

Online Appendix for

“The unbearable lightness of equilibria in a low interest rate environment”

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A.1 Derivation of results in Subsection 2.1

Proof of Proposition 1. Let $M^t = (M_t, \dots, M_0)$ denote the history of M_t . We consider fundamental solutions $f_{\pi_t}(M^t)$. Let $M_L^t = \left(\frac{e^{-rL}}{r}, \dots, \frac{e^{-rL}}{r}\right)$ denote a path along which M_t is in the transitory state. It follows that (with slight abuse of notation)

$$E\left(\frac{M_{t+1}}{\pi_{t+1}} \middle| M_L^t\right) = p \frac{\frac{e^{-rL}}{r}}{f_{\pi_{t+1}}\left(\frac{e^{-rL}}{r}, M_L^t\right)} + (1-p) \frac{r^{-1}}{f_{\pi_{t+1}}(r^{-1}, M_L^t)}. \quad (\text{A1})$$

Next, if $M_{t+1} = r^{-1}$, we have $E\left(\frac{M_{t+2}}{\pi_{t+2}} \middle| M_{t+1} = r^{-1}, M_L^t\right) = \frac{r^{-1}}{f_{\pi_{t+2}}(r^{-1}, r^{-1}, M_L^t)}$, so that after taking logs and re-arranging, (3) becomes

$$f_{\hat{\pi}_{t+2}}(0, 0, \hat{M}_L^t) = \max \left\{ -\mu, \psi f_{\hat{\pi}_{t+1}}(0, \hat{M}_L^t) \right\}, \quad (\text{A2})$$

where $f_{\hat{\pi}_t}(\cdot) := \log f_{\pi_t}(\cdot) - \log(\pi_*)$, $\mu := \log(r\pi_*)$, and $\hat{M}_t := \log M_t + \log r$, the latter being in log-deviation from its absorbing state. We have already established the support restriction $r\pi_* \geq 1$ in the main text after Proposition 1, which means $\mu \geq 0$. Because

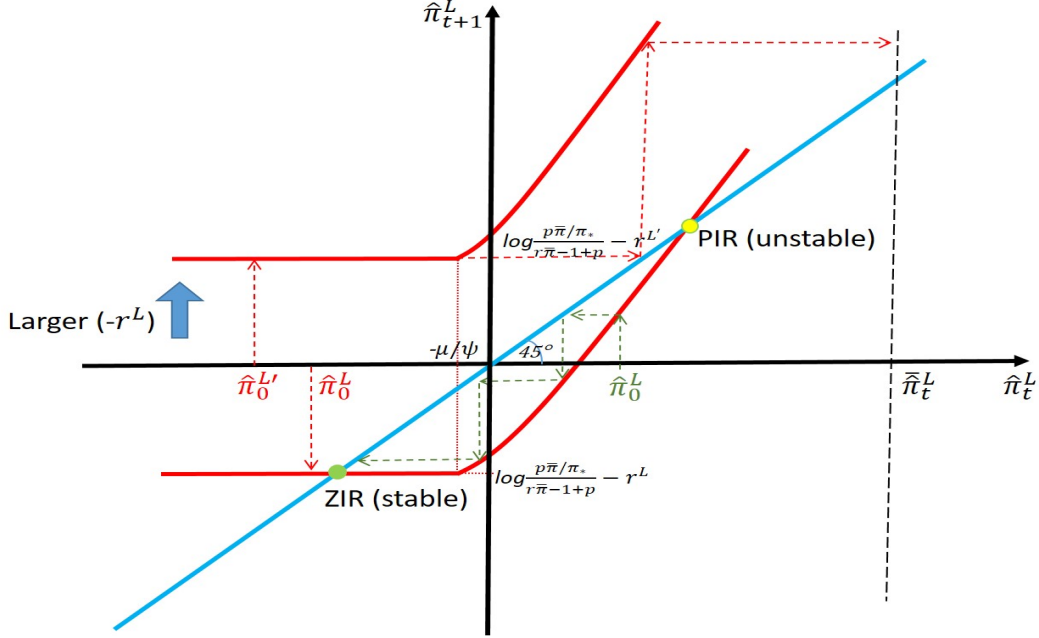


Figure A.9: Plot of (A4) for different values of $-r^{L'} > -r^L > 0$.

$\psi > 1$, the difference equation (A2) has two steady states, $-\mu$ and 0 , corresponding to ZIR and PIR, respectively. Moreover, the ZIR steady state is stable, while the PIR is unstable. Therefore, for stable equilibria we must have that $f_{\hat{\pi}_{t+1}}(0, \hat{M}_L^t) \leq 0$, for if $f_{\hat{\pi}_{t+1}}(0, \hat{M}_L^t) > 0$, $f_{\hat{\pi}_{t+s}}(0_{s \times 1}, \hat{M}_L^t)$ will grow exponentially without bound. So, a stable fundamental solution must have $f_{\hat{\pi}_{t+1}}(0, \hat{M}_L^t) \leq 0$, or equivalently $f_{\pi_{t+1}}(r^{-1}, M_L^t) \leq \pi_*$.

Setting $f_{\pi_{t+1}}(r^{-1}, M_L^t) = \bar{\pi} \leq \pi_*$ in (A1), substituting for $E\left(\frac{M_{t+1}}{\pi_{t+1}} \middle| M_L^t\right)$ in (3) and rearranging yields

$$\pi_{t+1}^L = \frac{\bar{\pi} p \max \left\{ 1, r \pi_* \left(\pi_t^L / \pi_* \right)^\psi \right\} e^{-r^L}}{r \bar{\pi} - (1-p) \max \left\{ 1, r \pi_* \left(\pi_t^L / \pi_* \right)^\psi \right\}}, \quad \pi_t^L < \pi_* \left(\frac{\bar{\pi} / \pi_*}{1-p} \right)^{1/\psi}, \quad (\text{A3})$$

where $\pi_t^L := f_{\pi_t}(M_L^t)$, for compactness of notation, and the bound on π_t^L is required for π_{t+1}^L to be positive. Take logs and define $\hat{\pi}_t^L := \log \pi_t^L - \log \pi_*$, then (A3) can be written as

$$\hat{\pi}_{t+1}^L = \begin{cases} \log \frac{p\bar{\pi}/\pi_*}{r\bar{\pi}-1+p} - r^L, & \hat{\pi}_t^L \leq -\frac{\mu}{\psi} \\ \log \frac{p\bar{\pi}/\pi_*}{\bar{\pi}/\pi_*(1-p)e^{\psi \hat{\pi}_t^L}} + \psi \hat{\pi}_t^L - r^L, & -\frac{\mu}{\psi} < \hat{\pi}_t^L < \bar{\hat{\pi}}_t^L = \frac{\log \bar{\pi}/\pi_* - \log(1-p)}{\psi}. \end{cases} \quad (\text{A4})$$

Figure A.9 plots (A4) against $\hat{\pi}_t^L$ together with the 45° line. We distinguish two cases. The first case is when the curve intersects with the 45° line, so that (A4) has two (generic) steady states. This happens when the kink in (A4) is below the 45° line. Noting that the kink is given by $(\hat{\pi}_t^L = -\frac{\mu}{\psi}; \hat{\pi}_{t+1}^L = \log \frac{p\bar{\pi}/\pi_*}{r\bar{\pi}-1+p} - r^L)$, then the condition $\hat{\pi}_t^L > \hat{\pi}_{t+1}^L$ becomes

$$-r^L \leq -\frac{\mu}{\psi} - \log \frac{p\bar{\pi}/\pi_*}{r\bar{\pi}-1+p} \leq -\frac{\log(r\pi_*)}{\psi} - \log \frac{p}{r\pi_*-1+p}, \quad (\text{A5})$$

where the second inequality holds because $-\log \frac{p\bar{\pi}/\pi_*}{r\bar{\pi}-1+p}$ is an increasing function of $\bar{\pi} \leq \pi_*$ and the definition of $\mu = -\log(r\pi_*)$. When the restriction (A5) on the support of the shock holds, then there clearly exist stable solutions to the model for arbitrary initial conditions $\hat{\pi}_0^L \leq \hat{\pi}^{PIR}$, where $\hat{\pi}^{PIR}$ is the high-inflation fixed point of the difference equation (A4).

Now consider the case when the support restriction (A5) does not hold. In this case, for any initial value $\hat{\pi}_0^L$ the solution of the difference equation (A4) will move along an explosive path while $\hat{\pi}_t^L$ is less than $\frac{\log \bar{\pi}/\pi_* - \log(1-p)}{\psi}$, and will eventually break down after a finite number of periods.

Finally, note how the transitory state resembles the simple case of the absorbing state in the main text, and Figure A.9 parallels Figure 1. At $\bar{\pi} = \pi_*$, the support restrictions simply implies that $\hat{\pi}^{ZIR} = \ln \frac{p}{r\pi_*-1+p} - r^L < -\frac{\mu}{\psi} < 0$. So, for an equilibrium to exist the intercept of the ZIR part of the red line must be negative, as in Figure 1. \square

Proof of Proposition 2. Sunspot solutions π_t may depend on ς_t and its lags. It is assumed that ς_t follows a first-order Markov chain, and so we may denote by π_t^ς the two different values that π_t can take depending on the outcome of the sunspot shock.¹ Letting $q_\varsigma := \Pr(\varsigma_{t+1} = 1 | \varsigma_t = \varsigma)$, (3) becomes

$$1 = \max \left\{ r^{-1}, \pi_* \left(\frac{\pi_t^{\varsigma_t}}{\pi_*} \right)^\psi \right\} \left(\frac{1 - q_{\varsigma_t}}{\pi_{t+1}^0} + \frac{q_{\varsigma_t}}{\pi_{t+1}^1} \right) \quad \varsigma_t = 0, 1. \quad (\text{A6})$$

This is a system of nonlinear difference equations in π_t^ς .

First, consider the case in which at least one of the initial values $\pi_t^{\varsigma_t}$ corresponds to a ZIR, which, wlog, we can set as $(\pi_t^0/\pi_*)^\psi \leq (r\pi_*)^{-1}$, since the labelling of ς_t is arbitrary. Under

¹These values may also vary over t if the solution is history dependent, which we do not rule out.

this assumption, (A6) yields $r = \left(\frac{1-q_0}{\pi_{t+1}^0} + \frac{q_0}{\pi_{t+1}^1} \right)$, which we can solve for π_{t+1}^0 and substitute back into (A6) with $\varsigma_t = 1$ to get

$$\pi_{t+1}^1 = \frac{\max \left\{ r^{-1}, \pi_* (\pi_t^1 / \pi_*)^\psi \right\} (q_1 - q_0)}{1 - q_0 - r \max \left\{ r^{-1}, \pi_* (\pi_t^1 / \pi_*)^\psi \right\} (1 - q_1)}.$$

This has almost exactly the same shape as (A3) that is plotted in Figure A.9. Hence, the same argument as above establishes the support restriction $r^{-1} \leq \pi_*$.

Second, suppose $\pi_t^{\varsigma_t}$ corresponds to a PIR for both ς_t , i.e., $(\pi_t^{\varsigma_t} / \pi_*)^\psi > (r\pi_*)^{-1}$. By the argument in the previous paragraph, if at any future date $\pi_{t+j}^{\varsigma_{t+j}}$ is a ZIR, then the support restriction for coherency $(r\pi_*)^{-1} \leq 1$ applies. So, the only case to consider is when $(\pi_t^{\varsigma_t} / \pi_*)^\psi > (r\pi_*)^{-1}$ for all t , i.e., the economy is always at a PIR. In this case, (A6) becomes $1 = (\pi_t / \pi_*)^\psi E_t (\pi_* / \pi_{t+1})$, with the additional restriction $\pi_t > \pi_* (r\pi_*)^{-1/\psi}$ for all t . Because $\psi > 1$, this equation has the unique stable solution $\pi_t = \pi_*$ for all t if and only if $r^{-1} \leq \pi_*$. \square

A.2 Derivation of results in Subsection 2.2

A.2.1 Coefficients of the canonical form

Coefficients in Example NK-TR

$$\begin{aligned} A_0 &= \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & -\lambda \\ \sigma\psi & 1 + \sigma\psi_x \end{pmatrix}, \quad B_0 = B_1 = \begin{pmatrix} -\beta & 0 \\ -\sigma & -1 \end{pmatrix}, \\ C_0 &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -\sigma\mu \end{pmatrix}, \quad C_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & \sigma & 0 \end{pmatrix}, \quad D_0 = D_1 = 0_{2 \times 4}, \end{aligned}$$

$a = (\psi, \psi_x)'$, $b = (0, 0)'$, $c = (0, 0, 1, \mu)'$ and $d = 0_{4 \times 1}$. \square

Coefficients in Example NK-OP

$$A_0 = \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & -\lambda \\ \frac{\lambda}{\gamma} & 1 \end{pmatrix}, \quad B_0 = \begin{pmatrix} -\beta & 0 \\ -\sigma & -1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} -\beta & 0 \\ 0 & 0 \end{pmatrix},$$

$$C_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -\sigma\mu \end{pmatrix}, \quad C_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D_0 = D_1 = 0_{2 \times 3},$$

$a = (0, -\sigma^{-1})'$, $b = (1, \sigma^{-1})'$, $c = (0, \sigma^{-1}, \mu)'$ and $d = 0_{3 \times 1}$. □

A.2.2 Proof of Proposition 3

In preparation for the proof of Proposition 3, we first establish a result that will be used in the proofs of both Propositions 3 and 7.

Proposition 9. *The NK-TR model given by (6) with $u_t = \nu_t = 0$ and ϵ_t a two-state Markov Chain with transition Kernel $K = \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix}$ can be written in the form $F(\mathbf{Y}) = \kappa(\mathbf{X})$, where \mathbf{Y} is a 2×1 vector containing the values of $\hat{\pi}_t$ in each of the two states, and $F(\cdot)$ is the piecewise linear function (9) with*

$$\begin{aligned} \mathcal{A}_{J_1} &= Q + \lambda\sigma \left(\psi I - \frac{\psi_x}{\lambda} (I - \beta K) \right), & J_1 &= \{1, 2\} & (PIR, PIR) \\ \mathcal{A}_{J_2} &= Q + \lambda\sigma \left(\psi I - \frac{\psi_x}{\lambda} (I - \beta K) \right) e_2 e_2', & J_2 &= \{2\} & (ZIR, PIR) \\ \mathcal{A}_{J_3} &= Q + \lambda\sigma \left(\psi I - \frac{\psi_x}{\lambda} (I - \beta K) \right) e_1 e_1', & J_3 &= \{1\} & (PIR, ZIR) \\ \mathcal{A}_{J_4} &= Q, & J_4 &= \emptyset & (ZIR, ZIR). \end{aligned} \tag{A7}$$

where e_i is the unit vector with 1 in position i ,

$$Q := I - K - \beta(I - K)K - \lambda\sigma K \tag{A8}$$

and

$$\begin{aligned} \det \mathcal{A}_{J_1} &= \sigma^2 \lambda^2 \left(\psi - 1 + \frac{\beta-1}{\lambda} \psi_x \right) \left(\psi - \psi_{p,q,\beta,\sigma\lambda} - \psi_x \frac{(1+\beta(1-p-q))}{\lambda} \right), \\ \det \mathcal{A}_{J_2} &= -\sigma^2 \lambda^2 \psi_{p,q,\beta,\sigma\lambda} \left(\psi - 1 + \frac{\beta-1}{\lambda} \psi_x \right) + \sigma(1-q) \left[\sigma\lambda\beta\psi_x \right. \\ &\quad \left. + \lambda(\beta(p+q-1) - 1 - \sigma\lambda) \left(\frac{\beta-1}{\lambda} \psi_x + \psi \right) \right], \\ \det \mathcal{A}_{J_3} &= -\sigma^2 \lambda^2 \left(\psi - \psi_{p,q,\beta,\sigma\lambda} - \psi_x \frac{(1+\beta(1-p-q))}{\lambda} \right) \\ &\quad - \sigma(1-q) \left[(1 - (p+q)\beta + \sigma\lambda - \beta^2(1-p-q))\psi_x \right. \\ &\quad \left. - \lambda\psi(1 + \sigma\lambda + \beta(1-p-q)) \right], \\ \det \mathcal{A}_{J_4} &= \sigma^2 \lambda^2 \psi_{p,q,\beta,\sigma\lambda}. \end{aligned} \tag{A9}$$

where $\psi_{p,q,\beta,\sigma\lambda}$ is given in (16).

Proof. Collect the $k = 2$ states of ϵ_t in the vector $\epsilon = (\epsilon^1, \epsilon^2)'$ and denote the corresponding states of $\hat{\pi}_t, \hat{x}_t, \hat{R}_t$ along a MSV solution by 2-dimensional vectors $\hat{\pi}, \hat{x}$ and \hat{R} , respectively, where $y = f(\epsilon)$ for some function $f(\cdot)$, and for each $y \in \{\hat{\pi}, \hat{x}, \hat{R}\}$. Because the dynamics are exogenous and determined completely by K , we have $E(y_{t+1}|\epsilon_t = \epsilon^i) = e_i'Ky$. Stacking the two conditioning states, we can write, with slight abuse of notation, $y_{t+1|t} = K\epsilon$. Substituting into (6a) with $u_t = 0$, we obtain

$$\hat{\pi} = \beta \overbrace{K\hat{\pi}}^{\hat{\pi}_{t+1|t}} + \lambda\hat{x}. \quad (\text{A10})$$

Similarly, from (6b) we obtain

$$\hat{x} = \overbrace{K\hat{x}}^{\hat{x}_{t+1|t}} - \sigma(\hat{R} - K\hat{\pi}) + \epsilon. \quad (\text{A11})$$

Combining the above two equations, we obtain

$$(I - K)\hat{\pi} = \beta(I - K)K\hat{\pi} - \lambda\sigma(\hat{R} - K\hat{\pi}) + \lambda\epsilon.$$

Substituting for $\hat{R} = \max\{-\mu\epsilon_2, \psi\hat{\pi} + \psi_x\hat{x}\}$, obtained from (6c) with $\nu_t = 0$, and for $\hat{x} = \lambda^{-1}(I - \beta K)\hat{\pi}$, and rearranging we get:

$$Q\hat{\pi} = -\lambda\sigma \max\left\{-\mu\epsilon_2, \left(\psi I - \frac{\psi_x}{\lambda}(I - \beta K)\right)\hat{\pi}\right\} + \lambda\epsilon. \quad (\text{A12})$$

This yields (A7). The determinants (A9) were derived using straightforward algebraic calculations (performed using Scientific Workplace). \square

Proof of Proposition 3. Setting $\psi_x = 0$ in (A9), we obtain

$$\det \mathcal{A}_{J_1} = \sigma^2 \lambda^2 (\psi - 1) (\psi - \psi_{p,q,\beta,\sigma\lambda}) > 0.$$

Since $\psi_{p,q,\beta,\sigma\lambda} \leq 1$, $\det \mathcal{A}_{J_1} > 0$, so coherency requires $\psi_{p,q,\beta,\sigma\lambda} > 0$ for $\det \mathcal{A}_{J_4} > 0$ from

(A9). However, in that case, from (A9) we get

$$\det \mathcal{A}_{J_2} = -\sigma^2 \lambda^2 \left(\psi_{p,q,\beta,\sigma\lambda} (\psi - 1) + \psi (1 - q) \left(1 + \frac{1 - \beta(p + q - 1)}{\sigma\lambda} \right) \right) < 0$$

because $\beta(p + q - 1) < 1$, violating the CC condition in the GLM Theorem. \square

Extension to $\psi_x \neq 0$ In this case, the Taylor principle becomes

$$\psi + \frac{\beta - 1}{\lambda} \psi_x > 1. \quad (\text{A13})$$

It is straightforward to show that the CC condition fails when (A13) holds for the absorbing case $q = 1$. Using this constraint in (A9), we obtain

$$\det \mathcal{A}_{J_2} = -\det \mathcal{A}_{J_4} \left(\psi + \frac{\beta - 1}{\lambda} \psi_x - 1 \right).$$

Thus, the two determinants must have opposite sign, violating the CC condition in the GLM Theorem.

It seems too complicated to prove this result analytically for $q < 1$, but we have verified it numerically for all the parametrizations we considered (see the replication code provided). \square

A.2.3 Proof of Proposition 4

Next, we establish a result that will be used in the proof of Propositions 4.

Proposition 10. *The NK-OP model given by (6) with (6c) replaced by (7) with $u_t = \nu_t = \psi_x = 0$ and ϵ_t a two-state Markov Chain with transition Kernel $K = \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix}$ can be written in the form (9) with*

$$\begin{aligned} \mathcal{A}_{J_1} &= \left(1 + \frac{\lambda^2}{\gamma} \right) I - \beta K, & J_1 &= \{1, 2\} \\ \mathcal{A}_{J_2} &= I - \beta K - e_1 e'_1 (K(I - \beta K) + \lambda \sigma K) + \frac{\lambda^2}{\gamma} e_2 e'_2, & J_2 &= \{2\} \\ \mathcal{A}_{J_3} &= I - \beta K - e_2 e'_2 (K(I - \beta K) + \lambda \sigma K) + \frac{\lambda^2}{\gamma} e_1 e'_1, & J_3 &= \{1\} \\ \mathcal{A}_{J_4} &= Q & J_4 &= \emptyset \end{aligned} \quad (\text{A14})$$

and

$$\begin{aligned}
\det \mathcal{A}_{J_1} &= \frac{(\gamma(1-\beta)+\lambda^2)(\gamma(1+(1-p-q)\beta)+\lambda^2)}{\gamma^2}, \\
\det \mathcal{A}_{J_2} &= -\frac{(\gamma(1-\beta)+\lambda^2)(\sigma\lambda\psi_{p,q,\beta,\sigma\lambda}+(1-q)(1+(1-p-q)\beta))+\sigma\lambda(1-q)(\gamma+\lambda^2)}{\gamma}, \\
\det \mathcal{A}_{J_3} &= -\frac{(\gamma(1-\beta)+\lambda^2)(\sigma\lambda\psi_{p,q,\beta,\sigma\lambda}+(1-p)(1+(1-p-q)\beta))+\sigma\lambda(1-p)(\gamma+\lambda^2)}{\gamma}, \\
\det \mathcal{A}_{J_4} &= \sigma^2\lambda^2\psi_{p,q,\beta,\sigma\lambda}.
\end{aligned} \tag{A15}$$

Proof. From (6b) and (7) we obtain

$$\hat{x} = \begin{cases} K\hat{x} - \sigma(-\mu - K\hat{\pi}) + \epsilon, & \text{if } K\hat{\pi} + \frac{1}{\sigma}(K\hat{x} - \hat{x} + \epsilon) \leq -\mu \quad (\text{ZIR}) \\ -\frac{\lambda}{\gamma}\hat{\pi}, & \text{if } K\hat{\pi} + \frac{1}{\sigma}(K\hat{x} - \hat{x} + \epsilon) > -\mu \quad (\text{PIR}) \end{cases} \tag{A16}$$

where the inequalities are element-wise. Substituting for \hat{x} using (A10) yields

$$(I - \beta K)\hat{\pi} = \begin{cases} K(I - \beta K)\hat{\pi} - \lambda\sigma(-\mu - K\hat{\pi}) + \lambda\epsilon, & (\text{ZIR}) \\ -\frac{\lambda^2}{\gamma}\hat{\pi}, & (\text{PIR}) \end{cases}$$

where ZIR occurs if and only if $K\hat{\pi} + \frac{1}{\lambda\sigma}((K - I)(I - \beta K)\hat{\pi} + \lambda\epsilon) \leq -\mu$ (element-wise).

Thus, for PIR,PIR we have

$$\mathcal{A}_{J_1} = \left(1 + \frac{\lambda^2}{\gamma}\right)I - \beta K$$

For ZIR,PIR, we have

$$\mathcal{A}_{J_2} = I - \beta K - e_1 e_1' (K(I - \beta K) + \lambda\sigma K) + \frac{\lambda^2}{\gamma} e_2 e_2',$$

and PIR,ZIR can be obtained symmetrically. For ZIR,ZIR, we have

$$\mathcal{A}_{J_4} = I - \beta K - (K(I - \beta K) + \lambda\sigma K) = Q.$$

This yields (A7). Finally, it is straightforward to verify (A15). \square

Proof of Proposition 4. First, observe that $\det \mathcal{A}_{J_1} > 0$ holds for all admissible values of the parameters $\beta, p, q \in [0, 1]$, and $\gamma, \lambda > 0$, since $\gamma(1 - \beta) + \lambda^2 > 0$ and $(1 + (1 - p - q)\beta) \geq 0$. Therefore, when $\theta > 1$ ($\psi_{p,1,\beta,\sigma\lambda} < 0$), the CC condition cannot hold because $\det \mathcal{A}_{J_4} < 0$. Turning to the case $\theta < 1$ ($\psi_{p,1,\beta,\sigma\lambda} > 0$) we immediately notice that both $\det \mathcal{A}_{J_2}$ and $\det \mathcal{A}_{J_3}$

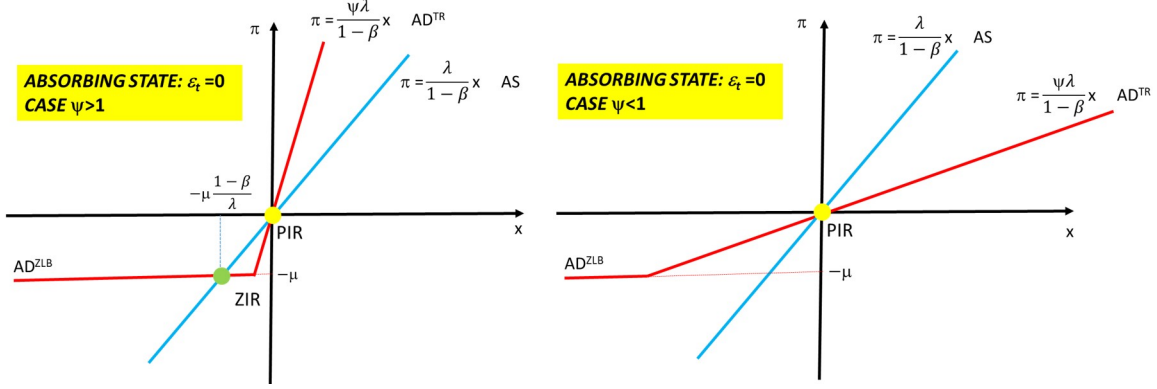


Figure A.10: The absorbing state in the NK-TR model

are negative, since the terms in the numerator of the fractions are all positive. \square

A.2.4 Proof of Proposition 5

We first look at the absorbing (or steady) state, where $\epsilon_t = 0$. Then, we need to solve

$$\hat{\pi} = \frac{\lambda}{1-\beta} \hat{x} \quad AS \quad ; \quad \hat{\pi} = \max \{-\mu, \psi \hat{\pi}\} = \max \begin{cases} \psi \frac{\lambda}{1-\beta} \hat{x} & AD^{TR} \\ -\mu & AD^{ZLB} \end{cases} . \quad (A17)$$

This is depicted in Figure A.10. It is immediately obvious that the necessary support restriction for existence of a solution is $\mu \geq 0$, i.e., $(r\pi_*)^{-1} \leq 1$. When this holds, there are two possible solutions: 1) PIR: $(\hat{\pi}, \hat{x}, \hat{R}) = (0, 0, 0)$; and 2) ZIR: $(\hat{\pi}, \hat{x}, \hat{R}) = (-\mu, -\mu \frac{(1-\beta)}{\lambda}, -\mu)$.

Next, turn to the transitory state. Here, there are four possibilities depending on the value of θ , and the equilibrium in the absorbing state. These are depicted in Figure A.11. The derivations of those cases is as follows.

The temporary state lasts for a random time T , after which the economy jumps to the absorbing state, because the model is completely forward-looking with no endogenous persistence. In the transitory state $\epsilon_t = -\sigma \hat{M}_{t+1|t} = \sigma p r^L < 0$, the equilibrium will be $(\hat{\pi}^L, \hat{x}^L)$ and with probability $(1-p)$ we are back in the absorbing state. The latter can be a PIR one or a ZIR one.

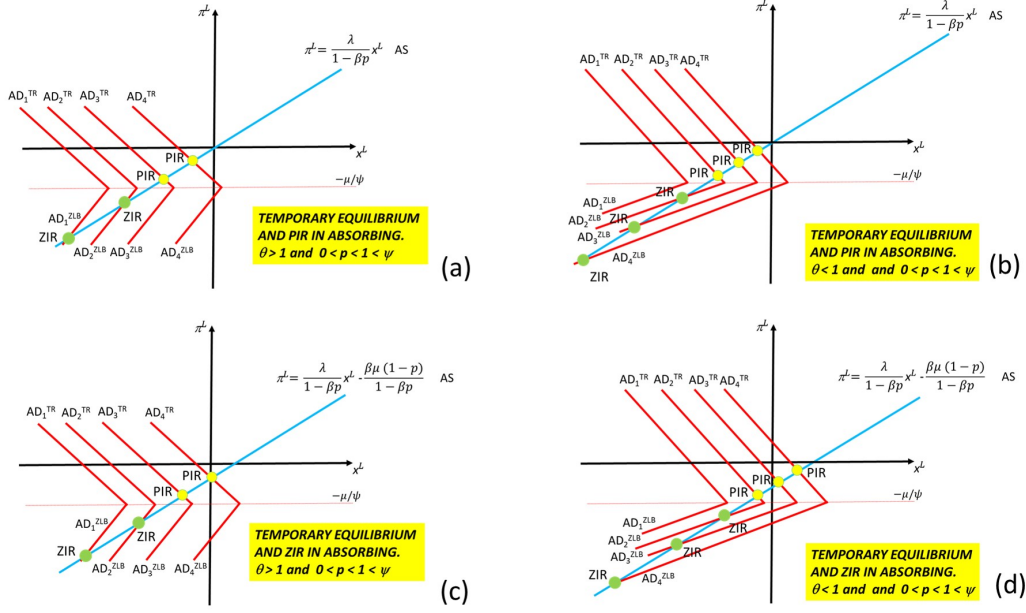


Figure A.11: The temporary state in the NK model when $\psi > 1$.

When the absorbing state is PIR, the system becomes

$$\hat{\pi}^L = \frac{\lambda}{1 - \beta p} \hat{x}^L \quad AS \quad (A18)$$

$$\hat{\pi}^L = \begin{cases} \frac{1-p}{\sigma(p-\psi)} \hat{x}^L + \frac{p(-r^L)}{(p-\psi)} & AD^{TR} \quad \text{for } \pi \geq -\frac{\mu}{\psi} \\ \frac{1-p}{\sigma p} \hat{x}^L - \frac{\mu}{p} + (-r^L) & AD^{ZLB} \quad \text{for } \pi \leq -\frac{\mu}{\psi} \end{cases} \quad (A19)$$

These curves are plotted in the top row of Figure A.11 for the cases $\theta > 1$ on the left, i.e., panel (a), where AS is flatter than AD^{ZLB} , and $\theta < 1$ on the right, i.e., panel (b), where AS is steeper than AD^{ZLB} .

When the absorbing state is ZIR, instead, expectations in the temporary equilibrium are different, so the system to solve for becomes

$$\hat{\pi}^L = \frac{\lambda}{1 - \beta p} \hat{x}^L - \frac{\beta \mu (1-p)}{1 - \beta p} \quad AS \quad (A20)$$

$$\hat{\pi}^L = \begin{cases} \hat{x}^L \frac{(1-p)}{\sigma(p-\psi)} + \frac{p(-r^L)}{p-\psi} + \frac{\mu(1-p)}{p-\psi} \left[\frac{(1-\beta)}{\lambda \sigma} + 1 \right] & AD^{TR} \quad \text{for } \pi \geq -\frac{\mu}{\psi} \\ \hat{x}^L \frac{1-p}{\sigma p} - \frac{\mu}{p} - r^L + \frac{\mu(1-p)}{p} \left[\frac{(1-\beta)}{\lambda \sigma} + 1 \right] & AD^{ZLB} \quad \text{for } \pi \leq -\frac{\mu}{\psi} \end{cases} \quad (A21)$$

These curves are plotted in the bottom row of Figure A.11 for the cases $\theta > 1$ on the left,

i.e., panel (c), where AS is flatter than AD^{ZLB} , and $\theta < 1$ on the right, i.e., panel (d), where AS is steeper than AD^{ZLB} .

Inspection of the graphs on the left of Figure A.11, where $\theta > 1$ for PIR absorbing (panel (a)) and ZIR absorbing (panel (c)) shows there is always a solution in both cases. We therefore conclude that when $\theta > 1$, the only necessary support restriction is $(r\pi_*)^{-1} \leq 1$ for existence of an equilibrium in the absorbing state. This proves (11a).

Next, turn to the case $\theta < 1$. Now it is clear that a further support restriction is needed on the value of the shock in the transitory state. The cutoff can be computed by finding the point where the AD and AS curves intersect at the kink of AD . There are two different points for the cases in Figure A.11: panel (b), PIR absorbing and panel (d), ZIR absorbing. From inspection, it is clear that the former is the least stringent condition, so it suffices to focus on that. Specifically, we equate (A18) with (A19) at $\hat{\pi}^L = -\frac{\mu}{\psi}$ to find the value of the shock $r^L = \bar{r}^L$ such that the equations have a solution for all $-r^L \leq -\bar{r}^L$. Hence, the cutoff can be found by solving:

$$-\frac{\mu}{\psi} \frac{1 - \beta p}{\lambda} = \sigma \frac{-(p - \psi) \frac{\mu}{\psi} + p \bar{r}^L}{1 - p},$$

which yields

$$-\bar{r}^L = \frac{\mu}{\psi} \frac{(1 - \beta p)(1 - p)}{p\lambda\sigma} - \frac{(p - \psi)}{p} \frac{\mu}{\psi} = \mu \left(\frac{\psi - p}{\psi p} + \frac{\theta}{\psi} \right),$$

which proves (11b). □

A.2.5 Proof of Proposition 6

We first look at the absorbing (or steady) state, where $\epsilon_t = 0$. Then, the system to solve is

$$\hat{\pi} = \frac{\lambda}{1 - \beta} \hat{x} \quad AS \quad ; \quad \hat{\pi} = \max \begin{cases} -\frac{\gamma}{\lambda} \hat{x} & AD^{OP} \\ -\mu & AD^{ZLB} \end{cases}. \quad (A22)$$

This is depicted in Figure A.12. In contrast with the NK-TR case, there are two inequalities to satisfy: the ZLB and the slackness condition on optimal policy, i.e., (7). In the NK-TR

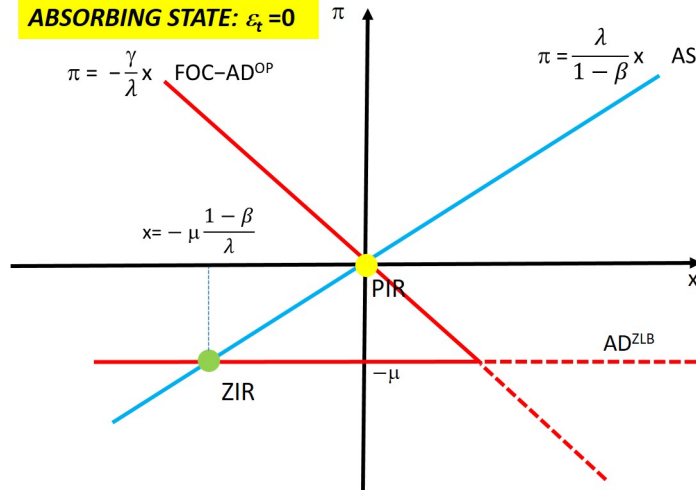


Figure A.12: The absorbing state in the NK-OP model.

case, there is only the former inequality, while the Taylor rule is expressed as equality, thus graphically a feasible point above the ZLB needs to be on the AD^{TR} line. Here instead, a feasible point can be below the first order conditions for optimal policy.² In Figure A.12 both the PIR and the ZIR are feasible steady states. The PIR equilibrium is feasible because it satisfies the ZLB constraint, i.e., is above the horizontal AD^{ZLB} ZLB line. The ZIR equilibrium is feasible because it satisfies the slackness condition on the first order conditions on optimal policy constraint, i.e., is below the AD^{OP} line.³ It is immediately obvious that the necessary support restriction for existence of a solution is $\mu \geq 0$, i.e., $r^{-1} \leq \pi_*$. When this holds, there are two possible solutions: 1) PIR: $(\hat{\pi}, \hat{x}, \hat{R}) = (0, 0, 0)$; and 2) ZIR: $(\hat{\pi}, \hat{x}, \hat{R}) = (-\mu, -\mu \frac{(1-\beta)}{\lambda}, -\mu)$.

Next, turn to the transitory state. Here, there are four possibilities depending on the value of θ , and the equilibrium in the absorbing state. These are depicted in Figure A.13. The derivations of those cases is as follows.

As before, the temporary state lasts for a random time T , after which the economy jumps to the absorbing state, because the model is completely forward-looking with no endogenous persistence. In the transitory state $\epsilon_t = -\sigma \hat{M}_{t+1|t} = \sigma pr^L < 0$, the equilibrium will be

²An alternative way to say the same thing is to note that the graph now shows that the AD is a correspondence and not a function, as in the case in the Taylor rule case.

³Note that there is an upper bound for the output gap defined jointly by optimal policy and the ZLB constraint. This value is given by the intersection of AD^{OP} and AD^{ZLB} hence: $\hat{x}^{UB} = \frac{\lambda\mu}{\gamma}$. If monetary authority tries to increase output further along the AD^{OP} then eventually it hits the ZLB constraint.

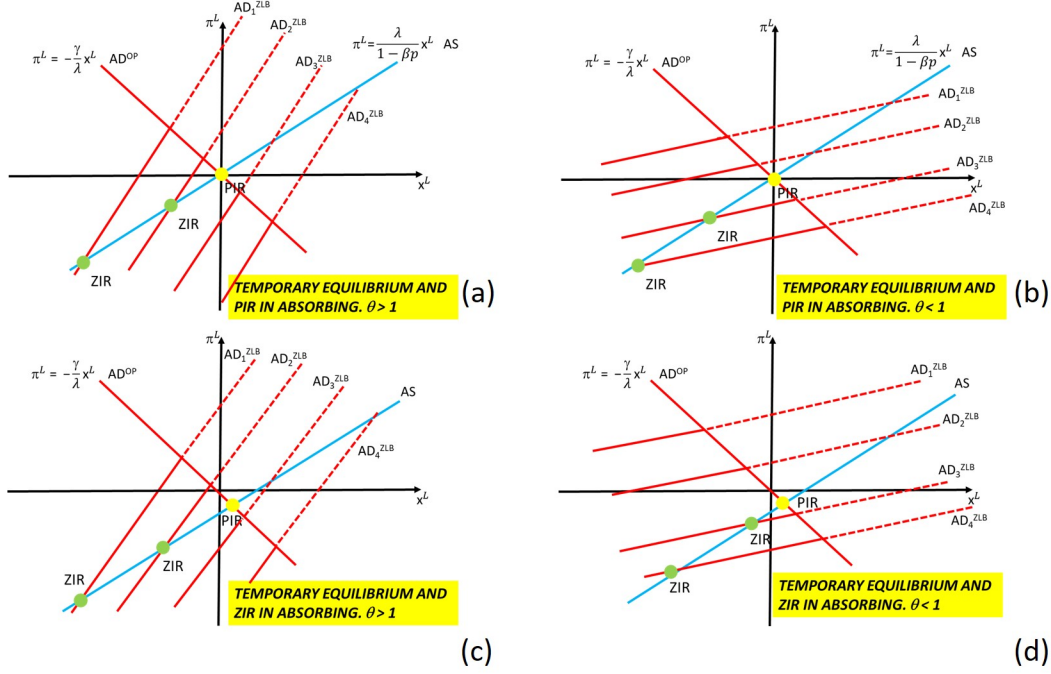


Figure A.13: The temporary state in the NK-OP model.

($\hat{\pi}^L, \hat{x}^L$) and with probability $(1 - p)$ we are back in the absorbing state. The latter can be a PIR one or a ZIR one.

When the absorbing state is PIR and the ZLB does not bind, the system becomes

$$\begin{aligned}
 \hat{\pi}^L &= \frac{\lambda}{1 - \beta p} \hat{x}^L & AS \\
 \hat{\pi}^L &= -\frac{\gamma}{\lambda} \hat{x}^L & AD^{OP} \\
 \hat{\pi}^L &> \hat{\pi}^{L,ZLB} = \hat{x}^L \frac{1 - p}{\sigma p} - \frac{\mu}{p} - r^L & AD^{ZLB}
 \end{aligned} \tag{A23}$$

When the absorbing state is PIR and the ZLB binds, then $\hat{\pi}^L = \hat{\pi}^{L,ZLB}$ and $\hat{\pi}^{L,ZLB}$ needs to be smaller than the one the central bank would have chosen to satisfy the first order conditions: $\hat{\pi}^L \leq -\frac{\gamma}{\lambda} \hat{x}^L$. The system becomes

$$\begin{aligned}
 \hat{\pi}^L &= \frac{\lambda}{1 - \beta p} \hat{x}^L & AS \\
 \hat{\pi}^L &\leq -\frac{\gamma}{\lambda} \hat{x}^L & AD^{OP} \\
 \hat{\pi}^L &= \hat{\pi}^{L,ZLB} = \hat{x}^L \frac{1 - p}{\sigma p} - \frac{\mu}{p} - r^L & AD^{ZLB}
 \end{aligned} \tag{A24}$$

The inequality in (A23) states that the equilibrium is above the AD^{ZLB} ; the inequality (A24) states you that the equilibrium is below the AD^{OP} . These curves are plotted in the top row of Figure A.13 for the cases $\theta > 1$ on the left, i.e., panel (a), where AS is flatter than AD^{ZLB} , and $\theta < 1$ on the right, i.e., panel (b), where AS is steeper than AD^{ZLB} . An increase in $-r^L$, i.e., an increase in the absolute value of the negative discount factor shock, shifts the AD^{ZLB} upwards. In both cases, there exists a threshold level of $(-r^L) = \frac{\mu}{p}$ such that the PIR coincides with the ZIR, that is, such that the intersection between AS and AD^{OP} coincides with the intersection between AS and AD^{ZLB} . Hence:

(i) when $\theta > 1$, there is unique equilibrium that is a ZIR if $-pr^L > \mu$ and a PIR if $-pr^L < \mu$;

(ii) when $\theta < 1$, there is no equilibrium if $-pr^L > \mu$ and 2 equilibria (both a ZIR and a PIR) if $-pr^L < \mu$.

When the absorbing state is ZIR, instead, expectations in the temporary equilibrium are different, and given by⁴

$$E_t(\hat{\pi}_{t+1}) = p\hat{\pi}^L - \mu(1-p),$$

$$E_t(\hat{x}_{t+1}) = p * (\hat{x}^L) + (1-p) * \left(-\mu \frac{(1-\beta)}{\lambda}\right) = p\hat{x}^L - \mu \frac{(1-\beta)(1-p)}{\lambda}.$$

When the absorbing state is ZIR and the ZLB does not bind, the system becomes

$$\begin{aligned} \hat{\pi}^L &= \frac{\lambda}{1-\beta p} \hat{x}^L - \frac{\beta\mu(1-p)}{1-\beta p} & AS \\ \hat{\pi}^L &= -\frac{\gamma}{\lambda} \hat{x}^L & AD^{OP} \\ \hat{\pi}^L &> \hat{\pi}^{L,ZLB} = \hat{x}^L \frac{1-p}{\sigma p} - r^L + \mu \left(\frac{1-p}{p} \frac{1-\beta}{\sigma\lambda} - 1 \right) & AD^{ZLB} \end{aligned}$$

In the ZLB, instead

$$\begin{aligned} \hat{\pi}^L &= \frac{\lambda}{1-\beta p} \hat{x}^L - \frac{\beta\mu(1-p)}{1-\beta p} & AS \\ \hat{\pi}^L &\leq -\frac{\gamma}{\lambda} \hat{x}^L & AD^{OP} \end{aligned}$$

⁴Note that this exactly as in the Taylor rule case, because the absorbing ZIR is not affected by the policy rule.

$$\hat{\pi}^L = \hat{\pi}^{L,ZLB} = \hat{x}^L \frac{1-p}{\sigma p} - r^L + \mu \left(\frac{1-p}{p} \frac{1-\beta}{\sigma \lambda} - 1 \right) \quad AD^{ZLB}$$

These curves are plotted in the bottom row of Figure A.13 for the cases $\theta > 1$ on the left, i.e., panel (c), where AS is flatter than AD^{ZLB} , and $\theta < 1$ on the right, i.e., panel (d), where AS is steeper than AD^{ZLB} . In both cases, r^L shifts the AD^{ZLB} and there exists a threshold level of $(-r^L) = \overline{(-r^L)} > \frac{\mu}{p}$ such that the PIR coincides with the ZIR, that is, such that the intersection between AS and AD^{OP} coincides with the intersection between AS and AD^{ZLB} .

Hence:

(i) when $\theta > 1$, there is unique equilibrium that is a ZIR if $(-r^L) > \overline{(-r^L)}$ and a PIR if $(-r^L) < \overline{(-r^L)}$;

(ii) when $\theta < 1$, there is no equilibrium if $(-r^L) > \overline{(-r^L)}$ and 2 equilibria (both a ZIR and a PIR) if $(-r^L) < \overline{(-r^L)}$.

When $\theta > 1$, thus, for PIR absorbing (panel (a)) and ZIR absorbing (panel (c)) there is always a solution in both cases. We therefore conclude that when $\theta > 1$, the only necessary support restriction is $r^{-1} \leq \pi_*$ for existence of an equilibrium in the absorbing state. This proves (12a). When $\theta < 1$, as evident from the graph and easy to prove, $\overline{(-r^L)} < \frac{\mu}{p}$. Thus, the relevant support restriction for coherency is given by $-r^L < \mu/p$, which is (12b). \square

A.2.6 Existence of sunspot equilibria in NK-TR model

Consider the NK-TR model in Proposition 5 with the additional restriction $\epsilon_t = 0$ and suppose there is a sunspot shock $\varsigma_t \in \{0, 1\}$ with transition matrix K . In this case, the vector of exogenous state variables in the canonical representation (5) can be written as $X_t = (1, \varsigma_t)'$. The model can be written as a piecewise linear system of equations $F(\mathbf{Y}) = \kappa$, where $F(\cdot)$ is given by (9) with \mathcal{A}_J given by Proposition 9 as before, since the sunspot shock affects the expectations in exactly the same way as a real shock would have. The RHS terms

κ can be obtained from (A12) with $\psi_x = 0$ and $\epsilon = 0$, that is,

$$\begin{aligned}\kappa_{J_1} &= 0_{2 \times 1}, & J_1 &= \{1, 2\} & (\text{PIR}, \text{PIR}) \\ \kappa_{J_2} &= \lambda \sigma \mu e_1, & J_2 &= \{2\} & (\text{ZIR}, \text{PIR}) \\ \kappa_{J_3} &= \lambda \sigma \mu e_2, & J_3 &= \{1\} & (\text{PIR}, \text{ZIR}) \\ \kappa_{J_4} &= \lambda \sigma \mu e_2, & J_4 &= \emptyset & (\text{ZIR}, \text{ZIR}).\end{aligned}$$

The four potential equilibria (solutions) are given by $\hat{\pi}_J := \mathcal{A}_J^{-1} \kappa_J$, i.e.,

$$\begin{aligned}\hat{\pi}_{J_1} &= 0_{2 \times 2}, & (\text{PIR}, \text{PIR}) \\ \hat{\pi}_{J_2} &= \mu \begin{pmatrix} \frac{a_q + \sigma \lambda - \sigma \lambda \psi}{\psi a_p + \sigma \lambda (\psi - \psi_{p,q,\beta,\sigma \lambda})} \\ \frac{a_q}{\psi a_p + \sigma \lambda (\psi - \psi_{p,q,\beta,\sigma \lambda})} \end{pmatrix}, & (\text{ZIR}, \text{PIR}) \\ \hat{\pi}_{J_3} &= \mu \begin{pmatrix} \frac{a_p}{\psi a_q + \sigma \lambda (\psi - \psi_{p,q,\beta,\sigma \lambda})} \\ \frac{a_p + \sigma \lambda - \sigma \lambda \psi}{\psi a_q + \sigma \lambda (\psi - \psi_{p,q,\beta,\sigma \lambda})} \end{pmatrix}, & (\text{PIR}, \text{ZIR}) \\ \hat{\pi}_{J_4} &= -\mu e_2, & (\text{ZIR}, \text{ZIR}),\end{aligned} \tag{A25}$$

where we used the definitions

$$\begin{aligned}a_q &:= (q-1)(\beta(1-p-q) + \sigma \lambda + 1) \leq 0 \\ a_p &:= (p-1)(\beta(1-p-q) + \sigma \lambda + 1) \leq 0,\end{aligned}$$

for compactness, and the fact that

$$\begin{aligned}a_p + a_q + \sigma \lambda &= (q-1)(\beta(1-p-q) + \sigma \lambda + 1) + \\ &\quad (p-1)(\beta(1-p-q) + \sigma \lambda + 1) + \sigma \lambda \\ &= (p+q-2)(\beta(1-p-q) + 1) + (p+q-1)\sigma \lambda \\ &= \sigma \lambda \psi_{p,q,\beta,\sigma \lambda}.\end{aligned}$$

Note that the PIR,PIR and ZIR,ZIR equilibria are actually sunspotless in the sense that they are completely independent of the sunspot process. (They don't depend on K , i.e., p, q). This is perfectly intuitive, because the sunspot would be effectively choosing over

two identical outcomes in each state. For existence of any of those equilibria, the support restriction is $\mu \geq 0$. So, it remains to show that there is no weaker condition that can support any of the other two equilibria ZIR,PIR or PIR,ZIR. That is, we need to check if any of the two sets of inequalities:

$$\left(\begin{array}{l} \frac{(a_q + \sigma\lambda - \sigma\lambda\psi)\mu}{\psi a_p + \sigma\lambda(\psi - \psi_{p,q,\beta,\sigma\lambda})} \leq -\frac{\mu}{\psi} \\ \frac{a_q\mu}{\psi a_p + \sigma\lambda(\psi - \psi_{p,q,\beta,\sigma\lambda})} > -\frac{\mu}{\psi} \end{array} \right) \quad \text{or} \quad \left(\begin{array}{l} \frac{a_p\mu}{\psi a_q + \sigma\lambda(\psi - \psi_{p,q,\beta,\sigma\lambda})} > -\frac{\mu}{\psi} \\ \frac{(a_p + \sigma\lambda - \sigma\lambda\psi)\mu}{\psi a_q + \sigma\lambda(\psi - \psi_{p,q,\beta,\sigma\lambda})} \leq -\frac{\mu}{\psi} \end{array} \right)$$

can be satisfied when $\mu < 0$.

Assuming $\mu < 0$ and cancelling out μ , we have

$$\left(\begin{array}{l} \frac{(a_q + \sigma\lambda - \sigma\lambda\psi)}{\psi a_p + \sigma\lambda(\psi - \psi_{p,q,\beta,\sigma\lambda})} \geq -\frac{1}{\psi} \\ \frac{a_q}{\psi a_p + \sigma\lambda(\psi - \psi_{p,q,\beta,\sigma\lambda})} < -\frac{1}{\psi} \end{array} \right) \quad \text{or} \quad \left(\begin{array}{l} \frac{a_p}{\psi a_q + \sigma\lambda(\psi - \psi_{p,q,\beta,\sigma\lambda})} < -\frac{1}{\psi} \\ \frac{(a_p + \sigma\lambda - \sigma\lambda\psi)}{\psi a_q + \sigma\lambda(\psi - \psi_{p,q,\beta,\sigma\lambda})} \geq -\frac{1}{\psi} \end{array} \right). \quad (\text{A26})$$

Given that $a_p \leq 0$ and $a_q \leq 0$, the bottom inequality on the LHS and the top inequality on the RHS both imply that $\psi a_p + \sigma\lambda(\psi - \psi_{p,q,\beta,\sigma\lambda}) > 0$ and $\psi a_q + \sigma\lambda(\psi - \psi_{p,q,\beta,\sigma\lambda}) > 0$, respectively. Then, in a ZIR,PIR equilibrium, the top inequality on the left of (A26) implies

$$\begin{aligned} \psi(a_q + a_p + \sigma\lambda - \sigma\lambda\psi) &\geq -\sigma\lambda(\psi - \psi_{p,q,\beta,\sigma\lambda}), \quad \text{or} \\ -\psi\sigma\lambda(\psi - \psi_{p,q,\beta,\sigma\lambda}) &\geq -\sigma\lambda(\psi - \psi_{p,q,\beta,\sigma\lambda}), \end{aligned} \quad (\text{A27})$$

which cannot hold, since $\psi > 1$ and $\psi_{p,q,\beta,\sigma\lambda} \leq 1$. An entirely symmetric argument can be used to rule out a PIR,ZIR – the top inequality on the RHS of (A26) also leads to (A28).

A.2.7 Relationship to Nakata and Schmidt (2019, Proposition 1)

The model in Nakata and Schmidt (2019) (henceforth NS) corresponds to (6a) with $u_t = 0$, (6b) with $\epsilon_t = -\sigma\hat{M}_{t+1|t}$ and (7). They denote their AD shock as $r_t^n := \mu - \hat{M}_{t+1|t}$, in our notation, and assume that it follows a two-state Markov process with support $\{r_L^n, r_H^n\}$, where $r_L^n < 0 < r_H^n$, and transition probabilities $\Pr(r_{t+1}^n = r_L^n | r_t^n = r_j^n) = p_j$ for $j \in \{L, H\}$. This translates in our notation to $0 > r_L^n = \mu + pr^L$, i.e., $-r^L p > \mu$, and $0 < r_H^n = \mu$, i.e., $r^{-1} < \pi_*$. The transition probabilities are in our notation $p_L = p$ and $p_H = 1 - q$. When the

‘high’ state is absorbing ($q = 1$), we have $p_H = 0$ in their notation.

Specializing to the case $p_H = 0$, NS Proposition 1 states that an equilibrium exists if and only if the following condition holds

$$p \leq p_L^* \quad \text{and} \quad 0 \leq p_H^*, \quad (\text{A28})$$

where

$$\begin{aligned} p_L^* &= \frac{-q_1 + \sqrt{q_1^2 - 4q_2q_0}}{2q_2}, \\ q_0 &= -(\lambda^2 + \gamma(1 - \beta)) \frac{1}{\sigma\lambda} < 0, \\ q_1 &= (\lambda^2 + \gamma(1 - \beta)) \left(\frac{1 + \beta}{\sigma\lambda} + 1 \right) = -q_0(1 + \beta + \sigma\lambda) > 0, \\ q_2 &= -(\lambda^2 + \gamma(1 - \beta)) \frac{\beta}{\sigma\lambda} = \beta q_0 < 0, \end{aligned}$$

so that

$$\begin{aligned} p_L^* &= \frac{-q_1 + \sqrt{q_1^2 - 4\beta q_0^2}}{2\beta q_0} = \frac{q_0(1 + \beta + \sigma\lambda) - q_0\sqrt{(1 + \beta + \sigma\lambda)^2 - 4\beta}}{2\beta q_0} \\ &= \frac{1 + \beta + \sigma\lambda - \sqrt{(1 + \beta + \sigma\lambda)^2 - 4\beta}}{2\beta}, \end{aligned} \quad (\text{A29})$$

and

$$p_H^* = \frac{-\phi_1 - \sqrt{\phi_1^2 - 4\phi_2\phi_0}}{2\phi_2}, \quad (\text{A30})$$

$$\begin{aligned} \phi_0 &= -\left(\frac{1-p}{\sigma\lambda} (1 - \beta p) - p \right) \frac{\mu}{\mu + pr^L} > 0, \\ \phi_1 &= -\frac{1 - \beta p + (1-p)\beta \frac{\mu}{\mu + pr^L}}{\sigma\lambda} - \frac{\lambda^2 + \left(1 - \beta \frac{\mu}{\mu + pr^L}\right) \gamma}{\lambda^2 + \gamma(1 - \beta)}, \\ \phi_2 &= -\frac{\beta}{\sigma\lambda} < 0. \end{aligned} \quad (\text{A31})$$

Substituting for p_L^* in the first inequality in (A28) using (A29), we obtain

$$p < \frac{1 + \beta + \sigma\lambda - \sqrt{(1 + \beta + \sigma\lambda)^2 - 4\beta}}{2\beta}. \quad (\text{A32})$$

This is equivalent to the condition $\theta > 1$ in (12a). Specifically, note that $\theta = \frac{(1-p)(1-\beta p)}{\sigma\lambda p} > 1$ is equivalent to

$$(1-p)(1-\beta p) - \sigma\lambda p > 0. \quad (\text{A33})$$

The discriminant of the quadratic equation $(1-p)(1-\beta p) - \sigma\lambda p = 0$ is $(1 + \beta + \sigma\lambda)^2 - 4\beta = (1 - \beta)^2 + 2\sigma\lambda + \sigma^2\lambda^2 + 2\sigma\beta\lambda > 0$, so the equation has real roots $p_1 \leq p_2$ given by

$$p_1 = \frac{1 + \beta + \sigma\lambda - \sqrt{(1 + \beta + \sigma\lambda)^2 - 4\beta}}{2\beta}, \quad p_2 = \frac{1 + \beta + \sigma\lambda + \sqrt{(1 + \beta + \sigma\lambda)^2 - 4\beta}}{2\beta}.$$

Thus, $\theta > 1$ is equivalent to $p < p_1 = p_L^*$, which is NS's condition (A32).

Next, turn to the second inequality ($p_H^* \geq 0$) in (A28). From (A30) and (A31), this is equivalent to

$$-\phi_1 \leq \sqrt{\phi_1^2 - 4\phi_2\phi_0}.$$

The inequality is obviously satisfied for $\phi_1 > 0$, and therefore, it is only a restriction on how negative ϕ_1 can be. In particular, it cannot fall below $-\sqrt{\phi_1^2 - 4\phi_2\phi_0}$, so, equivalently, when $\phi_1 < 0$, we must have $|\phi_1| \leq \sqrt{\phi_1^2 - 4\phi_2\phi_0}$, which is clearly equivalent to $\phi_2\phi_0 \leq 0$. Hence, the second condition of NS is equivalent to

$$\phi_2\phi_0 = \frac{\beta}{\sigma\lambda} \left(\frac{(1-p)(1-\beta p)}{\sigma\lambda} - p \right) \frac{\mu}{\mu + pr^L} = \frac{\beta p(\theta - 1)}{\sigma\lambda} \frac{\mu}{\mu + pr^L} \leq 0.$$

Since NS assumed $\mu + pr^L < 0$ and $\mu > 0$, it must be that $\theta > 1$. So, under NS's restrictions on the support $r_L^n < 0 < r_H^n$, the condition (A28) in NS Proposition 1 is equivalent to $\theta > 1$ in our Proposition 6.

A.3 Derivation of results in Subsection 2.4

Proof of Proposition 7. Proposition 9 expresses the model in the form (9) and gives $\det \mathcal{A}_{J_i}$, $i = 1, \dots, 4$. We need to find the range of parameters for which all $\det \mathcal{A}_{J_i}$ are of the same sign. Inspection of (A9) shows we need to consider the following two cases.

Case $\psi_{p,q,\beta,\sigma\lambda} > 0$. For CC we need all determinants to be positive. First, observe that $\psi_{p,q,\beta,\sigma\lambda} = p + q - 1 - \frac{(1-(p+q-1)\beta)(2-p-q)}{\sigma\lambda} \leq 1$, because $p + q - 1 \leq 1$ and $(1 - (p + q - 1)\beta)(2 - p - q) \geq 0$. Thus, $\det \mathcal{A}_{J_1} > 0$ implies

$$\psi < \psi_{p,q,\beta,\sigma\lambda} \quad \text{or} \quad \psi > 1. \quad (\text{A34})$$

For $\det \mathcal{A}_{J_2} > 0$ we need

$$\sigma\lambda\psi((1-p)(1-(p+q-1)\beta) - p\sigma\lambda) + \sigma^2\lambda^2\psi_{p,q,\beta,\sigma\lambda} > 0.$$

Now, observe that $\psi_{p,q,\beta,\sigma\lambda} > 0$ implies $(p+q-1)\lambda\sigma > (1-(p+q-1)\beta)(2-p-q)$, which, in turn, implies

$$(1-p)(1-(p+q-1)\beta) - p\sigma\lambda < -(1-q)(\lambda\sigma + (1-(p+q-1)\beta)) < 0,$$

Therefore, $\det \mathcal{A}_{J_2} > 0$ implies

$$\begin{aligned} \psi &< \frac{\sigma\lambda\psi_{p,q,\beta,\sigma\lambda}}{p\sigma\lambda - (1-p)(1-(p+q-1)\beta)} \\ &= \frac{(p+q-1)\sigma\lambda - (1-(p+q-1)\beta)(2-p-q)}{p\sigma\lambda - (1-p)(1-(p+q-1)\beta)} < 1, \end{aligned} \quad (\text{A35})$$

the last inequality following from

$$\begin{aligned} &(p+q-1)\sigma\lambda - (1-(p+q-1)\beta)(2-p-q) - p\sigma\lambda + (1-p)(1-(p+q-1)\beta) \\ &= -(1-q)\sigma\lambda - (1-(p+q-1)\beta)(1-q) < 0. \end{aligned} \quad (\text{A36})$$

An entirely symmetric argument applies for $\det \mathcal{A}_{J_3}$. Hence, combining (A35) and (A34), we obtain $\psi < \psi_{p,q,\beta,\sigma\lambda}$, which is (17b).

Case $\psi_{p,q,\beta,\sigma\lambda} < 0$. The CC now requires $\det \mathcal{A}_{J_i} < 0$ for all i . For $\det \mathcal{A}_{J_1} < 0$, we need $\psi_{p,q,\beta,\sigma\lambda} < \psi < 1$. Next, we turn to $\det \mathcal{A}_{J_2} < 0$

$$\sigma^2 \lambda^2 \psi_{p,q,\beta,\sigma\lambda} + \sigma \lambda \psi ((1-p)(1-(p+q-1)\beta) - p\sigma\lambda) < 0.$$

If $(1-p)(1-(p+q-1)\beta) - p\sigma\lambda < 0$, then

$$\psi > \frac{\sigma \lambda \psi_{p,q,\beta,\sigma\lambda}}{(p\sigma\lambda - (1-p)(1-(p+q-1)\beta))} = \frac{\psi_{p,q,\beta,\sigma\lambda}}{\left(p - \frac{(1-p)(1-(p+q-1)\beta)}{\sigma\lambda}\right)} < \psi_{p,q,\beta,\sigma\lambda}.$$

So, this condition is satisfied for all $\psi > \psi_{p,q,\beta,\sigma\lambda}$. Next, if $(1-p)(1-(p+q-1)\beta) - p\sigma\lambda > 0$, then

$$\begin{aligned} \frac{1}{\sigma^2 \lambda^2} \det \mathcal{A}_{J_2} &= \psi_{p,q,\beta,\sigma\lambda} + \psi \frac{(1-p)(1-(p+q-1)\beta)}{\sigma\lambda} \\ &< \psi_{p,q,\beta,\sigma\lambda} + \frac{(1-p)(1-(p+q-1)\beta)}{\sigma\lambda} - p < 0, \end{aligned}$$

where the first inequality follows from $\psi < 1$ and the second inequality follows from $\psi_{p,q,\beta,\sigma\lambda} < 0$ and (A36). An entirely symmetric argument applies for $\det \mathcal{A}_{J_3} < 0$. Hence, we have established that the CC condition in this case is $\psi_{p,q,\beta,\sigma\lambda} < \psi < 1$, which is (17a). \square

A.4 Derivation of results in Subsection 2.5

Derivation of equation (18). This is a simplified version of the New Keynesian model of bond market segmentation that appears in Ikeda et al. (2020) and Mavroeidis (2021), and is based on Chen et al. (2012). The economy consists of two types of households. A fraction ω_r of type ‘r’ households can only trade long-term government bonds. The remaining $1 - \omega_r$ households of type ‘u’ can purchase both short-term and long-term government bonds, the latter subject to a trading cost ζ_t . This trading cost gives rise to a term premium, i.e.,

a spread between long-term and short-term yields, that the central bank can manipulate by purchasing long-term bonds. The term premium affects aggregate demand through the consumption decisions of constrained households. This generates an UMP channel.

Households choose consumption to maximize an isoelastic utility function and firms set prices subject to Calvo frictions. These give rise to an Euler equation for output and a Phillips curve, respectively. Equation (18) can be derived from these Euler equations and an assumption about the policy rule for long-term asset purchases. For simplicity, we omit the AD shock ϵ_t from this derivation, as it is straightforward to add.

Up to a loglinear approximation, the relevant first-order conditions of the households' optimization problem can be written as

$$0 = E_t \left[-\frac{1}{\sigma} (\hat{c}_{t+1}^u - \hat{c}_t^u) + \hat{R}_t - \hat{\pi}_{t+1} \right], \quad (\text{A37})$$

$$\frac{\zeta}{1+\zeta} \hat{\zeta}_t = E_t \left[-\frac{1}{\sigma} (\hat{c}_{t+1}^u - \hat{c}_t^u) + \hat{R}_{L,t+1} - \hat{\pi}_{t+1} \right], \quad (\text{A38})$$

$$0 = E_t \left[-\frac{1}{\sigma} (\hat{c}_{t+1}^r - \hat{c}_t^r) + \hat{R}_{L,t+1} - \hat{\pi}_{t+1} \right], \quad (\text{A39})$$

where σ is the elasticity of intertemporal substitution, ζ is the steady state value of ζ_t , hatted variables denote log-deviations from steady state, \hat{c}_t^j is consumption of household $j \in \{u, r\}$, R_t is the short-term nominal interest rate, and $R_{L,t}$ is the gross yield on long-term government bonds from period $t-1$ to t . Goods market clearing yields

$$\hat{x}_t = \omega_r \hat{c}_t^r + (1 - \omega_r) \hat{c}_t^u, \quad (\text{A40})$$

where x_t is output, and we have assumed, for simplicity, that in steady state $c^u = c^r$, which implies $c^u = c^r = x$. Multiplying (A37) and (A39) by $(1 - \omega_r)$ and ω_r , respectively, and adding them yields

$$\hat{x}_t = E_t \hat{x}_{t+1} - \sigma E_t \left[(1 - \omega_r) \hat{R}_t + \omega_r \hat{R}_{L,t+1} - \hat{\pi}_{t+1} \right]. \quad (\text{A41})$$

Subtracting (A37) from (A38) yields

$$E_t \left(\hat{R}_{L,t+1} \right) = \hat{R}_t + \frac{\zeta}{1+\zeta} \hat{\zeta}_t, \quad (\text{A42})$$

which establishes that the term premium between long and short yields is proportional to $\hat{\zeta}_t$. Substituting for $E_t \left(\hat{R}_{L,t+1} \right)$ in (A41) using (A42) yields

$$\hat{x}_t = E_t \hat{x}_{t+1} - \sigma \left(\hat{R}_t + \omega_r \frac{\zeta}{1+\zeta} \hat{\zeta}_t \right) + \sigma E_t (\hat{\pi}_{t+1}) \quad (\text{A43})$$

Next, assume that the cost of trading long-term bonds depends on their supply, $b_{L,t}$, i.e.,

$$\hat{\zeta}_t = \rho_\zeta \hat{b}_{L,t}, \quad \rho_\zeta \geq 0.$$

Substituting for $\hat{\zeta}_t$ in (A43) yields the Euler equation

$$\hat{x}_t = E_t \hat{x}_{t+1} - \sigma \left(\hat{R}_t + \omega_r \frac{\zeta}{1+\zeta} \rho_\zeta \hat{b}_{L,t} \right) + \sigma E_t (\hat{\pi}_{t+1}). \quad (\text{A44})$$

Suppose that UMP follows the policy rule

$$\hat{b}_{L,t} = \alpha \min \left\{ \hat{R}_t^* + \mu, 0 \right\}, \quad (\text{A45})$$

where \hat{R}_t^* is the shadow rate prescribed by the Taylor rule (19), and $\alpha > 0$ is a factor of proportionality that can be interpreted as varying the intensity of UMP – a bigger α corresponds to a larger intervention for any given deviation of inflation and output from target. Substituting for $\hat{b}_{L,t}$ in (A44) using (A45), and using the fact that $\min \left\{ \hat{R}_t^* + \mu, 0 \right\} = \hat{R}_t^* - \max \left\{ \hat{R}_t^*, -\mu \right\} = \hat{R}_t^* - \hat{R}_t$ yields (18) with $\xi := \alpha \omega_r \frac{\zeta}{1+\zeta} \rho_\zeta$. \square

Proof of Proposition 8. The proof can follow the same steps as the proof of Proposition 7, but because of the absorbing state assumption, it is easier to proceed graphically. First, we look at the absorbing (or steady) state. The AS curve is the same as (A17), but the AD

curve is different:

$$\hat{\pi} = \frac{\lambda}{1-\beta} \hat{x} \quad AS \quad ; \quad \hat{\pi} = (1-\xi) \max \{-\mu, \psi \hat{\pi}\} + \xi \psi \hat{\pi} \quad AD$$

If the AS curve is everywhere steeper or everywhere flatter than the AD curve, then there will always be a unique steady state for any value of μ . This holds if and only if:

$$\xi\psi > 1, \quad \text{and} \quad \psi > 1, \quad \text{OR} \quad \xi\psi < 1, \quad \text{and} \quad \psi < 1.$$

The steady state is a PIR, and it is given by $\hat{\pi} = \hat{x} = \hat{R} = 0$ (because the value of the shock is zero at the absorbing state).

Suppose that in the transitory state $\epsilon_t = -\sigma \hat{M}_{t+1|t} = \sigma p r^L < 0$, for comparability with the standard NK model (this does not matter for the argument, since we only need to look at the slope of the AD curve). The MSV solution, if it exists, will be constant $(\hat{\pi}^L, \hat{x}^L)$ and with probability $(1-p)$ we are back in the absorbing state. The AS curve is given by (A18), but the AD curve (A19) now becomes

$$\hat{\pi}^L = \begin{cases} \frac{1-p}{\sigma(p-\psi)} \hat{x}^L - \frac{pr^L}{(p-\psi)} & AD^{TR} \text{ for } \pi > -\frac{\mu}{\psi} \\ \frac{1-p}{\sigma(p-\xi\psi)} \hat{x}^L - \frac{(1-\xi)\mu + pr^L}{(p-\xi\psi)} & AD^{ZLB} \text{ for } \pi \leq -\frac{\mu}{\psi}. \end{cases} \quad (A46)$$

Again, coherency requires that AD^{TR} and AD^{ZLB} be either both flatter or both steeper than AS. For AD^{TR}, AD^{ZLB} both to be flatter than AS we need

$$\psi < p - \frac{(1-p)(1-\beta p)}{\sigma\lambda} = \psi_{p,1,\beta,\sigma\lambda}, \quad \text{and} \quad \xi\psi < \psi_{p,1,\beta,\sigma\lambda}.$$

Alternatively, AD^{TR} and AD^{ZLB} must be both steeper than AS, which requires

$$\psi > \psi_{p,1,\beta,\sigma\lambda} \quad \text{and} \quad \xi\psi > \psi_{p,1,\beta,\sigma\lambda}.$$

Combining with the inequalities in the absorbing state, and using the fact that $\psi_{p,1,\beta,\sigma\lambda} \leq 0$ and $\xi > 0$, we obtain (20). \square

A.5 Derivation of results in Subsection 2.6

A.5.1 Coefficients in Example NK-ITR

The coefficients in the canonical representation of the model are:

$$A_0 = \begin{pmatrix} 1 & -\lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & -\lambda & 0 \\ 0 & 1 & \sigma \\ -\psi & -\psi_x & 1 \end{pmatrix}, \quad B_0 = B_1 = \begin{pmatrix} -\beta & 0 & 0 \\ -\sigma & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C_0 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -\sigma\mu \\ 0 & 0 & 0 & -\mu \end{pmatrix}, \quad C_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad D_0 = D_1 = 0_{3 \times 4},$$

$H_0 = 0_{3 \times 3}$, $H_1 = -\phi aa'$, $a = (0, 0, 1)'$, $b = 0_{3 \times 1}$, $c = (0, 0, 0, \mu)'$, $d = 0_{4 \times 1}$ and $h = 0_{3 \times 1}$.

A.5.2 Brute force method for checking coherency

To derive (22), first note that $E(Y_{t+1}|Y_t = \mathbf{Y}_t e_i, X_t = \mathbf{X} e_i) = (\mathbf{G}g'\mathbf{Y}_t e_i + \mathbf{Z})K'e_i$, because the support of Y_{t+1} conditional on $Y_t = \mathbf{Y}_t e_i$ is $\mathbf{G}g'\mathbf{Y}_t e_i + \mathbf{Z}$, recalling the definition $y_t := g'Y_t$. Substituting this and $Y_t = (\mathbf{G}y_{t-1} + \mathbf{Z})e_i$ into (21) yields (22).

We can solve the model backwards from some date T at which it is known that $\mathbf{Y}_T = \mathbf{G}_{J_0}y_{T-1} + \mathbf{Z}_{J_0}$, where $J_0 \in \mathcal{J}$ denotes the regime configuration across the exogenous states at T , and the set \mathcal{J} has 2^k elements. We will treat \mathbf{G}_{J_0} , \mathbf{Z}_{J_0} as known for the ensuing discussion. For example, if J_0 is PIR-only, i.e., the constraint never binds, \mathbf{G}_{J_0} , \mathbf{Z}_{J_0} can be obtained using the Blanchard and Kahn (1980) method. More generally, \mathbf{G}_{J_0} , \mathbf{Z}_{J_0} can be solved from the identities implied by (22), i.e.,

$$0 = A_{s_{t,i}} \mathbf{G} e_i + h_{s_{t,i}} + B \mathbf{G} K' e_i g' \mathbf{G} e_i, \quad \text{and} \quad (\text{A47})$$

$$0 = (A_{s_{t,i}} \mathbf{Z} + B_{s_{t,i}} \mathbf{G} K' e_i g' \mathbf{Z} + B_{s_{t,i}} \mathbf{Z} K' + C_{s_{t,i}} \mathbf{X} + D_{s_{t,i}} \mathbf{X} K') e_i, \quad (\text{A48})$$

for all $i = 1, \dots, k$.

Given $\mathbf{Y}_T = \mathbf{G}_{J_0} y_{T-1} + \mathbf{Z}_{J_0}$, we solve for \mathbf{Y}_{T-1} as a function of y_{T-2} from

$$\begin{aligned}
0 &= (A_{s_{T-1},i} \mathbf{Y}_{T-1} + B_{s_{T-1},i} \mathbf{Y}_T^i K' + C_{s_{T-1},i} \mathbf{X} + D_{s_{T-1},i} \mathbf{X} K') e_i + h_{s_{T-1},i} y_{T-2} \\
&= (A_{s_{T-1},i} \mathbf{Y}_{T-1} + B_{s_{T-1},i} (\mathbf{G}_{J_0} g' \mathbf{Y}_{T-1} e_i + \mathbf{Z}_{J_0}) K' + C_{s_{T-1},i} \mathbf{X} + D_{s_{T-1},i} \mathbf{X} K') e_i \\
&\quad + h_{s_{T-1},i} y_{T-2} \\
&= (A_{s_{T-1},i} + B_{s_{T-1},i} \mathbf{G}_{J_0} K' e_i g') \mathbf{Y}_{T-1} e_i \\
&\quad + (B_{s_{T-1},i} \mathbf{Z}_{J_0} K' + C_{s_{T-1},i} \mathbf{X} + D_{s_{T-1},i} \mathbf{X} K') e_i + h_{s_{T-1},i} y_{T-2}.
\end{aligned}$$

Since we can now treat $\mathbf{G}_{J_0}, \mathbf{Z}_{J_0}$ as fixed for solving backwards, given $J_0 \in \mathcal{J}$, the CC condition is that all of the 2^k determinants

$$\det \mathcal{A}_{J_0 J_1} = \prod_{i=1}^k \det (A_{s_{T-1},i} + B_{s_{T-1},i} \mathbf{G}_{J_0} K' e_i g'), \quad J_1 \in \mathcal{J} \quad (\text{A49})$$

should have the same sign:

$$\det \mathcal{A}_{J_0 J_1} \text{ has the same sign } \forall J_1 \in \mathcal{J}. \quad (\text{A50})$$

For example, if $k = 2$, then the determinants can be written as

$$\begin{aligned}
\det \mathcal{A}_{J_0 \{1,2\}} &= \det (A_1 + B_1 \mathbf{G}_{J_0} K' e_1 g') \det (A_1 + B_1 \mathbf{G}_{J_0} K' e_2 g') \quad (\text{P,P}) \\
\det \mathcal{A}_{J_0 \{2\}} &= \det (A_0 + B_0 \mathbf{G}_{J_0} K' e_1 g') \det (A_1 + B_1 \mathbf{G}_{J_0} K' e_2 g') \quad (\text{Z,P}) \\
\det \mathcal{A}_{J_0 \{1\}} &= \det (A_1 + B_1 \mathbf{G}_{J_0} K' e_1 g') \det (A_0 + B_0 \mathbf{G}_{J_0} K' e_2 g') \quad (\text{P,Z}) \\
\det \mathcal{A}_{J_0 \emptyset} &= \det (A_0 + B_0 \mathbf{G}_{J_0} K' e_1 g') \det (A_0 + B_0 \mathbf{G}_{J_0} K' e_2 g') \quad (\text{Z,Z}).
\end{aligned}$$

If the CC condition (A50) is violated, we need support restrictions. Otherwise, the solution will be given by

$$\begin{aligned}
\mathbf{Y}_{T-1} e_i &= - (A_{s_{T-1},i} + B_{s_{T-1},i} \mathbf{G}_{J_0} K' e_i g')^{-1} \\
&\quad \left[(B_{s_{T-1},i} \mathbf{Z}_{J_0} K' + C_{s_{T-1},i} \mathbf{X} + D_{s_{T-1},i} \mathbf{X} K') e_i + h_{s_{T-1},i} y_{T-2} \right] \quad (\text{A51})
\end{aligned}$$

for all $i = 1, \dots, k$, depending on which of the above satisfies the inequality implied by the

regime configuration J_1 . Collecting all the states, the solutions (A51) can be written as $\mathbf{Y}_{T-1} = \mathbf{G}_{J_0 J_1} y_{T-2} + \mathbf{Z}_{J_0 J_1}$, with

$$\begin{aligned}\mathbf{G}_{J_0 J_1, i} &:= - \left(A_{s_{T-1, i}} + B_{s_{T-1, i}} \mathbf{G}_{J_0} K' e_i g' \right)^{-1} h_{s_{T-1, i}}, \quad \text{and} \\ \mathbf{Z}_{J_0 J_1, i} &:= - \left(A_{s_{T-1, i}} + B_{s_{T-1, i}} \mathbf{G}_{J_0} K' e_i g' \right)^{-1} \\ &\quad \left(B_{s_{T-1, i}} \mathbf{Z}_{J_0} K' + C_{s_{T-1, i}} \mathbf{X} + D_{s_{T-1, i}} \mathbf{X} K' \right) e_i,\end{aligned}\tag{A52}$$

for all $i = 1, \dots, k$. Note that the double subscript in $\mathbf{G}_{J_0 J_1}$ and $\mathbf{Z}_{J_0 J_1}$ shows that there will be 2^k different solutions $J_1 \in \mathcal{J}$ at $T-1$ corresponding to each regime configuration $J_0 \in \mathcal{J}$ at T . So, there will be 2^{2k} different cases.

Substituting backwards to any date $t < T$, it is clear that the CC condition would be

$$\det \mathcal{A}_{J_0 \dots J_{T-t}} \text{ has the same sign } \forall J_{T-t} \in \mathcal{J},$$

where

$$\det \mathcal{A}_{J_0 \dots J_{T-t}} = \prod_{i=1}^k \det \left(A_{s_{t, i}} + B_{s_{t, i}} \mathbf{G}_{J_0 \dots J_{T-t-1}} K' e_i q' \right), \quad J_{T-t} \in \mathcal{J}$$

and the solution will be given by $\mathbf{Y}_t = \mathbf{G}_{J_0 \dots J_{T-t}} y_{t-1} + \mathbf{Z}_{J_0 \dots J_{T-t}}$, where $\mathbf{G}_{J_0 \dots J_{T-t}}$, $\mathbf{Z}_{J_0 \dots J_{T-t}}$ are computed recursively by

$$\begin{aligned}\mathbf{G}_{J_0 \dots J_{T-t}, i} &:= - \left(A_{s_{t, i}} + B_{s_{t, i}} \mathbf{G}_{J_0 \dots J_{T-t-1}} K' e_i g' \right)^{-1} h_{s_{t, i}}, \quad \text{and} \\ \mathbf{Z}_{J_0 \dots J_{T-t}, i} &:= - \left(A_{s_{t, i}} + B_{s_{t, i}} \mathbf{G}_{J_0 \dots J_{T-t-1}} K' e_i g' \right)^{-1} \\ &\quad \left(B_{s_{t, i}} \mathbf{Z}_{J_0 \dots J_{T-t-1}} K' + C_{s_{t, i}} \mathbf{X} + D_{s_{t, i}} \mathbf{X} K' \right) e_i.\end{aligned}\tag{A53}$$

At the end of this recursion at $t = 1$ we will have $2^{(T-1)k}$ paths. The initial condition y_0 will then pick the path(s) that satisfy the inequalities at all t . If the CC condition is satisfied at all t , then there will be a unique solution path for that particular y_0 . Otherwise, there may be 0 (incoherency) or multiple (incompleteness) solutions.

This suggests the following algorithm for checking the coherency of the model.

Algorithm (Coherency in model with endogenous states). Set a date $T > 1$.

1. For each possible regime configuration $J_0 \in \mathcal{J}$ (2^k elements):

- (a) Solve (A47) and (A48) to obtain \mathbf{G}_{J_0} and \mathbf{Z}_{J_0} .
- (b) For each $J_1 \in \mathcal{J}$ (2^k elements):
 - i. Compute $\det \mathcal{A}_{J_0 J_1}$ from (A49).
 - ii. If $\text{sign}(\det \mathcal{A}_{J_0 J_1})$ is different from previous J_1 , break the loop and go to next J_0 .
 - iii. Otherwise compute $\mathbf{G}_{J_0 J_1}$ and $\mathbf{Z}_{J_0 J_1}$ using (A52)
 - iv. Continue with a list of nested loops for each J_{T-t} , for $t = T - 2$ till $t = 1$.
- 2. If there is no $J_0 \in \mathcal{J}$ for which you reach $t = 1$, conclude that there is no equilibrium without support restrictions.
- 3. Otherwise, there will be a unique solution. The solution along any sequence i_t , $t = 1, \dots, T$ of exogenous shocks can be determined as follows:
 - (a) Pick a $\hat{J}_0, \dots, \hat{J}_{T-2} \in \mathcal{J}^{T-1}$.
 - (b) Find the (unique) $\hat{J}_{T-1} \in \mathcal{J}$ that ensures $\mathbf{G}_{J_0 \dots \hat{J}_{T-1}} y_0 + \mathbf{Z}_{J_0 \dots \hat{J}_{T-1}}$ satisfies the inequalities determined by regime \hat{J}_{T-1} .
 - (c) For $t = 2$ to $T - 1$,
 - i. Compute $y_{t-1} = g' \left(\mathbf{G}_{\hat{J}_0 \dots \hat{J}_{T-t+1}} y_{t-2} + \mathbf{Z}_{\hat{J}_0 \dots \hat{J}_{T-t+1}} \right) e_{i_t}$.
 - ii. If $\mathbf{G}_{\hat{J}_0 \dots \hat{J}_{T-t}} y_0 + \mathbf{Z}_{\hat{J}_0 \dots \hat{J}_{T-t}}$ satisfies the inequalities determined by regime \hat{J}_{T-t} , you have found the unique solution with regime configuration $\hat{J}_0, \dots, \hat{J}_{T-1}$.
 - iii. Otherwise, exit the loop, go back to 3.(a) and pick the next element in \mathcal{J}^{T-2} .

A.5.3 Derivation of the analytical results in Example NK-ITR

We proceed as for the proof of Proposition 5, but now the Taylor rule is given by $\hat{R}_t = \max \left(-\mu, \phi \hat{R}_{t-1} + \psi \hat{\pi}_t \right)$. First look at the steady state, where $\epsilon_t = 0$. Then, we need to solve a system as (A17), where the only difference now is the AD^{TR} equation given by $\hat{\pi} = (\phi + \psi) \frac{\lambda}{1-\beta} \hat{x}$. The graphical representation would be the same as Figure A.10, but with a steeper AD^{TR} . Whenever $(\phi + \psi) > 1$, the necessary support restriction for existence of

a solution is $\mu \geq 0$, i.e., $(r\pi_*)^{-1} \leq 1$. When this holds, there are two possible solutions: 1) PIR: $(\hat{\pi}, \hat{x}, \hat{R}) = (0, 0, 0)$; and 2) ZIR: $(\hat{\pi}, \hat{x}, \hat{R}) = (-\mu, -\mu \frac{(1-\beta)}{\lambda}, -\mu)$. However, in this case, the absorbing state admits endogenous dynamics because of the presence of the endogenous state variable \hat{R} . Outside the two steady states then the economy will travel along a stable trajectory that leads to one of the 2 steady states. Let's see under which condition the following solution exists: (i) when the shock disappears the economy will converge to the PIR along the stable manifold; (ii) the solution is MSV in the sense that it depends just on state variables; (iii) in the transitory state where $\epsilon_t = -\sigma \hat{M}_{t+1|t} = \sigma pr^L < 0$, the economy will be in a ZIR. Under these assumptions, once the shock disappears then we must be on the unique stable manifold that leads to the PIR, i.e., the 'intended steady state'. Assumption (i) hence is key because it pins down the expectations in the absorbing state. This is similar to the proof of Proposition 5. However, rather than jump to the intended steady state as when the model is forward-looking, we will arrive there inertially along the unique stable manifold. To find the MSV solution of the PIR system, we use undetermined coefficients and assume a solution of this form:

$$\hat{\pi}_t = \gamma_\pi \hat{R}_{t-1}; \quad \hat{x}_t = \gamma_x \hat{R}_{t-1}; \quad \hat{R}_t = \gamma_R \hat{R}_{t-1}. \quad (\text{A54})$$

Substituting in the [Example NK-ITR](#) system yields the following cubic equation in γ_R

$$\beta \gamma_R^3 + \gamma_R^2 (\psi \sigma - 1 - \beta - \beta \phi - \lambda \sigma) + \gamma_R (1 + \phi + \beta \phi + \lambda \sigma \phi) - \phi = 0 \quad (\text{A55})$$

Let us now assume that it exist a unique solution within the unit circle, i.e., $|\gamma_R| < 1$, as it would be in most applications.⁵ Then, the dynamics along the stable trajectory is given by the recursion:

$$\hat{\pi}_{t+j} = \gamma_\pi \gamma_R^j \hat{R}_{t-1}; \quad \hat{x}_{t+j} = \gamma_x \gamma_R^j \hat{R}_{t-1}; \quad \hat{R}_{t+j} = \gamma_R^{j+1} \hat{R}_{t-1}. \quad (\text{A56})$$

⁵Just as an example, using: $\psi = 1.5, \sigma = 1, \beta = 0.99, \phi = 0.8, \lambda = 0.02$, then the unique stable solution would be $\gamma_R = 0.35206$.

Note that if $\hat{R}_{t-1} = -\mu$, then simply

$$\hat{\pi}_t = -\gamma_\pi \mu; \quad \hat{x}_t = -\gamma_x \mu; \quad \hat{R}_t = -\gamma_R \mu,$$

hence, the system will never be in a ZIR when the shock vanishes, because $\hat{R}_t = -\gamma_R \mu > -\mu$ if $|\gamma_R| < 1$.

Next, turn to the transitory state. Here, we just want to study the situation in which the system is ZIR in the transitory state. In this case the system becomes

$$\hat{\pi}_t = \beta \hat{\pi}_{t+1|t} + \lambda \hat{x}_t, \quad \hat{x}_t = \hat{x}_{t+1|t} - \sigma (-\mu - \hat{\pi}_{t+1|t}) + \sigma p r^L.$$

Note that this system is completely forward looking and not inertial because it does not have endogenous state variables, since by assumption $\hat{R}_{t-1} = -\mu$. Hence we can follow the same steps we did for Proposition 5, because the MSV solution, if it exists, will be constant $(\hat{\pi}_t^L, \hat{x}_t^L)$ and with probability $(1-p)$ we are back on the manifold of the PIR absorbing state. The expectations thus are: $\hat{\pi}_{t+1|t} = p \hat{\pi}^L + (1-p)(\gamma_\pi(-\mu))$; $\hat{x}_{t+1|t} = p \hat{x}^L + (1-p)(\gamma_x(-\mu))$. Substitute into the above ZIR system to get

$$\hat{\pi}^L = \frac{\lambda}{1-\beta p} \hat{x}^L - \frac{\beta(1-p)\gamma_\pi}{1-\beta p} \mu \quad AS \quad (A57)$$

$$\hat{\pi}^L = \frac{1-p}{\sigma p} \hat{x}^L - \frac{\mu}{p} \left(1 - \frac{(1-p)(\gamma_x + \sigma \gamma_\pi)}{\sigma} \right) - r^L \quad AD^{ZLB} \quad (A58)$$

Note that if $\gamma_x = \gamma_\pi = 0$, we are back to equation (A18) and (A19) and figure A.11. In this case, a graph would be very similar, since the intercepts are different, but the slopes are not affected. The same reasoning therefore applies. For the solution to hold it must be that: $-\mu > \phi(-\mu) + \psi \hat{\pi} \Rightarrow \hat{\pi} \leq -\frac{\mu(1-\phi)}{\psi}$. To find the cutoff equates the two equations when $\hat{\pi} = -\frac{\mu(1-\phi)}{\psi}$, to get

$$-\bar{r}^L = \mu \left(\frac{\psi-p}{\psi p} + \frac{\theta}{\psi} + \frac{\phi}{\psi} (1-\theta) - (1-p) \frac{\lambda \gamma_x + \gamma_\pi [\beta(1-p) + \lambda \sigma]}{\lambda \sigma p} \right), \quad (A59)$$

which is (24).

A.5.4 Quasi differencing derivations

Premultiplying (5) by the $n_2 \times n$ matrix $(Q^{-1})_{22}^{-1} (Q^{-1})_{2\cdot}$, where $(Q^{-1})_{22}$ is the bottom right $n_2 \times n_2$ submatrix of Q^{-1} and $(Q^{-1})_{2\cdot}$ consists of the bottom n_2 rows of Q^{-1} , we get

$$0 = \tilde{A}_s \tilde{Y}_t + \tilde{Y}_{t+1|t} + \tilde{C}_s X_t + \tilde{D}_s X_{t+1|t} \quad (\text{A60})$$

$$s = 1_{\{\tilde{a}' \tilde{Y}_t + \tilde{b}' \tilde{Y}_{t+1|t} + \tilde{c}' X_t + \tilde{d}' X_{t+1|t} > 0\}},$$

where $\tilde{Y}_t = (Q^{-1})_{22}^{-1} (Q^{-1})_{2\cdot} Y_t = Y_{2t} + (Q^{-1})_{22}^{-1} (Q^{-1})_{21} Y_{1t}$, $\tilde{A}_s = (Q^{-1})_{22}^{-1} \Lambda_{s,22} (Q^{-1})_{22}$, $\tilde{C}_s = (Q^{-1})_{22}^{-1} (Q^{-1})_{2\cdot} C_s$ and $\tilde{D}_s = (Q^{-1})_{22}^{-1} (Q^{-1})_{2\cdot} D_s$, $\tilde{a}' = a' Q_2 (Q^{-1})_{22}$ and $\tilde{b}' = b' Q_2 (Q^{-1})_{22}$.

\tilde{Y}_t and \tilde{A}_s can be derived from:

$$\begin{aligned} \tilde{A}_s \tilde{Y}_t &= (Q^{-1})_{22}^{-1} (Q^{-1})_{2\cdot} \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \begin{pmatrix} \Lambda_{s,11} & \Lambda_{s,12} \\ 0 & \Lambda_{s,22} \end{pmatrix} \begin{pmatrix} (Q^{-1})_{1\cdot} Y_t \\ (Q^{-1})_{2\cdot} Y_t \end{pmatrix} \\ &= \begin{pmatrix} 0 & (Q^{-1})_{22}^{-1} \Lambda_{s,22} \end{pmatrix} \begin{pmatrix} (Q^{-1})_{1\cdot} Y_t \\ (Q^{-1})_{2\cdot} Y_t \end{pmatrix} \\ &= (Q^{-1})_{22}^{-1} \Lambda_{s,22} (Q^{-1})_{2\cdot} Y_t = \left[(Q^{-1})_{22}^{-1} \Lambda_{s,22} (Q^{-1})_{22} \right] \left[(Q^{-1})_{22}^{-1} (Q^{-1})_{2\cdot} Y_t \right]. \end{aligned}$$

\tilde{a} (and similarly \tilde{b}) follows from

$$\begin{aligned} a' Y_t &= a' \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \begin{pmatrix} (Q^{-1})_{1\cdot} Y_t \\ (Q^{-1})_{2\cdot} Y_t \end{pmatrix} = \begin{pmatrix} 0 & a' Q_2 \end{pmatrix} \begin{pmatrix} (Q^{-1})_{1\cdot} Y_t \\ (Q^{-1})_{2\cdot} Y_t \end{pmatrix} \\ &= a' Q_2 (Q^{-1})_{2\cdot} Y_t = a' Q_2 (Q^{-1})_{22} \left[(Q^{-1})_{22}^{-1} (Q^{-1})_{2\cdot} Y_t \right], \end{aligned}$$

where $a' Q_1 = 0$ follows by Assumption 2.

A.5.5 Proof of claims in Example ACS-STR

The model is:

$$\begin{aligned} \hat{R}_t &= \max \left(-\mu, \phi \hat{R}_{t-1} + \psi \hat{\pi}_t \right) \\ \hat{\pi}_{t+1|t} &= \hat{R}_t + \hat{M}_{t+1|t}. \end{aligned}$$

Let $Y_t = (\hat{\pi}_t, \hat{R}_{t-1})'$. At a PIR we have

$$\underbrace{\begin{pmatrix} \phi & \psi \\ 0 & 0 \end{pmatrix}}_{A_1} \underbrace{\begin{pmatrix} \hat{R}_{t-1} \\ \hat{\pi}_t \end{pmatrix}}_{B_1} + \underbrace{\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}}_{B_1} \underbrace{\begin{pmatrix} \hat{R}_t \\ \hat{\pi}_{t+1|t} \end{pmatrix}}_{D_1} + \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{D_1} \underbrace{\begin{pmatrix} \hat{M}_{t+1|t} \\ 1 \end{pmatrix}}_{D_1} = 0 \quad (\text{A61})$$

while at a ZIR we have

$$\underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}_{A_0} \underbrace{\begin{pmatrix} \hat{R}_{t-1} \\ \hat{\pi}_t \end{pmatrix}}_{B_0} + \underbrace{\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}}_{B_0} \underbrace{\begin{pmatrix} \hat{R}_t \\ \hat{\pi}_{t+1|t} \end{pmatrix}}_{D_0} + \underbrace{\begin{pmatrix} 0 & -\mu \\ 1 & 0 \end{pmatrix}}_{D_0} \underbrace{\begin{pmatrix} \hat{M}_{t+1|t} \\ 1 \end{pmatrix}}_{D_0} = 0.$$

Since $B_0 = B_1$ is clearly invertible and $A_0 = 0$, the matrices $B_0^{-1}A_0$ and $B_1^{-1}A_1$ clearly commute, satisfying the first part of Assumption 2. Because $B_0^{-1}A_0 = 0$, we may choose $Q\Lambda_1Q^{-1}$ as the Jordan decomposition of $B_1^{-1}A_1$, where

$$Q = \begin{pmatrix} 1 & 1 \\ -\frac{\phi}{\psi} & 1 \end{pmatrix}, \quad \Lambda_1 = \begin{pmatrix} 0 & 0 \\ 0 & -\psi - \phi \end{pmatrix}. \quad (\text{A62})$$

The occasionally binding constraint is $\phi\hat{R}_{t-1} + \psi\hat{\pi}_t + \mu > 0$, so $a = (\phi, \psi)'$, $b = 0$, $c = (0, \mu)$ in $s_t = 1_{\{a'Y_t + b'Y_{t+1|t} + c'X_t + d'X_{t+1|t} > 0\}}$. From (A62), we see that $Q_1 = \left(1, -\frac{\phi}{\psi}\right)'$, so $a'Q_1 = 0$, thus verifying the second part of Assumption 2.

The model can be written in the form (A60) with $\tilde{Y}_t = \hat{\pi}_t + \frac{\phi}{\psi}\hat{R}_{t-1}$, $\tilde{a} = (\psi + \phi) \frac{\psi}{\phi + \psi} = \psi$ and

$$-(\psi + \phi)\tilde{Y}_t + \tilde{Y}_{t+1|t} - \hat{M}_{t+1|t} = 0, \quad \text{if } \psi\tilde{Y}_t > -\mu, \quad (\text{A63})$$

$$\tilde{Y}_{t+1|t} - \hat{M}_{t+1|t} + \mu \frac{\phi + \psi}{\psi} = 0, \quad \text{if } \psi\tilde{Y}_t \leq -\mu. \quad (\text{A64})$$

This is a piecewise linear model. If \hat{M}_t follows a 2-state Markov Chain, it can be put in GLM

form (9) with

$$\begin{aligned}
\mathcal{A}_1 &= K - (\phi + \psi) I_2, & J_1 &= \{1, 2\} \text{ (PIR,PIR)} \\
\mathcal{A}_2 &= -(\phi + \psi) e_2 e'_2 + K, & J_2 &= \{2\} \text{ (ZIR,PIR)} \\
\mathcal{A}_3 &= -(\phi + \psi) e_1 e'_1 + K, & J_3 &= \{1\} \text{ (PIR,ZIR)} \\
\mathcal{A}_4 &= K, & J_4 &= \emptyset \text{ (ZIR,ZIR)}.
\end{aligned} \tag{A65}$$

The algebra to analyse its coherency properties is exactly the same as for the noninertial case with $\phi = 0$. Specifically, under Assumption 1, the CC condition of the GLM Theorem holds if and only if $\psi + \phi < p$, which nests the noninertial case $\psi < p$.⁶ This means that if $\phi > p$, then for all $\psi > 0$ this model will not be generically coherent, meaning that we will require support restrictions for existence of an equilibrium.

Finally, it is fairly straightforward to infer the support restriction $-r^L \leq \mu \frac{\psi + \phi - p}{\psi p}$ by following the steps in the proof of Proposition 5, i.e., by solving the model under all four regime configurations. For brevity, it suffices to give the solutions for the cases PIR,PIR and ZIR,PIR. For PIR,PIR, we have

$$\hat{\pi}_t = \begin{cases} -\frac{\phi}{\psi} \hat{R}_{t-1} + \frac{p}{\psi + \phi - p} r^L, & \text{if } \hat{M}_t = -r^L \\ -\frac{\phi}{\psi} \hat{R}_{t-1} & \text{if } \hat{M}_t = 0, \end{cases}$$

which requires the support restrictions

$$\psi \hat{\pi}_t + \phi \hat{R}_{t-1} = \begin{cases} \frac{\psi p}{\psi + \phi - p} r^L \geq -\mu, & \text{if } \hat{M}_t = -r^L \\ 0 \geq -\mu, & \text{if } \hat{M}_t = 0, \end{cases}$$

i.e.,

$$-r^L \leq \mu \frac{\psi + \phi - p}{\psi p}. \tag{A66}$$

For ZIR,PIR, the solution is

$$\hat{\pi}_t = \begin{cases} \frac{\psi + (1-p)\phi}{\psi p} \hat{R}_{t-1} - r^L, & \hat{M}_t = -r^L \\ -\frac{\phi}{\psi} \hat{R}_{t-1} & \hat{M}_t = 0. \end{cases}$$

⁶The latter can be derived from Proposition 7 with $\sigma = \infty$ and $q = 1$.

which requires the support restrictions

$$\psi\hat{\pi}_t + \phi\hat{R}_{t-1} = \begin{cases} -\frac{\psi+(1-p)\phi}{p}\mu - \psi r^L - \phi\mu \leq -\mu, & \text{if } \hat{M}_t = -r^L \\ 0 \geq -\mu, & \text{if } \hat{M}_t = 0, \end{cases}$$

which is also (A66). □

A.6 Derivation of the equilibria in Table 2

Here we derive the analytical expressions for the equilibria in Table 2. Assume to be in a period t , where the negative shock hits the economy, i.e., $\hat{M}_t = -r^L > 0$. To solve for the possible equilibria of

$$\hat{\pi}_{t+1|t} - \hat{M}_{t+1|t} - \max\{-\mu, \psi\hat{\pi}_t\} = 0, \quad (\text{A67})$$

one needs to solve for the expectations terms, that takes into account the possibility of ending up in the absorbing steady state. As we saw in the main text (see panel A in Figure 6), when $\psi > 1$, there are two possible steady state outcomes in the absorbing state: PIR where the economy is at the intended steady state inflation target, i.e., $(\hat{M}, \hat{\pi}, \hat{R}) = (0, 0, 0)$; ZIR where the economy steady state hits the ZLB constraint, i.e., $(\hat{M}, \hat{\pi}, \hat{R}) = (0, \hat{\pi}^{ZIR} = -\mu, -\mu)$. Hence, in the temporary state in t , agents might expect to end up in PIR or in ZIR. If the agents expect to end up in PIR in the absorbing state, then the expectations terms will be

$$E_t(\hat{\pi}_{t+1}) = p\hat{\pi} + (1-p)0 = p\hat{\pi}, \quad (\text{A68})$$

$$E_t(\hat{M}_{t+1}) = p(-r^L) + (1-p)0 = -pr^L, \quad (\text{A69})$$

and thus (A67) becomes

$$p\hat{\pi} = \max\{-\mu, \psi\hat{\pi}\} - pr^L. \quad (\text{A70})$$

Panel B in Figure 6 displays this equation in a graph. There are two changes with respect to Panel A that shows the absorbing state given by the equation $\hat{\pi} = \max\{-\mu, \psi\hat{\pi}\}$. First the blue line is flatter, because the slope is p rather than 1. Second, the negative \hat{r}_t (i.e., positive \hat{M}_t) shifts the red curve upwards. The two equilibria in Panel B survive only if the

real interest rate is not too low, in which case the red line shifts above the blue line and there is no possible equilibrium (incoherency). It is easy to show that the two equilibria in Panel B are given by

$$\hat{\pi}_t = \begin{cases} r^L \frac{p}{\psi-p}, & \text{if } \hat{M}_t = -r^L \in \left(0, \mu \frac{\psi-p}{\psi p}\right) \\ 0, & \text{if } \hat{M}_t = 0, \end{cases} \quad (\text{A71})$$

$$\hat{\pi}_t = \begin{cases} -r^L - \frac{\mu}{p}, & \text{if } \hat{M}_t = -r^L \in \left(0, \mu \frac{\psi-p}{\psi p}\right) \\ 0, & \text{if } \hat{M}_t = 0. \end{cases} \quad (\text{A72})$$

These are the (PIR, PIR) and (ZIR, PIR) equilibria in Table 2. The second one implies a liquidity trap equilibrium in the temporary state. If $r^L < -\frac{\psi-p}{\psi p}\mu$, there is no equilibrium.

If the agents expect to end up in ZIR in the absorbing state, instead, then the expectations terms will be

$$E_t(\hat{\pi}_{t+1}) = p\hat{\pi} + (1-p)(-\mu), \quad (\text{A73})$$

$$E_t(\hat{M}_{t+1}) = p(-r^L) + (1-p)0 = -pr^L, \quad (\text{A74})$$

and thus (A67) becomes

$$p\hat{\pi} - \mu(1-p) = \max\{-\mu, \psi\hat{\pi}\} - pr^L. \quad (\text{A75})$$

Panel C shows this case. With respect to Panel B, the blue line (LHS) now shifts down, because of the expectation of the possibility of a (permanent) liquidity trap equilibrium in the future (i.e., $(1-p)(-\mu)$). The two possible equilibria are

$$\hat{\pi}_t = \begin{cases} \frac{pr^L - (1-p)\mu}{\psi-p}, & \text{if } \hat{M}_t = -r^L \in \left(0, \mu \frac{\psi-1}{\psi}\right) \\ -\mu, & \text{if } \hat{M}_t = 0, \end{cases} \quad (\text{A76})$$

$$\hat{\pi}_t = \begin{cases} -r^L - \mu, & \text{if } \hat{M}_t = -r^L \in \left(0, \mu \frac{\psi-1}{\psi}\right) \\ -\mu, & \text{if } \hat{M}_t = 0. \end{cases} \quad (\text{A77})$$

These are the (PIR, ZIR) and (ZIR, ZIR) equilibria in Table 2. Again, the second one implies

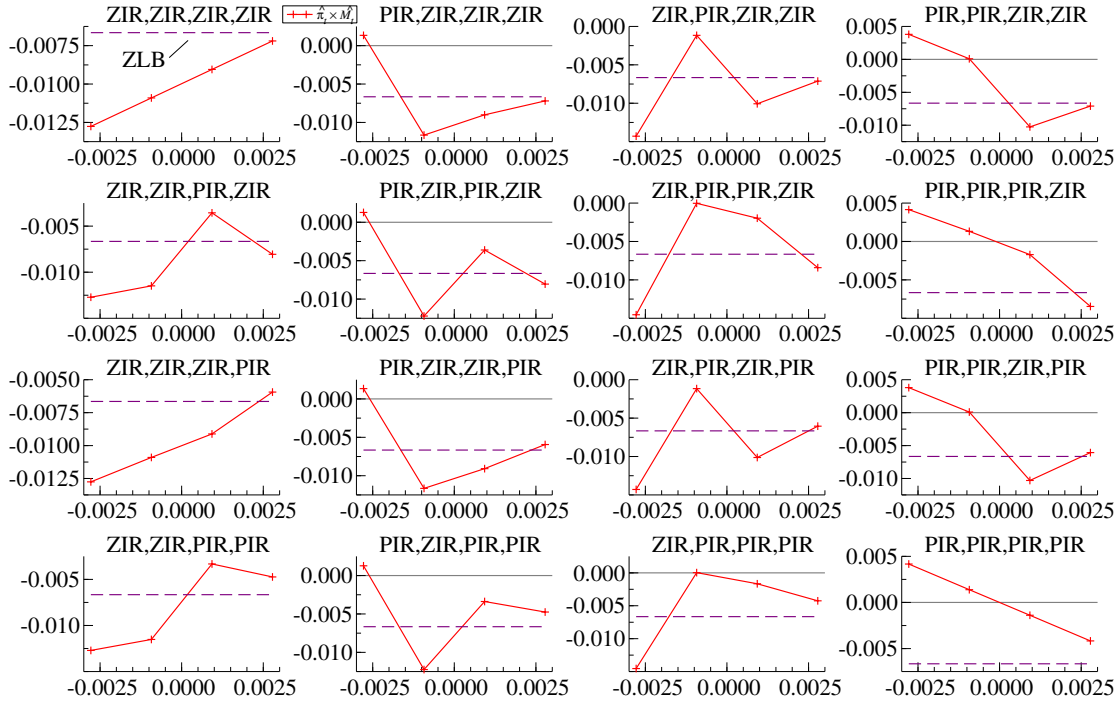


Figure A.14: The equilibria of model $\hat{\pi}_{t|t+1} = \max(-\mu, \psi \hat{\pi}_t) + \hat{M}_{t+1|t}$, when $\mu = 0.01$, $\psi = 1.5$ and \hat{M}_t follows a 4-state Markov Chain with mean 0, conditional st. dev. $\sigma = 0.007$, and autocorrelation $\rho = 0.9$.

a liquidity trap in the temporary state, and if $r^L < -\mu \frac{\psi-1}{\psi}$ there is no equilibrium.

A.7 Further numerical results on multiple equilibria

Figures A.14 and A.15 give solutions to the model of Section 3 with $k = 4$ and $k = 5$ states.

A.8 A model with ZLB on inflation expectations

In this section we exemplify how the coherency of a model with a second inequality constraint can be analysed using the methodology of this paper. We consider [Example ACS](#) with an addition ZLB on inflation expectation, motivated by [Gorodnichenko and Sergeyev \(2021\)](#). The model is given by

$$\max \{ \hat{\pi}_{t+1|t}, 0 \} = \max \{ -\mu, \psi \hat{\pi}_t \} + \hat{M}_{t+1|t}. \quad (\text{A78})$$

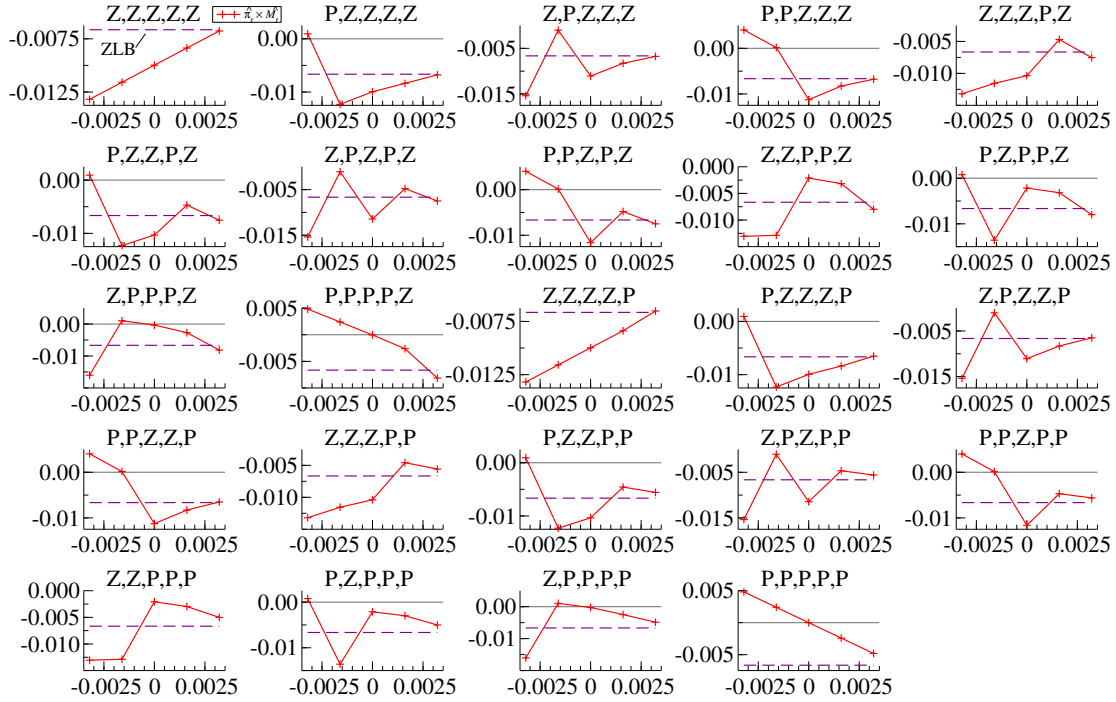


Figure A.15: The equilibria of model $\hat{\pi}_{t|t+1} = \max(-\mu, \psi \hat{\pi}_t) + \hat{M}_{t+1|t}$, when $\mu = 0.01$, $\psi = 1.5$ and \hat{M}_t follows a 5-state Markov Chain with mean 0, conditional st. dev. $\sigma = 0.007$, and autocorrelation $\rho = 0.9$.

Suppose \hat{M}_t follows a k -state Markov chain, with states m and transition probability kernel given by the $k \times k$ matrix K with $K_{ij} = \Pr(\hat{M}_{t+1} = m_j | \hat{M}_t = m_i)$.

$$E_t(\hat{\pi}_{t+1}) = K\pi, \quad E_t(\hat{M}_{t+1}) = Km,$$

and so the equation to be solved, (A78), can be written as

$$\max(K\pi, 0) = Km + \max(-\mu\iota_k, \psi\pi).$$

This is a system of piecewise linear equations with two inequality constraints $K\pi \geq 0$ and $\pi \geq -\mu/\psi$. This defines at most 4^k cones because some inequality combinations may be impossible. The following graph illustrates for the case $k = 2$ where the first state is transitory and persists with probability p and the second state is absorbing. The ZLB on the interest rate generates the inequalities we defined previously, i.e., $\pi_t > -\mu/\psi$ is a PIR and $\pi_t \leq -\mu$ is a ZIR. The ZLB on expectations is

$$E_t(\pi_{t+1}) \geq 0 \Rightarrow \begin{cases} p\pi_1 + (1-p)\pi_2 \geq 0, & \text{if } \hat{M}_t \text{ transitory} \\ \pi_2 \geq 0, & \text{if } \hat{M}_t \text{ absorbing.} \end{cases}$$

The possible regime configurations induced by combining the two inequalities are depicted in Figure A.16. We see that these inequalities split \Re^2 into 10 cones. Let Z,P denote the interest rate regimes ZIR and PIR, respectively, and Z_e, P_e the expectations regime. So, (P,P; P_e, P_e) denotes positive interest rate and positive expectations in both states. The model can then be written in canonical form (5) with \mathcal{A}_J for each cone defined as shown in Table 1.

We see by inspection that the CC condition in the GLM Theorem is violated, since some of the \mathcal{A}_J are evidently singular. Even if we restrict attention to $\pi_2 \geq 0$, i.e., regimes J_1 to J_4 , the determinants are (for $\psi > 1$): $\det \mathcal{A}_{J_1} = (\psi - 1)(\psi - p) > 0$, $\det \mathcal{A}_{J_2} = p(1 - \psi) < 0$, $\det \mathcal{A}_{J_3} = 0$ and $\det \mathcal{A}_{J_4} = \psi(\psi - 1) > 0$. So, this model is not generically coherent.

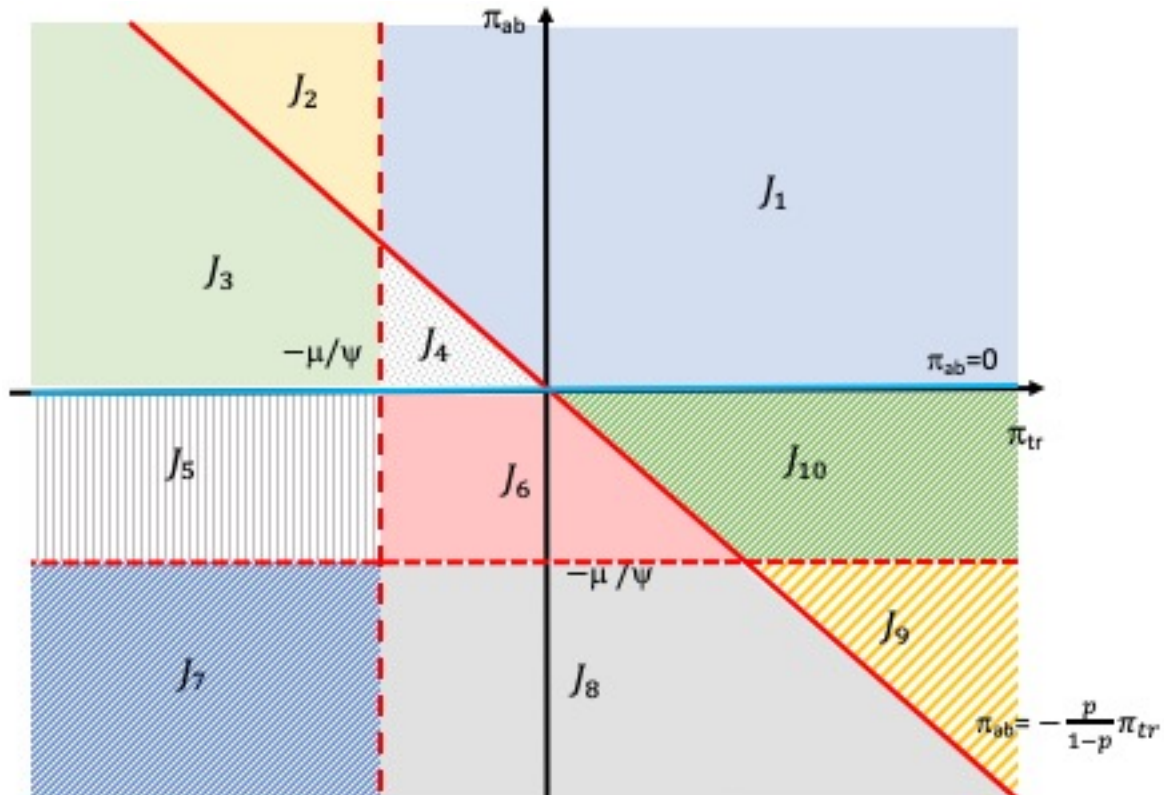


Figure A.16: Combination of regimes in [Example ACS](#) with additional ZLB on inflation expectations. Dotted lines delineate interest rate regimes. Solid lines delineate expectations regimes in transitory (red) and absorbing (blue) states. Regimes are denoted by J_i .

Coefficient matrix (2×2)	Int. rate regime	Infl. exp. regime
$\mathcal{A}_{J_1} = K - \psi I$	P,P	P_e, P_e
$\mathcal{A}_{J_2} = K - \psi e_2 e_2^T$	Z,P	P_e, P_e
$\mathcal{A}_{J_3} = e_2 e_2^T K - \psi e_2 e_2^T$	Z,P	Z_e, P_e
$\mathcal{A}_{J_4} = e_2 e_2^T K - \psi I$	P,P	Z_e, P_e
$\mathcal{A}_{J_5} = -\psi e_2 e_2^T$	Z,P	Z_e, Z_e
$\mathcal{A}_{J_6} = -\psi I$	P,P	Z_e, Z_e
$\mathcal{A}_{J_7} = 0$	Z,Z	Z_e, Z_e
$\mathcal{A}_{J_8} = -\psi e_1 e_1^T$	P,Z	Z_e, Z_e
$\mathcal{A}_{J_9} = e_1 e_1^T K - \psi e_1 e_1^T$	P,Z	P_e, Z_e
$\mathcal{A}_{J_{10}} = e_1 e_1^T K - \psi I$	P,P	P_e, Z_e

Table 1: Coefficients of canonical representation (5) of [Example ACS](#) with an additional ZLB on inflation expectations.

References

- Blanchard, O. J. and C. M. Kahn (1980). The solution of linear difference models under rational expectations. Econometrica 48, 1305–11.
- Chen, H., V. Cúrdia, and A. Ferrero (2012). The macroeconomic effects of large-scale asset purchase programmes. The Economic Journal 122, F289–F315.
- Gorodnichenko, Y. and D. Sergeyev (2021). Zero lower bound on inflation expectations. Mimeo.
- Ikeda, D., S. Li, S. Mavroeidis, and F. Zanetti (2020). Testing the effectiveness of unconventional monetary policy in japan and the united states. arXiv preprint, arXiv:2012.15158.
- Mavroeidis, S. (2021). Identification at the Zero Lower Bound. Econometrica 89(6), 2855–2885.
- Nakata, T. and S. Schmidt (2019). Conservatism and liquidity traps. Journal of Monetary Economics 104(C), 37–47.