GENERALISED GLOBAL SYMMETRIES

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Higher-form (p-form) symmetries are symmetries whose charged objects are extended operators of spacetime dimension p. They exist as a natural mathematical extension of the ordinary symmetries of quantum field theory, which are correspondingly 0-form. Exploring the implications of these generalised symmetries has proven to be an exciting area of research for theoretical and mathematical physics. We offer the original computation of the 1-form symmetry associated to a gauge theory with Abelian kinetic mixing of two $U(1)$ gauge sectors. We demonstrate that the symmetry is invariant under the strength of coupling and that explicit symmetry breaking occurs in the presence of both visible and hidden fermions. We also motivate a new potential signal for the detection of dark $U(1)$ sectors using the 1-form symmetry of the theory.

I. INTRODUCTION

Symmetries are well-established as a cornerstone of theoretical physics. The mathematical framework underlying the Standard Model (SM) is a quantum field theory (QFT) exhibiting a local $SU(3) \times SU(2) \times U(1)$ gauge symmetry, responsible for the three fundamental interactions in nature. Furthermore, general relativity, our most robust theory of gravitation, is underpinned by the external symmetry of general covariance. Modern field theories extending these paradigms also reveal abstract symmetries, e.g. supersymmetry (SUSY) in the SM or the mirror symmetry of string theory. The 2014 paper from Davide Gaiotto et al. [1] proposes a novel framework for describing canonical global symmetries of QFTs as topological operators. This generalised picture reveals higher-form (p-form) symmetries whose charged objects are extended operators, which are branes with spacetime dimension p. Characterising and understanding the implications of generalised symmetries represents an active and vibrant area of research.

The canonical properties of traditional (0-form) symmetries also appear in the study of higher-form symmetries. They describe Ward identities which induce selection rules on expectation values of charged operators. Generalised symmetries can also be spontaneously broken, either completely or into a subgroup. One unique property of higher-form symmetries is the existence of ‘t Hooft anomalies, which prevents them from being gauged by coupling to background fields. Another compelling feature of the framework is the mechanism of screening, which relates charged objects and is used to deduce higher-form symmetries of a theory.

Generalised symmetries and their ‘t Hooft anomalies are invariant under renormalisation group flows [2], thus the method of screening has shown merit in probing strongly coupled physics inaccessible with standard perturbation theory. The paper [3] deduces the higher-form symmetries of conformal theories of the Argyres-Douglas type using screening, which are isolated and strongly coupled thus lacking a Lagrangian description. Applications of screening to string theory have also been demonstrated, with [4] realising the 1-form symmetry of F-theory which contains non-perturbative effects from the presence of D7-branes and (p, q)-strings. The merit of these procedures lies not only in their elegance but, most critically, in their capacity to unveil new physics. For example, a recent research effort [5] shows that the higher-form symmetry structure of the SM restricts the possible flavour symmetries that could remain unbroken in an ultraviolet completion that includes magnetic charges.

In Section IV, we produce the original result of computing the 1-form symmetry of the $U(1) \times U(1)$ gauge theory containing kinetic mixing (KM) of gauge sectors exactly using a screening procedure. This computation reveals explicit breaking of the 1-form symmetry in the presence of matter charged under either $U(1)$ sector. The KM gauge theory describes a renormalisable coupling between a visibile and hidden $U(1)$ gauge sector [6]. At its conception in [7], it was shown that the charges of fermions in a given sector gain a small induced (millicharge) coupling to the gauge boson of the other sector under KM. The theory has garnered attention in dark matter models [8] as well as SUSY models [9]. We also use the 1-form symmetry to demonstrate that domain walls separating phases of kinetic mixing will gain an induced millicharge in the presence of matter. One potential implication stemming from this original insight for dark matter phenomenology is also explored.

The preliminary sections II-III motivate the background knowledge necessary to derive our new results. We employ a pedagogical approach inspired by [2], though offering original insights and examples designed to be more illustrative for a non-expert reader. In Section II, we connect 0-form symmetries to traditional symmetries in QFT and extend these ideas to p-form symmetries, focusing on the case of invertible $U(1)$ symmetries. In Section III, we derive the higher-form symmetries of Maxwell theory and introduce screening using quantum electrodynamics (QED). The appendices detail the key results of the differential geometric formulation of Maxwell theory, which is taken as a perquisite, as well formal, albeit interesting, proofs and discussions that are omitted from the main body of the report.

II. FORMALISM

A. 0-Form Symmetries; $G^{(0)} = U(1)$

To equip a QFT with a global symmetry is to introduce a symmetry group G, with a unitary representation $\pi : G \to \mathrm{U}(\mathcal{H})$ on the Hilbert space $\mathcal{H}$. That is, $\pi(g) \equiv U_g(t)$ is a unitary operator acting on the Hilbert space for every $g \in G$ at a given time t. Most critically, $U_g(t)$ must commute with the Hamiltonian, the generator of time evolution, which gives the condition

$$U_g(t) = U_g(t') \;\; \forall g \in G \; , \; t, t' \; , \tag{1}$$

for G to be a symmetry group of the theory. If spacetime is a d-dimensional manifold Σ_d, the operator $U_g(t)$ on $\mathcal{H}$ can be equivalently interpreted as $U_g(S_{d-1})$ with S_{d-1} a spatial slice of Σ_d at time t. A natural extension to this has $U_g(\Sigma_{d-1})$ defined on some arbitrary (non-spatial) slice Σ_{d-1} of spacetime, see Figure 1.

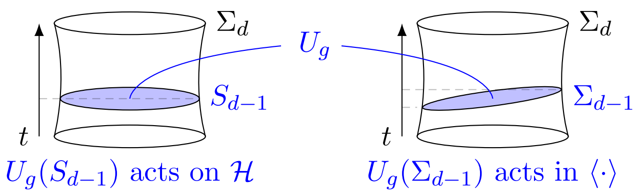


FIG. 1: The insertion of a symmetry operator on a spatial slice at t (left) $\mathcal{H}$ and an arbitrary submanifold Σ_{d-1} of spacetime Σ_d (right).

Correlation functions $\langle . \rangle$ in QFT are amplitudes describing the relationship of field excitations at different points in spacetime. The general operator $U_g(\Sigma_{d-1})$ pictured in Figure 1 inscribes spacetime points with different time coordinates, and hence cannot be an operator on $\mathcal{H}$ but is valid only inside such correlation functions. This discussion is condensed by the statement that U_g is a **codimension-1** operator, meaning that it is defined on $(d-1)$-dimensional submanifolds of spacetime. The symmetry constraint (1) can be described in this formalism by writing

$$U_g(S_{d-1}) = U_g(S'_{d-1}) \; ; \; \forall g \in G \tag{2}$$

where S'_{d-1} is some other spatial slice at time t'. Equation (2) has a natural generalisation for the operator $U_g(\Sigma_{d-1})$, that is

$$U_g(\Sigma_{d-1}) = U_g(\Sigma'_{d-1}) \; ; \; \forall g \in G \tag{3}$$

where $\Sigma'_{d-1} \subset \Sigma_{d-1}$ is related to Σ_{d-1} by the fact that they can be connected by a submanifold $\mathcal{M}_d$ of spacetime, see Figure 2.

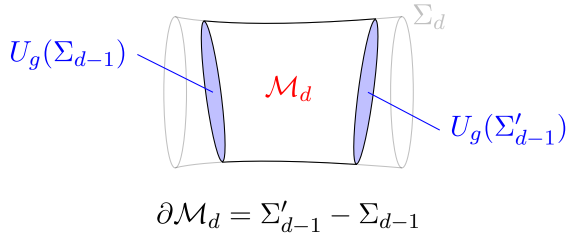


FIG. 2: Two submanifolds Σ_{d-1} and Σ'_{d-1} of spacetime forming the boundary of a manifold $\mathcal{M}_d \subset \Sigma_d$. Manifolds related in this way are said to be topologically equivalent.

Since Σ_{d-1} and Σ'_{d-1} are related by topological deformation, equation (3) corresponds to the statement that the operator U_g is **topological**. This property of U_g is hence a generalisation of the fact that $U_g(t)$ is time-independent. We also want to interpret the unitarity of symmetry operators U_g in this framework, which is required to make correlation functions invariant under symmetry transformations. A more involved proof [1] demonstrates that enforcing U_g to be **invertible** means that the Hilbert space operators $U_g(t)$ are indeed unitary. This is the statement that there exists another codimenion-1 operator $U_g^{-1}(\Sigma_{d-1})$ such that

$$U_g^{-1}(\Sigma_{d-1}) U_g(\Sigma_{d-1}) = 1 \tag{4}$$

where the 1 in (4) is the codimension-1 identity operator. The treatment thus far has been a natural extension of the traditional picture of symmetries and can be summarised by a single statement:

Definition 1. *A* ***0-form symmetry group*** $G^{(0)}$ *is composed of all* ***0-form symmetries*** $g \in G^{(0)}$*, each represented by a* ***codimension-1*** *operator* U_g *that is* ***topological*** *and invertible.*

The group structure $G^{(0)}$ and the corresponding representation is discussed later in this section in Example 1, with [2] recommended to the interested reader for a more thorough discussion. A final result that is central to the discussion of symmetries in QFT is the **Ward-Takahashi identity**

$$\langle \partial_\mu J^\mu(x) \varphi(y) \rangle = q \delta^{(4)}(x-y) \langle \varphi(x) \rangle \tag{5}$$

for a given field operator φ, which corresponds to the statement that φ is charged under $q \in \mathbb{Z}$ under $U(1)$.

Physically, the result may be interpreted as a QFT analogue of Noether's theorem in the presence of a charged object. A standard proof of (5) is included in the appendices for completeness, and for now the formula is reserved for future discussion. The operators $\varphi(x)$ charged under traditional symmetries are always defined on single points in spacetime x, which motivates an alternative interpretation of the symmetry operators:

Definition 2. *A 0-form symmetry is a symmetry that acts on operators $\varphi(\Sigma_0) \equiv \varphi(x)$ defined on a 0-dimensional submanifold $x \equiv \Sigma_0 \subset \Sigma_d$ of spacetime. Equivalently, a 0-form symmetry is one whose charged objects (defects) are* 0*-dimensional or* ***local****.*

We end the discussion of 0-form symmetries by applying the formalism to the simple example of a complex scalar QFT. The details of the computation presented are an original contribution to the literature, with the aim of motivating the validity of this generalised framework. The interested reader is also directed to [10] and [11] for a complete mathematical treatment of manifolds and topological operators, respectively.

Example 1. *Complex scalar QFT; $\varphi(x) \in \mathbb{C}$.*

Consider the QFT defined on an arbitrary 4-dimensional spacetime Σ_4 with a single complex scalar field φ and a Noether current J^μ associated to the global $U(1)$ symmetry. Without the explicit form of J^μ in terms of field operators $\{\varphi, \varphi^\dagger\}$, we can construct a **closed** 3-form on spacetime $j^{(3)}$, defined locally within some set of charts $\{\sigma_\alpha\}$ of Σ_4 as

$$\begin{aligned} j^{(3)} = J^0 dx^1 \wedge dx^2 \wedge dx^3 - J^1 dx^0 \wedge dx^2 \wedge dx^3 \\ + J^2 dx^0 \wedge dx^1 \wedge dx^3 - J^3 dx^0 \wedge dx^1 \wedge dx^2 \ , \end{aligned} \tag{6}$$

or equivalently $j^{(3)}_{\mu\nu\lambda} = \epsilon_{\mu\nu\lambda\rho} J^\rho$ in terms of the Noether current components. Checking that the 3-form is closed, i.e. $dj^{(3)} = 0$, amounts to

$$\begin{aligned} dj^{(3)} &= \partial_\rho j^{(3)}_{\mu\nu\lambda} dx^\rho \wedge dx^\mu \wedge dx^\nu \wedge dx^\lambda \\ &= \partial_\rho J^\rho \cdot \epsilon_{\mu\nu\lambda\rho} dx^\rho \wedge dx^\mu \wedge dx^\nu \wedge dx^\lambda = 0 \end{aligned} \tag{7}$$

which requires $\partial_\mu J^\mu = 0$ and hence this constraint may be viewed as a generalised continuity equation of the theory. The significance of the **generalised Noether current** $j^{(3)}$ is that it defines a representation $\pi : U(1) \to \mathcal{U}_{d-1}$ of $U(1)$ group elements as codimension-1 operators by

$$e^{i\alpha} \longmapsto U_g(\Sigma_3) = g^{\tilde{Q}(\Sigma_3)} \equiv e^{i\alpha \int_{\Sigma_3} j^{(3)}} \tag{8}$$

with $\tilde{Q}$ defined as the **generalised Noether charge**, which reduces to the traditional Noether charge $Q(t)$ when inserted on a spatial slice (volume in the 4-dimensional case) V, i.e.

$$\tilde{Q}(V) = \int_V dx^1 \wedge dx^2 \wedge dx^3 \cdot J^0 = \int_V d^3x \cdot J^0 \tag{9}$$

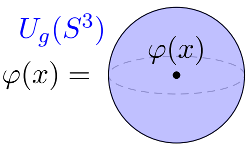
with the integrals of other terms in (6) vanishing since $dx^0 := 0$ on V. The expected operators on $\mathcal{H}$ are then automatically recovered in the representation π, since the Noether charge Q is the $U(1)$ generator in the traditional picture. We can also show that the symmetry operators operators defined by (8) are topological by construction:

$$\begin{aligned} U_g(\Sigma_3') &= \exp\left(i\alpha \int_{\Sigma_3'} j^{(3)}\right) \\ &= \exp\left(i\alpha \int_{\Sigma_3} j^{(3)}\right) \exp\left(i\alpha \int_{\underbrace{\Sigma_3' - \Sigma_3}_{=\partial\mathcal{M}_d}} j^{(3)}\right) \\ &= \exp\left(i\alpha \int_{\Sigma_3} j^{(3)}\right) \exp\left(i\alpha \int_{\partial\mathcal{M}_d} j^{(3)}\right) \\ &= \exp\left(i\alpha \int_{\Sigma_3} j^{(3)}\right) \exp\left(i\alpha \int_{\mathcal{M}_d} \underbrace{dj^{(3)}}_{:=0}\right) = U_g(\Sigma_3) \end{aligned}$$

where in the second equality we refer to Figure 2, and the fourth invokes generalised Stoke's theorem. The operators φ and $\varphi^\dagger$ are local defects (charged objects) charged under the 0-form $G^{(0)} = U(1)$ symmetry group, and to proceed we assert a 0-form analogue of the Ward-Takahashi identity (5) as

$$\langle dj^{(3)} \varphi(x) \rangle = q\delta^4(x) \langle \varphi(x) \rangle \tag{10}$$

with $\delta^4(x)$ the Dirac delta function 4-form centred at x, or formally the 4-form **Poincaré dual** to $\delta^{(4)}(x-y)$. We now implement the action of U_g on defect $\varphi(x)$ by surrounding the local point x with a 3-sphere[1] S^3, i.e.

$U_g(S^3)\varphi(x) =$ 

$$\begin{aligned} &= \exp\left(i\alpha \int_{S^3} j^{(3)}\right) \varphi(x) \\ &= \exp\left(i\alpha \int_{\mathcal{M}_4} dj^{(3)}\right) \varphi(x) = e^{iq\alpha} \varphi(x) \end{aligned} \tag{11}$$

which is the expected $U(1)$ transformation of field operator $\varphi(x)$. Note that the second equality uses

[1] S^3 is visualised diagrammatically as a 2-sphere for our purposes.

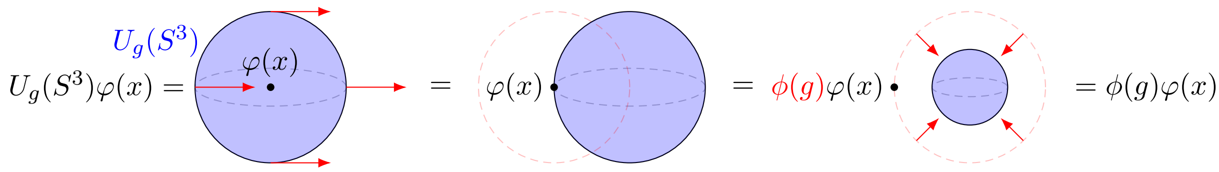


FIG. 3: When a symmetry operator U_g crosses over one of it's defects charged with q, it obtains a factor of $\phi(g) = \exp(iq\alpha)$.

generalised Stoke's theorem on some 4-manifold $\mathcal{M}_4$ such that $\partial\mathcal{M}_4 = S^3$ and the final equality uses the Ward-Takahashi identity (10). As is commonplace in literature, we also omit the $\langle . \rangle$ notation and assert that (11) is only valid in a correlation function. Another way of viewing this transformation law involves using the topological nature of $U_g(S^3)$, see Figure 3, and identifying the 'cost' $\phi(g)$ of crossing a symmetry operator $U_g(S^3)$ over one of its charged defects as

$$\phi(g) = e^{iq\alpha} \tag{12}$$

from (11). This result will be of use in Example 2. Topological moves in Figure 3 are highlighted in red, such as the final step where $U_g(S^3)$ is contracted to a point where it evaluates to the identity operator. We close by noting that the operators (8) are also invertible by construction, in that taking $g \mapsto g^{-1}$ (or equivalently $\alpha \mapsto -\alpha$) indeed produces the inverse operator $U_g^{-1}(\Sigma_{d-1})$ satisfying (4).

The recovery of ordinary properties of traditional symmetries from the 0-form symmetry formalism is a remarkable feature of this framework. For the QFT in Example 1, a complete treatment of these canonical properties is found in [12]. We are now also equipped to define QFTs on manifolds Σ_4 with non-trivial topology (not just Minkowski space), with Example 2 discussing one novel effect of this. However, the most profound insight of [1] is that this scheme can be easily generalised to 'higher-form' cases, which leads the reader to the subsequent discussion.

B. p-Form Symmetries; $G^{(p)} = U(1)$

Definition 3 stipulates the higher-form (p-form) analogue of Definition 1, which is recovered for the case $p = 0$.

Definition 3. *A* ***p-form symmetry group*** $G^{(p)}$ *is composed of all* ***p-form symmetries*** $g \in G^{(p)}$*, each representing a codimension-*$(p+1)$ *operator* U_g *that is topological and invertible.*

For a d-dimensional QFT on spacetime Σ_d with p-form symmetry group $G^{(p)} = U(1)$, the generalised Noether current $j^{(d-p-1)}$ is a closed $(d-p-1)$-form on spacetime with components defined by

$$j^{(d-p-1)}_{\mu_1 \dots \mu_{d-p-1}} = \epsilon_{\mu_1 \dots \mu_{d-p}} J^{\mu_{d-p}} \tag{13}$$

in analogy with (6), where J^μ are the $U(1)$ Noether current components. A codimension-$(p+1)$ operator $U_g(\Sigma_{d-p-1})$ is defined on any $(d-p-1)$-dimensional submanifold Σ_{d-p-1} of spacetime, and acts inside correlation functions. The generalised Noether charge $\tilde{Q}$ in this case is a scalar codimension-$(p+1)$ operator, defining a representation $\pi : U(1) \to \mathcal{U}_{d-p-1}$ of $U(1)$ group elements as codimension-$(p+1)$ operators by

$$\begin{aligned} e^{i\alpha} \longmapsto U_g(\Sigma_{d-p-1}) &= g^{\tilde{Q}(\Sigma_{d-p-1})} \\ &= \exp\left(i\alpha \int_{\Sigma_{d-p-1}} j^{(d-p-1)} \right) . \end{aligned} \tag{14}$$

The fact that the inserted operator $U_g(\Sigma_{d-p-1})$ is topological means that

$$U_g(\Sigma_{d-p-1}) = U_g(\Sigma'_{d-p-1}) \tag{15}$$

$\forall \Sigma'_{d-p-1}$ that are topologically equivalent to Σ_{d-p-1}. The steps taken in Example 1 can be applied to the symmetry operator $U_g(\Sigma_{d-p-1})$ in (14) to demonstrate that it is topological by construction, as required. The property of invertibility implies that there exists another topological codimension-$(p+1)$ operator $U_g^{-1}(\Sigma_{d-p-1})$ that satisfies:

$$U_g^{-1}(\Sigma_{d-p-1}) U_g(\Sigma_{d-p-1}) = 1 \tag{16}$$

where the 1 here is the codimension-$(p+1)$ identity operator. As with Example 1, the operators $U_g(\Sigma_{d-p-1})$ defined by (14) are also automatically invertible. The properties discussed up to this point should be unsurprising, though abstract, generalisations of the previous subsection. Interestingly, there exists an analogue of Definition 2 for p-form symmetries:

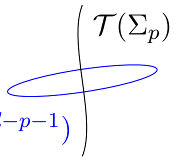
Definition 4. *A* p*-form symmetry is a symmetry that acts on* ***extended operators*** $\mathcal{T}(\Sigma_p)$ *defined on a* p*-dimensional submanifold* $\Sigma_p \subset \Sigma_d$ *of spacetime. Equivalently, a* p*-form symmetry is one whose charged objects (defects) are* p*-dimensional or* ***extended****.*

The Ward-Takahashi identity associated to a p-dimensional defect charged $q \in \mathbb{Z}$ under $U(1)$ is hence

$$\langle dj^{(d-p-1)} \mathcal{T}(\Sigma_p) \rangle = q \delta^{d-p}(\Sigma_p) \langle \mathcal{T}_q(\Sigma_p) \rangle \, . \tag{17}$$

with $\delta^{d-p}(\Sigma_p)$ the delta function $(d-p)$-form on Σ_p, in analogy to (10). The transformation law of operator $\mathcal{T}(\Sigma_p)$ under a p-form symmetry operator is then described by surrounding it with $U_g(S^{d-p-1})$, where S^{d-p-1} the $(d-p-1)$-sphere, i.e.

$$U_g(S^{d-p-1})\mathcal{T}(\Sigma_p) =$$



$$= e^{iq\alpha}\mathcal{T}(\Sigma_p) \qquad (18)$$

where this again holds most generally inside correlation functions. We also note that the derivation of (18) is a trivial extension of (11), see [2]. We can now make progress in identifying theories with higher-form $U(1)$ symmetries, the simplest example being Maxwell theory. We also alert the reader that the generalisation of this formalism to non-Abelian symmetries is straightforward, and for further reading on this topic we suggest [1,2].

III. MAXWELL THEORY

As a preamble, we reiterate to the reader that this section utilises the differential geometric formulation of Maxwell theory, as standard to modern literature. We hence invite the reader to Appendix A.4 for a short introduction to the topic, or to [10] for a more comprehensive description.

A. Pure Maxwell Theory

A **d-dimensional pure Maxwell theory** is a $U(1)$ gauge theory with gauge field A, which is a 1-form on spacetime Σ_d, with no matter fields. The **field strength** is a 2-form defined by

$$F = dA. \qquad (19)$$

The two equations governing the dynamics of the gauge field A are

$$dF = 0\ ,\ d\star F = 0\ , \qquad (20)$$

the first, which involves F, is the **Bianchi identity** and is automatic from equation (19) and the second is the equation of motion $d\star F = J$ in the absence of matter, i.e. $J = 0$. The equations in (20) suffice to identify the higher-form symmetries as the theory automatically exhibits two generalised Noether currents

$$\begin{aligned} j^{(d-p_M-1)} &\equiv F \in \Omega^2(\Sigma_d)\ , \\ j^{(d-p_E-1)} &\equiv \star F \in \Omega^{d-2}(\Sigma_d) \end{aligned} \qquad (21)$$

each satisfying the continuity equation $dj^{(d-p-1)} = 0$. The superscript $(d-p-1)$ of the generalised Noether current denotes its **degree** as a form, therefore one can identify $p_M = d-3$ and $p_E = 1$ from equation (21), since F is a 2-form and the action of the **Hodge star operator** is $\star : \Omega^p(\Sigma_d) \to \Omega^{d-p}(\Sigma_d)$. Therefore, the pure Maxwell theory exhibits two higher-form symmetries:

- The **magnetic $(d-3)$-form symmetry** $G_M^{(d-3)} = U(1)$ with generalised Noether current F.
- The **electric 1-form symmetry** $G_E^{(1)} = U(1)$ with generalised Noether current $\star F$.

In line with equation (14), these currents define representations of $U(1)$ group elements as topological and invertible operators defined by

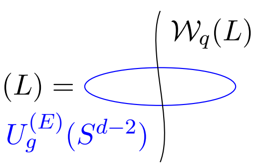
$$\begin{aligned} U_g^{(M)}(\Sigma_2) &= \exp\left(i\alpha\int_{\Sigma_2} F\right)\ , \\ U_g^{(E)}(\Sigma_{d-2}) &= \exp\left(i\alpha\int_{\Sigma_{d-2}} \star F\right) \end{aligned} \qquad (22)$$

that are codimension-$(d-2)$ (2-dimensional) and codimension-2 respectively. From Definition 4, it follows that the objects charged under the magnetic $(d-3)$-form symmetry and electric 1-form symmetry should be $(d-3)$-dimensional and 1-dimensional extended operators, respectively. These are known as **'t Hooft operators** $\mathcal{H}_m(\Sigma_{d-3})$ and **Wilson line operators** $\mathcal{W}_q(L)$, which are indexed by their magnetic and electric charges $m, q \in \mathbb{Z}$. The analytic form of the Wilson line operator $\mathcal{W}_q(L)$ is

$$\mathcal{W}_q(L) = \exp\left(iq\int_L A\right)\ ;\ q \in \mathbb{Z}\ . \qquad (23)$$

The physical relevance of the electric 1-form symmetry is best realised by analysing the transformation law (18) of the Wilson line operator with $U_g^{(E)}(S^{d-2})$ sat in a spatial slice of Σ_d, i.e.

$$e^{iq\alpha}\mathcal{W}_q(L) =$$



$$= \exp\Bigg(i\alpha\underbrace{\int_{S^{d-2}} \epsilon_{ijk}E^i dx^j \wedge dx^j}_{=\Phi_E}\Bigg)\mathcal{W}_q(L) \qquad (24)$$

where the first equality uses (18) and the second the local form of $\star F$ (see Appendix A.4) in terms of the electric field components E^i (and noting that $dx^0 := 0$ vanishes on S^{d-2} so any contributions from the magnetic field vanish). The expression in the

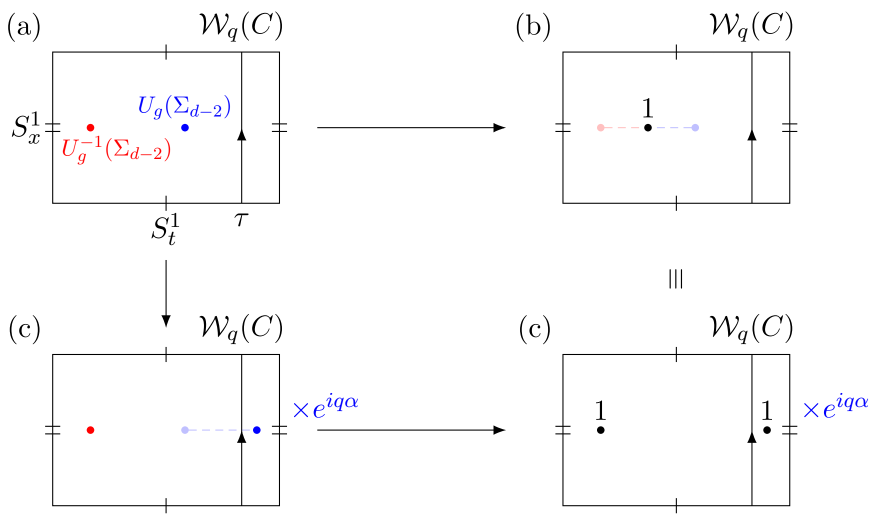


FIG. 4: The expectation value $\langle U_g^{-1}(\Sigma_{d-2})U_g(\Sigma_{d-2})\mathcal{W}_q(C)\rangle$ can be manipulated using two routes; (a)→(b) 'fuses' the symmetry operators to the identity using a topological move; (a)→(c)→(d) crosses one of the symmetry operators with the Wilson loop acquiring a factor of $e^{iq\alpha}$, then contracts both the symmetry operator and its inverse to the identity.

exponent of the final integral in (24) is equivalent to the electric flux Φ_E through the surface S^{d-2}. Stripping off the Wilson lines on either side hence leaves an operator statement of Gauss' law, that is $\Phi_E = q$. Therefore, using the 1-form symmetry in this way amounts to measuring the flux from a line defect $\mathcal{W}_q$ carrying a probe electric charge q. The Wilson line defined in (23) is also connected to the Aharonov–Bohm effect, which leads to the following physical interpretation:

Definition 5. *The Wilson line operator $\mathcal{W}_q(L)$ describes the phase acquired by an infinitely heavy probe particle of charge q under the 1-form symmetry $U(1)$, whose worldline is $L \subset \Sigma_d$ under gauge field A.*

The analytic form of the 't Hooft operator is not included in this report, and the physical interpretation presented in Definition 5 is extended to $\mathcal{H}_m(\Sigma_{d-3})$ by simple dimensional arguments. Specifically, the 't Hooft operator $\mathcal{H}_m(\Sigma_{d-3})$ describes the phase acquired by an infinitely heavy probe particle of magnetic charge m under $(d-3)$-form symmetry $U(1)$, whose **worldvolume** is $\Sigma_{d-3} \subset \Sigma_d$ under gauge field A. Applying a treatment identical to (24) to a 't Hooft operator also yields a magnetic Gauss' law $\Phi_M = m$. Magnetic charges, or **magnetic monopoles**, have been of interest in theoretical communities since 't Hooft and Polyakov showed that they must appear in certain non-Abelian gauge theories [13], and the above formalism elucidates their apperance in the generalised symmetry framework. One novel application of the higher-form symmetries presented here involves placing the pure Maxwell theory on a compact manifold with non-trivial topology. The steps taken here mirror the derivation in [14], with the result there applied to confinement in quantum chromodynamics with a large number of degrees of freedom.

Example 2. *Selection rules of Wilson Loops.*

Consider a spacetime $\Sigma_d = \Sigma_{d-2} \times S^1_x \times S^1_t$ with one spatial dimension x and time dimension t **compactified** to a circle S^1. We introduce a Wilson line $\mathcal{W}_q(C)$ on a loop C wrapping S^1_x and localised on $\Sigma_{d-2} \times S^1_t$ at coordinate $(\mathbf{r}, \tau)$. We note that the contour C cannot be contracted to a point and is hence referred to as a **non-trivial one-cycle** of Σ_d. We also introduce a 1-form symmetry operator (*dropping the E superscript to avoid clutter*) $U_g(\Sigma_{d-2})$ and its inverse $U_g^{-1}(\Sigma_{d-2})$ that wrap Σ_{d-2} and are hence local on $S^1_x \times S^1_t$, see panel (a) of Figure 4. Panel (a) describes the vacuum expectation value $\langle U_g^{-1}(\Sigma_{d-2})U_g(\Sigma_{d-2})\mathcal{W}_q(C)\rangle$ and is manipulated in two different ways which give the result $\langle \mathcal{W}_q(C)\rangle = e^{iq\alpha}\langle \mathcal{W}_q(C)\rangle$, or just $\langle \mathcal{W}_q(C)\rangle = 0$. This statement holds as a more general result, which is that a Wilson loop wrapping a non-trivial one-cycle of any compact manifold has a vanishing vacuum expectation value. This result was also derived in [15] using a change of variables in the functional integral, which is equivalent to using the 1-form symmetry. This example has demonstrated that higher-form symmetries of QFTs lead to **selection rules** on the expectation values of charged defects, which are also a central property of traditional symmetries.

B. Maxwell Theory with Matter

One modification to the d-dimensional pure Maxwell theory involves the introduction of electrically and magnetically charged excitations, that is a local

[2] Magnetic excitaitons have to be $(d-4)$-dimensional so that their worldvolumes are $(d-3)$-dimensional.

operator $\psi(x)$ (a fermion field) and a $(d-4)$-dimensional[2] extended operator $\mathcal{B}(\Sigma_{d-4})$, respectively. The **gauge invariance** of A is automatic from the definition of the field strength (19) and leads to the transformation law

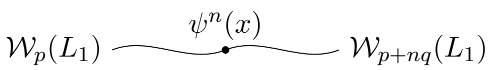
$$A \to A - d\Lambda \Longrightarrow \psi(x) \to e^{iq\Lambda(x)}\psi(x) \tag{25}$$

for the operator $\psi(x)$, with charge $q \in \mathbb{Z}$ under the **gauge group**[3] $\mathcal{G} = U(1)$ and Λ an arbitrary function on spacetime. Note that the transformation law (25) assumes the canonical Lagrangian prescription for QED with minimal coupling of matter to the gauge field A. There exists an analogous, though more complicated, transformation law for the dynamical magnetic excitation $\mathcal{B}(\Sigma_{d-4})$, which has charge $m \in \mathbb{Z}$ under $\mathcal{G} = U(1)$. The operators $\psi(x)$ and $\mathcal{B}(\Sigma_{d-4})$ are said to be **non-genuine** since they need to be attached to extended operators to be gauge invariant, and hence well-defined. For example, the field operator $\psi(x)$ attached to the end of a charge q Wilson line $\mathcal{W}_q(L)$ is gauge invariant, i.e.

$$\begin{aligned}\mathcal{W}_q(L)\psi(x) &\to \exp\left(iq\int_L (A - d\Lambda)\right)e^{iq\Lambda(x)}\psi(x)\\ &= \mathcal{W}_q(L)e^{-iq\Lambda(x)}e^{iq\Lambda(x)}\psi(x) = \mathcal{W}_q(L)\psi(x)\end{aligned} \tag{26}$$

where the second equality uses generalised Stoke's theorem and that $\partial L = x$. The other possible combination consists of $\psi(x)$ between line defects $\mathcal{W}_p(L_1)$ and $\mathcal{W}_{p+q}(L_2)$, with $\partial L_1 = x$, $\partial L_2 = x$, which is shown to be gauge invariant using an identical method to (26). Any QFT needs well-defined non-negative integer powers[4] of the field operator $\psi^n(x)$ as they describe physical quantities and appear in perturbative expansions. The operators $\psi^n(x)$ are non-genuine again, and have gauge invariant combinations

$$\mathcal{W}_q(L_1)\psi^n(x)\mathcal{W}_q(L_2) = \tag{27}$$



or written equivalently as

$$\mathcal{W}_p \overset{\psi^n}{\sim} \mathcal{W}_{p+nq}\ ;\ p \in \mathbb{Z} \tag{28}$$

where the notation used introduces the concept of **screening** $\sim$ of Wilson lines, that is the local operator $\psi^n(x)$ lives between these defects. In analogy to (28), the $(d-4)$-dimensional operators $\mathcal{B}^n$ furnish screenings of the $(d-3)$-dimensional 't Hooft operators $\mathcal{H}_m(\Sigma_{d-3})$, that is

$$\mathcal{H}_l \overset{\mathcal{B}^n}{\sim} \mathcal{H}_{l+nm}\ ;\ l \in \mathbb{Z}\,. \tag{29}$$

For the example of Wilson lines, we can demonstrate that the defects related by screening have the same charge under $G^{(1)}$ by 'measuring' the charge on either end using the 1-form symmetry operators. Figure 5 demonstrates this, with the two pictures equivalent as they are related by a topological deformation of $U_g^{(E)}$. In attempting to find the generalised symmetries of the theory, we note that the introduction of matter invalidates the central equations of (20). In addition, we can't use perturbation theory due to the strength of coupling. The symmetry groups are in fact exactly solvable using minimal algebraic manipulation, and the remainder of this section will outline this derivation, which simplifies the discussions present in the literature [1,2]. For the 1-form symmetry group $G_E^{(1)}$, we could write down the set, call it D_1, of **equivalence classes** $[\mathcal{W}_q]_\sim$, with the equivalence relation $\sim$ that defects are related by a screening from fermions:

$$D_1 = \{[\mathcal{W}_q]_\sim \mid q \in \mathbb{Z}\} = \frac{\mathbb{Z}}{q\mathbb{Z}} = \mathbb{Z}_q\,. \tag{30}$$

Equation (30) is the set of all Wilson lines, but aggregating those equivalent under screening from any ψ^n, which is exactly the additive integer group modulo q, i.e. $\mathbb{Z}_q$. Since each element of the group D_1 has a different charge it characterises a different representation of the symmetry group $G_E^{(1)}$, which originates from the transformation law (18). This amounts to the mathematical statement that $D_1 \cong \hat{G}_E^{(1)}$ is the **Pontryagin dual** of the group $G_E^{(1)}$, or that it contains all of its possible (different) representations. A more formal discussion of this is included in the literature [2], however most importantly are the two properties: for any group $G^{(p)}$, there exists an isomorphism between itself and the Pontryagin dual of its Pontryagin dual, i.e.

$$G^{(p)} \cong \widehat{\widehat{G^{(p)}}} \tag{31}$$

and if $G^{(p)}$ is a finite Abelian group, then there exists an isomorphism between itself and its Pontryagin dual, i.e.

$$G^{(p)} \cong \widehat{G^{(p)}}\,. \tag{32}$$

Equation 31 means that taking the Pontryagin dual of $D_1 \cong \widehat{G^{(1)}}$ recovers the 1-form symmetry group,

[3] Local symmetries are not treated in the 0-form framework, so this symmetry is distinct from the higher-form discussion.
[4] Formally a product of spinors and adjoint spinors, but we omit this distinction for simplicity.

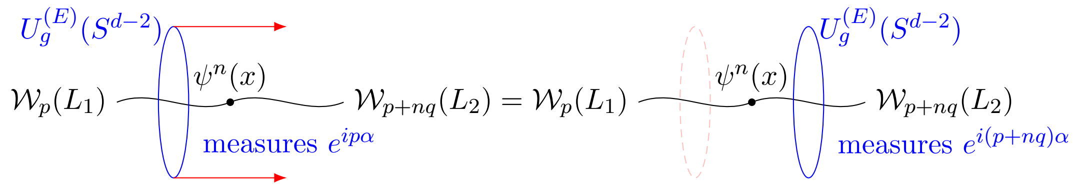


FIG. 5: Two Wilson lines related by screening must have the same charge under the 1-form symmetry as the symmetry operator can measure either side of the screened combination invariantly using a topological transformation (highlighted in red).

therefore

$$G_E^{(1)} \cong \hat{\mathbb{Z}}_q \cong \mathbb{Z}_q \tag{33}$$

which uses (32) and the fact that $\mathbb{Z}_q$ is finite Abelian. This phenomenon is **explicit breaking** of the electric 1-form symmetry, where $U(1)$ is broken into one of its subgroups $\mathbb{Z}_q$ in the introduction of matter. We note the distinction to **spontaneous symmetry breaking** which would give rise to a **Nambu–Goldstone boson**, which is lacking here. For a detailed exploration of this, see [1, 2]. We should also note that the screening relations (29) recover the magnetic $(d-3)$-form symmetry group as $G^{(d-3)} \cong \mathbb{Z}_m$. In recent efforts, screening has shown remarkable ability in solving the generalised symmetry groups of strongly-coupled theories, namely in [3] and [4]. In the following section, we will employ a screening procedure within a novel context, namely the $U(1)\times U(1)$ theory with **Abelian kinetic mixing**.

IV. ABELIAN KINETIC MIXING

A. 1-Form Symmetry

Many extensions of the SM incorporate a visible and hidden gauge sector, with communication of the sectors established by heavy messenger particles that are charged under both gauge groups. Integrating over the messenger particles generates an **effective field theory** with weak interactions between sectors due to the large mass of Φ. These interactions are generally non-renormalisable, though one renormalisable coupling is always possible if each gauge sector contains at least one $U(1)$ factor [6]. For example, hidden $U(1)$ groups are unavoidable in many string compactifications [16]. For the simplest gauge group $U(1)\times U(1)$, the coupling is characterised by a dimension-four renormalisable operator in the effective Lagrangian of the low energy theory [7], written in terms of a real symmetric matrix $\mathcal{K}$:

$$\begin{aligned}\mathcal{L} \supset& -\frac{\chi_1}{4}F\wedge F - \frac{\chi_2}{4}G\wedge G - \frac{\chi}{2}F\wedge G \\ &= -\frac{1}{4}\mathcal{F}^T\mathcal{K}\mathcal{F}\ ;\ \mathcal{K} = \begin{pmatrix}\chi_1 & \chi \\ \chi & \chi_2\end{pmatrix}\ ,\ \mathcal{F} = \begin{pmatrix}F \\ G\end{pmatrix}\end{aligned} \tag{34}$$

where F and G are the field strengths (2-forms) of the $U(1)$ gauge fields in the visible and hidden sector, respectively. The elements of $\mathcal{K}$ in (34) are related to the vacuum polarisation diagrams enforcing renormalisation of the gauge fields [17], and effects of $\mathcal{O}(\chi^2)$ may be neglected accordingly. This coupling was first introduced in [7] and, for obvious reasons, is referred to as **Abelian kinetic mixing**. To obtain the physical gauge fields, we start with an orthogonal diagonalisation, i.e. $\mathcal{K} = ODO^T$. For the Lagrangian in (34) to assume the canonical form

$$\mathcal{L} \supset -\frac{1}{4}\mathcal{F}^T\mathcal{F} = -\frac{1}{4}F\wedge F - \frac{1}{4}G\wedge G \tag{35}$$

the field strengths must be transformed as $\mathcal{F} \mapsto D^{1/2}O^T\mathcal{F} = \tilde{\mathcal{F}}$. Writing $\Omega = D^1/2O^T$ the charge $q \in \mathbb{Z}^2$ of a minimally coupled arbitrary fermion $\psi(x)$ must transform as $q \mapsto q\Omega^{-1} = \tilde{q}$. We compute the transformation to $\mathcal{O}(\chi)$ by diagonalising $\mathcal{K}$ in Mathematica, yielding

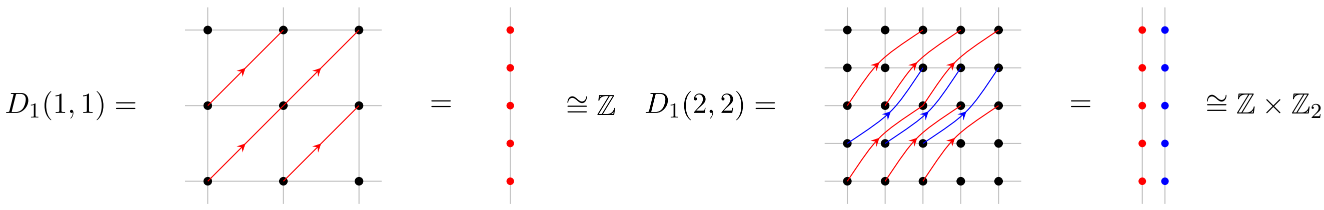
$$\begin{pmatrix}\tilde{q}_1 \\ \tilde{q}_2\end{pmatrix} = \begin{pmatrix}q_1 \\ q_2\end{pmatrix} - \begin{pmatrix}\epsilon_1 \\ 0\end{pmatrix}\ ;\ \epsilon_1 = q_2\chi\left(\frac{1+\chi_1}{1+\chi_2}\right)^{1/2} \tag{36}$$

with $\epsilon_1 \sim \mathcal{O}(\chi)$ referred to as a **millicharge** in the literature [7, 6]. As discovered in [7], the effect of (36) is that known fermions ($q_2 = 0$) would not gain a coupling to the paraphoton, that is the hidden $U(1)$ gauge boson, but hidden sector fermions ($q_2 \neq 0, q_1 = 0$) gain a small induced coupling to the photon. There is in fact a second possible diagonalisation [6] which leads to the millicharge shift:

$$\begin{pmatrix}\tilde{q}_1 \\ \tilde{q}_2\end{pmatrix} = \begin{pmatrix}q_1 \\ q_2\end{pmatrix} - \begin{pmatrix}0 \\ \epsilon_2\end{pmatrix}\ ;\ \epsilon_2 = q_1\chi\left(\frac{1+\chi_2}{1+\chi_1}\right)^{1/2} \tag{37}$$

which we reserve for later discussions. As an extension to the literature, we now solve the 1-form symmetry exactly using screening. While coupling between the sectors is weak, we will show that screening demonstrates that the 1-form symmetry is independent of the interaction strength exactly, i.e. not to some order of χ. Under (37), the fermion $\psi(x)$ now has an effective charge $\tilde{q}$ described by (36), so the charged line defects for the theory must be

$$\mathcal{W}_{\tilde{q}}(L) = \exp\left(i\tilde{q}\cdot\int_L \mathcal{A}\right)\ ;\ \mathcal{A} = \begin{pmatrix}A \\ B\end{pmatrix}\ , \tag{38}$$



(a) Screenings furnished by a $(1,1)$ -charged excitation ψ reduce the D_1-lattice to $\mathbb{Z}$.

(b) Screenings furnished by a $(2,2)$ -charged excitation ψ reduce the D_1-lattice to $\mathbb{Z} \times \mathbb{Z}_2$.

FIG. 6: Screening by a matter field identifies D_1-lattice points and collapses it into a subgroup, which is just a visual representation of explicit symmetry breaking.

with $F = dA$ and $G = dB$, so that the gauge invariant combinations furnishing screenings:

$$\mathcal{W}_{p\Omega^{-1}} \overset{\psi^n}{\sim} \mathcal{W}_{p\Omega^{-1}+nq\Omega^{-1}}$$
$$\implies \mathcal{W}_p \overset{\psi^n}{\sim} \mathcal{W}_{p+nq} \; ; \; p \in \mathbb{Z} \; , \; n \in \mathbb{N}^0 \; . \tag{39}$$

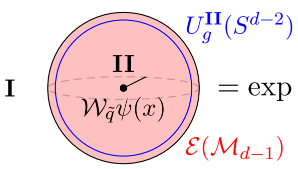
Since the transformations Ω^{-1} appear on both sides of the above, we can recover the familiar screening relation (28) with charges upgraded to be 2-dimensional vectors as shown. We visualise screenings from our arbitrary fermion as lines identifying $\mathbb{Z} \times \mathbb{Z}$ D_1 'lattice' points which reduce the group structure, see Figure 6(a) and Figure 6(b) as examples. The insertion of an excitation $\psi(x)$ with $q = (1,1)^T$ identifies D_1-lattice points along lines of 'gradient' 1, which collapses the lattice into a single $\mathbb{Z}$ line as shown in Figure 6(a). This arises from the screening relation (35), which states that a defect of charge $p \in \mathbb{Z}^2$ can be identified with a different defect of charge $p + n(1,1)$. The naive assumption would be that this structure holds for any charged insertion, i.e. $D_1 \cong \mathbb{Z}$ for any charge $q \in \mathbb{Z}^2$ of the fermion. However, if the greatest common divisor (gcd) of q_1 and q_2 is not equal to 1 (for example $q = (2,2)^T$ as shown in Figure 6(a)), screenings of a given lattice point 'skip' adjacent points and the collapsed D_1-lattice acquires an extra dimension. For an arbitrary insertion of charge $q = (q_1, q_2)^T$, the result is hence:

$$D_1 \cong \mathbb{Z} \times \mathbb{Z}_Q \Rightarrow G_E^{(1)} \cong U(1) \times \mathbb{Z}_Q \tag{40}$$

with $Q = \gcd(q_1, q_2)$, where have used the Pontryagin dual and asserted that it operates in a straightforward manner over a direct product. The reader is notified that $\mathbb{Z}_1$ is just the **trivial group**, that is it contains one element and hence does not contribute to $G_E^{(1)}$. Most critically, we see that matter fields lead to partial explicit breaking of the 1-form symmetry, breaking precisely one of the $U(1)$ symmetries into a subgroup, with the choice of broken $U(1)$ group a degree of freedom. This result is identical to that presented in [2] for a $U(1) \times U(1)$ theory without kinetic mixing, and this remarkable property can be owed to the intrinsic description of the higher-form symmetry framework. We also emphasize the merit of our original D_1-lattice visualization as a useful tool for computing higher-form symmetries.

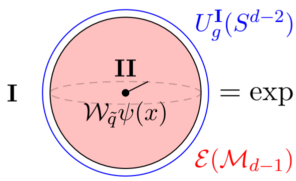
B. Domain Walls

We now address a unique problem consisting of a sharp domain wall $\mathcal{M}_{d-1}$ separating two regions of spacetime, one (domain **I**) with no kinetic mixing $\mathcal{K} = 1$ and one (domain **II**) with kinetic mixing $\mathcal{K} \neq 1$. Domain **II** can be thought of as a 'rotation' $\mathcal{K}$ of the vacuum polarisation in domain **I**, which would be generated from the usual dielectric effect from virtual particle–antiparticle pairs but with them now charged under two gauge sectors. A fermion $\psi(x)$ introduced to domain **II** acquires a millicharge shift $\tilde{q} = q\Omega^{-1}$ and is attached to a charge $\tilde{q}$ Wilson line. Measuring the charge inside **II** amounts to using the explicitly broken 1-form symmetry in (40) which has generalised Noether current $\Omega \star \mathcal{F}$:



$$= \exp\left(i\alpha \cdot \int_{S^{d-2}} \Omega \star \mathcal{F}\right) \mathcal{W}_{\tilde{q}}\psi(x)$$
$$= \exp\left(i\alpha \cdot \Omega q \Omega^{-1}\right) \mathcal{W}_{\tilde{q}}\psi(x) \tag{41}$$

using the transformation law of the Wilson line (18). On the other hand, measuring the charge in **I** amounts to using the unbroken 1-form symmetry with generalised Noether current $\star\mathcal{F}$:



$$= \exp\left(i\alpha \cdot \int_{S^{d-2}} \star\mathcal{F}\right) \mathcal{W}_{\tilde{q}}\psi(x)$$
$$= \exp\left(i\alpha \cdot q\Omega^{-1}\right) \mathcal{W}_{\tilde{q}}\psi(x) \; . \tag{42}$$

For charge to be conserved there must exist a surface charge $\mathcal{E}(\mathcal{M}_{d-1})$ on the domain wall that the unbroken 1-form symmetry operator $U_g^{\mathbf{I}}$ measures as having charge

$$\varepsilon = \Omega q \Omega^{-1} - q\Omega^{-1} \sim \begin{pmatrix} q_2\chi\left(\frac{1+\chi_2}{1+\chi_1}\right)^{1/2} \\ 0 \end{pmatrix} . \tag{43}$$

to $\mathcal{O}(\chi)$ using the transformation from (37). The domain wall therefore carries an induced positive surface charge of $\mathcal{O}(\chi)$ in the visible sector that can interact with photons, provided that it encloses a hidden fermion ($q_2 \neq 0$). From the electromagnetic wave equation $\Box A^\mu = -\mu_0 J^\mu$, interaction of an incident wave with the surface $J^\mu \sim \mathcal{O}(\chi)$ (assuming the hidden fermion is moving) would lead to a reflected wave of amplitude $A^\mu \sim \mathcal{O}(\chi)$ and hence intensity $\mathcal{I} \sim \mathcal{O}(\chi^2)$. In recent years, methods for directly detecting millicharged particles have been proposed such as [18], and our result represents a potentially new signal for searches of a 'dark' $U(1)$ sector.

V. CONCLUSION

In conclusion, we have shown that the framework of generalised symmetries provides an elegant and unified perspective of various physical phenomena. We have demonstrated that for the complex scalar QFT, a 0-form symmetry describes a generalised Ward-Takahashi identity which implements the transformation properties of scalar fields under the ordinary global $U(1)$ symmetry (11). We followed with a pedagogical discussion on extending to higher-form symmetries, with Maxwell theory serving as an example. The electric 1-form symmetry amounts to a generalised statement of Gauss' law (24) and explicit symmetry breaking occurs in the presence of charged matter. We recover the result from [14] that Wilson loops wrapping non-trivial one cycles have vanishing expectation values, which parallels the selection rules associated with traditional symmetries. Using the powerful method of screening, we find that the 1-form symmetry of a $U(1)\times U(1)$ gauge theory is exactly invariant under Abelian kinetic mixing of the gauge fields (40). We also show that the symmetry is explicitly broken by a fermion charged under either gauge sector. It is also demonstrated that domain walls separating regions of kinetic mixing incur a small, induced surface charge (43). We use this result to argue that measuring the reflected amplitude of an incident electromagnetic wave could probe the existence of a dark $U(1)$ sector. Developments to this report would examine this reflected intensity, in particular how the low-amplitude signal could be resolved among the background of incident waves. Computing the spontaneous symmetry breaking and 't Hooft anomalies could also hold promising original results for this theory. The formalism of generalised symmetries is one containing both mathematical depth and unique consequences for many field theories, and we eagerly anticipate the evolution of the field's application to high-energy physics in the coming years.

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A. APPENDICES

A.1 Abbreviations

In alphabetical order:

BSM: Beyond the Standard Model
KM: Kinetic mixing
SM: Standard Model
SUSY: Supersymmetry
QED: Quantum electrodynamics
QFT: Quantum field theory

A.2 Notation

In order of appearance:

U_g	a symmetry operator
Σ_d	spacetime, a d-dimensional manifold
S_q	a spatial q-dimensional manifold
$G^{(p)}$	a p-form symmetry group
$\varphi(x)$	a complex scalar field operator
d	the exterior derivative
$j^{(q)}$	the q-form generalised Noether current
$\wedge$	the exterior product
$\mathcal{U}_{d-q}$	the space of codimension-q operators
$\tilde{Q}$	the generalised Noether charge
S^q	the q-sphere manifold
A	the U(1) gauge field
F	the $U(1)$ gauge field strength
$\star$	the Hodge star operator
Ω^p	the space of p-forms
$\mathcal{H}_m$	a 't Hooft operator
$\mathcal{W}_q$	a Wilson line operator
$\psi(x)$	a fermion field operator
$\mathcal{B}$	a magnetic excitation operator
$\sim$	the equivalence relation of screening
$\hat{G}$	the Pontryagin dual of a group

A.3 Proof of the Ward-Takahashi Identity

The following is a standard derivation of the Ward-Takahashi identity of the form (5). In the path integral formulation, we may write the expectation value provided in the form

$$\langle \partial_\mu J^\mu(x)\varphi(y)\rangle = \mathcal{N}\int \mathcal{D}\varphi \cdot \partial_\mu J^\mu(x)\varphi(y)e^{iS[\varphi]} \quad (44)$$

with $S[\varphi]$ the action functional. The Noether current $J^\mu(x)$ is derived from the change of the Lagrangian under infinitesimal global $U(1)$ transformations, or equivalently from the action by

$$S[\varphi(x)+\epsilon q\varphi(x)] - S[\varphi(x)] = -\epsilon\int d^4x \cdot \partial_\mu J^\mu(x) \ , \quad (45)$$

which allows us to form the equality

$$\partial_\mu J^\mu(x) = -\frac{d}{d\epsilon}S[\varphi+\epsilon q\varphi]\Big|_{\epsilon=0} . \quad (46)$$

Substituting this into equation (44) and using integration by parts yields:

$$\begin{aligned}&\langle \partial_\mu J^\mu(x)\varphi(y)\rangle \\ &= -\mathcal{N}\int \mathcal{D}\varphi \cdot \frac{d}{d\epsilon}S[\varphi+\epsilon q\varphi]\varphi(y)e^{iS[\varphi]}\Big|_{\epsilon=0} \\ &= -\frac{1}{i}\frac{d}{d\epsilon}\mathcal{N}\int \mathcal{D}\varphi\cdot\varphi(y)e^{iS[\varphi+\epsilon q\varphi]}\Big|_{\epsilon=0} .\end{aligned} \quad (47)$$

We complete the derivation with a change of variables $\varphi(x)+\epsilon(x)q\varphi(x)\to\varphi(x)$, i.e.

$$\begin{aligned}&\langle \partial_\mu J^\mu(x)\varphi(y)\rangle \\ &= -\frac{1}{i}\frac{d}{d\epsilon}\mathcal{N}\int \mathcal{D}\varphi\cdot(\varphi(y)-\epsilon q\varphi(y))e^{iS[\varphi]}\Big|_{\epsilon=0} \\ &= \mathcal{N}\cdot\frac{1}{i}\delta^4(x-y)q\int \mathcal{D}\varphi\cdot\varphi(y)e^{iS[\varphi]} .\end{aligned} \quad (48)$$

The result is defined up to a multiplicative constant due to the nature of the Noether current $J^\mu(x)$. Observing that the final line of (48) is an expectation value in the path integral formulation and rescaling by an arbitrary factor of i, we obtain

$$\langle \partial_\mu J^\mu(x)\rangle = q\delta^{(4)}(x-y)\langle\varphi(x)\rangle \quad (49)$$

as required.

A.4 Differential Geometric Formulation of Maxwell Theory

Following the presentation in [10], we introduce the magnetic field strength B as a 2-form on spacetime Σ_4. In local coordinates (that is formally, in some set of charts) of the manifold, this object is defined as

$$B = B_x dz\wedge dy + B_y dx\wedge dz + B_z dy\wedge dx \ . \quad (50)$$

where B_i are the usual Cartesian components. The objects dx^i are 1-forms and are basis vectors of an abstract vector space known as the cotangent space $T_p^*(\Sigma_4)$of Σ_4. 1-forms combine under the **exterior product** to produce 2-forms $dx^i\wedge dx^j$, which can be interpreted as a cross product. For this reason, the magnetic field strength is itself a 2-form as it consists of a linear combination of such 2-forms. Similarly, we define the electric field E as a 1-form on spacetime, defined locally as

$$E = E_x dx + E_y dy + E_z dz \quad (51)$$

which is evidently a 1-form for the same reason. The field strength is defined as a 2-form written as

$$F = B + dt \wedge E \; . \tag{52}$$

By substituing equations (50) and (51) into the above formula, we can recover the local form of F, i.e.

$$\begin{aligned} F = B_x dz \wedge dy + B_y dx \wedge dz + B_z dy \wedge dx \\ + E_x dt \wedge dx + E_y dt \wedge dy + E_z dt \wedge dz \end{aligned} \tag{53}$$

if we assume that the exterior product is linear over the functions E_i. Similarly, we can define the dual field strength $\star F$. The operator $\star$ is the Hodge star operator, and operates as taking p-forms to $(d-p)$-forms with d the dimension of the spacetime. Therefore, $\star F$ is a 2-form on spacetime. We assert an equality for the action of $\star$ on 2-forms in 4-dimensional space, that is

$$\star(dx^\mu \wedge dx^\nu) = -\frac{1}{2}\epsilon^{\mu\nu\lambda\kappa} dx^\lambda \wedge dx^\kappa \tag{54}$$

with greek indices running over $(0, 1, 2, 3)$. We can then find the local form of the dual field strength, i.e.

$$\begin{aligned} \star F = -B_x dx \wedge dt - B_z dy \wedge dt - B_z dz \wedge dt \\ + E_x dy \wedge dz + E_y dz \wedge dx + E_z dx \wedge dy \; . \end{aligned} \tag{55}$$

Maxwell's equations can then be written in terms of objects (55) and (53) as

$$dF = 0 \; , \; d \star F = J \tag{56}$$

where J is a 3-form on spacetime with local coordinates:

$$\begin{aligned} J =& \rho dx \wedge dy \wedge dz - J_x dt \wedge dy \wedge dz \\ & - J_y dt \wedge dz \wedge dx - J_z dt \wedge dx \wedge dy \end{aligned} \tag{57}$$

in terms of the usual components of the four-current. The operator d is the **exterior derivative** and acts on p-forms by upgrading them to $(p+1)$-forms, by the action:

$$d\omega = d(\omega_\mu dx^\mu) = (\partial_\lambda \omega_\mu) dx^\lambda \wedge dx^\mu \; . \tag{58}$$

For an exterior product of 1-forms, the exterior derivative obeys a product rule, i.e.

$$d(\omega \wedge \mu) = d\omega \wedge \mu - \omega \wedge d\mu \tag{59}$$

which must be a 3-form as it consists of terms that exterior product of 2-forms and 1-forms. In addition, we have the important result:

$$d^2 = 0 \tag{60}$$

which tells us that when taking the exterior derivative of any p-form twice, the result must vanish. Applying the defining properties of the exterior derivative to the 2-forms (55) and (53) and comparing with (56) indeed recovers Maxwell's equations in the conventional vector calculus form. We also note the definition of the field strength in terms of the field A, which is a 1-form on spacetime, i.e.

$$F = dA \; . \tag{61}$$