FUNCTIONAL DIFFERENTIAL EQUATIONS AND LENS DESIGN IN GEOMETRICAL OPTICS

by

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ABSTRACT

The subject of this thesis is lens design using a system of functional differential equations arising from Fermat's Principle in geometrical optics. The emphasis is primarily on existence, uniqueness, and analyticity, properties of solutions to these equations, but some asymptotic methods are developed for special cases. Three specific lens problems are considered in detail: the first is an axial lens having two pairs of foci on the optical axis, the second is an axial lens which focuses light at two different frequencies to two distinct points, the third is a lens symmetric about an axis having foci not on said axis.

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Chapter 1

Introduction

Section 1. Prolegomena

The subject of this thesis is two-dimensional lens design using geometrical optics. In particular, three related problems in lens design are examined which, in essence, entail the determination of lens boundaries symmetric about an axis given the foci configuration, index of refraction, and lens thickness about said axis. The emphasis is on existence, uniqueness, and analyticity properties of solutions to these problems, but, for some special cases, asymptotic methods are developed which reduce the problems to solving first order nonlinear ordinary differential equations and algebraic relations.

Although lens design problems have been studied since Descartes (ca. 1637), much of the attention has been on numerical solutions and solutions under simplifying assumptions such as paraxial/Gaussian optics approximations, which in effect are linearised versions of lens problems. Questions concerning existence and uniqueness of solutions to 'full problems' in lens design have only recently been addressed. Motivated by questions posed by Sternberg (1955,1963,1977, 1985) in connection with microwave and acoustic scanning problems, Friedman & McLeod (1987) proved that there exist precisely two symmetric lenses (i.e. lenses having boundaries which are mirror images, cf. Section 2.5) with analytic boundaries for a given foci configuration, index of refraction, and axial lens thickness. Using a different method, Rogers (1988) established the same result. The symmetric lens result was generalised by Friedman & McLeod (1988) to a lens having two pairs of foci on the axis of symmetry. The present work, in part, builds on that of Friedman & McLeod and Rogers.

In this chapter, some general concepts in geometrical optics are discussed such as Fermat's Principle along with some different approaches to light ray geometry for inhomogeneous media. The transition to media with piecewise constant refractive indices is made in Chapter 2, where the governing system of functional differential equations is derived and general solution properties discussed.

The first problem is the design of a lens having two pairs of foci on the axis of symmetry. This problem is discussed in Chapter 3 and, using the work of Friedman & McLeod (1988), existence and uniqueness among analytic functions is established for various foci configurations. The result is sharpened to uniqueness among functions having continuous second derivatives, and questions are addressed regarding analytic dependence of the solution on parameters (such as foci locations) and solution stability. An asymptotic method is developed at the end of the chapter for the case when the two pairs of foci nearly coincide.

The second problem is discussed in Chapter 4 and entails the design of a lens which can focus two colours from a source to two distinct points. This problem is closely related to the first problem, and the analysis developed in Chapter 3 is used to establish existence, uniqueness, and analyticity results. In addition, a method for determining an asymptotic solution is described for the case when the source is nearly monochromatic.

In Chapter 5, the third problem is addressed. This problem concerns the design of an axisymmetric lens having foci not on the axis of symmetry. The nature of the functional arguments in the governing equations for this problem differs fundamentally from those considered in the earlier problems, and the analysis developed in Chapter 3 cannot be readily applied. Questions concerning existence, uniqueness, and analyticity are discussed in this chapter and an asymptotic method for solving the problem is described for the case when the foci are close to the optical axis.

The results in this thesis are summarized in Chapter 6 along with some comments regarding future work in the field. One appendix has been affixed to the text. This appendix is devoted primarily to an existence/uniqueness proof for a system of functional differential equations of a certain form. The governing equations for the first two problems are special cases of this form and thus the appendix provides alternative existence and uniqueness proofs for these problems.

Throughout this thesis certain notations concerning differentiability are used. The symbol C^n, $n \in \mathbb{Z}^+$ represents the set of functions having n continuous derivatives. It is made clear in context if the functions are vector valued as well as the appropriate range. By analytic it is always meant real analytic. The set of real analytic functions is denoted C^ω.

Each chapter begins with a short introduction to the material and then divided into sections. Equation numbering starts with each chapter. If an equation is referenced outside the chapter in which it appears, the equation number is preceded by the appropriate chapter number.

Section 2. The Transition to Geometrical Optics and Fermat's Principle

Light is an electromagnetic phenomenon and can be described by Maxwell's equations. The electromagnetic field associated with light is characterized by high frequencies of the order 10^{14}/sec. and correspondingly small wavelengths of the order 10^{-5}cm. It is reasonable to expect, in view of the magnitude of the wavelength, that, in regions with dimensions much greater than the wavelength where changes in the medium and electromagnetic field are gradual, an approximation which assumes a small wavelength will yield a close estimate of the actual behaviour of light. The branch of optics characterized by this approximation is known as <u>geometrical optics</u>.

The Maxwell equations for a non-conducting isotropic medium free from charges are

$$\underset{\sim}{\nabla}\wedge\underset{\sim}{H}(\underset{\sim}{r},t) - \partial\underset{\sim}{D}(\underset{\sim}{r},t)/\partial t = \underset{\sim}{0} , \tag{1}$$

$$\underset{\sim}{\nabla}\wedge\underset{\sim}{E}(\underset{\sim}{r},t) + \partial\underset{\sim}{B}(\underset{\sim}{r},t)/\partial t = \underset{\sim}{0} , \tag{2}$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{D}(\underset{\sim}{r},t) = 0 , \tag{3}$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{B}(\underset{\sim}{r},t) = 0 , \tag{4}$$

and the material equations are

$$\underset{\sim}{D}(\underset{\sim}{r},t) = \varepsilon(\underset{\sim}{r})\underset{\sim}{E}(\underset{\sim}{r},t) \tag{5}$$

$$\underset{\sim}{B}(\underset{\sim}{r},t) = \mu(\underset{\sim}{r})\underset{\sim}{H}(\underset{\sim}{r},t) \tag{6}$$

where t is time, $\underset{\sim}{r}$ is the position vector, $\underset{\sim}{H}$ is the magnetic field strength, $\underset{\sim}{E}$ is the electric field strength,

$\underset{\sim}{D}$ is the electric displacement, $\underset{\sim}{B}$ is the magnetic induction, ε is the permittivity, and μ is the magnetic permeability. The functions ε and μ, as indicated by equations (5) and (6), are assumed to be spatially dependent, and it is further assumed that they are in C^1. The vectors $\underset{\sim}{H}$, $\underset{\sim}{E}$, $\underset{\sim}{D}$ and $\underset{\sim}{B}$ are all assumed to be in C^1. The above equations have been normalized and the variables are non-dimensional. In particular, the speed of light, the permittivity, and the magnetic permeability in the vacuum are unity.

In a general time harmonic field, for an isotropic medium, the vectors $\underset{\sim}{E}$ and $\underset{\sim}{H}$ assume the form

$$\underset{\sim}{E}(\underset{\sim}{r},t) = \mathrm{Re}[\underset{\sim}{E}_0(\underset{\sim}{r})e^{-i\omega t}] \tag{7}$$

$$\underset{\sim}{H}(\underset{\sim}{r},t) = \mathrm{Re}[\underset{\sim}{H}_0(\underset{\sim}{r})e^{-i\omega t}] \ , \tag{8}$$

where $\underset{\sim}{E}_0$ and $\underset{\sim}{H}_0$ are in C^1 and ω is the angular frequency. Substituting (7) and (8) into equations (1) and (2), using the material equations (3) and (4), reduces Maxwell equations (1) and (2) to

$$\underset{\sim}{\nabla}\wedge\underset{\sim}{H}_0 + i\varepsilon\omega\ \underset{\sim}{E}_0 = \underset{\sim}{0} \tag{9}$$

$$\underset{\sim}{\nabla}\wedge\underset{\sim}{E}_0 - i\mu\omega\ \underset{\sim}{H}_0 = \underset{\sim}{0} \ . \tag{10}$$

It is assumed that $\underset{\sim}{r}$ is many wavelengths away from any sources or boundaries and only one wavefront passes through $\underset{\sim}{r}$ at a given time. It is further assumed that changes in the medium are gradual on the scale of the wavelength which is $O(\frac{1}{\omega})$. The complex amplitude functions $\underset{\sim}{E}_0$ and $\underset{\sim}{H}_0$ are assumed to have the form

$$\underset{\sim}{E}_0(\underset{\sim}{r}) = \underset{\sim}{e}(\underset{\sim}{r})e^{i\omega\psi(\underset{\sim}{r})} \tag{11}$$

$$\underset{\sim}{H}_0(\underset{\sim}{r}) = \underset{\sim}{h}(\underset{\sim}{r})e^{i\omega\psi(\underset{\sim}{r})} \tag{12}$$

where $\underset{\sim}{e}$ and $\underset{\sim}{h}$ are functions in C^1 which remain $O(1)$ as $\omega \to \infty$, and ψ is a real valued function in C^1 which remains $O(1)$ as $\omega \to \infty$.

The non-dimensional angular frequency is a large parameter in the above equations. On the length scale of centimetres, this parameter corresponds to a term of order 10^5 for visible light. Equations (11) and (12) are the starting points for WKB asymptotic expansions in the parameter ω. The first terms in these expansions closely resemble equations (11) and (12) and have been given the physical interpretation of being the principal transmitted wave through a medium (i.e. the portion of the wave which experiences no internal reflections) [cf. Bremmer, 1951].

The assumption is made that the functions $\underset{\sim}{e}$ and $\underset{\sim}{h}$ can be expressed as asymptotic expansions in inverse powers of ω such that these expansions are uniformly valid. It is further assumed that these expansions can be differentiated term by term to yield asymptotic expansions which correspond to the derivatives of the original functions and that the differentiated expansions are also uniformly valid. Under the above assumptions, the functions $\underset{\sim}{e}$ and $\underset{\sim}{h}$ can be written

$$\underset{\sim}{e}(\underset{\sim}{r}) = \underset{\sim}{e}_0(\underset{\sim}{r}) + \frac{1}{\omega}\,\underset{\sim}{e}_1(\underset{\sim}{r}) + \ldots \equiv \sum_{n\geq 0} \underset{\sim}{e}_n/\omega^n \tag{13}$$

$$\underset{\sim}{h}(\underset{\sim}{r}) = \underset{\sim}{h}_0(\underset{\sim}{r}) + \frac{1}{\omega}\,\underset{\sim}{h}_1(\underset{\sim}{r}) + \ldots \equiv \sum_{n\geq 0} \underset{\sim}{h}_n/\omega^n\ . \tag{14}$$

For physical reasons, it is assumed

$$\begin{aligned} \varepsilon(\underset{\sim}{r}) &> 1 \,, \\ \mu(\underset{\sim}{r}) &> 1 \,. \end{aligned} \tag{15}$$

Equations (11) and (12) written in terms of expansions (13) and (14) are

$$\underset{\sim}{E}_0(\underset{\sim}{r}) = e^{i\omega\psi} \sum_{n\geq 0} \underset{\sim}{e}_n/\omega^n \tag{16}$$

$$\underset{\sim}{H}_0(\underset{\sim}{r}) = e^{i\omega\psi} \sum_{n\geq 0} \underset{\sim}{h}_n/\omega^n \,, \tag{17}$$

and substituting these expressions into equations (9) and (10) gives

$$i[(\underset{\sim}{\nabla}\psi\wedge \sum_{n\geq 0}\underset{\sim}{h}_n/\omega^n) + \varepsilon \sum_{n\geq 0}\underset{\sim}{e}_n/\omega^n] + \underset{\sim}{\nabla}\wedge \sum_{n\geq 0}\underset{\sim}{h}_n/\omega^{n+1} = \underset{\sim}{0} \tag{18}$$

$$i[(\underset{\sim}{\nabla}\psi\wedge \sum_{n\geq 0}\underset{\sim}{e}_n/\omega^n) - \mu \sum_{n\geq 0}\underset{\sim}{h}_n/\omega^n] + \underset{\sim}{\nabla}\wedge \sum_{n\geq 0}\underset{\sim}{e}_n/\omega^{n+1} = \underset{\sim}{0} \,. \tag{19}$$

The function ψ is assumed to be non-zero and vary gradually on the scale of the wavelength so that, in particular, $|\nabla\psi|$ is $O(1)$. Similarly, the functions $\underset{\sim}{e}$ and $\underset{\sim}{h}$ are assumed to change gradually on the scale of the wavelength so that the derivatives of $\underset{\sim}{e}$ and $\underset{\sim}{h}$ are $O(1)$ in moduli.

Under the above conditions, the dominant terms in equations (18) and (19) are $O(1)$. Neglecting the $O(\frac{1}{\omega})$ terms in these equations gives

$$\underset{\sim}{\nabla}\psi\wedge\underset{\sim}{h}_0 + \varepsilon\underset{\sim}{e}_0 = \underset{\sim}{0} \tag{20}$$

$$\underset{\sim}{\nabla}\psi\wedge\underset{\sim}{e}_0 - \mu\underset{\sim}{h}_0 = 0 \,. \tag{21}$$

Eliminating $\underset{\approx}{h}_0$ between equations (20) and (21) for a non-trivial solution gives

$$\underset{\sim}{\nabla}\psi(\underset{\sim}{\nabla}\psi\cdot\underset{\approx}{e}_0) - |\nabla\psi|^2\underset{\approx}{e}_0 + \mu\varepsilon\underset{\approx}{e}_0 = 0 \; . \tag{22}$$

The inner product of $\underset{\sim}{\nabla}\psi$ and the terms in equation (20) implies

$$\underset{\sim}{\nabla}\psi\cdot\underset{\approx}{e}_0 = 0 \; , \tag{23}$$

and equation (22) becomes

$$|\underset{\sim}{\nabla}\psi|^2\underset{\approx}{e}_0 = \mu\varepsilon\underset{\approx}{e}_0 \; . \tag{24}$$

For a non-trivial solution, $\underset{\approx}{e}_0$ is not everywhere zero and equation (24) thus implies

$$|\underset{\sim}{\nabla}\psi|^2 = \mu\varepsilon \; . \tag{25}$$

The transition to geometrical optics is made when $\omega \to \infty$ so that

$$\begin{aligned} \underset{\approx}{e} &\sim \underset{\approx}{e}_0 \\ \underset{\approx}{h} &\sim \underset{\approx}{h}_0 \; . \end{aligned} \tag{26}$$

Using the Maxwell relation

$$n = \sqrt{\mu\varepsilon} \tag{27}$$

where n is the index of refraction [cf. Born & Wolf, 1986] equation (25) can be written

$$|\nabla\psi|^2 = n^2 \; . \tag{28}$$

Equation (28) is called the <u>eikonal equation</u> of geometrical optics.

Given suitable initial data, the eikonal equation can be solved using Charpit's method [cf. Forsyth, 1903; Ockendon & Tayler, 1986]. In particular, the characteristics must satisfy

$$\frac{dx}{dt} = \frac{\partial\psi}{\partial x} , \tag{29a}$$

$$\frac{dy}{dt} = \frac{\partial\psi}{\partial y} , \tag{29b}$$

$$\frac{d\psi}{dt} = n^2 , \tag{29c}$$

$$\frac{d}{dt} \left(\frac{\partial\psi}{\partial x}\right) = n \frac{\partial n}{\partial x} , \tag{29d}$$

$$\frac{d}{dt} \left(\frac{\partial\psi}{\partial y}\right) = n \frac{\partial n}{\partial y} . \tag{29e}$$

Here, t is some parameter on the characteristic. The characteristic projections of the eikonal equation are called light rays, and it is clear from system (29) that light rays are orthogonal to the level curves of ψ. A geometric interpretation of ψ for an optical system having two foci is given in figure 1. The definition of light rays is physically motivated by considering the time average Poynting vector which represents the average energy flux over some specified period of time. The Poynting vector, for an isotropic medium, is normal to the level curves of ψ [cf. Born & Wolf, 1986].

The optical path length from a point $\underset{\sim}{p}_0$ to some point $\underset{\sim}{p}_1$ is defined as

$$V(\underset{\sim}{p}_0,\underset{\sim}{p}_1) = \psi(\underset{\sim}{p}_1)-\psi(\underset{\sim}{p}_0) . \tag{30}$$

Since, on a characteristic,

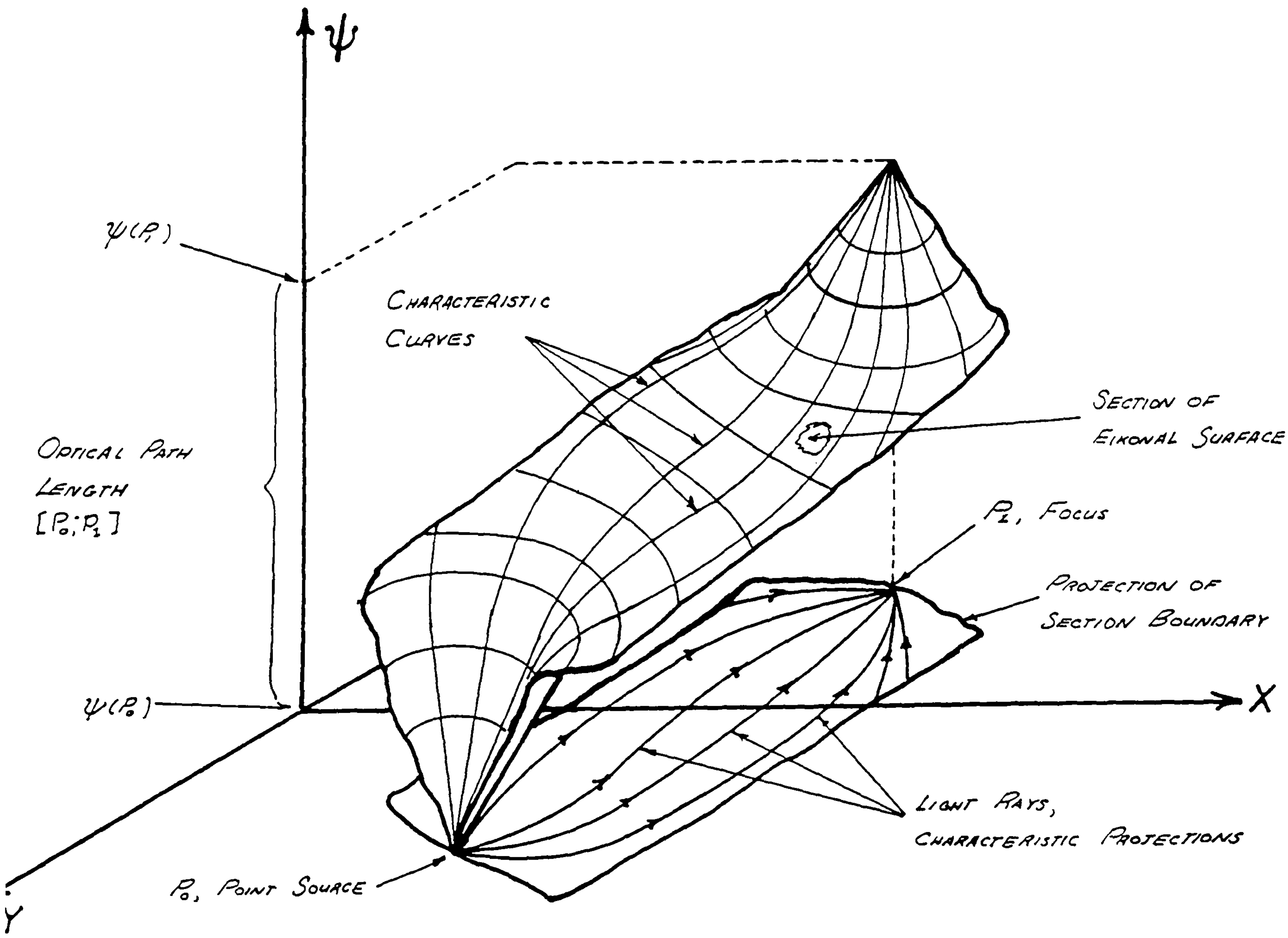


Figure 1: Geometric Interpretation of ψ

$$d\psi = n^2 dt = n\{\dot{x}^2 + \dot{y}^2\}^{\frac{1}{2}} dt \ ,$$

where $\dot{}$ denotes d/dt, the optical path length can be written

$$V(\underset{\sim}{p}_0, \underset{\sim}{p}_1) = \int_{\underset{\sim}{p}_0}^{\underset{\sim}{p}_1} n ds \ , \tag{31}$$

where s is arc length, and the path of integration is the light ray between $\underset{\sim}{p}_0$ and $\underset{\sim}{p}_1$. Equation (29c) ensures that $V(\underset{\sim}{p}_0, \underset{\sim}{p}_1)$ is positive provided the light rays are parametrized such that t increases from $\underset{\sim}{p}_0$ to $\underset{\sim}{p}_1$.

The eikonal equation and system (29) imply that

$$L = \frac{d\psi}{dt} = n(\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}} \ , \tag{32}$$

satisfies the Euler-Lagrange equations

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= 0 \ , \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} &= 0 \ , \end{aligned} \tag{33}$$

and this, in turn, implies that along a light ray

$$\delta \int_{\underset{\sim}{p}_0}^{\underset{\sim}{p}_1} n ds = 0 \ , \tag{34}$$

where δ is the first variation ($\underset{\sim}{p}_0, \underset{\sim}{p}_1$ fixed). Relation (34) is known as <u>Fermat's Principle</u>. Geometrical optics was originally developed using Fermat's Principle as an axiom. The connection between this approach and the electromagnetic theory of light was made by Sommerfeld & Runge (1911). The crucial link is the eikonal equation which, as shown above, leads to Fermat's Principle. Alternatively, the eikonal equation can be derived as the Hamilton-Jacobi equation for the Hamiltonian system associated with Fermat's Principle. In fact, the eikonal equation was originally derived from Fermat's Principle [Bruns, 1895].

Geometrical optics can be approached from several related but different mathematical directions. For instance, the Hamiltonian structure of the light ray equations can be exploited and results/techniques of classical mechanics used (e.g. canonical transformations, integral invariants etc.). This approach is discussed in detail by Born & Wolf (1986), Buchdahl (1970), Luneberg (1944), and Synge (1937a,1937b). The approach used in this thesis is based on a system of functional differential equations derived from Fermat's Principle by Ockendon et al. (1985). These equations are discussed in detail in the next chapter. The remainder of this chapter is devoted to a differential geometric interpretation which, though not used directly, nonetheless provides some insight into the problems and possible generalisations of the results.

Equation (34) is equivalent to the statement that a light ray path is a geodesic on some surface Λ having the first fundamental form

$$ds^2 = n^2(u,v)\{du^2 + dv^2\} , \qquad (35)$$

where u,v are curvilinear coordinates on Λ. The metric on Λ forms an isothermic system, and a conformal homeomorphism from Λ to $\mathbb{R}^2$ can be established so that light rays can be interpreted as images of geodesics under the homeomorphism. As Fermat's Principle does not contain information about the second fundamental form, it is clear that, for a particular optical system Λ is not uniquely determined and that light rays can be described in terms of intrinsic properties of Λ. Problems involving light ray configurations, in essence, can be viewed as geodesic problems in differential geometry.

Local existence and uniqueness results for geodesics [cf. Willmore, 1959] imply that for $\underset{\sim}{p}_0, \underset{\sim}{p}_1 \in \Lambda$ sufficiently close (in the metric) and n smooth, a unique geodesic contains both points. Globally, however, it is clear that this is not necessarily the case, and geodesics leaving $\underset{\sim}{p}_0$ may intersect again at $\underset{\sim}{p}_1$ (e.g. consider geodesics on a sphere). Let $W = \{\Gamma_a : \alpha \in (a,b)\}$ be a family of geodesics through $\underset{\sim}{p}_0$ where α is some direction parameter (e.g. the angle of the geodesic at $\underset{\sim}{p}_0$ with respect to some reference line). If all the geodesics in W intersect again at $\underset{\sim}{p}_1$, then

$$V_{\Gamma_\alpha}(\underset{\sim}{p}_0, \underset{\sim}{p}_1) = V_{\Gamma_\beta}(\underset{\sim}{p}_0, \underset{\sim}{p}_1) \tag{36}$$

for all $\Gamma_\alpha, \Gamma_\beta \in W$ [cf. Willmore 1959]. Relation (36) is a statement of the 'path length condition' in geometrical optics. It may be that only a finite number of geodesics intersect again at $\underset{\sim}{p}_1$. This occurs, for instance, when the geodesics form an envelope (cf. Section 2.6). If Γ_α and Γ_β intersect at $\underset{\sim}{p}_1$, relation (36) is not necessarily satisfied for this case.

Other results in intrinsic geometry can also be exploited for lens design problems. The Gaussian curvature is an intrinsic property and gives a local indication of geodesic behaviour. If the first fundamental form is given by equation (3), then the Gaussian curvature is

$$K = -\frac{1}{n^2} \underset{\sim}{\nabla}^2 \ell n\, n \; . \tag{37}$$

If $K < 0$ at $\underset{\sim}{p}_0$, then $\underset{\sim}{p}_0$ is hyperbolic and geodesics leaving p_0 diverge rapidly. If $K > 0$ at $\underset{\sim}{p}_0$ then the point is elliptic and geodesics may intersect again. The Gauss-Bonnet theorem can be used to extend this relation to a global result. In particular, if $K \leq 0$ in a region then two geodesics cannot intersect again. This can also be shown using Jacobi's equation for geodesics [cf. Struik, 1961; Willmore, 1959]. Another result which can be readily applied to geometrical optics is Jacobi's theorem [cf. Willmore, 1959] which gives necessary and sufficient conditions for the optical path length to be a local minimum.

The 'differential geometry approach' is useful in gaining insight into some of the lens design problems. For instance, the motivation behind Corollary 2.4 and the division of axial lenses into two classes is Jacobi's theorem. The problem, however, is that the results of classical intrinsic differential geometry are based on the assumption that the first fundamental form is smooth which is clearly not the case for the aforementioned lens problems. This is complicated further by the 'inverse nature' of these lens problems. Specifically, intrinsic differential geometry is concerned with determining geodesics given the first fundamental form;

lens design problems are concerned with determining the first fundamental form given certain properties of the geodesic configuration. Although the differential geometry approach is not followed directly in this thesis, the results can be viewed as extensions in the geometry of geodesics. The differential geometry associated with geometrical optics is discussed further by Born & Wolf (1986), Luneberg (1944) and van-Brunt (1987).

Chapter 2

The General Lens Equations and Axial Lenses

The purpose of this chapter is to introduce some basic definitions which will be used throughout the remainder of the thesis and a system of functional differential equations which will play a central role in the lens problems considered in the following chapters. The differential equations describe lens boundaries and follow directly from Fermat's Principle. The first section consists essentially of definitions, and the functional differential equations are derived in the second section. A special category of lenses (viz. axial lenses) is discussed in the third section along with some general solution properties which will be of use in later chapters. The fourth section concentrates on the linearised axial lens equations (i.e. paraxial optics), and the fifth section provides an example of an explicit solution to an axial lens problem. The sixth section discusses briefly caustics and aberrations in optical systems.

2.1 General Definitions and Assumptions

Let $S \subseteq \mathbb{R}^2$ be a simply connected region which is divided into three simply connected disjoint regions S_1, S_2, S_3 by curves $\gamma_f, \gamma_g \subseteq S$, i.e.

$$S_i \cap S_j = \emptyset \ , \qquad i \neq j$$

$$\bar{S}_1 \cap \bar{S}_2 = \bar{\gamma}_f$$

$$\bar{S}_2 \cap \bar{S}_3 = \bar{\gamma}_g$$

$$\bigcup_{i=1}^{3} S_i \cup \gamma_f \cup \gamma_g = S \ ,$$

where $\bar{S}_i$ denotes the closure of S_i, and let the index of refraction n be a piecewise constant function on S of the form

$$n(\underset{\sim}{r}) = \begin{cases} n_1, & \underset{\sim}{r} \in S_1 \\ n_2, & \underset{\sim}{r} \in S_2 \\ n_3, & \underset{\sim}{r} \in S_3 \ , \end{cases}$$

where $\underset{\sim}{r}$ is the position vector in $\mathbb{R}^2$ and the n_i are constants such that

$$n_1 \neq n_2 \ ; \qquad n_2 \neq n_3 \ ,$$

(see figure 1). The set S is termed the <u>optical system</u>. The set S_2 is defined as the <u>lens</u> of the optical system and the curves γ_f, γ_g as the <u>lens boundaries</u>.

The object and image spaces of the optical system are defined relative to the location of the particular source. It is assumed no sources are located in $\bar{S}_2$. The <u>object space</u> is the S_i containing the source and the

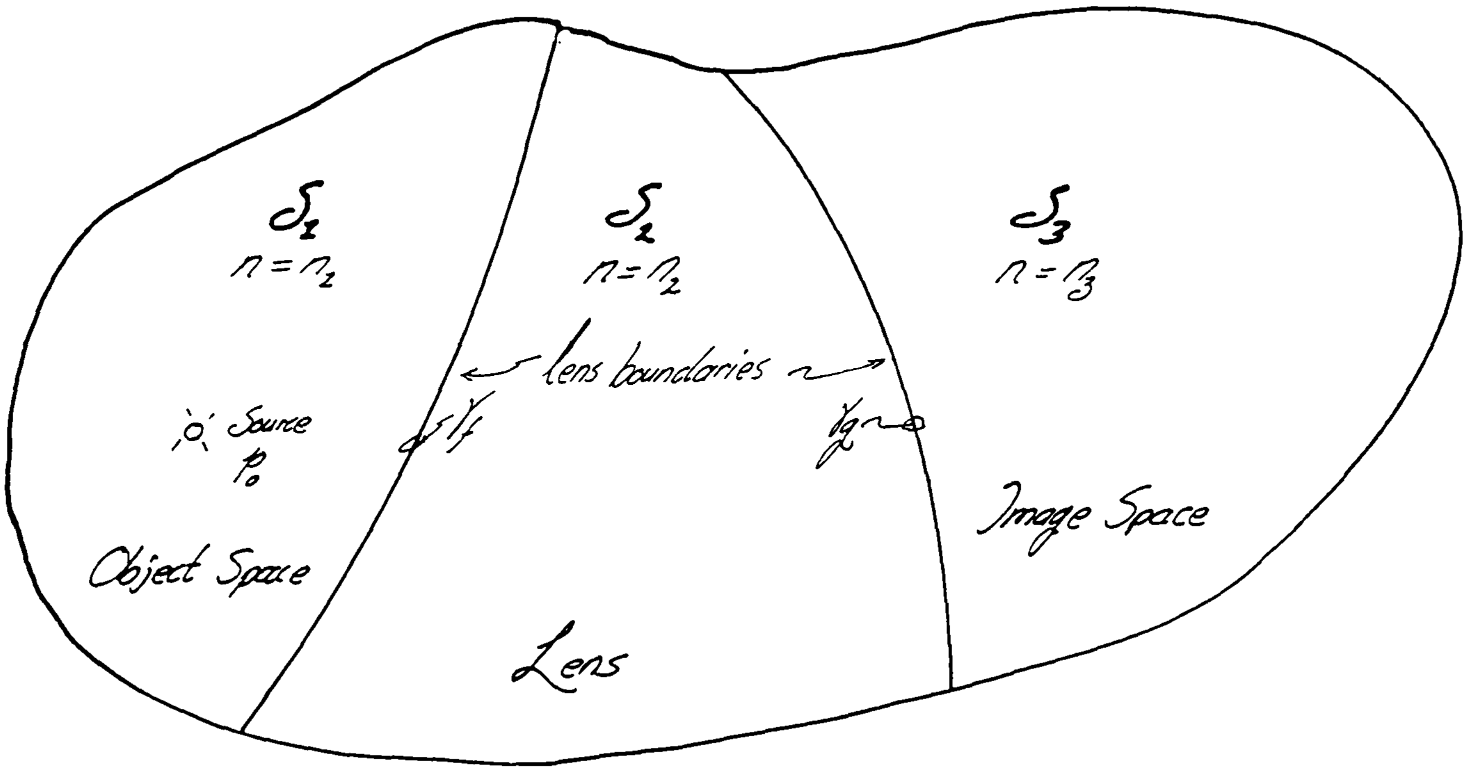


Figure 1: Optical System S

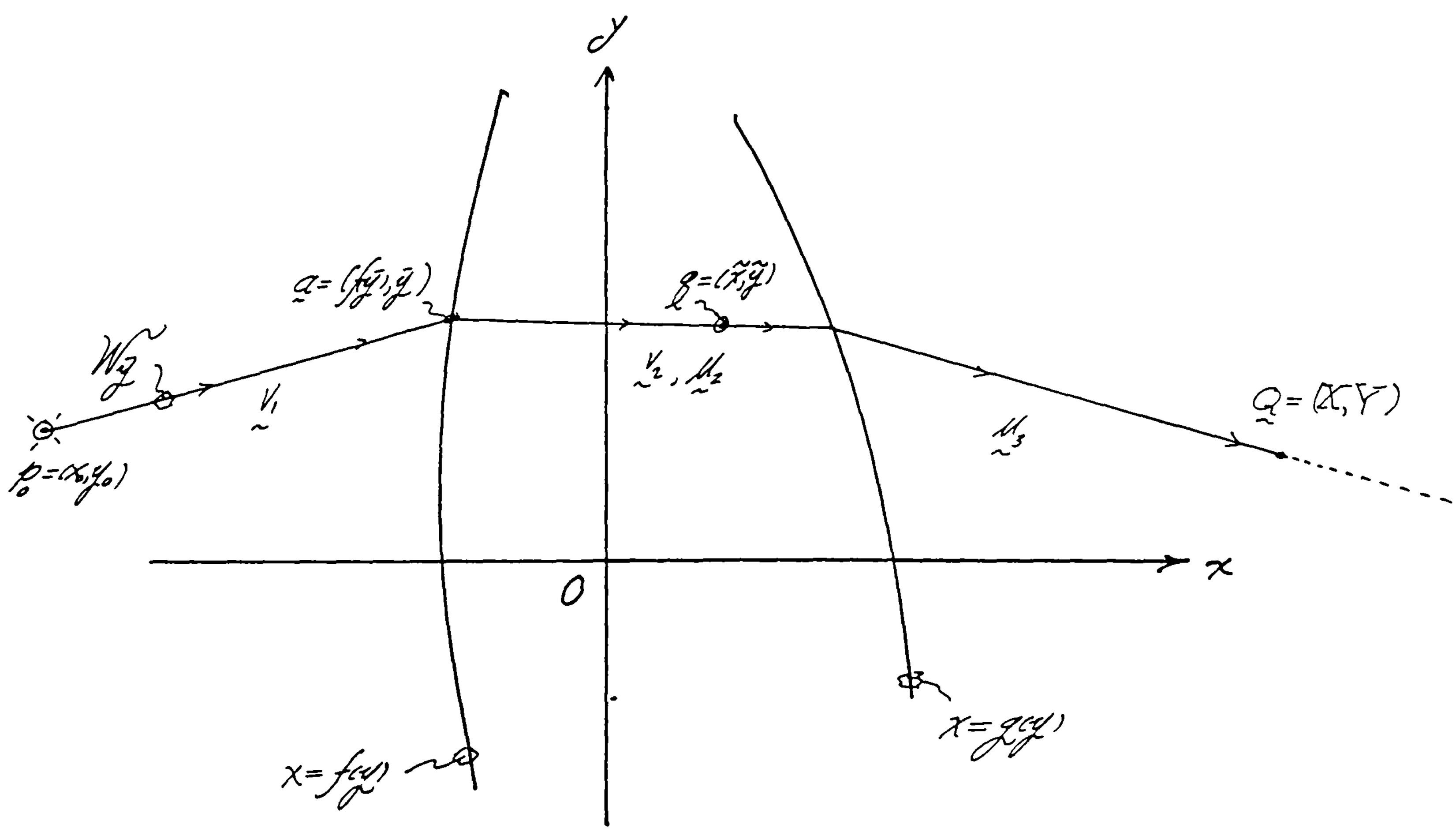


Figure 2: Optical System

<u>image space</u> is the S_i which is neither the object space nor the lens. It is assumed in this chapter, for definiteness, that only one source is present in S and that it is located at some point $\underset{\sim}{p}_0 \in S_1$. The object and image spaces are thus S_1 and S_3 respectively.

It is known from the previous chapter that light rays in homogeneous media travel in straight lines; consequently, the light rays in S must consist of line segments. Light rays radiating from $\underset{\sim}{p}_0$ may or may not intersect the lens, and those which do need not eventually refract into the image space. The light rays of interest in this and subsequent chapters are those which, leaving the object space, refract into the lens and thence refract into the image space intersecting each lens boundary exactly once. Such light rays are said to lie in the <u>field of the instrument</u>.

A point $\underset{\sim}{p}_1 \in S_3$ is a <u>focus</u> of $\underset{\sim}{p}_0 \in S_1$ if, for a given refractive index, there exist an infinite number of light rays in the field of the instrument intersecting both $\underset{\sim}{p}_0$ and $\underset{\sim}{p}_1$. The points $\underset{\sim}{p}_0$ and $\underset{\sim}{p}_1$ are refered to collectively as <u>foci</u> of the optical system. An <u>internal focus</u> of $\underset{\sim}{p}_0 \in S_1$ is any point $\underset{\sim}{q} \in \overline{S}_2$ such that an infinite number of light rays from $\underset{\sim}{p}_0$ intersect $\underset{\sim}{q}$. More general singularities formed by light rays are discussed in Section 6 of this chapter.

The lens boundaries γ_f, γ_g are assumed smooth unless otherwise noted. The image and object spaces are assumed to be in the vacuum, i.e. $n(\underset{\sim}{r})$ is of the form

$$n_1 = n_3 = 1$$

$$n_2 > 1 \ .$$

The subscript is dropped for the refractive index in the lens and, unless otherwise noted, n will always denote the refractive index in S_2.

2.2 General Ray Tracing and Lens Equations

There are two general problems associated with light ray geometry of optical interest. The ray tracing problem is the determination of the light ray configurations in an optical system given the source location and lens boundaries. The lens design problem is essentially the inverse problem, viz. the determination of lens boundaries which produce a light ray configuration having certain properties (e.g. foci). The design of most optical instruments, in practice, involves a combination of the two problems.

The general ray tracing equations are developed in this section for light rays in the field of the instrument. The equations follow directly from Fermat's Principle and are equivalent to Snell's law. The governing equations for the lens problem are then derived for light rays in an optical system having foci at specified points. A notable feature of these equations is that they are functional differential equations, i.e. they contain unknown functions having arguments which are also unknown. This system of functional differential equations will be the starting point for the investigation of various lens design problems in the chapters to follow.

Consider the optical system given in figure 2 having a source at $\underset{\sim}{p}_0 = (x_0, y_0) \in S_1$, and smooth lens boundaries described by $\underset{\sim}{\gamma}_f = (f(y), y)$, $\underset{\sim}{\gamma}_g = (g(y), y)$. Suppose γ_f and γ_g are given and let $W_{\bar{y}}$ be the light ray in the field of the instrument which intersects γ_f at some point $\underset{\sim}{a} = (f(\bar{y}), \bar{y})$. Fermat's Principle implies that for any $\underset{\sim}{q} = (\tilde{x}, \tilde{y}) \in S_2 \cup \gamma_g$ on $W_{\bar{y}}$, the optical path length from $\underset{\sim}{p}_0$ to $\underset{\sim}{q}$ is an extremum, i.e.

$$\frac{d}{dy}\{|\underset{\sim}{v}_1| + n|\underset{\sim}{v}_2|\}\Big|_{y=\bar{y}} = 0 \,, \tag{1}$$

where

$$\begin{aligned} \underset{\sim}{v}_1 &= \underset{\sim}{\gamma}_f - \underset{\sim}{p}_0 \\ \underset{\sim}{v}_2 &= \underset{\sim}{q} - \underset{\sim}{\gamma}_f \,. \end{aligned} \tag{2}$$

Equation (1), in detail, is

$$\frac{(f(\bar{y})-x_0)f'(\bar{y}) + \bar{y} - y_0}{|\underset{\sim}{v}_1(\bar{y})|} - \frac{n\{(\tilde{x}-f(\bar{y}))f'(\bar{y}) + \tilde{y} - y\}}{|\underset{\sim}{v}_2(\bar{y})|} = 0 \,, \tag{1'}$$

and, as can readily be seen, is equivalent to Snell's Law

$$\left\{\frac{v_1(\bar{y})}{|\underset{\sim}{v}_1(\bar{y})|} - n\,\frac{v_2(\bar{y})}{|v_2(\bar{y})|}\right\} \wedge \underset{\sim}{N}_f(\bar{y}) = 0 \,,$$

where $\underset{\sim}{N}_f$ is a normal to γ_f at $\underset{\sim}{a}$. Given $\bar{y}$, equation (1) describes the points $q \in S_2 \cup \gamma_g$ on $W_{\bar{y}}$ and, in particular, the point $\underset{\sim}{b} = (g(\hat{y}), \hat{y})$ at which the light ray exits the lens and refracts into the image space. Fermat's Principle can be applied again to describe the refraction of the light ray at γ_g. Let $\underset{\sim}{Q} = (X, Y)$ be some point in $S_3 \cap W_{\bar{y}}$. Fermat's Principle implies

$$\frac{d}{dy}\{n|\underset{\sim}{\mu}_2| + |\underset{\sim}{\mu}_3|\}\Big|_{y=\hat{y}} = 0 \ , \tag{4}$$

where

$$\underset{\sim}{\mu}_2 = \underset{\sim}{\gamma}_g - \underset{\sim}{a}$$
$$\underset{\sim}{\mu}_3 = \underset{\sim}{Q} - \underset{\sim}{\gamma}_g \ . \tag{5}$$

Equation (4) is equivalent to Snell's Law at γ_g and describes the points $\underset{\sim}{Q} \in S_3 \cup W_{\bar{y}}$ on the light ray exiting the lens. Equations (1) and (4) can be used to derive a one parameter family of lines parametrized by $\bar{y}$ (or $\hat{y}$) describing the light rays in the image space. Equations (1) and (4) thus provide the solution to the ray tracing problem.

The lens problem, as mentioned above, is essentially the 'inverse' of the ray tracing problem. The lens boundaries γ_f, γ_g are to be determined given certain properties of the light ray configuration. The discussion of the lens problem in this and the following chapters is restricted primarily to light ray configurations having specified foci. Let $\underset{\sim}{p}_0 = (x_0, y_0) \in S_1$, $\underset{\sim}{p}_1 = (x_1, y_1) \in S_3$ be given foci for the optical system. The object and image spaces have yet to be determined but it is assumed that there exist neighbourhoods Np_0, Np_1 around $\underset{\sim}{p}_0$ and $\underset{\sim}{p}_1$ respectively such that $Np_0 \subseteq S_1$, $Np_1 \subseteq S_3$.

Any light ray in the field of the instrument is described completely by its point of intersection with either γ_f or γ_g. Let $W(y)$ be some light ray in the field of the instrument and $(f(y),y), (g(\xi),\xi)$ be the points at which it intersects γ_g and γ_f respectively. The variables

y and ξ are not independent but, as the ray tracing problem (and Snell's Law) suggests, it is convenient nonetheless to retain both variables. Following Ockendon et al. (1985), Fermat's Principle implies that, for any $W(y)$, the optical path length between $\underset{\sim}{p}_0$ and $(g(\xi),\xi)$ is an extremum, i.e.

$$\left.\frac{\partial}{\partial y}\right|_{\xi \text{ fixed}} \{d_1 + nd_2\} = 0 \,, \tag{6}$$

where

$$\begin{aligned} d_1^2 &= (f(y) - x_0)^2 + (y - y_0)^2 \\ d_2^2 &= (g(\xi) - f(y))^2 + (\xi - y)^2 \,. \end{aligned} \tag{7}$$

Similarly, the optical path length between $(f(y),y)$ and $\underset{\sim}{p}_1$ is an extremum so that

$$\left.\frac{\partial}{\partial \xi}\right|_{y \text{ fixed}} \{nd_2 + d_3\} = 0 \,, \tag{8}$$

where

$$d_3^2 = (x - g(\xi))^2 + (y_1 - \xi)^2 \,. \tag{9}$$

As with the ray tracing problem, equations (6) and (8) are equivalent to Snell's Law at γ_f and γ_g respectively. Since ξ and y are dependent variables, the partial derivative in (6) signifies that ξ and consequently $g(\xi)$ are held fixed. The partial derivative in equation (8) means that y and $f(y)$ are held fixed. The qualifications 'ξ fixed', 'y fixed' on the partial derivatives will be dropped to simplify notation unless there is some possibility of confusion.

Using the optical path length function

$$V \triangleq d_1 + nd_2 + d_3 \ , \tag{10}$$

equations (6) and (8) can be written in the compact form

$$\frac{\partial V}{\partial y} = 0 \tag{11}$$

$$\frac{\partial V}{\partial \xi} = 0 \ . \tag{12}$$

If y is regarded as the independent variable and ξ as some function of y, equations (11) and (12) imply

$$\frac{d}{dy} V = \frac{\partial}{\partial y} V + \xi'(y) \frac{\partial V}{\partial \xi} = 0 \ , \tag{13}$$

which gives the so-called 'path length condition' for a lens with foci, i.e.,

$$V = \text{const.} \ . \tag{14}$$

This relation is not independent of (11) and (12) but could be used instead of either one.

The lens boundaries for any optical system having foci at $\underset{\sim}{p}_0$ and $\underset{\sim}{p}_1$ must satisfy equations (11) and (12). In detail, these equations are

$$\frac{(f(y)-x_0)f'(y) + y-y_0}{\{(f(y)-x_0)^2 + (y-y_0)^2\}^{\frac{1}{2}}} - \frac{n\{(g(\xi)-f(y))f'(y) + \xi-y}{\{(g(\xi)-f(y))^2 + (\xi-y)^2\}^{\frac{1}{2}}} = 0 \ , \tag{11'}$$

$$\frac{n\{(g(\xi)-f(y))g'(\xi) + \xi-y\}}{\{(g(\xi)-f(y))^2 + (\xi-y)^2\}^{\frac{1}{2}}} - \frac{(x-g(\xi))g'(\xi) + y_1-\xi}{\{(x_1-g(\xi))^2 + (y_1-\xi)^2\}^{\frac{1}{2}}} = 0 \ , \tag{12'}$$

the ' denoting differentiation with respect to the indicated argument. The function $\xi(y)$ is unknown and occurs as an argument in the unknown function g and its derivative; equations (11) and (12) thus form a system of first order nonlinear functional differential equations for the lens boundaries.

The 'initial value problem', in analogy with ordinary differential equations, is the determination of an $f(y)$, $g(y)$ and $\xi(y)$ satisfying equations (11) and (12) given some 'initial light ray', i.e. given explicitly or through symmetry the values $f(y^*)$, $g(\xi(y^*))$, and $\xi(y^*)$ for some y^*. As equations (11) and (12) contain three unknown functions, it is evident that the initial value problem for f,g,ξ does not have a unique solution and probable that further restrictions (e.g. symmetry, multiple foci etc.) can be imposed on the optical system. The non-uniqueness of solutions is discussed in the next section for a special case.

2.3 The Axial Lens

An <u>axial lens</u> is defined as a lens having boundaries γ_f, γ_g such that they are symmetric with respect to a common axis, called the <u>optical axis</u>, and the optical system with these boundaries has foci at $\underset{\sim}{p}_0, \underset{\sim}{p}_1$ on the optical axis. In this and subsequent chapters, the cartesian axes are assumed oriented such that the x-axis is the optical axis. Thus, if the lens boundaries are represented parametrically by $\gamma_f = (f(y),y)$, $\gamma_g = (g(y),y)$, then the lens is an axial lens if f and g are even in y and $y_0 = y_1 = 0$ (see figure 3). An immediate consequence of the axial symmetry is that ξ must be odd in y.

The lens problem addressed in this section is the determination of a γ_f, γ_g satisfying equations (11) and (12) given the lens thickness along the optical axis, i.e. given

$$\begin{aligned} f(0) &= a_0 \\ g(0) &= b_0 \ . \end{aligned} \tag{15}$$

The symmetry of the optical system and equations (11) and (12) evaluated at y = 0 imply that there exists a light ray between $\underset{\sim}{p}_0 = (x_0, 0)$ and $\underset{\sim}{p}_1 = (x_1, 0)$ along the optical axis so that, in particular,

$$\xi(0) = 0 \ .$$

The above problem will be referred to as the <u>axial lens problem</u>. It is assumed, unless otherwise noted, that

$$x_0 < a_0 < 0 < b_0 < x_1 \ . \tag{17}$$

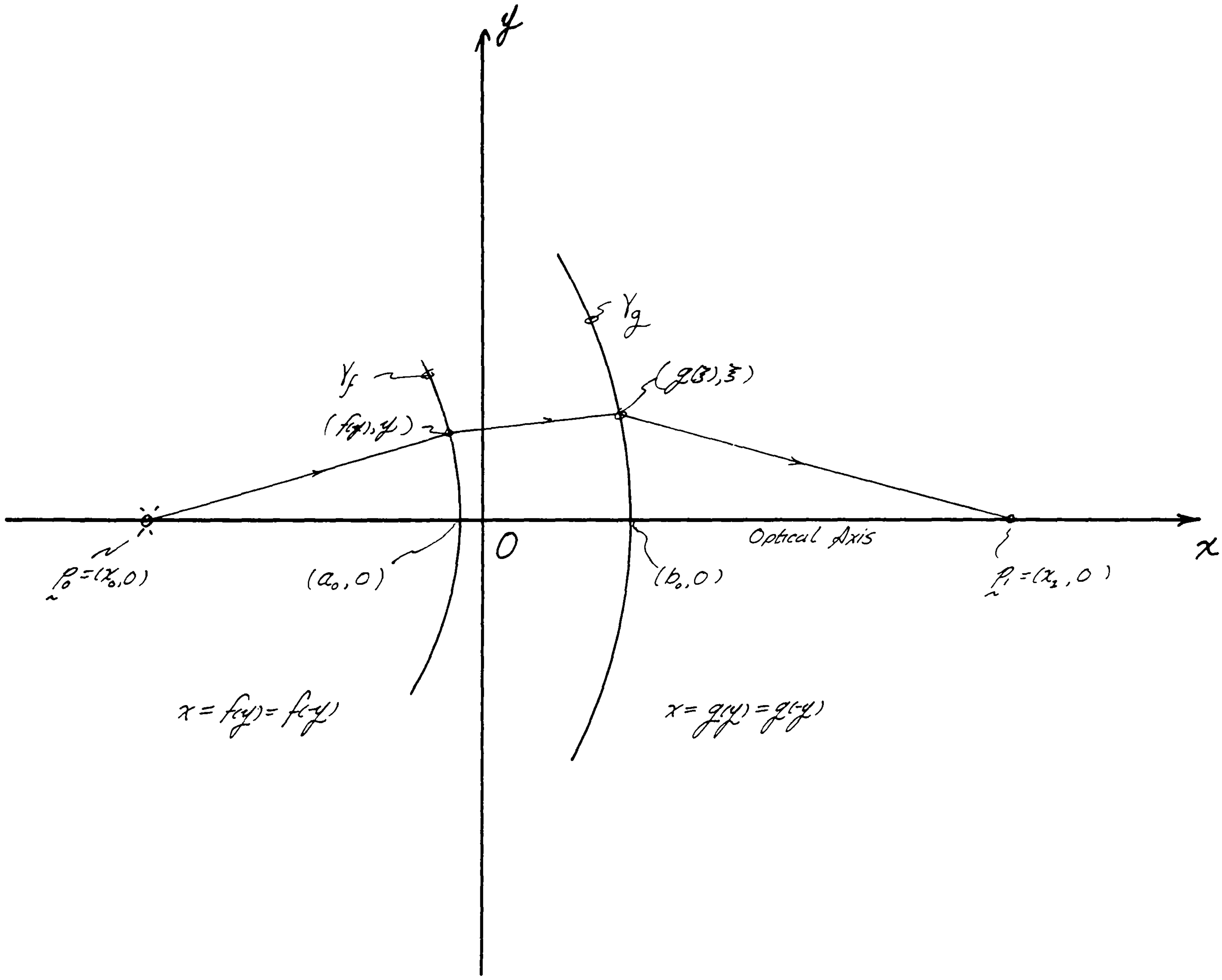


Figure 3: Axial Lens

The axial lens problem is central in lens design as most lenses for optical instruments are required to have boundaries which are surfaces of revolution. This symmetry is desirable for several reasons, e.g. symmetric imaging properties, relative simplicity of manufacture etc.. Of the three lens problems discussed in the following chapters, two are special axial lens problems, and a certain axial lens, the aplanatic lens, plays a major role in understanding the third lens problem which involves an optical system having foci off the optical axis. In this section, some of the general properties of axial lenses relevant to these problems are discussed.

The symmetry conditions for an axial lens imposed on an optical system in effect define an initial light ray but do not resolve the non-uniqueness problem discussed in the previous section. The next theorem shows that $\xi(y)$ can be an arbitrary odd function in y whose derivative does not vanish, and this in turn implies that the axial lens problem has an infinite number of solutions satisfying the initial conditions. Other types of relations can be used to establish this result (e.g. a relation involving only $g(\xi)$ and $f(y)$). The following theorem is useful not only because it implies an infinite number of solutions exist to the general axial lens problem, but also because it gives a general existence/uniqueness proof for special axial lenses such as the Herschel and aplanatic lenses.

Theorem 1

Suppose

$$\xi(y) = H(y,f(y),g(\xi)) \ , \tag{18}$$

where H is some specified C^2 function of $y,f(y),g(\xi)$ defined in a neighbourhood of $(0,f(0),g(0))$ such that

$$\left.\frac{\partial H}{\partial y}\right|_{(0,f(0),g(0))} \neq 0 \ . \tag{19}$$

There exists a unique local solution $(f(y),g(y),\xi(y))$ to the axial lens problem satisfying initial conditions (15) and equation (18). If H is analytic in $y,f(y),g(\xi)$, then the local solution is analytic.

Proof:

The axial lens equations can be written

$$\begin{aligned} f'(y) &= Z(y,\xi,f(y),g(\xi)) \\ g'(\xi) &= W(y,\xi,f(y),g(\xi)) \ , \end{aligned} \tag{20}$$

where Z and W are functions analytic in $y,\xi,f(y),g(\xi)$ in some neighbourhood of $(0,0,a_0,b_0)$. Equation (18) implies

$$\begin{aligned} \xi'(y) &= \frac{\dfrac{\partial H}{\partial y} + f'(y)\,\dfrac{\partial H}{\partial f}}{1 - g'(\xi)\,\dfrac{\partial H}{\partial g}} \\ &\triangleq E(y,\xi,f(y),g(\xi)) \ . \end{aligned} \tag{21}$$

Relations (20) and (21) can be used to formulate the following system:

$$f'(y) = \tilde{Z}(y,\xi(y),f(y),h(y))$$

$$h'(y) = \tilde{W}(y,\xi(y),f(y),h(y)) \tag{22}$$

$$\xi'(y) = \tilde{E}(y,\xi(y),f(y),h(y)) ,$$

where

$$h(y) = g(\xi(y)) . \tag{23}$$

Since E has continuous partial derivatives, the right hand side of the above system is Lipschitz continuous in some neighbourhood of $(0,0,a_0,b_0)$. The Picard-Lindelöf theorem implies that a local solution $(f(y),h(y),\xi(y))$ exists. Relation (19) implies that ξ has an inverse for y small so that equation (23) can be solved for $g(y)$. If H is analytic in $y,f(y),g(\xi)$, then $\tilde{Z}$, $\tilde{W}$, and $\tilde{E}$ are analytic in $y,\xi,f(y),h(y)$, and Cauchy's theorem implies that there exists a unique solution for y small and that it is analytic.

□

Although an infinite number of solutions exist for the axial lens problem, some general solution properties can nevertheless be gleaned directly from equations (11) and (12). As these properties will be used in subsequent chapters, it is convenient to list them as theorems, corollaries etc. for future reference.

Suppose f,g,ξ is a solution of the axial lens problem for

$$y \in B_{\delta_1} \triangleq \{y : |y| < \delta_1\} , \tag{24}$$

where every light ray intersecting $(f(B_{\delta_1}), B_{\delta_1})$ lies in the field of the instrument, and let

$$B_{\delta_2} \stackrel{\Delta}{=} \{\xi(y) : y \in B_{\delta_1}\} . \qquad (25)$$

Theorem 2: (Smoothness and Invertibility of $\xi(y)$)

If $f \in C^k(B_{\delta_1})$, $g \in C^k(B_{\delta_2})$ for $k \geq 1$, then $\xi \in C^{k-1}(B_{\delta_1})$. Moreover, if $k \geq 2$ then $\xi'(y) \neq 0$ for all $y \in B_{\delta_1}$.

Proof:

Since $f \in C^1(B_{\delta_1})$, the light rays in $\overline{S}_2$ can be represented by a one parameter family of lines continuously dependent on the parameter y. These lines intersect $g \in C^1(B_{\delta_2})$ transversely by hypothesis so that the function $\xi(y)$ must be a continuous function of y. Equation (11) implies

$$\xi'(y) = -\frac{\partial^2 V}{\partial y^2} \Big/ \frac{\partial^2 V}{\partial y \partial \xi} , \qquad (26)$$

so that, for $k \geq 2$, the derivative of ξ at $\bar{y}$ exists provided

$$\lim_{y \to \bar{y}} \left| \frac{\partial^2 V}{\partial y^2} \middle| \frac{\partial^2 V}{\partial Y \partial \xi} \right| < \infty . \qquad (27)$$

The partial derivatives in equation (26) can be written

$$\begin{aligned}\frac{\partial^2 V}{\partial y^2} = {} & f''(y)\left\{\frac{f(y)-x_0}{d_1} - \frac{n(g(\xi)-f(y))}{d_2}\right\} \\ & + \frac{n}{d_2^3}\{(g(\xi)-f(y)) - f'(y)(\xi-y)\}^2 \qquad (28) \\ & + \frac{1}{d_1^3}\{(f(y)-x_0) - f'(y)y\}^2 ,\end{aligned}$$

$$\frac{\partial^2 V}{\partial y \partial \xi} = \frac{-n}{d_2^2}\{(g(\xi)-f(y))-f'(y)(\xi-y)\}\{(g(\xi)-f(y))-g'(\xi)(\xi-y)\} , \qquad (29)$$

and it is readily seen that both derivatives are finite functions continuous in y. The partial derivative given by equation (29) is non-zero since the light rays, by hypothesis, are in the field of the instrument (in particular, the lens does not close and no light rays are tangent to the lens boundaries for $y \in B_{\delta_1}$); consequently, $\xi(y) \in C'(B_{\delta_1})$. For $k > 2$, the result follows by repeated differentiation of equation (26) and induction.

Equation (12) implies

$$\frac{\partial^2 V}{\partial \xi^2}\, \xi'(y) + \frac{\partial^2 V}{\partial \xi \partial y} = 0 \ , \tag{30}$$

and from the above discussion

$$\frac{\partial^2 V}{\partial y \partial \xi} < 0 \ . \tag{31}$$

Since

$$\begin{aligned} \frac{\partial^2 V}{\partial \xi^2} = g''(\xi)\left\{\frac{n(g(\xi)-f(y))}{d_2} - \frac{x_1-g(\xi)}{d_3}\right\} \\ + \frac{n}{d_2^3}\{(g(\xi)-f(y)) - g'(\xi)(\xi-y)\}^2 \\ + \frac{1}{d_3^3}\{(x_1-g(\xi)) + g'(\xi)\xi\}^2 \ , \end{aligned} \tag{32}$$

$\frac{\partial^2 V}{\partial \xi^2}$ is finite for $y \in B_{\delta_1}$ and $\xi'(y)$ cannot vanish for $y \in B_{\delta_1}$.

□

Corollary 3.

If $f \in C^{\omega}(B_{\delta_1})$, $g \in C^{\omega}(B_{\delta_2})$, then $\xi \in C^{\omega}(B_{\delta_1})$.

Proof:

Equation (26) implies

$$\xi'(y) = E(y,\xi) \ , \tag{33}$$

where E is a function analytic in y and ξ. Cauchy's theorem implies that $\xi(y)$ must be analytic.

□

If f and g are smooth, Theorem 2 means that $\xi(y)$ defines a diffeomorphism from γ_f to γ_g, and the optical path length function is smooth. The invertible property of $\xi(y)$ is physically obvious from the geometry of light rays since $\underset{\sim}{p}_1$ can be interpreted as the source and $\underset{\sim}{p}_0$ as the focus.

Theorem 2 implies that $\xi(y)$ is strictly monotonic, and axial lenses, consequently, can be classified according to the sign of $\xi'(y)$. Let A be the set of axial lenses with smooth boundaries, and

$$\begin{aligned} A^+ &\stackrel{\Delta}{=} \{S_2 \in A : \xi'(y) > 0\} \\ A^- &\stackrel{\Delta}{=} \{S_2 \in A : \xi'(y) < 0\} . \end{aligned} \tag{34}$$

There is a fundamental difference between the light ray configurations induced by lenses in A^+ and those induced by lenses in A^-. If $S_2 \in A^+$, then no light rays in the field of the instrument intersect the optical axis in $\overline{S}_2$ except the light ray along the axis. If $S_2 \in A^-$, then every light ray in the field of the instrument intersects the optical axis in S_2. In terms of differential geometry, the light ray configurations differ in that the configuration induced by the lens in A^- has a conjugate point in S_2. Jacobi's theorem for geodesics on a smooth surface [cf. Willmore, 1959] suggests the following corollary:

Corollary 4

The optical path length between $\underset{\sim}{p}_0$ and $\underset{\sim}{p}_1$ is a local minimum if $S_2 \in A^+$. If $S_2 \in A^-$, the optical path length is a local maximum.

Proof:

If $S_2 \in A^+$ then for all $y \in B_{\delta_1}$

$$\frac{\partial^2 V}{\partial y^2} > 0 \; ; \tag{35}$$

consequently, the optical path length between $\underset{\sim}{p}_0$ and any point on $(g(B_{\delta_2}), B_{\delta_2})$ must be a minimum. Since no conjugate points occur on γ_g or in the image space, the optical path length between $\underset{\sim}{p}_0$ and $\underset{\sim}{p}_1$ represents a local minimum.

If $S_2 \in A^-$, then

$$\frac{\partial^2 V}{\partial y^2} < 0 \; , \tag{36}$$

and the optical path length between $\underset{\sim}{p}_0$ and any point on $(g(B_{\delta_2}), B_{\delta_2})$ is a local maximum. As no conjugate points occur on γ_g or in the image space, the optical path length between $\underset{\sim}{p}_0$ and $\underset{\sim}{p}_1$ is a local maximum.

$\square$

Even though there exist an infinite number of solutions to the axial lens problem, some information concerning lens shapes can still be deduced. In particular, if $S \in A^-$ the following theorem shows that f and g must have monotonic derivatives of opposite sign, i.e. the lens must be 'double convex'.

Theorem 5

If $S_2 \in A$, then at least one of the following inequalities is true for $y \in B_{\delta_1}$, $y > 0$

$$f'(y) > 0 \tag{37}$$

$$g'(\xi) < 0 \ . \tag{38}$$

If $S_2 \in A^-$ then both inequalities are satisfied for all $y \in B_{\delta_1}$, $y > 0$ and, in addition, for all $y \in B_{\delta_1}$

$$f''(y) > 0 \tag{39}$$

$$g''(y) < 0 \ . \tag{40}$$

Proof:

Equations (11) and (12) imply

$$f'(y)L_1 = \frac{n(\xi - y)}{d_2} - \frac{y}{d_1} \tag{41}$$

$$g'(\xi)L_2 = \frac{-n(\xi - y)}{d_2} - \frac{\xi}{d_3} \ , \tag{42}$$

where

$$L_1 \triangleq \frac{f(y) - x_0}{d_1} - \frac{n\{g(\xi) - f(y)\}}{d_2} \tag{43}$$

$$L_2 \triangleq \frac{n\{g(\xi) - f(y)\}}{d_2} - \frac{x_1 - g(\xi)}{d_3} \ . \tag{44}$$

Since $n > 1$ for all $y \in B_{\delta_1}$

$$\begin{aligned} L_2 &< 0 \\ L_2 &> 0 \ , \end{aligned} \tag{45}$$

and the first part of the theorem follows immediately from equations (41) and (42). If $S_2 \in A^-$, then for $y > 0$, ξ must be negative and therefore inequalities (37) and (38) must both be satisfied. In addition, since $\xi' < 0$ for all $y \in B_{\delta_1}$, equations (28)-(32) and inequality (36) give the inequalities

$$f''(y)L_1 < 0$$
$$g''(\xi)L_2 < 0 \, , \tag{46}$$

which yield (39) and (40).

□

If $S_2 \in A^-$, the above theorem implies that $f(y)$ $(g(y))$ is strictly monotonic increasing (decreasing) for $y > 0$, and $f'(y)$ $(g'(y))$ is strictly monotonic increasing (decreasing) for $y \in B_{\delta_1}$. If $S_2 \in A^+$, no similar statement can be made, and examples of lenses can be concocted having lens boundaries which do not have strictly monotonic derivatives.

Given two lenses $S_2^+ \in A^+$, $S_2^- \in A^-$ having the same foci and satisfying the same initial conditions, the next theorem shows that the lens boundaries of S_2^- must lie between those of S_2^+ intersecting only at the optical axis.

Theorem 6

If $S_2^+ \in A^+$ and $S_2^- \in A^-$ satisfy the same axial lens problem then for $y \in B_{\delta_1}$, $y > 0$

$$f_+(y) < f_-(y)$$
$$g_+(y) > g_-(y) \tag{47}$$

$$f_+'(y) < f_-'(y)$$
$$g_+'(y) > g_-'(y) \, , \tag{48}$$

where $f_+(y), g_+(y)$ describe the lens boundaries of S_2^+ and $f_-(y), g_-(y)$ describe those of S_2^-.

Proof:

Theorem 2 and equations (28) and (32) imply

$$\begin{aligned} f_+''(0) &< f_-''(0) \\ g_+''(0) &> g_-''(0) \, , \end{aligned} \tag{49}$$

so that relations (47) are valid for y sufficiently small. Suppose there exists a $y \in B_{\delta_1}$, $y > 0$ such that (47) is false. There must then exist a point $\hat{y}$ such that

$$\begin{aligned} f_+(\hat{y}) &= f_-(\hat{y}) \\ f'(\hat{y}) &\geq f_-'(\hat{y}) \, . \end{aligned} \tag{50}$$

Relations (50) imply that the light ray entering S_2^+ at $(f(\hat{y}), \hat{y})$ must intersect the optical axis in S_2^+ which contradicts the hypothesis. The other inequality in (47) can be proved in a similar manner.

Inequalities (49) imply that (48) is valid for y sufficiently small. Suppose there exists a point $\hat{y} \in B_{\delta_1}$, $\hat{y} > 0$ such that

$$f_+'(\hat{y}) \geq f_-'(\hat{y}) \, . \tag{51}$$

Inequalities (47) and (51) imply

$$\frac{(f_+(y)-x_0)f_+'(y)+y}{\{(f_+(y)-x_0)^2+y^2\}^{\frac{1}{2}}} > \frac{(f_-(y)-x_0)f_-'(y)+y}{\{(f_-(y)-x_0)^2+y^2\}^{\frac{1}{2}}} \tag{52}$$

which, using equation (11) implies that $S_2^+ \in A^-$ contradicting the hypothesis. The other inequality in (48) can be proved in a similar manner.

$\square$

The aperture of a lens (i.e. maximum y for which a solution is valid) is often an important design consideration [cf. Sternberg, 1963]. Theorem 6 implies that a convex solution in A^+ has an aperture at least as large as any solution in A^-.

2.4 The Paraxial Approximation

The properties of axial lenses discussed in the previous section with the exception of Theorem 1 were global in character, and no assumptions were made regarding the size of δ_1 defining the neighbourhood B_{δ_1}. In this section, the axial lens problem is examined for y sufficiently small, and relations are derived for an approximate solution.

Suppose δ_1 in relation (24) is small compared to the foci distances from the lens boundaries, i.e. for $O(1)$ distances $a_0 - x_0$, $x_1 - b_1$,

$$\delta_1 << 1 .$$

The lens boundaries are assumed smooth, and Theorem 2 implies that, for δ_1 small, f, g, and ξ can be written

$$f(y) = a_0 + a_2y^2 + O(\delta_1^4)$$

$$g(y) = b_0 + b_2y^2 + O(\delta_1^4) \qquad (53)$$

$$\xi(y) = c_1y + O(\delta_1^3) ,$$

where a_2, b_2, and c_1 are constants. The functions

$$f_p(y) = a_0 + a_2 y^2$$

$$g_p(y) = b_0 + b_2 y^2 \tag{54}$$

$$\xi_p(y) = c_1 y$$

form what is known as the <u>paraxial</u> or <u>Gaussian optics</u> solution to the axial lens problem. The strip along the axis

$$P \stackrel{\Delta}{=} \{(s,y) : x \in \mathbb{R}, y \in B_{\delta_1}\} \tag{55}$$

is referred to as a <u>paraxial region</u>.

Since the functions $f(y)$, $g(y)$ are even in y, it is clear that for δ_1 small, light rays entering the lens at $(f(y),y)$, $y \in B_{\delta_1}$ are in the field of the instrument. Theorem 2 indicates that $\xi(y)$ is $O(y)$, and substituting relations (53) into equations (11) and (12) and equating $O(\delta_1)$ terms gives the following relations for the paraxial coefficients:

$$-2(n-1)a_2 \ell_2 \ell_1 + \ell_2 + n\ell_1 = n\ell_1 c_1 \tag{56}$$

$$2(n-1)b_2 \ell_2 \ell_3 + \ell_2 + n\ell_3 = n\ell_3 / c_1 , \tag{57}$$

where

$$\ell_1 = a_0 - x_0$$

$$\ell_2 = b_0 - a_0 \tag{58}$$

$$\ell_3 = x_1 - b_0 .$$

Aside from the obvious practical value of the paraxial solution as an approximation, it is also useful as a guide to indicate non-uniqueness or non-existence of solutions to various axial lens problems. As equations (56) and (57) contain three unknown quantities, it is evident that the axial lens problem without further restrictions has an infinite number of paraxial solutions (as expected from Theorem 1). The existence and uniqueness of the paraxial solution subject to various conditions is considered in subsequent chapters, and it forms the starting point for local existence/uniqueness investigations.

It is clear from equations (56), (57) and Theorem 2 that the paraxial solution determines whether or not the lens is in A^+ or A^-. In particular, the sign of c_1 determines membership in A^+ or A^-. Theorem 2 implies

$$\begin{aligned} a_2 &\neq \frac{1}{2} c_f \triangleq \frac{1}{2(n-1)} \{\frac{n}{\ell_2} + \frac{1}{\ell_1}\} \\ b_2 &\neq \frac{1}{2} c_g \triangleq \frac{-1}{2(n-1)} \{\frac{n}{\ell_2} + \frac{1}{\ell_3}\} , \end{aligned} \tag{59}$$

and it is evident from equations (56) and (57) that if $S_2 \in A$

$$a_2 = \frac{1}{2} c_f + (\frac{n}{n-1})^2 \{\frac{1}{2c_g - b_2}\} . \tag{60}$$

If $S_2 \in A^-$, then

$$\begin{aligned} a_2 &> \frac{1}{2} c_f \\ b_2 &< \frac{1}{2} c_g , \end{aligned} \tag{61}$$

and if it is in A^+, the above inequalities are reversed. Equation (60) implies that a_2 and b_2 lie on a branch of a hyperbola with asymptotes given by $\frac{1}{2}c_f$ and $\frac{1}{2}c_g$. The branch of the hyperbola characterizes the lens in A^+ or A^-. The characterization is illustrated in figure 4. Equation (60) implies that axial lenses may have paraxial lens shapes which are 'meniscus', 'plano-convex', or 'double convex' and that no axial lens has 'divergent' or 'planar' paraxial lens shapes (figure 5).

Owing to its relative simplicity and usefulness, paraxial optics is perhaps the most developed and widely used branch of geometrical optics. Paraxial optics is essentially the linearisation of Snell's law for light rays in a paraxial region, and the study of it in 2-D is equivalent to the study of the group $Sl(2;\mathbb{R})$ of 2×2 real matrices of determinant 1 which is isomorphic to the symplectic group $S_p(2;\mathbb{R})$. Classical accounts of paraxial optics can be found in most optics texts [e.g. Born & Wolf, 1986; Luneberg, 1944]. The group theoretic aspects of paraxial optics (and geometrical optics in general) are discussed in Dragt et al. (1986), Guillemin & Sternberg (1984), and Hertzberger (1943).

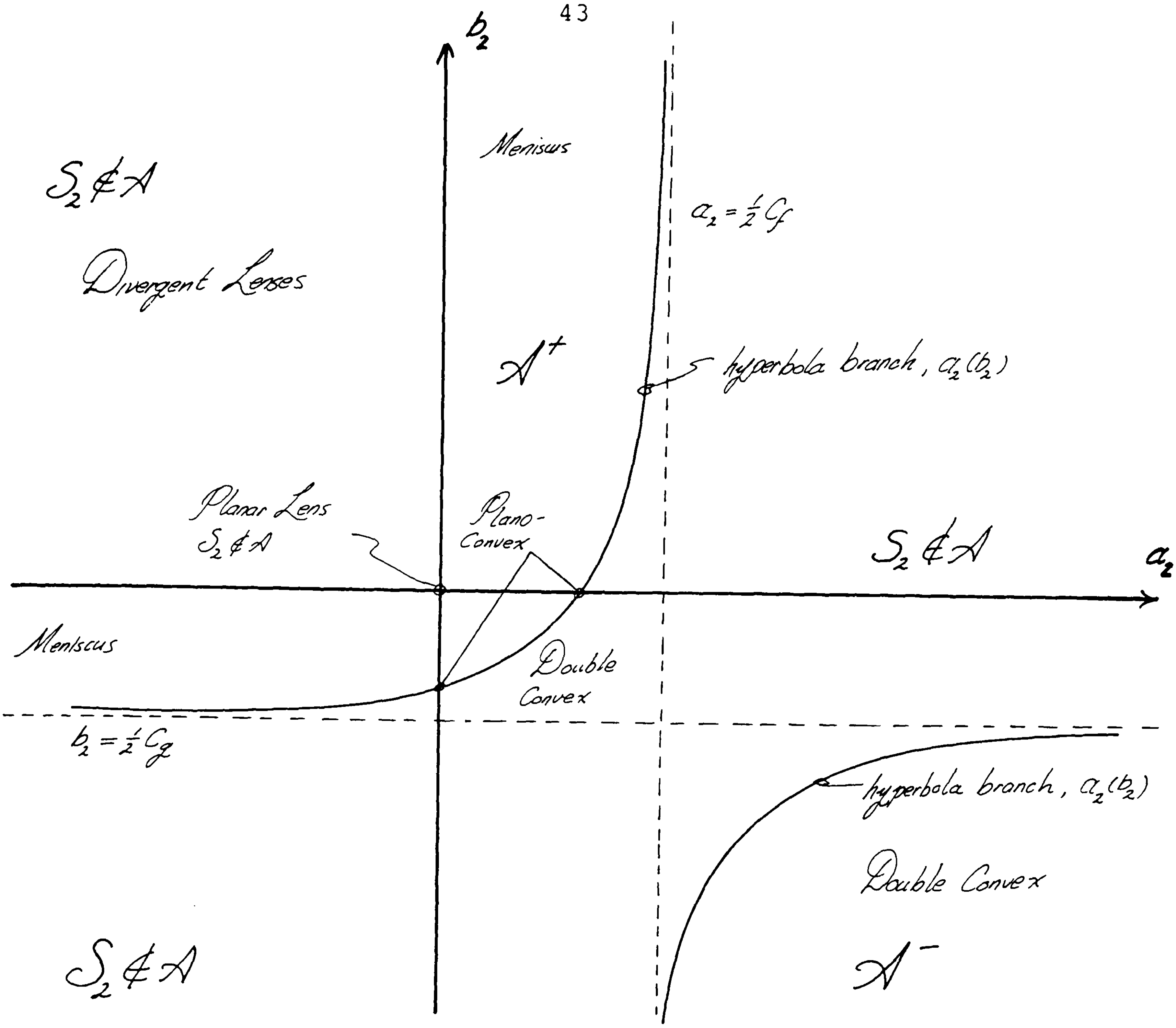
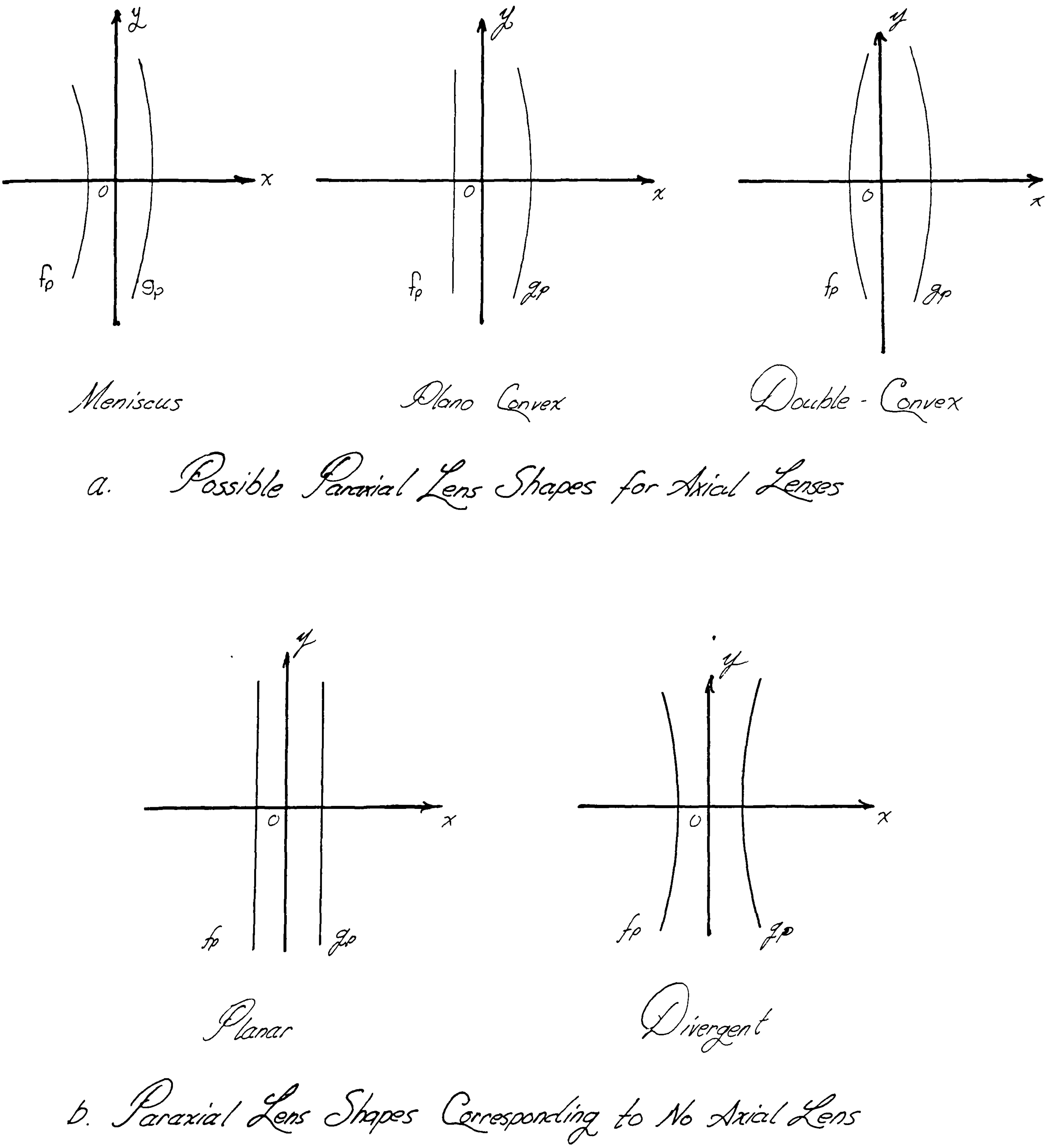


Figure 4: Characterization of Lens by a_2 and b_2



<u>Figure 5</u>: Paraxial Lens Shapes

2.5 The Symmetric Lens with Symmetric Foci and Cartesian Ovals

An axial lens is termed <u>symmetric</u> if the lens boundaries γ_f, γ_g are reflections of each other through the y axis, i.e. if

$$f(y) = -g(y) \ . \tag{62}$$

In this section, a particularly simple example of a symmetric lens is studied to illustrate some of the concepts of Sections 3 and 4 and serve as an example in subsequent chapters.

Let $S_2 \in A$ be a symmetric lens and suppose the foci for the optical system are symmetric with respect to the y axis. Since $a_2 = b_2$, $\ell_1 = \ell_3$, the paraxial solution can readily be calculated from equations (56) and (57) giving the relations

$$\begin{aligned} &2(n-1)b_2\ell_1\ell_2 + \ell_2 + n\ell_1 = n\ell_1 c_1 \\ &c_1^2 = 1 \ . \end{aligned} \tag{63}$$

The paraxial solution this suggests that two solutions to the full problem exist, one in A^+ and the other in A^-. In fact, it is evident from symmetry that either

$$\xi(y) = y \tag{64}$$

or

$$\xi(y) = -y \ . \tag{64'}$$

Differential equations (11) and (12) can be used to solve the full problem [cf. Friedman & McLeod 1987] but, for this special case, it is easier to solve the problem algebraically using the path length condition

$$2d_1 + nd_2 = 2\ell_1 + n\ell_2 \triangleq K_0 \ . \qquad (65)$$

If the lens is in A^+, then equation (64) and the path length condition give the quadratic equation

$$f^2(y)(n^2-1) + f(y)(2x_0-k_0 n) + k_0^2/4 - y^2 - x_0^2 = 0 ,$$

so that

$$f(y) = \frac{(x_0 + a_0 n) - \{(x_0-a_0)^2 + \frac{n+1}{n-1}\, y^2\}^{\frac{1}{2}}}{n+1} \ . \qquad (66)$$

The sign in front of the radical is determined by the initial condition and the solution is unique provided it is stipulated that the lens is in A^+. The solution represents a section of a hyperbola and is illustrated in figure 6a.

Suppose the solution is required to be in A^-. It is clear from the symmetry of the problem that an internal focus must exist at the origin and that the problem is essentially the determination of a single curve γ_g such that light rays leaving the origin focus at $(x_1, 0)$. The solution again can be obtained algebraically and the curve γ_g is a section of a fourth degree curve known as a Cartesian oval. Specifically, the path length condition and equation (64') imply

$$K_0^4-2K_0^2x_1^2+x_1^4+4x_1^2 g(y)\{K_0^2-x_1^2\}+g^2(y)\{4x_1^2-2(K_0^2(n^2+1)+x_1^2(n^2-1)\}$$
$$-2y^2\{K_0^2(n^2+1)+x_1^2(n^2-1)\} + 4(n^2-1)x_1 g^3(y)+4(n^2-1)x_1 g(y)y^2$$
$$+(n^2-1)^2 g^4(y)+2(n^2-1)g^2(y)y^2+(n^2-1)^1 y^4 = 0 , \qquad (67)$$

where

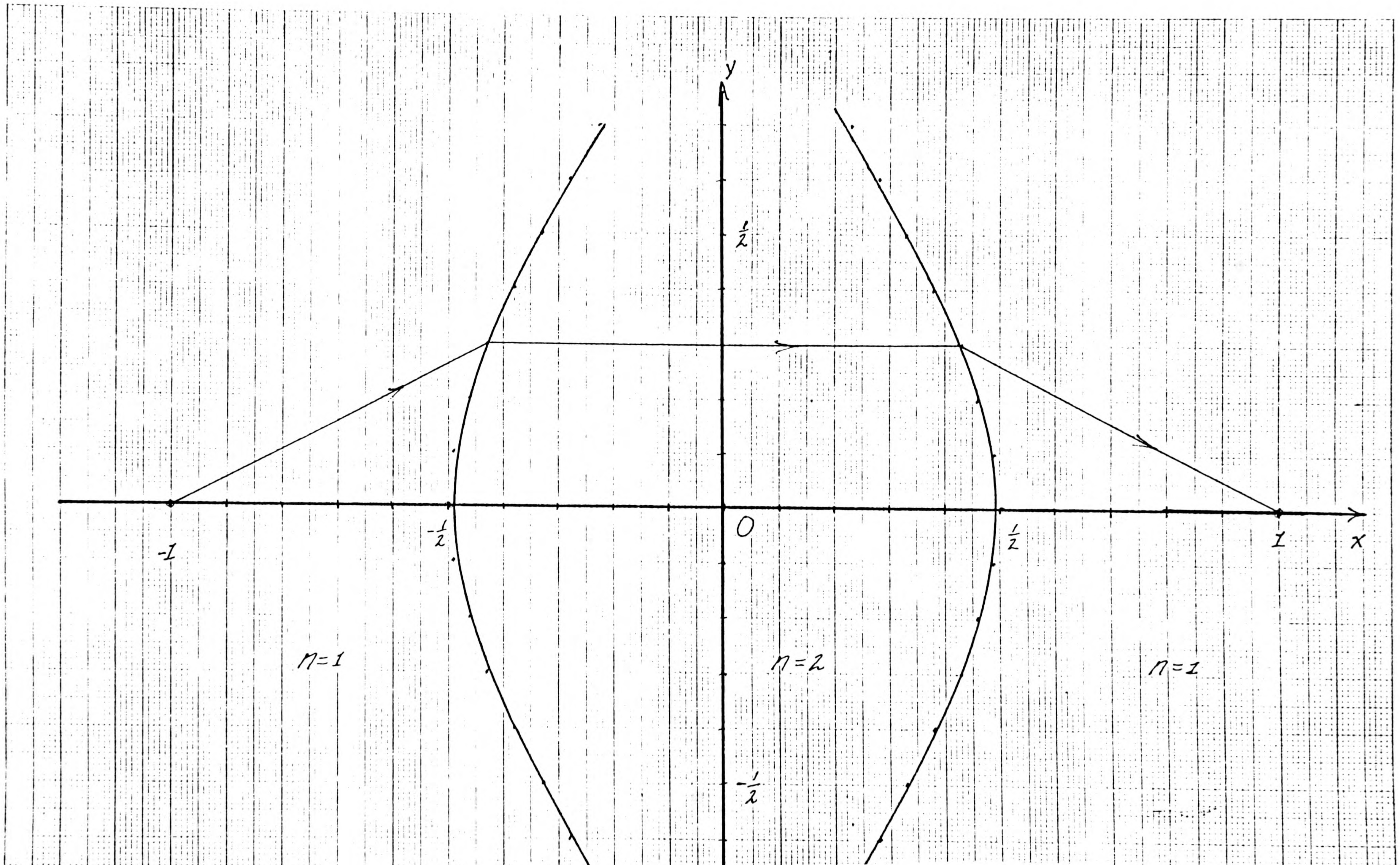


Figure 6a: Symmetric Lens Solution in A^+

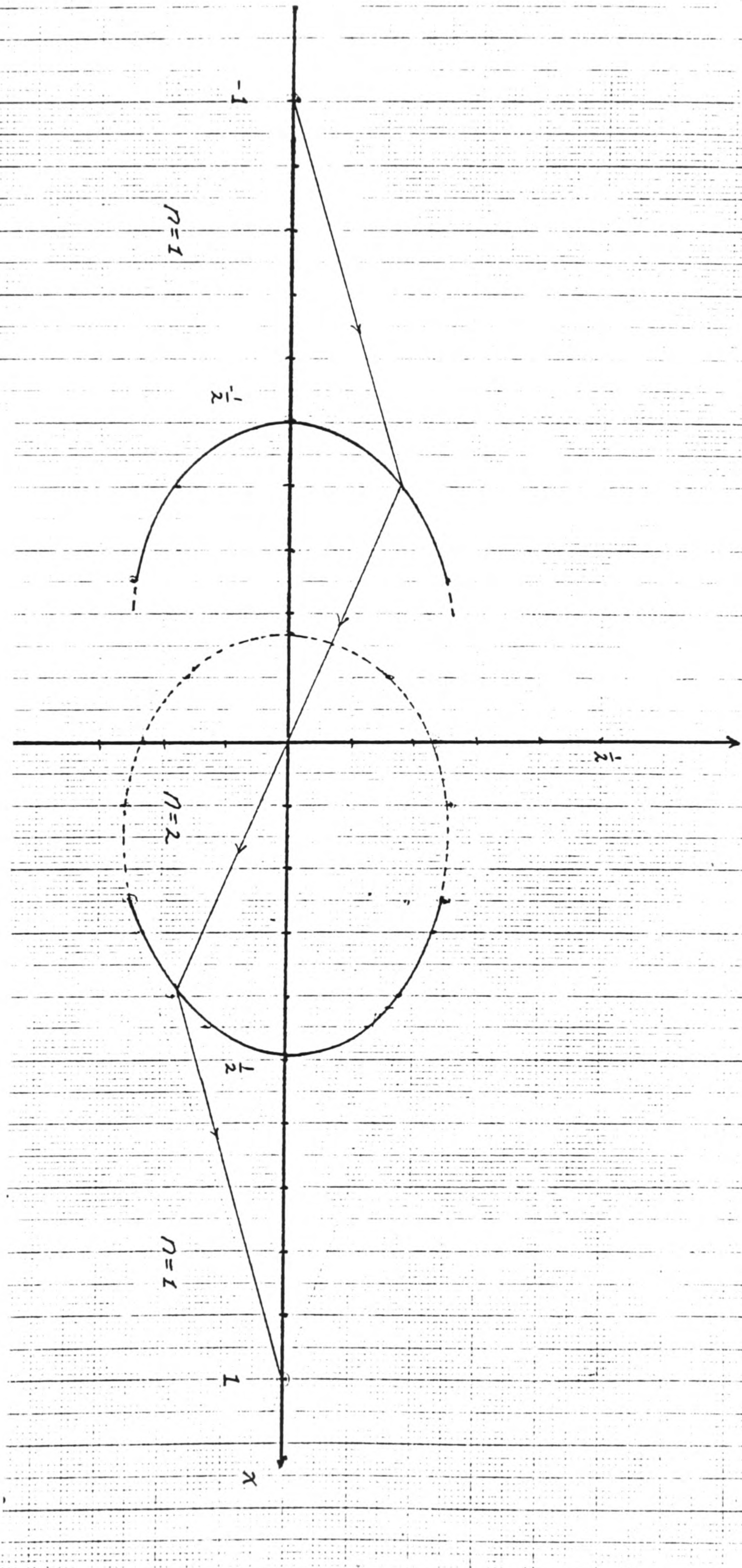


Figure 6b: Symmetric Lens Solution in A^-

$$K_0 = b_0(n-1) + x_1 \ . \tag{68}$$

The solution is illustrated in figure 6b.

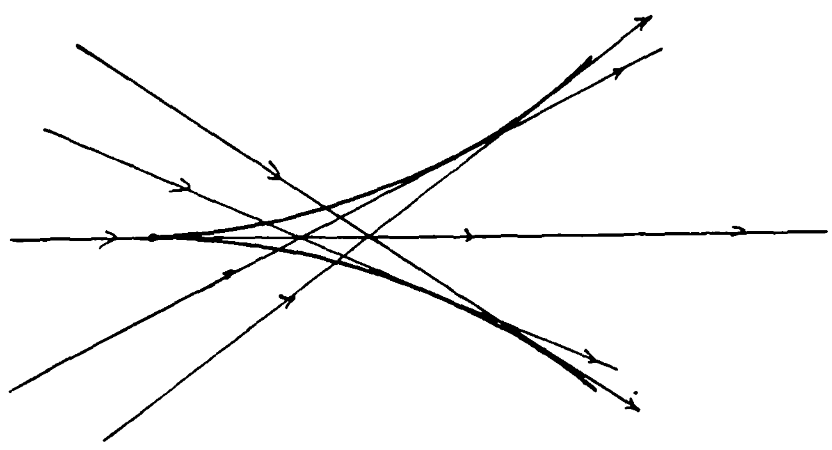
The algebraic solution to the symmetric lens with symmetric foci dates back to Descartes (ca. 1637) and is virtually the only known explicit solution to an axial lens problem. The symmetric lens with symmetric foci also has other special properties (e.g. aplanatism) which will be discussed in the following chapters. Further comments concerning the solution and cartesian ovals can be found in Cornbleet (1984) and Luneberg (1944).

2.6 Light Ray Singularities

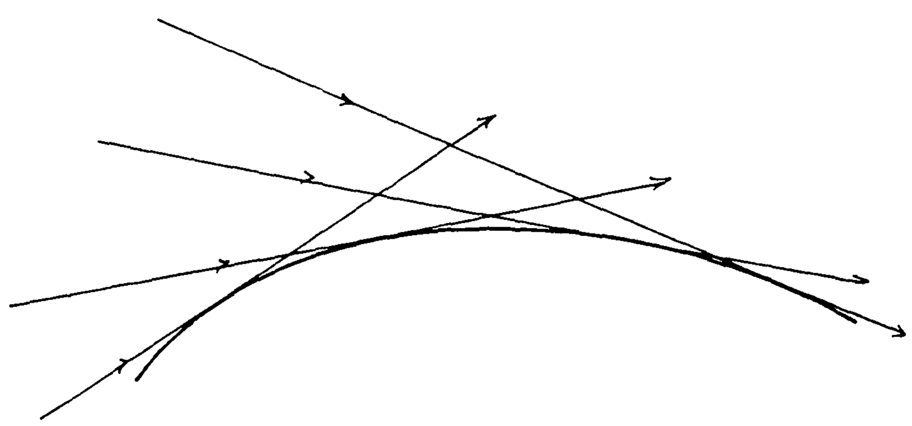
A light ray singularity or caustic corresponds to an envelope formed by light rays (the focus being a degenerate case), and it appears physically as a bright curve. A common example of this phenomenon is the nephroid formed on the inside of a tea cup by sunlight. Caustics are of considerable interest in lens design since often they destroy image quality. These singularities arise frequently in optical systems as a result of imperfect source location with respect to design location. Since, in practice, most lens boundaries can be only approximated, the departures of the approximation from the exact solution also create caustics. A major aspect of practical lens design is the mitigation of the effects these singularities have on image quality.

The purpose of this section is to introduce and briefly discuss some of the basic qualitative notions about light ray singularities, and provide some motivation for the discussions in the subsequent chapters on eliminating particular singularities. The emphasis is primarily on 2-D light ray singularities which occur in the paraxial region.

Light rays, as discussed in the previous chapter, correspond to the characteristic projections of the eikonal equation. Caustics correspond to singularities of the projection mapping and form a boundary where optical path length function becomes multi-valued (i.e. where light rays intersect each other). It is readily deduced from Theorem 2 that if the source for an axial lens were perturbed slightly or the lens boundaries deformed by small smooth perturbations, a diffeomorphism can be established between the resulting family of light rays and the original family. It is known from Whitney's singularity theory [cf. Arnol'd, 1984] that in 2-D the only singularities of the projective mapping which are stable under small diffeomorphic perturbations consist of folds and cusps (figure 7). The focus, in particular, is not stable, and a small perturbation of either the source or a lens surface, in general, converts this singularity into a cusp. Further discussion of caustics in the context of singularity theory can be found in Arnol'd (loc. cit.; 1978), Berry & Upstill (1980), and Poston & Stewart (1978) among several other works on the



a. Cusp



b. Fold

Figure 7: Stable Singularities in 2-D

subject. It suffices for this discussion to note that, in general, light rays in 2-D optical systems form caustics which look locally like either a fold or a cusp.

Caustics are light ray envelopes and, consequently, correspond to wave front evolutes. In studying the qualitative properties of caustics, it is often convenient to examine the geometry of wave fronts under small perturbations and deduce caustic properties from evolute properties. The tip of a cusp, for instance, corresponds to an umbilic point on the wave front. The absence of an umbilic point implies no cusp is present.

Let $S_2 \in A$ with foci at $\underset{\sim}{p}_0, \underset{\sim}{p}_1$ and suppose that the source is perturbed a small distance η along the optical axis to $\bar{\underset{\sim}{p}}_0 = (x_0 + \eta, 0)$. The wave fronts created by the displaced source must be even, and Theorem 1 implies that a diffeomorphism exists between the perturbed wavefronts and the original ones. The caustic must therefore be even and confined to some $O(\eta)$ neighbourhood of $\underset{\sim}{p}_2$. The perturbed wavefronts have an umbilic point where they intersect the optical axis; consequently, the only stable caustic produced by S_2 contains a cusp even in g having a tip on the optical axis. The caustic may have other cusp singularities away from the optical axis depending on the number of wavefront umbilic points. If the source were moved off the axis to $\bar{\underset{\sim}{p}}_0 = (x_0, \eta)$, then the wavefronts would no longer be even in y but the caustic would still be confined to some $O(\eta)$ region around $\underset{\sim}{p}_1$. If η is sufficiently small then this caustic will also contain a cusp singularity having a tip on the line x_1 = const. The location of the cusp tip for either case can be approximated to within $O(\eta^3)$ using paraxial optics. In particular, if the source is perturbed slightly to $\bar{\underset{\sim}{p}}_0 = (x_0 + \eta\alpha, \eta\beta)$ where η is some small positive number and α, β are constants, the cusp tip is given by

$$\bar{X}_1 = b_0 - A/\{AC - n/2\}$$

$$\bar{Y}_1 = B\{1 - AC/\{AC - n/2\}\} , \tag{69}$$

where for the paraxial coefficients a_2, b_2 and $\bar{\ell}_2 = \ell_1 + \eta\alpha$

$$A = \{2\bar{\ell}_1\ell_2 a_2(1-n) + \ell_2 + n\bar{\ell}_1\}/n\bar{\ell}_1$$

$$B = -\eta\ell_2\beta/n\bar{\ell}_1 \tag{70}$$

$$C = 2b_2(n-1) + n/\ell_2 .$$

Equations (62) and (63), in effect, define the paraxial or Gaussian focus of S_2 for the perturbed source. Figure 8 shows typical caustics for small on-axis and off-axis source perturbations.

The above approach is useful for ascertaining some of the crude properties of the caustic, but it will not yield directly an explicit equation for the caustic in terms of the lens boundaries without deriving a relation between said boundaries and image space wavefronts. Rather than use wavefronts, it is more direct to use equations (1) and (4). Either approach is algebraically awkward. If $(\tilde{x}, \tilde{y})$ represents the coordinates of the caustic in the image space, then equation (4) implies

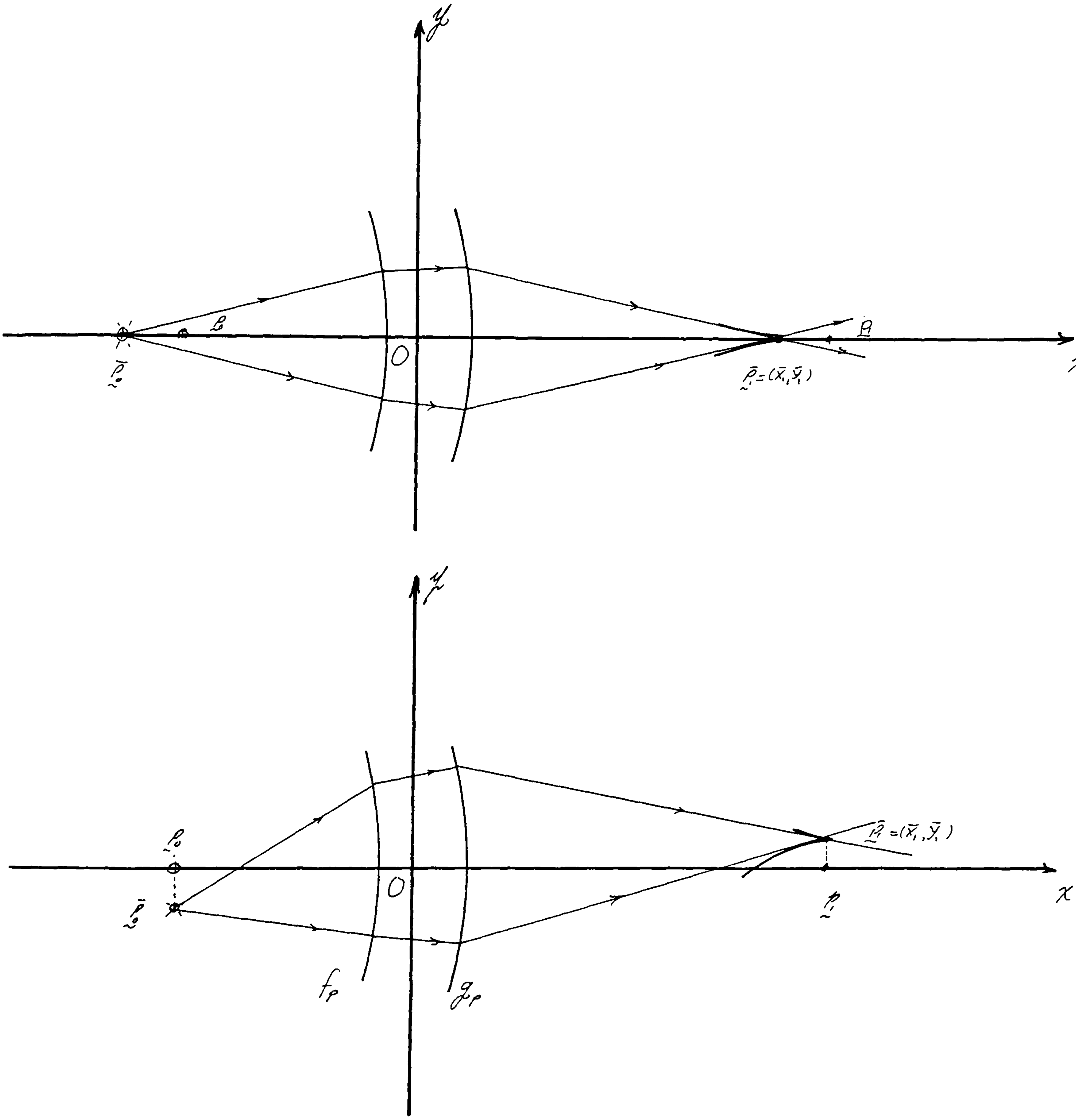


Figure 8: Paraxial Caustic Shapes

$$G(\tilde{x},\tilde{y};y) = m(y)\{\tilde{x}-g(\xi(y))\} + \xi(y)-\tilde{y} = 0 \ , \qquad (71)$$

where m is some smooth function of y known in terms of $g(\xi(y))$ and $\xi(y)$. Equation (71) describes a one parameter family of lines parametrized by y representing light rays in the image space, and as $(\tilde{x},\tilde{y})$ are coordinates on the envelope of these lines, the equation

$$\frac{\partial}{\partial y} G(\tilde{x},\tilde{y};y) = 0 \qquad (72)$$

must also be satisfied. The caustic curve can thus be calculated using the ray tracing equations and equations (71) and (72).

The paraxial solution, as mentioned above, can be used to locate the cusp tip, but it is necessary to consider higher order terms to estimate the shape of the caustic in a paraxial region. Suppose, for example, one desires to estimate the shape in a paraxial region of the caustic created by shifting the source from $\underset{\sim}{p}_0$ to $\bar{\underset{\sim}{p}}_0 = (x_0 + \eta\alpha, 0)$. Let

$$\begin{aligned} f(y) &= a_0 + a_2y^2 + a_4y^4 + O(y^6) \\ g(y) &= b_0 + b_2y^2 + b_4y^4 + O(y^6) \\ \xi(y) &= c_1y + c_3y^3 + O(y^5) \\ m(y) &= m_1y + m_3y^3 + O(y^5) \ . \end{aligned} \qquad (73)$$

Equations (71) and (72) imply the parametric relation

$$\begin{aligned} \tilde{x} &= -\frac{1}{m_1}\{c_1-b_0m_1\} + \frac{3y^2}{m_1}\{c_1\frac{m_3}{m_1} + m_1b_2c_1^2-c_3\} + O(y^4) \\ \tilde{y} &= 2y^3\{c_1\frac{m_3}{m_1} + m_1b_2c_1^2 - c_3\} + O(y^5) \ , \end{aligned} \qquad (74)$$

where m_1 is non-zero since x_0 is $O(1)$. It can be seen from equation (74) that, in general, the caustic is a semi-cubical cusp in a paraxial region. Equations (1), (4) and (71) can be used to write the m_i, c_i in terms of the lens boundary coefficients. A tedious calculation gives

$$
\begin{aligned}
c_1 &= \frac{1}{n\bar{\ell}_1}\{2\bar{\ell}_1\ell_2a_2(1-n) + \ell_2\} + 1 \\
c_3 &= \frac{1}{n\bar{\ell}_1}\{4\bar{\ell}_1\ell_2a_4(1-n) + 2\bar{\ell}_1a_2b_2c_1^2(1-n) + b_2c_1^2 \\
&\qquad -2a_2^2\bar{\ell}_1(1-n) - a_2(1+\ell_2/\bar{\ell}_1) \\
&\qquad + \frac{1}{2\bar{\ell}_1\ell_2}\{(2a_2\bar{\ell}_1 + 1)(c_1-1)(\bar{\ell}_1(c_1-1)-n\ell_2)\} \qquad (75) \\
m_1 &= \frac{1}{\ell_2}\{2b_2\ell_2(n-1) + n(c_1-1)\} \\
m_3 &= \frac{1}{\ell_2}\{2\ell_2b_2c_3(n-1) + 4\ell_2b_4c_1^3(n-1) + nc_3 + 2nb_2(b_2c_1^2-a_2) \\
&\qquad + \frac{nm_1^2}{2}(2b_2\ell_2+c_1-1) - \frac{1}{2\ell_2}\{(2b_2+m_1)((b_2c_1^2-a_2)2\ell_2 \\
&\qquad + (c_1-1)^2\}\}.
\end{aligned}
$$

Although the cusp is, in general, semi-cubical, equations (74) and (75) imply that if the a_i, b_i are such that

$$c_1\frac{m_3}{m_1} + m_1b_2c_1^2 - c_3 = 0 , \qquad (76)$$

the paraxial caustic will be confined to an $O(y^4)$ region as opposed to an $O(y^2)$ region, and the imaging capabilities of the system are improved. As there is some degree of freedom in the choice of axial lenses having foci at $\underset{\sim}{p}_0$ and $\underset{\sim}{p}_1$, it is expected that there exist lenses in A which reduce the size of the caustic produced by shifting the source along the optical axis. These lenses are discussed in the next chapter.

If the source were moved slightly off the optical axis, an approach similar to the above can be used to estimate the paraxial cusp. The $O(y^4)$ terms are replaced by $O(\eta^4)$ terms and the caustic is no longer even in y. As with the above example, it is expected that there are lenses which reduce the size of the caustic produced by off-axis perturbations. Lenses having this property are discussed in Chapter 5.

Paraxial caustics in three-dimensional optical systems form the subject of geometrical aberration theory. The leading order behaviour of these caustics is described by the Seidel or primary aberrations. The Seidel aberrations are classified according to the possible perturbations of a paraxial image space wave front and are called spherical aberration, coma, astigmatism, distortion, and curvature of field. Of the five Seidel aberrations, the first three affect the focusing capabilities of the instrument. The remaining aberrations are related to the form and position of the image (i.e. they affect the relative proximities of point sources 'focused' in the image space). Spherical aberration is essentially the three dimensional version of the cusp described by equation (74), i.e. it appears geometrically as a surface of revolution generated by a semi-cubical cusp. If the perturbed source remains on the optical axis, only spherical aberration occurs.

The coma aberration is the three dimensional extension of the caustic produced by perturbing the source slightly off the optical axis. The caustic surface resembles a hyperbolic umbilic. Astigmatism does not correspond to any caustic in two dimensions, and if the plane π containing the source and the optical axis were examined, no caustic except a focus would be present. If, however, the plane orthogonal to π is examined, another focus would be present so that in three dimensions astigmatism gives rise to two images (viz. tangential and sagittal). The Seidel and higher order aberrations are discussed in detail by Born & Wolf (1986), Buchdahl (1954), Luneberg (1944), and Welford (1986).

Catastrophe theory can be applied to paraxial caustics in three dimensions, and it can be shown [cf. Berry & Upstill, loc. cit.] that the only stable caustic surface is the hyperbolic umbilic. This result is deduced by considerations analogous to those used above for caustics in two dimensions, viz., umbilic points on the image space wavefronts and symmetry.

Chapter 3

The Multi-Foci Lens

An axial lens having two pairs of foci on the optical axis is investigated in this chapter. The governing equations are presented in the first section along with the paraxial solution. The primary subject of the second section is an existence theorem for local solutions which are analytic. The work follows that of Friedman & McLeod (1987,1988). Uniqueness among analytic functions for a specified paraxial solution is also established in this section, and the local solution is analytically continued in the third section. In the fourth section, the uniqueness result of Section 2 is sharpened to uniqueness among functions in C^2, and the stability of the solution is investigated in the fifth section. An asymptotic method is developed in the sixth section for the case when the distance between the foci in the object space is small. The crucial advantage of this method is that it reduces the functional differential equations to ordinary differential equations. The leading order solution corresponds to a lens with special imaging properties called the Herschel lens. The Herschel lens and the ordinary differential equation describing it are discussed in Section 7.

3.1 The Multi-Foci Lens Problem

An axial lens having two pairs of foci on the optical axis is examined in this section. The governing equations for this lens are derived and the paraxial solution investigated.

Let S_2 be an axial lens having boundaries γ_f, γ_g satisfying initial condition (2.15), and suppose the lens focuses $\underset{\sim}{p}_0 = (x_0, 0)$ to $\underset{\sim}{p}_1 = (x_1, 0)$ and $\underset{\sim}{q}_0 = (\bar{x}_0, 0)$ to $\underset{\sim}{q}_1(\bar{x}_1, 0)$ (see figure 1). If $V(\underset{\sim}{a}, \underset{\sim}{b})$ denotes the optical path length between $\underset{\sim}{a} \in S_1$ and $\underset{\sim}{b} \in S_3$, $(g(\xi), \xi)$ is the exit point for a light ray from $\underset{\sim}{p}_0$ entering the lens at $(f(y), y)$, and $(g(\phi), \phi)$ is the exit point for a light ray leaving $\underset{\sim}{q}_0$ entering the lens at $(f(y), y)$, the optical path lengths for these families of light rays are given by

$$V(\underset{\sim}{p}_0, \underset{\sim}{p}_1) = d_1 + nd_2 + d_3 ,$$
$$V(\underset{\sim}{q}_0, \underset{\sim}{q}_1) = \bar{d}_1 + n\bar{d}_2 + \bar{d}_3 , \tag{1}$$

where

$$d_1^2 = (f(y) - x_0)^2 + y^2 ,$$
$$d_2^2 = (g(\xi) - f(y))^2 + (\xi - y)^2 , \tag{2}$$
$$d_3^2 = (x_1 - g(\xi))^2 + \xi^2 ,$$

$$\bar{d}_1^2 = (f(y) - \bar{x}_0)^2 + y^2 ,$$
$$\bar{d}_2^2 = (g(\phi) - f(y))^2 + (\phi - y)^2 , \tag{3}$$
$$\bar{d}_3^2 = (\bar{x}_1 - g(\phi))^2 + \phi^2 .$$

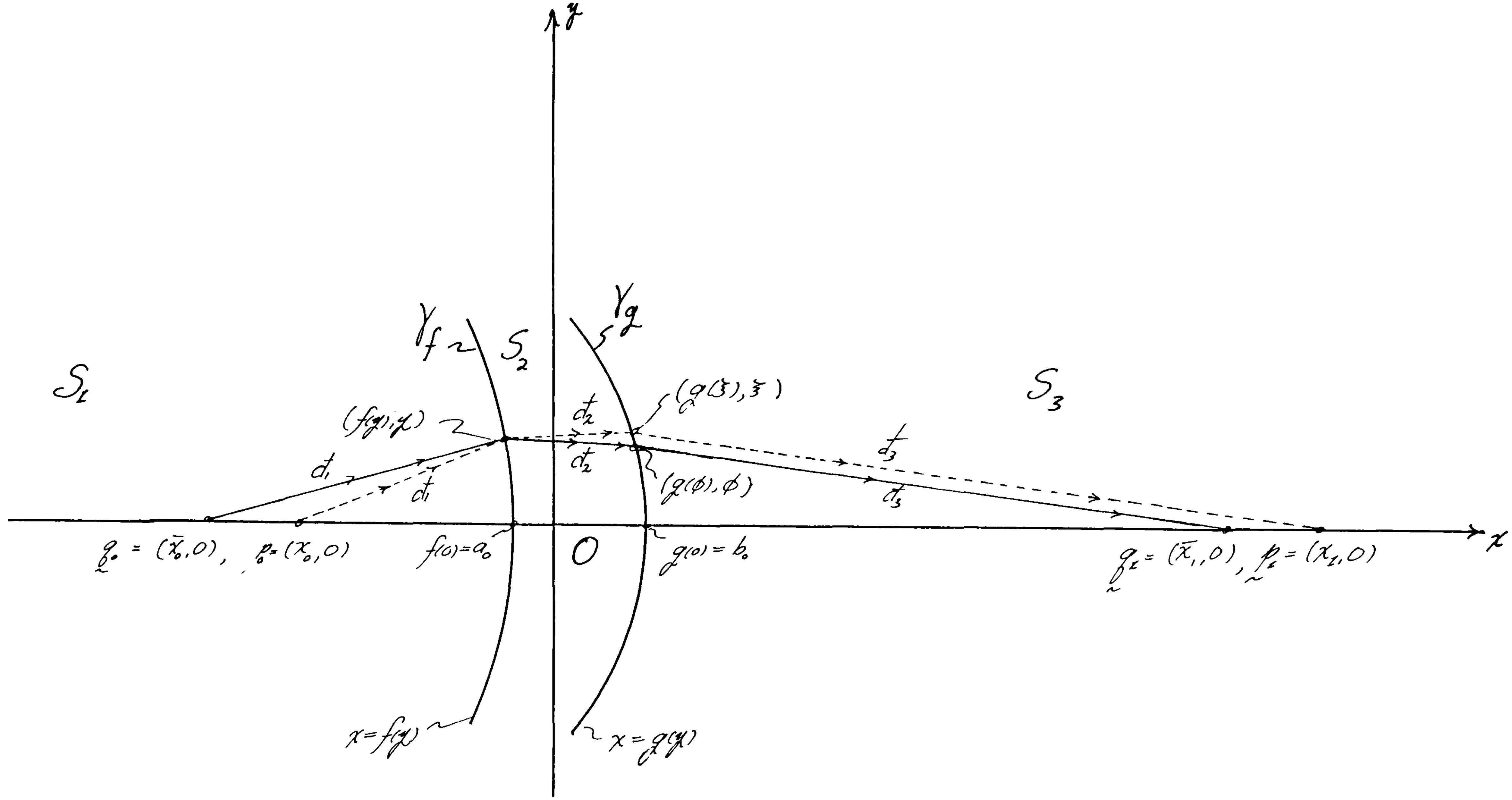


Figure 1: Multi-Foci Optical System

The lens boundaries are thus required to satisfy

$$\frac{\partial V}{\partial y}(\underset{\sim}{p}_0,\underset{\sim}{p}_1) = \frac{(f(y)-x_0)f'(y)+y}{d_1} - \frac{n\{(g(\xi)-f(y))f'(y)+\xi-y\}}{d_2}$$

$$= 0 \qquad (4a)$$

$$\frac{\partial V}{\partial \xi}(\underset{\sim}{p}_0,\underset{\sim}{p}_1) = \frac{n\{(g(\xi)-f(y))g'(\xi)+\xi-y\}}{d_2} + \frac{-(x_1-g(\xi))g'(\xi)+\xi}{d_3}$$

$$= 0 , \qquad (4b)$$

$$\frac{\partial V}{\partial y}(\underset{\sim}{q}_0,\underset{\sim}{q}_1) = \frac{(f(y)-\bar{x}_0)f'(y)+y}{\bar{d}_1} - \frac{n\{(g(\phi)-f(y))f'(y)+\phi-y\}}{\bar{d}_2}$$

$$= 0 , \qquad (4c)$$

$$\frac{\partial V}{\partial \phi}(\underset{\sim}{q}_0,\underset{\sim}{q}_1) = \frac{n\{(g(\phi)-f(y))g'(\phi)+\phi-y\}}{\bar{d}_2} + \frac{-(\bar{x}_1-g(\phi))g'(\phi)+\phi}{\bar{d}_3}$$

$$= 0 , \qquad (4d)$$

where ' denotes differentiation with respect to the indicated argument.

Unlike the general axial lens problem discussed in Chapter 2, the above axial lens problem is described by four first order nonlinear functional differential equations containing the four unknown functions $f(y)$, $g(y)$, $\xi(y)$ and $\phi(y)$. The two remaining initial conditions for the problem are given by symmetry with respect to the optical axis, i.e.

$$\begin{aligned} \xi(0) &= 0 , \\ \phi(0) &= 0 . \end{aligned} \qquad (5)$$

The above problem will be termed the <u>Multi-Foci Lens Problem</u>.

Although the definitions of A^+ and A^- given in Section 2.3 consider only one pair of foci, it is clear that the definition can be extended for multiple foci. In particular, if either ξ' or ϕ' is negative, the lens will have the characteristics of an axial lens in A^- (e.g. the lens shape must be double convex). A multi-foci lens is said to be in A^+ if the inequalities

$$\xi'(y) > 0 \ ,$$

$$\phi'(y) > 0 \ ,$$

are both satisfied. The lens is in A^- otherwise.

Assuming the lens boundaries are smooth, Theorem 2.2 implies that ξ and ϕ must also be smooth, and f, g, ξ, ϕ can be written

$$\begin{aligned} f(y) &= a_0 + a_2 y^2 + O(y^4) \ , \\ g(y) &= b_0 + b_2 y^2 + O(y^4) \ , \\ \xi(y) &= c_1 y + O(|y|^3) \ , \\ \phi(y) &= e_1 y + O(|y|^3) \ . \end{aligned} \tag{6}$$

Substituting these expressions into system (4) yields the following equations for the paraxial coefficients a_2, b_2, c_1, e_1 (cf. Section 2.4):

$$\begin{aligned} -2(n-1)a_2 \ell_2 \ell_1 + \ell_2 + n\ell_1 &= n\ell_1 c_1 \ , \\ 2(n-1)b_2 \ell_2 \ell_3 + \ell_2 + n\ell_3 &= n\ell_3 / c_1 \ , \\ -2(n-1)a_2 \ell_2 \bar{\ell}_1 + \ell_2 + n\bar{\ell}_1 &= n\bar{\ell}_1 e_1 \ , \\ 2(n-1)b_2 \ell_2 \bar{\ell}_3 + \ell_2 + n\bar{\ell}_3 &= n\bar{\ell}_3 / e_1 \ , \end{aligned} \tag{7}$$

where

$$\ell_1 = a_0 - x_0 \, ,$$

$$\ell_2 = b_0 - a_0 \, ,$$

$$\ell_3 = x_1 - b_0 \, , \tag{8}$$

$$\bar{\ell}_1 = a_0 - \bar{x}_0 \, ,$$

$$\bar{\ell}_3 = \bar{x}_1 - b_0 \, .$$

Eliminating a_2, b_2 from system (7), the paraxial coefficients c_1, e_1 are given by

$$c_1 = e_1 - S \, , \tag{9}$$

$$c_1^2 + Sc_1 - \frac{S}{T} = 0 \, , \tag{10}$$

where

$$S = -\frac{\ell_2}{n}\left\{\frac{1}{\ell_1} - \frac{1}{\bar{\ell}_1}\right\} \, ,$$

$$T = \frac{\ell_2}{n}\left\{\frac{1}{\ell_3} - \frac{1}{\bar{\ell}_3}\right\} \, . \tag{11}$$

It is assumed that the $\underset{\sim}{p}_i, \underset{\sim}{q}_i$ are distinct so that neither S nor T vanish, and it is evident that a paraxial solution exists only if

$$S^2 + 4\,\frac{S}{T} \geq 0 \, . \tag{12}$$

There are two possible types of foci configuration. For definiteness, let

$$\bar{\ell}_1 > \ell_1 \, . \tag{13}$$

The foci may be 'staggered', i.e.

$$\ell_3 > \bar{\ell}_3 ,$$

as shown in figure 1, or the foci may be 'nested',

$$\ell_3 < \bar{\ell}_3 .$$

If the foci are staggered, T is negative and equation (10) always has real solutions. Since S and T have the same sign one paraxial solution corresponds to a lens in A^+, the other to a lens in A^-. Inequality (13) and equations (4a) and (4c) imply

$$\phi(y) < \xi(y) , \quad y \neq 0 , \tag{14}$$

so that, for staggered foci,

$$c_1 > e_1 > 0 , \quad S_2 \in A^+ , \tag{15}$$

$$e_1 < c_1 < 0 , \quad S_2 \in A^- . \tag{16}$$

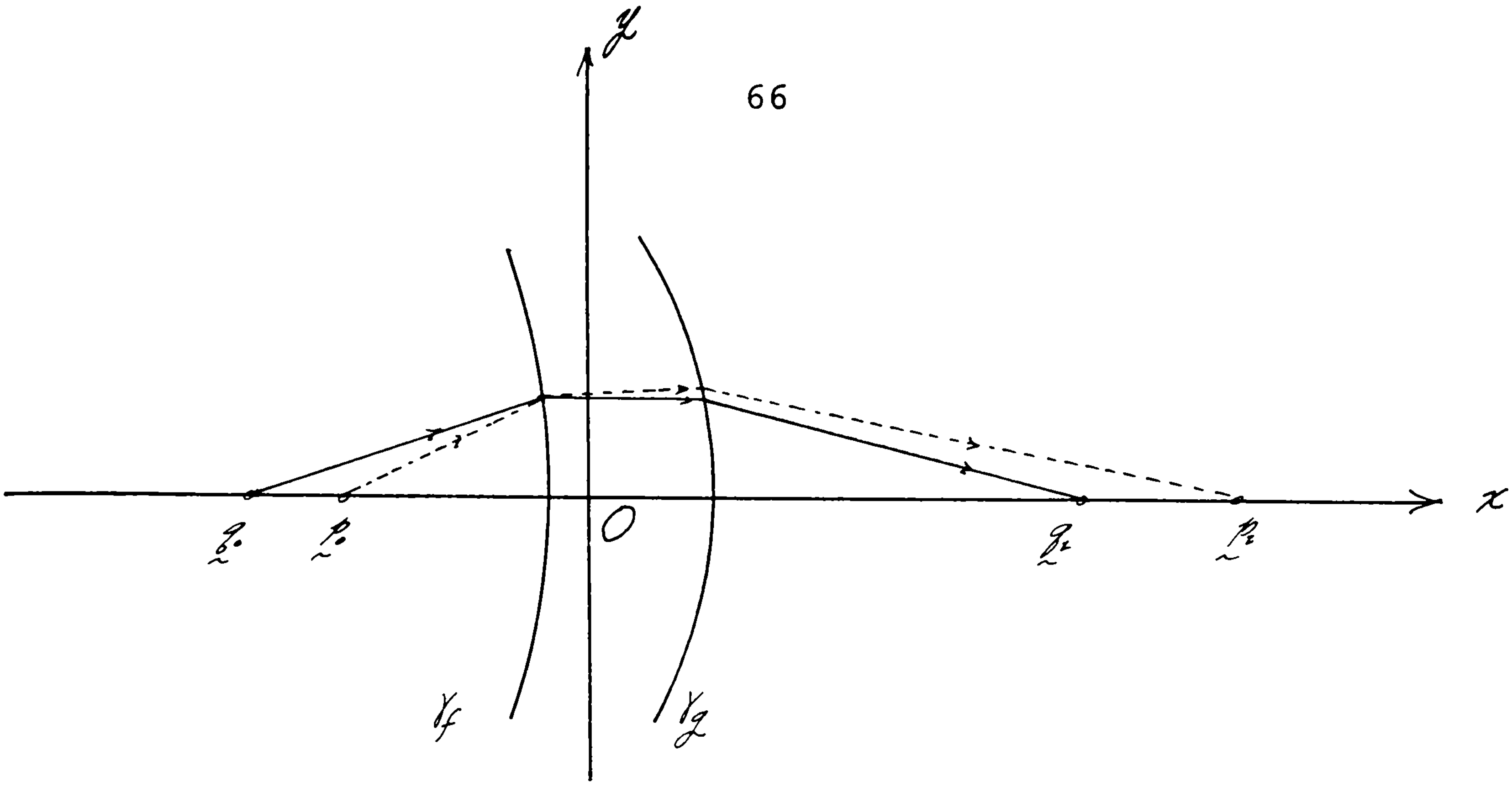
If the foci are nested, then a paraxial solution exists only if inequality (12) is satisfied, i.e. only if

$$4n^2 \leq \ell_2^2\{\frac{1}{\ell_1} - \frac{1}{\bar{\ell}_1}\}\{\frac{1}{\ell_3} - \frac{1}{\bar{\ell}_3}\} . \tag{17}$$

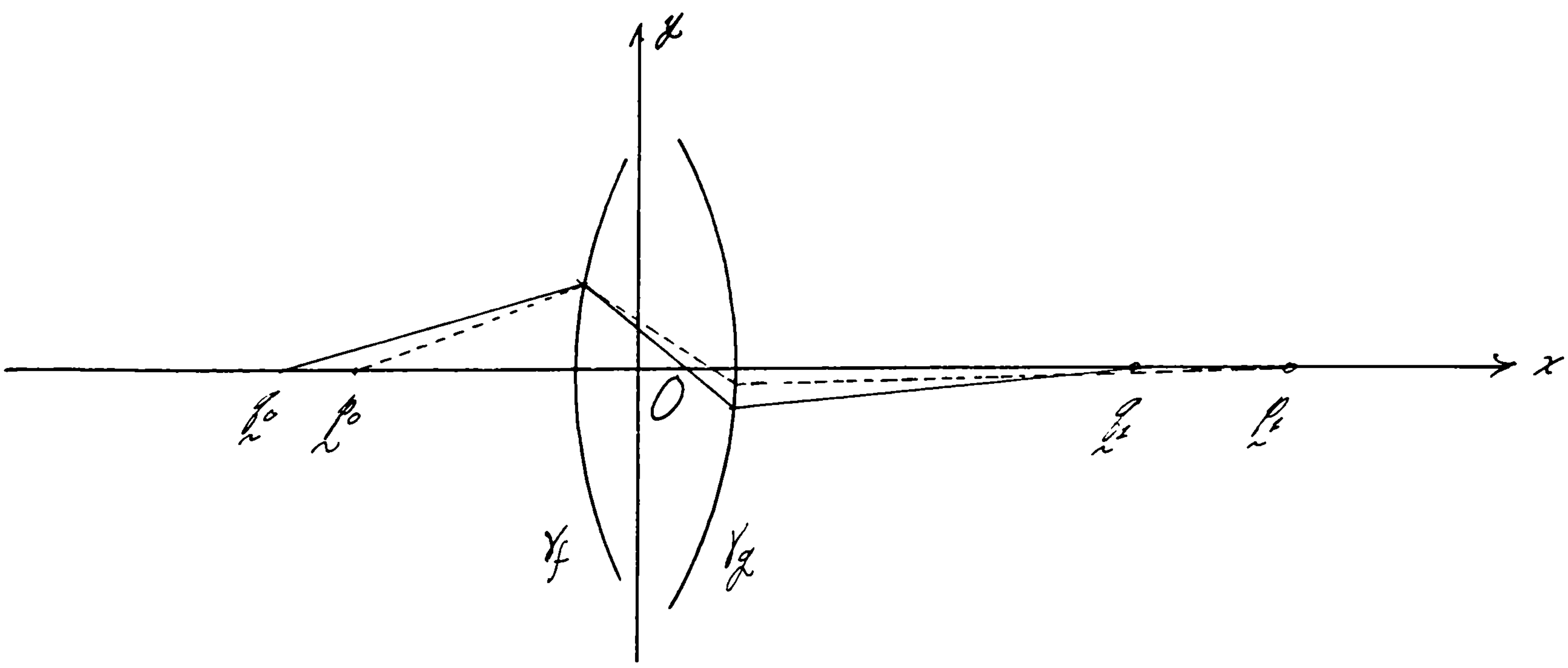
The two solutions to equation (10) are positive, and equation (9) gives the inequality

$$c_1 > 0 > e_1 , \tag{18}$$

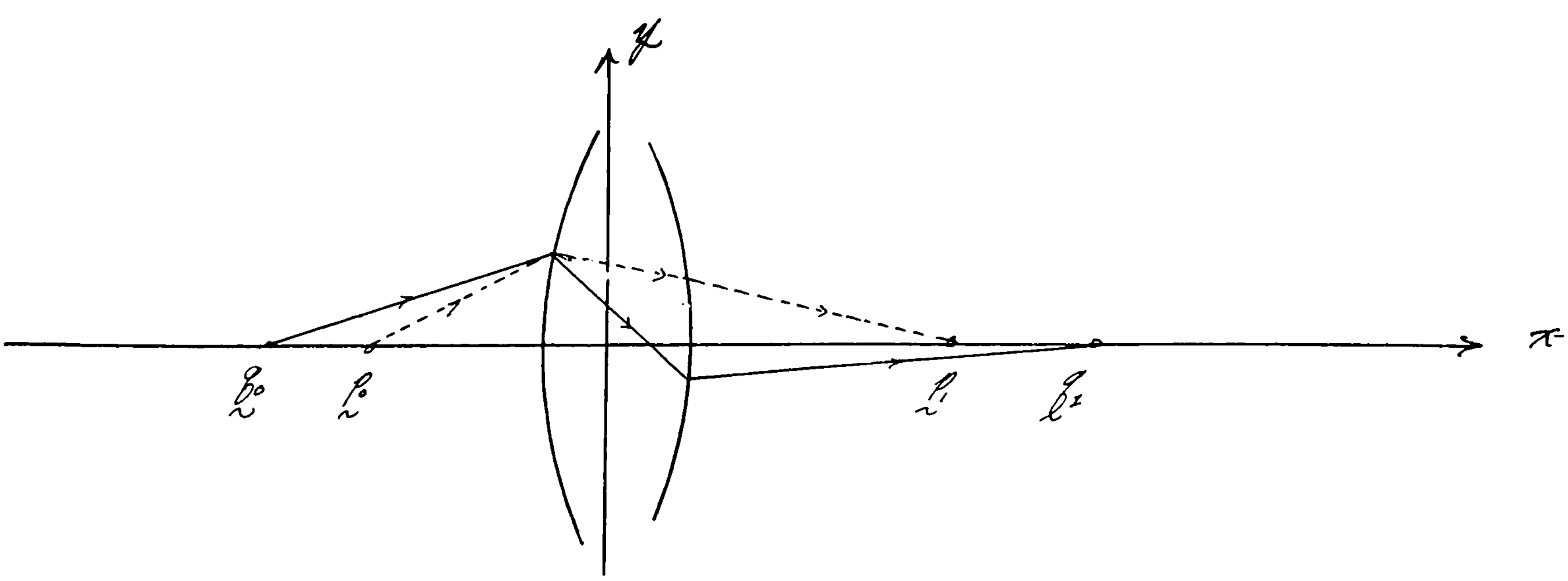
so that the corresponding lens must be in A^-. Figure 2 shows the paraxial light ray geometry associated with the above cases. Figure 3 depicts the general parameter region (n fixed) for which a paraxial solution exists.



<u>Figure 2a</u>: Staggered Foci, $S_2 \in A^+$



<u>Figure 2b</u>: Staggered Foci, $S_2 \in A^-$



<u>Figure 2c</u>: Nested Foci, $S_2 \in A^-$

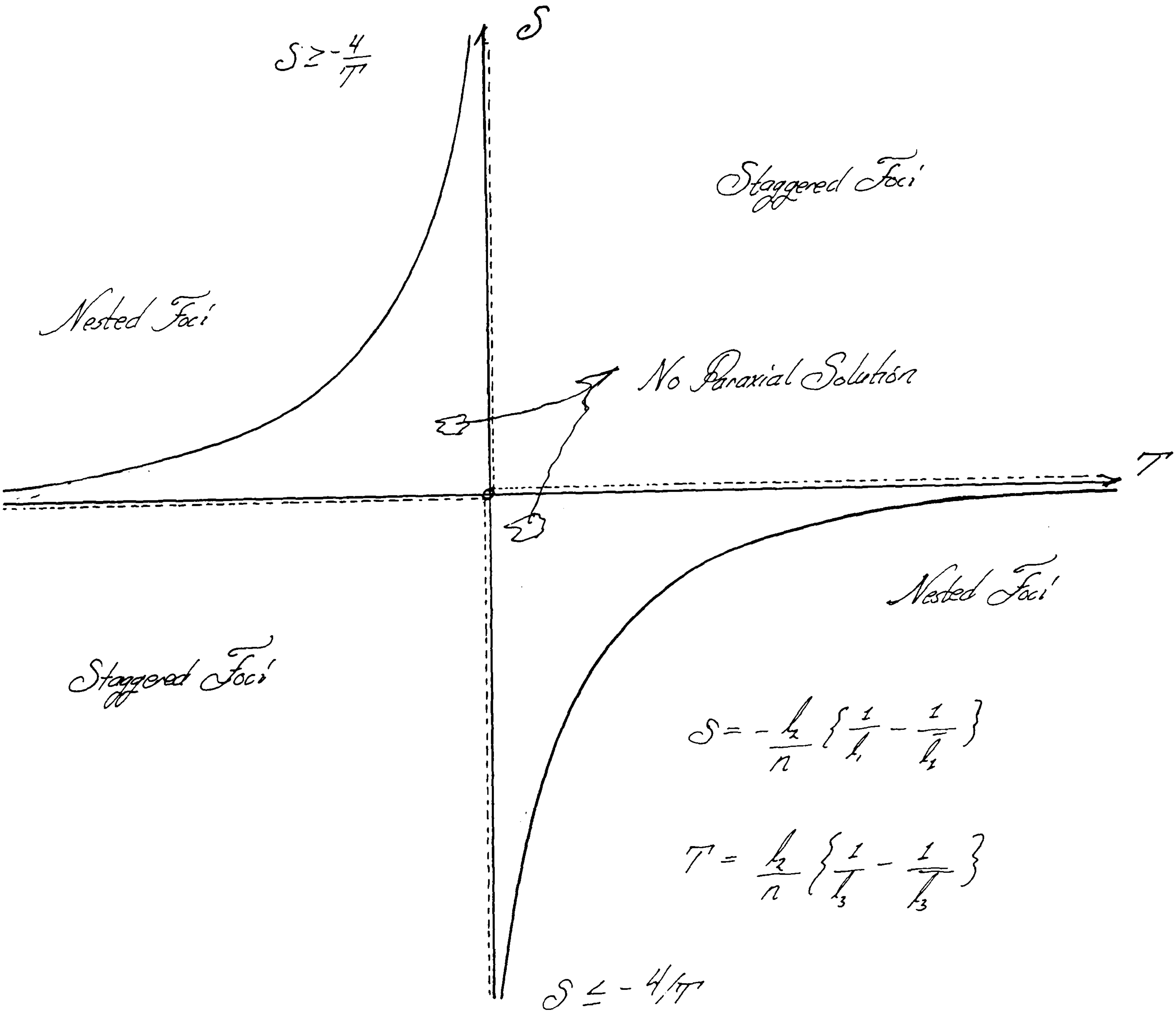


Figure 3: Parameter Region for Which a Paraxial Solution Exists

3.2 Existence and Uniqueness of a Local Analytic Solution

Existence and uniqueness questions concerning local analytic solutions to the multi-foci lens problem are addressed in this section. These questions have been studied by Friedman & McLeod (1987,1988) and their work is followed closely in this section. It is shown in this section that a local solution exists which is analytic for each paraxial solution. The analytic continuation of the solutions is discussed in the next section.

The primary result of this section is the following theorem:

Theorem 1

If paraxial solutions to the multi-foci problem exist and satisfy initial conditions (2.15), then for each paraxial solution there exists a unique local solution which is analytic. In particular, if the foci are staggered, there exist precisely two local analytic solutions corresponding to lenses in A^- and A^+.

Proof:

The proof of existence essentially involves an application of the Schauder fixed-point theorem to a suitable transformation having fixed points corresponding to solutions of system (4). The paraxial solution plays a crucial role in selecting the appropriate form of the transformation domain and in establishing various inequalities necessary for the proof. Since, in general, two paraxial solutions exist, uniqueness is possible only after one of the two paraxial solutions is selected. Uniqueness then follows since the derivatives of a local analytic solution can be calculated uniquely at $y = 0$. The proof is divided into two sections, viz., "Existence" and "Uniqueness". The former division is subdivided into three sections corresponding to the two possible light ray geometries for staggered foci and the light ray geometry for nested foci. The existence proofs for these light ray geometries are similar and only the case when the foci are staggered and $S_2 \in A^+$ is given in detail.

Existence:

The strategy is to define a mapping

$$M : (f,g,\xi,\phi) \to (\tilde{f},\tilde{g},\tilde{\xi},\tilde{\phi})$$

such that:

a. A fixed point of M corresponds to a solution of system (4).

b. M maps a compact convex subset of its domain into itself.

c. M is continuous.

If a mapping M can be found which satisfies the above properties, then the Schauder fixed-point theorem can be applied to show existence.

Case 1: Staggered Foci, $S_2 \in A^+$

The mapping M will be defined using $\xi^{-1}(y)$ rather than $\xi(y)$. Theorem 2.2 shows that the existence of one implies the other so that no new conditions on the solution are required. The choice of $\xi^{-1}(y)$ is motivated by inequality (15) and, as will be seen below, is used to ensure that M is well-defined. Let

$$\hat{d}_1^2 = (f(\xi^{-1})-x_0)^2 + (\xi^{-1})^2 ,$$

$$\hat{d}_2^2 = (g(y)-f(\xi^{-1}))^2 + (y-\xi^{-1})^2 , \tag{19}$$

$$\hat{d}_3^2 = (x_1-g(y))^2 + y^2 ,$$

and define M by the following relations:

$$-\frac{(f(\xi^{-1})-x_0)f'(\xi^{-1})+\sigma_1\xi^{-1}+\tau_1\tilde{\xi}^{-1}}{\hat{d}_1} - \frac{n\{(g(y)-f(\xi^{-1}))f'(\xi^{-1})+y-\xi^{-1}\}}{\hat{d}_2}$$

$$= 0 , \tag{20}$$

$$\frac{n\{(g(y)-f(\xi^{-1}))\tilde{g}'(y)+y-\tilde{\xi}^{-1}}{\hat{d}_2} + \frac{-(x_1-g(y))\tilde{g}'(y)+y}{\hat{d}_3}$$

$$= 0 , \tag{21}$$

$$\frac{\{(f(y)-\bar{x}_0)\tilde{f}'(y)+y\}}{\bar{d}_1} - \frac{n\{(g(\phi)-f(y))\tilde{f}'(y)+\tilde{\phi}-y\}}{\bar{d}_2}$$

$$= 0 , \tag{22}$$

$$\frac{n\{(g(\phi)-f(y))g'(\phi)+\phi-y\}}{\bar{d}_2} + \frac{-(\bar{x}_1-g(\phi))g'(\phi)+\sigma_2\phi+\tau_2\tilde{\phi}}{\bar{d}_3}$$

$$= 0 , \tag{23}$$

where σ_i, τ_i are constants satisfying

$$\tau_i + \sigma_i = 1 , \tag{24}$$

$$2a_2(1-n) + \frac{n}{\ell_2} + \frac{\sigma_1}{\ell_1} = 0 , \tag{25}$$

$$2b_2(n-1) + \frac{n}{\ell_2} + \frac{\sigma_2}{\bar{\ell}_3} = 0 , \tag{26}$$

and

$$\tilde{f}(0) = a_0$$
$$\tilde{g}(0) = b_0 \ . \tag{27}$$

It is clear that any fixed point of the mapping defined above corresponds to a solution of system (4). Before investigating M further to see if the other Schauder fixed-point theorem conditions are satisfied, it is necessary to define the domain of M so that any fixed points correspond to local analytic solutions and M is well-defined for y sufficiently small.

Let

$$B_\delta \triangleq \{y : |y| < \delta\} ,$$

and δ_1, δ_2 be two small positive numbers satisfying

$$\frac{\delta_2}{\delta_1} = \nu c_1 \ , \quad \frac{\delta_1}{\delta_2} = \frac{\nu}{e_1} \ , \tag{28}$$

where ν is a number such that

$$0 < \nu < 1 \ . \tag{29}$$

The Banach space X is defined as the space of all functions (f,g,ξ,ϕ) such that f,ξ,ϕ are analytic in B_{δ_1} and g is analytic in B_{δ_2}. The norm for X is given by

$$\|(f,g,\xi,\phi)\| \triangleq \sup_{B_{\delta_1}}|f| + \sup_{B_{\delta_2}}|g| + \sup_{B_{\delta_1}}|\xi| + \sup_{B_{\delta_1}}|\phi| \ .$$

A solution to system (4) is expected in the form

$$
\begin{aligned}
f(y) &= a_0 + a_2 y^2 + F(y) \\
g(y) &= b_0 + b_2 y^2 + G(y) \\
\xi(y) &= c_1 y + E(y) \\
\phi(y) &= e_1 y + P(y) ,
\end{aligned} \tag{30}
$$

where the a_i, b_i, c_1, e_1 are the paraxial coefficients, $F(y), G(y)$ are $O(|y|^3)$ analytic functions, and $E(y), P(y)$ are $O(y^2)$ analytic functions as $|y| \to 0$. Thus, it is convenient thus to define the domain of M in terms of the form given in (30). Let R_1, R_2, C, D be positive constants and define the domain $X_{R_1R_2CD}$ of M as follows:

$X_{R_1R_2CD} = \{(f,g,\xi,\phi) \in X :$

a. f, g, ξ, and ϕ can be written in the form of (30)

b. F, G, E, P satisfy the inequalities

$$|F(y)| \le R_1|y|^3 , \qquad |F'(y)| \le 3R_1 y^2 , \quad y \in B_{\delta_1} \tag{31}$$

$$|G(y)| \le R_2|y|^3 , \qquad |G'(y)| \le 3R_2 y^2 , \quad y \in B_{\delta_2} \tag{32}$$

$$|E(y)|, |P(y)| \le Cy^2 , \quad |E'(y)|, |P'(y)| \le D|y| , \quad y \in B_{\delta_1} \}. \tag{33}$$

Relations (28) and (29) ensure that $g(\phi)$ and $f(\xi^{-1})$ are analytic for δ_1 sufficiently small, $y \in B_{\delta_1}$. Specifically, since $\delta_1 e_1 = \delta_2 \nu$, $\nu < 1$,

$$\phi(B_{\delta_1}) \subseteq \phi(B_{\delta_2}) ,$$

for δ_1 small, and, consequently, $g(\phi)$ is analytic in y for $y \in B_{\delta_1}$. Since $\nu\delta_1 c_1 = \delta_2$,

$$B_{\delta_2} \subseteq \xi(B_{\delta_2}) ,$$

for δ_1 small and thus $f(\xi^{-1})$ is analytic for $y \in B_{\delta_2}$.

Cauchy's theorem for ordinary differential equations implies that for a given $(f,g,\xi,\phi) \in X_{R_1R_2CD}$, M determines $\tilde{f},\tilde{g},\tilde{\xi}^{-1},\tilde{\phi}$ uniquely and that these functions are analytic for δ_1 small. The transformed functions can thus be written

$$\begin{aligned} \tilde{f}(y) &= a_0 + \tilde{a}_1 y + \tilde{a}_2 y^2 + \tilde{F}(y) , \\ \tilde{g}(y) &= b_0 + \tilde{b}_1 y + \tilde{b}_2 y^2 + \tilde{G}(y) , \\ \tilde{\xi}^{-1}(y) &= \tilde{h}_0 + \tilde{h}_1 y + \tilde{H}(y) , \\ \tilde{\phi}(y) &= \tilde{e}_0 + \tilde{e}_1 y + \tilde{P}(y) , \end{aligned} \tag{34}$$

where $\tilde{F}(y),\tilde{G}(y)$ are $O(|y|^3)$ analytic functions and $\tilde{H}(y),\tilde{P}(y)$ are $O(y^2)$ analytic functions. Substituting $y = 0$ into equations (20)-(23) gives

$$\tilde{a}_1 = \tilde{b}_1 = \tilde{h}_0 = \tilde{e}_0 = 0 ,$$

and substituting relations (34) (after the appropriate compositions) into equations (20)-(23) yields

$$\begin{aligned} \tilde{a}_2 &= a_2 \\ \tilde{b}_2 &= b_2 \\ \tilde{h}_1 &= \frac{1}{c_1} \\ \tilde{e}_1 &= e_1 . \end{aligned}$$

Since $\tilde{h}_1 \neq 0$, the implicit function theorem implies that the function $\tilde{\xi}^{-1}$ can be inverted to $\tilde{\xi}$ analytic for $y \in \delta_1$,

δ_1 small. The mapping M thus defines a transformation from $X_{R_1R_2CD}$ to X. The transformed functions are of the form

$$\begin{aligned}
\tilde{f}(y) &= a_0 + a_2y^2 + \tilde{F}(y) \\
\tilde{g}(y) &= b_0 + b_2y^2 + \tilde{G}(y) \\
\tilde{\xi}(y) &= c_1y + \tilde{E}(y) \\
\phi(y) &= e_1y + \tilde{P}(y) \ ,
\end{aligned} \tag{35}$$

and any fixed point of M corresponds to a local analytic solution of system (4).

The set $X_{R_1R_2CD}$ is compact and convex. Condition b will be satisfied if it can be shown that there exist positive constants R_1, R_2, C, D such that

$$x \in X_{R_1R_2CD} \Longrightarrow M(x) \in X_{R_1R_2CD} \ . \tag{36}$$

Substituting (35) into equation (20) gives

$$F(\xi^{-1})(1-n) + H(y)\{2a_2(1-n) + \frac{n}{\ell_2} + \frac{\sigma_1}{\ell_1}\}\{1 + O(y^2)\}$$
$$+ \frac{\tau_1}{\ell_1}\tilde{H}(y) = O(|y|^3) \ ,$$

where the $O(y^i)$ terms are known in terms of f, g, ξ, and ϕ. Using equations (24) and (25) along with paraxial relations (7), the above expression can be reduced to

$$F'(\xi^{-1})(n-1) - \frac{nc_1}{\ell_2}\tilde{H}(y)(1 + O(y^2)) = O(|y|^3) \ . \tag{37}$$

Inequality (31) and the above relation imply

$$|\tilde{H}(y)| \le \frac{\ell_2}{c_1^3}(1 - \frac{1}{n})3R_1y^2 + O(|y|^3) \ ,$$

so that for δ_1 sufficiently small

$$|\tilde{H}(y)| < \frac{3R_1\ell_2y^2}{c_1^3} ,$$
$$|\tilde{E}(y)| \le 3R_1\ell_2y^2 . \tag{38}$$

The $O(y^2)$ terms in equation (21) are related by

$$(n-1)\tilde{G}'(y) - \frac{n}{\ell_2} \tilde{H}(y)(1 + O(y^2)) = O(|y|^3) ,$$

and, using equation (37), this expression reduces to

$$\tilde{G}'(y) = \frac{F'(\xi^{-1})}{c_1} \{1 + O(y^2)\} + O(|y|^3) . \tag{39}$$

Inequality (32) and equation (39) imply

$$|\tilde{G}'(y)| \le \frac{3R_1y^2}{c_1^3} + O(|y|^3) , \tag{40}$$

and, by integration,

$$|\tilde{G}(y)| \le \frac{R_1|y|^3}{c_1^3} + O(y^4) . \tag{41}$$

A bound on $\tilde{E}'(y)$ can be obtained by differentiating equation (20) with respect to y and equating $O(y)$ terms, i.e.

$$(\xi^{-1})'F''(\xi^{-1})(n-1) - \frac{nc_1}{\ell_2} \tilde{H}'(y)(1 + O(y^2)) = O(y^2) . \tag{42}$$

The function F is analytic and consequently there exist positive numbers N, r such that

$$|F^{(k)}(y)| \le \frac{Nk!|y|^k}{r^k} , \qquad |y| < r ,$$

so that, in particular, there exists a number $\Lambda > 0$ such that

$$|F''(\xi^{-1})| \le R_1\Lambda|y| . \tag{43}$$

Equation (42) and inequality (43) imply there exists a number $\tilde{\Lambda} > 0$ such that

$$|\tilde{H}'(y)| \leq \tilde{\Lambda} R_1 |y| \quad , \tag{44}$$

for δ_1 small. Using the relation

$$\frac{d}{dy}\{\tilde{\xi}^{-1}(\xi)\} = 1 \quad ,$$

inequality (44) implies there exists a positive number Γ_1 such that

$$|\tilde{E}'(y)| \leq \Gamma_1 R_1 |y| \tag{45}$$

for δ_1 small.

Similarly, substituting (35) into equations (22) and (23) and equating the $O(y^2)$ terms gives

$$(1 - \frac{1}{n})\tilde{F}'(y) + \frac{\tilde{P}(y)}{\ell_2}(1 + O(y^2)) = O(|y|^3) \quad ,$$

$$(1 - \frac{1}{n})G'(\phi) + \frac{\tilde{P}(y)}{e_1\ell_2}(1 + O(y^2)) = O(|y|^3) \quad ,$$

from which the inequalities

$$|\tilde{P}(y)| \leq 3\ell_2 R_2 e_1^3 y^2 \quad , \tag{46}$$

$$|\tilde{P}'(y)| \leq \Gamma_2 R_2 |y| \quad , \tag{47}$$

$$|\tilde{F}(y)| \leq R_2 e_1^3 |y|^3 \quad , \tag{48}$$

$$|\tilde{F}'(y)| \leq 3R_2 e_1^3 y^2 \quad , \tag{49}$$

can be derived for δ_1 sufficiently small.

Since $e_1 < c_2$, positive numbers R_1, R_2 exist such that

$$R_2 e_1^3 < R_1 < R_2 c_1^3 \ . \tag{50}$$

If δ_1 is sufficiently small and the R_i satisfy inequality (50), inequalities (40),(41),(48) and (49) imply that inequalities (31) and (32) are satisfied for $\tilde{F}$ and $\tilde{G}$. Inequalities (38),(45),(46) and (47) imply that positive numbers C and D exist such that (33) is satisfied for $\tilde{E}$ and $\tilde{P}$. Consequently, there exist numbers R_1, R_2, C, D such that relation (36) is satisfied.

The mapping M is continuous and satisfies conditions a and b. The existence of a local analytic solution follows from the Schauder fixed-point theorem.

Case 2: Staggered Foci, $S_2 \in A^-$

The existence proof for case 1 can be modified slightly to prove existence of a local solution when the foci are staggered and the lens is in A^-. Since, for this case,

$$|e_1| > |c_1| ,$$

the roles of ξ and ϕ (and their paraxial coefficients) must be interchanged to ensure a well-defined mapping. The proof then follows mutatis mutandis from the proof of case 1.

Case 3: Nested

If the foci are nested and paraxial solutions exist, then inequality (18) is satisfied and the lens must be in A^-. Once one of the two possible paraxial solutions is chosen, existence of a local analytic solution can be established by a slight modification of the above existence proof. In this case, the roles of ξ and ϕ are determined by $\max(|e_1|,|c_1|)$.

Uniqueness:

System (4) can be used directly to show uniqueness of a local analytic solution once one of the paraxial solutions is chosen. Since the solution is analytic for y small, it suffices to show that all the derivatives of f, g, ξ, and ϕ at $y = 0$ can be calculated uniquely. The uniqueness proof follows by induction.

The first derivatives of f and g at $y = 0$ can be calculated uniquely by evaluating system (4) at $y = 0$. The paraxial solution provides $f''(0), g''(0)$ as well as $\xi'(0), \phi'(0)$. Suppose $f^{(k+1)}(0), g^{(k+1)}(0), \xi^{(k)}(0)$ and $\phi^{(k)}(0)$ are known for $k \le m-1$. Differentiating equations (4a)-(4d) m times and evaluating at $y = 0$ yields

$$
\begin{aligned}
&(n-1) f^{(m+1)}(0) + \frac{n}{\ell_2}\, \xi^{(m)}(0) = A_{1,m}(0) \\
&(n-1) g^{(m+1)}(0) c_1^m + \frac{n^2 \xi^{(m)}(0)}{\ell_2 c_1} = A_{2,m}(0) \\
&(n-1) f^{(m+1)}(0) + \frac{n\phi^{(m)}(0)}{\ell_2} = A_{3,m}(0) \\
&(n-1) g^{(m+1)}(0) e_1^m + \frac{n^2 \phi^{(m)}(0)}{\ell_2 e_1} = A_{4,m}(0) \ ,
\end{aligned}
\qquad (51)
$$

where

$$
\begin{aligned}
A_{i,m}(y) = A_{i,m}(y;\ & \xi(y), \xi'(y), \ldots, \xi^{(m-1)}(y); \phi(y), \phi'(y), \ldots, \phi^{(m-1)}(y) \\
& f(y), f'(y), \ldots, f^{(m)}(y); g(\xi), g'(\xi), \ldots, g^{(m)}(\xi); \\
& g(\phi), g'(\phi), \ldots, g^{(m)}(\phi)) \ ,
\end{aligned}
$$

is known at $y = 0$. Since $n > 1$, $e_1 \neq c_1$,

$$\det \begin{vmatrix} n-1 & 0 & \frac{n}{\ell_2} & 0 \\ 0 & (n-1)c_1^m & \frac{n^2}{\ell_2 c_1} & 0 \\ n-1 & 0 & 0 & \frac{n}{\ell_2} \\ 0 & (n-1)e_1^m & 0 & \frac{n^2}{\ell_2 e_1} \end{vmatrix}$$

$$= \frac{n^2(n-1)^2}{\ell_2 c_1 e_1} \{e_1^{m+1} - c_1^{m+1}\} \neq 0 \;;$$

consequently, system (51) can be solved uniquely for $f^{(m+1)}(0), g^{(m+1)}(0), \xi^{(m)}(0), \phi^{(m)}(0)$. The local analytic solution is thus unique.

□

Remark 1. Although the even properties of f and g required for an axial lens are not used directly in the proof, it is evident from system (4) that if $(f(y),g(y),\xi(y),\phi(y))$ is a solution, then $(f(-y),g(-y),-\xi(-y),-\phi(-y))$ must also be a solution. The uniqueness of the analytic solution implies that f,g,ξ and ϕ have the requisite symmetry properties for an axial lens.

Remark 2. The above proof, as mentioned earlier, is very close to that given by Friedman & McLeod (1988). Unlike their proof, however, there are no assumptions regarding the foci configuration and relative distances between foci. The crucial assumptions are that a paraxial solution exists and the foci are distinct.

Remark 3. The symmetric lens is a special case of the multi-foci lens problem. The existence/uniqueness of solutions to this problem has been proved by Friedman

& McLeod (1987) using an approach similar to the above proof. Rogers (1988) has given a different proof of this result.

Remark 4. The above proof shows uniqueness among analytic functions but does not prove uniqueness among smooth functions. This is addressed in Section 4.

3.3 Analytic Continuation

The existence of a unique local analytic solution to the multi-foci lens problem for each paraxial solution was established in the previous section. In this section, it will be shown that the local analytic solution can be analytically continued until either the lens closes or a light ray becomes tangent to the lens. The geometric ideas for this continuation follow those of Friedman & McLeod (1988). The process for analytic continuation is described only for the case when the foci are staggered and the lens is in A^+. The other cases can be addressed in virtually the same manner.

The key to analytic continuation of a local solution lies in the relations

$$B_{\delta_1} \neq B_{\delta_2}$$

$$B_{\delta_1} \subseteq B_{\delta_2} \quad \text{or} \quad B_{\delta_2} \subseteq B_{\delta_1} ,$$

where the B_{δ_i} are as defined in Section 2. It is assumed, for definiteness, that the lens is oriented so that

$$B_{\delta_2} \subseteq B_{\delta_1} ,$$

and that the lens is in A^+ (see figure 4). For the analytic continuation, system (4) is written

$$f'(y) = \frac{H_1(y,\phi,f(y),g(\phi))}{L_1(y,\phi,f(y),g(\phi))} , \tag{52a}$$

$$g'(\xi) = \frac{H_2(y,\xi,f(y),g(\xi))}{L_2(y,\xi,f(y),g(\xi))} , \tag{52b}$$

$$\xi'(y) = -\frac{\partial^2 V}{\partial y^2}\left\{\frac{\partial^2 V}{\partial y\partial\xi}\right\}^{-1} = -\frac{\partial^2 V}{\partial y\partial\xi}\left\{\frac{\partial^2 V}{\partial\xi^2}\right\}^{-1} , \tag{52c}$$

$$\phi'(y) = -\frac{\partial^2 \bar{V}}{\partial y^2}\left\{\frac{\partial^2 \bar{V}}{\partial y\partial\phi}\right\}^{-1} = -\frac{\partial^2 \bar{V}}{\partial y\partial\phi}\left\{\frac{\partial^2 \bar{V}}{\partial\phi^2}\right\}^{-1} , \tag{52d}$$

where

$$H_1 = \frac{n(\phi-y)}{\bar{d}_2} - \frac{y}{\bar{d}_1} ,$$

$$L_1 = \frac{(f(y)-\bar{x}_0)}{\bar{d}_1} - \frac{n\{g(\phi)-f(y)\}}{\bar{d}_2} ,$$

$$H_2 = -\left\{\frac{\xi}{d_3} + \frac{n(\xi-y)}{d_2}\right\} ,$$

$$L_2 = \frac{n\{g(\xi)-f(y)\}}{d_2} - \frac{x_1-g(\xi)}{d_3} ,$$

$$V \triangleq V(\underset{\sim}{p}_0,\underset{\sim}{p}_1) ,$$

$$\bar{V} \triangleq V(\underset{\sim}{q}_0,\underset{\sim}{q}_1) .$$

The strategy for the analytic continuation is first to show that g can be continued from δ_2 to a point arbitrarily close to $\xi(\delta_1)$ using equation (52b). Once this is established, equation (52a) can be used to

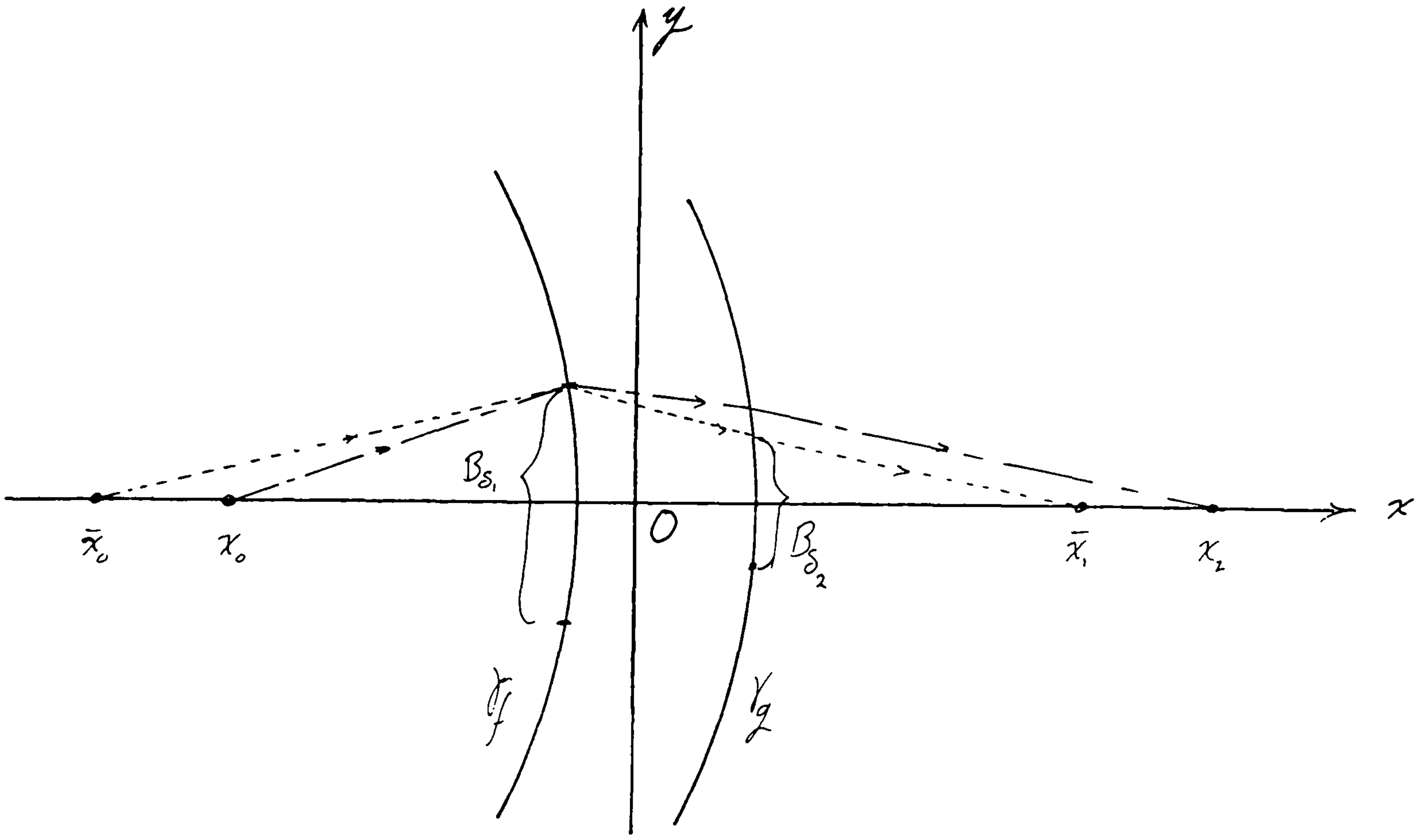


Figure 4: Multi-Foci Lens

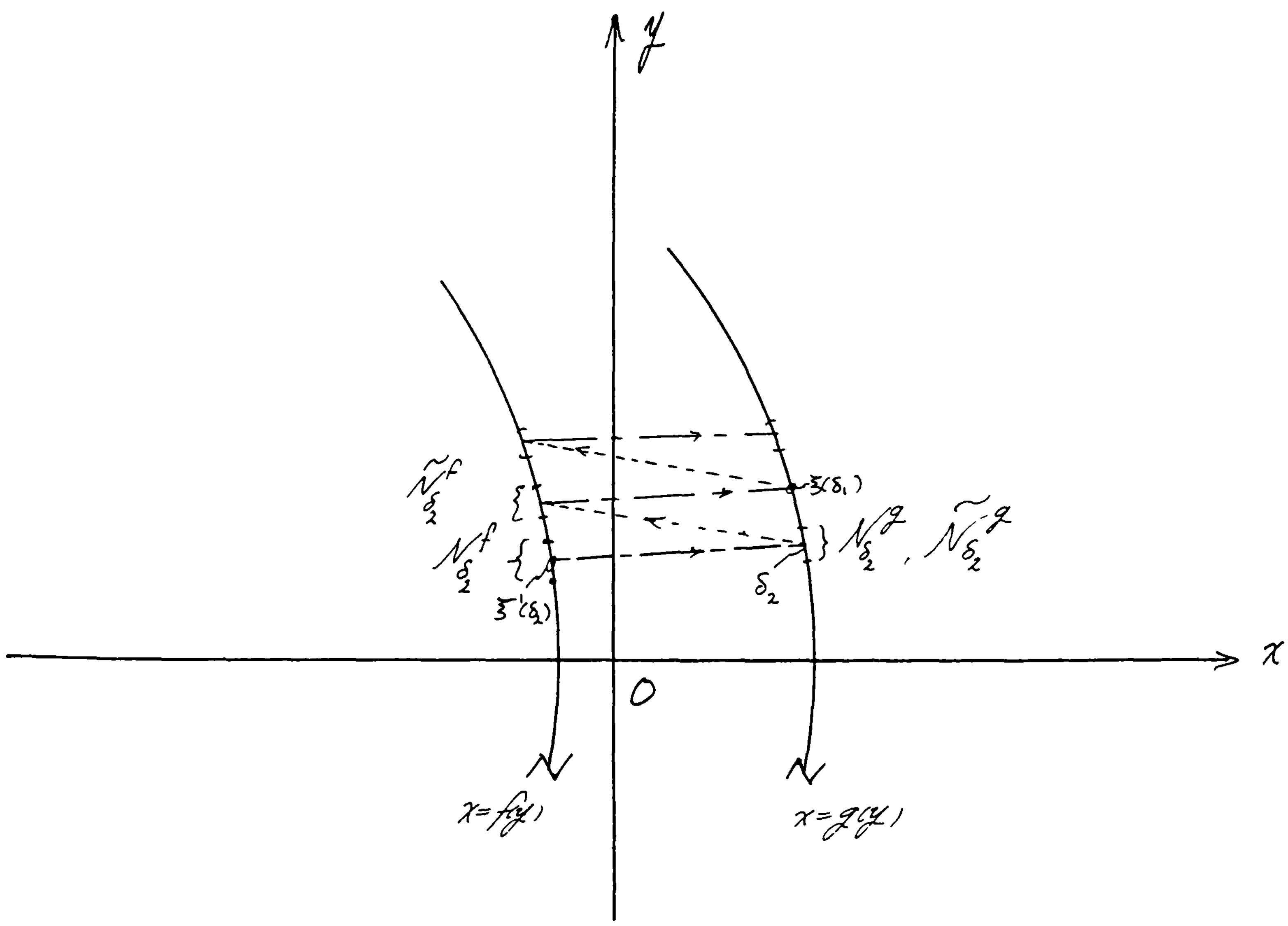


Figure 5: Analytic Continuation

extend f from B_{δ_1} to a point close to $\phi^{-1}(\xi(\delta_1))$. Equations (52e) and (52d) are used to continue ξ and ϕ. The process relies directly on Cauchy's theorem and can be repeated until one of the functions describing f', g', ξ' or ϕ' is no longer analytic in its arguments. A geometric interpretation of this procedure is given in figure 5.

Let

$$N^f_{\delta_2} \triangleq \{y : |y-\xi^{-1}(\delta_2)| < \varepsilon_{\delta_2}\} ,$$

$$N^g_{\delta_2} \triangleq \{y : \xi^{-1}(y) \in N^f_{\delta_2}\} ,$$

where ε_{δ_2} is some positive small number such that

$$N^f_{\delta_2} \subseteq B_{\delta_2} .$$

The above definitions imply

$$N^g_{\delta_2} \not\subseteq B_{\delta_2}$$

$$N^g_{\delta_2} \cap B_{\delta_2} \neq \emptyset .$$

The functions f, ξ, and ϕ are, by hypothesis, analytic in B_{δ_1}. Since H_2, L_2 are analytic in their arguments and L_2 cannot vanish $(n > 1)$, equation (52b) and Cauchy's theorem imply that g is analytic in $N^g_{\delta_2}$ for ε_{δ_2} small. Another point in $N^f_{\delta_2}$ between $\xi^{-1}(\delta_2)$ and $\xi^{-1}(\delta_2) + \varepsilon_{\delta_2}$ can be selected and the procedure repeated. In this manner, g can be extended from δ_2 to some point y^* arbitrarily close to $\xi(\delta_1)$.

Let

$$\tilde{N}^g_{\delta_2} \triangleq \{y : |y-\delta_2| < \tilde{\varepsilon}_{\delta_2}\}$$

$$\tilde{N}^f_{\delta_2} \triangleq \{y : \phi^{-1}(y) \in \tilde{N}^g_{\delta_2}\} ,$$

where $\tilde{\varepsilon}_{\delta_2}$ is some small positive number. The functions g and ϕ are analytic in $\tilde{N}^g_{\delta_2}$ from the above argument for $\tilde{\varepsilon}_{\delta_2}$ sufficiently small, and

$$\tilde{N}^f_{\delta_2} \not\subseteq B_{\delta_1}$$

$$\tilde{N}^f_{\delta_2} \cap B_{\delta_1} \neq \emptyset .$$

Equation (52a) and Cauchy's theorem imply that f is analytic in $\tilde{N}^f_{\delta_2}$ and can ths be continued. The function ξ can be continued using equation (52c). This process can be repeated and f continued up to a point arbitrarily close to $\phi^{-1}(\xi(\delta_1))$ provided Cauchy's theorem can be applied to equations (52a) and (52c). Having continued f and ξ, g and ϕ can then be continued using equations (52b) and (52d) respectively provided Cauchy's theorem can be applied. Once g and ϕ are continued, f and ξ can be continued further using equations (52a) and (52c). The cycle can be repeated ad nauseam until Cauchy's theorem can no longer be applied, i.e. until one of the functions describing f', g', ξ', or ϕ' is no longer analytic in its arguments. Equations (2.28), (2.29), and (2.32) from the proof of Theorem 2.2 imply that a 'breakdown' will occur only if either the lens closes or a light ray from either family becomes tangent to the lens.

3.4 Uniqueness of the Local Solution among C^2 Functions

It was shown in Section 2 that there exists a local analytic solution to the multi-foci lens problem which is unique among analytic functions once one of the two possible paraxial solutions is selected. It is conceivable, however, that non-analytic solutions to the problem exist. This question is addressed in this section, and it will be shown that the local analytic solution of Section 2 is in fact unique among functions in C^2. Before discussing the uniqueness of solutions to the problem, a general uniqueness theorem for a class of functional differential equations is given. The analysis in the proof of this theorem is a blend of that used by Rogers (1988) and Osgood's uniqueness theorem. After the theorem is established, it will be applied specifically to the multi-foci lens problem.

<u>Theorem 2</u>

Given the functional differential equation problem

$$\underset{\sim}{Z}'(y) = \underset{\sim}{H}(y;\underset{\sim}{Z}(z_1),\dots,\underset{\sim}{Z}(z_m);\underset{\sim}{Z}'(z_1),\dots,\underset{\sim}{Z}'(z_m))$$
$$\underset{\sim}{Z}(0) = \underset{\sim}{0}\ , \tag{53}$$

where ' denotes differentiation with respect to the indicated argument, $n,m \in \mathbb{Z}^+$

$1 \le i \le n$,

$0 \le m \le n$,

$$\underset{\sim}{Z}(y) = \begin{pmatrix} z_1(y) \\ \vdots \\ z_n(y) \end{pmatrix} , \quad \underset{\sim}{Z}'(y) = \begin{pmatrix} z_1'(y) \\ \vdots \\ z_n'(y) \end{pmatrix} , \quad \underset{\sim}{H} = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$$

at the h_i satisfy the following conditions:

a. There exists a unique admissible solution*

$\underset{\sim}{Z}'(0)$ to the algebraic system

$$z_i'(0) = h_i(0;\underbrace{0,\ldots,0}_{m \text{ times}};\underbrace{\underset{\sim}{Z}'(0),\ldots,\underset{\sim}{Z}'(0)}_{m \text{ times}}) \ , \quad 1 \le i \le n \tag{54}$$

for which

$$|z_i'(0)| \le 1 \ , \quad 1 \le i \le m \ ; \tag{55}$$

b. There are positive constants ε,ν such that,

for some given vector norm $\|\cdot\|$, in

$$N_\varepsilon \triangleq \{(y;\underset{\sim}{Z}(z_1),\ldots,\underset{\sim}{Z}(z_m);\underset{\sim}{Z}'(z_1),\ldots,\underset{\sim}{Z}'(z_m)) : \frac{1}{\nu}(|y|+\sum_{i=1}^{m}\|\underset{\sim}{Z}(z_i)\|)+\sum_{i=1}^{m}\|\underset{\sim}{Z}'(z_i)-\underset{\sim}{Z}'(0)\| < \varepsilon\} \ , \tag{56}$$

the h_i are uniformly Lipschitz continuous with

respect to their dependence on the $\underset{\sim}{Z}'(z_i)$, i.e. for

$$\underset{\sim}{Z}(z_i) \triangleq \underset{\sim}{Z}_i \ , \quad \underset{\sim}{Z}'(z_i) \triangleq \underset{\sim}{Z}_i' \ ,$$

$$\underset{\sim}{a} = (y;\underset{\sim}{Z}_1,\ldots,\underset{\sim}{Z}_m;\underset{\sim}{Z}_1',\ldots,\underset{\sim}{Z}_m') \ ,$$

$$\hat{\underset{\sim}{a}} = (y;\underset{\sim}{Z}_1,\ldots,\underset{\sim}{Z}_m;\hat{\underset{\sim}{Z}}_1',\ldots,\hat{\underset{\sim}{Z}}_m') \ ,$$

$$\underset{\sim}{H} \triangleq \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \ ,$$

*i.e. if multiple solutions exist there must be enough 'side conditions' to restrict the solutions to (54),(55) down to just one solution. This is the admissible solution.

there exist positive constants L_i such that

$$\|\underset{\sim}{H}(\hat{\underset{\sim}{a}})-\underset{\sim}{H}(a)\| \le \sum_{i=1}^{m} L_i \|\hat{\underset{\sim}{Z}}_i'-\underset{\sim}{Z}_i'\| ; \tag{57}$$

c. The L_i satisfy the inequality

$$0 < \sum_{i=1}^{m} L_i \triangleq \theta < 1 ; \tag{58}$$

d. In N_ε,

$$|h_i| \le 1 , \quad 1 \le i \le m , \tag{59}$$

and for some positive constants Λ_1, Λ_2,

$$\|H\| \le \Lambda_1 , \tag{60}$$

$$\|H'\| \le \Lambda_2 ; \tag{61}$$

e. In N_ε, the h_i are uniformly Lipschitz continuous with respect to their dependence on the $\underset{\sim}{Z}_i$, i.e. for

$$\underset{\sim}{a} = (y;\underset{\sim}{Z}_1,\ldots,\underset{\sim}{Z}_m;\underset{\sim}{Z}_1',\ldots,\underset{\sim}{Z}_m') \in N_\varepsilon ,$$

$$\hat{\underset{\sim}{a}} = (y;\hat{\underset{\sim}{Z}}_1,\ldots,\hat{\underset{\sim}{Z}}_m;\underset{\sim}{Z}_1',\ldots,\underset{\sim}{Z}_m') \in N_\varepsilon ,$$

there exist positive constants K_i such that

$$\|\underset{\sim}{H}(\hat{a})-H(\underset{\sim}{a})\| \le \sum_{i=1}^{m} K_i \|\hat{\underset{\sim}{Z}}_i-\underset{\sim}{Z}_i\| ; \tag{62}$$

if a local solution exists to (53) then it is unique (among C^2 functions).

<u>Proof</u>:

Suppose that $\underset{\sim}{Z}(y)$ and $\hat{\underset{\sim}{Z}}(y)$ are two local solutions and that there exists a $y_0 \neq 0$ such that

$$\|\hat{\underset{\sim}{z}}(y_0)-\underset{\sim}{z}(y_0)\| > 0 \ , \tag{63}$$

for

$$(y_0;\underset{\sim}{z}(z_1(y_0)),\ldots,\underset{\sim}{z}(z_m(y_0));\underset{\sim}{z}'(z_1(y_0)),\ldots,\underset{\sim}{z}'(z_m(y_0))) \in N_\varepsilon$$

$$(y_0;\hat{\underset{\sim}{z}}(\hat{z}_1(y_0)),\ldots,\hat{\underset{\sim}{z}}(\hat{z}_m(y_0));\hat{\underset{\sim}{z}}'(\hat{z}_1(y_0)),\ldots,\hat{\underset{\sim}{z}}'(\hat{z}_m(y_0))) \in N_\varepsilon \ .$$

It is assumed for definiteness that $y_0 > 0$ since a change in variables $\mu = -y$ can be made to handle the opposite case. Let

$$\tau(y) \overset{\Delta}{=} \sup_{-y\le\eta\le y} \|\hat{\underset{\sim}{z}}(\eta)-\underset{\sim}{z}(\eta)\| \tag{64}$$

$$\sigma(y) \overset{\Delta}{=} \sup_{-y\le\eta\le y} \|\hat{\underset{\sim}{z}}'(\eta)-\underset{\sim}{z}'(\eta)\| \ . \tag{65}$$

Relations (64) and (65) yield the inequality

$$\frac{d}{dy}\,\tau(y) \le \sigma(y) \ , \tag{66}$$

and if it can be shown that

$$\sigma(y) < \Gamma\tau(y) \ , \tag{67}$$

where Γ is some positive constant, then it can be shown that $\tau(y)$ cannot vanish for $y < y_0$ and, consequently, one of the solutions cannot meet the initial conditions at $y = 0$ (i.e. inequality (67) gives a contradiction and the theorem is established).

The term $\hat{\underset{\sim}{z}}'(y)-\underset{\sim}{z}'(y)$ can be written

$$\hat{\underset{\sim}{z}}'(y)-\underset{\sim}{z}'(y) = \sum_{i=1}^{m} \underset{\sim}{\mathcal{K}}_i + \sum_{i=1} \underset{\sim}{\mathcal{V}}_i \ ,$$

where

$$\underset{\sim}{\mathcal{H}}_i \triangleq \underset{\sim}{H}(y;\underset{\sim}{Z}(z_1),\ldots,\underset{\sim}{Z}(z_{i-1}),\hat{\underset{\sim}{Z}}(\hat{z}_i),\ldots,\hat{\underset{\sim}{Z}}(\hat{z}_m);\hat{\underset{\sim}{Z}}'(\hat{z}_1),\ldots,\hat{\underset{\sim}{Z}}'(\hat{z}_m))$$

$$- \underset{\sim}{H}(y;\underset{\sim}{Z}(z_1),\ldots,\underset{\sim}{Z}(z_i),\hat{\underset{\sim}{Z}}(\hat{z}_{i+1}),\ldots,\hat{\underset{\sim}{Z}}(\hat{z}_m);\hat{\underset{\sim}{Z}}'(\hat{z}_1),\ldots,\hat{\underset{\sim}{Z}}'(\hat{z}_m)) ,$$

$$\underset{\sim}{\mathcal{U}}_i \triangleq \underset{\sim}{H}(y;\underset{\sim}{Z}(z_1),\ldots,\underset{\sim}{Z}(z_m);\underset{\sim}{Z}'(z_1),\ldots,\underset{\sim}{Z}'(z_{i-1}),\hat{\underset{\sim}{Z}}'(\hat{z}_i),\ldots,\hat{\underset{\sim}{Z}}'(\hat{z}_m))$$

$$- \underset{\sim}{H}(y;\underset{\sim}{Z}(z_1),\ldots,\underset{\sim}{Z}(z_m);\underset{\sim}{Z}'(z_1),\ldots,\underset{\sim}{Z}'(z_i),\hat{\underset{\sim}{Z}}'(z_{i+1}),\ldots,\hat{\underset{\sim}{Z}}'(\hat{z}_m)) ,$$

so that

$$\sigma(y) \leq \sup_{-y\leq\eta\leq y} \sum_{i=1}^{m} \{||\underset{\sim}{\mathcal{H}}_i|| + ||\underset{\sim}{\mathcal{U}}_i||\} . \tag{68}$$

Condition e implies

$$||\underset{\sim}{\mathcal{H}}_i|| \leq K_i||\hat{\underset{\sim}{Z}}(\hat{z}_i)-\underset{\sim}{Z}(z_i)||$$

$$\leq K_i\{ ||\hat{\underset{\sim}{Z}}(\hat{z}_i)-\underset{\sim}{Z}(\hat{z}_i)|| + ||\underset{\sim}{Z}(\hat{z}_i)-\underset{\sim}{Z}(z_i)||\} ,$$

and since

$$|z_i| \leq |y| , \quad 1 \leq i \leq m$$

by inequality (59),

$$\sup_{-y\leq\eta\leq y} ||\underset{\sim}{\mathcal{H}}_i|| \leq K_i\{\tau(y) + \sup_{-y\leq\eta\leq y} ||\underset{\sim}{Z}(\hat{z}_i)-\underset{\sim}{Z}(z_i)||\} . \tag{69}$$

Using the mean value theorem,

$$||\underset{\sim}{Z}(\hat{z}_i)-\underset{\sim}{Z}(z_i)|| = ||\underset{\sim}{Z}'(y^*)|| \; |\hat{z}_i-z_i| ,$$

where y^* is between $\hat{z}_i$ and z_i. Inequality (60) implies

$$\sup_{-y\leq\eta\leq y} ||\underset{\sim}{Z}(\hat{z}_i)-\underset{\sim}{Z}(z_i)|| \leq \Lambda_1 C\tau(y) , \tag{70}$$

where C is some positive constant. The $\underset{\sim}{\mathcal{H}}_i$ terms can thus be bounded by

$$\sup_{-y\le\eta\le y} \|\underset{\sim}{\mathcal{K}}_i\| \le K_i\{1 + C\Lambda_1\}\tau(y) \ . \tag{71}$$

Condition b implies

$$\|\underset{\sim}{\mathcal{U}}_i\| \le L_i \|\hat{\underset{\sim}{Z}}'(\hat{z}_i) - \underset{\sim}{Z}'(z_i)\| ,$$

and inequality (59) implies

$$\sup_{-y\le\eta\le y} \|\underset{\sim}{\mathcal{U}}_i\| \le L_i\{\sigma(y) + \sup_{-y\le\eta\le y} \|\underset{\sim}{Z}'(\hat{z}_i) - \underset{\sim}{Z}'(z_i)\|\} . \tag{72}$$

The last term in inequality (72) can be bounded using the mean value theorem and inequality (61), i.e.,

$$\sup_{-y\le\eta\le y} \|\underset{\sim}{Z}'(\hat{z}_i) - \underset{\sim}{Z}'(z_i)\| \le$$

$$\sup_{\substack{-y\le\eta\le y\\ -y\le\eta^*\le y}} \|\underset{\sim}{Z}''(\eta^*)\| \, |\hat{z}_i - z_i| \le \Lambda_2 C\tau(y) \ ; \tag{73}$$

consequently,

$$\sup_{-y\le\eta\le y} \|\underset{\sim}{\mathcal{U}}_i\| \le L_i\{\sigma(y) + \Lambda_2 C\tau(y)\} \ . \tag{74}$$

Inequalities (68),(71) and (74) imply

$$\sigma(y) \le \sum_{i=1}^{m} K_i\{1+C\Lambda_1\}\tau(y) + \sum_{i=1}^{m} L_i\{\sigma(y)+\Lambda_2 C\tau(y)\} \ .$$

Condition c and the above inequality imply

$$0 < (1-\theta)\sigma(y) \le \{(1+C\Lambda_1)\sum_{i=1}^{m} K_i + \Lambda_2 C\theta\}\tau(y) \ ,$$

so that inequality (67) is satisfied for any Γ such that

$$\Gamma > \frac{1}{1-\theta}\{(1 + C\Lambda_1)\sum_{i=1}^{m} K_i + \Lambda_2 C\theta\} \ . \tag{75}$$

Inequality (67) implies that for $y < y_0$, there is a positive number $\tilde{K}$ such that

$$\tau(y) > \tilde{K}e^{\Gamma y} \ , \tag{76}$$

so that, in particular,

$$\tau(0) \neq 0 \ , \tag{77}$$

which contradicts the hypothesis.

□

Remark 1. Condition a, the algebraic compatibility condition is essential to the proof since N_ε cannot be defined for arbitrarily small ε without imposing existence and uniqueness conditions on the algebraic system. For the multi-foci lens problem, this condition translates to the existence of paraxial solutions and the selection of a partdular one.

Remark 2. Inequality (59) restricts the functional arguments so that in N_ε, $|z_i| \leq |y|$, $1 \leq i \leq m$. This requirement is crucial in order to obtain inequalities (69) and (72). A counter example if (59) is not satisfied has been given by Kato & McLeod (1971).

Remark 3.. Condition c ensures that the derivatives of the z_i are uniquely solvable in N_ε. Inequality (58) implies that the h_i essentially form a contraction for this determination.

Remark 4. Inequality (61) restricts the admissible solutions to those having second derivatives. This requirement can be relaxed to the assumption that the z_i are in C^1 and the h_i are uniformly Lipschitz continuous in N_ε with respect to y. This generalisation is discussed further in Appendix I.

Remark 5. Using the above assumptions in Theorem 2, it can be shown that a solution to (53) exists. The proof is essentially a generalisation of the Picard-Lindelöf theorem and is given in Appendix I.

Theorem 3

Any solution (f,g,ξ,ϕ) to the multi-foci lens problem which is in C^2 must be analytic at $y = 0$.

Proof:

Since $f,g \in C^2$, Theorem 2.2 implies that $\xi'(y)$ and $\phi'(y)$ cannot change sign; consequently, any solution to the problem which is in C^2 must correspond to a lens in either A^+ or A^-. If the foci are nested, the lens must be in A^-. Once the foci configuration is specified there are only two possible cases. It suffices to show that for each case the solution is analytic. As the proofs are similar for each case, only the proof for staggered foci, $S_2 \in A^+$ is given in detail. Comments concerning the other cases are given near the end of the proof.

Assume that the foci are staggered and the solution corresponds to a lens in A^+. The first step of the proof is to show that the functional differential equations describing the problem can be written in the form of system (53). Let

$$\begin{array}{llll} z_1(y) = y & & z_0(y) = f(y)-a_0 & \\ z_2(y) = \xi^{-1}(\phi(y)) & & z_0(y) = f'(y) & \\ & & & (78) \\ z_3(y) = \xi(y) & & z_7(y) = g(\xi)-b_0 & \\ z_4(y) = \phi(y) & & z_8(y) = g'(\xi)\ . & \end{array}$$

The argument y is suppressed for the remainder of this section unless there is some possibility of confusion.

It is evident that

$$h_1 = 1$$

$$h_5 = z_6 \tag{79}$$

$$h_7 = z_8 z_3' \ .$$

The remaining h_i's can be calculated using system (4). In terms of the new variables, the optical path length functions are

$$V \triangleq V(z_1,z_3,z_5,z_7) = \{(z_5 + \ell_1)^2 + z_1^2\}^{\frac{1}{2}}$$

$$+ n\{(z_7-z_5 + \ell_2)^2 + (z_3-z_1)^2\}^{\frac{1}{2}}$$

$$+ \{(\ell_3-z_7(z_2))^2 + z_4^2\}^{\frac{1}{2}}$$

$$\bar{V} \triangleq \bar{V}(z_1,z_4,z_5,z_7(z_2)) = \{(z_5 + \bar{\ell}_1)^2 + z_1^2\}^{\frac{1}{2}}$$

$$+ n\{(z_7(z_2)-z_5 + \ell_2)^2 + (z_4-z_1)^2\}^{\frac{1}{2}}$$

$$+ \{(\bar{\ell}_3-z_7(z_2))^2 + z_4^2\}^{\frac{1}{2}}$$

$$U \triangleq V(z_1(z_2),z_3(z_2),z_5(z_2),z_7(z_2)) = \{(z_5(z_2) + \ell_1)^2 + z_2^2\}^{\frac{1}{2}}$$

$$+ n\{(z_7(z_2)-z_5(z_2) + \ell_2)^2 + (z_4-z_2)^2\}^{\frac{1}{2}}$$

$$+ \{(\ell_3-z_7(z_2))^2 + z_3^2\}^{\frac{1}{2}} \ .$$

Let

$$\underline{\mu} \triangleq (\mu_1,\mu_2,\ldots,\mu_8) = (z_1,z_2,z_3,z_4,z_5,z_5(z_2),z_7,z_7(z_2)) \ ,$$

$$V_i = \frac{\partial V}{\partial \mu_i} \ , \qquad V_{ij} = \frac{\partial^2 V}{\partial \mu_i \partial \mu_j} \quad \text{etc.} \ .$$

It is clear from the above expressions that $V, \bar{V}$ and U are analytic in the μ_k, $k = 1,2,\ldots,8$. System (4) (i.e. Snell's Law) implies

$$V_1 + z_6 V_5 = 0 \tag{80}$$

$$V_3 + z_8 V_7 = 0 \tag{81}$$

$$\bar{V}_1 + z_6 \bar{V}_5 = 0 \tag{82}$$

$$\bar{V}_4 + z_8(z_2)\bar{V}_8 = 0 \tag{83}$$

$$U_2 + z_6(z_2)U_6 = 0 \tag{84}$$

$$U_4 + z_8(z_2)U_8 = 0 . \tag{85}$$

Differentiating equation (84) with respect to y and solving for $z_2'(y)$ yields

$$h_2 = \frac{-\{U_{24}+z_6(z_2)U_{64}\}z_4'}{U_{22}+2z_6(z_2)U_{26}+z_8(z_2)U_{28}+z_6'(z_2)U_6+z_6^2(z_2)U_{66}+z_6(z_2)z_8(z_2)U_{68}} \tag{86}$$

Similarly, differentiating equations (80) and (83) with respect to y and solving for $z_3'(y), z_4'(y)$ respectively gives

$$h_3 = \frac{-\{V_{11}+2z_6V_{15}+z_6'V_5+z_6^2V_{55}\}}{V_{13}+z_6V_{53}+z_8V_{17}+z_6z_8V_{57}} , \tag{87}$$

$$h_4 = \frac{-\{\bar{V}_{41}+z_6\bar{V}_{45}+z_8(z_2)z_2'\bar{V}_{48}+z_8(z_2)z_2'\bar{V}_8+z_8(z_2)(\bar{V}_{81}+z_6\bar{V}_{85}+z_8(z_2)z_2'\bar{V}_{88})\}}{\bar{V}_{44}+z_8(z_2)\bar{V}_{84}} . \tag{88}$$

Expressions for h_6 and h_8 can be obtained by differentiating equations (82) and (81) respectively, i.e.

$$h_6 = -\frac{1}{\bar{V}_5}\{\bar{V}_{11} + z_4'\bar{V}_{14} + 2z_6\bar{V}_{15} + z_8(z_2)z_2'\bar{V}_{18} + z_6\{z_4'\bar{V}_{54} + z_6\bar{V}_{55} + z_8(z_2)z_2'\bar{V}_{58}\}\} , \tag{89}$$

$$h_8 = -\frac{1}{V_7}\{V_{31} + z_3'V_{33} + z_6V_{35} + 2z_8z_3'V_{37} + z_3\{V_{17} + z_6V_{75} + z_8z_3'V_{77}\}\} \tag{90}$$

The functional differential equations describing the multi-foci lens can thus be written in the form of (53) where $n = 8$, $m = 2$.

The partial derivatives in equations (86)-(90) are analytic in $\underset{\sim}{Z}(z_i)$, $i = 1,2$, and if it can be shown that the denominators in the expressions for the h_i do not vanish at $y = 0$, then the continuity of the $\underset{\sim}{Z}(z_i), \underset{\sim}{Z}'(z_i)$ implies that there exists an ε such that the h_i are analytic in the variables $\underset{\sim}{Z}(z_i), \underset{\sim}{Z}'(z_i)$ for $(y;\underset{\sim}{Z}(z_1),\underset{\sim}{Z}(z_2);\underset{\sim}{Z}'(z_1),\underset{\sim}{Z}'(z_2)) \in N_\varepsilon$. At

$$\underset{\sim}{a}_0 \overset{\Delta}{=} (0;0,0;\underset{\sim}{Z}'(0),\underset{\sim}{Z}'(0)) \in N_\varepsilon ,$$

the aforesaid denominators are non-zero, i.e.

$$U_{22}+2z_6(z_2)U_{26}+z_8(z_2)U_{28}+z_6'(z_2)U_6+z_6^2(z_2)U_{66}+z_6(z_2)z_8(z_2)U_{68}$$

$$= U_{22}+z_6'(0)U_6 = \frac{nc_1}{\ell_2} > 0 ,$$

$$V_{13}+z_6V_{53}+z_8V_{17}+z_6z_8V_{57} = V_{13} = \frac{-n}{\ell_2} < 0 ,$$

$$\bar{V}_{44}+z_8(z_2)\bar{V}_{84} = \bar{V}_{44} = \frac{n}{\bar{\ell}_2} + \frac{1}{\bar{\ell}_3} > 0 ,$$

$$\bar{V}_5 = 1-n < 0 ,$$

$$V_7 = n-1 > 0 ,$$

so that for ε sufficiently small, the h_i are analytic in N_ε. Conditions b and e of Theorem 2 are thus satisfied.

The solution, by hypothesis, is in C^2. Since ξ and ϕ are in C^2 and the h_i are analytic in their variables, f and g must be in C^3. There exists, consequently, a paraxial solution corresponding to the solution (f,g,ξ,ϕ) in C^2. There are two solutions to the algebraic system (54) but the additional restriction that $S_2 \epsilon A^+$ eliminates one of the solutions so that system (54) has a unique admissible solution. Since $S_2 \epsilon A^+$

$$0 < z_4 < z_3 \,, \quad y > 0 \,,$$

and, consequently,

$$0 < z_2 < y \,, \quad y > 0 \tag{91}$$

$$0 < z_2'(0) < 1 \,. \tag{92}$$

Condition a of Theorem 2 is thus satisfied.

The h_i at $\underset{\sim}{a}_0$ are

$$h_1 = 1 \qquad h_5 = 0$$

$$h_2 = \frac{e_1}{c_1} < 1 \qquad h_6 = 2a_2$$

$$h_3 = c_1 \qquad h_7 = 0$$

$$h_4 = e_1 \qquad h_8 = 2b_2 \,,$$

and it is clear that inequalities (59) and (60) are satisfied for ε sufficiently small. The functions f and g are C^3 so that the z_i must be at least C^2; consequently, there exists a constant such that inequality (70) is satisfied. The system satisfies condition d.

The remaining condition to satisfy in order to apply Theorem 2 is c. Since L_1 can be chosen to be zero, it suffices to show that in some vector norm, for ε sufficiently small, there exists an L_2 such that

$$\|\underset{\sim}{H}(\hat{\underset{\sim}{a}})-\underset{\sim}{H}(\underset{\sim}{a})\| \le L_2 \, \|\hat{\underset{\sim}{z}}_2'-\underset{\sim}{z}_2'\| \qquad L_2 < 1 \tag{93}$$

(cf. inequalities (57),(58)). The only h_i containing a $z_i'(z_2)$ term is h_2. At $\underset{\sim}{a}_0$,

$$\frac{\partial h_2}{\partial z_2'(z_2)} = \frac{(n-1)e_1\ell_2}{nc_1^2} \; ,$$

so that a vector norm of the form

$$\|\underset{\sim}{V}\| \overset{\Delta}{=} \max(\lambda_1|V_1|,\lambda_2|V_2|,\ldots,\lambda_8|V_8|) \; ,$$

where

$$\underset{\sim}{V} = (V_1,V_2,\ldots,V_8) \; ,$$

and the λ_i are positive numbers such that

$$0 < \lambda_2 < \frac{nc_1^2}{(n-1)e_1\ell_2} \; ,$$

ensures that (93) is satisfied for ε sufficiently small.

The conditions of Theorem 2 are thus satisfied and it follows that the local solution is unique among C^2 functions. Theorem 1 implies that a local analytic solution exists; therefore, Theorem 3 follows for the case where the lens is in A^+.

If the solution is required to be in A^- and the foci are staggered, then the result can be obtained using

$z_2 = \phi^{-1}(\xi)$ instead of $\xi^{-1}(\phi)$. The effect of this variable change is to ensure that inequalities (55) and (59) are satisfied. The proof is essentially the same as the above (mutatis mutandis). If the foci are nested, the result can be obtained using $z_2 = a^{-1}(b)$ where, in N_ε,

$$a = \max(|\xi|,|\phi|)$$

$$b = \min(|\xi|,|\phi|) .$$

□

3.5 Analytic Dependence on Parameters and Stability

The dependence of the solution to the multi-foci lens problem on parameters such as foci locations is investigated in this section. The primary motivation for the investigation is to examine the stability of the global solution. It will be shown in this section that the solution depends analytically on the parameters corresponding to the foci locations, the index of refraction, and the axial lens thickness (i.e. on the parameters $x_0, \bar{x}_0, x_1, \bar{x}_1, n, a_0, b_0$ in system (4)). This analytic dependence, in turn, can be exploited to show that the global solution constructed in Section 3 is stable under small perturbations of the aforesaid parameters.

Let δ,ζ be positive numbers and

$$B_\delta \triangleq \{y : |y| < \delta\}$$

$$\tilde{B}_\zeta \triangleq \{\lambda : |\lambda-\hat{\lambda}| < \zeta\}$$

$$\tilde{N}_\varepsilon \triangleq N_\varepsilon \times \tilde{B}_\varepsilon ,$$

where $\hat{\lambda}$ is some number and N_ε is defined by (56). The next theorem shows that a local solution which is analytic in y is also analytic in a parameter λ provided the h_i are analytic in λ.

Theorem 4

Suppose for $\lambda \in \tilde{B}_\varepsilon$ there exists a solution $\underset{\sim}{z}$ to

$$z_i'(y) = h_i(y,\lambda;\underset{\sim}{z}(z_1),\ldots,\underset{\sim}{z}(z_m);\underset{\sim}{z}'(z_1),\ldots,\underset{\sim}{z}'(z_m)) \tag{94}$$

$$\underset{\sim}{z}(0) = \underset{\sim}{0}\ , \quad 0 \le m \le n\ , \quad 1 \le i \le n\ ,$$

where in $\tilde{N}_\varepsilon$ the h_i satisfy the conditions of Theorem 2 and, in addition, are analytic in λ. If the z_i are analytic in y for $y \in B_\delta$, then there exists a neighbourhood $J \subseteq B_\delta \times \tilde{B}_\varepsilon$ such that the z_i are analytic in y and λ.

Proof:

If $\underset{\sim}{z}$ is a solution to (94) and the h_i satisfy the conditions of Theorem 2, then it can be shown (cf. Appendix I) that the sequence

$$\underset{\sim}{z}^{(0)}(y) = y\underset{\sim}{z}'(0)$$

$$\underset{\sim}{z}^{(k+1)}(y) = \int_0^y H(s,\lambda;\underset{\sim}{z}^{(k)}(z_1^{(k)}),\ldots,\underset{\sim}{z}^{(k)}(z_m^{(k)}); \underset{\sim}{z}^{(k)}(z_1^{(k)}),\ldots,\underset{\sim}{z}^{(k)}(z_m^{(k)}))ds \tag{95}$$

$$k \in \mathbb{Z}^+\ ,$$

converges uniformly for y sufficiently small, and that

$$\lim_{k\to\infty} \underset{\sim}{z}^{(k)}(y) = \underset{\sim}{z}(y)\ . \tag{96}$$

Since $\underset{\sim}{H}$ is analytic in λ, $\underset{\sim}{Z}$ can be written as the limit of a uniformly convergent sequence of functions analytic in λ for $\lambda \in \tilde{B}_\varepsilon$. Consequently, there exists a neighbourhood $J \subseteq B_\delta \times \tilde{B}_\delta$ wherein $\underset{\sim}{Z}$ is analytic in both y and λ.

□

The above theorem can be extended for several parameters. It is clear from the proof of Theorem 3 that the system of equations describing the multi-foci lens satisfies the conditions of Theorem 2 and that the h_i are analytic in the parameters $x_0, \bar{x}_0, x_1, \bar{x}_1, a_0, b_0, n$. If the foci are staggered, it is evident from Section 2 that an analytic local solution exists if the parameters are perturbed a small amount because a paraxial solution corresponding to the perturbed state exists. If the foci are nested, a similar statement can be made provided the parameters satisfy

$$4n^2 < \ell_2^2\{\frac{1}{\ell_1} - \frac{1}{\bar{\ell}_1}\}\{\frac{1}{\ell_3} - \frac{1}{\bar{\ell}_3}\} \tag{97}$$

(cf. inequality (17)). The above inequality ensures that a paraxial solution exists for sufficiently small perturbations of the parameter. If

$$4n^2 = \ell_2^2\{\frac{1}{\ell_1} - \frac{1}{\bar{\ell}_1}\}\{\frac{1}{\ell_3} - \frac{1}{\bar{\ell}_3}\} , \tag{98}$$

then only one paraxial solution exists, and a small perturbation of a parameter could result in complex paraxial coefficients (consequently, no real analytic solution would exist for the perturbed problem).

Theorem 4 implies that a local solution to system (4) is analytic in y and its parameters provided either the foci are staggered or inequality (97) is satisfied.

If one of the parameters (say the location of a source) were perturbed slightly, the question arises which, roughly speaking, asks whether or not the resulting lens 'looks' approximately the same as the original. More precisely, let λ be some parameter (e.g. focus location, index of refraction etc.) and $(f(y,\lambda),g(y,\lambda),\zeta(y,\lambda),\phi(y,\lambda))$ a solution to system (4). Suppose λ is perturbed slightly to λ^*, i.e.

$$|\lambda-\lambda^*| \sim O(\eta) ,$$

where η is some small parameter, and $(f(y,\lambda^*),g(y,\lambda^*),\xi(y,\lambda^*),\phi(y,\lambda^*))$ is the solution to the perturbed problem. The question is whether or not

$$\|(f(y,\lambda^*),g(y,\lambda^*),\xi(y,\lambda^*),\phi(y,\lambda^*))-(f(y,\lambda),g(y,\lambda),\xi(y,\lambda),\phi(y,\lambda))\| \sim O(\eta) , \tag{99}$$

for $y \sim O(1)$. It is assumed, of course, that the allowable perturbed solutions have paraxial solutions corresponding to the same branch of the governing quadratic equation (e.g. if the original lens is in A^+, the allowable perturbed solution must also be in A^+). It is clear from Theorem 4 that the local solution satisfies (99). The primary concern is if the global solution constructed in Section 3 also satisfies (99). This stability question has been addressed by Rogers (loc. cit.) for the general system of Theorem 2. A different (albeit less general) proof can be given to show that the solution is stable in the above sense for η sufficiently small. The proof exploits the analytic properties of the local solution.

Corollary 5

Let λ be some parameter

$$\bar{I}_\delta \triangleq \{y : 0 \le y \le \delta\} ,$$

and $\underset{\sim}{Z}$ be a solution to (95) analytic in y for $y \in \bar{I}_\delta$. If the h_i are analytic in λ for

$$\underset{\sim}{a}(y) \in A_\delta \triangleq \{\underset{\sim}{a} : y \in \bar{I}_\delta\}$$

$$\underset{\sim}{a}(y) \triangleq (y,\lambda;\underset{\sim}{Z}(z_1(y)),\ldots,\underset{\sim}{Z}(z_m(y));\underset{\sim}{Z}'(z_1(y)),\ldots,\underset{\sim}{Z}'(z_m(y)) ,$$

then there exists a $\zeta > 0$ such that $\underset{\sim}{Z}$ is analytic in y and λ for $(y,\lambda) \in \bar{I}_\delta \times B_\zeta$.

Proof:

The proof follows along the lines of an analogous result in the theory of ordinary differential equations [cf. Lefschetz 1963, Chapter 2]. The proof is sketched here.

The set $\bar{I}_\delta$ is compact and therefore the analytic continuation of the local solution in $\bar{I}_\delta$ can be constructed in a finite number of steps. Let $\bigcup_{j=1}^{k} I_j$ be a chain for the continuation such that $0 \in I_1$, $\delta \in I_k$. Since $I_j \cap I_{j+1} \ne \emptyset$, $j = 1,2,\ldots,k-1$, points $y_1,\ldots,y_{k-1}$ can be chosen such that $y_j \in I_j \cap I_{j+1}$. By hypothesis, the h_i are analytic in λ for $a \in A_\delta$, and Theorem 4 can be applied (mutatis mutandis) to each I_j; consequently, there exists a ζ for each I_j such that the z_i are analytic in $I_j \times \tilde{B}_{\zeta_j}$. As k is finite, a ζ can be chosen so small that the z_i are analytic in $\bar{I}_\delta \times \tilde{B}_\zeta$.

$\square$

The formulation of the multi-foci problem in Section 4 can be used to show that the conditions of Corollary 5 are satisfied for the global solution constructed in Section 3. In particular, $V, \bar{V}$ and U are analytic in $x_0, \bar{x}_0, x_1, \bar{x}_1, a_0, b_0, n$, and the h_i are analytic in these parameters for $y \in \bar{I}_\delta$ if the z_i are analytic in y. Corollary 5, in turn, implies that the solution is stable in the above sense.

The results of this and the previous sections can be combined and summarized in terms of the paraxial solutions. The advantage of this formulation is that once the parameters are given (e.g. foci locations, lens thickness etc.) the paraxial coefficients can be calculated from the simple relations given in Section 1. The existence, uniqueness, and stability of the system depends on whether or not the coefficients are real and the number of solutions to the paraxial equations (i.e. whether or not equation (98) is satisfied). The following Theorem is a statement of this.

Theorem 6.

If two paraxial solutions exist to system (4), then there exists a local solution corresponding to each paraxial solution. Once one of the paraxial solutions is selected, the corresponding local solution is unique among C^2 functions and is analytic. The local solution can be analytically continued until either the lens closes or a light ray becmes tangent to the lens boundaries. The global solution is stable with respect to small perturbations of the initial data, foci locations, and index of refraction.

3.6 An Asymptotic Solution for a Limiting Case

The previous sections of this chapter discussed questions concerning existence, uniqueness, and analyticity of solutions to the multi-foci lens problem. In this section, methods for constructing solutions are briefly discussed. The limiting case when the foci in the object space coincide and the foci in the image space coincide (i.e. $\bar{x}_0 \to x_0$, $\bar{x}_1 \to x_1$) is examined, and an algorithm for an asymptotic expansion is developed using the distance between the object space foci as the 'small parameter'. The crucial advantage of this method is that it requires the solution of ordinary differential equations as opposed to functional differential equations. The leading order term in this expansion corresponds to a special lens known as the "Herschel Lens" and is discussed further in the next section.

The uniqueness proof given in Section 2 suggests one obvious method for constructing a solution, viz. the direct calculation of derivatives at $y = 0$. Theorem 1 implies the solution (f,g,ξ,ϕ) can be written

$$
\begin{aligned}
f(y) &= \sum_{k=0}^{\infty} a_{2k} y^{2k} \\
g(y) &= \sum_{k=0}^{\infty} b_{2k} y^{2k} \\
\xi(y) &= \sum_{k=0}^{\infty} c_{2k+1} y^{2k+1} \\
\phi(y) &= \sum_{k=0}^{\infty} e_{2k+1} y^{2k+1} ,
\end{aligned}
$$

where a_2, b_2, c_1, e_1, are known from the paraxial solution. After the appropriate compositions, these expressions can be substituted into system (4) and, equating terms of equal order in y, the coefficients can be determined ad nauseam. The essence of the aforesaid uniqueness proof is that, after the paraxial coefficients are determined, the other coefficients can be determined from a set of linear equations (cf. system (51)). Although in principle the solution can be constructed using this method, it is algebraically awkward because of the nonlinear occurence of f,g,ξ,ϕ in the equations of system (4) and the functional arguments.

Another method for constructing a solution is given by the sequence of successive approximations introduced in Section 5 (equation (95)). This sequence is discussed at length in Appendix I in connection with an alternative existence/uniqueness proof. The sequence converges uniformly and, in principle, a solution can be constructed using this method. Like the previous method, however, it is awkward to use owing to the complicated expressions for the h_i (cf. equations (86)-(90)) and the functional arguments.

Rather than attack the full problem, the limiting case is considered when the foci nearly coincide, i.e., when $\bar{x}_0 \to x_0$, $\bar{x}_1 \to x_1$. Let

$$\bar{x}_0 = x_0 + \eta\alpha_0$$

$$\bar{x}_1 = x_1 + \eta\alpha_1 ,$$

where η is some positive number such that

$$\eta << 1 ,$$

for $O(1)\ell_i$, and the α_i are constants. Equation (10) for the paraxial solution implies

$$c_1 = \pm\{\frac{\ell_3\alpha_0}{\ell_1\alpha_1}\}^{\frac{1}{2}} + O(\eta) ,$$

from which it can be seen that α_0 and α_1 must be of the same order for a smooth solution as $\eta \to 0$. Specifically, if $\frac{\alpha_0}{\alpha_1}$ is $O(\eta)$ then c_1 is $O(\sqrt{\eta})$ and consequently b_2 is $O(1/\sqrt{\eta})$ (equation (7)). If $\frac{\alpha_0}{\alpha_1}$ is $O(\frac{1}{\eta})$ then c_1 is $O(1/\sqrt{\eta})$ and consequently a_2 is $O(1/\sqrt{\eta})$. It is clear from the discussion in Section 1 that a paraxial solution exists only if α_0 and α_1 have the same sign (i.e. the foci are staggered).

It was shown in Chapter 2 that pairs of foci satisfy the 'path length condition' which can be derived from the governing equations. The system of equations

$$\frac{\partial V}{\partial \xi} = 0 \tag{100a}$$

$$V = \ell_1 + n\ell_2 + \ell_3 \tag{100b}$$

$$\frac{\partial \bar{V}}{\partial y} = 0 \tag{100c}$$

$$\bar{V} = \ell_1 - \eta\alpha_0 + n\ell_2 + \ell_3 + \eta\alpha_1 \ , \tag{100d}$$

consequently is equivalent to system (4). Motivated by the stability/analyticity theorem in the previous section, let

$$\begin{aligned} f(y) &= f_0(y) + \eta f_1(y) + \eta^2 f_2(y) + O(\eta^3) \\ g(y) &= g_0(y) + \eta g_1(y) + \eta^2 g_2(y) + O(\eta^3) \\ \xi(y) &= \xi_0(y) + \eta \xi_1(y) + \eta^2 \xi_2(y) + O(\eta^3) \\ \phi(y) &= \phi_0(y) + \eta \phi_1(y) + \eta^2 \phi_2(y) + O(\eta^3) \ . \end{aligned} \tag{101}$$

The $d_i, \bar{d}_i$ can thus be written

$$\begin{aligned} d_i &= d_{i0} + \eta d_{i1} + \eta^2 d_{i2} + O(\eta^3) \\ \bar{d}_i &= \bar{d}_{i0} + \eta \bar{d}_{i1} + \eta^2 \bar{d}_{i2} + O(\eta^3) \end{aligned} \tag{102}$$

for i = 1,2,3. Clearly,

$$\begin{aligned} \xi_0(y) &= \phi_0(y) \\ d_{i0} &= \bar{d}_{i0} \ . \end{aligned} \tag{103}$$

Let

$$\Delta_k = \bar{d}_{1k} - d_{1k} + \eta(\bar{d}_{2k} - d_{2k}) + \bar{d}_{3k} - d_{3k} , \quad k \in \mathbb{Z}^+ . \quad (104)$$

Substituting relations (102) into equations (100b) and (100c), and equating terms of equal order in η gives the relations

$$\Delta_1 = \alpha_1 - \alpha_0 \quad (105)$$

$$\Delta_k = 0 , \quad k \geq 2 . \quad (106)$$

Using

$$g(\xi) = g_0(\xi_0) + \eta\tilde{g}_1 + \eta^2\tilde{g}_2 + \dots ,$$

$$g(\phi) = g_0(\xi_0) + \eta\hat{g}_1 + \eta^2\hat{g}_2 + \dots ,$$

where

$$\tilde{g}_1 \triangleq g_1(\xi_0) + \xi_1 g_0'(\xi_0)$$

$$\tilde{g}_2 \triangleq g_2(\xi_0) + \xi_1 g_1'(\xi_0) + \xi_2 g_0'(\xi_0) + \frac{\xi_1^2}{2} g_0''(\xi_0)$$

$$\hat{g}_1 \triangleq g_1(\xi_0) + \phi_1 g_0'(\xi_0)$$

$$\hat{g}_2 \triangleq g_2(\xi_0) + \phi_1 g_1'(\xi_0) + \phi_2 g_0'(\xi_0) + \frac{\phi_1^2}{2} g_0''(\xi_0) , \quad \text{etc.},$$

the $d_{ij}, \bar{d}_{ij}$ can be calculated. For $j = 0,1,2,$

$$d_{10} = \{(f_0(y)-x_0)^2 + y^2\}^{\frac{1}{2}} ,$$

$$d_{11} = \frac{f_1(y)}{d_{10}} \{f_0(y)-x_0\} ,$$

$$d_{12} = \frac{(f_0(y)-x_0)}{d_{10}} \left\{f_2(y) - \frac{(f_0(y)-x_0)f_1(y)}{d_{10}^2}\right\} ,$$

$$d_{20} = \{(g_0(\xi_0)-f_0(y))^2 + (\xi_0-y)^2\}^{\frac{1}{2}} ,$$

$$d_{21} = \frac{1}{d_{20}} \{(g_0(\xi_0)-f_0(y))(\tilde{g}_1-f_1(y))+(\xi_0-y)\xi_1\} ,$$

$$d_{22} = \frac{1}{2d_{20}} \{2(g_0(\xi_0)-f_0(y))(\tilde{g}_2-f_2(y)) + 2(\xi_0-y)\xi_2 + (\tilde{g}_1-f_1))^2 + \xi_1^2\}$$

$$- \frac{1}{2d_{20}^3} \{(g_0(\xi_0)-f_0(y))(\tilde{g}_1-f_1(y))+(\xi_0-y)\xi_1\}^2 ,$$

$$d_{30} = \{(x_1-g_0(\xi_0))^2 + \xi_0^2\}^{\frac{1}{2}} ,$$

$$d_{31} = \frac{1}{d_{30}} \{-(x_1-g_0(\xi_0))\tilde{g}_1 + \xi_0\xi_1\} ,$$

$$d_{32} = \frac{1}{2d_{30}} \{-2(x_1-g_0(\xi_0))\tilde{g}_2 + \tilde{g}_1^2 + 2\xi_0\xi_2 + \xi_1^2\}$$

$$- \frac{1}{2d_{30}^3} \{-(x_1-g_0(\xi_0))\tilde{g}_1 + \xi_0\xi_1\}^2 ,$$

$$\bar{d}_{10} = d_{10} ,$$

$$\bar{d}_{11} = d_{11} - \frac{\alpha_0(f_0(y)-x_0)}{d_{10}} ,$$

$$\bar{d}_{12} = d_{12} + \frac{1}{2d_{10}} \{d_{11}^2 + \alpha_0^2\} - \frac{1}{2d_{10}^3} \{d_{10}d_{11} - (f_0(y)-x_0)\alpha_0\}^2 ,$$

$$\bar{d}_{20} = d_{20} ,$$

$$\bar{d}_{21} = \frac{1}{d_{20}} \{(g_0(\xi_0)-f_0(y))(\hat{g}_1-f_1(y))+(\xi_0-y)\phi_1\} ,$$

$$\bar{d}_{22} = \frac{1}{2d_{20}} \{2(g_0(\xi_0)-f_0(y))(g_2-f_2(y)) + 2(\xi_0-y)\phi_2 + (\hat{g}_1-f_1(y))^2 + \phi_1^2\}$$

$$- \frac{1}{2d_{20}^3} \{(g_0(\xi_0)-f_0(y))(\hat{g}_1-f_1(y))+(\xi_0-y)\phi_1\}^2 ,$$

$$\bar{d}_{30} = d_{30} ,$$

$$\bar{d}_{31} = \frac{1}{d_{30}} \{-(x_1-g_0(\xi_0))\hat{g}_1 + \xi_0\phi_1 + \alpha_1(x_1-g_0(\xi_0))\} ,$$

$$\bar{d}_{32} = \frac{1}{2d_{30}} \{-2(x_1-g_0(\xi_0))\hat{g}_2 + \hat{g}_1^2 + 2\xi_0\phi_2 + \phi_1^2 - 2\alpha_1\hat{g}_1 + \alpha_1^2\} ,$$

$$+ \frac{1}{2d_{30}^3} \{-(x_1-g_0(\xi_0))\hat{g}_1 + \xi_0\phi_1 + \alpha_1(x_1-g_0(\xi_0))\}^2 ,$$

so that

$$\Delta_0 = 0$$

$$\Delta_1 = \frac{-(f_0(y)-x_0)\alpha_0}{d_{10}} + \frac{(x_1-g_0(\xi_0))\alpha_1}{d_{30}}$$
$$+ (\phi_1-\xi_1)\{\frac{n}{d_{20}}((g_0(\xi_0)-f_0(y))g_0'(\xi_0) + \xi_0-y)$$
$$+ \frac{1}{d_{30}}(-(x_1-g_0(\xi_0))g_0'(\xi_0) + \xi_0)\} , \quad (107)$$

$$\Delta_2 = \Gamma(y,\xi_0,\xi_1,\phi_1,f_0(y),f_1(y),g_0(\xi_0),g_0'(\xi_0),g_0''(\xi_0),g_1(\xi_0))$$
$$+ (\phi_2-\xi_2)\{\frac{n}{d_{20}}((g_0(\xi_0)-f_0(y))g_0'(\xi_0) + \xi_0-y)$$
$$+ \frac{1}{d_{30}}(-(x_1-g_0(\xi_0))g_0'(\xi_0) + \xi_0)\} , \quad (108)$$

where Γ_2 is known in terms of the variables indicated.

Equation (100a) implies

$$\frac{n}{d_{20}}\{(g_0(\xi_0)-f_0(y))g_0'(\xi_0)+\xi_0-y\} + \frac{1}{d_{30}}\{-(x_1-g_0(\xi_0))g_0'(\xi_0)+\xi_0\}$$
$$= 0 , \quad (109)$$

and, consequently, the last terms in equations (107) and (108) vanish giving the relations

$$1 - \frac{(f_0(y)-x_0)}{d_{10}} = M_H\{1 - \frac{x_1-g_0(\xi_0)}{d_{30}}\} , \quad (110)$$

$$\Gamma_2 = 0 , \quad (111)$$

where

$$M_H \triangleq \frac{\alpha_1}{\alpha_0} > 0 . \quad (112)$$

Although equation (110) is derived by equating $O(\eta)$ terms, the functions f_1,g_1,ξ_1,ϕ_1 do not appear in this expression. Equation (110) involves only terms of the leading order solution.

The leading order problem is

$$\frac{(f_0(y)-x_0)f_0'(y)+y}{d_{10}} - \frac{n\{(g_0(\xi_0)-f_0(y))f_0'(y)+\xi_0-y\}}{d_{20}} = 0 \ , \quad (113)$$

$$d_{10} + nd_{20} + d_{30} = \ell_1 + n\ell_2 + \ell_3 \ . \quad (114)$$

Equations (113) and (114) describe the general axial lens, and the problem is degenerate in that an infinite number of solutions exist (cf. Section 2.3). Equation (110), however, provides an additional relation so that ξ_0 is known in terms of g_0, f_0 and y, i.e.

$$\xi_0(y) = \pm\{x_1-g_0(\xi_0)\}\{(1-\frac{1}{M_H}(1-\frac{(f_0(y)-x_0)}{d_{10}}))^{-1}-1\}^{\frac{1}{2}} \ , \quad (115)$$

and it is clear from Theorem 2.1 that precisely two local solutions exist to the leading order problem. The solutions are analytic and correspond to a lens in A^+ and one in A^-. Relation (110) is known as the Herschel condition and axial lenses satisfying this relation are called Herschel lenses. It is evident from equations (114) and (115) that ξ_0 and $g(\xi_0)$ can be eliminated in equation (113). The leading order problem can thus be reduced to one involving a first order nonlinear differential equation for $f_0(y)$ and algebraic relations for g_0 and ξ_0. This ordinary differential equation is derived in the next section where the Herschel lens is further discussed.

Once the leading order solution is obtained, f_1, g_1, ξ_1, ϕ_1 can be determined using relation (111) (which does not involve f_2, g_2, ξ_2, ϕ_2) and the relations

$$\frac{\partial}{\partial y}\{d_{11} + nd_{21}\} = 0 \tag{116}$$

$$d_{11} + nd_{21} + d_{31} = 0 \tag{117}$$

$$\bar{d}_{11} + n\bar{d}_{21} + \bar{d}_{31} = \alpha_1 - \alpha_0 \ . \tag{118}$$

As with the leading order solution, $g_1(\xi_0), \xi_1, \phi_1$ can be eliminated from (116) using equations (111), (117) and (118), and the problem reduced to a linear ordinary differential equation for $f_1(y)$.

The procedure can be repeated to get higher order terms. In particular, f_k, g_k, ξ_k, ϕ_k can be calculated from

$$\frac{\partial}{\partial y}\{d_{2k} + nd_{2k}\} = 0 \tag{119}$$

$$d_{1k} + nd_{2k} + d_{3k} = 0 \tag{120}$$

$$\bar{d}_{1k} + n\bar{d}_{2k} + \bar{d}_{3k} = 0 \tag{121}$$

$$\Delta_{k+1} = 0 \ . \tag{122}$$

Equation (122) is akin to an orthogonality relation derived from the Fredholm alternative, and provides a relation between ξ_k and ϕ_k. The functions g_k, ξ_k, and ϕ_k can be eliminated from (119) using equations (120)-(122). As above, the problem is reduced to a linear first order differential equation for f_k. A flowchart of the process is given in figure 6. In summary, once the Herschel lens is determined, the higher order terms can be calculated by quadrature and algebra.

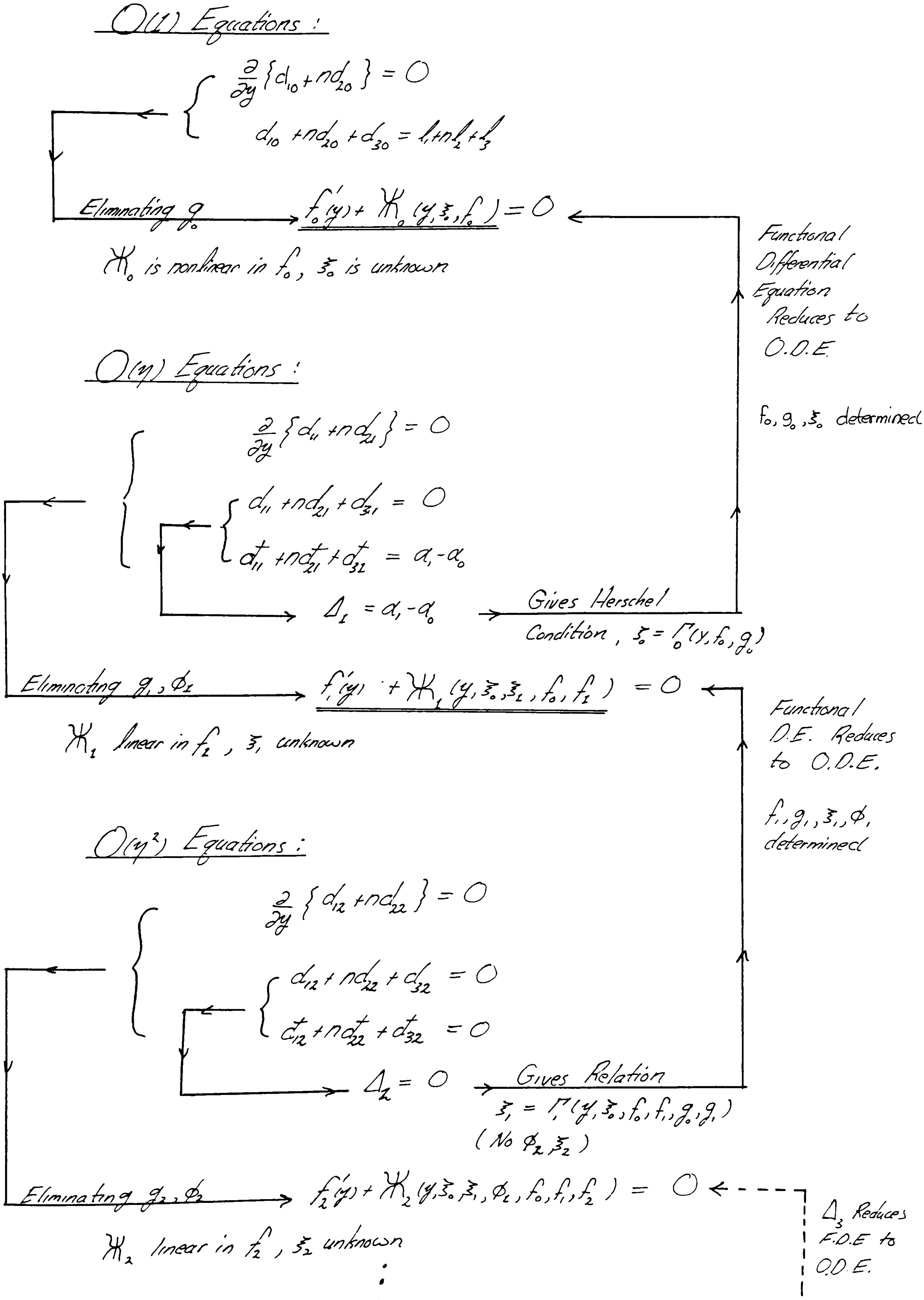


Figure 6: Flowchart For Asymptotic Method

3.7 The Herschel Lens

The Herschel lens introduced in Section 6 is examined in this section, and the leading order problem is reduced to a first order nonlinear ordinary differential equation. The limiting lens as $\eta \to 0$ has been examined by Friedman & McLeod (1988) who concluded that, in general, it does not correspond to a symmetric lens (i.e. $f(y) \neq -g(y)$), and by van-Brunt (1988) who showed that as $\eta \to 0$ the solution is asymptotic to a Herschel lens.

The Herschel condition, oddly enough, was actually 'discovered' by J. Herschel (1821). A remarkable feature of equation (110) is that the foci separation parameters α_0, α_1 occur only in the ratio M_H. This means crudely that the leading order solution for α_0, α_1 is also that for any O(1) numbers $\hat{\alpha}_0, \hat{\alpha}_1$ provided

$$\hat{\alpha}_1/\hat{\alpha}_0 = M_H ,$$

and, as a result, the Herschel lens has special "imaging" properties on the optical axis. These properties are discussed below in the context of Herschel's problem.

Let S_2 be an axial lens having foci at $\underset{\sim}{p}_0 = (x_0, 0)$ $\underset{\sim}{p}_1 = (x_1, 0)$. In general, if the source were perturbed by some $O(\eta)$ translation along the optical axis, the light rays leaving the perturbed source would converge in some $O(\eta)$ neighbourhood of $\underset{\sim}{p}_1$. The light rays in the image space would envelope a semi-cubical parabola symmetric about the optical axis and confined in some region with $O(\eta)$ dimensions. Suppose it is required that the perturbed source at $x_0 + \eta\alpha_0$ 'nearly focus' to some specified point $x_1 + \eta\alpha_1$. By 'nearly focus' it is meant that light rays leaving the perturbed source converge in some $O(\eta^2)$ neighbourhood N of $x_1 + \eta\alpha_1$.

The optical path length between $\underset{\sim}{q}_0$ and any point $\hat{\underset{\sim}{q}}$ on the optical axis in N is given by

$$V(\underset{\sim}{q}_0,\hat{\underset{\sim}{q}}) = \ell_1 + n\ell_2 + \ell_3 + \eta(\alpha_1-\alpha_0) + O(\eta^2) \ . \tag{123}$$

The optical path length can also be written

$$\begin{aligned} V(\underset{\sim}{q}_0,\hat{\underset{\sim}{q}}) &= V(\underset{\sim}{p}_0,\underset{\sim}{p}_1) + \eta\alpha_0 \frac{\partial V(\underset{\sim}{p}_0,\underset{\sim}{p}_1)}{\partial x_0} + \eta\alpha_1 \frac{\partial V(\underset{\sim}{p}_0,\underset{\sim}{p}_1)}{\partial x_1} + O(\eta^2) \\ &= \ell_1 + n\ell_2 + \ell_3 + \eta\{\alpha_0 \frac{\partial V(\underset{\sim}{p}_0,\underset{\sim}{p}_1)}{\partial x_0} + \alpha_1 \frac{\partial V(\underset{\sim}{p}_0,\underset{\sim}{p}_1)}{\partial x_1}\} \\ &\quad + O(\eta^2) \ . \end{aligned} \tag{124}$$

Since light rays leaving $\underset{\sim}{q}_0$ are required to converge in N, the $O(\eta)$ terms in equation (124) must be constant. Equation (123) implies

$$\alpha_1-\alpha_0 = \alpha_0 \frac{\partial V(\underset{\sim}{p}_0,\underset{\sim}{p}_1)}{\partial x_0} + \alpha_1 \frac{\partial V(\underset{\sim}{p}_0,\underset{\sim}{p}_1)}{\partial x_1} \ , \tag{125}$$

which yields the Herschel condition.

The above derivation of the Herschel condition implies that all points on the optical axis in an $O(\eta)$ neighbourhood of N_0 of $\underset{\sim}{p}_0$ have conjugate points in some $O(\eta)$ neighbourhood N_1 of $\underset{\sim}{p}_1$. If the source is moved to any $\hat{\underset{\sim}{q}}_0 = (x_0+\eta\hat{\alpha}_0, 0) \in N_0$, $\hat{\underset{\sim}{q}}$ will nearly focus at $\hat{q}_1 = (x_1+\eta\hat{\alpha}_1, 0) \in N_1$, where $\hat{\alpha}_1 = M_H\hat{\alpha}_0$. Physically, this means that a small segment of the optical axis containing

$\underset{\sim}{p}_0$, is 'imaged' to a small line segment of the optical axis containing $\underset{\sim}{p}_1$; the line segment in the object space is magnified by a factor of M_H in the image space and, in general, the only (perfect) pair of foci is $\underset{\sim}{p}_0, \underset{\sim}{p}_1$. An example of a Herschel lens for $M_H = 1$ is the symmetric lens with symmetric foci discussed in Chapter 2.

Given the Herschel condition, the functional differential equations for an axial lens can be reduced to a first order nonlinear ordinary differential equation and algebraic relations. To mitigate algebraic manipulation, let

$$w_0 = \frac{y}{d_1} = \sin\psi_0$$

$$w_1 = -\xi/d_3 = \sin\psi_1 \ ,$$

where ψ_0, ψ_1 are the angles between the light ray and the optical axis in the object and image spaces respectively (figure 7). Using the new variables, the d_i are given by

$$d_1 = \frac{(f(w_0)-x_0)}{(1-w_0^2)^{\frac{1}{2}}}$$

$$d_2 = \{(g(w_1)-f(w_0))^2 + (d_3w_1 + d_1w_0)^2\}^{\frac{1}{2}}$$

$$d_3 = \frac{(x_1-g(w_1))}{(1-w_1^2)^{\frac{1}{2}}} \ ,$$

and the Herschel condition is

$$1 - (1-w_0^2)^{\frac{1}{2}} = M_H\{1 - (1-w_1^2)^{\frac{1}{2}}\} \ .$$

The Herschel condition provides an algebraic relation between w_0 and w_1. The functional argument is thus known. Using the path length condition

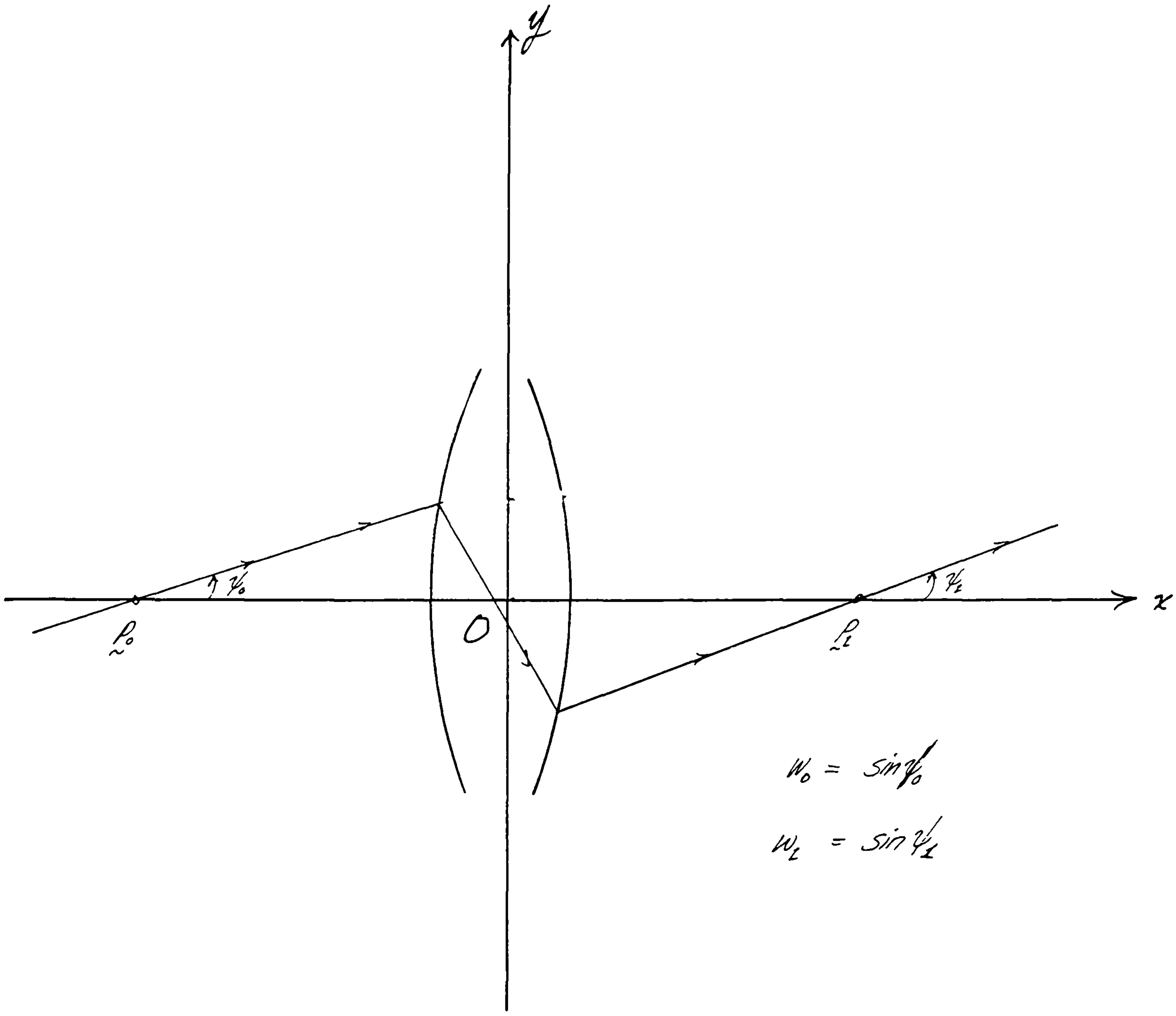


Figure 7: Geometric Interpretation of New Coordinates w_0, w_1

$$d_1 + nd_2 + d_3 = \ell_1 + n\ell_2 + \ell_3 \overset{\Delta}{=} K ,$$

and the Herschel condition, the function g can be determined in terms of f and w_0. Snell's law at γ_f (i.e. equation (113)) yields the following first order nonlinear ordinary differential equation:

$$\frac{df(w_0)}{dw_0} = \left\{\frac{n}{d_2}\left(\frac{w_1d_3+w_0d_1}{(1-w_0^2)^{\frac{1}{2}}} - g(w_1)+f(w_0)\right) + \frac{1}{(1-w_0^2)^{\frac{1}{2}}}\right\}^{-1} \times$$

$$\left\{-\frac{w_0d_1}{(1-w_0^2)^{\frac{1}{2}}} + \frac{n(w_1d_3+w_0d_1)d_1}{d_2(1-w_0^2)}\right\} , \qquad (126)$$

with the initial condition

$$f(0) = a_0 .$$

The Herschel lens problem can thus be converted into a problem involving an ordinary differential equation and algebraic relations. Cauchy's theorem or Theorem 2.1 can be used to show that two analytic local solutions exist for (126). The Herschel lens problem can also be written as a coupled system of ordinary differential equations using Snell's law at γ_f ang γ_g. This formulation is discussed briefly in Born & Wolf (1986).

Chapter 4

Dispersive Stigmatic Axial Lenses

The index of refraction for a given isotropic medium is, in general, a function of position and frequency. The geometrical optics approximation neglects the wave character of light, and the index of refraction is considered a function solely of position. Geometrical optics provides an accurate approximation in regions having dimensions much greater than the wavelengths of light provided the source is nearly monochromatic or the refractive index for the particular medium is not sensitive to frequency changes. If a polychromatic source is present, however, then even at the crude level of the paraxial optics approximation (i.e. Gaussian optics) the influence of frequency on the focusing properties of an optical system can be detected for many common optical materials. For instance, in the visible light spectrum, dense flint glass has a refractive index of approximately 1.74 for red light and approximately 1.84 for violet light. Other materials such as hard crown glass and fluorite are less sensitive to frequency in the visible light spectrum. The variation of the refractive index with frequency is called <u>dispersion</u>.

Dispersion is not governed by any 'simple' physical law and a large body of literature including many diverse theories has been published attempting to describe the phenomenon. The theories (still in use) date from

Cauchy (1830) to modern quantum theories (e.g. London 1973), and the theoretical (and empirical) formulae vary considerably in structure (cf. Welford, 1986 for further comments). Welford (loc. cit.) comments that for optical design, precision up to five decimal places is often required and that the mathematical form of the dependence is still in question.

Roughly speaking, two types of dispersions occur, viz. normal and anomalous. Normal dispersion refers to small changes in the refractive index for small changes in frequency; anomalous dispersion refers to large changes in the refractive index for small changes in frequency. Anomalous dispersion occurs when the frequency is close to the absorption frequency of the material. Because of the high absorption losses, materials with an absorption frequency in the visible light spectrum have limited optical design value. In the visible light spectrum, normal dispersion occurs for most optical materials, and the refractive index, in general, increases as frequency increases.

An axial lens which focuses light from the same source at two different frequencies to different foci in the image space is studied in this chapter. The governing equations and the paraxial solution are discussed in the first section. The problem is similar to the multi-foci lens problem of the previous chapter, and the analysis of Chapter 3 is modified to establish existence, uniqueness and analyticity in the second section.

A method by which the local solution can be analytically continued is described in the third section. In the fourth section a limiting case is investigated and the leading order solution, called the 'chromatic lens', is described. The chromatic lens has special imaging properties similar to the Herschel lens described in Section 3.7.

The object and image spaces, as in the previous chapters, are assumed in the vacuum where $n = 1$ for all frequencies. The lens material is assumed such that only normal dispersion occurs in any frequency range under consideration (e.g. visible light). The index of refraction is assumed to be some known continuous function of frequency.

4.1 The Dispersive Stigmatic Axial Lens Problem

Let S_2 be an axial lens having boundaries γ_f, γ_g satisfying initial condition (2.25), and suppose the lens focuses $\underset{\sim}{p}_0 = (x_0, 0)$ to $\underset{\sim}{p}_1 = (x_1, 0)$ for some given frequency ω. Suppose further that at some other frequency $\bar{\omega}$, the lens focuses p_0 to some other point $\underset{\sim}{q}_1 = (\bar{x}_1, 0)$ (see figure 1). Since the medium has a piecewise constant

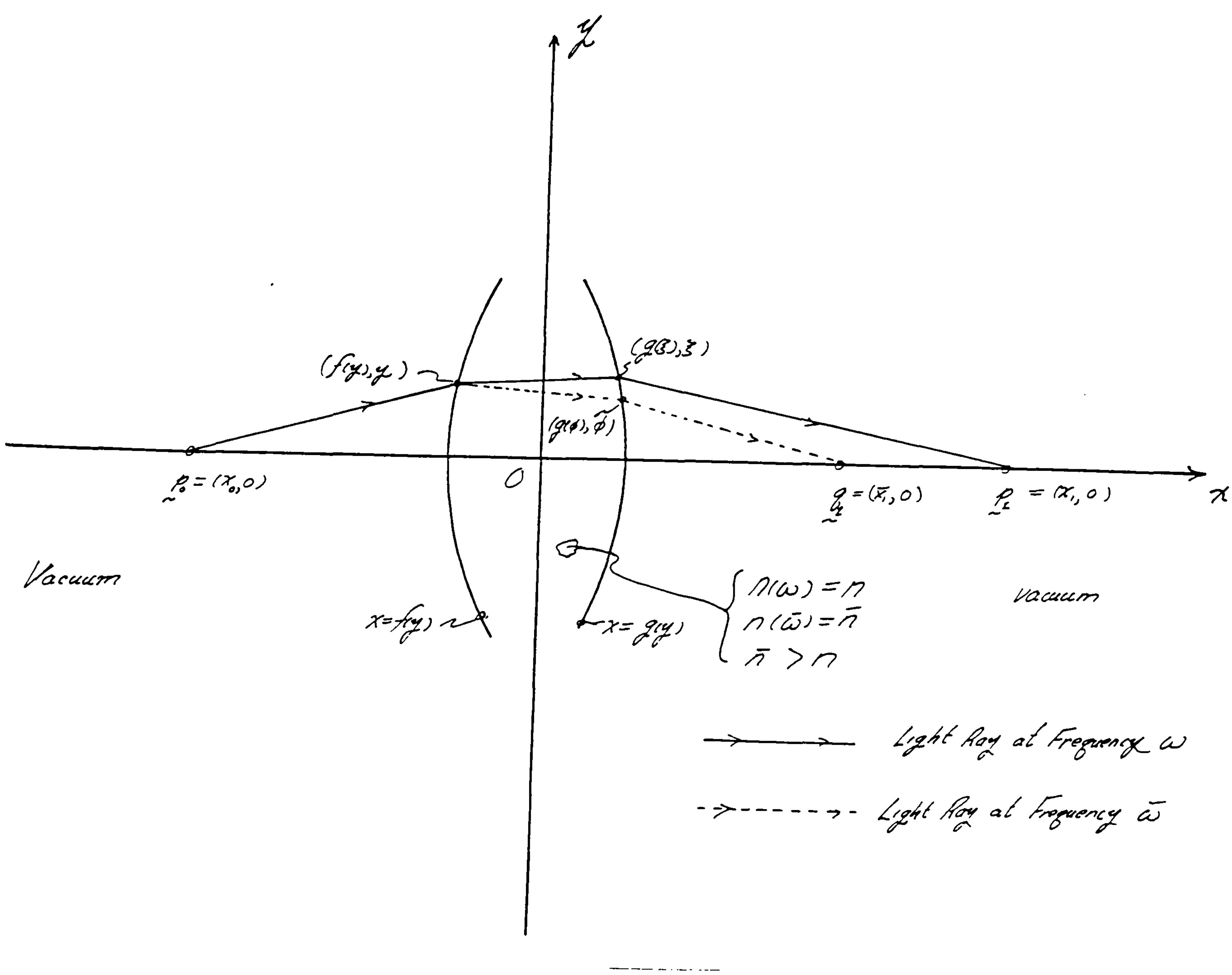


Figure 1: Dispersive Stigmatic Axial Lens

index of refraction, dispersion is noticeable only at the lens boundaries. The object and image spaces are assumed in the vacuum so that n = 1; whereas, the refractive index for S_2 is frequency dependent. Let

$$n(\omega) \triangleq n$$
$$n(\bar{\omega}) \triangleq \bar{n} \,. \tag{1}$$

It is assumed for definiteness that

$$\bar{n} > n \,. \tag{2}$$

Consider the light rays at both frequencies leaving $\underset{\sim}{p}_0$, entering the lens at $(f(y),y) \in \gamma_f$. The light rays are collinear in the object space but after refraction at γ_f the light ray paths split (figure 1). Let $(g(\xi),\xi)$ be the lens exit point for the light ray at frequency ω and $(g(\phi),\phi)$ be the lens exit point for the light ray at frequency $\bar{\omega}$. The optical path lengths between the foci are given by

$$V = d_1 + nd_2 + d_3 \,, \tag{3}$$

$$\bar{V} = d_1 + \bar{n}\bar{d}_2 + \bar{d}_3 \,, \tag{4}$$

where

$$d_1 = \{(f(y)-x_0)^2 + y^2\}^{\frac{1}{2}}$$
$$d_2 = \{(g(\xi)-f(y))^2 + (\xi-y)^2\}^{\frac{1}{2}} \tag{5}$$
$$d_3 = \{(x_1-g(\xi))^2 + \xi^2\}^{\frac{1}{2}}$$

$$\bar{d}_2 = \{(g(\phi)-f(y))^2 + (\phi-y)^2\}^{\frac{1}{2}} \tag{6}$$
$$\bar{d}_3 = \{(\bar{x}_0-g(\phi))^2 + \phi^2\}^{\frac{1}{2}} \,.$$

The 'lens equations' of Chapter 2 (i.e. Snell's Law) gives the following system of equations':

$$\frac{\partial V}{\partial y} = \frac{(f(y)-x_0)f'(y)+y}{d_1} - \frac{n\{(g(\xi)-f(y))f'(y)+\xi-y\}}{d_2} = 0 \,, \qquad (7a)$$

$$\frac{\partial V}{\partial \xi} = \frac{n\{(g(\xi)-f(y))g'(\xi)+\xi-y\}}{d_2} + \frac{-(x_1-g(\xi))g'(\xi)+\xi}{d_3} = 0 \,, \qquad (7b)$$

$$\frac{\partial \bar{V}}{\partial y} = \frac{(f(y)-x_0)f'(y)+y}{d_1} - \frac{\bar{n}\{(g(\phi)-f(y))f'(y)+\phi-y\}}{\bar{d}_2} = 0 \,, \qquad (7c)$$

$$\frac{\partial \bar{V}}{\partial \phi} = \frac{\bar{n}\{(g(\phi)-f(y))g'(\phi)+\phi-y\}}{\bar{d}_2} + \frac{-(\bar{x}_1-g(\phi))g'(\phi)+\phi}{\bar{d}_3} = 0 \,. \qquad (7d)$$

The above system of first order nonlinear functional differential equations is similar to system (4) of Chapter 3, and in the following sections of this chapter the analysis of Chapter 3 will be exploited to establish existence, uniqueness, and analyticity. The remaining initial conditions for system (7) are given by symmetry with respect to the optical axis, i.e. since f and g are even in y,

$$\begin{aligned} \xi(0) &= 0 \\ \phi(0) &= 0 \,. \end{aligned} \qquad (8)$$

System (7) along with initial conditions (2.25) and (8) is termed the Dispersive Stigmatic Lens Problem.

The lens boundaries γ_f, γ_g are assumed smooth and Theorem 2.2 implies that ξ and ϕ are also smooth. The functions f, g, ξ and ϕ can thus be written

$$
\begin{aligned}
f(y) &= a_0 + a_2y^2 + O(y^4) \\
g(y) &= b_0 + b_2y^2 + O(y^4) \\
\xi(y) &= c_1y + O(|y|^3) \\
\phi(y) &= e_1y + O(|y|^3) \ .
\end{aligned}
\tag{9}
$$

Substituting these expressions into system (7) and equating O(y) terms gives the following equations for the paraxial coefficients a_2, b_2, c_1, e_1 (cf. Section 2.4):

$$-2(n-1)a_2\ell_1\ell_2 + \ell_2 + n\ell_1 = n\ell_1c_1 \ , \tag{10}$$

$$2(n-1)b_2\ell_2\ell_3 + \ell_2 + n\ell_3 = n\ell_3/c_1 \ , \tag{11}$$

$$-2(\bar{n}-1)a_2\ell_1\ell_2 + \ell_2 + \bar{n}\ell_1 = \bar{n}\ell_1e_1 \ , \tag{12}$$

$$2(n-1)b_2\ell_2\bar{\ell}_3 + \ell_2 + \bar{n}\bar{\ell}_3 = \bar{n}\bar{\ell}_3/e_1 \ , \tag{13}$$

where

$$
\begin{aligned}
\ell_1 &= a_0 - x_0 \\
\ell_2 &= b_0 - a_0 \\
\ell_3 &= x_1 - b_0 \\
\bar{\ell}_3 &= \bar{x}_1 - b_0 \ .
\end{aligned}
\tag{14}
$$

Eliminating a_2 from equations (10) and (12) gives

$$e_1 = \frac{n(\bar{n}-1)c_1}{\bar{n}(n-1)} - \left(1 + \frac{\ell_2}{\ell_1}\right)\frac{(\bar{n}-n)}{(n-1)} \ , \tag{15}$$

and eliminating b_2 from equations (11) and (13) gives

$$\frac{n(\bar{n}-1)e_1}{\bar{n}(n-1)} - \left\{\bar{n}-n + \ell_2\left(\frac{\bar{n}-1}{\ell_3} - \frac{n-1}{\bar{\ell}_3}\right)\right\}\frac{e_1c_1}{\bar{n}(n-1)} - c_1 = 0 \ ; \tag{16}$$

consequently, c_1 must be a solution to the quadratic equation

$$Ac_1^2 + Bc_1 + C = 0 \ , \tag{17}$$

where

$$\begin{aligned} A &= n(\bar{n}-1)\{\bar{n}-n + \ell_2(\frac{\bar{n}-1}{\ell_3} - \frac{n-1}{\bar{\ell}_3})\} \\ B &= -(\bar{n}-n)\{n(\bar{n}-1) + \bar{n}(n-1) + (1+\frac{\ell_2}{\ell_1})(\bar{n}-n+\ell_2(\frac{\bar{n}-1}{\ell_3} - \frac{n-1}{\bar{\ell}_3}))\} \\ C &= (\bar{n}-n)n(\bar{n}-1)(1+\frac{\ell_2}{\ell_1}) \ . \end{aligned} \tag{18}$$

A real paraxial solution exists provided the discriminant of (17) is non-negative, i.e. provided

$$\begin{aligned} D \triangleq\ & (\bar{n}-n)(\bar{n}(n-1)+n(\bar{n}-1))^2 + (\bar{n}-n)(1+\frac{\ell_2}{\ell_1})^2(\bar{n}-n+\ell_2(\frac{\bar{n}-1}{\ell_3} - \frac{n-1}{\bar{\ell}_3}))^2 \\ & + 2(1+\frac{\ell_2}{\ell_1})(\bar{n}-n+\ell_2(\frac{\bar{n}-1}{\ell_3} - \frac{n-1}{\bar{\ell}_3}))(\bar{n}-n)(\bar{n}(n-1)+n(\bar{n}-1))-2n^2(\bar{n}-1)^2) \\ & \geq 0 \ , \end{aligned} \tag{19}$$

where inequality (2) has been used. It is clear that if $\bar{\ell}_3$ is small enough, A is negative and consequently inequality (19) is satisfied. As $\bar{\ell}_3$ approaches ℓ_3 A is positive and a paraxial solution may or may not exist depending on the values of the other parameters.

An important case to consider is when $\ell_3 = \bar{\ell}_3$ (i.e. $\bar{x}_1 = x_1$ in figure 1). This case corresponds to the correction of dramatic aberrations using a single lens. The problem was addressed by Schwarzschild (1905) (English translation, Schulz, 1983) who showed that for $\bar{n}$-n small, no paraxial solution exists. The result can be deduced from inequality (19). For

$$\mu \stackrel{\Delta}{=} \bar{n}-n \ , \tag{20}$$

inequality (19) becomes

$$\mu 4n^2(n-1)^2(1-L) + \mu^2 4n(n-1)(2n-1)(1-L)$$
$$+ \mu^3\{(2n-1+L)^2 - 4Ln^2\} \geq 0 \ , \tag{21}$$

where

$$L = (1 + \frac{\ell_2}{\ell_1})(1 + \frac{\ell_2}{\ell_3}) \ . \tag{22}$$

Since $L > 1$, inequality (19) will not be satisfied for μ arbitrarily small. On the other hand, if μ is $O(1)$ and L is sufficiently large, inequality (19) can be satisfied. The latter case can be realized if, for instance, one focus is close to the lens and $\omega, \bar{\omega}$ are not close (or anomalous dispersion occurs). In practice, chromatic aberrations are mitigated using two lenses (the achromatic doublet) composed of different material (cf. Welford, 1986 or Born & Wolf (1986)).

Schwarzchild's result and inequality (19) suggest that for the ℓ_i, $n, \bar{n}$ fixed and μ small, there exists a 'critical' $\bar{\ell}_3$ (say $\bar{\ell}_3^*$) such that no paraxial solution exists for $\bar{\ell}_3 > \bar{\ell}_3^*$. The critical value can be estimated using inequality (19) to $O(\mu^2)$, i.e.,

$$\bar{\ell}_3^* \leq \ell_3 - \frac{\mu \ell_3}{(n-1)\ell_2}\{1 + \frac{\ell_2}{\ell_3} - \frac{1}{1+\ell_2/\ell_1}\} + O(\mu^2) \ . \tag{23}$$

A typical plot of $\bar{\ell}_3^*$ and μ is given in figure 2.

In general, there exist two paraxial solutions provided inequality (19) is satisfied. If μ is small and

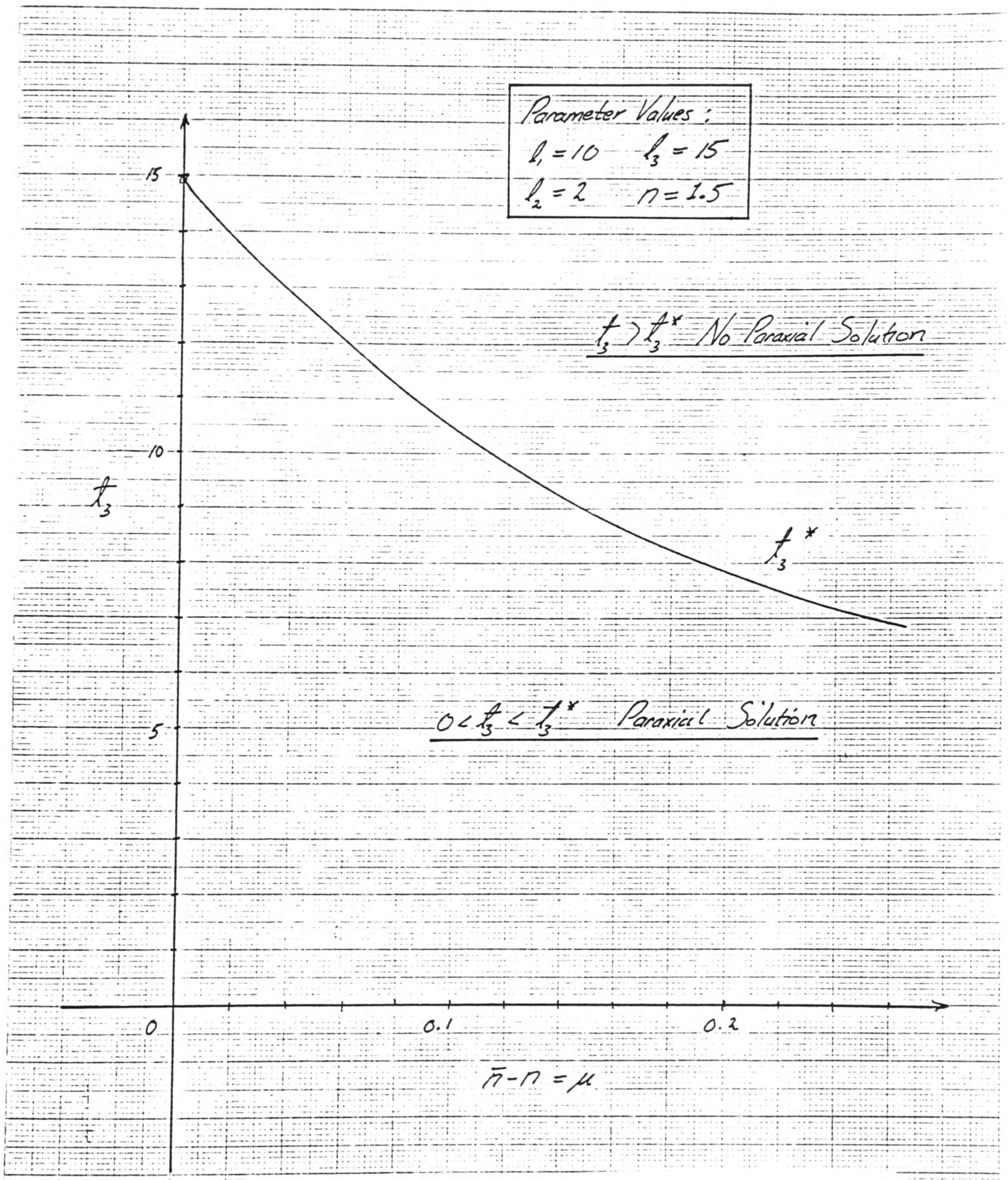


Figure 2: Example of $\bar{\ell}_3^*$ Dependence on μ

$$0 < \bar{\ell}_3 < \frac{(n-1)\ell_3}{n-1 + (1 + \ell_3/\ell_2)\mu} , \qquad (24)$$

then one solution corresponds to a lens in A^+ and the other to a lens in A^-. If μ is small and

$$\frac{(n-1)\ell_3}{n-1 + (1 + \ell_3/\ell_2)\mu} \leq \bar{\ell}_3 \leq \bar{\ell}_3^* , \qquad (25)$$

then all solutions correspond to lenses in A^+. The paraxial shapes may be meniscus or double convex.

4.2 Existence, Uniqueness, and Analyticity

The goal of this section is to establish the existence, uniqueness, and analyticity of solutions to the dispersive stigmatic lens problem. The analysis of Chapter 3 and Appendix I can be exploited to establish the existence of a local analytic solution provided a paraxial solution exists. Once one of the two possible paraxial solutions is selected uniqueness follows. The analytic continuation of the solutions is discussed in the next section.

Theorem 1

If paraxial solutions to the dispersive stigmatic lens problem exist satisfying initial conditions (2.15), then for each paraxial solution there exists a local solution to system (7) satisfying (2.15) which is unique among functions in C^2 and analytic. In particular, if two distinct paraxial solutions exist satisfying (2.15) then there exist precisely two local solutions to system (7) satisfying (2.15) in C^2 and both these solutions are analytic.

Proof:

The existence and analyticity of local solutions can be shown using either the method of Friedman & McLeod (1988) as in the proof of Theorem 3.1 or Corollary 5 in Appendix I (a generalisation of Cauchy's theorem for functional differential equations). Suppose, for definiteness,

$$0 < e_1 < c_1 \,. \tag{26}$$

Using the method of Theorem 3.1, let $M : (f,g,\xi,\phi) \to (\tilde{f},\tilde{g},\tilde{\xi},\tilde{\phi})$ be a mapping defined by

$$\frac{-\bar{n}}{\bar{d}_2}\{(g(\phi)-f(y))\tilde{f}'(y)+\tilde{\phi}-y\} + \frac{1}{d_1}\{(f(y)-x_0)\tilde{f}'(y)+y\}$$
$$= 0 \tag{27a}$$

$$\frac{\bar{n}}{\bar{d}_2}\{(g(\phi)-f(y))g'(\phi)+\phi-y\} + \frac{1}{\bar{d}_3}\{-(x_1-g(\phi))g'(\phi)+\sigma_1\phi+\tau_1\tilde{\phi}\}$$
$$= 0\,, \tag{27b}$$

$$\frac{-n}{\hat{d}_2}\{(g(y)-f(\xi^{-1}))f'(\xi^{-1})+y-\xi^{-1}\}$$
$$+\frac{1}{\hat{d}_1}\{(f(\xi^{-1})-x_0)f'(\xi^{-1})+\sigma_2\xi^{-1}+\tau_2\tilde{\xi}^{-1}\}$$
$$= 0 \tag{27c}$$

$$\frac{n}{\hat{d}_2}\{(g(y)-f(\xi^{-1}))\tilde{g}'(y)+y-\tilde{\xi}^{-1}\} - \frac{1}{\hat{d}_3}\{-(x_1-g(y))\tilde{g}'(y)+y\}$$
$$= 0 \tag{27d}$$

$$\tilde{f}(0) = a_0\,,$$
$$\tilde{g}(0) = b_0\,, \tag{28}$$

where,

$$\hat{d}_1^2 = (f(\xi^{-1})-x_0)^2 + (\xi^{-1})^2 \ ,$$

$$\hat{d}_2^2 = (g(y)-f(\xi^{-1}))^2 + (y-\xi^{-1})^2 \ , \tag{29}$$

$$\hat{d}_3^2 = (x_1-g(y))^2 + y^2 \ ,$$

$$\tau_i + \sigma_i = 1 \ , \quad i = 1,2$$

$$\sigma_1 = -\bar{\ell}_3\{2b_2(\bar{n}-1) + n/\ell_2\} \tag{30}$$

$$\sigma_2 = \ell_1\{2a_2(n-1) + n/\ell_2\} \ .$$

It is clear that a fixed point of M corresponds to a solution of system (7). A slight modification of the analysis in the proof of Theorem 3.1 can be used to show that the conditions of the Schauder fixed point theorem are satisfied. The existence of a local solution and analyticity follow mutatis mutandis from the proof of Theorem 3.1.

If inequality (26) is not satisfied an existence and analyticity proof can still be established by a modification of the above mapping. It can be seen from the proof of Theorem 3.1 that $\max(|c_1|,|e_1|)$ determines whether ξ^{-1} or ϕ^{-1} is used in the relations defining the mapping. The choice is critical as it is used to establish that the mapping is well-defined for analytic f,g,ξ,ϕ (cf. comments concerning cases 2 & 3 in the proof of Theorem 3.1).

Uniqueness among analytic functions can be shown using a slight modification of the uniqueness proof for Theorem 3.1. It is required, however, to establish uniqueness among functions in C^2. Theorem 3.2 can be used to establish this uniqueness provided system (7) can be converted into a system of the form (3.53) satisfying the conditions of said theorem. Suppose inequality (26) is satisfied and

$$c_1 \le 1 \; . \tag{31}$$

Let

$$\begin{aligned} z_1(y) &= y \; , & z_5(y) &= f(y)-a_0 \; , \\ z_2(y) &= \xi^{-1}(\phi(y)) \; , & z_6(y) &= f'(y) \; , \\ z_3(y) &= \xi(y) \; , & z_7(y) &= g(\xi)-b_0 \; , \\ z_4(y) &= \phi(y) \; , & z_8(y) &= g'(\xi) \; , \end{aligned} \tag{32}$$

and, suppressing arguments in y,

$$V \triangleq V(z_1,z_3,z_5,z_7) = \{(z_5 + \ell_1)^2 + z_1^2\}^{\frac{1}{2}}$$

$$+ n\{(z_7-z_5+\ell_2)^2 + (z_3-z_1)^2\}^{\frac{1}{2}} + \{(\ell_3-z_7(z_2))^2 + z_4^2\}^{\frac{1}{2}} \; ,$$

$$\bar{V} = \bar{V}(z_1,z_4,z_5,z_7(z_2)) = \{(z_5+\ell_1)^2 + z_1^2\}^{\frac{1}{2}}$$

$$+ \bar{n}\{(z_7(z_2)-z_5+\ell_2)^2 + (z_4-z_1)^2\}^{\frac{1}{2}} + \{(\bar{\ell}_3-z_7(z_2))^2 + z_4^2\}^{\frac{1}{2}} \; ,$$

$$U \triangleq V(z_1(z_2),z_3(z_2),z_5(z_2),z_7(z_2)) = \{(z_5(z_2)+\ell_1)^2 + z_2^2\}^{\frac{1}{2}}$$

$$+ n\{(z_7(z_2)-z_5(z_2)+\ell_2)^2 + (z_4-z_2)^2\}^{\frac{1}{2}}$$

$$+ \{(\ell_3-z_7(z_2))^2 + z_4^2\}^{\frac{1}{2}} \; .$$

Using the new variables, system (7) can be written in the form (3.53), i.e.,

$$z_i'(y) = h_i(\underset{\sim}{Z}(z_1),\underset{\sim}{Z}(z_2);\underset{\sim}{Z}'(z_1),\underset{\sim}{Z}'(z_2))$$
$$\underset{\sim}{Z}(0) = 0 \tag{33}$$

where $i = 1,2,\ldots,8$, and

$$\underset{\sim}{Z}(y) = \begin{bmatrix} z_1(y) \\ \vdots \\ z_8(y) \end{bmatrix}, \qquad \underset{\sim}{Z}'(y) = \begin{bmatrix} z_1'(y) \\ \vdots \\ z_8'(y) \end{bmatrix}.$$

The h_i can be determined in a manner analogous to that described in the proof of Theorem 3.3 and are given by

$$h_1 = 1,$$

$$h_2 = \frac{-\{U_{24} + z_6(z_2)U_{64}\}z_4'}{U_{22}+2z_6(z_2)U_{26}+z_8(z_2)U_{28}+z_6'(z_2)U_6+z_6^2(z_2)U_{66}+z_6(z_2)z_8(z_2)U_{68}},$$

$$h_3 = \frac{-\{V_{11} + 2z_6V_{15} + z_6'V_5 + z_6^2V_{55}\}}{V_{13} + z_6V_{53} + z_8V_{17} + z_6z_8V_{57}},$$

$$h_4 = -\{\bar{V}_{41}+z_6\bar{V}_{45}+z_8(z_2)\{z_2'\bar{V}_{48}+z_2'\bar{V}_8+\bar{V}_{81}+z_6\bar{V}_{85}+z_8(z_2)z_2'\bar{V}_{88}\}\},$$

$$h_5 = z_6,$$

$$h_6 = -\frac{1}{\bar{V}_5}\{\bar{V}_{11}+z_4'\bar{V}_{14}+2z_6\bar{V}_{15}+z_8(z_2)z_2'\bar{V}_{18}+z_6(z_4'\bar{V}_{54}+z_6\bar{V}_{55}+z_8(z_2)z_2'V_{58})$$

$$h_7 = z_8z_3',$$

$$h_8 = -\frac{1}{V_7}\{V_{31}+z_3'V_{33}+z_6V_{35}+2z_8z_3'V_{37}+z_3(V_{17}+z_6V_{15}+z_8z_3'V_{17})\},$$

where

$$\zeta = (\zeta_1,\ldots,\zeta_8) = (z_1,z_2,z_3,z_4,z_5,z_5(z_2),z_7,z_7(z_2))$$

$$V_i \triangleq \frac{\partial V}{\partial \zeta_i}; \qquad V_{ij} = \frac{\partial V}{\partial \zeta_i \partial \zeta_j} \quad \text{etc.}\,.$$

Inequalities (26) and (31) ensure that condition a of Theorem 3.2 is satisfied once one of the two possible paraxial solutions is specified. As with the system for the multi-foci lens problem, the h_i are analytic in the $z_j(z_i), z_j'(z_i)$ variables, and using the same method as in the proof of Theorem 3.3, it can be shown that the h_i satisfy the remaining conditions of Theorem 3.2 (see also Corollary 4 of Appendix I).

The uniqueness result also follows when inequalities (26) and (31) are not satisfied. These inequalities were required to show only that condition a of Theorem 3.2 is satisfied. If either (or both) of these inequalities are not satisfied, the system can be reformulated (i.e. the z_i redefined) so that condition a is satisfied. Depending on the values of c_1 and e_1, it may be necessary to reverse the light ray paths to effect this.

□

Remark 1. Since system (7) can be reformulated as a system satisfying the conditions of Theorem 3.2, Corollary 5 of Appendix I provides an alternative proof of Theorem 1. A local solution can thus be constructed using the method of successive approximations described in Appendix I.

Remark 2. No assumptions regarding the size of $\bar{n}-n$ and $\ell_3-\bar{\ell}_3$ are required for the above proof. The essential requirement is that a paraxial solution exists (i.e. inequality (19) is satisfied) and $\bar{\ell}_3 > 0$. The latter condition ensures that the focus for light rays at frequency $\bar{\omega}$ is in the image space. It is possible to design a lens having the focus for light rays at frequency $\bar{\omega}$ on the optical axis in $S_2 \cup \gamma_g$. As $\underset{\sim}{q}_1$ is no longer in the image space, equations (7c) and (7d) are not valid. The existence proof for this case is simpler than the above proof. It can be shown (cf. Chapter 2) that if $\underset{\sim}{q}_1$ is in $S_2 \cup \gamma_g$, then γ_f must be a cartesian oval and this curve can be determined algebraically from the path length condition for light rays at frequency $\bar{\omega}$. The function f(y) is thus known, and the path length condition for the light rays at frequency ω can be used to reduce equation (7b) to a first order nonlinear ordinary differential equation for $g(\xi)$.

4.3 Analytic Continuation

The existence, uniqueness, and analyticity of a local solution to the dispersive stigmatic lens was established in the previous section. Once one of the possible paraxial solutions is selected, Theorem 1 states that for some number $\delta > 0$ and

$$y \in B_\delta \overset{\Delta}{=} \{y : |y| < \delta\} ,$$

there exists a solution (f,g,ξ,ϕ) to system (7) satisfying (2.15) which is unique among C^2 functions and analytic. In this section, it will be shown that this local analytic solution can be analytically continued until either the lens closes or a light ray at either frequency becomes tangent to a lens boundary. The geometric ideas for this continuation are similar to those in Section 3.3. As with the analytic continuation of the local solution to the multi-foci lens problem, the key to the continuation lies in the fact that for some $\delta, y \in B_\delta$, the ranges of ξ and ϕ do not coincide. If

$$\xi(B_\delta) \overset{\Delta}{=} B_{\delta_1} ,$$

$$\phi(B_\delta) \overset{\Delta}{=} B_{\delta_2} ,$$

then either $B_{\delta_1} \subseteq B_{\delta_2}$ or $B_{\delta_2} \subseteq B_{\delta_1}$ but $B_{\delta_1} \neq B_{\delta_2}$. It is assumed for definiteness that

$$B_{\delta_2} \subseteq B_{\delta_1} ,$$

and that the lens is in A^+. The analytic continuation for the other cases is similar and will not be discussed.

The following lemma shows that $\xi(y)$ and $\phi(y)$ can not coincide for $|y| > 0$ if the light rays are in the field of the instrument (i.e. no lens closure or tangent light rays):

Lemma 2

Suppose (f,g,ξ,ϕ) is a solution to system (7) satisfying (2.15) for $y \in B_{y*}$, where $y* > 0$ is some number, and

$$\bar{\ell}_3 < \ell_3 , \tag{34}$$

then

$$\xi(y) = \phi(y) \tag{35}$$

if and only if

$$y = 0 .$$

Proof:

If $y = 0$ then (35) follows immediately from axial symmetry. Suppose for some $\hat{y} \in B_{y*}$ equation (35) is satisfied and, without loss of generality, $\hat{y} > 0$. If (35) is satisfied at $\hat{y}$ then no dispersion occurs at $(f(\hat{y}),\hat{y})$ and the light rays at both frequencies from the source to $(f(\hat{y}),\hat{y})$ must coincide between the source and $(g(\xi(\hat{y})),\xi(\hat{y}))$ and be collinear with the normal to γ_f at $(f(\hat{y}),\hat{y})$, i.e.,

$$(f(\hat{y})-x_0)f'(\hat{y}) + \hat{y} = 0 \tag{36}$$

(see figure 3). Equation (36) implies

$$f'(\hat{y}) < 0 \; , \tag{37}$$

for $y > 0$, and Theorem 2.4 implies

$$g'(\xi(\hat{y})) < 0 \; . \tag{38}$$

Since $\bar{n} > n$, inequality (38) implies that the light ray at frequency $\bar{\omega}$ must be between the normal to γ_g at $(g(\xi(\hat{y})),\xi(\hat{y}))$ and the light ray at frequency ω; consequently, the light ray at frequency $\bar{\omega}$ must lie above the one at frequency ω in the image (figure 3). Since these light rays cannot intersect in the image space, inequality (34) cannot be satisfied.

□

Remark 3. The analytic continuation of the local solution to the multi-foci lens problem in Section 3.3 does not require a statement analogous to the above lemma as it is evident from Snell's law that ξ and ϕ cannot be equal unless $y = 0$.

Lemma 2 implies that if $|c_1| > |e_1|$, then for all $y \in B_{y^*}$, $y > 0$

$$\xi(y) > \phi(y) \; , \tag{39}$$

so that for all $y \in B_{y^*}$

$$\begin{aligned} &\phi(B_y) \subseteq \xi(B_y) \\ &\phi(B_y) \neq \xi(B_y) \; . \end{aligned} \tag{40}$$

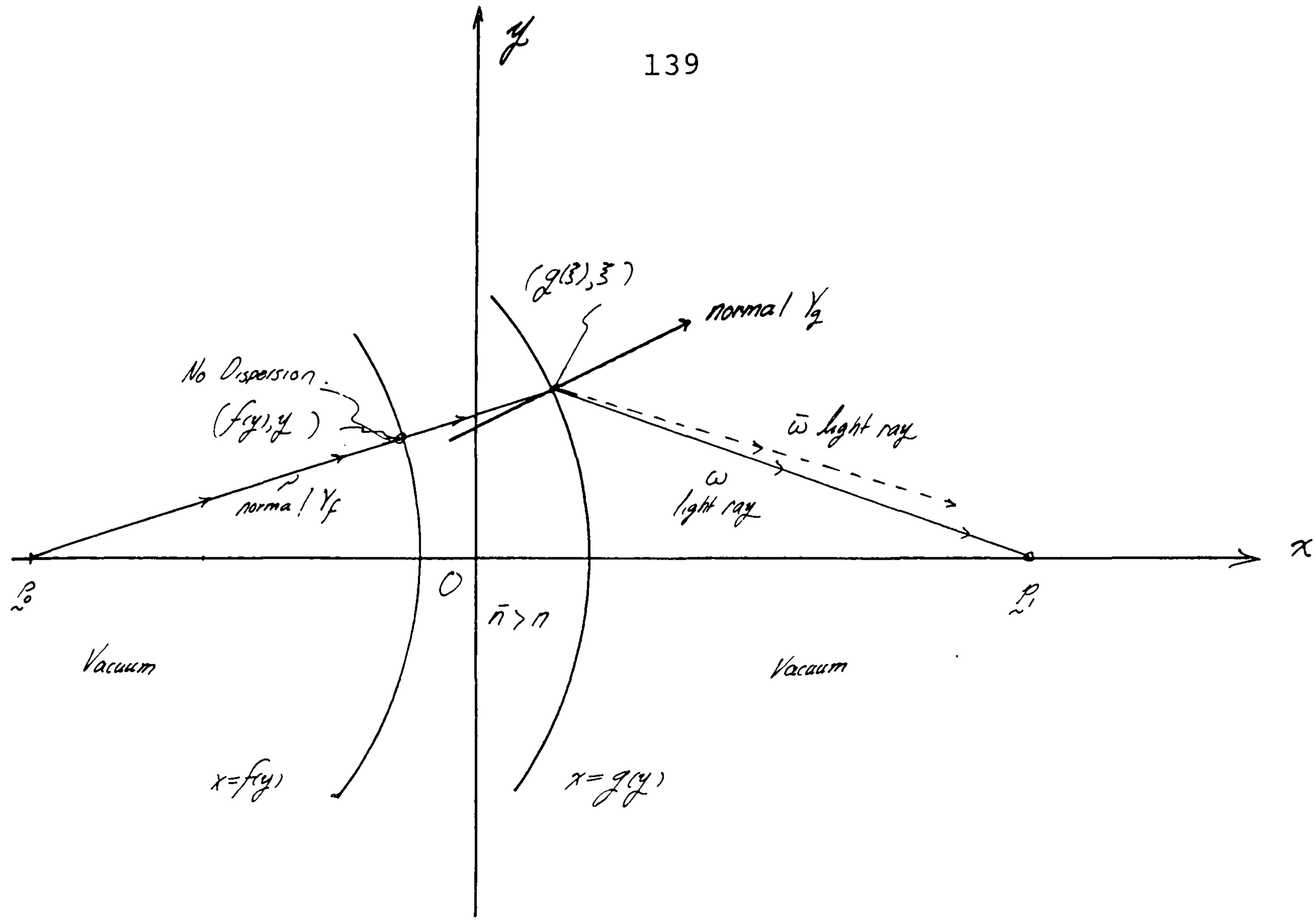


Figure 3: Geometrical Interpretation of Lemma 2

Relations (40) suggest that the same general mechanism for continuation used in Section 3.3 can also be used to continue the local solution of Section 2.

System (7) can be written

$$f'(y) = \frac{H_1(y,\phi,f(y),g(\phi))}{L_1(y,\phi,f(y),g(\phi))} \triangleq A_1 , \tag{41a}$$

$$g'(\xi) = \frac{H_2(y,\xi,f(y),g(\xi))}{L_2(y,\xi,f(y),g(\xi))} \triangleq A_2 , \tag{41b}$$

$$\xi'(y) = -\frac{\partial^2 v}{\partial y^2}\left\{\frac{\partial^2 v}{\partial y \partial \xi}\right\}^{-1} = -\frac{\partial^2 v}{\partial y \partial \xi}\left\{\frac{\partial^2 v}{\partial \xi^2}\right\}^{-1} \triangleq A_3 \tag{41c}$$

$$\phi'(y) = -\frac{\partial^2 \bar{v}}{\partial y^2}\left\{\frac{\partial^2 \bar{v}}{\partial y \partial \phi}\right\}^{-1} = -\frac{\partial^2 \bar{v}}{\partial y \partial \phi}\left\{\frac{\partial^2 v}{\partial \phi^2}\right\}^{-1} \triangleq A_4 , \tag{41d}$$

where

$$H_1 = \frac{\bar{n}(\phi-y)}{\bar{d}_2} - \frac{y}{d_1} ,$$

$$L_1 = \frac{f(y)-x_0}{d_1} - \frac{\bar{n}\{g(\phi)-f(y)\}}{\bar{d}_2} ,$$

$$H_2 = -\{\frac{\xi}{d_3} + \frac{n(\xi-y)}{d_2}\}$$

$$L_2 = \frac{n\{g(\xi)-f(y)\}}{d_2} - \frac{(x_1-g(\xi))}{d_3} .$$

Equation (41b) can be used to analytically continue g from δ_2 to a point arbitrarily close to $\xi(\delta) = \delta_1$. After this is established, equation (41a) can be used to extend f from δ_1 to a point close to $\phi^{-1}(\delta_1)$. The functions ξ and ϕ can be continued using equations (41c) and (41d) respectively. Like the analytic continuation described in Section 3.3, the analyticity of the A_i in their arguments is exploited and the process relies on repeated application of Cauchy's theorem. Lemma 2 ensures that each application extends the domain of analyticity. The solution can thus be continued until one of the A_i is no longer analytic in its variables. The functions A_1 and A_2 are analytic in their arguments for all y provided the L_i do not vanish. It was shown in the proof of Theorem 2.4 that the L_i cannot vanish since $\bar{n},n > 1$. The functions A_3 and A_4 are analytic in their variables provided the second derivatives in the denominators of (41c) and (41d) do not vanish. The proof of Theorem 2.2 implies that these derivatives vanish only if the lens closes or a light ray at either frequency becomes tangent to a lens boundary. As the details of the continuation are virtually the same as those described in Section 3.3, they are omitted. A geometric interpretation of the procedure is given in figure 4.

Since system (7) can be reformulated as a system satisfying the conditions of Theorem 3.2, the results concerning stability and analytic dependence of the solution on parameters discussed in Section 3.5 (i.e. $x_0,\bar{x}_0,x_1,\bar{x}_1,n,a_0,b_0$) can be used for the dispersive stigmatic lens problem. If for instance, $\bar{\ell}_3 < \bar{\ell}_3^*$ and $\bar{n}$-n is sufficiently small, then the solution will be analytic in $\bar{\ell}_3$.

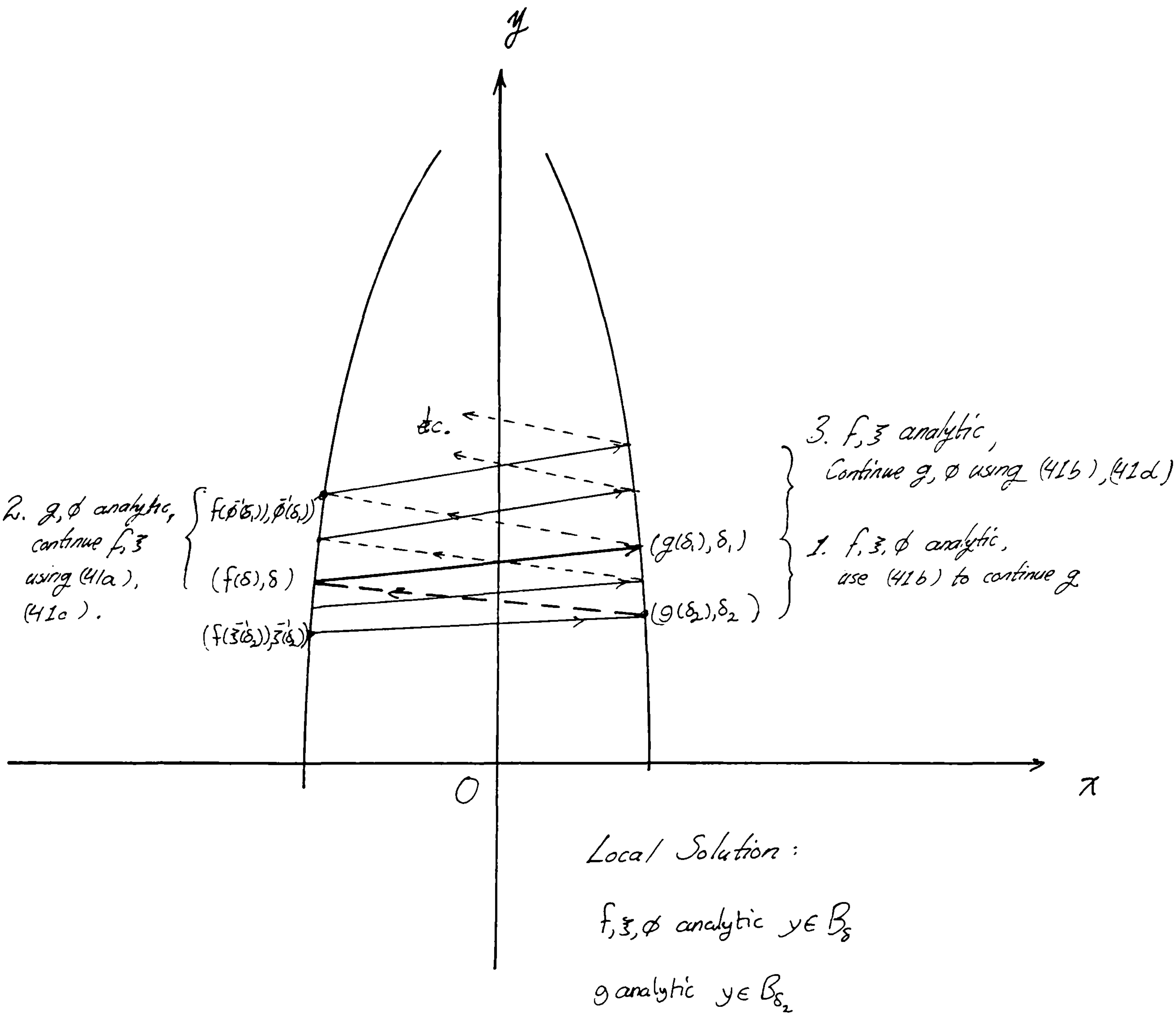


Figure 4: Analytic Continuation of Local Solution

4.4 The Chromatic Lens

It was shown in Section 1 that if $\mu \stackrel{\Delta}{=} \bar{n}-n$ is small, an $\bar{\ell}_3 < \bar{\ell}_3^*$ can be chosen such that a (real) paraxial solution exists and $\ell_3-\bar{\ell}_3$ is $O(\mu)$ (cf. inequality (23)). Theorem 1 implies that an analytic solution exists for this case. In this section, the case is investigated when

$$\mu = \bar{n}-n = \eta N \,,$$
$$\ell_3-\bar{\ell}_3 = \eta L \,, \qquad \bar{\ell}_3 < \ell_3^* \,, \tag{42}$$

for constants N,L, and some positive number η such that

$$\eta << 1 \,, \tag{43}$$

for $O(1)\ell_i$. Since

$$c_1 = \frac{1 \pm \left\{\frac{\ell_2}{\ell_3\ell_1}\right\}^{\frac{1}{2}} \left\{\frac{L}{N}\ (n-1)\left(\frac{\ell_1+\ell_2}{\ell_3}\right) - (\ell_3+\ell_2+\ell_1)\right\}^{\frac{1}{2}}}{\{1 + \ell_2(1 - \frac{L}{N}\ (n-1))/\ell_3^2\}}$$
$$+ O(\sqrt{\eta})$$

as $\eta \to 0$, the constants N and L must be of the same order to ensure that a paraxial solution exists and, as $\eta \to 0$, the solution is smooth. If L/N is $O(\eta)$ then no paraxial solution exists. If L/N is $O(1/\eta)$ then c_1 is $O(1/\sqrt{\eta})$ and consequently a_2 is $O(1/\sqrt{\eta})$. Motivated by the asymptotic method of Section 3.6, the quantity

$$\bar{V}-V = \eta\ell_2 N - \eta L \tag{44}$$

is investigated as $\eta \to 0$. Attention is concentrated on the leading order solution as $\eta \to 0$ and a condition analogous to the Herschel condition is derived which can be used to reduce the functional differential equations describing the leading order solution to an ordinary differential equation for g and algebraic relations for the other functions. The leading order solution is termed (for want of a better appellation) the "chromatic lens", and, as described below, has special imaging properties similar to those of the Herschel lens.

Let

$$\begin{aligned} f(y) &= f_0(y) + \eta f_1(y) + O(\eta^2) \ , \\ g(y) &= g_0(y) + \eta g_1(y) + O(\eta^2) \ , \\ \xi(y) &= \xi_0(y) + \eta \xi_1(y) + O(\eta^2) \ , \\ \phi(y) &= \phi_0(y) + \eta \phi_1(y) + O(\eta^2) \ , \end{aligned} \tag{45}$$

$$\begin{aligned} d_i &= d_{i0} + \eta d_{i1} + O(\eta^2) \ , \quad i = 1,2,3 \\ \bar{d}_i &= \bar{d}_{i0} + \eta \bar{d}_{i1} + O(\eta^2) \ , \quad i = 1,2,3 \ . \end{aligned} \tag{46}$$

Clearly,

$$\phi_0(y) = \xi_0(y) \ ,$$

and using

$$g(\xi) = g_0(\xi_0) + \eta \tilde{g}_1 + O(\eta^2) \ ,$$

$$g(\phi) = g_0(\xi_0) + \eta \hat{g}_1 + O(\eta^2) \ ,$$

where

$$\tilde{g}_1 \triangleq g_1(\xi_0) + \xi_1 g_0'(\xi_0) \ ,$$

$$\hat{g}_1 \triangleq g_1(\xi_0) + \phi_1 g_0'(\xi_0) \ ,$$

the $d_{ij}, \bar{d}_{ij}$ for $j = 0,1$ are given by

$$d_{10} = \{(f_0(y)-x_0)^2 + y^2\}^{\frac{1}{2}} \ ,$$

$$d_{11} = \frac{f_1(y)}{d_{10}} (f_0(y)-x_0) \ ,$$

$$d_{20} = \{(g_0(\xi_0)-f_0(y))^2 + (\xi_0-y)^2\}^{\frac{1}{2}}$$

$$d_{21} = \frac{1}{d_{20}} \{(g_0(\xi_0)-f_0(y))(\tilde{g}_1-f_1(y)) + \xi_1(\xi_0-y)\}$$

$$d_{30} = \{(x_1-g_0(\xi_0))^2 + \xi_0^2\}^{\frac{1}{2}}$$

$$d_{31} = \frac{1}{d_{30}} \{-(x_1-g_0(\xi_0))\tilde{g}_1 + \xi_0\xi_1\}$$

$$\bar{d}_{10} = d_{10}$$

$$\bar{d}_{11} = d_{11}$$

$$\bar{d}_{20} = d_{20}$$

$$\bar{d}_{21} = \frac{1}{d_{20}} \{(g_0(\xi_0)-f_0(y))(\hat{g}_1-f_1(y)) + \phi_1(\xi_0-y)\}$$

$$\bar{d}_{30} = d_{30}$$

$$\bar{d}_{31} = \frac{1}{d_{30}} \{-(x_1-g_0(\xi_0))\hat{g}_1 + \xi_0\phi_1 - L(x_1-g_0(\xi_0))\}.$$

Substituting the above expressions for the $d_i, \bar{d}_i$ into equation (44) yields

$$\bar{V}-V = \bar{d}_{10}-d_{10} + n(\bar{d}_{20}-d_{20}) + \bar{d}_{30}-d_{30}$$
$$+ \eta\{Nd_{20}+\bar{d}_{11}-d_{11}+n(\bar{d}_{21}-d_{21})+\bar{d}_{31}-d_{31}\} + O(\eta^2) = \eta(N\ell_2-L) \ , \tag{47}$$

so that, equating $O(\eta)$ terms,

$$N\ell_2 - L = Nd_{20} - \frac{L(x_1 - g_0(\xi_0))}{d_{30}}$$

$$+ (\phi_1 - \xi_1)\left\{\frac{n}{d_{20}}\{(g_0(\xi_0) - f(y))g_0'(\xi_0) + \xi_0 - y\}\right.$$

$$\left. + \frac{1}{d_{30}}\{-(x_1 - g_0(\xi_0))g_0'(\xi_0) + \xi_0\}\right\} . \qquad (48)$$

Since

$$\frac{\partial}{\partial \xi_0}\{d_{10} + nd_{20} + d_{30}\} = 0 ,$$

equation (48) reduces to

$$d_{20} - \ell_2 = M_c\left(1 - \frac{x_1 - g_0(\xi_0)}{d_{30}}\right) , \qquad (49)$$

where

$$M_c \overset{\Delta}{=} -L/N . \qquad (50)$$

Equation (49) is termed the <u>chromatic condition</u>, and the axial lens satisfying this condition along with the equations

$$\frac{\partial}{\partial y}\{d_{10} + nd_{20}\} = 0 , \qquad (51a)$$

$$d_{10} + nd_{20} + d_{30} = \ell_1 + n\ell_2 + \ell_3 , \qquad (51b)$$

is termed the <u>chromatic lens</u>.

Relation (49) can be solved for ξ_0 in terms of f_0 and g_0, and it is clear from Theorem 2.1 that analytic solutions to (51) exist which satisfy the chromatic condition. The chromatic condition is analogous to the Herschel condition of Section (3.6), and, like the Herschel condition, it is an 'orthogonality' type relation which can be used to reduce system (51) containing a functional differential equation to a system comprised of a first order nonlinear differential equation and algebraic relations.

Let

$$w_0 = \frac{y}{d_{10}} = \sin\psi_0$$

$$w_1 = \frac{-\xi_0}{d_{30}} = \sin\psi_1 \; ,$$

where ψ_0, ψ_1 are the angles between the light ray and the optical axis in the object and image spaces respectively (figure 3.7). In terms of the new variables, the d_{i0} are

$$d_{10} = \frac{f_0(w_0)-x_0}{(1-w_0^2)^{\frac{1}{2}}} \; ,$$

$$d_{20} = \{(g_0(w_1)-f_0(w_0))^2 + (d_{30}w_1 + d_{10}w_0)^2\}^{\frac{1}{2}} \; ,$$

$$d_{30} = \frac{(x_1-g_0(w_1))}{(1-w_1^2)^{\frac{1}{2}}} \; ,$$

and Snell's law at g (i.e. equation (51a)) yields

$$\frac{dg_0(w_1)}{dw_1} = \frac{-\{\frac{n}{d_{20}}(d_{30}w_1 + d_{10}w_0) + w_1\}d_{30}}{(1-w_1^2)^{\frac{1}{2}}\{\frac{n}{d_{20}}\{(g_0(w_1)-f_0(w_0))(1-w_1^2)^{\frac{1}{2}}-w_1(d_{30}w_1+d_{10}w_0)\}-1\}} \; . \tag{52}$$

The chromatic condition is

$$d_{20} = \ell_2 + M_c\{1-(1-w_1^2)^{\frac{1}{2}}\} = d_{20}(w_1) \; . \tag{53}$$

The path length condition (equation 51b) and the chromatic condition imply that d_{10} is known in terms of w_1 and $g_0(w_1)$. The function $f_0(w_0)$ can be eliminated using the relation

$$f_0(w_0) = (1-w_0^2)^{\frac{1}{2}}d_{10} + x_0 \; , \tag{54}$$

and the chromatic condition implies

$$w_0^2 d_{10}^2\{d_{30}^2 w_1^2 + (g(w_1)-x_0)^2\}-Cd_{30}d_{10}w_1w_0 + \frac{C^2}{4} - d_{10}^2(g(w_1)-x_0)^2 = 0 \ , \tag{55}$$

where

$$C = C(w_1,g_0(w_1)) \triangleq d_{20}^2-d_{10}^2-d_{30}^2w_1^2-(g_0(w_1)-x_0)^2 \ . \tag{56}$$

Since $\bar{\ell}_3 \leq \bar{\ell}_3^*$, inequality (23) ensures that (55) has real solutions for w_1 sufficiently small. The variable w_0 (and consequently $f_0(w_0)$) is known in terms of w_1 and $g(w_1)$, and equation (52) can be written

$$\frac{dg_0(w_1)}{dw_1} = Z(w_1,g_0(w_1)) \ ,$$

where $Z(w_1,g(w_1))$ is a known function of w_1 and $g(w_1)$. The chromatic lens problem can thus be reduced to a first order nonlinear ordinary differential equation for g_0 and algebraic relations for f_0,w_0.

The chromatic lens, aside from being the leading order approximation to the dispersive stigmatic lens problem, is of interest owing to its special imaging properties. By construction the chromatic lens has (perfect) foci at $\underset{\sim}{p}_0,\underset{\sim}{p}_1$, and the chromatic condition implies

$$\bar{V} = \text{const.} + O(\eta^2) \ ,$$

i.e. $\underset{\sim}{q}_1$ is the conjugate point of $\underset{\sim}{p}_0$ for light rays at frequency $\bar{\omega}$. The constants L and N appear in the chromatic condition only in the ratio M_c, and if these constants were replaced by some members $\tilde{L},\tilde{N}$ such that

$$M_c = -\tilde{L}/\tilde{N}$$

the solution to (52) would not be changed. Physically this means that the chromatic lens is capable of 'nearly focusing' a small interval of the spectrum. Indeed, the chromatic condition can be derived in a manner similar to the derivation of the Herschel condition in Section 3.7.

Chapter 5

The Off-Axis Lens

The lenses considered in Chapters 3 and 4 had foci on the optical axis. In this chapter, a lens is examined which is symmetric about the x-axis but the foci are not on the axis of symmetry. The governing equations are introduced in the first section and contrasted with the axial lens equations of the previous chapters. The special case when the foci are close to the optical axis is discussed in the second section, and an approximate solution analogous to the paraxial solution is developed. An asymptotic approximation similar to that for the multi-foci lens is described in the third section. A relation known as the Abbe sine condition is developed in this section, and it is shown that the leading order solution corresponds to a special axial lens called the aplanatic lens. The aplanatic lens is discussed further in the fourth section, and the Abbe sine condition is used to reduce the functional differential equations describing the aplanatic lens to an ordinary differential equation for a lens boundary and algebraic relations for the other functions.

5.1 The Off-Axis Lens Problem

A system of functional differential equations for a lens symmetric about the x-axis having foci not on said axis, i.e. an 'off-axis' lens, is introduced in this section. As a consequence of asymmetric foci locations, the functional arguments in this system are of a nature different from those in the axial lens systems considered earlier. In contrast with axial lens systems, the functional arguments for this system are 'non-local' (i.e. a light ray entering near the optical axis need not exit the lens near this axis). The non-local nature of these functional arguments create problems which, as discussed below, give rise to some degree of choice for a local solution.

Let S_2 be a lens having boundaries γ_f, γ_g symmetric with respect to the x-axis satisfying the axial thickness condition,

$$\begin{aligned} f(0) &= a_0 , \\ g(0) &= b_0 , \end{aligned} \tag{1}$$

and suppose the lens focuses $\underset{\sim}{p}_0 = (x_0, y_0)$ to $\underset{\sim}{p}_1 = (x_1, y_1)$ (see figure 1). The symmetry of the optical system with respect to the x-axis implies that S_2 also focuses $\bar{\underset{\sim}{p}}_0 = (x_0, -y_0)$ to $\bar{\underset{\sim}{p}}_1 = (x_1, -y_1)$. If $(g(\xi), \xi)$ is the exit point for a light ray leaving $\underset{\sim}{p}_0$ entering the lens at $(f(y), y)$, and $(g(\phi), \phi)$ is the exit point for a light ray leaving $\bar{\underset{\sim}{p}}_0$ entering the lens at $(f(y), y)$, then

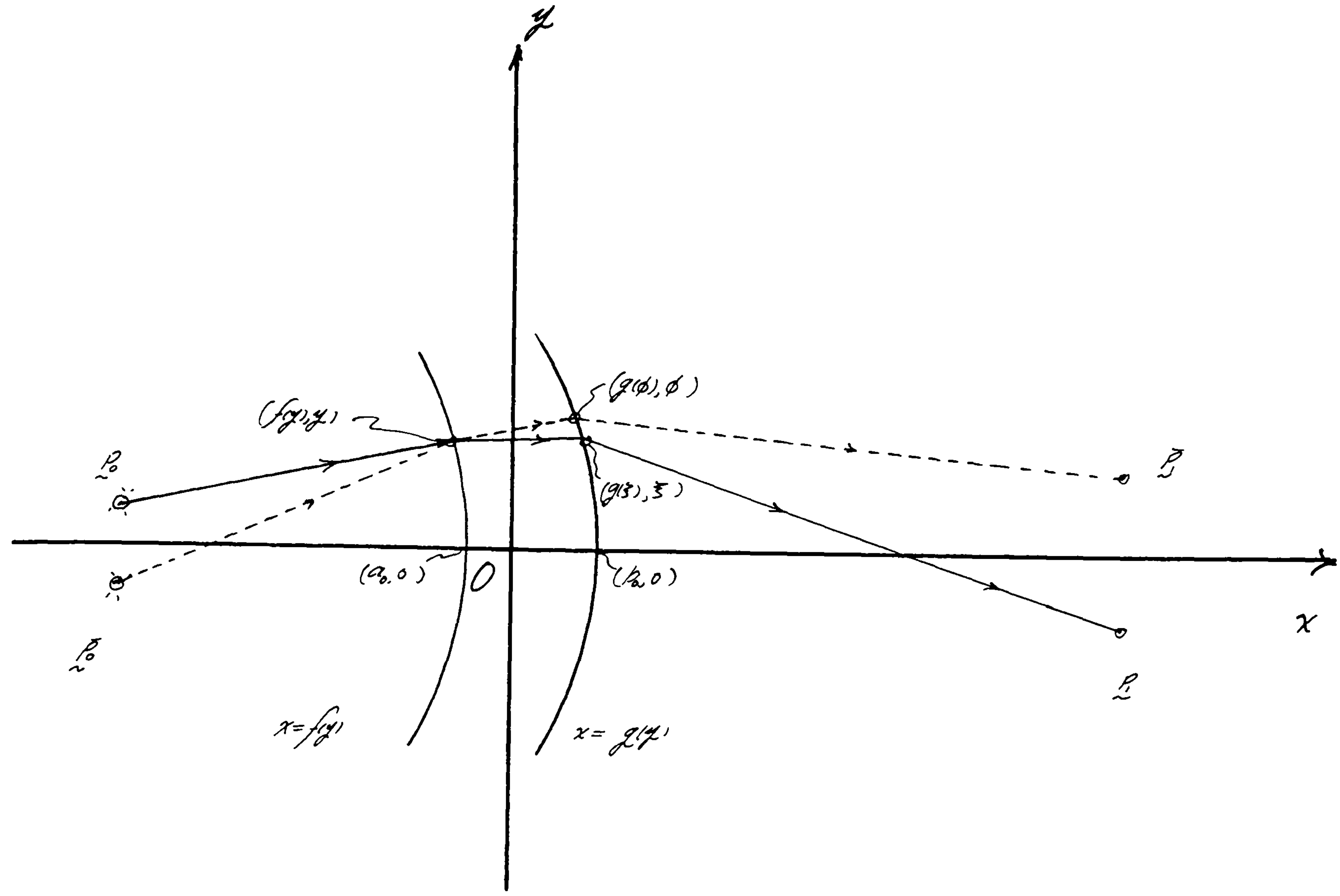


Figure 1: Off-Axis Lens

$$\phi(y) = -\xi(-y) \;, \tag{2}$$

and the optical path lengths for these families of light rays are given by

$$\begin{aligned} V &\triangleq V(\underset{\sim}{p}_0,\underset{\sim}{p}_1) = d_1 + nd_2 + d_3 \;, \\ \bar{V} &\triangleq V(\bar{\underset{\sim}{p}}_0,\bar{\underset{\sim}{p}}_1) = \bar{d}_1 + n\bar{d}_2 + \bar{d}_3 \;, \end{aligned} \tag{3}$$

where

$$\begin{aligned} d_1^2 &= (f(y)-x_0)^2 + (y-y_0)^2 \;, \\ d_2^2 &= (g(\xi)-f(y))^2 + (\xi-y)^2 \;, \\ d_3^2 &= (x_1-g(\xi))^2 + (y_1-\xi)^2 \;, \end{aligned} \tag{4}$$

$$\bar{d}_1^2 = (f(y)-x_0)^2 + (y+y_0)^2 ,$$

$$\bar{d}_2^2 = (g(\phi)-f(y))^2 + (\phi-y)^2 , \qquad (5)$$

$$\bar{d}_3^2 = (x_1-g(\phi))^2 + (y_1+\phi)^2 .$$

The lens boundaries must satisfy

$$\frac{\partial V}{\partial y} = \frac{(f(y)-x_0)f'(y)+y-y_0}{d_1} - \frac{n\{(g(\xi)-f(y))'f(y)+\xi-y\}}{d_2} = 0 , \qquad (6a)$$

$$\frac{\partial V}{\partial \xi} = \frac{n\{(g(\xi)-f(y))g'(\xi)+\xi-y\}}{d_2} - \frac{(x_1-g(\xi))g'(\xi)+y_1-\xi}{d_3} = 0 , \qquad (6b)$$

and the symmetry conditions $f(y) = f(-y)$, $g(y) = g(-y)$. Unlike the systems formulated for the multi-foci and stigmatic dispersive lenses, axial symmetry can be used to generate two additional functional differential equations, viz.,

$$\frac{\partial \bar{V}}{\partial y} = \frac{(f(y)-x_0)f'(y)+y+y_0}{\bar{d}_1} - \frac{n\{(g(\phi)-f(y))f'(y)+\phi-y\}}{\bar{d}_2} = 0 \qquad (6c)$$

$$\frac{\partial \bar{V}}{\partial \phi} = \frac{n\{(g(\phi)-f(y))g_1'(\phi)+\phi-y\}}{\bar{d}_2} \quad \frac{(x_1-g(\phi))g'(\phi)+y_1+\phi}{\bar{d}_3} = 0 . \qquad (6d)$$

It can readily be deduced from Snell's Law and axial symmetry that $y_0 = 0$ if and only if $y_1 = 0$. If $y_0 = 0$, the lens is thus an axial lens as discussed in Section 2.3. It is assumed unless otherwise noted that $y_0 \neq 0$. System (6) with initial conditions (1) is termed the <u>off-axis lens problem</u>.

The multi-foci and stigmatic dispersive lenses have foci on the axis of symmetry, and it is clear that axial symmetry considerations cannot generate additional independent differential equations analogous to (6c) and (6d). Symmetry considerations enter into these problems when the initial values for ξ and ϕ are derived. Specifically, axial symmetry implies that for sources on the optical axis there exists a light ray along this axis and, consequently, both ξ and ϕ vanish at $y = 0$. The proof of Theorem 3.1 (existence/uniqueness theorem for the multi-foci lens), it is recalled, does not even require the assumptions that f and g are even in y, yet this symmetry is nonetheless imposed on the solution as a consequence of the uniqueness (once the paraxial solution is specified) and the symmetry of the governing functional differential equations (cf. Remark 1, Chapter 3). If the foci are not on the optical axis, it is obvious from Snell's Law that ξ does not vanish at $y = 0$. System (6) evaluated at $y = 0$ provides only two relations for the three unknown quantities $\xi(0), g(\xi(0)), g'(\xi(0))$; consequently, $\xi(0)$ cannot be determined uniquely from system (6). Equivalently, system (6) implies the path length condition

$$d_1 + nd_2 + d_3 = k , \tag{7}$$

but k is an unknown constant.

Let $\delta > 0$ be some number and

$$B_\delta \triangleq \{y : |y| < \delta\}$$

$$N_\delta \triangleq \{\xi(y) : y \in B_\delta\}$$

$$\bar{N}_\delta \triangleq \{\phi(y) : y \in B_\delta\} .$$

For the multi-foci and stigmatic dispersive lens problems, it is clear that $N_\delta \cap \bar{N}_\delta \neq \emptyset$ for δ arbitrarily small. The existence of this non-empty set for all δ plays a crucial role in the local analysis used to establish the existence/uniqueness results of Chapters 3 and 4. This property was required to derive the well-defined mapping M in the proof of Theorem 3.1. If the method of Appendix I is used, it is this property which ensures that the optical system can be oriented so that $|\xi|, |\phi| \le |y|$. For the off-axis lens problem, however, it is evident that

$$N_\delta \cap \bar{N}_\delta = \emptyset \tag{8}$$

for δ sufficiently small. Relation (8) means that N_δ and $\bar{N}_\delta$ do not contain zero if δ is small, and this creates problems since g is required to satisfy an initial condition at $y = 0$.

The first consequence of relation (8) has been discussed above, viz., $\xi(0)$ cannot be determined uniquely. Were (8) false for all δ small, $\xi(0)$ would necessarily be zero. The second consequence of (8) is that system (6) is effectively two functional differential equations for three unknown functions. Even if $\xi(0)$ were given, it is clear that an infinite number of solutions can be constructed. Some suitable, smooth, even $f(y)$ can be specified and equations (6a) and (6b) reduced to an ordinary differential equation for $g(\xi)$. The existence of a solution would follow from Peano's theorem. A reflection in the x-axis of the γ_g determined above would complete the 'local solution' (see figure 2). The solution is required to satisfy initial conditions (1), and relation (8) gives rise to another problem in that the 'centre' of γ_g can be any arbitrary even function satisfying the initial condition. In particular, if the local solution constructed above is defined for $y \in B_\delta$, then $g(y)$ for $|y| \leq \min(|\xi(\delta)|, |\xi(-\delta)|)$ can be any smooth even function satisfying the initial condition which can be 'patched' smoothly into the local solution (see figure 2). The situation is roughly analogous to that which occurs in the simpler functional differential equation problem

$$\begin{aligned} z'(y) &= z(y-\eta) \ , \\ z(0) &= 0 \ , \end{aligned} \tag{9}$$

where η is some given positive number. It is clear that (9) does not define a function uniquely for y small since $\eta \neq 0$ and the initial condition is given at $y = 0$. If, for instance, $\eta < 0$ and solutions for $y \geq 0$ are desired, $z(y)$ must be specified on the interval $[\eta, 0]$ before (9) defines a function uniquely [cf. Hale (1977) for further details].

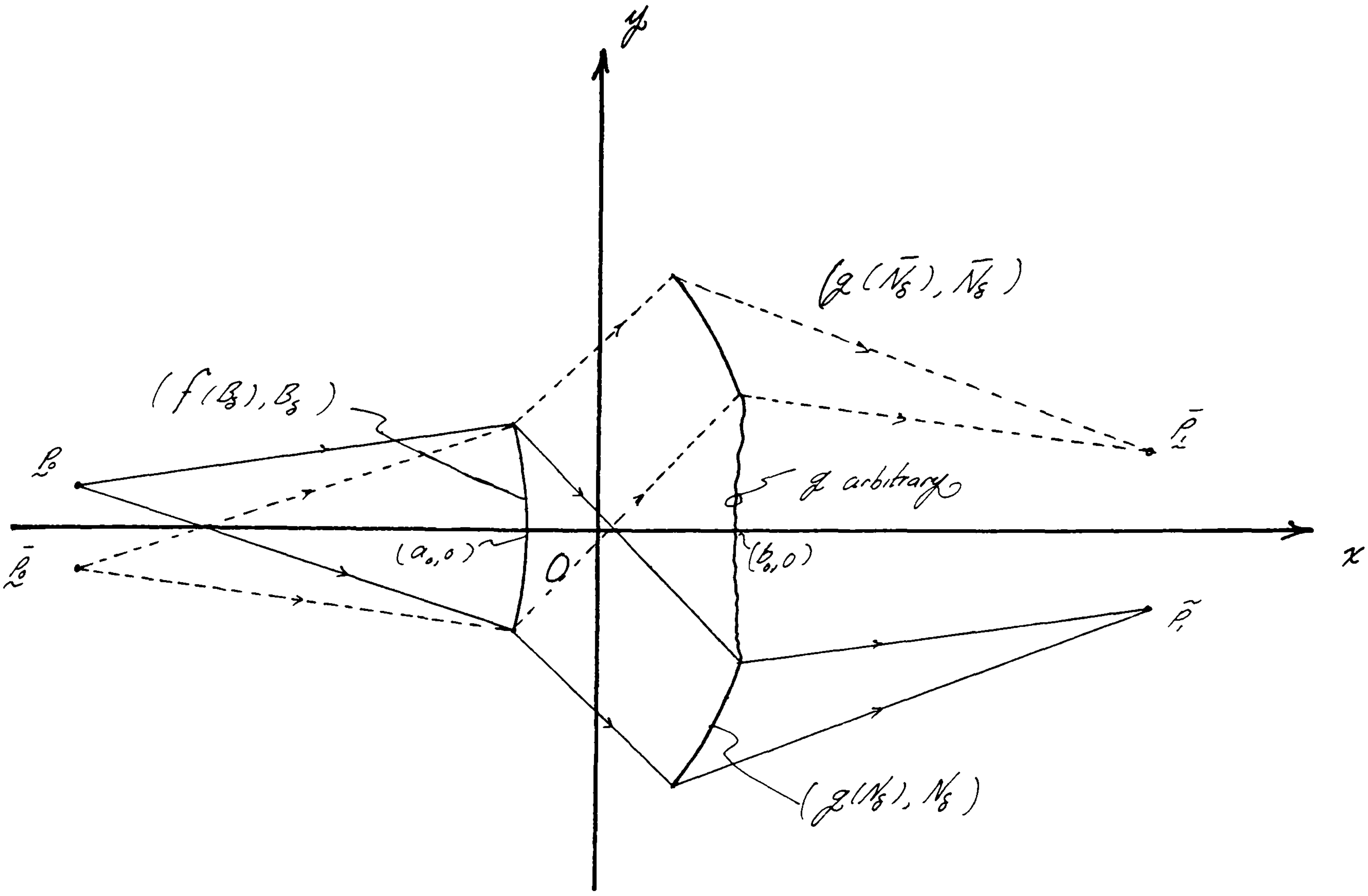


Figure 2: Local Solution to the Off-Axis Lens Problem

5.2 The Off-Axis Lens Problem for Small y_0, y_1

It was shown in the previous section that the off-axis lens problem has an infinite number of solutions for y small. The special case when the foci are nearly on the optical axis is examined in this section. It is clear from the discussion in Section 1 that additional conditions can be placed on the solutions. A useful local solution is one which allows continuation to a smooth global solution and is stable under small perturbations of y_0 and y_1. These requirements motivate the choice of a local solution form.

Let

$$\ell_1 = a_0 - x_0 ,$$
$$\ell_2 = b_0 - a_0 , \tag{10}$$
$$\ell_3 = x_1 - b_0 ,$$

and suppose the off-axis distances are small compared to the ℓ_i, i.e.

$$y_0 = \eta Y_0 ,$$
$$y_1 = \eta Y_1 ,$$

where Y_0, Y_1 are $O(1)$ numbers and

$$\eta << 1 ,$$

for $O(1)$ ℓ_i. A solution smooth for y small is desired, and it is evident from Snell's Law that Y_0 and Y_1 must be of the same order. If, for instance, Y_1/Y_0 were $O(\eta)$ then $\xi'(0)$ would be $O(\frac{1}{\eta})$ etc..

The non-uniqueness of local solutions to the off-axis lens problem implies that further restrictions can be placed on local solutions. For the purposes of lens design, it is desirable to have a local solution which can be continued to a smooth global solution stable under small perturbations of the foci. The multi-foci lens of Chapter 3 has these properties. The analyticity of the local solution to the multi-foci lens problem in the variable y affords a means whereby the local solution can be continued to a global analytic solution; analyticity in the foci location parameters (i.e. x_0, x_1) ensures that the solution is stable under small perturbations of the foci along the optical axis. Motivated by this example, a local solution to the off-axis problem is assumed in the form

$$
\begin{aligned}
f(y,\eta) &= \sum_{i,j} a_{i,j} y^i \eta^j \\
g(y,\eta) &= \sum_{i,j} b_{i,j} y^i \eta^j \qquad (11)\\
\xi(y,\eta) &= \sum_{i,j} c_{i,j} y^i \eta^j .
\end{aligned}
$$

Axial symmetry implies

$$
\begin{aligned}
f(y,\eta) &= f(-y,\eta) = f(y,-\eta) , \\
g(y,\eta) &= g(-y,\eta) = g(y,-\eta) , \qquad (12)\\
\xi(y,\eta) &= -\xi(-y,-\eta) ,
\end{aligned}
$$

and initial condition (1) implies

$$f(0,\eta) = a_0$$
$$g(0,\eta) = b_0 \; ; \qquad (13)$$

consequently,

$$a_{0,0} = a_0 \; ,$$
$$b_{0,0} = b_0 \; , \qquad (14)$$

and f,g,ξ can be written

$$f(y,\eta) = a_{0,0} + a_{2,0}y^2 + a_{4,0}y^4 + a_{2,2}y^2\eta^2 + \ldots$$
$$g(y,\eta) = b_{0,0} + b_{2,0}y^2 + b_{4,0}y^4 + b_{2,2}y^2\eta^2 + \ldots \qquad (15)$$
$$\xi(y,\eta) = c_{1,0}y + c_{0,1}\eta + c_{3,0}y^3 + c_{2,1}y^2\eta + c_{1,2}y\eta^2 + c_{0,3}\eta^3 + \ldots$$

Although the paraxial solution was defined in Chapter 2 only for axial lenses, the notion can be extended to the off-axis lens problem when the foci are close to the optical axis. Suppose y is $O(\eta)$. The functions f,g,ξ can be approximated by

$$f(y,\eta) = a_{0,0} + a_{2,0}y^2 + O(\eta^4)$$
$$g(y,\eta) = b_{0,0} + b_{2,0}y^2 + O(\eta^4) \qquad (16)$$
$$\xi(y,\eta) = c_{1,0}y + c_{0,2}\eta + O(\eta^3) \; .$$

Substituting relations (16) into equations (6a),(6b), and equating coefficients of y and η yields

$$
\begin{aligned}
&-2a_{2,0}(n-1)\ell_2\ell_1 + \ell_2 + n\ell_1 = n\ell_1 c_{1,0} \\
&2b_{2,0}(n-1)\ell_2\ell_3 + \ell_2 + n\ell_3 = n\ell_3/c_{1,0} \\
&\frac{nc_{0,1}}{\ell_2} = \frac{-Y_0}{\ell_1} \\
&2b_{2,0}(n-1)\ell_2\ell_3 + \ell_2 + n\ell_3 = \frac{Y_1\ell_2}{c_{0,1}} ,
\end{aligned}
\tag{17}
$$

which can be solved for $a_{2,0}, b_{2,0}, c_{1,0}, c_{0,1}$, viz.,

$$
\begin{aligned}
a_{2,0} &= \frac{1}{2\ell_1\ell_2(n-1)}\left\{\frac{nY_0\ell_3}{Y_1} + \ell_2 + n\ell_1\right\} \\
b_{2,0} &= \frac{-1}{2\ell_2\ell_3(n-1)}\left\{\frac{nY_1\ell_1}{Y_0} + \ell_2 + n\ell_3\right\} \\
c_{1,0} &= \frac{-Y_0\ell_3}{Y_1\ell_1} \\
c_{0,1} &= \frac{-Y_0\ell_2}{n\ell_1} .
\end{aligned}
\tag{18}
$$

In contrast with the paraxial solutions for the multi-foci and stigmatic dispersive lenses, the above solution does not involve quadratic expressions, i.e. the coefficients are uniquely determined. For η small the lens shape can be double convex or meniscus depending on the parameters. If Y_0 and Y_1 have the same sign, then the lens must be double convex.

Although it is known that an infinite number of local solutions exist, the existence of a local solution in y and η has not been established. The methods of Chapter 3 and theorems of Appendix I cannot be applied readily to the problem owing to relation (8). In order to apply the fixed point method of Chapter 3, a well-defined mapping M must be determined. The problem is that $\xi(0,\eta)$ vanishes only for $\eta = 0$, and a relation analogous to (3.28) which ensures M is well-defined for $\eta \neq 0$ does not exist. The theorems of Appendix I require $|\xi| \leq |y|$ for y sufficiently small. This relation cannot be satisfied for $\eta \neq 0$.

A natural direction to pursue for an existence proof is a Cauchy-Kowalevsky type argument. The Cauchy-Kowalevsky theorem has been generalised for some types of functional differential equations by Walter (1985) and Yamanaka & Tamaki (1980) (see also Tutschke, 1989), and it may be possible to extend the theorem so that it includes functional arguments of the type in the off-axis lens problem. Even if this generalisation is established, however, the problem of initial data for the off-axis lens problem remains. Initial data for $f(0,\eta)$, $g(0,\eta)$ can be specified but once this is done $\xi(0,\eta)$ cannot be specified. In particular, the coefficients c_{0j} can be calculated from system (6) and the initial values $f(0,\eta)$, $g(0,\eta)$. This suggests a power series representation for $\xi(0,\eta)$, but it is not known if this series converges.

5.3 Asymptotic Approximation

If η is small, an asymptotic approximation similar to that in Section 3.6 can be developed. A method for obtaining successive terms of this approximation is outlined in this section, and a relation analogous to the Herschel condition is derived which allows the leading order problem to be formulated as an ordinary differential equation and algebraic relations. The leading order solution corresponds to an axial lens with special imaging properties known as the aplanatic lens. This relationship was conjectured by Ockendon & Howison (1986). The method follows that given by van-Brunt (1988). The aplanatic lens and the corresponding ordinary differential equation are discussed further in the next section.

Suppose η is small and

$$\begin{aligned} f(y,\eta) &= \sum_{i=0}^{\infty} \eta^i f_i(y) \\ g(y,\eta) &= \sum_{i=0}^{\infty} \eta^i g_i(y) \\ \xi(y,\eta) &= \sum_{i=0}^{\infty} \eta^i \xi_i(y) \,. \end{aligned} \tag{19}$$

Since $\xi(y,-\eta) = \phi(y,\eta)$

$$\phi(y,\eta) = \sum_{i=0}^{\infty} \eta^i \phi_i(y) = \sum_{i=0}^{\infty} (-1)^i \eta^i \xi_i(y) \,. \tag{20}$$

Let

$$\begin{aligned} d_k &= \sum_{i=0}^{\infty} \eta^i d_{ki} \,, \\ \bar{d}_k &= \sum_{i=0}^{\infty} \eta^i \bar{d}_{ki} \,, \end{aligned}$$

for k = 1,2,3. Using

$$g(\xi) = g_0(\xi_0) + \eta\tilde{g}_1 + \eta^2\tilde{g}_2 + \dots ,$$

$$g(\phi) = g_0(\xi_0) + \eta\hat{g}_1 + \eta^2\hat{g}_2 + \dots ,$$

where

$$\tilde{g}_1 \triangleq g_1(\xi_0) + \xi_1 g_0'(\xi_0) ,$$

$$\tilde{g}_2 \triangleq g_2(\xi_0) + \xi_1 g_1'(\xi_0) + \xi_2 g_0'(\xi_0) + \frac{\xi_1^2}{2} g_0''(\xi_0) , \text{ etc.,}$$

$$\hat{g}_1 \triangleq g_1(\xi_0) + \phi_1 g_0'(\xi_0)$$

$$\hat{g}_2 \triangleq g_2(\xi_0) + \phi_1 g_1'(\xi_0) + \phi_2 g_0'(\xi_0) + \frac{\phi_1^2}{2} g_0''(\xi_0) , \text{ etc.,}$$

the $d_{ki}, \bar{d}_{ki}$ can be calculated. For i = 0,1,2,

$$d_{10} = \{(f_0(y)-x_0)^2 + y^2\}^{\frac{1}{2}} ,$$

$$d_{11} = \frac{1}{d_{10}} \{(f_0(y)-x_0)f_1(y)-yY_0\} ,$$

$$d_{20} = \{(g_0(\xi_0)-f_0(y))^2 + (\xi_0-y)^2\}^{\frac{1}{2}}$$

$$d_{21} = \frac{1}{d_{20}} \{(g_0(\xi_0)-f_0(y))(\tilde{g}_1-f_1(y)) + (\xi_0-y)\xi_1\}$$

$$d_{30} = \{(x_1-g_0(\xi_0))^2 + \xi_0^2\}^{\frac{1}{2}}$$

$$d_{31} = \frac{-1}{d_{30}} \{(x_1-g_0(\xi_0))\tilde{g}_1 + (Y_1-\xi_1)\xi_0\}$$

$$\bar{d}_{10} = d_{10}$$

$$\bar{d}_{11} = \frac{1}{d_{10}} \{(f_0(y)-x_0)f_1(y) + yY_0\}$$

$$\bar{d}_{20} = d_{20}$$

$$\bar{d}_{21} = \frac{1}{d_{20}} \{(g_0(\xi_0)-f_0(y))(\hat{g}_1-f_1(y)) + (\xi_0-y)\phi_1\}$$

$$\bar{d}_{30} = d_{30}$$

$$\bar{d}_{31} = \frac{-1}{d_{30}} \{(x_1 - g_0(\xi_0))\hat{g}_1 + \xi_0(Y_1 - \phi_1)\} .$$

Substituting relations (19) into system (6) and equating O(1) terms yields the leading order problem:

$$\frac{(f_0(y)-x_1)f_0'(y)+y}{d_{10}} - \frac{n\{(g_0(\xi_0)-f_0(y))f_0'(y)+\xi_0-y\}}{d_{20}}$$

$$= \frac{\partial}{\partial y} \{d_{10} + nd_{20} + d_{30}\} = 0$$

$$\frac{n\{(g_0(\xi_0)-f_0(y))g_0'(\xi_0)+\xi_0-y\}}{d_{20}} + \frac{-(x_1-g_0(\xi_0))g_0'(\xi_0)+\xi_0}{d_{30}}$$

$$= \frac{\partial}{\partial \xi_0} \{d_{10} + nd_{20} + d_{30}\} = 0 \tag{21}$$

$$f_0(0) = a_0$$

$$g_0(0) = b_0$$

$$\xi_0(0) = 0 .$$

It is clear from (21) that the leading order solution corresponds to an axial lens. It is known from Section 2.3 that an infinite number of solutions exist to (21). The leading order problem is thus degenerate. In the previous section, however, it was shown that if the solution is analytic in y and η, then the curvatures of the lens boundaries are uniquely determined at y = 0 (in fact all the coefficients of the double series are uniquely determined). If (19) is viewed as a rearrangement of double series (11), the uniqueness of the analytic solution implies that there must be an additional relation for the leading order problem.

Although the optical path length from $\underset{\sim}{p}_0$ to $\underset{\sim}{p}_1$ is unknown

$$\bar{V}-V = \bar{d}_1-d_1 + n(\bar{d}_2-d_2) + \bar{d}_3-d_3 = 0 \ , \tag{22}$$

and it is this equation which provides the additional relation for the leading order solution. Substituting the expressions for the $d_{ki}, \bar{d}_{ki}$ into equation (22) and collecting terms of the same order in η yields

$$\bar{d}_{11}-d_{11} + n(\bar{d}_{21}-d_{21}) + \bar{d}_{31}-d_{31} = 2\left\{\frac{yY_0}{d_{10}} + \frac{\xi_0 Y_1}{d_{30}}\right\}$$
$$+ (\phi_1-\xi_1)\frac{\partial}{\partial\xi_0}\{d_{10} + nd_{20} + d_{30}\} = 0 \ .$$

System (21) implies the term containing ϕ_1 and ξ_1 in the above expression vanishes yielding a relation involving only f_0, g_0, ξ_0, viz.,

$$\frac{y}{d_{10}} = \frac{M_s\xi_0}{d_{30}} \ , \tag{23}$$

where

$$M_s \triangleq \frac{-\gamma_1}{\gamma_0} \ . \tag{24}$$

Relation (23) is known as the Abbe sine condition, and axial lenses satisfying it are called aplanatic lenses. Theorem 2.1 implies that system (21) has a unique analytic local solution satisfying (23) and the initial conditions. Like the Herschel condition, the Abbe sine condition can be used to reduce the functional differential equations of the leading order problem to a first order nonlinear ordinary differential equation. This formulation is discussed in the next section along with the special imaging properties of aplanatic lenses.

The above derivation of the Abbe sine condition is similar to that of the Herschel condition given in Section 3.6. The Abbe sine condition, like the Herschel condition, is akin to an orthogonality condition derived using the Fredholm alternative. System (21) is degenerate but the $O(\eta)$ terms of equation (22) provide an 'orthogonality condition', viz., the Abbe sine condition which uniquely determines the leading order solution. The relationship between the Herschel and Abbe sine conditions is discussed further in Section 5.

After the leading order problem has been solved, the higher order terms in the approximation can be determined in a fashion similar to that described in Section 3.6. The $O(\eta)$ terms of equations (6a) and (6b) yield two linear differential equations for $f_1(y)$ and $g_1(\xi_0)$ containing ξ_1. The $O(\eta^2)$ terms of equation (22) provide an additional relation which allows ξ_1 to be eliminated from the differential equations. The mechanism for obtaining higher order terms is very close to that described in Section 3.6.

5.4 The Aplanatic Lens

Aplanatic lenses are of interest not only because they are the leading order solution to the off-axis lens problem, but also because they have a special imaging property. The imaging property is discussed briefly in this section, and the governing functional differential equations are reduced to a first order nonlinear ordinary differential equation and algebraic relations. The aplanatic lens problem has been formulated as a system of ordinary differential equations by Wasserman & Wolf (1949) (see also Born & Wolf, 1986) and as uncoupled ordinary differential equations by Head (1959).

The Abbe sine condition was originally derived by Clausius (1864) and Helmholz (1874) using thermodynamical arguments. The thermodynamical approach, roughly speaking, considers the energy transmitted from the object to the image space under the imaging conditions (described below) for aplanatic lenses. The energy transmitted is termed "pure temperature radiation" (as opposed to "luminescence") because the nature of the optical system is not changed by this radiation (e.g. no chemical changes occur) (cf. Drude, 1917). The second law of thermodynamics ensures that no temperature differences can develop in the (ideal) system* and this leads to an energy balance between the object and image planes which results in the Abbe sine condition. The derivation of the Abbe sine condition

*Indeed, the admission of temperature changes would enable a perpetual motion machine to be constructed.

using the thermodynamical approach of Clausius and Helmholz is given in English by Drude (1917) (see also Kline 1970). It is of interest to note that this derivation does not involve the lens boundaries or the refractive index of the lens. In this sense, the optical system between the foci is a 'black box'. In the next section the Abbe sine condition is derived from Fermat's principle without specific knowledge of the optical system between the foci.

Abbe first stated the sine relation without proof or reference [Abbe (1873)], but six years later he gave a proof using geometrical optics [Abbe (1879)]. It was through Abbe's work that the relation became well known and understood in connection with image formation. Geometrical optics derivations of the Abbe sine condition can be found in most optics texts (e.g. Born & Wolf, 1986; Longhurst, 1967; Sommerfeld, 1964). Welford (1976) gives a brief account of the history of the Abbe sine condition and aplanatic lenses.

The ratio M_s in the Abbe sine condition (23) is analogous to the ratio M_H in the Herschel condition (3.110). Since the Abbe sine condition contains Y_0, Y_1 only in the ratio M_s, it is clear that the aplanatic lens satisfying (23) is the leading order solution to any off-axis lens problem with foci

$$\begin{aligned} \hat{\underset{\sim}{p}}_0 &= (x_0, \eta\hat{Y}_0) \,, \\ \hat{\underset{\sim}{p}}_1 &= (x_1, \eta\hat{Y}_1) \,, \end{aligned} \tag{24}$$

satisfying

$$\frac{Y_1}{Y_0} = -M_s \ . \tag{25}$$

Since the aplanatic lens has (perfect) foci at $(x_0,0),(x_1,0)$, it can be shown using a method similar to that used in Section 3.7 that for any $\hat{\underset{\sim}{p}}_0, \hat{\underset{\sim}{p}}_1$ satisfying (24) and (25)

$$V(\hat{\underset{\sim}{p}}_0,\hat{\underset{\sim}{p}}_1) = \ell_1 + n\ell_2 + \ell_3 + O(\eta^2) \ . \tag{26}$$

Relation (26) means that light rays from $\hat{\underset{\sim}{p}}_0$ converge in some $O(\eta^2)$ neighbourhood of $\hat{\underset{\sim}{p}}_1$. Thus, for η small, the aplanatic lens nearly focuses $\hat{\underset{\sim}{p}}_0$ to $\hat{\underset{\sim}{p}}_1$ in the same sense as described for the Herschel lens. Physically this means that a small line segment orthogonal to the optical axis passing through $(x_0,0)$ in the object space is imaged to a small line segment orthogonal to the optical axis passing through $(x_1,0)$ in the image space. The line segment in the object space is magnified by a factor of M_s in the image space; the only (perfect) foci are $(x_0,0),(x_1,0)$.

In general, if $\underset{\sim}{p}_0 = (x_0,0)$, $\underset{\sim}{p}_1 = (x_1,0)$ are the foci of an axial lens having smooth boundaries and the source at $\underset{\sim}{p}_0$ is perturbed off the optical axis to some point $\hat{\underset{\sim}{p}}_0 = (x_0,\eta Y_0)$, the light rays leaving $\hat{\underset{\sim}{p}}_0$ form a caustic in some $O(\eta)$ neighbourhood of $\hat{\underset{\sim}{p}}_1$, and this is detrimental to image quality. For the three-dimensional case, this singularity is called primary coma (cf. Section 2.6). If the lens is aplanatic, however, then the light rays leaving $\hat{\underset{\sim}{p}}_0$ converge in some $O(\eta^2)$ neighbourhood of $\hat{\underset{\sim}{p}}_1 = (x_1,\eta Y_1)$ where Y_1 can be determined knowing M_s and Y_0. Although $\hat{\underset{\sim}{p}}_0$ and $\hat{\underset{\sim}{p}}_1$ are not foci, the caustic produced by an aplanatic lens is contained in an $O(\eta^2)$ neighbourhood and the image quality, consequently, is improved. The aplanatic lens is obviously of practical interest in lens design. The Abbe sine condition, like the Herschel (and chromatic) condition is a global relation in that no restrictions on the magnitude of y are imposed. It is thus useful in practical lens design problems where wide angle (as opposed to paraxial) pencils of light rays are involved such as the design of microscope lenses (this problem motivated Abbe's work). Aplanatic lenses have been studied by several authors, e.g., Head (loc. cit.), Luneberg (1944), Martin (1944), Sternberg (1955,1963), Vaskas (1957), Wassermann & Wolf (loc. cit.), and Welford (1976,1986). An example of aplanatic lens is the symmetric lens with symmetric foci. This lens is aplanatic for $|M_s| = 1$. The solution in A^+ (the hyperbola) corresponds to an aplanatic lens with $M_s = -1$; the solution in A^- (the cartesian oval) corresponds to an aplanatic lens with $M_s = 1$.

The Abbe sine condition can be used to reduce the functional differential equations describing the aplanatic lens to an ordinary differential equation for f and an algebraic relation for g. The method is essentially the same as that used for the Herschel lens. Let

$$w_0 = y/d_1$$

$$w_1 = -\xi/d_3 \; .$$

Using the new variables, the d_i are given by

$$d_1 = \frac{f(w_0)-x_0}{\{1-w_0^2\}^{\frac{1}{2}}}$$

$$d_2 = \{(g(w_1)-f(w_0))^2 + (w_1 d_3 + w_0 d_1)^2\}^{\frac{1}{2}}$$

$$d_3 = \frac{x_1-g(w_1)}{\{1-w_1^2\}^{\frac{1}{2}}} ,$$

and the Abbe sine condition is

$$w_0 = -M_s w_1 . \qquad (27)$$

The path length condition and Abbe sine condition imply

$$g^2(w_1)\left\{n^2 + \frac{M_s^2}{M_s^2-w_0^2}\left(\frac{n^2 w_0^2}{M_s^2} - 1\right)\right\}$$
$$-2g(w_1)\left\{n^2 f(w_0) + \frac{x_1 M_s^2}{M_s^2-w_0^2}\left(\frac{n^2 w_0^2}{M_s^2} - 1\right) - \frac{M_s}{(M_s^2-w_0^2)^{\frac{1}{2}}}\left(\frac{n^2 w_0^2 d_1}{M_s} + d_1 - k\right)\right\}$$
$$+ \frac{x_1 M_s^2}{M_s^2-w_0^2}\left\{\frac{n^2 w_0^2}{M_s^2} - 1\right\} - \frac{2x_1 M_s}{(M_s^2-w_0^2)^{\frac{1}{2}}}\left\{\frac{n^2 w_0^2 d_1}{M_s} + d_1 - k\right\} + n^2\left\{f^2(w_0) + w_0^2 d_1^2\right\}$$
$$- (k-d_1)^2 = 0 , \qquad (28)$$

where

$$k = \ell_1 + n\ell_2 + \ell_3 ,$$

so that $g(w_1)$ is known in terms of $f(w_0)$ and w_0. Quadratic equation (28) and the initial condition for g at $w_0 = w_1 = 0$ determine g uniquely in terms of w_0 and $f(w_0)$. Snell's law at γ_f gives the following first order nonlinear ordinary differential equation:

$$\frac{df(w_0)}{dw_0} = \frac{w_1 d_1\{1 + \dfrac{n(M_s d_1 - d_3)}{d_2 M_s (1-w_0^2)^{\frac{1}{2}}}\}}{1 + \dfrac{n}{d_2}\{\dfrac{w_0}{M_s}(M_s\ {}_1 - d_3) + (f(w_0)-g(w_1))(1-w_0^2)^{\frac{1}{2}}\}} , \qquad (29)$$

with the initial condition

$$f(0) = a_0.$$

The aplanatic lens problem can thus be formulated as an ordinary differential equation and algebraic relations. Cauchy's theorem (or Theorem 2.1) can be used to show that a unique analytic solution exists to the problem.

5.5 A General Imaging Condition

The derivation of the Abbe sine condition is similar to that given in Section 3.6 for the Herschel condition, and the imaging properties of the aplanatic lens are similar to those of the Herschel lens. These similarities suggest that the Abbe sine and Herschel conditions are special cases of a more general imaging condition. In this section, an imaging condition is derived which includes the Abbe sine and Herschel conditions as special cases. The condition is derived for the general case when the refractive index is not necessarily a piecewise constant function of position.

Suppose S is an optical system such that $\underset{\sim}{p}_0 = (x_0, y_0)$, $\underset{\sim}{p}_1 = (x_1, y_1)$ are foci, the refractive index $n(\underset{\sim}{r})$ is a smooth function of position in some neighbourhood N_0, N_1 of $\underset{\sim}{p}_0, \underset{\sim}{p}_1$ respectively, and the optical path length function $V(\underset{\sim}{r}_0, \underset{\sim}{r}_1)$ is smooth for $\underset{\sim}{r}_0 \in N_0$, $\underset{\sim}{r}_1 \in N_1$. Let η be some positive number such that

$$\eta << V(\underset{\sim}{p}_0, \underset{\sim}{p}_1) = K \ ,$$

where K is some constant (cf. Section 1.2). In general, if the source $\underset{\sim}{p}_0$ is perturbed a small distance (in the metric) to $\underset{\sim}{q}_0 = (x_0 + \eta\Delta x_0,\ y_0 + \eta\Delta y_0) \in N_0$, the light rays leaving $\underset{\sim}{q}_0$ converge in some $O(\eta)$ neighbourhood of $\underset{\sim}{p}_1$. Suppose, however, it is required that the light rays leaving $\underset{\sim}{q}_0$ converge in some $O(\eta^2)$ neighbourhood of a specified point $\underset{\sim}{q}_1 = (x_1 + \eta\Delta x_1,\ y_1 + \eta\Delta y_1) \in N_1$, i.e., suppose that $\underset{\sim}{q}_0$ is nearly focused to $\underset{\sim}{q}_1$. Since the light rays converge in an $O(\eta^2)$ neighbourhood of $\underset{\sim}{q}_1$

$$V(\underset{\sim}{q}_0,\underset{\sim}{q}_1) = K + \eta\Delta K + O(\eta^2) \ , \tag{30}$$

where ΔK is some constant (independent of light ray choice). The optical path length $V(\underset{\sim}{q}_0,\underset{\sim}{q}_1)$ can also be written

$$V(\underset{\sim}{q}_0,\underset{\sim}{q}_1) = V(\underset{\sim}{p}_0,\underset{\sim}{p}_1)+\eta\{\Delta x_0 \frac{\partial V}{\partial x_0} +\Delta y_0 \frac{\partial V}{\partial y_0} + \Delta x_1 \frac{\partial V}{\partial x_1} + \Delta y_1 \frac{\partial V}{\partial y_1}\}\Big|_{(\underset{\sim}{p}_0,\underset{\sim}{p}_1)}$$
$$+ O(\eta^2) \ , \tag{31}$$

so that equating $O(\eta)$ terms in (30) and (31) yields

$$\{\Delta x_0 \frac{\partial V}{\partial x_0} + \Delta y_0 \frac{\partial V}{\partial y_0} + \Delta x_1 \frac{\partial V}{\partial x_1} + \Delta y_1 \frac{\partial V}{\partial y_1}\}\Big|_{(\underset{\sim}{p}_0,\underset{\sim}{p}_1)} = \Delta K \ . \tag{32}$$

Equation (32) is thus a general imaging condition for a light ray leaving $\underset{\sim}{q}_0$ to nearly focus to $\underset{\sim}{q}_1$. The eikonal equation and the equations for its characteristic projections (cf. Section 1.2) can be used to determine the partial derivatives in (32) in terms of light ray angles. In particular, if α_0,α_1 denote the angles between the light ray and the x-axis at $\underset{\sim}{p}_0,\underset{\sim}{p}_1$ respectively, then

$$\frac{\partial V}{\partial x_0} = -n_0\cos\alpha_0 \ , \quad \frac{\partial V}{\partial x_1} = n_1\cos\alpha_1 \ ,$$
$$\frac{\partial V}{\partial y_1} = -n_0\sin\alpha_0 \ , \quad \frac{\partial V}{\partial y_1} = n_1\sin\alpha_1 \ ,$$

where

$$n_0 = n(\underset{\sim}{p}_0) \ , \quad n_1 = n(\underset{\sim}{p}_1) \ ,$$

(see figure 3). If the directions of one light ray through $\underset{\sim}{p}_0$ and $\underset{\sim}{p}_1$ are known, then ΔK can be determined, i.e. if

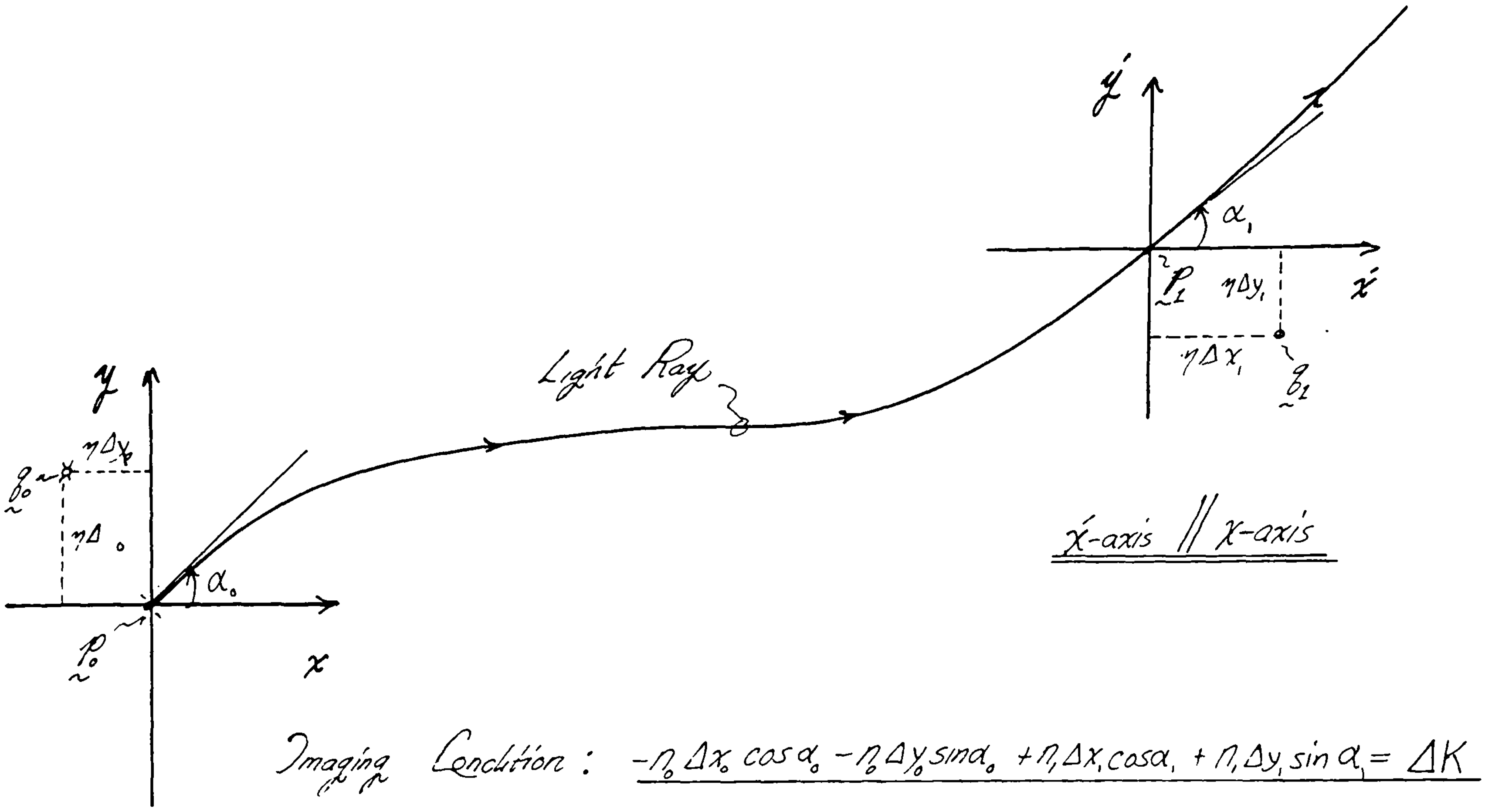


Figure 3: Geometric Interpretation of Imaging Condition

it is known that a light ray leaves $\underset{\sim}{p}_0$ at angle $\hat{\alpha}_0$ and intersects $\underset{\sim}{p}_1$ at angle $\hat{\alpha}_1$, then

$$\Delta K = -n_0\Delta x_0 a - n_0\Delta y_0 b + n_1\Delta x_1 c + n_1\Delta y_1 d \ , \tag{33}$$

where

$$a = \cos\hat{\alpha}_0 \ , \quad c = \cos\hat{\alpha}_1 \ ,$$

$$b = \sin\hat{\alpha}_0 \ , \quad d = \sin\hat{\alpha}_1 \ .$$

The imaging condition can thus be written as the hyperplane equation

$$(\underset{\sim}{A}-\underset{\sim}{P})\cdot\underset{\sim}{N} = 0 \ , \tag{34}$$

where

$$\underset{\sim}{A} = (-n_0\cos\alpha_0, -n_0\sin\alpha_0, n_1\cos\alpha_1, n_1\sin\alpha_1)$$

$$\underset{\sim}{P} = (-n_0 a, -n_0 b, n_1 c, n_1 d)$$

$$\underset{\sim}{N} = (\Delta x_0, \Delta y_0, \Delta x_1, \Delta y_1) \ .$$

Suppose, for instance, $\hat{\alpha}_0 = \hat{\alpha}_1 = 0$ (as is the case for axial lenses) so that

$$\underset{\sim}{P} = (-n_0, 0, n_1, 0) \ . \tag{35}$$

If $\Delta y_0 = \Delta y_1 = 0$, equation (34) reduces to

$$n_0\Delta x_0(1-\cos\alpha_0) = n_1\Delta x_1(1-\cos\alpha_1) \ , \tag{36}$$

which is the general Herschel condition. Alternatively, if $\Delta x_0 = \Delta x_1 = 0$, then equation (34) reduces to

$$n_0 \Delta y_0 \sin\alpha_0 = n_1 \Delta y_1 \sin\alpha_1 \ , \tag{37}$$

which is the general Abbe sine condition. In general, an optical system cannot satisfy both the Abbe sine and Herschel conditions. An optical system satisfies both conditions only if $|\alpha_0| = |\alpha_1|$, and this implies the $\Delta x_i, \Delta y_i$, and n_i must be such that

$$\frac{n_0}{n_1} = \frac{\Delta x_1}{\Delta x_0} = \left|\frac{\Delta y_1}{\Delta y_0}\right| \ . \tag{38}$$

An important feature of the aplanatic and Herschel lenses is that they nearly focus (i.e. image) line segments. The aplanatic lens, for example, images a small line segment orthogonal to the optical axis containing $\underset{\sim}{p}_0$ to another small line segment orthogonal to said axis containing $\underset{\sim}{p}_1$. The line segment in the object space is magnified by some factor M_s. This type of imaging is an approximation to the Maxwellian ideal image formation [cf. Maxwell, 1858; Welford, 1986] and is particularly desirable in three dimensions since geometric similarity is preserved. Equation (34) implies that this type of imaging can be realized only for $\hat{\underset{\sim}{N}} = (\Delta\hat{x}_0, \Delta\hat{y}_0, \Delta\hat{x}_1, \Delta\hat{y}_1)$ such that

$$\hat{\underset{\sim}{N}} = \rho \underset{\sim}{N} \ , \tag{39}$$

where ρ is some real number.

The imaging condition is a relation satisfied by all optical systems having the requisite imaging properties such that V is C^2 in $N_0 \times N_1$. A notable feature of the imaging condition is that it involves only terms of the object and image states, i.e., only terms evaluated at $\underset{\sim}{p}_0$ and $\underset{\sim}{p}_1$ appear in equation (34). In this sense, the optical system between the foci is a 'black box' and may consist of several lenses, inhomogeneous media etc.. The Abbe sine condition (23), for example, does not involve the refractive index of the lens. In order to obtain the uniqueness result for the aplanatic lens, it was assumed that the optical system is symmetric about the x-axis, the refractive index is a piecewise constant function of position, only one lens exists between the foci, and the lens intersects the optical axis at $(a_0,0),(b_0,0)$. The general Abbe sine condition (37) was derived under the assumption that the light ray leaving $\underset{\sim}{p}_0$ at $\alpha_0 = 0$ intersects $\underset{\sim}{p}_1$ at $\alpha_1 = 0$.

The imaging condition can be used to filter out some types of optical system without involving copious restrictions. For example, suppose the optical system is symmetric with respect to the x-axis and that the foci are located on said axis. It is clear that equation (35) is satisfied. Since the optical system is symmetric relation (34) must be satisfied for

$$\underset{\sim}{\bar{A}} = (-n_0\cos\alpha_0, n_0\sin\alpha_0, n_1\cos\alpha_1, -n_1\sin\alpha_1) \ .$$

The above relation implies that both the Abbe sine and Herschel conditions are satisfied; consequently, only optical systems which satisfy (38) have these imaging and symmetry properties.

Chapter 6

Conclusions

Three specific lens design problems have been addressed in this thesis. The investigations centred around a system of functional differential equations arising from Fermat's Principle. The emphasis was primarily on questions concerning existence, uniqueness and analyticity of solutions to these problems. Some asymptotic methods were developed for special cases of these problems which in effect reduce the functional differential equations to ordinary differential equations.

Some basic notions of geometrical optics were discussed in Chapter 1. In Chapter 2, definitions pertaining to lenses were given and the general functional differential equations derived. Some general solution properties for axial lenses relevant to the lens problems were also discussed in Chapter 2 along with the notion of paraxial solutions which play a major role in the local analysis.

The multi-foci lens problem was the subject of Chapter 3. In this chapter, the existence, uniqueness, and analyticity of local solutions was established. The existence proof was based on the work of Friedman & McLeod [1987,1988]. Using the ideas of Rogers (1988), it was shown that once the paraxial solution is selected, the local solution is unique among functions in C^2. A method for analytically continuing the local solution was described and it was shown that the solution can be continued until either the lens closes or a light ray from either family becomes tangent to a lens boundary. The analytic occurence of parameters such as the foci location in the functional differential equations was exploited to show that the solution is analytic in said parameters, and this result was used to show the stability of the global solution with respect to small perturbations of the parameters. Motivated by this stability result, an asymptotic method was investigated for the special case when the foci in the object space are close and the foci in the image space are close. The crucial advantage of the method is that the functional differential equations for the successive terms can be reduced to ordinary differential equations using an orthogonality condition. For the leading order problem this relation is the Herschel condition.

A new type of lens, the dispersive stigmatic lens, was introduced in Chapter 4. The governing equations for this lens are similar to those of the multi-foci lens, and the analysis developed in Chapter 3 was modified to establish analogous existence, uniqueness, and analyticity results. An asymptotic method similar to that for the multi-foci lens was described for the case when the image space foci are close and the difference between the indices of refraction small. Like the other asymptotic method, this reduces the functional differential equations to ordinary differential equations.

In Chapter 5, the off-axis lens problem was studied. The functional arguments in the governing equations are of a nature different from those in the previous lens problems in that they are non-local. The non-local character of these arguments precludes the direct application of the methods used in the previous chapters to establish existence, uniqueness, and analyticity. It was shown that an infinite number of solutions exist to the problem. The special case when the foci are close to the optical axis was discussed and, under the assumption that the solution is analytic in y and the off-axis distance parameter η, an approximate solution analogous to a paraxial solution was developed. An asymptotic method similar to that used in Chapter 3 was developed for this special case. As with the other asymptotic methods, this method reduces the functional differential equations to ordinary differential equations. Following a conjecture by Ockendon & Howison (1986), it was shown that the leading order solution corresponds to a well-known axial lens, the aplanatic lens. The final section investigates a more general imaging condition which includes the Abbe sine and Herschel conditions.

The existence of a solution to the off-axis problem which is analytic in y and η remains an open question. One possible approach was described in Chapter 5 provided the problem of assigning initial data could be resolved. If such a solution does exist, then $f(y,0), g(y,0), \xi(y,0)$ correspond to the aplanatic lens solution, and it may be possible to reformulate the problem using the aplanatic lens solution as initial data.

If the foci are not close to the optical axis, the imaging condition developed in Chapter 5 might be used to construct lenses of optical interest. The off-axis problem with the imaging condition might lead to a class of lenses which can image a small portion of a surface in the object space to a small portion of a surface in the image space.

Asymptotic methods were developed for solutions to the three problems considered in this thesis yet little was said concerning the nature of these approximations. These solutions may be merely asymptotic representations or they may converge and be a rearrangement of a double series. The analysis of Section 3.5 suggests that the asymptotic series for the multi-foci and dispersive stigmatic lens problems are in fact rearrangements of convergent double series. The nature of the asymptotic solution to the off-axis lens problem is linked to the existence problem. If a general Cauchy-Kowalevsky theorem can be established, it may be possible to resolve these questions.

The analysis of the previous chapters concentrated primarily on existence, uniqueness, and analyticity. General shape characteristics are of central importance in lens design. The paraxial solutions provided some local shape characteristics (i.e. magnitudes and signs of curvatures at $y = 0$); the global shape characteristics pose the problems for practical lens design.

Basic questions such as whether or not the lens closes and the maximum aperture are important design considerations. Other properties such as the monotonicity of lens boundaries are important questions.

Another direction for future work is boundary value problems for lenses. The lens problems considered in this thesis were essentially 'initial value problems'. A crucial design consideration, however, is often aperture size rather than axial lens thickness and this leads to a so-called boundary value problem since the lens boundaries must be even and satisfy an aperture condition. For the aplanatic and off axis lenses, this boundary value problem has been investigated by Sternberg (1955,1963,1977), Sternberg & Goldberg (1961), Sternberg, Goldstein & Drinkard (1982), Sternberg, Grafton & Childs (1985) and Sternberg & Grafton (1988) in connection with acoustic and microwave scanning problems. The above authors have approached the problem primarily from a numerical standpoint, and the existence/uniqueness theory remains to be developed.

Appendix 1

A Generalisation of the Picard-Lindelöf Theorem

A system of functional differential equations was introduced in Section 3.4 in connection with a uniqueness result given by Theorem 3.2, and it was shown that the multi-foci lens problem could be formulated as such a system. In this appendix, it will be shown that there exists a solution to this system unique among functions in C^1. The proof of this result is motivated by the proof of the Picard-Lindelöf theorem in the theory of ordinary differential equations. The result for functional differential equations was first published by Rogers (1988). The existence/uniqueness proof given below follows the analysis of Rogers, but has been modified to amplify the close connection with the Picard-Lindelöf theorem. The notation used in Theorem 3.2 is retained throughout the appendix, and some corollaries have been given to simplify the application of the theorem in special cases. The last corollary is a generalisation of Cauchy's theorem and this provides an alternative proof for the existence, uniqueness, and analyticity results of Chapter 3.

<u>Theorem 1</u> (<u>Existence/Uniqueness</u>)

Given the functional differential equation problem

$$z_i'(y) = h_i(y; \underset{\sim}{z}(z_1), \ldots, \underset{\sim}{z}(z_m); \underset{\sim}{z}'(z_1), \ldots, \underset{\sim}{z}'(z_m)) \tag{1}$$

$$\underset{\sim}{z}(0) = 0 \ ,$$

where the h_i satisfy conditions a-e of Theorem 3.2 (with the possible exception of inequality (3.61)) and the additional condition:

f. In N_ε the h_i are uniformly Lipschitz continuous with respect to y, i.e., for

$$\underset{\sim}{a} = (y;\underset{\sim}{Z}(z_1),\ldots,\underset{\sim}{Z}(z_m);\underset{\sim}{Z}'(z_1),\ldots,\underset{\sim}{Z}'(z_m)) \ ,$$

$$\hat{\underset{\sim}{a}} = (y;\underset{\sim}{Z}(z_1),\ldots,\underset{\sim}{Z}(z_m);\underset{\sim}{Z}'(z_1),\ldots,\underset{\sim}{Z}'(z_m)) \ ,$$

there exists a positive constant M such that

$$\|\underset{\sim}{H}(\hat{\underset{\sim}{a}})-\underset{\sim}{H}(\underset{\sim}{a})\| \le M|\hat{y}-y| \ ; \tag{2}$$

there exists a local solution to system (1) which is unique among functions in C^1.

<u>Proof</u>:

In the theory of ordinary differential equations, the Picard-Lindelöf existence/uniqueness theorem is proved using the method of successive approximations. Guided by this theorem, a 'natural' strategy is to define the sequence

$$\underset{\sim}{Z}^{(0)}(y) = y\underset{\sim}{Z}'(0) \ ,$$

$$\underset{\sim}{Z}^{(k+1)}(y) = \int_0^y H(s;\underset{\sim}{Z}^{(k)}(z_1^{(k)}),\ldots,\underset{\sim}{Z}^{(k)}(z_m^{(k)}); \underset{\sim}{Z}^{(k)\prime}(z_1^{(k)}),\ldots,\underset{\sim}{Z}^{(k)\prime}(z_m^{(k)}))\,ds \ , \tag{3}$$

and show that it converges uniformly for $y \in \bar{B}_\delta \triangleq \{y : |y| \le \delta\}$.

Let

$$\tau_k = \sup_{y\in\bar{B}_\delta} \|\underset{\sim}{Z}^{(k+1)}(y)-\underset{\sim}{Z}^{(k)}(y)\| \tag{4}$$

$$\sigma_k = \sup_{y\in\bar{B}_\delta} \|Z^{(k+1)\prime}(y)-Z^{(k)\prime}(y)\| \ . \tag{5}$$

Uniform convergence of the sequence defined by (3) follows if it can be shown that

$$\lim_{k\to\infty} \tau_k = 0 \ . \tag{6}$$

Since

$$\tau_k \le \delta\sigma_k \ , \tag{7}$$

relation (6) will be established if it can be shown that there exists a μ, $0 < \mu < 1$ such that

$$\sigma_{k+1} \le \mu\sigma_k \tag{8}$$

(i.e. if a contraction mapping can be established).

If inequality (3.61) were satisfied and a δ could be chosen so small that for all $k \in \mathbb{Z}^+$

$$(y;\underset{\sim}{z}^{(k)}(z_1^{(k)}),\ldots,\underset{\sim}{z}^{(k)}(z_m^{(k)});\underset{\sim}{z}^{(k)'}(z_1^{(k)}),\ldots,\underset{\sim}{z}^{(k)}(z_m^{(k)})) \in N_\varepsilon \ , \tag{9}$$

then the proof of Theorem 3.2 (and in particular inequality 3.67) could be used to derive

$$\sigma_{k+1} \le \Gamma\tau_k < \Gamma\delta\sigma_k \ , \tag{10}$$

where Γ satisfies inequality (3.75). It is clear from the above inequality that a suitably small δ ensures that (8) is satisfied. The following lemma shows that inequality (3.61) can be replaced by the less restrictive condition f and the assumption that the solution is in C^1.

<u>Lemma 2</u>

Under the conditions of Theorem 1, for any solution $\underset{\sim}{z}$ there exists a $\delta > 0$, $\Lambda_3 > 0$ such that

$$\|\underset{\sim}{Z}'(\hat{y}) - \underset{\sim}{Z}(y)\| \le \Lambda_3|\hat{y}-y| \ , \tag{11}$$

for $\hat{y}, y \in \bar{B}_\delta$.

<u>Proof</u>:

Inequality (11) is trivial for $y = \hat{y}$. Suppose $\hat{y} \neq y$, and for definiteness

$$-\delta \le y < \hat{y} \le \delta \ .$$

The term $\underset{\sim}{Z}'(\hat{y}) - \underset{\sim}{Z}'(y)$ can be decomposed in a manner similar to that used in Theorem 3.2, i.e.

$$\begin{aligned}
\underset{\sim}{Z}'(\hat{y})-\underset{\sim}{Z}'(y) &= \underset{\sim}{H}(\hat{y};\underset{\sim}{Z}(z_1(\hat{y})),\ldots,\underset{\sim}{Z}(z_m(\hat{y}));\underset{\sim}{Z}'(z_1(\hat{y})),\ldots,\underset{\sim}{Z}'(z_m(\hat{y})))\\
&\quad -\underset{\sim}{H}(y;\underset{\sim}{Z}(z_1(y)),\ldots,\underset{\sim}{Z}(z_m(y));\underset{\sim}{Z}'(z_1(y)),\ldots,\underset{\sim}{Z}'(z_m(y)))\\
&= \underset{\sim}{H}(\hat{y};\underset{\sim}{Z}(z_1(\hat{y})),\ldots,\underset{\sim}{Z}(z_m(\hat{y}));\underset{\sim}{Z}'(z_1(\hat{y})),\ldots,\underset{\sim}{Z}'(z_m(\hat{y})))\\
&\quad -\underset{\sim}{H}(y;\underset{\sim}{Z}(z_1(\hat{y})),\ldots,\underset{\sim}{Z}(z_m(\hat{y}));\underset{\sim}{Z}'(z_1(\hat{y})),\ldots,\underset{\sim}{Z}'(z_m(\hat{y})))\\
&\quad +\sum_{i=1}^{m}\underset{\sim}{\sigma}_i + \sum_{i=1}^{m}\underset{\sim}{Q}_i \ ,
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
\underset{\sim}{\sigma}_i \triangleq\ & \underset{\sim}{H}(y;\underset{\sim}{Z}(z_1(y)),\ldots,\underset{\sim}{Z}(z_{i-1}(y)),\underset{\sim}{Z}(z_i(\hat{y})),\ldots,\underset{\sim}{Z}(z_m(\hat{y}));\\
&\qquad \underset{\sim}{Z}'(z_1(\hat{y})),\ldots,\underset{\sim}{Z}'(z_m(\hat{y})))\\
&- \underset{\sim}{H}(y;\underset{\sim}{Z}(z_1(y)),\ldots,\underset{\sim}{Z}(z_i(y)),\underset{\sim}{Z}(z_{i+1}(\hat{y})),\ldots,\underset{\sim}{Z}(z_m(\hat{y}));\\
&\qquad \underset{\sim}{Z}'(z_1(\hat{y})),\ldots,\underset{\sim}{Z}'(z_m(\hat{y})))
\end{aligned}$$

$$\begin{aligned}
\underset{\sim}{Q}_i \triangleq\ & \underset{\sim}{H}(y;\underset{\sim}{Z}(z_1(y)),\ldots,\underset{\sim}{Z}(z_m(y));\underset{\sim}{Z}'(z_1(y)),\ldots,\underset{\sim}{Z}'(z_{i-1}(y)),\\
&\qquad \underset{\sim}{Z}'(z_i(\hat{y})),\ldots,\underset{\sim}{Z}'(z_m(\hat{y})))\\
&- \underset{\sim}{H}(y;\underset{\sim}{Z}(z_1(y)),\ldots,\underset{\sim}{Z}(z_m(y));\underset{\sim}{Z}'(z_1(y)),\ldots,\underset{\sim}{Z}'(z_i(y)),\\
&\qquad Z'(z_{i+1}(\hat{y})),\ldots,Z'(z_m(\hat{y}))) \ .
\end{aligned}$$

The solution $\underset{\sim}{Z}$ is assumed in C^1 so that for δ sufficiently small the arguments of $\underset{\sim}{H}$ in the above decomposition all lie in N_ε. The first difference on the right hand side of equation (12) can be bounded using condition f. The $\underset{\sim}{g}_i$ can be bounded in a manner similar to that used in Theorem 3.2, viz., condition e implies

$$\begin{aligned} \|\underset{\sim}{g}_i\| &\le K_i \; \|\underset{\sim}{Z}(z_1(\hat{y}))-\underset{\sim}{Z}(z_i(y))\| \\ &\le K_i\Lambda_1 |z_i(\hat{y})-z_i(y)| \;, \qquad (13) \end{aligned}$$

the last inequality following from the mean value theorem and inequality (3.60).

Let

$$\psi(\rho) \overset{\Delta}{=} \sup_{\substack{-\delta \le y < \hat{y} \le \delta \\ \hat{y}-y \le \rho}} \|\underset{\sim}{Z}'(\hat{y})-Z'(y)\|$$

The $\underset{\sim}{Q}_i$ can be bounded using inequality (3.57)

$$\begin{aligned} \|\underset{\sim}{Q}_i\| &\le L_i \; \|\underset{\sim}{Z}'(z_i(\hat{y})) - \underset{\sim}{Z}'(z_i(y))\| \\ &\le L_i\psi(\rho) \; . \qquad (14) \end{aligned}$$

Equation (12) thus yields the inequality

$$\psi(\rho) \le M\rho + \Lambda_1\rho \sum_{i=1}^{m} K_i + \theta\psi(\rho) \; .$$

Condition c requires $\theta < 1$; consequently

$$0 < \psi(\rho)(1-\theta) \le \rho\{M + \Lambda_1 \sum_{i=1}^{m} K_i\} \; . \qquad (15)$$

Inequality (11) follows using

$$\Lambda_3 = \sup_{-\delta \le y < \hat{y} \le \delta} \frac{\|\underset{\sim}{z}'(\hat{y})-\underset{\sim}{z}'(y)\|}{|\hat{y}-y|}$$

$$= \sup_{0 < \rho \le 2\delta} \frac{\psi(\rho)}{\rho} \le \frac{1}{(1-\theta)} \{M + \Lambda_1 \sum_{i=1}^{m} k_i\} .$$

□

Inequality (3.61) was used in conjunction with the mean value theorem to establish only inequality (3.73). It is clear from the proof of Theorem 3.2 that (3.74) can be derived using (11) instead of (3.61) (i.e. the $\underset{\sim}{Y}_i$ can be bounded using (11)).

If relation (9) is satisfied for some δ, then the inequality

$$\sigma_{k+1} \le \hat{\Gamma}\delta\sigma_k \ , \ \forall k \in \mathbb{Z}^+ , \tag{16}$$

where

$$\hat{\Gamma} = \frac{1}{(1-\theta)} \{(1 + C\Lambda_1) \sum_{i=1}^{m} K_i + \Lambda_3\theta C\} , \tag{17}$$

can be derived readily using the proof of Theorem 3.2. The following lemma shows that relation (9) is satisfied for δ suitably small.

<u>Lemma 3</u>

There exists a $\delta > 0$ such that, under the conditions of Theorem 1, the $\underset{\sim}{z}^{(k)}, \underset{\sim}{z}^{(k)'}$ as defined by (3) satisfy relation (9) for all $k \in \mathbb{Z}^+$.

<u>Proof</u>:

It is necessary to show that there exists a $\delta > 0$ such that

$$\frac{1}{\nu} \{ |y| + \sum_{i=1}^{m} \|\underset{\sim}{z}^{(k)}(z_i^{(k)})\| \} + \sum_{i=1}^{m} \|\underset{\sim}{z}^{(k)'}(z^{(k)})-\underset{\sim}{z}'(0)\| < \varepsilon , \tag{18}$$

for $y \in \bar{B}_\delta$, $k \in \mathbb{Z}^+$. The proof follows using induction. If $k = 1$

$$\sup_{y \in \bar{B}_\delta} \|\underset{\sim}{Z}^{(1)}(y)\| \leq \Lambda_1 \delta \ , \tag{19}$$

$$\sup_{y \in \bar{B}_\delta} \|\underset{\sim}{Z}^{(1)'}(y)\| \leq M\delta + C\Lambda_1 \delta \sum_{i=1}^{m} K_i \ , \tag{20}$$

so that any δ satisfying

$$\frac{1}{\nu} \{\delta + m\Lambda_1 \delta\} + m\{M\delta + C\Lambda_1 \delta \sum_{i=1}^{m} K_i\} < \varepsilon \ , \tag{21}$$

will satisfy (18) for $k = 1$.

Suppose relation (18) is satisfied for some positive integer k. Condition d and equation (3) imply

$$\sup_{y \in \bar{B}_\delta} \|\underset{\sim}{Z}^{(k+1)}(y)\| \leq \Lambda_1 \delta \ . \tag{22}$$

The $Z^{(k+1)'}(z_i) - Z'(0)$ terms can be bounded using a special case of the decomposition introduced in Lemma 2, viz.

$$\begin{aligned}
\underset{\sim}{Z}^{(k+1)'}(y) - \underset{\sim}{Z}'(0) &= \underset{\sim}{H}(y; \underset{\sim}{Z}^{(k)}(z_1^{(k)}), \ldots, \underset{\sim}{Z}^{(k)}(z_m^{(k)}); \\
&\qquad \underset{\sim}{Z}^{(k)'}(z_1^{(k)}), \ldots, \underset{\sim}{Z}^{(k)'}(z_m^{(k)})) \\
&\quad - \underset{\sim}{H}(0; \underbrace{0, \ldots, 0}_{m \text{ times}}; \underbrace{\underset{\sim}{Z}'(0), \ldots, \underset{\sim}{Z}'(0)}_{m \text{ times}}) \\
&= \underset{\sim}{H}(y; \underset{\sim}{Z}^{(k)}(z_1^{(k)}), \ldots, \underset{\sim}{Z}^{(k)}(z_m^{(k)}); \\
&\qquad \underset{\sim}{Z}^{(k)'}(z_1^{(k)}), \ldots, \underset{\sim}{Z}^{(k)'}(z_m^{(k)})) \\
&\quad - \underset{\sim}{H}(0, \underset{\sim}{Z}^{(k)}(z_1^{(k)}), \ldots, \underset{\sim}{Z}^{(k)}(z_m^{(k)}); \\
&\qquad \underset{\sim}{Z}^{(k)'}(z_1^{(k)}), \ldots, \underset{\sim}{Z}^{(k)'}(z_m^{(k)})) \\
&\quad + \sum_{i=1}^{m} \underset{\sim}{Q}_i^0 + \sum_{i=1}^{m} \underset{\sim}{Q}_i \ ,
\end{aligned} \tag{23}$$

where

$$\underset{\sim}{\sigma}_i^0 = \underset{\sim}{H}(0\,;0,\ldots,0,\underset{\sim}{Z}^{(k)}(z_i^{(k)}),\ldots,\underset{\sim}{Z}^{(k)}(z_m^{(k)});$$
$$\underset{\sim}{Z}^{(k)'}(z_1^{(k)}),\ldots,\underset{\sim}{Z}^{(k)'}(z_m^{(k)}))$$
$$-\ \underset{\sim}{H}(0;0,\ldots,0,\underset{\sim}{Z}^{(k)}(z_{i+1}^{(k)}),\ldots,\underset{\sim}{Z}^{(k)}(z_m^{(k)});$$
$$\underset{\sim}{Z}^{(k)'}(z_1^{(k)}),\ldots,\underset{\sim}{Z}^{(k)'}(z_m^{(k)}))$$

$$\underset{\sim}{Q}_i^0 = \underset{\sim}{H}(0;0,\ldots,0;\underset{\sim}{Z}'(0),\ldots,\underset{\sim}{Z}'(0),\underset{\sim}{Z}^{(k)'}(z_i^{(k)}),\ldots,\underset{\sim}{Z}^{(k)}(z_m^{(k)}))$$
$$-\ \underset{\sim}{H}(0;0,\ldots,0;\underset{\sim}{Z}'(0),\ldots,\underset{\sim}{Z}'(0),\underset{\sim}{Z}^{(k)'}(z_{i+1}^{(k)}),\ldots,\underset{\sim}{Z}^{(k)'}(z_m^{(k)}))\ .$$

Since, by hypothesis,

$$(y;\underset{\sim}{Z}^{(k)}(z_1^{(k)}),\ldots,\underset{\sim}{Z}^{(k)}(z_m^{(k)});\underset{\sim}{Z}^{(k)'}(z_1^{(k)}),\ldots,\underset{\sim}{Z}^{(k)'}(z_m^{(k)})) \in N_\varepsilon\ ,$$

conditions a-f can be applied and equation (23) can be used to obtain bounds on $Z^{(k+1)'}(y) - \underset{\sim}{Z}'(0)$, i.e.

$$\sup_{y\in\bar{B}_\delta} \|\underset{\sim}{Z}^{(k+1)'}(y)-\underset{\sim}{Z}'(0)\| \le M\delta + C\Lambda_1\delta\sum_{i=1}^{m}K_i + \sup_{y\in\bar{B}_\delta}\theta\|\underset{\sim}{Z}^{(k)'}(y)-\underset{\sim}{Z}'(0)\|\ . \tag{24}$$

Using the relation

$$\sigma_j \le \hat{\Gamma}\delta\sigma_{j-1}\ , \quad j = 1,2,\ldots,k-1\ , \tag{25}$$

the last term on the right hand side of inequality (24) can be bounded, i.e.,

$$\sup_{y\in\bar{B}_\delta}\|\underset{\sim}{Z}^{(k)'}(y)-\underset{\sim}{Z}'(0)\| \le \sum_{i=1}^{k}\sup_{y\in\bar{B}_\delta}\|\underset{\sim}{Z}^{(i)'}(y)-Z^{(i-1)'}(y)\|$$
$$\le \sup_{y\in\bar{B}_\delta}\|\underset{\sim}{Z}^{(1)'}(y)-\underset{\sim}{Z}'(0)\|\sum_{i=1}^{k-1}(\delta\hat{\Gamma})^i\ .$$

A number $\delta > 0$ can be selected such that

$$0 < \delta\hat{\Gamma} < \theta \tag{26}$$

$$\sup_{y\in\bar{B}_\delta} \|\underset{\sim}{z}^{(1)'}(y)-\underset{\sim}{z}'(0)\| \overset{\Delta}{=} \Delta < \frac{\varepsilon}{2m}(1-\theta) \ . \tag{27}$$

Inequality (24) can thus be replaced by

$$\sup_{y\in\bar{B}_\delta} \|\underset{\sim}{z}^{(k+1)'}(y)-\underset{\sim}{z}'(0)\| \le M\delta + C\Lambda_1\delta \sum_{i=1}^{m} K_i + \frac{\Delta}{1-\theta} ,$$

so that

$$\frac{1}{\nu}\{|y| + \sum_{i=1}^{m} \|\underset{\sim}{z}^{(k+1)}(z_i^{(k+1)})\|\} + \sum_{i=1}^{m}\|\underset{\sim}{z}^{(k+1)'}(z^{(k+1)})-\underset{\sim}{z}'(0)\|$$

$$\le \frac{1}{\nu}\{\delta+m\Lambda_1\delta\} + m\{M\delta + C\Lambda_1\delta \sum_{i=1}^{m} K_i + \frac{\Delta}{1-\theta}\}$$

$$< \frac{1}{\nu}\{\delta + m\Lambda_1\delta\} + m\{M\delta + C\Lambda_1\delta \sum_{i=1}^{m} K_i\} + \frac{\varepsilon}{2} \ . \tag{28}$$

A $\delta > 0$ thus can be chosen so small that inequalities (26), (27), and

$$\delta\{\frac{1}{\nu}\{1 + m\Lambda_1\} + m\{M + C\Lambda_1 \sum_{i=1}^{m} K_i\}\} < \frac{\varepsilon}{2} , \tag{29}$$

are satisfied; therefore, there exists a $\delta > 0$ such that relation (9) is satisfied for all $k \in \mathbb{Z}^+$.

□

Relation (9) and inequality (26) are satisfied; consequently, the inequality

$$\sigma_{k+1} \le \theta\sigma_k , \tag{30}$$

can be established. Since $0 < \theta < 1$,

$$\lim_{k\to\infty} \sigma_k = 0$$

$$\lim_{k\to\infty} \tau_k = 0 ,$$

and, by virtue of the definitions of τ_k and σ_k, the sequence defined by (3) converges uniformly in $\bar{B}_\delta$. Since the h_i

are continuous in their variables, the solution to system (1) is in C^1.

□

Theorem 1 can be viewed as a generalisation of the Picard-Lindelöf theorem. If the h_i have continuous partial derivatives with respect to y, $\underset{\sim}{Z}(z_i), \underset{\sim}{Z}'(z_i)$ in N_ε, then the Lipschitz continuity requirements are automatically satisfied by virtue of the convexity of N_ε and the mean value theorem. More interesting perhaps is that this condition also implies condition c for some suitably chosen vector norm. The following corollary to Theorem 1 is a less general existence/uniqueness result but has the decisive merit of simpler conditions to verify.

Corollary 4

Suppose the h_i in system (1) satisfy condition a and inequalities (3.59),(3.60) of Theorem 3.2 and have second partial derivatives with respect to y, the $\underset{\sim}{Z}(z_i)$, and the $\underset{\sim}{Z}'(z_i)$, for $0 \le i \le m$, $(y;\underset{\sim}{Z}(z_1),\ldots,\underset{\sim}{Z}(z_m);$ $\underset{\sim}{Z}'(z_1),\ldots,\underset{\sim}{Z}'(z_m)) \in N_\varepsilon$. There exists a local solution to (1) unique among functions in C^1.

Proof:

It is clear that all the Lipschitz conditions of Theorem 1 (i.e. conditions b,e and f) are satisfied since the first partial derivatives of the h_i exist and are continuous in N_ε. Once condition c is verified, the corollary will follow from Theorem 1.

The h_i are functions of 2mn+1 variables. Let

$$\underset{\sim}{a} \in \mathbb{R}^{2mn+1}$$

$$\underset{\sim}{a} = (a_1, a_2, \dots, a_{2mn+1}) \in N_\varepsilon$$

$$y = a_1$$

$$z_k(z_i) = a_{n(i-1)+k+1}$$

$$z_k'(z_i) = a_{j+n(i-1)+k} \ ,$$

where

$$1 \le k \le n$$

$$0 \le i \le m$$

$$j \overset{\Delta}{=} 1+mn \ .$$

Under the above notation

$$\underset{\sim}{Z}(z_i) = (a_{n(i-1)+1}, \dots, a_{ni+1})^T \in \mathbb{R}^n$$

$$Z'(z_i) = (a_{j+n(i-1)}, \dots, a_{j+ni})^T \in \mathbb{R}^n \ .$$

It is required to show that, for some vector norm $||\cdot||$ and

$$\underset{\sim}{a} = (a_1; \dots, a_{2mn+1})$$

$$\hat{\underset{\sim}{a}} = (a_1; a_2, \dots, a_{mn+1}; \hat{a}_{mn+2}, \dots, \hat{a}_{2mn+1}) \ ,$$

there exist constants $L_i \ge 0$ such that

$$||\underset{\sim}{H}(\hat{\underset{\sim}{a}}) - \underset{\sim}{H}(\underset{\sim}{a})|| \le \sum_{i=1}^{m} L_i \, ||\underset{\sim}{Z}(\hat{z}_i) - \underset{\sim}{Z}'(z_i)|| \tag{31}$$

and

$$\sum_{i=1}^{m} L_i < 1 \ . \tag{32}$$

Let

$$\underset{\sim}{D}(\underset{\sim}{a}) \triangleq \begin{pmatrix} \frac{\partial h_1(\underset{\sim}{a})}{\partial a_{j+1}} & \frac{\partial h_1(\underset{\sim}{a})}{\partial a_{j+2}} & \cdots & \frac{\partial h_1(\underset{\sim}{a})}{\partial a_{j+mn}} \\ \frac{\partial h_2(\underset{\sim}{a})}{\partial a_{j+1}} & \frac{\partial h_2(\underset{\sim}{a})}{\partial a_{j+2}} & \cdots & \frac{\partial h_2(\underset{\sim}{a})}{\partial a_{j+mn}} \\ \vdots & & & \\ \frac{\partial h_n(\underset{\sim}{a})}{\partial a_{j+1}} & \frac{\partial h_n(\underset{\sim}{a})}{\partial a_{j+2}} & \cdots & \frac{\partial h_n(\underset{\sim}{a})}{\partial a_{j+mn}} \end{pmatrix} ,$$

$$\underset{\sim}{a}^0 \triangleq (\underbrace{0;0,\ldots,0}_{m \text{ times}};\underbrace{\underset{\sim}{Z}'(0),\ldots,\underset{\sim}{Z}'(0)}_{m \text{ times}}) .$$

By virtue of condition a, the elements of $\underset{\sim}{D}$ can be calculated at $\underset{\sim}{a} = \underset{\sim}{a}_0$. The mean value theorem implies

$$\|\underset{\sim}{H}(\hat{\underset{\sim}{a}})-\underset{\sim}{H}(\underset{\sim}{a})\| = \|\underset{\sim}{D}(\underset{\sim}{a}^0)(\hat{a}_{j+1}-a_{j+1},\ldots,\hat{a}_{j+mn}-a_{j+mn})^T\| + O(m\alpha^2) ,$$

where

$$\alpha = \max(\hat{a}_{j+1}-a_{j+1},\ldots,\hat{a}_{j+mn}-a_{j+mn}) ,$$

so that

$$\|\underset{\sim}{H}(\hat{\underset{\sim}{a}})-\underset{\sim}{H}(\underset{\sim}{a})\| \le \sum_{i=1}^{m} \|\underset{\sim}{D}(\underset{\sim}{a}^0)(\hat{\underset{\sim}{Z}}(z_i)-\underset{\sim}{Z}'(z_i))\| + O(\varepsilon^2) . \tag{33}$$

Let

$$B \triangleq \max\left(\left|\frac{\partial h_k(\underset{\sim}{a}^0)}{\partial a_{j+i}}\right|\right)$$

$$1 \le k \le n , \quad 1 \le i \le mn ,$$

and, for $\underset{\sim}{v} = (v_1, v_2, \ldots, v_n)$, define the norm

$$\|\underset{\sim}{v}\| \triangleq \max(\lambda_1 |v_1|, \lambda_2 |v_2|, \ldots, \lambda_n |v_n|) \ ,$$

where the λ_i are positive constants. Inequality (33) implies

$$\|\underset{\sim}{H}(\hat{\underset{\sim}{a}}) - \underset{\sim}{H}(\underset{\sim}{a})\| \le mB \sum_{i=1}^{m} \|\hat{\underset{\sim}{Z}}'(z_i) - \underset{\sim}{Z}'(z_i)\| + O(\varepsilon^2) \ ,$$

and the corollary follows if the λ_i are chosen such that

$$\max(\lambda_1, \lambda_2, \ldots, \lambda_n) < 1/mB \ .$$

□

Cauchy's theorem for ordinary differential equations states that the initial value problem

$$z'(y) = h(y,z)$$

$$z(0) = 0 \ ,$$

has a unique analytic local solution in a neighbourhood of $y = 0$ provided h is analytic in the variables y and z in some neighbourhood of $(0,0)$. The classic proof of this theorem uses the method of majorants (cf. Petrovski 1966). The result is a special case of the Picard-Lindelöf theorem, and an alternative proof of Cauchy's theorem can be gleaned from the method of successive approximations. The following corollary is a generalisation of Cauchy's theorem for system (1).

Corollary 5

Suppose the h_i in system (1) satisfy condition a and inequalities (3.59), (3.60) of Theorem 3.2 and are analytic in the variables y, $\underset{\sim}{Z}(z_i)$, $\underset{\sim}{Z}'(z_i)$, $0 \le i \le m$ for $(y;\underset{\sim}{Z}(z_1),\ldots,\underset{\sim}{Z}(z_m);\underset{\sim}{Z}'(z_1),\ldots,\underset{\sim}{Z}'(z_m)) \in N_\varepsilon$. There exists a local solution to system (1) unique among functions in C^1 and that solution is analytic in y.

Proof:

The existence and uniqueness of a solution follow immediately from Corollary 4. The only thing to show is that the solution is analytic for y sufficiently small. It was shown in Theorem 1 that the sequence defined by (3) converges uniformly to the solution. If the h_i are analytic in the aforesaid variables, then the z_i can be written as a double series

$$z_i(y) = \sum_{k=0}^{\infty} s_i^{(k)}(y) \ , \tag{34}$$

where the $s_i^{(k)}(y)$ terms are analytic in y, i.e.

$$s_i^{(k)}(y) = \sum_{j=0}^{\infty} \alpha_i^{(j)} y^j \ . \tag{35}$$

Since series (34) converges uniformly for y small and there exists a δ such that for all i series (35) converges for $|y| < \delta$, series (34) must converge to an analytic function.

$\square$

It was shown in Section 3.4 that the governing equations for the multi-foci lens problem can be written in the form of system (1). The h_i, as noted in Section 3.4, are analytic in their arguments, and the system thus satisfies the conditions of Corollary 5 once one of the paraxial solutions is specified. Corollary 5 thus gives an alternative existence, uniqueness, and analyticity proof for this problem. The lens boundaries are unique among C^2 functions since the formulation of the system requires expressions for $f''(y), g''(y)$ (i.e. to ensure the z_i are C^1, it must be assumed f and g are C^2).

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