



Low Regularity of Self-Similar Solutions of Two-Dimensional Riemann Problems with Shocks for the Isentropic Euler System

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Abstract

We are concerned with the low regularity of self-similar solutions of two-dimensional Riemann problems for the isentropic Euler system. We establish a general framework for the analysis of the local regularity of such solutions for a class of two-dimensional Riemann problems for the isentropic Euler system, which includes the regular shock reflection problem, the Prandtl reflection problem, the Lighthill diffraction problem, and the four-shock Riemann problem. We prove that the velocity is not in H^1 in the subsonic domain for the self-similar solutions of these problems in general. This indicates that the self-similar solutions of the Riemann problems with shocks for the isentropic Euler system are of much more complicated structure than those for the Euler system for potential flow; in particular, the velocity is not necessarily continuous in the subsonic domain. The proof is based on a regularization of the isentropic Euler system to derive the transport equation for the vorticity, a renormalization argument extended to the case of domains with boundary, and DiPerna-Lions-type commutator estimates.

Keywords Low regularity · Transonic shock · Free boundary · Compressible flow · Potential flow · Self-similar · Conservation laws · Shock reflection-diffraction · Regular shock reflection · Prandtl reflection · Lighthill diffraction · Nonlinear · Fine properties

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1 Introduction

We are concerned with the regularity of self-similar solutions of the Riemann problems with shocks for the isentropic Euler system in a general setting, including several fundamental shock problems such as the regular shock reflection problem, the Prandtl reflection problem, the Lighthill diffraction problem, and the four-shock Riemann problem. In 1860, Riemann first considered a special initial value problem with two constant states separated at the origin for the one-dimensional isentropic Euler system in [Rie60] – now known as the Riemann problem; this Riemann problem has played a fundamental role in the mathematical theory of hyperbolic systems of conservation laws, since its solutions are building blocks and asymptotic attractors of general global entropy solutions. Since then, a systematic theory of one-dimensional hyperbolic systems of conservation laws has been established; see [Bre00, BLY99, CH89, Daf16, Gli65, Lax57, LY99] and the reference cited therein. However, multi-dimensional Riemann problems are much more complicated and completely different from the one-dimensional case. Even for the two-dimensional Riemann problem with four constant states given in the four quadrants for the Euler system, nineteen genuinely different configurations have been identified; see [CCY95, CCY00, CCHLW25, LL98, ZZ90, Zhe06]. Since then, rigorous global results for the 2-D four-quadrant Riemann problem for the Euler system were only done by Li-Zheng in [LZ09, LZ10]. See also [CQ12, Ser09] for the 2-D Riemann problem for Chaplygin gases.

On the other hand, the regular shock reflection problem is a different type of multi-dimensional Riemann problems – a lateral Riemann problem that involves the wedge boundaries. Shock reflection-diffraction phenomena were first presented by Ernst Mach [Mac78] in 1878, and experimental, computational, and asymptotic analysis has shown that various patterns of shock reflection may occur, including regular and Mach reflection; see [CF18] and the references cited therein.

More recently, the existence of global regular reflection solutions was established for the Euler system for potential flow in [CF10, CF18], and further properties of these solutions were proved in [BCF09, CFX20]. In particular, the solution has a high regularity in the subsonic domain Ω (where the solution is not a constant state); see Figs. 2–3 in which density ρ and velocity \mathbf{v} are in $C^\alpha(\bar{\Omega}) \cap C^\infty(\Omega)$ for some $\alpha \in (0, 1)$. More precisely, solution (ρ, \mathbf{v}) is in a weighted Hölder space which implies that (ρ, \mathbf{v}) is in the Sobolev space $W^{1,p}(\Omega)$ for some $p > 2$. In addition, the regularity of the curved reflected-diffracted shock was shown to be C^∞ in the interior and $C^{2,\alpha}$ up to the endpoints, which is also expected to hold for the case of the isentropic Euler system. See also [BCF13, BCF24, EL08] for the Prandtl reflection-diffraction configuration, [CDX14] for the Lighthill diffraction problem, and [CCHLW25] for the four-shock Riemann problem.

In contrast, a remarkable phenomenon was first observed by Serre in [Ser07] which showed by a formal calculation that, in the case of the isentropic Euler system, the regular shock reflection solutions develop *vortical singularity*, specifically that the vorticity cannot be in $L^2(\Omega)$. This implies that the velocity cannot be in $W^{1,2}(\Omega)$, i.e., $H^1(\Omega)$, which is lower than the regularity of the velocity for the case of potential flow discussed above. This in particular allows that the velocity can be discontinuous,

which at least can not be excluded by the Sobolev embeddings. Since the calculation is formal, it is important to find out whether the low regularity indeed necessarily holds for the regular shock reflection solutions.

In this paper, under the natural assumptions on the regularity of the self-similar solutions in the subsonic domain Ω near the shocks, we rigorously prove that the velocity is not in $H^1(\Omega)$ indeed. The argument is based on the vortical singularity calculation in [Ser07]. We first apply this calculation to the regularized solutions carefully for which the calculation can be rigorously justified; however, the additional error terms appear due to the regularization. Then we develop DiPerna-Lions-type commutator estimates to control the error terms when the regularization parameter tends to zero. With this, we then prove the lower regularity property of the self-similar solutions, by employing renormalization argument. These self-similar solutions include the above-mentioned shock reflection problems: the regular shock reflection problem, the Prandtl reflection problem, the Lighthill diffraction problem, and the four-shock Riemann problem.

This indicates that the self-similar solutions of the Riemann problems with shocks for the isentropic Euler system have much more complicated structure than the corresponding solutions for the Euler system for potential flow; in particular, the velocity is not necessarily continuous in Ω , at least their continuity can not be obtained directly by the Sobolev embeddings. On the other hand, this argument allows the possibility that $(\rho, \mathbf{v}) \in W^{1,p}(\Omega)$ for some or even all $p \in [1, 2)$, in which case there are no shocks in Ω .

This paper is organized as follows: In § 2, we derive the isentropic Euler system in self-similar coordinates and then present the notion of entropy solutions in the new coordinates. In § 3, we first formulate a general framework for analyzing the low regularity of entropy solutions of the Riemann problems in Definition 3.1 and then establish our main theorem of this paper, Theorem 3.2, for the entropy solutions. Then, in § 4, we employ the main theorem, Theorem 3.2, for the general framework established in § 3 to several fundamental transonic shock problems including the regular shock reflection problem, the Prandtl reflection problem, the Lighthill diffraction problem, and the Riemann problem with four-shock interactions in § 4.1–§ 4.4, respectively. The general framework and the main theorem, as well as ideas and approaches, developed in this paper should be useful for solving other similar low regularity problems for solutions of nonlinear partial differential equations.

2 The Isentropic Euler System and Entropy Solutions with Shocks

In this section, we first derive the isentropic Euler system in self-similar coordinates and then present the notion of entropy solutions, as well as the Rankine-Hugoniot conditions and the corresponding entropy condition across a shock (as a free boundary), in self-similar coordinates.

2.1 The Isentropic Euler System

As in [Daf16, Ser07] (see also [CF18]), the isentropic Euler equations consist of the conservation laws of mass and momentum:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0, \end{cases} \quad (2.1)$$

where ρ is the density, $\mathbf{u} = (u_1, u_2)$ is the velocity, and p is the pressure. The constitutive relation between pressure p and density ρ is through the γ -law relation: $p = \frac{\rho^\gamma}{\gamma}$ after scaling, and the adiabatic exponent $\gamma > 1$ is a given constant.

If an initial-boundary value problem is invariant under the self-similar scaling:

$$(\mathbf{u}, p, \rho) = (\mathbf{u}, p, \rho)\left(\frac{\mathbf{x}}{t}\right), \quad (2.2)$$

we introduce self-similar variables $\boldsymbol{\xi} = (\xi_1, \xi_2) = \frac{\mathbf{x}}{t} \in \mathbb{R}^2$ and the pseudo-velocity $\mathbf{v} = \mathbf{u} - \boldsymbol{\xi}$. Then we obtain the isentropic Euler system for self-similar flow $(\rho, \mathbf{v}) = (\rho, \mathbf{v})(\boldsymbol{\xi})$ in the form:

$$\operatorname{div}(\rho \mathbf{v}) + 2\rho = 0, \quad (2.3)$$

$$\operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) + 3\rho \mathbf{v} + \nabla p = 0. \quad (2.4)$$

If $(\rho, \mathbf{v}) \in C^1$, then we can combine the equations above to rewrite (2.4) as

$$(\mathbf{v} \cdot \nabla) \mathbf{v} + \mathbf{v} + \nabla h(\rho) = 0, \quad (2.5)$$

where $h(\rho) = \frac{\rho^{\gamma-1}-1}{\gamma-1}$ is the enthalpy. Note that the speed of sound is $c = \rho^{\frac{\gamma-1}{2}} = \sqrt{(\gamma-1)h(\rho) + 1}$.

2.2 Entropy Solutions and the Rankine-Hugoniot Conditions

We consider solutions with shocks, which satisfy (2.3)–(2.4) in the following weak sense:

Definition 2.1 Let $\Lambda \subset \mathbb{R}^2$ be a domain. Then $(\rho, \mathbf{v}) \in L^\infty(\Lambda)$ is an entropy solution of system (2.3)–(2.4) if the following conditions hold:

(i) (ρ, \mathbf{v}) is a weak solution: For any test functions $\phi \in C_c^\infty(\Lambda)$ and $\zeta \in C_c^\infty(\Lambda; \mathbb{R}^2)$,

$$\int_{\Lambda} (\rho \mathbf{v} \cdot \nabla \phi - 2\rho \phi) \, d\boldsymbol{\xi} = 0, \quad (2.6)$$

$$\int_{\Lambda} (\rho \mathbf{v} \otimes \mathbf{v} : D\zeta - 3\rho \mathbf{v} \cdot \zeta + p \operatorname{div} \zeta) \, d\xi = 0, \tag{2.7}$$

where we have used the notation $A : B = \sum_{i,j=1}^2 a_{ij} b_{ij}$ for 2×2 matrices A and B .

(ii) (ρ, \mathbf{v}) satisfies the entropy condition: For any non-negative test function $\psi \in C_c^\infty(\Lambda)$,

$$\int_{\Lambda} \left(\left(\frac{1}{2} \rho |\mathbf{v}|^2 + \rho e(\rho) + p(\rho) \right) \mathbf{v} \cdot \nabla \psi - 2(\rho |\mathbf{v}|^2 + \rho e(\rho) + p(\rho)) \psi \right) \, d\xi \geq 0, \tag{2.8}$$

where the internal energy $e(\rho)$ is defined by $p(\rho) = \rho^2 e'(\rho)$, i.e., $e(\rho) = \frac{\rho^{\gamma-1}}{\gamma(\gamma-1)}$ for the polytropic case with $p(\rho) = \frac{\rho^\gamma}{\gamma}$.

Suppose that S is a smooth curve in domain Λ . An entropy solution of (2.3)–(2.4), which is C^1 in the open domains near S and C^0 up to S on both sides, satisfies the following Rankine-Hugoniot (R-H) conditions on S :

$$[\rho \mathbf{v} \cdot \boldsymbol{\nu}] = 0, \quad [(\rho \mathbf{v} \cdot \boldsymbol{\nu}) \mathbf{v} + p \boldsymbol{\nu}] = 0 \quad \text{on } S, \tag{2.9}$$

where $\boldsymbol{\nu}$ is a unit normal to S , and $[\cdot]$ denotes the difference of the concerned quantity across S . Denote by $(\rho^\pm, \mathbf{v}^\pm)$ the values of solution (ρ, \mathbf{v}) on the \pm sides of S . Assume that $\rho^\pm > 0$ on S and some of (ρ, \mathbf{v}) are discontinuous across S .

If $\rho^+ \mathbf{v}^+ \cdot \boldsymbol{\nu} = 0$ on S , then $\rho^- \mathbf{v}^- \cdot \boldsymbol{\nu} = 0$ on S by the first equality in (2.9), and $[p] = 0$ from the second equality in (2.9) which implies that $[\rho] = 0$. This discontinuity is called a vortex sheet. In this case, $[\mathbf{v} \cdot \boldsymbol{\tau}] \neq 0$ on S , unless the solution is continuous across S , where $\boldsymbol{\tau}$ is a unit tangent vector to S .

If $\rho^+ \mathbf{v}^+ \cdot \boldsymbol{\nu} \neq 0$ on S , then $\rho^- \mathbf{v}^- \cdot \boldsymbol{\nu} \neq 0$ on S by the first equality in (2.9). This discontinuity may form a shock. In this case, $\rho \mathbf{v} \cdot \boldsymbol{\nu}$ is continuous across S from the first equality in (2.9), and $[\mathbf{v} \cdot \boldsymbol{\tau}] = 0$ and $[p] = -\rho \mathbf{v} \cdot \boldsymbol{\nu} [\mathbf{v} \cdot \boldsymbol{\nu}]$ from the second equality in (2.9). This implies that $[\mathbf{v} \cdot \boldsymbol{\nu}] \neq 0$, unless the solution is continuous across S , so that $[\rho] \neq 0$ from the first equality in (2.9). Also, it follows from $\rho^\pm > 0$ on S that $(\mathbf{v}^+ \cdot \boldsymbol{\nu})(\mathbf{v}^- \cdot \boldsymbol{\nu}) > 0$ on S .

Thus, we have shown that the following properties hold, in addition to (2.9), in the case $\rho^\pm > 0$ on S :

Shock: $(\mathbf{v}^+ \cdot \boldsymbol{\nu})(\mathbf{v}^- \cdot \boldsymbol{\nu}) > 0, \quad [\mathbf{v} \cdot \boldsymbol{\tau}] = 0, \quad [\mathbf{v} \cdot \boldsymbol{\nu}] \neq 0, \quad [\rho] \neq 0; \tag{2.10}$

Vortex sheet: $\mathbf{v}^+ \cdot \boldsymbol{\nu} = \mathbf{v}^- \cdot \boldsymbol{\nu} = 0, \quad [\mathbf{v} \cdot \boldsymbol{\tau}] \neq 0, \quad [\rho] = 0. \tag{2.11}$

Furthermore, the entropy condition (2.8) is required across shock S separating the smooth states $(\rho^\pm, \mathbf{v}^\pm)$ defined in domains Λ^\pm with $\rho^\pm > 0$. Then it follows from the direct calculation through (2.8)–(2.10) and the choice of orientation of the unit normal $\boldsymbol{\nu}$ on S to point from Λ^- to Λ^+ that, on S ,

If $\mathbf{v}^- \cdot \boldsymbol{\nu} > 0$, then $\mathbf{v}^- \cdot \boldsymbol{\nu} > \mathbf{v}^+ \cdot \boldsymbol{\nu} > 0$, $\rho^- < \rho^+$, $\mathbf{v}^- \cdot \boldsymbol{\nu} > c^-$, $\mathbf{v}^+ \cdot \boldsymbol{\nu} < c^+$, (2.12)

where the last two inequalities are shown e.g., in [Ser07, Theorem 2.2] (the argument is given there for steady solutions; the proof applies to the self-similar case because the self-similar Rankine-Hugoniot conditions are the same), since $p(\rho)$ is increasing and convex, implied by $p(\rho) = \frac{\rho^\gamma}{\gamma}$ with $\gamma > 1$.

We also give a definition of entropy solutions of system (2.3)–(2.4) in Λ with the slip boundary conditions:

$$\mathbf{v} \cdot \boldsymbol{\nu} = 0 \quad \text{on } \partial\Lambda, \quad (2.13)$$

where $\boldsymbol{\nu}$ is the outer normal to $\partial\Lambda$. The motivation is the following: Suppose that $(\rho, \mathbf{v}) \in C^1$ and $\partial\Lambda \in \text{Lip}$ satisfy (2.3)–(2.4) in Λ and (2.13). Then it follows that (2.6) and

$$\int_{\Lambda} (\rho \mathbf{v} \otimes \mathbf{v} : D\zeta - 3\rho \mathbf{v} \cdot \zeta + p \operatorname{div} \zeta) \, d\xi - \int_{\partial\Lambda} p \zeta \cdot \boldsymbol{\nu} \, dl = 0 \quad (2.14)$$

are satisfied for any test function $\zeta \in C_c^\infty(\mathbb{R}^2; \mathbb{R}^2)$. Note that here the test function is not required to vanish on $\partial\Lambda$.

From now on, we define the r -neighborhood of a curve Γ as

$$\mathcal{N}_r(\Gamma) := \{\xi : \operatorname{dist}\{\xi, \Gamma\} < r\}. \quad (2.15)$$

Based on that, we define the notion of entropy solutions of the boundary value problem (2.3)–(2.4) and (2.13) as follows:

Definition 2.2 Let $\Lambda \subset \mathbb{R}^2$ be a domain with a Lipschitz boundary $\partial\Lambda$. Let $(\rho, \mathbf{v}) \in L^\infty(\Lambda)$, and let $\rho \in BV_{\text{loc}}(\Lambda \cap \mathcal{N}_r(\partial\Lambda))$ for some $r > 0$. Then (ρ, \mathbf{v}) is an entropy solution of system (2.3)–(2.4) with slip boundary condition (2.13) if (2.6), (2.14), and (2.8) are satisfied for any test functions $\phi \in C_c^\infty(\mathbb{R}^2)$, $\zeta \in C_c^\infty(\mathbb{R}^2; \mathbb{R}^2)$, and $\psi \in C_c^\infty(\Lambda)$ with $\psi \geq 0$, respectively.

3 Low Regularity of Self-Similar Solutions of the Riemann Problems With Shocks for the Isentropic Euler System

In this section, we first formulate a general framework for analyzing the low regularity of self-similar entropy solutions of admissible structure of the Riemann problems in Definition 3.1, motivated by the solutions of the physically fundamental Riemann problems described in § 4 below. Then we establish our main theorem, Theorem 3.2, for the entropy solutions (ρ, \mathbf{v}) in the general framework, which will be applied to understanding the low regularity of the solutions of the Riemann problems including the regular shock reflection problem, the Prandtl reflection problem, the Lighthill diffraction problem, and the Riemann problem with four shock interactions. These

are achieved by carefully analyzing the vorticity function $\omega := \partial_1 v_2 - \partial_2 v_1$ for the pseudo-velocity $\mathbf{v} = (v_1, v_2)$.

More precisely, as shown in Fig. 1, we consider a Riemann problem in a self-similar wedge domain Λ in \mathbb{R}^2 with the wedge-vertex being the origin, which also includes the case of the whole space. That is, in polar coordinates,

$$\text{either } \Lambda = \mathbb{R}^2 \text{ or } \Lambda = \{(r, \theta) : \theta^- < \theta < \theta^+\}. \tag{3.1}$$

For a Riemann problem in a domain Λ whose boundary contains a wedge-boundary, the initial data are the given constant velocity and density in each sub-sector of Λ such that the Rankine-Hugoniot conditions hold on the lines separating them. Motivated by the expected configurations of self-similar solutions of the Riemann problems in § 4, we consider solutions in self-similar coordinates of the following form, in terms of the density and pseudo-velocity (ρ, \mathbf{v}) :

Definition 3.1 We say that the vector function (ρ, \mathbf{v}) on Λ is an entropy solution of the Riemann problem of admissible structure if $(\rho, \mathbf{v}) \in L^\infty(\Lambda)$, with $\rho \in BV_{\text{loc}}(\Lambda \cap \mathcal{N}_r(\partial\Lambda))$ for some $r > 0$, is an entropy solution of system (2.3)–(2.4) with the slip boundary condition (2.13) in the sense of Definition 2.2, which satisfies the following properties:

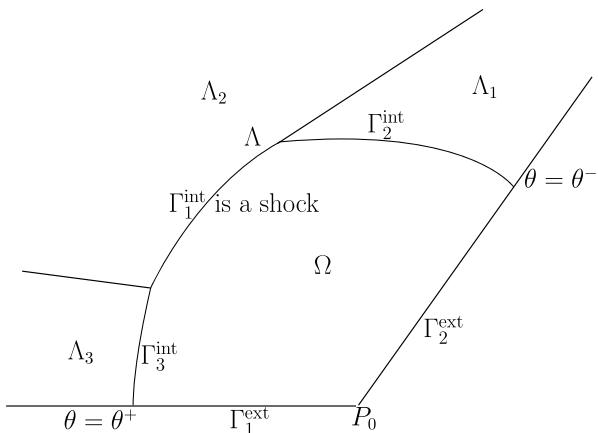
- (i) There exist an open, bounded, connected set $\Omega \subset \Lambda$, an integer $M \geq 1$, and open connected sets $\Lambda_i, i = 1, \dots, M$, pairwise disjoint, such that

$$\overline{\Lambda} \setminus \Omega = \cup_{i=1}^M \overline{\Lambda}_i,$$

and (ρ, \mathbf{v}) is a constant state in each Λ_i , i.e., $(\rho, \mathbf{v})(\xi) = (\rho_i, \mathbf{u}_i - \xi)$ in Λ_i , where $\rho_i > 0$ is a constant and \mathbf{u}_i is a constant vector.

- (ii) If, for $i \neq j$, sets Λ_i and Λ_j have common boundary within Λ , i.e., if $\partial\Lambda_i \cap \partial\Lambda_j \cap \Lambda \neq \emptyset$, then the corresponding constant states are not equal to each other: $(\rho_i, \mathbf{v}_i) \neq (\rho_j, \mathbf{v}_j)$.

Fig. 1 The Riemann problem in a general setting



(iii) $\partial\Omega$ is Lipschitz. Denote

$$\Gamma^{\text{ext}} = \partial\Omega \cap \partial\Lambda, \quad \Gamma^{\text{int}} = \partial\Omega \cap \Lambda,$$

so that $\partial\Omega = \Gamma^{\text{ext}} \cup \Gamma^{\text{int}}$. It is possible that $\Gamma^{\text{ext}} = \emptyset$ (in particular, this is true when $\partial\Lambda = \emptyset$).

- (iv) If $\partial\Lambda \neq \emptyset$, then $\Gamma^{\text{ext}} = \cup_{i=1}^{N_1} \overline{\Gamma_i^{\text{ext}}}$, where $N_1 \geq 1$ and each Γ_i^{ext} is a relatively open segment of straight line. Segments $\{\Gamma_i^{\text{ext}}\}_{i=1}^{N_1}$ are disjoint and, if Γ_i^{ext} and Γ_j^{ext} have a common endpoint, then the interior angle (for Ω) is within $(0, 2\pi)$ at that point. Since Λ is a wedge domain with the wedge-vertex at the origin, only one pair among segments $\{\Gamma_i^{\text{ext}}\}$ may have a common endpoint that is $\mathcal{P}_0 = (0, 0)$.
- (v) $\overline{\Gamma^{\text{int}}} = \cup_{i=1}^{N_2} \overline{\Gamma_i^{\text{int}}}$, where $N_2 \geq 1$, each Γ_i^{int} is a relatively open segment of curve, and segments $\{\Gamma_i^{\text{int}}\}_{i=1}^{N_2}$ are disjoint. Each Γ_i^{int} is C^2 in its relative interior and C^1 up to the endpoints. Moreover, $\mathcal{P}_0 \notin \overline{\Gamma^{\text{int}}}$.
- (vi) $\mathbf{v} \in C(\mathcal{N}_\sigma(\Gamma^{\text{int}}) \cap \overline{\Omega})$ for some $\sigma > 0$.
- (vii) Γ_1^{int} is a shock and $\mathbf{v} \cdot \boldsymbol{\nu} \leq -C^{-1}$ on Γ_1^{int} , and $\mathbf{v} \cdot \boldsymbol{\nu} \leq 0$ on Γ_i^{int} for $i = 2, \dots, N_2$, where \mathbf{v} is taken from the Ω -side and $\boldsymbol{\nu}$ denotes the outer unit normal with respect to Ω .
- (viii) There exists a point \hat{P} in the relative interior of Γ_1^{int} such that the curvature of Γ_1^{int} is non-zero at \hat{P} , and $(\mathbf{v} \cdot \boldsymbol{\tau})(\hat{P}) \neq 0$, where $\boldsymbol{\tau}(\hat{P})$ is a unit tangent vector to Γ_1^{int} at \hat{P} .

Then we have the following main theorem.

Theorem 3.2 (Main Theorem). Let (ρ, \mathbf{v}) be a solution of the Riemann problem of admissible structure in the sense of Definition 3.1. Assume that (ρ, \mathbf{v}) satisfy

- (i) $(\rho, \mathbf{v}) \in C^1((\mathcal{N}_\sigma(\Gamma^{\text{int}}) \cap \overline{\Omega}) \setminus \partial_p \Gamma^{\text{int}}) \cap C^{0,1}(\mathcal{N}_\sigma(\Gamma^{\text{int}}) \cap \overline{\Omega})$ for some $\sigma > 0$, where $\partial_p \Gamma^{\text{int}}$ denotes the set of endpoints of the curve segments Γ_i^{int} for $i = 1, \dots, N_2$.
- (ii) $|\mathbf{v}| \leq C_0$ and $C_0^{-1} \leq \rho \leq C_0$ in Ω for some $C_0 \geq 1$.
- (iii) The flow is subsonic on Γ_1^{int} from the Ω -side: $|\mathbf{v}| < c$ on Γ_1^{int} , where $c = \rho^{\frac{\gamma-1}{2}}$ is the speed of sound.

Then $\mathbf{v} \notin H^1(\Omega)$.

Proof We divide the proof into seven steps. In the first six steps, we prove that both ρ and \mathbf{v} can not be in $H^1(\Omega)$:

$$(\rho, \mathbf{v}) \notin H^1(\Omega). \tag{3.2}$$

This will be proved by showing that, under the assumption that

$$(\rho, \mathbf{v}) \in H^1(\Omega), \tag{3.3}$$

the calculations of vortical singularity can rigorously be justified, which leads to the contradiction. Then, in Step 7, we prove that \mathbf{v} is not in $H^1(\Omega)$ indeed: $\mathbf{v} \notin H^1(\Omega)$.

1. In Steps 1–6 of this proof, we assume that (3.3) holds. Then vorticity $\omega = \nabla \times \mathbf{v}$ satisfies $\omega \in L^2(\Omega)$ and

$$X := \frac{\omega}{\rho} \in L^2(\Omega), \tag{3.4}$$

where we have used (3.3) and assumption (ii). In addition, using (3.3) and assumptions (i)–(ii), we see that

$$\begin{aligned} & \text{The left-hand sides of equations (2.3) and (2.5) are in } L^2(\Omega), \\ & \text{and the equations hold a.e. in } \Omega. \end{aligned} \tag{3.5}$$

2. We first formally derive the equations and identities that vorticity ω satisfies, and then prove them rigorously.

Taking the curl of (2.5), we formally obtain the equation:

$$\mathbf{v} \cdot \nabla \omega + (1 + \operatorname{div} \mathbf{v}) \omega = 0. \tag{3.6}$$

Combining with the first equation of (2.3), we formally have

$$\mathbf{v} \cdot \nabla \left(\frac{\omega}{\rho} \right) = \frac{\omega}{\rho}. \tag{3.7}$$

If $f \in C^1(\mathbb{R})$, multiplying the last equation by $f'(\frac{\omega}{\rho})$ and then combining with the first equation of (2.3), we formally derive

$$\operatorname{div} \left(\rho f \left(\frac{\omega}{\rho} \right) \mathbf{v} \right) = \rho g \left(\frac{\omega}{\rho} \right), \tag{3.8}$$

where

$$g(s) := s f'(s) - 2f(s). \tag{3.9}$$

Now we are going to show that Eq. (3.8) holds under the present assumptions in the weak sense defined as follows: Using notation (3.4), Eq. (3.8) can be written as

$$\operatorname{div} (\rho f(X) \mathbf{v}) = \rho g(X) \tag{3.10}$$

for any $f \in C^1(\mathbb{R})$ with g defined by (3.9). From the weak form (2.6) of the conservation law of mass, which holds for all ϕ pecified in Definition 2.2, and the regularity assumption (3.3), we obtain that $\rho \mathbf{v} \cdot \boldsymbol{\nu} = 0$ holds \mathcal{H}^1 -a.e. on Γ^{ext} in the sense of traces. Then, using assumption (ii), we recover condition (2.13) on $\partial\Lambda \cap \partial\Omega$:

$$\mathbf{v} \cdot \boldsymbol{\nu} = 0 \quad \mathcal{H}^1\text{-a.e. on } \Gamma^{\text{ext}}. \tag{3.11}$$

We next show that, under the present assumptions, (3.10) holds weakly in Ω in the sense that

$$\int_{\Omega} (\rho f(X) \mathbf{v} \cdot \nabla \zeta + \rho g(X) \zeta) d\xi - \int_{\Gamma^{\text{int}}} \rho f(X) \mathbf{v} \cdot \boldsymbol{\nu} \zeta dl = 0 \tag{3.12}$$

for all $\zeta \in C^\infty(\mathbb{R}^2)$

for any $f \in C^1(\mathbb{R})$ with $\|f'\|_{C^{0,1}(\mathbb{R})} < \infty$.

Notice that the boundary integral in (3.12) is taken along a part of $\partial\Omega$. Formally, the boundary integral along the remaining part Γ^{ext} of $\partial\Omega$ is expected to vanish by (3.11). Eq. (3.12) shows that

$$\rho f(X) \mathbf{v} \cdot \boldsymbol{\nu} = 0 \quad \text{on } \Gamma^{\text{ext}} \text{ in the weak sense.}$$

Furthermore, the integrand in the boundary integral in (3.12) is well-defined by assumption (i) and Definition 3.1(v).

3. To prove (3.12), we introduce smooth approximations of (ρ, \mathbf{v}) . Since we work in the bounded domain Ω and need the boundary condition $\mathbf{v} \cdot \boldsymbol{\nu} = 0$ on Γ^{ext} and other properties to hold for the approximating functions for the argument below, we first construct a specific sequence of smooth approximations of (ρ, \mathbf{v}) .

We first extend \mathbf{v} by the reflection across the straight boundary segments Γ_i^{ext} , $i = 1, \dots, N_1$, so that the normal component is extended by the odd reflection and condition (3.11) is used to conclude that the extended function remains in H^1 , and the tangential component is extended by the even reflection.

Denote by $\boldsymbol{\nu}_i^{\text{ext}}$ the unit inner normal vector to Γ_i^{ext} with respect to Λ , $i = 1, \dots, N_1$, respectively, and by $\boldsymbol{\tau}_i^{\text{ext}}$ the unit tangential vectors to Γ_i^{ext} .

For each $r > 0$ and $i = 1, \dots, N_1$, we first define an extension of \mathbf{v} from $\mathcal{N}_r(\Gamma_i^{\text{ext}}) \cap \Omega$ across Γ_i^{ext} by reflection. However, there is the following issue: Denote

$$W_i^r := \{ \boldsymbol{\xi} - s \boldsymbol{\nu}_i^{\text{ext}} : \boldsymbol{\xi} \in \Gamma_i^{\text{ext}}, s \in (-r, 0), \boldsymbol{\xi} + s \boldsymbol{\nu}_i^{\text{ext}} \in \Omega \},$$

the image of $\mathcal{N}_r(\Gamma_i^{\text{ext}}) \cap \Omega$ under the reflection across Γ_i^{ext} . We note that, if $\mathcal{P}_0 = (0, 0)$ is a common endpoint of Γ_i^{ext} and Γ_j^{ext} , and the interior angle $\theta_{\mathcal{P}_0}$ of Λ at \mathcal{P}_0 is larger than π , i.e., $\theta_{\mathcal{P}_0} \in (\pi, 2\pi)$, then $W_i^r \cap W_j^r \neq \emptyset$ for any $r > 0$. Moreover, if $\theta_{\mathcal{P}_0} \in (\frac{3}{2}\pi, 2\pi)$, then $W_i^r \cap \Omega \neq \emptyset$ for any $r > 0$. Of course, in general, the extension of \mathbf{v} by reflection across Γ_i^{ext} into W_i^r does not match with \mathbf{v} in $W_i^r \cap \Omega$ or with extension of \mathbf{v} by reflection across Γ_j^{ext} in $W_i^r \cap W_j^r$, so the region needs to be restricted.

In the case above, for each $r > 0$, we have

$$(W_i^r \cap \Omega) \setminus B_{Lr}(\mathcal{P}_0) = \emptyset, \quad (W_i^r \cap W_j^r) \setminus B_{Lr}(\mathcal{P}_0) = \emptyset$$

$$\text{for some } L = L(\theta_{\mathcal{P}_0}) \geq 0. \tag{3.13}$$

In fact, this is true with $L = \operatorname{cosec}(\theta_{\mathcal{P}_0})$ if $\theta_{\mathcal{P}_0} \in (\pi, 2\pi)$ and $L=0$ if $\theta_{\mathcal{P}_0} \in (0, \pi]$. Also, there exists $r_1 > 0$ such that, for $i = 1, \dots, N_1$,

$$\Gamma_i^{\text{ext}} \cap B_{(L+1)r_1}(\mathcal{P}_0) = \emptyset \quad \text{if } \mathcal{P}_0 \text{ is not an endpoint of } \Gamma_i^{\text{ext}}. \tag{3.14}$$

Now, for each $r \in (0, r_1]$, define

$$V_i^r := W_i^r \setminus \overline{B_{Lr}(\mathcal{P}_0)},$$

which is an open connected set. We extend \mathbf{v} to V_i^r by

$$\mathbf{v}(\boldsymbol{\xi} - s\boldsymbol{\nu}_i^{\text{ext}}) := -(\mathbf{v} \cdot \boldsymbol{\nu}_i^{\text{ext}})(\boldsymbol{\xi} + s\boldsymbol{\nu}_i^{\text{ext}})\boldsymbol{\nu}_i^{\text{ext}} + (\mathbf{v} \cdot \boldsymbol{\tau}_i^{\text{ext}})(\boldsymbol{\xi} + s\boldsymbol{\nu}_i^{\text{ext}})\boldsymbol{\tau}_i^{\text{ext}} \tag{3.15}$$

for all $\boldsymbol{\xi} \in \Gamma_i^{\text{ext}}$ and $s \in (0, r)$ such that $\boldsymbol{\xi} + s\boldsymbol{\nu}_i^{\text{ext}} \in \Omega \setminus \overline{B_{Lr}(\mathcal{P}_0)}$. From the definitions of W_i^r and V_i^r , since segment Γ_i^{ext} lies on the line passing through \mathcal{P}_0 , the expression above defines \mathbf{v} on the whole region V_i^r . Also, it follows that, for all $\boldsymbol{\xi} \in \Gamma_i^{\text{ext}}$ and $s \in (0, r)$ such that $\boldsymbol{\xi} + s\boldsymbol{\nu}_i^{\text{ext}} \in \Omega \setminus \overline{B_{Lr}(\mathcal{P}_0)}$,

$$\begin{aligned} (\mathbf{v} \cdot \boldsymbol{\nu}_i^{\text{ext}})(\boldsymbol{\xi} - s\boldsymbol{\nu}_i^{\text{ext}}) &:= -(\mathbf{v} \cdot \boldsymbol{\nu}_i^{\text{ext}})(\boldsymbol{\xi} + s\boldsymbol{\nu}_i^{\text{ext}}), \\ (\mathbf{v} \cdot \boldsymbol{\tau}_i^{\text{ext}})(\boldsymbol{\xi} - s\boldsymbol{\nu}_i^{\text{ext}}) &:= (\mathbf{v} \cdot \boldsymbol{\tau}_i^{\text{ext}})(\boldsymbol{\xi} + s\boldsymbol{\nu}_i^{\text{ext}}), \end{aligned} \tag{3.16}$$

that is, $\mathbf{v} \cdot \boldsymbol{\nu}_i^{\text{ext}}$ is extended by the odd reflection, and $\mathbf{v} \cdot \boldsymbol{\tau}_i^{\text{ext}}$ is extended by the even reflection across Γ_i^{ext} .

Similarly, for each $i = 1, \dots, N_1$ and $r \in (0, r_1]$, we extend ρ to V_i^r by the even reflection across Γ_i^{ext} : For all $\boldsymbol{\xi} \in \Gamma_i^{\text{ext}}$ and $s \in (0, r)$ such that $\boldsymbol{\xi} + s\boldsymbol{\nu}_i^{\text{ext}} \in \Omega \setminus \overline{B_{Lr}(\mathcal{P}_0)}$,

$$\rho(\boldsymbol{\xi} - s\boldsymbol{\nu}) := \rho(\boldsymbol{\xi} + s\boldsymbol{\nu}), \tag{3.17}$$

where we recall that Γ_i^{ext} is a relatively open line segment.

Using (3.3) and (ii), and noting that $\mathbf{v} \cdot \boldsymbol{\nu} = 0$ to make the odd extension of $\mathbf{v} \cdot \boldsymbol{\nu}$ across Γ_i^{ext} , we find that the extended (ρ, \mathbf{v}) satisfies $(\rho, \mathbf{v}) \in H^1((\Omega \cup \overline{V_i^{r_1}})^0) \cap L^\infty(\Omega \cup \overline{V_i^{r_1}})$ and $\rho \geq C_0^{-1}$ on $\Omega \cup \overline{V_i^{r_1}}$.

Combining the extensions for $i = 1, \dots, N_1$, and using (3.13)–(3.14), we obtain that, for each $r \in (0, r_1]$, (ρ, \mathbf{v}) is extended to the domain:

$$\Omega_r^{\text{ext}} = (\Omega \cup (\cup_{i=1}^{N_1} \overline{V_i^r}))^0, \tag{3.18}$$

and the extension satisfies $(\rho, \mathbf{v}) \in H^1(\Omega_r^{\text{ext}}) \cap L^\infty(\Omega_r^{\text{ext}})$ and $\rho \geq C_0^{-1}$ on Ω_r^{ext} .

Moreover, by (3.5) and the explicit structure (3.15)–(3.17) of the extension, it follows that property (3.5) holds for the extended (ρ, \mathbf{v}) in Ω_r^{ext} for any $r \in (0, r_1]$:

The left-hand sides of Eqs. (2.3) and (2.5) are in $L^2(\Omega_r^{\text{ext}})$, and the equations hold in Ω_r^{ext} . (3.19)

Indeed, if $\xi \in \Gamma^{\text{ext}}$ and $s \in (0, r)$, and if (ρ, \mathbf{v}) is differentiable at $\xi + s\nu$, then clearly (ρ, \mathbf{v}) is differentiable at $\xi - s\nu$, and (2.3) and (2.5) hold at $\xi - s\nu$, which can be seen by the explicit calculation or using the standard symmetries of the isentropic Euler system (2.3)–(2.4).

Thus, we have shown the following results in this step:

Lemma 3.3 Under assumptions (i)–(ii) of Theorem 3.2 and (3.3), there exists $r_1 > 0$ depending only on Ω such that, for any $r \in (0, r_1]$, there is an extension of (ρ, \mathbf{v}) into Ω_r^{ext} , still denoted as (ρ, \mathbf{v}) , so that

- (a) $(\rho, \mathbf{v}) \in H^1(\Omega_r^{\text{ext}}) \cap L^\infty(\Omega_r^{\text{ext}})$ with $\rho \geq C_0^{-1}$ in Ω_r^{ext} ;
- (b) For any $\xi \in \Gamma^{\text{ext}}$ and $s \in (0, r)$ such that $\xi + s\nu \in \Omega \setminus \overline{B_{Lr}(\mathcal{P}_0)}$ for L from (3.13), the following odd and even reflection properties hold:

$$\begin{cases} (\mathbf{v} \cdot \nu)(\xi - s\nu) = -(\mathbf{v} \cdot \nu)(\xi + s\nu), \\ (\mathbf{v} \cdot \tau)(\xi - s\nu) = (\mathbf{v} \cdot \tau)(\xi + s\nu), \\ \rho(\xi - s\nu) = \rho(\xi + s\nu). \end{cases} \tag{3.20}$$

- (c) Property (3.19) holds.

4. We now show that Eq. (3.10) is satisfied in the weak sense:

Lemma 3.4 Under the assumptions of Theorem 3.2 and (3.3), Eq. (3.10) holds weakly in the sense (3.12) for each $f \in C^1(\mathbb{R})$ with $\|f'\|_{C^{0,1}(\mathbb{R})} < \infty$ and g defined by (3.9).

This can be proved as follows: We reduce r_1 to obtain, in addition to (3.14), that, for $j = 1, \dots, N_2$,

$$\Gamma_j^{\text{int}} \cap B_{(L+1)r_1}(\mathcal{P}_0) = \emptyset \quad \text{and} \quad r_1 < \sigma, \tag{3.21}$$

where we have used that $\mathcal{P}_0 \notin \overline{\Gamma^{\text{int}}}$ by Definition 3.1(v), σ is from assumption (i) of Theorem 3.2, and L is from (3.13). Fix $f \in C^1(\mathbb{R})$ with $\|f'\|_{C^{0,1}(\mathbb{R})} \leq C$, and let g be defined by (3.9).

4.1. We first prove (3.12) in the case when the smooth function ζ satisfies that there exists $r \in (0, r_1)$ such that

$$\zeta \equiv 0 \quad \text{in } \mathcal{N}_r(\Gamma^{\text{int}}) \cup B_{2Lr}(\mathcal{P}_0). \tag{3.22}$$

Thus, we need to consider (ρ, \mathbf{v}) only in the region:

$$\Omega^r := \Omega \setminus (\overline{\mathcal{N}_r(\Gamma^{\text{int}})} \cup \overline{B_{2Lr}(\mathcal{P}_0)}). \tag{3.23}$$

Note that $\Omega^r \Subset \Omega_r^{\text{ext}}$. Let

$$\delta := \text{dist}(\partial\Omega_r^{\text{ext}}, \Omega^r).$$

Then $\delta \in (0, r]$. We now mollify (\mathbf{v}, ω) and ρ in Ω^r , by using the extension of (ρ, \mathbf{v}) into Ω_r^{ext} constructed in Lemma 3.3 and the corresponding extension of $\omega = \partial_1 v_2 - \partial_2 v_1$. In order to achieve that the mollified \mathbf{v} satisfies the boundary condition (3.11), we use $\eta \in C_c^\infty(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} \eta(\boldsymbol{\xi}) \, d\boldsymbol{\xi} = 1$ of form $\eta(\boldsymbol{\xi}) = g(|\boldsymbol{\xi}|)$ for some $g \in C_c^\infty(\mathbb{R})$, for example, the standard mollifier. Then we define $\eta_\varepsilon(\boldsymbol{\xi}) = \frac{1}{\varepsilon^2} \eta(\frac{\boldsymbol{\xi}}{\varepsilon})$ for $\varepsilon > 0$, and denote $F_\varepsilon := F * \eta_\varepsilon$ for various functions F , specifically

$$\mathbf{v}_\varepsilon = \mathbf{v} * \eta_\varepsilon, \quad \omega_\varepsilon = \omega * \eta_\varepsilon, \quad \rho_\varepsilon = \rho * \eta_\varepsilon \quad (h(\rho))_\varepsilon = h(\rho) * \eta_\varepsilon \quad \text{for } \varepsilon \in (0, \frac{\delta}{2}).$$

Now, since $(\rho, \mathbf{v}) \in H^1(\Omega_r^{\text{ext}}) \cap L^\infty(\Omega_r^{\text{ext}})$ with $\rho \geq C^{-1}$ by Lemma 3.3(a)–(b), and $\Omega^r \Subset \Omega_r^{\text{ext}}$, we obtain that, for each $\varepsilon \in (0, \frac{\delta}{2})$,

$$\begin{aligned} \rho_\varepsilon, \mathbf{v}_\varepsilon, (h(\rho))_\varepsilon &\in H^1(\Omega^r) \cap L^\infty(\Omega^r), \\ \|(\rho_\varepsilon, \mathbf{v}_\varepsilon)\|_{L^\infty(\Omega^r)} &\leq \|(\rho, \mathbf{v})\|_{L^\infty(\Omega_r^{\text{ext}})}, \\ (\rho_\varepsilon, \mathbf{v}_\varepsilon) &\rightarrow (\rho, \mathbf{v}) \text{ in } H^1(\Omega^r), \quad \omega_\varepsilon \rightarrow \omega \text{ in } L^2(\Omega^r) \quad \text{as } \varepsilon \rightarrow 0. \end{aligned} \tag{3.24}$$

Moreover, from (3.15)–(3.16) and $\eta(\boldsymbol{\xi}) = g(|\boldsymbol{\xi}|)$, we have

$$\mathbf{v}_\varepsilon \cdot \boldsymbol{\nu} = 0 \quad \text{on } \cup_{i=1}^{N_1} \Gamma_i^{\text{ext}} \cap \partial\Omega^r \quad \text{for all } \varepsilon \in (0, \delta). \tag{3.25}$$

Now, from (3.19), we see that, in Ω^r ,

$$\text{div}(\rho_\varepsilon \mathbf{v}_\varepsilon) + 2\rho_\varepsilon + r_\varepsilon^{(1)} = 0, \tag{3.26}$$

$$(\mathbf{v}_\varepsilon \cdot \nabla) \mathbf{v}_\varepsilon + \mathbf{v}_\varepsilon + \nabla(h(\rho))_\varepsilon + r_\varepsilon^{(2)} = 0, \tag{3.27}$$

where

$$r_\varepsilon^{(1)} = (\text{div}(\rho \mathbf{v}))_\varepsilon - \text{div}(\rho_\varepsilon \mathbf{v}_\varepsilon), \quad r_\varepsilon^{(2)} = ((\mathbf{v} \cdot \nabla) \mathbf{v})_\varepsilon - (\mathbf{v}_\varepsilon \cdot \nabla) \mathbf{v}_\varepsilon.$$

The functions on the left-hand side of (3.26)–(3.27) are smooth. Taking the curl of (3.27), we obtain

$$\mathbf{v}_\varepsilon \cdot \nabla \omega_\varepsilon + (1 + \text{div} \mathbf{v}_\varepsilon) \omega_\varepsilon + \text{curl} r_\varepsilon^{(2)} = 0 \quad \text{in } \Omega^r. \tag{3.28}$$

Denote

$$X(\varepsilon) := \frac{\omega_\varepsilon}{\rho_\varepsilon}.$$

Using (3.24) and the lower bound of ρ in Ω_r^{ext} by Lemma 3.3(a), we obtain that, for $\varepsilon \in (0, \frac{\delta}{2})$,

$$X^{(\varepsilon)} \in L^2(\Omega^r), \quad X^{(\varepsilon)} \rightarrow X \text{ in } L^2(\Omega^r) \text{ as } \varepsilon \rightarrow 0. \quad (3.29)$$

We apply the definition of $X^{(\varepsilon)}$ in the first equality, along with Eqs. (3.26) and (3.28) in the second equality, to compute:

$$\begin{aligned} \rho_\varepsilon \mathbf{v}_\varepsilon \cdot \nabla X^{(\varepsilon)} &= \mathbf{v}_\varepsilon \cdot \nabla \omega_\varepsilon - \nabla \rho_\varepsilon \cdot \mathbf{v}_\varepsilon X^{(\varepsilon)} \\ &= \rho_\varepsilon X^{(\varepsilon)} + r_\varepsilon^{(1)} - \operatorname{curl} r_\varepsilon^{(2)}. \end{aligned}$$

From this and (3.26), we have

$$\begin{aligned} \operatorname{div}(\rho_\varepsilon f(X^{(\varepsilon)}) \mathbf{v}_\varepsilon) &= f'(X^{(\varepsilon)}) \rho_\varepsilon \mathbf{v}_\varepsilon \cdot \nabla X^{(\varepsilon)} + f(X^{(\varepsilon)}) \operatorname{div}(\rho_\varepsilon \mathbf{v}_\varepsilon) \\ &= \rho_\varepsilon (X^{(\varepsilon)} f'(X^{(\varepsilon)}) - 2f(X^{(\varepsilon)})) + \mathcal{R}_\varepsilon, \end{aligned} \quad (3.30)$$

where

$$\mathcal{R}_\varepsilon = f'(X^{(\varepsilon)})(r_\varepsilon^{(1)} - \operatorname{curl} r_\varepsilon^{(2)}) - f(X^{(\varepsilon)}) r_\varepsilon^{(1)}. \quad (3.31)$$

Recalling definition (3.9) of $g(\cdot)$, we rewrite (3.30) as

$$\operatorname{div}(\rho_\varepsilon f(X^{(\varepsilon)}) \mathbf{v}_\varepsilon) = \rho_\varepsilon g(X^{(\varepsilon)}) + \mathcal{R}_\varepsilon \quad \text{in } \Omega^r.$$

Let $\zeta \in C^\infty(\mathbb{R}^2)$ satisfy (3.22). Multiply the last equation by ζ , integrate over Ω , and integrate by parts via (3.22) and (3.25) to obtain

$$\int_{\Omega^r} (\rho_\varepsilon f(X^{(\varepsilon)}) \mathbf{v}_\varepsilon \cdot \nabla \zeta + (\rho_\varepsilon g(X^{(\varepsilon)}) + \mathcal{R}_\varepsilon) \zeta) \, d\xi = 0, \quad (3.32)$$

where we have used (3.22)–(3.23) to restrict the domain to Ω^r .

4.2. To send $\varepsilon \rightarrow 0$ in (3.32), we note the following facts: Since $f \in C^1(\mathbb{R})$ with $\|f'\|_{C^{0,1}(\mathbb{R})} < \infty$, then, by (3.9), $g \in C(\mathbb{R})$ with $\operatorname{Lip}(g) < \infty$ on \mathbb{R} . It follows from (3.4) and (3.29) that, for all $\varepsilon \in (0, \frac{\delta}{2})$,

$$\|f(X^{(\varepsilon)})\|_{L^2(\Omega^r)} + \|f'(X^{(\varepsilon)})\|_{L^\infty(\Omega^r)} + \|g(X^{(\varepsilon)})\|_{L^2(\Omega^r)} \leq C, \quad (3.33)$$

and

$$(f(X^{(\varepsilon)}), g(X^{(\varepsilon)})) \rightarrow (f(X), g(X)) \quad \text{in } L^2(\Omega^r) \text{ as } \varepsilon \rightarrow 0. \quad (3.34)$$

Next, we show that

$$\mathcal{R}_\varepsilon \rightarrow 0 \quad \text{in } L^1(\Omega^r) \text{ as } \varepsilon \rightarrow 0. \quad (3.35)$$

From (3.31) and (3.33) and the fact that Ω^r is a bounded domain, in order to prove (3.35), it suffices to show that

$$r_\varepsilon^{(1)} \rightarrow 0 \quad \text{in } L^2(\Omega^r) \text{ as } \varepsilon \rightarrow 0, \tag{3.36}$$

$$\text{curl} r_\varepsilon^{(2)} \rightarrow 0 \quad \text{in } L^1(\Omega^r) \text{ as } \varepsilon \rightarrow 0. \tag{3.37}$$

To show (3.36), we first note that $\Omega^r \subseteq \Omega_r^{\text{ext}}$ and $\rho \mathbf{v} \in H^1(\Omega_r^{\text{ext}}) \cap L^\infty(\Omega_r^{\text{ext}})$ by Lemma 3.3(a), so that

$$(\text{div}(\rho \mathbf{v}))_\varepsilon - \text{div}(\rho \mathbf{v}) \rightarrow 0 \quad \text{in } L^2(\Omega^r) \text{ as } \varepsilon \rightarrow 0.$$

Thus, it remains to show that

$$\text{div}(\rho_\varepsilon \mathbf{v}_\varepsilon) - \text{div}(\rho \mathbf{v}) \rightarrow 0 \quad \text{in } L^2(\Omega^r) \text{ as } \varepsilon \rightarrow 0. \tag{3.38}$$

We first show that

$$\text{div}(\rho_\varepsilon \mathbf{v}_\varepsilon) - \text{div}(\rho \mathbf{v}_\varepsilon) \rightarrow 0 \quad \text{in } L^2(\Omega^r) \text{ as } \varepsilon \rightarrow 0.$$

Indeed,

$$\text{div}(\rho_\varepsilon \mathbf{v}_\varepsilon) - \text{div}(\rho \mathbf{v}_\varepsilon) = \nabla(\rho_\varepsilon - \rho) \cdot \mathbf{v}_\varepsilon + (\rho_\varepsilon - \rho)(\text{div} \mathbf{v}_\varepsilon - \text{div} \mathbf{v}) + (\rho_\varepsilon - \rho)\text{div} \mathbf{v}.$$

In the argument below, we use (3.24). We see that $\nabla(\rho_\varepsilon - \rho) \cdot \mathbf{v}_\varepsilon \rightarrow 0$ in $L^2(\Omega^r)$ as $\varepsilon \rightarrow 0$, because $\nabla \rho_\varepsilon \rightarrow \nabla \rho$ in $L^2(\Omega^r)$ and \mathbf{v}_ε is uniformly bounded in $L^\infty(\Omega^r)$. Also, $(\rho_\varepsilon - \rho)(\text{div} \mathbf{v}_\varepsilon - \text{div} \mathbf{v}) \rightarrow 0$ in $L^2(\Omega^r)$, because $\rho - \rho_\varepsilon$ is uniformly bounded in $L^\infty(\Omega^r)$ and $\text{div} \mathbf{v}_\varepsilon \rightarrow \text{div} \mathbf{v}$ in $L^2(\Omega^r)$. Finally, $(\rho_\varepsilon - \rho)\text{div} \mathbf{v} \rightarrow 0$ in $L^2(\Omega^r)$, because $\rho - \rho_\varepsilon$ is uniformly bounded in $L^\infty(\Omega^r)$ and converges to zero *a.e.* in Ω^r , and $\text{div} \mathbf{v} \in L^2(\Omega^r)$, so that

$$\int_{\Omega_r} (\rho_\varepsilon - \rho)^2 (\text{div} \mathbf{v})^2 \, d\xi \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

by the dominated convergence theorem. The convergence:

$$\text{div}(\rho \mathbf{v}_\varepsilon) - \text{div}(\rho \mathbf{v}) \rightarrow 0 \quad \text{in } L^2(\Omega^r) \text{ as } \varepsilon \rightarrow 0$$

can be shown similarly. This completes the proof of (3.38), which leads to (3.36).

Now we show (3.37). Note that

$$\text{curl} r_\varepsilon^{(2)} = \partial_{\xi_1}((\mathbf{v} \cdot \nabla v_2)_\varepsilon - \mathbf{v}_\varepsilon \cdot \nabla(v_2)_\varepsilon) - \partial_{\xi_2}((\mathbf{v} \cdot \nabla v_1)_\varepsilon - \mathbf{v}_\varepsilon \cdot \nabla(v_1)_\varepsilon).$$

Then (3.37) follows from Lemma A.1 in Appendix A, applied in Ω_r^{ext} with $p = q = 2$, $b = v_j$, and $u = \partial_j v_k$ for the corresponding $j, k = 1, 2$, and $i = 3 - k$.

Combining the results above, (3.35) is now proved. Then, sending $\varepsilon \rightarrow 0$ in (3.32) and using (3.34)–(3.35), we obtain

$$\int_{\Omega} (\rho f(X)\mathbf{v} \cdot \nabla \zeta + \rho g(X)\zeta) \, d\xi = 0,$$

which is equivalent to (3.12), by using (3.22).

4.3. Next, we prove (3.12) in the case when the smooth function ζ satisfies:

$$\zeta \equiv 0 \quad \text{in } \Omega \setminus \mathcal{N}_r(\Gamma^{\text{int}}) \text{ for some } r \in (0, r_1). \tag{3.39}$$

Then, by (3.21) and assumption (i) of Theorem 3.2, Eq. (3.10) and the boundary conditions (3.11) hold classically on $\text{supp}(\zeta) \cap \Omega$ (for the equation, the argument is given from (3.6) to (3.8)). Multiplying (3.10) by ζ , integrating over Ω , and integrating by parts in the first term with the use of (3.11), we obtain (3.12) for ζ satisfying (3.39).

4.4. Combining the two previous cases, we obtain (3.12) for all smooth ζ satisfying

$$\zeta \equiv 0 \quad \text{in } \Omega \cap B_r(\mathcal{P}_0) \text{ for some } r > 0. \tag{3.40}$$

This in particular implies the following: if $\mathcal{P}_0 \notin \partial\Omega$, then (3.12) holds for all smooth ζ , since we can modify ζ outside Ω in this case so that the modified function ζ satisfies (3.40) for some $r > 0$, and this modification clearly does not affect (3.12) for ζ .

Thus, the remaining proof is for the case that $\mathcal{P}_0 \in \partial\Omega$. Note that this means that \mathcal{P}_0 is a common endpoint of some of Γ_i^{ext} and Γ_j^{ext} .

4.5. Now we consider the case that \mathcal{P}_0 is a common endpoint of some of Γ_i^{ext} and Γ_j^{ext} , and fix $\zeta \in C^\infty(\mathbb{R}^2)$. Let $\psi \in C^\infty(\mathbb{R}^2)$ be such that

$$0 \leq \psi \leq 1 \text{ on } \mathbb{R}^2, \quad \psi \equiv 0 \text{ on } B_1, \quad \psi \equiv 1 \text{ on } \mathbb{R}^2 \setminus B_2.$$

Let $\psi^r(\xi) = \psi(\frac{\xi}{r})$ for $r > 0$. In particular,

$$\begin{aligned} \psi^r &\equiv 0 \text{ in } B_r, & \psi^r &\equiv 1 \text{ in } \mathbb{R}^2 \setminus B_{2r}, \\ \phi^r &\leq \frac{C}{r}, & \text{supp}(D\psi^r) &\subset B_{2r}. \end{aligned} \tag{3.41}$$

Here and below, the universal constant C is independent of r , which may be different at different occurrence. Then, for any small $r > 0$, function $\zeta\psi^r$ satisfies (3.40), so (3.12) holds with the test function $\zeta\psi^r$ instead of ζ . Thus, we have

$$\begin{aligned} \int_{\Omega} (\rho f(X)(\mathbf{v} \cdot \nabla \zeta) + \rho g(X)\zeta)\psi^r \, d\xi &+ \int_{\Omega} \rho f(X)(\mathbf{v} \cdot \nabla \psi^r)\zeta \, d\xi \\ &- \sum_{i=1}^{N_2} \int_{\Gamma_i^{\text{int}}} \rho f(X)(\mathbf{v} \cdot \boldsymbol{\nu})\zeta\psi^r \, dl = 0. \end{aligned} \tag{3.42}$$

We estimate the second integral in (3.42), by using (3.4), (3.41), the boundedness of \mathbf{v} by assumption (ii), and that $f'(X)$ is bounded on \mathbb{R} :

$$\begin{aligned} \left| \int_{\Omega} \rho f(X)(\mathbf{v} \cdot \nabla \psi^r) \zeta \, d\xi \right| &\leq \frac{C}{r} \int_{\Omega \cap B_{2r}} (1 + |X|) \, d\xi \\ &\leq C \left(\int_{\Omega \cap B_{2r}} (1 + |X|^2) \, d\xi \right)^{\frac{1}{2}} \rightarrow 0 \quad \text{as } r \rightarrow 0. \end{aligned}$$

Notice that $\psi^r \equiv 1$ in $\mathbb{R}^2 \setminus B_{2r}$ and $|\psi^r| \leq 1$ in B_{2r} , (ρ, \mathbf{v}) is bounded and $X \in L^2(\Omega)$, and $|(f(X), g(X))| \leq C(1 + |X|)$ from the assumptions of $f(X)$ and (3.9). Then, denoting by L_r the difference between the first term in (3.12) and the first term in (3.42), we have

$$\begin{aligned} |L_r| &= \left| \int_{\Omega \cap B_{2r}} (\rho f(X)(\mathbf{v} \cdot \nabla \zeta) + \rho g(X)\zeta)(1 - \psi_r) \, d\xi \right| \\ &\leq C \int_{\Omega \cap B_{2r}} (1 + |X|) \, d\xi \rightarrow 0 \quad \text{as } r \rightarrow 0. \end{aligned}$$

For the boundary integral, we obtain that, for each $i \in \{1, \dots, N_2\}$,

$$\int_{\Gamma_i^{\text{int}}} \rho f(X)(\mathbf{v} \cdot \boldsymbol{\nu}) \zeta \psi^r \, dl \rightarrow \int_{\Gamma_i^{\text{int}}} \rho f(X)(\mathbf{v} \cdot \boldsymbol{\nu}) \zeta \, dl \quad \text{as } r \rightarrow 0.$$

Indeed, $\mathcal{P}_0 \notin \overline{\Gamma_i^{\text{int}}}$ by Definition 3.1(v), so that $\psi^r \equiv 1$ on Γ_i^{int} if $r < \frac{1}{2} \text{dist}(\mathcal{P}_0, \Gamma_i^{\text{int}})$.

Combining the convergence facts shown above and sending $r \rightarrow 0$ in (3.42), we conclude (3.12).

5. We now show that vorticity ω (and thus X) is smooth and not identically zero on Γ_1^{int} . Recall that Γ_1^{int} denotes a relatively open curve segment and Γ_1^{int} is a shock by Definition 3.1(vii).

Lemma 3.5 Under the assumptions of Theorem 3.2, vorticity ω on Γ_1^{int} from the Ω -side is continuous and not identically zero on Γ_1^{int} , and is bounded on $\overline{\Gamma_1^{\text{int}}}$.

This can be proved as follows: First, it follows directly from the regularity of Γ_1^{int} in Definition 3.1(v) and assumption (i) of Theorem 3.2 that ω on Γ_1^{int} from the Ω -side is continuous on Γ_1^{int} and bounded on $\overline{\Gamma_1^{\text{int}}}$. Then, in the rest of the proof, it suffices to show that ω is not identically zero on Γ_1^{int} .

Since Eqs. (2.3)–(2.4) and the Rankine-Hugoniot conditions (2.9) are invariant under the coordinate rotation and translation, at any fixed point $P \in \Gamma_1^{\text{int}}$, we can choose the coordinates ξ such that the ξ_1 -direction is tangent to Γ_1^{int} at P . Then Γ_1^{int} is the graph of a function f_s locally, i.e., $\Gamma_1^{\text{int}} = \{\xi : \xi_2 = f_s(\xi_1)\}$ locally near P .

Thus, f_s is in C^2 in a neighborhood of P by Definition 3.1(v), and we can obtain that $f'_s = 0$ at point P by choosing the appropriate coordinate system. Note that $(f'_s, -1)$ is a normal of Γ_1^{int} , so that the Rankine-Hugoniot conditions (2.9) become

$$\begin{cases} (\rho v_1 - \rho_1 v_1^-) f'_s - (\rho v_2 - \rho_1 v_2^-) = 0, \\ (\rho v_1^2 + p - \rho_1 (v_1^-)^2 - p_1) f'_s - (\rho v_1 v_2 - \rho_1 v_1^- v_2^-) = 0, \\ (\rho v_1 v_2 - \rho_1 v_1^- v_2^-) f'_s - (\rho v_2^2 + p - \rho_1 (v_2^-)^2 - p_1) = 0, \end{cases} \tag{3.43}$$

where $\mathbf{v}^- = (v_1^-, v_2^-) = (u_1^- - \xi_1, u_2^- - \xi_2)$ with some constants u_1^- and u_2^- , and $\mathbf{v} = (v_1, v_2)$. Taking the tangential derivative $\partial_\tau := \partial_{\xi_1} + f'_s \partial_{\xi_2}$ of the Rankine-Hugoniot conditions (3.43) along Γ_1^{int} and using the condition that $f'_s = 0$ at P , we obtain that, at point P ,

$$\begin{cases} (\rho v_1 - \rho_1 v_1^-) f''_s - (\rho v_2) \xi_1 = 0, \\ (\rho v_1^2 + p - \rho_1 (v_1^-)^2 - p_1) f''_s - (\rho v_1 v_2) \xi_1 - \rho_1 v_2^- = 0, \\ (\rho v_2^2) \xi_1 + c^2 \rho \xi_1 = 0. \end{cases}$$

By Eqs. (2.3)–(2.4) and the definition: $\omega = (v_1)_{\xi_2} - (v_2)_{\xi_1}$, it follows from a straightforward but long calculation that

$$\begin{aligned} (v_1)_{\xi_1} &= -1 - \frac{c^2 + v_2^2}{\rho |\mathbf{v}|^2} v_1 \rho \xi_1 + \frac{c^2 - v_2^2}{\rho |\mathbf{v}|^2} v_2 \rho \xi_2 - \frac{v_1 v_2}{|\mathbf{v}|^2} \omega, \\ (v_1)_{\xi_2} &= -\frac{c^2 - v_1^2}{\rho |\mathbf{v}|^2} v_2 \rho \xi_1 - \frac{c^2 - v_2^2}{\rho |\mathbf{v}|^2} v_1 \rho \xi_2 + \frac{v_1^2}{|\mathbf{v}|^2} \omega, \\ (v_2)_{\xi_1} &= -\frac{c^2 - v_1^2}{\rho |\mathbf{v}|^2} v_2 \rho \xi_1 - \frac{c^2 - v_2^2}{\rho |\mathbf{v}|^2} v_1 \rho \xi_2 - \frac{v_2^2}{|\mathbf{v}|^2} \omega, \\ (v_2)_{\xi_2} &= -1 + \frac{c^2 - v_1^2}{\rho |\mathbf{v}|^2} v_1 \rho \xi_1 - \frac{c^2 + v_1^2}{\rho |\mathbf{v}|^2} v_2 \rho \xi_2 + \frac{v_1 v_2}{|\mathbf{v}|^2} \omega. \end{aligned}$$

Therefore, by a long computation, we obtain that $(\rho_{\xi_1}, \rho_{\xi_2}, \omega)$ satisfy the linear system at P :

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} \rho_{\xi_1} \\ \rho_{\xi_2} \\ \omega \end{pmatrix} = \begin{pmatrix} d_1 f''_s \\ d_2 f''_s \\ 0 \end{pmatrix}, \tag{3.44}$$

where

$$\begin{aligned} (a_1, b_1, c_1) &= ((c^2 - 2v_1^2 - v_2^2)v_2, (c^2 - v_2^2)v_1, \rho v_2^2), \\ (a_2, b_2, c_2) &= (2(c^2 - v_1^2)v_1 v_2, -(c^2 - v_2^2)(v_2^2 - v_1^2), 2\rho v_1 v_2^2), \\ (a_3, b_3, c_3) &= (3v_1^2 v_2^2 + v_1^2 c^2 + v_2^4 - v_2^2 c^2, -2(c^2 - v_2^2)v_1 v_2, -2\rho v_2^3), \end{aligned}$$

and

$$d_1 = -|\mathbf{v}|^2(\rho v_1 - \rho_1 v_1^-), \quad d_2 = -|\mathbf{v}|^2(\rho v_1^2 + p - \rho_1(v_1^-)^2 - p_1).$$

Notice that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \rho v_2(c^2 - v_2^2)^2 |\mathbf{v}|^4 \neq 0,$$

where we have used the fact that $v_2 = \mathbf{v} \cdot \boldsymbol{\nu} \neq 0$ at P by the entropy condition and the ellipticity assumption that $|\mathbf{v}| < c$ on Γ_1^{int} in Theorem 3.2. Therefore, from (3.44),

$$\omega = \frac{d_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} - d_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}}{\rho v_2(c^2 - v_2^2)^2 |\mathbf{v}|^4} f_s''.$$

Since $f_s' = 0$ at P , it follows from (3.43) that

$$\rho v_2 - \rho_1 v_2^- = 0, \quad v_1 = v_1^-, \quad \rho v_2^2 + p - \rho_1(v_2^-)^2 - p_1 = 0 \quad \text{at } P. \quad (3.45)$$

Then

$$\omega = \frac{v_1((\rho - \rho_1)v_2^2 + p - p_1)}{\rho v_2} f_s''. \quad (3.46)$$

Notice that $\mathbf{v} \cdot \boldsymbol{\nu} \leq -C^{-1}$ on Γ_1^{int} by Definition 3.1(vii), where \mathbf{v} on Γ_1^{int} is taken from the Ω -side, and $\boldsymbol{\nu}$ is the outer normal with respect to Ω . By (2.10), this implies that $\mathbf{v} \cdot \boldsymbol{\nu} < 0$ on Γ_1^{int} for \mathbf{v} taken from the $\Lambda \setminus \Omega$ -side. Using the entropy condition (2.12) on Γ_1^{int} , we obtain that $\rho > \rho_1$ and $p > p_1$ on Γ_1^{int} . Moreover, $v_1(\hat{P}) = (\mathbf{v} \cdot \boldsymbol{\tau})(\hat{P}) \neq 0$ for point $\hat{P} \in \Gamma_1^{\text{int}}$ specified in Definition 3.1(viii). Therefore, $\omega \neq 0$ at \hat{P} . Then the vorticity is not identically zero on Γ_1^{int} .

6. Using Lemma 3.4, we can formally choose $f(s) = s^2$ with $g(s) = 0$ by (3.9), and use $\zeta \equiv 1$ to obtain that, by (3.12),

$$0 = - \int_{\Gamma_{\text{shock}}} \left| \frac{\omega}{\rho} \right|^2 \rho \mathbf{v} \cdot \boldsymbol{\nu} \, dl > 0.$$

The strict inequality above follows from Definition 3.1(vii) and Lemma 3.5. This (formally) shows that assumption (3.3) leads to a contradiction. A minor technical point is that function $f(s) = s^2$ does not satisfy the assumption of Lemma 3.4: $\|f'\|_{C^{0,1}(\mathbb{R})} < \infty$.

To make this rigorously, we approximate $f(s) = s^2$ by the functions that satisfy the assumptions of Lemma 3.4 and verify the limit process of this approximation. More specifically, for any $M > 1$, define $f_M \in C^1(\mathbb{R})$ by $f_M(0) = 0$ and $f_M'(t) = 2 \min\{|t|, M\} \text{sign}(t)$. Then

$$f_M(t) = \begin{cases} t^2 & \text{if } |t| \leq M, \\ M^2 + 2M(|t| - M) & \text{if } |t| > M. \end{cases}$$

It follows that $\|f'_M\|_{C^{0,1}(\mathbb{R})} < \infty$ and the function defined by (3.9) is

$$g_M(t) = \begin{cases} 0 & \text{if } |t| \leq M, \\ 2(M^2 - M|t|) & \text{if } |t| > M. \end{cases} \tag{3.47}$$

Now (3.12) holds with f_M, g_M , and any $\zeta \in C^\infty(\mathbb{R}^2)$. Choosing $\zeta \equiv 1$, then we have

$$\int_{\Omega} \rho g_M(X) \, d\xi = \int_{\Gamma^{\text{int}}} \rho f_M(X) (\mathbf{v} \cdot \boldsymbol{\nu}) \, dl. \tag{3.48}$$

We send $M \rightarrow \infty$. Clearly, $g_M(X) \rightarrow 0$ pointwise in Ω . Also, from (3.47), we obtain

$$|g_M(t)| \leq 2t^2 \quad \text{for all } M > 1 \text{ and } t \in \mathbb{R}.$$

Thus, using that $X \in L^2(\Omega)$ and $\rho \in L^\infty(\Omega)$, we have

$$\int_{\Omega} \rho g_M(X) \, d\xi \rightarrow 0 \quad \text{as } M \rightarrow \infty,$$

by the dominated convergence theorem. Also, since $X \in L^\infty(\Gamma^{\text{int}})$ by assumptions (i)–(ii), then it follows from the explicit form of f_M that

$$\int_{\Gamma^{\text{int}}} \rho f_M(X) (\mathbf{v} \cdot \boldsymbol{\nu}) \, dl = \int_{\Gamma^{\text{int}}} \rho |X|^2 (\mathbf{v} \cdot \boldsymbol{\nu}) \, dl \quad \text{for all } M \geq \|X\|_{L^\infty(\Gamma^{\text{int}})}.$$

Thus, sending $M \rightarrow \infty$ in (3.48), we obtain

$$\int_{\Gamma^{\text{int}}} \rho |X|^2 (\mathbf{v} \cdot \boldsymbol{\nu}) \, dl = 0.$$

Since $\rho \in [C_0^{-1}, C_0]$ by assumption (ii) and $\mathbf{v} \cdot \boldsymbol{\nu} \leq 0$ on Γ_i^{int} , for $i = 2, \dots, N_2$, from Definition 3.1(vii), we obtain

$$\int_{\Gamma_1^{\text{int}}} \rho |X|^2 (\mathbf{v} \cdot \boldsymbol{\nu}) \, dl \geq 0.$$

This is a contradiction since $\rho \in [C_0^{-1}, C_0]$, $\mathbf{v} \cdot \boldsymbol{\nu} \leq -C^{-1}$ on Γ_1^{int} by Definition 3.1(vii), and ω (and thus X) is continuous and not identically zero on Γ_1^{int} by Lemma 3.5.

This completes the proof of the fact that, under the assumptions of the theorem, it is not possible that (3.3) holds.

7. We now complete the proof as claimed. On the contrary, assume that

$$\mathbf{v} \in H^1(\Omega). \tag{3.49}$$

Then, combining (3.49) with the assumptions in the theorem, we have

$$\mathbf{v} \in H^1(\Omega) \cap L^\infty(\Omega), \quad C^{-1} \leq \rho \leq C, \quad \tilde{C}^{-1} \leq p \leq \tilde{C}, \tag{3.50}$$

where $C, \tilde{C} \geq 1$, and we recall that $p = \frac{\rho^\gamma}{\gamma}$. We show that, with these properties, we can obtain from the weak form of the Euler system (2.6)–(2.7) that

$$\int_{\Omega} ((\rho \mathbf{v} \cdot \nabla \mathbf{v} + \rho \mathbf{v}) \cdot \zeta - p \operatorname{div} \zeta) \, d\xi = 0 \tag{3.51}$$

for any $\zeta = (\zeta_1, \zeta_2) \in C_c^1(\Omega; \mathbb{R}^2)$. To show (3.51), we compute for $j = 1, 2$, using (3.50):

$$\begin{aligned} \int_{\Omega} \rho \sum_{i=1}^2 v_i v_j \partial_i \zeta_j \, d\xi &= \int_{\Omega} \rho \sum_{i=1}^2 v_i \partial_i (v_j \zeta_j) \, d\xi - \int_{\Omega} \rho \sum_{i=1}^2 v_i (\partial_i v_j) \zeta_j \, d\xi \\ &= \int_{\Omega} \rho \mathbf{v} \cdot \nabla (v_j \zeta_j) \, d\xi - \int_{\Omega} \rho (\mathbf{v} \cdot \nabla v_j) \zeta_j \, d\xi. \end{aligned} \tag{3.52}$$

Next, by the properties in (3.50), we can use the test function $\phi = v_j \zeta_j$ in (2.6) to obtain

$$\int_{\Lambda} \rho \mathbf{v} \cdot \nabla (v_j \zeta_j) \, d\xi = \int_{\Omega} 2\rho v_j \zeta_j \, d\xi.$$

Substituting this into the last expression of (3.52) for $j = 1, 2$, we have

$$\int_{\Omega} \rho \mathbf{v} \otimes \mathbf{v} : D\zeta \, d\xi = \int_{\Omega} (2\rho v - \rho(\mathbf{v} \cdot \nabla) \mathbf{v}) \cdot \zeta \, d\xi.$$

Substituting this into (2.7), we obtain (3.51). From (3.51) and the fact that $\rho \mathbf{v} \cdot \nabla \mathbf{v} + \rho \mathbf{v} \in L^2(\Omega)$ by (3.50), it follows from (3.51) that the weak derivative of p exists in Ω and that $\nabla p \in L^2(\Omega)$. Then, using (3.50), we have

$$p \in H^1(\Omega) \cap L^\infty(\Omega).$$

From this, since $\rho = (\gamma p)^{1/\gamma}$ and $\tilde{C}^{-1} \leq p \leq \tilde{C}$, we obtain

$$\rho \in H^1(\Omega) \cap L^\infty(\Omega).$$

Combining this with (3.49), we obtain that $(\rho, \mathbf{v}) \in H^1(\Omega)$, which contradicts (3.2). Then we conclude that (3.49) is false. □

4 Applications to Transonic Shock Problems

In this section, we employ the general framework of Definition 3.1 and the main theorem, Theorem 3.2, established in § 3 to analyze the low regularity of entropy solutions of several transonic shock problems including the regular shock reflection problem, the Prandtl reflection problem, the Lighthill diffraction problem, and the Riemann problem with four-shock interactions.

First of all, in verifying the conditions of Definition 3.1, condition (viii) is often difficult to be verified directly. The following lemma is useful for that; in fact, it is used in all the applications we describe below.

Lemma 4.1 Let (ρ, \mathbf{v}) be a solution of the Riemann problem that satisfies conditions (i)–(vii) of Definition 3.1 and assumption (i) of Theorem 3.2. Assume that

- (a) $\Gamma_1^{\text{int}} = \partial\Omega \cap \partial\Lambda_j$ for some $j \in \{1, \dots, M\}$, where Λ_j are defined in Definition 3.1(i).
- (b) Γ_1^{int} is not a segment of straight line. In particular, denoting by P_1 and P_2 the end-points of Γ_1^{int} , there exists a point $P^* \in \overline{\Gamma_1^{\text{int}}} \setminus \{P_1\}$ such that $\tau(P_1) \neq \pm\tau(P^*)$, where $\tau(\cdot)$ is a unit tangent vector to Γ_1^{int} at a point.
- (c) $(\mathbf{v} \cdot \tau)(P_1) \neq 0$.

Then condition (viii) of Definition 3.1 is satisfied.

Proof By re-indexing sets Λ_j , we can assume that $\Gamma_1^{\text{int}} = \partial\Omega \cap \partial\Lambda_1$. Then (ρ_1, \mathbf{v}_1) is the uniform state in Λ_1 , where ρ_1 is constant and $\mathbf{v}_1 = (u_1^{(1)}, u_2^{(1)}) - \xi$ with a constant state $(u_1^{(1)}, u_2^{(1)})$. This state is called state (1).

We show the existence of a point \hat{P} in the relative interior of Γ_1^{int} such that the curvature of Γ_1^{int} is non-zero at \hat{P} and $(\mathbf{v} \cdot \tau)(\hat{P}) \neq 0$. Denote by S the line tangential to $\overline{\Gamma_1^{\text{int}}}$ at P_1 .

Denote by Q the intersection point of line S and line L through center $O_1 = (u_1^{(1)}, u_2^{(1)})$ of state (1) perpendicular to S . Note that, for any point P , $\mathbf{v}_1 = O_1 - P$ so that $(\mathbf{v}_1 \cdot \tau_S)(Q) = 0$ and $(\mathbf{v}_1 \cdot \tau_S)(P) \neq 0$ for all $P \in S \setminus Q$, where we recall that \mathbf{v}_1 is the pseudo-velocity of the uniform state in Λ_1 which also defines \mathbf{v}_1 on the whole \mathbb{R}^2 . Denote by \hat{Q} the point on S such that Γ_1^{int} coincides with S between points P_1 and \hat{Q} , but not on any larger interval extended through \hat{Q} . Note that it is possible that $\hat{Q} = P_1$, but $\hat{Q} \neq P^*$ since the tangential line to $\overline{\Gamma_1^{\text{int}}}$ at P^* is not parallel to L by our condition (b).

If $\hat{Q} \neq Q$, then $(\mathbf{v} \cdot \tau)(\hat{Q}) = (\mathbf{v}_1 \cdot \tau)(\hat{Q}) \neq 0$, where \mathbf{v} is the velocity on Γ_1^{int} from the Ω -side and we have used (2.10). Also, from the definition of \hat{Q} , in any neighborhood of \hat{Q} , there exists a point $P \in \Gamma_1^{\text{int}}$ with nonzero curvature. Thus, it follows from the C^2 -regularity of Γ_1^{int} in its relative interior (by condition (v) of Definition 3.1) and the continuity of \mathbf{v} in Ω near and up to Γ_1^{int} (by assumption (i)

of Theorem 3.2) that there exists a point $\hat{P} \in (\Gamma_1^{\text{int}})^0$ with non-zero curvature and $(\mathbf{v} \cdot \boldsymbol{\tau})(\hat{P}) \neq 0$.

Therefore, the remaining possibility is that $\hat{Q} = Q$. Note that $(\mathbf{v}_1 \cdot \boldsymbol{\tau}_S)(Q) = 0$. Moreover, using condition (c), the Rankine-Hugoniot conditions, and the regularity of Γ_1^{int} and (ρ, \mathbf{v}) given in condition (v) of Definition 3.1 and assumption (i) of Theorem 3.2, we see that $(\mathbf{v}_1 \cdot \boldsymbol{\tau}_S)(P_1) = (\mathbf{v} \cdot \boldsymbol{\tau}_S)(P_1) \neq 0$. This implies that $Q \neq P_1$. Also, since $\hat{Q} \neq P^*$ as we discussed above, then $Q \neq P^*$ for the present case. Thus, for the present case $Q = \hat{Q}$, it follows that Q is an interior point of Γ_1^{int} , and the part of Γ_1^{int} between P_1 and Q lies on the straight line S . In particular, line S is tangential to Γ_1^{int} at Q . We now shift and rotate the coordinates to have the origin at Q and the coordinates ξ_1 and ξ_2 to be along S and L , respectively. Then $O_1 = (0, \tilde{v}_1)$ for some $\tilde{v}_1 \in \mathbb{R}$ (in fact, $\tilde{v}_1 \neq 0$ by condition (v) of Definition 3.1, but this will not be used below). To fix notation, let the ξ_1 -axis along S be oriented so that $P_1 = (\xi_{P_1}, 0)$ with $\xi_{P_1} < 0$. Since $S = \{\xi_2 = 0\}$ is tangential to Γ_1^{int} at $Q = (0, 0)$, then curve $\Gamma_1^{\text{int}} \cap B_r(Q)$ is a graph for some $r > 0$: There exists $f \in C^2(\mathbb{R})$ such that

$$\Gamma_1^{\text{int}} \cap B_r(Q) = \{(\xi_1, f(\xi_1)) : \xi_1 \in (a, b)\}$$

for some $a < 0, b > 0$, and $f \equiv 0$ on $(a, 0)$, where the last assertion holds because Γ_1^{int} lies on S between P_1 and Q . Thus, $f'(0) = f''(0) = 0$ so that $|f'(\xi_1)| \leq O(\varepsilon)\xi_1$ for all $\xi_1 \in (-\varepsilon, \varepsilon)$, where $O(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0^+$. For any $P = (\xi_1, f(\xi_1)) \in \Gamma_1^{\text{int}} \cap B_r(Q)$, we have

$$\mathbf{v}_1(P) = O_1 - P = (-\xi_1, \tilde{v}_2 - f(\xi_1)), \quad \boldsymbol{\tau}(P) = \frac{(1, f'(\xi_1))}{\sqrt{1 + (f'(\xi_1))^2}}.$$

Then

$$(\mathbf{v}_1 \cdot \boldsymbol{\tau})(P) = \frac{-\xi_1 + (\tilde{v}_2 - f(\xi_1))f'(\xi_1)}{\sqrt{1 + (f'(\xi_1))^2}} = \frac{-\xi_1 + O(\varepsilon)\xi_1}{\sqrt{1 + (f'(\xi_1))^2}} \neq 0$$

for $\xi_1 \in (0, \varepsilon)$ if ε is small. Thus, $\mathbf{v} \cdot \boldsymbol{\tau} = \mathbf{v}_1 \cdot \boldsymbol{\tau} \neq 0$ at any $P = (\xi_1, f(\xi_1))$ with $\xi_1 \in (0, \varepsilon)$. Since $\hat{Q} = Q$, i.e., for every $\varepsilon > 0$, there exists a point $\xi_1 \in (0, \varepsilon)$ such that Γ_1^{int} has non-zero curvature at $P = (\xi_1, f(\xi_1))$, it follows that there exists a point \hat{P} , at which the tangential velocity and the curvature of Γ_1^{int} are nonzero, in the present case. This completes the proof. \square

4.1 Lower Regularity of the Regular Shock Reflection Solutions for the Isentropic Euler System

The first Riemann problem we address is the regular shock reflection problem for the isentropic Euler system (2.1). When a plane incident shock $S_0 := \Gamma_{\text{shock}}^0$ hits a two-dimensional wedge, a shock reflection-diffraction configuration takes shape. The incident shock S_0 separates two constant states: state (0) with velocity $\mathbf{u}^{(0)} = (0, 0)$

and density ρ_0 ahead of the shock, and state (1) with velocity $\mathbf{u}^{(1)} = (u_1^{(1)}, 0)$ and density ρ_1 behind the shock, where $\rho_1 > \rho_0$, and $u_1^{(1)} > 0$ is determined by (ρ_0, ρ_1, γ) through the Rankine-Hugoniot conditions on S_0 . The incident shock S_0 moves in the direction of the x_1 -axis and hits the wedge vertex at the initial time. The slip boundary condition $\mathbf{u} \cdot \boldsymbol{\nu} = 0$ is prescribed on the wedge boundary, where \mathbf{u} is the velocity of gas. Since state (1) does not satisfy the boundary condition, the shock reflection-diffraction configuration occurs at later time, which is self-similar. Depending on the flow parameters and the wedge angle, there may be various patterns of shock reflection-diffraction configurations, including Regular Reflection and Mach Reflection.

The regular reflection problem is a lateral Riemann problem in the region

$$\Lambda = \mathbb{R}_+^2 \setminus \{ \mathbf{x} : x_1 > 0, 0 < x_2 < x_1 \tan \theta_w \},$$

where $\mathbb{R}_+^2 = \mathbb{R}^2 \cap \{x_1 > 0\}$. We seek $(\rho, \mathbf{u})(\mathbf{x}, t)$ satisfying system (2.1) in Λ with the boundary condition $\mathbf{u} \cdot \boldsymbol{\nu} = 0$ on $\partial\Lambda$ and the initial data:

$$(\rho, \mathbf{u})(\mathbf{x}, 0) = \begin{cases} (\rho_0, \mathbf{u}^{(0)}) & \text{if } \mathbf{x} \in \Lambda \cap \{x_1 < 0\}, \\ (\rho_1, \mathbf{u}^{(1)}) & \text{if } \mathbf{x} \in \Lambda \cap \{x_1 > 0\}. \end{cases}$$

This initial-boundary value problem is invariant under scaling (2.2), so we seek a self-similar solution $(\rho, \mathbf{v}) = (\rho, \mathbf{v})(\boldsymbol{\xi})$, where the self-similar variables $\boldsymbol{\xi}$ and the pseudo-velocity $\mathbf{v} = \mathbf{u} - \boldsymbol{\xi}$ are introduced in § 2.1.

First, consider the problem with an assumption on the symmetry with respect to the x_1 -axis. Then we can consider only the upper half-plane $\{x_2 > 0\}$ and prescribe the slip boundary condition $\mathbf{u} \cdot \boldsymbol{\nu} = 0$ on the symmetry line $\{x_2 = 0\}$, so that there is only one reflection point P_0 to be considered (see Figs. 2–3).

Fig. 2 Supersonic regular reflection

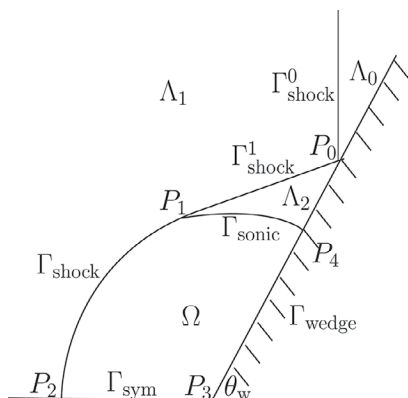
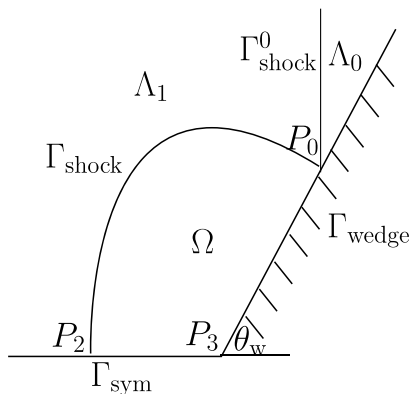


Fig. 3 Subsonic regular reflection



The regular shock reflection-diffraction configuration is characterized by the fact that the reflection occurs at point P_0 of the intersection of the incident shock with the wedge boundary. Figures 2–3 show the structure of regular shock reflection-diffraction configurations in self-similar coordinates.

A necessary condition for the existence of a regular reflection-diffraction configuration is the existence of the constant state (2) and the reflected shock line such that state (2) satisfies both the slip boundary condition on the wedge and the Rankine-Hugoniot conditions (2.9)–(2.10) on the shock with state (1) across the reflected shock line written at P_0 . These conditions lead to a system of algebraic equations for the constant velocity and density of state (2). Moreover, the entropy condition (2.8) becomes an inequality in terms of the parameters of states (1) and (2).

It is well-known (see e.g. [CF18, Chap. 18] for the full Euler system case; the argument for the isentropic Euler system case is similar) that, given the parameters of states (0) and (1), there exists a detachment angle $\theta_w^d \in (0, \frac{\pi}{2})$ such that the system of algebraic equations for parameters of state (2) has two solutions for each wedge angle $\theta_w \in (\theta_w^d, \frac{\pi}{2})$ such that the entropy condition (2.8) is satisfied for the resulting two-shock configuration. These two solutions become equal when $\theta_w = \theta_w^d$. Thus, two types of two-shock configurations occur at P_0 for each $\theta_w \in (\theta_w^d, \frac{\pi}{2})$. For such θ_w , state (2) with the smaller density is called a weak state (2). It is expected that the weak state (2) is physical, while the strong state (2) is not stable as the wedge angle tends to $\frac{\pi}{2}$ (as shown in [CF10] for the potential flow case). In the case of the potential flow model, the global existence of regular shock reflection solutions for all $\theta_w \in (\theta_w^d, \frac{\pi}{2})$ with (ρ, \mathbf{u}) at P_0 determined by the weak state (2) has been established in [CF10, CF18]. For the full or isentropic Euler system, the existence of regular reflection solutions is an outstanding open problem. From now on, state (2) always refers to the weak state (2), which is unique for each $\theta_w \in (\theta_w^d, \frac{\pi}{2})$.

Furthermore, it follows from the Rankine-Hugoniot conditions (2.9)–(2.10) on the straight shock $S_1 := \Gamma_{\text{shock}}^1$ between states (1) and (2), the slip boundary condition on the wedge for state (2), $\mathbf{v}_1(\boldsymbol{\xi}) = (u_1^{(1)}, 0) - \boldsymbol{\xi}$, and $\mathbf{v}_2(\boldsymbol{\xi}) = \mathbf{u}_2 - \boldsymbol{\xi}$ that

The shock line S_1 between states (1) and (2) is not vertical for all $\theta_w \in (\theta_w^d, \frac{\pi}{2})$. (4.1)

Moreover, from the entropy condition (2.12) on S_1 , we have

$$\rho_2 > \rho_1. \tag{4.2}$$

Depending on the wedge angle, state (2) can be either supersonic or subsonic at P_0 , i.e., either $|\mathbf{v}_2(P_0)| > c_2$ or the opposite inequality holds, where $c_2 = \rho_2^{\frac{\gamma-1}{2}}$ is the (constant) speed of sound of state (2). Moreover, for θ_w near $\frac{\pi}{2}$ (resp. for θ_w near θ_w^d), state (2) is supersonic (resp. subsonic) at P_0 . The type of state (2) at P_0 for a given wedge angle θ_w determines the type of reflection, supersonic or subsonic, as shown in Figs. 2 and 3, respectively.

Definition 4.2 $(\rho, \mathbf{v}) \in L^\infty(\Lambda)$, with $\rho \in BV_{loc}(\Lambda \cap \mathcal{N}_r(\partial\Lambda))$ for some $r > 0$, is called an entropy solution of the regular shock reflection problem if (ρ, \mathbf{v}) is an entropy solution of system (2.3)–(2.4) with slip boundary condition (2.13) in the sense of Definition 2.2, which satisfies the asymptotic conditions:

$$\lim_{R \rightarrow \infty} \|(\rho, \mathbf{v}) - (\bar{\rho}, \bar{\mathbf{v}})\|_{0, \Lambda \setminus B_R(0)} = 0,$$

where

$$(\bar{\rho}, \bar{\mathbf{v}}) = \begin{cases} (\rho_0, \mathbf{v}_0) & \text{for } \xi_1 > \xi_1^0, \\ (\rho_1, \mathbf{v}_1) & \text{for } \xi_1 < \xi_1^0, \end{cases}$$

and $\xi_1^0 > 0$ is the location of the incident shock S_0 on the self-similar plane.

Next, we define the points and lines in Figs. 2–3. The incident shock S_0 is line $\{\xi_1 = \xi_1^0\}$ with $\xi_1^0 = \frac{\rho_1 u_1^{(1)}}{\rho_1 - \rho_0} > 0$. The center, $O_2 = \mathbf{u}^{(2)} = (u_1^{(2)}, u_2^{(2)})$, of the sonic circle $B_{c_2}(O_2)$ of state (2) lies on the wedge boundary between the reflection point P_0 and the wedge vertex P_3 for both the supersonic and subsonic cases.

Then, for the supersonic case, i.e., when $|\mathbf{v}_2(P_0)| = |P_0 O_2| > c_2$ so that $P_0 \notin \overline{B_{c_2}(O_2)}$, we denote by P_4 the upper point of intersection of $\partial B_{c_2}(O_2)$ with the wedge boundary such that $O_2 \in P_3 P_4$. Also, the sonic circle $\partial B_{c_2}(O_2)$ of state (2) intersects line S_1 , and one of the points of intersection, $P_1 \in \Lambda$, is such that segment $P_0 P_1$ is outside $B_{c_2}(O_2)$. Denote the arc of $\partial B_{c_2}(O_2)$ by $\Gamma_{sonic} = P_1 P_4$. The curved part of the reflected-diffracted shock is $\Gamma_{shock} = P_1 P_2$, where $P_2 \in \{\xi_2 = 0\}$. Then we denote the line segments $\Gamma_{sym} := P_2 P_3$ and $\Gamma_{wedge} := P_3 P_4$. The lines and curves Γ_{shock} , Γ_{sonic} , Γ_{sym} , and Γ_{wedge} do not have common points, except their endpoints P_1, \dots, P_4 . Thus, $\Gamma_{shock} \cup \Gamma_{sonic} \cup \Gamma_{sym} \cup \Gamma_{wedge}$ is a closed curve without self-intersection. Denote by Ω the bounded domain restricted by this curve.

For the subsonic/sonic case, i.e., when $|\mathbf{v}_2(P_0)| = |P_0O_2| \leq c_2$ so that $P_0 \in \overline{B_{c_2}(O_2)}$, the curved reflected-diffracted shock is $\Gamma_{\text{shock}} = P_0P_2$, which does not have common interior points with the line segments $\Gamma_{\text{sym}} = P_2P_3$ and $\Gamma_{\text{wedge}} = P_0P_3$. Then $\Gamma_{\text{shock}} \cup \Gamma_{\text{sym}} \cup \Gamma_{\text{wedge}}$ is a closed curve without self-intersection, and Ω is the bounded domain restricted by this curve.

Furthermore, in some parts of the argument below, it is convenient to extend problem (2.3)–(2.4) and (2.13), given in Λ by even reflection about the ξ_1 -axis, i.e., defining

$$(\rho^{\text{ext}}, \mathbf{v}^{\text{ext}})(-\xi_1, \xi_2) := (\rho^{\text{ext}}, \mathbf{v}^{\text{ext}})(\xi_1, \xi_2) \quad \text{for any } \boldsymbol{\xi} = (\xi_1, \xi_2) \in \Lambda.$$

Then $(\rho^{\text{ext}}, \mathbf{v}^{\text{ext}})$ is defined in region Λ^{ext} obtained from Λ by adding the reflected region Λ^- , i.e., $\Lambda^{\text{ext}} = \Lambda \cup \{(\xi_1, 0) : \xi_1 < 0\} \cup \Lambda^-$. In a similar way, region Ω and curves $\Gamma_{\text{shock}} \subset \partial\Omega$ and P_0P_2 can be extended into the corresponding region Ω^{ext} and curves $\Gamma_{\text{shock}}^{\text{ext}} \subset \partial\Omega^{\text{ext}}$ and $P_0P_2P_0^{\text{ext}}$.

Now we give the definition of a global regular shock reflection solution. The intuition for the definition is the following: The regular shock reflection solution is an entropy solution of the regular shock reflection problem in the sense of Definition 4.2 which has the structure shown in Figs. 2–3, where (ρ, \mathbf{v}) coincides with those of states (0), (1), and (2) in their respective regions. As we discussed above, the necessary condition is the existence of state (2), which means that the wedge angle satisfies $\theta_w \in (\theta_w^d, \frac{\pi}{2})$. Moreover, it is expected that the solution is relatively regular in Ω . However, we show below that it is not possible that $\mathbf{v} \in H^1(\Omega)$. On the other hand, from the physical/computational experiments and the theoretical results in the case of potential flow, it is expected that Γ_{shock} is a smooth curve and \mathbf{v} is smooth near and up to $\Gamma_{\text{shock}} \cup \Gamma_{\text{sonic}}$ in Ω . Moreover, the regularity discussed is expected for the shock reflection-diffraction configuration extended to $\{\xi_1 < 0\}$ by the even reflection about the ξ_1 -axis (since this is the original shock reflection-diffraction configuration). In particular, the extended shock curve $\Gamma_{\text{shock}}^{\text{ext}}$ is smooth, which shows that Γ_{shock} must be orthogonal to Γ_{sym} at P_2 . Then, noting that the pseudo-velocity of state (1) is $\mathbf{v}_1(\boldsymbol{\xi}) = (u_1^{(1)}, 0) - \boldsymbol{\xi}$ with $u_1^{(1)} > 0$ and $\{P_2\} = \overline{\Gamma_{\text{shock}}} \cap \overline{\Gamma_{\text{sym}}} \subset \{\xi_1 < 0\}$, it follows that vector \mathbf{v}_1 on Γ_{shock} near P_2 points into Ω . It is then expected that this holds on the whole shock Γ_{shock} , unless the jumps of the velocity and the density across the reflected-diffracted shock are degenerate at some points (which are not expected). Then it follows from (2.10) for Γ_{shock} that \mathbf{v} on Γ_{shock} from the Ω -side also points into Ω . On the sonic arc (for the supersonic reflection), the jump of the velocity is not expected, and velocity \mathbf{v}_2 of state (2) on Γ_{sonic} points into Ω (as \mathbf{v}_2 points along the radial direction of the sonic circle of state (2) towards its center $O_2 \in P_3P_4$). Therefore, it is expected that

$$\mathbf{v} \cdot \boldsymbol{\nu} \leq -C^{-1} \quad \text{on } \overline{\Gamma_{\text{shock}}} \cup \overline{\Gamma_{\text{sonic}}} \tag{4.3}$$

for some $C > 0$, where \mathbf{v} on the curves is taken from the Ω -side and $\boldsymbol{\nu}$ is the outer normal with respect to Ω .

Based on the remarks above, we define the notion of regular shock reflection solutions:

Definition 4.3 Fix the wedge angle $\theta_w \in (\theta_w^d, \frac{\pi}{2})$, and let domain $\Lambda = \Lambda(\theta_w)$ as defined above. An entropy solution (ρ, \mathbf{v}) of the regular shock reflection problem in the sense of Definition 4.2 is called a regular shock reflection solution if (ρ, \mathbf{v}) satisfies the following additional properties:

If state (2) for θ_w is supersonic at P_0 , i.e., $|\mathbf{v}_2(P_0)| > c_2$, the solution has the supersonic reflection structure as on Fig. 2. If state (2) for θ_w is subsonic or sonic at P_0 , i.e., $|\mathbf{v}_2(P_0)| \leq c_2$, the solution has the subsonic reflection structure as on Fig. 3. More specifically,

- (i) The extended reflected-diffracted shock curve $P_0P_2P_0^{\text{ext}}$ is C^1 up to its endpoints.
- (ii) (ρ, \mathbf{v}) is continuous in $\bar{\Omega} \cap \mathcal{N}_r(\Gamma_{\text{shock}} \cup \Gamma_{\text{sonic}})$ for the supersonic reflection, and in $\bar{\Omega} \cap \mathcal{N}_r(\Gamma_{\text{shock}})$ for the subsonic or sonic reflection, for some $r > 0$.
- (iii) The solution coincides with states (0), (1), and (2) in their respective regions: for the supersonic reflection case,

$$(\rho, \mathbf{v}) = \begin{cases} (\rho_0, \mathbf{v}_0) & \text{for } \xi_1 > \xi_1^0 \text{ and } \xi_2 > \xi_1 \tan \theta_w, \\ (\rho_1, \mathbf{v}_1) & \text{for } \xi_1 < \xi_1^0 \text{ and above curve } P_0P_1P_2, \\ (\rho_2, \mathbf{v}_2) & \text{in } P_0P_1P_4, \end{cases}$$

where $\xi_1^0 > 0$ is the location of the incident shock S_0 on the self-similar plane; and for the subsonic or sonic reflection case,

$$(\rho, \mathbf{v}) = \begin{cases} (\rho_0, \mathbf{v}_0) & \text{for } \xi_1 > \xi_1^0 \text{ and } \xi_2 > \xi_1 \tan \theta_w, \\ (\rho_1, \mathbf{v}_1) & \text{for } \xi_1 < \xi_1^0 \text{ and above curve } P_0P_2, \end{cases}$$

and $\lim_{\xi \in \Omega, \xi \rightarrow P_0} (\rho, \mathbf{v})(\xi) = (\rho_2, \mathbf{v}_2)(P_0)$.

- (iv) (4.3) holds for some $C > 0$, where \mathbf{v} on the curves is taken from the Ω -side and \mathbf{v} is the outer normal with respect to Ω .
- (v) The flow is pseudo-subsonic in Ω on and near Γ_{shock} , except for the sonic point P_1 for the supersonic or sonic reflections, i.e., $|\mathbf{v}| < c$ on $\bar{\Gamma}_{\text{shock}} \setminus \{P_1\}$ for supersonic and sonic reflections, and on $\bar{\Gamma}_{\text{shock}}$ for strictly subsonic reflections, where \mathbf{v} on Γ_{shock} is computed from the Ω -side.

Remark 4.4 The curve, Γ_{shock} , cannot be a straight segment. Indeed, if the shock is a straight segment, then it lies on a vertical line passing through P_2 , since the tangent line to Γ_{shock} at P_2 is vertical by condition (i) of Definition 4.3. On the other hand, the tangent to Γ_{shock} at P_1 for the supersonic reflection and at P_0 for the subsonic reflection is tangent to the straight shock S_1 between state (1) and (2), where, for the subsonic case, this follows from the property: $\lim_{\xi \in \Omega, \xi \rightarrow P_0} (\rho, \mathbf{v})(\xi) = (\rho_2, \mathbf{v}_2)(P_0)$

in Definition 4.3(iii) by using the Rankine-Hugoniot conditions (2.9)–(2.10) on the shock. As we have shown above, the straight shock between states (1) and (2) is not vertical. This shows that Γ_{shock} cannot lie in a straight line.

Remark 4.5 For the supersonic reflections, it follows from conditions (ii)–(iii) of Definition 4.3 and the Rankine-Hugoniot conditions that (ρ, \mathbf{v}) is continuous across the sonic arc $P_1P_4 = \overline{\Gamma_{\text{sonic}}}$:

$$\lim_{\xi \in \Omega, \xi \rightarrow P} (\rho, \mathbf{v})(\xi) = (\rho_2, \mathbf{v}_2(P)) \quad \text{for any } P \in \overline{\Gamma_{\text{sonic}}}. \tag{4.4}$$

Theorem 4.6 Let (ρ, \mathbf{v}) be a regular shock reflection solution in the sense of Definition 4.3. Assume that (ρ, \mathbf{v}) satisfies the following:

- (i) The reflected-diffracted shock $\Gamma_{\text{shock}} = P_1P_2$ is C^2 in its relative interior, and C^1 up to endpoints P_1 , where P_1 is replaced by P_0 if the subsonic shock reflection occurs at P_0 ;
- (ii) for some $\sigma > 0$,
 $(\rho, \mathbf{v}) \in C^1(\mathcal{N}_\sigma(\Gamma_{\text{shock}} \cup \Gamma_{\text{sonic}}) \cap \overline{\Omega}) \setminus \{P_1\}) \cap C^{0,1}(\mathcal{N}_\sigma(\Gamma_{\text{shock}} \cup \Gamma_{\text{sonic}}) \cap \overline{\Omega})$
 where P_1 is replaced by P_0 if the subsonic shock reflection occurs at P_0 ;
- (iii) $|\mathbf{v}| \leq C_0$ and $C_0^{-1} \leq \rho \leq C_0$ in Ω for some $C_0 \geq 1$.

Then $\mathbf{v} \notin H^1(\Omega)$.

The proof of Theorem 4.6 will be given for the more general non-symmetric case below (Theorem 4.10), based on the observation in Remark 4.8.

Next, for the non-symmetric regular shock reflection problem, based on the argument near the reflection point P_0 for the symmetric case above, there are four configurations depending on whether the solution at P_0 and P_1 is subsonic or supersonic. As two examples of them, Figs. 4 and 5 exhibit the structures of two supersonic regular reflections and two subsonic regular reflections, respectively. In light of Definition 4.3, we define the regular reflection solution for the non-symmetric case. Let the wedge angle $\theta_w = \theta_w^1 + \theta_w^2$ with $\theta_w^i \in (\theta_w^d, \frac{\pi}{2})$, where θ_w^i is the angle between Γ_w^i and the ξ_1 -axis for $i = 1, 2$. Let

$$\Lambda = \mathbb{R}_+^2 \setminus \{ \xi : \xi_1 > 0, -\xi_1 \tan \theta_w^2 < \xi_2 < \xi_1 \tan \theta_w^1 \}.$$

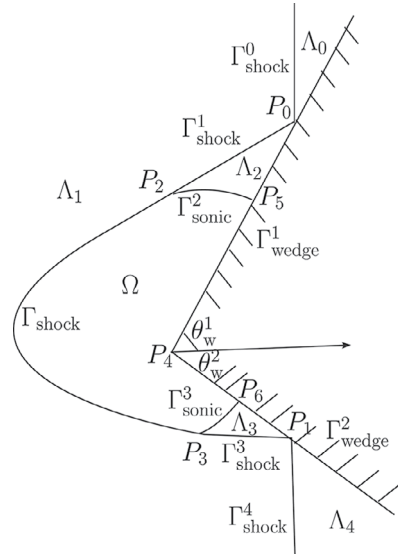
Let $\xi_1^0 > 0$ be the location of the incident shock S_0 on the self-similar plane.

We call $(\rho, \mathbf{v}) \in L^\infty(\Lambda)$, with $\rho \in BV_{\text{loc}}(\Lambda \cap \mathcal{N}_r(\partial\Lambda))$ for some $r > 0$, an entropy solution of the non-symmetric regular shock reflection problem if (ρ, \mathbf{v}) is an entropy solution of system (2.3)–(2.4) with the slip boundary condition (2.13) in the sense of Definition 2.2, which satisfies the asymptotic conditions:

$$\lim_{R \rightarrow \infty} \|(\rho, \mathbf{v}) - (\bar{\rho}, \bar{\mathbf{v}})\|_{0, \Lambda \setminus B_R(0)} = 0,$$

where

Fig. 4 Non-symmetric supersonic regular reflection



$$(\bar{\rho}, \bar{\mathbf{v}}) = \begin{cases} (\rho_0, \mathbf{v}_0) & \text{for } \xi_1 > \xi_1^0, \\ (\rho_1, \mathbf{v}_1) & \text{for } \xi_1 < \xi_1^0. \end{cases}$$

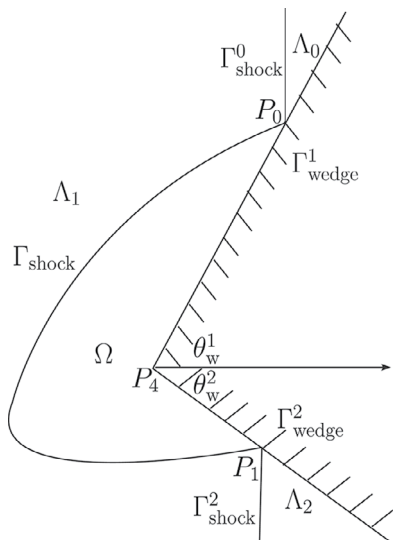
Then we can define the non-symmetric regular shock reflection solution.

Definition 4.7 For given angles θ_w^i with $\theta_w^i \in (\theta_w^d, \frac{\pi}{2})$ for $i = 1, 2$, an entropy solution (ρ, \mathbf{v}) is called a non-symmetric regular shock reflection solution of the non-symmetric regular shock reflection problem if (ρ, \mathbf{v}) satisfies the following additional properties:

- (i) If state (2) at P_0 and state (3) at P_1 are both supersonic, i.e., $|\mathbf{v}_2(P_0)| > c_2$ and $|\mathbf{v}_2(P_1)| > c_3$, the solution has the supersonic reflection structure as in Fig. 4 at both points P_0 and P_1 . If state (2) at P_0 and state (3) at P_1 are both subsonic or sonic, i.e., $|\mathbf{v}_2(P_0)| \leq c_2$ and $|\mathbf{v}_2(P_1)| \leq c_3$, the solution has the subsonic reflection structure as in Fig. 5.
- (ii) The reflected-diffracted shock curve P_0P_1 is C^1 up to its endpoints.
- (iii) (ρ, \mathbf{v}) is continuous in $\bar{\Omega} \cap \mathcal{N}_r(\Gamma_{shock} \cup \Gamma_{sonic}^2 \cup \Gamma_{sonic}^3)$ for the supersonic reflections, and in $\bar{\Omega} \cap \mathcal{N}_r(\Gamma_{shock})$ for the subsonic reflection, for some $r > 0$.
- (iv) The solution coincides with states (0), (1), (2), and (3) in their respective regions:

$$(\rho, \mathbf{v}) = \begin{cases} (\rho_0, \mathbf{v}_0) & \text{for } \xi_1 > \xi_1^0, \quad \xi_2 > \xi_1 \tan \theta_w^1, \\ (\rho_0, \mathbf{v}_0) & \text{for } \xi_1 > \xi_1^0, \quad \xi_2 < -\xi_1 \tan \theta_w^2, \\ (\rho_1, \mathbf{v}_1) & \text{for } \xi_1 < \xi_1^0, \quad \text{and the left to curve } P_0P_1, \\ (\rho_2, \mathbf{v}_2) & \text{in } \Lambda_2 \text{ if state (2) at } P_0 \text{ is supersonic,} \\ (\rho_3, \mathbf{v}_3) & \text{in } \Lambda_3 \text{ if state (3) at } P_1 \text{ is supersonic,} \end{cases}$$

Fig. 5 Non-symmetric subsonic regular reflection



where $\xi_1^0 > 0$ is the location of the incident shock S_0 on the self-similar plane, and

$$\lim_{\xi \in \Omega, \xi \rightarrow P_0} (\rho, \mathbf{v})(\xi) = (\rho_2, \mathbf{v}_2)(P_0) \quad \text{if state (2) at } P_0 \text{ is subsonic or sonic,}$$

$$\lim_{\xi \in \Omega, \xi \rightarrow P_1} (\rho, \mathbf{v})(\xi) = (\rho_3, \mathbf{v}_3)(P_1) \quad \text{if state (3) at } P_1 \text{ is subsonic or sonic.}$$

- (v) (4.3) holds for some $C > 0$, where Γ_{sonic} is replaced by $\Gamma_{\text{sonic}}^2 \cup \Gamma_{\text{sonic}}^3$, \mathbf{v} on the curves is taken from the Ω -side, and \mathbf{v} is the outer normal with respect to Ω .
- (vi) The flow is pseudo-subsonic in Ω on and near Γ_{shock} , except for the sonic points P_2 and P_3 if they exist, point P_0 if state (2) is sonic at P_0 , and point P_1 if state (3) is sonic at P_1 .

Remark 4.8 Similarly as stated in Remark 4.4, curve Γ_{shock} cannot be a straight segment, which will be addressed in more detail in Step 3 of the proof of Theorem 4.10 below. Moreover, if $\theta_w^1 = \theta_w^2$, the symmetric regular shock reflection problem is a special case of the non-symmetric regular shock reflection problem, by defining $(\rho^{\text{ext}}, \mathbf{v}^{\text{ext}})(\xi_1, -\xi_2) := (\rho^{\text{ext}}, \mathbf{v}^{\text{ext}})(\xi_1, \xi_2)$ for any $\xi = (\xi_1, \xi_2) \in \Lambda$. Therefore, we only prove the low regularity of the non-symmetric regular reflection solution below, since the low regularity for the symmetric case follows directly.

Remark 4.9 For the supersonic reflections, it follows from conditions (ii)–(iv) of Definition 4.7 and the Rankine-Hugoniot conditions that (ρ, \mathbf{v}) is continuous across the sonic arcs $\Gamma_{\text{sonic}}^2 \cup \Gamma_{\text{sonic}}^3$:

$$\lim_{\xi \in \Omega, \xi \rightarrow P} (\rho, \mathbf{v})(\xi) = (\rho_2, \mathbf{v}_i)(P) \quad \text{for any } P \in \overline{\Gamma_{\text{sonic}}^i} \text{ if } |\mathbf{v}_i(P_0)| > c_i \text{ for } i = 2, 3.$$

Theorem 4.10 Let (ρ, \mathbf{v}) be a non-symmetric regular shock reflection solution in the sense of Definition 4.7. Assume that (ρ, \mathbf{v}) satisfies the following:

- (i) The reflected-diffracted shock $\Gamma_{\text{shock}} = P_2P_3$ is C^2 in its relative interior, and C^1 up to endpoints P_2 and P_3 , where P_2 (or P_3) is replaced by P_0 (or P_1) if the subsonic shock reflection occurs at P_0 (or P_1);
- (ii) For some $\sigma > 0$, $(\rho, \mathbf{v}) \in C^1((\mathcal{N}_\sigma(\Gamma_{\text{shock}} \cup \Gamma_{\text{sonic}}^1 \cup \Gamma_{\text{sonic}}^2) \cap \bar{\Omega}) \setminus \{P_2, P_3\}) \cap C^{0,1}(\mathcal{N}_\sigma(\Gamma_{\text{shock}} \cup \Gamma_{\text{sonic}}^1 \cup \Gamma_{\text{sonic}}^2) \cap \bar{\Omega})$, where P_2 (or P_3) is replaced by P_0 (or P_1) if the subsonic shock reflection occurs at P_0 (or P_1);
- (iii) $|\mathbf{v}| \leq C_0$ and $C_0^{-1} \leq \rho \leq C_0$ in Ω for some $C_0 \geq 1$.

Then $\mathbf{v} \notin H^1(\Omega)$.

Proof It suffices to check that the assumptions of Theorem 3.2 are satisfied. We divide the proof into four steps:

1. Since the cases with one supersonic reflection and one subsonic reflection can be treated similarly, we focus only on the non-symmetric supersonic reflection solution (see Fig. 4) and the non-symmetric subsonic reflection solution (see Fig. 5) in the proof below.

2. We first show that the non-symmetric regular reflection solutions satisfy conditions (i)–(vii) of Definition 3.1 and assumptions (i)–(iii) of Theorem 3.2. The properties described in Definition 3.1(i) with $M=5$ for the non-symmetric supersonic reflection solutions and with $M=3$ for the non-symmetric subsonic reflection solutions by Definition 4.7(iv), and with Λ_j being the region of possible state (0), (1), (2), or (3). Moreover, condition (ii) of Definition 3.1 also holds for the regular shock reflection solutions because states (0), (1), (2), and (3) are four different constant states, specifically $\rho_0 < \rho_1 < \rho_2$ and $\rho_0 < \rho_1 < \rho_3$.

The property in Definition 3.1(iii) holds, i.e., $\partial\Omega$ is a Lipschitz curve, by Definition 4.7(ii) and the fact that the rest of $\partial\Omega$ consists of the straight segments and the arcs of the circles for the supersonic reflection, and the angles of vertices $P_j, j = 1, 2, \dots, 6$, for the non-symmetric supersonic reflection and of vertices P_0, P_1 , and P_4 for the non-symmetric subsonic reflection are all within $(0, \pi)$.

For the non-symmetric supersonic reflection, $N_1 = 2$ and $N_2 = 3$, with $\Gamma_1^{\text{ext}} = \Gamma_{\text{wedge}}^1 = P_4P_5$, $\Gamma_2^{\text{ext}} = \Gamma_{\text{wedge}}^2 = P_4P_6$, $\Gamma_1^{\text{int}} = \Gamma_{\text{shock}} = P_2P_3$, $\Gamma_2^{\text{int}} = \Gamma_{\text{sonic}}^2 = P_2P_5$, and $\Gamma_3^{\text{int}} = \Gamma_{\text{sonic}}^3 = P_3P_6$.

For the non-symmetric subsonic reflection, $N_1 = 2$ and $N_2 = 1$, with $\Gamma_1^{\text{ext}} = \Gamma_{\text{wedge}}^1 = P_0P_4$, $\Gamma_2^{\text{ext}} = \Gamma_{\text{wedge}}^2 = P_1P_4$, and $\Gamma_1^{\text{int}} = \Gamma_{\text{shock}} = P_0P_1$.

In both the non-symmetric supersonic and subsonic reflection cases, all the requirements of Definition 3.1(iv)–(v) hold, by the regularity of Γ_{shock} given in Definition 4.7(ii) and the facts that the angles in the corner points of Ω are within $(0, \pi)$ for the non-symmetric regular reflection except for point P_4 , where condition (3.1) holds with $\theta^- = \theta_w^1$ and $\theta^+ = 2\pi - \theta_w^2$. Also, point $P_4 = (0, 0)$ for the non-symmet-

ric regular reflection solution is point $\mathcal{P}_0 = (0, 0)$ as described in Definition 3.1(iv). Moreover, P_4 is the common point of $\Gamma_1^{\text{ext}} = \Gamma_{\text{wedge}}^1$ and $\Gamma_2^{\text{ext}} = \Gamma_{\text{wedge}}^2$ for both the regular supersonic and subsonic reflections, as described in Definition 3.1(iv).

Now we show that the requirements of Definition 3.1(vii) hold. Clearly, $\Gamma_1^{\text{int}} = \Gamma_{\text{shock}}$ is a shock, and $\mathbf{v} \cdot \boldsymbol{\nu} \leq -C^{-1}$ where \mathbf{v} on Γ_1^{int} is taken from the Ω -side and $\boldsymbol{\nu}$ is the outer normal with respect to Ω by property (v) of Definition 4.7 of the non-symmetric regular reflection solution. Also, for the non-symmetric supersonic reflection, $\Gamma_2^{\text{int}} = \Gamma_{\text{sonic}}^2$ and $\Gamma_3^{\text{int}} = \Gamma_{\text{sonic}}^3$ are an arc of the sonic circles of state (2) and state (3), respectively, which imply that $\mathbf{v}_2 \cdot \boldsymbol{\nu} = -|\mathbf{v}_2| = -c_2 < 0$ on Γ_{sonic}^2 and $\mathbf{v}_3 \cdot \boldsymbol{\nu} = -|\mathbf{v}_3| = -c_3 < 0$ on Γ_{sonic}^3 . Using the boundary conditions that $\mathbf{v} = \mathbf{v}_2$ on Γ_{sonic}^2 and $\mathbf{v} = \mathbf{v}_3$ on Γ_{sonic}^3 by Definition 4.7(iv), we obtain that

$$\mathbf{v} \cdot \boldsymbol{\nu} = -c_2 < 0 \text{ on } \Gamma_{\text{sonic}}^2, \quad \mathbf{v} \cdot \boldsymbol{\nu} = -c_3 < 0 \text{ on } \Gamma_{\text{sonic}}^3.$$

Thus, the requirements of Definition 3.1(vii) hold.

Next, since $\overline{\Gamma^{\text{int}}} = \overline{\Gamma_{\text{shock}}} \cup \overline{\Gamma_{\text{sonic}}^1} \cup \overline{\Gamma_{\text{sonic}}^2}$, assumptions (ii)–(iii) of Theorem 4.10 imply that assumptions (i)–(ii) of Theorem 3.2 hold for the regular shock reflection solution.

Therefore, we have shown that conditions (i)–(vii) of Definition 3.1 and assumptions (i)–(ii) of Theorem 3.2 hold.

3. It remains to show that the requirements in Definition 3.1(viii) hold. This is achieved by the use of Lemma 4.1. Thus, it remains to show that conditions (a)–(c) of Lemma 4.1 are satisfied below.

Condition (a) is satisfied since Λ_1 is the region of state (1) as shown in Figs. 4–5 above, and $\Gamma_{\text{shock}} = \partial\Lambda_1 \cap \partial\Omega$ in both the supersonic and subsonic cases.

We now check condition (b) of Lemma 4.1. Assume that Γ_{shock} is a straight segment. Recall that $S_1 = \Gamma_{\text{shock}}^1$ is the straight shock between states (1) and (2), i.e., the line passing through points P_0 and P_2 for the supersonic reflection, and through P_0 for the subsonic reflection. Using Definition 4.7(iv), we obtain that $(\rho, \mathbf{v})(P_2) = (\rho_2, \mathbf{v}_2)(P_2)$ for the non-symmetric supersonic reflection, and $(\rho, \mathbf{v})(P_0) = (\rho_2, \mathbf{v}_2)(P_0)$ for the non-symmetric subsonic reflection, where (ρ, \mathbf{v}) in both cases is computed from the Ω -side. Thus, the tangent line to $\Gamma_1^{\text{int}} = \Gamma_{\text{shock}}$ at the upper endpoint, i.e., at P_2 for the non-symmetric supersonic reflection and at P_0 for the non-symmetric subsonic reflection, is line S_1 . Similarly, we have the tangent line to $\Gamma_1^{\text{int}} = \Gamma_{\text{shock}}$ at the lower endpoint, i.e., at P_3 for the non-symmetric supersonic reflection and at P_1 for the non-symmetric subsonic reflection, is line $S_2 = \Gamma_{\text{shock}}^2$. Hence, if Γ_{shock} is a straight segment, then Γ_{shock} lies on both lines S_1 and S_2 ; in particular, these lines coincide. It follows that $S_1 = S_2$ is the line passing through P_0 and P_1 , and Γ_{shock} lies within interval P_0P_1 . However, the wedge is convex since $\theta_w^i \in (0, \frac{\pi}{2})$ for $i = 1, 2$, so P_0P_1 lies within the wedge, thus outside Λ . It follows that $\Gamma_{\text{shock}} \subset P_0P_1$ lies outside Λ , which contradicts the structure of regular reflection-diffraction configuration. That is, the assumption that Γ_{shock} is a straight segment leads to a contradiction, which verifies (b) of Lemma 4.1.

Next, we show condition (c) of Lemma 4.1 is satisfied.

Consider first the case of reflections that are supersonic at P_0 , i.e., $|\mathbf{v}_2(P_0)| > c_2$; see Fig. 4. Then we need to show that $(\mathbf{v} \cdot \boldsymbol{\tau}_{\text{shock}})(P_2) \neq 0$. Assume this is not true, then $(\mathbf{v} \cdot \boldsymbol{\tau}_{\text{shock}})(P_2) = 0$. Note that $\boldsymbol{\tau}_{\text{shock}}(P_2) = \boldsymbol{\tau}_{S_1}$ and $\mathbf{v}|_{\Omega}(P_2) = \mathbf{v}_2(P_2)$ by Definition 4.7(i, iv) so that $\mathbf{v}_2(P_2) \cdot \boldsymbol{\tau}_{S_1} = 0$ and

$$|\mathbf{v}(P_2) \cdot \boldsymbol{\nu}_{\text{shock}}| = |\mathbf{v}_2(P_2) \cdot \boldsymbol{\nu}_{S_1}| = |\mathbf{v}_2(P_2)| = c_2,$$

which contradicts the last inequality in (2.12). Thus condition (c) of Lemma 4.1 is proved in the case when $|\mathbf{v}_2(P_0)| > c_2$.

In the case of reflections that are sonic at P_0 , i.e., $|\mathbf{v}_2(P_0)| = c_2$ (see Fig. 5), the argument is the same as above with only notational change: we use point P_0 here instead of point P_2 .

It remains to consider the case of reflections that are subsonic at P_0 , i.e., $|\mathbf{v}_2(P_0)| < c_2$; see Fig. 5. Assume that $(\mathbf{v} \cdot \boldsymbol{\tau}_{\text{shock}})(P_0) = 0$. Recall that $(\rho, \mathbf{v})(P_0) = (\rho_2, \mathbf{v}_2)(P_0)$ for the subsonic reflection and S_1 is tangent to Γ_{shock} at P_0 , as we have shown above. Thus, from $(\mathbf{v} \cdot \boldsymbol{\tau}_{\text{shock}})(P_0) = 0$, we obtain that $(\mathbf{v}_2 \cdot \boldsymbol{\tau}_{S_1})(P_0) = 0$. Since S_1 is the line shock between states (1) and (2), the last equality implies by (2.10) that $(\mathbf{v}_1 \cdot \boldsymbol{\tau}_{S_1})(P_0) = 0$, so that line L through centers $O_1 = (u_1^{(1)}, 0)$ and $O_2 = (u_1^{(2)}, u_2^{(2)})$ of states (1) and (2) (which is orthogonal to S_1) intersects S_1 at P_0 . Since $P_0 \in \Gamma_{\text{wedge}}$, $O_2 \in L \cap \Gamma_{\text{wedge}}$ (where $O_2 \in \Gamma_{\text{wedge}}$ because $\mathbf{v}_2 \cdot \boldsymbol{\nu} = 0$ on Γ_{wedge}), and $L \cap \Gamma_{\text{wedge}} = \{P_0\}$, it follows that $O_2 = P_0$ so that $\mathbf{v}_2(P_0) = \mathbf{0}$. Since $\mathbf{v}_2(P_0) = \mathbf{0}$, it follows from the Rankine-Hugoniot conditions (2.9)–(2.10) on S_1 between states (1) and (2) that $\mathbf{v}_1(P_0) = \mathbf{0}$, that is, $O_1 = P_0$. However, this is not true since $O_1 = (u_1^{(1)}, 0)$ for $u_1^{(1)} > 0$, while $P_0 = (l \cos \theta_w^1, l \sin \theta_w^1)$ for some $l > 0$. This contradiction shows that $(\mathbf{v} \cdot \boldsymbol{\tau}_{\text{shock}})(P_0) \neq 0$, i.e., condition (c) of Lemma 4.1 holds for the subsonic reflection at P_0 .

Now all the conditions of Lemma 4.1 are verified for the regular reflection solutions. Applying Lemma 4.1, we obtain that Definition 3.1(viii) holds.

4. Now, all the conditions of Theorem 3.2 are verified for the regular reflection solutions. Then the conclusion of Theorem 4.10 follows from Theorem 3.2. \square

4.2 Lower Regularity of the Prantl-Meyer Reflection Solutions for Supersonic Flows Past a Solid Ramp

The second example is the Prandtl reflection problem for the isentropic Euler system (2.1). This is of a self-similar structure that occurs when a 2-D supersonic flow with density $\rho_\infty > 0$ and velocity $\mathbf{v}_\infty = (u_\infty, 0)$, $u_\infty > 0$, along the wedge-axis hits the wedge in the direction at $t=0$. See Figs. 6–7; also see Bae-Chen-Feldman [BCF13, BCF24] and Elling-Liu [EL08].

Consider the problem in the self-similar coordinates $\boldsymbol{\xi} = (\xi_1, \xi_2)$ in the region:

$$\Lambda = \mathbb{R}_+^2 \setminus \{\boldsymbol{\xi} : \xi_2 > \max(0, \xi_1 \tan \theta_w)\}.$$

We seek global entropy solutions of the boundary value problem in the sense of Definition 2.2.

First, a similar argument as made for the regular shock reflection at the reflection point P_0 (see Figs. 2–3) yields that, at wedge-vertex P_1 , for a given uniform incoming flow $(\rho_\infty, \mathbf{v}_\infty)$, there is a detachment angle $\theta_w^d \in (0, \frac{\pi}{2})$ such that the system of algebraic Eqs. (2.9) and (2.13) for state (O) has two solutions for each wedge-angle $\theta_w \in (0, \theta_w^d)$ such that the entropy condition is satisfied for the two-shock configuration. The weak state (O) with the smaller density is expected to be physical. So we always refer to state (O) as the weak state (O) . Similarly, it follows from the Rankine-Hugoniot conditions (2.9) and the slip boundary condition (2.13) that

$$\textit{The shock line } \Gamma_{\text{shock}}^O \textit{ between states } (O) \textit{ and } (\infty) \textit{ is not parallel to } \Gamma_{\text{wedge}} \textit{ for all } \theta_w \in (0, \theta_w^d). \tag{4.5}$$

Depending on the wedge-angle, state (O) can be either supersonic or subsonic at P_1 . It determines the type of the reflection, supersonic or subsonic, as shown in Figs. 6 and 7, respectively.

Second, by a straightforward computation, we know that there exists a unique constant state (N) , which determines the normal reflection of state (∞) from the wedge boundary Γ_{wedge} so that state (N) satisfies the slip boundary condition (2.13) along Γ_{wedge} and the Rankine-Hugoniot conditions (2.9) along a straight line Γ_{shock}^N , which lies in Λ and is parallel to Γ_{wedge} .

Hinted by the solution structures given in [BCF13, BCF24, EL08] for the potential flow, for any given wedge-angle $\theta_w \in (0, \theta_w^d)$, an entropy solution (ρ, \mathbf{v}) of the boundary value problem in the sense of Definition 2.2 is called a regular Prandtl-

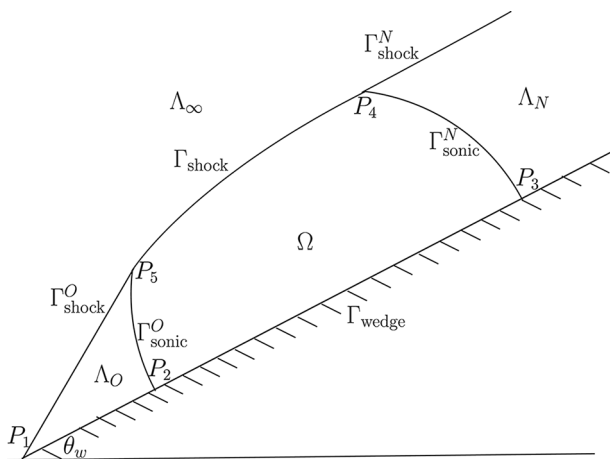
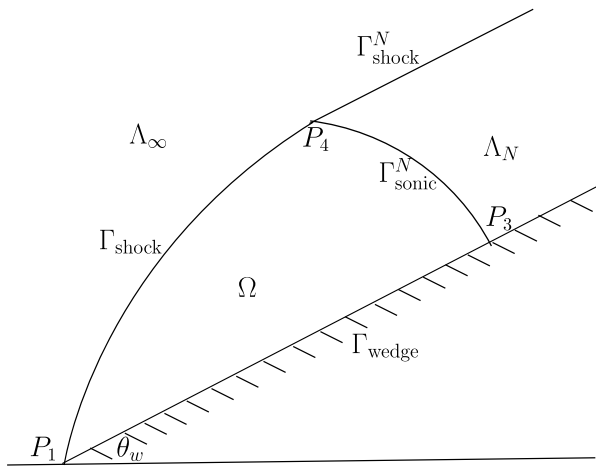


Fig. 6 Prandtl supersonic shock reflection

Fig. 7 Prandtl subsonic shock reflection



Meyer reflection solution for the isentropic Euler system (2.3)–(2.4) if (ρ, \mathbf{v}) satisfies the following further properties:

- (i) If state (O) at P_1 is supersonic, i.e., $|\mathbf{v}_O(P_1)| > c_O$, the solution has the supersonic reflection structure as in Fig. 6 at point P_1 . If state (O) at P_1 is subsonic or sonic, i.e., $|\mathbf{v}_O(P_1)| \leq c_O$, the solution has the subsonic reflection structure as in Fig. 7.
- (ii) The reflected shock curve Γ_{shock} (i.e., P_4P_5 for the supersonic reflection and P_1P_4 for the subsonic reflection) is C^1 up to its endpoints and is C^2 in its relative interior.
- (iii) (ρ, \mathbf{v}) is continuous in $\bar{\Omega} \cap \mathcal{N}_r(\Gamma_{\text{shock}} \cup \Gamma_{\text{sonic}}^N \cup \Gamma_{\text{sonic}}^O)$ when $|\mathbf{v}_O(P_1)| > c_O$, and in $\bar{\Omega} \cap \mathcal{N}_r(\Gamma_{\text{shock}} \cup \Gamma_{\text{sonic}}^N)$ when $|\mathbf{v}_O(P_1)| \leq c_O$ for some $r > 0$.
- (iv) The solution coincides with states (∞) , (O) , and (N) in their respective regions. Specifically,

$$(\rho, \mathbf{v}) = \begin{cases} (\rho_\infty, \mathbf{v}_\infty) & \text{in } \Lambda_\infty, \\ (\rho_N, \mathbf{v}_N) & \text{in } \Lambda_N, \\ (\rho_O, \mathbf{v}_O) & \text{in } \Lambda_O \text{ if state } (O) \text{ at } P_1 \text{ is supersonic,} \end{cases}$$

and

$$\lim_{\xi \in \Omega, \xi \rightarrow P_1} (\rho, \mathbf{v})(\xi) = (\rho_O, \mathbf{v}_O)(P_1) \quad \text{if state } (O) \text{ at } P_1 \text{ is subsonic or sonic.}$$

- (v) $\mathbf{v} \cdot \nu \leq -C^{-1}$ on $\overline{\Gamma_{\text{shock}}} \cup \overline{\Gamma_{\text{sonic}}^N} \cap \overline{\Gamma_{\text{sonic}}^O}$ for some $C > 0$, where \mathbf{v} on the curves is taken from the Ω -side and ν is the outer normal with respect to Ω .

- (vi) The flow is pseudo-subsonic in Ω on and near Γ_{shock} , except for the sonic points P_4 and P_5 if they exist, or point P_1 if state (O) is sonic at P_1 .

We remark that, for the supersonic reflections, it follows from (ii)–(iv) of the definition above and the Rankine-Hugoniot conditions that (ρ, \mathbf{v}) is continuous across the sonic arcs Γ_{sonic}^N and Γ_{sonic}^O .

Then, following the arguments as for the proof of Theorem 4.10, we have the following theorem on the lower regularity of the regular Prandtl-Meyer shock reflection solutions:

Theorem 4.11 Let (ρ, \mathbf{v}) be a regular Prandtl-Meyer shock reflection solution for the isentropic Euler system such that

- (i) $(\rho, \mathbf{v}) \in C^1((\mathcal{N}_\sigma(\Gamma_{\text{shock}} \cup \Gamma_{\text{sonic}}^N \cup \Gamma_{\text{sonic}}^O) \cap \bar{\Omega}) \setminus \{P_4, P_5\}) \cap C^{0,1}(\mathcal{N}_\sigma(\Gamma_{\text{shock}} \cup \Gamma_{\text{sonic}}^N \cup \Gamma_{\text{sonic}}^O) \cap \bar{\Omega})$ for some $\sigma > 0$, where P_5 is replaced by P_1 if the subsonic shock reflection occurs at P_1 ;
- (ii) $|\mathbf{v}| \leq C_0$ and $C_0^{-1} \leq \rho \leq C_0$ in Ω for some $C_0 \geq 1$.

Then $\mathbf{v} \notin H^1(\Omega)$.

Proof For the supersonic reflection, $M=3$, $N_1 = 1$, and $N_2 = 3$ with $\Lambda_1 = \Lambda_N$, $\Lambda_2 = \Lambda_\infty$, $\Lambda_3 = \Lambda_O$, $\Gamma_1^{\text{ext}} = \Gamma_{\text{wedge}} = P_2P_3$, $\Gamma_1^{\text{int}} = \Gamma_{\text{shock}} = P_4P_5$, $\Gamma_2^{\text{int}} = \Gamma_{\text{sonic}}^N = P_3P_4$, and $\Gamma_3^{\text{int}} = \Gamma_{\text{sonic}}^O = P_2P_5$.

For the subsonic reflection, $M=2$, $N_1 = 1$, and $N_2 = 2$ with $\Lambda_1 = \Lambda_N$, $\Lambda_2 = \Lambda_\infty$, $\Gamma_1^{\text{ext}} = \Gamma_{\text{wedge}} = P_1P_3$, $\Gamma_1^{\text{int}} = \Gamma_{\text{shock}} = P_1P_4$, and $\Gamma_2^{\text{int}} = \Gamma_{\text{sonic}}^N = P_3P_4$.

Then we can follow the proof of Theorem 4.10 to show that the regular solutions satisfy the assumptions of Theorem 3.2 similarly. The only (slight) difference is in showing that condition (b) of Lemma 4.1 holds. We need to show that Γ_{shock} is not a straight segment. Let us first consider the supersonic Prandtl reflection case; see Fig. 6. Arguing as in the proof of Theorem 4.10, we show that, at point P_4 (resp. P_5), curve Γ_{shock} is tangential to line Γ_{shock}^N (resp. Γ_{shock}^O). If Γ_{shock} is a straight segment, we obtain that Γ_{shock}^N and Γ_{shock}^O lie in the same line. However, Γ_{shock}^N is parallel to Γ_{wedge} with a positive distance from it, while Γ_{shock}^O passes through point $P_1 \in \Gamma_{\text{wedge}}$. This contradiction shows that Γ_{shock} is not a straight segment. In the case of subsonic or sonic Prandtl reflection as in Fig. 7, we argue similarly, except that we consider point P_1 instead of point P_5 and use the fact that, in the subsonic or sonic case, Γ_{shock} is parallel to Γ_{shock}^O at P_1 by the last equality in condition (iv) of the definition of admissible solutions. Then we conclude in the same way as in the case of supersonic Prandtl reflection. Thus, we do not repeat the similar arguments for the proof. □

4.3 Lower Regularity of the Shock Diffraction Solutions of the Lighthill Problem

The third problem is the Lighthill problem, i.e., the shock diffraction problem, for the isentropic Euler system. As discussed in [CFHX20] for the potential flow and shown in Fig. 8, the Lighthill shock diffraction problem arises as a straight incident shock passes through a wedge stepping down.

Initially, we consider two piecewise constant Riemann data with the left state (1): $(\rho_1, u_1^{(1)}, 0)$ for $u_1^{(1)} > 0$ and the right state (0): $(\rho_0, 0, 0)$, separated by a vertical shock, which hits the wedge at the wedge-corner P_1 . Then the Lighthill shock diffraction problem evolves in a self-similar structure as time goes on. In the self-similar coordinates ξ , the incident shock Γ_{shock}^1 is given by $\xi_1 = \xi_1^0$. By a straightforward calculation, it follows from the Rankine-Hugoniot conditions (2.9) and the entropy condition (2.12) that the location of the incident shock satisfies $0 < \xi_1^0 < c_1$, where $c_1 = \rho_1^{\frac{\gamma-1}{2}}$ is the sonic speed of state (1). Thus, as shown in Fig. 8, the incident shock Γ_{shock}^1 interacts with the sonic circle Γ_{sonic} of state (1) and becomes a transonic shock Γ_{shock} , and then Γ_{shock} meets wedge Γ_{wedge}^0 perpendicularly.

Let Λ_0, Λ_1 , and Ω be defined, as shown in Fig. 8. Let $\Lambda = \overline{\Lambda_0 \cup \Lambda_1 \cup \Omega}$. An entropy solution (ρ, \mathbf{v}) of the boundary value problem in Λ in the sense of Definition 2.2 is called a regular shock diffraction solution of the Lighthill problem for the isentropic Euler system if (ρ, \mathbf{v}) satisfies the following further properties:

- (i) The diffracted shock curve $\Gamma_{\text{shock}} = P_2P_3$ is C^1 up to its endpoints and is C^2 in its relative interior. Its tangent is perpendicular to Γ_{wedge}^0 at P_2 and is vertical at P_3 .
- (ii) (ρ, \mathbf{v}) is continuous in $\overline{\Omega} \cap \mathcal{N}_r(\Gamma_{\text{shock}} \cup \Gamma_{\text{sonic}})$ for some $r > 0$.
- (iii) The solution coincides with states (0) and (1) in their respective regions. Specifically,

$$(\rho, \mathbf{v}) = \begin{cases} (\rho_0, \mathbf{v}_0) & \text{in } \Lambda_0, \\ (\rho_1, \mathbf{v}_1) & \text{in } \Lambda_1, \end{cases}$$

where $\mathbf{v}_0 = -\xi$ and $\mathbf{v}_1 = (u_1^{(1)}, 0) - \xi$.

- (iv) $\mathbf{v} \cdot \nu \leq -C^{-1}$ on $\overline{\Gamma_{\text{shock}}} \cup \overline{\Gamma_{\text{sonic}}}$ for some $C > 0$, where \mathbf{v} on the curves is taken from the Ω -side and ν is the outer normal with respect to Ω .
- (v) The flow is pseudo-subsonic in Ω on and near Γ_{shock} except for the sonic points P_3 .

Let $M=2, N_1 = 2$, and $N_2 = 2$ with $\Lambda_1 = \Lambda_0, \Lambda_2 = \Lambda_1, \Gamma_1^{\text{ext}} = \Gamma_{\text{wedge}}^1 = P_1P_4, \Gamma_2^{\text{ext}} = \Gamma_{\text{wedge}}^0 = P_1P_2, \Gamma_1^{\text{int}} = \Gamma_{\text{shock}} = P_2P_3$, and $\Gamma_2^{\text{int}} = \Gamma_{\text{sonic}} = P_3P_4$. Then, following the proof of Theorem 4.10, we have the following theorem on the lower regularity of the regular shock diffraction solutions.

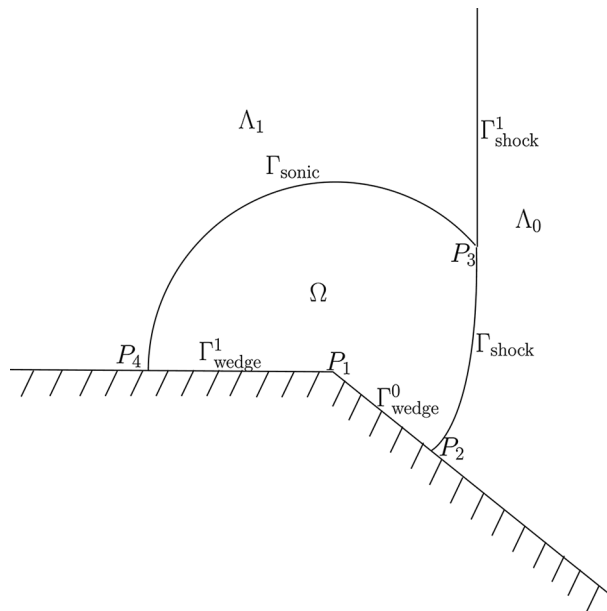
Theorem 4.12 Let (ρ, \mathbf{v}) be a regular shock diffraction solution of the Lighthill problem for the isentropic Euler system and satisfy the following:

- (i) $\rho, \mathbf{v} \in C^1((\mathcal{N}_\sigma(\Gamma_{\text{shock}} \cup \Gamma_{\text{sonic}}) \cap \overline{\Omega}) \setminus \{P_3\}) \cap C^{0,1}(\mathcal{N}_\sigma(\Gamma_{\text{shock}} \cup \Gamma_{\text{sonic}}) \cap \overline{\Omega})$ for some $\sigma > 0$;
- (ii) $|\mathbf{v}| \leq C_0$ and $C_0^{-1} \leq \rho \leq C_0$ in Ω for some $C_0 \geq 1$.

Then $\mathbf{v} \notin H^1(\Omega)$.

Since the proof argument is similar to the one for Theorem 4.11, we omit the details.

Fig. 8 Lighthill shock diffraction problem



4.4 Lower Regularity of the Riemann Solutions with Four-Shock Interactions

The final problem is the Riemann problem with four-shock interaction structure for the isentropic Euler system. As discussed in [CCHLW25] for the Riemann problem with four-shock interactions for the potential flow, initially in the x -coordinates, the scale-invariant domains $\Lambda_i \subset \mathbb{R}^2$, for $i = 1, 2, 3, 4$, are defined by

$$\begin{aligned} \Lambda_1 &= \{ \mathbf{x} \in \mathbb{R}^2 : -\theta_{14} < \theta < \theta_{12} \}, & \Lambda_2 &= \{ \mathbf{x} \in \mathbb{R}^2 : \theta_{12} < \theta < \pi - \theta_{32} \}, \\ \Lambda_3 &= \{ \mathbf{x} \in \mathbb{R}^2 : \pi - \theta_{32} < \theta < \pi + \theta_{34} \}, & \Lambda_4 &= \{ \mathbf{x} \in \mathbb{R}^2 : \pi + \theta_{34} < \theta < 2\pi - \theta_{14} \}, \end{aligned}$$

where θ is the polar angle of point $\mathbf{x} \in \mathbb{R}^2$ and the four parameters $\theta_{12}, \theta_{32}, \theta_{34}, \theta_{14} \in (0, \frac{\pi}{2})$. On each Λ_i , suitable constant states (i) with values (ρ_i, \mathbf{v}_i) are given for the Riemann initial data such that any two neighboring states are connected by exactly one planar shock discontinuity, which satisfies the Rankine-Hugoniot conditions and the entropy condition for the isentropic Euler system. As introduced in [CCHLW25], under the symmetry assumption that $\theta_{12} = \theta_{14} = \theta_1$, $\theta_{32} = \theta_{34} = \theta_2$, and states in Λ_2 and Λ_4 are the same constants, and under a structure assumption that one forward shock is generated between states (1) and (j) and one backward shock is generated between states (3) and (j), governed by the Rankine-Hugoniot conditions (2.9) and the entropy condition (2.12), we expect the Riemann problem develops a Riemann solution with the structure as shown in Fig. 9.

On the symmetry line that is the dashed line shown in Fig. 9, the velocity satisfies (2.13) with \mathbf{v} being the vertical direction. At the interaction points P_1 and P_4 , based on an argument similar to the one for the regular shock reflection problem at the reflection point P_0 , with the symmetry line and state (2) in Fig. 9 corresponding to

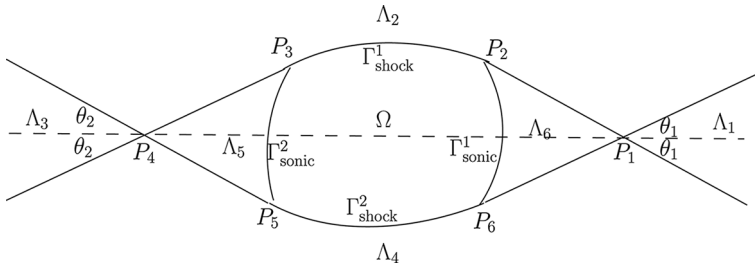


Fig. 9 Riemann problem with four-shock interactions

Γ_{wedge} and state (1) in Fig. 2, there exists a detachment angle $\theta^d \in (0, \frac{\pi}{2})$ depending on the data such that, for $\theta_1, \theta_2 \in (0, \theta_w^d)$, there exist two constant states (5) and (6) in Λ_5 and Λ_6 , respectively, satisfying the Rankine-Hugoniot conditions (2.9) and the entropy condition (2.12). One corresponds to the strong shock and the other to the weak shock. We always select the weak one because it expects to be stable in general. Then there exists a sonic angle $\theta^s \in (0, \theta^d]$ such that the state at P_1 (or P_4) is pseudo-supersonic if $\theta_1 \in (0, \theta^s)$ (or $\theta_2 \in (0, \theta^s)$) and is pseudo-subsonic if $\theta_1 \in (\theta^s, \theta^d]$ (or $\theta_2 \in (\theta^s, \theta^d]$).

When $\theta_1 \in (0, \theta^s)$ at P_1 , there exist points P_2 and P_6 on Γ_{sonic}^1 (the sonic curve of state (6)) such that shocks P_1P_2 and P_1P_6 are straight shocks and the state in Λ_6 with boundaries P_1P_2 , Γ_{sonic}^1 , and P_1P_6 is the constant state (6). Similarly, when $\theta_2 \in (0, \theta^s)$ at P_4 , there exist points P_3 and P_5 on Γ_{sonic}^2 (the sonic curve of state (5)) such that shocks P_3P_4 and P_4P_5 are straight shocks and the state in Λ_5 with boundaries P_3P_4 , Γ_{sonic}^2 , and P_4P_5 is the constant state (5). On the other hand, when $\theta_1 \in [\theta^s, \theta^d]$ at P_1 (or when $\theta_2 \in [\theta^s, \theta^d]$ at P_4), the sonic arc, the straight shocks, and Λ_6 (or Λ_5) disappear near point P_1 (or P_4), and state (6) (or state (5)) is the limit of solution (ρ, \mathbf{v}) in Ω as the point tends to P_1 (or P_4). It is direct to see that

$$\text{The tangents to } \Gamma_{\text{shock}}^1 \text{ at } P_2 \text{ and } P_3 \text{ are not parallel to each other,} \quad (4.6)$$

where P_2 is replaced by P_1 if state (6) is subsonic and P_3 is replaced by P_4 if state (5) is subsonic.

Let Λ_i for $i = 1, \dots, 6$, if they exist, and let Ω be defined as shown in Fig. 9. Let $\Lambda = \mathbb{R}^2$. An entropy solution (ρ, \mathbf{v}) of the boundary value problem in Λ in the sense of Definition 2.2 is called a regular Riemann solution with four-shock interactions for the isentropic Euler system if (ρ, \mathbf{v}) satisfies the following further properties:

- (i) The shock curves $\Gamma_{\text{shock}}^1 = P_2P_3$ and $\Gamma_{\text{shock}}^2 = P_5P_6$ are C^1 up to their endpoints and are C^2 in their relative interiors. Their tangents satisfy (4.6).
- (ii) (ρ, \mathbf{v}) is continuous in $\bar{\Omega} \cap \mathcal{N}_r(\Gamma_{\text{shock}}^1 \cup \Gamma_{\text{sonic}}^1 \cup \Gamma_{\text{shock}}^2 \cup \Gamma_{\text{sonic}}^2)$ for some $r > 0$.
- (iii) The solution coincides with the corresponding constant states in their respective regions Λ_i , for $i = 1, \dots, 6$, if they exist: Specifically,

$$(\rho, \mathbf{v}) = \begin{cases} (\rho_1, \mathbf{v}_1) & \text{in } \Lambda_1, \\ (\rho_2, \mathbf{v}_2) & \text{in } \Lambda_2, \\ (\rho_3, \mathbf{v}_3) & \text{in } \Lambda_3, \\ (\rho_2, \mathbf{v}_2) & \text{in } \Lambda_4, \\ (\rho_5, \mathbf{v}_5) & \text{in } \Lambda_5 \text{ if state (5) at } P_4 \text{ is supersonic,} \\ (\rho_6, \mathbf{v}_6) & \text{in } \Lambda_6 \text{ if state (6) at } P_1 \text{ is supersonic,} \end{cases}$$

where

$$\lim_{\xi \in \Omega, \xi \rightarrow P_4} (\rho, \mathbf{v})(\xi) = (\rho_5, \mathbf{v}_5)(P_4) \quad \text{if state (5) at } P_4 \text{ is subsonic or sonic,}$$

$$\lim_{\xi \in \Omega, \xi \rightarrow P_1} (\rho, \mathbf{v})(\xi) = (\rho_6, \mathbf{v}_6)(P_1) \quad \text{if state (6) at } P_1 \text{ is subsonic or sonic.}$$

(iv) $\mathbf{v} \cdot \nu \leq -C^{-1}$ on $\overline{\Gamma_{shock}^1} \cup \overline{\Gamma_{sonic}^1} \cup \overline{\Gamma_{shock}^2} \cup \overline{\Gamma_{sonic}^2}$ for some $C > 0$, where \mathbf{v} on the curves is taken from the Ω -side and ν is the outer normal with respect to Ω .

(v) The flow is pseudo-subsonic in Ω on and near $\Gamma_{shock}^1 \cup \Gamma_{shock}^2$, except for possible sonic points P_2, P_3, P_5 , and P_6 .

If state (5) at P_4 and state (6) at P_1 are both supersonic, then $M=6, N_1 = 0$, and $N_2 = 4$ with Λ_i (for $i = 1, \dots, 6$) being given as in Fig. 9, $\Gamma_1^{int} = \Gamma_{shock}^1 = P_2P_3$, $\Gamma_2^{int} = \Gamma_{shock}^2 = P_5P_6$, $\Gamma_3^{int} = \Gamma_{sonic}^1 = P_2P_6$, and $\Gamma_4^{int} = \Gamma_{sonic}^2 = P_3P_5$.

If state (5) at P_4 and state (6) at P_1 are both subsonic, then $M=4, N_1 = 0$, and $N_2 = 2$ with Λ_i (for $i = 1, 2, 3, 4$) being given as in Fig. 9, Λ_5 and Λ_6 disappearing, $\Gamma_1^{int} = \Gamma_{shock}^1 = P_1P_4$, and $\Gamma_2^{int} = \Gamma_{shock}^2 = P_4P_1$.

If state (5) at P_4 is supersonic and state (6) at P_1 is subsonic, then $M=5, N_1 = 0$, and $N_2 = 2$ with Λ_i (for $i = 1, \dots, 5$) being given as in Fig. 9, Λ_6 disappearing, $\Gamma_1^{int} = \Gamma_{shock}^1 = P_1P_3$, and $\Gamma_2^{int} = \Gamma_{shock}^2 = P_5P_1$, and $\Gamma_3^{int} = \Gamma_{sonic}^2 = P_3P_5$.

If state (5) at P_4 is subsonic and state (6) at P_1 is supersonic, then $M=5, N_1 = 0$ and $N_2 = 3$ with Λ_i (for $i = 1, \dots, 6$) being given as in Fig. 9, Λ_5 disappearing, $\Gamma_1^{int} = \Gamma_{shock}^1 = P_2P_4$, and $\Gamma_2^{int} = \Gamma_{shock}^2 = P_4P_6$, and $\Gamma_3^{int} = \Gamma_{sonic}^1 = P_2P_6$.

Then, following the proof of Theorem 4.10, we have the following theorem on the lower regularity of the regular Riemann solution, whose proof is omitted since it is similar to the one for Theorem 4.10:

Theorem 4.13 Let (ρ, \mathbf{v}) be a regular Riemann solution with four-shock interactions for the isentropic Euler system and further satisfy:

(i) for some $\sigma > 0, (\rho, \mathbf{v}) \in C^1((\mathcal{N}_\sigma(\Gamma_{shock}^1 \cup \Gamma_{shock}^2 \cup \Gamma_{sonic}^1 \cup \Gamma_{sonic}^2) \cap \overline{\Omega}) \setminus \{P_2, P_3, P_5, P_6\}) \cap C^{0,1}(\mathcal{N}_\sigma(\Gamma_{shock}^1 \cup \Gamma_{shock}^2 \cup \Gamma_{sonic}^1 \cup \Gamma_{sonic}^2) \cap \overline{\Omega})$, where $\Gamma_{sonic}^i = \emptyset$ when it does not exist for $i = 1, 2$;

(ii) $|\mathbf{v}| \leq C_0$ and $C_0^{-1} \leq \rho \leq C_0$ in Ω for some $C_0 \geq 1$.

Then $\mathbf{v} \notin H^1(\Omega)$.

Appendix A. DiPerna-Lions-Type Commutator Estimates

In this appendix, we show the DiPerna-Lions-type commutator estimates (cf. [DL89]), which have been used in the proof of Theorem 3.2 in § 3.

Lemma A.1 Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and $n \geq 2$. For $\varepsilon > 0$, define $\eta_\varepsilon(\mathbf{x}) = \frac{1}{\varepsilon^n} \eta(\frac{\mathbf{x}}{\varepsilon})$ for a mollifier kernel $\eta \in C_c^\infty(\mathbb{R}^n)$ satisfying $\eta(\mathbf{x}) \geq 0$ and $\int_{\mathbb{R}^n} \eta(\mathbf{x}) \, d\mathbf{x} = 1$. Let $b \in W_{loc}^{1,p}(\Omega)$ and $u \in L_{loc}^q(\Omega)$ for $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} \leq 1$. Then, for any $i = 1, \dots, n$,

$$\partial_{x_i}((bu)_\varepsilon - b_\varepsilon u_\varepsilon) \rightarrow 0 \quad \text{in } L_{loc}^1(\Omega) \text{ as } \varepsilon \rightarrow 0^+, \tag{A1}$$

where we have used the notation $F_\varepsilon := F * \eta_\varepsilon$.

Proof Let $\text{supp}(\eta) \subset B_M$. Denote

$$A_\varepsilon[u, b] := \partial_{x_i}((bu)_\varepsilon - b_\varepsilon u_\varepsilon).$$

We divide the proof into two steps.

1. We first show that, for all open $\Omega' \Subset \Omega$ and $\varepsilon \in (0, \frac{1}{2M} \text{dist}(\Omega', \partial\Omega))$,

$$\|A_\varepsilon[u, b]\|_{L^1(\Omega')} \leq C(n, \eta, \Omega') \|\nabla b\|_{L^p(\Omega)} \|u\|_{L^q(\Omega)}. \tag{A2}$$

Notice that

$$\begin{aligned} A_\varepsilon[u, b] &= \partial_{x_i}((bu)_\varepsilon - bu_\varepsilon) + \partial_{x_i}((b_\varepsilon - b)u_\varepsilon) \\ &= (\partial_{x_i}(bu)_\varepsilon - b\partial_{x_i}u_\varepsilon) - b_{x_i}u_\varepsilon + u_\varepsilon\partial_{x_i}(b_\varepsilon - b) + (b_\varepsilon - b)\partial_{x_i}u_\varepsilon \\ &=: \sum_{m=1}^4 I_{\varepsilon,m}. \end{aligned}$$

We now show that, for $\varepsilon \in (0, \frac{1}{2M} \text{dist}(\Omega', \partial\Omega))$,

$$\|I_{\varepsilon,m}\|_{L^1(\Omega')} \leq C(n, \eta, \Omega') \|\nabla b\|_{L^p(\Omega)} \|u\|_{L^q(\Omega)} \quad \text{for } m = 1, \dots, 4. \tag{A3}$$

First, we estimate $I_{\varepsilon,1} = \partial_{x_i}(bu)_\varepsilon - b\partial_{x_i}(u)_\varepsilon$. For $\mathbf{x} \in \Omega'$,

$$\begin{aligned} I_{\varepsilon,1}(\mathbf{x}) &= \int_{\mathbb{R}^n} \partial_{x_i}(\eta_\varepsilon(\mathbf{x} - \mathbf{y})) (b(\mathbf{y}) - b(\mathbf{x}))u(\mathbf{y}) \, d\mathbf{y} \\ &= \int_{B_M} \eta_{x_i}(\mathbf{y}) \frac{b(\mathbf{x} - \varepsilon\mathbf{y}) - b(\mathbf{x})}{\varepsilon} u(\mathbf{x} - \varepsilon\mathbf{y}) \, d\mathbf{y}. \end{aligned}$$

Then we use the estimate of the difference quotient for a Sobolev function to obtain that, for any $\varepsilon \in (0, \frac{1}{2M} \text{dist}(\Omega', \partial\Omega))$,

$$\left\| \frac{b(\cdot - \varepsilon \mathbf{y}) - b(\cdot)}{\varepsilon} \right\|_{L^p(\Omega')} \leq C(n, \Omega') \|\nabla b\|_{L^p(\Omega)} |\mathbf{y}| \quad \text{for each } \mathbf{y} \in B_M.$$

Noting that $\|u(\cdot - \varepsilon \mathbf{y})\|_{L^q(\Omega')} \leq \|u\|_{L^q(\Omega)}$ for \mathbf{y} and ε as above, we have

$$\|I_{\varepsilon,1}\|_{L^1(\Omega')} \leq C(n, \Omega') \|\nabla b\|_{L^p(\Omega)} \|u\|_{L^q(\Omega)} \int_{B_M} |\eta_{x_i}(\mathbf{y})| \, d\mathbf{y},$$

which is (A3) for $m = 1$.

For $I_{\varepsilon,2} = -b_{x_i} u_\varepsilon$ and $I_{\varepsilon,3} = u_\varepsilon \partial_{x_i}(b_\varepsilon - b)$, (A3) follows from the standard estimates.

Finally, we estimate $I_{\varepsilon,4} = (b_\varepsilon - b) \partial_{x_i} u_\varepsilon$. For $\mathbf{x} \in \Omega'$,

$$\begin{aligned} I_{\varepsilon,4}(\mathbf{x}) &= \int_{\mathbb{R}^n \times \mathbb{R}^n} \eta_\varepsilon(\mathbf{x} - \mathbf{y}) \partial_{x_i}(\eta_\varepsilon(\mathbf{x} - \mathbf{z})) (b(\mathbf{y}) - b(\mathbf{x})) u(\mathbf{z}) \, d\mathbf{y} d\mathbf{z} \\ &= \int_{B_M \times B_M} \eta(\mathbf{y}) \eta_{x_i}(\mathbf{z}) \frac{b(\mathbf{x} - \varepsilon \mathbf{y}) - b(\mathbf{x})}{\varepsilon} u(\mathbf{x} - \varepsilon \mathbf{z}) \, d\mathbf{y} d\mathbf{z}. \end{aligned}$$

Then we complete the proof of (A3) for $I_{\varepsilon,4}$ similar as for $I_{\varepsilon,1}$ above, by using the L^p -estimate of the difference quotient for the b -term, and the L^q -estimate of the shift for the u -term. Then estimate (A2) holds.

2. Now we prove the convergence (A1). We first note that (A1) holds if u has the higher regularity $u \in W_{\text{loc}}^{1,q}(\Omega)$ and b as assumed. Indeed, in this case, $\partial_{x_i}(bu) \in L_{\text{loc}}^1(\Omega)$. Since Ω is bounded,

$$\begin{aligned} A_\varepsilon[u, b] &= (\partial_{x_i}(bu)) * \eta_\varepsilon - (\partial_{x_i} b)_\varepsilon u_\varepsilon - b_\varepsilon (\partial_{x_i} u)_\varepsilon \\ &\rightarrow \partial_{x_i}(bu) - \partial_{x_i} b u - b \partial_{x_i} u = 0 \quad \text{in } L_{\text{loc}}^1(\Omega) \text{ as } \varepsilon \rightarrow 0. \end{aligned}$$

For general u and b as assumed, the same convergence is obtained by approximation, and using (A2), via a standard argument that we briefly describe now. First, let $p \in [1, \infty)$. To include the case that $q = \infty$, we argue as follows: Choose open $\Omega' \Subset \Omega$, then choose open Ω'' such that $\Omega' \Subset \Omega'' \Subset \Omega$. Let $(b_k, u_k) \in C^\infty(\Omega)$ be such that $b_k \rightarrow b$ in $W_{\text{loc}}^{1,p}(\Omega)$ and $u_k \rightarrow u$ in $L_{\text{loc}}^1(\Omega)$. Replacing $\{u_k\}$ by its subsequence if necessary, we obtain

$$\|\nabla b_k\|_{L^\infty(\Omega'')} \|u - u_k\|_{L^1(\Omega'')} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \tag{A4}$$

Then, for each $\varepsilon \in (0, \frac{1}{2} \text{dist}(\Omega', \partial\Omega''))$, using the bi-linearity of $A_\varepsilon[\cdot, \cdot]$ and estimate (A2) on sets $\Omega' \Subset \Omega''$ for (p, q) and $(\infty, 1)$ respectively, we have

$$\begin{aligned}
& \|A_\varepsilon[u, b]\|_{L^1(\Omega')} \\
&= \|A_\varepsilon[u_k, b_k] + A_\varepsilon[u, b - b_k] + A_\varepsilon[u - u_k, b_k]\|_{L^1(\Omega')} \\
&\leq \|A_\varepsilon[u_k, b_k]\|_{L^1(\Omega')} + C(\|\nabla b - \nabla b_k\|_{L^p(\Omega'')} \|u\|_{L^q(\Omega'')} + \|\nabla b_k\|_{L^\infty(\Omega'')} \|u - u_k\|_{L^1(\Omega'')}).
\end{aligned}$$

Recalling that $\|A_\varepsilon[u_k, b_k]\|_{L^1(\Omega')} \rightarrow 0$ as $\varepsilon \rightarrow 0$ for each k and using (A4), we obtain that $\|A_\varepsilon[u, b]\|_{L^1(\Omega')} \rightarrow 0$ as $\varepsilon \rightarrow 0$. In the remaining case $(p, q) = (\infty, 1)$, we argue similarly, interchanging p and q (resp. b and u), and, instead of (A4), we replace $\{b_k\}$ by its subsequence if necessary to obtain

$$\|\nabla b - \nabla b_k\|_{L^1(\Omega'')} \|u_k\|_{L^\infty(\Omega'')} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Then we modify the rest of the argument correspondingly to conclude the proof. \square

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Declarations

Conflict of Interest The authors declare no competing interests.

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