(sn^{2/3}), n^{-1/3}I_n(sn^{2/3}), N_n(sn^{2/3})\right)_{0 \leq s \leq T},$$

and so, by Theorem 1.6 and various technical theorems (e.g. the continuous mapping theorem, the bounded convergence theorem), it holds that this converges, as $n \rightarrow \infty$, to

$$\frac{\mathbb{E}\left[q^{N(T)}f(X(s), I(s) : 0 \leq s \leq t)\mathbb{1}_{E'(T,1/T)}\right]}{\mathbb{E}\left[q^{N(T)}\mathbb{1}_{E'(T,1/T)}\right]},$$

where

$$E'(T, \varepsilon) = \{\exists x \in [T, T + \varepsilon] : X(x) - I(x) = 0 \text{ and } N(x) = N(T)\}.$$

Now recall that, conditionally on (X, I) , we have that $N(T) \sim \text{Poi}\left(\int_0^T (X(u) - I(u)) du\right)$,

and that $N(T + 1/T) - N(T)$ is conditionally independent of $N(T)$. Hence, by the tower law,

$$\begin{aligned}
& \frac{\mathbb{E}\left[q^{N(T)} f(X(s), I(s) : 0 \leq s \leq t) \mathbb{1}_{E'(T, 1/T)}\right]}{\mathbb{E}\left[q^{N(T)} \mathbb{1}_{E'(T, 1/T)}\right]} \\
&= \frac{\mathbb{E}\left[\mathbb{E}\left[q^{N(T)} f(X(s), I(s) : 0 \leq s \leq t) \mathbb{1}_{E'(T, 1/T)} \mid X, I\right]\right]}{\mathbb{E}\left[\mathbb{E}\left[q^{N(T)} \mathbb{1}_{E'(T, 1/T)} \mid X, I\right]\right]} \\
&= \frac{\mathbb{E}\left[f(X(s), I(s) : 0 \leq s \leq t) \mathbb{E}\left[q^{N(T)} \mid X, I\right] \mathbb{E}\left[\mathbb{1}_{E'(T, 1/T)} \mid X, I\right]\right]}{\mathbb{E}\left[\mathbb{E}\left[q^{N(T)} \mid X, I\right] \mathbb{E}\left[\mathbb{1}_{E'(T, 1/T)} \mid X, I\right]\right]} \\
&= \frac{\mathbb{E}\left[f(X(s), I(s) : 0 \leq s \leq t) \exp\left((q-1) \int_0^T (X(u) - I(u)) du\right) \mathbb{E}\left[\mathbb{1}_{E'(T, 1/T)} \mid X, I\right]\right]}{\mathbb{E}\left[\exp\left((q-1) \int_0^T (X(u) - I(u)) du\right) \mathbb{E}\left[\mathbb{1}_{E'(T, 1/T)} \mid X, I\right]\right]} \\
&= \frac{\mathbb{E}\left[f(X(s), I(s) : 0 \leq s \leq t) \exp\left((q-1) \int_0^T (X(u) - I(u)) du\right) \mathbb{1}_{E'(T, 1/T)}\right]}{\mathbb{E}\left[\exp\left((q-1) \int_0^T (X(u) - I(u)) du\right) \mathbb{1}_{E'(T, 1/T)}\right]}. \quad (6.12)
\end{aligned}$$

Dividing the top and bottom of (6.12) by the non-random expression $\mathfrak{D}_q(T)$ in the statement of Conjecture 6.2—see (6.1)—we obtain that

$$\begin{aligned}
& \frac{\mathbb{E}\left[f(X(s), I(s) : 0 \leq s \leq t) \exp\left((q-1) \int_0^T (X(u) - I(u)) du\right) \mathbb{1}_{E'(T, 1/T)}\right]}{\mathbb{E}\left[\exp\left((q-1) \int_0^T (X(u) - I(u)) du\right) \mathbb{1}_{E'(T, 1/T)}\right]} \\
&= \frac{\mathbb{E}\left[R_q(T) f(X(s), I(s) : 0 \leq s \leq t) \mathbb{1}_{E'(T, 1/T)}\right]}{\mathbb{E}\left[R_q(T) \mathbb{1}_{E'(T, 1/T)}\right]}. \quad (6.13)
\end{aligned}$$

Then, assuming Conjecture 6.2, it holds that there is an almost surely positive random variable R_q such that $R_q(T) \xrightarrow{P} R_q$ as $T \rightarrow \infty$ (with $R_q > 0$ a.s.), with this process being uniformly integrable. Indeed, we have that jointly

$$\left(R_q(T), f(X(s), I(s) : 0 \leq s \leq t), \mathbb{1}_{\{E'(T, 1/T)\}}\right) \xrightarrow{P} \left(R_q, f(X(s), I(s) : 0 \leq s \leq t), 1\right)$$

as $T \rightarrow \infty$ (because marginal convergence in probability implies joint convergence in probability). Then, by the continuous mapping theorem, as $T \rightarrow \infty$

$$R_q(T) f(X(s), I(s) : 0 \leq s \leq t) \mathbb{1}_{\{E'(T, 1/T)\}} \xrightarrow{P} R_q f(X(s), I(s) : 0 \leq s \leq t),$$

and of course this process is still uniformly integrable (since for each T it is bounded in absolute value by $R_q(T)$ times the bound on $|f|$). By setting $f \equiv 1$, we have that

$$R_q(T) \mathbb{1}_{\{E'(T, 1/T)\}} \xrightarrow{P} R_q$$

as $T \rightarrow \infty$, and again we still have uniform integrability. Thus, applying Vitali's convergence

theorem, we have that the quantity on the right-hand side of (6.13) converges as $T \rightarrow \infty$ to

$$\frac{\mathbb{E}[R_q f(X(s), I(s) : 0 \leq s \leq t)]}{\mathbb{E}[R_q]},$$

where the denominator is necessarily positive.

Then, by (6.2) and (6.10), we have firstly that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \mathbb{E}_{n,p,q} [f(n^{-1/3} X_n(\lfloor sn^{2/3} \rfloor), n^{-1/3} I_n(\lfloor sn^{2/3} \rfloor) : 0 \leq s \leq t)] \\ & \leq \limsup_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}_{n,p,q} [f(n^{-1/3} X_n(\lfloor sn^{2/3} \rfloor), n^{-1/3} I_n(\lfloor sn^{2/3} \rfloor) : 0 \leq s \leq t) \mathbb{1}_{E'_n(T,1/T)}] \\ & \quad + \underbrace{\limsup_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}_{n,p,q} [f(n^{-1/3} X_n(\lfloor sn^{2/3} \rfloor), n^{-1/3} I_n(\lfloor sn^{2/3} \rfloor) : 0 \leq s \leq t) \mathbb{1}_{E'_n(T,1/T)^c}]}_{=0} \\ & \leq \limsup_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{\mathbb{E}_{n,\bar{p}} [q^{N_n(\lfloor Tn^{2/3} \rfloor)} f(X_n(\lfloor s \rfloor), I_n(\lfloor s \rfloor) : 0 \leq s \leq tn^{2/3}) \mathbb{1}_{E'_n(T,1/T)}]}{\mathbb{E}_{n,\bar{p}} [q^{N_n(\lfloor Tn^{2/3} \rfloor)} \mathbb{1}_{E'_n(T,1/T)}]} \\ & \quad \times \frac{\limsup_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} p_n(T, 1/T)}{\liminf_{T \rightarrow \infty} \liminf_{n \rightarrow \infty} r_n(T, 1/T)} \\ & \leq \mathbb{E}[R_q f(X(s), I(s) : 0 \leq s \leq t)], \end{aligned}$$

and similarly that

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \mathbb{E}_{n,p,q} [f(n^{-1/3} X_n(\lfloor sn^{2/3} \rfloor), n^{-1/3} I_n(\lfloor sn^{2/3} \rfloor) : 0 \leq s \leq t)] \\ & \geq \liminf_{T \rightarrow \infty} \liminf_{n \rightarrow \infty} \mathbb{E}_{n,p,q} [f(n^{-1/3} X_n(\lfloor sn^{2/3} \rfloor), n^{-1/3} I_n(\lfloor sn^{2/3} \rfloor) : 0 \leq s \leq t) \mathbb{1}_{E'_n(T,1/T)}] \\ & \quad + \underbrace{\liminf_{T \rightarrow \infty} \liminf_{n \rightarrow \infty} \mathbb{E}_{n,p,q} [f(n^{-1/3} X_n(\lfloor sn^{2/3} \rfloor), n^{-1/3} I_n(\lfloor sn^{2/3} \rfloor) : 0 \leq s \leq t) \mathbb{1}_{E'_n(T,1/T)^c}]}_{\geq 0} \\ & \geq \liminf_{T \rightarrow \infty} \liminf_{n \rightarrow \infty} \frac{\mathbb{E}_{n,\bar{p}} [q^{N_n(\lfloor Tn^{2/3} \rfloor)} f(X_n(\lfloor s \rfloor), I_n(\lfloor s \rfloor) : 0 \leq s \leq tn^{2/3}) \mathbb{1}_{E'_n(T,1/T)}]}{\mathbb{E}_{n,\bar{p}} [q^{N_n(\lfloor Tn^{2/3} \rfloor)} \mathbb{1}_{E'_n(T,1/T)}]} \\ & \quad \times \liminf_{T \rightarrow \infty} \liminf_{n \rightarrow \infty} p_n\left(T, \frac{1}{T}\right) \times \liminf_{T \rightarrow \infty} \liminf_{n \rightarrow \infty} r_n\left(T, \frac{1}{T}\right) \\ & = \mathbb{E}[R_q f(X(s), I(s) : 0 \leq s \leq t)]. \end{aligned}$$

Since we have that the lim sup and lim inf in n of the same quantity are bounded, respectively, above and below by the same value, this must be a limit in n , completing the proof. \square

Remark. Note that one can apply Vitali's convergence theorem directly to the numerator and denominator of (6.12), since the quantities inside the expectations are uniformly bounded in T (and so uniformly integrable) and converge in distribution (because they converge in

probability). However, as we will see in Proposition 7.1, $\int_0^T (X(u) - I(u)) du \xrightarrow{p} \infty$, so, for $q < 1$, both the numerator and the denominator converge to 0. As such, we need to find an appropriate compensating function.

This shows that (conditionally on the conjecture) the exploration process converges. While this is of some independent interest, we're more interested in using it as a lemma to show that the sequence of component sizes, appropriately rescaled, converges. This certainly holds in the vague topology, as we now show. (In the following corollary, we will use the term “twisted” to refer to a measure change. This might more normally be referred to as a “tilted” process, but we reserve the word “tilted” for the tilted excursions from Definition 2.16, and related objects and distributions.)

Corollary 6.4. *Suppose Conjecture 6.2 holds. For $C_i^{(n)}$, let $L(C_i^{(n)}) = k - 1$ if the first vertex of $C_i^{(n)}$ to be explored is chosen as v_k . Let*

$$\Theta_n = \left\{ \left(n^{-2/3} |C_i^{(n)}|, n^{-1/3} L(C_i^{(n)}) \right) : i \geq 1 \right\},$$

and, if \mathfrak{E} is the set of excursions of the reflected limit process (i.e. the Erdős–Rényi process “twisted” by R_q), let

$$\Theta = \{ (|\mathbf{e}|, L(\mathbf{e})) : \mathbf{e} \in \mathfrak{E} \},$$

where $|\mathbf{e}|$ and $L(\mathbf{e})$ are the length and left end-point of \mathbf{e} , respectively. Then

$$\Theta_n \xrightarrow{d} \Theta$$

in the vague topology.

We omit the proof; it follows from Theorem 6.3 above by the proof of [6, Thm. 8] (or the proof of Lemma 5.5).

To convert this into convergence in l^2 , we would need to use Proposition 5.7, which imposes three criteria on the limit object. We know (e.g. by [6, § 3.4]) that all three criteria hold for the untwisted object. But then, applying the definition of the measure change via R_q , these criteria transfer to the twisted object straightforwardly. Thus we have established the following.

Corollary 6.5. *If Conjecture 6.2 holds, then, for $G_n \sim \mathcal{G}(n, p, q)$ with $q < 1$ and $p = 1/n + qn^{-4/3}$, as $n \rightarrow \infty$,*

$$n^{-2/3} \mathbf{C}_n \xrightarrow{d} \mathbf{C}$$

in l^2 , where \mathbf{C} is the well-defined ordered list of excursion lengths of the reflected limit process (again, the Erdős–Rényi process “twisted” by R_q).

Note that this is all consistent with Conjecture 3.6 about the form of the limit distribution of component sizes. Indeed suppose, $\int_0^T \exp\left\{(q-1) \int_0^T (X(u) - I(u)) du\right\} - (\log T)/2$ converges in probability, in such a way that Conjecture 6.2 holds with

$$\mathfrak{D}_q(T) \propto \exp\left\{\frac{q-1}{2} \log T\right\} = T^{(q-1)/2}. \quad (6.14)$$

If so, then we would have shown Conjecture 3.6.

In fact, the convergence in probability does hold (even for $q < 1$, by the continuous mapping theorem), and, for $q > 1$ (instead of $q < 1$), we have that the equivalent of Conjecture 6.2 holds, for $\mathfrak{D}_q(T)$ as in (6.14). We will see this in the next chapter.

6.3 Problems for $q > 1$

There are key steps that are missing to make the argument in the previous section work for $q > 1$. We discuss the problems here.

6.3.1 Approximate independence

In Section 6.1 we showed a result that gives us an “approximate independence” for earlier and later stages of the random cluster model exploration process, if $q < 1$. The proof relied on bounding that exploration process by the process for the Erdős–Rényi graph, but this method cannot be applied for $q > 1$, because here the same arguments used to make the comparison (those in Section 4.1) show that the Erdős–Rényi process provides a *lower* bound, rather than an upper bound.

Though we did not prove this, a possible approach may involve the “fundamental lemma” given in [16], which we present here.

Lemma 6.6 ([16, Lma. 3.1]). *Let $G_n \sim \mathcal{G}(n, p, q)$, for any feasible values of the parameters. Let $r \in (0, 1)$. Independently colour each component of G_n red with probability r , and green with probability $1-r$; let $G_n^{(1)}$ be the red subgraph, and $G_n^{(2)}$ the green subgraph. Let N_1 be the number*

of red vertices. Then, conditionally on $N_1 = n_1$, and up to (a random) relabelling of the vertices:

$$G_n^{(1)} \sim \mathcal{G}(n_1, p, qr); \text{ and}$$

$$G_n^{(2)} \sim \mathcal{G}(n - n_1, p, q(1 - r)),$$

with these two subgraphs being (conditionally) independent.

The utility of this for $q \in (1, 2)$ is that, by setting $r = 1/2$, we obtain two graphs with $q \in (0, 1)$, which we have already considered. The authors of [16] typically take instead the choice of $r = 1/q$, which makes the red subgraph an instance of the Erdős–Rényi model; potentially we too could use this approach, but taking $r = 1/2$ makes the red and green subgraphs exchangeable, slightly simplifying some of our argument. On the other hand, while our method has shortcomings (as we will see), these would not be resolved by the choice of $r = 1/q$.

We now attempt to characterise the distribution of N_1 .

Lemma 6.7. *In the setting of Lemma 6.6, take $q \in (1, 2)$, $p = q/n + \lambda n^{-4/3}$, and $r = 1/2$. Then, as $n \rightarrow \infty$ (and working under $\mathbb{P}_{n,p,q}$ for each n),*

$$N_1 = \frac{n}{2} + O_p(n^{2/3}).$$

The $O_p(n^{2/3})$ requires some control on the component sizes; otherwise, informally, a single very large component, or “too many” medium-sized components, could make the variance of N_1 too large. Of course, we do not know the exact distribution of the component sizes (it is part of what we are aiming to prove), but a result of Łuczak and Łuczak [43] gives us enough control to get the result above. We state that result here.

Theorem 6.8 ([43, Thm. 17, (ii) and (iii)]). *Let $0 < q < 2$, let $np/q = 1 + \varepsilon$ where $\varepsilon = \varepsilon(n)$ is a function (possibly taking negative values) tending to 0 as $n \rightarrow \infty$. Let C_1 be the largest component of $G_n \sim \mathcal{G}(n, p, q)$.*

(a) *Let $a \in \mathbb{R}$ and let $\varepsilon n^{1/3} \rightarrow a$ as $n \rightarrow \infty$. Then, for $n \rightarrow \infty$,*

$$|C_1| = \Theta_p(n^{2/3}).$$

(b) If $\varepsilon n^{1/3} \rightarrow \infty$, then there exists a function

$$l = l(n, p) = (1 + o(1)) \frac{2\varepsilon n}{2 - q} \in \mathbb{N}$$

such that

$$\frac{|C_1| - l}{\sqrt{2n/\varepsilon q}} \xrightarrow{d} \mathcal{N}(0, 1).$$

Remark. Theorem 3.2, given above, and Theorem 6.8, are both simplifications of Theorem 17 in [43]. In fact, the authors give a local limit theorem for case (b); however, the central limit theorem is implied by their proof. Note in this case that $|C_1| = \Theta_p(\varepsilon n)$.

We first state and prove a weaker result giving control on N_1 , with a (relatively) easy proof.

Lemma 6.9. *In the setting of Lemma 6.7,*

$$N_1 = \frac{n}{2} + o_p(n).$$

Proof. Let I_1, I_2, \dots be an i.i.d. sequence of Bern(1/2) random variables, independent of G_n . Then we can set

$$N_1 = \sum_i |C_i| I_i.$$

where C_i is the i -th-largest component of G_n . We can condition on G_n and apply the law of total variance to conclude that, working under the measure $\mathbb{P} = \mathbb{P}_{n,p,q}$,

$$\begin{aligned} \text{Var}(N_1) &= \mathbb{E} \left[\text{Var} \left(\sum_i |C_i| I_i \mid G_n \right) \right] + \text{Var} \left(\mathbb{E} \left[\sum_i |C_i| I_i \mid G_n \right] \right) \\ &= \mathbb{E} \left[\frac{1}{4} \sum_i |C_i|^2 \right] + \text{Var} \left(\frac{n}{2} \right) \\ &= \frac{1}{4} \mathbb{E} \left[\sum_i |C_i|^2 \right]. \end{aligned}$$

However, Theorem 6.8 tells us that, with high probability, $|C_1| = \Theta_p(n^{2/3})$. So let B_n be the event that $|C_1| \geq n^{3/4}$; we know, then, that B_n^c is true with high probability. Note also that, as the components are in order,

$$\sum_i |C_i|^2 \leq |C_1| \sum_i |C_i| = n |C_1|.$$

Then we have that

$$\begin{aligned} \mathbb{E} \left[\sum_i |C_i|^2 \right] &\leq n \mathbb{E}[|C_1|] = n \mathbb{P}(B_n) \mathbb{E}[|C_1| \mid B_n] + n \mathbb{P}(B_n^c) \mathbb{E}[|C_1| \mid B_n^c] \\ &\leq n^2 \mathbb{P}(B_n) + n^{7/4} = o(n^2). \end{aligned}$$

Since $\mathbb{E}[N_1] = n/2$ trivially, the result follows by Chebyshev's inequality. \square

This bound is obviously weaker than the one that we will obtain from Lemma 6.7, though we will need to “bootstrap” from it: that is, we can combine this with a further use of the *a priori* bounds on the component sizes from Theorem 6.8 to obtain the stronger result.

Proof of Lemma 6.7. Equivalently to the statement of the lemma, we show that for any sequence $\omega_n \rightarrow \infty$ as $n \rightarrow \infty$, we have that

$$\mathbb{P}_{n,p,q} \left(\left| N_1 - \frac{n}{2} \right| \geq \omega_n n^{2/3} \right) \rightarrow 0.$$

(This equivalence is given by [37, Lemma 3].) We observe, that, as $G_n^{(1)}$ and $G_n^{(2)}$ are exchangeable, the distribution of N_1 is symmetric about $n/2$, so it is sufficient to show that

$$\mathbb{P}_{n,p,q} \left(N_1 \geq \frac{n}{2} + \omega_n n^{2/3} \right) \rightarrow 0$$

as $n \rightarrow \infty$. We also restrict, without loss of generality, to considering $\omega_n = o(n^{1/3})$.

We prove this by contradiction. Suppose there were to exist $\delta > 0$ and an increasing sequence $(n_k)_{k \in \mathbb{N}}$ in \mathbb{N} such that, for all k , if we define

$$A_k = \left\{ N_1 \geq \frac{n_k}{2} + \omega_{n_k} n_k^{2/3} \right\}$$

then

$$\mathbb{P}_{n_k,p,q}(A_k) \geq \delta.$$

for all k . Define a new probability measure \mathbb{Q}_k by

$$\mathbb{Q}_k(\cdot) = \mathbb{P}_{n_k,p,q}(\cdot \mid A_k).$$

We now let $N_1(k)$ represent N_1 distributed according to $\mathbb{P}_{n_k, p, q}$, and define

$$\varepsilon(k) = \frac{2N_1(k)p}{q} - 1 = \frac{2N_1(k)}{q} \left(\frac{q}{n_k} + \frac{\lambda}{n_k^{4/3}} \right) - 1,$$

so that, almost surely under \mathbb{Q}_k ,

$$\varepsilon(k) \geq \frac{2\omega_{n_k}}{n_k^{1/3}} + \frac{\lambda}{qn_k^{1/3}} + \frac{2\lambda\omega_{n_k}}{qn_k^{2/3}},$$

and in particular $\varepsilon(k)n_k^{1/3} \rightarrow \infty$ almost surely under any coupling of the \mathbb{Q}_k s. Now, from Lemma 6.9, we also have that $\varepsilon(k) \rightarrow 0$ in probability, so that, by Skohorod's Representation Theorem, there exists a measure \mathbb{Q} that couples the \mathbb{Q}_k s so that in fact $\varepsilon(k) \rightarrow 0$ almost surely as $k \rightarrow \infty$. Then the random vector $\varepsilon = (\varepsilon(k))_{k \in \mathbb{N}}$ satisfies the conditions of Theorem 6.8, part (b), applied to the red graph, so, if $C_1^{(n)}$ is the largest component of the red subgraph,

$$|C_1^{(n)}| = \Theta_p(\varepsilon(k)N_1(k)).$$

Note that, working under \mathbb{Q} , it holds that $\varepsilon(k)N_1(k) = \omega_p(n^{2/3})$. Thus, under \mathbb{Q} , and working with C_1 (the largest component in G),

$$|C_1| = \omega_p(n^{2/3}).$$

That is to say, for all (large) $M > 0$

$$\mathbb{P}_{n_k, p, q}(|C_1| \leq Mn^{2/3} \mid A_k) = \mathbb{Q}_k(|C_1| \leq Mn^{2/3}) \rightarrow 0,$$

which implies, as $\mathbb{P}_{n_k, p, q}(A_k^c) \leq 1 - \delta$, that for all $M > 0$,

$$\limsup_{n \rightarrow \infty} \mathbb{P}_{n_k, p, q}(|C_1| \leq Mn^{2/3}) \leq 1 - \delta.$$

However, this contradicts Theorem 6.8 applied to G , which says that

$$|C_1| = \Theta_p(n^{2/3});$$

since each A_k has positive probability, this applies under the measure \mathbb{Q} as well. Thus the claim is shown. \square

This bound on N_1 tells us that the red and green subgraphs, although containing a random number of vertices, are both within the critical window for the random cluster graph established by Luczak and Łuczak [43]. However, they are at random (dependent) positions in the critical window, and we don't have much control over where precisely they fall. This therefore does not appear to be quite enough to show that the following conjecture holds.

Conjecture 6.10. *Proposition 6.1 can be extended to the case $q \in (1, 2)$.*

One might try to find a proof that involved breaking down the exploration into explorations of the red and green subgraphs, and claiming that, since Proposition 6.1 holds for these, the same result holds for the overall graph. This simplification, however, omits the fact that we need to identify, at a given time in the exploration of the overall graph, how far we might be through the exploration each of the subgraphs—we need to know that, as we take the limit in n and then in T , we are progressing through the red and green sub-explorations appropriately quickly. We were unable to resolve this problem.

6.3.2 Integrability and the limit in n

In our argument in Section 6.2 above, we dealt with quantities of the form

$$q^{N_n(\lfloor Tn^{2/3} \rfloor)},$$

where we needed the convergence of their expectation (or, rather, their expectations when multiplied by a bounded function) when $n \rightarrow \infty$. For $q \in (0, 1)$ we have no issues here, by the bounded convergence theorem and the known convergence in distribution. For $q \in (1, 2)$, we observe that, by the tower law,

$$\mathbb{E}_{n, \tilde{p}} \left[q^{N_n(\lfloor Tn^{2/3} \rfloor)} M_n \right] = \mathbb{E}_{n, \tilde{p}} \left[\exp \left\{ \log(\tilde{p}q + 1 - \tilde{p}) \sum_{i=1}^{\lfloor Tn^{2/3} \rfloor} (X_n(i) - I_n(i)) \right\} M_n \right],$$

for any sequence of uniformly bounded random variables $(M_n)_{n \in \mathbb{N}}$ defined on the same probability spaces. It is sufficient to show that the sequence inside the expectation is uniformly integrable. Since M_n is uniformly bounded, it is in turn sufficient to show that

$$\left(\exp \left\{ (q-1) \sum_{i=1}^{\lfloor Tn^{2/3} \rfloor} (X_n(i) - I_n(i)) \right\} \right)_{n \in \mathbb{N}}$$

is uniformly integrable. If we can show this, then the conjecture below would be true.

Conjecture 6.11. Let $p = 1/n + \lambda n^{-4/3}$, let $G_n \sim \mathcal{G}(n, 1/n + \lambda n^{-4/3})$, and let (X_n, I_n, N_n) be the usual depth-first exploration process. If $(M_n)_{n \in \mathbb{N}}$ is a sequence of uniformly bounded $\sigma(X_n)$ -measurable random variables that converge in distribution to M , then for all $q \in (1, 2)$, as $n \rightarrow \infty$,

$$\mathbb{E}_{n, \tilde{p}}[q^{N_n(\lfloor Tn^{2/3} \rfloor)} M_n] \rightarrow \mathbb{E}[q^{N(T)} M].$$

Chapter 7

$q > 1$: Convergence and integrability

Problem 2 (in the heuristic proof given in Section 3.2) required us to find a function $\mathfrak{D}_q(t)$ such that

$$\frac{\exp\left\{(q-1)\int_0^t(X(u)-I(u))\,du\right\}}{\mathfrak{D}_q(t)} \xrightarrow{d} R_q > 0$$

a.s. (where (X, I, N) is the limit process given by Theorem 1.6), and for that to have sufficient uniform integrability properties that convergence of the expectation holds too. This problem remains unresolved in the case $q < 1$, but turns out to be solvable in the case $q > 1$. We thus obtain a candidate limit object in that setting.

Specifically, we will prove the following propositions.

Proposition 7.1. *Let X be the appropriately rescaled limit of the exploration process for the random graph $\mathcal{G}(n, 1/n + \lambda n^{-4/3})$, and let I be the running infimum of X . Then there exists a random variable L_λ such that, as $t \rightarrow \infty$,*

$$\int_0^t (X(u) - I(u))\,du - \frac{1}{2} \log t \xrightarrow{p} L_\lambda.$$

Proposition 7.2. *For all $q \in (1, 2)$, and for all $\lambda \in \mathbb{R}$, the collection of random variables*

$$\left(\exp\left\{(q-1)\left[\int_0^t (X(u) - I(u))\,du - \frac{1}{2} \log t\right]\right\} \right)_{t \geq 0}$$

is uniformly integrable.

The point here is that, for any bounded continuous functional f , we can conclude that, as

$T \rightarrow \infty$,

$$\begin{aligned} & \mathbb{E} \left[f(X(s), I(s) : 0 \leq s \leq t) \exp \left\{ (q-1) \left[\int_0^T (X(u) - I(u)) du - \frac{1}{2} \log t \right] \right\} \right] \\ & \rightarrow \mathbb{E}[f(X(s), I(s) : 0 \leq s \leq t) \exp\{(q-1)L_\lambda\}], \end{aligned}$$

which is exactly what we needed in our heuristic proof. (Though this solves Problem 2, recall that Problem 1 remains to be solved in the $q > 1$ case.)

Remark. As an aside, these results essentially tell us that the number of surplus edges discovered up to time t in the limit of the Erdős–Rényi exploration process (in the critical window) grows as $(1/2) \log t$. We will therefore also prove this corollary, which is a weak law of large numbers for that process:

Corollary 7.3. *For (X, I, N) the Aldous limit process, and with any $\lambda \in \mathbb{R}$,*

$$\frac{N(t)}{\log t} \xrightarrow{p} \frac{1}{2}$$

as $t \rightarrow \infty$.

To prove these propositions, we will use results of Janson and Spencer [40], which we discussed in Section 2.4—these consider the Erdős–Rényi exploration’s limit process as a point process of excursion lengths. Their approach applies more naturally to the process $X - I$ considered up until a given local time at 0, as opposed to a given “real” time. Thus, in Section 7.1 we prove analogues of Propositions 7.1 and 7.2 in this local time setting. Section 7.2 is then devoted to converting these results back to the real-time setting. In Section 7.3 we discuss in more detail where and why the proofs in this chapter fail for $q < 1$, outlining what future work would be needed.

7.1 Results in terms of local time

Recall from Section 2.4 the point process $\Xi^{(\lambda)}$ which represents the component sizes of the scaling limit of $\mathcal{G}(n, 1/n + \lambda n^{-4/3})$. The main result we will use to analyse this was already given as Theorem 2.29(a); we restate it here for convenience.

Theorem 7.4 ([40, Thm. 4.1]). *The intensity measure of $\Xi^{(\lambda)}$ has density*

$$\Lambda^{(\lambda)}(x) := \frac{1}{\sqrt{2\pi}} x^{-5/2} \Psi(x^{3/2}) \exp \left\{ -\frac{x^3}{6} + \frac{x^2 \lambda}{2} - \frac{x \lambda^2}{2} \right\},$$

where, if A is the area under a standard Brownian excursion,

$$\Psi(t) := \mathbb{E}[e^{tA}]. \quad (7.1)$$

Before we go on, we will make some useful observations about the function Ψ , which we will need in our analysis. First, a technical lemma.

Lemma 7.5. *Suppose $\sum_{k=0}^{\infty} a_k x^k$ is a power series that converges for all $x \in \mathbb{R}$, with $a_k > 0$ for all sufficiently large k . Suppose also that $a_k \sim b_k$ as $k \rightarrow \infty$. Then $\sum_{k=0}^{\infty} b_k x^k$ converges for all $x \in \mathbb{R}$, and as $x \rightarrow \infty$,*

$$\sum_{k=0}^{\infty} b_k x^k \sim \sum_{k=0}^{\infty} a_k x^k.$$

We defer the proof of this lemma to Appendix A (p. 181).

In [40, (3.4)], the authors show that

$$\Psi(t) \sim \frac{1}{2} t^2 e^{t^2/24} \quad (7.2)$$

as $t \rightarrow \infty$. We extend their proof to show that the k -th derivative of Ψ is asymptotically equivalent to the k -th derivative of the right-hand side of (7.2).

Lemma 7.6. *For all $k \in \mathbb{N}$,*

$$\frac{d^k \Psi(t)}{dt^k} \sim \frac{1}{2} (12)^{-k} t^{2+k} e^{t^2/24}$$

as $t \rightarrow \infty$.

Proof. We have

$$\Psi(t) = \sum_{\ell=0}^{\infty} \frac{\mathbb{E}[A^\ell]}{\ell!} t^\ell,$$

whenever the sum is well-defined. It is also given in [40] that $\mathbb{E}[A^\ell] \sim \sqrt{18\ell} (12e)^{-\ell/2} \ell^{\ell/2}$. Thus, by Lemma 7.5 applied in conjunction with Lemma 4.1(ii) of [38] (which says that, for $\gamma < 1$ and $b \in \mathbb{R}$,

$$\sum_{k=1}^{\infty} \frac{k^b k^{\gamma k} x^k}{k!} \sim (1-\gamma)^{-1/2} (e^\gamma x)^{b/(1-\gamma)} \exp\{(1-\gamma)(e^\gamma x)^{1/(1-\gamma)}\}$$

as $x \rightarrow \infty$), the power series converges for all $t > 0$, and hence has infinite radius of convergence. By a standard result of analysis, this means we may differentiate the power series

term-by-term, to obtain (after reparametrising the sum) that

$$\Psi^{(k)}(t) = \sum_{\ell=0}^{\infty} \frac{\mathbb{E}[A^{\ell+k}]}{\ell!} t^{\ell}.$$

We note that, by the asymptotic behaviour of $\mathbb{E}[A^{\ell}]$ given above,

$$\mathbb{E}[A^{\ell+k}] \sim \sqrt{18}(\ell+k)(12e)^{-(\ell+k)/2}(\ell+k)^{(\ell+k)/2} \sim \sqrt{18}(12)^{-k/2}\ell^{1+k/2}\ell^{\ell/2}(12e)^{-\ell/2}.$$

Then, applying Lemma 7.5 and [38, Lma. 4.1(ii)] again, we obtain the claimed result. \square

7.1.1 Towards a limit

We now study the pointset that is the limit of rescaled component sizes of $\mathcal{G}(n, 1/n + \lambda n^{-4/3})$, which we called $\Xi^{(\lambda)}$. Let D_i be the i -th-largest element of $\Xi^{(\lambda)}$, which [6] proved was well-defined. Then let $\tilde{A}_i(D_i)$ denote the area under the associated excursion. Note that $\tilde{A}_i(x)$ is distributed as the area under a *tilted* Brownian excursion of length x ; see Definition 2.16 and Theorem 2.25. Also note that, given $\Xi^{(\lambda)}$, the random variables \tilde{A}_i are independent.

Let $L(t)$ be the local time at 0 of the process $X - I$, evaluated at time t . (We may define this, e.g., by

$$L(t) = \lim_{n \rightarrow \infty} 2^{-n} U(t, 2^{-n})$$

where $U(t, \varepsilon)$ denotes the number of upcrossings by $X - I$ of the interval $[0, \varepsilon]$ completed before time t —see, for instance, [56, (14.8)].) It can be shown that $L(t) = -I(t)$: for reflected Brownian motion, the equivalent statement is a result of Lévy (see [56, Thm. 14.7]), from which the result for our process with quadratic drift follows by the Cameron–Martin–Girsanov formula, which establishes that the two processes are absolutely continuous with respect to one another. (See also Aldous [6, p. 846].) This representation of the local time of a reflected process as the infimum of the unreflected process will be useful for calculations in Section 7.2.

Now, for each i , let $L_i(D_i)$ denote the local time at 0 at which D_i occurs in the reflected process (it is clear from the definition of $L(t)$ that this function is constant throughout values of t belonging to any excursion interval). By [6, (72)], we have that $L_i(x) \sim \text{Exp}(x)$, and that, moreover, the $L_i(D_i)$ are conditionally independent across i , given $\Xi^{(\lambda)}$. Moreover, the random variables \tilde{A}_i can depend only on the excursion lengths of $X - I$; so, conditionally on $\Xi^{(\lambda)}$,

$$(L_i(D_i))_{i \in \mathbb{N}} \perp\!\!\!\perp (\tilde{A}_i(D_i))_{i \in \mathbb{N}}.$$

(Aldous [6, pp. 846–847] uses this fact implicitly in his proof of the density of the excursion measure of his process with respect to that of reflected Brownian motion. Alternatively, it can be shown by a limit argument from the discrete object.)

Let τ denote the right-continuous inverse local time, which satisfies

$$\tau(\ell) = \sum_{i=1}^{\infty} D_i \mathbb{1}_{\{L_i(D_i) \leq \ell\}}.$$

Let $\check{\tau}$ denote the *left*-continuous version instead; i.e., define

$$\check{\tau}(\ell) = \sum_{i=1}^{\infty} D_i \mathbb{1}_{\{L_i(D_i) < \ell\}}.$$

Then we take

$$\mathfrak{J}(\ell) := \int_0^{\tau(\ell)} (X(u) - I(u)) \, du = \sum_{i=1}^{\infty} \tilde{A}_i(D_i) \mathbb{1}_{\{L_i(D_i) \leq \ell\}}, \quad (7.3)$$

and

$$\check{\mathfrak{J}}(\ell) := \int_0^{\check{\tau}(\ell)} (X(u) - I(u)) \, du = \sum_{i=1}^{\infty} \tilde{A}_i(D_i) \mathbb{1}_{\{L_i(D_i) < \ell\}}. \quad (7.4)$$

Since there are only countably many excursions, $\mathfrak{J} = \check{\mathfrak{J}}$ on $[0, \infty)$ except at countably many points (these points being random).

It will be useful to prove some properties of $\tilde{A}(x)$, the area under a tilted Brownian excursion of length x . Define $\Phi = \log \Psi$, and note that, from (7.2) and Lemma 7.6,

$$\Phi(t) = \frac{t^2}{24} + O(\log t); \quad (7.5)$$

$$\Phi'(t) = \frac{\Psi'(t)}{\Psi(t)} \sim \frac{t}{12}; \quad (7.6)$$

and

$$\Phi''(t) = \frac{\Psi''(t)}{\Psi(t)} - \left(\frac{\Psi'(t)}{\Psi(t)} \right)^2 = o(t^2). \quad (7.7)$$

Note also that as A is non-negative, Ψ is an increasing function, and so Φ is too.

Lemma 7.7. *Let $\tilde{A}(x)$ be the area under a tilted Brownian excursion of length x , and let $\check{\Psi}_x$ be*

its moment generating function. Then

$$\tilde{\Psi}_x(t) = \exp\{\Phi((t+1)x^{3/2}) - \Phi(x^{3/2})\}. \quad (7.8)$$

Proof. Let $A(x)$ be the area under a Brownian excursion of length x (so $A = A(1)$). Then

$$\begin{aligned} \tilde{\Psi}_x(t) &= \mathbb{E}\left[e^{t\tilde{A}(x)}\right] = \frac{\mathbb{E}\left[e^{(t+1)A(x)}\right]}{\mathbb{E}\left[e^{A(x)}\right]} = \frac{\mathbb{E}\left[e^{(t+1)x^{3/2}A}\right]}{\mathbb{E}\left[e^{x^{3/2}A}\right]} = \frac{\Psi((t+1)x^{3/2})}{\Psi(x^{3/2})} \\ &= \exp\{\Phi((t+1)x^{3/2}) - \Phi(x^{3/2})\}, \end{aligned}$$

where the third equality follows by Brownian scaling. \square

We now have all the machinery we need to show that $\mathfrak{J}(\ell) - \mathbb{E}[\mathfrak{J}(\ell)]$ has an almost sure limit as $\ell \rightarrow \infty$, and similarly for $\check{\mathfrak{J}}$. To simplify the proof a little, we will state and prove the following lemma.

Lemma 7.8. *For all $\ell \in [0, \infty)$, $\mathfrak{J}(\ell) = \check{\mathfrak{J}}(\ell)$ a.s.*

Proof. By definition, $\mathfrak{J}(\ell) \geq \check{\mathfrak{J}}(\ell)$. Now

$$\begin{aligned} \mathbb{E}[\mathfrak{J}(\ell) - \check{\mathfrak{J}}(\ell)] &= \mathbb{E}\left[\sum_{i=1}^{\infty} \tilde{A}_i(D_i) \mathbf{1}_{\{L_i(D_i)=\ell\}}\right] \\ &= \sum_{i=1}^{\infty} \mathbb{E}\left[\mathbb{E}[\tilde{A}_i(D_i) \mathbf{1}_{\{L_i(D_i)=\ell\}} \mid D_i]\right] \\ &= \sum_{i=1}^{\infty} \mathbb{E}\left[\mathbb{E}[\tilde{A}_i(D_i) \mid D_i] \mathbb{P}(L_i(D_i) = \ell \mid D_i)\right] = 0, \end{aligned}$$

by the conditional independence of $\tilde{A}_i(D_i)$ and $L_i(D_i)$ and the fact that $L_i(x) \sim \text{Exp}(x)$. So we have a non-negative random variable with expectation 0, which must therefore be almost surely equal to 0, completing the proof. \square

We next investigate the asymptotics of the expectation of the process \mathfrak{J} . As is conventional, let γ represent the Euler–Mascheroni constant ($\gamma = 0.57721 \dots$ [1, p. 3]). Further, define the constant

$$\begin{aligned} C_\lambda &= \frac{\gamma}{4} + \int_1^\infty \frac{1}{\sqrt{2\pi}} x^{-1} \Psi'(x^{3/2}) \exp\left\{-\frac{x^3}{6} + \frac{x^2\lambda}{2} - \frac{x\lambda^2}{2}\right\} dx \\ &\quad + \int_0^1 \frac{1}{\sqrt{2\pi}} x^{-1} \left(\Psi'(x^{3/2}) \exp\left\{-\frac{x^3}{6} + \frac{x^2\lambda}{2} - \frac{x\lambda^2}{2}\right\} - \sqrt{\frac{\pi}{8}}\right) dx. \end{aligned}$$

Proposition 7.9. C_λ is a finite constant. Moreover, let $f(\ell) = \mathbb{E}[\mathcal{J}(\ell)]$; then, as $\ell \rightarrow \infty$,

$$f(\ell) = \frac{1}{4} \log \ell + C_\lambda + o(1).$$

Remark. Note that $f(\ell) = \mathbb{E}[\check{\mathcal{J}}(\ell)]$ also, by Lemma 7.8.

To prove this, we first need a technical lemma, whose proof we defer to Appendix A (p. 180).

Lemma 7.10. As $\ell \rightarrow \infty$,

$$\int_0^1 x^{-1}(1 - e^{-\ell x}) dx = \gamma + \log \ell + O(\ell^{-1}e^{-\ell}).$$

With this in hand, we can prove the proposition.

Proof of Proposition 7.9. Using the tower law, we have that

$$\begin{aligned} f(\ell) &= \mathbb{E} \left[\mathbb{E} \left[\sum_{i=1}^{\infty} \tilde{A}_i(D_i) \mathbb{1}_{\{L_i(D_i) \leq \ell\}} \mid \Xi^{(\lambda)} \right] \right] \\ &= \mathbb{E} \left[\sum_{i=1}^{\infty} \mathbb{E}[\tilde{A}_i(D_i) \mid D_i] \mathbb{E}[\mathbb{1}_{\{L_i(D_i) \leq \ell\}} \mid D_i] \right] \\ &= \mathbb{E} \left[\sum_{i=1}^{\infty} \mathbb{E}[\tilde{A}(D_i) \mid D_i] (1 - \exp\{-\ell D_i\}) \right], \end{aligned} \quad (7.9)$$

where we use the previously-noted conditional independence for the second line. For the expectation of the area term, we use Lemma 7.7; we have that $\mathbb{E}[\tilde{A}(x)] = \tilde{\Psi}'_x(0)$, where

$$\tilde{\Psi}'_x(t) = x^{3/2} \Phi'((t+1)x^{3/2}) \tilde{\Psi}_x(t).$$

Hence,

$$\mathbb{E}[\tilde{A}(x)] = x^{3/2} \Phi'(x^{3/2}), \quad (7.10)$$

and, substituting into (7.9), we have that

$$f(\ell) = \mathbb{E} \left[\sum_{i=1}^{\infty} D_i^{3/2} \Phi'(D_i^{3/2}) (1 - \exp\{-\ell D_i\}) \right].$$

By Theorem 7.4, we obtain

$$\begin{aligned} f(\ell) &= \int_0^\infty x^{3/2} \Phi'(x^{3/2}) (1 - e^{-\ell x}) \Lambda^{(\lambda)}(x) dx \\ &= \int_0^\infty \frac{1}{\sqrt{2\pi}} x^{-1} \Psi'(x^{3/2}) (1 - e^{-\ell x}) \exp\left\{-\frac{x^3}{6} + \frac{x^2\lambda}{2} - \frac{x\lambda^2}{2}\right\} dx, \end{aligned} \quad (7.11)$$

noting from (7.6) that $\Psi'(t) = \Phi'(t)\Psi(t)$ for all t .

We will split this integral into two parts, integrating separately over the intervals $(0, 1]$ and $(1, \infty)$. (Note that the integrand is non-negative, so both these component integrals are non-negative.) We start with the integral over $(1, \infty)$. Define

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} x^{-1} \Psi'(x^{3/2}) \exp\left\{-\frac{x^3}{6} + \frac{x^2\lambda}{2} - \frac{x\lambda^2}{2}\right\};$$

via Lemma 7.6, this is asymptotically equivalent to

$$\frac{1}{24\sqrt{2\pi}} x^{7/2} \exp\left\{-\frac{x^3}{8} + \frac{x^2\lambda}{2} - \frac{x\lambda^2}{2}\right\},$$

so φ is integrable over $(1, \infty)$. Hence, using the dominated convergence theorem with φ as a dominating function, it holds that, as $\ell \rightarrow \infty$,

$$\begin{aligned} \int_1^\infty \frac{1}{\sqrt{2\pi}} x^{-1} \Psi'(x^{3/2}) (1 - e^{-\ell x}) \exp\left\{-\frac{x^3}{6} + \frac{x^2\lambda}{2} - \frac{x\lambda^2}{2}\right\} dx \\ \rightarrow \int_1^\infty \frac{1}{\sqrt{2\pi}} x^{-1} \Psi'(x^{3/2}) \exp\left\{-\frac{x^3}{6} + \frac{x^2\lambda}{2} - \frac{x\lambda^2}{2}\right\} dx. \end{aligned} \quad (7.12)$$

For the integral over $(0, 1]$, note that, as $x \downarrow 0$,

$$\Psi'(x^{3/2}) \exp\left\{-\frac{x^3}{6} + \frac{x^2\lambda}{2} - \frac{x\lambda^2}{2}\right\} = \Psi'(0) + O(x) = \mathbb{E}[A] + O(x),$$

since Ψ is the moment generating function of A . Indeed, Spencer [57, (4.3)] gives that $\mathbb{E}[A] = \sqrt{\pi/8}$, so that

$$\Psi'(x^{3/2}) \exp\left\{-\frac{x^3}{6} + \frac{x^2\lambda}{2} - \frac{x\lambda^2}{2}\right\} = \sqrt{\frac{\pi}{8}} + O(x). \quad (7.13)$$

We therefore consider the decomposition

$$\begin{aligned} & \int_0^1 \frac{1}{\sqrt{2\pi}} x^{-1} \Psi'(x^{3/2}) (1 - e^{-\ell x}) \exp\left\{-\frac{x^3}{6} + \frac{x^2 \lambda}{2} - \frac{x \lambda^2}{2}\right\} dx \\ &= \frac{1}{4} \int_0^1 x^{-1} (1 - e^{-\ell x}) dx \\ & \quad + \int_0^1 \frac{1}{\sqrt{2\pi}} x^{-1} \left(\Psi'(x^{3/2}) \exp\left\{-\frac{x^3}{6} + \frac{x^2 \lambda}{2} - \frac{x \lambda^2}{2}\right\} - \sqrt{\frac{\pi}{8}} \right) (1 - e^{-\ell x}) dx. \end{aligned}$$

We have already established in Lemma 7.10 that

$$\frac{1}{4} \int_0^1 x^{-1} (1 - e^{-\ell x}) dx = \frac{\gamma}{4} + \frac{1}{4} \log \ell + o(1). \quad (7.14)$$

Meanwhile (7.13) (together with results about the continuity of moment generating functions within their radius of convergence) allows us to conclude that there is a bound, uniform in ℓ , on the second integral, so we may apply the bounded convergence theorem to conclude that

$$\begin{aligned} & \int_0^1 \frac{1}{\sqrt{2\pi}} x^{-1} \left(\Psi'(x^{3/2}) \exp\left\{-\frac{x^3}{6} + \frac{x^2 \lambda}{2} - \frac{x \lambda^2}{2}\right\} - \sqrt{\frac{\pi}{8}} \right) (1 - e^{-\ell x}) dx \\ & \rightarrow \int_0^1 \frac{1}{\sqrt{2\pi}} x^{-1} \left(\Psi'(x^{3/2}) \exp\left\{-\frac{x^3}{6} + \frac{x^2 \lambda}{2} - \frac{x \lambda^2}{2}\right\} - \sqrt{\frac{\pi}{8}} \right) dx. \end{aligned} \quad (7.15)$$

Putting together (7.12), (7.14) and (7.15), and noting that they sum to (7.11), gives the result. \square

We next wish to show that \mathfrak{J} and $\check{\mathfrak{J}}$, compensated by their means, converge almost surely. To do this, we set up some notation. Noting that $\tau(0) = 0$ a.s., let

$$J(k) = \int_{\tau((k-1)^2)}^{\tau(k^2)} (X(u) - I(u)) du.$$

Then, for $\ell \in \mathbb{N}$, we have that

$$\mathfrak{J}(\ell^2) = \sum_{k=1}^{\ell} J(k)$$

almost surely, and that

$$f(\ell^2) = \sum_{k=1}^{\ell} \mathbb{E}[J(k)].$$

We now establish a bound for $\mathbb{E}[J(k)]$.

Lemma 7.11. Let $g(k) = \mathbb{E}[J(k)]$. As $k \rightarrow \infty$, $g(k) = O(k^{-1})$.

We defer the proof of this lemma to Appendix A (p. 182).

Lemma 7.12. $\text{Var}(J(k))$ is finite for all k , and, as $k \rightarrow \infty$,

$$\text{Var}(J(k)) = O(k^{-13/4}).$$

Proof. We only explicitly prove the asymptotic result; that all the relevant quantities are finite can be read off from the bounds obtained to prove the asymptotics.

We have that

$$\begin{aligned} \text{Var}(J(k)) &= \mathbb{E}[J(k)^2] - \mathbb{E}[J(k)]^2 \\ &= \mathbb{E} \left[\left(\sum_{i=1}^{\infty} \tilde{A}_i(D_i) \mathbb{1}_{\{(k-1)^2 < L_i(D_i) \leq k^2\}} \right)^2 \right] - \mathbb{E} \left[\sum_{i=1}^{\infty} \tilde{A}_i(D_i) \mathbb{1}_{\{(k-1)^2 < L_i(D_i) \leq k^2\}} \right]^2 \\ &= \mathfrak{X}_1(k) + \mathfrak{X}_2(k) - \mathfrak{X}_3(k), \end{aligned} \quad (7.16)$$

where we define

$$\mathfrak{X}_1(k) = \mathbb{E} \left[\sum_{i=1}^{\infty} \tilde{A}_i(D_i)^2 \mathbb{1}_{\{(k-1)^2 < L_i(D_i) \leq k^2\}} \right], \quad (7.17)$$

$$\mathfrak{X}_2(k) = \mathbb{E} \left[\sum_{i=1}^{\infty} \sum_{\substack{j=1 \\ j \neq i}}^{\infty} \tilde{A}_i(D_i) \mathbb{1}_{\{(k-1)^2 < L_i(D_i) \leq k^2\}} \tilde{A}_j(D_j) \mathbb{1}_{\{(k-1)^2 < L_j(D_j) \leq k^2\}} \right], \quad (7.18)$$

and

$$\mathfrak{X}_3(k) = \mathbb{E} \left[\sum_{i=1}^{\infty} \tilde{A}_i(D_i) \mathbb{1}_{\{(k-1)^2 < L_i(D_i) \leq k^2\}} \right]^2. \quad (7.19)$$

We start with $\mathfrak{X}_1(k)$, where we have that

$$\begin{aligned} \mathfrak{X}_1(k) &= \mathbb{E} \left[\sum_{i=1}^{\infty} \tilde{A}_i(D_i)^2 \mathbb{1}_{\{(k-1)^2 < L_i(D_i) \leq k^2\}} \right] \\ &= \mathbb{E} \left[\sum_{i=1}^{\infty} D_i^3 \left[\Phi''(D_i^{3/2}) + \Phi'(D_i^{3/2})^2 \right] \left(e^{-(k-1)^2 D_i} - e^{-k^2 D_i} \right) \right] \\ &= \int_0^{\infty} x^3 \left[\Phi''(x^{3/2}) + \Phi'(x^{3/2})^2 \right] \left(e^{-(k-1)^2 x} - e^{-k^2 x} \right) \Lambda^{(\lambda)}(x) dx \\ &= \int_0^{\infty} \frac{1}{\sqrt{2\pi}} x^{1/2} \Psi''(x^{3/2}) \left(e^{-(k-1)^2 x} - e^{-k^2 x} \right) \exp \left\{ -\frac{x^3}{6} + \frac{x^2 \lambda}{2} - \frac{x \lambda^2}{2} \right\} dx \end{aligned}$$

(by the tower law and Theorem 7.4, since we have that

$$\mathbb{E}[\tilde{A}(x)^2] = \tilde{\Psi}_x''(0) = x^3[\Phi''(x^{3/2}) + \Phi'(x^{3/2})^2]$$

from Lemma 7.7).

We will split this integral into two parts, integrating over $[0, k^{-3/2}]$ and $(k^{-3/2}, \infty)$. We start with the latter, noting that the integral over $(k^{-3/2}, \infty)$ is bounded above by

$$\frac{e^{-k^{-3/2}(k-1)^2}}{\sqrt{2\pi}} \int_{k^{-3/2}}^{\infty} x^{1/2} \Psi''(x^{3/2}) \exp\left\{-\frac{x^3}{6} + \frac{x^2\lambda}{2} - \frac{x\lambda^2}{2}\right\} dx = O(e^{-k^{1/2}});$$

the last equality holds as the integrand here is, from Lemma 7.6, asymptotically equivalent to

$$\frac{1}{288} x^{13/2} \exp\left\{-\frac{x^3}{8} + \frac{x^2\lambda}{2} - \frac{x\lambda^2}{2}\right\},$$

a function integrable over $(0, \infty)$.

For the left-hand integral, over $[0, k^{-3/2}]$, we note that, for $k \geq 2$, the function

$$r_k(x) := e^{-(k-1)^2x} - e^{-k^2x} \tag{7.20}$$

attains its maximum over $x \in \mathbb{R}$ at

$$x_{\max} = \frac{2}{2k-1} \log\left(\frac{k}{k-1}\right); \tag{7.21}$$

if $k = 1$, it approaches its supremum over $x \in [0, \infty)$ as $x_{\max} \rightarrow \infty$. Substituting (7.21) into the expression for r_k , we obtain that, as $k \rightarrow \infty$,

$$r_k(x_{\max}) = O\left(\frac{1}{k}\right),$$

so we can find a constant C_1 independent of k such that, for all $k \geq 1$ and for all $x \in [0, \infty)$, we have $r_k(x) \leq C_1 k^{-1}$ (including the $k = 1$ case by taking $C_1 \geq 1$). Meanwhile, we can find a constant C_2 such that, for all $x \in [0, 1]$,

$$\frac{1}{\sqrt{2\pi}} \Psi''(x^{3/2}) \exp\left\{-\frac{x^3}{6} + \frac{x^2\lambda}{2} - \frac{x\lambda^2}{2}\right\} \leq C_2,$$

and clearly we can choose C_2 to be independent of k .

Applying these bounds, and integrating, gives us that

$$\int_0^{k^{-3/2}} \frac{1}{\sqrt{2\pi}} x^{1/2} \Psi''(x^{3/2}) (e^{-(k-1)^2 x} - e^{-k^2 x}) \exp\left\{-\frac{x^3}{6} + \frac{x^2 \lambda}{2} - \frac{x \lambda^2}{2}\right\} dx \leq \frac{2C_1 C_2}{3} k^{-13/4}.$$

Thus, overall, we have that

$$\mathfrak{X}_1(k) = O(k^{-13/4}). \quad (7.22)$$

We return to (7.16). To evaluate $\mathfrak{X}_2(k)$, we will additionally require Theorem 2.29(b).

This gives us the Palm distribution for the point process; we have that

$$\begin{aligned} \mathfrak{X}_2(k) = & \int_0^\infty \int_0^\infty x^{3/2} \Phi'(x^{3/2}) (e^{-(k-1)^2 x} - e^{-k^2 x}) \Lambda^{(\lambda)}(x) \\ & \times y^{3/2} \Phi'(y^{3/2}) (e^{-(k-1)^2 y} - e^{-k^2 y}) \Lambda^{(\lambda-x)}(y) dy dx. \end{aligned}$$

If we evaluate this together with $\mathfrak{X}_3(k)$ (as defined in (7.19)), we obtain the expression

$$\begin{aligned} \mathfrak{X}_2(k) - \mathfrak{X}_3(k) = & \int_0^\infty \int_0^\infty x^{3/2} \Phi'(x^{3/2}) (e^{-(k-1)^2 x} - e^{-k^2 x}) \Lambda^{(\lambda)}(x) y^{3/2} \Phi'(y^{3/2}) \\ & \times (e^{-(k-1)^2 y} - e^{-k^2 y}) \Lambda^{(\lambda)}(y) \left(\frac{\Lambda^{(\lambda-x)}(y)}{\Lambda^{(\lambda)}(y)} - 1 \right) dy dx, \end{aligned} \quad (7.23)$$

where

$$\frac{\Lambda^{(\lambda-x)}(y)}{\Lambda^{(\lambda)}(y)} - 1 = \exp\left\{-\frac{xy(x+y-2\lambda)}{2}\right\} - 1. \quad (7.24)$$

We note from (7.22), and the fact that (7.16) is a non-negative expression, that, for some constant $C_3 > 0$, this expression must be bounded below by $-C_3 k^{-13/4}$. If $\lambda \leq 0$, in fact (7.24) is bounded above by 0 for all $x, y > 0$; we have thus established, via (7.16), that $\text{Var}(J(k)) = O(k^{-13/4})$, proving the claim for non-positive λ .

Otherwise, if $\lambda > 0$, we note that (7.24) is negative if $x > 2\lambda$ or $y > 2\lambda$; meanwhile, on $[0, 2\lambda] \times [0, 2\lambda]$, we can find $C_4 > 0$ (depending only on λ) such that for all $x, y \in [0, 2\lambda]$,

$$\frac{\Lambda^{(\lambda-x)}(y)}{\Lambda^{(\lambda)}(y)} - 1 \leq C_4 xy.$$

Applying this in (7.23) gives an upper bound on $\mathfrak{X}_2(k) - \mathfrak{X}_3(k)$ of

$$\mathfrak{X}_2(k) - \mathfrak{X}_3(k) \leq C_4 \left(\frac{1}{\sqrt{2\pi}} \int_0^{2\lambda} \Psi''(x^{3/2}) (e^{-(k-1)^2 x} - e^{-k^2 x}) \exp\left\{-\frac{x^3}{6} + \frac{x^2 \lambda}{2} - \frac{x \lambda^2}{2}\right\} dx \right)^2,$$

since the integrand now separates into two identical expressions in x and y . Similarly to our treatment of $\mathfrak{X}_1(k)$, we integrate separately over $[0, \lambda k^{-3/2}]$ and $(\lambda k^{-3/2}, 2\lambda)$ to obtain that the integral (before squaring) is $O(k^{-5/2})$, so we have that

$$-C_3 k^{-13/4} \leq \mathfrak{X}_2(k) - \mathfrak{X}_3(k) \leq C_5 k^{-5},$$

for constants $C_3 > 0$ and $C_5 > 0$. Substituting this, and (7.22), into (7.16) gives the claim. \square

We now come to the main result in this subsection, which is a local-time analogue of Proposition 7.1. Recall that

$$\begin{aligned} \mathfrak{J}(\ell) &:= \int_0^{\tau(\ell)} (X(u) - I(u)) \, du = \sum_{i=1}^{\infty} \tilde{A}_i(D_i) \mathbb{1}_{\{L_i(D_i) \leq \ell\}}, \\ \check{\mathfrak{J}}(\ell) &:= \int_0^{\check{\tau}(\ell)} (X(u) - I(u)) \, du = \sum_{i=1}^{\infty} \tilde{A}_i(D_i) \mathbb{1}_{\{L_i(D_i) < \ell\}} \end{aligned}$$

and

$$f(\ell) = \mathbb{E}[\mathfrak{J}(\ell)] = \mathbb{E}[\check{\mathfrak{J}}(\ell)].$$

Proposition 7.13. *The quantities $\mathfrak{J}(\ell) - f(\ell)$ and $\check{\mathfrak{J}}(\ell) - f(\ell)$ converge almost surely, as $\ell \rightarrow \infty$, to the same limit.*

Proof. We start with the claim about \mathfrak{J} . It is sufficient to show that, for $k \in \mathbb{N}$, the sequence $\mathfrak{J}(k^2) - f(k^2)$ converges almost surely as $k \rightarrow \infty$. Indeed,

$$\begin{aligned} \mathfrak{J}(\lfloor \sqrt{\ell} \rfloor^2) - f(\lfloor \sqrt{\ell} \rfloor^2) - g(\lceil \sqrt{\ell} \rceil) &\leq \mathfrak{J}(\ell) - f(\ell) \\ &\leq \mathfrak{J}(\lceil \sqrt{\ell} \rceil^2) - f(\lceil \sqrt{\ell} \rceil^2) + g(\lceil \sqrt{\ell} \rceil), \end{aligned}$$

whence the result follows by a standard sandwiching argument, since we already know that $g(k) \rightarrow 0$ as $k \rightarrow \infty$ (by Lemma 7.11).

It remains to show that

$$\sum_{j=1}^k (J(j) - g(j))$$

converges almost surely as $k \rightarrow \infty$. Define the event

$$E_j = \{ |J(j) - g(j)| > j^{-13/12} \}.$$

Then, if $\mathbb{P}(E_j \text{ infinitely often}) = 0$, it holds, by the Weierstrass M -test, that we do indeed have almost sure convergence (since we already know that $J(j) - g(j)$ is a.s. finite for all j).

But, by Chebyshev's inequality and Lemma 7.12,

$$\mathbb{P}(E_j) \leq j^{13/6} \text{Var}(J(j)) = O(j^{-13/12}),$$

so that

$$\sum_{j=1}^{\infty} \mathbb{P}(E_j) < \infty.$$

Thus, $\mathbb{P}(E_j \text{ infinitely often}) = 0$, by the first Borel–Cantelli lemma. Thus $\mathfrak{J}(\ell) - f(\ell)$ does indeed converge almost surely to some limit.

An almost-identical argument shows that $\check{\mathfrak{J}}(\ell) - f(\ell)$ converges almost surely. Note also that \mathfrak{J} and $\check{\mathfrak{J}}$ are, almost surely, equal except at only countably many points in $[0, \infty)$; if the limits had different values, it is a standard result of analysis that there would exist some M such that the functions were distinct for all $\ell \geq M$, giving a contradiction. Thus the limits must be almost surely equal. \square

Call the limit L'_λ , so that, almost surely,

$$\mathfrak{J}(\ell) - f(\ell) \rightarrow L'_\lambda$$

and

$$\check{\mathfrak{J}}(\ell) - f(\ell) \rightarrow L'_\lambda.$$

At present we know very little about L'_λ .

For our heuristic proof method, it would be useful to know that, for all $q \in (0, 2)$ and for all $\lambda \in \mathbb{R}$,

$$\mathbb{E}\left[e^{(q-1)(\mathfrak{J}(\ell)-f(\ell))}\right] \rightarrow \mathbb{E}\left[e^{(q-1)L'_\lambda}\right]$$

as $\ell \rightarrow \infty$, to derive similar results in real time. (Note that it is non-trivial that the right-hand side even exists; if it does, however, for some $q > 1$ and also for some $q < 1$, it implies that all moments of L'_λ exist.)

To see why this is useful, recall that we need to consider quantities of the form q^N , where N is some number of surplus edges discovered by the Aldous limit process, up to a certain point in its “exploration”. Suppose $\mathfrak{N}(\ell)$, then, is the number of surplus edges discovered up

to the point where the local time at 0 first hits ℓ . Then, conditionally on X and I ,

$$\mathfrak{N}(\ell) \sim \text{Poi}\left(\int_0^{\tilde{\tau}(\ell)} (X(u) - I(u)) du\right),$$

so

$$\mathbb{E}[q^{\mathfrak{N}(\ell)} \mid X, I] = \exp\{(q-1)\mathfrak{J}(\ell)\},$$

and so we need the expectation of the right-hand side to be well-behaved as $\ell \rightarrow \infty$.

Fortunately the necessary results will hold as long as a suitable uniform integrability property holds, which we prove in the next subsection.

7.1.2 Uniform integrability

We seek a local time analogue of Proposition 7.2. To do so, we first seek a bound on the moment generating function of quantities of the form

$$\sum_{x \in \Xi^{(\lambda)}} \phi(x)$$

for general (Borel-measurable) non-negative functions $\phi : [0, \infty) \rightarrow [0, \infty)$ (recalling that $\Xi^{(\lambda)}$ is the point process of component sizes in the limit process). A bound is given for certain point processes by Błaszczyszyn and Yogeshwaran [14, Lma. 3.1], which, for reference, we give shortly (in slightly adapted form). To state it, we first need a definition.

Definition 7.14 ([14, Defn. 1.2]). A point process Ξ on \mathbb{R}^d is α -weakly sub-Poisson if, for any disjoint Borel subsets $(B_i)_{i \in \mathbb{N}}$ of \mathbb{R}^d ,

$$\mathbb{E}\left[\prod_i \Xi(B_i)\right] \leq \prod_i \mathbb{E}[\Xi(B_i)].$$

Remark. The terminology “ α -weakly sub-Poisson” comes from [14], in which α^k is used to denote the k -th-order moment measure of the point process.

Lemma 7.15 ([14, Lma. 3.1]). *Let Ξ be a simple point process on \mathbb{R}^d with intensity Λ , which is α -weakly sub-Poisson. Let $\phi : \mathbb{R}^d \rightarrow [0, \infty)$ be Lebesgue measurable. Then*

$$\mathbb{E}\left[\exp\left\{\sum_{x \in \Xi} \phi(x)\right\}\right] \leq \exp\left\{\int_{\mathbb{R}^d} (e^{\phi(x)} - 1)\Lambda(x) dx\right\}.$$

However, this result is not directly applicable in our case because our point process is only α -weakly sub-Poisson in the case where $\lambda \leq 0$. We thus modify their results appropriately.

We first note the following bound on $\Lambda^{(\lambda)}(x)$.

Lemma 7.16. *For any $x, y \geq 0$, and any $\lambda \in \mathbb{R}$,*

$$\Lambda^{(\lambda-y)}(x) \leq \Lambda^{(\lambda)}(x) \exp\left\{\frac{x\lambda^2}{2}\right\} =: \tilde{\Lambda}^{(\lambda)}(x). \quad (7.25)$$

Proof. We established in the previous section (or can see easily by algebraic manipulations) that

$$\Lambda^{(\lambda-y)}(x) = \Lambda^{(\lambda)}(x) \exp\left\{-\frac{xy(x+y-2\lambda)}{2}\right\}. \quad (7.26)$$

We first note that, if $\lambda \leq 0$, the exponential is bounded above by 1 for all $x, y \geq 0$, and the result follows immediately. So we assume $\lambda > 0$.

The argument of the exponential above can be written as

$$-\frac{x}{2}y^2 + \frac{x(2\lambda-x)}{2}y,$$

a quadratic in y that, for $x \geq 0$, is maximised at $y = (2\lambda-x)/2$. However, if $x > 2\lambda$, then this is negative; since we only consider $y \geq 0$, the function is in fact decreasing for all (relevant) y , and the exponent is in fact always negative. So, for $x > 2\lambda$, we have

$$\Lambda^{(\lambda-y)}(x) \leq \Lambda^{(\lambda)}(x) \leq \Lambda^{(\lambda)}(x) \exp\left\{\frac{x\lambda^2}{2}\right\},$$

the second inequality being trivial. Meanwhile, if $x \leq 2\lambda$, we substitute in $y = (2\lambda-x)/2$ to give that

$$\Lambda^{(\lambda-y)}(x) \leq \Lambda^{(\lambda)}(x) \exp\left\{\frac{x\lambda^2}{2}\right\}. \quad \square$$

We set up some notation for the results that follow. As usual, let $\lambda \in \mathbb{R}$; let $\phi : (0, \infty) \rightarrow [0, \infty)$ be a Borel-measurable function, and let

$$V_{\lambda, \phi} = \sum_{i=1}^{\infty} \phi(D_i). \quad (7.27)$$

For all $k \in \mathbb{N}$, define the sets of compositions of k

$$\mathfrak{S}_k = \{(j, \mathbf{i}) : j \in [k], \mathbf{i} = (i_1, \dots, i_j), i_1, \dots, i_k \in \mathbb{N}, i_1 + \dots + i_j = k\}$$

(so, in particular, we require $i_1, \dots, i_k \geq 1$), and define

$$\mathfrak{S} = \{(k, j, \mathbf{i}) : k \in \mathbb{N}, (j, \mathbf{i}) \in \mathfrak{S}_k\}.$$

Finally, for all $(k, j, \mathbf{i}) \in \mathfrak{S}$, define

$$\begin{aligned} I^{(\lambda, \phi)}(k, j, \mathbf{i}) \\ = \int_0^\infty \cdots \int_0^\infty \phi(x_1)^{i_1} \cdots \phi(x_j)^{i_j} \Lambda^{(\lambda)}(x_1) \Lambda^{(\lambda-x_1)}(x_2) \cdots \Lambda^{(\lambda-x_1-\cdots-x_{j-1})}(x_j) dx_j \cdots dx_1, \end{aligned}$$

and, similarly, define

$$\tilde{I}^{(\lambda, \phi)}(k, j, \mathbf{i}) = \int_0^\infty \cdots \int_0^\infty \phi(x_1)^{i_1} \cdots \phi(x_j)^{i_j} \tilde{\Lambda}^{(\lambda)}(x_1) \cdots \tilde{\Lambda}^{(\lambda)}(x_j) dx_j \cdots dx_1,$$

where $\tilde{\Lambda}$ is defined in Lemma 7.16.

Before we go on, we note the following corollary to Lemma 7.16, whose proof is immediate upon inspecting the formulae for I and \tilde{I} , and applying that lemma.

Corollary 7.17. *For all $\lambda \in \mathbb{R}$ and for all measurable $\phi : (0, \infty) \rightarrow [0, \infty)$,*

$$I^{(\lambda, \phi)}(k, j, \mathbf{i}) \leq \tilde{I}^{(\lambda, \phi)}(k, j, \mathbf{i}).$$

We now show that moments of $V_{\lambda, \phi}$ (and $\tilde{V}_{\lambda, \phi}$) may be expressed in a convenient way in terms of the integrals defined above.

Lemma 7.18. *Fix $\lambda \in \mathbb{R}$ and $\phi : (0, \infty) \rightarrow [0, \infty)$, where ϕ is measurable. Let $\tilde{\Xi}^{(\lambda)}$ be a Poisson point process with intensity function $\tilde{\Lambda}^{(\lambda)}$, and let*

$$\tilde{V}_{\lambda, \phi} = \sum_{x \in \tilde{\Xi}^{(\lambda)}} \phi(x).$$

Then there exists a collection of non-negative constants $(c_{j, \mathbf{i}}^{(k)} : (k, j, \mathbf{i}) \in \mathfrak{S})$, which do not depend on λ or ϕ , such that:

(I) *for all $k \in \mathbb{N}$, the k -th moment of $V_{\lambda, \phi}$ may be expressed as*

$$\sum_{(j, \mathbf{i}) \in \mathfrak{S}_k} c_{j, \mathbf{i}}^{(k)} I^{(\lambda, \phi)}(k, j, \mathbf{i});$$

(II) for all $k \in \mathbb{N}$, the k -th moment of $\tilde{V}_{\lambda, \phi}$ may be expressed as

$$\sum_{(j, \mathbf{i}) \in \mathfrak{S}_k} c_{j, \mathbf{i}}^{(k)} \tilde{I}^{(\lambda, \phi)}(k, j, \mathbf{i}).$$

Remark. The proof here is similar to that of [40, Cor. 8.7], which gives an exact formula for the integral for the case where $\phi(x) = \mathbb{1}_{\{x \in B\}}$, for a Borel set B ; the hypotheses of our statement are significantly more general, and the conclusion is consequently much weaker.

Proof. We prove this by induction on k ; our inductive hypothesis will be that, for each k , both (I) and (II) hold, and that, for $(j, \mathbf{i}) \in \mathfrak{S}_k$, the constants $c_{j, \mathbf{i}}^{(k)}$ do not depend on either λ or ϕ . For $k = 1$, this clearly holds by Campbell's theorem (see, e.g., [22, (9.5.2)]).

Now suppose it holds for $k = 1, \dots, k' - 1$. Then let

$$F(x, \Xi) = \phi(x) V(\Xi)^{k'-1},$$

where we treat V as a function of a generic point process Ξ given by

$$V(\Xi) = \sum_{x \in \Xi} \phi(x).$$

Since F is a non-negative function, [40, Thm. 8.2] gives

$$\begin{aligned} \mathbb{E} \left[(V(\Xi^{(\lambda)}))^{k'} \right] &= \mathbb{E} \left[\sum_{i=1}^{\infty} \phi(D_i) (V(\Xi^{(\lambda)}))^{k'-1} \right] \\ &= \mathbb{E} \left[\sum_{i=1}^{\infty} F(D_i, \Xi^{(\lambda)}) \right] \\ &= \int_0^{\infty} \phi(x) \mathbb{E} \left[\left(\phi(x) + \sum_{y \in \Xi^{(\lambda-x)}} \phi(y) \right)^{k'-1} \right] \Lambda^{(\lambda)}(x) dx \\ &= \sum_{i=1}^{k'} \binom{k'-1}{i-1} \int_0^{\infty} \phi(x)^i \mathbb{E} \left[\left(\sum_{y \in \Xi^{(\lambda-x)}} \phi(y) \right)^{k'-i} \right] \Lambda^{(\lambda)}(x) dx, \end{aligned}$$

using the binomial theorem. We note that the term in the sum where $i = k'$ can be expressed as

$$\int_0^{\infty} \phi(x)^{k'} \Lambda^{(\lambda)}(x) dx;$$

for the other terms, we apply the inductive hypothesis. Noting that binomial coefficients are all non-negative, we have (I). Since the constants $c_{j, \mathbf{i}}^{(k')}$ are generated only by the constants for $k = k' - 1$, and by the binomial theorem, they do not depend on λ or on ϕ .

Moreover, as $\tilde{\Xi}^{(\lambda)}$ is a Poisson point process, it holds that

$$\mathbb{E}\left[\left(V(\tilde{\Xi}^{(\lambda)})\right)^{k'}\right] = \sum_{i=1}^{k'} \binom{k'-1}{i-1} \int_0^\infty \phi(x)^i \mathbb{E}\left[\left(\sum_{y \in \tilde{\Xi}^{(\lambda)}} \phi(y)\right)^{k'-i}\right] \tilde{\Lambda}^{(\lambda)}(x) dx,$$

which similarly gives us (II); since the constants are generated in an identical way as in (I), using the constants for $k = k' - 1$ and the binomial theorem, they must also be the same. \square

Lemma 7.19. *We may bound*

$$\mathbb{E}[\exp\{V_{\lambda,\phi}\}] \leq \exp\left\{\int_0^\infty (e^{\phi(x)} - 1) \tilde{\Lambda}^{(\lambda)}(x) dx\right\}. \quad (7.28)$$

This proof follows the strategy of the proof from [14] of our Lemma 7.15; as previously noted, that result is not directly applicable because our point process is only α -weakly sub-Poisson (as in Definition 7.14) in the case where $\lambda \leq 0$.

Proof. We start by noting that, since $\tilde{\Xi}^{(\lambda)}$ is a Poisson point process, an exact formula for $\mathbb{E}[\exp\{\tilde{V}_{\lambda,\phi}\}]$ is known (see, for instance, [22, (9.4.17)]); we have that

$$\mathbb{E}[\exp\{\tilde{V}_{\lambda,\phi}\}] = \exp\left\{\int_0^\infty (e^{\phi(x)} - 1) \tilde{\Lambda}^{(\lambda)}(x) dx\right\}. \quad (7.29)$$

where the right-hand side of (7.29) is exactly the right-hand side of (7.28);

Now we consider $V_{\lambda,\phi}$, for which we use the expansion of the moment generating function: namely, that

$$\mathbb{E}[\exp\{V_{\lambda,\phi}\}] = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \mathbb{E}[V_{\lambda,\phi}^k]. \quad (7.30)$$

Since $V_{\lambda,\phi}$ is non-negative, so are all its moments. Moreover, by part (I) of Lemma 7.18, we have that

$$\mathbb{E}[V_{\lambda,\phi}^k] = \sum_{(j,\mathbf{i}) \in \mathfrak{S}_k} c_{j,\mathbf{i}}^{(k)} I^{\lambda,\phi}(k, j, \mathbf{i}),$$

where the $c_{j,\mathbf{i}}^{(k)}$ are all non-negative. Thus we may apply Corollary 7.17, followed by part (II) of Lemma 7.18, to obtain that

$$\mathbb{E}[V_{\lambda,\phi}^k] \leq \sum_{(j,\mathbf{i}) \in \mathfrak{S}_k} c_{j,\mathbf{i}}^{(k)} \tilde{I}^{\lambda,\phi}(k, j, \mathbf{i}) = \mathbb{E}[\tilde{V}_{\lambda,\phi}^k].$$

Substituting this bound into (7.30) tells us that

$$\mathbb{E}[\exp\{V_{\lambda,\phi}\}] \leq 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \mathbb{E}[\tilde{V}_{\lambda,\phi}^k] = \mathbb{E}[\exp\{\tilde{V}_{\lambda,\phi}\}].$$

Combining with (7.29) gives the result. \square

Proposition 7.20. *Fix $q \in (1, 2)$. Then, for all $\ell > 0$, the quantity $\mathbb{E}[e^{(q-1)(\mathfrak{J}(\ell)-f(\ell))}]$ is finite; moreover, there exists $r > 1$ such that $e^{(q-1)(\mathfrak{J}(\ell)-f(\ell))}$ is bounded in L^r as $\ell \rightarrow \infty$.*

Proof. We first show that there exists $r > 1$ such that

$$G(\ell, r) := \mathbb{E} \left[\exp \left\{ \sum_{i=1}^{\infty} r(q-1) \tilde{A}_i(D_i) \mathbf{1}_{\{L_i(D_i) \leq \ell\}} \right\} \right]$$

is finite, which in particular implies the first part of the claim. Using the tower law, and the conditional independence of the \tilde{A}_i and L_i terms, we have that

$$\begin{aligned} G(\ell, r) &= \mathbb{E} \left[\prod_{i=1}^{\infty} \mathbb{E} \left[\exp \{ r(q-1) \tilde{A}_i(D_i) \mathbf{1}_{\{L_i(D_i) \leq \ell\}} \} \mid D_i \right] \right] \\ &= \mathbb{E} \left[\prod_{i=1}^{\infty} \left(\exp \{ \Phi(r' D_i^{3/2}) - \Phi(D_i^{3/2}) \} (1 - e^{-\ell D_i}) + e^{-\ell D_i} \right) \right], \end{aligned}$$

where we take $r' := r(q-1) + 1 > 1$; this follows by Lemma 7.7. We now apply the explicit formula from Lemma 7.19 using

$$\phi(x) = \log \left((1 - e^{-\ell x}) \exp \{ \Phi(r' x^{3/2}) - \Phi(x^{3/2}) \} + e^{-\ell x} \right) \geq 0,$$

thus obtaining that

$$G(\ell, r) \leq \exp \left\{ \int_0^{\infty} (1 - e^{-\ell x}) (\Psi(r' x^{3/2}) - \Psi(x^{3/2})) \frac{1}{\sqrt{2\pi}} x^{-5/2} \exp \left(-\frac{x^3}{6} + \frac{x^2 \lambda}{2} \right) dx \right\}$$

(using the fact that $\Phi = \log \Psi$).

We split $\log G$ into two parts; let

$$\mathfrak{Y}_1(\ell, r) = \int_0^1 (1 - e^{-\ell x}) (\Psi(r' x^{3/2}) - \Psi(x^{3/2})) \frac{1}{\sqrt{2\pi}} x^{-5/2} \exp \left(-\frac{x^3}{6} + \frac{x^2 \lambda}{2} \right) dx$$

and

$$\mathfrak{Y}_2(\ell, r) = \int_1^{\infty} (1 - e^{-\ell x}) (\Psi(r' x^{3/2}) - \Psi(x^{3/2})) \frac{1}{\sqrt{2\pi}} x^{-5/2} \exp \left(-\frac{x^3}{6} + \frac{x^2 \lambda}{2} \right) dx,$$

so that $\log G(\ell, r) = \mathfrak{Y}_1(\ell, r) + \mathfrak{Y}_2(\ell, r)$. We start with $\mathfrak{Y}_2(\ell, r)$, first bounding the $1 - e^{-\ell x}$ term above by 1. This gives a bound on the integrand that is independent of ℓ ; moreover, we note from (7.2) that this bound is, as $x \rightarrow \infty$, asymptotically equivalent to

$$\frac{1}{2\sqrt{2\pi}} r'^2 x^{1/2} \exp\left\{\frac{(r'^2 - 4)x^3}{24} + \frac{x^2 \lambda}{2}\right\}.$$

This is integrable if $(r')^2 - 4 < 0$, which holds for $r > 1$ if and only if $r < 1/(q - 1)$. We can choose such an r , since $q < 2$; take, for instance,

$$r = \frac{q}{2(q - 1)},$$

which gives $r' = (q + 2)/2$. Hence, we have that $\mathfrak{Y}_2(\ell, r)$ is finite for all $\ell > 0$, and moreover that $\mathfrak{Y}_2(\ell, r) = O(1)$ as $\ell \rightarrow \infty$.

We now consider \mathfrak{Y}_1 ; we will continue to use the choice of r (and hence that of r') that we just established. Note that, as $x \downarrow 0$, it holds that

$$\Psi(r'x^{3/2}) - \Psi(x^{3/2}) = \frac{\mathbb{E}[A]q}{2} x^{3/2} + O(x^3);$$

thus, we may choose $C_6 > 0$ (independent of ℓ) such that, for all $x \in (0, 1)$,

$$(\Psi(r'x^{3/2}) - \Psi(x^{3/2})) \frac{1}{\sqrt{2\pi}} x^{-5/2} \exp\left(-\frac{x^3}{6} + \frac{x^2 \lambda}{2}\right) \leq \frac{q\mathbb{E}[A]}{2\sqrt{2\pi}} x^{-1} + C_6 \sqrt{x} \quad (7.31)$$

by considering the behaviour of the whole function near x .

Then we have that

$$\begin{aligned} \mathfrak{Y}_1\left(\ell, \frac{q}{2(q-1)}\right) &\leq \frac{q\mathbb{E}[A]}{2\sqrt{2\pi}} \int_0^1 x^{-1}(1 - e^{-\ell x}) dx + C_6 \int_0^1 \sqrt{x}(1 - e^{-\ell x}) dx \\ &= \frac{q\mathbb{E}[A]}{2\sqrt{2\pi}} \log \ell + O(1), \end{aligned}$$

using Lemma 7.10 for the asymptotics of the first integral.

Thus we have shown that $\log G(\ell, r)$ is finite for all $\ell > 0$ for this choice of r , and moreover that

$$\log G\left(\ell, \frac{q}{2(q-1)}\right) \leq \frac{q\mathbb{E}[A]}{2\sqrt{2\pi}} \log \ell + O(1),$$

which shows that

$$\mathbb{E}\left[e^{r(q-1)(\mathfrak{J}(\ell)-f(\ell))}\right] \leq \ell^{q\mathbb{E}[A]/(2\sqrt{2\pi})} \exp\{O(1) - qf(\ell)/2\}. \quad (7.32)$$

We note moreover, from Proposition 7.9, that

$$f(\ell) = \frac{\mathbb{E}[A]}{\sqrt{2\pi}} \log \ell + O(1);$$

substituting into (7.32) gives us that, as $\ell \rightarrow \infty$,

$$\mathbb{E}\left[e^{r(q-1)(\mathfrak{J}(\ell)-f(\ell))}\right] = O(1),$$

since the terms in ℓ cancel. Hence the quantity $e^{(q-1)(\mathfrak{J}(\ell)-f(\ell))}$ is bounded in L^r for $r = q/(2(q-1))$. \square

We note that the above argument may be adapted to show that, for $q = 2$, the expectation $\mathbb{E}\left[e^{(q-1)(\mathfrak{J}(\ell)-f(\ell))}\right]$ is finite if $\lambda < 0$. This fails for $\lambda \geq 0$ when $q = 2$, and also for all values of λ when $q > 2$. This is consistent with our expectation that it should be possible to extend a proof for $q < 2$ to the case where $q = 2$ and $\lambda < 0$, but not to the other cases. This expectation comes from the facts that:

- by Theorem 18 of [43], if $q = 2$, it is only when $\lambda < 0$ that the largest components are of order $\Theta_p(n^{2/3})$, and
- there is a completely different critical behaviour for $q > 2$, as given by [16] and [58].

Corollary 7.21. *Fix $q \in (1, 2)$. Then, for all $\ell > 0$, the quantity $\mathbb{E}\left[e^{(q-1)(\check{\mathfrak{J}}(\ell)-f(\ell))}\right]$ is finite; moreover, there exists $r > 1$ such that $e^{(q-1)(\check{\mathfrak{J}}(\ell)-f(\ell))}$ is bounded in L^r as $\ell \rightarrow \infty$.*

Proof. This is immediate from Lemma 7.8—i.e. the fact that, for all $\ell \geq 0$, a.s. $\check{\mathfrak{J}}(\ell) = \mathfrak{J}(\ell)$. \square

Recall that L'_λ is the almost sure limit of $\mathfrak{J}(\ell) - f(\ell)$ as $\ell \rightarrow \infty$, whose existence was established in Proposition 7.13. We now give our local-time analogue of Proposition 7.2.

Corollary 7.22. *As $\ell \rightarrow \infty$, the collection of random variables*

$$\left(e^{(q-1)(\check{\mathfrak{J}}(\ell)-f(\ell))}\right)_{\ell \geq 0}$$

is uniformly integrable; moreover,

$$\mathbb{E}\left[e^{(q-1)(\mathfrak{J}(\ell)-f(\ell))}\right] \rightarrow \mathbb{E}\left[e^{(q-1)L'_\lambda}\right].$$

The equivalent result holds for $\check{\mathfrak{J}}$.

Proof. The first part follows immediately from the fact that a collection of random variables bounded in L^r for $r > 1$ is uniformly integrable; the second then follows from this and Proposition 7.13, via Vitali's convergence theorem. \square

7.2 Converting to “real” time

We now wish to prove Propositions 7.1 and 7.2 using the local-time analogues that we have established (Proposition 7.13 and Corollary 7.22).

For this, we will need a way to convert between the two settings. Fortunately, we have one: we can use the fact that, as described at the start of Section 7.1, the local time at t is given by $-I(t)$. Meanwhile, we have already constructed τ and $\check{\tau}$, the right- and left-continuous inverse local times, respectively. Thus we have that

$$\check{\tau}(-I(t)) \leq t \leq \tau(-I(t)). \tag{7.33}$$

We first establish some control on the behaviour of $I(t)$. Recall that we fix $\lambda \in \mathbb{R}$, and define, for $B(t)$ a Brownian motion,

$$X(t) = B(t) - \frac{t^2}{2} + \lambda t;$$

$$I(t) = \inf_{0 \leq u \leq t} X(u).$$

Lemma 7.23. *As $t \rightarrow \infty$,*

$$I(t) = -\frac{t^2}{2} + \lambda t + O_p(t^{1/2}).$$

In the proof, we make use of a comparison with two Brownian motions with a linear drift:

$$X^{(t)}(x) = B(x) + \frac{t^2}{2} + (\lambda - t)x$$

and

$$X_{(t)}(x) = B(x) + \left(\lambda - \frac{t}{2}\right)x,$$

with the same B as above. Note that, for $0 \leq x \leq t$,

$$X_{(t)}(x) \leq X(x) \leq X^{(t)}(x),$$

and so, if we define

$$I^{(t)}(x) = \inf_{0 \leq u \leq x} X^{(t)}(u)$$

and similarly for $I_{(t)}$, we have that, for all $x \in [0, t]$,

$$I_{(t)}(x) \leq I(x) \leq I^{(t)}(x).$$

We will use these bounds in our proof.

Proof. Fix $a, \mu \in \mathbb{R}$. Let

$$X^{(a,\mu)}(x) = B(x) + a - \mu x,$$

and let

$$I^{(a,\mu)}(x) = \inf_{0 \leq u \leq x} X^{(a,\mu)}(u).$$

We seek the mean and variance of $I^{(a,\mu)}(x)$. We first consider the case where $a = 0$.

Recall that

$$\text{Erf}(x) = \begin{cases} \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt & \text{for } x \geq 0; \\ -\text{Erf}(-x) & \text{for } x < 0, \end{cases}$$

and $\text{Erfc}(x) = 1 - \text{Erf}(x)$. We established in (4.32) that

$$\begin{aligned} \mathbb{E}[I^{(0,\mu)}(x)] &= -\frac{1}{2}\mu x + \left(\frac{1}{2\mu} + \frac{1}{2}\mu x\right) \text{Erf}\left(-\mu\sqrt{\frac{x}{2}}\right) - \sqrt{\frac{x}{2\pi}} \exp\left\{-\frac{\mu^2 x}{2}\right\} \\ &= -\mu x - \frac{1}{2\mu} + \left(\frac{1}{2\mu} + \frac{1}{2}\mu x\right) \text{Erfc}\left(\mu\sqrt{\frac{x}{2}}\right) - \sqrt{\frac{x}{2\pi}} \exp\left\{-\frac{\mu^2 x}{2}\right\}. \end{aligned} \tag{7.34}$$

Similarly, we calculated the second moment in (4.33), obtaining

$$\begin{aligned} \mathbb{E}[I^{(0,\mu)}(x)^2] &= x + \frac{\mu^2 x^2}{2} + \left(\frac{1}{\mu} + \mu x\right) \sqrt{\frac{x}{2\pi}} \exp\{-\mu^2 x/2\} + \operatorname{Erf}\left(-\mu\sqrt{\frac{x}{2}}\right) \left[\frac{1}{2\mu^2} - x - \frac{\mu^2 x^2}{2}\right] \\ &= 2x - \frac{1}{2\mu^2} + \mu^2 x^2 + \left(\frac{1}{\mu} + \mu x\right) \sqrt{\frac{x}{2\pi}} \exp\{-\mu^2 x/2\} + \operatorname{Erfc}\left(\mu\sqrt{\frac{x}{2}}\right) \left[\frac{1}{2\mu^2} - x - \frac{\mu^2 x^2}{2}\right]. \end{aligned}$$

Then

$$\begin{aligned} \operatorname{Var}(I^{(0,\mu)}(x)) &= 3x - \frac{3}{4\mu^2} + \left(\frac{1}{\mu^2} + \frac{\mu^2 x^2}{2} + \frac{x}{2}\right) \operatorname{Erfc}\left(\mu\sqrt{\frac{x}{2}}\right) - \mu x \sqrt{\frac{x}{2\pi}} \exp\left\{-\frac{\mu^2 x}{2}\right\} \\ &\quad - \left[\left(\frac{\mu x}{2} + \frac{1}{2\mu}\right) \operatorname{Erfc}\left(\mu\sqrt{\frac{x}{2}}\right) - \sqrt{\frac{x}{2\pi}} \exp\left\{-\frac{\mu^2 x}{2}\right\}\right]^2. \end{aligned} \quad (7.35)$$

Clearly taking $a \neq 0$ simply induces a shift in the distribution of $I^{(a,\mu)}(x)$ compared to when $a = 0$, so it must hold that

$$\mathbb{E}[I^{(a,\mu)}(x)] = a + \mathbb{E}[I^{(0,\mu)}(x)]$$

and

$$\operatorname{Var}(I^{(a,\mu)}(x)) = \operatorname{Var}(I^{(0,\mu)}(x)).$$

Now we note that

$$X^{(t)} = X^{(t^2/2, t-\lambda)} \quad \text{and} \quad X_{(t)} = X^{(0, t/2-\lambda)},$$

so we may use (7.34) to conclude that, as $t \rightarrow \infty$,

$$\mathbb{E}[I(t)] \leq \mathbb{E}[I^{(t)}(t)] = \frac{t^2}{2} - (t-\lambda)t + O\left(\frac{1}{t}\right) = \lambda t - \frac{t^2}{2} + O\left(\frac{1}{t}\right)$$

and

$$\mathbb{E}[I(t)] \geq \mathbb{E}[I_{(t)}(t)] = -\left(\frac{t}{2} - \lambda\right)t + O\left(\frac{1}{t}\right) = \lambda t - \frac{t^2}{2} + O\left(\frac{1}{t}\right),$$

so indeed

$$\mathbb{E}[I(t)] = \lambda t - \frac{t^2}{2} + O\left(\frac{1}{t}\right). \quad (7.36)$$

Similarly, we note using (7.35) that

$$\text{Var}(I_{(t)}(t)), \text{Var}(I^{(t)}(t)) = 3t + O\left(\frac{1}{t^2}\right).$$

Hence, using Chebyshev's inequality, we have that

$$I_{(t)}(t), I^{(t)}(t) = \lambda t - \frac{t^2}{2} + O_p(t^{1/2}).$$

Since $I_{(t)}(t) \leq I(t) \leq I^{(t)}(t)$, this completes the proof. \square

We are now in a position where we can prove Proposition 7.1, one of the main results of this section. We first restate it, for convenience.

Proposition 7.1. *There exists a random variable L_λ such that, as $t \rightarrow \infty$,*

$$\int_0^t (X(u) - I(u)) du - \frac{1}{2} \log t \xrightarrow{p} L_\lambda.$$

Proof. We first note that $I(t) \rightarrow -\infty$ almost surely as $t \rightarrow \infty$ (e.g. since $X(t)$ does). So, from Proposition 7.13, it holds that

$$\int_0^{\tau(-I(t))} (X(u) - I(u)) du - f(-I(t)) \rightarrow L'_\lambda$$

and

$$\int_0^{\check{\tau}(-I(t))} (X(u) - I(u)) du - f(-I(t)) \rightarrow L'_\lambda$$

almost surely. We recall (7.33), namely that

$$\check{\tau}(-I(t)) \leq t \leq \tau(-I(t)),$$

Thus, by a sandwich argument, it holds that

$$\int_0^t (X(u) - I(u)) du - f(-I(t)) \rightarrow L'_\lambda \tag{7.37}$$

almost surely.

We now assess $f(-I(t))$, noting that, by Proposition 7.9,

$$f(-I(t)) = \frac{1}{4} \log(-I(t)) + C_\lambda + o(1),$$

where the $o(1)$ term converges to 0 almost surely. Meanwhile, it holds, by Lemma 7.23, that

$$\begin{aligned} f(-I(t)) &= \frac{1}{4} \log\left(\frac{t^2}{2} - \lambda t + O_p(t^{1/2})\right) + C_\lambda + o_p(1) \\ &= \frac{1}{4} \log\left(\frac{t^2}{2} \left[1 + O_p\left(\frac{1}{t}\right)\right]\right) + C_\lambda + o_p(1) \\ &= \frac{1}{2} \log t - \frac{1}{4} \log 2 + C_\lambda + o_p(1). \end{aligned}$$

Putting this together with (7.37) gives the result, where we take

$$L_\lambda = L'_\lambda + \frac{1}{4} \log 2 - C_\lambda. \quad \square$$

Remark. We believe this convergence holds almost surely as well, though did not show this. The evidence (while admittedly somewhat weak) starts with the fact that the analogous convergence is almost sure in the local-time setting, and there's no real reason to expect that that should fail when we convert to "real" time. In the local-time setting, each excursion is added in its entirety to the integral as it occurs, whereas in "real" time we add each excursion progressively; however, we wouldn't expect this to cause too many problems, especially as the excursions get smaller and smaller (in particular, we know that $X(t) - I(t) \rightarrow 0$ almost surely, by Proposition 4.13). All this being said, convergence in probability is strong enough for our subsequent results.

Before we go on, we prove Corollary 7.3, which we recall is a weak law of large numbers for the number of surplus edges in the limit process.

Proof of Corollary 7.3. Conditionally on X and I , we know that

$$N(t) \sim \text{Poi}\left(\int_0^t (X(u) - I(u)) du\right).$$

Fix an arbitrary $\varepsilon > 0$. Then

$$\begin{aligned} \mathbb{P}\left(\left|\frac{2N(t)}{\log t} - 1\right| > \varepsilon\right) &\leq \mathbb{P}\left(\left|\frac{2\int_0^t(X(u)-I(u))du}{\log t} - 1\right| > \frac{\varepsilon}{2}\right) \\ &\quad + \mathbb{P}\left(\left|\frac{2N(t)}{\log t} - 1\right| > \varepsilon \mid \left|\frac{2\int_0^t(X(u)-I(u))du}{\log t} - 1\right| \leq \frac{\varepsilon}{2}\right). \end{aligned}$$

That the first term on the right-hand side converges to 0 as $t \rightarrow \infty$ follows from Proposition 7.1, so it suffices to show that the second term also converges to 0. We note that we can write this second term as

$$\begin{aligned} &\mathbb{P}\left(\frac{2N(t)}{\log t} > 1 + \varepsilon \mid \left|\frac{2\int_0^t(X(u)-I(u))du}{\log t} - 1\right| \leq \frac{\varepsilon}{2}\right) \\ &\quad + \mathbb{P}\left(\frac{2N(t)}{\log t} < 1 - \varepsilon \mid \left|\frac{2\int_0^t(X(u)-I(u))du}{\log t} - 1\right| \leq \frac{\varepsilon}{2}\right). \end{aligned} \tag{7.38}$$

To simplify notation slightly, suppose that $(P(\lambda))_{\lambda>0}$ is a family of random variables such that $P(\lambda) \sim \text{Poi}(\lambda)$. We note that, as a standard property of the Poisson distribution, we may couple these such that $P(\lambda_1) \leq P(\lambda_2)$ whenever $\lambda_1 \leq \lambda_2$. Further, we let Λ be a random variable independent of the family $(P(\lambda))_{\lambda>0}$.

Then, taking $\Lambda = \int_0^t(X(u)-I(u))du$, the first term in (7.38) is equal to

$$\begin{aligned} &\mathbb{P}\left(P(\Lambda) > (1 + \varepsilon)\frac{1}{2}\log t \mid \left(1 - \frac{\varepsilon}{2}\right)\frac{1}{2}\log t \leq \Lambda \leq \left(1 + \frac{\varepsilon}{2}\right)\frac{1}{2}\log t\right) \\ &\quad \leq \mathbb{P}\left(P\left(\left(1 + \frac{\varepsilon}{2}\right)\frac{1}{2}\log t\right) > (1 + \varepsilon)\frac{1}{2}\log t\right), \end{aligned}$$

using the coupling established for the family $(P(\lambda))_{\lambda>0}$. We then have that, by Chebyshev's inequality,

$$\mathbb{P}\left(P\left(\left(1 + \frac{\varepsilon}{2}\right)\frac{1}{2}\log t\right) > (1 + \varepsilon)\frac{1}{2}\log t\right) \leq \frac{(1 + \varepsilon/2)(\log t)/2}{\varepsilon^2(\log t)^2/16} = \frac{8(1 + \varepsilon/2)}{\varepsilon^2 \log t} \rightarrow 0$$

as $t \rightarrow \infty$. A similar argument shows that the second term in (7.38) also converges to zero, completing the proof. \square

We now proceed towards a proof of Proposition 7.2—i.e., to show that

$$\left(\exp\left\{(q-1)\left[\int_0^t(X(u)-I(u))du - \frac{1}{2}\log t\right]\right\}\right)_{t \geq 0}$$

is uniformly integrable. We first give a very weak bound on the random variables in the family.

Lemma 7.24. Fix $\lambda \in \mathbb{R}$ and $\theta > 1$, not depending on t . Then as $t \rightarrow \infty$,

$$\mathbb{E} \left[\exp \left\{ \theta \int_0^t (X(u) - I(u)) du \right\} \right] = O \left(t^{3/2} e^{\theta^2 t^3 / 2 + \lambda \theta t^2} \right).$$

This is far from optimal (indeed, we will later show that the quantity on the left is of polynomial order in t), but we will need to “bootstrap” from this bound in order to reach that conclusion. The proof of Lemma 7.24 is obtained via comparison to Brownian motion with linear drift; we defer the proof to Appendix A (p. 183).

Now, to show the uniform integrability, we will consider separately the cases where $-I(t)$ is “large” and “small”, starting with the small case.

Lemma 7.25. *The collection of random variables*

$$\left(\exp \left\{ (q-1) \left(\int_0^t (X(u) - I(u)) du - \frac{1}{2} \log t \right) \right\} \mathbb{1}_{\{I(t) \geq -\alpha t^2\}} \right)_{t \geq 0}$$

is bounded in L^r , for the same r as in Proposition 7.20.

Proof. As in the statement of the lemma, let $r > 1$ take the value given by Proposition 7.20, and let $\theta = r(q-1)$. Fix $\alpha > 1/2$. Then

$$\begin{aligned} & \mathbb{E} \left[\exp \left\{ \theta \int_0^t (X(u) - I(u)) du - \frac{\theta}{2} \log t \right\} \mathbb{1}_{\{I(t) \geq -\alpha t^2\}} \right] \\ & \leq \mathbb{E} \left[e^{\theta(\mathfrak{J}(-I(t)) - f(\alpha t^2))} \mathbb{1}_{\{I(t) \geq -\alpha t^2\}} \right] \exp \left\{ \theta f(\alpha t^2) - \frac{\theta}{2} \log t \right\} \\ & \leq \mathbb{E} \left[e^{\theta(\mathfrak{J}(\alpha t^2) - f(\alpha t^2))} \mathbb{1}_{\{I(t) \geq -\alpha t^2\}} \right] \exp \left\{ \theta f(\alpha t^2) - \frac{\theta}{2} \log t \right\}, \end{aligned}$$

since \mathfrak{J} is increasing in its argument. But then

$$\mathbb{E} \left[e^{\theta(\mathfrak{J}(\alpha t^2) - f(\alpha t^2))} \mathbb{1}_{\{I(t) \geq -\alpha t^2\}} \right] \leq \mathbb{E} \left[e^{\theta(\mathfrak{J}(\alpha t^2) - f(\alpha t^2))} \right] \leq \sup_{\ell > 0} \mathbb{E} \left[e^{\theta(\mathfrak{J}(\ell) - f(\ell))} \right] < \infty,$$

by Proposition 7.20. Meanwhile, by Proposition 7.9, as $t \rightarrow \infty$,

$$\begin{aligned} \exp \left\{ \theta f(\alpha t^2) - \frac{\theta}{2} \log t \right\} &= \exp \left\{ \frac{\theta}{4} \log(\alpha t^2) + C_\lambda + o(1) - \frac{\theta}{2} \log t \right\} \\ &= \exp \{ \log \alpha + C_\lambda + o(1) \}; \end{aligned}$$

since f is also well-defined and finite for all $t > 0$, this implies that

$$\sup_{t > 0} \exp \left\{ \theta f(\alpha t^2) - \frac{\theta}{2} \log t \right\} < \infty.$$

Thus we can find a uniform bound on

$$\mathbb{E} \left[\exp \left\{ \theta \int_0^t (X(u) - I(u)) du - \frac{\theta}{2} \log t \right\} \mathbb{1}_{\{I(t) \geq -at^2\}} \right]$$

for all $t > 0$, and so

$$\exp \left\{ (q-1) \int_0^t (X(u) - I(u)) du - \frac{q-1}{2} \log t \right\} \mathbb{1}_{\{I(t) \geq -at^2\}}$$

is bounded in L^r . □

The large $-I(t)$ case is much harder. For it, we will prove the following lemma.

Lemma 7.26. *Fix $\alpha > 0$, $\theta > 0$ and $t > 0$. Then the random variables*

$$\exp \left\{ \theta \int_0^t (X(u) - I(u)) du \right\}$$

and

$$\mathbb{1}_{\{I(t) \leq -at^2\}}$$

are negatively correlated.

This will allow us to separate the indicator function and the exponential, and bound them separately.

To prove Lemma 7.26, we will quote a version of the FKG inequality, by Ma and Privault [45], for which we need the following definition.

Definition 7.27. Let $a, b \in \mathbb{R}$ with $a < b$. For continuous functions $f_1, f_2 : [a, b] \rightarrow \mathbb{R}$, define that $f_1 \preceq f_2$ if and only if, for all s, u with $a \leq s \leq u \leq b$,

$$f_1(u) - f_1(s) \leq f_2(u) - f_2(s).$$

Theorem 7.28 ([45, Thm. 3.1]). *Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ in which Ω is the set of continuous functions from $[0, 1]$ to \mathbb{R} , and in which we equip Ω with the partial order \preceq . Let X, Y be real-valued random variables on this space such that, if $f_1 \preceq f_2$, then $X(f_1) \leq X(f_2)$ almost surely, and similarly for Y . Then*

$$\text{Cov}(X, Y) \geq 0.$$

Remark. If a random variable X satisfies the hypothesis of the theorem, we say it is an *increasing function* of the state space. We note, by a simple scaling argument, that Theorem 7.28 also

holds for continuous functions on $[a, b]$, for any $a, b \in \mathbb{R}$ with $a < b$.

Proof of Lemma 7.26. We can (almost surely) consider the random variables in the statement as random variables on the space of continuous paths over $[0, t]$, with the probability measure being chosen so that the paths have the same law as $(X(s))_{0 \leq s \leq t}$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be this probability space. We now seek to apply the extension of Theorem 7.28 to functions on $[0, t]$.

Define real-valued random variables on the space by

$$R_1 : f \mapsto \exp \left\{ \theta \int_0^t \left(f(u) - \inf_{0 \leq s \leq u} f(s) \right) du \right\}$$

and

$$R_2 : f \mapsto -\mathbb{1}_{\{\inf_{0 \leq s \leq t} f(s) \leq -at^2\}}.$$

It then suffices, by Theorem 7.28, to show that R_1 and R_2 are both increasing functions of the state space.

Suppose $f_1, f_2 \in \Omega$ with $f_1 \preceq f_2$, noting that this is precisely the condition that $f_2 - f_1$ is a non-decreasing function of its argument. Since we only need to show almost surely that, for $i = 1, 2$, $R_i(f_1) \leq R_i(f_2)$, we assume that $f_1(0) = f_2(0) = 0$, so that

$$f_1(u) \leq f_2(u) \quad \text{for all } u \in [0, t]. \quad (7.39)$$

Then, by Lemma 4.9 (taking $d = \tilde{d} = 0$), it holds that, for all $u \in [0, t]$,

$$f_1(u) - \inf_{0 \leq s \leq u} f_1(s) \leq f_2(u) - \inf_{0 \leq s \leq u} f_2(s),$$

and so $R_1(f_1) \leq R_1(f_2)$. Meanwhile, applying (7.39) again, we see that, if $\inf_{0 \leq s \leq t} f_2(s) \leq -at^2$, then $\inf_{0 \leq s \leq t} f_1(s) \leq -at^2$. So $R_2(f_1) \leq R_2(f_2)$. \square

We already have Lemma 7.24 to bound the expectation of the exponential, so we need a bound on the large-deviations behaviour of $I(t)$. We apply the following explicit bound, which appears in [23], and is a special case of Schilder's theorem.

Lemma 7.29 ([23, Lma. 5.2.1]). *Let $B(t)$ be a Brownian motion. Then for all $t > 0$ and $\delta > 0$,*

$$\mathbb{P} \left(\sup_{0 \leq u \leq t} |B(u)| \geq \delta \right) \leq 4e^{-\delta^2/(2t)}.$$

Lemma 7.30. Let $\lambda \in \mathbb{R}$. Fix $\delta > 0$, and let $c > 1/2 + \delta$. Then, for sufficiently large t (in terms of δ and λ),

$$\mathbb{P}(I(t) \leq -ct^2) \leq 4e^{-(c-1/2-\delta)^2 t^3/2}.$$

Proof. If $\lambda \geq 0$, take $t \geq 2\lambda$; if $\lambda < 0$, take $t \geq -\lambda/\delta$. Then, for all such t , we have from Lemma 7.29 that

$$\begin{aligned} \mathbb{P}(I(t) \leq -ct^2) &= \mathbb{P}\left(\exists u \in [0, t] \text{ s.t. } B(u) + \lambda u - \frac{u^2}{2} \leq -ct^2\right) \\ &\leq \mathbb{P}\left(\exists u \in [0, t] \text{ s.t. } B(u) \leq -\left(c - \frac{1}{2} - \delta\right)t^2\right) \\ &\leq \mathbb{P}\left(\sup_{0 \leq u \leq t} |B(u)| \geq \left(c - \frac{1}{2} - \delta\right)t^2\right) \leq 4e^{-(c-\frac{1}{2}-\delta)^2 t^3/2}. \quad \square \end{aligned}$$

We are now ready to bound our exponentiated random variable restricted to the case of large $-I(t)$.

Lemma 7.31. Fix $\theta > 0$, and take $\alpha > \theta + 1$. Then, as $t \rightarrow \infty$,

$$\mathbb{E}\left[\exp\left\{\theta \int_0^t (X(u) - I(u)) du\right\} \mathbb{1}_{\{I(t) \leq -\alpha t^2\}}\right] \rightarrow 0.$$

Proof. By Lemma 7.26, it holds that

$$\begin{aligned} &\mathbb{E}\left[\exp\left\{\theta \int_0^t (X(u) - I(u)) du\right\} \mathbb{1}_{\{I(t) \leq -\alpha t^2\}}\right] \\ &\leq \mathbb{E}\left[\exp\left\{\theta \int_0^t (X(u) - I(u)) du\right\}\right] \mathbb{P}(I(t) \leq -\alpha t^2). \end{aligned}$$

By Lemma 7.24,

$$\mathbb{E}\left[\exp\left\{\theta \int_0^t (X(u) - I(u)) du\right\}\right] = O\left(t^{3/2} e^{\theta^2 t^3/2 + \lambda \theta t^2}\right).$$

Meanwhile,

$$\mathbb{P}(I(t) \leq -ct^2) \leq 4e^{-(\alpha-1)^2 t^3/2}$$

by Lemma 7.30, taking $\delta = 1/2$ (since $\alpha > 1$). So

$$\mathbb{E}\left[\exp\left\{\theta \int_0^t (X(u) - I(u)) du\right\} \mathbb{1}_{\{I(t) \leq -\alpha t^2\}}\right] = O\left(t^{3/2} e^{[\theta^2 - (\alpha-1)^2] t^3/2 + \lambda \theta t^2}\right) \rightarrow 0,$$

since $\alpha - 1 > \theta$. □

Finally, we can prove the second main proposition of this chapter.

Proof of Proposition 7.2. Taken together, we can use Lemmas 7.25 and 7.31 to show that

$$\left(\exp \left\{ (q-1) \left[\int_0^t (X(u) - I(u)) du \right] - (1/2) \log t \right\} \right)_{t \geq 0}$$

is bounded in L^r for the same r as in Proposition 7.20. Since this $r > 1$, we have that the collection is uniformly integrable. \square

7.3 $q < 1$

We discuss here what in the argument above fails for $q < 1$. Proposition 7.1 doesn't refer to q at all, so the missing piece is in extending Proposition 7.2 to $q \in (0, 1)$.

Essentially, the problem here is finding an analogue of Lemma 7.19. We noted before that that was an adaptation of a result from [14] about quantities like

$$\mathbb{E} \left[\exp \left\{ \sum_{x \in \Xi} \phi(x) \right\} \right],$$

where Ξ is a point process, and ϕ is a non-negative function. For $q < 1$, the form of the expression in Proposition 7.2 indicates that we will need to consider quantities of the form

$$\mathbb{E} \left[\exp \left\{ - \sum_{x \in \Xi} \phi(x) \right\} \right]$$

instead (since the argument of the exponential is multiplied by $q - 1$, which is now negative).

In fact, Błaszczyszyn and Yogeshwaran [14] have an analogous result that applies to such quantities, which we state here.

Definition 7.32 ([14, Defn. 1.2]). A point process Ξ on \mathbb{R}^d is ν -weakly sub-Poisson if, for any Borel subset $B \subseteq \mathbb{R}^d$,

$$\mathbb{P}(\Phi(B) = 0) \leq \exp\{-\mathbb{E}[\Phi(B)]\}. \quad (7.40)$$

Remark. As with α -weakly sub-Poisson processes, the notation comes from [14], in which $\nu(B)$ is used to denote the void probability, $\mathbb{P}(\Phi(B) = 0)$.

Lemma 7.33 ([14, Lma. 3.2]). Let Ξ be a simple point process on \mathbb{R}^d with intensity Λ . Let

$\phi : \mathbb{R}^d \rightarrow [0, \infty)$ be Lebesgue measurable. Then

$$\mathbb{E} \left[\exp \left\{ - \sum_{x \in \Xi} \phi(x) \right\} \right] \leq \exp \left\{ \int_{\mathbb{R}^d} (e^{-\ell(x)} - 1) \Lambda(x) dx \right\}$$

if and only if Ξ is ν -weakly sub-Poisson.

In our case, there is no value of λ for which $\Xi^{(\lambda)}$ is ν -weakly sub-Poisson, and in fact it isn't possible to find an approximation of this property by using a result like Lemma 7.16, either. So we would need a completely different approach to find an appropriate bound.

Chapter 8

Other approaches

We complete our proofs related to the random cluster model in this chapter, with some results of a somewhat different flavour to the heuristic proof offered in Section 3.2, but which may, with further research, provide an alternative approach to proving that the rescaled component sizes of the random cluster model converge, and thence that a scaling limit exists. The two sections of this chapter are largely independent of each other.

In Section 8.1 we outline an alternative potential proof strategy from the heuristic one we have been following, as well as some results which suggest it may be plausible to follow this route. Then, in Section 8.2, we show that, if the sequence of component sizes converges appropriately, then the scaling limit must follow.

8.1 A limit for component sizes

We recall that, in Proposition 3.7, we observed that $\mathbb{P}_{n,p,q}$ can be written as a measure change from $\mathbb{P}_{n,\tilde{p}}$ by $q^{s(G_n)}$, where $s(G_n)$ is the number of surplus edges in G_n , and

$$\tilde{p} = \frac{p}{p + q(1-p)}.$$

We also observed that, when p is in the critical window for the random cluster model, \tilde{p} is in the critical window for Erdős–Rényi.

In our heuristic proof in Section 3.2, we then split $s(G_n)$ into surplus edges discovered before and after a time $\lfloor Tn^{2/3} \rfloor$ in the exploration process (for some $T > 0$), so we could apply Aldous’s limit theorem (quoted here as Theorem 1.6); we argued that the post- $\lfloor Tn^{2/3} \rfloor$ part was approximately independent of the part captured by the Aldous limit, and, since it appeared on the top and bottom of a fraction, it cancelled. The problem when $q > 1$ was

showing that this approximate independence held; when $q < 1$, the problem was that, to make the approximate independence valid, we needed to take the limit in T after the limit in n , and, even allowing for the fact that we could multiply the top and bottom of the fraction by some compensating function of T , and could find a candidate function, we couldn't show that the appropriate uniform integrability held to allow convergence of expectations.

An alternative approach would be to consider directly compensating the expression involving $q^{s(G_n)}$, without the split. For this, we would need that there was some function $\mathfrak{D}(n, q)$ such that

$$\frac{q^{s(G_n)}}{\mathfrak{D}(n, q)}$$

is uniformly integrable, as $n \rightarrow \infty$, and has a non-degenerate limit as $n \rightarrow \infty$.

We provide some calculations that suggest this is plausible, adapting an approach of Noy, Rasendrasahina, Ravelomanana and Rué [49]. First, we set up some notation.

Definition 8.1. The *excess* of a connected graph G is defined by

$$x(G) := (e(G) - |G|)_+,$$

where $a_+ := \max\{a, 0\}$. The excess of a general graph G is the sum of the excess of its components (or, equivalently, $e(G) - |G| + t(G)$, where $t(G)$ is the number of tree components in G).

A connected graph (or equivalently a component of a graph) is *complex* if it has non-zero excess; that is, if it has strictly more edges than vertices. It is *unicyclic* if it has equal numbers of edges and vertices (or, equivalently, if it contains exactly one cycle).

Remark. It can be easily checked that, when $s(G)$ denotes the number of surplus edges in a graph G , then

$$s(G) = x(G) + u(G) + p(G), \tag{8.1}$$

where $u(G)$ is the number of unicyclic components, and $p(G)$ the number of complex components, in G .

We now quote the theorem of Noy, Rasendrasahina, Ravelomanana and Rué [49] that we will use.

Theorem 8.2. For $n \in \mathbb{N}$, let

$$m = m(n) = \left\lfloor \frac{n}{2} (1 + \lambda n^{-1/3}) \right\rfloor.$$

For $k, r \in \mathbb{N}$, let $p_k(n, m, r)$ be the probability that $\mathcal{G}(n, m)$ has total excess r , with k unicyclic components. There exist constants $C_1 > 0$ and $R > 0$ such that, for all $n \in \mathbb{N}$, for all $k \in \mathbb{N}$ and for all $r \geq R$, then

$$p_k(n, m, r) \leq e^{-(\log n)/6} \frac{((\log n)/6)^k}{k!} \times C_1 \exp\left\{\lambda r^{2/3} - \left(\log\left(\frac{4}{3}\right) - \frac{1}{12}\right)r\right\}.$$

This is a slightly stronger version of the second part of [49, Thm. 4.3], which is in fact implied by their proof of that result.

Note that Theorem 8.2 concerns $\mathcal{G}(n, m)$, not $\mathcal{G}(n, p)$. This is why our calculations are only indicative, not conclusive, for results about $\mathcal{G}(n, p)$, and hence about $\mathcal{G}(n, p, q)$, the model of interest. (Later we will make some remarks about how we might approach $\mathcal{G}(n, m, q)$, where this result is more directly relevant.)

Before we go on, we will simplify our notation slightly: let N_n be the number of surplus edges in G_n . (This conflicts slightly with our previous notation for exploration processes: note that N_n here is equal to $N_n(n)$ there.) We now state our indicative lemma.

Lemma 8.3. *Let*

$$m = m(n) = \left\lfloor \frac{n}{2} (1 + \lambda n^{-1/3}) \right\rfloor.$$

There exists $\bar{q} \in (1, 2)$ such that, for all $q \in (0, \bar{q})$,

$$\sup_{n \in \mathbb{N}} n^{-(q-1)/6} \mathbb{E}_{n, m} [q^{N_n}] < \infty.$$

Proof. Let

$$\mu := \log\left(\frac{4}{3}\right) - \frac{1}{12} > 0,$$

(the value is approximately 0.204); assume $\varepsilon < \mu$. Then, routinely (since, for all sufficiently large r , we have $\lambda r^{2/3} \leq \varepsilon r$), it follows from Theorem 8.2 that there exists a constant $C_2 > 0$ such that, for all n, k and r ,

$$p_k(n, m, r) \leq e^{-(\log n)/6} \frac{((\log n)/6)^k}{k!} \times C_2 e^{(-\mu+\varepsilon)r} \quad (8.2)$$

(noting that C_2 depends on ε).

Take

$$\bar{q} = e^{\mu/2} \approx 1.11$$

(in particular, $\bar{q} \in (1, 2)$), and pick $q \in (1, \bar{q})$. Choose ε satisfying

$$0 < \varepsilon < \mu - 2 \log q = 2 \log \bar{q} - 2 \log q.$$

Let X_n , U_n and P_n be the excess, number of unicyclic components and number of complex components of G_n . Then, by (8.1),

$$\mathbb{E}_{n,m}[q^{N_n}] = \mathbb{E}_{n,m}[q^{X_n+U_n+P_n}] \leq \mathbb{E}_{n,m}[q^{2X_n+U_n}],$$

since the number of complex components is bounded above by the excess (each complex component has excess at least one). So, by (8.2)

$$\begin{aligned} \mathbb{E}_{n,m}[q^{N_n}] &\leq \sum_{k,r} q^{2r+k} p_k(n, m, r) \leq C_2 \sum_{k=0}^{\infty} q^k e^{-(\log n)/6} \frac{((\log n)/6)^k}{k!} \sum_{r=0}^{\infty} \exp\{r[2 \log q - \mu + \varepsilon]\} \\ &= C_2 \exp\left\{\frac{1}{6}(q-1) \log n\right\} (1 - \exp\{2 \log q - \mu + \varepsilon\})^{-1}, \end{aligned}$$

since $2 \log q - \mu + \varepsilon < 0$, where for the sum in k we used the formula for the probability generating function of the Poisson distribution. Since

$$\exp\left\{\frac{1}{6}(q-1) \log n\right\} = n^{(q-1)/6},$$

the result holds.

Observing that the case $q = 1$ is trivial, we now suppose we take $q \in (0, 1)$. Then we have, using (8.1) that

$$\mathbb{E}_{n,m}[q^{N_n}] \leq \mathbb{E}_{n,m}[q^{X_n+U_n}].$$

Take any $\varepsilon \in (0, \mu)$; then

$$\mathbb{E}_{n,m}[q^{N_n}] \leq C_2 \exp\left\{\frac{1}{6}(q-1) \log n\right\} (1 - \exp\{\log q - \mu + \varepsilon\})^{-1}.$$

The result follows as before. □

Remark. Note that in the above, for $q > 1$, we took the bound $P_n \leq X_n$ —we bounded the number of complex components above by the excess. It is the tail behaviour in r of $p_k(n, m, r)$ (where r represents the excess) that governs whether or not the sums converge, and consequently constrains what values of q we can take. We may therefore be able to increase the value of \bar{q} if we could find a better bound for P_n that didn't rely on the excess; alternatively, it

may be possible to evaluate or bound directly the probability that $\mathcal{G}(n, m)$ has a given number of unicyclic components, a given excess and a given number of complex components.

Conjecture 8.4. *Lemma 8.3 can be extended to all $q \in (0, 2)$.*

Conjecture 8.5. *Let $p = 1/n + \lambda n^{-4/3}$. Then, as $n \rightarrow \infty$, for all $q \in (0, 2)$,*

$$\mathbb{E}_{n,p}[q^{N_n}] = O(n^{-(q-1)/6}).$$

This doesn't immediately follow from Lemma 8.3 by averaging over m , because the fact that the number of surplus edges is in the exponent means that an atypical number of edges might dominate the expectation, and Theorem 8.2 doesn't give us enough control to ensure that it doesn't. Theorem 8.2 was, we recall, essentially the second part of [49, Thm. 4.3]; while the *first* part of that result gives us tighter bounds on $p_k(n, m, r)$, it is only for a more restricted range of k , and so doesn't quite give enough control either.

However, suppose Conjecture 8.5 holds. In that case, there is still some additional machinery needed to get a proof of convergence of the (rescaled) sequence of component sizes. Hence, in the rest of this section, we first, in Subsection 8.1.1, develop some technical results similar to those from Section 7.1, but working with component sizes, instead of with local time within the exploration process. Then, in the following subsection, Subsection 8.1.2, we outline our potential alternative proof strategy, which will rely on Conjecture 8.5 and a further conjecture; Subsection 8.1.3 will address how we might go about proving that conjecture. We close the section with some very brief remarks on adapting this method for $\mathcal{G}(n, m, q)$, in Subsection 8.1.4.

8.1.1 Janson and Spencer, revisited

In Section 7.1, we considered, for the limit process, the components in time order in the exploration. Specifically, we considered the total area under components that had appeared by a given local time at 0, say ℓ , and we let $\ell \rightarrow \infty$. We showed that, appropriately compensated, this approached a limit random variable, and that the moment generating function also converged in a certain range of its argument.

It will be useful in the sequel to have a version of this in which we consider components in decreasing order of size. Specifically, using the notation from the previous chapter, we want

to consider the random variable given by

$$\sum_{i=1}^{\infty} \tilde{A}_i(D_i) \mathbb{1}_{\{D_i \geq \varepsilon\}},$$

where D_i is the i -th-largest element in the point process $\Xi^{(\lambda)}$, and show that this, when appropriately compensated, converges in probability as $\varepsilon \downarrow 0$, with suitable uniform integrability properties.

First let us calculate the mean of this random variable. Abusing notation slightly, we (re)define

$$\mathfrak{J}(\varepsilon) = \sum_{i=1}^{\infty} \tilde{A}_i(D_i) \mathbb{1}_{\{D_i \geq \varepsilon\}}$$

and

$$f(\varepsilon) = \mathbb{E}[\mathfrak{J}(\varepsilon)].$$

Proposition 8.6. *Define*

$$\begin{aligned} A_\lambda &= \int_1^\infty \frac{1}{\sqrt{2\pi}} x^{-1} \Psi'(x^{3/2}) \exp\left\{-\frac{x^3}{6} + \frac{x^2\lambda}{2} - \frac{x\lambda^2}{2}\right\} dx \\ &\quad + \int_0^1 \frac{1}{\sqrt{2\pi}} x^{-1} \left[\Psi'(x^{3/2}) \exp\left\{-\frac{x^3}{6} + \frac{x^2\lambda}{2} - \frac{x\lambda^2}{2}\right\} - \sqrt{\frac{\pi}{8}} \right] dx. \end{aligned}$$

Then $A_\lambda < \infty$, and, moreover, as $\varepsilon \downarrow 0$,

$$f(\varepsilon) = -\frac{1}{4} \log \varepsilon + A_\lambda + o(1).$$

Proof. We assume without loss of generality that $\varepsilon < 1$. As in the previous chapter, we apply the tower law (cf. (7.9) and Theorem 7.4 (cf. (7.11))) to obtain that

$$\begin{aligned} f(\varepsilon) &= \mathbb{E} \left[\sum_{i=1}^{\infty} D_i^{3/2} \Phi'(D_i^{3/2}) \mathbb{1}_{\{D_i \geq \varepsilon\}} \right] \\ &= \int_\varepsilon^\infty \frac{1}{\sqrt{2\pi}} x^{-1} \Psi'(x^{3/2}) \exp\left\{-\frac{x^3}{6} + \frac{x^2\lambda}{2} - \frac{x\lambda^2}{2}\right\} dx. \end{aligned}$$

(Recall that we gave asymptotics for Ψ in (7.2) and Lemma 7.6, and for Φ in (7.5), (7.6) and (7.7).) We note that the integral is finite, and that the integral over $(1, \infty)$ is the first term

in the expression for A_λ . Moreover, recalling that $\Psi'(0) = \sqrt{\pi/8}$, we note that, as $x \downarrow 0$, the integrand behaves as

$$\frac{1}{4x} + O(1).$$

So we note that

$$\begin{aligned} & \int_\varepsilon^1 \frac{1}{\sqrt{2\pi}} x^{-1} \Psi'(x^{3/2}) \exp\left\{-\frac{x^3}{6} + \frac{x^2\lambda}{2} - \frac{x\lambda^2}{2}\right\} dx \\ &= \frac{1}{4} \int_\varepsilon^1 \frac{1}{x} dx + \int_\varepsilon^1 \frac{1}{\sqrt{2\pi}} x^{-1} \left[\Psi'(x^{3/2}) \exp\left\{-\frac{x^3}{6} + \frac{x^2\lambda}{2} - \frac{x\lambda^2}{2}\right\} - \sqrt{\frac{\pi}{8}} \right] dx. \end{aligned}$$

The second integrand is bounded on $[0, 1]$, so converges to the other term in the expression for A_λ . Meanwhile the first integral is equal to $-(\log \varepsilon)/4$. \square

We will now want some control on the variance, akin to Lemma 7.12 and Proposition 7.13.

We (re)define, for each $k \in \mathbb{N}$

$$J(k) = \sum_{i=1}^{\infty} \tilde{A}_i(D_i) \mathbf{1}_{\{k^{-3} \leq D_i < (k-1)^{-3}\}}$$

(interpreting 0^{-3} as ∞), and let

$$g(k) = \mathbb{E}[J(k)].$$

We note that, by similar calculations to those above

$$g(k) = \int_{k^{-3}}^{(k-1)^{-3}} \frac{1}{\sqrt{2\pi}} x^{-1} \Psi'(x^{3/2}) \exp\left\{-\frac{x^3}{6} + \frac{x^2\lambda}{2} - \frac{x\lambda^2}{2}\right\} dx = O\left(\frac{1}{k}\right).$$

Lemma 8.7. *The variance $\text{Var}(J(k))$ is finite for all k , and, as $k \rightarrow \infty$,*

$$\text{Var}(J(k)) = O(k^{-4}).$$

Proof. We follow the proof of Lemma 7.12 almost directly. We note that if we define

$$\begin{aligned} \mathfrak{X}_1(k) &= \mathbb{E} \left[\sum_{i=1}^{\infty} \tilde{A}_i(D_i)^2 \mathbf{1}_{\{k^{-3} \leq D_i < (k-1)^{-3}\}} \right], \\ \mathfrak{X}_2(k) &= \mathbb{E} \left[\sum_{i=1}^{\infty} \sum_{\substack{j=1 \\ j \neq i}}^{\infty} \tilde{A}_i(D_i)^2 \mathbf{1}_{\{k^{-3} \leq D_i < (k-1)^{-3}\}} \tilde{A}_j(D_j)^2 \mathbf{1}_{\{k^{-3} \leq D_j < (k-1)^{-3}\}} \right], \text{ and} \\ \mathfrak{X}_3(k) &= \mathbb{E}[J(k)]^2 = \mathbb{E} \left[\tilde{A}_i(D_i) \mathbf{1}_{\{k^{-3} \leq D_i < (k-1)^{-3}\}} \right]^2, \end{aligned}$$

then

$$\text{Var}(J(k)) = \mathfrak{X}_1(k) + \mathfrak{X}_2(k) - \mathfrak{X}_3(k). \quad (8.3)$$

Here we may express

$$\mathfrak{X}_1(k) = \int_{k^{-3}}^{(k-1)^{-3}} \frac{1}{\sqrt{2\pi}} x^{1/2} \Psi''(x^{3/2}) \exp\left\{-\frac{x^3}{6} + \frac{x^2\lambda}{2} - \frac{x\lambda^2}{2}\right\} dx.$$

Note that the integrand has a sufficiently light tail at ∞ that $\mathfrak{X}_1(1) < \infty$. Moreover, the (non-negative) integrand is bounded above on $[0, 1]$ by $F_1 x^{1/2}$, for some $F_1 > 0$, so, for $k \geq 2$

$$\mathfrak{X}_1(k) \leq \frac{2F_1}{3} \left(\frac{1}{(k-1)^{9/2}} - \frac{1}{k^{9/2}} \right) = O\left(\frac{1}{k^{11/2}}\right). \quad (8.4)$$

Then, for the other two terms, we have that

$$\begin{aligned} & \mathfrak{X}_2(k) - \mathfrak{X}_3(k) \\ &= \int_{k^{-3}}^{(k-1)^{-3}} \int_{k^{-3}}^{(k-1)^{-3}} x^{3/2} \Phi'(x^{3/2}) \Lambda^{(\lambda)}(x) y^{3/2} \Phi'(y^{3/2}) \Lambda^{(\lambda)}(y) \left(\frac{\Lambda^{(\lambda-x)}(y)}{\Lambda^{(\lambda)}(y)} - 1 \right) dy dx \end{aligned} \quad (8.5)$$

(cf. (7.23)) where again

$$\frac{\Lambda^{(\lambda-x)}(y)}{\Lambda^{(\lambda)}(y)} - 1 = \exp\left\{-\frac{xy(x+y-2\lambda)}{2}\right\} - 1. \quad (8.6)$$

We note immediately (from (8.3) and (8.4)) that this is bounded below, for some $F_2 > 0$, by $-F_2 k^{-11/2}$; we can also check that it is finite for all k , proving the finiteness claim in the lemma. Since the integrand in (8.5) is non-positive when $\lambda \leq 0$, in this case we are done. Otherwise (i.e. if $\lambda > 0$), note that (8.6) is negative if (but not only if) $x > 2\lambda$ and $y > 2\lambda$; let K_λ be the smallest $k \in \mathbb{N} \setminus \{1\}$ such that $k^{-3} \leq 2\lambda$ (i.e. $K_\lambda = \lceil 8\lambda^3 \rceil$). Then we can find $F_3 > 0$ (depending only on λ) such that, for all $x, y \in [0, (K_\lambda - 1)^{-3}]$,

$$\frac{\Lambda^{(\lambda-x)}(y)}{\Lambda^{(\lambda)}(y)} - 1 \leq F_3 xy,$$

so that we may bound, for $k \geq K_\lambda$, and upon substituting for Φ' and Λ ,

$$\mathfrak{X}_2(k) - \mathfrak{X}_3(k) \leq F_3 \left(\frac{1}{\sqrt{2\pi}} \int_{k^{-3}}^{(k-1)^{-3}} \Psi''(x^{3/2}) \exp\left\{-\frac{x^3}{6} + \frac{x^2\lambda}{2} - \frac{x\lambda^2}{2}\right\} dx \right)^2 = O\left(\frac{1}{k^4}\right),$$

since the integrand is bounded on $[0, 1]$; this completes the proof. \square

Proposition 8.8. *The quantity $\mathfrak{J}(\varepsilon) - f(\varepsilon)$ converges almost surely, as $\varepsilon \downarrow 0$.*

Proof. We follow the proof of Proposition 7.13. First, assume WLOG that $\varepsilon < 1$; then if we define

$$\underline{k}(\varepsilon) = \lfloor \varepsilon^{-3} \rfloor \quad \text{and} \quad \bar{k}(\varepsilon) = \lceil \varepsilon^{-3} \rceil,$$

we see that

$$\mathfrak{J}(\bar{k}(\varepsilon)^{-3}) - f(\bar{k}(\varepsilon)^{-3}) - g(\bar{k}(\varepsilon)) \leq \mathfrak{J}(\varepsilon) - f(\varepsilon) \leq \mathfrak{J}(\underline{k}(\varepsilon)^{-3}) - f(\underline{k}(\varepsilon)^{-3}) + g(\bar{k}(\varepsilon)).$$

Since $g(k) \rightarrow 0$ as $k \rightarrow \infty$, it is sufficient by a sandwiching argument to prove that, as $k \rightarrow \infty$ through \mathbb{N} ,

$$\mathfrak{J}(k^{-3}) - f(k^{-3}) = \sum_{j=1}^k (J(j) - g(j)) \tag{8.7}$$

converges almost surely. For $j \in \mathbb{N}$, let E_j be the event

$$E_j = \{ |J_j - g(j)| > j^{-4/3} \}.$$

Then, using Lemma 8.7 and Chebyshev's inequality, we can show that

$$\mathbb{P}(E_j) = O(j^{-4/3})$$

as $j \rightarrow \infty$. But then $\sum_j \mathbb{P}(E_j) < \infty$, so by the first Borel–Cantelli lemma, it must be that $\mathbb{P}(E_j \text{ infinitely often}) = 0$. By the Weierstrass M -test, since all the $J(j)$ s are a.s. finite, the sum in (8.7) converges almost surely. \square

Call the limit random variable L_λ^\downarrow . As with L_λ and L'_λ in the previous chapter, we know very little about this random variable other than that it exists.

We now seek to show the uniform integrability property that we had in the local time setting (cf. Subsection 7.1.2, and in particular Corollary 7.22).

Proposition 8.9. *Fix $q \in (1, 2)$. As $\varepsilon \downarrow 0$, the collection of random variables*

$$\left(e^{(q-1)(\mathfrak{J}(\varepsilon) - f(\varepsilon))} \right)_{\varepsilon > 0}$$

is uniformly integrable; moreover,

$$\mathbb{E}\left[e^{(q-1)(\mathcal{J}(\varepsilon)-f(\varepsilon))}\right] \rightarrow \mathbb{E}\left[e^{(q-1)L_\lambda^\downarrow}\right] < \infty.$$

Proof. By the proof of Corollary 7.22, it suffices to show that the random variables are bounded in L^r , for some $r > 1$. To show this, we follow the proof of Proposition 7.20.

Define

$$G^\downarrow(\varepsilon, r) = \mathbb{E}\left[\exp\left\{\sum_{i=1}^{\infty} r(q-1)\tilde{A}_i(D_i)\mathbb{1}_{\{D_i \geq \varepsilon\}}\right\}\right]$$

By the tower law, and using the fact that $(A_i(D_i))_{i \geq 1}$ are conditionally independent given $(D_i)_{i \geq 1}$, it holds that

$$\begin{aligned} G^\downarrow(\varepsilon, r) &= \mathbb{E}\left[\prod_{i=1}^{\infty} \left(\exp\left\{\Phi(r'D_i^{3/2}) - \Phi(D_i^{3/2})\right\}\mathbb{1}_{\{D_i \geq \varepsilon\}} + \mathbb{1}_{\{D_i < \varepsilon\}}\right)\right] \\ &= \mathbb{E}\left[\exp\left\{\sum_{i=1}^{\infty} \left(\Phi(r'D_i^{3/2}) - \Phi(D_i^{3/2})\right)\mathbb{1}_{\{D_i \geq \varepsilon\}}\right\}\right], \end{aligned}$$

where we take $r' = r(q-1) + 1 > 1$ as in the last chapter. Using Lemma 7.19 with

$$\phi(x) = \left(\Phi(r'x^{3/2}) - \Phi(x^{3/2})\right)\mathbb{1}_{\{x \geq \varepsilon\}},$$

we obtain

$$G^\downarrow(\varepsilon, r) = \exp\left\{\int_{\varepsilon}^{\infty} \left[\Psi(r'x^{3/2}) - \Psi(x^{3/2})\right] \frac{1}{\sqrt{2\pi}} x^{-5/2} \exp\left(-\frac{x^3}{6} + \frac{x^2\lambda}{2}\right) dx\right\}.$$

Assume WLOG that $\varepsilon < 1$, and split the integral into two integrals, over $[\varepsilon, 1)$ and $[1, \infty)$. The latter can be shown to be finite (by almost exactly the same argument as in the proof of Proposition 7.20) if we choose r' so that $(r')^2 - 4 < 0$; as in that proof, we therefore take $r = q/(2(q-1))$, so that $r' = (q+2)/2$. It remains to consider the other integral, which governs the behaviour as $\varepsilon \downarrow 0$. Using the bound on the integrand given by (7.31), there is a constant $F_4 > 0$ such that

$$\begin{aligned} \int_{\varepsilon}^1 \left[\Psi(r'x^{3/2}) - \Psi(x^{3/2})\right] \frac{1}{\sqrt{2\pi}} x^{-5/2} \exp\left(-\frac{x^3}{6} + \frac{x^2\lambda}{2}\right) dx &\leq \int_{\varepsilon}^1 \left(\frac{q}{8}x^{-1} + F_4\right) dx \\ &= -\frac{q}{32} \log \varepsilon + F_4(1 - \varepsilon). \end{aligned}$$

So we have that

$$G^\downarrow\left(\varepsilon, \frac{q}{2(q-1)}\right) \leq \exp\left\{-\frac{q}{8} \log \varepsilon + O(1)\right\}.$$

Hence,

$$\mathbb{E}\left[e^{r(q-1)(\mathcal{J}(\varepsilon)-f(\varepsilon))}\right] \leq \exp\left\{-\frac{q}{8} \log \varepsilon + \frac{r(q-1)}{4} \log \varepsilon + O(1)\right\},$$

by Proposition 8.6. But $r(q-1)/4 = q/8$, which completes the proof. \square

8.1.2 An alternative proof strategy

We now outline an alternative possible proof strategy for convergence of the sequence of component sizes in l^2 , for $\mathcal{G}(n, p, q)$. It will use the machinery from the previous section, and will be conditional on Conjecture 8.5 and on one further conjecture, which we now state.

Conjecture 8.10. *Let $C_i^{(n)}$ be the i -th largest component of $G_n \sim \mathcal{G}(n, p)$, where $p = 1/n + \lambda n^{-4/3} + O(n^{-2})$. Let $\tilde{A}_i^{(n)}(|C_i^{(n)}|)$ be the area corresponding to the component $C_i^{(n)}$. Then there exists $c \in \mathbb{R}$ such that, for all $\delta > 0$,*

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}_{n,p} \left(\left| \sum_i n^{-1} \tilde{A}_i^{(n)}(|C_i^{(n)}|) \mathbb{1}_{\{n^{-2/3}|C_i^{(n)}| < \varepsilon\}} - \frac{1}{4} \log(n^{2/3} \varepsilon) + c \right| \geq \delta \right) = 0.$$

This is a slightly abstract condition; why we need this specifically will become clear in our conditional proof.

Theorem 8.11. *Let \mathbf{C}_n be the decreasing sequence of component sizes in G_n , and let \mathbf{C} , under the measure \mathbb{P} , have the law of the limit in distribution of $n^{-2/3} \mathbf{C}_n$ when $G_n \sim \mathcal{G}(n, 1/n + \lambda n^{-4/3}/q)$.*

Now, instead, let $G_n \sim \mathcal{G}(n, p, q)$, where $q \in (1, 2)$ and

$$p = \frac{q}{n} + \frac{\lambda}{n^{4/3}}.$$

Assume Conjectures 8.5 and 8.10 hold. Then, as $n \rightarrow \infty$, the sequence of sequences $(\mathbf{C}_n)_{n \in \mathbb{N}}$ converges in distribution in l^2 . Moreover, when \mathbf{C} is as above, the distributional limit can be described as a measure change for \mathbf{C} . Specifically, let $\tilde{\mathbb{P}}$ be defined by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \frac{\exp\{(q-1)L_\lambda^\downarrow\}}{\mathbb{E}\left[\exp\{(q-1)L_\lambda^\downarrow\}\right]},$$

where \mathbb{E} denotes expectation under \mathbb{P} , and $\tilde{L}_\lambda^\downarrow$ is the random variable whose existence is established by Proposition 8.8, defined on the same probability space as \mathbf{C} in the natural way. Then the limit

random variable is \mathbf{C} under the measure $\tilde{\mathbb{P}}$.

Proof. Much like in Proposition 3.7, it holds that, for any continuous bounded real-valued function $f : l^2 \rightarrow \mathbb{R}$,

$$\mathbb{E}_{n,p,q}[f(n^{-2/3}\mathbf{C}_n)] = \frac{\mathbb{E}_{n,\tilde{p}}[f(n^{-2/3}\mathbf{C}_n)q^{N_n}]}{\mathbb{E}_{n,\tilde{p}}[q^{N_n}]}, \quad (8.8)$$

where $\tilde{p} = p/(p+q(1-p))$. We previously noted, in (3.9), that $\tilde{p} = 1/n + \tilde{\lambda}n^{-4/3} + O(n^{-2})$, so we have quantities that are in the critical window. By the portmanteau theorem, it is sufficient to show that the right-hand side converges for all f .

Now, let $\tilde{A}_i^{(n)}(|C_i^{(n)}|)$ be the area corresponding to the component $C_i^{(n)}$ in G_n . Fix some arbitrary $d > 0$. Then define, for $n \in \mathbb{N}$ and $\varepsilon > 0$,

$$X(n, \varepsilon) = \sum_i n^{-1} \tilde{A}_i^{(n)}(|C_i^{(n)}|) \mathbb{1}_{\{n^{-2/3}|C_i^{(n)}| \geq \varepsilon \vee n^{-2/3}\}} + \frac{1}{4} \log(\varepsilon \vee (dn^{-2/3})),$$

where $x \vee y = \max\{x, y\}$. Now, as $n \rightarrow \infty$, under $\mathbb{E}_{n,\tilde{p}}$, by Theorem 1.6, we have

$$X(n, \varepsilon) \xrightarrow{d} \sum_i \tilde{A}_i(D_i) \mathbb{1}_{\{D_i \geq \varepsilon\}} + \frac{1}{4} \log \varepsilon =: X(\infty, \varepsilon). \quad (8.9)$$

(This isn't quite immediate from the continuous mapping theorem, say, because the indicator function introduces a discontinuity; however, it can be deduced from Skohorod's representation theorem and the fact, as implied by Janson and Spencer [40], that almost surely there is no $i \in \mathbb{N}$ such that $D_i = \varepsilon$.) Then, by Propositions 8.6 and 8.8 we see that, as $\varepsilon \downarrow 0$,

$$X(\infty, \varepsilon) \xrightarrow{d} \tilde{L}_\lambda^\downarrow$$

for some random variable $\tilde{L}_\lambda^\downarrow$. (This random variable differs from L_λ^\downarrow by the non-random constant A_λ identified in Proposition 8.6.)

Now note that, as $\varepsilon \downarrow 0$

$$\begin{aligned} X(n, \varepsilon) &\xrightarrow{d} \sum_i n^{-1} \tilde{A}_i^{(n)}(|C_i^{(n)}|) \mathbb{1}_{\{n^{-2/3}|C_i^{(n)}| \geq n^{-2/3}\}} + \frac{1}{4} \log(n^{-2/3}) \\ &= \sum_i n^{-1} \tilde{A}_i^{(n)}(|C_i^{(n)}|) - \frac{1}{6} \log n + c \\ &=: X(n, 0), \end{aligned}$$

where $c = (\log d)/4$ (indeed, this convergence holds almost surely). Since $d > 0$ was arbitrary,

$c \in \mathbb{R}$ is arbitrary.

We seek to use the principle of accompanying laws to show that this also converges in distribution, as $n \rightarrow \infty$, to $\tilde{L}_\lambda^\downarrow$; specifically we use [12, Thm. 3.2], which requires us to show that, for all $\delta > 0$,

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}_{n, \tilde{p}}(|X(n, 0) - X(n, \varepsilon)| \geq \delta) = 0; \quad (8.10)$$

by that theorem, (8.10) is enough to show that $X(n, 0) \xrightarrow{d} \tilde{L}_\lambda^\downarrow$. Here, assuming WLOG that $\varepsilon > n^{-2/3}$,

$$X(n, 0) - X(n, \varepsilon) = \sum_i n^{-1} \tilde{A}_i^{(n)}(|C_i^{(n)}|) \mathbb{1}_{\{n^{-2/3} |C_i^{(n)}| < \varepsilon\}} - \frac{1}{6} \log n - \frac{1}{4} \log \varepsilon + c.$$

We suppose, as asserted by Conjecture 8.10, that (8.10) holds, so that indeed $X(n, 0) \xrightarrow{d} \tilde{L}_\lambda^\downarrow$ for some value of c .

We return to (8.8). Note that, conditionally on the processes X_n and I_n ,

$$N_n \sim \text{Bin}\left(\sum_i \tilde{A}_i^{(n)}(|C_i^{(n)}|), \tilde{p}\right)$$

under $\mathbb{P}_{n, \tilde{p}}$. So, by the tower law,

$$\begin{aligned} \mathbb{E}_{n, \tilde{p}}[f(n^{-2/3} \mathbf{C}_n) q^{N_n}] &= \mathbb{E}_{n, \tilde{p}}\left[f(n^{-2/3} \mathbf{C}_n) (1 - \tilde{p} + \tilde{p}q)^{\sum_i \tilde{A}_i^{(n)}(|C_i^{(n)}|)}\right] \\ &= \mathbb{E}_{n, \tilde{p}}[f(n^{-2/3} \mathbf{C}_n) (1 - \tilde{p} + \tilde{p}q)^{nX(n, 0) + n(\log n)/6 - nc}] \\ &= (e^{-c(q-1)} n^{(q-1)/6} + o(1)) \mathbb{E}_{n, \tilde{p}}[f(n^{-2/3} \mathbf{C}_n) (1 - \tilde{p} + \tilde{p}q)^{nX(n, 0)}], \end{aligned}$$

since

$$(1 - \tilde{p} + \tilde{p}q)^{-nc} = e^{-c(q-1)} + O(n^{-1/3})$$

and

$$(1 - \tilde{p} + \tilde{p}q)^{n(\log n)/6} = n^{(q-1)/6} + O\left(\frac{\log n}{n^{1/3}}\right).$$

Hence, in (8.8), we divide top and bottom by $e^{-c(q-1)} n^{(q-1)/6}$ to obtain that

$$\mathbb{E}_{n, p, q}[f(n^{-2/3} \mathbf{C}_n)] = (1 + o(1)) \frac{\mathbb{E}_{n, \tilde{p}}[f(n^{-2/3} \mathbf{C}_n) (1 - \tilde{p} + \tilde{p}q)^{nX(n, 0)}]}{\mathbb{E}_{n, \tilde{p}}[(1 - \tilde{p} + \tilde{p}q)^{nX(n, 0)}]}$$

(where the denominator is achieved by setting $f \equiv 1$ in the calculations above). We claimed

that $X(n, 0)$ converges in distribution as $n \rightarrow \infty$. Note that

$$(1 - \tilde{p} + \tilde{p}q)^{nX(n,0)} = \exp\{nX(n, 0) \log(1 - \tilde{p} + \tilde{p}q)\}.$$

By the continuous mapping theorem and similar results on limits in distribution, we can show that this converges in distribution to

$$\exp\{(q-1)\tilde{L}_\lambda^\downarrow\},$$

and by the continuous mapping theorem again,

$$f(n^{-2/3}\mathbf{C}_n)(1 - \tilde{p} + \tilde{p}q)^{nX(n,0)} \xrightarrow{d} f(\mathbf{C}) \exp\{(q-1)\tilde{L}_\lambda^\downarrow\}, \quad (8.11)$$

since $n^{-2/3}\mathbf{C}_n \xrightarrow{d} \mathbf{C}$ by Theorem 1.5.

Now, for any $r > 1$, note that

$$\begin{aligned} & \mathbb{E}_{n,\tilde{p}}[(1 - \tilde{p} + \tilde{p}q)^{rnX(n,0)}] \\ &= (1 - \tilde{p} + \tilde{p}q)^{-rn(\log n)/6 + crn} \mathbb{E}_{n,\tilde{p}}\left[\exp\left\{r \sum \tilde{A}_i^{(n)}(|G_i^{(n)}|) \log(1 - \tilde{p} + \tilde{p}q)\right\}\right] \\ &\leq (1 + o(1))n^{(\tilde{q}-1)/6} \mathbb{E}_{n,\tilde{p}}[\tilde{q}^{N_n}], \end{aligned} \quad (8.12)$$

where

$$\tilde{q} := r(q-1) + 1 > q$$

(since $(r-1)(q-1) > 0$). Choose r sufficiently close to 1 that $\tilde{q} < 2$. Then, assuming Conjecture 8.5, we have that (8.12) is bounded as $n \rightarrow \infty$, so the family

$$\left((1 - \tilde{p} + \tilde{p}q)^{nX(n,0)}\right)_{n \in \mathbb{N}}$$

is uniformly integrable as it is bounded in L^r . Consequently, it follows from (8.11) and Vitali's convergence theorem that, as $n \rightarrow \infty$,

$$\mathbb{E}_{n,\tilde{p}}[f(n^{-2/3}\mathbf{C}_n)(1 - \tilde{p} + \tilde{p}q)^{nX(n,0)}] \rightarrow \mathbb{E}[f(\mathbf{C}) \exp\{(q-1)\tilde{L}_\lambda^\downarrow\}]$$

and (by setting $f \equiv 1$ again) that

$$\mathbb{E}_{n,\tilde{p}}[(1 - \tilde{p} + \tilde{p}q)^{nX(n,0)}] \rightarrow \mathbb{E}[\exp\{(q-1)\tilde{L}_\lambda^\downarrow\}].$$

Since $\tilde{L}_\lambda^\downarrow$ differs only from L_λ^\downarrow by a deterministic constant, the result follows. \square

8.1.3 Proving Conjecture 8.10

One of the missing conjectures, Conjecture 8.10, is the criterion in the principle of accompanying laws. To prove it, one approach might be to apply Chebyshev's inequality, provided we could show that, for some c ,

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \rightarrow \infty} \left| \mathbb{E}_{n,p} \left[n^{-1} \tilde{A}_i^{(n)}(|C_i^{(n)}|) \mathbb{1}_{\{n^{-2/3}|C_i^{(n)}| < \varepsilon\}} \right] - \frac{1}{4} \log(n^{2/3} \varepsilon) - c \right| = 0. \quad (8.13)$$

This would mean it was sufficient to bound the variance of the sum of the areas.

In fact, we can note by the tower law that, as $n \rightarrow \infty$,

$$\mathbb{E}_{n,p} \left[n^{-1} \tilde{A}_i^{(n)}(|C_i^{(n)}|) \mathbb{1}_{\{n^{-2/3}|C_i^{(n)}| < \varepsilon\}} \right] = \frac{p}{n} \mathbb{E}_{n,p} [N_n^{<\varepsilon}] = (1 + O(n^{-1/3})) \mathbb{E}_{n,p} [N_n^{<\varepsilon}],$$

where $N^{<\varepsilon}$ is the number of surplus edges deriving from components of size less than $\varepsilon n^{2/3}$. Thus, an equivalent statement to (8.13) is obtained by replacing the left-hand side of that equation by $\mathbb{E}_{n,p} [N_n]$.

In the rest of this section, we will look at how we might prove (8.13), specifically by again attempting to exploit (8.1). We write

$$N_n^{<\varepsilon} = U_n^{<\varepsilon} + E_n^{<\varepsilon} + C_n^{<\varepsilon}, \quad (8.14)$$

where $U_n^{<\varepsilon}$, $E_n^{<\varepsilon}$ and $C_n^{<\varepsilon}$ are the number of unicyclic components of size $< \varepsilon n^{2/3}$, the excess contributed by components of size $< \varepsilon n^{2/3}$, and the number of complex components of size $< \varepsilon n^{2/3}$, respectively, in G_n . We now state and prove some results here that go partway towards showing the final result about $N_n^{<\varepsilon}$. We will throughout take $p = 1/n + \lambda n^{-4/3}$.

Proposition 8.12. *There is a constant C such that*

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \rightarrow \infty} \left| \mathbb{E}_{n,p} [U_n^{<\varepsilon}] - \frac{1}{4} \log(n^{2/3} \varepsilon) - C \right| = 0$$

Our proof of this result is essentially the approach taken by Durrett [25, § 2.7], with additional precision in our approximations.

Proof. Let $U_{n,k}$ denote the number of unicyclic components of size k in G_n . Then, as the

smallest possible unicyclic component has size 3,

$$\mathbb{E}_{n,p}[U_n^{<\varepsilon}] = \sum_{k=3}^{\lceil \varepsilon n^{2/3} \rceil - 1} \mathbb{E}_{n,p}[U_{n,k}]. \quad (8.15)$$

Let u_k be the number of connected unicyclic graphs on $[k]$. Then it follows by linearity of expectation that

$$\mathbb{E}_{n,p}[U_{n,k}] = \binom{n}{k} u_k \left(\frac{1 + \lambda n^{-1/3}}{n} \right)^k \left(1 - \frac{1 + \lambda n^{-1/3}}{n} \right)^{k(n-k) + \binom{k}{2} - k},$$

an expression obtained in [25]. Similarly, using Cayley's formula, if $T_{n,k}$ is the number of size- k trees in G_n , we see (again as obtained in [25]) that

$$\mathbb{E}_{n,p}[T_{n,k}] = \binom{n}{k} k^{k-2} \left(\frac{1 + \lambda n^{-1/3}}{n} \right)^{k-1} \left(1 - \frac{1 + \lambda n^{-1/3}}{n} \right)^{k(n-k) + \binom{k}{2} - (k-1)}, \quad (8.16)$$

so that

$$\mathbb{E}_{n,p}[U_{n,k}] = \mathbb{E}_{n,p}[T_{n,k}] \frac{u_k}{k^{k-2}} \left(\frac{1 + \lambda n^{-1/3}}{n} \right) \left(1 - \frac{1 + \lambda n^{-1/3}}{n} \right)^{-1}. \quad (8.17)$$

We therefore start by analysing (8.16).

Let δ_k satisfy

$$\frac{1}{k!} = \frac{e^k}{k^{k+1/2} \sqrt{2\pi}} (1 + \delta_k).$$

Since, by Stirling's approximation,

$$k! = e^{-k} k^{k+1/2} \sqrt{2\pi} \left(1 + O\left(\frac{1}{k}\right) \right),$$

we have that $\delta_k = O(k^{-1})$. So

$$\begin{aligned} \mathbb{E}_{n,p}[T_{n,k}] &= n \left[\prod_{j=1}^{k-1} \left(1 - \frac{j}{n} \right) \right] \frac{e^k}{k^{5/2} \sqrt{2\pi}} (1 + \lambda n^{-1/3})^{k-1} \left(1 - \frac{1 + \lambda n^{-1/3}}{n} \right)^{kn - k^2/2 - 3k/2 + 1} \\ &\quad \times (1 + \delta_k). \end{aligned}$$

We note that all k in (8.15) are bounded above by $\varepsilon n^{2/3}$, which is crudely bounded (for all sufficiently large n) by $n^{7/10}$. So we take this bound on k . Then we may see by routine algebra

that, as $n \rightarrow \infty$,

$$\left(1 - \frac{1 + \lambda n^{-1/3}}{n}\right)^{kn - k^2/2 - 3k/2 + 1} = \exp\left\{-k + \frac{k^2}{2n} - \frac{k\lambda}{n^{1/3}} + \frac{k^2\lambda}{2n^{4/3}} + O(n^{-3/10})\right\},$$

where the $O(n^{-3/10})$ is uniform in k (that is, if we write

$$\left(1 - \frac{1 + \lambda n^{-1/3}}{n}\right)^{kn - k^2/2 - 3k/2 + 1} = \exp\left\{-k + \frac{k^2}{2n} - \frac{k\lambda}{n^{1/3}} + \frac{k^2\lambda}{2n^{4/3}} + a(n, k)\right\},$$

there is a constant $D > 0$ independent of n and k such that, for all sufficiently large n , and for all choices of $k \leq n^{7/10}$, we have $|a(n, k)| \leq Cn^{-3/10}$. Similarly we can show that

$$\prod_{j=1}^{k-1} \left(1 - \frac{j}{n}\right) = \exp\left\{-\frac{k^2}{2n} - \frac{k^3}{6n^2} + O(n^{-1/5})\right\},$$

the big- O term again being uniform in k . So

$$\begin{aligned} \mathbb{E}_{n,p}[T_{n,k}] &= n \frac{1}{k^{5/2} \sqrt{2\pi}} (1 + \lambda n^{-1/3})^{-1} \exp\left\{k \log(1 + \lambda n^{-1/3}) - \frac{\lambda k}{n^{1/3}} + \frac{\lambda k^2}{2n^{4/3}} - \frac{k^3}{6n^2}\right\} \\ &\quad \times (1 + \delta_k)(1 + O(n^{-1/5})), \end{aligned}$$

with the same uniformity.

We return to (8.17). Let δ'_k satisfy

$$u_k = \sqrt{\frac{\pi}{8}} k^{k-1/2} (1 + \delta'_k);$$

here, by Bollobás [15, Proof of Cor. 5.19], $\delta'_k = O(k^{-1/2})$. Then

$$\mathbb{E}_{n,p}[U_{n,k}] = \frac{1}{4k} \exp\left\{k \log(1 + \lambda n^{-1/3}) - \frac{\lambda k}{n^{1/3}} + \frac{\lambda k^2}{2n^{4/3}} - \frac{k^3}{6n^2}\right\} (1 + \delta''_k)(1 + O(n^{-1/5})),$$

where $\delta''_k = \delta_k + \delta'_k + \delta_k \delta'_k = O(k^{-1/2})$. Expanding the log, and summing up, we get

$$\begin{aligned} \mathbb{E}_{n,p}[U_n^{<\varepsilon}] &= (1 + O(n^{-1/5})) \sum_{k=3}^{\lceil \varepsilon n^{2/3} \rceil - 1} \frac{1}{4k} \exp\left\{-\frac{\lambda^2 k}{2n^{2/3}} + \frac{\lambda k^2}{2n^{4/3}} - \frac{k^3}{6n^2}\right\} \\ &\quad + (1 + O(n^{-1/5})) \sum_{k=3}^{\lceil \varepsilon n^{2/3} \rceil - 1} \frac{\delta''_k}{4k} \exp\left\{-\frac{\lambda^2 k}{2n^{2/3}} + \frac{\lambda k^2}{2n^{4/3}} - \frac{k^3}{6n^2}\right\}. \end{aligned} \tag{8.18}$$

(Throughout, all $O(n^{-1/5})$ terms have been uniform in k , so we can take them outside the sum.)

We start with the first term in (8.18), which we may express as

$$\begin{aligned} & \sum_{k=3}^{\lceil \varepsilon n^{2/3} \rceil - 1} \frac{1}{4k} + \sum_{k=3}^{\lceil \varepsilon n^{2/3} \rceil - 1} \frac{1}{4k} \left[\exp \left\{ -\frac{\lambda^2 k}{2n^{2/3}} + \frac{\lambda k^2}{2n^{4/3}} - \frac{k^3}{6n^2} \right\} - 1 \right] \\ &= \frac{1}{4} \left[\log(\varepsilon n^{2/3}) - \frac{3}{2} + \gamma + O(n^{-2/3}) \right] + \sum_{k=3}^{\lceil \varepsilon n^{2/3} \rceil - 1} \frac{1}{4k} \left[\exp \left\{ -\frac{\lambda^2 k}{2n^{2/3}} + \frac{\lambda k^2}{2n^{4/3}} - \frac{k^3}{6n^2} \right\} - 1 \right], \end{aligned}$$

where the big- O term is in n , treating ε as constant (since we will take the limit in n before the limit in ε , this term will have vanished before we consider the limit in ε), and, as before, γ is the Euler–Mascheroni constant. Now define

$$f_n(x) = \frac{1}{4\lfloor x \rfloor} \left[\exp \left\{ -\frac{\lambda^2 \lfloor x \rfloor}{2n^{2/3}} + \frac{\lambda \lfloor x \rfloor^2}{2n^{4/3}} - \frac{\lfloor x \rfloor^3}{6n^2} \right\} - 1 \right] \mathbb{1}_{\{x < \lceil \varepsilon n^{2/3} \rceil\}}$$

for all $x \in [3, \infty)$, so that

$$\sum_{k=3}^{\lceil \varepsilon n^{2/3} \rceil - 1} \frac{1}{4k} \left[\exp \left\{ -\frac{\lambda^2 k}{2n^{2/3}} + \frac{\lambda k^2}{2n^{4/3}} - \frac{k^3}{6n^2} \right\} - 1 \right] = \int_3^{\lceil \varepsilon n^{2/3} \rceil} f_n(x) dx.$$

Set $v = \lfloor x \rfloor / n^{2/3}$. Then the argument of the exponential is the cubic

$$-\frac{\lambda^2}{2}v + \frac{\lambda}{2}v^2 - \frac{1}{6}v^3,$$

which, by standard calculus, we may show is everywhere strictly decreasing in v . Since it evaluates to 0 at $v = 0$, we have that $f_n(x) \leq 0$ for all $x \in [3, \infty)$. Thus

$$\begin{aligned} 0 &\leq \int_3^{\lceil \varepsilon n^{2/3} \rceil} (-f_n(x)) dx \leq \int_3^{\varepsilon n^{2/3} + 1} \frac{1}{4(x-1)} \left[1 - \exp \left\{ -\frac{\lambda^2 x}{2n^{2/3}} + \frac{\lambda x^2}{2n^{4/3}} - \frac{x^3}{6n^2} \right\} \right] dx \\ &= \int_{3n^{-2/3}}^{\varepsilon + n^{-2/3}} \frac{1}{4(u - n^{-2/3})} \left[1 - \exp \left\{ -\frac{\lambda^2}{2}u + \frac{\lambda}{2}u^2 - \frac{1}{6}u^3 \right\} \right] du \\ &\rightarrow \int_0^\varepsilon \frac{1}{4u} \left[1 - \exp \left\{ -\frac{\lambda^2}{2}u + \frac{\lambda}{2}u^2 - \frac{1}{6}u^3 \right\} \right] du \end{aligned}$$

as $n \rightarrow \infty$, using the fact that the cubic is decreasing in the first line, and taking $u = xn^{-2/3}$ for the substitution in the second. This is finite, as the integrand tends to $\lambda^2/8$ as $u \downarrow 0$; thus, the integral tends to 0 as $\varepsilon \downarrow 0$.

Returning to (8.18), it remains to show that there is a constant $d \in \mathbb{R}$ such that

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \rightarrow \infty} \left| \sum_{k=3}^{\lceil \varepsilon n^{2/3} \rceil - 1} \frac{\delta_k''}{4k} \exp \left\{ -\frac{\lambda^2 k}{2n^{2/3}} + \frac{\lambda k^2}{2n^{4/3}} - \frac{k^3}{6n^2} \right\} - d \right| = 0. \quad (8.19)$$

Redefine

$$f_n(x) = \frac{\delta''_{\lfloor x \rfloor}}{4\lfloor x \rfloor} \exp\left\{-\frac{\lambda^2 \lfloor x \rfloor}{2n^{2/3}} + \frac{\lambda \lfloor x \rfloor^2}{2n^{4/3}} - \frac{\lfloor x \rfloor^3}{6n^2}\right\} \mathbb{1}_{\{x < \lceil \varepsilon n^{2/3} \rceil\}},$$

so that

$$\sum_{k=3}^{\lceil \varepsilon n^{2/3} \rceil - 1} \frac{\delta''_k}{4k} \exp\left\{-\frac{\lambda^2 k}{2n^{2/3}} + \frac{\lambda k^2}{2n^{4/3}} - \frac{k^3}{6n^2}\right\} = \int_3^{\lceil \varepsilon n^{2/3} \rceil} f_n(x) dx.$$

Now, as $n \rightarrow \infty$,

$$f_n(x) \rightarrow \frac{\delta''_{\lfloor x \rfloor}}{4\lfloor x \rfloor} =: f(x).$$

Moreover, we have argued that the argument of the exponent is always negative, so that, for all n , and for $x \geq 3$,

$$|f_n(x)| \leq \frac{|\delta''_{\lfloor x \rfloor}|}{4\lfloor x \rfloor} = g(x),$$

where

$$\int_3^\infty g(x) dx = \sum_{k=3}^\infty \frac{|\delta''_k|}{4k} < \infty$$

as $\delta''_k = O(k^{-1/2})$. So, by the dominated convergence theorem,

$$\sum_{k=3}^{\lceil \varepsilon n^{2/3} \rceil - 1} \frac{\delta''_k}{4k} \exp\left\{-\frac{\lambda^2 k}{2n^{2/3}} + \frac{\lambda k^2}{2n^{4/3}} - \frac{k^3}{6n^2}\right\} \rightarrow \sum_{k=3}^\infty \frac{\delta''_k}{4k}$$

as $n \rightarrow \infty$. (Note that this constant does not depend at all on ε .) This shows (8.19), and hence the result follows. \square

Having shown this, and comparing the result just obtained with (8.13) and (8.14), it would remain, in order to get (8.13), to show that there are constants D_1 and D_2 such that

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \rightarrow \infty} |\mathbb{E}_{n,p}[|E_n^{<\varepsilon} - D_1|]| = 0$$

and

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \rightarrow \infty} |\mathbb{E}_{n,p}[|C_n^{<\varepsilon} - D_2|]| = 0.$$

For the other two parts, however, we have only been able to show that the expectations of $E_n^{<\varepsilon}$ and $C_n^{<\varepsilon}$ are $O(1)$ as $n \rightarrow \infty$. In both cases, we can do so by bounding above by E_n and C_n , the total excess and total number of complex components, respectively. The expectation of the latter converges, by [44, Thm. 2]; we bound the former in the next lemma.

Lemma 8.13. *If $p = 1/n + \lambda n^{-4/3}$, then $\mathbb{E}_{n,p}[E_n] = O(1)$ as $n \rightarrow \infty$.*

Proof. We note that E_n may alternatively be expressed as

$$E_n = \sum_i \left(e(G_i^{(n)}) - |C_i^{(n)}| \right)_+,$$

where $C_i^{(n)}$, as usual, denotes the i -th largest component of G_n —in other words, it is the number of surplus edges in each component, after the first such edge. Writing in this way lets us check, by careful enumeration of cases, that adding edges to a graph can only increase its excess. By combining this with the standard coupling of $\mathcal{G}(n, p)$ across values of p , we see that $E_n = E_n(\lambda)$ is a weakly increasing function of λ . So we restrict without loss of generality to $\lambda \geq 1$.

For $n, m \in \mathbb{N}$ (with $m \leq \binom{n}{2}$), let $G_{n,m} \sim \mathcal{G}(n, m)$: the uniform random graph with n vertices and m edges. By essentially the same argument as above, the expected excess of $G_{n,m}$ must be a weakly increasing function of m . Then it is standard that

$$G_n \stackrel{d}{=} G_{n, M_n(\lambda)}$$

where $M_n(\lambda) \sim \text{Bin}\left(\binom{n}{2}, p\right)$, independently of $(G_{n,m})_{m \in \mathbb{N}}$. Let $E_{n,m}$ denote the excess of $G_{n,m}$. Then it holds that

$$\begin{aligned} \mathbb{E}_{n,p}[E_n] &= \mathbb{E}_{n,p} \left[E_n \mid M_n(\lambda) > \frac{n}{2} + \lambda n^{2/3} \right] \mathbb{P}_{n,p} \left(M_n(\lambda) > \frac{n}{2} + \lambda n^{2/3} \right) \\ &\quad + \mathbb{E}_{n,p} \left[E_n \mid M_n(\lambda) \leq \frac{n}{2} + \lambda n^{2/3} \right] \mathbb{P}_{n,p} \left(M_n(\lambda) \leq \frac{n}{2} + \lambda n^{2/3} \right) \\ &\leq \left(\binom{n}{2} - n \right) \mathbb{P}_{n,p} \left(M_n(\lambda) > \frac{n}{2} + \lambda n^{2/3} \right) + \mathbb{E}[E_{n, n/2 + \lambda n^{2/3}}]. \end{aligned} \quad (8.20)$$

By a result of Janson, Knuth, Łuczak and Pittel [39, Thm. 6], the second term here converges as $n \rightarrow \infty$, so in particular is $O(1)$.

The first term, meanwhile, is a product of an $O(n^2)$ term and a binomial probability, where $\mathbb{E}_{n,p}[M_n(\lambda)] = \binom{n}{2}(1/n + \lambda n^{-4/3}) = n/2 + \lambda n^{2/3}/2 + O(1)$. We apply a Chernoff bound, noting (e.g. from [47, (4.2)]) that

$$\mathbb{P}_{n,p} \left(M_n(\lambda) \geq (1 + \delta) \binom{n}{2} p \right) \leq e^{-\mu \delta^2 / 3}$$

as long as $\delta \in (0, 1)$. Here we wish to take

$$\delta = \frac{n/2 + \lambda n^{2/3} - \binom{n}{2} p}{\binom{n}{2} p} = \frac{\lambda n^{2/3}/2 + O(1)}{n/2 + O(n^{2/3})} = \lambda n^{-1/3} + O(n^{2/3}),$$

and hence $\delta \in (0, 1)$ for sufficiently large n , since we restricted to $\lambda \geq 1$. So

$$\mathbb{P}_{n,p}\left(M_n(\lambda) > \frac{n}{2} + \lambda n^{2/3}\right) \leq \exp\left\{-\frac{1}{3}\binom{n}{2}p(\lambda n^{-1/3} + O(n^{-2/3}))^2\right\} = \exp\left\{-\frac{1}{6}n^{1/3}\right\} = o(n^2),$$

in particular. So the first term in (8.20) tends to 0 as $n \rightarrow \infty$, completing the proof. \square

As an aside, while we couldn't prove the result we wanted, we note that we *can* use essentially the “ingredients” above to prove the following result.

Theorem 8.14. *Let $p = 1/n + \lambda n^{-4/3}$. Then, as $n \rightarrow \infty$,*

$$\mathbb{E}_{n,p}[N_n] = \frac{1}{6} \log n + O(1).$$

We use the decomposition in (8.1) again. The only result missing to turn the results above into a proof of this theorem is a bound on $\mathbb{E}_{n,p}[U_{n,k}]$ for large k , which can be used to prove that

$$\mathbb{E}_{n,p}[U_n] = \frac{1}{6} \log n + O(1).$$

We state this below.

Lemma 8.15. *Fix $\lambda \in \mathbb{R}$. Then there exists a constant $C > 0$ such that, for all $n \in \mathbb{N}$ with*

$$\frac{1}{n} + \frac{\lambda}{n^{4/3}} \in (0, 1),$$

and for all $k \in \mathbb{N}$ with $k \leq n$,

$$\mathbb{E}_{n,p}[U_{n,k}] \leq C \frac{1}{k} \exp\left\{k \log(1 + \lambda n^{-1/3}) - \frac{\lambda k}{n^{1/3}} + \frac{\lambda k^2}{2n^{4/3}} - \frac{k^3}{6n^2}\right\}.$$

We omit the proof; it follows the strategy of the first part of the proof of Proposition 8.12.

We have mostly discussed showing (8.13) in this subsection. Recall, however, that this alone would not be enough to give Conjecture 8.10, as, to apply Chebyshev's inequality, we would need a bound on the variance that we do not currently possess.

8.1.4 Adapting for $\mathcal{G}(n, m, q)$

We briefly address how we might adapt this proof strategy (i.e. the proof of Theorem 8.11) for $\mathcal{G}(n, m, q)$. Much of it translates in a straightforward manner—and of course we already have an equivalent to Conjecture 8.5, in the form of Lemma 8.3. The problem of finding an

equivalent to Conjecture 8.10 remains unsolved, however.

8.2 A conditional scaling limit for the random cluster model

Suppose we have convergence in distribution of an ordered sequence of component sizes for $\mathcal{G}(n, q/n + \lambda n^{-4/3}, q)$ (by the method outlined in the previous section, by a proof along the lines of the heuristic one in Section 3.2, or in some other way). In this section, we show that this, applying methods of Addario-Berry, Broutin and Goldschmidt [3], is sufficient to obtain a form of metric space scaling limit for the components.

We have already noted in Lemma 3.5 that, conditionally on the ordered sequence of component sizes, a random cluster graph and an Erdős–Rényi graph have identical (conditional) distribution. In particular, this implies that $\mathcal{G}(n, p)$ and $\mathcal{G}(n, p, q)$, conditioned on being connected, likewise have identical (conditional) distribution. However, we note that, starting with a critical random cluster graph, the Erdős–Rényi graph to which we are comparing will be sub- or supercritical, depending on whether $q < 1$ or $q > 1$. Hence, if we condition on the typical $n^{2/3}$ sizes of the components for the random cluster graph, we will be conditioning on a large-deviations behaviour for the Erdős–Rényi graph.

However, much of the machinery developed in [3] can be modified to get us the result we need. We first recall some notation from Chapter 2. We define G_m^p to have the distribution of $\mathcal{G}(m, p)$, conditional on being connected. (Informally, this is equivalent to a component of $\mathcal{G}(n, p)$ —or, by Lemma 3.5, a component of $\mathcal{G}(n, p, q)$ —conditioned on having size m .) We take the tree \tilde{T}_m^p to be a random tree on $[m]$ chosen such that, for any such tree T ,

$$\mathbb{P}(\tilde{T}_m^p = T) \propto (1 - p)^{-a(T)},$$

where $a(T)$ is the area of the tree as defined in Definition 2.13.

We borrow some further definitions from [3, p. 387], and define, for a tree T with depth-first walk X , and for a countable subset $\mathcal{Q} \subseteq (\mathbb{Z}^{\geq 0})^2$, that $G^X(T, \mathcal{Q})$ is the graph formed by adding to T the edges corresponding to the points in $\mathcal{Q} \cap X$, where we define

$$\mathcal{S} \cap f := \{(x, y) \in \mathcal{S} : 0 \leq y \leq f(x)\}.$$

(We discussed how such points can correspond to edges in Section 2.3, and gave an illustrated example in Figure 2.2.) Finally, we say that \mathcal{Q}^p is a *Binomial pointset of probability p* , which is

a random subset of $(\mathbb{Z}^{\geq 0})^2$ in which each point is included independently with probability p .

Our adaptation of the proof in Addario-Berry, Broutin and Goldschmidt [3] begins with their Lemma 18, which we may apply directly.

Lemma 8.16 ([3, Lma. 18]). *Let $p \in (0, 1)$. Let \tilde{T}_m^p be a tree on $[m]$ sampled in such a way that $\mathbb{P}(\tilde{T}_m^p = T) \propto (1-p)^{-a(T)}$, and let $(\tilde{X}_m(i))_{0 \leq i < m}$ be the associated depth-first walk. Take \mathcal{Q}^p independent of \tilde{T}_m^p . Then*

$$G_m^p \stackrel{d}{=} G^X(\tilde{T}_m^p, \mathcal{Q}^p).$$

Here, informally, we are using this to consider a component of size m within a graph of size n . Where we differ from [3] is that we will consider p so that $np \rightarrow q$ instead of $np \rightarrow 1$.

To that end, we note the following lemma:

Lemma 8.17. *Let $p = p(m)$ be such that $mp^{2/3} \rightarrow q^{2/3}\sigma$ as $m \rightarrow \infty$. Pick a labelled tree \tilde{T}_m^p on $[m]$ in such a way that*

$$\mathbb{P}(\tilde{T}_m^p = T) \propto (1-p)^{-a(T)}$$

and let \tilde{X}_m be the associated depth-first walk. Let \mathcal{Q}^p be independent. Let

$$\mathcal{P}_m = \left\{ \left(i \frac{\sigma}{m}, j \sqrt{\frac{\sigma}{m}} \right) : (i, j) \in \mathcal{Q}^p \right\}.$$

Then

$$\begin{aligned} & \left(\sqrt{\frac{\sigma}{m}} \tilde{X}_m \left(\lfloor \left(\frac{m}{\sigma} \cdot \rfloor \right) \right), \mathcal{P}_m \cap \left(\sqrt{\frac{\sigma}{m}} \tilde{X}_m \left(\lfloor \left(\frac{m}{\sigma} \cdot \rfloor \right) \right) \right) \right) \\ & \xrightarrow{d} \left(q^{-1/3} \tilde{\mathbf{e}}^{(\sigma q^{2/3})}(q^{2/3}), \mathcal{P}^q \cap \left(q^{-1/3} \tilde{\mathbf{e}}^{(\sigma q^{2/3})}(q^{2/3}) \right) \right) \end{aligned} \quad (8.21)$$

as $n \rightarrow \infty$, where \mathcal{P}^q is a homogeneous Poisson point process of rate q on $[0, \infty)^2$, and \mathcal{P}^q is independent of $\tilde{\mathbf{e}}^{(\sigma)}$. Convergence in the first co-ordinate is in $\mathbb{D}([0, \sigma], [0, \infty))$ equipped with the Skorokhod topology, and in the second co-ordinate is in the sense of Hausdorff distance.

Proof. This directly follows from the fact that, when we replace σ by $q^{2/3}\sigma$ in Lemma 19 of [3], we have that if

$$\mathcal{P}_{m,q} = \left\{ \left(i \frac{\sigma q^{2/3}}{m}, j q^{1/3} \sqrt{\frac{\sigma}{m}} \right) : (i, j) \in \mathcal{Q}^p \right\},$$

then

$$\left(q^{1/3} \sqrt{\frac{\sigma}{m}} \tilde{X}_m \left(\left\lfloor \left(\frac{m}{\sigma q^{2/3}} \right) \cdot \right\rfloor \right), \mathcal{P}_{m,q} \cap \left(q^{1/3} \sqrt{\frac{\sigma}{m}} \tilde{X}_m \left(\left\lfloor \left(\frac{m}{\sigma q^{2/3}} \right) \cdot \right\rfloor \right) \right) \right) \xrightarrow{d} (\tilde{\mathbf{e}}^{(\sigma,q)}, \mathcal{P}^1 \cap \tilde{\mathbf{e}}^{(\sigma,q)}); \quad (8.22)$$

the left-hand side of (8.21) is exactly the left-hand side of (8.22), except with distances in the y -direction rescaled by $q^{-1/3}$, and in the x -direction rescaled by $q^{-2/3}$. \square

To simplify notation, let us define the random excursion $\tilde{\mathbf{e}}^{(\sigma,q)} : [0, \sigma] \rightarrow [0, \infty)$, by

$$\tilde{\mathbf{e}}^{(\sigma,q)} \stackrel{d}{=} q^{-1/3} \tilde{\mathbf{e}}^{(\sigma q^{2/3})}(q^{2/3} \cdot), \quad (8.23)$$

so that the right-hand side of (8.21) becomes

$$(\tilde{\mathbf{e}}^{(\sigma,q)}, \mathcal{P}^q \cap \tilde{\mathbf{e}}^{(\sigma,q)}).$$

We will analogously (cf. Defn. 2.23) define the random metric space

$$\mathcal{M}^{(\sigma,q)} = g(2\tilde{\mathbf{e}}^{(\sigma,q)}, \mathcal{P}^q),$$

using the function g defined at Definition 2.22 which is the continuous analogue to G^X above.

We can then adapt Theorem 22 of [3]—given as Theorem 2.24 here—to get a limit for a component of $\mathcal{G}(n, p, q)$ in its critical window.

Theorem 8.18. *Let $\sigma > 0$. Let $m = m(n) \in \mathbb{Z}^{\geq 0}$ be a sequence of integers such that $n^{-2/3}m \rightarrow \sigma$ as $n \rightarrow \infty$, and let $p = p(n) \in (0, 1)$ be such that $pn \rightarrow q$. Then, as $n \rightarrow \infty$,*

$$n^{-1/3} G_m^p \xrightarrow{d} \mathcal{M}^{(\sigma,q)},$$

in the Gromov–Hausdorff distance.

The proof is nearly identical to the proof of Theorem 22 of [3], except relying on our Lemma 8.17 instead of [3, Lma. 19]. (The proof also relies on Lemma 16 of the same paper, but that applies exactly to our setting, simply by replacing σ by $q^{2/3}\sigma$ in its hypotheses.) Consequently, we omit this proof.

Having considered the component picture, it remains to consider the entire graph. The following theorem may be proved using the first part of the proof of [3, Thm. 24].

Theorem 8.19. Let $G_n \sim \mathcal{G}(n, p, q)$, for $q \in (1, 2)$ and $p = q/n + \lambda n^{-4/3}$. For $i \geq 1$, let $C_i^{(n)}$ be the i -th largest component of G_n , with ties broken arbitrarily. Define

$$\mathbf{C}_n = (|C_1^{(n)}|, |C_2^{(n)}|, \dots),$$

where, if $C_k^{(n)}$ does not exist, we say that $|C_k^{(n)}| = 0$. Suppose it holds that

$$n^{-2/3} \mathbf{C}_n \xrightarrow{d} \mathbf{C} = (C_1, C_2, \dots)$$

in l_{\searrow}^2 , the space of decreasing non-negative sequences $\mathbf{x} = (x_1, x_2, \dots)$ such that

$$\sum_{i=1}^{\infty} x_i^2 < \infty,$$

equipped with the l^2 norm. Let $\mathbf{M}^{(n)} = (M_1^{(n)}, M_2^{(n)}, \dots)$ be the sequence in which $M_i^{(n)}$ is $C_i^{(n)}$ treated as a metric space equipped with the graph distance. Then it holds that

$$(n^{-2/3} \mathbf{C}_n, n^{-1/3} \mathbf{M}_n) \xrightarrow{d} (\mathbf{C}, \mathbf{M}),$$

where $\mathbf{M} = (M_1, M_2, \dots)$ is a sequence of metric spaces such that, conditional on \mathbf{Z} , we have $(M_i)_{i \in \mathbb{N}}$ all independent, and $M_i \stackrel{d}{=} \mathcal{M}^{(C_i, q)}$. Convergence in the first co-ordinate is still in l_{\searrow}^2 , while convergence in the second co-ordinate is in the product topology, the distance between the individual metric spaces in the sequence being taken to be Gromov–Hausdorff distance.

Note here we have convergence of the sequence of metric spaces in the product topology, rather than in the metric d_{seq} defined in (2.9). It is potentially possible to extend the result to this metric.

Chapter 9

Outlook

We conclude by summarising what remains, following the results in this thesis, in order to prove the existence of the scaling limit of the random cluster model. We also discuss some possibilities for future related research.

9.1 Completing the proof of a scaling limit for $q \in (0, 2)$

The results in Section 8.2 show that a full proof of the scaling limit of this model rests on showing that the sequence of component sizes converges, which we formally state as a conjecture now.

Conjecture 9.1. *Let $G_n \sim \mathcal{G}(n, p, q)$, for $q \in (1, 2)$ and $p = q/n + \lambda n^{-4/3}$. For $i \geq 1$, let $C_i^{(n)}$ be the i -th largest component of G_n , with ties broken arbitrarily. Define*

$$\mathbf{C}_n = (|C_1^{(n)}|, |C_2^{(n)}|, \dots),$$

where, if $C_i^{(n)}$ does not exist, we say that $|C_i^{(n)}| = 0$. Then there exists a random sequence \mathbf{C} such that

$$n^{-2/3} \mathbf{C}_n \xrightarrow{d} \mathbf{C}$$

in l^2_{\searrow} , the space of decreasing non-negative sequences $\mathbf{x} = (x_1, x_2, \dots)$ such that

$$\sum_{i=1}^{\infty} x_i^2 < \infty,$$

equipped with the l^2 norm.

We have outlined two major possible proof strategies, one of which directed the bulk of

the work in this thesis—the one which we described in Section 3.2. We recall that this strategy would provide a proof of Conjecture 9.1 for *some* values of $q \in (0, 2) \setminus \{1\}$ if either the first of the following conjectures holds, or both of the other two do. (All further conjectures in this section are restated from earlier in the thesis, preserving their numbering.)

Conjecture 6.2. Fix $\lambda \in \mathbb{R}$, and define (X, I, N) to be real-valued processes on $[0, \infty)$ with distributions given by:

$$\begin{aligned} X(t) &= B(t) + \lambda t - \frac{t^2}{2}; \\ I(t) &= \inf_{0 \leq s \leq t} X(s); \\ (N(t))_{t \geq 0} &\text{ is a Poisson process of rate } X(t) - I(t) \end{aligned}$$

(where $B(t)$ is a Brownian motion). Then, for $q \in (0, 1)$, there exists a function $\mathfrak{D}_q(T) > 0$ such that the random process $(R_q(T))_{T \geq 0}$, where

$$R_q(T) := \frac{1}{\mathfrak{D}_q(T)} \exp \left\{ (q-1) \int_0^T (X(u) - I(u)) du \right\}, \quad (6.1)$$

is uniformly integrable, and converges in probability as $T \rightarrow \infty$ to a random variable that is almost surely positive.

Remark. This, if proven, would give us Conjecture 9.1 in the range $q \in (0, 1)$.

Conjecture 6.10. Fix $q \in (1, 2)$ and $\lambda \in \mathbb{R}$, and define $p = p(n, \lambda, q) = q/n + \lambda n^{-4/3}$. Let the event $E_n(T, \varepsilon)$ be defined by

$$\begin{aligned} E_n(T, \varepsilon) &= \{X_n(k) = I_n(k) \text{ for some } k \in [\lfloor Tn^{2/3} \rfloor, \lfloor (T + \varepsilon)n^{2/3} \rfloor]\} \\ &\quad \cap \{N_n(\lfloor (T + \varepsilon)n^{2/3} \rfloor) = N_n(\lfloor Tn^{2/3} \rfloor)\}, \end{aligned}$$

where (X_n, I_n, N_n) is the exploration process defined by Algorithm A, applied to the graph $G_n \sim \mathcal{G}(n, p, q)$.

Then

$$\lim_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}_{n,p,q} \left(E_n \left(T, \frac{1}{T} \right)^c \right) = 0.$$

Conjecture 6.11. Let $p = 1/n + \lambda n^{-4/3}$, let $G_n \sim \mathcal{G}(n, 1/n + \lambda n^{-4/3})$, and let (X_n, I_n, N_n) be the usual depth-first exploration process. If $(M_n)_{n \in \mathbb{N}}$ is a sequence of uniformly bounded $\sigma(X_n)$ -measurable random variables that converge in distribution to M , then for all $q \in (1, 2)$, as $n \rightarrow$

∞ ,

$$\mathbb{E}_{n,\tilde{p}}[q^{N_n}(\lfloor Tn^{2/3} \rfloor)M_n] \rightarrow \mathbb{E}[q^{N(T)}M].$$

Remark. Conjecture 6.10, which we state more explicitly here than on its previous appearance, taken together with Conjecture 6.11, would be sufficient to prove Conjecture 9.1 in the range $q \in (1, 2)$.

Aside from the approach from Section 3.2, there is the approach in Theorem 8.11. We recall that this relied on two conjectures, which we restate.

Conjecture 8.5. *Let $p = 1/n + \lambda n^{-4/3}$. Then, as $n \rightarrow \infty$, for all $q \in (0, 2)$,*

$$\mathbb{E}_{n,p}[q^{N_n}] = O(n^{-(q-1)/6}).$$

Conjecture 8.10. *Let $C_i^{(n)}$ be the i -th largest component of $G_n \sim \mathcal{G}(n, p)$, where $p = 1/n + \lambda n^{-4/3} + O(n^{-2})$. Let $\tilde{A}_i^{(n)}(|C_i^{(n)}|)$ be the area corresponding to the component $C_i^{(n)}$. Then there exists $c \in \mathbb{R}$ such that, for all $\delta > 0$,*

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}_{n,p} \left(\left| \sum_i n^{-1} \tilde{A}_i^{(n)}(|C_i^{(n)}|) \mathbb{1}_{\{n^{-2/3}|C_i^{(n)}| < \varepsilon\}} - \frac{1}{4} \log(n^{2/3} \varepsilon) + c \right| \geq \delta \right) = 0.$$

If both of the above could be proven, then we would have that Conjecture 9.1 holds, and hence that a scaling limit exists. (This is irrespective, therefore, of whether we can prove either of Conjectures 6.2 and 6.10.)

Thus an obvious possible direction for future research would be to prove any of the above conjectures. Alternatively, there may be some other approach not addressed in this thesis, by which one could prove Conjecture 9.1.

Note Conjecture 9.1 makes no assertion about the distribution of \mathbf{C} . We provided a conjecture for this in Conjecture 3.6. We supposed that, under the measure \mathbb{P} , \mathbf{C} was the sequence of excursion lengths, in decreasing order, of $X - I$, where, for a Brownian motion $B(t)$, we have $X(t) = B(t) + \lambda t/q - t^2/2$, and $I(t) = \inf_{0 \leq u \leq t} X(u)$; then, we let L_λ be the limit in probability, as $t \rightarrow \infty$, of $\int_0^t (X(u) - I(u)) du - (\log t)/2$. We then conjectured that \mathbf{C} has the law that it does under $\tilde{\mathbb{P}}$, where

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \frac{\exp\{(q-1)L_\lambda\}}{\mathbb{E}[\exp\{(q-1)L_\lambda\}]}, \quad (9.1)$$

in which \mathbb{E} denotes expectation under \mathbb{P} .

This conjecture follows naturally from the heuristic proof in Section 3.2, and we have found good evidence for it via our results in Chapter 8. In Section 8.1, we found evidence

for an alternative measure change, where instead we took a random variable corresponding to summing the area under the excursions in decreasing order of length, and compensating appropriately. We conjecture that these two random variables differ only by a constant, and so the two measure changes are equivalent.

9.2 $q \geq 2$

Apart from a brief reference in Chapter 3, we have not discussed what happens when $q \geq 2$. In the case of the strict inequality $q > 2$, taking again $p = \kappa/n$, we know that the phase transition occurs at

$$\kappa = 2 \frac{q-1}{q-2} \log(q-1)$$

[16, (1.4)], and, for this value of κ , we w.h.p. see something “like” either the sub- or super-critical regimes, with non-vanishing probability of either case ([16, Thm. 2.3] or [58, pp. 202–203]). Both papers use slightly informal statements along these lines (though only [58] asserts explicitly that the probability of neither case vanishes). To state this more formally (taking a lead from [16, p. 299], we mean (e.g.) that, for any $\omega = \omega(n) \rightarrow \infty$ as $n \rightarrow \infty$, we have that there exist constants $c = c(q)$ and $d = d(q)$ such that, if we let

$$E_n^{(1)} = \left\{ \frac{\left| |C_1^{(n)}| - c \log n \right|}{\log \log n} < \frac{1}{\omega} \right\}$$

and

$$E_n^{(2)} = \left\{ \left| \frac{|C_1^{(n)}|}{n} - d \right| < \frac{1}{\omega} \right\},$$

then $\mathbb{P}(E_n^{(1)} \cup E_n^{(2)}) \rightarrow 1$ as $n \rightarrow \infty$, with $\limsup_{n \rightarrow \infty} \mathbb{P}(E_n^{(i)}) > 0$ for $i = 1, 2$.

Thus the critical behaviour in this setting is of a completely different nature.

There is perhaps more possible interest when $q = 2$. Here, the phase transition (when $p = \kappa/n$) occurs at $\kappa = 2$ [16, (1.4)]; moreover, there are a number of possible near-critical behaviours, which we enumerate below (adapted from Theorem 18 of Łuczak and Łuczak [43]).

Theorem 9.2 ([43, Thm. 18]). *Let $q = 2$, and let $p = (1 + \varepsilon)2/n$, for some $\varepsilon = \varepsilon(n)$. Let $G_n \sim \mathcal{G}(n, p, 2)$, and let $C_i^{(n)}$ be the i -th largest component of G_n . Then, as $n \rightarrow \infty$:*

(1) if $\varepsilon n^{1/3} \rightarrow -\infty$, w.h.p. G_n has no complex components, and

$$|C_1^{(n)}| = \frac{2}{\varepsilon^2} \left(\log(n|\varepsilon|^3) - \frac{5}{2} \log \log(n|\varepsilon|^3) + O_p(1) \right);$$

(2) if $\varepsilon n^{1/3} \rightarrow \lambda < 0$, then $|C_1^{(n)}|, |C_2^{(n)}| = \Theta_p(n^{2/3})$, with G_n having $O_p(1)$ complex components all of size $\Theta_p(1)$;

(3) if $\varepsilon = o(n^{1/3})$ but $\varepsilon n^{1/2} \rightarrow -\infty$, then w.h.p. $|C_1^{(n)}|$ is the only complex component,

$$|C_1^{(n)}| = \Theta_p\left(\sqrt{\frac{n}{|\varepsilon|}}\right),$$

and

$$|C_2^{(n)}| = O_p\left(|\varepsilon n| \log\left(\frac{1}{n|\varepsilon|^3}\right)\right);$$

(4) if $\varepsilon n^{1/2} \rightarrow c \in \mathbb{R}$, then w.h.p. $|C_1^{(n)}|$ is the only complex component, $|C_1^{(n)}| = \Theta_p(n^{3/4})$, and $|C_2^{(n)}| = O_p(\sqrt{n} \log n)$; and

(5) if $\varepsilon n^{1/2} \rightarrow \infty$ but $\varepsilon = o(1)$, then w.h.p. $|C_1^{(n)}|$ is the only complex component,

$$|C_1^{(n)}| = (1 + o(1))\sqrt{3\varepsilon n} + O_p\left(\sqrt{\frac{n}{\varepsilon}}\right)$$

(where the $o(1)$ term is a deterministic error), and

$$|C_2^{(n)}| = (1 + o_p(1)) \frac{\log(n^2 \varepsilon^3)}{3\varepsilon}.$$

Remark. Parts of the result above have been quoted earlier in this thesis. In [43] the authors gave many of the big-O-in-probability statements above as local limit theorems, which are in some sense more precise than the big-O statements, but do not directly imply them; their proof method shows that the above results do, in fact, hold.

We conjecture that everything above holds in case (2) above.

Conjecture 9.3. *Conjecture 9.1, about a limit for the component sizes, also holds in the case where $q = 2$ and $\lambda < 0$.*

Remark. We have already noted that there is evidence for this, such as the fact that Proposition 7.20 extends to this case; we note this formally here.

Proposition 9.4. *Theorem 8.19, which shows that a metric space scaling limit holds if the rescaled component sizes converge, also holds in the case where $q = 2$ and $\lambda < 0$.*

We do not write out the proof, simply noting that there is nothing in the proof of Theorem 8.19 that requires $q < 2$.

In the other cases, it may be that more can be shown than was shown in Theorem 9.2, for instance about the distributions of smaller component sizes. We do not make any specific conjectures, but note that it is potentially an avenue for future research.

9.3 Universality

Results in the literature (e.g. [10, 11, 24]) imply that there is, in fact, a universality class of random graph models “surrounding” the Erdős–Rényi model whose exploration processes converge to the process found by [6], including critical versions of the configuration model and the rank-one inhomogeneous random graphs. It would be natural to expect that a similar result holds for graphs related to the mean-field random cluster model—for instance, biasing instances of a graph G distributed according to the configuration model by $q^{c(G)}$, where $c(G)$, as before, denotes the number of components of G ; or, alternatively, the equivalent for rank-one inhomogeneous random cluster graphs. If Conjecture 9.1 could be shown, it would be a natural direction to try to prove that the scaling limit is universal, in the sense that it holds for “similar” models such as these.

Appendix A

Additional proofs

Here we include certain proofs of (mostly technical) results that, for overall clarity, were omitted from the main body of this thesis. For reference, we will restate each result itself before the corresponding proof.

Lemma 5.3. *Let $m = (1 + \lambda n^{-1/3} O(n^{-2/3}))(n-1)/2$. There is a universal constant C for all m' satisfying*

$$|m' - m + \lfloor n^{2/3} t \rfloor| \leq n^{1/3} \log n,$$

and for all sufficiently large n , giving that, if

$$\tilde{M}_n(t) \sim \text{Bin}\left(\binom{n - \lfloor n^{2/3} t \rfloor}{2}, p\right),$$

then

$$\left| \mathbb{P}(\tilde{M}_n(t) = m') - \frac{1}{\sqrt{2\pi m}} \right| \leq C \frac{(\log n)^2}{n^{5/6}}. \quad (5.2)$$

(Originally stated on page 82.)

Proof. By (5.6), we have that there exists $C_1 > 0$ such that

$$\left| \mathbb{P}(\tilde{M}_n(t) - m') - \mathbb{P}\left(\text{Poi}\left(\binom{n'}{2} p\right) = m'\right) \right| \leq \frac{C_1}{n} \quad (A.1)$$

for all sufficiently large n , and for all valid choices of m' . Let

$$a'_n = \binom{n'}{2} p - m',$$

so that there exists $C_2 > 0$ such that, for all sufficiently large n and for all valid m' ,

$$|a'_n| \leq \left| \binom{n - \lfloor n^{2/3}t \rfloor}{2} p - m + \lfloor n^{2/3}t \rfloor \right| + |m - \lfloor n^{2/3}t \rfloor - m'| \leq C_2 n^{1/3} \log n,$$

by (5.8) and the fact that

$$\begin{aligned} & \binom{n - \lfloor n^{2/3}t \rfloor}{2} p - m + \lfloor n^{2/3}t \rfloor \\ &= \binom{n}{2} p - m + \lfloor n^{2/3}t \rfloor - \frac{\lfloor n^{2/3}t \rfloor (2n - 1) + \lfloor n^{2/3}t \rfloor^2}{2} \left(\frac{1}{n} + O(n^{-4/3}) \right) = O(n^{1/3}), \end{aligned}$$

which doesn't depend on m' . (Again, taking crude bounds, we can observe that, for sufficiently large n , this constrains $|a'_n/m'| \leq 1/2$, say.) Then, similarly to our calculations in the proof of Theorem 5.2, we have that

$$\begin{aligned} \mathbb{P}\left(\text{Poi}\left(\binom{n'}{2} p\right) = m'\right) &= \frac{\exp\{-m' - a'_n\} (m' + a'_n)^{m'}}{m'!} \\ &= \frac{1}{\sqrt{2\pi m'}} \exp\left\{-a'_n + m' \log\left(1 + \frac{a'_n}{m'}\right)\right\} \\ &\leq \frac{1}{\sqrt{2\pi m'}} \end{aligned}$$

(using Stirling's formula and the fact that $\log(1+x) \leq x$ for all $x > -1$); likewise, we see that there are constants $C_3, \dots, C_6 > 0$ such that, for all appropriate n and m' ,

$$\begin{aligned} \mathbb{P}\left(\text{Poi}\left(\binom{n'}{2} p\right) = m'\right) &\geq \frac{1}{\sqrt{2\pi m'}} \exp\left\{-a'_n + m' \log\left(1 + \frac{a'_n}{m'}\right)\right\} \left(1 + \frac{C_3}{m'}\right)^{-1} \\ &\geq \frac{1}{\sqrt{2\pi m'}} \exp\left\{-C_4 \frac{(a'_n)^2}{m'}\right\} \left(1 + \frac{C_3}{m'}\right)^{-1} \\ &\geq \frac{1}{\sqrt{2\pi m'}} \left(1 - C_5 \frac{(\log n)^2}{n^{1/3}}\right) \geq \frac{1}{\sqrt{2\pi m'}} - C_6 \frac{(\log n)^2}{n^{5/6}}. \end{aligned}$$

Thus

$$\left| \mathbb{P}\left(\text{Poi}\left(\binom{n'}{2} p\right) = m'\right) - \frac{1}{\sqrt{2\pi m'}} \right| \leq C_6 \frac{(\log n)^2}{n^{5/6}} \quad (\text{A.2})$$

for all such n and m' .

It can be similarly observed that there exists $C_7 > 0$ such that, for all appropriate n and m' ,

$$\left| \frac{1}{\sqrt{2\pi m'}} - \frac{1}{\sqrt{2\pi m}} \right| \leq \frac{C_7}{m^{5/6}}.$$

Putting this together with (A.1) and (A.2) gives (5.2). \square

Lemma 4.9. Let $f, \tilde{f} : [0, \infty) \rightarrow \mathbb{R}$ be continuous functions, and $d, \tilde{d} \in \mathbb{R}$, such that, for all s , $f(s) + d \leq \tilde{f}(s) + \tilde{d}$, and such that $\tilde{f} - f$ is a (weakly) increasing function on $[0, \infty)$. Define:

$$m(s) = \min \left\{ \inf_{0 \leq u \leq s} f(u), -d \right\};$$

$$\tilde{m}(s) = \min \left\{ \inf_{0 \leq u \leq s} \tilde{f}(u), -\tilde{d} \right\}.$$

Let $y(s) = f(s) - m(s)$ and $\tilde{y}(s) = \tilde{f}(s) - \tilde{m}(s)$. Then $y(s) \leq \tilde{y}(s)$ for all $s \geq 0$.

(Originally stated on page 55.)

Proof. It is evident that m, \tilde{m}, y and \tilde{y} all inherit their continuity from f and \tilde{f} .

Now suppose for some s that $\tilde{y}(s) = 0$, so that $\tilde{f}(s) = \tilde{m}(s)$. Then, for $u \leq s$,

$$\tilde{f}(u) \geq \tilde{f}(s);$$

also, since $u \leq s$, and $\tilde{f} - f$ is increasing,

$$f(u) - \tilde{f}(u) \geq f(s) - \tilde{f}(s).$$

Hence, for all $u \leq s$,

$$f(u) \geq f(s);$$

that is, $f(s) = m(s)$, and hence $y(s) = 0$. So for such values of s , we have $\tilde{y}(s) \geq y(s)$.

Now suppose that $\tilde{y}(s) > 0$; note that, by construction, $y(s) > 0$. Then there are two cases: either $\{u \leq s : y(u) = 0\}$ is empty, or it is non-empty. If it is empty, then, $y(u) > 0$ for $u \in [0, s]$. Thus it holds that $m(s) = -d$. (The proof is by contradiction. Note $m(s) \leq -d$. Then if $m(s) < -d$, then there exists $u \leq s$ such that $f(u) = m(s)$. But then, by the continuity of \tilde{f} , there exists $v < u$ such that $\tilde{f}(v) = -d$; at the infimum v^* of such points, by continuity $f(v^*) = -d$, but $m(v^*) = -d$ here too, giving a contradiction.)

Note that in this case, since $y(u) > 0$ for all $u \in [0, s]$, by the first part $\tilde{y}(u) > 0$ for all $u \in [0, s]$, and so by the same argument $\tilde{m}(s) = -\tilde{d}$. Then

$$y(s) = f(s) + d \leq \tilde{f}(s) + \tilde{d} = \tilde{y}(s).$$

Now suppose there exists $u \leq s$ such that $y(u) = 0$. Let $u^* = \sup \{u \leq s : y(u) = 0\}$; note that $y(u^*) = 0$, by continuity. Now for $u \in (u^*, s]$, we have $y(u) > 0$; this implies that m is

constant on $[u^*, s]$ (in a similar contradiction argument to that above). So

$$y(s) = y(s) - y(u^*) = f(s) - f(u^*).$$

But now $\tilde{y}(u) > 0$ on $(u^*, s]$ as well, and by the same argument $\tilde{m}(u)$ is constant on $[u^*, s]$, so $\tilde{y}(s) - \tilde{y}(u^*) = \tilde{f}(s) - \tilde{f}(u^*)$. So

$$y(s) = f(s) - m(s) - f(u^*) + m(u^*) = f(s) - f(u^*) \leq \tilde{f}(s) - \tilde{f}(u^*) = \tilde{y}(s) - \tilde{y}(u^*) \leq \tilde{y}(s)$$

as \tilde{y} is a non-negative function by definition, where the first inequality holds because $\tilde{f} - f$ is an increasing function. \square

Lemma 7.10. As $\ell \rightarrow \infty$,

$$\int_0^1 x^{-1}(1 - e^{-\ell x}) dx = \gamma + \log \ell + O(\ell^{-1}e^{-\ell}).$$

(Originally stated on page 117.)

Proof. The integrand is bounded over $(0, 1]$, so, by the Bounded Convergence Theorem,

$$\int_0^1 x^{-1}(1 - e^{-\ell x}) dx = \lim_{\varepsilon \downarrow 0} \int_\varepsilon^1 x^{-1}(1 - e^{-\ell x}) dx.$$

Evaluating this integral, we have that

$$\begin{aligned} \int_\varepsilon^1 x^{-1}(1 - e^{-\ell x}) dx &= -\log \varepsilon - \int_{\varepsilon \ell}^{\ell} \frac{e^{-u}}{u} du \\ &= -\log \varepsilon - E_1(\varepsilon \ell) + E_1(\ell) \end{aligned}$$

(substituting $u = \ell x$ in the first line), where E_1 is defined as

$$E_1(x) = \int_x^\infty \frac{e^{-u}}{u} du.$$

Now, by [1, eqn. 5.1.11] (and standard results about power series), it holds that, as $u \downarrow 0$,

$$E_1(x) = -\gamma - \log x + O(x). \tag{A.3}$$

Thus, it holds that (for fixed ℓ , as $\varepsilon \downarrow 0$),

$$\int_{\varepsilon}^1 x^{-1}(1 - e^{-\ell x}) dx = -\log \varepsilon + \gamma + \log(\varepsilon \ell) + O(\varepsilon) + E_1(\ell);$$

taking the limit in ε gives

$$\int_0^1 x^{-1}(1 - e^{-\ell x}) dx = \gamma + \log \ell + E_1(\ell).$$

The result then follows from the fact that, for $x > 0$,

$$E_1(x) \leq \frac{1}{x} \int_x^{\infty} e^{-u} du = x^{-1} e^{-x}. \quad \square$$

Lemma 7.5. Suppose $\sum_{k=0}^{\infty} a_k x^k$ is a power series that converges for all $x \in \mathbb{R}$, with $a_k > 0$ for all sufficiently large k . Suppose also that $a_k \sim b_k$ as $k \rightarrow \infty$. Then $\sum_{k=0}^{\infty} b_k x^k$ converges for all $x \in \mathbb{R}$, and as $x \rightarrow \infty$,

$$\sum_{k=0}^{\infty} b_k x^k \sim \sum_{k=0}^{\infty} a_k x^k.$$

(Originally stated on page 113.)

Proof. We first check the series converges. It is sufficient to consider $x > 0$, and it is also sufficient to show convergence of the sequence of partial sums $s_K(x) = \sum_{k=L}^{\infty} b_k x^k$, where L is chosen so that $b_k > 0$ for all $K \geq L$. Then, we have a monotone sequence $(s_K)_{K \geq L}$. Moreover, since $b_k/a_k \rightarrow 1$ as $k \rightarrow \infty$, there exists M such that, for all $k \geq L$, we have that $b_k/a_k \leq M$. Thus, for all $K \geq L$,

$$s_K(x) \leq M \sum_{k=L}^K a_k x^k \leq M \sum_{k=L}^{\infty} a_k x^k < \infty,$$

and we have a monotone sequence bounded above, which therefore converges.

Now fix $\varepsilon > 0$. Then there exists $K_{\varepsilon} \in \mathbb{N}$ (with $K_{\varepsilon} \geq L$) such that, for all $k > K_{\varepsilon}$,

$$\left| \frac{b_k}{a_k} - 1 \right| \leq \varepsilon.$$

Then

$$\frac{\sum_{k=0}^{\infty} b_k x^k}{\sum_{k=0}^{\infty} a_k x^k} \leq 1 + \varepsilon + \frac{\sum_{k=0}^{K_{\varepsilon}} (b_k - (1 + \varepsilon)a_k) x^k}{\sum_{k=0}^{\infty} a_k x^k}. \quad (\text{A.4})$$

But we note that, since $a_k > 0$ for all $k \geq K_{\varepsilon}$,

$$x^{-K_{\varepsilon}} \sum_{k=0}^{\infty} a_k x^k \geq a_{K_{\varepsilon}+1} x + a_{K_{\varepsilon}} - O(x^{-1}) \rightarrow \infty$$

as $x \rightarrow \infty$. Hence, if we divide through the numerator and the denominator of the fraction on the right-hand side of (A.4) by x^{K_ε} , we see that that fraction converges to 0. It therefore holds that

$$\limsup_{x \rightarrow \infty} \frac{\sum_{k=0}^{\infty} b_k x^k}{\sum_{k=0}^{\infty} a_k x^k} \leq 1 + \varepsilon$$

for all $\varepsilon > 0$. By an analogous argument, we can show that

$$\liminf_{x \rightarrow \infty} \frac{\sum_{k=0}^{\infty} b_k x^k}{\sum_{k=0}^{\infty} a_k x^k} \geq 1 - \varepsilon,$$

thus completing the result. □

Lemma 7.11. Fix $\lambda \in \mathbb{R}$. Let

$$g(k) = \mathbb{E} \left[\int_{\tau((k-1)^2)}^{\tau(k^2)} (X(u) - I(u)) du \right],$$

where, for a Brownian motion $B(t)$, we take $X(t) = B(t) + \lambda t - t^2/2$, and $I(t) = \inf_{0 \leq u \leq t} X(u)$, and define τ to be the right-continuous inverse local time at 0 of $X - I$. Then $g(k) = O(k^{-1})$.

(Originally stated on page 120.)

Proof. By a method similar to that used in the proof of Proposition 7.9, we have that

$$g(k) = \int_0^{\infty} \frac{1}{\sqrt{2\pi}} x^{-1} (e^{-(k-1)^2 x} - e^{-k^2 x}) \Psi'(x^{3/2}) \exp\left\{-\frac{x^3}{6} + \frac{x^2 \lambda}{2} - \frac{x \lambda^2}{2}\right\} dx. \quad (\text{A.5})$$

We split this integral at 1, noting first that the integral over $[1, \infty)$ is bounded above by

$$\frac{1}{\sqrt{2\pi}} e^{-(k-1)^2} \int_1^{\infty} x^{-1} \Psi'(x^{3/2}) \exp\left\{-\frac{x^3}{6} + \frac{x^2 \lambda}{2} - \frac{x \lambda^2}{2}\right\} dx = O(e^{-k^2 + 2k}),$$

as the integrand is now integrable and independent of k . Meanwhile, if D (independent of k) is chosen such that

$$\frac{1}{\sqrt{2\pi}} \Psi'(x^{3/2}) \exp\left\{-\frac{x^3}{6} + \frac{x^2 \lambda}{2} - \frac{x \lambda^2}{2}\right\} \leq D$$

for all $x \in [0, 1]$, then the section of the integral in (A.5) over $(0, 1)$ is bounded above by

$$\begin{aligned} D \int_0^1 x^{-1} (e^{-(k-1)^2} - e^{-k^2 x}) dx &= D \left[\int_0^1 x^{-1} (1 - e^{-k^2 x}) dx - \int_0^1 x^{-1} (1 - e^{-(k-1)^2}) dx \right] \\ &= D \left[2 \log k - 2 \log(k-1) + O(k^{-2} e^{-(k-1)^2 x}) \right] = O(k^{-1}) \end{aligned}$$

as $k \rightarrow \infty$, by Lemma 7.10. □

For the next lemma, take X and I as in the statement of Lemma 7.11.

Lemma 7.24. *Fix $\lambda \in \mathbb{R}$ and $\theta > 1$, not depending on t . Then as $t \rightarrow \infty$,*

$$\mathbb{E} \left[\exp \left\{ \theta \int_0^t (X(u) - I(u)) du \right\} \right] = O \left(t^{3/2} e^{\theta^2 t^3 / 2 + \lambda \theta t^2} \right).$$

(Originally stated on page 139.)

For the proof we will need the following result taken from Williams [62].

Theorem A.1 ([62, Thm. 1]). *Let $Y^{(\mu)}$ be a Brownian motion with drift μ , started at 0, reflected (from below) at 0, and define, for $c > 0$*

$$\tau^{(\mu)}(c) = \inf \{ t \geq 0 : Y^{(\mu)}(t) = c \}.$$

Fix $\beta > 0$. Then

$$\mathbb{E} \left[e^{-\beta \tau^{(\mu)}(c)} \right] = \frac{e^{\mu c}}{\cosh(\gamma c) + \mu \sinh(\gamma c) / \gamma},$$

where $\gamma = \sqrt{\mu^2 + 2\beta}$.

We will also need the following lemmas:

Lemma A.2. *In the setting of Theorem A.1, if $c > t|\mu|$,*

$$\mathbb{P} \left(\sup_{0 \leq u \leq t} Y^{(\mu)}(u) \geq c \right) \leq \frac{2c}{c + \mu t} e^{-\mu^2 t / 2 + \mu c - c^2 / (2t)}.$$

Proof. We note that, for all $\beta > 0$,

$$\mathbb{P} \left(\sup_{0 \leq u \leq t} Y^{(\mu)}(u) \geq c \right) = \mathbb{P} \left(\tau^{(\mu)}(c) \leq t \right) \leq e^{\beta t} \mathbb{E} \left[e^{-\beta \tau^{(\mu)}(c)} \right], \quad (\text{A.6})$$

by Markov's inequality. Take

$$\beta = \frac{c^2}{2t^2} - \frac{\mu^2}{2} > 0$$

for the range of values of c specified. Then, as defined in Theorem A.1, $\gamma = c/t$, and thus

$$\begin{aligned} \mathbb{E} \left[e^{-\beta \tau^{(\mu)}(c)} \right] &= \frac{ce^{\mu c}}{c \cosh(c^2/t) + \mu t \sinh(c^2/t)} = \frac{2ce^{\mu c}}{(c + \mu t)e^{c^2/t} + (c - \mu t)e^{-c^2/t}} \\ &\leq \frac{2c}{c + \mu t} e^{\mu c - c^2/t}, \end{aligned}$$

since $c - \mu t > 0$ by the constraint we have taken on c . Applying this bound in (A.6), and substituting again for β , gives the result. \square

Lemma A.3. *In the setting of Lemma A.2, we have that, for all $\alpha > 0$,*

$$\mathbb{E} \left[\exp \left\{ \alpha \int_0^t Y^{(\mu)}(u) du \right\} \right] \leq e^{2\alpha t^2 |\mu|} + 4\alpha t \sqrt{2\pi t} e^{\alpha t^2 (2\mu + \alpha t)/2}.$$

Proof. We first note that

$$\int_0^t Y^{(\mu)}(u) du \leq t \sup_{0 \leq u \leq t} Y^{(\mu)}(u).$$

So, using the fact that, if X is a non-negative random variable, then $\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X \geq x) dx$, we have that

$$\begin{aligned} \mathbb{E} \left[\exp \left\{ \alpha \int_0^t Y^{(\mu)}(u) du \right\} \right] &\leq 1 + \int_1^\infty \mathbb{P} \left(\exp \left\{ \alpha t \sup_{0 \leq u \leq t} Y^{(\mu)}(u) \right\} \geq v \right) dv \\ &= 1 + \alpha t \int_0^\infty e^{\alpha t c} \mathbb{P} \left(\sup_{0 \leq u \leq t} Y^{(\mu)}(u) \geq c \right) dc, \end{aligned} \quad (\text{A.7})$$

substituting $c = (\alpha t)^{-1} \log v$.

We may now use Lemma A.2 to obtain a bound on the integral over $(2|\mu|t, \infty)$: namely,

$$\begin{aligned} \alpha t \int_{2|\mu|t}^\infty e^{\alpha t c} \mathbb{P} \left(\sup_{0 \leq u \leq t} Y^{(\mu)}(u) \geq c \right) dc &\leq \alpha t \int_{2|\mu|t}^\infty \frac{2c}{c + \mu t} e^{\alpha t c - \mu^2 t/2 + \mu c - c^2/(2t)} dc \\ &= \frac{2c\alpha t}{c + \mu t} e^{\alpha t^2 (2\mu + \alpha t)/2} \int_{2|\mu|t}^\infty e^{-(c-t(\mu+\alpha t))^2/(2t)} dc \\ &\leq 4\alpha t \sqrt{2\pi t} e^{\alpha t^2 (2\mu + \alpha t)/2}, \end{aligned}$$

since, for all $\mu \in \mathbb{R}$ and for all $c > 2|\mu|t$, we have that $c/(c + \mu t) \leq 2$.

Meanwhile, we have for the integral over $[0, 2|\mu|t]$ that

$$\alpha t \int_0^{2|\mu|t} e^{\alpha t c} \mathbb{P} \left(\sup_{0 \leq u \leq t} Y^{(\mu)}(u) \geq c \right) dc \leq \alpha t \int_0^{2|\mu|t} e^{\alpha t c} dc = e^{2\alpha t^2 |\mu|} - 1. \quad \square$$

Proof of Lemma 7.24. In the setting of Lemma A.3, take $\mu = \lambda$ and $x = 0$. Then, by Lemma 4.9, we can couple X and $Y^{(\lambda)}$ such that, for all $u \geq 0$,

$$Y^{(\lambda)}(u) \geq X(u) - I(u).$$

The result follows from Lemma A.3. \square

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This thesis is the culmination of over eight years spent studying at the University of Oxford. I've therefore picked up an extremely long list of people that I ought to thank. But if I printed the whole list it would be highly self-indulgent, and even less likely to be read. I'm reminded that the Oscars have an orchestra waiting to play out the winners if they talk for more than 45 seconds in their speeches. Perhaps these sections of theses should have something similar.

I've given my academic acknowledgements on page v, but, more widely, the Department of Statistics is a wonderful place full of wonderful people: e.g. the Stats Choir, the Friday board games crowd, the cryptic crossword club, the StatsBioDem reading group, the Thursday morning cake-bakers and -eaters, and everyone involved in the Mental Health Awareness Weeks. The fact that so much social activity has continued during COVID-19 is testament to what extraordinary people normally work within 24–29 St Giles'. Special northern solidarity goes to the amazing Beverley (and her delicious cookies, of course).

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I can see the metaphorical conductor impatiently waving their baton, even though my list is incomplete. I have three people left to thank. They have all been a colossal help to me, in very different ways, and I found it impossible to decide who should appear before or after whom. Therefore, as befits a probability thesis, their order was selected at random.

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