Asymptotic Dimension of One-Relator Groups and Coxeter Groups



Panagiotis Tselekidis
The Queens College
University of Oxford

A thesis submitted for the degree of
Doctor of Philosophy

2023

To my family

for its unwavering support

that made this thesis feasible.

Acknowledgements

I am deeply indebted to my supervisor, Prof. Panos Papasoglu for his invaluable advice, continuous encouragement, and patience during my DPhil studies. His knowledge and plentiful experience have inspired me in all the time of my academic research.
I am thankful to The A. G. Leventis Foundation that supported me financially during the pandemic. I would also like to offer my special thanks to Dr. Mark Hagen, Dr. Richard Wade, Dr. David Hume, Dr. Anthony Genevois, Dr. Marco Linton, and Prof. Dawid Kielak for their very useful comments on my papers.
Finally, I would like to express my gratitude to my family. Without its tremendous understanding and encouragement in the past years, it would be impossible for me to complete my studies.

Abstract

The *asymptotic dimension* is a large scale analogue of topological dimension which proved to be a very useful tool in Group Theory and Topology (see [39]).

In Chapter 1, we prove a new inequality for the asymptotic dimension of HNN-extensions. We deduce that the asymptotic dimension of every finitely generated one relator group is at most two, confirming a conjecture of A.Dranishnikov. As further corollaries, we calculate the asymptotic dimension of RAAGs, and we give a new upper bound for the asymptotic dimension of fundamental groups of graphs of groups.

In Chapter 2, we give a new upper bound for the asymptotic dimension of RACGs. To be more precise, let W_Γ be the right-angled Coxeter group with defining graph Γ. We show that the asymptotic dimension of W_Γ is smaller than or equal to $dim_{CC}(\Gamma)$, the clique-connected dimension of the graph.

In Chapter 3, we give some new results for the asymptotic dimension of Artin and Coxeter groups. If A_Γ (W_Γ) is the Artin (Coxeter) group with defining graph Γ we denote by $Sim(\Gamma)$ the number of vertices of the largest clique in Γ. We show that $asdim A_\Gamma \leq Sim(\Gamma)$, if $Sim(\Gamma) = 2$. We conjecture that the inequality holds for every Artin group. We prove that if for all free of infinity Artin (Coxeter) groups the conjecture holds, then it holds for all Artin (Coxeter) groups. As a corollary, we show that $asdim W_\Gamma \leq Sim(\Gamma)$ for all Coxeter groups, which is the best known upper bound for the asymptotic dimension of Coxeter Groups. As a further corollary, we show that the asymptotic dimension of any large type Artin groups with $Sim(\Gamma) = 3$ is at most three.

The fourth chapter is inspired by a theorem of topological dimension theory which states that every topological space X of dimension n can be separated by a subset Y of dimension equal to or smaller than $n - 1$. We give an analogous theorem in the asymptotic dimension case. To be

more precise, we prove that every geodesic metric space of asymptotic dimension n containing a bi-infinite geodesic can be coarsely separated by a subset S of asymptotic dimension equal to or smaller than $n-1$. Moreover, we define asymptotic Cantor manifolds and we state some questions related to them.

Contents

0 Introduction. 8

0.1 Asymptotic Dimension. 8

0.2 Graphs of Groups and One-Relator groups. 9

0.3 RAAGs and RACGs. 10

0.4 Artin and Coxeter Groups. 11

0.5 Coarse Separation. 13

0.6 Statement of Originality. 14

1 Asymptotic dimension of Graphs of Groups and One-Relator groups. 15

1.1 Asymptotic dimension of HNN-extensions. 18

1.1.1 Right-angled Artin groups. 30

1.2 Asymptotic dimension of one-relator groups. 33

1.2.1 One relator groups with asymptotic dimension two. 35

1.3 Graphs of Groups. 37

2 A new upper bound for the Asymptotic Dimension of RACGs. 39

2.1 Preliminaries. 41

2.1.1 The clique connected dimension of a graph. 42

2.2 Basic Lemmas. 43

2.3 An increasing property of $dim_{CC}(*)$. 45

2.4 Asymptotic dimension of RACGs. 47

2.4.1 Graph Products of Finite Groups. 49

2.5 Corollaries of the main result. 49

3 Asymptotic dimension of Artin groups and a new upper bound for Coxeter groups. 53

3.1 Preliminaries. 55

3.2 The case $Sim(\Gamma) = 2$. 56

3.2.1 Graph-Groups. 58

3.3 The main results. 58

4 Coarsely separation of groups and spaces. 62

4.1 Weak Coarse Separation of Spaces. 64

4.2 Coarse Separation of Spaces. 72

4.3 Some questions. 74

Chapter 0

Introduction.

This thesis consists of five (main) chapters each of which is a paper of mine. Every chapter has its own detailed introduction. Chapter 1 (see [55]) is an accepted paper of mine. Chapter 2 (see [56]) is submitted to a journal. Chapter 3 (see [57]) can be found on arXiv. Chapter 4 is a finished article but not uploaded anywhere yet.

0.1 Asymptotic Dimension.

In 1993, M. Gromov introduced the notion of the asymptotic dimension of metric spaces (see [22]) as an invariant of finitely generated groups. It can be shown that if two metric spaces are quasi isometric then they have the same asymptotic dimension. The asymptotic dimension $asdimX$ of a metric space X is defined as follows: $asdimX \leq n$ if and only if for every $R > 0$ there exists a uniformly bounded covering $\mathcal{U}$ of X such that the R-multiplicity of $\mathcal{U}$ is smaller than or equal to $n+1$ (i.e. every R-ball in X intersects at most $n+1$ elements of $\mathcal{U}$).
There are many equivalent ways to define the asymptotic dimension of a metric space. It turns out that the asymptotic dimension of an infinite tree is 1 and the asymptotic dimension of $\mathbb{E}^n$ is n.

Let G be a finitely generated group. We define $asdimG = asdimC(G,S)$, where $C(G,S)$ is a Cayley graph of G with respect to a finite generating set S. One of the most important properties of the asymptotic dimension is that passes to subgroups. To be more precise, let G, H be two finitely generated groups. If H is a subgroup of G, then $asdimH \leq asdimG$.

In 1998, the asymptotic dimension achieved particular prominence in geometric group theory after a paper of Guoliang Yu, (see [39]) which proved the Novikov higher

signature conjecture for manifolds whose fundamental group has finite asymptotic dimension.
Unfortunately, not all finitely presented groups have finite asymptotic dimension. For example, Thompson's group F has infinite asymptotic dimension since it contains $\mathbb{Z}^n$ for all n.
However, we know for many classes of groups that they have finite asymptotic dimension, for instance, hyperbolic, relatively hyperbolic, Mapping Class Groups of surfaces and one relator groups have finite asymptotic dimension (see [4], [33], [6], [32] [55]). The exact computation of the asymptotic dimension of groups, or finding the optimal upper bound is more delicate.
Another remarkable result is that of Buyalo and Lebedeva (see [8]) where in 2006 they established the following equality for hyperbolic groups:

$$asdimG = dim\partial_\infty G + 1.$$

0.2 Graphs of Groups and One-Relator groups.

Definition 0.2.1. G is one relator group, if it has a presentation of the form

$$G = \langle a_1, \ldots, a_n \mid r \rangle,$$

where r is a word on $\{a_1, \ldots, a_n\}$.

Some examples of one-relator groups include Surface groups, Baumslag-Solitar groups, and $\pi_1(S^3 \backslash$some knots$)$.

One relator groups is a classical topic in group theory, which has been studied since 1930s. As one relator groups are some of the most important finitely generated groups, it is reasonable to ask about their asymptotic dimension.

Theorem 0.2.1. *(Dranishnikov, Matsnev, 2006)([4],[32]) One relator groups have finite asymptotic dimension.*

They also gave an upper bound, which is half of the length of the relation. However, the upper bound that Dranishnikov and Matsnev gave was far from optimal. In 2007, Dranishnikov conjectured the following ([11]):
Conjecture 1: One relator groups have asymptotic dimension at most two.

In Chapter 1, we prove,

Theorem 0.2.2. *([55]) One relator groups have asymptotic dimension at most two.*

We further prove that,

Theorem 0.2.3. *([55]) Let G be a finitely generated one relator group. Then one of the following is true:*
(i) *G is finite cyclic, and $asdim\, G = 0$*
(ii) *G is a nontrivial free group or a free product of a nontrivial free group and a finite cyclic group, and $asdim\, G = 1$*
(iii) *G is an infinite freely indecomposable not cyclic group or a free product of a nontrivial free group and an infinite freely indecomposable not cyclic group, and $asdim\, G = 2$.*

We also obtain a general theorem for the asymptotic dimension of fundamental groups of graphs of groups, improving the previously known theorem of Dranishnikov and Bell ([2])

Theorem 0.2.4. *([55]) Let $(\mathbb{G}, Y)$ be a finite graph of finitely generated groups with vertex groups $\{G_v \mid v \in Y^0\}$ and finitely generated edge groups $\{G_e \mid e \in Y^1_+\}$. Then the following inequality holds:*

$$asdim\pi_1(\mathbb{G}, Y, \mathbb{T}) \leq max_{v\in Y^0, e\in Y^1_+}\{asdim G_v, asdim\, G_e + 1\}.$$

Using this theorem we can see, for example, that the asymptotic dimension of a graph of surface groups (genus ≥ 2) with free edge groups is two.

0.3 RAAGs and RACGs.

RAAGs and RACGs touch upon a number of areas of mathematics, such as representation theory, combinatorics, topology, and geometry. They are often considered as a playground for many open problems in geometric group theory. RAAGs and RACGs are the simplest examples of Artin and Coxeter groups; in these the edges are labeled by 2.

The asymptotic dimension of RAAGs is completely known, while for RACGs there is still some mystery. In Chapter 1, we show the following,

Theorem 0.3.1. *([55]) $asdim A_\Gamma = Sim(\Gamma)$, for all RAAGs groups.*

We note that this has been proven earlier by N.Wright [38] by different methods.

Dranishnikov proved (see [14]) that the asymptotic dimension of RACGs is bounded from above by the dimension of their Davis complex.

However, in many cases Dranishnikov's bound is far from optimal. For example, if the defining graph Γ is a clique with n vertices, then by Dranishnikov's result we have that $asdim W_\Gamma \leq n$, however, $asdim W_\Gamma = 0$.

In Chapter 2, we provide a new upper bound for the asymptotic dimension of RACGs treating some of the cases in which Dranishnikov's bound fails to be optimal.

We show,

Theorem 0.3.2. *([57]) Let W_Γ be the right-angled Coxeter group with connected defining graph Γ. Then,*

$$asdim W_\Gamma \leq dim_{CC}(\Gamma).$$

If Γ is not connected, then $asdim W_\Gamma \leq max\{1, dim_{CC}(\Gamma)\}$.

This theorem is also true for Coxeter groups such that every clique in their defining graphs defines a finite Coxeter subgroup.

By $dim_{CC}(\Gamma)$ we denote the clique-connected dimension of a graph. The clique-connected dimension of a finite graph can be described as an index showing how connected is the graph modulo cliques. For example, if Γ is a clique, then $dim_{CC}(\Gamma) = 0$.

We further show that if $dim_{CC}(\Gamma) \leq 2$, then $asdim W_\Gamma = dim_{CC}(\Gamma)$.

Let Γ be a simplicial graph. We label every vertex v of the graph by a *vertex group* G_v. The *graph product* $\mathbb{GP}(\Gamma, G_v)_{v \in V(\Gamma)}$ is the free product of the vertex groups modulo the relations $xyx^{-1}y^{-1}$ where $x \in G_v$, $y \in G_w$, and v, w are connected by an edge.

We generalize Theorem 0.3.2 to the following:

Theorem 0.3.3. *([57]) Let Γ be a finite connected simplicial graph along with a collection of finite vertex groups. Then,*

$$asdim\, \mathbb{GP}(\Gamma, G_v)_{v \in V(\Gamma)} \leq dim_{CC}(\Gamma).$$

If Γ is not connected, then $asdim\, \mathbb{GP}(\Gamma, G_v)_{v \in V(\Gamma)} \leq max\{1, dim_{CC}(\Gamma)\}$.

0.4 Artin and Coxeter Groups.

Artin groups are easily defined via presentations, like Coxeter groups, but they are often poorly understood. There are numerous conjectures about them, for example, it is not known, in general, whether they are torsion free.

Some examples of Artin groups are the free non-abelian groups, free abelian groups, and RAAGs. In recent years, Artin and Coxeter groups proved to be a very rich source of examples and counterexamples of interesting phenomena in geometric group theory.

They both are defined via finite simplicial graphs, by labeling each edge of the graph by a natural number.

The following is an important question in the subject posed by Dranishnikov ([11]):
Question 1: Do Artin groups have finite asymptotic dimension?

Considering this question we make the following:
Conjecture 2:*([56])*
1. $asdimA_\Gamma \leq Sim(\Gamma)$, for all Artin groups,
2. $asdimW_\Gamma \leq Sim(\Gamma)$, for all Coxeter groups,
where Γ is the defining graph, and $Sim(\Gamma)$ the number of vertices of the largest clique in Γ.

In Chapter 3, we give the following partial results regarding the first part of Conjecture 2,

Proposition 0.4.1. *([56]) Let A_Γ be an Artin group with $Sim(\Gamma) = 2$. Then,*

$$asdimA_\Gamma = 2.$$

Proposition 0.4.2. *([56]) Let A_Γ be an Artin group of large type with $Sim(\Gamma) = 3$. Then,*

$$asdimA_\Gamma \leq 3.$$

Proposition 0.4.3. *Let A_Γ be an Artin group such that every label is at most four, and with $Sim(\Gamma) = 3$. Then,*

$$asdimA_\Gamma \leq 3.$$

We also prove the following,

Theorem 0.4.4. *([56]) If for all Artin (Coxeter) groups defined by cliques the conjecture holds, then it holds for all Artin (Coxeter) groups.*

It is known by an isometric embedding theorem of Januszkiewicz (see [52]) that Coxeter groups have finite asymptotic dimension. In particular, Januszkiewicz's theorem shows that for any Coxeter group W_Γ with defining graph Γ we have the following upper bound: $asdimW_\Gamma \leq \sharp V(\Gamma)$.
Dranishnikov notes that this can be improved to the following: $asdimW_\Gamma \leq ch(\Gamma)$, where $ch(\cdot)$ is the chromatic number of a graph. In general, $Sim(\Gamma) \leq ch(\Gamma)$.

As an application of Theorem 0.4.4, we prove the second part of conjecture 2, namely,

Theorem 0.4.5. *([56]) $asdimW_\Gamma \leq Sim(\Gamma)$, for all Coxeter groups.*

This is the best known upper bound for the asymptotic dimension of Coxeter groups, improving the previous bound due to Dranishnikov.

0.5 Coarse Separation.

It is a well-known theorem of topological dimension theory that every topological space X of dimension $n \geq 1$ can be separated by a subset Y of dimension equal to or smaller than $n-1$ (see [23]). By separation we mean that the space $X \setminus Y$ contains at least two connected components.
In Chapter 4, inspired by this result, we ask whether an asymptotic analogue exists. We introduce an asymptotic analogue of topological separation, the *weak coarse separation*, and we prove the following,

Theorem 0.5.1. *Let X be a geodesic metric space with finite asymptotic dimension containing a bi-infinite geodesic. If $asdimX = n > 0$, then there exists a subspace Y of asymptotic dimension strictly less than n which coarsely separates X.*

We also give examples of geodesic metric spaces which cannot be (weakly) coarsely separated by any subset.

We recall that a topological space X of dimension n is a *Cantor n-manifold* if X can not be separated by a closed $(n-2)$-dimensional subset $(n \geq 2)$.
We give an asymptotic analogue of Cantor manifolds and we asks some questions about them.

0.6 Statement of Originality.

Originality is claimed for every result of this thesis unless otherwise stated.

Chapter 1

Asymptotic dimension of Graphs of Groups and One-Relator groups.

The inequalities of G.Bell and A.Dranishnikov (see [2] and [14]) play a key role on finding an upper bound for the asymptotic dimension of groups. However, in some cases the upper bounds that the inequalities of G.Bell and A.Dranishnikov provide are quite far from being optimal. An example is the asymptotic dimension of one relator groups.

In this chapter we prove some new inequalities that can be a useful tool for the computation of the asymptotic dimension of groups. As an application, we give the optimal upper bound for the asymptotic dimension of one relator groups which was conjectured by A.Dranishnikov. As a further corollary, we calculate the exact asymptotic dimension of any right-angled Artin group; this has been proven earlier by N.Wright [38] by different methods.

The first inequality and one of the main results we prove is the following:

Theorem 1.0.1. *Let $G*_N$ be an HNN-extension of the finitely generated group G over a finitely generated subgroup N. We have the following inequality*

$$asdim\, G*_N \leq max\{asdim G, asdim N + 1\}.$$

Next, we calculate the asymptotic dimension of the right-angled Artin groups. To be more precise, let Γ be a finite simplicial graph; we denote by $A(\Gamma)$ the *right-angled Artin group* (RAAG) associated to the graph Γ. We set

$Sim(\Gamma) = max\{n \mid \Gamma$ contains the 1-skeleton of the standard $(n-1)$-simplex $\Delta^{n-1}\}$.

Then by applying Theorem 1.0.1 we obtain the following:

Theorem 1.0.2. *Let Γ be a finite simplicial graph. Then,*

$$asdim A(\Gamma) = Sim(\Gamma).$$

In 2005, G.Bell, and A.Dranishnikov (see [3]) gave a proof that the asymptotic dimension of one relator groups is finite and also they gave an upper bound, namely the length of the relator plus one. Let $G = \langle S \mid r \rangle$ be a finitely generated one relator group such that $|\, r \,| = n$. Then

$$asdim\, G \leq n + 1.$$

To prove this upper bound G. Bell, and A. Dranishnikov used an inequality for the asymptotic dimension of HNN-extensions (see [2]).
In particular, let G be a finitely generated group and let N be a subgroup of G. Then,

$$asdim\, G*_N \leq asdim\, G + 1.$$

In 2006, D. Matsnev (see [32]) proved a sharper upper bound for the asymptotic dimension of one relator groups. D. Matsnev proved the following: let $G = \langle S \mid r \rangle$ be a one relator group then

$$asdim\, G \leq \lceil \tfrac{length(r)}{2} \rceil.$$

Here by $\lceil a \rceil$ $(a \in \mathbb{R})$ we denote the minimal integer greater than or equal to a.

Applying Theorem 1.0.1 we answer a conjecture of A.Dranishnikov (see [11]) giving the optimal upper bound for the asymptotic dimension of one relator groups.

Theorem 1.0.3. *Let G be a finitely generated one relator group. Then*

$$asdim\, G \leq 2.$$

We note that R. C. Lyndon (see [29]) has shown that the *cohomological dimension* of a torsion-free one-relator group is smaller than or equal to 2. Our result can be seen as a large scale analog of this.
We note that the large scale geometry of one relator groups can be quite complicated, for example one relator groups can have very large isoperimetric functions (see e.g. [34]).

It is worth noting that L.Sledd showed that the Assouad-Nagata dimension of any finitely generated $C'(1/6)$ group is at most two (see [36]).

Theorem 1.0.3 combined with the results of M.Kapovich and B.Kleiner (see [24]) leads us to a description of the boundary of hyperbolic one relator groups.

We determine also the one relator groups that have asymptotic dimension exactly two. We prove that every infinite finitely generated one relator group G that is not a free group or a free product of a free group and a finite cyclic group has asymptotic dimension equal to 2 (Proposition 1.2.5).
We obtain the following:
Corollary. *Let G be finitely generated freely indecomposable one relator group which is not cyclic. Then*

$$asdim\, G = 2.$$

Moreover, we describe the finitely generated one relator groups in the following corollary:
Corollary. *Let G be a finitely generated one relator group. Then one of the following is true*:
(i) *G is finite cyclic, and $asdim\, G = 0$*
(ii) *G is a nontrivial free group or a free product of a nontrivial free group and a finite cyclic group, and $asdim\, G = 1$*
(iii) *G is an infinite freely indecomposable not cyclic group or a free product of a nontrivial free group and an infinite freely indecomposable not cyclic group, and $asdim\, G = 2$.*

Using Theorem 1.0.1 and an inequality of A.Dranishnikov about the asymptotic dimension of amalgamated products (see [14]) we obtain a more general theorem for the asymptotic dimension of fundamental groups of graphs of groups.

Theorem 1.0.4. *Let $(\mathbb{G}, Y)$ be a finite graph of groups with finitely generated vertex groups $\{G_v \mid v \in Y^0\}$ and finitely generated edge groups $\{G_e \mid e \in Y^1_+\}$. Then the following inequality holds:*

$$asdim \pi_1(\mathbb{G}, Y, \mathbb{T}) \leq max_{v \in Y^0, e \in Y^1_+}\{asdim G_v, asdim\, G_e + 1\}.$$

Using the previous theorem we can obtain for example, that the asymptotic dimension of a graph of surface groups (genus ≥ 2) with free edge groups is two. Theorem 1.0.4 says that the asymptotic dimension doesn't *jump* as long as there exists a vertex

group with asymptotic dimension greater than the asymptotic dimension of any edge group.

1.1 Asymptotic dimension of HNN-extensions.

Let X be a metric space and $\mathcal{U}$ a covering of X, we say that the covering $\mathcal{U}$ is *d-bounded* or *d-uniformly bounded* if $sup_{U\in\mathcal{U}}\{\operatorname{diam} U\} \leq d$. The *Lebesgue number* $L(\mathcal{U})$ of the covering $\mathcal{U}$ is defined as follows:

$$L(\mathcal{U}) = sup\{\lambda \mid \text{if } A \subseteq X \text{ with } \operatorname{diam} A \leq \lambda \text{ then there exists } U \in \mathcal{U} \text{ s.t. } A \subseteq U\}.$$

We recall that the order $ord(\mathcal{U})$ of the cover $\mathcal{U}$ is the smallest number n (if it exists) such that each point of the space belongs to at most n sets in the cover.
For a metric space X, we say that $(\mathrm{r},\mathrm{d})-dimX \leq n$ if for $r > 0$ there exists a d-bounded cover $\mathcal{U}$ of X with $ord(\mathcal{U}) \leq n+1$ and with Lebesgue number $L(\mathcal{U}) > r$. We refer to such a cover as an (r,d)-*cover* of X.
The following proposition is due to G.Bell and A.Dranishnikov (see [2]).

Proposition 1.1.1. *For a metric space X, $asdimX \leq n$ if and only if there exists a function $d(r)$ such that $(r, d(r)) - dimX \leq n$ for all $r > 0$.*

We recall that a family $\{X_i\}$ of subsets of X satisfies the inequality $asdimX_i \leq n$ *uniformly* if for every $R > 0$ there exists a D-bounded covering $\mathcal{U}_i$ of X_i with $R - mult(\mathcal{U}_i) \leq n+1$, for every i. By $R - mult(\mathcal{U})$ we denote the R-multiplicity of $\mathcal{U}$ (i.e. the maximal number of elements of the cover which can be intersected by an R-ball).
For the proofs of the following theorems 1.1.2 and 1.1.3 see [1].

Theorem 1.1.2. *(Infinite Union Theorem) Let $X = \cup_a X_a$ be a metric space where the family $\{X_a\}$ satisfies the inequality $asdimX_a \leq n$ uniformly. Suppose further that for every $r > 0$ there is a subset $Y_r \subseteq X$ with $asdimY_r \leq n$ such that $d(X_a \setminus Y_r, X_b \setminus Y_r) \geq r$ whenever $X_a \neq X_b$. Then $asdimX \leq n$.*

Theorem 1.1.3. *(Finite Union Theorem) For every metric space presented as a finite union $X = \cup_i X_i$ we have*

$$asdimX = max\{asdimX_i\}.$$

A partition of a metric space X is a presentation as a union $X = \cup_i W_i$ such that $Int(W_i) \cap Int(W_j) = \varnothing$ whenever $i \neq j$. We denote by ∂W_i the topological boundary of W_i and by $Int(W_i)$ the topological interior. We have that $\partial W \cap Int(W) = \varnothing$. The boundary can be written as

$$\partial W_i = \{x \in X \mid d(x, W_i) = d(x, X \setminus W_i) = 0\}.$$

For the proof of the following theorem see [14].

Theorem 1.1.4. *(Partition Theorem) Let X be a geodesic metric space. Suppose that for every $R > 0$ there is $d > 0$ and a partition $X = \cup_i W_i$ with $asdim W_i \leq n$ uniformly in i, and such that $(R, d) - dim(\cup_i \partial W_i) \leq n - 1$, where ∂W_i is taken with the metric restricted from X. Then $asdim X \leq n$.*

Let G be a finitely generated group, N a subgroup of G and $\phi : N \to G$ a monomorphism. We set $\overline{G} = G*_N$ the HNN-extension of G over the subgroup N with respect to the monomorphism ϕ. We fix a finite generating set S for the group G, then $\overline{S} = S \cup \{t, t^{-1}\}$ is a finite generating set for the group $\overline{G}$. We set $C(\overline{G}) = Cay(\overline{G}, \overline{S})$ (Cayley graph).

Normal forms for HNN-extensions.
We note that there exist two types of *normal forms* for HNN-extensions, the right normal form and the left normal form. We are going to use both of them in this chapter.
Right normal form: Let S_N and $S_{\phi(N)}$ be sets of representatives of *right cosets* of G/N and of $G/\phi(N)$ respectively. Then every $w \in \overline{G}$ has a unique normal form $w = gt^{\epsilon_1}s_1t^{\epsilon_2}s_2...t^{\epsilon_k}s_k$ where $g \in G$, $\epsilon_i \in \{-1, 1\}$, if $\epsilon_i = 1$ then $s_i \in S_N$, if $\epsilon_i = -1$ then $s_i \in S_{\phi(N)}$, and if $s_i = 1$ then $\epsilon_i\epsilon_{i+1} > 0$. We say that the *length* of the right normal form of w is k.
Left normal form: Let ${}_NS$ and ${}_{\phi(N)}S$ be sets of representatives of *left cosets* of G/N and of $G/\phi(N)$ respectively. Then every $w \in \overline{G}$ has a unique normal form $w = s_1t^{\epsilon_1}s_2t^{\epsilon_2}...s_kt^{\epsilon_k}g$ where $g \in G$, $\epsilon_i \in \{-1, 1\}$, if $\epsilon_i = 1$ then $s_i \in_{\phi(N)} S$, if $\epsilon_i = -1$ then $s_i \in_N S$, and if $s_i = 1$ then $\epsilon_{i-1}\epsilon_i > 0$. We say that the *length* of the left normal form of w is k.

We observe that the *length* of the right and left normal forms of an element coincide; we denote this length by $l(w)$.

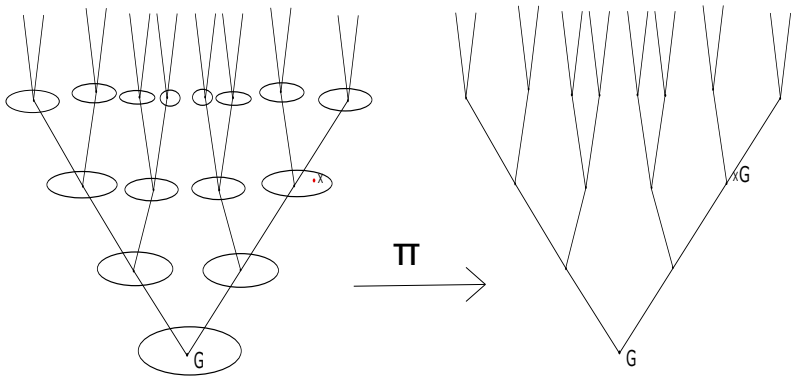


Figure 1.1: An illustration of the projection $\pi : C(\overline{G}, S) \to T$.

Convention: When we write a normal form we mean the right normal form, unless otherwise stated.

The group $\overline{G} = G*_N$ acts on its Bass-Serre tree T. There is a natural projection $\pi : G*_N \to T$ defined by the action: $\pi(g) = gG$.

Lemma 1.1.5. *The map $\pi : \overline{G} \to T$ extends to a simplicial map from the Cayley graph, $\pi : C(\overline{G}, S) \to T$ which is 1-Lipschitz.*

Proof. Let $g \in \overline{G}$ and $s \in \overline{S}$. Then the vertex g is mapped to the vertex $\pi(g) = \pi(gs) = gG$.

If $s \in S$, then the edge $[g, gs]$ is mapped to the vertex $\pi(g) = \pi(gs) = gG$.

If $s \in \{t, t^{-1}\}$, without loss of generality we may assume that $s = t$, then the edge $[g, gs]$ is mapped to the edge $[\pi(g), \pi(gs)] = [gG, gtG]$ of T.

We observe that the simplicial map $\pi : C(\overline{G}) \to T$ is 1-Lipschitz. □

The base vertex G separates T into two parts $T_- \setminus G$ and $T_+ \setminus G$, where

$$\pi^{-1}(T_+) = \{w \in \overline{G} \mid \text{if } w = gt^{\epsilon_1}s_1t^{\epsilon_2}s_2...t^{\epsilon_k}s_k \text{ is the normal form of } w \text{ then } \epsilon_1 = 1\}$$

and similarly

$$\pi^{-1}(T_-) = \{w \in \overline{G} \mid \text{if } w = gt^{\epsilon_1}s_1t^{\epsilon_2}s_2...t^{\epsilon_k}s_k \text{ is the normal form of } w \text{ then } \epsilon_1 = -1\}.$$

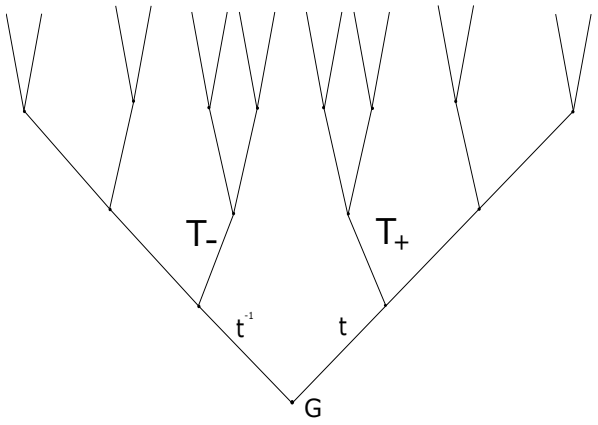


Figure 1.2: An illustration of T_+ and T_-.

We note that both $T_+ \setminus G$ and $T_- \setminus G$ are unions of connected components of T and $\pi^{-1}(T_+)$ and $\pi^{-1}(T_-)$ are unions of connected components of $C(\overline{G})$, see Figure 1.2.

We consider the Bass-Serre tree T as a metric space with the simplicial metric $\overline{d}$. If Y is a graph, we denote by Y^0 or $V(Y)$ the vertices of Y.
For $u \in T^0$ we denote by $|\,u\,|$ the distance to the vertex with label G. We note that the distance of the vertex wG from G in the Bass-Serre tree T equals to the length $l(w)$ of the normal form of w, $|\,wG\,| = l(w)$.

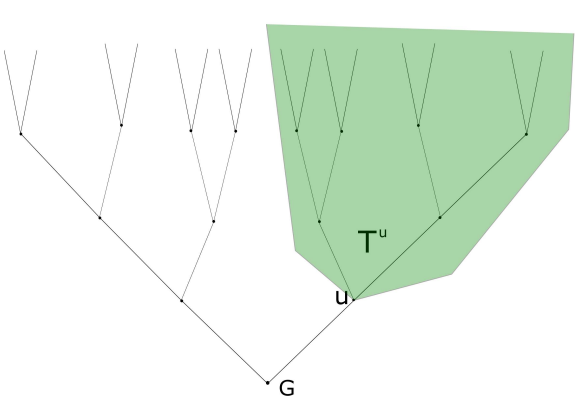
We recall that a *full subgraph* of a graph Γ is a subgraph formed from a subset of vertices V and from all of the edges that have both endpoints in the subset V.
If A is a subgraph of Γ we define the *edge closure* $E(A)$ of A to be the full subgraph of Γ formed from $V(A)$. Obviously, $V(E(A)) = V(A)$.

We fix some notation on the Bass-Serre tree T and on the Cayley graph.

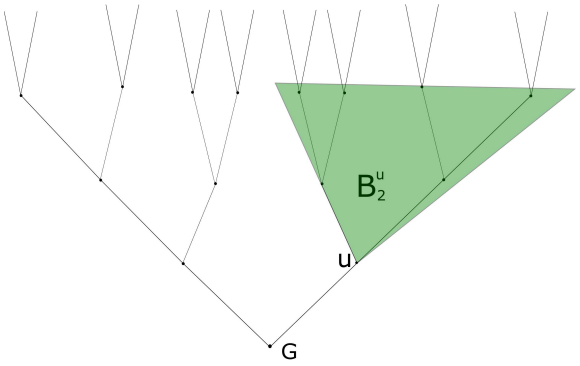
In the tree T. We denote by B_r^T the r-ball in T centered at G ($r \in \mathbb{N}$). There is a partial order on vertices of T defined as follows: $v \leq u$ if and only if v lies in the geodesic segment $[G, u]$ joining the base vertex G with u. For $u \in T^0$ of nonzero level (i.e. $u \neq G$) and $r > 0$ we set

$$T^u = E(\{v \in T^0 \mid u \leq v\}),\ B_r^u = E(\{v \in T^u \mid\mid v\mid \leq \mid u\mid + r\}).$$

For every vertex $u \in T^0$ represented by a coset $g_u G$ we have the equality $B_r^u = g_u B_r^T \cap T^u$. We also observe that $B_r^u = E(\{v \in T^u \mid \text{s.t. } \overline{d}(v, u) \leq r\})$. See figures



(a) Here we have an illustration of T^u



(b) Here we have an illustration of B_r^u, where $r = 2$.

Figure 1.3

1.3a and 1.3b for an illustration of the sets T^u and B_r^u

In the Cayley graph. For $R \in \mathbb{N}$, let

$$M_R = \{g \in \overline{G} \mid dist(g, N \cup \phi(N)) = R\}.$$

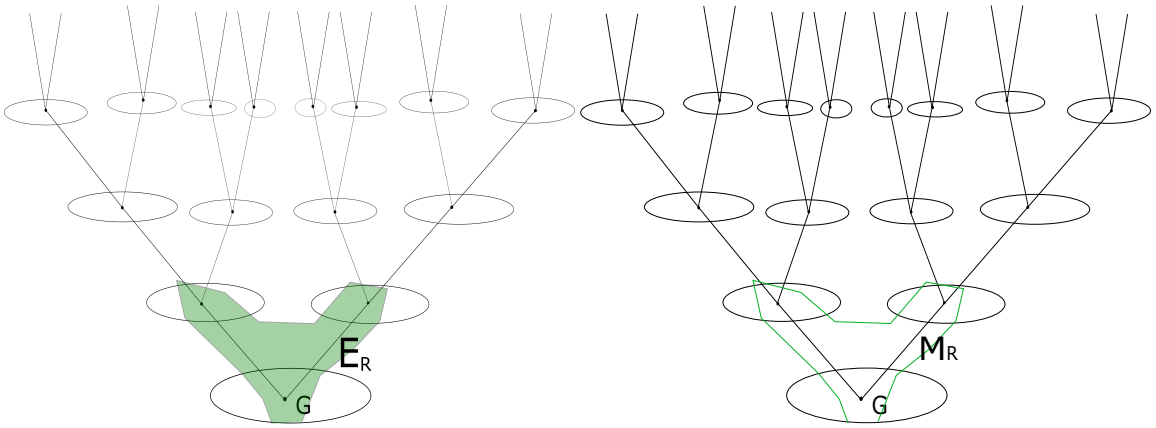
Let $u = g_u G$, we set $M_R^u = g_u M_R \cap \pi^{-1}(T^u)$. We observe that $\pi(M_R^u) \subseteq B_R^u$ since π is 1-Lipschitz.

Let $u = g_u G$, we set $E_R = E(N_R(N \cup \phi(N)))$ and

$$E_R^u = g_u E_R \cap \pi^{-1}(T^u).$$

Obviously, $M_R^u \subseteq E_R^u \subseteq \pi^{-1}(B_R^u)$. By $N_R(A)$ we denote the R-neighbourhood of a set A (i.e. all the points of distance at most R from A).

Convention: We associate every $u \in T^0$ to an element $g_u \in \overline{G}$ such that the following two conditions hold:
(i) $u = g_u G$.
(ii) if the left normal form of g_u is $s_1 t^{\epsilon_1} s_2 t^{\epsilon_2} ... s_k t^{\epsilon_k} g$ then $g = 1_{\overline{G}}$.



(a) An illustration of E_R. (b) An illustration of M_R.

Figure 1.4

We see that in this way we may define a bijective map from T^0 to the set $\mathcal{G}_T$ which consists of the elements of $\overline{G}$ such that conditions (i) and (ii) hold.

Proposition 1.1.6. *If $4 < 4R \leq r$, and the distinct vertices $u, u' \in T^0$ satisfy $|u|, |u'| \in \{nr \mid n \in \mathbb{N}\}$, then*

$$d(M_R^u, M_R^{u'}) \geq 2R.$$

Proof. We distinguish two cases. See figures 1.5a and 1.5b for the case 1 and case 2 respectively.

Case 1: $|u| \neq |u'|$. We recall that every path γ in $C(\overline{G})$ projects to a path $\pi(\gamma)$ in the tree T. Then since

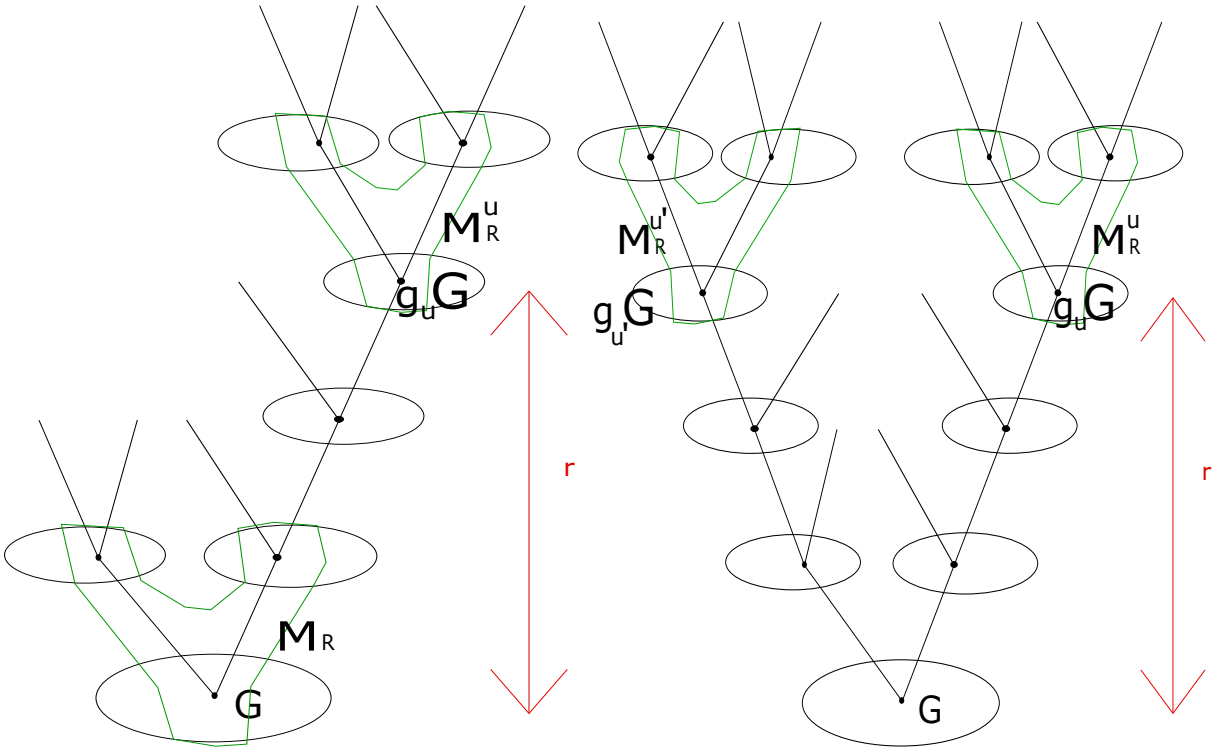
$$M_R^u = g_u M_R \cap \pi^{-1}(T^u) \subseteq \pi^{-1}(B_R^u),$$

$$M_R^{u'} = g_{u'} M_R \cap \pi^{-1}(T^{u'}) \subseteq \pi^{-1}(B_R^{u'})$$

and π is 1-Lipschitz we have that

$$d(M_R^u, M_R^{u'}) \geq \overline{d}(B_R^u, B_R^{u'})) \geq r - R \geq 3R.$$

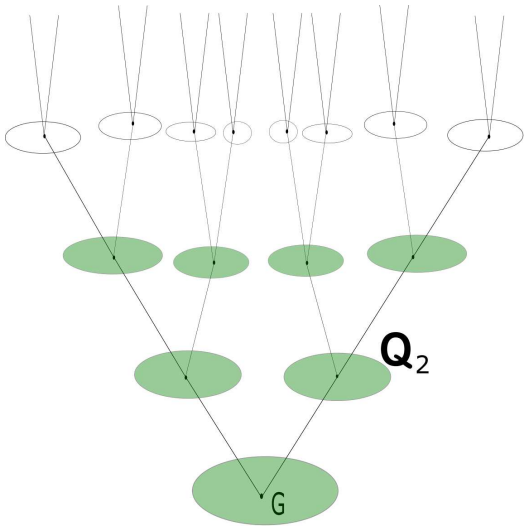
Case 2: $|u| = |u'|$ $(u \neq u')$. We denote by ζ_0 the last vertex of the common geodesic segment $[G, \zeta_0]$ of the geodesics $[G, u]$ and $[G, u']$. We observe that $\overline{d}(u, \zeta_0), \overline{d}(u', \zeta_0) \geq 1$.
Let $x \in M_R^u$, $y \in M_R^{u'}$ and let γ be a geodesic from x to y. Then the path $\pi(\gamma)$ passes through the vertices u, u' and ζ_0. So the geodesic γ intersects both $g_u(N \cup \phi(N))$ and $g_{u'}(N \cup \phi(N))$. Hence



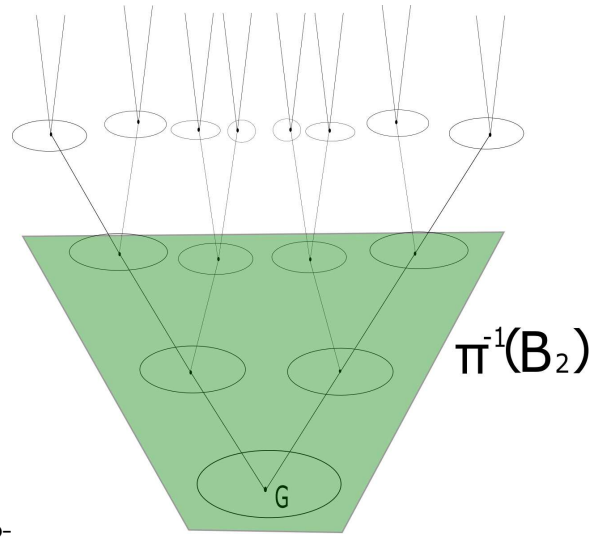
(a) An illustration of case 1 of proposition 1.1.6.

(b) An illustration of case 2 of proposition 1.1.6.

Figure 1.5: In figure (a) we have $u' = G$



(a) An illustration of Q_m, for $m = 2$ (proposition 1.1.8).



(b) An illustration of $\pi^{-1}(B_r)$, where $r = 2$.

Figure 1.6: We note that $Q_m = V(\pi^{-1}(B_r))$.

$$d(x,y) \geq dist(x, g_u(N \cup \phi(N))) + dist(y, g_{u'}(N \cup \phi(N))) + length([\zeta_0, u']) + length([\zeta_0, u]) \geq R + R + 2 = 2(R+1).$$

□

For $w \in G*_N$, we denote by $\| w \|$ the distance from w to $1_{\overline{G}}$ in the Cayley graph $Cay(\overline{G}, \overline{S})$.

Lemma 1.1.7. *Let $w = gt^{\epsilon_1}s_1t^{\epsilon_2}s_2...t^{\epsilon_k}s_k$ be the normal form of w. Then*

$$\| w \| \geq d(s_k, N) \text{ if } \epsilon_k = 1 \text{ and } \| w \| \geq d(s_k, \phi(N)) \text{ if } \epsilon_k = -1.$$

Proof. Without loss of generality we assume that $\epsilon_k = 1$. Let

$$w = (\prod_{i_0=1}^{m_0} s_{i_0})t^{\epsilon_1}(\prod_{i_1=1}^{m_1} s_{i_1})t^{\epsilon_2}...t(\prod_{i_k=1}^{m_k} s_{i_k})$$

be a shortest presentation of w in the alphabet $\overline{S}$ (we note that $s_{i_j} \notin \{t, t^{-1}\}$). We set $\prod_{i_j=1}^{m_j} s_{i_j} = g_j$ for every $j \in \{1, ..., k\}$. Then $w = gt^{\epsilon_1}g_1t^{\epsilon_2}s_2...tg_k = w_0tg_k$.
The first step when we rewrite w in normal form starting from the previous presentation is to write $g_k = ns_k$ (where $n \in N$). Then

$$\| w \| \geq \| g_k \| = \| ns_k \| = d(ns_k, 1) = d(s_k, n^{-1}) \geq d(s_k, N).$$

□

We note that there exists an amalgamated product analogue of the following proposition proved by A.Dranishnikov in [14].

Proposition 1.1.8. *Suppose that* $asdim\, G \leq n$. *Let*

$$Q_m = \{w \in \overline{G} \mid l(w) \leq m\}.$$

Then $asdim Q_m \leq n$, *for every* $m \in \mathbb{N}$.

Proof. We set $P_\lambda = \{w \in \overline{G} \mid l(w) = \lambda\}$. To prove the statement of the proposition it is enough to show that $asdim P_\lambda \leq n$, for every $\lambda \in \mathbb{N}$. Indeed, since

$$Q_m = \cup_{i=0}^{m} P_i$$

by the Finite Union Theorem we obtain that $asdim Q_m \leq n$.

Claim: For $\lambda \in \mathbb{N}$ we have $asdim P_\lambda \leq n$.
Proof of claim: We use induction on λ. We have $P_0 = G$, so $asdim P_0 \leq n$. We observe that $P_\lambda \subseteq P_{\lambda-1}tG \cup P_{\lambda-1}t^{-1}G$. Using the Finite Union Theorem it suffices to show that $asdim(P_\lambda \cap P_{\lambda-1}tG) \leq n$ and $asdim(P_\lambda \cap P_{\lambda-1}t^{-1}G) \leq n$, we show the first.

To show that $asdim(P_\lambda \cap P_{\lambda-1}tG) \leq n$ we use the Infinite Union Theorem. For $r > 0$ we set $Y_r = P_{\lambda-1}tN_r(N)$. We claim that

$$Y_r \subseteq N_{r+1}(P_{\lambda-1}).$$

Indeed, if $z \in Y_r$ then $z = z_0 t z_1$, where $z_0 \in P_{\lambda-1}$ and $z_1 \in N_r(N)$. Since $z_1 \in N_r(N)$ there exists $n \in N$ with $d(n, z_1) \leq$ r.
So $z = z_0 t n n^{-1} z_1 = z_0 \phi(n) t n^{-1} z_1$, and

$$d(z, P_{\lambda-1}) \leq d(z, z_0\phi(n)) = \| tn^{-1}z_1 \| \leq \| t \| + \| n^{-1}z_1 \| \leq 1 + r.$$

Hence $Y_r \sim_{q.i.} P_{\lambda-1}$, so $asdim Y_r \leq n$.

We consider the family xtG where $x \in P_{\lambda-1}$. For $xtG \neq ytG$, we have $d(xtG \setminus Y_r, ytG \setminus Y_r) = d(xtg, yth) = \| g^{-1}t^{-1}x^{-1}yth \|$, where $g, h \in G \setminus N_r(N)$. The first step when we rewrite $g^{-1}t^{-1}x^{-1}yth$ in normal form is to replace $h = ns_k$, where $n \in N$ and $s_k \in S_N$, so $g^{-1}t^{-1}x^{-1}yth = g^{-1}t^{-1}x^{-1}y\phi(n)ts_k$.
Since $h \in G \setminus N_r(N)$ we have that $\|s_k\| = \|n^{-1}h\| \geq d(h, N) \geq r$.

By lemma 1.1.7 we obtain that $\| g^{-1}t^{-1}x^{-1}y\phi(n)ts_k \| \geq \| s_k \| \geq r$.

Finally, by observing that xtG and G are isometric we deduce that $asdim(xtG) \leq n$ uniformly. Since all the conditions of the Infinite Union Theorem hold we have that

$$asdim(P_\lambda \cap P_{\lambda-1}tG) \leq n$$

for every $\lambda \in \mathbb{N}$.

□

We observe that $E(Q_m) = \pi^{-1}(B_m^T)$ and $Q_m = \overline{G} \cap \pi^{-1}(B_m^T)$.

For $w \in \overline{G}$, we set $T^w = T^{\pi(w)}$, where $\pi(w) = wG$.

We note that there was an attempt to prove the following theorem in [31], however, there is a gap in that proof. We give a few details about this gap right after the proof of Theorem 1.1.9.

Theorem 1.1.9. *Let $G*_N$ be an HNN-extension of the finitely generated group G over a finitely generated subgroup N. We have the following inequality*

$$asdim\, G*_N \leq max\{asdimG, asdimN + 1\}.$$

Proof. Let $n = max\{asdimG, asdimN+1\}$. We denote by $\pi : C(\overline{G}, S) \to T$ the map of Lemma 1.1.5.
We recall that we denote by $l(g)$ the length of the normal form of g.

We will use the Partition Theorem (Thm 1.1.4). Let $R, r \in \mathbb{N}$ be such that $R > 1$ and $r > 4R$.

We set,

$$U_r = E[(\pi^{-1}(B_{r-1}^T) \cap E(\{g \in \overline{G} \mid d(g, N \cup \phi(N)) \geq R\})) \cup (\bigcup_{u \in \partial B_r^T} E_R^u)],$$

where $E_R^u = g_u E(N_R(N \cup \phi(N))) \cap \pi^{-1}(T^u)$.
We recall that $M_R = \{g \in \overline{G} \mid d(g, N \cup \phi(N)) = R\}$.
Let A_R be the collection of the edges between the elements of $M_R \subseteq U_r$. We have that $A_R \subseteq U_r$. We define V_r to be the set obtained by removing the interior of the edges of A_R from U_r.
Formally we have that

$$V_r = U_r \setminus \{interior(e) \mid e \in A_R\}.$$

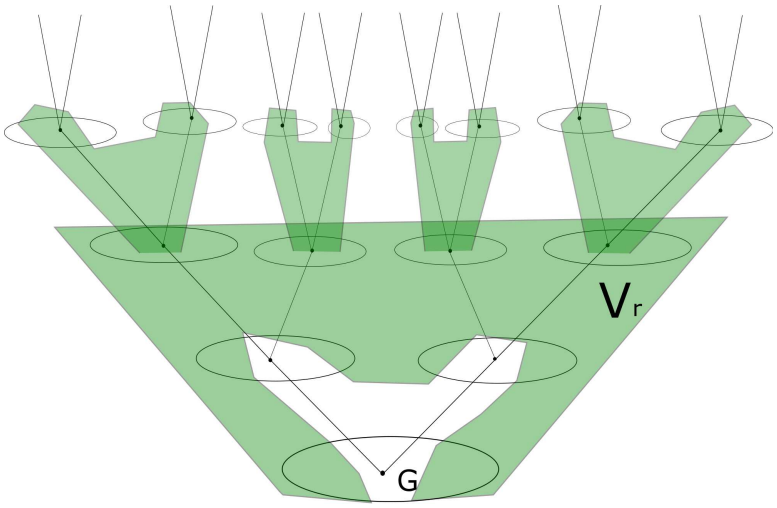


Figure 1.7: An illustration of V_r.

See figure 1.7 for an illustration of the set V_r. We observe that the sets U_r and V_r are subgraphs of $C(\overline{G})$, $\partial U_r = \partial V_r$ and $V_r \cap \overline{G} = U_r \cap \overline{G}$. Obviously, $\bigcup_{u \in \partial B_r^T} E_R^u \subseteq V_r$. We also have

$$V_r \cap \overline{G} = (\overline{G} \cap \pi^{-1}(B_{r-1}^T) \cap E(\{g \in \overline{G} \mid d(g, N \cup \phi(N)) \geq R\})) \cup (\overline{G} \cap \bigcup_{u \in \partial B_r^T} E_R^u).$$

To be more precise,

$$V_r \cap \overline{G} = \{wx \in \overline{G} \mid d(w, N \cup \phi(N)) \geq R \text{ and if } w = g_0 t^{\epsilon_1} g_1 ... t^{\epsilon_k} g_k \text{ is the normal form of } w \text{ then } k \leq r-1 \text{ or } g_k = 1 \text{ and } k = r \text{ , if } x \neq 1 \text{ then } k = r,\ g_k = 1 \text{ , } d(x, N \cup \phi(N)) \leq R\}.$$

For every vertex $u \in T^0$ satisfying $|u| \in \{nr \mid n \in \mathbb{N}\}$, we define

$$V_r^u = g_u V_r \cap \pi^{-1}(T^u).$$

Obviously, the sets V_r^u are subgraphs of $C(\overline{G})$ and $V_r^u \nsubseteq \overline{G}$. We observe that $V_r \subseteq \pi^{-1}(B_{r+R}^T)$, so $V_r^u \subseteq \pi^{-1}(B_{r+R}^u)$. Obviously, for every h such that $h = g_1 t^{\epsilon_1} g_2 ... t^{\epsilon_r}$ is the *left normal form* of h, $u \leq g_u h G$ and $|u| + r = |g_u h G|$ we have that:

$$(g_u M_R \cap \pi^{-1}(T^{g_u G})) \cup (g_u h M_R \cap \pi^{-1}(T^{g_u h G})) \subseteq V_r^u.$$

We also observe that

$$g_u M_R \cap \pi^{-1}(T^{g_u G}) \subseteq \partial V_r^u.$$

This can also be written as:

$$M_R^u = M_R^{g_u G} \subseteq \partial V_r^u.$$

We set $V_r^G = V_r$.

We consider the partition

$$C(\overline{G}) = \pi^{-1}(T) = (\bigcup_{|u| \in \{nr | n \in \mathbb{N}_+ \cup \{0\}\}} V_r^u) \cup E(N_R(N \cup \phi(N))) \tag{1.1}$$

We set,

$$Z = (\bigcup_{|u| \in \{nr | n \in \mathbb{N}_+\} \cup \{0\}} \partial V_r^u) \cup \partial E(N_R(N \cup \phi(N))).$$

We observe that if $V_r^u \cap V_r^v \neq \varnothing$, then either $u \leq v$ and $|u| + r = |v|$ or $u \geq v$ and $|v| + r = |u|$. If $V_r^u \cap V_r^v \neq \varnothing$ such that $u \leq v$ and $|u| + r = |v|$, then

$$V_r^u \cap V_r^v = M_R^v.$$

We deduce that

$$Z = (\bigcup_{|u| \in \{nr | n \in \mathbb{N}_+\}} M_R^u)) \cup M_R.$$

We will show that there exists $d > 0$ such that $(R, d) - dim Z \leq n - 1$.
Since M_R is quasi isometric to $N_R(N \cup \phi(N))$, which is quasi isometric to $N \cup \phi(N)$, we have that $asdim M_R \leq n - 1$. Then for $R > 0$ there exists a (R, d)-covering $\mathcal{U}$ of M_R with $ord(\mathcal{U}) \leq n$.
In view of the proposition 1.1.6 we have that the following covering

$$\mathcal{V} = \mathcal{U} \cup \bigcup_{|u| \in \{nr | n \in \mathbb{N}_+\}} (g_u \mathcal{U} \cap M_R^u)$$

is a (R, d)-covering of Z with $ord(\mathcal{V}) \leq n$. We conclude that $(R, d) - dim Z \leq n - 1$.

Next, we will show that $asdim V_r^u \leq n$ and $asdim N_R(N \cup \phi(N)) \leq n$ uniformly. This will complete our proof, since all the conditions of the Partition Theorem are satisfied.
It suffices to show that $asdim V_r^u \leq n$ uniformly and

$$asdim N_R(N \cup \phi(N)) \leq n.$$

We observe that $V_r \subseteq \pi^{-1}(B_{r+R}^T) \subseteq N_1(Q_{r+R})$, so by the proposition 1.1.8 we have that $asdim V_r^u \leq n$. Since there are at most two isometric classes for the sets V_r^u (for $u \neq G$) of our partition, we conclude that $asdim V_r^u \leq n$ uniformly.
Finally, $asdim N_R(N \cup \phi(N)) \leq n - 1$ since $N_R(N \cup \phi(N))$ is quasi isometric to $N \cup \phi(N)$.

By the Partition Theorem (Thm 1.1.4), $asdim C(\overline{G}) = asdim \pi^{-1}(T) \leq n$. $\square$

We give now a few details about the gap we found in [31]. We use the same notation as on pages 2276-2279 of [31]. We are not going to redefine all the symbols we use. We consider the HNN-extension $A*_C = \langle S_A | R_A,\ ct = tf(c)\rangle$ with respect to a monomorphism f. The idea was to construct a partition for $\pi^{-1}(K_1)$. The building block of this partition is the set $V_r = X_+ \cap (\cap_{|u|=r} X_-^u)$, where $u = g_u C$ are vertices of K_1.

The problem is that the set V_r is empty when the index $[A : C] \geq 2$. One can find two vertices u, v of K_1 (where $|u| = |v| = r$), such that $X_-^v \cap X_-^u = \varnothing$. To see that, it requires to investigate how the dual graph K *behaves* under translations. For example, $K^u = g_u K_1$ when g_u (u is a vertex of K_1) has a normal form ending with t, while $K^u = g_u K_0$ when g_u (u is a vertex of K_1) has a normal form ending with t^{-1}.

1.1.1 Right-angled Artin groups.

We use the following theorem of G.Bell, A.Dranishnikov, and J.Keesling (see [5]).

Theorem 1.1.10. *If A and B are non-trivial finitely generated groups, then*

$$asdim\, A * B = max\{1, asdim A, asdim B\}.$$

Let Γ be a finite simplicial graph with n vertices, the *right-angled Artin group* (RAAG) $A(\Gamma)$ associated to the graph Γ has the following presentation:

$$A(\Gamma) = \langle s_u\ ,\ (u \in V(\Gamma)) \mid [s_u, s_v]\ ,\ ([u, v] \in E(\Gamma))\rangle.$$

By $[s_u, s_v] = s_u s_v s_u^{-1} s_v^{-1}$ we mean the commutator.
We set $Val(\Gamma) = max\{valency(u) \mid u \in V(\Gamma)\}$. By $valency(u)$ of a vertex u we denote the number of edges incident to the vertex u.
Clearly, $Val(\Gamma) \leq rank(A(\Gamma)) - 1$.

If Γ is a simplicial graph, we denote by $1 - skel(\Gamma)$ the 1-skeleton of Γ. Recall that a *full subgraph* of a graph Γ is a subgraph formed from a subset of vertices V and from all of the edges that have both endpoints in the subset V.

Conventions: Let Γ be simplicial graph, $u \in V(\Gamma)$ and $e \in E(\Gamma)$. We denote by:
(i) $\Gamma \setminus \{u\}$ the full subgraph of Γ formed from $V(\Gamma) \setminus \{u\}$.
(ii) $\Gamma \setminus e$ the subgraph of Γ such that $V(\Gamma \setminus e) = V(\Gamma)$ and $E(\Gamma \setminus e) = E(\Gamma) \setminus \{e\}$.

Lemma 1.1.11. *Let Γ be a finite simplicial graph. Then*

$$asdim A(\Gamma) \leq Val(\Gamma) + 1.$$

Proof. Since Theorem 1.1.10 holds we observe that it suffices to show the statement of the lemma 1.1.11 for connected simplicial graphs. We assume that Γ is a connected simplicial graph.
We use induction on the $rank(A(\Gamma))$. For $rank(A(\Gamma)) = 1$ we have that $A(\Gamma)$ is the integers $\mathbb{Z}$, so the statement holds. We assume that the statement holds for every $k \leq n$ and we show that it holds for $n+1$ with $n+1 \geq 2$.

Let Γ be a simplicial graph with $n+1$ vertices. We remove a vertex u from the graph Γ such that $valency(u) = Val(\Gamma) = m \geq 1$. Let us denote by v_i ($i \in \{1, ..., m\}$) the vertices of Γ which are adjacent to u.
We set $\Gamma' = \Gamma \setminus \{u\}$. Obviously, $Val(\Gamma') \leq Val(\Gamma)$. We note that Γ' is might not be connected which doesn't affect the proof since Theorem 1.1.10 holds. Thus we may assume that Γ' is connected.
We denote by Y the full subgraph of Γ formed from $\{v_1, \ldots, v_m\}$.

We observe that the RAAG $A(\Gamma)$ is an HNN-extension of the RAAG $A(\Gamma')$. To be more precise, we have that

$$A(\Gamma) = A(\Gamma') *_{A(Y)} .$$

By Theorem 1.1.9 we obtain that

$$asdimA(\Gamma) \leq max\{asdimA(\Gamma'), asdimA(Y) + 1\}.$$

We observe that $Val(Y) \leq Val(\Gamma) - 1$, so by the induction ($rank(A(Y)) \leq n$) we obtain

$$asdimA(Y) \leq Val(Y) + 1 \leq Val(\Gamma).$$

Since $rankA(\Gamma') = n$, by the induction we deduce that

$$asdimA(\Gamma') \leq Val(\Gamma') + 1 \leq Val(\Gamma) + 1.$$

Combining the three previous inequalities we obtain:

$$asdimA(\Gamma) \leq max\{Val(\Gamma) + 1, Val(\Gamma) + 1\} = Val(\Gamma) + 1.$$

□

Using the previous lemma we can compute the exact asymptotic dimension of $A(\Gamma)$. We note that this has already been computed by N.Wright [38] using different

methods.

We set

$Sim(\Gamma) = max\{n \mid \Gamma \text{ contains the 1-skeleton of the standard } (n-1)\text{-simplex } \Delta^{n-1}\}$.

Obviously if $\Gamma' \subseteq \Gamma$, then $Sim(\Gamma') \leq Sim(\Gamma)$.

Theorem 1.1.12. *Let Γ be a finite simplicial graph. Then,*

$$asdimA(\Gamma) = Sim(\Gamma).$$

Proof. Since theorem 1.1.10 holds we observe that it suffices to show the statement of Theorem 1.1.12 for connected simplicial graphs. We assume that Γ is a connected simplicial graph.

Claim 1. $Sim(\Gamma) \leq asdimA(\Gamma)$.

Proof of claim 1: Let $Sim(\Gamma) = n$. We observe that $\mathbb{Z}^n = A(S_{n-1}) \leq A(\Gamma)$. It follows that

$$n = asdim\mathbb{Z}^n \leq asdimA(\Gamma).$$

Claim 2. $asdimA(\Gamma) \leq Sim(\Gamma)$.

Proof of claim 2: We use induction on the $rank(A(\Gamma))$, for $rank(A(\Gamma)) = 1$ we have that $A(\Gamma)$ is the integers $\mathbb{Z}$, so the statement holds. We assume that the statement holds for every $r \leq m$, we will show that holds for $m+1$ as well. Let Γ be a connected simplicial graph with $m+1$ vertices.

Let $Sim(\Gamma) = n$, then Γ contains the 1-skeleton of the standard $(n-1)$-simplex S_{n-1} ($S_{n-1} = 1 - skel(\Delta^{n-1})$).

Case 1. $\Gamma = S_{n-1}$.

Then $m + 1 = n$, so by lemma 1.1.11 we have

$asdimA(S_{n-1}) \leq Val(S_{n-1}) + 1$. By observing that $Val(S_{n-1}) = n - 1$ we obtain that

$$asdimA(S_{n-1}) \leq n = Sim(\Gamma).$$

Case 2. $S_{n-1} \subsetneqq \Gamma$.

We will remove a vertex $u \in V(S_{n-1})$. Let's denote by v_i ($i \in \{1, ..., k\}$) the vertices of Γ which are adjacent to u . We set $\Gamma' = \Gamma \setminus \{u\}$. Obviously $Sim(\Gamma') \leq n$.

We denote by Y the full subgraph of Γ formed from $\{v_1, \ldots, v_k\}$.

We observe that the RAAG $A(\Gamma)$ is an HNN-extension of the RAAG $A(\Gamma')$. To be more precise, we have that

$$A(\Gamma) = A(\Gamma') *_{A(Y)} .$$

By Theorem 1.1.9 we obtain that

$$asdimA(\Gamma) \leq max\{asdimA(\Gamma'), asdimA(Y) + 1\}. \tag{1.2}$$

Since $Sim(\Gamma') \leq n$ and $rank(\Gamma') \leq m$, by the inductive assumption we have that:

$$asdimA(\Gamma') \leq Sim(\Gamma') \leq n. \tag{1.3}$$

We observe that $Sim(Y) \leq n - 1$ and $rank(Y) \leq m$, then by the induction we obtain

$$asdimA(Y) + 1 \leq Sim(Y) + 1 \leq n. \tag{1.4}$$

by (1.2), (1.3) and (1.4) we conclude that

$$asdimA(\Gamma) \leq n = Sim(\Gamma).$$

□

1.2 Asymptotic dimension of one-relator groups.

Theorem 1.2.1. *Let G be a finitely generated one relator group. Then*

$$asdimG \leq 2.$$

Proof. Let $G = \langle S \mid r \rangle$ be a presentation of G where S is finite and r is a cyclically reduced word in $S \bigcup S^{-1}$. To omit trivial cases, we assume that S contains at least two elements and $\mid r \mid > 0$, (we denote by $\mid r \mid$ the length of the relator r in the free group $F(S)$).

We may assume that every letter of S appears in r. Otherwise our group G is isomorphic to a free product $H * F$ of a finitely generated one relator group H with relator r and generating set $S_H \subseteq S$ consisting of all letters which appear in r and a free group F with generating set the remaining letters of S. We recall that the asymptotic dmension of any finitely generated non-abelian free group is equal to one. Then $asdimG = max\{asdimH, asdimF\} = max\{asdimH, 1\}$ (see [2]).

We denote by $\epsilon_r(s)$ the exponent sum of a letter $s \in S$ in a word r and by $oc_r(s)$ the minimum number of the positions of appearance of the elements of the set $\{s^k$ for some $0 \neq k \in \mathbb{Z}\}$ in a cyclically reduced word r. For example, if $r = abcab^{10}a^{-2}c^{-1}$, then $oc_r(a) = 3$, $oc_r(b) = 2$, $oc_r(c) = 2$ and $\epsilon_r(c) = 0$.

We observe that if there exists $b \in S$ such that $oc_r(b) = 1$, then the group G is free (see [30], theorem 5.1, page 198), so $asdimG = 1$. From now on we assume that for every $s \in S$ we have that $oc_r(s) \geq 2$ (so $\mid r \mid \geq 4$).

The proof is by induction on the length of r. We observe that if $\mid r \mid = 4$ then the statement of the theorem holds since by the result of D.Matsnev [32] we have that $asdimG \leq 2$.
We assume that the statement of the theorem holds for all one relator groups with relator length smaller than or equal to $\mid r \mid -1$.

We follow the arguments from the book of Lyndon-Schupp (see [30], theorem 5.1, page 198) and the book of Wise (see [9], construction 18.5). We distinguish two cases. In the first case, there exists a letter $a \in S$ such that $\epsilon_r(a) = 0$. In the second case, for every letter $s \in S$ we have $\mid \epsilon_r(s) \mid \geq 1$.

In the first case, by combining the proof of theorem 5.1 from [30] and the construction 18.5 from [9] it follows that $G \simeq G_1 *_F$, where G_1 is a one relator group with with relation r' such that $\mid r' \mid < \mid r \mid$, and F is a finitely generated free subgroup of G_1. By the fact that $\mid r' \mid < \mid r \mid$ and the inductive assumption we have that $asdimG_1 \leq 2$. To conclude we apply the inequality for HNN-extensions (Theorem 1.1.9):
$asdimG \leq max\{asdimG_1, asdimF + 1\} = max\{asdimG_1, 2\} = 2$.

In the second case, by the proof of theorem 5.1 from [30] it follows that G can be embedded either in an HNN extension $\Gamma = H *_F$, or in a free product $\Gamma = \langle t \rangle * \Gamma'$.

In the first subcase, H is a one relator group with with relation r' such that $\mid r' \mid < \mid r \mid$, and F is a finitely generated free subgroup of H. By using the inductive assumption for r' we obtain that $asdimG \leq asdim\Gamma \leq 2$.

In the second subcase, Γ' is a one relator group with with relation r' such that $\mid r' \mid < \mid r \mid$. Since $asdim(G_1 * G_2) = max\{1, asdimG_1, asdimG_2\}$ holds (see [2]) we have that $asdim\Gamma = max\{1, asdim\Gamma'\}$. Then by the inductive assumption for r' we have that $asdim\Gamma' \leq 2$. Finally, we conclude that

$$asdimG \leq asdim\Gamma \leq 2.$$

□

1.2.1 One relator groups with asymptotic dimension two.

We recall that a nontrivial group H is *freely indecomposable* if H can not be expressed as a free product of two non-trivial groups.

A natural question derived from Theorem 1.2.1 is, which one relator groups have asymptotic dimension two.
In this subsection, we will show that the asymptotic dimension of every finitely generated one relator group that is not a free group or a free product of a free group and a finite cyclic group is exactly two.

We will use following propositions 1.2.2 and 1.2.3 from [19] and [37] respectively.

Proposition 1.2.2. *Let G be an infinite finitely generated one relator group with torsion. If G has more than one end, then G is a free product of a nontrivial free group and a freely indecomposable one relator group.*

Proposition 1.2.3. *Let G be a torsion free infinite finitely generated group. If G is virtually free, then it is free.*

Lemma 1.2.4. *Let G be an infinite finitely generated one relator group such that is not a free group or a free product of a nontrivial free group and a freely indecomposable one relator group. Then G is not virtually free.*

Proof. If G has torsion, by proposition 1.2.2 we have that G has exactly one end, so G can not be virtually free. If G is torsion free, by proposition 1.2.3 we obtain that G is free and this is a contradiction by the assumption of the lemma. □

We note that every finite one relator group is cyclic. To see that it is enough to observe that every one relator group with at least two generators has infinite abelianization.

The following proposition is the main result of this subsection.

Proposition 1.2.5. *Let G be a finitely generated one relator group that is not a free group or a free product of a free group and a finite cyclic group. Then*

$$asdim\, G = 2.$$

Proof. By Theorem 1.2.1 we have that $asdim\, G \leq 2$. If G is finite then it is cyclic. If G is infinite we have that $1 \leq asdim\, G$. By a theorem of Gentimis ([20]) we have that $asdim\, G = 1$ if and only if G is virtually free.
We assume that G is an infinite virtually free group. So if G is torsion free then by proposition 1.2.3 we obtain that G is free. If G has torsion then by lemma 1.2.2 G is a free product of a nontrivial free group and a freely indecomposable one relator group G_1. Observe that if G_1 is an infinite noncyclic group, then by lemma 1.2.4 G_1 is not virtually free, so G is not virtually free either, which is a contradiction.
We conclude that $asdim\, G = 2$. □

Corollary 1. *Let G be a finitely generated freely indecomposable one relator group which is not cyclic. Then*

$$asdim\, G = 2.$$

The following proposition can be found in [30] (proposition 5.13, page 107).

Proposition 1.2.6. *Let $G = \langle x_1, \ldots, x_n | r \rangle$ be a finitely generated one relator group, where r is of minimal length under $Aut(F(\{x_1, \ldots, x_n\}))$ and contains exactly the generators $x_1, \ldots, x_k$ for some k, $0 \leq k \leq n$. Then G is isomorphic to the free product $G_1 * G_2$, where $G_1 = \langle x_1, \ldots, x_k | r \rangle$ is freely indecomposable and G_2 is free with basis $\{x_{k+1}, \ldots, x_n\}$.*

Using the above results we sum up to following corollary which describes the finitely generated one relator groups.

Corollary 2. *Let G be a finitely generated one relator group. Then one of the following is true:*
(i) *G is finite cyclic, and $asdim\, G = 0$*
(ii) *G is a nontrivial free group or a free product of a nontrivial free group and a finite cyclic group, and $asdim\, G = 1$*
(iii) *G is an infinite freely indecomposable not cyclic group or a free product of a nontrivial free group and an infinite freely indecomposable not cyclic group, and $asdim\, G = 2$.*

We could further describe the boundaries of hyperbolic one relator groups. We recall the following result of Buyalo and Lebedeva (see [8]) for hyperbolic groups:

$$asdim G = dim \partial_\infty G + 1.$$

Let G be an infinite finitely generated hyperbolic one relator group that is not virtually free. By T.Gentimis ([20]) we obtain that $asdimG \neq 1$,so $asdimG = 2$. Using the previous equality we obtain that G has one dimensional boundary.
Applying a theorem of M. Kapovich and B. Kleiner (see [24]) we can describe the boundaries of hyperbolic one relator groups.

Proposition 1.2.7. *Let G be a hyperbolic one relator group. Then $asdim\, G = 0, 1 or$ 2.*
(i) *If $asdim\, G = 0$, then G is finite.*
(ii) *If $asdim\, G = 1$, then G is virtually free and the boundary is a Cantor set.*
(iii) *If $asdim\, G = 2$ providing that G does not split over a virtually cyclic subgroup, then one of the following holds:*
1. $\partial_\infty G$ is a Menger curve.
2. $\partial_\infty G$ is a Sierpinski carpet.
3. $\partial_\infty G$ is homeomorphic to S^1.

1.3 Graphs of Groups.

We will prove a general theorem for the asymptotic dimension of fundamental groups of finite graphs of groups.

Theorem 1.3.1. *Let $(\mathbb{G}, Y)$ be a finite graph of groups with finitely generated vertex groups $\{G_v \mid v \in Y^0\}$ and finitely generated edge groups $\{G_e \mid e \in Y^1_+\}$. Then the following inequality holds:*

$$asdim\pi_1(\mathbb{G}, Y, \mathbb{T}) \leq max_{v\in Y^0, e\in Y^1_+}\{asdimG_v, asdim\, G_e + 1\}.$$

Proof. We use induction on the number $\sharp E(Y)$ of edges of the graph Y. For $\sharp E(Y) = 1$ we distinguish two cases. The first case is when the fundamental group $\pi_1(\mathbb{G}, Y, \mathbb{T})$ is an amalgamated product, so the statement of the theorem follows by the following inequality of A.Dranishnikov (see [14])

$$asdimA *_C B \leq max\{asdimA, asdimB, asdimC + 1\}.$$

The second case is when the fundamental group $\pi_1(\mathbb{G}, Y, \mathbb{T})$ is an HNN-extension, so the statement of the theorem follows by Theorem 1.1.9.

We assume that the statement of the theorem holds for $\sharp E(Y) \leq m$. Let $(\mathbb{G}, Y)$ be a finite graph of groups with $\sharp E(Y) = m + 1$. We denote by $\mathbb{T}$ a maximal tree of Y.

We distinguish two cases:

Case 1: $Y = \mathbb{T}$. We remove a terminal edge $e' = [v, u]$ from the graph Y such that the full subgraph of Y denoted by Γ and formed from the vertices $V(Y) \setminus \{u\}$ is connected. We observe that Γ is also a tree which we denote by $\mathbb{T}'$.

Then $\pi_1(\mathbb{G}, Y, \mathbb{T}) = \pi_1(\mathbb{G}, \Gamma, \mathbb{T}') *_{G_{e'}} G_u$, so by the inequality for amalgamated products of A.Dranishnikov (see [14]), we have

$$asdim\pi_1(\mathbb{G}, Y, \mathbb{T}) \leq max\{asdim\pi_1(\mathbb{G}, \Gamma, \mathbb{T}'), asdim\, G_u, asdimG_{e'} + 1\}.$$

Since $\sharp E(\Gamma) = m$, by the inductive assumption we obtain that

$$asdim\pi_1(\mathbb{G}, \Gamma, \mathbb{T}') \leq max_{v \in Y^0 \setminus \{u\}, e \in Y^1_+ \setminus \{e'\}}\{asdimG_v, asdim\, G_e + 1\},$$

so

$$asdim\pi_1(\mathbb{G}, Y, \mathbb{T}) \leq max_{v \in Y^0, e \in Y^1_+}\{asdimG_v, asdim\, G_e + 1\}.$$

Case 2: $\mathbb{T} \subsetneqq Y$. We remove from Y an edge $e' = [v, u]$ which doesn't belong to $\mathbb{T}$. Since the tree $\mathbb{T}$ is a maximal tree of Y and $e' \notin E(\mathbb{T})$ we have that the graph $\Gamma = Y \setminus e'$ is connected and $\mathbb{T} \subseteq \Gamma$.
Then $\pi_1(\mathbb{G}, Y, \mathbb{T}) = \pi_1(\mathbb{G}, \Gamma, \mathbb{T}) *_{G_{e'}}$, so by the inequality for HNN-extensions (Theorem 1.1.9) we have

$$asdim\pi_1(\mathbb{G}, Y, \mathbb{T}) \leq max\{asdim\pi_1(\mathbb{G}, \Gamma, \mathbb{T}), asdimG_{e'} + 1\}.$$

Since $\sharp E(\Gamma) = m$, by the inductive assumption we obtain that

$$asdim\pi_1(\mathbb{G}, \Gamma, \mathbb{T}) \leq max_{v \in Y^0, e \in Y^1_+ \setminus \{e'\}}\{asdimG_v, asdim\, G_e + 1\},$$

so

$$asdim\pi_1(\mathbb{G}, Y, \mathbb{T}) \leq max_{v \in Y^0, e \in Y^1_+}\{asdimG_v, asdim\, G_e + 1\}.$$

□

We obtain as a corollary the following proposition.

Proposition 1.3.2. *Let $(\mathbb{G}, Y)$ be a finite graph of groups with vertex groups $\{G_v \mid v \in Y^0\}$ and edge groups $\{G_e \mid e \in Y^1_+\}$. We assume that*

$$max_{e \in Y^1_+}\{asdim\, G_e\} < max_{v \in Y^0}\{asdimG_v\} = n.$$

Then, $asdim\pi_1(\mathbb{G}, Y, \mathbb{T}) = n$.

As an immediate corollary of Proposition 1.3.2 we obtain that the asymptotic dimension of a graph of one ended hyperbolic groups with n-dimensional boundary with free edge groups is $n + 1$.

Chapter 2

A new upper bound for the Asymptotic Dimension of RACGs.

Coxeter groups touch upon a number of areas of mathematics, such as representation theory, combinatorics, topology, and geometry. They are often considered as a playground for many open problems in Geometric Group Theory.

It is known by an isometric embedding theorem of Januszkiewicz (see [52]) that Coxeter groups have finite asymptotic dimension. In particular, Januszkiewicz's theorem shows that for any Coxeter group W_Γ with defining graph Γ we have the following upper bound: $asdim W_\Gamma \leq \sharp V(\Gamma)$.
A lower bound for the asymptotic dimension of Coxeter groups was given by Dranishnikov in [13], $vcd(W_\Gamma) \leq asdim W_\Gamma$.

Right-angled Coxeter groups (RACGs) are the simplest examples of Coxeter groups; in these, the only relations between distinct generators are commuting relations. In other words, RACGs are the Coxeter groups defined by RAAGs. Dranishnikov proved (see [14]) that the asymptotic dimension of RACGs is bounded from above by the dimension of their Davis complex.

Question. *Is possible to determine the asympotitic dimension of a RACG from its defining graph?*

In many cases Dranishnikov's bound is far from being optimal. For example, if the defining graph Γ is a clique with n vertices, then by Dranishnikov's result we have that $asdim W_\Gamma \leq n$, however, $asdim W_\Gamma = 0$.
The aim of this chapter is to provide a new upper bound for the asymptotic dimension of RACGs treating some of the cases in which Dranishnikov's bound fails to

be optimal. The main result of this Chapter and its corollaries make some progress towards the previous question.

We prove the following:

Theorem 2.0.1. *Let W_Γ be the right-angled Coxeter group with connected defining graph Γ. Then*

$$asdim W_\Gamma \leq dim_{CC}(\Gamma).$$

If Γ is not connected, then $asdim W_\Gamma \leq max\{1, dim_{CC}(\Gamma)\}$.

The clique-connected dimension of a finite graph, $dim_{CC}(\Gamma)$ (see subsection 2.1.1), can be described as an index showing how connected is the graph modulo cliques. For example, if Γ is a clique, then $dim_{CC}(\Gamma) = 0$. In the case of Γ being a clique, we have that W_Γ is finite, so $asdim W_\Gamma = dim_{CC}(\Gamma)$. We will further show that if $dim_{CC}(\Gamma) \leq 2$, then $asdim W_\Gamma = dim_{CC}(\Gamma)$ (see proposition 2.5.2).

Since there are cases where the dimension of the Davis complex $\Sigma(W_\Gamma)$ is smaller than $dim_{CC}(\Gamma)$ and other cases where $dim_{CC}(\Gamma) < dim\Sigma(W_\Gamma)$, we have as a corollary:

Theorem 2.0.2. *Let W_Γ be the right-angled Coxeter group with connected defining graph Γ. Then*

$$asdim W_\Gamma \leq min\{dim_{CC}(\Gamma), dim\Sigma(W_\Gamma)\}.$$

If Γ is not connected then $asdim W_\Gamma \leq max\{1, min\{dim_{CC}(\Gamma), dim\Sigma(W_\Gamma)\}\}$.

As a corollary of theorem 2.0.1 we prove that:

Proposition 2.0.3. *Let W_Γ be the right-angled Coxeter group with connected defining graph Γ. If $dim_{CC}(\Gamma) \leq 2$, then $asdim W_\Gamma = dim_{CC}(\Gamma)$.*

Let Γ be a simplicial graph. We label every vertex v of the graph by a *vertex group* G_v. The *graph product* $\mathbb{GP}(\Gamma, G_v)_{v\in V(\Gamma)}$ is the free product of the vertex groups modulo the relations $xyx^{-1}y^{-1}$ where $x \in G_v$, $y \in G_w$, and v, w are connected by an edge.

We generalize theorem 2.0.1 to the following:

Theorem 2.0.4. *Let Γ be a finite connected simplicial graph along with a finite collection of finite vertex groups. Then*

$$asdim\, \mathbb{GP}(\Gamma, G_v)_{v\in V(\Gamma)} \leq dim_{CC}(\Gamma).$$

If Γ is not connected, then $asdim\, \mathbb{GP}(\Gamma, G_v)_{v\in V(\Gamma)} \leq max\{1, dim_{CC}(\Gamma)\}$.

2.1 Preliminaries.

By finite simplicial labeled graph we mean a finite simplicial graph Γ such that every edge $[a, b]$ is labeled by a natural number $m_{ab} > 1$. The Coxeter group associated to Γ is the group W_Γ given by the following presentation:

$$W_\Gamma = \langle V(\Gamma) | a^2 \text{ for all } a \in V(\Gamma) \text{ and } \underbrace{ab\dots}_{m_{ab}} = \underbrace{ba\dots}_{m_{ab}} \text{ when a, b are connected by an edge } \rangle.$$

A Coxeter group is called *right-angled Coxeter group* (RACG) if $m_{ab} = 2$ when a, b are connected by an edge.

We say a simplicial graph is *complete* or equivalently *a clique* if any two vetrices are connected by an edge. An *n-clique* is the complete graph on n vertices.
We recall that the *full subgraph* defined by a subset V of the vertices of a graph Γ is a subgraph of Γ formed from V and from all of the edges that have both endpoints in the subset V. If G is a subgraph of Γ, we denote by $FS^\Gamma(G)$ the full-subgraph of Γ defined by $V(G)$.
The *simplicial closure* of Γ is the flag complex $SC(\Gamma)$ defined by Γ.

We recall the definition of a *parabolic subgroup* of a Coxeter group. Let Γ be a finite simplicial labeled graph and W_Γ be the Coxeter group associated to Γ. Let X be a subset of $V(\Gamma)$, we denote by Γ_X the full subgraph of Γ formed from X, and by G_X the subgroup of W_Γ generated by X (we see X as a subset of the natural generating set of W_Γ). We consider the graph Γ_X as a labeled graph inheriting its labeling from Γ.
It is known that G_X is a Coxeter group, it is actually equal to W_{Γ_X}, the Coxeter group associated to Γ_X see([49]).
The subgroup $G_X = W_{\Gamma_X}$ is called *standard parabolic subgroup* of W_Γ.

The following theorem is proved by Dranishnikov in [14].

Theorem 2.1.1. *For any finitely generated groups A and B with a common finitely generated subgroup C we have:*

$$asdimA *_C B \leq max\{asdimA, asdimB, asdimC + 1\}.$$

The following theorem is a generalisazion of theorem 2.1.1. It was proved by the author in [55].

Theorem 2.1.2. *Let $(\mathbb{G}, Y)$ be a finite graph of groups with vertex groups $\{G_v \mid v \in Y^0\}$ and edge groups $\{G_e \mid e \in Y^1_+\}$. Then the following inequality holds:*

$$asdim\, \pi_1(\mathbb{G}, Y, \mathbb{T}) \leq max_{v \in Y^0, e \in Y^1_+}\{asdim G_v, asdim\, G_e + 1\}.$$

We also need a theorem for free products from [5], see also [2].

Theorem 2.1.3. *Let A, B be two non-trivial finitely generated groups. Then:*

$$asdim\, A * B = max\{asdim A, asdim B, 1\}.$$

2.1.1 The clique connected dimension of a graph.

Let Γ be a connected simplicial graph. We say that a finite subset S of $V(\Gamma)$ is a *vertex cut* of Γ if $FS^{\Gamma}(S)$ separates the graph (i.e. $\Gamma \setminus FS^{\Gamma}(S)$ contains at least two connected components) and no proper subset of S does that.

Definition 2.1.1. Let Γ be a simplicial graph and let $\mathcal{C}_{\Gamma} = \{C_1, \ldots, C_i, \ldots\}$ be a collection of distinct cliques of Γ. We say $\mathcal{C}_{\Gamma}$ is a *clique twin* of Γ if the following conditions are satisfied:
i) Each C_i is a maximal clique in Γ i.e. there is no other clique in Γ containing C_i.
ii) If C is a clique of Γ, then there is a clique in $\mathcal{C}_{\Gamma}$ containing C.

We note that the last condition can be replaced by the following:
$\cup_{C_i \in \mathcal{C}_{\Gamma}} SC(C_i) = SC(\Gamma)$.
Observe that a clique twin of a graph Γ is actually a "covering" of Γ with maximal cliques. We further observe, that if a clique twin exists, then it is equal to the set of the maximal cliques of the graph.

Definition 2.1.2. Let Γ be a simplicial graph which has at least one clique twin $\mathcal{C}_{\Gamma}$. We set

$$m_C(\Gamma) = min\{\sharp\mathcal{C}_{\Gamma}| \text{ where } \mathcal{C}_{\Gamma} \text{ is a clique twin of } \Gamma\}.$$

Definition 2.1.3. Let Γ be a connected simplicial graph. We set

$$CC(\Gamma) = min\{m_C(FS^{\Gamma}(S))| \text{ where } S \text{ is a vertex cut of } \Gamma\}.$$

To treat the case when the set on the right-hand side is empty, we insist that $min\{\varnothing\} = 0$. The number $CC(\Gamma)$ "measures" how connected is the graph Γ modulo its cliques. If S is a vertex cut of Γ such that $CC(\Gamma) = m_C(FS^{\Gamma}(S))$, we say that S

is a *minimal vertex cut* of Γ.

Observe that we can generalize the previous definition to all simplicial graphs, by setting $CC(\Gamma) = min\{CC(E) \mid E \text{ is a component of } \Gamma\}$. Finally, we can define $dim_{CC}(*)$.

Definition 2.1.4. Let Γ be a simplicial graph. We set

$$dim_{CC}(\Gamma) = sup\{CC(G)| \text{ where } G \text{ is a full subgraph of } \Gamma\}.$$

The clique-connected dimension $dim_{CC}(\Gamma)$ "measures" how connected is the graph Γ modulo its cliques by taking into account all the full-subgraphs of Γ.

2.2 Basic Lemmas.

Lemma 2.2.1. *(Existence of clique twins.)*
Let Γ be a finite simplicial graph. Then there exists at least one clique twin of Γ.

Proof. We use induction on the number of vertices of the graph. Obviously, the lemma is true if the graph is just a vertex. We assume the lemma is true for any graph with less than $N+1$ vertices. Let Γ be a finite simplicial graph with $N+1$ vertices.
If the graph is disconnected the lemma follows by the inductive hypothesis and the fact that the clique twins of the components of the graph forms a clique twin of Γ.
So we assume that the graph is connected. We choose an arbitrary vertex say v and we consider the graph $\Gamma_v = \Gamma \setminus v$. By inductive hypothesis, there exists a clique twin $\mathcal{C}_{\Gamma_v}$ of Γ_v. Since the graph is connected the link of v in Γ is non-empty ($lk_\Gamma(v) \neq \varnothing$). We enumerate the vertices of the link, $lk_\Gamma(v) = \{v_1, v_2, \ldots, v_k\}$. For every C in $\mathcal{C}_{\Gamma_v}$ we set:

$\overline{C}$ = either C (if $C \cup v$ doesn't define a clique of Γ), or the clique defined by C and v (otherwise).

For every v_i in $lk_\Gamma(v)$ we define:

$\mathcal{C}_i$ to be the collection of all maximal cliques in Γ containing both v and v_i.

Observe then that the union $\mathcal{C}_\Gamma = (\cup_i \mathcal{C}_i) \cup \{\overline{C}|C \in \mathcal{C}_{\Gamma_v}\}$ satisfies the conditions of definition 2.1.1, so it is a clique twin of Γ.

□

We will see that every finite graph has a unique clique twin. The proof of the previous lemma actually give us a description of how to construct the clique twin of every finite graph.

Lemma 2.2.2. *Let Γ be a connected finite simplicial graph. Then Γ is a clique if and only if $CC(\Gamma) = 0 = dim_{CC}(\Gamma)$.*

Proof. We assume that Γ is a clique, then obviously there is no vertex cut of Γ. So by definition 2.1.3 we have that $CC(\Gamma) = 0$.

We now prove the other direction, so we assume that $CC(\Gamma) = 0$. If Γ is not a clique there exist two vertices a, b such that they are not connected by an edge, then $V(\Gamma) \setminus \{a, b\}$ separate the graph. Obviously then we may find a vertex cut S of Γ, so by lemma 2.2.1 $CC(\Gamma) > 0$. □

Lemma 2.2.3. *(Uniqness of clique twins.)*
Let Γ be a finite simplicial graph. Then there exists exactly one clique twin of Γ.

Proof. By lemma 2.2.1 there exists a clique twin $\mathcal{C}_\Gamma$ of Γ. We assume that there exists another clique twin of the graph, say $\mathcal{C}'_\Gamma$. By condition (ii) of definition of clique twins we observe that for every $C' \in \mathcal{C}'_\Gamma$ there exists a $C \in \mathcal{C}_\Gamma$ containing C'. By condition (i) we obtain that if $C' \subseteq C$, then $C' = C$. Thus $\mathcal{C}'_\Gamma \subseteq \mathcal{C}_\Gamma$. Using the same argument we conclude that $\mathcal{C}'_\Gamma = \mathcal{C}_\Gamma$.

□

Lemma 2.2.4. *(Monotonicity of $dim_{CC}(*)$.)*
Let Γ be a simplicial graph and let G be a full subgraph of Γ. Then

$$dim_{CC}(G) \leq dim_{CC}(\Gamma).$$

Proof. Since G is a full subgraph of Γ we have that every full subgraph of G is also a full subgraph of Γ. The lemma follows by the definition of dim_{CC}. □

If G is not full subgraph the previous lemma is not true. The reader may notice some similarities between the proof of the following lemma and the paper [10].

Lemma 2.2.5. *Let Γ be a connected simplicial graph such that $CC(\Gamma) \geq 2$. Then $CC(G) \leq 1$ for every G proper full subgraph of Γ if and only if Γ is a k-cycle ($k \geq 4$).*

Proof. Suppose that $CC(G) \leq 1$ for every G proper full subgraph of Γ. Let S be a vertex cut of Γ such that $m_C(FS^\Gamma(S)) \geq 2$ and let $\mathcal{C}_S = \{C_1, \ldots, C_k\}$ be the clique twin of $FS^\Gamma(S)$.

Let E_1, E_2 be two of the components of $\Gamma \setminus FS^{\Gamma}(S)$. Observe that there are vertices $v_1, v_2 \in FS^{\Gamma}(S)$ such that they are not connected by an edge in Γ. Obviously, they belong to distinct cliques. We may assume that $v_1 \in C_1 \setminus C_2$ and $v_2 \in C_2 \setminus C_1$. We note that for every $s \in S$ and every component E_i there exists an edge connecting s with E_i (it follows from the fact that S is a vertex cut of the graph). Thus there exist edge paths $p_i \subseteq E_i \cup C_1 \cup C_2$ connecting v_1 with v_2 such that $p_i \cap (C_1 \cup C_2) = \{v_1, v_2\}$. We may assume that these paths have the minimum possible length.
Observe that the length of these edge paths is at least two. Trivially, the union $p_1 \cup p_2$ is k-cycle where $k \geq 4$.
It remains to show that $p_1 \cup p_2$ is a full subgraph of Γ. Indeed, it follows from the choice of v_1 and v_2, the fact that p_1, p_2 are of minimum length and that $p_1 \setminus (C_1 \cup C_2)$, $p_2 \setminus (C_1 \cup C_2)$ belong to distinct components of $\Gamma \setminus FS^{\Gamma}(S)$.
Obviously, $CC(p_1 \cup p_2) = 2$. By the hypothesis of lemma we conclude that $\Gamma = p_1 \cup p_2$.

We assume that Γ is a k-cycle ($k \geq 4$). Let G proper full subgraph of Γ. Then there is a vertex v of Γ such that G is a full subgraph of $\Gamma \setminus v$. We observe that if we remove a vertex from Γ, then the resulting graph Γ' is a concatenation of edges. Trivially, $CC(\Gamma') = 1$ and $CC(G) \leq 1$ for every G full subgraph of Γ'.

□

2.3 An increasing property of $dim_{CC}(*)$.

Lemma 2.2.4 will play a vital role in the proof our main theorem but it is not enough for a complete proof; we need something stronger. The main result of this chapter is an interesting increasing property of $dim_{CC}(*)$. We will show that $dim_{CC}(G) < dim_{CC}(\Gamma)$, for some full subgraphs G of Γ.

Lemma 2.3.1. *Let Γ be a finite simplicial graph and let G be a full subgraph of Γ. Then $m_C(G) \leq m_C(\Gamma)$.*

Proof. By lemma 2.2.3, there exists unique clique twins $\mathcal{C}_G$ of G and $\mathcal{C}_\Gamma$ of Γ. By condition (ii) of definition of clique twins we observe that for every $C_G \in \mathcal{C}_G$ there exists a $C_\Gamma \in \mathcal{C}_\Gamma$ containing C_G. Since G is a full subgraph of Γ this correspondence is 1-1 meaning there are no two distinct cliques of $\mathcal{C}_G$ contained in the same clique of $\mathcal{C}_\Gamma$. Thus $m_C(G) = \sharp\mathcal{C}_G \leq \sharp\mathcal{C}_\Gamma = m_C(\Gamma)$.

□

Proposition 2.3.2. *Let Γ be a finite simplicial graph. Then*

$$CC(\Gamma) < m_C(\Gamma).$$

Proof. It suffices to show the proposition only for connected graphs. If Γ is a clique, then the proposition holds. We assume that the graph is not a clique. We denote by $\mathcal{C}_\Gamma$ the unique clique twin of Γ.
Since Γ is connected there exists at least two cliques of $\mathcal{C}_\Gamma$ intersecting each other. Let C_1 be a clique of $\mathcal{C}_\Gamma$ such that there exists another element of $\mathcal{C}_\Gamma$ intersecting C_1. Let F be a subclique of C_1 of maximal cardinality such that there exists $C_2 \in \mathcal{C}_\Gamma$ and $C_1 \cap C_2 = F$.
We set $C_i^c = C_i \setminus F$ $(i = 1, 2)$ and $\Gamma' = \Gamma \setminus (C_1^c \cup C_2^c)$. Obviously, all of them are full subgraphs of Γ.
Claim 1: Γ' separates Γ.
If not, then there exists an edge $e = [v_1, v_2]$ where $v_i \in C_i^c$. But then the clique $F' = FS^\Gamma(F \cup v_1) \subseteq C_1$ has strictly larger cardinality from F, and the clique $C_3 = FS^\Gamma(F' \cup v_2)$ intersects Γ on F', which is a contradiction by the choice of F.
Claim 2: $m_C(\Gamma') < m_C(\Gamma)$.
We denote by $\mathcal{C}_{\Gamma'}$ the unique clique twin of Γ'. Observe that distinct elements of $\mathcal{C}_{\Gamma'}$ belong to distinct elements of $\mathcal{C}_\Gamma$. Thus if the claim is not true (i.e. $\mid \mathcal{C}_{\Gamma'} \mid = \mid \mathcal{C}_\Gamma \mid$), there are distinct elements $B_1, B_2 \in \mathcal{C}_{\Gamma'}$ such that $B_i \subseteq C_i$ $(i = 1, 2)$. By the definition of Γ' we have that $B_i \subseteq F$, but both B_1 and B_2 are maximal in Γ', thus $B_1 = F = B_2$, which is a contradiction.

By claim 1 and the fact that Γ is finite, there exists a vertex cut $S \subseteq \Gamma'$. Then $CC(\Gamma) \leq m_C(FS^\Gamma(S)) \leq m_C(\Gamma') < m_C(\Gamma)$

□

Proposition 2.3.3. *Let Γ be a connected finite simplicial graph and S be a minimal vertex cut of Γ. Then*

$$CC(FS^\Gamma(S)) < CC(\Gamma).$$

Proof. By proposition 2.3.2 we have that $CC(FS^\Gamma(S)) < m_C(FS^\Gamma(S)) = CC(\Gamma)$. □

The next theorem is the main result of this section.

Theorem 2.3.4. *Let Γ be a connected finite simplicial graph and S be a minimal vertex cut of Γ. Then for every full subgraph G of $FS^\Gamma(S)$ we have the following:*

$$dim_{CC}(G) < dim_{CC}(\Gamma).$$

In particular, $dim_{CC}(FS^\Gamma(S)) < dim_{CC}(\Gamma)$.

Proof. Let H be a full subgraph of G, observe that H is a full subgraph of Γ as well. By proposition 2.3.2 and lemma 2.3.1 we have that $CC(H) < m_C(H) \leq m_C(G) \leq m_C(FS^{\Gamma}(S)) = CC(\Gamma)$. Thus $CC(H) < CC(\Gamma)$.

□

2.4 Asymptotic dimension of RACGs.

Theorem 2.4.1. *Let W_Γ be the right-angled Coxeter group with connected defining graph Γ. Then W_Γ is the fundamental group of a graph of groups such that $asdim(G_v) \leq dim_{CC}(\Gamma)$ for every vertex group and $asdim(G_e) < dim_{CC}(\Gamma)$ for every edge group. In particular,*

$$asdim W_\Gamma \leq dim_{CC}(\Gamma).$$

If Γ is not connected, then $asdim W_\Gamma \leq max\{1, dim_{CC}(\Gamma)\}$.

Proof. We will use induction on $\sharp V(\Gamma)$. If $\sharp V(\Gamma) = 1$, the theorem is obviously true. We assume that for any graph with $\sharp V(\Gamma) < N + 1$ the theorem holds. Let Γ be a graph such that $\sharp V(\Gamma) = N + 1$.

By theorem 2.1.3 it is enough to prove the inequality only for RACGs with connected defining graphs. So we assume that the graph is connected. If the graph Γ is a clique, then by lemma 2.2.2 the theorem holds. So we further assume that the graph is not a clique.

Since the graph is not a clique there is a subset of its vertices separating it, thus there is at least one vertex cut of Γ. Let S be a minimal vertex cut of the graph and let $E_1, \ldots, E_k$ be the connected components of $\Gamma \setminus FS^{\Gamma}(S)$. Observe that since S is a vertex cut we have that for every vertex s of S and every component E_i there exists at least one edge connecting them.

We set $\overline{E_i} = FS^{\Gamma}(E_i \cup S)$, observe that $\overline{E_i}$ is a connected full subgraph of Γ, and thus by lemma 2.2.4 $dim_{CC}(\overline{E}_i) \leq dim_{CC}(\Gamma)$. By the inductive hypothesis $asdim W_{\overline{E}_i} \leq dim_{CC}(\overline{E}_i)$, so

$$asdim W_{\overline{E}_i} \leq dim_{CC}(\Gamma), \tag{2.1}$$

where $W_{\overline{E}_i}$ is the parabolic subgroup of W_Γ defined by $\overline{E}_i$ (of course $W_{\overline{E}_i}$ is RACG). Using theorem 2.3.4 we have $dim_{CC}(FS^{\Gamma}(S)) < dim_{CC}(\Gamma)$.

We distinguish two cases.

Case 1: $FS^{\Gamma}(S)$ is connected.
By the inductive hypothesis $asdimW_{FS^{\Gamma}(S)} \leq dim_{CC}(FS^{\Gamma}(S))$, then

$$asdimW_{FS^{\Gamma}(S)} < dim_{CC}(\Gamma), \tag{2.2}$$

where $W_{FS^{\Gamma}(S)}$ is the parabolic subgroup of W_{Γ} defined by $FS^{\Gamma}(S)$ (of course $W_{FS^{\Gamma}(S)}$ is RACG).
Finally, observe that W_{Γ} can be obtained from $W_{\overline{E}_i}$ after a finite sequence of amalgamated product over $W_{FS^{\Gamma}(S)}$. To be more precise,

$$W_{\Gamma} = (W_{\overline{E}_1} \underset{W_{FS^{\Gamma}(S)}}{*} W_{\overline{E}_2}) \underset{W_{FS^{\Gamma}(S)}}{*} \dots W_{\overline{E}_k}. \tag{2.3}$$

In other words, W_{Γ} is the fundamental group of a graph of groups with vertex groups $W_{\overline{E}_i}$ and $W_{FS^{\Gamma}(S)}$, and edge groups isomorphic to $W_{FS^{\Gamma}(S)}$. Applying theorem 2.1.1 or theorem 2.1.2 we conclude that $asdimW_{\Gamma} \leq dim_{CC}(\Gamma)$.
Case 2: $FS^{\Gamma}(S)$ is not connected.
By equality (2.3) we observe that to complete the proof of the theorem it's enough to show that $asdimW_{FS^{\Gamma}(S)} < dim_{CC}(\Gamma)$.

Let $C_1, \dots, C_{\lambda}$ be the connected components of $FS^{\Gamma}(S)$ ($\lambda \geq 2$). Without loss of generality we may assume that the $dim_{CC}(C_i) \leq dim_{CC}(C_{\lambda})$, for any i. By the inductive hypothesis $asdimW_{C_i} \leq dim_{CC}(C_{\lambda})$. So by theorem 2.1.3

$$asdimW_{FS^{\Gamma}(S)} = max\{1, dim_{CC}(C_{\lambda})\}. \tag{2.4}$$

We distinguish two subcases.
Case 2(a): $dim_{CC}(C_{\lambda}) \geq 1$.
Then $asdimW_{FS^{\Gamma}(S)} = dim_{CC}(C_{\lambda})$. Since $dim_{CC}(C_{\lambda}) \leq dim_{CC}(FS^{\Gamma}(S)) < dim_{CC}(\Gamma)$, we conclude that $asdimW_{FS^{\Gamma}(S)} < dim_{CC}(\Gamma)$.
Case 2(b): $dim_{CC}(C_{\lambda}) = 0$.
Then $asdimW_{FS^{\Gamma}(S)} = 1$. Since S is a minimal vertex cut, and $FS^{\Gamma}(S)$ is not connected we obtain that $dim_{CC}(\Gamma) \geq 2$. Thus $asdimW_{FS^{\Gamma}(S)} < dim_{CC}(\Gamma)$.

□

We observe that the previous theorem is also true for Coxeter groups such that every clique in their defining graphs defines a finite Coxeter subgroup.

As a corollary of theorem 2.4.1 we have:

Theorem 2.4.2. *Let W_Γ be the right-angled Coxeter group with defining graph Γ. Then*

$$asdim W_\Gamma \leq min\{dim_{CC}(\Gamma), dim\Sigma(W_\Gamma)\}.$$

If Γ is not connected then $asdim W_\Gamma \leq max\{1, min\{dim_{CC}(\Gamma), dim\Sigma(W_\Gamma)\}\}$.

2.4.1 Graph Products of Finite Groups.

We recall the definition of Graph Products of Groups. Let Γ be a simplicial graph. We label every vertex v of the graph by a *vertex group* G_v.
The *graph product* $\mathbb{GP}(\Gamma, G_v)_{v\in V(\Gamma)}$ is the free product of the vertex groups modulo the relations $xyx^{-1}y^{-1}$ where $x \in G_v$, $y \in G_w$, and v, w are connected by an edge.

Graph products generalize free products, direct products, RAAGs, and RACGs.

We set $G = \mathbb{GP}(\Gamma, G_v)_{v\in V(\Gamma)}$. For a full subgraph X of Γ we define the graph product $G\mid_X$ as $\mathbb{GP}(X, G_v)_{v\in V(X)}$.

Theorem 2.4.3. *Let Γ be a finite connected simplicial graph along with a finite collection of finite vertex groups. Then*

$$asdim\, \mathbb{GP}(\Gamma, G_v)_{v\in V(\Gamma)} \leq dim_{CC}(\Gamma).$$

If Γ is not connected, then $asdim\, \mathbb{GP}(\Gamma, G_v)_{v\in V(\Gamma)} \leq max\{1, dim_{CC}(\Gamma)\}$.

Proof. The proof of this theorem is almost identical to the proof of theorem 2.4.1. We only need the fact that if X is a full subgraph of Γ, then $G\mid_X$ is a subgroup of $\mathbb{GP}(\Gamma, G_v)_{v\in V(\Gamma)}$ (see [21]). The subgroups $G\mid_X$ are the analogues of the parabolic subgroups of RACGs.

□

2.5 Corollaries of the main result.

Proposition 2.5.1. *Let Γ be a simplicial graph such that $dim_{CC}(\Gamma) \geq 2$. Then the right-angled Coxeter group W_Γ defined by Γ contains a one-ended parabolic subgroup.*

Proof. Γ contains a full subgraph G such that $CC(G) \geq 2$. We assume that G is a minimal full subgraph of Γ such that $CC(G) \geq 2$. Trivially, G is connected. Since G is minimal we have that $CC(G') \leq 1$ for every G' proper full subgraph of G, so by lemma 2.2.5 G is a k-cycle ($k \geq 4$).
By theorem 8.7.2 of [48] we have that the parabolic subgroup W_G of W_Γ defined by G is one-ended. □

Proposition 2.5.2. *Let W_Γ be the right-angled Coxeter group with connected defining graph Γ. If $dim_{CC}(\Gamma) \leq 2$, then $asdimW_\Gamma = dim_{CC}(\Gamma)$.*

Proof. If $dim_{CC}(\Gamma) = 0$, then Γ is a clique, so W_Γ is finite. Then $asdimW_\Gamma = 0$.

If $dim_{CC}(\Gamma) = 1$, then by theorem 2.4.1 we have $asdimW_\Gamma \leq 1$. By lemma 2.2.2 Γ is not a clique and thus there are two vertices a, b which are not connected by an edge. This means that W_Γ contains $\mathbb{Z}_2 * \mathbb{Z}_2$ as a parabolic subgroup, so $asdimW_\Gamma = 1$.

If $dim_{CC}(\Gamma) = 2$, then by theorem 2.4.1 we have $asdimW_\Gamma \leq 2$. By proposition 2.5.1 we have that there exists an one-ended parabolic subgroup W_G of W_Γ. Then by the main theorem of [20] we obtain that $2 \leq asdimW_G$. So $asdimW_\Gamma = 2$.

□

Corollary 3. *Let W_Γ be the right-angled Coxeter group with connected defining graph Γ. Then W_Γ is finite if and only if $dim_{CC}(\Gamma) = 0$.*

Proof. Suppose that W_Γ is finite. Then Γ is a clique, indeed, otherwise W_Γ contains $\mathbb{Z}_2 * \mathbb{Z}_2$ as a parabolic subgroup, so $asdimW_\Gamma > 0$. Which is a contradiction. Since Γ is not a clique by lemma 2.2.2 we obtain $dim_{CC}(\Gamma) = 0$.

The other direction follows by the previous proposition. □

When Γ is connected and has clique-connected dimension one, the graph looks like a "thick" tree.

Proposition 2.5.3. *Let W_Γ be the right-angled Coxeter group with connected defining graph Γ. Then W_Γ is a virtually free group if and only if $dim_{CC}(\Gamma) = 1$.*

Proof. We assume that W_Γ is virtually free. If $dim_{CC}(\Gamma) \geq 2$, then by proposition 2.5.1 W_Γ contains an one-ended parabolic subgroup. Since one-ended groups have asymptotic dimension at least two (see [20]) we have that $asdimW_\Gamma \geq 2$. By the fact that the asymptotic dimension of virtually free grous is one (see [20]) we have a contradiction.
If $dim_{CC}(\Gamma) = 0$, then Γ is a clique. In that case, W_Γ is finite, which is a contradiction.

Suppose that $dim_{CC}(\Gamma) = 1$, then by proposition 2.5.2 we have $asdimW_\Gamma = 1$. Applying Gentimis' theorem for virtually free groups (see [20]), we conclude that W_Γ is virtually free.

□

Remark 1. It is known by M.Lohrey and G.Senizergues (see [28]) that a RACG is virtually free if and only if its defining graph Γ is chordal (i.e. does not contain k-cycles as a full subgraph for $k \geq 4$). We note that, in their paper they consider finite groups as virtually free groups.

Remark 2. It is easy to see that a graph Γ is chordal if and only if $dim_{CC}(\Gamma) \leq 1$. Indeed, if a graph Γ is chordal and $dim_{CC}(\Gamma) \geq 2$, then there exists a full subgraph Γ' such that $CC(\Gamma') \geq 2$. Thus, by lemma 2.2.5 we have a contradiction.

Now suppose that $dim_{CC}(\Gamma) \leq 1$ and that Γ is connected (w.l.o.g). Since cliques are complete graphs we further assume that Γ is not a clique. By lemma 2.2.2 we have that $dim_{CC}(\Gamma) = 1$. Then by the definition of the clique connected dimension and the fact that $CC(k\text{-cycle}) = 2$ $(k \geq 4)$, we have that Γ is chordal. This proves our statement.

One can also show the fact in remark 2 by using theorem 2.1 from [10], and the observation that vertex cuts are minimal vertex separators in the sense of Dirac (see [10]).

As a corollary of proposition 2.5.3 we obtain the following:

Proposition 2.5.4. *Let W_Γ be the right-angled Coxeter group with connected defining graph Γ. Then $asdim(W_\Gamma) \geq 2$ if and only if $dim_{CC}(\Gamma) \geq 2$.*

Proof. We suppose that $asdim(W_\Gamma) \geq 2$, then by theorem 2.4.1 we have that $dim_{CC}(\Gamma) \geq 2$.

Conversely, we assume that $dim_{CC}(\Gamma) \geq 2$, then by the previous proposition and the fact that the only groups having asymptotic dimension one are the virtually free groups (see [20]) we have that $asdim(W_\Gamma) \neq 1$. Obviously, $asdim(W_\Gamma) \neq 0$, otherwise we have a contradiction by corollary 3.

□

Observe that proposition 2.5.3 and corollary 3 can be rephrased as follows:

Corollary 3: $asdimW_\Gamma = 0$ if and only if $dim_{CC}(\Gamma) = 0$.

Proposition 2.5.3: $asdimW_\Gamma = 1$ if and only if $dim_{CC}(\Gamma) = 1$.

We know by proposition 2.5.2 that if $dim_{CC}(\Gamma) = 2$, then $asdimW_\Gamma = 2$. One may ask whether the converse is true. We note that by the previous proposition if $asdim(W_\Gamma) = 2$, then $dim_{CC}(\Gamma) \geq 2$.

Question. *Is there any connected graph such that the RACG defined by the this graph has asymptotic dimension two while the clique connected dimension of the graph is greater than two?*

The answer is yes. We will construct a graph X with clique connected dimension equal to three while $asdimW_X = 2$. Let X_1, X_2 and X_3 be 4-cycles with vertices

$\{v_1^1, \ldots, v_4^1\}, \{v_1^2, \ldots, v_4^2\}$ and $\{v_1^3, \ldots, v_4^3\}$. We join the vertices v_j^i, v_j^{i+1} with edges. The resulting graph X has clique connected dimension equal to three. The graph X is actually the 1-skeleton of a cube complex, thus $Sim(X) = 2$.
We recall that by $Sim(\Gamma)$ we denote the number of vertices of a maximal clique of Γ. It turns out that the dimension of the Davis complex of W_Γ is equal to $Sim(\Gamma)$.
Using the Dranishnikov's upper bound (see [14]) and the fact that W_X is one ended we obtain that $asdimW_X = 2$.

Thus an analogue of corollary 3 and proposition 2.5.3 for asymptotic dimension two doesn't exist. However, we have the following:

Proposition 2.5.5. *Let W_Γ be the right-angled Coxeter group with connected defining graph Γ. If $asdimW_\Gamma = 2$, then there exists a full subgraph G of Γ such that $dim_{CC}(G) = 2$ and $asdimW_G = 2$.*

Proof. By theorem 2.4.1, $dim_{CC}(\Gamma) \geq 2$. By the proof of proposition 2.5.1, Γ contains a k-cycle G as a full subgraph ($k \geq 4$). Trivially, $dim_{CC}(G) = 2$.

□

Chapter 3

Asymptotic dimension of Artin groups and a new upper bound for Coxeter groups.

Artin groups are easily defined via presentations, like Coxeter groups, but they are often very poorly understood. There are numerous conjectures about them, for example, it is not known in general whether they are torsion free. While Artin groups remain much more mysterious than their Coxeter relatives, a driving theme behind current research has been to show that these groups are as well behaved as Coxeter groups.

Some examples of Artin groups are the non-abelian free groups, free abelian groups, and RAAGs. In recent years, Artin and Coxeter groups proved to be a very rich source of examples and counterexamples of interesting phenomena in geometric group theory. For example, the Boltyanski compactum cannot be a boundary of a Coxeter group (see [15]).

In this chapter we are interested in the geometry of Artin and Coxeter groups and in particular in the asymptotic dimension of these groups. The asymptotic dimension achieved particular prominence in geometric group theory after a paper of Guoliang Yu, (see [39]) which proved the Novikov higher signature conjecture for manifolds whose fundamental group has finite asymptotic dimension. It is known for many classes of groups that they have finite asymptotic dimension, for instance, hyperbolic, relative hyperbolic, hierarchically hyperbolic, mapping class groups of surfaces, and one relator groups have finite asymptotic dimension (see [4], [33], [58], [6], [32], [55]). However, there are still many open questions regarding asymptotic dimension, for example, do all Artin groups have finite asymptotic dimension ([11]) ? It is also unknown whether $Out(F_n)$ groups have finite asymptotic dimension, when $n > 2$

([11]).

Recently, it was proved in [53] that Artin groups being both of large and of hyperbolic type are hierarchically hyperbolic. As a consequence, these Artin groups have finite asymptotic dimension. However, no specific bound was obtained.

The purpose of this Chapter is to give some new results about the asymptotic dimension of Artin and Coxeter groups. To be more precise, let Γ be a finite simplicial labeled graph, we denote by A_Γ (W_Γ) the *Artin group* (*Coxeter group*) associated to the graph Γ. We set

$Sim(\Gamma) = max\{n \mid \Gamma$ contains the 1-skeleton of the standard $(n-1)$-simplex $\Delta^{n-1}\}$.

We conjecture that:

Conjecture 3.0.1. $asdimA_\Gamma \leq Sim(\Gamma)$, for all Artin groups.

Conjecture 3.0.2. $asdimW_\Gamma \leq Sim(\Gamma)$, for all Coxeter groups.

It is known that if A_Γ is a RAAG, then $asdimA_\Gamma = Sim(\Gamma)$ see ([55] and [38]). Since when $Sim(\Gamma) = 1$ the group A_Γ is a free group, $asdim(A_\Gamma) = 1$. We prove that:

Theorem 3.0.1. *Let A_Γ be an Artin group with $Sim(\Gamma) = 2$. Then*

$$asdimA_\Gamma = 2.$$

Theorem 3.0.2. *Let A_Γ be an Artin group of large type with $Sim(\Gamma) = 3$. Then*

$$asdimA_\Gamma \leq 3.$$

Theorem 3.0.3. *Let A_Γ be an Artin group such that every label is at most four, and with $Sim(\Gamma) = 3$. Then,*

$$asdimA_\Gamma \leq 3.$$

It is known by an isometric embedding theorem of Januszkiewicz (see [52]) that Coxeter groups have finite asymptotic dimension. In particular, Januszkiewicz's theorem shows that for any Coxeter group W_Γ with defining graph Γ we have the following upper bound: $asdimW_\Gamma \leq \sharp V(\Gamma)$.
Dranishnikov notes that this can be improved to the following: $asdimW_\Gamma \leq ch(\Gamma)$, where $ch(\cdot)$ is the chromatic number of a graph. In general, $Sim(\Gamma) \leq ch(\Gamma)$.

Using theorem 3.0.1 we prove that:

Theorem 3.0.4. *If for all free of infinity Artin (Coxeter) groups the conjecture 3.0.1 (conjecture 3.0.2) holds, then it holds for all Artin (Coxeter) groups.*

We recall that an Artin (Coxeter) group is *free of infinity* if the corresponding graph is complete. Using theorem 3.0.4 we show that conjecture 3.0.2 is true.

Theorem 3.0.5. *Let W_Γ be a Coxeter group. Then*

$$asdim W_\Gamma \leq Sim(\Gamma).$$

Since $Sim(\Gamma) \leq ch(N(W_\Gamma)$ and there are graphs such that the inequality is strict, theorem 3.0.5 improves the previously best known upper bound.

By a finite simplicial *G-labeled graph* we mean a finite simplicial graph Γ such that every edge $[a, b]$ is labeled by a cyclically reduced word $r_{a,b}$ (in terms of letters-vertices a, b). The word $r_{a,b}$ is neither of the form $b^k ab^\lambda$ nor $a^k ba^\lambda$.

The *Graph group* associated to a finite simplicial G-labeled graph Γ is the group G_Γ given by the following presentation:

$$G_\Gamma = \langle a, b | r_{a,b} \text{ when } a, b \text{ are connected by an edge } \rangle.$$

Obviously, Graph groups generalize Artin groups.

Proposition 3.0.6. *Let Γ be a finite simplicial G-graph such that $Sim(\Gamma) = 2$. We further assume that Γ is a forest. Then*

$$asdim\, G_\Gamma \leq 2.$$

3.1 Preliminaries.

By a finite simplicial labeled graph we mean a finite simplicial graph Γ such that every edge $[a, b]$ is labeled by a natural number $m_{ab} > 1$. The Artin group associated to Γ is the group A_Γ given by the following presentation:

$$A_\Gamma = \langle a, b \in V(\Gamma) | \underbrace{ab\ldots}_{m_{ab}} = \underbrace{ba\ldots}_{m_{ab}} \text{ when a, b are connected by an edge } \rangle.$$

Convention. We will use the same symbol m_{ab} both for the label of the edge $[a, b]$ and the relation $\underbrace{ab\ldots}_{m_{ab}} \cdot (\underbrace{ba\ldots}_{m_{ab}})^{-1}$ of the group.

We call an Artin group *right-angled Artin group* (RAAG) if $m_{ab} = 2$ when a, b are connected by an edge. We call an Artin group *free of infinity* if the corresponding graph is complete. An Artin group is of *large type* if every label m_{ab} is at least three.

We call a Coxeter group *free of infinity* if the corresponding graph is complete.

There is another equivalent way to define $Sim(\Gamma)$, in particular, it can be viewed as the larger natural number n such that Γ contains a complete subgraph Γ' with n vertices.

3.2 The case $Sim(\Gamma) = 2$.

Let Γ be a finite simplicial labeled graph, such that $Sim(\Gamma) = 2$. We distinguish two cases, either the graph contains a cycle or not. We will handle these cases separately.

Proposition 3.2.1. *Let Γ be a finite simplicial labeled graph, such that $Sim(\Gamma) = 2$. We further assume that Γ is a forest. Then*

$$asdim\, A_\Gamma = 2 \text{ and } asdim\, W_\Gamma \leq 1.$$

Proof. (*Artin groups*) We observe that if Γ is not connected, then $A_\Gamma = *_{i=1}^{n} A_{\Gamma_i}$, where Γ_i are the connected components of Γ. By the previous observation and theorem 2.1.3 we may consider only connected graphs. So we assume that Γ is connected, thus Γ is a tree.

We observe that if Γ is a tree, then A_Γ is the fundamental group of graph of groups where the vertex groups are one-relator parabolic subgroups of A_Γ and the edge groups are infinite cyclic parabolic subgroups. By theorem 1.3.1 and theorem 0.2.3 we have that $asdim\, A_\Gamma \leq 2$.

Suppose that A_Γ is the artin group defined by a single edge. Obviously, A_Γ is one relator group. Since A_Γ is not a free group and it is torsion free (see [30] page 200), by theorem 0.2.3 we obtain that $asdim\, A_\Gamma = 2$. Since every Artin group with $Sim(\Gamma) = 2$ contains a one relator parabolic subgroup we conclude that the proposition is true.

(*Coxeter groups*) If Γ is a tree, then W_Γ is the fundamental group of graph of finite parabolic subgroups. By theorem 1.3.1 we have that $asdim\, W_\Gamma \leq 1$.
If Γ is not connected, then $asdim\, W_\Gamma = max\{1, asdim\, W_E\}$, where E is a connected component of Γ. Obviously, $asdim\, W_\Gamma \leq 1$.

□

In the second case we have that the graph contains a cycle, however, it is possible to contain only cycles of length larger than or equal to four.

Theorem 3.2.2. *Let Γ be a finite simplicial labeled graph, such that $Sim(\Gamma) = 2$. Then*

$$asdim\, A_\Gamma = 2 \text{ and } asdim\, W_\Gamma \leq 2.$$

Proof. Again we may assume that Γ is connected. We will use induction on the number n_C of simple cycles contained in the graph. If $n_C = 0$, then by proposition 3.2.1 the statement of this theorem is true. We assume that the statement is true for all $n_C \leq n$. Let Γ be a finite simplicial labeled graph such that $n_C = n + 1$. By the previous observation we know that the graph contains a cycle γ of length $m + 1 \geq 4$.

Let $\{v_1, \ldots, v_{m+1}\}$ be the vertex set of γ and $\{e_1, \ldots, e_{m+1}\}$ be the edges of the cycle where $e_i = [v_i, v_{i+1}]$. We set $\Gamma_1 = \Gamma \setminus \{v_1\}$. Obviously, this graph has at least one edge, and contains at least one cycle less than Γ. If Γ_1 is connected, then by induction $asdim\, A_{\Gamma_1} = 2$ ($asdim\, W_{\Gamma_1} \leq 2$). If Γ_1 is not connected, then $asdim\, A_{\Gamma_1} = asdim\, A_E$ ($asdim\, W_{\Gamma_1} = max\{1, asdim\, W_E\}$), where E is a component of Γ_1. Then by induction (for the Artin case $Sim(E) = 2$), $asdim\, A_E = 2$ ($asdim\, W_E \leq 2$), thus $asdim\, A_{\Gamma_1} = 2$ ($asdim\, W_{\Gamma_1} \leq 2$).

Let $U = \{u_1, \ldots, u_k\}$ be all the vertices of Γ such that for each u_j there exists an edge connecting it with v_1. It is easy to observe that the full subgraph Γ_U of Γ formed by U contains no edge. Thus the parabolic subgroup A_{Γ_U} (W_{Γ_U}) of A_Γ (W_Γ) is the free group with k generators F_k (is the virtually free group $*_{i=1}^{k}\mathbb{Z}_2$).

Let Γ_{v_1} be the full subgraph of Γ formed by $U \cup \{v_1\}$, we observe that contains no cycle. It is not hard to see that Γ_{v_1} is connected and contains at least one edge, thus by the previous proposition $asdim\, A_{\Gamma_{v_1}} = 2$ ($asdim\, W_{\Gamma_{v_1}} \leq 1$.).

It is easy to check that the Artin (Coxeter) group A_Γ (W_Γ) is an amalgamated product of its parabolic subgroups A_{Γ_1} (W_{Γ_1}), A_{Γ_U} (W_{Γ_U}) and $A_{\Gamma_{v_1}}$ ($W_{\Gamma_{v_1}}$). To be more precise, the following equalities are a direct consequence of the presentation of A_Γ and W_Γ:

$$A_\Gamma = A_{\Gamma_1} \underset{A_{\Gamma_U}}{*} A_{\Gamma_{v_1}} \text{ and } W_\Gamma = W_{\Gamma_1} \underset{W_{\Gamma_U}}{*} W_{\Gamma_{v_1}}$$

Then by theorem 1.3.1 (or using Dranishnikov's theorem (see [14])) we conclude that the theorem holds. □

3.2.1 Graph-Groups.

In fact we may obtain a more general result than proposition 3.2.1.

By a finite simplicial *G-labeled graph* we mean a finite simplicial graph Γ such that every edge $[a, b]$ is labeled by a cyclically reduced word $r_{a,b}$ (in terms of letters-vertices a, b). The word $r_{a,b}$ is neither of the form $b^k ab^\lambda$ nor $a^k ba^\lambda$.

The *Graph group* associated to a finite simplicial G-labeled graph Γ is the group G_Γ given by the following presentation:

$$G_\Gamma = \langle a, b | r_{a,b} \text{ when } a, b \text{ are connected by an edge } \rangle.$$

Obviously, Artin groups are Graph groups.

Proposition 3.2.3. *Let Γ be a finite simplicial G-graph, such that $Sim(\Gamma) = 2$. We further assume that Γ is a forest, then*

$$asdim\, G_\Gamma \leq 2.$$

Proof. Again we assume that Γ is connected, thus Γ is a tree. Suppose that G_Γ is a graph group defined by a single edge. Obviously, G_Γ is one relator group. By theorem 0.2.3 we obtain that $asdim\, G_\Gamma \leq 2$.

It is important to note that if a is a vertex of Γ, then $\langle a \rangle$ is an infinite cyclic subgroup of G_Γ. This follows by the Freiheitssatz (see [23], thm 5.1, page 198) and induction on the number of vertices. Since $Sim(\Gamma) = 2$, the graph contains at least an edge.

We observe that if Γ is a tree, then G_Γ is the fundamental group of graph of groups where the vertex groups are one-relator Graph subgroups of G_Γ and the edge groups are infinite cyclic subgroups. By theorem 1.3.1 and theorem 0.2.3 we have that $asdim\, G_\Gamma \leq 2$..

□

3.3 The main results.

Theorem 3.3.1. *If for all free of infinity Artin (Coxeter) groups the conjecture 3.0.1 (conjecture 3.0.2) holds, then it holds for all Artin (Coxeter) groups.*

Proof. We will show the theorem only for Artin groups since the proof for Coxeter groups is similar to that for Artin groups.

We observe that the free of infinity Artin groups are those defined by a complete graph. By theorem 2.1.3 it suffices to prove the statement only for Artin groups

defined by a connected graph. Since conjecture 3.0.2 is true for graphs with $Sim(\Gamma) = 1, 2$, it also suffices to prove the statement only for Coxeter groups defined by a connected graph.

Let Γ be a finite simplicial connected labeled graph. We will use induction on $Sim(\Gamma)$. We distinguish two cases.

Case 1. The graph has at most two vertices.

Then A_Γ is either a free group or an one relator group. In the first subcase the asymptotic dimension is one and in the second subcase by theorem 3.2.2 the asymptotic dimension is two. In particular, in both subcases the conjecture 3.0.1 holds.

Case 2. The graph has at least three vertices.

For $Sim(\Gamma) = 1$ and 2 the conjecture 3.0.1 is true. We assume that the statement of this theorem is true for any $Sim(\Gamma) \leq n$, where $n \geq 2$. Let Γ be a finite simplicial connected labeled graph with $Sim(\Gamma) = n + 1$.

It is vital to use another induction here, in particular, we use induction on $n_{V(\Gamma)} = \sharp V(\Gamma)$. If $n_{V(\Gamma)} = Sim(\Gamma)$, then the graph is a complete graph thus the group A_Γ is free of infinity, so $asdim A_\Gamma \leq n + 1$. We assume that $asdim A_\Gamma \leq n + 1$ for every graph with $n_{V(\Gamma)} < N + 1$, we note that $N \geq Sim(\Gamma)$. Let Γ be a finite simplicial connected labeled graph with $Sim(\Gamma) = n + 1$ and $n_{V(\Gamma)} = N + 1$.

Let $\{C_1, \ldots, C_k\}$ be the complete subgraphs of Γ such that $Sim(C_i) = Sim(\Gamma)$, we will call these subgraphs *maximal complete subgraphs*. We set $n_C = \sharp\{C_1, \ldots, C_k\}$. We further distinguish two subcases.

Subcase 2.1 $n_C = 1$.

Let C be the only maximal complete subgraph of Γ. Since $n_{V(\Gamma)} > Sim(\Gamma)$, there exist vertices $v_1, v_2 \in C$ and $x \notin C$ such that there is no edge connecting v_1 with x. We set $\Gamma_1 = \Gamma \setminus \{v_1\}$, $\Gamma_x = \Gamma \setminus \{x\}$ and $\Gamma_{1x} = \Gamma \setminus \{v_1, x\}$ and we observe that all these graphs form the following parabolic subgroups of A_Γ, A_{Γ_1} A_{Γ_x} and $A_{\Gamma_{1x}}$. As a direct consequence of the presentation of A_Γ we have that

$$A_\Gamma = A_{\Gamma_1} \underset{A_{\Gamma_{1x}}}{*} A_{\Gamma_x}. \tag{3.1}$$

(This observation has been used in [51].) It is easy to see that $Sim(\Gamma_1), Sim(\Gamma_{1x}) < n+1$, while $Sim(\Gamma_x) = n+1$ but $n_{V(\Gamma_x)} < n_{V(\Gamma)} = N+1$. Thus $asdim\, A_{\Gamma_1}, asdim\, A_{\Gamma_{1x}} < n+1$, while $asdim\, A_{\Gamma_x} \leq n+1$. Applying theorem 1.3.1 we conclude that $asdim\, A_\Gamma \leq n + 1$.

(*For Coxeter groups:*) If Γ_1 is not connected, then $asdim\, W_{\Gamma_1} = max\{1, asdim\, W_E\}$, where E is a connected component of Γ_1. We have that $Sim(E) < n + 1$, thus by

induction, $asdim\, W_E \leq n$. Since $n \geq 2$, we conclude that $asdim\, W_{\Gamma_1} < n+1$.
Similarly, if Γ_{1x}, or (and) Γ_x are not connected we can show that $asdim\, W_{\Gamma_{1x}} < n+1$ and $asdim\, W_{\Gamma_x} \leq n+1$.

Subcase 2.2 $n_C > 1$.

This case is a bit more complicate than the previous. The goal is again to write the A_Γ as an amalgamated product, similar to the equality (3.1).

Recall that $S_C = \{C_1, \ldots, C_{n_C}\}$ is the set of the maximal complete subgraphs of Γ. Observe that there exists a vertex v_2 of C_2 and an edge $e = [v_0, v_1]$ of C_1 such that there is no edge between v_2 and v_1. We will divide S_C into two disjoint subsets, S_1 and S_2, the first set consists of the maximal subgraphs containing v_1 and the second set containing the rest of the elements. After possibly relabeling the set S_C we may assume that $S_2 = \{C_2, \ldots, C_\sigma\}$. We fix an element v_i for every $C_i \in S_2$ such that there is no edge between v_1 and v_i (such elements exist and they are not equal to v_0).

We set $\Gamma_1 = \Gamma \setminus \{v_1\}$, $\Gamma_\sigma = \Gamma \setminus \{v_2, \ldots, v_\sigma\}$ and $\Gamma_{1\sigma} = \Gamma \setminus \{v_1, \ldots, v_\sigma\}$. We observe that all these graphs form the following parabolic subgroups of A_Γ, A_{Γ_1} A_{Γ_σ} and $A_{\Gamma_{1\sigma}}$. As a direct consequence of the presentation of A_Γ we have that

$$A_\Gamma = A_{\Gamma_1} \underset{A_{\Gamma_{1\sigma}}}{*} A_{\Gamma_\sigma}. \tag{3.2}$$

Observe that $Sim(\Gamma_{1\sigma}) < n+1$, while $Sim(\Gamma_1) = Sim(\Gamma_\sigma) = n+1$, however, $n_{V(\Gamma_\sigma)}, n_{V(\Gamma_1)} < n_{V(\Gamma)} = N+1$. So we obtain $asdim\, A_{\Gamma_{1\sigma}} < n+1$, while $asdim\, A_{\Gamma_\sigma}, asdim\, A_{\Gamma_1} \leq n+1$. By theorem 1.3.1 we conclude that $asdim\, A_\Gamma \leq n+1$.
(*For Coxeter groups:*) If Γ_1 is not connected, then $asdim\, W_{\Gamma_1} = max\{1, asdim\, W_E\}$, where E is a connected component of Γ_1. We have that $Sim(E) \leq n+1$, thus by induction, $asdim\, W_E \leq n+1$. Since $n \geq 2$, we conclude that $asdim\, W_{\Gamma_1} \leq n+1$.
Similarly, if $\Gamma_{1\sigma}$, or (and) Γ_σ are not connected we can show that $asdim\, W_{\Gamma_{1\sigma}} < n+1$ and $asdim\, W_{\Gamma_\sigma} \leq n+1$.

This completes the proof that $asdim\, A_\Gamma \leq n+1$ in any subcase when $Sim(\Gamma) = n+1$, thus by induction the theorem holds.

□

In following theorem we give the best known upper bound for the asymptotic dimension of Coxeter groups, improving the previous bound due to Dranishnikov.

Theorem 3.3.2. *Let Γ be a finite simplicial labeled graph, and let W_Γ be the Coxeter group defined by Γ. Then*

$$asdim W_\Gamma \leq Sim(\Gamma).$$

Proof. By an isometric embedding theorem of Januszkiewicz (see [52]) we have that $asdim W_\Gamma \leq \sharp V(\Gamma)$. Observe that if Γ is complete graph, then $\sharp V(\Gamma) = Sim(\Gamma)$. This means that the conjecture 3.0.2 is true for complete graphs.
Thus by theorem 3.3.1 we conclude that conjecture 3.0.2 is true for any Coxeter group.
□

As further corollaries of theorem 3.3.1, we have:

Proposition 3.3.3. *Let A_Γ be an Artin group of large type with $Sim(\Gamma) = 3$. Then*

$$asdim A_\Gamma \leq 3.$$

Proof. By [25] every Artin group of large type defined by a triangle splits as a graph of free groups, thus by theorem 1.3.1 $asdim A_\Gamma = 2$. By the proof of theorem 3.3.1 and the fact that every parabolic subgroup of a large type Artin group is also a large type Artin group, we conclude that the proposition is true.
□

Proposition 3.3.4. *Let A_Γ be an Artin group such that every label is at most four, and with $Sim(\Gamma) = 3$. Then,*

$$asdim A_\Gamma \leq 3.$$

Proof. By the proof of theorem 3.3.1 it is enough to show the statement only for Artin group defined by a triangle. We distinguish four cases.
Case 1. All sides are labelled by two.
Then $A_\Gamma = \mathbb{Z}^3$, so it holds.
Case 2. Two sides are labelled by two.
Then A_Γ is the HNN-extension of an Artin group defined by an edge, thus the statement also holds.
Case 3. One side is labelled by two.
If the triangle has two sides labelled by four then by [26] splits as a graph of free groups, so the statement holds.
Otherwise, A_Γ is of finite type, in particular, of $\mathcal{A}_3$ or $\mathcal{B}_3$. Then by [7] $asdim A_\Gamma \leq 3$, thus the statement holds.
Case 4. No side is labelled by two.
Then we have a triangle Artin group of large type, thus by [25] the statement of the proposition holds.
□

Chapter 4

Coarsely separation of groups and spaces.

The asymptotic dimension theory bears a great deal of resemblance to dimension theory in topology, so it is natural to ask whether results of topological dimension theory have an asymptotic analogue.

It is known that every topological space X of dimension $n \geq 1$ can be separated by a subset Y of dimension equal to or smaller than $n-1$ (see [23]). By separation we mean that the space $X \setminus Y$ contains at least two connected components. One may ask whether there exists an asymptotic analogue of that for metric spaces.

The aim of this chapter is to show that for geodesic metric spaces containing a bi-infinite geodesic an asymptotic analogue exists.

Let X be a metric space and $K \subseteq X$. We say that a component C of $X \setminus N_R(K)$ is *deep* if C is not contained in $N_{R'}(K)$ for any $R' > 0$. We recall that by $N_R(A)$ we denote the R-neighbourhood of a set A (i.e. all the points of distance at most R from A).
We say that K *weakly coarsely separates* X if there is an $R > 0$ such that $X \setminus N_R(K)$ contains at least two deep components of X and at least one of them is connected.
We say that K *coarsely separates* X if there is an $R > 0$ such that $X \setminus N_R(K)$ contains at least two connected deep components of X. We say that a finitely generated group G can be (weakly) coarsely separated by a set Y if there exists a finite generating set S such that $Y \subseteq Cay(G, S)$ and Y (weakly) coarsely separates the Cayley graph.

Convension: In what follows we consider only metric spaces with finite asymptotic dimension.

We show that:

Theorem 4.0.1. *Let X be a geodesic metric space containing a bi-infinite geodesic. If $asdimX = n > 0$, then there exists a subspace Y of asymptotic dimension strictly less than n which coarsely separates X.*

Firstly, we show that:

Theorem 4.0.2. *Every finitely generated group G can be weakly coarsely separated by a set Y of asymptotic dimension strictly less than $asdimG$ (provided that $asdimG > 0$).*

The proof of this theorem can be used to show that

Theorem 4.0.3. *Every finitely generated group G can be coarsely separated by a set Y of asymptotic dimension strictly less than $asdimG$ (provided that $asdimG > 0$).*

The strategy of proof of theorem 4.0.1 is almost identical to that of theorem 4.0.3. Thus we only prove theorem 4.0.3. and theorem 4.0.2.

We also give an example of a geodesic metric space which cannot be coarsely separated by any subset.

We note that for proper metric spaces the analogue of Theorem 4.0.1 might follows from [44]. We recall that a topological space X of dimension n is a *Cantor n-manifold* if X cannot be separated by a closed $(n-2)$-dimensional subset $(n \geq 2)$. We give the following asymptotic analogue of Cantor manifolds:

Definition 4.0.1. Let X be a metric space with $asdimX = n$, where $n \geq 2$. We say that X is an *n-asymptotic Cantor manifold* if X cannot be coarsely separated by a k-asymptotic dimensional subset, where $k \leq n-2$.

In 1947 (see [40]) P.S. Aleksandrov proved the following theorem:

Theorem 4.0.4. *For $n \geq 2$, every n-dimensional Hausdorff topological space X contains an n-dimensional Cantor manifold.*

We ask whether an asymptotic analogue exists, and we give some trivial examples of groups that cannot be asymptotic Cantor manifolds

4.1 Weak Coarse Separation of Spaces.

Let X be a metric space and $\mathcal{U}$ a covering of X, we denote by $mesh(\mathcal{U}) = sup\{diam(U) \mid U \in \mathcal{U}\}$. We say that the covering is *D-uniformly bounded* (or D-bounded) if $mesh(\mathcal{U}) \leq D$. For $G = \langle S \rangle$ a finitely generated group we define by S_d (or $\partial B_d(1_G)$) the *boundary* of $B_d(1_G)$ in $Cay(G, S)$, namely, $S_d = \{g \in G \mid \| g \| = d\}$.

The following theorem shows that every finitely generated group G can be weakly coarsely separated by a subset Y of asymptotic dimension strictly less than $asdimG$.

Theorem 4.1.1. *Let G be a finitely generated group with $asdim(G) = n$, $n \geq 1$. Then there exists a set Γ which has asymptotic dimension strictly less than n, and weakly coarsely separates G.*

We need two auxiliary lemmas.

Lemma 4.1.2. *Let $G = \langle S \rangle$ be a finitely generated group with $asdim(G) = n$. Then for every $R \in \mathbb{N} \setminus 0$ there exist D_R,$d_R > 0$ such that for every $d \geq d_R$ there exists a D_R-uniformly bounded family $\overline{\mathcal{U}}_d$, such that:*
(i) $(R-1)$-mult$(\overline{\mathcal{U}}_d) \leq n$.
(ii) The union $\Gamma_{R,d} = \bigcup \overline{\mathcal{U}}_d$ separates the group G into at least two connected components, one of which is unbounded and one of which is bounded.
(iii) One of the bounded components contains the ball with center 1_G and radius $\frac{d}{2}$ of $Cay(G, S)$ denoted by $B_{\frac{d}{2}}(1_G)$.

Proof. For $R > 0$ there exists a uniformly bounded cover $\overline{\mathcal{U}}$ of G with R-mult$(\overline{\mathcal{U}})$ $= n + 1$ and mesh$(\overline{\mathcal{U}})$= D_R ($D_R > 10$).
We choose $d > d_R \geq 10^9(R + D_R + 1)$. Let $X = Cay(G, S)$.

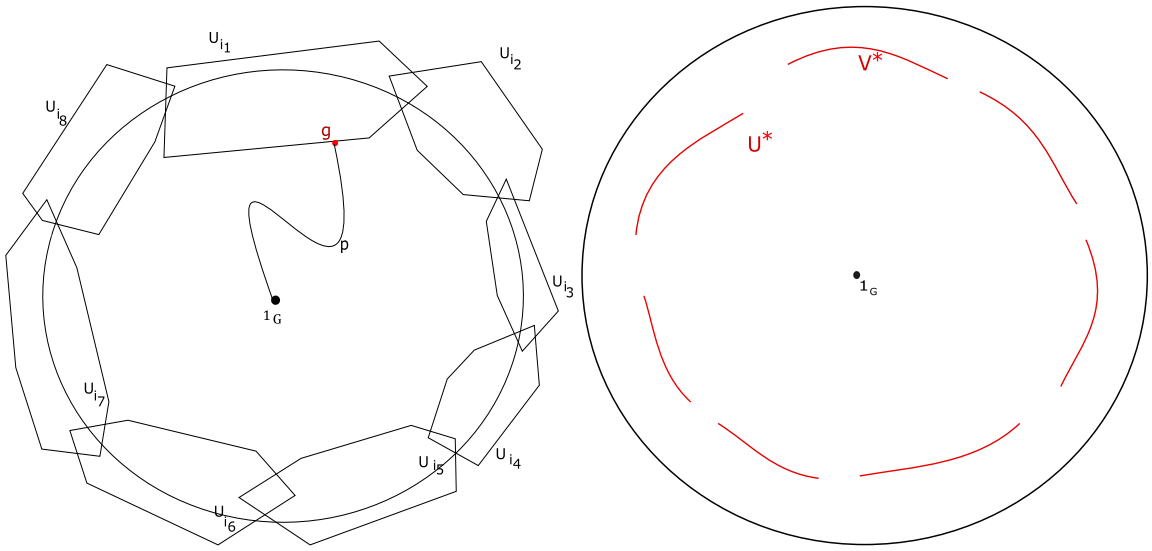
We define $\overline{\mathcal{U}}_{S_d} = \{U \in \overline{\mathcal{U}} \mid U \cap S_d \neq \varnothing\}$ Let $U \in \overline{\mathcal{U}}_{S_d}$ then we denote by

$U^* = \{g \in U \cap G$ such that there exists a path $p : [0, l] \longrightarrow X$ from 1_G to g and g is the first element of the path in $\bigcup \overline{\mathcal{U}}_{S_d}\}$

Remark: We may assume that the path $p : [0, l] \longrightarrow X$ is simple. Obviously, we can omit loops in X.

We say that U^* is the *descendant* of U. Let:

$$\overline{\mathcal{U}}_d = \{U^* \mid U \in \overline{\mathcal{U}}_{S_d}\}. \tag{4.1}$$



(a) $\overline{\mathcal{U}}_{S_d}$ (b) Here we have the union $\bigcup \overline{\mathcal{U}}_d$

Figure 4.1

We observe that mesh$(\overline{\mathcal{U}}_d) \leq D_R$.

Claim 1: $X \setminus \bigcup \overline{\mathcal{U}}_d$ has at least two connected components, one of which is bounded and one of which is unbounded. The bounded component contains the ball $B_{d-(D_R+2)}(1_G)$.

Proof: We remark that $\bigcup \overline{\mathcal{U}}_d \subseteq B_{d+D_R}$ where $B_{d+D_R} = B_{d+D_R}(1_G)$, so $X \setminus B_{d+D_R}$ is a subset of $X \setminus \bigcup \overline{\mathcal{U}}_d$. Now it is easy to see that $X \setminus \bigcup \overline{\mathcal{U}}_d$ contains $B_{d-(D_R+2)}(1_G)$ and at least one unbounded connected subset C of X.

We show that $B_{d-(D_R+2)}(1_G)$ and C are disconnected in $X \setminus \bigcup \overline{\mathcal{U}}_d$. It suffices to show that every path $p : [0, l] \longrightarrow X$ from 1_G to $c \in C$ meets $\bigcup \overline{\mathcal{U}}_d$.

The path p meets the sphere S_d, so there exists $U \in \overline{\mathcal{U}}_{S_d}$ in which the path p has his first element in $\bigcup \overline{\mathcal{U}}_{S_d}$. Let's denote that element by $p_0 \in U$. Obviously $p_0 \in U^*$.

Claim 2: $(R-1)$-$mult(\overline{\mathcal{U}}_d) \leq n$.

Proof: Suppose that claim 2 does not hold. Then there exist $U_1^*, \ldots, U_{n+1}^*$ elements in $\overline{\mathcal{U}}_d$ and a ball $B_{R-1}(x)$ which intersects all these elements.

If $u \in U_1^*$ then by definition there exists a path $\gamma : [0, l] \longrightarrow X$ from the identity to u and its first element into $\bigcup \overline{\mathcal{U}}_{S_d}$ is u. Let's denote by w the element $\gamma(l-1) \in G$. Then we have that $w \notin \bigcup \overline{\mathcal{U}}_{S_d}$ and $d_S(v, w) = 1$, so

$$w \in B_R(x). \tag{4.2}$$

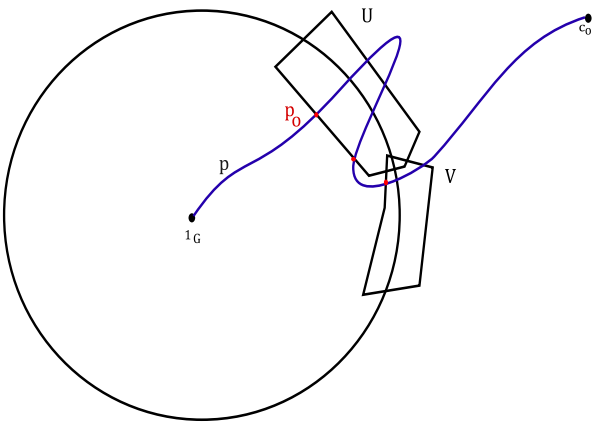


Figure 4.2: For $\mathcal{U}, V \in \overline{\mathcal{U}}_{S_d}$

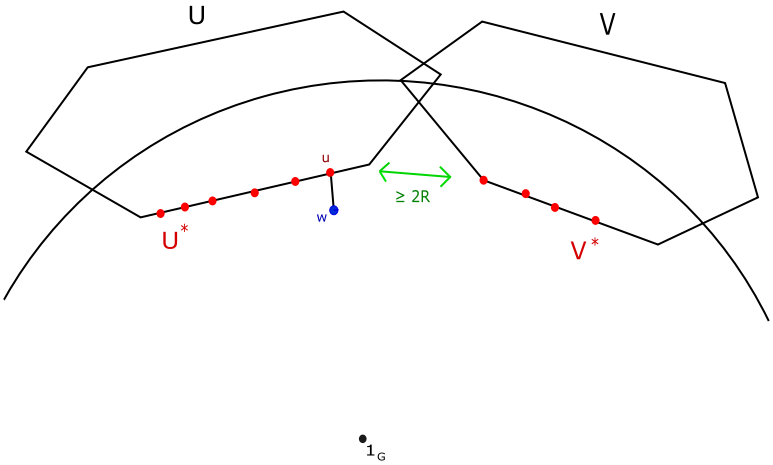


Figure 4.3

In addition to, there exists $V \in \overline{\mathcal{U}} \setminus \overline{\mathcal{U}}_{S_d}$ such that $w \in V$, so the ball $B_R(x)$ intersects $n+2$ discrete elements of $\overline{\mathcal{U}}$. This is a contradiction.

These two claims together prove the statement of the lemma.

□

Lemma 4.1.3. *Let $G = \langle S \rangle$ be a finitely generated group with $asdim(G) = n$. Let R, D_R, and $d \geq d_R$ be as the lemma 4.1.2.*
Assume E is a bounded subset of $Cay(G, S)$ which separates $Cay(G, S)$ into at least two connected components, one of which is bounded and one of which is unbounded. Assume further that one bounded connected component of $Cay(G, S) \setminus E$ contains the

ball $B_{\frac{d}{2}}$ *and*

$$E \subseteq B(\frac{3d}{2}) \setminus B(\frac{3d}{4}+1). \tag{4.3}$$

Then for every $R' \leq R$ *there exists a* $D_{R'}$*-uniformly bounded family* $\overline{\mathcal{U}}_{R'}$ *such that:*
(i) The $D_{R'}$*-neighbourhood of* E *contains the union* $\bigcup\overline{\mathcal{U}}_{R'}$*.*
(ii) $(R'-1)$*-mult*$(\overline{\mathcal{U}}_{R'}) \leq n$*.*
(iii) The union $\bigcup\overline{\mathcal{U}}_{R'}$ *separates the* $Cay(G,S)$ *into at least two connected components, one of which is bounded and one of which is unbounded.*
(iv) One of the bounded components contains the ball $B_{\frac{d}{2}}$*.*

Proof. The proof is exactly the same as the proof of the previous lemma. We do the same process by replacing the sphere S_d by E. The only difference between them is the proof of claim 1, thus we will prove only the analogue of claim 1. We have the same symbolism as that of proof of lemma 4.1.2 except from $\overline{\mathcal{U}}_{S_d}$, the analogue of $\overline{\mathcal{U}}_{S_d}$ will be denoted by $\overline{\mathcal{U}}_E$. We also replace of R by R'. The family $\overline{\mathcal{U}}_d$ consists of the *descendants* of $\overline{\mathcal{U}}_E$. We can assume that $D_{R'} = D_R = D$.

Claim 1: $X \setminus \bigcup\overline{\mathcal{U}}_d$ contains at least two connected components, one of which is bounded and one of which is unbounded. The bounded contains the ball $B_{\frac{d}{2}}$.

Proof: We remark that $\bigcup\overline{\mathcal{U}}_d \subseteq B(\frac{3d}{2}+D) \setminus B(\frac{3d}{4}-D)$, so $X \setminus \bigcup\overline{\mathcal{U}}_d$ contains the ball $B_{\frac{d}{2}}$ and at least one unbounded connected subset C of X.
We have that the bounded set E separates $Cay(G,S)$ into at least two connected components, one of which is bounded and one of which is unbounded. Thus, there exists an unbounded connected subset C' of C such that C' is contained in an unbounded connected component F of $X \setminus E$.

We show that $B_{\frac{d}{2}}$ and C' are disconnected in $X \setminus \bigcup\overline{\mathcal{U}}_d$. It suffices to show that every path $p : [0,l] \longrightarrow X$ from 1_G to $c \in C'$ meets $\bigcup\overline{\mathcal{U}}_d$.
Since the subset E separates $B(\frac{d}{2})$ and F the path p meets E, there exists $U \in \overline{\mathcal{U}}_E$ in which the path p has it's first element in $\bigcup\overline{\mathcal{U}}_E$. Let's denote that element by $p_0 \in U$. Obviously $p_0 \in U^*$. This proves claim 1.

□

Convention: When we have a star $(*)$ as superscript to a set U, namely U^*, then U is related to the proofs of the previous two lemmas and U^* is a *descendant* of U.

Proof. (*Theorem 4.1.1*)

We follow some steps in order to construct the wanted sequence of subsets $\{\Gamma_k\}_{k\in\mathbb{N}}$ of G. Let S be a finite generating set of G.

CONSTRUCTION I:

Step 1: We choose $R_1 + 1 > 100$. By lemma 4.1.2 there exist $d_{R_1} > 0$ and a D_{R_1}-uniformly bounded family $\overline{\mathcal{U}}_{d_{R_1}}$ with R_1-mult$(\overline{\mathcal{U}}_{d_{R_1}}) \leq n$. We take $d_{R_1} \geq 10^9(R_1 + 2D_{R_1} + 1)$. The family $\overline{\mathcal{U}}_{d_{R_1}}$ was obtained from a D_{R_1}-uniformly bounded covering $\overline{\mathcal{U}}_1$ of G. We set $\Gamma_0^1 = \bigcup \overline{\mathcal{U}}_{d_{R_1}}$.

Step 2: For $R_2 + 1 > 10^9(R_1 + 2D_{R_1} + 1)$ we take

$$d_{R_2} \geq 10^9(d_{R_1} + R_2 + 2(D_{R_2} + D_{R_1}) + 2).$$

By lemma 4.1.2 there exists a D_{R_2}-uniformly bounded family $\overline{\mathcal{U}}^1_{d_{R_2}}$ with R_2-mult$(\overline{\mathcal{U}}^1_{d_{R_2}}) \leq n$. The family $\overline{\mathcal{U}}^1_{d_{R_2}}$ was obtained from a D_{R_2}-uniformly bounded covering $\overline{\mathcal{U}}_2$ of G. We set $\Gamma^1_{R_2} = \bigcup \overline{\mathcal{U}}^1_{d_{R_2}}$.
The conditions of lemma 4.1.3 are satisfied for $\Gamma^1_{R_2}$. Applying the proof of that lemma using the cover $\overline{\mathcal{U}}_1$ we obtain a D_{R_1}-uniformly bounded family $\overline{\mathcal{U}}^2_{d_{R_2}}$ with R_1-mult$(\overline{\mathcal{U}}^2_{d_{R_2}}) \leq n$. We set $\Gamma^2_{R_2} = \bigcup \overline{\mathcal{U}}^2_{d_{R_2}}$.

To every U_i^{1*} in $\overline{\mathcal{U}}^1_{d_{R_2}}$ we associate a family,

$$U_i^{1*} \longrightarrow \{U^{2*}_{(i,s)}, s \in I_i^1\}. \tag{4.4}$$

This family consists of all $U^{2*}_{(i,s)} \in \overline{\mathcal{U}}^2_{d_{R_2}}$ such that $U^2_{(i,s)} \cap U_i^{1*} \neq \emptyset$. (The set $U^{2*}_{(i,s)}$ is a descendant of $U^2_{(i,s)}$, and $U^2_{(i,s)}$ is obtained by procedures of the previous lemmas.)

We observe that:
(i) $\bigcup \overline{\mathcal{U}}^2_{d_{R_2}} = \bigcup \{U^{2*}_{(i,s)} \forall i, s\}$
(ii) R_1-mult$(\{U^{2*}_{(i,s)} \forall i, s\}) \leq n$
(iii) For $U^{1*}_{i_1}, \ldots, U^{1*}_{i_{n+1}} \in \overline{\mathcal{U}}^1_{d_{R_2}}$ the sets $\bigcup \{U^{2*}_{(i_1,s)}, s \in I^2_{i_1}\}, \ldots, \bigcup \{U^{2*}_{(i_{n+1},s)}, s \in I^2_{i_{n+1}}\}$ can not be intersected by a $(R_2 - D_{R_1})$-ball.
(iv) We set $\overline{\mathcal{U}}^2_2 = \{U^{2*}_{(i,s)} \forall i, s\}$, then

$$mesh(\overline{\mathcal{U}}^2_2) \leq D_{R_2} + 2D_{R_1}. \tag{4.5}$$

We set $\Gamma^2_0 = \bigcup \overline{\mathcal{U}}^2_2$.
.
.

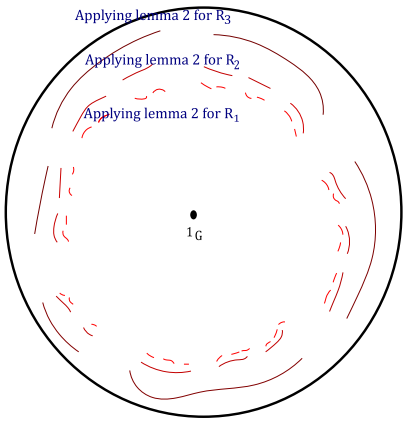


Figure 4.4: Here we have an illustration of the result of step 3 for n=1.

.

Step n: We take:

$$R_n \geq 10^9(R_{n-1} + 2\sum_{s=1}^{n-1} D_{R_s} + n) \tag{4.6}$$

and

$$d_{R_n} \geq 10^9(d_{R_{n-1}} + R_n + 2\sum_{s=1}^{n-1} D_{R_s} + n). \tag{4.7}$$

By lemma 4.1.2 there exists a D_{R_n}-uniformly bounded family $\overline{\mathcal{U}}^1_{d_{R_n}}$ with R_n-mult$(\overline{\mathcal{U}}^1_{d_{R_n}}) \leq n$. We set $\Gamma^1_{R_n} = \bigcup \overline{\mathcal{U}}^1_{d_{R_n}}$.
The conditions of lemma 4.1.3 are satisfied for $\Gamma^1_{R_n}$. Applying the proof of that lemma using the cover $\overline{\mathcal{U}}_{n-1}$ we obtain a $D_{R_{n-1}}$-uniformly bounded family $\overline{\mathcal{U}}^2_{d_{R_n}}$ with R_{n-1}-mult$(\overline{\mathcal{U}}^2_{d_{R_n}}) \leq n$. By $\overline{\mathcal{U}}^2_n$ we denote the analogue of $\overline{\mathcal{U}}^2_2$ of the step 2. We set $\Gamma^2_{R_n} = \bigcup \overline{\mathcal{U}}^2_n$.

We continue by applying the lemma 4.1.3 (n-2) more times and we end up having a D_{R_1}-uniformly bounded family such that R_1-mult$(\overline{\mathcal{U}}^n_{d_{R_n}}) \leq n$.
For every $k \in \{1, \ldots, n\}$ and every fixed $(i_1, \ldots, i_k)$ we set

$$U^{n,(i_1,\ldots,i_k)} = \bigcup_{(i_{k+1},\ldots,i_n)} \{U^{n*}_{(i_1,\ldots,i_n)}, \text{ and } i_s \in I^n_{(i_1,\ldots,i_{s-1})} \text{ for every } s \in \{k+1, \ldots n\}\}.$$

We set $D_s = D_{R_s}$. We define the families:

$$\mathcal{V}^n_k = \{U^{n,(i_1,\ldots,i_k)} \text{ for every possible } (i_1, \ldots, i_k)\}$$

Observe that,

$$mesh(\mathcal{V}^n_k) \leq D_{n-k+1} + 2\sum_{m=1}^{n-k} D_m.$$

and

$$(R_{n-k+1} - \sum_{m=1}^{n-k} D_m)\text{-mult}(\mathcal{V}_k^n) \leq n.$$

We set $\Gamma_0^n = \bigcup \overline{\mathcal{U}}_n^n$, where $\overline{\mathcal{U}}_n^n = \{U^{n*}_{(i_1,\ldots,i_n)}$ for all $(i_1,\ldots,i_n)\}$. Here the step n ends.

Using the previous steps we constructed a sequence:

$$\{\Gamma_0^n = \bigcup \overline{\mathcal{U}}_n^n\}_{n\in\mathbb{N}}, \tag{4.8}$$

and for every step n and every $k \in \{1,\ldots,n\}$ the families:

$$\mathcal{V}_k^n = \{U^{n,(i_1,\ldots,i_k)} \text{ for every possible } (i_1,\ldots,i_k)\}.$$

CONSTRUCTION II:

Let $c : \mathbb{R} \to Cay(G,S)$ be a bi-infinite geodesic such that $c(0) = 1_G$. We choose a sequence of elements of G onto the geodesic c. The first element is $x_1 = 1_G$, the second element is $x_2 = c(10^9(d_{R_1} + d_{R_2} + 2))$, so $d_S(x_1,x_2) = 10^9(d_{R_1} + d_{R_2} + 2)$. The n-nth element is:

$$x_n = c(10^9(n + d_{R_n} + d_{R_{n-1}} + d_S(x_1, x_{n-1})). \tag{4.9}$$

and $d_S(x_n,x_{n-1}) = 10^9(n + d_{R_n} + d_{R_{n-1}} + d_S(x_1,x_{n-1}))$.
We denote by Γ_n the transportation of Γ_0^n by x_n.

$$\Gamma_n = x_n \cdot \Gamma_0^n \tag{4.10}$$

Let $\Gamma = \bigcup_{k=1}^{\infty} \Gamma_k$. Obviously, Γ weakly coarsely separates $Cay(G,S)$. It remains to show the following,

Claim: asdim$\Gamma \leq n-1$.

Proof: Let $R > R_1$, then there exists a natural number n_0 such that $n_0 - 1 > 10^9 R$.

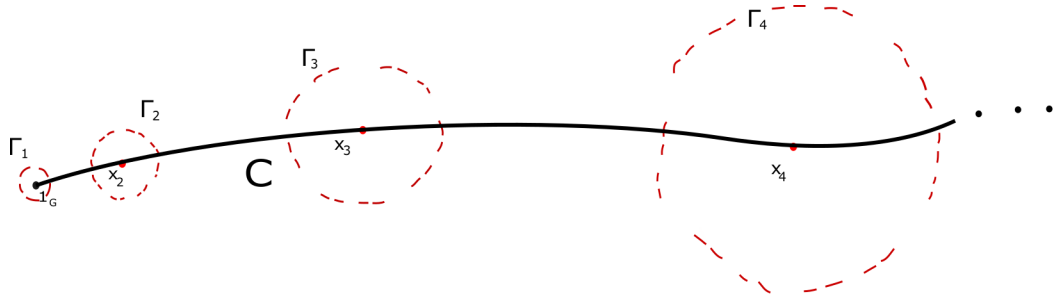


Figure 4.5: This is the picture we have.

We observe that for every $k \in \{1, \dots, n\}$ each family $x_n \cdot \mathcal{V}_k^n$ is a $(D_{n-k+1} + 2\sum_{m=1}^{n-k} D_m)$-uniformly bounded covering of Γ_n and

$$(R_{n-k+1} - \sum_{m=1}^{n-k} D_m)\text{-mult}(x_n \cdot \mathcal{V}_k^n) \leq n.$$

For every $n \geq 2n_0 + 1$ we have that

$$R_{n-n_0} - \sum_{m=1}^{n-n_0} D_m > n_0 > R$$

We set $V_0 = \bigcup_{n=1}^{2n_0+1} \Gamma_n$. We will construct a uniformly bounded covering $\overline{\mathcal{V}}$ of Γ with R-multiplicity less or equal to n. We consider the covering:

$$\overline{\mathcal{V}} = \{V_0\} \bigcup_{n=2(n_0+1)}^{\infty} \{x_n \cdot \mathcal{V}_{n-(2n_0+1)}^n\}.$$

Every element $V \in \overline{\mathcal{V}}$ has $diam(V) \leq max\{diamV_0, (D_{2(n_0+1)} + 2\sum_{m=1}^{2n_0+1} D_m)\}$, so the cover $\overline{\mathcal{V}}$ is indeed uniformly bounded.

To complete the proof we need to show that every R-ball meets at most n element of the covering.

First, we have that V_0 and $\overline{\mathcal{V}} \setminus \{V_0\}$ are $(d_S(x_{2(n_0+1)}, x_{2n_0+1}) - (d_{R_{2(n_0+1)}} + d_{R_{2n_0+1}}))$-disjoint. Observe that:

$$d_S(x_{2(n_0+1)}, x_{2n_0+1}) - (d_{R_{2(n_0+1)}} + d_{R_{2n_0+1}}) > n_0 > 10^9 R,$$

so there is no R-ball that meets both V_0 and an element of $\overline{\mathcal{V}} \setminus \{V_0\}$.

Second, let $2n_0 + 1 < n_1 < n_2$. If $x_{n_1} \cdot V^{n,(i_1,\dots,i_{2n_0+1})} \in x_{n_1} \cdot \mathcal{V}_{n_1-(2n_0+1)}^{n_1}$ and $x_{n_2} \cdot V^{n,(i_1,\dots,i_{2n_0+1})} \in x_{n_2} \cdot \mathcal{V}_{n_2-(2n_0+1)}^{n_2}$, then these two sets are $(d_S(x_{n_1}, x_{n_2}) - (d_{R_{n_1}} + d_{R_{n_2}}))-$disjoint. We have:

$$d_S(x_{n_1}, x_{n_2}) - (d_{R_{n_1}} + d_{R_{n_2}}) \geq d_S(x_{n_2-1}, x_{n_2}) - (d_{R_{n_2-1}} + d_{R_{n_2}}) > n_2 > 10^9 R, \quad (4.11)$$

so there is no R-ball that meets both those elements.

Finally, let $2n_0+1 < n$. We have that $(R_{2(n_0+1)} - \sum_{m=1}^{2n_0+1} D_m)\text{-mult}(x_n \cdot \mathcal{V}_{n-(2n_0+1)}^n) \leq n$. Since

$$R_{2(n_0+1)} - \sum_{m=1}^{2n_0+1} D_m > n_0 > 10^9 R. \quad (4.12)$$

we conclude that $R\text{-mult}(x_n \cdot \mathcal{V}_{n-(2n_0+1)}^n) \leq n$.

□

The proof of the next theorem is almost identical to that of theorem 4.1.1.

Theorem 4.1.4. *Let X be a geodesic metric space containing a bi-infinite geodesic. If $asdimX = n > 0$, then there exists a subspace Y of asymptotic dimension strictly less than n which weakly coarsely separates X.*

The following example shows that theorem 4.1.4 cannot be generalized to any geodesic metric space.

Example 4.1.1. Let L be the infinite line $[0, \infty)$ with the graph metric. The asymptotic dimension of L is one. We can observe that if Y weakly coarsely separates L, then all the deep components of $L \setminus Y$ intersect each other since one of the deep connected components must be an unbounded interval.

4.2 Coarse Separation of Spaces.

Theorem 4.2.1. *Every finitely generated group G can be coarsely separated by a set Y of asymptotic dimension strictly less than $asdimG$ (provided that $asdimG > 0$).*

Proof. Let $\{\Gamma_n\}_n$ be the sequence of constructed in the proof of theorem 4.1.1 and let $c : \mathbb{R} \longrightarrow Cay(G, S)$ be the bi-infinite geodesic from Construction II of the proof of theorem 4.1.1.
We observe that the subsets Γ_n look like spheres of G. The idea is to construct another sequence $\{T_n\}_n$ of subsets of G using the same strategy but instead of starting from spheres of G we start from "tubes-cylinders" of G. Then by combining these two sequences we construct a subset Y of asymptotic dimension strictly less than $asdimG$ (provided that $asdimG > 0$) which coarsely separates G.
Discription of the sequence $\{T_n\}_n$:
We show how to construct T_1, then using the same strategy as that of Γ_n one can construct the rest of the sequence.

We consider the sequence of numbers $\{R_n\}_n$ as defined in the proof of theorem 4.1.1. We start with a large enough neighborhood of a long enough geodesic segment c_1 of c. Then using the same idea as that of lemma 4.1.2 and lemma 4.1.3 we obtain a D_{R_1}-uniformly bounded family $\overline{\mathcal{W}}_{d'_{R_1}}$ with R_1-mult$(\overline{\mathcal{W}}_{d'_{R_1}}) \leq asdim\, G$. Here the role of d_{R_1} will be played by $d'_{R_1} = \dfrac{d_{R_1}}{2^7}$. The family $\overline{\mathcal{W}}_{d'_{R_1}}$ was obtained from the same D_{R_1}-uniformly bounded covering $\overline{\mathcal{U}}_1$ of G as that used in the proof of theorem 4.1.1.

We set $T_1 = \bigcup \overline{\mathcal{W}}_{d'_{R_1}}$. The set T_1 looks like a "tube-cylinder" of G. Observe that T_1 separates the $Cay(G, S)$ into at least two connected components, one of which is bounded and one of which is unbounded. One of the bounded components, say Σ_1 contains a large enough neighborhood of a long enough geodesic segment c_1 of $c([0, \infty))$.
Construction of Y:
We define Y in steps. We may choose the end points of the segments c_n to be x_n and

x_{n+1} as these have been defined in the proof of theorem 4.1.1. We will describe only the step 1 since every other step n is the analogue of step 1.

Step 1:

(a) We remove the points of T_1 from the balls $B_{\frac{d_{R_1}}{2^3}}(x_1)$ and $B_{\frac{d_{R_2}}{2^3}}(x_2)$.

(b) We remove the points of Γ_1 and Γ_2 from $N_{\frac{d_{R_1}}{2^9}}(c([0, \| x_2 \|_S]))$. By $\| x \|_S$ we denote the distance of x from 1_G, and by $N_r(A)$ we denote the r-neighborhood of A. Here the step 1 ends.

We set $T_i' = T_i \setminus (B_{\frac{d_{R_i}}{2^3}}(x_i) \cup B_{\frac{d_{R_{i+1}}}{2^3}}(x_{i+1}))$ and

$\Gamma_i' = \Gamma_i \setminus (N_{\frac{d_{R_{i-1}}}{2^9}}(c([\| x_{i-1} \|_S, \| x_i \|_S])) \cup N_{\frac{d_{R_i}}{2^9}}(c([\| x_i \|_S, \| x_{i+1} \|_S])))$.

Then we set

$$Y = \cup_{i \in \mathbb{N}}(T_i' \cup \Gamma_i')$$

We observe that $Cay(G, S) \setminus Y$ contains at least two deep components one of which contains $c : [0, \infty) \longrightarrow Cay(G, S)$ and one of which contains $c : (-\infty, -N] \longrightarrow Cay(G, S)$ for some $N > 0$.

By the construction of Y it follows that

$$Y \subseteq \cup_{i \in \mathbb{N}}(T_i' \cup \Gamma_i).$$

Since $asdim \cup T_n' < asdim\, G$ and $asdim \cup \Gamma_n < asdim\, G$, we conclude that:

$$asdim\, Y < asdim\, G.$$

□

By the proof of the previous theorem we can generalize to the following:

Theorem 4.2.2. *Let X be a geodesic metric space containing a bi-infinite geodesic. If $asdim X = n > 0$, then there exists a subspace Y of asymptotic dimension strictly less than n which coarsely separates X.*

The following example shows that theorem 4.2.2 cannot be generalized to any geodesic metric space.

Example 4.2.1. Let A be the graph obtained by the wedge sum of infinitely many pointed intervals $\{([0, a_n], 0)\}$, where $a_n \to \infty$. The asymptotic dimension of A with the graph metric is one. We can observe that if Y coarsely separates A, then all the deep connected components of $A \setminus Y$ must contain 0, which is impossible.

Obviously, the previous example can generalized to any asymptotic dimension.

4.3 Some questions.

We recall that a topological space X is a *Cantor n-manifold* if X can not be separated by a closed $(n-2)$-dimensional subset $(n \geq 2)$. The concept of Cantor manifolds was introduced by P.S. Urysohn (see [47]). Some examples of n-dimensional Cantor manifolds are the n-dimensional closed balls and the n-dimensional Euclidean space.

A maximal n-dimensional Cantor manifold in an n-dimensional compact space X is called a *dimensional component* of X. Each n-dimensional Cantor manifold of a compact Hausdorff space X is contained in a unique dimensional component of X. In 1947 (see [40]) P.S. Aleksandrov proved the following theorem:

Theorem 4.3.1. *For $n \geq 2$, every n-dimensional Hausdorff topological space X contains an n-dimensional Cantor manifold.*

It is natural to ask if there exists an analogue for the asymptotic dimension. We give the following definitions to make this precise.

Definition 4.3.1. Let X be a metric space with $asdimX = n$, where $n \geq 2$. We say that X is an *n-asymptotic Cantor manifold* if X cannot be coarsely separated by a k-asymptotic dimensional subset, where $k \leq n-2$.

Definition 4.3.2. Let X be a metric space with $asdimX = n$, where $n \geq 2$. We say that $Y \subseteq X$ is an *asymptotic component* of X, if Y is an n-asymptotic Cantor manifold and it is maximal in the following sense: for every other n-asymptotic Cantor manifold $A \subseteq X$ such that $Y \subseteq A$ there exists $R > 0$ such that $A \subseteq N_R(Y)$.

Question 2. Do asymptotic Cantor manifolds exist?

The short answer to this question is, yes. For example, consider a collection of balls B_r $(r \geq 0)$ in the Cayley graph of $\mathbb{Z}^n$ (with respect to the usual generating set) with center at $(2^{r+2}, 0, \ldots, 0)$ and radius r. Then connect those balls by straight lines. The geodesic metric space obtained by this procedure (graph metric) is an n-asymptotic Cantor manifold.
One could alter question 2 and ask about asymptotic Cantor manifolds with specific properties.

Question 3. Do asymptotic components always exist?

Question 4. Do asymptotic components of a group G correspond to subgroups ?

We give two examples of groups that are not asymptotic Cantor manifolds.

Example 4.3.1. Let A, B be finitely generated groups with $max\{asdimA, asdimB\} = n$, and let C a subgroup of them such that $asdimC \leq n-2$. Then the group C coarsely separates the amalgamated product $A *_C B$, and $asdimA *_C B = n$.

Example 4.3.2. Let G a finitely generated group with $asdimG = n$, and let N be a subgroup such that $asdimN \leq n-2$. Then the group N coarsely separates the HNN-extension $G*_N$, and $asdimG*_N = n$.

Corollary 4. *Let G a finitely generated group with $asdimG = n$. If the group G is an n-asymptotic Cantor manifold, then it does not split over a subgroup H with $asdimH \leq n-2$.*

Bibliography

[1] G Bell, A Dranishnikov, *On asymptotic dimension of groups*, Algebr. Geom. Topol. 1 (2001) 57-71 MR1808331

[2] G. Bell, and A. Dranishnikov, *On asymptotic dimension of groups acting on trees*. Geometriae Dedicata 103 (2004), 89-101

[3] G. C. Bell and A. N. Dranishnikov, *Asymptotic dimension in Bedlewo*, (2005) ArXiv:math.GR/0507570

[4] G. Bell and A. Dranishnikov. *Asymptotic dimension*. Topology Appl., 155(12):1265-1296, 2008.

[5] G. C. Bell, A. N. Dranishnikov, and J. E. Keesling, *On a formula for the asymptotic dimension of free products*, Fundamenta Mathematicae 183.1 (2004): 39-45.

[6] Bestvina, Mladen, Ken Bromberg, and Koji Fujiwara, *Constructing group actions on quasi-trees and applications tomapping class groups*, Publications mathematiques de l'IHES 122.1 (2015): 1-64.

[7] Gregory C. Bell, Koji Fujiwara, *The asymptotic dimension of a curve graph is finite* , Journal of the London Mathematical Society, Volume 77, Issue 1, February 2008, Pages 33–50, https://doi.org/10.1112/jlms/jdm090

[8] S. Buyalo, N. Lebedeva, *Dimension of locally and asymptotically self-similar spaces*, arXiv.org:math.GT/0509433 (2005)

[9] D. Wise, *The Structure of Groups with a Quasiconvex Hierarchy* (AMS-209)

[10] G. A. Dirac. *On rigid circuit graphs*. Abhandlungen aus dem Mathematischen Seminar der Universitat Hamburg, 25:71-76, 1961.

[11] Site: https://sites.google.com/a/scu.edu/rscott/pggt, *Problems in Geometric Group Theory*, Asymptotic Dimension – Dranishnikov.

[12] A.N.Dranishnikov, *Asymptotic topology*, Uspekhi Mat. Nauk, 55:6(336) (2000), 71–116; Russian Math. Surveys, 55:6 (2000), 1085–1129

[13] A.Dranishnikov, *Cohomological approach to asymptotic dimension*. Geom Dedicata 141, 59 (2009). https://doi.org/10.1007/s10711-008-9343-0

[14] A. Dranishnikov, *On asymptotic dimension of amalgamated products and right-angled Coxeter groups*, Algebr. Geom. Topol. 8 (2008), no. 3, 1281-1293.

[15] A. Dranishnikov, *On the virtual cohomological dimensions of Coxeter groups*, Proceedings of the American Mathematical Society, Volume 125, Number 7, July 1997, Pages 1885–1891 S 0002-9939(97)04106-3

[16] C DrutÂ¸u, M Sapir, Tree-graded spaces and asymptotic cones of groups, Topology 44

[17] A. Dranishnikov and J. Smith, *Asymptotic dimension of discrete groups*, Fund. Math. 189 (2006), no. 1, 27-34. MR2213160

[18] M.Dunwoody, *The accessibility of finitely presented groups*, Invent. Math. 81 (1985), No 3, 449-457.

[19] J. Fischer, A. Karrass, and D. Solitar, *On one-relator groups having elements of finite order*, Proc. Amer. Math. Soc 33 (1972), 297-301.

[20] Thanos Gentimis, *Asymptotic dimension of finitely presented groups*, Proceedings of the American Mathematical Society 136, no. 12 (2008): 4103-4110.

[21] E. R. Green. *Graph Products of Groups*. PhD thesis, The University of Leeds, 1990.

[22] M.Gromov, *Asymptotic invariants of infinite groups*, in Geometric Group Theory, v. 2, 1-295, London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, Cambridge, 1993.

[23] Hurewicz, Witold, and Henry. Wallman. *Dimension Theory*. Princeton: Princeton UP, 1941. Print. Princeton Mathematical Ser. ; 4.

[24] M. Kapovich, B. Kleiner, *Hyperbolic groups with low-dimensional boundary*, Ann. Sci. Ecole Norm. Sup. 33:5 (2000), 647-669.

[25] Kasia Jankiewicz, *Residual finiteness of certain 2-dimensional Artin groups* (2020) arXiv:2006.04717

[26] Kasia Jankiewicz, *Splittings of triangle Artin groups* (2021) arXiv:2103.01343

[27] N.Lebedeva, *Dimensions of products of hyperbolic spaces*, (2006) ArXiv: math.DG/0609463.

[28] M.Lohrey and G. Senizergues, *When Is a Graph Product of Groups Virtually-Free?*, Communications in Algebra,35, no.2 ,617-621 (2007).

[29] R. C. Lyndon, *Cohomology theory of groups with a single defining relation*, Ann. of Math. 52 (1950), 650-665.

[30] R. Lyndon and P. Schupp, *Combinatorial group theory*, Springer-Verlag, 1977.

[31] J. M. Mackay and A. Sisto, *Embedding relatively hyperbolic groups in products of trees*, Algebr. Geom. Topol. 13 (2013), no. 4, 2261-2282.

[32] D. Matsnev, *Asymptotic dimension of one relator groups*, (2006) ArXiv: math.GR/0607525

[33] Denis Osin, *Asymptotic dimension of relatively hyperbolic groups*, International Mathematics Research Notices, 2005, no. 35, 2143-2161.

[34] A. N. Platonov. *An isoperimetric function of the Baumslag Gersten group. Vestnik Moskov.* Mekh., 3:1217, 70, 2004. Translation in Moscow Univ. Math. Bull. 59 (2004)

[35] J.P.Serre, *Trees*, Springer-Verlag (1980).

[36] L.Sledd, *Assouad-Nagata dimension of C'(1/6) groups*, (2019) ArXiv:1909.06646

[37] Stallings, John. *Groups of dimension 1 are locally free.* Bull. Amer. Math. Soc. 74 (1968), no. 2, 361-364.

[38] N.Wright, *Finite asymptotic dimension for CAT(0) cube complexes*, Geom. Topol. 16 (2012), 527–554.

[39] Guoliang Yu, *The Novikov conjecture for groups with finite asymptotic dimension*, Ann. of Math. (2) 147 (1998), no. 2, 325-355.

[40] P.S. Aleksandrov, *On the dimension of normal spaces* Proc. Royal. Soc. London Ser. A , 189 (1947) pp. 11–39

[41] Arthur C. Bartels, *Squeezing and higher algebraic K-theory*, K-Theory 28 (2003), no. 1, 19–37. MR MR1988817 (2004f:19006)

[42] Gunnar Carlsson and Boris Goldfarb, *On homological coherence of discrete groups*, J. Algebra 276 (2004), no. 2, 502–514. MR MR2058455 (2005a:20078)

[43] A. Dranishnikov, S. Ferry, and S. Weinberger, *An etale approach to the Novikov conjecture*, Preprint of MPIfM2005-30, on-line at http://www.mpim-bonn.mpg.de/html/preprints/preprints.html (2005)

[44] A. Dranishnikov, M. Zarichnyi, *Universal spaces for asymptotic dimension*, Topology Appl. 140 (2–3) (2004) 203–225, MR MR2074917 (2005e:54032).

[45] L. Ji, *Asymptotic dimension of arithmetic groups*, preprint, 2003

[46] Lizhen Ji, *Asymptotic dimension and the integral K-theoretic Novikov conjecture for arithmetic groups*, Journal of Differential Geometry 68 (2004), no. 3, 535–544.

[47] P.S. Urysohn, *Works on topology and other areas of mathematics* , 1 , Moscow-Leningrad (1951) (In Russian)

[48] Michael W. Davis, *The geometry and topology of Coxeter groups*, London Mathematical Society Monographs Series, vol. 32, Princeton University Press, Princeton, NJ, 2008. MR 2360474

[49] Elements de mathematique. Fasc. XXXIV. Groupes et algebres de Lie. Chapitre IV: Groupes de Coxeter et systemes de Tits. Chapitre V: Groupes engendres par des reflexions. Chapitre VI: Systemes de racines. Actualites Scientifiques et Industrielles, No. 1337, Hermann, Paris, 1968.

[50] E. Godelle, L. Paris. $K(\pi, 1)$ *and word problems for infinite type Artin-Tits groups, and applications to virtual braid groups.* Preprint, arXiv:1007.1365.

[51] E. Godelle, L. Paris. *Basic Questions on Artin-Tits groups.* arXiv:1105.1048

[52] Tadeusz Januszkiewicz, *For Coxeter groups $z^{|g|}$ is a coefficient of a uniformly bounded representation*, Fund. Math. 174 (2002), no. 1, 79–86. MR MR1925487 (2003f:20061)

[53] M.Hagen, A.Martin, A.Sisto, *Extra-large type Artin groups are hierarchically hyperbolic* (2021)arXiv:2109.04387

[54] L. Paris. *Universal cover of Salvetti's complex and topology of simplicial arrangements of hyperplanes*. Trans. Amer. Math. Soc. 340 (1993), no. 1, 149–178.

[55] Panagiotis Tselekidis, *Asymptotic Dimension of Graphs of Groups and One Relator Groups*, (2022), to appear in the journal of Algebraic and Geometric Topology, arXiv: 1905.07925

[56] Panagiotis Tselekidis, *Asymptotic dimension of Artin groups and a new upper bound for Coxeter groups*, (2022), arXiv:2203.05746

[57] Panagiotis Tselekidis, *A new upper bound for the Asymptotic Dimension of RACGs*, (2021) (submitted), arXiv:2112.09440

[58] Jason Behrstock, Mark F. Hagen, and Alessandro Sisto. *Asymptotic dimension and small cancellation for hierarchically hyperbolic spaces and groups*. Proc. Lond. Math. Soc. (3), 114(5):890–926, 2017

arXiv:math.GT/2004.11650 (2020).

[59] Mladen Bestvina and Geoffrey Mess. *The boundary of negatively curved groups*. J. Amer. Math. Soc., 4(3):469–481, 1991.

[60] Martin R. Bridson and Andr´e Haefliger. *Metric spaces of non-positive curvature*, volume 319 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1999.

[61] John R. Stallings. *On torsion-free groups with infinitely many ends*. Ann. of Math. (2), 88:312–334, 1968