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Quantum quasi-neutral limits and isothermal Euler equations

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Abstract

We provide a rigorous justification of the semiclassical quasi-neutral and the quantum many-body limits to the isothermal Euler equations. We consider the nonlinear Schrödinger–Poisson–Boltzmann system under a quasi-neutral scaling and establish the convergence of its solutions to the isothermal Euler equations. Different from the previous results that dealt with the linear Poisson equations, the system under our consideration accounts for the exponential nonlinearity in the potential. A modulated energy method is adopted, allowing us to derive the stability estimates and asymptotics. Furthermore, we focus our analysis on the many-body quantum problem via the von Neumann equation and establish a mean-field limit in one dimension by using Serfaty's functional inequalities, thus connecting the quantum many-body dynamics with the macroscopic hydrodynamic equations. A refined analysis of the quasi-neutral scaling for the massless systems is presented, and the well-posedness of the underlying quantum dynamics is established. Moreover, the construction of well-prepared admissible initial data is obtained. Our results provide a rigorous mathematical analysis for the derivation of quantum hydrodynamic models and their limits, contributing to the broader understanding of interactions between quantum mechanics and compressible fluid dynamics.

1. Introduction

We are concerned with the rigorous justification of the semi-classical and quantum many-body quasi-neutral limits to the macroscopic systems of nonlinear partial differential equations. Consider the following d-dimensional (d-D) *Schrödinger–Poisson–Boltzmann system with quasi-neutral scaling*:

$$\begin{cases} i\hbar\partial_t\psi_{\varepsilon,\hbar} = -\frac{\hbar^2}{2}\Delta\psi_{\varepsilon,\hbar} + V_{\varepsilon,\hbar}\psi_{\varepsilon,\hbar}, & x\in\mathbb{T}^d,\ t>0, \\ -\varepsilon\Delta V_{\varepsilon,\hbar} = |\psi_{\varepsilon,\hbar}|^2 - e^{V_{\varepsilon,\hbar}}, & x\in\mathbb{T}^d,\ t>0, \\ \psi_{\varepsilon,\hbar}|_{t=0} = \psi^{\mathrm{in}}_{\varepsilon,\hbar}, & x\in\mathbb{T}^d. \end{cases} \tag{1.1}$$

The unknown $\psi_{\varepsilon,\hbar}(t,\cdot)$ is a time-dependent element of the Hilbert space $\mathfrak{H} := L^2(\mathbb{T}^d)$ such that $\int_{\mathbb{T}^d}|\psi_{\varepsilon,\hbar}(t,x)|^2\,\mathrm{d}x = 1$. The symbol $\hbar$ represents the Planck constant and should be regarded as a small parameter. The 1-body quantum Hamiltonian $H_{\varepsilon,\hbar}$ is the differential operator given by

$$H_{\varepsilon,\hbar} := -\frac{\hbar^2}{2}\Delta + V_{\varepsilon,\hbar},$$

so that (1.1) can be written as

$$\begin{cases} i\hbar\partial_t\psi_{\varepsilon,\hbar} = H_{\varepsilon,\hbar}\psi_{\varepsilon,\hbar}, & x\in\mathbb{T}^d,\ t>0,\\ -\varepsilon\Delta V_{\varepsilon,\hbar} = |\psi_{\varepsilon,\hbar}|^2 - m_{\varepsilon,\hbar}, & x\in\mathbb{T}^d,\ t>0,\\ \psi_{\varepsilon,\hbar}|_{t=0} = \psi^{\text{in}}_{\varepsilon,\hbar}, & x\in\mathbb{T}^d, \end{cases}$$

where $m_{\varepsilon,\hbar} := e^{V_{\varepsilon,\hbar}}$.

Notice that this system is a variant of the much more well studied Schrödinger–Poisson system:

$$\begin{cases} i\hbar\partial_t\psi_{\varepsilon,\hbar} = -\frac{\hbar^2}{2}\Delta\psi_{\varepsilon,\hbar} + V_{\varepsilon,\hbar}\psi_{\varepsilon,\hbar}, & x\in\mathbb{T}^d,\ t>0,\\ -\varepsilon\Delta V_{\varepsilon,\hbar} = |\psi_{\varepsilon,\hbar}|^2 - 1, & x\in\mathbb{T}^d,\ t>0,\\ \psi_{\varepsilon,\hbar}|_{t=0} = \psi^{\text{in}}_{\varepsilon,\hbar}, & x\in\mathbb{T}^d. \end{cases} \tag{1.2}$$

The fundamental difference between (1.1) and (1.2) is encapsulated in the exponential nonlinearity in system (1.1), *i.e.* from the fact that the dynamics are coupled with the so called *Poisson–Boltzmann* equation, instead of the *linear Poisson* equation. This nonlinearity is the source of several mathematical obstructions: for instance, while the well-posedness theory of the linear Poisson equation:

$$-\Delta V = h - 1$$

is classical even for a probability measure h, a well-posedness theory has been developed only relatively recently for the Poisson–Boltzmann equation (see [6, 15]):

$$-\Delta V = h - e^V,$$

where $h\in L^p$ for arbitrary $p>1$ and dimension $d\geqslant 1$. Thus, the classical well-posedness results for the Schrödinger–Poisson system (established for instance in [3, 9]) are still inadequate for the present settings. Therefore, one of our purposes in this paper is to prove the well-posedness of system (1.1). For this purpose, we rely heavily on the elliptic PDE theory for the Poisson-Boltzmann equation developed in [15], where h is a given bounded function. The well-posedness of system (1.1) is summarized in the following theorem:

Theorem 1.1. *Let $d\in\{2,3\}$. Assume that*

$$\psi^{\text{in}}_{\varepsilon,\hbar}\in C^\infty(\mathbb{T}^d) \qquad \text{with } \int_{\mathbb{T}^d}\left|\psi^{\text{in}}_{\varepsilon,\hbar}(x)\right|^2\mathrm{d}x = 1. \tag{1.3}$$

Then, for each fixed $\varepsilon>0$ and $\hbar>0$, there exists a unique solution $\psi_{\varepsilon,\hbar}\in\mathrm{Lip}([0,T];H^2(\mathbb{T}^d))$ of the Cauchy problem:

$$\begin{cases} i\hbar\partial_t\psi_{\varepsilon,\hbar}(t,x) = -\frac{\hbar^2}{2}\Delta\psi_{\varepsilon,\hbar}(t,x) + V_{\varepsilon,\hbar}(t,x)\,\psi_{\varepsilon,\hbar}(t,x), & x\in\mathbb{T}^d,\ t>0,\\ -\varepsilon\Delta V_{\varepsilon,\hbar}(t,x) = |\psi_{\varepsilon,\hbar}(t,x)|^2 - e^{V_{\varepsilon,\hbar}(t,x)}, & x\in\mathbb{T}^d,\ t>0,\\ \psi_{\varepsilon,\hbar}|_{t=0} = \psi^{\text{in}}_{\varepsilon,\hbar}, & x\in\mathbb{T}^d. \end{cases} \tag{1.4}$$

As the main focus of this paper is on the semi-classical and quantum many-body limits, the proof of theorem 1.1 is postponed to section 5. Our first main result is the rigorous derivation of the isothermal Euler equations:

$$\begin{cases} \partial_t\rho + \mathrm{div}\,(\rho u) = 0,\\ \partial_t(\rho u) + \mathrm{div}\,(\rho u\otimes u) + \nabla\rho = 0, \end{cases} \tag{1.5}$$

as a combined semi-classical quasi-neutral limit. Before we state this main result, we remark that this problem has already been studied in the context of the usual Schrödinger–Poisson system (1.2) by Puel [25] for pure states and later by Rosenzweig [27] for general states where the density operator formulation of the Schrödinger–Poisson system, also known as the Hartree equations, is considered. Motivated by Rosenzweig's approach, we make use of the modulated energy method. However, the modulated energy has to be modified in comparison to [27]: The modulated energy used in [27] is the quantum analogue of the quantity introduced in [4], while the modulated energy studied in this paper is the quantum analogue of the quantity introduced in [18]. Jüngel-Wang [21] obtained combined semi-classical and quasi-neutral limits in the bipolar defocusing nonlinear Schrödinger–Poisson system in the whole space. The electron and current density defined by the solution of the Schrödinger–Poisson system converge to the solution of the compressible Euler equations with nonlinear pressure. We also refer to Golse–Paul [11] for the validity of the joint mean-field and classical limit of the quantum N-body dynamics leading to the pressureless Euler–Poisson system for factorized initial data whose first marginal has a monokinetic Wigner measure. Given a solution $\psi_{\varepsilon,\hbar}$ to system (1.1), consider the time-dependent trace-class operator $R_{\varepsilon,\hbar}(t) := |\psi_{\varepsilon,\hbar}(t)\rangle\langle\psi_{\varepsilon,\hbar}(t)|$ (here the bra-ket $|\psi\rangle\langle\psi|$ is the rank-1 projection on $\mathfrak{H}$ defined by $\varphi\mapsto\langle\psi,\varphi\rangle\psi$ in the conventional notation). The crux of the argument reduces to obtaining an evolution estimate on the following quantity, called *the modulated energy*:

$$\begin{aligned}\mathcal{E}_{\varepsilon,\hbar}(t) :=& \frac{1}{2}\sum_{j=1}^{d}\operatorname{tr}\big((i\hbar\partial_{x_j}+u^j)^2R_{\varepsilon,\hbar}(t)\big)+\frac{\varepsilon}{2}\int_{\mathbb{T}^d}|\nabla V_{\varepsilon,\hbar}(t,x)|^2\,\mathrm{d}x\\ &+\int_{\mathbb{T}^d}\Big(m_{\varepsilon,\hbar}(t,x)\log\big(\frac{m_{\varepsilon,\hbar}(t,x)}{\rho(t,x)}\big)-m_{\varepsilon,\hbar}(t,x)+\rho(t,x)\Big)\,\mathrm{d}x,\end{aligned}\tag{1.6}$$

where (ρ,u) is the solution of (1.5) and ρ denotes the macroscopic density solving the isothermal Euler equations, and u^j denotes the jth component of the vector field u. The *total energy*, denoted by $\mathcal{F}_{\varepsilon,\hbar}(t)$ and defined explicitly in (2.4) below, is a conserved quantity obtained from $\mathcal{E}_{\varepsilon,\hbar}(t)$ by plugging in $\rho=1$ and $u=0$. The quantum current $J_{\varepsilon,\hbar}$ and quantum density $\rho_{\varepsilon,\hbar}$ are defined by

$$J_{\varepsilon,\hbar}(t,x) := \hbar\,\mathrm{Im}\big(\overline{\psi}_{\varepsilon,\hbar}(t,x)\nabla\psi_{\varepsilon,\hbar}(t,x)\big),\qquad \rho_{\varepsilon,\hbar}(t,x) := |\psi_{\varepsilon,\hbar}|^2(t,x).$$

Our first main result is described in the following theorem:

Theorem 1.2 *Let $d\in\{2,3\}$. Let $(\rho,u)\in C^1([0,T];H^{s-1}(\mathbb{T}^d))\times\big(C([0,T);H^s(\mathbb{T}^d))\cap C^1([0,T)\times\mathbb{T}^d)\big)$ be the solution of (1.5) with initial data $(\rho_0,u_0)\in\big(H^s(\mathbb{T}^d)\cap\mathcal{P}(\mathbb{T}^d)\big)\times H^s(\mathbb{T}^d)$ and $\rho_0>0$ for some $s>0$ sufficiently large and $T>0$. Let $\psi_{\varepsilon,\hbar}(t,x)$ be the unique solution in theorem 1.1 with the initial data $\psi^{\mathrm{in}}_{\varepsilon,\hbar}(x)$ satisfying (1.3). Assume that there is some constant $\mathcal{F}_0$ such that $\mathcal{F}_{\varepsilon,\hbar}(0)\leqslant\mathcal{F}_0$ uniformly in $(\varepsilon,\hbar)$. Let $\mathcal{E}_{\varepsilon,\hbar}(t)$ be given by (1.6). Then*

$$\mathcal{E}_{\varepsilon,\hbar}(t)\leqslant e^{Ct}\left(\mathcal{E}_{\varepsilon,\hbar}(0)+\sqrt{\varepsilon}+\hbar^2\right)\qquad\text{for all } t\in[0,T],$$

where $C=C(\|u\|_{L^\infty_t W^{2,\infty}_x},\|\log\rho\|_{W^{1,\infty}_t H^1_x},\|\nabla(u\cdot\nabla\log\rho)\|_{L^\infty_t L^2_x},T,\mathcal{F}_0)$. Consequently,

$$\sup_{t\in[0,T]}\mathcal{E}_{\varepsilon,\hbar}(t)\underset{\varepsilon+\hbar\to0}{\longrightarrow}0$$

provided $\mathcal{E}_{\varepsilon,\hbar}(0)\underset{\varepsilon+\hbar\to0}{\longrightarrow}0$, and

$$(\rho_{\varepsilon,\hbar},J_{\varepsilon,\hbar})(t,\cdot)\rightharpoonup(\rho,\rho u)(t,\cdot)$$

for the narrow topology of signed Radon measures on $\mathbb{T}^d$, as $\varepsilon+\hbar\to0$ locally uniformly in time.

The initial data $\psi^{\mathrm{in}}_{\varepsilon,\hbar}$ can be constructed such that $\mathcal{E}_{\varepsilon,\hbar}(0)\underset{\varepsilon+\hbar\to0}{\longrightarrow}0$. We will elaborate on this in section 4.

The stability estimate and the implied weak convergence are valid for arbitrarily large-time intervals. However, the existence and uniqueness of a smooth solution to the isothermal Euler equations are known to be true only for a short-time interval.

Our second main result concerns the N-body problem. Given a configuration $X_N=(x_1,\cdots,x_N)\in\mathbb{T}^N$, denote by μ_{X_N} the empirical measure centered at X_N, *i.e.*,

$$\mu_{X_N} := \frac{1}{N}\sum_{j=1}^{N}\delta_{x_j}.$$

In variance with the quasi-neutral quantum many-body limit established in [27], our main result in this context is limited to the 1D settings. Let V_{ε,X_N} be the solution to the equation:

$$-\varepsilon\Delta V_{\varepsilon,X_N}=\mu_{X_N}-m_{\varepsilon,X_N}\qquad\text{with}\quad m_{\varepsilon,X_N}=e^{V_{\varepsilon,X_N}}.$$

The N-body quantum Hamiltonian is the differential operator acting on the Hilbert space $\mathfrak{H}^{\otimes N} = L^2(\mathbb{T}^N)$ given by

$$\begin{aligned}\mathscr{H}_{\varepsilon,\hbar,N}(X_N) &:= -\frac{\hbar^2}{2}\sum_{k=1}^{N}\Delta_{x_k} + N\int_{\mathbb{T}} V_{\varepsilon,X_N}(x)\, m_{\varepsilon,X_N}(x)\,\mathrm{d}x \\ &\quad + \frac{N}{2\varepsilon}\int_{\mathbb{T}\times\mathbb{T}} K(x-y)\,(\mu_{X_N} - m_{\varepsilon,X_N})^{\otimes 2}(\mathrm{d}x\mathrm{d}y) \\ &:= \mathscr{K}_{N,\hbar} + \mathscr{I}_{\varepsilon,X_N} + \mathscr{V}_{\varepsilon,X_N}.\end{aligned}$$

Here, K designates the Green function of the Laplacian. In 1D, the Green function admits the explicit formula $K(x) = \frac{x^2-|x|}{2}$. Note that the terms $\mathscr{I}_{\varepsilon,X_N}$ and $\mathscr{V}_{\varepsilon,X_N}$ are viewed as a multiplication operator in the variable X_N. The Cauchy problem for the N-body von Neumann equation is given by

$$\begin{cases} i\hbar\partial_t R_{\varepsilon,\hbar,N}(t) = [\mathscr{H}_{\varepsilon,\hbar,N}, R_{\varepsilon,\hbar,N}(t)], \\ R_{\varepsilon,\hbar,N}|_{t=0} = R^{\mathrm{in}}_{\varepsilon,\hbar,N}. \end{cases} \tag{1.7}$$

The unknown $R_{\varepsilon,\hbar,N}$ is a time-dependent symmetric density operator, and $[\cdot,\cdot]$ designates the commutator defined by $[A,B] := AB - BA$ for any given operators A and B. By a *density operator*, we mean a bounded operator R such that R is self-adjoint non-negative ($R = R^* \geqslant 0$) and $\mathrm{tr}(R) = 1$. By a *symmetric operator* on $\mathfrak{H}^{\otimes N}$ we mean an operator R_N such that, for all permutations $\sigma \in \mathfrak{S}_N$ ($\mathfrak{S}_N$ is the symmetric group on N elements),

$$U_\sigma R_N U_\sigma^* = R_N,$$

where U_σ is the operator on $\mathfrak{H}^{\otimes N}$ defined by

$$(U_\sigma \Psi_N)(x_1,\cdots,x_N) := \Psi_N(x_{\sigma^{-1}(1)},\cdots,x_{\sigma^{-1}(N)}) \qquad \text{for any } \Psi_N \in \mathfrak{H}^{\otimes N}.$$

We denote the class of density operators on $\mathfrak{H}$ by $\mathcal{D}(\mathfrak{H})$ and the class of symmetric density operators on $\mathfrak{H}^{\otimes N}$ by $\mathcal{D}_s(\mathfrak{H}^{\otimes N})$. If $R_N \in \mathcal{D}_s(\mathfrak{H}^{\otimes N})$ with kernel $k(X_N, Y_N)$, then $\rho_N \in L^1(\mathbb{T}^N)$ is denoted as the function defined by $\rho_N(X_N) := k(X_N, X_N)$. It is well known that ρ_N is a symmetric probability density on $\mathbb{T}^N$ (see the footnote pp 61–62 in [10] for more details). We call ρ_N the *density* associated with R_N. Hereafter, we denote by $\rho_{\varepsilon,\hbar,N}(t)$ the density associated with the operator $R_{\varepsilon,\hbar,N}(t)$. With these definitions, we can define the following modulated energy:

$$\begin{aligned}\mathcal{E}_{\varepsilon,\hbar,N}(t) &:= \frac{1}{2N}\sum_{j=1}^{N}\mathrm{tr}\big((i\hbar\partial_{x_j} + u(t,x_j))^2 R_{\varepsilon,\hbar,N}(t)\big) \\ &\quad + \frac{1}{2\varepsilon}\int_{\mathbb{T}^N}\Big(\int_{\mathbb{T}\times\mathbb{T}} K(x-y)\,(\mu_{X_N} - m_{\varepsilon,X_N})^{\otimes 2}(\mathrm{d}x\mathrm{d}y)\Big)\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N \\ &\quad + \int_{\mathbb{T}^N}\Big(\int_{\mathbb{T}}\big(m_{\varepsilon,X_N}(x)\log\big(\frac{m_{\varepsilon,X_N}(x)}{\rho(t,x)}\big) - m_{\varepsilon,X_N}(x) + \rho(t,x)\big)\,\mathrm{d}x\Big)\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N,\end{aligned}$$

where (ρ,u) is the solution of (1.5). The total energy of the system is denoted by $\mathcal{F}_{\varepsilon,\hbar,N}(t)$ (see (3.2) below). We can also define an N-body analogue of the current that is denoted by $J_{\varepsilon,\hbar,N}$ (see definition 3.1 below). Let $R_{\varepsilon,\hbar,N:1}(t)$ be its first marginal density (see definition 3.1 below for the definition of the marginal). Denote the density function and the current of $R_{\varepsilon,\hbar,N:1}(t)$ by $\rho_{\varepsilon,\hbar,N:1}(t,\cdot)$ and $J_{\varepsilon,\hbar,N:1}(t,\cdot)$, respectively.

Our second main result is summarized below:

Theorem 1.3 *Let $(\rho,u) \in C^1([0,T];H^{s-1}(\mathbb{T})) \times \big(C([0,T];H^s(\mathbb{T})) \cap C^1([0,T)\times\mathbb{T})\big)$ be the solution of (1.5) with initial data $(\rho_0,u_0) \in \big(H^s(\mathbb{T}) \cap \mathcal{P}(\mathbb{T})\big) \times H^s(\mathbb{T})$ and $\rho_0 > 0$ for some $s > 0$ sufficiently large and $T > 0$. Let $R^{\mathrm{in}}_{\varepsilon,\hbar,N} \in \mathcal{D}_s(\mathfrak{H}^{\otimes N})$ and $\mathrm{tr}\big((-\Delta_N)^2 R^{\mathrm{in}}_{\varepsilon,\hbar,N}\big) < \infty$. Let $R_{\varepsilon,\hbar,N}(t)$ be the solution to (1.7) (ensured by lemma 5.2). Assume that there is some constant $\mathcal{F}_0$ such that $\mathcal{F}_{\varepsilon,\hbar,N}(0) \leqslant \mathcal{F}_0$ uniformly in $(\varepsilon,\hbar,N)$. Then*

$$\mathcal{E}_{\varepsilon,\hbar,N}(t) \leqslant e^{Ct}\Big(\mathcal{E}_{\varepsilon,\hbar,N}(0) + N^{-\lambda} + \sqrt{\varepsilon} + \frac{e^{\frac{1}{\varepsilon}}}{\varepsilon N^2} + \hbar^2\Big) \qquad \text{for all } t \in [0,T],$$

where $C = C(T, \|u\|_{L_t^\infty W_x^{2,\infty}}, \|\log(\rho)\|_{W_{t,x}^{1,\infty}}, \mathcal{F}_0)$ *and* $\lambda > 0$*. Consequently,*

$$\sup_{t\in[0,T]} \mathcal{E}_{\varepsilon,\hbar,N}(t) \underset{\varepsilon+\hbar+\frac{1}{N}\to 0}{\longrightarrow} 0,$$

provided $\mathcal{E}_{\varepsilon,\hbar,N}(0) \underset{\varepsilon+\hbar+\frac{1}{N}\to 0}{\to} 0$ *and* $\varepsilon = \varepsilon(N)$ *is such that* $\frac{e^{\frac{1}{\varepsilon}}}{\varepsilon N^2} \underset{N\to\infty}{\to} 0$*, and*

$$(\rho_{\varepsilon,\hbar,N:1}, J_{\varepsilon,\hbar,N:1})(t,\cdot) \rightharpoonup (\rho, \rho u)(t,\cdot)$$

for the narrow topology of signed Radon measures on $\mathbb{T}$*, as* $\varepsilon + \hbar + \frac{1}{N} \to 0$ *locally uniformly in time.*

A curious feature of the proof of theorem 1.3 is that it exploits the commutator estimates of Serfaty [28]. These commutator estimates also played a decisive role in the derivation of the incompressible Euler equations as a mean-field quasi-neutral limit in [20] or as a quantum many-body quasi-neutral limit in [27]. However, this approach, which is a reminiscent of the *renormalized energy* method, is absent from the literature on the mean-field quasi-neutral limit of massless Vlasov–Poisson (VP)-like equations; in this sense, it reflects a novelty of the method. The constraint that $\varepsilon(N)$ vanishes sufficiently slowly in N is natural. Note, however, that, in our setting, the vanishing rate of ε is logarithmic in N, whereas it is polynomial in N in the usual quasi-neutral limit (considered, for example, in [27]). This slower decay rate emerges from the L^∞ bound on $e^{V_{\varepsilon,X_N}}$ in the argument.

To put our results in an appropriate context, we close this introduction by providing a brief overview of the classical quasi-neutral limit. There has been an extensive study of the classical quasi-neutral limit from the VP equations with massless electrons (VPMEs), also known as the VP system for ions, that is a mathematical model from plasma physics, which can be described through the following kinetic equations:

$$\begin{cases} \partial_t f + \xi\cdot\nabla_x f - \nabla V\cdot\nabla_\xi f = 0, \\ -\Delta V = \int_{\mathbb{R}^d} f(t,x,\xi)\,\mathrm{d}\xi - e^V, \end{cases} \tag{1.8}$$

with $f(0,x,\xi) = f^0 \geqslant 0$ and $\int_{\mathbb{T}^d\times\mathbb{R}^d} f^0(x,\xi)\,\mathrm{d}x\mathrm{d}\xi = 1$. The unknown $f\colon [0,T]\times\mathbb{T}^d\times\mathbb{R}^d \to \mathbb{R}$ is a time-dependent probability density $f(t,\cdot,\cdot) \in \mathcal{P}(\mathbb{T}^d\times\mathbb{R}^d)$. This system is a variant of the well-known VP system that reads

$$\begin{cases} \partial_t f + \xi\cdot\nabla_x f - \nabla V\cdot\nabla_\xi f = 0, \\ -\Delta V = \int_{\mathbb{R}^d} f(t,x,\xi)\,\mathrm{d}\xi - 1, \end{cases} \tag{1.9}$$

with $f(0,x,\xi) = f^0 \geqslant 0$ and $\int_{\mathbb{T}^d\times\mathbb{R}^d} f^0(x,\xi)\,\mathrm{d}x\mathrm{d}\xi = 1$.

As before, the notable difference between (1.8) and (1.9) is reflected in the fact that the Poisson equation coupled to the Vlasov equation is nonlinear in (1.8), where it is linear and explicitly solvable in (1.9). The variance between (1.8) and (1.9) is demonstrated in a decisive manner already at the level of well-posedness: the well-posedness theory on the entire space and the periodic case of the VP system has been established in the 1990s in Lions–Perthame [22] and Batt–Rein [1], respectively; also see [16] for an overview of recent developments in the well-posedness theory of the Vlasov equations of this type. Only much later, the question of existence and uniqueness for the case of the VPME system (1.8) has been settled: the existence of global weak solutions has been proved in [19] in the periodic 1D case. Later, global well-posedness has been proved on the torus in [15]. Endowed with this well-posedness, it is a natural question to consider hydrodynamical limits associated with this system, particularly the quasi-neutral limit. As was done in the quantum regime considered before, we write the VP system and the VPME system with quasi-neutral scaling. The VPME system with quasi-neutral scaling reads

$$\begin{cases} \partial_t f_\varepsilon + \xi\cdot\nabla_x f_\varepsilon - \nabla V_\varepsilon\cdot\nabla_\xi f_\varepsilon = 0, \\ -\varepsilon\Delta V_\varepsilon = \int_{\mathbb{R}^d} f_\varepsilon(t,x,\xi)\,\mathrm{d}\xi - e^{V_\varepsilon}, \end{cases} \tag{1.10}$$

with $f_\varepsilon(0,x,\xi) = f_\varepsilon^0 \geqslant 0$ and $\int_{\mathbb{T}^d\times\mathbb{R}^d} f_\varepsilon^0(x,\xi)\,\mathrm{d}x\mathrm{d}\xi = 1$, while the VP system with quasi-neutral scaling reads

$$\begin{cases} \partial_t f_\varepsilon + \xi\cdot\nabla_x f_\varepsilon - \nabla V_\varepsilon\cdot\nabla_\xi f_\varepsilon = 0, \\ -\varepsilon\Delta V_\varepsilon = \int_{\mathbb{R}^d} f_\varepsilon(t,x,\xi)\,\mathrm{d}\xi - 1, \end{cases} \tag{1.11}$$

with $f_\varepsilon(0,x,\xi) = f_\varepsilon^0 \geqslant 0$ and $\int_{\mathbb{T}^d\times\mathbb{R}^d} f_\varepsilon^0(x,\xi)\,\mathrm{d}x\mathrm{d}\xi = 1$.

In the classical quasi-neutral limit, one seeks to study the convergence of f_ε as $\varepsilon \to 0$. Formal considerations suggest, as the limit system, the incompressible Euler equations in the 2D VP case and the isothermal Euler equations in the VPME case. More precisely, given a solution $f_\varepsilon(t,x,\xi)$ to system (1.10) or (1.11), consider the density

$$\rho_\varepsilon(t,x) := \int_{\mathbb{R}^d} f_\varepsilon(t,x,\xi)\,\mathrm{d}\xi \tag{1.12}$$

and the current

$$J_\varepsilon(t,x) := \int_{\mathbb{R}^d} \xi f_\varepsilon(t,x,\xi)\,\mathrm{d}\xi. \tag{1.13}$$

Then, in the VPME case, one expects the limit system to be the isothermal Euler equations (1.5) in the sense that the convergences: $\rho_\varepsilon(t,\cdot) \to \rho(t,\cdot)$ and $J_\varepsilon(t,\cdot) \to (\rho u)(t,\cdot)$ are propagated in time. In the VP case, the expected limit system is the incompressible Euler equations:

$$\begin{cases} \rho = 1, \\ \partial_t u + u\cdot\nabla_x u + \nabla_x p = 0, \\ \operatorname{div} u = 0. \end{cases} \tag{1.14}$$

Therefore, the quasi-neutral limit offers a derivation of fluid equations from kinetic equations. The rigorous justification of the quasi-neutral limit for VP has been initiated in Brenier–Grenier [5] and Brenier [4]. See also Golse–Saint–Raymond [12, 13] for an approach involving compactness arguments. Since we are mostly concerned with the quasi-neutral limit for quantum analogues of VPME, we will not dwell further on this limit for VP. Various works have been dedicated to the study of the derivation of (1.5) from (1.10). The quasi-neutral limit has been studied in Han-Kwan [18] by using a modulated energy approach inspired by Brenier's method [4]. As already explained, this method will also play an important role in the present work. It should be mentioned that this approach can be extended to cover other variants of the Vlasov system (1.8), including in the presence of magnetic fields. The 1D quasi-neutral limit, along with the combined 1D mean-field quasi-neutral limit, has been analyzed by Han-Kwan–Iacobelli in [19, 20]. Higher dimensions have been studied by Griffin–Pickering–Iacobelli in [14], for mollified empirical measures, with a mollification constant depending on the parameter ε.

Theorem 1.2 is the 1-body semi-classical analogue of the main result in [18], while theorem 1.3 is the N-body quantum analogue of the main result in [18]. The latter limit, being a singular limit, requires the use of the functional inequalities in [28], and thus part of the novelty of the argument in the proof is reflected in the observation that these functional inequalities can be incorporated within the framework of quantum massless dynamics. The fact that we are working in the quantum regime introduces technical difficulties at several different levels: the well-posedness of the underlying dispersive dynamics (addressed in theorem 1.1), the construction of admissible initial data (addressed in section 4), and, most importantly, the calculation of the quantum analogues of the modulated energy (addressed through theorems 1.2 and 1.3).

The paper is organized as follows. In section 2, we calculate the time derivative of the modulated energy $\mathcal{E}_{\varepsilon,\hbar}$ and, as a result, obtain a Grönwall estimate for this quantity. In section 3, we renormalize this argument by using the functional inequalities discovered in [28], thus obtaining a stability estimate for $\mathcal{E}_{\varepsilon,\hbar,N}$. In section 4, the well-prepared initial data $\psi^{\mathrm{in}}_{\varepsilon,\hbar}$ and $R^{\mathrm{in}}_{\varepsilon,\hbar,N}$ are constructed for which $\mathcal{E}_{\varepsilon,\hbar}(0) \underset{\varepsilon+\hbar\to 0}{\to} 0$ and $\mathcal{E}_{\varepsilon,\hbar,N}(0) \underset{\varepsilon+\hbar+\frac{1}{N}\to 0}{\to} 0$, respectively. Finally, in section 5, we prove the well-posedness of system (1.1), based on [15]. As will be explained, the well-posedness of system (1.7) is a direct consequence of the Kato-Rellich theorem.

2. The 1-body semi-classical limit

This section concerns the derivation of the isothermal Euler equations:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) + \nabla\rho = 0, \end{cases} \tag{2.1}$$

as a limit of (1.1) when $\varepsilon+\hbar\to 0$. Observe that an alternative formulation of (2.1) is

$$\begin{cases}\partial_t \log\rho + \operatorname{div} u + u\cdot\nabla\log\rho = 0,\\ \partial_t u + u\cdot\nabla_x u + \nabla\log\rho = 0.\end{cases} \tag{2.2}$$

The well-posedness theory of the isothermal Euler equations is summarized in the following theorem, which can be found in [23].

Theorem 2.1. *Let $(\rho_0,u_0)\in(H^s(\mathbb{T}^d))^2$ for some $s>\frac{d}{2}+1$, with $\rho_0(x)>0$ for $x\in\mathbb{T}^d$. Then there is $T_*>0$ such that there exists a unique classical solution*

$$(\rho,u)\in C^1([0,T_*);H^{s-1}(\mathbb{T}^d))\times(C([0,T_*);H^s(\mathbb{T}^d))\cap C^1([0,T_*)\times\mathbb{T}^d))$$

of the Cauchy problem for system (2.2) with the initial data:

$$(\rho,u)\,|_{t=0} = (\rho_0(x),u_0(x))\,. \tag{2.3}$$

Moreover, $\log\rho\in C([0,T_*);H^s(\mathbb{T}^d))\cap C^1([0,T_*)\times\mathbb{T}^d)$.

Remark 2.2. The well-posedness of the Cauchy problem (2.2) and (2.3) is stated in [18] on the whole space. However, the proof extends with no difficulties to the periodic settings considered here. See also [23].

We briefly review the elliptic theory necessary for the study of the nonlinear elliptic equation governing V, as developed in [15]. First, we recap the existence and uniqueness theory and the Schauder-type estimates, as summarized in the following:

Lemma 2.3 ([15], proposition 3.1). *Assume that $d\in\{2,3\}$ and $h\in L^\infty(\mathbb{T}^d)\cap\mathcal{P}(\mathbb{T}^d)$.Then there exist both a unique $\tilde V\in H^1(\mathbb{T}^d)$ with zero mean and $\hat V\in H^1(\mathbb{T}^d)$ satisfying*

$$-\Delta\tilde V = h-1,\quad -\Delta\hat V = 1-e^{\hat V+\tilde V}.$$

Furthermore, the following estimates hold:

(i) *For all $\alpha\in(0,1)$,*

$$\|\tilde V\|_{C^{1,\alpha}} \leqslant C_{\alpha,d}\left(1+\|h\|_\infty\right),$$
$$\|\hat V\|_{C^{1,\alpha}} \leqslant C_{\alpha,d}\exp\left(C_{\alpha,d}(1+\|h\|_{\frac{d+2}{d}})\right),$$

for some constant $C_{\alpha,d}>0$.

(ii) *For all $\alpha\in(0,\frac{1}{5}]$ if $d=3$ and $\alpha\in(0,1)$ if $d=2$,*

$$\|\hat V\|_{C^{2,\alpha}} \leqslant C_{\alpha,d}\exp\exp\left(C_{\alpha,d}(1+\|h\|_{\frac{d+2}{d}})\right),$$

for some constant $C_{\alpha,d}>0$.

Lemma 2.4 ([17], proposition 3.2). *Assume that $\rho\in\mathcal{P}(\mathbb{T})$ is a probability measure. Then there exist both a unique $\tilde V\in H^1(\mathbb{T})$ with zero mean and $\hat V\in H^1(\mathbb{T})$ satisfying*

$$-\tilde V'' = \rho-1,\quad -\hat V'' = 1-e^{\tilde V+\hat V}.$$

Moreover, the following estimates hold:

$$\|\tilde V+\hat V\|_\infty\leqslant 1,\qquad \|\hat V'\|_{\mathrm{Lip}}\leqslant 1.$$

Remark 2.5 It can be checked directly that, subject to the assumptions of lemmas 2.3 or 2.4, $e^{\tilde V(x)+\hat V(x)}$ is a probability density, *i.e.*, $\int_{\mathbb{T}^d} e^{\tilde V(x)+\hat V(x)}\,\mathrm{d}x = 1$.

Remark 2.6 According to lemma 2.1 in [2], given $h\in L^\infty(\mathbb{T}^d)\cap\mathcal{P}(\mathbb{T}^d)$, there exists a unique solution $V\in H^1(\mathbb{T})$ of the problem: $-\Delta V = h-e^V$. Therefore, if h and $(\tilde V,\hat V)$ are as in lemma 2.3, we can represent the unique solution V as $V=\tilde V+\hat V$. The same remark applies when $d=1$ and $\rho\in\mathcal{P}(\mathbb{T})$ is a probability measure. Hereafter, we denote $V:=\tilde V+\hat V$, where $\tilde V$ and $\hat V$ are the solutions guaranteed by lemmas 2.3 and 2.4, respectively.

Remark 2.7 If we further assume that $h \in C^\infty(\mathbb{T}^d)$ in lemma 2.3, by the Schauder estimates and a standard iteration argument, then we see that $V \in C^\infty(\mathbb{T}^d)$ with the estimate:

$$\|V\|_{C^{k,\alpha}} \leqslant C$$

for some $C = C(k, \alpha, d, \|h\|_\infty)$.

Recall that the modulated energy is given by

$$\mathcal{E}_{\varepsilon,\hbar}(t) := \underbrace{\frac{1}{2}\sum_{j=1}^{d} \operatorname{tr}\big((i\hbar\partial_{x_j} + u^j)^2 R_{\varepsilon,\hbar}(t)\big)}_{:=\mathcal{K}_{\varepsilon,\hbar}(t)} + \underbrace{\frac{\varepsilon}{2}\int_{\mathbb{T}^d} |\nabla V_{\varepsilon,\hbar}(t,x)|^2 \,\mathrm{d}x}_{:=\mathcal{V}_{\varepsilon,\hbar}(t)}$$
$$+ \int_{\mathbb{T}^d} \Big(m_{\varepsilon,\hbar}(t,x)\log\big(\frac{m_{\varepsilon,\hbar}(t,x)}{\rho(t,x)}\big) - m_{\varepsilon,\hbar}(t,x) + \rho(t,x)\Big)\,\mathrm{d}x$$

and the total energy is given by

$$\mathcal{F}_{\varepsilon,\hbar}(t) := \frac{1}{2}\operatorname{tr}\big(-\hbar^2\Delta R_{\varepsilon,\hbar}(t)\big) + \frac{\varepsilon}{2}\int_{\mathbb{T}^d} |\nabla V_{\varepsilon,\hbar}(t,x)|^2 \,\mathrm{d}x + \int_{\mathbb{T}^d} V_{\varepsilon,\hbar}(t,x)\, m_{\varepsilon,\hbar}(t,x)\,\mathrm{d}x. \tag{2.4}$$

Remark 2.8. It is classical that, given probability densities $\mu, \nu \in \mathcal{P}(\mathbb{T}^d) \cap L^1(\mathbb{T}^d)$, the associated relative entropy is non-negative: $\int_{\mathbb{T}^d} \mu \log\big(\frac{\mu}{\nu}\big)\,\mathrm{d}x \geqslant 0$. This explains why the last terms in the definitions of $\mathcal{E}_{\varepsilon,\hbar}(t)$ and $\mathcal{F}_{\varepsilon,\hbar}(t)$ are non-negative and, as a result, $\mathcal{E}_{\varepsilon,\hbar}(t) \geqslant 0$.

Remark 2.9. Note that

$$\mathcal{K}_{\varepsilon,\hbar}(t) = \frac{1}{2}\int_{\mathbb{T}^d} |(i\hbar\nabla + u)\psi_{\varepsilon,\hbar}|^2(t,x)\,\mathrm{d}x$$

and that $R_{\varepsilon,\hbar}(t)$ is determined by the Cauchy problem for the Hartree-type system:

$$\begin{cases} i\hbar\partial_t R_{\varepsilon,\hbar}(t) = [H_{\varepsilon,\hbar}, R_{\varepsilon,\hbar}(t)], \\ -\varepsilon\Delta V_{\varepsilon,\hbar} = |\psi_{\varepsilon,\hbar}|^2 - e^{V_{\varepsilon,\hbar}}, \\ \psi_{\varepsilon,\hbar}|_{t=0} = \psi^{\mathrm{in}}_{\varepsilon,\hbar}. \end{cases}$$

It is instructive to invoke the Hartree formulation and the trace, since it clarifies the underlying algebraic structure. With this notation, the quantum current can be equivalently defined by duality as the unique vector field $J_{\varepsilon,\hbar}$ such that, for each $a \in W^{1,\infty}(\mathbb{T}^d;\mathbb{R}^d)$,

$$\int_{\mathbb{T}^d} a(x)\cdot J_{\varepsilon,\hbar}(t,x)\,\mathrm{d}x = \frac{1}{2}\sum_k \operatorname{tr}\big((-i\hbar\partial_{x_k}) \vee a^k R_{\varepsilon,\hbar}(t)\big).$$

Before proving the stability estimate for $\mathcal{E}_{\varepsilon,\hbar}(t)$ as stated in theorem 1.2, we observe that the density is governed by an evolution equation and the conservation of the total energy $\mathcal{F}_{\varepsilon,\hbar}(t)$. In the forthcoming calculation, we frequently use the anticommutator which is denoted by $\vee$ and defined by $A \vee B := AB + BA$ for any given operators A, B.

For any (scalar- or vector-valued) $f \in C^\infty(\mathbb{T}^d)$ viewed as a multiplication operator, we will repeatedly use

$$\frac{i}{\hbar}\Big[-\frac{\hbar^2}{2}\Delta, f\Big] = \frac{1}{2}\sum_{k=1}^{d}(-i\hbar\partial_{x_k}) \vee (\partial_{x_k} f), \tag{2.4a}$$

$$\frac{i}{\hbar}\big[f, -\hbar^2\Delta\big] = \sum_{k=1}^{d}(-i\hbar\partial_{x_k}) \vee (-\partial_{x_k} f). \tag{2.4b}$$

Indeed, $[f, -i\hbar\partial_{x_k}] = i\hbar\,\partial_{x_k} f$ and $[A^2, f] = A \vee [A, f]$.

Lemma 2.10. *Let $R_{\varepsilon,\hbar}(t)$ and $\rho_{\varepsilon,\hbar}(t,\cdot)$ be as in theorem 1.2. Then*

$$\partial_t \rho_{\varepsilon,\hbar} + \operatorname{div} J_{\varepsilon,\hbar} = 0. \tag{2.5}$$

Proof. For any $a \in C^\infty(\mathbb{T}^d;\mathbb{R})$, we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} a(x)\rho_{\varepsilon,\hbar}(t,x)\,\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}\big(aR_{\varepsilon,\hbar}(t)\big)\\ &= \frac{1}{i\hbar}\mathrm{tr}\big(a[-\frac{\hbar^2}{2}\Delta + V_{\varepsilon,\hbar},\, R_{\varepsilon,\hbar}(t)]\big)\\ &= \frac{i}{\hbar}\mathrm{tr}\big([-\frac{\hbar^2}{2}\Delta,\, a]R_{\varepsilon,\hbar}(t)\big)\\ &= \frac{1}{2}\sum_{k=1}^{d}\mathrm{tr}\big((-i\hbar\partial_{x_k})\vee(\partial_{x_k}a)R_{\varepsilon,\hbar}(t)\big)\\ &= \int_{\mathbb{T}^d}\nabla a(x)\cdot J_{\varepsilon,\hbar}(t,x)\,\mathrm{d}x,\end{aligned}$$

where we have used the equations in remark 2.9. □

Lemma 2.11. *Let $R_{\varepsilon,\hbar}(t)$ be as in theorem 1.2. Then $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{\varepsilon,\hbar}(t) = 0$.*

Proof. We first compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}\left(-\hbar^2\Delta R_{\varepsilon,\hbar}(t)\right) &= \frac{1}{i\hbar}\mathrm{tr}\big(-\hbar^2\Delta[-\frac{1}{2}\hbar^2\Delta + V_{\varepsilon,\hbar}, R_{\varepsilon,\hbar}(t)]\big)\\ &= \frac{i}{\hbar}\mathrm{tr}\big([-\frac{1}{2}\hbar^2\Delta + V_{\varepsilon,\hbar}, -\hbar^2\Delta]R_{\varepsilon,\hbar}(t)\big)\\ &= \frac{i}{\hbar}\mathrm{tr}\big([V_{\varepsilon,\hbar}, -\hbar^2\Delta]R_{\varepsilon,\hbar}(t)\big),\end{aligned}$$

where the equations in remark 2.9 have been used.
By (2.4b) with $f = V_{\varepsilon,\hbar}$,

$$\frac{i}{\hbar}\left[V_{\varepsilon,\hbar}, -\hbar^2\Delta\right] = \sum_{k=1}^{d}(-i\hbar\partial_{x_k})\vee(-\partial_{x_k}V_{\varepsilon,\hbar}),$$

so that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}\left(-\hbar^2\Delta R_{\varepsilon,\hbar}(t)\right) &= \frac{i}{2\hbar}\mathrm{tr}\big([V_{\varepsilon,\hbar}, -\hbar^2\Delta]R_{\varepsilon,\hbar}(t)\big)\\ &= \frac{1}{2}\sum_{k=1}^{d}\mathrm{tr}\big((-i\hbar\partial_{x_k})\vee(-\partial_{x_k}V_{\varepsilon,\hbar})R_{\varepsilon,\hbar}(t)\big)\\ &= -\int_{\mathbb{T}^d}(\nabla V_{\varepsilon,\hbar}\cdot J_{\varepsilon,\hbar})(t,x)\,\mathrm{d}x. \end{aligned}\tag{2.6}$$

Define

$$G(V_{\varepsilon,\hbar}) = V_{\varepsilon,\hbar}e^{V_{\varepsilon,\hbar}} - e^{V_{\varepsilon,\hbar}}.$$

Then it follows from (2.5) that

$$\begin{aligned}-\int_{\mathbb{T}^d}\nabla V_{\varepsilon,\hbar}\cdot J_{\varepsilon,\hbar}\,\mathrm{d}x &= \int_{\mathbb{T}^d}\mathrm{div}J_{\varepsilon,\hbar}\cdot V_{\varepsilon,\hbar}\,\mathrm{d}x = -\int_{\mathbb{T}^d}\partial_t\rho^\varepsilon\cdot V_{\varepsilon,\hbar}\,\mathrm{d}x\\ &= \varepsilon\int_{\mathbb{T}^d}\Delta\partial_t V_{\varepsilon,\hbar}\cdot V_{\varepsilon,\hbar}\,\mathrm{d}x - \int_{\mathbb{T}^d}e^{V_{\varepsilon,\hbar}}\partial_t V_{\varepsilon,\hbar}\cdot V_{\varepsilon,\hbar}\,\mathrm{d}x\\ &= -\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d}|\nabla V_{\varepsilon,\hbar}|^2\,\mathrm{d}x - \int_{\mathbb{T}^d}G'(V_{\varepsilon,\hbar})\partial_t V_{\varepsilon,\hbar}\,\mathrm{d}x\\ &= -\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d}|\nabla V_{\varepsilon,\hbar}|^2\,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d}G(V_{\varepsilon,\hbar})\,\mathrm{d}x.\end{aligned}\tag{2.7}$$

Notice that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d}e^{V_{\varepsilon,\hbar}(t,x)}\,\mathrm{d}x = 0.$$

Then, combining (2.6) with (2.7), we conclude the proof. □

Lemma 2.12. *Let $u:\mathbb{T}^d\to\mathbb{R}^d$ be a (sufficiently regular) vector field. For any $1\leqslant j\leqslant d$, let $\Pi_j := i\hbar\partial_{x_j}+u^j$. Then, for any $1\leqslant j,k\leqslant d$,*

$$\Pi_j\vee\left(\Pi_k\vee\partial_{x_k}u^j\right)=2\Pi_j\partial_{x_k}u^j\Pi_k+2\partial_{x_k}u^j\Pi_k\Pi_j+\Pi_j\vee\left(i\hbar\partial_{x_kx_k}u^j\right).$$

Proof. Denote $\Pi_j=i\hbar\partial_{x_j}+u^j$, and observe the following identity:

$$\Pi_k\vee\partial_{x_k}u^j=2\partial_{x_k}u^j\Pi_k+\left[\Pi_k,\partial_{x_k}u^j\right]=2\partial_{x_k}u^j\Pi_k+i\hbar\partial_{x_kx_k}u^j.$$

Therefore, we have

$$\begin{aligned}\Pi_j\vee\left(\Pi_k\vee\partial_{x_k}u^j\right)&=\Pi_j\vee\left(2\partial_{x_k}u^j\Pi_k+i\hbar\partial_{x_kx_k}u^j\right)\\&=2\Pi_j\partial_{x_k}u^j\Pi_k+2\partial_{x_k}u^j\Pi_k\Pi_j+\Pi_j\vee\left(i\hbar\partial_{x_kx_k}u^j\right).\end{aligned}$$

□

Proof of theorem 1.2. The proof is divided into five steps.
1. Note that

$$\begin{aligned}\mathcal{E}_{\varepsilon,\hbar}(t)=\mathcal{F}_{\varepsilon,\hbar}(t)&+\frac{1}{2}\sum_j\mathrm{tr}\left((i\hbar\partial_{x_j})\vee u^jR_{\varepsilon,\hbar}(t)\right)+\frac{1}{2}\int_{\mathbb{T}^d}\left(\rho_{\varepsilon,\hbar}|u|^2\right)(t,x)\,\mathrm{d}x\\&+\int_{\mathbb{T}^d}\left((m_{\varepsilon,\hbar}\log(1/\rho))(t,x)+\rho(t,x)\right)\mathrm{d}x.\end{aligned}$$

Therefore, by the conservation of energy (lemma 2.11), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{\varepsilon,\hbar}(t)&=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_j\mathrm{tr}\left((i\hbar\partial_{x_j})\vee u^jR_{\varepsilon,\hbar}(t)\right)+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d}\left(\rho_{\varepsilon,\hbar}|u|^2\right)(t,x)\,\mathrm{d}x\\&+\int_{\mathbb{T}^d}\partial_t\left(m_{\varepsilon,\hbar}\log(1/\rho)\right)(t,x)\,\mathrm{d}x+\int_{\mathbb{T}^d}\partial_t\rho(t,x)\,\mathrm{d}x:=\sum_{j=1}^4I^j(t).\end{aligned}$$

2. We first claim that the following identity holds:

$$\begin{aligned}I^1(t)+I^2(t)=&\frac{1}{2}\sum_j\mathrm{tr}\left((i\hbar\partial_{x_j}+u^j)\vee(\partial_tu^j+(u\cdot\nabla u)^j)R_{\varepsilon,\hbar}(t)\right)\\&-\frac{1}{4}\sum_{j,k}\mathrm{tr}\left((i\hbar\partial_{x_j}+u^j)\vee((i\hbar\partial_{x_k}+u^k)\vee(\partial_{x_k}u^j))R_{\varepsilon,\hbar}(t)\right)\\&+\int_{\mathbb{T}^d}\rho_{\varepsilon,\hbar}(t,x)\,\nabla V_{\varepsilon,\hbar}(t,x)\cdot u(t,x)\,\mathrm{d}x.\end{aligned}\tag{2.8}$$

We compute:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_j\mathrm{tr}\left(i\hbar\partial_{x_j}\vee u^jR_{\varepsilon,\hbar}(t)\right)+\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}\left(|u|^2R_{\varepsilon,\hbar}(t)\right)\\&=\mathrm{tr}\left((\sum_ji\hbar\partial_{x_j}\vee u^j+|u|^2)\partial_tR_{\varepsilon,\hbar}(t)\right)+\mathrm{tr}\left(\partial_t(\sum_ji\hbar\partial_{x_j}\vee u^j+|u|^2)R_{\varepsilon,\hbar}(t)\right)\\&=\frac{1}{i\hbar}\mathrm{tr}\left((\sum_ji\hbar\partial_{x_j}\vee u^j+|u|^2)[H_{\varepsilon,\hbar}(t),R_{\varepsilon,\hbar}(t)]\right)+\mathrm{tr}\left(\partial_t(\sum_ji\hbar\partial_{x_j}\vee u^j+|u|^2)R_{\varepsilon,\hbar}(t)\right)\\&=\frac{i}{\hbar}\mathrm{tr}\left([H_{\varepsilon,\hbar}(t),\sum_ji\hbar\partial_{x_j}\vee u^j+|u|^2]R_{\varepsilon,\hbar}(t)\right)+\mathrm{tr}\left(\partial_t(\sum_ji\hbar\partial_{x_j}\vee u^j+|u|^2)R_{\varepsilon,\hbar}(t)\right)\\&=\mathrm{tr}\left((\partial_t+\frac{i}{\hbar}[H_{\varepsilon,\hbar}(t),\cdot])(\sum_ji\hbar\partial_{x_j}\vee u^j+|u|^2)R_{\varepsilon,\hbar}(t)\right)\\&=\sum_j\mathrm{tr}\left((\partial_t+\frac{i}{\hbar}[-\frac{\hbar^2}{2}\Delta,\cdot])(i\hbar\partial_{x_j}\vee u^j+\frac{1}{2}u^j\vee u^j)R_{\varepsilon,\hbar}(t)\right)\end{aligned}$$

$$+\sum_j \mathrm{tr}\big(\big(\frac{i}{\hbar}[V_{\varepsilon,\hbar},\cdot]\big)(i\hbar\partial_{x_j}\vee u^j+\frac{1}{2}u^j\vee u^j)R_{\varepsilon,\hbar}(t)\big)$$
$$=:\sum_j \mathrm{tr}\big(J_{1,j}R_{\varepsilon,\hbar}(t)\big)+\sum_j \mathrm{tr}\big(J_{2,j}R_{\varepsilon,\hbar}(t)\big).$$

We start with $J_{1,j}$. Using the Leibniz rule: $[A,B\vee C]=[A,B]\vee C+[A,C]\vee B$, we can write $J_{1,j}$ as

$$\begin{aligned}J_{1,j}&=\big(\partial_t+\frac{i}{\hbar}[-\frac{\hbar^2}{2}\Delta,\cdot]\big)\big((\frac{1}{2}u^j+i\hbar\partial_{x_j})\vee u^j\big)\\&=\big((\partial_t+\frac{i}{\hbar}[-\frac{\hbar^2}{2}\Delta,\cdot])(\frac{1}{2}u^j+i\hbar\partial_{x_j})\big)\vee u^j+\big(\frac{1}{2}u^j+i\hbar\partial_{x_j}\big)\vee\big((\partial_t+\frac{i}{\hbar}[-\frac{\hbar^2}{2}\Delta,\cdot])u^j\big)\\&=\big((\partial_t+\frac{i}{\hbar}[-\frac{\hbar^2}{2}\Delta,\cdot])u^j\big)\vee u^j+i\hbar\partial_{x_j}\vee\big((\partial_t+\frac{i}{\hbar}[-\frac{\hbar^2}{2}\Delta,\cdot])u^j\big)\\&=\big(u^j+i\hbar\partial_{x_j}\big)\vee\big((\partial_t+\frac{i}{\hbar}[-\frac{\hbar^2}{2}\Delta,\cdot])u^j\big)\\&=\big(u^j+i\hbar\partial_{x_j}\big)\vee\big(\partial_t u^j+\sum_k u^k\partial_{x_k}u^j\big)+\big(u^j+i\hbar\partial_{x_j}\big)\vee\big((\frac{i}{\hbar}[-\frac{\hbar^2}{2}\Delta,\cdot]-\sum_k u^k\partial_{x_k})u^j\big).\end{aligned}\tag{2.9}$$

Notice that

$$\frac{i}{\hbar}\big[-\frac{\hbar^2}{2}\partial_{x_kx_k},u^j\big]=-\frac{i\hbar}{2}\partial_{x_k}\vee[\partial_{x_k},u^j]=-\frac{1}{2}i\hbar\partial_{x_k}\vee\partial_{x_k}u^j,$$

which shows that the right-hand side of (2.9) is

$$\big(u^j+i\hbar\partial_{x_j}\big)\vee\big(\partial_t u^j+\sum_k u^k\partial_{x_k}u^j\big)-\frac{1}{2}\sum_k\big(u^j+i\hbar\partial_{x_j}\big)\vee\big((i\hbar\partial_{x_k}+u^k)\vee\partial_{x_k}u^j\big),$$

so that

$$\begin{aligned}\sum_j\mathrm{tr}\big(J_{1,j}R_{\varepsilon,\hbar}(t)\big)&=\sum_j\mathrm{tr}\big((u^j+i\hbar\partial_{x_j})\vee(\partial_t u^j+(u\cdot\nabla u)^j)\,R_{\varepsilon,\hbar}(t)\big)\\&\quad-\frac{1}{2}\sum_{k,j}\mathrm{tr}\big((u^j+i\hbar\partial_{x_j})\vee((i\hbar\partial_{x_k}+u^k)\vee\partial_{x_k}u^j)R_{\varepsilon,\hbar}(t)\big).\end{aligned}\tag{2.10}$$

As for $J_{2,j}$, we see that

$$\big[V_{\varepsilon,\hbar},u^j\vee u^j\big]=0,\qquad -\big[V_{\varepsilon,\hbar},\partial_{x_j}\vee u^j\big]=2\partial_{x_j}V_{\varepsilon,\hbar}u^j,$$

which yields

$$J_{2,j}=2\partial_{x_j}V_{\varepsilon,\hbar}(t,x)\,u^j(t,x).$$

Then

$$\sum_j\mathrm{tr}\big(J_{2,j}R_{\varepsilon,\hbar}(t)\big)=2\int_{\mathbb{T}^d}\rho_{\varepsilon,\hbar}(t,x)\nabla V_{\varepsilon,\hbar}(t,x)\cdot u(t,x)\,\mathrm{d}x.\tag{2.11}$$

Combining (2.10) with (2.11), we obtain (2.8).
3. We manipulate the last term on the right-hand side of (2.8):

$$\begin{aligned}&\int_{\mathbb{T}^d}\rho_{\varepsilon,\hbar}(t,x)\,\nabla V_{\varepsilon,\hbar}(t,x)\cdot u(t,x)\,\mathrm{d}x\\&\quad=\int_{\mathbb{T}^d}\big(e^{V_{\varepsilon,\hbar}(t,x)}-\varepsilon\Delta V_{\varepsilon,\hbar}(t,x)\big)\nabla V_{\varepsilon,\hbar}(t,x)\cdot u(t,x)\,\mathrm{d}x\\&\quad=\int_{\mathbb{T}^d}e^{V_{\varepsilon,\hbar}(t,x)}\nabla V_{\varepsilon,\hbar}(t,x)\cdot u(t,x)\,\mathrm{d}x-\varepsilon\int_{\mathbb{T}^d}\Delta V_{\varepsilon,\hbar}(t,x)\nabla V_{\varepsilon,\hbar}(t,x)\cdot u(t,x)\,\mathrm{d}x\\&\quad=\int_{\mathbb{T}^d}\nabla e^{V_{\varepsilon,\hbar}(t,x)}\cdot u(t,x)\,\mathrm{d}x-\varepsilon\int_{\mathbb{T}^d}\big(\nabla:(\nabla V_{\varepsilon,\hbar}\otimes\nabla V_{\varepsilon,\hbar})\cdot u\big)(t,x)\,\mathrm{d}x\\&\qquad+\frac{\varepsilon}{2}\int_{\mathbb{T}^d}\big(\nabla|\nabla V_{\varepsilon,\hbar}|^2\cdot u\big)(t,x)\,\mathrm{d}x\end{aligned}$$

$$= -\int_{\mathbb{T}^d} e^{V_{\varepsilon,\hbar}(t,x)} \operatorname{div} u(t,x)\,\mathrm{d}x + \varepsilon \int_{\mathbb{T}^d} Du(t,x) : (\nabla V_{\varepsilon,\hbar} \otimes \nabla V_{\varepsilon,\hbar})(t,x)\,\mathrm{d}x$$
$$- \frac{\varepsilon}{2} \int_{\mathbb{T}^d} |\nabla V_{\varepsilon,\hbar}(t,x)|^2 \operatorname{div} u(t,x)\,\mathrm{d}x. \tag{2.12}$$

In (2.12), we have used $Du := \frac{1}{2}(\nabla u + (\nabla u)^\top)$, the symmetrized gradient. Next, using lemma 2.10, we can prove the following identities:

$$I^3(t) + I^4(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^d} m_{\varepsilon,\hbar}(t,x) \log(1/\rho(t,x))\,\mathrm{d}x + \int_{\mathbb{T}^d} \partial_t \rho(t,x)\,\mathrm{d}x$$
$$= \int_{\mathbb{T}^d} \partial_t e^{V_{\varepsilon,\hbar}(t,x)} \log(1/\rho(t,x))\,\mathrm{d}x - \int_{\mathbb{T}^d} \frac{e^{V_{\varepsilon,\hbar}(t,x)}}{\rho(t,x)} \partial_t \rho(t,x)\,\mathrm{d}x + \int_{\mathbb{T}^d} \partial_t \rho(t,x)\,\mathrm{d}x$$
$$= \int_{\mathbb{T}^d} \Big(-\frac{e^{V_{\varepsilon,\hbar}(t,x)}}{\rho(t,x)} + 1 \Big) \partial_t \rho(t,x)\,\mathrm{d}x + \varepsilon \int_{\mathbb{T}^d} \partial_t \Delta V_{\varepsilon,\hbar}(t,x) \log(1/\rho(t,x))\,\mathrm{d}x$$
$$- \int_{\mathbb{T}^d} \operatorname{div} J_{\varepsilon,\hbar}(t,x) \log(1/\rho(t,x))\,\mathrm{d}x, \tag{2.13}$$

$$\frac{1}{2} \sum_j \operatorname{tr}\big((u^j + i\hbar\partial_{x_j}) \vee \partial_{x_j} \log(\rho(t,x)) R_{\varepsilon,\hbar}(t) \big)$$
$$= \int_{\mathbb{T}^d} u(t,x) \cdot \nabla \log(\rho(t,x)) \rho_{\varepsilon,\hbar}(t,x)\,\mathrm{d}x - \int_{\mathbb{T}^d} J_{\varepsilon,\hbar}(t,x) \cdot \nabla \log(\rho(t,x))\,\mathrm{d}x$$
$$= \int_{\mathbb{T}^d} e^{V_{\varepsilon,\hbar}(t,x)} u(t,x) \cdot \nabla \log(\rho(t,x))\,\mathrm{d}x - \varepsilon \int_{\mathbb{T}^d} \Delta V_{\varepsilon,\hbar}(t,x) u(t,x) \cdot \nabla \log(\rho(t,x))\,\mathrm{d}x$$
$$- \int_{\mathbb{T}^d} J_{\varepsilon,\hbar}(t,x) \cdot \nabla \log(\rho(t,x))\,\mathrm{d}x. \tag{2.14}$$

Therefore, gathering (2.8) with (2.12)–(2.13) yields

$$I^1(t) + I^2(t) + I^3(t) + I^4(t)$$
$$= \frac{1}{2} \sum_j \operatorname{tr}\big((i\hbar\partial_{x_j} + u^j)) \vee (\partial_t u^j + (u \cdot \nabla u)^j) R_{\varepsilon,\hbar}(t) \big)$$
$$- \frac{1}{4} \sum_{j,k} \operatorname{tr}\big((i\hbar\partial_{x_j} + u^j) \vee ((i\hbar\partial_{x_k} + u^k) \vee (\partial_{x_k} u^j)) R_{\varepsilon,\hbar}(t) \big)$$
$$- \int_{\mathbb{T}^d} e^{V_{\varepsilon,\hbar}(t,x)} \operatorname{div} u(t,x)\,\mathrm{d}x + \varepsilon \int_{\mathbb{T}^d} Du(t,x) : \big(\nabla V_{\varepsilon,\hbar} \otimes \nabla V_{\varepsilon,\hbar} \big)(t,x)\,\mathrm{d}x$$
$$- \frac{\varepsilon}{2} \int_{\mathbb{T}^d} |\nabla V_{\varepsilon,\hbar}(t,x)|^2 \operatorname{div} u(t,x)\,\mathrm{d}x$$
$$+ \int_{\mathbb{T}^d} \Big(-\frac{e^{V_{\varepsilon,\hbar}(t,x)}}{\rho(t,x)} + 1 \Big) \partial_t \rho(t,x)\,\mathrm{d}x + \varepsilon \int_{\mathbb{T}^d} \partial_t \Delta V_{\varepsilon,\hbar}(t,x) \log(1/\rho(t,x))\,\mathrm{d}x$$
$$- \int_{\mathbb{T}^d} \operatorname{div} J_{\varepsilon,\hbar}(t,x) \log(1/\rho(t,x))\,\mathrm{d}x.$$

Thanks to equation (2.14), we can rewrite the last term on the right-hand side of the last identity to obtain

$$I^1(t) + I^2(t) + I^3(t) + I^4(t)$$
$$= \frac{1}{2} \sum_j \operatorname{tr}\big((i\hbar\partial_{x_j} + u^j) \vee (\partial_t u^j + (u \cdot \nabla u)^j) R_{\varepsilon,\hbar}(t) \big)$$
$$- \frac{1}{4} \sum_{j,k} \operatorname{tr}\big((i\hbar\partial_{x_j} + u^j) \vee ((i\hbar\partial_{x_k} + u^k) \vee (\partial_{x_k} u^j)) R_{\varepsilon,\hbar}(t) \big)$$
$$- \int_{\mathbb{T}^d} e^{V_{\varepsilon,\hbar}(t,x)} \operatorname{div} u(t,x)\,\mathrm{d}x + \varepsilon \int_{\mathbb{T}^d} Du(t,x) : \big(\nabla V_{\varepsilon,\hbar} \otimes \nabla V_{\varepsilon,\hbar} \big)(t,x)\,\mathrm{d}x$$
$$- \frac{\varepsilon}{2} \int_{\mathbb{T}^d} |\nabla V_{\varepsilon,\hbar}(t,x)|^2 \operatorname{div} u(t,x)\,\mathrm{d}x$$
$$+ \int_{\mathbb{T}^d} \Big(-\frac{e^{V_{\varepsilon,\hbar}(t,x)}}{\rho(t,x)} + 1 \Big) \partial_t \rho(t,x)\,\mathrm{d}x + \varepsilon \int_{\mathbb{T}^d} \partial_t \Delta V_{\varepsilon,\hbar}(t,x) \log(1/\rho(t,x))\,\mathrm{d}x$$

$$+\varepsilon\int_{\mathbb{T}^d}\Delta V_{\varepsilon,\hbar}(t,x)u(t,x)\cdot\nabla\log(\rho(t,x))\,\mathrm{d}x+\frac{1}{2}\sum_j\mathrm{tr}\big((u^j+i\hbar\partial_{x_j})\vee(\partial_{x_j}\log(\rho))R_{\varepsilon,\hbar}(t)\big)$$
$$-\int_{\mathbb{T}^d}e^{V_{\varepsilon,\hbar}(t,x)}u(t,x)\cdot\nabla\log(\rho(t,x))\,\mathrm{d}x.$$

Rearranging the terms leads to

$$\begin{aligned}
&I^1(t)+I^2(t)+I^3(t)+I^4(t)\\
&=-\int_{\mathbb{T}^d}e^{V_{\varepsilon,\hbar}(t,x)}\big(\partial_t\log(\rho(t,x))+\mathrm{div}\,u(t,x)+u(t,x)\cdot\nabla\log(\rho(t,x))\big)\,\mathrm{d}x+\int_{\mathbb{T}^d}\partial_t\rho(t,x)\,\mathrm{d}x\\
&+\frac{1}{2}\sum_j\mathrm{tr}\big((i\hbar\partial_{x_j}+u^j)\vee(\partial_t u^j+(u\cdot\nabla u)^j+\partial_{x_j}\log(\rho(t,x)))R_{\varepsilon,\hbar}(t)\big)\\
&+\varepsilon\int_{\mathbb{T}^d}\Delta V_{\varepsilon,\hbar}(t,x)u(t,x)\cdot\nabla\log(\rho(t,x))\,\mathrm{d}x+\varepsilon\int_{\mathbb{T}^d}\partial_t\Delta V_{\varepsilon,\hbar}(t,x)\log(1/\rho(t,x))\,\mathrm{d}x\\
&-\frac{1}{4}\sum_{j,k}\mathrm{tr}\big((i\hbar\partial_{x_j}+u^j)\vee((i\hbar\partial_{x_k}+u^k)\vee(\partial_{x_k}u^j))R_{\varepsilon,\hbar}(t)\big)\\
&+\varepsilon\int_{\mathbb{T}^d}Du(t,x):\big(\nabla V_{\varepsilon,\hbar}\otimes\nabla V_{\varepsilon,\hbar}\big)(t,x)\,\mathrm{d}x-\frac{\varepsilon}{2}\int_{\mathbb{T}^d}|\nabla V_{\varepsilon,\hbar}(t,x)|^2\mathrm{div}\,u(t,x)\,\mathrm{d}x.
\end{aligned}\tag{2.15}$$

Using (2.2), we see that the first three terms in (2.15) vanish, which yields

$$\begin{aligned}
&I^1(t)+I^2(t)+I^3(t)+I^4(t)\\
&=\varepsilon\int_{\mathbb{T}^d}\Delta V_{\varepsilon,\hbar}(t,x)u(t,x)\cdot\nabla\log(\rho(t,x))\,\mathrm{d}x+\varepsilon\int_{\mathbb{T}^d}\partial_t\Delta V_{\varepsilon,\hbar}(t,x)\log(1/\rho(t,x))\,\mathrm{d}x\\
&-\frac{1}{4}\sum_{j,k}\mathrm{tr}\big((i\hbar\partial_{x_j}+u^j)\vee((i\hbar\partial_{x_k}+u^k)\vee(\partial_{x_k}u^j))R_{\varepsilon,\hbar}(t)\big)\\
&+\varepsilon\int_{\mathbb{T}^d}Du(t,x):\big(\nabla V_{\varepsilon,\hbar}\otimes\nabla V_{\varepsilon,\hbar}\big)(t,x)\,\mathrm{d}x-\frac{\varepsilon}{2}\int_{\mathbb{T}^d}|\nabla V_{\varepsilon,\hbar}(t,x)|^2\mathrm{div}\,u(t,x)\,\mathrm{d}x\\
&=:\sum_{k=1}^5\mathcal{T}_k.
\end{aligned}\tag{2.16}$$

4. We proceed by estimating separately each of the summands $\mathcal{T}_k$ on the right-hand side in (2.16). We have

$$\begin{aligned}
\mathcal{T}_1&=\varepsilon\int_{\mathbb{T}^d}\Delta V_{\varepsilon,\hbar}(t,x)\,u(t,x)\cdot\nabla\log(\rho(t,x))\,\mathrm{d}x\\
&=-\varepsilon\int_{\mathbb{T}^d}\nabla V_{\varepsilon,\hbar}(t,x)\cdot\nabla\big(u(t,x)\cdot\nabla\log(\rho(t,x))\big)\,\mathrm{d}x\\
&\leqslant\sqrt{\varepsilon}\,\big\|\sqrt{\varepsilon}\nabla V_{\varepsilon,\hbar}\big\|_{L^\infty_tL^2_x}\|\nabla(u\cdot\nabla\log(\rho))\|_{L^\infty_tL^2_x}\\
&\leqslant C\sqrt{\varepsilon}\sqrt{\mathcal{F}_0}\,\|\nabla(u\cdot\nabla\log(\rho))\|_{L^\infty_tL^2_x},
\end{aligned}\tag{2.17}$$

where we have used that $\|\sqrt{\varepsilon}\nabla V_{\varepsilon,\hbar}\|_{L^\infty_tL^2_x}$ is uniformly bounded in $(\varepsilon,\hbar)$ by $\mathcal{F}_0$ thanks to lemma 2.11 and the assumption that $\mathcal{F}_{\varepsilon,\hbar}(0)\leqslant\mathcal{F}_0$. Furthermore, integrating by parts in time yields

$$\begin{aligned}
\int_0^t\mathcal{T}_2(s)\,\mathrm{d}s&=-\varepsilon\int_0^t\int_{\mathbb{T}^d}\partial_s\Delta V_{\varepsilon,\hbar}(s,x)\log(\rho(s,x))\,\mathrm{d}s\mathrm{d}x\\
&=\varepsilon\int_0^t\int_{\mathbb{T}^d}\Delta V_{\varepsilon,\hbar}\partial_s\log(\rho(s,x))\,\mathrm{d}s\mathrm{d}x-\varepsilon\int_{\mathbb{T}^d}\nabla\log(\rho(0,x))\nabla V_{\varepsilon,\hbar}(0,x)\,\mathrm{d}x\\
&\quad+\varepsilon\int_{\mathbb{T}^d}\nabla\log(\rho(t,x))\nabla V_{\varepsilon,\hbar}(t,x)\,\mathrm{d}x\\
&\leqslant\sqrt{\varepsilon}\,\|\log(\rho)\|_{W^{1,\infty}_tH^1_x}\int_0^t\big\|\sqrt{\varepsilon}\nabla V_\varepsilon(\tau,\cdot)\big\|_2\,\mathrm{d}\tau\\
&\quad+\sqrt{\varepsilon}\,\|\log(\rho)\|_{L^\infty_tH^1_x}\big(\big\|\sqrt{\varepsilon}\nabla V_{\varepsilon,\hbar}(t,\cdot)\big\|_2+\big\|\sqrt{\varepsilon}\nabla V_{\varepsilon,\hbar}(0,\cdot)\big\|_2\big)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant 2\sqrt{\varepsilon}\,\|\log(\rho)\|_{W_t^{1,\infty}H_x^1}\, t\sqrt{\mathcal{F}_0} + 4\sqrt{\varepsilon}\,\|\log(\rho)\|_{L_t^\infty H_x^1}\sqrt{\mathcal{F}_0} \\
&\leqslant 6\sqrt{\varepsilon}\,\|\log(\rho)\|_{W_t^{1,\infty}H_x^1}(t+1)\sqrt{\mathcal{F}_0}.
\end{aligned} \tag{2.18}
$$

By lemma 2.12, we have

$$
\begin{aligned}
-4\mathcal{T}_3 &= \sum_{j,k}\operatorname{tr}\big(2\Pi_j\partial_{x_k}u^j\Pi_k R_{\varepsilon,\hbar}(t)\big) + \sum_{j,k}\operatorname{tr}\big(2\partial_{x_k}u^j\Pi_k\Pi_j R_{\varepsilon,\hbar}(t)\big) + \sum_{j,k}\operatorname{tr}\big(\Pi_j \vee i\hbar\partial_{x_kx_k}u^j R_{\varepsilon,\hbar}(t)\big) \\
&:= \mathcal{T}_3^1 + \mathcal{T}_3^2 + \mathcal{T}_3^3.
\end{aligned}
$$

To estimate $\mathcal{T}_3^1$, we note that

$$
\begin{aligned}
\mathcal{T}_3^1 &= 2\sum_{j,k}\int_{\mathbb{T}^d}\left(\Pi_j\psi_{\varepsilon,\hbar}\right)\left(\partial_{x_k}u^j\overline{\Pi_k\psi_{\varepsilon,\hbar}}\right)(t,x)\,\mathrm{d}x \\
&\leqslant \|\nabla u\|_{L_t^\infty L_x^\infty}\sum_{j,k}\Big(\int_{\mathbb{T}^d}\left|\Pi_j\psi_{\varepsilon,\hbar}\right|^2(t,x)\,\mathrm{d}x + \int_{\mathbb{T}^d}\left|\Pi_k\psi_{\varepsilon,\hbar}\right|^2(t,x)\,\mathrm{d}x\Big) \\
&\leqslant 20\,\|\nabla u\|_{L_t^\infty L_x^\infty}\,\mathcal{K}_{\varepsilon,\hbar}(t).
\end{aligned} \tag{2.19}
$$

In addition, we have

$$
\begin{aligned}
\mathcal{T}_3^2 &= 2\sum_{j,k}\int_{\mathbb{T}^d}\Pi_k\left(\psi_{\varepsilon,\hbar}\partial_{x_k}u^j\right)(t,x)\,\overline{\Pi_j\psi_{\varepsilon,\hbar}}(t,x)\,\mathrm{d}x \\
&= 2\sum_{j,k}\int_{\mathbb{T}^d}\Pi_k\psi_{\varepsilon,\hbar}(t,x)\,\partial_{x_k}u^j\,\overline{\Pi_j\psi_{\varepsilon,\hbar}}(t,x)\,\mathrm{d}x \\
&\quad + 2\sum_{j,k}\int_{\mathbb{T}^d}\psi_{\varepsilon,\hbar}(t,x)\,i\hbar\,\partial_{x_kx_k}u^j(t,x)\,\overline{\Pi_j\psi_{\varepsilon,\hbar}}(t,x)\,\mathrm{d}x.
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
\mathcal{T}_3^2 &\leqslant \|\nabla u\|_{L_t^\infty L_x^\infty}\sum_{j,k}\Big(\int_{\mathbb{T}^d}\left|\Pi_k\psi_{\varepsilon,\hbar}\right|^2(t,x)\,\mathrm{d}x + \int_{\mathbb{T}^d}\left|\Pi_j\psi_{\varepsilon,\hbar}\right|^2(t,x)\,\mathrm{d}x\Big) \\
&\quad + \sum_{j,k}\Big(\hbar^2\left\|\nabla^2 u\right\|_{L_t^\infty L_x^\infty} + \int_{\mathbb{T}^d}\left|\Pi_j\psi_{\varepsilon,\hbar}\right|^2(t,x)\,\mathrm{d}x\Big) \\
&\leqslant 20\big(\|\nabla u\|_{L_t^\infty L_x^\infty} + 1\big)\mathcal{K}_{\varepsilon,\hbar}(t) + 20\left\|\nabla^2 u\right\|_{L_t^\infty L_x^\infty}\hbar^2.
\end{aligned} \tag{2.20}
$$

To estimate $\mathcal{T}_3^3$, we note that

$$
\mathcal{T}_3^3 \leqslant \sum_{j,k}\big(\operatorname{tr}(\Pi_j^2 R_{\varepsilon,\hbar}(t)) + \hbar^2\|\nabla^2 u\|_{L_t^\infty L_x^\infty}\big) \leqslant 20\big(\mathcal{K}_{\varepsilon,\hbar}(t) + \hbar^2\|\nabla^2 u\|_{L_t^\infty L_x^\infty}\big). \tag{2.21}
$$

Gathering (2.19)–(2.21), we obtain the inequality:

$$
|\mathcal{T}_3| \leqslant C\left(\mathcal{K}_{\varepsilon,\hbar}(t) + \hbar^2\right), \tag{2.22}
$$

where $C = C(\|u\|_{L_t^\infty W_x^{2,\infty}})$. Finally, observe that

$$
|\mathcal{T}_4| \leqslant 20\varepsilon\,\|\nabla u\|_{L_{t,x}^\infty}\int_{\mathbb{T}^d}|\nabla V_{\varepsilon,\hbar}(t,x)|^2\,\mathrm{d}x \leqslant 40\,\|\nabla u\|_{L_{t,x}^\infty}\,\mathcal{V}_{\varepsilon,\hbar}(t), \tag{2.23}
$$

$$
|\mathcal{T}_5| \leqslant \frac{\varepsilon}{2}\,\|\operatorname{div} u\|_{L_{t,x}^\infty}\int_{\mathbb{T}^d}|\nabla V_{\varepsilon,\hbar}(t,x)|^2\,\mathrm{d}x \leqslant \|\operatorname{div} u\|_{L_{t,x}^\infty}\,\mathcal{V}_{\varepsilon,\hbar}(t). \tag{2.24}
$$

Gathering (2.17)–(2.24), we have

$$
\mathcal{E}_{\varepsilon,\hbar}(t) \leqslant \mathcal{E}_{\varepsilon,\hbar}(0) + C\Big(\int_0^t \mathcal{E}_{\varepsilon,\hbar}(s)\,\mathrm{d}s + 6\sqrt{\varepsilon}\,(t+1)\sqrt{\mathcal{F}_0} + t\hbar^2\Big),
$$

where $C = C(\|u\|_{L_t^\infty W_x^{2,\infty}}, \|\log(\rho)\|_{W_t^{1,\infty}H_x^1}, \|\nabla(u\cdot\nabla\log(\rho))\|_{L_t^\infty L_x^2})$. The inequality of theorem 1.2 follows now by the Grönwall inequality.

5. Using the convergence:

$$\sup_{t\in[0,T]} \mathcal{E}_{\varepsilon,\hbar}(t) \underset{\varepsilon+\hbar\to 0}{\longrightarrow} 0$$

proved in Step 4, we can now establish the convergence of $\rho_{\varepsilon,\hbar}$ and $J_{\varepsilon,\hbar}$. The convergence

$$\varepsilon \int_{\mathbb{T}^d} |\nabla V_{\varepsilon,\hbar}(t,x)|^2 \,\mathrm{d}x \underset{\varepsilon+\hbar\to 0}{\longrightarrow} 0$$

implies

$$\sup_{t\in[0,T]} \|\rho_{\varepsilon,\hbar}(t,\cdot) - m_{\varepsilon,\hbar}(t,\cdot)\|_{\dot{H}^{-1}} \underset{\varepsilon+\hbar\to 0}{\longrightarrow} 0.$$

To prove the convergence of $\rho_{\varepsilon,\hbar}$, we need the following Csiszár–Kullback–Pinsker inequality:

Lemma 2.13 ([31]). *Let $(\rho,m) \in \left(\mathcal{P}(\mathbb{T}^d)\cap L^1(\mathbb{T}^d)\right)^2$. Then*

$$\|\rho - m\|_1 \leqslant \sqrt{2\int_{\mathbb{T}^d} \log\left(\frac{m(x)}{\rho(x)}\right) m(x)\,\mathrm{d}x}.$$

By virtue of lemma 2.13, the convergence

$$\int_{\mathbb{T}^d} m_{\varepsilon,\hbar}(t,x) \log\left(\frac{m_{\varepsilon,\hbar}(t,x)}{\rho(t,x)}\right)\mathrm{d}x \underset{\varepsilon+\hbar\to 0}{\longrightarrow} 0$$

implies

$$\sup_{t\in[0,T]} \|m_{\varepsilon,\hbar}(t,\cdot) - \rho(t,\cdot)\|_1 \underset{\varepsilon+\hbar\to 0}{\longrightarrow} 0.$$

Thus, by the triangle inequality, we obtain

$$\sup_{t\in[0,T]} \|\rho_{\varepsilon,\hbar}(t,\cdot) - \rho(t,\cdot)\|_{\dot{H}^{-1}} \underset{\varepsilon+\hbar\to 0}{\longrightarrow} 0. \tag{2.25}$$

The convergence of $J_{\varepsilon,\hbar}$ is established as follows: Given a Lipschitz vector field $b \in W^{1,\infty}(\mathbb{T}^d;\mathbb{R}^d)$, we have

$$\begin{aligned}\int_{\mathbb{T}^d} \big(J_{\varepsilon,\hbar}(t,x) - \rho(t,x)u(t,x)\big)b(x)\,\mathrm{d}x &= \int_{\mathbb{T}^d} \big(\rho_{\varepsilon,\hbar}(t,x) - \rho(t,x)\big)u(t,x)b(x)\,\mathrm{d}x \\ &\quad + \int_{\mathbb{T}^d} \big(J_{\varepsilon,\hbar}(t,x) - \rho_{\varepsilon,\hbar}(t,x)u(t,x)\big)b(x)\,\mathrm{d}x.\end{aligned}$$

The first integral on the right-hand side above tends to 0 as $\varepsilon+\hbar\to 0$ by (2.25), while the second integral tends to 0 due to the convergence of the kinetic part: $\sup_{t\in[0,T]} \mathcal{K}_{\varepsilon,\hbar}(t) \underset{\varepsilon+\hbar\to 0}{\longrightarrow} 0$. Indeed, we have

$$\begin{aligned}\left|\int_{\mathbb{T}^d} \big(J_{\varepsilon,\hbar}(t,x) - \rho_{\varepsilon,\hbar}(t,x)u(t,x)\big)b(x)\,\mathrm{d}x\right| &= \left|\frac{1}{2}\sum_{j=1}^{d} \mathrm{tr}\big((i\hbar\partial_{x_j} + u^j)\vee b^j R_{\varepsilon,\hbar}(t)\big)\right| \\ &\leqslant 2\|b\|_\infty \sqrt{\mathcal{K}_{\varepsilon,\hbar}(t)},\end{aligned}$$

by the Cauchy–Schwarz inequality. This completes the proof of theorem 1.2. □

3. The N-body problem

This section concerns the N-body quantum mean-field limit. Being a triple limit, this limit is naturally more involved and imposes additional considerations, namely the dimensionality and an asymptotic relation between ε and N. Unless otherwise stated, throughout this section, we take $d = 1$. Let us start by recalling the basic notation from the theory of density operators.

Definition. Let $R_N \in \mathcal{D}_s(\mathfrak{H}^{\otimes N})$.

(i) For each $1 \leqslant k \leqslant N$, define *the kth marginal of* R_N, as the unique element $R_{N:k} \in \mathcal{D}_s(\mathfrak{H}^{\otimes k})$, such that

$$\mathrm{tr}_{\mathfrak{H}^{\otimes k}}\big(A_k R_{N:k}\big) = \mathrm{tr}_{\mathfrak{H}^{\otimes N}}\big((A_k \otimes I^{\otimes (N-k)}) R_N\big)$$

for all bounded operators A_k on $\mathfrak{H}^{\otimes k}$.

(ii) The *current* of R_N, denoted by $J_{\hbar,N:1}$, is the unique signed Radon measure on $\mathbb{T}$ such that, for all $a \in W^{1,\infty}(\mathbb{T})$,

$$\int_{\mathbb{T}} a(x) J_{\hbar,N:1}(\mathrm{d}x) = \frac{1}{2}\mathrm{tr}\big(a \vee (-i\hbar\partial_x) R_{N:1}\big).$$

We remark that $\rho_{\varepsilon,\hbar,N:1}$ can equivalently be defined as

$$\rho_{\varepsilon,\hbar,N:1}(t,x) = \int_{\mathbb{T}^{N-1}} \rho_{\varepsilon,\hbar,N}(x,x_2,\cdots,x_N)\ \mathrm{d}x_2\cdots\mathrm{d}x_N.$$

Recall that the quantum Hamiltonian is given by

$$\begin{aligned}\mathscr{H}_{\varepsilon,\hbar,N}(X_N) &= -\frac{\hbar^2}{2}\sum_{k=1}^N \Delta_{x_k} + N\int_{\mathbb{T}} V_{\varepsilon,X_N}(x)\, m_{\varepsilon,X_N}(x)\,\mathrm{d}x \\ &\quad + \frac{N}{2\varepsilon}\int_{\mathbb{T}\times\mathbb{T}} K(x-y)\,(\mu_{X_N} - m_{\varepsilon,X_N})^{\otimes 2}(\mathrm{d}x\mathrm{d}y) \\ &=: \mathscr{K}_{N,\hbar} + \mathscr{I}_{\varepsilon,X_N} + \mathscr{V}_{\varepsilon,X_N},\end{aligned}$$

where

$$-\varepsilon V''_{\varepsilon,X_N} = \mu_{X_N} - m_{\varepsilon,X_N} \qquad \text{with}\ \ m_{\varepsilon,X_N} = e^{V_{\varepsilon,X_N}}. \tag{3.1}$$

The associated von Neumann equation is

$$\begin{cases} i\hbar\partial_t R_{\varepsilon,\hbar,N} = [\mathscr{H}_{\varepsilon,\hbar,N}, R_{\varepsilon,\hbar,N}], \\ R_{\varepsilon,\hbar,N}(0) = R^{\mathrm{in}}_{\varepsilon,\hbar,N}. \end{cases}$$

Recall that the modulated energy is given by

$$\begin{aligned}\mathcal{E}_{\varepsilon,\hbar,N}(t) &= \underbrace{\frac{1}{2N}\sum_{j=1}^N \mathrm{tr}\big((i\hbar\partial_{x_j} + u(t,x_j))^2 R_{\varepsilon,\hbar,N}(t)\big)}_{:=\mathcal{K}_{\varepsilon,\hbar,N}(t)} \\ &\quad + \underbrace{\begin{aligned}&\int_{\mathbb{T}^N}\int_{\mathbb{T}}\Big(m_{\varepsilon,X_N}(x)\log\big(\frac{m_{\varepsilon,X_N}(x)}{\rho(t,x)}\big) - m_{\varepsilon,X_N}(x) + \rho(t,x)\Big)\,\mathrm{d}x\,\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N \\ &+ \frac{1}{2\varepsilon}\int_{\mathbb{T}^N}\Big(\int_{\mathbb{T}\times\mathbb{T}} K(x-y)\,(\mu_{X_N} - m_{\varepsilon,X_N})^{\otimes 2}(\mathrm{d}x\mathrm{d}y)\Big)\,\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N,\end{aligned}}_{:=\mathcal{V}_{\varepsilon,\hbar,N}(t)}\end{aligned}$$

and the total energy is given by

$$\begin{aligned}\mathcal{F}_{\varepsilon,\hbar,N}(t) &:= \frac{1}{2N}\mathrm{tr}\big(-\hbar^2\sum_{j=1}^N \Delta_{x_j} R_{\varepsilon,\hbar,N}(t)\big) \\ &\quad + \int_{\mathbb{T}^N}\Big(\int_{\mathbb{T}} V_{\varepsilon,X_N}(x)\, m_{\varepsilon,X_N}(x)\ \mathrm{d}x\Big)\,\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N \\ &\quad + \frac{1}{2\varepsilon}\int_{\mathbb{T}^N}\Big(\int_{\mathbb{T}\times\mathbb{T}} K(x-y)\,(\mu_{X_N} - m_{\varepsilon,X_N})^{\otimes 2}(\mathrm{d}x\mathrm{d}y)\Big)\,\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N.\end{aligned} \tag{3.2}$$

As in the case of 1-body dynamics, the conservation of energy holds, and $\rho_{\varepsilon,\hbar,N:1}$ evolves according to an evolution equation.

Lemma 3.2. *Let $R_{\varepsilon,\hbar,N}(t)$ be as in theorem 1.3. Then*

(i) $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{\varepsilon,\hbar,N}(t)=0$,

(ii) $\partial_t\rho_{\varepsilon,\hbar,N:1}+\partial_x J_{\varepsilon,\hbar,N:1}=0$.

The proof is omitted, since it is similar to the previous considerations. In variance with the quasi-neutral limit for VP, the quasi-neutral limit for VPME necessitates an L^∞–bound on $e^{V_{\varepsilon,X_N}}$, which is not uniformly bounded in ε. Therefore, in order to derive the mean-field limit, the vanishing rate of $\varepsilon=\varepsilon(N)$ with respect to N is slower than in the usual quasi-neutral mean-field limit. This bound is included in the following lemma:

Lemma 3.3 ([17], proposition 3.2). *Let $(\tilde{V}_{\varepsilon,X_N},\hat{V}_{\varepsilon,X_N})$ be the solution of the system:*

$$\begin{cases}-\varepsilon\tilde{V}''_{\varepsilon,X_N}=\mu_{X_N}-1,\\-\varepsilon\hat{V}''_{\varepsilon,X_N}=1-e^{\tilde{V}_{\varepsilon,X_N}+\hat{V}_{\varepsilon,X_N}}\end{cases}\tag{3.3}$$

ensured by lemma 2.4. Then $V_{\varepsilon,X_N}=\tilde{V}_{\varepsilon,X_N}+\hat{V}_{\varepsilon,X_N}$ satisfies the estimate:

$$\|V_{\varepsilon,X_N}\|_\infty\leqslant\frac{1}{\varepsilon}.$$

We also make use of the following lemma that provides the stability with respect to the perturbations by measures:

Lemma 3.4 ([17], proposition 3.2). *Let $h_i\in\mathcal{P}(\mathbb{T})$ $(i=1,2)$ be probability measures. Let $(\tilde{V}_i,\hat{V}_i)\in(H^1(\mathbb{T}))^2$ be the solution of the system:*

$$\begin{cases}-\varepsilon\tilde{V}''_i=h_i-1,\\-\varepsilon\hat{V}''_i=1-e^{\tilde{V}_i+\hat{V}_i}.\end{cases}$$

Then

$$\|\tilde{V}'_1-\tilde{V}'_2\|_2+4\sqrt{\varepsilon}\|\hat{V}'_1-\hat{V}'_2\|_2\leqslant\frac{1}{\varepsilon}W_1(h_1,h_2),$$

where W_1 denotes the 1-Wasserstein distance.

An additional significant ingredient in the renormalization argument is the following asymptotic positivity, coercivity inequality, and commutator estimates, all of which are fundamental discoveries due to [28]. We state these inequalities only for the 1D periodic case, although the results hold in greater generality. Denote

$$\mathscr{E}(X_N,\mu):=\int_{\mathbb{T}\times\mathbb{T}}K(x-y)(\mu_{X_N}-\mu)^{\otimes2}(\mathrm{d}x\mathrm{d}y)$$

so that

$$\mathscr{V}_{\varepsilon,X_N}=\frac{N}{2\varepsilon}\mathscr{E}(X_N,m_{\varepsilon,X_N}).$$

Lemma 3.5 ([28], Corollary 3.5). *Let $\mu\in L^\infty(\mathbb{T})\cap\mathcal{P}(\mathbb{T})$ and $X_N\in\mathbb{T}^N\setminus\triangle_N$. Then*

(i) $\mathscr{E}(X_N,\mu)+\frac{1+\|\mu\|_\infty}{N^2}\geqslant0$;

(ii) *There are some $\lambda,C>0$ such that, for all $\varphi\in W^{1,\infty}(\mathbb{T})$,*

$$\left|\int_{\mathbb{T}}(\mu_{X_N}-\mu)\varphi(x)\,\mathrm{d}x\right|\leqslant C\|\nabla\varphi\|_\infty N^{-\lambda}+\|\nabla\varphi\|_2\left(\mathscr{E}(X_N,\mu)+\frac{1+\|\mu\|_\infty}{N^2}\right)^{\frac{1}{2}}.$$

Lemma 3.6 ([28], proposition 1.1). *Let the assumptions of lemma 3.5 hold, and assume further that $u:\mathbb{T}\to\mathbb{R}$ is Lipschitz. Then*

$$\left|\int_{\mathbb{T}\times\mathbb{T}\setminus\triangle}(u(x)-u(y))K'(x-y)(\mu_N-\mu)^{\otimes 2}(\mathrm{d}x\mathrm{d}y)\right|\leqslant C\Big(\mathscr{E}(X_N,\mu)+\frac{1+\|\mu\|_\infty}{N^2}\Big),$$

where $C=C(\|u\|_{W^{1,\infty}})$.

The significance of the above-mentioned results for what concerns the classical problem of deriving a Vlasov-like equation as a mean-field limit is beyond the scope of this paper. We refer to [29] (and especially Chapter 2) for an exhaustive discussion. The following simple lemma concerns the Lipschitz continuity of V_{ε,X_N} with respect to the configuration X_N and related properties.

Lemma 3.7. *Let V_{ε,X_N} and m_{ε,X_N} be the solutions guaranteed by lemma 2.4 to the equation*

$$-\varepsilon V''_{\varepsilon,X_N}=\mu_{X_N}-m_{\varepsilon,X_N}\qquad \text{with } \ m_{\varepsilon,X_N}=e^{V_{\varepsilon,X_N}}. \tag{3.4}$$

Then the following statements hold:

(i) *The functions $X_N\mapsto V_{\varepsilon,X_N}(x)$ and $X_N\mapsto m_{\varepsilon,X_N}(x)$ are Lipschitz continuous uniformly in x. Moreover, for each $1\leqslant j\leqslant N$ and every fixed $x\in\mathbb{T}$, the map: $x_j\mapsto V_{\varepsilon,X_N}(x)$ is Lipschitz with constant $\frac{2}{\varepsilon^{\frac{3}{2}}N}$. In particular,*

$$\left|\partial_{x_j}V_{\varepsilon,X_N}(x)\right|\leq\frac{2}{\varepsilon^{\frac{3}{2}}N}\qquad \text{for a.e. } x_j.$$

(ii) *For any fixed X_N, the functions: $x\mapsto V_{\varepsilon,X_N}(x)$ and $x\mapsto m_{\varepsilon,X_N}(x)$ are Lipschitz.*

Proof. This can be seen as follows:

(i) Let $X_N=(x_1^0,\cdots,x_j,\cdots,x_N^0)$ and $Y_N=(x_1^0,\cdots,y_j,\cdots,x_N^0)$, where x_k^0 is fixed for any $k\neq j$. Thanks to the Sobolev embedding, we have

$$\begin{aligned}|V_{\varepsilon,X_N}(x)-V_{\varepsilon,Y_N}(x)|&\leqslant\|\tilde V_{\varepsilon,X_N}-\tilde V_{\varepsilon,Y_N}\|_\infty+\|\hat V_{\varepsilon,X_N}-\hat V_{\varepsilon,Y_N}\|_\infty\\&\leqslant\|\tilde V_{\varepsilon,X_N}-\tilde V_{\varepsilon,Y_N}\|_\infty+\|\hat V_{\varepsilon,X_N}-\hat V_{\varepsilon,Y_N}\|_{H^1}.\end{aligned}\tag{3.5}$$

Clearly, $x_j\mapsto\tilde V_{\varepsilon,X_N}(x)$ is Lipschitz with the estimate:

$$|\tilde V_{\varepsilon,X_N}(x)-\tilde V_{\varepsilon,Y_N}(x)|\leqslant\frac{1}{\varepsilon N}|x_j-y_j|.$$

Moreover, by lemma 3.4, we have the estimate:

$$\|\hat V_{\varepsilon,X_N}-\hat V_{\varepsilon,Y_N}\|_{H^1}\leqslant\varepsilon^{-\frac{3}{2}}W_1(\mu_{X_N},\mu_{Y_N})\leqslant\frac{1}{\varepsilon^{\frac{3}{2}}N}|x_j-y_j|.$$

Together with inequality (3.5), we deduce

$$|V_{\varepsilon,X_N}(x)-V_{\varepsilon,Y_N}(x)|\leqslant\frac{2}{\varepsilon^{\frac{3}{2}}N}\left|x_j-y_j\right|\qquad\text{for all } x\in\mathbb{T}.$$

Hence, for every fixed $x\in\mathbb{T}$, the map: $x_j\mapsto V_{\varepsilon,X_N}(x)$ is Lipschitz with constant $\frac{2}{\varepsilon^{\frac{3}{2}}N}$. By Rademacher's theorem, it is differentiable for *a.e.* x_j, and

$$\left|\partial_{x_j}V_{\varepsilon,X_N}(x)\right|\leqslant\frac{2}{\varepsilon^{\frac{3}{2}}N}\qquad\text{for } a.e.\ x_j.$$

Consequently, $X_N\mapsto m_{\varepsilon,X_N}(x)$ is also Lipschitz uniformly in x.

(ii) This is an immediate consequence of lemma 2.4, the fact that $x\mapsto\tilde V_{\varepsilon,X_N}$ is Lipschitz uniformly in X_N, and the decomposition $V_{\varepsilon,X_N}=\tilde V_{\varepsilon,X_N}+\hat V_{\varepsilon,X_N}$.

□

We now renormalize the argument given in section 2 in order to establish the quantum mean-field limit.

Proof of theorem 1.3. We divide the proof into five steps:
1. Thanks to the conservation of total energy (lemma 3.2), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{\varepsilon,\hbar,N}(t) &= \frac{1}{2N}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}\Big(\sum_{j=1}^{N} i\hbar\partial_{x_j}\vee u(t,x_j)R_{\varepsilon,\hbar,N}(t)\Big) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}}|u|^2(t,x)\,\rho_{\varepsilon,\hbar,N:1}(t,x)\,\mathrm{d}x\\ &\quad+\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^N}\Big(\int_{\mathbb{T}} m_{\varepsilon,X_N}(x)\log(1/\rho(t,x))\,\mathrm{d}x\Big)\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N + \int_{\mathbb{T}}\partial_t\rho(t,x)\,\mathrm{d}x\\ &=: \sum_{j=1}^{4} I^j(t).\end{aligned}$$

2. In this step, we establish the following identity (compare with (2.8)):

$$\begin{aligned}&\frac{1}{2N}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}\Big(\sum_{j=1}^{N} i\hbar\partial_{x_j}\vee u(t,x_j)R_{\varepsilon,\hbar,N}(t)\Big) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d}|u|^2(t,x)\,\rho_{\varepsilon,\hbar,N:1}(t,x)\,\mathrm{d}x\\ &\quad= \frac{1}{2}\mathrm{tr}\big((i\hbar\partial_x+u)\vee(\partial_t u+u\partial_x u)R_{\varepsilon,\hbar,N:1}(t)\big)\\ &\quad-\frac{1}{4}\mathrm{tr}\big((i\hbar\partial_x+u)\vee((i\hbar\partial_x+u)\vee\partial_x u)R_{\varepsilon,\hbar,N:1}(t)\big)\\ &\quad+\frac{1}{N}\sum_{j=1}^{N}\int_{\mathbb{T}^N}\partial_{x_j}\big(\mathscr{I}_{\varepsilon,X_N}+\mathscr{V}_{\varepsilon,X_N}\big)u(t,x_j)\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N. \end{aligned}\tag{3.6}$$

We proceed in a manner similar to Step 2 in the proof of theorem 1.2. To make the equation lighter, we omit the dependency on time whenever there is no ambiguity.

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_j\mathrm{tr}\big(i\hbar\partial_{x_j}\vee u(x_j)R_{\varepsilon,\hbar,N}\big)+\frac{\mathrm{d}}{\mathrm{d}t}\sum_j\mathrm{tr}\big(|u|^2(x_j)R_{\varepsilon,\hbar,N}\big)\\ &=\sum_j\mathrm{tr}\big((i\hbar\partial_{x_j}\vee u(x_j)+|u|^2(x_j))\partial_t R_{\varepsilon,\hbar,N}\big)+\sum_j\mathrm{tr}\big(\partial_t(i\hbar\partial_{x_j}\vee u(x_j)+|u|^2(x_j))R_{\varepsilon,\hbar,N}\big)\\ &=\frac{1}{i\hbar}\sum_j\mathrm{tr}\big((i\hbar\partial_{x_j}\vee u(x_j)+|u|^2(x_j))[\mathscr{H}_{\varepsilon,\hbar,N},R_{\varepsilon,\hbar,N}]\big)+\sum_j\mathrm{tr}\big(\partial_t(i\hbar\partial_{x_j}\vee u(x_j)+|u|^2(x_j))R_{\varepsilon,\hbar,N}\big)\\ &=\frac{i}{\hbar}\mathrm{tr}\big([\mathscr{H}_{\varepsilon,\hbar,N},\sum_j(i\hbar\partial_{x_j}\vee u(x_j)+|u|^2(x_j))]R_{\varepsilon,\hbar,N}\big)+\mathrm{tr}\big(\partial_t(\sum_j(i\hbar\partial_{x_j}\vee u(x_j)+|u|^2(x_j)))R_{\varepsilon,\hbar,N}\big)\\ &=\mathrm{tr}\big((\partial_t+\frac{i}{\hbar}[\mathscr{H}_{\varepsilon,\hbar,N},\cdot])(\sum_j(i\hbar\partial_{x_j}\vee u(x_j)+|u|^2(x_j)))R_{\varepsilon,\hbar,N}\big)\\ &=\mathrm{tr}\big((\partial_t+\frac{i}{\hbar}[\mathscr{K}_{N,\hbar},\cdot])(\sum_j(i\hbar\partial_{x_j}\vee u(x_j)+\frac{1}{2}u(x_j)\vee u(x_j)))R_{\varepsilon,\hbar,N}\big)\\ &\quad+\mathrm{tr}\big((\frac{i}{\hbar}[\mathscr{I}_{\varepsilon,X_N}+\mathscr{V}_{\varepsilon,X_N},\cdot])(\sum_j(i\hbar\partial_{x_j}\vee u(x_j)+\frac{1}{2}u(x_j)\vee u(x_j)))R_{\varepsilon,\hbar,N}\big)\\ &=:\sum_j\mathrm{tr}\big(J_{1,j}R_{\varepsilon,\hbar,N}\big)+\sum_j\mathrm{tr}(J_{2,j}R_{\varepsilon,\hbar,N}).\end{aligned}$$

We now simplify the expression for $J_{1,j}$ and $J_{2,j}$. We write $J_{1,j}$ as

$$\begin{aligned}J_{1,j}&=\big(\partial_t+\frac{i}{\hbar}[-\frac{\hbar^2}{2}\Delta_N,\cdot]\big)\big((\frac{1}{2}u(x_j)+i\hbar\partial_{x_j})\vee u(x_j)\big)\\ &=\big((\partial_t+\frac{i}{\hbar}[-\frac{\hbar^2}{2}\Delta_N,\cdot]\big)\big(\frac{1}{2}u(x_j)+i\hbar\partial_{x_j}\big)\big)\vee u(x_j)\\ &\quad+\big(\frac{1}{2}u(x_j)+i\hbar\partial_{x_j}\big)\vee\big((\partial_t+\frac{i}{\hbar}[-\frac{\hbar^2}{2}\Delta_N,\cdot])u(x_j)\big)\\ &=\big((\partial_t+\frac{i}{\hbar}[-\frac{\hbar^2}{2}\Delta_N,\cdot])u(x_j)\big)\vee u(x_j)+i\hbar\partial_{x_j}\vee\big((\partial_t+\frac{i}{\hbar}[-\frac{\hbar^2}{2}\Delta_N,\cdot])u(x_j)\big),\end{aligned}$$

so that

$$J_{1,j} = \big(u(x_j) + i\hbar\partial_{x_j}\big) \vee \big((\partial_t + \frac{i}{\hbar}[-\frac{\hbar^2}{2}\Delta_N, \cdot])u(x_j)\big)$$
$$= \big(u(x_j) + i\hbar\partial_{x_j}\big) \vee \big(\partial_t u(x_j) + u(x_j)\partial_{x_j}u(x_j)\big)$$
$$+ \big(u(x_j) + i\hbar\partial_{x_j}\big) \vee \big(\frac{i}{\hbar}[-\frac{\hbar^2}{2}\Delta_N, u(x_j)] - u(x_j)\partial_{x_j}u(x_j)\big). \tag{3.7}$$

Notice that

$$\frac{i}{\hbar}\big[-\frac{\hbar^2}{2}\Delta_N, u(x_j)\big] = -\frac{i\hbar}{2}\partial_{x_j} \vee \big[\partial_{x_j}, u(x_j)\big] = -\frac{1}{2}i\hbar\partial_{x_j} \vee \partial_{x_j}u(x_j),$$

which shows that the right-hand side of (3.7) is equal to

$$\big(u(x_j) + i\hbar\partial_{x_j}\big) \vee \big(\partial_t u(x_j) + u(x_j)\partial_{x_j}u(x_j)\big) - \frac{1}{2}\big(u(x_j) + i\hbar\partial_{x_j}\big) \vee \big((i\hbar\partial_{x_j} + u(x_j)) \vee \partial_{x_j}u(x_j)\big),$$

so that

$$\frac{1}{N}\sum_j \mathrm{tr}\big(J_{1,j}R_{\varepsilon,\hbar,N}\big) = \frac{1}{N}\sum_j \mathrm{tr}\big((u(x_j) + i\hbar\partial_{x_j}) \vee (\partial_t u(x_j) + u(x_j)\partial_{x_j}u(x_j))R_{\varepsilon,\hbar,N}\big)$$
$$- \frac{1}{2N}\sum_j \mathrm{tr}\big((u(x_j) + i\hbar\partial_{x_j}) \vee ((i\hbar\partial_{x_j} + u(x_j)) \vee \partial_{x_j}u(x_j))R_{\varepsilon,\hbar,N}\big). \tag{3.8}$$

Turning to the calculation of $J_{2,j}$, we observe
$\frac{i}{\hbar}[\mathscr{I}_{\varepsilon,X_N} + \mathscr{V}_{\varepsilon,X_N}, i\hbar\partial_{x_j} \vee u(x_j)] = -2[\mathscr{I}_{\varepsilon,X_N} + \mathscr{V}_{\varepsilon,X_N}, u(x_j)\partial_{x_j}]$
so that

$$\mathrm{tr}\big(J_{2,j}R_{\varepsilon,\hbar,N}\big) = -2\,\mathrm{tr}\big([\mathscr{I}_{\varepsilon,X_N} + \mathscr{V}_{\varepsilon,X_N}, u(t,x_j)\partial_{x_j}]R_{\varepsilon,\hbar,N}\big)$$
$$= 2\int_{\mathbb{T}^N} \partial_{x_j}\big(\mathscr{I}_{\varepsilon,X_N} + \mathscr{V}_{\varepsilon,X_N}\big)u(t,x_j)\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N,$$

and hence

$$\frac{1}{2N}\sum_j \mathrm{tr}\big(J_{2,j}R_{\varepsilon,\hbar,N}\big) = \frac{1}{N}\sum_{j=1}^N \int_{\mathbb{T}^N} \partial_{x_j}\big(\mathscr{I}_{\varepsilon,X_N} + \mathscr{V}_{\varepsilon,X_N}\big)u(t,x_j)\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N. \tag{3.9}$$

The combination of (3.8) with (3.9) yields (3.6).
3. We simplify the last term in (3.6) as

$$\frac{1}{N}\sum_{j=1}^N \int_{\mathbb{T}^N} \partial_{x_j}\big(\mathscr{I}_{\varepsilon,X_N} + \mathscr{V}_{\varepsilon,X_N}\big)u(t,x_j)\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N$$
$$= \frac{1}{2\varepsilon}\sum_{j=1}^N \int_{\mathbb{T}^N} u(t,x_j)\partial_{x_j}\Big(\int_{\mathbb{T}\times\mathbb{T}} K(x-y)\,(\mu_{X_N} - m_{\varepsilon,X_N})^{\otimes 2}\,(\mathrm{d}x\mathrm{d}y)\Big)\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N$$
$$+ \sum_{j=1}^N \int_{\mathbb{T}^N} u(t,x_j)\partial_{x_j}\Big(\int_{\mathbb{T}} m_{\varepsilon,X_N}(x)\,V_{\varepsilon,X_N}(x)\,\mathrm{d}x\Big)\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N. \tag{3.10}$$

Recall that, by lemma 3.7, $X_N \mapsto V_{\varepsilon,X_N}$ and $X_N \mapsto m_{\varepsilon,X_N}$ are Lipschitz so that the forthcoming calculations are justified. First, since

$$\partial_{x_j}\Big(\frac{1}{N^2}\sum_{k\neq l} K(x_k - x_l)\Big) = \frac{2}{N^2}\sum_{k:\,k\neq j} K'(x_j - x_k),$$

we have

$$\partial_{x_j}\Big(\int_{\mathbb{T}\times\mathbb{T}} K(x-y)\,(\mu_{X_N} - m_{\varepsilon,X_N})^{\otimes 2}\,(\mathrm{d}x\mathrm{d}y)\Big)$$
$$= \frac{2}{N^2}\sum_{k:\,k\neq j} K'(x_j - x_k) - 2\int_{\mathbb{T}} (K\star m_{\varepsilon,X_N})\,(x)\,\partial_{x_j}\mu_{X_N}\,(\mathrm{d}x)$$

$$
\begin{aligned}
&- 2\int_{\mathbb{T}}\left(K\star\partial_{x_j} m_{\varepsilon,X_N}\right)(x)\,\mu_{X_N}(\mathrm{d}x) + 2\int_{\mathbb{T}}\left(K\star\partial_{x_j} m_{\varepsilon,X_N}\right)(x)\,m_{\varepsilon,X_N}(x)\,\mathrm{d}x \\
&:= \sum_{k=1}^{4} L_k^j. \qquad (3.11)
\end{aligned}
$$

We start with $L_1^j + L_2^j$. Note that

$$
\int_{\mathbb{T}} K\star m_{\varepsilon,X_N}\partial_{x_j}\mu_{X_N}(\mathrm{d}x) = -\frac{1}{N}\int_{\mathbb{T}} K\star m_{\varepsilon,X_N}\delta'_{x_j}(\mathrm{d}x) = \frac{1}{N}\left(K'\star m_{\varepsilon,X_N}\right)(x_j),
$$

so that

$$
\begin{aligned}
&\frac{1}{\varepsilon}\sum_{j=1}^{N} u(t,x_j)\left(\frac{2}{N^2}\sum_{k:\,k\neq j} K'(x_j-x_k) - 2\int_{\mathbb{T}}\left(K\star m_{\varepsilon,X_N}\right)\partial_{x_j}\mu_{X_N}(\mathrm{d}x)\right) \\
&\quad = \frac{2}{\varepsilon}\int_{\mathbb{T}\times\mathbb{T}\setminus\triangle} u(t,x)\,K'(x-y)\left(\mu_{X_N}-m_{\varepsilon,X_N}\right)(\mathrm{d}y)\,\mu_{X_N}(\mathrm{d}x)\,.
\end{aligned}
$$

We can symmetrize the right-hand side of the last identity as

$$
\begin{aligned}
&\frac{2}{\varepsilon}\int_{\mathbb{T}\times\mathbb{T}\setminus\triangle} u(t,x)\,K'(x-y)\left(\mu_{X_N}-m_{\varepsilon,X_N}\right)(\mathrm{d}x)\left(\mu_{X_N}-m_{\varepsilon,X_N}\right)(\mathrm{d}y) \\
&\qquad + \frac{2}{\varepsilon}\int_{\mathbb{T}\times\mathbb{T}\setminus\triangle} u(t,x)\,K'(x-y)\left(\mu_{X_N}-m_{\varepsilon,X_N}\right)(\mathrm{d}y)\,m_{\varepsilon,X_N}(\mathrm{d}x) \\
&\quad = \frac{1}{\varepsilon}\int_{\mathbb{T}\times\mathbb{T}\setminus\triangle}\left(u(t,x)-u(t,y)\right)K'(x-y)\left(\mu_{X_N}-m_{\varepsilon,X_N}\right)(\mathrm{d}x)\left(\mu_{X_N}-m_{\varepsilon,X_N}\right)(\mathrm{d}y) \\
&\qquad + \frac{2}{\varepsilon}\int_{\mathbb{T}\times\mathbb{T}\setminus\triangle} u(t,x)\,K'(x-y)\left(\mu_{X_N}-m_{\varepsilon,X_N}\right)(\mathrm{d}y)\,m_{\varepsilon,X_N}(\mathrm{d}x)\,.
\end{aligned}
$$

The second term on the right-hand side of the last identity is recast as

$$
\begin{aligned}
&-\frac{2}{\varepsilon}\int_{\mathbb{T}}\left(K'\star\left(m_{\varepsilon,X_N}u\right)\right)(t,y)\left(\mu_{X_N}-m_{\varepsilon,X_N}\right)(\mathrm{d}y) \\
&= 2\int_{\mathbb{T}}\left(K'\star\left(m_{\varepsilon,X_N}u\right)\right)(t,y)\,V''_{\varepsilon,X_N}(t,y)\,\mathrm{d}y \\
&= 2\int_{\mathbb{T}}\left(\left(m_{\varepsilon,X_N}u\right)(t,y) - \int_{\mathbb{T}}\left(m_{\varepsilon,X_N}u\right)(t,x)\,\mathrm{d}x\right)V'_{\varepsilon,X_N}(t,y)\,\mathrm{d}y \\
&= 2\int_{\mathbb{T}}\left(\partial_x m_{\varepsilon,X_N}u\right)(t,x)\,\mathrm{d}x \\
&= -2\int_{\mathbb{T}}\partial_x u(t,x)\,m_{\varepsilon,X_N}(x)\,\mathrm{d}x.
\end{aligned}
$$

Thus, we have proved the identity

$$
\begin{aligned}
\frac{1}{\varepsilon}\sum_{j=1}^{N} u(t,x_j)\left(L_1^j+L_2^j\right) &= \frac{1}{\varepsilon}\int_{\mathbb{T}\times\mathbb{T}\setminus\triangle}\left(u(t,x)-u(t,y)\right)K'(x-y)\left(\mu_{X_N}-m_{\varepsilon,X_N}\right)^{\otimes 2}(\mathrm{d}x\mathrm{d}y) \\
&\quad - 2\int_{\mathbb{T}}\partial_x u(t,x)\,m_{\varepsilon,X_N}(x)\,\mathrm{d}x. \qquad (3.12)
\end{aligned}
$$

We continue with the calculation of $L_3^j + L_4^j$. We have

$$
\begin{aligned}
\frac{1}{\varepsilon}\left(L_3^j+L_4^j\right) &= -\frac{2}{\varepsilon}\int_{\mathbb{T}} K\star\partial_{x_j} m_{\varepsilon,X_N}\left(\mu_{X_N}-m_{\varepsilon,X_N}\right)(\mathrm{d}x) \\
&= 2\int_{\mathbb{T}} K\star\partial_{x_j} m_{\varepsilon,X_N}V''_{\varepsilon,X_N}(\mathrm{d}x) \\
&= -2\int_{\mathbb{T}}\partial_{x_j}\left(m_{\varepsilon,X_N}\right)V_{\varepsilon,X_N}(\mathrm{d}x) \\
&= -2\partial_{x_j}\left(\int_{\mathbb{T}} m_{\varepsilon,X_N}(x)\,V_{\varepsilon,X_N}(x)\,\mathrm{d}x\right),
\end{aligned}
$$

where the last identity is due to the observation that

$$\int_{\mathbb{T}} m_{\varepsilon,X_N}(x)\,\partial_{x_j} V_{\varepsilon,X_N}(x)\,\mathrm{d}x = \partial_{x_j}\int_{\mathbb{T}} m_{\varepsilon,X_N}(x)\,\mathrm{d}x = 0.$$

Therefore, we have

$$\frac{1}{\varepsilon}\sum_{j=1}^{N} u(t,x_j)\big(L_3^j + L_4^j\big) = -2\sum_{j=1}^{N} u(t,x_j)\partial_{x_j}\Big(\int_{\mathbb{T}} m_{\varepsilon,X_N}(x)\,V_{\varepsilon,X_N}(x)\,\mathrm{d}x\Big). \tag{3.13}$$

Substituting (3.12) and (3.13) into (3.10), we obtain

$$\begin{aligned}
&\frac{1}{N}\sum_{j=1}^{N}\int_{\mathbb{T}^N}\partial_{x_j}\big(\mathscr{I}_{\varepsilon,X_N}+\mathscr{V}_{\varepsilon,X_N}\big)u(t,x_j)\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N\\
&=\frac{1}{2\varepsilon}\int_{\mathbb{T}^N}\Big(\int_{\mathbb{T}\times\mathbb{T}\setminus\triangle}\big(u(t,x)-u(t,y)\big)K'(x-y)\,(\mu_{X_N}-m_{\varepsilon,X_N})^{\otimes 2}(\mathrm{d}x\mathrm{d}y)\Big)\,\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N\\
&\quad-\int_{\mathbb{T}^N}\Big(\int_{\mathbb{T}}\partial_x u(t,x)\,m_{\varepsilon,X_N}(x)\,\mathrm{d}x\Big)\,\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N,
\end{aligned}$$

so that, in view of (3.6),

$$\begin{aligned}
I^1(t)+I^2(t)=&\frac{1}{2}\mathrm{tr}\big((i\hbar\partial_x+u)\vee(\partial_t u+u\partial_x u)\,R_{\varepsilon,\hbar,N:1}(t)\big)\\
&-\frac{1}{4}\mathrm{tr}\big((i\hbar\partial_x+u)\vee(i\hbar\partial_x+u)\vee(\partial_x u)\,R_{\varepsilon,\hbar,N:1}(t)\big)\\
&-\int_{\mathbb{T}^N}\Big(\int_{\mathbb{T}}\partial_x u(t,x)\,m_{\varepsilon,X_N}(x)\,\mathrm{d}x\Big)\,\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N\\
&+\frac{1}{2\varepsilon}\int_{\mathbb{T}^N}\Big(\int_{\mathbb{T}\times\mathbb{T}\setminus\triangle}\big(u(t,x)-u(t,y)\big)K'(x-y)\,(\mu_{X_N}-m_{\varepsilon,X_N})^{\otimes 2}(\mathrm{d}x\mathrm{d}y)\Big)\,\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N.
\end{aligned} \tag{3.14}$$

Next we compute the term

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^N}\Big(\int_{\mathbb{T}} m_{\varepsilon,X_N}(x)\log(1/\rho(t,x))\,\mathrm{d}x\Big)\,\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N+\int_{\mathbb{T}}\partial_t\rho(t,x)\,\mathrm{d}x.$$

We have

$$\begin{aligned}
I^3(t)+I^4(t)=&\int_{\mathbb{T}^N}\int_{\mathbb{T}}\frac{\mathrm{d}}{\mathrm{d}t}\big(m_{\varepsilon,X_N}(x)\,\rho_{\varepsilon,\hbar,N}(t,X_N)\big)\log(1/\rho(t,x))\,\mathrm{d}x\mathrm{d}X_N\\
&-\int_{\mathbb{T}^N}\int_{\mathbb{T}} m_{\varepsilon,X_N}(x)\,\rho_{\varepsilon,\hbar,N}t(t,X_N)\Big(\frac{\partial_t\rho}{\rho}\Big)(t,x)\,\mathrm{d}x\mathrm{d}X_N+\int_{\mathbb{T}}\partial_t\rho(t,x)\,\mathrm{d}x\\
=&\int_{\mathbb{T}}\Big(1-\frac{\int_{\mathbb{T}^N}e^{V_{\varepsilon,X_N}}\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N}{\rho(t,x)}\Big)\partial_t\rho(t,x)\,\mathrm{d}x\\
&+\int_{\mathbb{T}}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_{\mathbb{T}^N}\big(\mu_{X_N}+\varepsilon V''_{\varepsilon,X_N}\big)\,\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N\Big)\log(1/\rho(t,x))\,\mathrm{d}x.
\end{aligned} \tag{3.15}$$

Thanks to lemma 3.2 and noticing that $\int_{\mathbb{T}^N}\rho_{\varepsilon,\hbar,N}(t,X_N)\mu_{X_N}(x)\,\mathrm{d}X_N=\rho_{\varepsilon,\hbar,N:1}(t,x)$, we have

$$\begin{aligned}
I^3(t)+I^4(t)=&\int_{\mathbb{T}}\Big(1-\frac{\int_{\mathbb{T}^N}e^{V_{\varepsilon,X_N}}\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N}{\rho(t,x)}\Big)\partial_t\rho(t,x)\,\mathrm{d}x\\
&+\varepsilon\int_{\mathbb{T}}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_{\mathbb{T}^N}V''_{\varepsilon,X_N}\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N\Big)\log(1/\rho(t,x))\,\mathrm{d}x\\
&-\int_{\mathbb{T}}\partial_x J_{\varepsilon,\hbar,N:1}(t,x)\log(1/\rho(t,x))\,\mathrm{d}x.
\end{aligned} \tag{3.16}$$

Also, observe that

$$
\begin{aligned}
&\frac{1}{2}\mathrm{tr}\Big((i\hbar\partial_x + u) \vee \partial_x \log(\rho)\, R_{\varepsilon,\hbar,N:1}(t) \Big) \\
&\quad = \int_{\mathbb{T}} u(t,x)\,\partial_x \log(\rho(t,x))\,\rho_{\varepsilon,\hbar,N:1}(t,x)\,\mathrm{d}x - \int_{\mathbb{T}} J_{\varepsilon,\hbar,N:1}(t,x)\,\partial_x \log(\rho(t,x))\,\mathrm{d}x \\
&\quad = \int_{\mathbb{T}} \Big(\int_{\mathbb{T}^N} e^{V_{\varepsilon,X_N}(x)} \rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N \Big) u(t,x)\,\partial_x \log(\rho(t,x))\,\mathrm{d}x \\
&\quad - \varepsilon \int_{\mathbb{T}} \Big(\int_{\mathbb{T}^N} V''_{\varepsilon,X_N}(x)\,\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N \Big) u(t,x)\,\partial_x \log(\rho(t,x))\,\mathrm{d}x \\
&\quad - \int_{\mathbb{T}} J_{\varepsilon,\hbar,N:1}(t,x)\,\partial_x \log(\rho(t,x))\,\mathrm{d}x.
\end{aligned}
$$

Consequently, we have proved

$$
\begin{aligned}
\sum_{j=1}^{4} I^j(t) &= \frac{1}{2}\mathrm{tr}\Big((i\hbar\partial_x + u) \vee (\partial_t u + u\partial_x u + \partial_x \log(\rho)) R_{\varepsilon,\hbar,N:1}(t)\Big) \\
&- \frac{1}{4}\mathrm{tr}\Big((i\hbar\partial_x + u) \vee ((i\hbar\partial_x + u) \vee (\partial_x u)) R_{\varepsilon,\hbar,N:1}(t)\Big) \\
&- \int_{\mathbb{T}^N} \Big(\int_{\mathbb{T}} \partial_x u(t,x)\, m_{\varepsilon,X_N}(x)\,\mathrm{d}x \Big) \rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N \\
&+ \frac{1}{2\varepsilon} \int_{\mathbb{T}^N} \Big(\int_{\mathbb{T}\times\mathbb{T}\setminus\triangle} (u(t,x) - u(t,y)) K'(x-y) (\mu_{X_N} - m_{\varepsilon,X_N})^{\otimes 2}(\mathrm{d}x\mathrm{d}y) \Big) \rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N \\
&+ \int_{\mathbb{T}} \Big(-\frac{\int_{\mathbb{T}^N} e^{V_{\varepsilon,X_N}} \rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N}{\rho(t,x)} + 1 \Big) \partial_t \rho(t,x)\,\mathrm{d}x \\
&+ \varepsilon \int_{\mathbb{T}} \frac{\mathrm{d}}{\mathrm{d}t} \Big(\int_{\mathbb{T}^N} V''_{\varepsilon,X_N} \rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N \Big) \log(1/\rho(t,x))\,\mathrm{d}x \\
&- \int_{\mathbb{T}} \Big(\int_{\mathbb{T}^N} e^{V_{\varepsilon,X_N}(x)} \rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N \Big) u(t,x)\partial_x \log(\rho(t,x))\,\mathrm{d}x \\
&+ \varepsilon \int_{\mathbb{T}} \Big(\int_{\mathbb{T}^N} V''_{\varepsilon,X_N}(x) \rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N \Big) u(t,x)\partial_x \log(\rho(t,x))\,\mathrm{d}x
\end{aligned}
$$

By the equation, we have

$$
\frac{1}{2}\mathrm{tr}\Big((i\hbar\partial_x + u) \vee (\partial_t u + u\partial_x u + \partial_x \log(\rho))\, R_{\varepsilon,\hbar,N:1}(t)\Big) = 0,
$$

so that

$$
\begin{aligned}
\sum_{j=1}^{4} I^j(t) &= \varepsilon \int_{\mathbb{T}} \Big(\int_{\mathbb{T}^N} V''_{\varepsilon,X_N}(x)\,\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N \Big) u(t,x)\,\partial_x \log(\rho(t,x))\,\mathrm{d}x \\
&+ \varepsilon \int_{\mathbb{T}} \frac{\mathrm{d}}{\mathrm{d}t} \Big(\int_{\mathbb{T}^N} V''_{\varepsilon,X_N} \rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N \Big) \log(1/\rho(t,x))\,\mathrm{d}x \\
&- \frac{1}{4}\mathrm{tr}\Big((i\hbar\partial_x + u) \vee ((i\hbar\partial_x + u) \vee (\partial_x u)) R_{\varepsilon,\hbar,N:1}(t)\Big) \\
&+ \frac{1}{2\varepsilon} \int_{\mathbb{T}^N} \Big(\int_{\mathbb{T}\times\mathbb{T}\setminus\triangle} (u(t,x) - u(t,y)) K'(x-y) (\mu_{X_N} - m_{\varepsilon,X_N})^{\otimes 2}(\mathrm{d}x\mathrm{d}y) \Big) \rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N \\
&- \int_{\mathbb{T}^N} \Big(\int_{\mathbb{T}} \partial_x u(t,x)\, m_{\varepsilon,X_N}(x)\,\mathrm{d}x \Big) \rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N \\
&+ \int_{\mathbb{T}} \Big(-\frac{\int_{\mathbb{T}^N} e^{V_{\varepsilon,X_N}} \rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N}{\rho(t,x)} + 1 \Big) \partial_t \rho(t,x)\,\mathrm{d}x \\
&- \int_{\mathbb{T}} \Big(\int_{\mathbb{T}^N} e^{V_{\varepsilon,X_N}(x)} \rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N \Big) u(t,x)\partial_x \log(\rho(t,x))\,\mathrm{d}x.
\end{aligned}
$$

Furthermore, by the equation: $\partial_t\rho + \partial_x(\rho u) = 0$, the last three terms in the identity above cancel out. Then we finally obtain

$$
\begin{aligned}
\sum_{j=1}^{4} I^j(t) = &\ \varepsilon \int_{\mathbb{T}^N} \Big(\int_{\mathbb{T}} V''_{\varepsilon,X_N}(x) u(t,x) \partial_x \log \rho(t,x)\, \mathrm{d}x \Big) \rho_{\varepsilon,\hbar,N}(t,X_N)\, \mathrm{d}X_N \\
&+ \varepsilon \int_{\mathbb{T}} \frac{\mathrm{d}}{\mathrm{d}t} \Big(\int_{\mathbb{T}^N} V''_{\varepsilon,X_N}(x) \rho_{\varepsilon,\hbar,N}(t,X_N)\, \mathrm{d}X_N \Big) \log(1/\rho(t,x))\, \mathrm{d}x \\
&- \frac{1}{4} \mathrm{tr}\Big((i\hbar\partial_x + u) \vee (i\hbar\partial_x + u) \vee (\partial_x u)) R_{\varepsilon,\hbar,N:1}(t) \Big) \\
&+ \frac{1}{2\varepsilon} \int_{\mathbb{T}^N} \Big(\int_{\mathbb{T}\times\mathbb{T}\setminus\triangle} (u(t,x) - u(t,y)) K'(x-y) (\mu_{X_N} - m_{\varepsilon,X_N})^{\otimes 2}(\mathrm{d}x\mathrm{d}y) \Big) \rho_{\varepsilon,\hbar,N}(t,X_N)\, \mathrm{d}X_N \\
=: &\sum_{k=1}^{4} J_k.
\end{aligned}
$$

4. By lemma 3.5 (ii), for $\lambda > 0$, we have

$$
\begin{aligned}
|J_1| &\leqslant \int_{\mathbb{T}^N} \left| \int_{\mathbb{T}} (\mu_{X_N} - m_{\varepsilon,X_N}(x))\, u(t,x)\, \partial_x \log(\rho(t,x))\, \mathrm{d}x \right| \rho_{\varepsilon,\hbar,N}(t,X_N)\, \mathrm{d}X_N \\
&\leqslant C \|u\partial_x \log(\rho)\|_{L^\infty_t W^{1,\infty}_x} N^{-\lambda} \\
&\quad + C \|u\partial_x \log(\rho)\|_{L^\infty_t \dot{H}^1_x} \int_{\mathbb{T}^N} \left(\mathscr{E}(X_N, m_{\varepsilon,X_N}) + \frac{1 + \|m_{\varepsilon,X_N}\|_\infty}{N^2} \right)^{\frac{1}{2}} \rho_{\varepsilon,\hbar,N}(t,X_N)\, \mathrm{d}X_N.
\end{aligned}
$$

Thanks to lemma 3.3, we obtain the inequality:

$$
\begin{aligned}
\left(\mathscr{E}(X_N, m_{\varepsilon,X_N}) + \frac{1 + \|m_{\varepsilon,X_N}\|_\infty}{N^2} \right)^{\frac{1}{2}} &\leqslant \varepsilon + \frac{1}{\varepsilon} \int_{\mathbb{T}\times\mathbb{T}} K(x-y) (\mu_{X_N} - m_{\varepsilon,X_N})^{\otimes 2}(\mathrm{d}x\mathrm{d}y) + \frac{1 + \|m_{\varepsilon,X_N}\|_\infty}{\varepsilon N^2} \\
&\leqslant \varepsilon + \frac{1}{\varepsilon} \int_{\mathbb{T}\times\mathbb{T}} K(x-y) (\mu_{X_N} - m_{\varepsilon,X_N})^{\otimes 2}(\mathrm{d}x\mathrm{d}y) + \frac{1 + e^{\frac{1}{\varepsilon}}}{\varepsilon N^2}.
\end{aligned}
$$

Therefore, it follows that

$$
|J_1| \leqslant C \|u\partial_x \log(\rho)\|_{L^\infty_t W^{1,\infty}_x} \left(\varepsilon + N^{-\lambda} + \mathcal{V}_{\varepsilon,\hbar,N}(t) + \frac{2e^{\frac{1}{\varepsilon}}}{\varepsilon N^2} \right). \tag{3.17}
$$

Next, note that

$$
\begin{aligned}
\int_0^t J_2(s)\, \mathrm{d}s &= -\varepsilon \int_0^t \int_{\mathbb{T}} \partial_s \Big(\int_{\mathbb{T}^N} V''_{\varepsilon,X_N}(x) \rho_{\varepsilon,\hbar,N}(s,X_N)\, \mathrm{d}X_N \Big) \log(\rho(s,x))\, \mathrm{d}x\mathrm{d}s \\
&= \varepsilon \int_0^t \int_{\mathbb{T}^N} \Big(\int_{\mathbb{T}} V''_{\varepsilon,X_N}(x) \partial_s \log(\rho(s,x))\, \mathrm{d}x \Big) \rho_{\varepsilon,\hbar,N}(s,X_N)\, \mathrm{d}X_N \mathrm{d}s \\
&\quad - \varepsilon \int_{\mathbb{T}^N} \Big(\int_{\mathbb{T}} \log(\rho(0,x)) V''_{\varepsilon,X_N}(x)\, \mathrm{d}x \Big) \rho_{\varepsilon,\hbar,N}(0,X_N)\, \mathrm{d}X_N \\
&\quad + \varepsilon \int_{\mathbb{T}^N} \Big(\int_{\mathbb{T}} \log(\rho(t,x)) V''_{\varepsilon,X_N}(x)\, \mathrm{d}x \Big) \rho_{\varepsilon,\hbar,N}(t,X_N)\, \mathrm{d}X_N.
\end{aligned}
$$

Once again, in view of lemma 3.5(ii) and lemma 3.3 for each $\varphi \in W^{1,\infty}(\mathbb{T}^d)$, we have the estimate:

$$
\begin{aligned}
&\left| \varepsilon \int_{\mathbb{T}^N} \Big(\int_{\mathbb{T}} V''_{\varepsilon,X_N}(x) \varphi(x)\, \mathrm{d}x \Big) \rho_{\varepsilon,\hbar,N}(s,X_N)\, \mathrm{d}X_N \right| \\
&\quad \leqslant \varepsilon \int_{\mathbb{T}^N} \left| \int_{\mathbb{T}} V''_{\varepsilon,X_N}(x) \varphi(x)\, \mathrm{d}x \right| \rho_{\varepsilon,\hbar,N}(s,X_N)\, \mathrm{d}X_N \\
&\quad \leqslant C \|\varphi\|_{W^{1,\infty}} N^{-\lambda} + \|\varphi\|_{H^1} \int_{\mathbb{T}^N} \left(\mathscr{E}(X_N, m_{\varepsilon,X_N}) + \frac{2e^{\frac{1}{\varepsilon}}}{N^2} \right)^{\frac{1}{2}} \rho_{\varepsilon,\hbar,N}(s,X_N)\, \mathrm{d}X_N \\
&\quad \leqslant C \|\varphi\|_{W^{1,\infty}} N^{-\lambda} + \sqrt{\varepsilon} \|\varphi\|^2_{H^1} + \sqrt{\varepsilon} \mathcal{V}_{\varepsilon,\hbar,N}(s) + \frac{2e^{\frac{1}{\varepsilon}}}{N^2\sqrt{\varepsilon}} \\
&\quad \leqslant C \|\varphi\|_{W^{1,\infty}} N^{-\lambda} + \sqrt{\varepsilon} \|\varphi\|^2_{H^1} + \sqrt{\varepsilon} \mathcal{F}_0 + \frac{2e^{\frac{1}{\varepsilon}}}{N^2\sqrt{\varepsilon}},
\end{aligned} \tag{3.18}
$$

where the last inequality is due to the conservation of energy (lemma 3.2) and the assumption that $\mathcal{F}_{\varepsilon,\hbar,N}(0) \leqslant \mathcal{F}_0$. Utilizing (3.18) with $\varphi = \partial_s \log(\rho(s,x))$, $\log(\rho(0,x))$, and $\log(\rho(t,x))$, respectively, we see that

$$\int_0^t J_2(s)\,\mathrm{d}s \leqslant C\Big(N^{-\lambda} + \sqrt{\varepsilon} + \frac{e^{\frac{1}{\varepsilon}}}{\sqrt{\varepsilon}N^2}\Big), \tag{3.19}$$

where $C = C(\|\log(\rho)\|_{W^{1,\infty}_{t,x}}, \mathcal{F}_0, T)$. As for J_3, we write the eigenfunction decomposition of $R_{\varepsilon,\hbar,N:1}$:

$$R_{\varepsilon,\hbar,N:1}(t) = \sum_{k=1}^{\infty} \lambda_k |\psi_k\rangle\langle\psi_k| \qquad \text{with } \lambda_k \geqslant 0, \sum_{k=1}^{\infty}\lambda_k = 1, \text{ and } \psi_k \in \mathfrak{H}.$$

Therefore, recalling $\Pi = i\hbar\partial_x + u$, we have

$$\begin{aligned} -4J_3 =& 2\sum_{k=1}^{\infty}\lambda_k \int_{\mathbb{T}} (\Pi\psi_k)\big(\partial_x u \overline{\Pi\psi_k}\big)(t,x)\,\mathrm{d}x + 2\sum_{k=1}^{\infty}\lambda_k \int_{\mathbb{T}} (\Pi\psi_k)\partial_x u\big(\overline{\Pi\psi_k}\big)(t,x)\,\mathrm{d}x \\ &+ 2\sum_{k=1}^{\infty}\lambda_k \int_{\mathbb{T}} \psi_k\, i\hbar\partial_{xx}u\big(\overline{\Pi\psi_k}\big)(t,x)\,\mathrm{d}x + \mathrm{tr}\big(\Pi \vee i\hbar\partial_{xx}u R_{\varepsilon,\hbar,N:1}\big). \end{aligned} \tag{3.21}$$

Arguing exactly as in Step 4 of the proof of theorem 1.2, we can estimate the four terms in (3.21) and obtain

$$|J_3| \leqslant \Big|\mathrm{tr}\big((i\hbar\partial_x + u) \vee ((i\hbar\partial_x + u)\vee(\partial_x u))R_{\varepsilon,\hbar,N:1}(t)\big)\Big| \leqslant C\left(\mathcal{K}_{\varepsilon,\hbar,N}(t) + \hbar^2\right), \tag{3.22}$$

where $C = C(\|u\|_{L^\infty_t W^{2,\infty}_x})$. Finally, applying lemmas 3.3 and 3.6, we have

$$|J_4| \leqslant C\Big(\mathcal{V}_{\varepsilon,\hbar,N}(t) + \frac{1+e^{\frac{1}{\varepsilon}}}{N^2\varepsilon}\Big), \tag{3.23}$$

where $C = C(\|u\|_{L^\infty_t W^{1,\infty}_x})$. Thus, gathering inequalities (3.17) and (3.19)–(3.23), we find

$$\mathcal{E}_{\varepsilon,\hbar,N}(t) \leqslant \mathcal{E}_{\varepsilon,\hbar,N}(0) + C\Big(\int_0^t \mathcal{E}_{\varepsilon,\hbar,N}(s)\,\mathrm{d}s + N^{-\lambda} + \sqrt{\varepsilon} + \frac{e^{\frac{1}{\varepsilon}}}{\varepsilon N^2} + \hbar^2\Big),$$

where $C = C\big(T, \|u\|_{L^\infty_t W^{2,\infty}_x}, \|\log(\rho)\|_{W^{1,\infty}_{t,x}}, \mathcal{F}_0\big)$. It follows from the Grönwall inequality and lemma 3.5(i) that

$$\mathcal{E}_{\varepsilon,\hbar,N}(t) \leqslant e^{Ct}\Big(\mathcal{E}_{\varepsilon,\hbar,N}(0) + N^{-\lambda} + \sqrt{\varepsilon} + \frac{e^{\frac{1}{\varepsilon}}}{\varepsilon N^2} + \hbar^2\Big). \tag{3.24}$$

5. Define $\tilde{m}_{\varepsilon,\hbar,N}(t,x) := \int_{\mathbb{T}^N} m_{\varepsilon,X_N}(x)\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N$. According to lemma 3.3, we have

$$\|m_{\varepsilon,X_N}\|_\infty \leqslant e^{\frac{1}{\varepsilon}}$$

so that, in view of lemma 3.5(ii), we find

$$\sup_{t\in[0,T]} W_1\left(\rho_{\varepsilon,\hbar,N:1}(t,\cdot), \tilde{m}_{\varepsilon,\hbar,N}(t,\cdot)\right) \leqslant \frac{C}{N^\lambda} + \Big(\sup_{t\in[0,T]}\mathcal{E}_{\varepsilon,\hbar,N}(t) + \frac{Ce^{\frac{1}{\varepsilon}}}{\varepsilon N^2}\Big)^{\frac{1}{2}}. \tag{3.25}$$

In addition, thanks to lemma 2.13, we see that

$$\begin{aligned} \int_{\mathbb{T}} |\tilde{m}_{\varepsilon,\hbar,N} - \rho|(t,x)\,\mathrm{d}x &\leqslant \int_{\mathbb{T}^N}\Big(\int_{\mathbb{T}} |m_{\varepsilon,X_N} - \rho|(t,x)\,\mathrm{d}x\Big)\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N \\ &\leqslant \int_{\mathbb{T}^N}\sqrt{2\int_{\mathbb{T}} m_{\varepsilon,X_N}(x)\log\Big(\frac{m_{\varepsilon,X_N}(x)}{\rho(t,x)}\Big)\mathrm{d}x}\,\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N \\ &\leqslant \sqrt{2\int_{\mathbb{T}^N}\Big(\int_{\mathbb{T}} m_{\varepsilon,X_N}(x)\log\Big(\frac{m_{\varepsilon,X_N}(x)}{\rho(t,x)}\Big)\mathrm{d}x\Big)\rho_{\varepsilon,\hbar,N}(t,X_N)\,\mathrm{d}X_N}, \end{aligned}$$

where the last inequality is by the Cauchy–Schwarz inequality and the fact that $\rho_{\varepsilon,\hbar,N}$ is a probability density. Thus, it follows from lemma 3.5(i) that

$$\sup_{t\in[0,T]} \|(\tilde{m}_{\varepsilon,\hbar,N}-\rho)(t,\cdot)\|_1 \leqslant C\Big(\sup_{t\in[0,T]} \mathcal{E}_{\varepsilon,\hbar,N}(t)+\frac{1+e^{\frac{1}{\varepsilon}}}{\varepsilon N^2}\Big)^{\frac{1}{2}} \tag{3.26}$$

for some constant $C>0$ independent of $(\varepsilon,\hbar,N)$. Hence, thanks to the assumption on $\varepsilon=\varepsilon(N)$, it follows from inequalities (3.24)–(3.26) that

$$\sup_{t\in[0,T]} W_1\left(\rho_{\varepsilon,\hbar,N:1}(t,\cdot),\rho(t,\cdot)\right) \underset{\varepsilon+\hbar+\frac{1}{N}\to 0}{\longrightarrow} 0.$$

The convergence:

$$J_{\varepsilon,\hbar,N:1}(t,\cdot) \underset{\varepsilon+\hbar+\frac{1}{N}\to 0}{\longrightarrow} (\rho u)(t,\cdot)$$

is deduced by the same argument presented in Step 5 in the proof of theorem 1.2. This completes the proof of theorem 1.3. □

4. Well-prepared initial data

We start by constructing well-prepared initial data for the 1-body problem, *i.e.* the initial data such that $\mathcal{E}_{\varepsilon,\hbar}(0)\to 0$ as $\varepsilon+\hbar\to 0$. Our construction of the initial data is summarized in the following lemma:

Lemma 4.1. *Let $\rho_0\in H^s(\mathbb{T}^d)\cap\mathcal{P}(\mathbb{T}^d)$, $\rho_0>0$, and $u_0=\nabla U_0\in H^s(\mathbb{T}^d)$ for some potential $U_0\in H^{s+1}(\mathbb{T}^d)$ with $s>1$ sufficiently large. Let $V_0=\log(\rho_0)$. Set*

$$\psi^{\text{in}}_{\varepsilon,\hbar} := \sqrt{e^{V_0}-\varepsilon\Delta V_0}\,e^{\frac{iU_0(x)}{\hbar}}. \tag{4.1}$$

Then, for this choice of $\psi^{\text{in}}_{\varepsilon,\hbar}$,

$$\mathcal{E}_{\varepsilon,\hbar}(0) \underset{\varepsilon+\hbar\to 0}{\longrightarrow} 0.$$

Proof. Clearly, for ε sufficiently small, $e^{V_0}-\varepsilon\Delta V_0$ is positive and ΔV_0 has mean 0, so that $|\psi^{\text{in}}_{\varepsilon,\hbar}|^2$ is a probability density. Moreover, it follows from the construction that the Poisson–Boltzmann equation

$$-\varepsilon\Delta V_0 = \left|\psi^{\text{in}}_{\varepsilon,\hbar}\right|^2 - e^{V_0}$$

is satisfied. To show that $\mathcal{E}_{\varepsilon,\hbar}(0) \underset{\varepsilon+\hbar\to 0}{\to} 0$, we start by controlling the kinetic part. Put $\varrho_\varepsilon := \sqrt{e^{V_0}-\varepsilon\Delta V_0}$ so that

$$\mathcal{K}_{\varepsilon,\hbar}(0) = \frac{1}{2}\int_{\mathbb{T}^d}\left|(i\hbar\nabla+u_0)\psi^{\text{in}}_{\varepsilon,\hbar}\right|^2(x)\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{T}^d}\left|(i\hbar\nabla+u_0(x))\,\varrho_\varepsilon(x)\,e^{\frac{iU_0(x)}{\hbar}}\right|^2\mathrm{d}x.$$

We expand the right-hand side of the last identity. First, notice that

$$\begin{aligned}\hbar^2\int_{\mathbb{T}^d}\left|\nabla\left(\varrho_\varepsilon e^{\frac{iU_0(x)}{\hbar}}\right)\right|^2\mathrm{d}x &= \int_{\mathbb{T}^d}|\varrho_\varepsilon\nabla U_0|^2(x)\,\mathrm{d}x + 2\hbar^2\Re\int_{\mathbb{T}^d}\nabla\varrho_\varepsilon(x)\,e^{-\frac{iU_0(x)}{\hbar}}\,\varrho_\varepsilon(x)\,\nabla e^{\frac{iU_0(x)}{\hbar}}\,\mathrm{d}x\\ &\quad+\hbar^2\int_{\mathbb{T}^d}|\nabla\varrho_\varepsilon|^2(x)\,\mathrm{d}x,\\ \int_{\mathbb{T}^d}\left|u_0(x)\,\varrho_\varepsilon e^{\frac{iU_0(x)}{\hbar}}\right|^2\mathrm{d}x &= \int_{\mathbb{T}^d}|\varrho_\varepsilon u_0|^2(x)\,\mathrm{d}x,\end{aligned}$$

and

$$\begin{aligned}&i\hbar\int_{\mathbb{T}^d}u_0(x)\,\varrho_\varepsilon(x)\,e^{-\frac{iU_0(x)}{\hbar}}\nabla\left(\varrho_\varepsilon(x)\,e^{\frac{iU_0(x)}{\hbar}}\right)\mathrm{d}x - i\hbar\int_{\mathbb{T}^d}\varrho_\varepsilon(x)\,e^{\frac{iU_0(x)}{\hbar}}u_0(x)\,\nabla\left(\varrho_\varepsilon(x)\,e^{-\frac{iU_0(x)}{\hbar}}\right)\mathrm{d}x\\ &\quad= i\hbar\int_{\mathbb{T}^d}u_0(x)\,\varrho^2_\varepsilon(x)\,e^{-\frac{iU_0(x)}{\hbar}}\nabla\left(e^{\frac{iU_0(x)}{\hbar}}\right)\mathrm{d}x - i\hbar\int_{\mathbb{T}^d}\varrho^2_\varepsilon(x)\,e^{\frac{iU_0(x)}{\hbar}}u_0(x)\,\nabla\left(e^{-\frac{iU_0(x)}{\hbar}}\right)\mathrm{d}x\\ &\quad= -2\int_{\mathbb{T}^d}\varrho^2_\varepsilon(x)\,\nabla U_0(x)\cdot u_0(x)\,\mathrm{d}x.\end{aligned}$$

Thus, expanding the square yields

$$\mathcal{K}_{\varepsilon,\hbar}(0) = \int_{\mathbb{T}^d} |\varrho_\varepsilon (\nabla U_0 - u_0)|^2 (x)\, \mathrm{d}x + 2\hbar^2 \Re \int_{\mathbb{T}^d} \nabla \varrho_\varepsilon(x) e^{-\frac{iU_0(x)}{\hbar}} \varrho_\varepsilon(x) \nabla e^{\frac{iU_0(x)}{\hbar}}\, \mathrm{d}x$$
$$+ \hbar^2 \int_{\mathbb{T}^d} |\nabla \varrho_\varepsilon(x)|^2\, \mathrm{d}x. \tag{4.2}$$

Since $u_0 = \nabla U_0$ and the second integrand is purely imaginary, then the first and the second term in (4.2) vanishes identically. Clearly, $\|\nabla \varrho_\varepsilon\|_2$ is uniformly bounded in ε so that the third term in (4.2) are bounded by

$$\hbar^2 \|\nabla \varrho_\varepsilon\|_2^2 \underset{\varepsilon+\hbar\to 0}{\longrightarrow} 0.$$

To conclude, this implies

$$\mathcal{K}_{\varepsilon,\hbar}(0) \underset{\varepsilon+\hbar\to 0}{\longrightarrow} 0.$$

Finally, it is direct that the last term in (1.6) vanishes identically and the second term $\mathcal{V}_{\varepsilon,\hbar}(0)$ satisfies

$$\mathcal{V}_{\varepsilon,\hbar}(0) = \frac{\varepsilon}{2} \int_{\mathbb{T}^d} |\nabla V_0(x)|^2\, \mathrm{d}x \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Therefore, we conclude

$$\mathcal{E}_{\varepsilon,\hbar}(0) \underset{\varepsilon+\hbar\to 0}{\to} 0.$$

□

The construction of $R^{\mathrm{in}}_{\varepsilon,\hbar,N}$ proceeds as follows:

Lemma 4.2. *Let $\rho_0 \in H^s(\mathbb{T}) \cap \mathcal{P}(\mathbb{T})$, $\rho_0 > 0$, and $u_0 = U_0' \in H^s(\mathbb{T})$ for some potential $U_0 \in H^{s+1}(\mathbb{T})$ with some $s > 1$ sufficiently large. Let $V_0 = \log(\rho_0)$. Let $\psi^{\mathrm{in}}_{\varepsilon,\hbar} := \sqrt{e^{V_0} - \varepsilon V_0''} e^{\frac{iU_0}{\hbar}}$ and $R^{\mathrm{in}}_{\varepsilon,\hbar} := |\psi^{\mathrm{in}}_{\varepsilon,\hbar}\rangle\langle\psi^{\mathrm{in}}_{\varepsilon,\hbar}|$. Set $R^{\mathrm{in}}_{\varepsilon,\hbar,N} = R^{\mathrm{in}\otimes N}_{\varepsilon,\hbar}$. Then there exists some $\Lambda > 0$ such that*

$$\mathcal{E}_{\varepsilon,\hbar,N}(0) \underset{\varepsilon+\hbar+\frac{1}{N}\to 0}{\longrightarrow} 0,$$

provided that $\varepsilon = \varepsilon(N)$ is chosen such that $\frac{1}{\varepsilon^2 N^\Lambda} \underset{N\to\infty}{\to} 0$.

Proof. Note that

$$\mathcal{K}_{\varepsilon,\hbar,N}(0) = \frac{1}{2N} \sum_{j=1}^{N} \mathrm{tr}\left((i\hbar\partial_{x_j} + u_0(x_j))^2 R^{\mathrm{in}}_{\varepsilon,\hbar,N}\right) = \frac{1}{2}\mathrm{tr}\left((i\hbar\partial_x + u_0)^2 R^{\mathrm{in}}_{\varepsilon,\hbar}\right). \tag{4.3}$$

The same considerations demonstrated in lemma 4.1 show that the right-hand side of (4.3) tends to 0 as $\varepsilon + \hbar + \frac{1}{N} \to 0$. Note also that $\rho_{\varepsilon,\hbar,N}(0, X_N) \equiv \left(|\psi^{\mathrm{in}}_{\varepsilon,\hbar}|^2\right)^{\otimes N} := \rho_\varepsilon^{\otimes N}$. To show that the entropy part vanishes asymptotically, it follows from lemma 3.4 that

$$\int_{\mathbb{T}} m_{\varepsilon,X_N}(x) \log\left(\frac{m_{\varepsilon,X_N}(x)}{\rho_0(x)}\right) \mathrm{d}x = \int_{\mathbb{T}} m_{\varepsilon,X_N}(x) (V_{\varepsilon,X_N} - V_0)(x)\, \mathrm{d}x \leqslant \|V_{\varepsilon,X_N} - V_0\|_\infty$$
$$\leqslant \|V_{\varepsilon,X_N} - V_0\|_{H^1} \leqslant \frac{5}{4\varepsilon^{\frac{3}{2}}} W_1(\mu_{X_N}, \rho_\varepsilon).$$

Thus, owing to remark 4.3 below, we see that

$$\int_{\mathbb{T}^N} \left(\int_{\mathbb{T}} m_{\varepsilon,X_N}(x) \log\left(\frac{m_{\varepsilon,X_N}(x)}{\rho_0(x)}\right) \mathrm{d}x\right) \rho_N(0, X_N)\, \mathrm{d}X_N$$
$$\leqslant \frac{5}{4\varepsilon^{\frac{3}{2}}} \int_{\mathbb{T}^N} W_1(\mu_{X_N}, \rho_\varepsilon) \rho_\varepsilon^{\otimes N}(X_N)\, \mathrm{d}X_N \leqslant \frac{5}{4N^\Lambda \varepsilon^{\frac{3}{2}}}. \tag{4.4}$$

Next, we treat the interaction part $\mathcal{V}_{\varepsilon,\hbar,N}(0)$. Given a configuration $X_N \in \mathbb{T}^N$ and $\eta > 0$, consider the truncated empirical measure

$$\mu_{X_N}^{(\eta)} = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}^{(\eta)},$$

where $\delta_x^{(\eta)}$ designates the η-truncation of δ_x[4]. Denote by $V_{\varepsilon,X_N,\eta}$ the solution of

$$-\varepsilon V''_{\varepsilon,X_N,\eta} = \mu_{X_N}^{(\eta)} - e^{V_{\varepsilon,X_N,\eta}}.$$

According to lemma 3.4, we have

$$\begin{aligned}\varepsilon\|\hat{V}'_{\varepsilon,X_N,\eta} - \hat{V}'_0\|_2^2 &\leqslant \frac{1}{4\varepsilon^2}W_1^2(\mu_{X_N}^{(\eta)},\rho_\varepsilon)\\ &\leqslant \frac{1}{2\varepsilon^2}\big(W_1^2(\mu_{X_N},\rho_\varepsilon) + W_1^2(\mu_{X_N}^{(\eta)},\mu_{X_N})\big)\\ &\leqslant \frac{1}{2\varepsilon^2}\big(W_1^2(\mu_{X_N},\rho_\varepsilon)+\eta^2\big),\end{aligned}$$

where the last inequality is due to the general estimate $W_1(\chi_\eta \star \mu,\mu) \leqslant \eta$ (see e.g. lemma 7.1 in [14]). From the same considerations, we have

$$\varepsilon\|\tilde{V}'_{\varepsilon,X_N,\eta} - \tilde{V}'_0\|_2^2 \leqslant \frac{2}{\varepsilon}W_1^2(\mu_{X_N}^{(\eta)},\rho_\varepsilon) \leqslant \frac{4}{\varepsilon}\big(W_1^2(\mu_{X_N},\rho_\varepsilon)+\eta^2\big),$$

which implies

$$\varepsilon||V'_{\varepsilon,X_N,\eta} - V'_0||_2^2 \leqslant \frac{20}{\varepsilon^2}\left(W_1^2\left(\mu_{X_N},\rho_\varepsilon\right)+\eta^2\right).$$

Integrating with respect to $\rho_{\varepsilon,\hbar,N}(0,X_N)$, we have

$$\varepsilon\int_{\mathbb{T}^N}\|V'_{\varepsilon,X_N,\eta} - V'_0\|_2^2\,\rho_{\varepsilon,\hbar,N}(0,X_N)\ \mathrm{d}X_N \leqslant \frac{20}{\varepsilon^2}\Big(\int_{\mathbb{T}^N}W_1^2\left(\mu_{X_N},\rho_\varepsilon\right)\rho_{\varepsilon,\hbar,N}(0,X_N)\ \mathrm{d}X_N + \eta^2\Big).$$

Again, we invoke remark 4.3 below to find that there is some $\Lambda > 0$ such that

$$\int_{\mathbb{T}^N}W_1^2(\mu_{X_N},\rho_\varepsilon)\,\rho_{\varepsilon,\hbar,N}(0,\mathrm{d}X_N) = O\Big(\frac{1}{N^\Lambda}\Big). \tag{4.5}$$

If $\varepsilon = \varepsilon(N)$ is chosen such that $\frac{1}{\varepsilon^2 N^\Lambda} \to 0$ as $N \to \infty$, and $\eta = \eta(\varepsilon)$ such that $\frac{\eta(\varepsilon)}{\varepsilon} \underset{\varepsilon\to 0}{\to} 0$, we conclude

$$\varepsilon\int_{\mathbb{T}^N}\|V'_{\varepsilon,X_N,\eta} - V'_0\|_2^2\,\rho_{\varepsilon,\hbar,N}(0,X_N)\ \mathrm{d}X_N \underset{\varepsilon+\hbar+\frac{1}{N}\to 0}{\longrightarrow} 0,$$

and hence

$$\varepsilon\int_{\mathbb{T}^N}\|V'_{\varepsilon,X_N,\eta}\|_2^2\,\rho_{\varepsilon,\hbar,N}(0,X_N)\ \mathrm{d}X_N \underset{\varepsilon+\hbar+\frac{1}{N}\to 0}{\longrightarrow} 0.$$

Therefore, for this choice of ε, we have

$$\mathcal{V}_{\varepsilon,\hbar,N}(0) = \frac{1}{N}\int_{\mathbb{T}^N}\mathscr{V}_{\varepsilon,X_N}\,\rho_{\varepsilon,\hbar,N}(0,X_N)\ \mathrm{d}X_N \underset{\varepsilon+\hbar+\frac{1}{N}\to 0}{\longrightarrow} 0,$$

because of the relation (see (1.29) in [28]):

$$\lim_{\eta\to 0}\varepsilon\int_{\mathbb{T}^N}\|V'_{\varepsilon,X_N,\eta}\|_2^2\,\rho_{\varepsilon,\hbar,N}(0,X_N)\ \mathrm{d}X_N = \frac{1}{N}\int_{\mathbb{T}^N}\mathscr{V}_{\varepsilon,X_N}\,\rho_{\varepsilon,\hbar,N}(0,X_N)\ \mathrm{d}X_N.$$

□

Remark 4.3. The large-deviation-type inequality used in (4.4) and (4.5) and its various extensions appear, for instance, in [7]. Here is a short way to deduce this estimate from lemma 3.5(ii) for the specific choice $\rho_0 \equiv 1$. Indeed, integrating the inequality in lemma 3.5(ii) with respect to X_N, we obtain that, for some constant $\lambda, C > 0$,

[4] For instance, we can take $\delta_x^{(\eta)} = \chi_\eta \star \delta_x$, where χ is a smooth probability density supported on $B(0,\frac{1}{4}) \setminus B(0,\frac{3}{16})$ and $\chi_\eta(y) := \frac{1}{\eta}\chi(\frac{y}{\eta})$.

$$\int_{\mathbb{T}^N} W_1^2(\mu_{X_N},1)\,\mathrm{d}X_N \leqslant C\Big(N^{-2\lambda}+\int_{\mathbb{T}^N}\mathcal{E}\left(X_N,1\right)\mathrm{d}X_N+\frac{2}{N^2}\Big). \tag{4.6}$$

A direct calculation reveals

$$\mathcal{E}\left(X_N,1\right)=\frac{1}{N^2}\sum_{1\leqslant i,j\leqslant N}K\left(x_i-x_j\right)-\int_{\mathbb{T}}K\left(x\right)\mathrm{d}x.$$

Furthermore, utilizing that $K(0)=0$ and K is even, we have

$$\frac{1}{N^2}\sum_{1\leqslant i,j\leqslant N}\int_{\mathbb{T}^N}K\left(x_i-x_j\right)\mathrm{d}X_N=\frac{N-1}{N}\int_{\mathbb{T}}K\left(x\right)\mathrm{d}x,$$

which shows

$$\left|\int_{\mathbb{T}^N}\mathcal{E}\left(X_N,1\right)\mathrm{d}X_N\right|=\frac{\left|\int_{\mathbb{T}}K(x)\,\mathrm{d}x\right|}{N}\leqslant\frac{1}{N}. \tag{4.7}$$

Substituting (4.7) into (4.6) yields

$$\int_{\mathbb{T}^N}W_1^2(\mu_{X_N},1)\,\mathrm{d}X_N=O\big(\frac{1}{N^\Lambda}\big)$$

with $\Lambda=\min\{2\lambda,1\}$.

5. Well-posedness theory

This section is qualitative in nature, and we take $\varepsilon=\hbar=1$, for simplicity. We start with the von Neumann equation (1.7). Recall Kato's perturbation theory, which is the main ingredient of studying the existence theory of linear equations of Schrödinger-type.

Theorem 5.1 ([30], theorem 6.4). *Let $T,D(T)\subset\mathfrak{H}$ be an (essentially) self-adjoint operator and $S,D(S)\subset\mathfrak{H}$ a symmetric operator such that $D(T)\subset D(S)$. Suppose that there exist $0<a<1$ and $b>0$ such that, for each $\varphi\in D(T)$,*

$$\|S\varphi\|^2\leqslant a\|T\varphi\|^2+b\|\varphi\|^2. \tag{5.1}$$

Then $T+S$ is (essentially) self-adjoint and $D(T+S)=D(T)$. In the case when T is essentially self-adjoint, $D(\overline{T})\subset D(\overline{S})$ and $\overline{T+S}=\overline{T}+\overline{S}$, where $\overline{T}$ and $\overline{S}$ stand for the closures of T and S respectively.

The combination of theorem 5.1 and lemma 2.4 yields the following conclusion (with the notation $\mathscr{H}_N=\mathscr{H}_{1,1,N}$):

Lemma 5.2. *Let $X_N\in\mathbb{T}^N$, and let $(\tilde{V}_{X_N},\hat{V}_{X_N})$ be the solution to the system:*

$$\begin{cases}-\tilde{V}_{X_N}''=\mu_{X_N}-1,\\ -\hat{V}_{X_N}''=1-e^{\tilde{V}_{X_N}+\hat{V}_{X_N}},\end{cases}$$

guaranteed by lemma 2.4. Then $\mathscr{H}_N$ is self-adjoint on $H^2(\mathbb{T}^N)$. Consequently, if $R_N^{\mathrm{in}}\in\mathcal{D}_s(\mathfrak{H}^{\otimes N})$ is such that $\operatorname{tr}\left((-\Delta_N)^2R_N^{\mathrm{in}}\right)<\infty$, then there exists a unique solution of the Cauchy problem

$$\begin{cases}i\partial_tR_N(t)=[\mathscr{H}_N,R_N(t)],\\ R_N(0)=R_N^{\mathrm{in}}.\end{cases} \tag{5.2}$$

Proof. Recall the notation:

$$\mathscr{V}_{X_N}:=\frac{N}{2}\int_{\mathbb{T}\times\mathbb{T}}K\left(x-y\right)\left(\mu_{X_N}-m_{X_N}\right)^{\otimes2}(\mathrm{d}x\mathrm{d}y)\qquad\text{with }\ m_{X_N}=e^{V_{X_N}},$$

$$\mathscr{I}_{X_N}=N\int_{\mathbb{T}}V_{X_N}\left(x\right)m_{X_N}\left(x\right)\mathrm{d}x.$$

By lemma 2.4, we see that $\|m_{X_N}\|_\infty \leqslant C$ for some effective constant $C > 0$ so that

$$\begin{aligned}|\mathscr{V}_{X_N}| &\leqslant \frac{N}{2}\left|\int_{\mathbb{T}} K\star(\mu_{X_N}-m_{X_N})(\mu_{X_N}-m_{X_N})(\mathrm{d}x)\right| \\ &\leqslant N(C+\|K\star m_{X_N}\|_\infty\|m_{X_N}\|_1) \\ &\leqslant N\big(C+\|K\|_\infty\|m_{X_N}\|_1^2\big) = N(C+1),\end{aligned}$$

where K stands for the Green function of the Laplacian on $\mathbb{T}$. The estimate is uniform in X_N so that $X_N \mapsto \mathscr{V}_{X_N} \in L^\infty(\mathbb{T}^N)$. From the same considerations, $X_N \mapsto \mathscr{I}_{X_N} \in L^\infty(\mathbb{T}^N)$. That $\mathscr{H}_N$ is self-adjoint follows from theorem 5.1. The existence is now immediate via Stone's theorem, and the uniqueness follows by linearity of the equation. □

Next, we are concerned with the existence and uniqueness theory for system (1.1). As already remarked, the case of the Schrödinger–Poisson system (1.2) with $-\Delta V = \rho - 1$ is classical; see, for instance, [3, 9]. We plan to first decouple the equations and then apply a fixed-point argument. For this to succeed, we need the following lemma, showing that $\nabla\hat{V}$ is stable with respect to $\tilde{V}$ in the L^2-norm, which is crucial in order to be able to control properly the terms contributed by the nonlinearity:

Lemma 5.3 ([15], lemma 3.9). *Let $d \in \{2,3\}$ and $h_i \in L^\infty(\mathbb{T}^d)\cap\mathcal{P}(\mathbb{T}^d)$ for $i = 1,2$. Consider the system:*

$$\begin{cases}-\Delta\tilde{V}_i = h_i - 1,\\ -\Delta\hat{V}_i = 1 - e^{\tilde{V}_i+\hat{V}_i}.\end{cases}$$

Then

$$\|\nabla\hat{V}_1 - \nabla\hat{V}_2\|_2^2 \leqslant C\|\tilde{V}_1 - \tilde{V}_2\|_2^2,$$

where $C = C_d(\max_i\|\tilde{V}_i\|_\infty + \max_i\|\hat{V}_i\|_\infty)$.

As a corollary, we have the following lemma, which will be useful in several instances in the sequel:

Lemma 5.4 *For each $r \in (0,\frac{1}{4}]$, let $\chi_r := \frac{1}{r^d}\chi(\frac{x}{r})$, where $\chi \geqslant 0$ is a smooth radially symmetric function on $\mathbb{T}^d$ with* $\mathrm{supp}(\chi) \subset B(0,1)$ *and $\int_{\mathbb{T}^d}\chi(x)\,\mathrm{d}x = 1$. With the same hypothesis and notation of lemma 5.3, it holds that*

$$\|\chi_r\star(V_1 - V_2)\|_\infty \leqslant C\|h_1 - h_2\|_1,$$

where $C = C(r,d,\max_i\|h_i\|_\infty)$.

Proof. Notice first that

$$\|\chi_r\star(V_1-V_2)\|_\infty \leqslant \|\chi_r\star(\tilde{V}_1-\tilde{V}_2)\|_\infty + \|\chi_r\star(\hat{V}_1-\hat{V}_2)\|_\infty.$$

We first have

$$\|\chi_r\star(\tilde{V}_1-\tilde{V}_2)\|_\infty \leqslant \|\chi_r\star K\|_\infty\|h_1-h_2\|_1 \leqslant C_{r,d}\|h_1-h_2\|_1. \tag{5.3}$$

Furthermore, lemma 2.3 implies that

$$\begin{aligned}\|\chi_r\star(\hat{V}_1-\hat{V}_2)\|_\infty^2 &\leqslant \|K\|_2^2\|e^{\hat{V}_1+\tilde{V}_1} - e^{\hat{V}_2+\tilde{V}_2}\|_2^2\\ &\leqslant \|K\|_2^2\|e^{\hat{V}_1+\tilde{V}_1} - e^{\hat{V}_1+\tilde{V}_2}\|_2^2 + \|K\|_2^2\|e^{\hat{V}_1+\tilde{V}_2} - e^{\hat{V}_2+\tilde{V}_2}\|_2^2\\ &\lesssim \|e^{\hat{V}_1}\|_\infty^2\|e^{\tilde{V}_1} - e^{\tilde{V}_2}\|_2^2 + \|e^{\tilde{V}_2}\|_\infty^2\|e^{\hat{V}_1} - e^{\hat{V}_2}\|_2^2\\ &\lesssim_{d,\max_i\|h_i\|_\infty} \|e^{\tilde{V}_1} - e^{\tilde{V}_2}\|_2^2 + \|e^{\hat{V}_1} - e^{\hat{V}_2}\|_2^2\\ &\lesssim_{d,\max_i\|h_i\|_\infty} \|\tilde{V}_1-\tilde{V}_2\|_2^2 + \|\hat{V}_1-\hat{V}_2\|_2^2,\end{aligned} \tag{5.4}$$

where the last inequality in (5.4) is due to the mean value theorem as applied to the function: $x \mapsto e^x$. The first term on the right-hand side of (5.4) is

$$\|\tilde{V}_1-\tilde{V}_2\|_2^2 \leqslant \|h_1-h_2\|_1^2.$$

Therefore, by the Poincaré inequality and lemma 5.3, we have

$$\|\hat{V}_1 - \hat{V}_2\|_2^2 \lesssim_d \|\nabla\hat{V}_1 - \nabla\hat{V}_2\|_2^2 \\ \leqslant C(\max_i \|h_i\|_\infty, d)\|\tilde{V}_1 - \tilde{V}_2\|_2^2 \leqslant C(\max_i \|h_i\|_\infty, d)\|h_1 - h_2\|_1^2. \tag{5.5}$$

Gathering (5.3)–(5.5) yields the desired inequality. □

We need also to observe that the L^2-norm of e^V is bounded by means of the L^2-norm of ρ, as encapsulated in the following lemma:

Lemma 5.5. *Let $\rho \in L^\infty(\mathbb{T}^d) \cap \mathcal{P}(\mathbb{T}^d)$, and let V be the unique solution of*

$$-\Delta V(x) = \rho(x) - e^{V(x)}.$$

Then

$$\|e^V\|_2 \leqslant \|\rho\|_2.$$

Proof. Multiplying the equation by e^V and then integrating, we have

$$-\int_{\mathbb{T}^d} \Delta V(x)\, e^{V(x)}\, \mathrm{d}x = \int_{\mathbb{T}^d} \left(\rho(x)\, e^{V(x)} - e^{2V(x)}\right) \mathrm{d}x.$$

Integrating the left-hand side by parts, we recognize that

$$-\int_{\mathbb{T}^d} \Delta V(x)\, e^{V(x)}\, \mathrm{d}x = \int_{\mathbb{T}^d} |\nabla V(x)|^2\, e^{V(x)}\, \mathrm{d}x \geqslant 0.$$

Therefore, it follows that

$$\int_{\mathbb{T}^d} \rho(x)\, e^{V(x)}\, \mathrm{d}x \geqslant \int_{\mathbb{T}^d} e^{2V(x)}\, \mathrm{d}x.$$

Using the Cauchy–Schwarz inequality, we deduce

$$\|e^V\|_2^2 \leqslant \|e^V\|_2 \|\rho\|_2.$$

This completes the proof. □

To prove the well-posedness of the Schrödinger–Poisson–Boltzmann system, we first study the Cauchy problem of the regularized equation. We recall the following existence and uniqueness result for a Schrödinger equation with a time-dependent potential (one can also consult [26] for similar results).

Lemma 5.6 ([8], theorem 2). *Let $V : [0,T] \times \mathbb{T}^d \to \mathbb{R}$ be a function such that*

(i) $V \in C([0,T]; C^\infty(\mathbb{T}^d))$,

(ii) $\partial_x^\alpha V \in L^\infty([0,T]; C(\mathbb{T}^d))$ *for all* $\alpha \in \mathbb{N}$.

Let $\psi^{\mathrm{in}} \in C^\infty(\mathbb{T}^d)$. Then the Cauchy problem

$$\begin{cases} i\partial_t \psi(t,x) = -\frac{1}{2}\Delta\psi(t,x) + V(t,x)\,\psi(t,x), & x \in \mathbb{T}^d, t > 0, \\ \psi|_{t=0} = \psi^{\mathrm{in}}, & x \in \mathbb{T}^d, \end{cases}$$

has a unique solution $\psi \in C^1([0,T]; C^\infty(\mathbb{T}^d))$.

Lemma 5.7 *Let $\psi^{\mathrm{in}} \in C^\infty(\mathbb{T}^d)$ with $\int_{\mathbb{T}^d} |\psi^{\mathrm{in}}(x)|^2\, \mathrm{d}x = 1$. For each $r \in (0, \frac{1}{4}]$, let $\chi_r := \frac{1}{r^d}\chi(\frac{x}{r})$, where $\chi \geqslant 0$ is a smooth radially symmetric function on $\mathbb{T}^d$ with $\mathrm{supp}(\chi) \subset B(0,1)$ and $\int_{\mathbb{T}^d} \chi(x)\, \mathrm{d}x = 1$. Then, for any $T > 0$ and $r \in (0, \frac{1}{4}]$ fixed, there exists a unique solution $\psi_r \in C^1([0,T]; C^\infty(\mathbb{T}^d))$ to the regularized Cauchy problem:*

$$\begin{cases} i\partial_t \psi_r(t,x) = -\frac{1}{2}\Delta\psi_r(t,x) + (\chi_r \star V_r)(t,x)\,\psi_r(t,x), & x \in \mathbb{T}^d, t > 0, \\ -\Delta V_r(t,x) = \left(\chi_r \star |\psi_r|^2\right)(t,x) - e^{V_r(t,x)}, & x \in \mathbb{T}^d, t > 0, \\ \psi_r|_{t=0} = \psi^{\mathrm{in}}, & x \in \mathbb{T}^d. \end{cases} \tag{5.6}$$

Proof. We first regularize the equation in time and study the well-posedness of the resulting equation via a fixed-point argument. Afterwards, we apply a compactness argument with respect to the mollification in time. We divide the proof into three steps:

1. *Well-posedness for the time regularized system.* Set

$$\mathfrak{X} := \left\{ \psi \in C\left([0,T]; L^2(\mathbb{T}^d)\right) : \|\psi(t,\cdot)\|_2 = 1 \text{ for any } t \in [0,T] \right\}.$$

Fix a standard mollifier ζ_δ on $\mathbb{R}$. In this step, we prove that the time regularized system:

$$\begin{cases} i\partial_t \psi_{r,\delta}(t,x) = -\frac{1}{2}\Delta\psi_{r,\delta}(t,x) + (\chi_r \star_x V_{r,\delta})(t,x)\,\psi_{r,\delta}(t,x), & x \in \mathbb{T}^d,\ t > 0, \\ -\Delta V_{r,\delta}(t,x) = \left(\zeta_\delta \star_t \chi_r \star_x |\psi_{r,\delta}|^2\right)(t,x) - e^{V_{r,\delta}(t,x)}, & x \in \mathbb{T}^d,\ t > 0, \\ \psi_{r,\delta}|_{t=0} = \psi^{\text{in}}, & x \in \mathbb{T}^d \end{cases} \tag{5.7}$$

is well posed. For each fixed $\phi \in \mathfrak{X}$, we consider the linear Schrödinger equation with time dependent interaction

$$\begin{cases} i\partial_t \psi_{r,\delta}(t,x) = -\frac{1}{2}\Delta\psi_{r,\delta}(t,x) + (\chi_r \star_x V_{r,\delta})(t,x)\,\psi_{r,\delta}(t,x), & x \in \mathbb{T}^d,\ t > 0, \\ -\Delta V_{r,\delta}(t,x) = \left(\zeta_\delta \star_t \chi_r \star_x |\phi|^2\right)(t,x) - e^{V_{r,\delta}(t,x)}, & x \in \mathbb{T}^d,\ t > 0, \\ \psi_{r,\delta}|_{t=0} = \psi^{\text{in}}, & x \in \mathbb{T}^d. \end{cases} \tag{5.8}$$

When there is no ambiguity, we omit t and x from $\star_t$ and $\star_x$, respectively. By lemma 2.3, for each $t \in (0,T]$, there is a unique solution $V_{r,\delta}(t,\cdot) \in C^{2,\alpha}(\mathbb{T}^d)$ to the problem

$$-\Delta V_{r,\delta}[\phi](t,x) = \left(\zeta_\delta \star \chi_r \star |\phi|^2\right)(t,x) - e^{V_{r,\delta}[\phi](t,x)}.$$

By remark 2.7, $\chi_r \star V$ verifies assumption (ii) of theorem 5.6. In addition, we notice that $\chi_r \star V_{r,\delta}$ is Lipschitz in time due to lemma 5.4 applied with $h_1(t,\cdot) = (\zeta_\delta \star \chi_r \star |\psi_{r,\delta}|^2)(t,\cdot)$ and $h_2(s,\cdot) = (\zeta_\delta \star \chi_r \star |\psi_{r,\delta}|^2)(s,\cdot)$ for any $t,s \in [0,T]$. Thus, assumption (i) of theorem 5.6 is also satisfied. Then theorem 5.6 implies the existence and uniqueness of a solution $\Psi_{r,\delta} \in C^1\left([0,T]; C^\infty(\mathbb{T}^d)\right)$ of the Cauchy problem for the linear equation:

$$\begin{cases} i\partial_t \psi_{r,\delta}(t,x) = -\frac{1}{2}\Delta\psi_{r,\delta}(t,x) + (\chi_r \star V_{r,\delta}[\phi])(t,x)\,\psi_{r,\delta}(t,x), & x \in \mathbb{T}^d,\ t > 0, \\ \psi_{r,\delta}|_{t=0} = \psi^{\text{in}}, & x \in \mathbb{T}^d. \end{cases}$$

We aim to prove that the operator: $\phi \mapsto \Psi_{r,\delta}[\phi]$ is a contraction from $\mathfrak{X}$ to $\mathfrak{X}$ and thereby conclude via the Banach fixed point theorem. Since $\chi_r \star V_{r,\delta}[\phi]$ is real-valued, it is readily checked that the L^2-norm of $\Psi_{r,\delta}[\phi]$ is conserved, *i.e.*, $\|\Psi_{r,\delta}[\phi](t,\cdot)\|_2 = 1$, so that $\Psi_{r,\delta}[\phi] \in \mathfrak{X}$ indeed. Given $\phi_1, \phi_2 \in \mathfrak{X}$, set $\Psi_1 := \Psi_{r,\delta}[\phi_1], \Psi_2 := \Psi_{r,\delta}[\phi_2]$, and $V_1 := V_{r,\delta}[\phi_1], V_2 := V_{r,\delta}[\phi_2]$. We compute

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|\Psi_1(t,\cdot) - \Psi_2(t,\cdot)\|_2^2 \\ &= 2\Re\Big(-i\int_{\mathbb{T}^d} \partial_t(\Psi_1(t,x) - \Psi_2(t,x))\,\overline{(\Psi_1(t,x) - \Psi_2(t,x))}\,\mathrm{d}x\Big) \\ &\quad = \Re\Big(-i\int_{\mathbb{T}^d} \Delta(\Psi_2(t,x) - \Psi_1(t,x))\,\overline{(\Psi_1(t,x) - \Psi_2(t,x))}\,\mathrm{d}x\Big) \\ &\qquad + 2\Re\Big(-i\int_{\mathbb{T}^d} ((\chi_r \star V_1)(t,x)\,\Psi_1(t,x) - (\chi_r \star V_2)(t,x)\,\Psi_2(t,x))\,\overline{(\Psi_1(t,x) - \Psi_2(t,x))}\,\mathrm{d}x\Big) \\ &=: I + J. \end{aligned}$$

Integrating by parts, we see that

$$-\int_{\mathbb{T}^d} \Delta(\Psi_2(t,x) - \Psi_1(t,x))\,\overline{(\Psi_1(t,x) - \Psi_2(t,x))}\,\mathrm{d}x = \int_{\mathbb{T}^d} |\nabla(\Psi_1 - \Psi_2)|^2(t,x)\,\mathrm{d}x,$$

so that $I = 0$. Furthermore, we have

$$\begin{aligned}
&\int_{\mathbb{T}^d} \left((\chi_r \star V_1)(t,x)\,\Psi_1(t,x) - (\chi_r \star V_2)(t,x)\,\Psi_2(t,x) \right) \overline{\left(\Psi_1(t,x) - \Psi_2(t,x)\right)}\,\mathrm{d}x \\
&\quad = \int_{\mathbb{T}^d} (\chi_r \star V_1)(t,x)\left(\Psi_1(t,x) - \Psi_2(t,x)\right)\overline{\left(\Psi_1(t,x) - \Psi_2(t,x)\right)}\,\mathrm{d}x \\
&\qquad + \int_{\mathbb{T}^d} \Psi_2(t,x)\left(\chi_r \star (V_1 - V_2)\right)(t,x)\,\overline{\left(\Psi_1(t,x) - \Psi_2(t,x)\right)}\,\mathrm{d}x \\
&\quad = \int_{\mathbb{T}^d} (\chi_r \star V_1)(t,x)\,|\Psi_1(t,x) - \Psi_2(t,x)|^2\,\mathrm{d}x \\
&\qquad + \int_{\mathbb{T}^d} \Psi_2(t,x)(\chi_r \star (V_1 - V_2))(t,x)\,\overline{\left(\Psi_1(t,x) - \Psi_2(t,x)\right)}\,\mathrm{d}x. \qquad (5.9)
\end{aligned}$$

Clearly, one has

$$\Re\Big(i\int_{\mathbb{T}^d} (\chi_r \star V_1)(t,x)\,|\Psi_1(t,x) - \Psi_2(t,x)|^2\,\mathrm{d}x\Big) = 0,$$

so it suffices to deal with the second integral on the right-hand side of (5.9).

Notice that

$$\begin{aligned}
&\int_{\mathbb{T}^d} \Psi_2(t,x)\left(\chi_r \star (V_1 - V_2)\right)(t,x)\,\overline{\left(\Psi_1(t,x) - \Psi_2(t,x)\right)}\,\mathrm{d}x \\
&\quad \leqslant \frac{1}{2}\int_{\mathbb{T}^d} |\Psi_2|^2(t,x)\,|\chi_r \star (V_1 - V_2)|^2(t,x)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^d} \left|\overline{\Psi_1(t,x) - \Psi_2(t,x)}\right|^2\mathrm{d}x. \qquad (5.10)
\end{aligned}$$

In order to control the first term on the right-hand side of (5.10), we invoke lemma 5.4 with $h_1 = (\zeta_\delta \star \chi_r \star |\phi_1|^2)(t,x)$ and $h_2 = (\zeta_\delta \star \chi_r \star |\phi_2|^2)(t,x)$ to see that

$$\begin{aligned}
\|(\chi_r \star (V_1 - V_2))(t,\cdot)\|_\infty^2 &\leqslant C_{r,d}\sup_{t\in[0,T]} \||\phi_1(t,\cdot) - \phi_2(t,\cdot)|\,(|\phi_1(t,\cdot)| + |\phi_2(t,\cdot)|)\|_1^2 \\
&\leqslant C_{r,d}\sup_{t\in[0,T]} \|\phi_1(t,\cdot) - \phi_2(t,\cdot)\|_2^2.
\end{aligned}$$

We have thus proved

$$\int_{\mathbb{T}^d} |\Psi_2|^2(t,x)\,|\chi_r \star (V_1 - V_2)|^2(t,x)\,\mathrm{d}x \leqslant C_{r,d}\sup_{t\in[0,T]} \|\phi_1(t,\cdot) - \phi_2(t,\cdot)\|_2^2 \qquad (5.11)$$

for some constant $C_{r,d} > 0$. Inequalities (5.10) and (5.11) entail

$$\begin{aligned}
\|\Psi_1(t,\cdot) - \Psi_2(t,\cdot)\|_2^2 &\leqslant C_{r,d}\,t\sup_{\tau\in[0,T]} \|\phi_1(\tau,\cdot) - \phi_2(\tau,\cdot)\|_2^2 + \frac{1}{2}\int_0^t \|\Psi_1(\tau,\cdot) - \Psi_2(\tau,\cdot)\|_2^2\,\mathrm{d}\tau \\
&\leqslant C_{r,d}\,T\sup_{\tau\in[0,T]} \|\phi_1(\tau,\cdot) - \phi_2(\tau,\cdot)\|_2^2 + \frac{1}{2}\int_0^t \|\Psi_1(\tau,\cdot) - \Psi_2(\tau,\cdot)\|_2^2\,\mathrm{d}\tau.
\end{aligned}$$

As a result,

$$\sup_{t\in[0,T]} \|\Psi_1(t,\cdot) - \Psi_2(t,\cdot)\|_2 \leqslant C_{r,d}\sqrt{T}e^{\frac{T}{2}}\sup_{t\in[0,T]} \|\phi_1(t,\cdot) - \phi_2(t,\cdot)\|_2,$$

which shows that, for $T = T_{r,d}$ sufficiently small, $\phi \mapsto \Psi_{r,\delta}[\phi]$ is a contraction, thereby ensuring the existence and uniqueness of a fixed point $\psi_{r,\delta}$. Note that, by construction of Ψ, we see that this solution belongs to $C^1([0,T];C^\infty(\mathbb{T}^d))$. The global existence follows by a standard iteration argument together with the conservation of the L^2-norm.

2. *Compactness in* δ. In this step, we aim to remove the regularization in time, which will enable us to prove that the system:

$$\begin{cases}
i\partial_t\psi_r(t,x) = -\frac{1}{2}\Delta\psi_r(t,x) + (\chi_r \star V_r)(t,x)\,\psi_r(t,x), & x\in\mathbb{T}^d,\ t>0, \\
-\Delta V_r(t,x) = \left(\chi_r \star |\psi_r|^2\right)(t,x) - e^{V_r(t,x)}, & x\in\mathbb{T}^d,\ t>0, \\
\psi_r(0,x) = \psi^{\mathrm{in}}, & x\in\mathbb{T}^d
\end{cases} \qquad (5.12)$$

is well-posed. Consider the solution $\psi_{r,\delta}$ of (5.7) constructed in Step 1. We obtain that, for any $t,s\in[0,T]$,

$$\|\psi_{r,\delta}(t,\cdot)-\psi_{r,\delta}(s,\cdot)\|_{H^2}\leqslant|t-s|\big(\|\Delta\psi_{r,\delta}\|_{L^\infty_tL^2_x}+\|\Delta^2\psi_{r,\delta}\|_{L^\infty_tL^2_x}$$
$$+\|(\chi_r\star V_{r,\delta})\psi_{r,\delta}\|_{L^\infty_tL^2_x}+\|\Delta((\chi_r\star V_{r,\delta})\psi_{r,\delta}))\|_{L^\infty_tL^2_x}\big). \tag{5.13}$$

We start by propagating the Sobolev norms of $\psi_{r,\delta}$ uniformly in δ.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Delta^2\psi_{r,\delta}(t,\cdot)\right\|_2^2=2\Re\Big(\int_{\mathbb{T}^d}\Delta^2\partial_t\psi_{r,\delta}(t,x)\,\Delta^2\overline{\psi_{r,\delta}}(t,x)\;\mathrm{d}x\Big)$$
$$=-2\Re\Big(i\int_{\mathbb{T}^d}\Delta^2\Big(-\frac{1}{2}\Delta\psi_{r,\delta}(t,x)+(\chi_r\star V_{r,\delta})(t,x)\,\psi_{r,\delta}(t,x)\Big)\Delta^2\overline{\psi_{r,\delta}}(t,x)\;\mathrm{d}x\Big).$$

Integration by parts reveals

$$\Re\Big(i\int_{\mathbb{T}^d}\Delta^2\Delta\psi_{r,\delta}(t,x)\,\Delta^2\overline{\psi_{r,\delta}}(t,x)\;\mathrm{d}x\Big)=\Re\Big(i\int_{\mathbb{T}^d}\left|\nabla\Delta^2\psi_{r,\delta}\right|^2(t,x)\;\mathrm{d}x\Big)=0,$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Delta^2\psi_{r,\delta}(t,\cdot)\right\|_2^2=2\Re\Big(-i\int_{\mathbb{T}^d}\Delta^2\left((\chi_r\star V_{r,\delta})\,\psi_{r,\delta}\right)(t,x)\,\Delta^2\overline{\psi_{r,\delta}}(t,x)\;\mathrm{d}x\Big).$$

We can write

$$\Delta^2\big((\chi_r\star V_{r,\delta})\,\psi_{r,\delta}\big)=\sum_{k=0}^{4}c_k\nabla^k(\chi_r\star V_{r,\delta})\,\nabla^{4-k}\psi_{r,\delta}$$

for some constants $c_k\in\mathbb{N}$, so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Delta^2\psi_{r,\delta}(t,\cdot)\right\|_2^2\lesssim\sum_{k=0}^{4}\int_{\mathbb{T}^d}\left|\nabla^k(\chi_r\star V_{r,\delta})\,\nabla^{4-k}\psi_{r,\delta}\Delta^2\psi_{r,\delta}\right|(t,x)\;\mathrm{d}x:=\sum_{k=0}^{4}I_k. \tag{5.14}$$

We proceed by estimating each one of I_k:

$$I_0=\int_{\mathbb{T}^d}\left|(\chi_r\star V_{r,\delta})(t,x)\right|\left|\Delta^2\psi_{r,\delta}(t,x)\right|^2\mathrm{d}x\leqslant\left\|V_{r,\delta}\right\|_{L^\infty_{t,x}}\|\Delta^2\psi_{r,\delta}(t,\cdot)\|_2^2,$$
$$I_1=\int_{\mathbb{T}^d}\left|\nabla(\chi_r\star V_{r,\delta})(t,x)\right|\left|\nabla\Delta\psi_{r,\delta}(t,x)\,\Delta^2\psi_{r,\delta}(t,x)\right|\mathrm{d}x$$
$$\leqslant\left\|\nabla V_{r,\delta}\right\|_{L^\infty_{t,x}}\left(\|\nabla\Delta\psi_{r,\delta}(t,\cdot)\|_2^2+\|\Delta^2\psi_{r,\delta}(t,\cdot)\|_2^2\right),$$
$$I_2=\int_{\mathbb{T}^d}\left|\Delta(\chi_r\star V_{r,\delta})(t,x)\right|\left|\left(\Delta\psi_{r,\delta}\Delta^2\psi_{r,\delta}\right)(t,x)\right|\mathrm{d}x$$
$$\leqslant\|\Delta V_{r,\delta}\|_{L^\infty_{t,x}}\left(\|\Delta\psi_{r,\delta}(t,\cdot)\|_2^2+\|\Delta^2\psi_{r,\delta}(t,\cdot)\|_2^2\right).$$

Using lemma 2.3, we see that

$$\|V_{r,\delta}\|_{L^\infty_tC^{2,\alpha}_x}\leqslant C_{r,d}.$$

Thus, summarizing the above estimate, we have

$$I_0+I_1+I_2\leqslant C_{r,d}\left(\|\Delta\psi_{r,\delta}(t,\cdot)\|_2^2+\|\Delta^2\psi_{r,\delta}(t,\cdot)\|_2^2\right).$$

In order to bound I_3 and I_4, note that

$$-\Delta^2V_{r,\delta}=\zeta_\delta\star\Delta\chi_r\star|\psi_{r,\delta}|^2-\Delta(e^{V_{r,\delta}})=\zeta_\delta\star\Delta\chi_r\star|\psi_{r,\delta}|^2-e^{V_{r,\delta}}\left|\nabla V_{r,\delta}\right|^2-e^{V_{r,\delta}}\Delta V_{r,\delta},$$

and therefore lemma 2.3 entails

$$\|\Delta^2V_{r,\delta}\|_{L^\infty_{t,x}}\leqslant C_{r,d}.$$

Thus, we obtain the bounds:

$$I_3\leqslant C_{r,d}\left(\|\nabla\psi_{r,\delta}\|_2^2+\|\Delta^2\psi_{r,\delta}\|_2^2\right),$$

$$I_4 \leqslant C_{r,d}\left(1+\|\Delta^2\psi_{r,\delta}\|_2^2\right).$$

To conclude, we have proved the estimate:

$$\sum_{k=0}^{4} I_k \leqslant C_{r,d}\left(1+\|\Delta^2\psi_{r,\delta}(t,\cdot)\|_2^2\right),$$

so that, in view of (5.14), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta^2\psi_{r,\delta}(t,\cdot)\|_2^2 \leqslant C_{r,d}\left(1+\|\Delta^2\psi_{r,\delta}(t,\cdot)\|_2^2\right),$$

which yields the estimate:

$$\|\psi_{r,\delta}(t,\cdot)\|_{H^4}^2 \leqslant C \tag{5.15}$$

for some $C = C(r,d,\|\psi^{\text{in}}\|_{H^4})$. Substituting (5.15) into (5.13) produces the inequality:

$$\|\psi_{r,\delta}(t,\cdot) - \psi_{r,\delta}(s,\cdot)\|_{H^2} \leqslant C|t-s|, \tag{5.16}$$

where $C = C(r,d,\|\psi^{\text{in}}\|_{H^4})$.

3. *Extraction of a solution*. By the Arzela-Ascoli theorem and (5.16), there exist a function $\psi_r \in C([0,T];H^2(\mathbb{T}^d))$ and a subsequence δ_k such that

$$\|\psi_{r,\delta_k} - \psi_r\|_{C_tH_x^2} \underset{k\to\infty}{\longrightarrow} 0.$$

Moreover, inequality (5.16) ensures that $\psi_r \in \mathrm{Lip}([0,T];H^2(\mathbb{T}^d))$. We are left to verify that ψ_r is the asserted solution. Denote by V_r the solution of

$$-\Delta V_r = \chi_r \star |\psi_r|^2 - e^{V_r}.$$

Lemma 5.3 and Poincaré's inequality imply that $\|\hat{V}_1 - \hat{V}_2\|_2^2 \leqslant C\|\widetilde{V}_1 - \widetilde{V}_2\|_2^2$ and hence

$$\begin{aligned}
\|V_{r,\delta_k}(t,\cdot) - V_r(t,\cdot)\|_2 &\leqslant C\|(\zeta_{\delta_k}\star\chi_r\star|\psi_{r,\delta_k}|^2)(t,\cdot) - (\chi_r\star|\psi_r|^2)(t,\cdot)\|_2 \\
&\leqslant C\|(\zeta_{\delta_k}\star\chi_r\star|\psi_{r,\delta_k}|^2)(t,\cdot) - (\zeta_{\delta_k}\star\chi_r\star|\psi_r|^2)(t,\cdot)\|_2 \\
&\quad + C\|(\zeta_{\delta_k}\star\chi_r\star|\psi_r|^2)(t,\cdot) - (\chi_r\star|\psi_r|^2)(t,\cdot)\|_2.
\end{aligned}$$

The first term is bounded by

$$\begin{aligned}
C\sup_{t\in[0,T]}\|\chi_r\star|\psi_{r,\delta_k}|^2 - \chi_r\star|\psi_r|^2\|_2 &\lesssim_r \sup_{t\in[0,T]}\||\psi_{r,\delta_k}|^2(t,\cdot) - |\psi_r|^2(t,\cdot)\|_1 \\
&\lesssim_r \sup_{t\in[0,T]}\|\psi_{r,\delta_k}(t,\cdot) - \psi_r(t,\cdot)\|_2 \underset{k\to\infty}{\longrightarrow} 0.
\end{aligned}$$

In addition, note that

$$(\zeta_{\delta_k}\star\chi_r\star|\psi_r|^2)(t,x) \underset{k\to\infty}{\longrightarrow} (\chi_r\star|\psi_r|^2)(t,x) \qquad \text{pointwise a.e. in } (t,x),$$

and, by Lebesgue's dominated convergence theorem, we have

$$\|(\zeta_{\delta_k}\star\chi_r\star|\psi_r|^2)(t,\cdot) - (\chi_r\star|\psi_r|^2)(t,\cdot)\|_2 \underset{k\to\infty}{\longrightarrow} 0.$$

Consequently, we obtain

$$\|V_{r,\delta_k}(t,\cdot) - V_r(t,\cdot)\|_2 \underset{k\to\infty}{\longrightarrow} 0.$$

As a result, we can pass to the limit as $k\to\infty$ in order to conclude that, for any $\varphi \in C_0^\infty((0,T)\times\mathbb{T}^d)$,

$$\begin{aligned}
\int_{[0,T]\times\mathbb{T}^d} i\psi_r(t,x)\,\partial_t\varphi(t,x)\,\mathrm{d}x\mathrm{d}t &= -\frac{1}{2}\int_{[0,T]\times\mathbb{T}^d}\Delta\psi_r(t,x)\,\varphi(t,x)\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_{[0,T]\times\mathbb{T}^d}(\chi_r\star V_r)(t,x)\,\psi_r(t,x)\,\varphi(t,x)\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Since $\psi_r \in \mathrm{Lip}([0,T];H^2(\mathbb{T}^d))$, we can integrate the term on the left-hand side by parts in order to conclude that $\psi_r \in \mathrm{Lip}([0,T];H^2(\mathbb{T}^d))$ verifies the Cauchy problem (5.12). Finally, by lemma 5.6 and the uniqueness, we can prove that $\psi_r \in C^1([0,T];C^\infty(\mathbb{T}^d))$. The uniqueness for (5.6) follows from the standard (L^2) Grónwall argument.

□

We continue by showing the compactness in r of the family of solutions $\{\psi_r\}_{r>0}$ constructed above, which will enable us to extract a converging subsequence, thereby proving the existence of a solution to the original equation. As remarked in [14], the advantage of using a double regularization for V_r is because this regularization procedure ensures the conservation of the following time-dependent quantity $\mathcal{F}_r$ defined in the next lemma. The proof is omitted, since it is similar to the proof of the conservation of total energy that has been given in section 2.

Lemma 5.8. *Let $\psi_r(t,x) \in \mathrm{Lip}([0,T];H^2(\mathbb{T}^d))$ be the solution to (5.6). Let*

$$\mathcal{F}_r(t) := \frac{1}{2}\int_{\mathbb{T}^d} |\nabla\psi_r(t,x)|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^d} |\nabla V_r(t,x)|^2\,\mathrm{d}x + \int_{\mathbb{T}^d} V_r(t,x)\,e^{V_r(t,x)}\,\mathrm{d}x.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_r(t) = 0. \tag{5.17}$$

We will also need the following lemma.

Lemma 5.9 ([14], proposition 4.4). *Let $d \in \{2,3\}$. For each $i = 1,2$, let $h_i \in L^\infty(\mathbb{T}^d) \cap \mathcal{P}(\mathbb{T}^d)$, and let $(\tilde V_i, \hat V_i)$ be the solution of the system:*

$$\begin{cases} -\Delta \tilde V_i = h_i - 1, \\ -\Delta \hat V_i = 1 - e^{\tilde V_i + \hat V_i}. \end{cases}$$

Then

$$\|\nabla\tilde V_1 - \nabla\tilde V_2\|_2^2 \leqslant \max_i \|h_i\|_\infty W_2^2(h_1,h_2),$$
$$\|\nabla\hat V_1 - \nabla\hat V_2\|_2^2 \leqslant C\max_i \|h_i\|_\infty W_2^2(h_1,h_2),$$

where $C = C_d(1 + \max_i \|h_i\|_{\frac{d+2}{d}})$, and W_2 designates the 2-Wasserstein distance.

Proof of theorem 1.1. We divide the proof into five steps.

1. *Uniform bound in r on $\sup_{t\in[0,T]} \|\psi_r(t,\cdot)\|_{H^1(\mathbb{T}^d)}$.* Consider the mollified system:

$$\begin{cases} i\partial_t\psi_r(t,x) = -\frac{1}{2}\Delta\psi_r(t,x) + (\chi_r \star V_r)(t,x)\,\psi_r(t,x), & x\in\mathbb{T}^d,\ t>0, \\ -\Delta V_r(t,x) = (\chi_r \star |\psi_r|^2)(t,x) - e^{V_r(t,x)}, & x\in\mathbb{T}^d,\ t>0, \\ \psi_r|_{t=0} = \psi^{\mathrm{in}}, & x\in\mathbb{T}^d, \end{cases} \tag{5.18}$$

and denote by $\psi_r \in C^1([0,T];C^\infty(\mathbb{T}^d))$ the solution to this system ensured thanks to lemma 5.6. Using lemma 5.8, we have

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{T}^d} |\nabla\psi_r(t,x)|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^d} |\nabla V_r(t,x)|^2\,\mathrm{d}x + \int_{\mathbb{T}^d} V_r(t,x)\,e^{V_r(t,x)}\,\mathrm{d}x \\ &\quad = \frac{1}{2}\int_{\mathbb{T}^d} |\nabla\psi^{\mathrm{in}}(x)|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^d} |\nabla V_r(0,x)|^2\,\mathrm{d}x + \int_{\mathbb{T}^d} V_r(0,x)\,e^{V_r(0,x)}\,\mathrm{d}x \\ &\quad \leqslant \frac{1}{2}\|\nabla\psi^{\mathrm{in}}\|_2^2 + \frac{1}{2}\|\nabla V_r(0,\cdot)\|_2^2 + \|V_r(0,\cdot)\|_\infty, \end{aligned} \tag{5.19}$$

where we have used that $\int_{\mathbb{T}^d} e^{V_r(0,x)}\,\mathrm{d}x = 1$ in the last inequality. By the Sobolev embedding, we have

$$\|\chi_r \star |\psi^{\mathrm{in}}|^2\|_{\frac{d+2}{d}} \leqslant \||\psi^{\mathrm{in}}|^2\|_{\frac{d+2}{d}} \leqslant C_S\|\psi^{\mathrm{in}}\|_{H^1(\mathbb{T}^d)}^2,$$

where C_S stands for the Sobolev constant. Therefore, by lemma 2.3, we see that

$$\|\nabla \hat{V}_r(0,\cdot)\|_2^2 \lesssim_{d,\|\psi^{\rm in}\|_{H^1(\mathbb{T}^d)}} 1.$$

In addition, using the Hölder inequality, we obtain

$$\|\nabla \tilde{V}_r(0,\cdot)\|_2^2 \leqslant 1 + C_S\|\psi^{\rm in}\|_{H^1(\mathbb{T}^d)}^2.$$

Therefore, we have

$$\|\nabla V_r(0,\cdot)\|_2^2 \leqslant 2\|\nabla \tilde{V}_r(0,\cdot)\|_2^2 + 2\|\nabla \hat{V}_r(0,\cdot)\|_2^2 \lesssim_{d,\|\psi^{\rm in}\|_{H^1(\mathbb{T}^d)}} 1,$$

and similarly

$$\|V_r(0,\cdot)\|_\infty \lesssim_{d,\|\psi^{\rm in}\|_{H^1(\mathbb{T}^d)}} 1,$$

which, in view of (5.19), shows

$$\sup_{t\in[0,T]} \|\psi_r(t,\cdot)\|_{H^1(\mathbb{T}^d)} \leqslant \mathcal{M}_{\rm in}, \tag{5.20}$$

for some constant $\mathcal{M}_{\rm in} = \mathcal{M}_{\rm in}(\|\psi^{\rm in}\|_{H^1(\mathbb{T}^d)}, d)$.

2. *Uniform bound in r on* $\sup_{t\in[0,T]} \|\psi_r(t,\cdot)\|_{H^2(\mathbb{T}^d)}$. We compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} \frac{1}{2}|\Delta\psi_r(t,x)|^2\,\mathrm{d}x &= \Re\Big(\int_{\mathbb{T}^d} \Delta\partial_t\psi_r(t,x)\,\Delta\overline{\psi_r}(t,x)\,\mathrm{d}x\Big)\\
&= \Re\Big(-i\int_{\mathbb{T}} \Delta\Big(-\frac{1}{2}\Delta\psi_r(t,x) + (\chi_r\star V_r)(t,x)\,\psi_r(t,x)\Big)\Delta\overline{\psi_r}(t,x)\,\mathrm{d}x\Big)\\
&= \Re\Big(-i\int_{\mathbb{T}^d} \Delta\left((\chi_r\star V_r)(t,x)\,\psi_r(t,x)\right)\Delta\overline{\psi_r}(t,x)\,\mathrm{d}x\Big)\\
&= \Re\Big(-i\int_{\mathbb{T}^d} (\chi_r\star V_r)(t,x)\,|\Delta\psi_r(t,x)|^2\,\mathrm{d}x\Big)\\
&\quad + \Re\Big(-i\int_{\mathbb{T}^d} \Delta(\chi_r\star V_r)(t,x)\,\psi_r(t,x)\Delta\overline{\psi_r}(t,x)\,\mathrm{d}x\Big)\\
&\quad + 2\Re\Big(-i\int_{\mathbb{T}^d} \nabla(\chi_r\star V_r)(t,x)\nabla\psi_r(t,x)\Delta\overline{\psi_r}(t,x)\,\mathrm{d}x\Big)\\
&= \Re\Big(-i\int_{\mathbb{T}^d} \Delta(\chi_r\star V_r)(t,x)\psi_r(t,x)\Delta\overline{\psi_r}(t,x)\,\mathrm{d}x\Big)\\
&\quad + 2\Re\Big(-i\int_{\mathbb{T}^d} \nabla(\chi_r\star V_r)(t,x)\nabla\psi_r(t,x)\Delta\overline{\psi_r}(t,x)\,\mathrm{d}x\Big)\\
&=: I + J.
\end{aligned}$$

First, note that

$$\begin{aligned}
|J| &\leqslant 2\int_{\mathbb{T}^d} \left|\nabla(\chi_r\star V_r)(t,x)\,\nabla\psi_r(t,x)\,\Delta\overline{\psi_r}(t,x)\right|\mathrm{d}x\\
&\leqslant \int_{\mathbb{T}^d} |\nabla(\chi_r\star V_r)(t,x)\,\nabla\psi_r(t,x)|^2\,\mathrm{d}x + \int_{\mathbb{T}^d} |\Delta\psi_r(t,x)|^2\,\mathrm{d}x\\
&\leqslant \|(\chi_r\star\nabla V_r)(t,\cdot)\|_4^2\Big(\int_{\mathbb{T}^d} |\nabla\psi_r(t,x)|^4\,\mathrm{d}x\Big)^{\frac{1}{2}} + \int_{\mathbb{T}^d} |\Delta\psi_r(t,x)|^2\,\mathrm{d}x\\
&\leqslant \|\nabla V_r(t,\cdot)\|_4^2\Big(\int_{\mathbb{T}^d} |\nabla\psi_r(t,x)|^4\,\mathrm{d}x\Big)^{\frac{1}{2}} + \int_{\mathbb{T}^d} |\Delta\psi_r(t,x)|^2\,\mathrm{d}x\\
&\lesssim \left(\|\nabla\tilde{V}_r(t,\cdot)\|_4^2 + \|\nabla\hat{V}_r(t,\cdot)\|_4^2\right)\Big(1 + \int_{\mathbb{T}^d} |\Delta\psi_r(t,x)|^2\,\mathrm{d}x\Big) + \int_{\mathbb{T}^d} |\Delta\psi_r(t,x)|^2\,\mathrm{d}x,
\end{aligned} \tag{5.21}$$

where we have used the Sobolev inequality, according to which $\|\nabla\psi\|_4^2 \lesssim 1 + \|\Delta\psi\|_2^2$, in order to derive the last inequality. To bound the first term in the above inequality, observe that

$$\|\nabla\tilde{V}_r(t,\cdot)\|_4^2 \leqslant \|\psi_r(t,\cdot)\|_4^2 + 1 \leqslant C_S\left(1 + \|\nabla\psi_r(t,\cdot)\|_2^2\right) \lesssim_{\mathcal{M}_{\rm in},d} 1, \tag{5.22}$$

where the last inequality is due to the Sobolev inequality and (5.20) in Step 1. In addition, lemma 2.3 entails

$$\|\nabla\hat{V}_r(t,\cdot)\|_4 \lesssim_{\mathcal{M}_{\text{in}},d} 1. \tag{5.23}$$

Hence, inequalities (5.21)–(5.23) imply

$$J \lesssim_{\mathcal{M}_{\text{in}},d} 1 + \|\Delta\psi_r(t,\cdot)\|_2^2. \tag{5.24}$$

As for I, we first observe

$$\|\Delta V_r(t,\cdot)\|_2 \leqslant \|\left(\chi_r \star |\psi_r|^2\right)(t,\cdot)\|_2 + \|e^{V_r(t,\cdot)}\|_2 \leqslant \|\psi_r(t,\cdot)\|_4^2 + \|\psi_r(t,\cdot)\|_4^2 = 2\|\psi_r(t,\cdot)\|_4^2,$$

where the second inequality is due to the estimate: $\|e^{V_r}\|_2 \leqslant \|(\chi_r \star |\psi|^2)(t,\cdot)\|_2$ proved in lemma 5.5. Therefore, it follows from the Sobolev embedding and (5.20) that

$$\|\Delta V_r(t,\cdot)\|_2 \lesssim_{\mathcal{M}_{\text{in}},d} 1. \tag{5.25}$$

We proceed by observing that

$$\begin{aligned} I &\leqslant \int_{\mathbb{T}^d} \left|(\chi_r \star \Delta V_r)(t,x)\,\psi_r(t,x)\,\Delta\overline{\psi_r}(t,x)\right| \mathrm{d}x \\ &\leqslant \int_{\mathbb{T}^d} |(\chi_r \star \Delta V_r)(t,x)\,\psi_r(t,x)|^2 \,\mathrm{d}x + \int_{\mathbb{T}^d} \left|\Delta\overline{\psi_r}(t,x)\right|^2 \mathrm{d}x \\ &\leqslant \|\psi_r(t,\cdot)\|_\infty^2 \|(\chi_r \star \Delta V_r)(t,\cdot)\|_2^2 + \|\Delta\psi_r(t,\cdot)\|_2^2 \\ &\leqslant \|\psi_r(t,\cdot)\|_\infty^2 \|\Delta V_r(t,\cdot)\|_2^2 + \|\Delta\psi_r(t,\cdot)\|_2^2 \\ &\lesssim_{\mathcal{M}_{\text{in}},d} \left(1 + \|\Delta\psi_r(t,\cdot)\|_2^2\right), \end{aligned} \tag{5.26}$$

where we have used (5.25) and the interpolation inequality:

$$\|\Psi\|_\infty \leqslant C_d \left(\|\Psi\|_2 + \|\Delta\Psi\|_2\right) \tag{5.27}$$

for the last inequality in (5.26). Thus, gathering (5.26) and (5.24), we have proved

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi_r(t,\cdot)\|_{H^2(\mathbb{T}^d)}^2 \leqslant C\|\psi_r(t,\cdot)\|_{H^2(\mathbb{T}^d)}^2$$

for some $C = C(\|\psi^{\text{in}}\|_{H^1}, d)$, which, owing to the Grönwall inequality, implies

$$\|\psi_r(t,\cdot)\|_{H^2(\mathbb{T}^d)} \leqslant e^{Ct}\|\psi^{\text{in}}\|_{H^2(\mathbb{T}^d)} \leqslant \mathcal{M}'_{\text{in}}, \tag{5.28}$$

where $C = C(\mathcal{M}_{\text{in}}, d)$ and $\mathcal{M}'_{\text{in}} = \mathcal{M}'_{\text{in}}(T, d, \|\psi^{\text{in}}\|_{H^2(\mathbb{T}^d)})$.

3. *Uniform bound in r on* $\sup_{t\in[0,T]} \|\psi_r(t,\cdot)\|_{H^4(\mathbb{T}^d)}$. We follow a similar approach to the one taken in Step 2 in the proof of theorem 5.6, keeping track of the constants to ensure that they are uniform in r. From the same calculations leading to (5.14), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta^2\psi_r(t,\cdot)\|_2^2 \lesssim \sum_{k=0}^{4} \int_{\mathbb{T}^d} \left|\nabla^k (\chi_r \star V_r)\,\nabla^{4-k}\psi_r \Delta^2\psi_r\right|(t,x)\;\mathrm{d}x := \sum_{k=0}^{4} I_k.$$

We proceed by estimating each one of I_k:

$$\begin{aligned} I_0 &= \int_{\mathbb{T}^d} \left|(\chi_r \star V_r)(t,x)\right| \left|\Delta^2\psi_r(t,x)\right|^2 \mathrm{d}x \leqslant \|V_r\|_{L^\infty_{t,x}} \|\Delta^2\psi_r(t,\cdot)\|_2^2, \\ I_1 &= \int_{\mathbb{T}^d} \left|\nabla(\chi_r \star V_r)(t,x)\right| \left|\nabla\Delta\psi_r(t,x)\,\Delta^2\psi_r(t,x)\right| \mathrm{d}x \\ &\leqslant \|\nabla V_r\|_{L^\infty_{t,x}} \left(\|\nabla\Delta\psi_r(t,\cdot)\|_2^2 + \|\Delta^2\psi_r(t,\cdot)\|_2^2\right), \\ I_2 &= \int_{\mathbb{T}^d} \left|\Delta(\chi_r \star V_r)(t,x)\right| \left|\left(\Delta\psi_r\Delta^2\psi_r\right)(t,x)\right| \mathrm{d}x \\ &\leqslant \|\Delta V_r\|_{L^\infty_{t,x}} \left(\|\Delta\psi_r(t,\cdot)\|_2^2 + \|\Delta^2\psi_r(t,\cdot)\|_2^2\right). \end{aligned}$$

Using lemma 2.3, (5.28), and the Sobolev embedding $H^2(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$, we see that

$$\|V_r\|_{L^\infty_{t,x}} + \|\Delta V_r\|_{L^\infty_{t,x}} \leqslant \|V_r\|_{L^\infty_{t,x}} + \|\psi_r\|^2_{L^\infty_{t,x}} + \left\|e^{V_r}\right\|_\infty \lesssim_{T,d,\|\psi^{\text{in}}\|_{H^2(\mathbb{T}^d)}} 1, \tag{5.29}$$

so that

$$I_0 + I_1 + I_2 \lesssim_{T,d,\|\psi^{\text{in}}\|_{H^2(\mathbb{T}^d)}} \left(\|\Delta\psi_r(t,\cdot)\|_2^2 + \|\Delta^2\psi_r(t,\cdot)\|_2^2\right). \tag{5.30}$$

In order to bound I_3 and I_4, note that

$$-\nabla\Delta V_r = \chi_r \star \nabla|\psi_r|^2 - e^{V_r}\nabla V_r$$

and

$$-\Delta^2 V_r = \chi_r \star \Delta|\psi_r|^2 - \Delta\left(e^{V_r}\right) = \chi_r \star \Delta|\psi_r|^2 - e^{V_r}|\nabla V_r|^2 - e^{V_r}\Delta V_r.$$

Therefore, we have

$$\begin{aligned}
I_4 &\leqslant \|\Delta^2\psi_r(t,\cdot)\|_2^2 + \|(\chi_r \star \Delta^2 V_r)(t,\cdot)\|_\infty^2 \leqslant \|\Delta^2\psi_r(t,\cdot)\|_2^2 + \|\Delta^2 V_r(t,\cdot)\|_\infty^2 \\
&\leqslant \|\Delta^2\psi_r(t,\cdot)\|_2^2 + \|(\chi_r \star \Delta|\psi_r|^2)(t,\cdot)\|_\infty + \|(e^{V_r}|\nabla V_r|^2)(t,\cdot)\|_\infty + \|(e^{V_r}\Delta V_r)(t,\cdot)\|_\infty \\
&:= I_{40} + I_{41} + I_{42} + I_{43}.
\end{aligned}$$

First, using (5.27), we obtain

$$\begin{aligned}
I_{41} &\leqslant \|\Delta|\psi_r|^2(t,\cdot)\|_\infty = 2\|\psi_r(t,\cdot)\|_\infty\|\Delta\psi_r(t,\cdot)\|_\infty + 2\|\nabla\psi_r(t,\cdot)\|_\infty^2 \\
&\leqslant \|\psi_r(t,\cdot)\|_\infty^2 + \|\Delta\psi_r(t,\cdot)\|_\infty^2 + 2\|\nabla\psi_r(t,\cdot)\|_\infty^2 \\
&\lesssim_d 1 + \|\Delta\psi_r(t,\cdot)\|_2^2 + \|\Delta^2\psi_r(t,\cdot)\|_2^2 + \|\nabla\psi_r(t,\cdot)\|_2^2 + \|\nabla\Delta\psi_r(t,\cdot)\|_2^2 \\
&\lesssim_d 1 + \|\Delta^2\psi_r(t,\cdot)\|_2^2.
\end{aligned} \tag{5.31}$$

Secondly, note that, by lemma 2.3 (or alternatively by (5.29)), we have

$$\|e^{V_r}\|_\infty \lesssim_{T,d,\|\psi^{\text{in}}\|_{H^2(\mathbb{T}^d)}} 1, \qquad \|\Delta V_r(t,\cdot)\|_\infty \lesssim_{T,d,\|\psi^{\text{in}}\|_{H^2(\mathbb{T}^d)}} 1.$$

It follows that

$$I_{42} + I_{43} \lesssim_{T,d,\|\psi^{\text{in}}\|_{H^2(\mathbb{T}^d)}} 1 + \|\nabla V_r\|_\infty^2 + \|\Delta V_r\|_\infty^2 \lesssim_{T,d,\|\psi^{\text{in}}\|_{H^2(\mathbb{T}^d)}} 1. \tag{5.32}$$

Gathering (5.31) with (5.32) shows that

$$I_4 \lesssim_{T,d,\|\psi^{\text{in}}\|_{H^2(\mathbb{T}^d)}} \left(1 + \left\|\Delta^2\psi_r(t,\cdot)\right\|_2^2\right).$$

It follows from the same considerations that

$$I_3 \lesssim_{T,d,\|\psi^{\text{in}}\|_{H^2(\mathbb{T}^d)}} \left(1 + \left\|\Delta^2\psi_r(t,\cdot)\right\|_2^2\right),$$

which implies

$$I_3 + I_4 \lesssim_{T,d,\|\psi^{\text{in}}\|_{H^2(\mathbb{T}^d)}} \left(1 + \left\|\Delta^2\psi_r(t,\cdot)\right\|_2^2\right). \tag{5.33}$$

Thus, the combination of (5.30) with (5.33) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Delta^2\psi_r(t,\cdot)\right\|_2^2 \lesssim_{T,d,\|\psi^{\text{in}}\|_{H^2(\mathbb{T}^d)}} \left(1 + \left\|\Delta^2\psi_r(t,\cdot)\right\|_2^2\right)$$

and therefore

$$\|\psi_r(t,\cdot)\|_{H^4(\mathbb{T}^d)} \leqslant e^{Ct}\|\psi^{\text{in}}\|_{H^4(\mathbb{T}^d)} \leqslant \mathcal{M}''_{\text{in}}, \tag{5.34}$$

where $C = C(T,d,\|\psi^{\text{in}}\|_{H^4(\mathbb{T}^d)})$ and $\mathcal{M}''_{\text{in}} = \mathcal{M}''_{\text{in}}(T,d,\|\psi^{\text{in}}\|_{H^4(\mathbb{T}^d)})$.

4. *Compactness in r.* We utilize the preceding estimates in order to extract a converging subsequence. We have

$$\begin{aligned}\|\psi_r(t,\cdot)-\psi_r(s,\cdot)\|_{H^2}^2 &\leqslant 2\Big\|\int_t^s \partial_\tau\psi_r(\tau,\cdot)\,\mathrm{d}\tau\Big\|_2^2+2\Big\|\int_t^s \partial_\tau\Delta\psi_r(\tau,\cdot)\,\mathrm{d}\tau\Big\|_2^2\\ &\leqslant 2|t-s|^2\Big(\sup_{t\in[0,T]}\|\partial_t\psi_r(t,\cdot)\|_2^2+\sup_{t\in[0,T]}\|\partial_t\Delta\psi_r(t,\cdot)\|_2^2\Big).\end{aligned}\tag{5.35}$$

Using (5.18) and (5.28), we have

$$\begin{aligned}\|\partial_t\psi_r(t,\cdot)\|_2 &\leqslant \|\Delta\psi_r(t,\cdot)\|_2+\|(\chi_r\star V_r)(t,\cdot)\psi_r(t,\cdot)\|_2\\ &\leqslant C\big(T,\|\psi^{\mathrm{in}}\|_{H^2},d\big)+\|(\chi_r\star V_r)(t,\cdot)\psi_r(t,\cdot)\|_2.\end{aligned}$$

In addition, recall that, by (5.20),

$$\|(\chi_r\star V_r)(t,\cdot)\psi_r(t,\cdot)\|_2\leqslant\|(\chi_r\star V_r)(t,\cdot)\|_\infty\leqslant\|V_r(t,\cdot)\|_\infty\leqslant C\big(\|\psi^{\mathrm{in}}\|_{H^1},d\big).$$

It follows that

$$\sup_{t\in[0,T]}\|\partial_t\psi_r(t,\cdot)\|_2\lesssim_{T,d,\|\psi^{\mathrm{in}}\|_{H^2}}1.\tag{5.36}$$

Moreover, we have

$$\begin{aligned}\|\partial_t\Delta\psi_r(t,\cdot)\|_2 &\leqslant \|\Delta^2\psi_r(t,\cdot)\|_2+\|\Delta(\chi_r\star V_r\psi_r)(t,\cdot)\|_2\\ &\leqslant \|\Delta^2\psi_r(t,\cdot)\|_2+\|V_r(t,\cdot)\|_\infty\|\Delta\psi_r(t,\cdot)\|_2+\|\Delta V_r(t,\cdot)\|_\infty+2\|\nabla V_r(t,\cdot)\|_\infty\|\nabla\psi_r(t,\cdot)\|_2.\end{aligned}$$

Thanks to (5.29), (5.34), and the last inequality above, we see that

$$\sup_{t\in[0,T]}\|\partial_t\Delta\psi_r(t,\cdot)\|_2^2\lesssim_{T,d,\|\psi^{\mathrm{in}}\|_{H^4}}1.\tag{5.37}$$

Substituting (5.36) and (5.37) into (5.35) gives

$$\|\psi_r(t,\cdot)-\psi_r(s,\cdot)\|_{H^2(\mathbb{T}^d)}\leqslant C|t-s|,$$

where $C=C(T,d,\|\psi^{\mathrm{in}}\|_{H^4})$. Therefore, it follows from the Arzela–Ascoli theorem that there are both a sequence $r_k\to0$ as $k\to\infty$ and some $\psi\in C([0,T];H^2(\mathbb{T}^d))$ such that

$$\|\psi_{r_k}(t,\cdot)-\psi(t,\cdot)\|_{C([0,T];H^2(\mathbb{T}^d))}\underset{k\to\infty}{\longrightarrow}0$$

with the estimate:

$$\sup_{t\in[0,T]}\|\psi(t,\cdot)\|_{H^2}\leqslant\mathcal{M}'_{\mathrm{in}}\tag{5.38}$$

where $\mathcal{M}'_{\mathrm{in}}$ is as in (5.28).

5. *$\psi(t,\cdot)$ is a solution.* It remains to show that the limit $\psi(t,\cdot)$ is a solution to system (1.4). Note that $\psi(t,\cdot)\in L^\infty(\mathbb{T}^d)$, because of the embedding $H^2(\mathbb{T}^d)\hookrightarrow L^\infty(\mathbb{T}^d)$. Therefore, by lemma 2.3, there exists a unique solution V of the equation:

$$-\Delta V=|\psi|^2-e^V.$$

To show that ψ is a solution, the key component is to show that

$$\|(\chi_{r_k}\star V_{r_k})(t,\cdot)\psi_{r_k}(t,\cdot)-V(t,\cdot)\psi(t,\cdot)\|_2\underset{k\to\infty}{\longrightarrow}0.$$

By the triangle inequality, we have

$$\begin{aligned}&\|(\chi_{r_k}\star V_{r_k})(t,\cdot)\psi_{r_k}(t,\cdot)-V(t,\cdot)\psi(t,\cdot)\|_2\\ &\quad\leqslant\|(\chi_{r_k}\star V_{r_k})(t,\cdot)\psi_{r_k}(t,\cdot)-(\chi_{r_k}\star V)(t,\cdot)\psi(t,\cdot)\|_2\\ &\qquad+\|(\chi_{r_k}\star V)(t,\cdot)\psi(t,\cdot)-V(t,\cdot)\psi(t,\cdot)\|_2\\ &\quad\leqslant\|(\chi_{r_k}\star V)(t,\cdot)(\psi_{r_k}(t,\cdot)-\psi(t,\cdot))\|_2\end{aligned}$$

$$+\|(\chi_{r_k}\star V_{r_k}-\chi_{r_k}\star V)(t,\cdot)\psi_{r_k}(t,\cdot)\|_2$$
$$+\|\psi(t,\cdot)((\chi_{r_k}\star V)(t,\cdot)-V(t,\cdot))\|_2:=\mathcal{I}+\mathcal{J}+\mathcal{L}.$$

The term $\mathcal{I}$ is estimated as

$$\mathcal{I}\leqslant\|V(t,\cdot)\|_\infty\|\psi_{r_k}(t,\cdot)-\psi(t,\cdot)\|_2\leqslant\left(\|\tilde{V}(t,\cdot)\|_\infty+\|\hat{V}(t,\cdot)\|_\infty\right)\|\psi_{r_k}(t,\cdot)-\psi(t,\cdot)\|_2.$$

By the Sobolev embedding and (5.38), we have

$$\begin{aligned}\|\tilde{V}(t,\cdot)\|_\infty&\leqslant\|K\|_2\left(1+\|\psi(t,\cdot)\|_4^2\right)\\&\leqslant\|K\|_2\left(1+C_S\sup_{t\in[0,T]}\|\psi(t,\cdot)\|^2_{H^1(\mathbb{T}^d)}\right)\lesssim_{T,d,\|\psi^{\rm in}\|_{H^2}}1.\end{aligned}$$

In addition, lemma 2.3 and (5.38) entail

$$\sup_{t\in[0,T]}\|\hat{V}(t,\cdot)\|_\infty\lesssim_{T,d,\|\psi^{\rm in}\|_{H^2}}1.$$

Thus, we obtain the estimate:

$$\mathcal{I}\lesssim_{T,d,\|\psi^{\rm in}\|_{H^2}}\sup_{t\in[0,T]}\|\psi_{r_k}(t,\cdot)-\psi(t,\cdot)\|_2\underset{k\to\infty}{\longrightarrow}0.$$

To treat $\mathcal{J}$, we note that

$$\mathcal{J}\leqslant\|(\chi_{r_k}\star V_{r_k}-\chi_{r_k}\star V)(t,\cdot)\psi_{r_k}(t,\cdot)\|_2\leqslant\|\psi_{r_k}(t,\cdot)\|_\infty\|(V_{r_k}-V)(t,\cdot)\|_2.$$

Using lemma 5.9, (5.28), and (5.38), we see that

$$\|V_{r_k}(t,\cdot)-V(t,\cdot)\|_2\lesssim_{T,d,\|\psi^{\rm in}\|_{H^2}}W_2\left((\chi_{r_k}\star|\psi_{r_k}|^2)(t,\cdot),|\psi|^2(t,\cdot)\right)\underset{k\to\infty}{\to}0.$$

In addition, we have

$$\|\psi_{r_k}(t,\cdot)\|_\infty\leqslant\|\psi_{r_k}(t,\cdot)\|_{H^2(\mathbb{T}^d)}\lesssim_{T,d,\|\psi^{\rm in}\|_{H^2}}1,$$

which eventually yields

$$\mathcal{J}\underset{k\to\infty}{\longrightarrow}0.$$

Finally, again by (5.38), we obtain

$$\begin{aligned}\mathcal{L}&\leqslant\|\psi(t,\cdot)\|_\infty\|(\chi_{r_k}\star V)(t,\cdot)-V(t,\cdot)\|_2\\&\lesssim_{T,d,\|\psi^{\rm in}\|_{H^2}}\|(\chi_{r_k}\star V)(t,\cdot)-V(t,\cdot)\|_2\underset{k\to\infty}{\to}0.\end{aligned}$$

This completes the proof of theorem 1.1.

□

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Data availability statement

No new data were created or analysed in this study.

Conflict of interest

The authors declare that they have no conflict of interest. The authors also declare that this manuscript has not been previously published.

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