

ROUGH SOLUTIONS OF THE 3-D COMPRESSIBLE EULER EQUATIONS

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ABSTRACT. We prove the local-in-time well-posedness for the solution of the compressible Euler equations in 3-D, for the Cauchy data of the velocity, density and vorticity $(v, \varrho, \mathfrak{w}) \in H^s \times H^s \times H^{s'}$, $2 < s' < s$. The result extends the sharp result of Smith-Tataru [34] and Wang [45], established in the irrotational case, i.e. $\mathfrak{w} = 0$, which is known to be optimal for $s > 2$. At the opposite extreme, in the incompressible case, i.e. with a constant density, the result is known to hold for $\mathfrak{w} \in H^s$, $s > 3/2$ and fails for $s \leq 3/2$, see [4]. We therefore conjecture that the optimal result should be $(v, \varrho, \mathfrak{w}) \in H^s \times H^s \times H^{s'}$, $s > 2$, $s' > \frac{3}{2}$. We view our work here as an important step in proving the conjecture. The main difficulty in establishing sharp well-posedness results for general compressible Euler flow is due to the highly nontrivial interaction between the sound waves, governed by quasilinear wave equations, and vorticity which is transported by the flow. To overcome this difficulty, we separate the dispersive part of sound wave from the transported part, and gain regularity significantly by exploiting the nonlinear structure of the system and the geometric structures of the acoustical spacetime.

1. INTRODUCTION

We consider the local well-posedness problem of the compressible Euler flow in 3 space dimension with Cauchy data. By the classical theory for quasi-linear hyperbolic systems, attributed to Friedrichs, Gaarding, Kato and Lax, etc, it is known the solutions are at least well-posed for a short time if the velocity and density are in the Sobolev space H^s with $s > \frac{3}{2} + 1$. The result is far from what would be needed to control solutions for long time. This paper continues the efforts made by many authors in recent years to derive optimal well-posedness results for physically relevant equations, such as quasi-linear wave equations [18, 34, 45] and Einstein vacuum equations [20, 42, 23].

The optimal well-posedness theory for quasi-linear hyperbolic system is much more established in 1-D. Due to the works of Glimm [13], Bressan and coworkers (see [5, 6] and references therein), the small data solution can be globally well-posed in the scale-invariant space of BV, whereas BV is improper for the multi-dimensional case due to the results of Brenner [7, 8]. The quest for the proper functional space to understand the long time behavior of solutions in the multi-dimensional space is still a work in progress. Due to the energy method, the H^s Sobolev space is a natural choice.

Various scale invariant results in H^s spaces are known in the case of geometric semilinear equations verifying some version of the null conditions. The best known among them are those concerning finite energy solutions such as the 2+1 wave maps [32, 31] and Maxwell Klein-Gordon equations in 4+1 spacetime [28, 24], etc.

The compressible Euler flow, on the other hand, is quasi-linear and does not verify null conditions. It is thus an open question that what is the optimal well-posedness result in this case. The irrotational isentropic case, which reduces to a nonlinear wave equation, is better understood. We know that in that case the initial value problem is locally well-posed if the initial velocity

and density are in H^s , $s > 2$, due to the results of [34] and [45]. This result is sharp due to the counter example of Lindblad in [25].

Dispersion is an important phenomenon typical for multi-dimensional waves. Analytically the dispersive properties of waves can be captured by L^p type Strichartz estimates, in contrast with the L^2 -norm used in establishing the classical local well-posedness results. Meanwhile, the geometry of characteristic surfaces, which is deeply linked to dispersion and the propagation of the flow becomes significantly more complicated for the compressible Euler flow. Both the dispersion and the characteristic surfaces are better understood for the general quasi-linear wave equations, including the irrotational isentropic Euler flow. In that case powerful analytic tools were developed, which blend the paradifferential calculus with the geometric analysis on the characteristic surfaces, to derive adequate Strichartz estimates in rough (dynamical) spacetimes, see [30], [2, 3], [36, 37, 34], [17]-[21], and [42, 43, 45]. Such methods work particularly well for the irrotational fluid. Nevertheless, in the general compressible Euler flow, the interaction between acoustical waves governed by quasi-linear wave equations and vorticity governed by transport equations makes the problem significantly more complex. We also note, at the other extreme, that the incompressible Euler flow is ill-posed if the vorticity $\mathbf{w} \in H^s$ with $s \leq \frac{3}{2}$ due to the result of Bourgain-Li [4]. The separate analysis of well-posedness results for both the irrotational and the incompressible Euler equations leads us to conjecture the following optimal local well-posedness result.

Conjecture 1. The solution of velocity, density¹ and vorticity (v, ϱ, \mathbf{w}) of the 3-D compressible Euler equations is well-posed for $t \in (0, T]$ with some $T > 0$ if the data satisfy the boundedness condition in (1.10) and $(\partial v, \partial \varrho, \mathbf{w}) \in H^{s-1} \times H^{s-1} \times H^{s'}$ with $s > 2$ and $s' > \frac{3}{2}$.

As an important step to attack the conjecture, in this paper we achieve the local well-posedness for the general compressible Euler flow with $2 < s' < s$. The weak regularity assumption on v and ϱ forces us to exploit the dispersive behavior of the sound wave. Nevertheless the transported vorticity flow strongly counteracts with the dispersion due to the rough data. We introduce the decomposition of the velocity into the term $(I - \Delta_e)^{-1} \text{curl} \mathbf{w}$, with Δ_e the Laplace-Beltrami operator of the Euclidean metric, and a wave function verifying a better wave equation than the wave equation for v . By carrying out a series of cancellations to treat the equation for the wave function, we manage to propagate the H^s -energy and complete the linearization for the wave functions by using the $H^{s'-\frac{1}{2}}$, $s' > 2$ norm for the vorticity, which is consistent with the conjectured regularity. The energy propagation of the vorticity typically requires $\text{curl} \mathbf{w} \in C^{0,0+}$ initially, stronger than our assumption by $\frac{1}{2}$ -derivative. To overcome this issue, we take the full nonlinear structure of the vorticity flow into account and perform trilinear analysis. However, affected by the rough vorticity flow, the rough Ricci tensor field potentially leads to the formation of caustics along the outgoing characteristic surfaces. We control the geometry of the characteristic surfaces by uncovering the geometric cancellations on the angular derivatives of the Ricci tensor and the second fundamental form. Nevertheless, similar to the issue of the critical scaling on the characteristic surfaces, c.f. addressed in [23, Section 1] for Einstein vacuum equations with bounded L^2 -curvature data, it is still a goal out of reach to obtain a result for rougher data.

1.1. Basic set-up and the main result. We consider the compressible Euler equations of 3 space dimension for a perfect fluid under a barotropic equation of state, that is, the pressure p is a function of the density $\rho : \mathbb{R}^{1+3} \rightarrow (0, \infty)$,

$$p = p(\rho). \quad (1.1)$$

¹Here ϱ is the normalized density to be defined in (1.2).

We can fix a constant background density $\bar{\rho} > 0$. Define the normalized density

$$\varrho = \ln(\rho/\bar{\rho}) \quad (1.2)$$

and the sound speed

$$c = \sqrt{\frac{dp}{d\rho}}.$$

Clearly, due to (1.1), $c = c(\varrho)$.

Let v be the velocity of the compressible fluid $v : \mathbb{R}^{3+1} \rightarrow \mathbb{R}^3$. We define the acoustical metric \mathbf{g} by

$$\mathbf{g} := -dt \otimes dt + c^{-2} \sum_{a=1}^3 (dx^a - v^a dt) \otimes (dx^a - v^a dt), \quad (1.3)$$

and regard $\mathbb{R}^3 \times [0, T]$ with $T > 0$ as the acoustical spacetime $(\mathcal{M}, \mathbf{g})$. (See [9] for a slightly different set-up of the acoustical metric.) Let $\Sigma_t = \mathbb{R}^3 \times \{t\}$. The inverse metric \mathbf{g}^{-1} can be written as

$$\mathbf{g}^{-1} = -\mathbf{T} \otimes \mathbf{T} + c^2 \Sigma_{a=1}^3 \partial_a \otimes \partial_a,$$

where \mathbf{T} is the future directed, time-like unit normal of Σ_t . We will denote the components of \mathbf{g}^{-1} by $\mathbf{g}^{\alpha\beta}$, the components of the Ricci curvature of \mathbf{g} by $\mathbf{R}_{\mu\nu}$ or by \mathbf{Ric} for short.²

In the Cartesian coordinates, \mathbf{T} is written as

$$\mathbf{T} = \partial_t + v^a \partial_a.$$

The induced metric on Σ_t takes the form $g_{ij} = c^{-2} \delta_{ij}$, where δ_{ij} is the kronecker delta. Define the second fundamental form of Σ_t

$$k_{ij} = -\frac{1}{2} \mathcal{L}_{\mathbf{T}} g_{ij}, \quad \text{Tr} k = g^{ij} k_{ij},^3$$

where \mathcal{L}_X denotes the Lie derivative by the vector field X . Let $\overset{\circ}{k}_{ij} = -\frac{1}{2} \mathcal{L}_{\mathbf{T}} \delta_{ij}$. Thus $\text{Tr } \overset{\circ}{k} := \delta^{ij} \overset{\circ}{k}_{ij} = -\partial_i v^i$.

Now we introduce the compressible Euler equations with (1.1) for ϱ and v ,

$$\begin{cases} \mathbf{T} \varrho = -\text{div } v \\ \mathbf{T} v^i = -c^2 \delta^{ia} \partial_a \varrho, \end{cases} \quad (1.4)$$

where $\text{div } v = \partial_i v^i$, ϱ is the normalized density function in (1.2).

Let ϵ_i^{jk} , $i, j, k = 1, 2, 3$, be the standard volume form on \mathbb{R}^3 . We define the vorticity to be $\mathfrak{w}_i = \epsilon_i^j \partial_j v^k$.⁴ We may employ the normalized vorticity $\Omega = e^{-\varrho} \mathfrak{w}$ for convenience. There hold for Ω the equations

$$\text{div } \Omega = -\Omega^a \partial_a \varrho, \quad (1.5)$$

$$\mathbf{T} \Omega^i = \Omega^a \partial_a v^i, \quad (1.6)$$

where (1.5) follows directly from $\text{div } \mathfrak{w} = 0$.

The compressible Euler equations (1.4) can be reduced to

$$\square_{\mathbf{g}} v^i = -e^{\varrho} c^2 \text{curl } \Omega^i + \mathcal{Q}^i, \quad (1.7)$$

$$\square_{\mathbf{g}} \varrho = \mathcal{Q}^0, \quad (1.8)$$

²We fix the convention that $\partial_0 = \partial_t$. For a tensor field, we set indices in Greek letters, such as α, β, μ, ν , to range from 0 to 3, and set the indices in Latin letters, such as i, j, k, l, m, n, a, b , to range from 1 to 3.

³We adopt Einstein summation convention in this article.

⁴The indices of the tensor fields and differentiation are lifted and lowered by the Euclidean metric.

where $\square_{\mathbf{g}}$ is the Laplace-Beltrami operator of the Lorentzian metric \mathbf{g} , and the quadratic terms are

$$\begin{aligned}\mathcal{Q}^i &:= -(1 + c^{-1}c')\mathbf{g}^{\alpha\beta}\partial_\alpha\varrho\partial_\beta v^i + 2e^\varrho\epsilon^i_{ab}\mathbf{T}v^a\Omega^b, \\ \mathcal{Q}^0 &:= -3c^{-1}c'\mathbf{g}^{\alpha\beta}\partial_\alpha\varrho\partial_\beta\varrho + 2\sum_{1\leq a<b\leq 3}(\partial_a v^a\partial_b v^b - \partial_b v^a\partial_a v^b),\end{aligned}$$

where c' is the first derivative of c . See the equations from the work of Luk-Speck [26, Page 13].

By using (1.6) we derive for $\mathfrak{C}^i = e^{-\varrho}\text{curl}\Omega^i$ the following transport equation

$$\mathbf{T}\mathfrak{C}^i = -2\delta_{jk}\epsilon^{iab}\partial_a v^j\partial_b\Omega^k e^{-\varrho} + \epsilon^{aj}_k\partial_a v^i\partial_j\Omega^k e^{-\varrho}, \quad (1.9)$$

with the derivation given in Section 3. (See also [26, (2.3.4.b)].)

Assume there hold

$$|v, \varrho| \leq C_1, \quad c \geq c_0 > 0, \quad \text{at } t = 0 \quad (1.10)$$

where $C_1, c_0 > 0$ are constants. $c_0 > 0$ is used in particular to ensure the uniform hyperbolicity of the compressible Euler system. We remark that the lower bound c_0 can be determined by the bound C_1 on $|\varrho(0)|$ if the pressure is assumed to satisfy the Gamma-law, i.e. $p(\rho) = A\rho^\gamma$, with constants $A, \gamma > 0$.

Let ∂ represent the spatial derivative $\partial_i, i = 1, 2, 3$ and let $\boldsymbol{\partial}$ denote ∂ and \mathbf{T} . Now we state the main result of this paper.

Theorem 1.1. *Let s and s' be fixed and $2 < s' < s$. For the given data set of $(v, \varrho, \mathfrak{w})$ satisfying the assumption of (1.10) and any $M > 0$ such that*

$$\|(\partial v, \partial \varrho, \mathfrak{w})(0)\|_{H^{s-1}(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3) \times H^{s'}(\mathbb{R}^3)} \leq M < \infty,$$

there exist positive constants T_ and M_1 depending merely on C_1, c_0, M, s, s' and a unique set of the solution with $(\partial v, \partial \varrho, \mathfrak{w}) \in C(I_*, H^{s-1}) \times C(I_*, H^{s-1}) \times C(I_*, H^{s'})$ for the 3-D compressible Euler equations in (1.4) and (1.6), satisfying the estimates*

$$\begin{aligned}\|\partial v, \boldsymbol{\partial} \varrho\|_{L^2_{I_*} L^\infty_x} &\leq M_1, \\ \|(\partial v, \partial \varrho, \mathfrak{w})(t)\|_{H^{s-1}(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3) \times H^{s'}(\mathbb{R}^3)} + |v, \varrho| &\leq M_1, \quad t \in I_*,\end{aligned} \quad (1.11)$$

where $I_* = [0, T_*]$ and $0 < s' - 2 < (\frac{s-2}{5})^2$.⁵

Remark 1.2. Theorem 1.1 holds if the first assumption in (1.10) is replaced by the boundedness of $\|v - \bar{v}\|_{L^2(\Sigma_0)} + \|\varrho - \bar{\varrho}\|_{L^2(\Sigma_0)}$ with \bar{v} and $\bar{\varrho}$ constant states, since the bound of $|v, \varrho|$ in (1.10) can be obtained by using this L^2 bound, the assumption for $(\partial v, \partial \varrho)(0)$ in Theorem 1.1 and Sobolev embedding.

Remark 1.3. The assumption on ∂v in Theorem 1.1 actually is merely on $\text{div } v$, due to the standard elliptic estimates for the Hodge operator $(\text{div}, \text{curl})$ on vector fields in \mathbb{R}^3 . By direct comparison estimates on the initial slice, the assumption and the result on $\partial \varrho$ in Theorem 1.1 can be stated for $\mathbf{T}v$ instead, due to (1.4) and (1.10). Theorem 1.1 can also be stated with \mathfrak{w} replaced by Ω .

Remark 1.4. Since the classical local well-posedness holds for $s > \frac{5}{2}$, we focus on the case $2 < s < \frac{5}{2}$.

Our result can be directly extended to the case with dynamical entropy by treating the divergence of the entropy gradient analogous to $\text{curl } \mathfrak{w}$.

⁵The upper bound of s' here is merely chosen for convenience. One may extend the energy estimate to $2 < s' < s$. Since the ultimate challenge is to prove the above result for $s' > \frac{3}{2}$, such an extension, requiring higher regularity assumption on data, is not of our interest.

1.2. Motivation of the problem. The classical local well-posedness of the compressible Euler equations in n -D can be obtained by using energy method for the Cauchy data $v, \varrho \in H^s$, $s > \frac{n}{2} + 1$. See [27, Chapter 2] and also the earlier work by Kato [15] for an alternative approach. For incompressible Euler equations, Kato and Ponce in [16] proved the classical local well-posedness for data in the general Sobolev space $v \in W^{s,p}(\mathbb{R}^n) = (I - \Delta_e)^{-\frac{s}{2}} L^p(\mathbb{R}^n)$ with $s > n/p + 1$ and $1 < p < \infty$. In 3-D for the incompressible case, the assumption of the datum that $\mathfrak{w} \in H^\alpha$, $\alpha > \frac{3}{2}$ is sharp, since in [4] the solution is proved to be strongly ill-posed if $\alpha = \frac{3}{2}$.

Recall the general quasilinear wave equations considered in [18, 34, 45]

$$\square_{h(\phi)} \phi = Q(\partial\phi, \partial\phi) \quad (1.12)$$

with $h(\phi)$ a Lorentzian metric depending on ϕ and the right hand side being quadratic forms of $\partial\phi$. By using the energy and iteration methods, the classical local well-posedness holds for $(\phi(0), \partial_t \phi(0)) \in H^s \times H^{s-1}$ with $s > \frac{5}{2}$ (see [14]), which is of the same level as the classical result for the compressible Euler equations. Since the late 90s, based on establishing Strichartz estimates with loss for wave equations with rough coefficients, there had been vast improvements to $s > 2 + \frac{1}{4}$ achieved in [2, 3, 36], to $s > 2 + \frac{1}{6}$ in [37, 17], and to $s > 2 + \frac{2-\sqrt{3}}{2}$ achieved in [18], and for $s > 2$ for Einstein vacuum equations in [19]-[21] and [42, 43]. Apart from the improvements over the Sobolev exponents, the commuting vector field approach for proving Strichartz estimate was introduced in [17]. The rather physical approach further showed its power in [18] where a fundamental decomposition of a Ricci component (proposed in [17]) was used for improving the regularity of the causal geometry. With the help of the improvement in the control of causal geometry, the sharp local well-posedness for the solution of (1.12) was proved in [34] by constructing parametrix and wave packets, and by the vector field approach in [45], for the initial data $(\phi(0), \partial_t \phi(0)) \in H^s \times H^{s-1}$ for $s > 2$.

In comparison with the equation (1.12), we unify (1.7) and (1.8) into the following equation for $\Phi = (v, \varrho)$,

$$\square_{\mathbf{g}(\Phi)} \Phi = Q(\partial\Phi, \partial\Phi) - (e^e c^2 \operatorname{curl} \Omega, 0).^6 \quad (1.13)$$

Here the term $Q(\partial\Phi, \partial\Phi)$ denotes a finite sum of $\mathcal{N}(\Phi) \partial\Phi \partial\Phi$, with \mathcal{N} smooth functions of their variables, and $\partial\Phi$ representing the terms of ∂v or $\partial \varrho$. It represents \mathcal{Q}^i and \mathcal{Q}^0 in (1.7) and (1.8).

Note that if $\Omega = 0$, the above equation takes the form of

$$\square_{\mathbf{g}(\Phi)} \Phi = Q(\partial\Phi, \partial\Phi).$$

Therefore the local well-posedness for the irrotational (isentropic) compressible Euler flow holds for $(v(0), \varrho(0)) \in H^s \times H^s$ for any $s > 2$ by the results of [34] and [45]. Note the proof of the sharp local well-posedness result for (1.12) is based on the energy method and, via a bootstrap argument, the Strichartz estimate for the solution of the linearized wave equation $\square_{h(\phi)} \psi = 0$ within a short life-span. The latter relies crucially on the regularity of the spacetime metric. If Ω (or \mathfrak{w}) is non-zero but sufficiently smooth, the term $e^e c^2 \operatorname{curl} \Omega$ in (1.13) can be treated as a good inhomogeneous term of the linearized wave equation for both the energy method and the Strichartz estimate, which does not in turn influence the regularity of the spacetime metric. This case can be incorporated in the regime either given in [34] or in [45]. See [12] which assumes $\operatorname{curl} \Omega \in H^{s-1} \cap C^{0,\alpha}$, $0 < \alpha < 1$ for the data, and the same regularity for the divergence of the nontrivial initial entropy gradient. In terms of H^γ -Sobolev norms, this assumption at least needs a $\frac{1}{2}$ -derivative more than assumed in Theorem 1.1.

⁶If this is the equation of v , the last term on the right of (1.13) takes the first value in the bracket, and the term is 0 for the equation of ϱ .

Similar to the result of [4], lowering the regularity of the datum for Ω below a certain level would significantly changes the behaviour of the solution and the regularity of the spacetime metric. In Theorem 1.1, the regularity of vorticity datum is set such that the term of $\text{curl } \Omega$ in (1.7) counteracts crucially with the dispersion from the wave equation. It forces us to understand better the interaction between the dispersive and transported properties in the velocity. By introducing the decomposition of the velocity into two parts with the two properties and uncovering cancellations, merely with the bound of $\|\Omega\|_{C^0[0,T]H_x^{s'-\frac{1}{2}}}$, $2 < s' < s$, we reduce the proof of the main theorem to establishing for the linear wave equation in the acoustical spacetime the Strichartz estimate comparable to the sharp one in the Minkowski space. To solve Conjecture 1, there remains the obstruction for proving the Strichartz estimate due to the weak regularity of the acoustical null hypersurfaces. In the non-irrotational case, the rough transported part of the velocity in particular leaves a serious defect to form caustics on the outgoing null hypersurfaces. The regularity of data in Theorem 1.1 reaches the borderline for the defect to be absorbed by the structural cancellations in the nonlinearity and geometry. With lower regularity of $\Omega(0)$ than in Theorem 1.1, we encounter the same difficulty from the rough null hypersurfaces as in improving the resolution of the bounded L^2 -curvature conjecture for Einstein vacuum equations in [23]. For the latter, there is also a gap of $\frac{1}{2}$ -derivative between the critical Sobolev exponent for the Einstein vacuum equations and the result by Klainerman-Rodnianski-Szeftel in [23].

1.3. Main idea of the proof for Theorem 1.1.

1.3.1. A quick guide to the main difficulties. v, ρ and Ω in the compressible Euler equations exhibit different physical and analytic properties. To achieve a result with the data $v, \rho \in H^s$, $s > 2$, it is necessary to establish the Strichartz estimate which gains the spatial regularity by taking advantage of the dispersive property in time-variable of the quantities. It works particularly well for the solutions of wave equations. To have a brief idea of how the vorticity derivative is involved in the analysis, we will apply the energy method to the geometric wave equations (1.7) and (1.8). Through this procedure, we can see the regularity requirement on the Cauchy data by following the approach in [45].

Under the bootstrap assumption that $\|\partial\Phi\|_{L_t^1[0,T]L_x^\infty}$ is bounded, applying the standard energy argument to (1.13) implies ⁷

$$\|\partial\Phi(t)\|_{H_x^{s-1}} \lesssim \|\partial\Phi(0)\|_{H_x^{s-1}} + \|e^{\rho}c^2 \text{curl } \Omega\|_{L_t^1H_x^{s-1}} + \|Q(\partial\Phi, \partial\Phi)\|_{L_t^1H_x^{s-1}}, \quad s \geq 2. \quad (1.14)$$

Due to the standard product estimate, the last term on the right hand side can be controlled by the energy bound, provided that one can control $\|\partial\Phi\|_{L_t^1[0,T]L_x^\infty}$ in terms of the initial data. Assuming the absence of $\text{curl } \Omega$, the estimate was established in [34, 45] by proving the sharp Strichartz estimate of the solution ψ of the linear wave equation in the acoustical spacetime $(\mathbb{R}^3 \times [0, T], \mathbf{g})$,

$$\square_{\mathbf{g}(\Phi)}\psi = 0, \quad (1.15)$$

which implies

$$\|\partial\Phi\|_{L_t^q[0,T]L_x^\infty} \lesssim \|\partial\Phi(0)\|_{H_x^{1+\epsilon}} + \|\square_{\mathbf{g}(\Phi)}\Phi\|_{L_t^1[0,T]H_x^{1+\epsilon}}, \quad \epsilon > 0, \quad (1.16)$$

where $q > 2$ is sufficiently close to 2.⁸ By choosing a small life-span T , the estimate (1.16) can close the bootstrap argument with the help of the standard product estimate if the last term is

⁷We call C a universal constant if it depends merely on the bound of s, s' and M in Theorem 1.1, the constants C_1 and c_0 in (1.10). $A \lesssim B$ means there exists a universal constant $C > 0$ such that $A \leq CB$. We denote $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

⁸We refer to Theorem 4.3 for the precise version, with the relation between ϵ, q specified therein.

quadratic in $\partial\Phi$. The proof of the Strichartz estimates depends crucially on the regularity of the spacetime metric. It at least requires the bound on $\|\partial\Phi\|_{C_t^0[0,T]H_x^{1+}} + \|\partial\Phi\|_{L_t^2[0,T]L_x^\infty}$.

Consider the term of $\text{curl}\Omega$ on the right hand side of (1.14). Note that with a normalization $\text{curl}\Omega$ verifies the transport equation (1.9). We can at best expect the regularity of $\text{curl}\Omega$ is preserved in time. This means one has to assume $\text{curl}\Omega \in H_x^{s-1}$ when $t = 0$, in order to complete the estimate of (1.14). However, our assumption of the initial data is that $\text{curl}\Omega \in H_x^{s'-1}$, $2 < s' < s$, with s' arbitrarily close to 2, which is insufficient for using (1.14) to derive (1.11). Therefore we need a different strategy.

The other serious difficulty arises from the energy propagation of the vorticity, or more precisely, in bounding the norm $\|\text{curl}\Omega(t)\|_{H_x^\alpha}$, $\alpha \geq 1$, for $0 < t \leq T$. To understand the issue, we consider the normalized transport equation (1.9) for $\text{curl}\Omega$, which symbolically reads

$$\mathbf{T}\mathfrak{C} = \partial v \partial \Omega, \quad \text{with } \mathfrak{C} = e^{-\varrho} \text{curl}\Omega. \quad (1.17)$$

Applying (3.4) to $F = G = \mathfrak{C}$, and using $\text{Tr } \overset{\circ}{k} = -\text{div } v$, we obtain in view of (1.17) that

$$\|\mathfrak{C}(t)\|_{H_x^\alpha} \lesssim \|\mathfrak{C}(0)\|_{H_x^\alpha} + \int_0^t \|\partial\Omega \cdot \partial v\|_{H_x^\alpha} dt', \quad \text{with } 1 \leq \alpha \leq s' - 1.$$

By the standard product estimate

$$\|\partial\Omega \cdot \partial v\|_{H_x^\alpha} \lesssim \|\partial\Omega\|_{L_x^\infty} \|\partial v\|_{H_x^\alpha} + \|\partial\Omega\|_{H_x^\alpha} \|\partial v\|_{L_x^\infty}, \quad 1 \leq \alpha \leq s' - 1,$$

we derive

$$\|\mathfrak{C}(t)\|_{H_x^\alpha} \lesssim \|\mathfrak{C}(0)\|_{H_x^\alpha} + \int_0^t (\|\partial\Omega\|_{L_x^\infty} \|\partial v\|_{H_x^\alpha} + \|\partial v\|_{L_x^\infty} \|\partial\Omega\|_{H_x^\alpha}) dt', \quad \text{with } 1 \leq \alpha \leq s' - 1. \quad (1.18)$$

In view of (1.5) and the elliptic estimate for the div - curl Hodge system, we obtain

$$\|\partial\Omega\|_{H_x^\alpha} \lesssim \|\mathfrak{C}\|_{H_x^\alpha} + l.o.t.. \quad (1.19)$$

Thus by applying Gronwall's inequality and the above estimate to (1.18), to bound $\|\partial\Omega(t)\|_{H_x^\alpha}$, with $1 \leq \alpha \leq s' - 1$, we need the bound $\|\partial\Omega\|_{L_t^1[0,T]L_x^\infty}$. The latter can be bounded by requiring $\text{curl}\Omega(0) \in C_x^{0,0+}$, by using (1.9) and the elliptic theory for the Hodge operator $(\text{div}, \text{curl})$. This assumes an additional $\frac{1}{2}$ -derivative than Theorem 1.1 in terms of Sobolev embedding.

Note that without the bound of $\|\text{curl}\Omega\|_{H_x^1}$ or equivalently $\|\mathfrak{C}\|_{H_x^1}$, we lose the bound of $\|\square_{\mathbf{g}(\Phi)}\Phi\|_{L_t^1H_x^1}$ in view of (1.13). If using (1.14) with $s = 2$ and (1.16), we would lose both the bounds of $\|\partial\Phi\|_{C_t^0[0,T]H_x^1}$ and $\|\partial\Phi\|_{L_t^2[0,T]L_x^\infty}$, which are the prerequisites to establish the Strichartz estimate for the linearized wave (1.15). Based on the above treatment, in order to close the energy argument, one may have to assume $\text{curl}\Omega(0) \in H^{s-1} \cap C^{0,0+}$.

To prove the Strichartz estimate, in particular the key dispersive estimate (see Theorem 4.12) for (1.15), we can see a more crucial obstruction from the influence of rough vorticity on the regularity of the characteristic surfaces. The proof relies on the derivative estimates of the optical function u of the acoustical spacetime. The optical function, defined by the solution of the Eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$, can be constructed by level hypersurfaces, formed by generating null geodesic congruences satisfying certain initial conditions, which are acoustical null cones C_u .

Nearly no analysis can be carried out on C_u without preventing the formation of caustics, which quantitatively is controlled by the null area expansion $\text{tr}\chi = -\mathbf{b}\square_{\mathbf{g}}u$, with $\mathbf{b}^{-1} = \mathbf{T}(u)$. Denote the normalized null geodesic generator by $L = -\mathbf{b}\mathbf{D}u$, where \mathbf{D} is the Levi-Civita connection of \mathbf{g} . $\text{tr}\chi$ satisfies the Raychaudhuri equation given in (7.12), where the Ricci component $\mathbf{R}_{LL} = \mathbf{R}_{\alpha\beta}L^\alpha L^\beta$ is the highest order term on the right hand side. Note $\text{curl}\Omega$ appears in the equation

(7.30) for \mathbf{R}_{LL} as one of the main terms. To achieve the crucial $L_t^2 L_x^\infty$ control of $\text{tr}\chi$ ⁹ for proving the decay estimate in Theorem 4.12, we need the L^∞ bound of $\text{curl}\Omega$ along each null cone C_u . This can be directly obtained if assuming and propagating via the transport equation the $C^{0,0+}$ data of $\text{curl}\Omega$. In this case the causal geometry can be controlled by the analysis in [45, Section 5 and 6]. However, with merely the control of H_x^{2+} bound of $\Omega(t)$, we can only obtain the $H^{\frac{1}{2}+}$ bound on a null cone for $\partial\Omega(t, u)$ by the standard trace inequality, while we need H^{1+} bound to have the L^∞ control of $\partial\Omega(t, u)$.

The above mechanism is based on treating the velocity identically as for the irrotational case. We list below the regularity it needs for the vorticity to summarize the above discussion.

- (a) For achieving the energy estimate for v, ϱ in (1.11), it requires the initial datum of the vorticity to be bounded in H^s norm;
- (b) For bounding the energy of vorticity, it requires the initial datum of $\text{curl}\Omega$ to be bounded in $C^{0,0+}$;
- (c) For controlling the characteristic surfaces, it requires the initial datum of $\text{curl}\Omega$ to be bounded in $C^{0,0+}$.

With the much weaker assumption on the datum of the vorticity in Theorem 1.1, we will give the main ingredients of our approach based on a deeper understanding of the behaviour of the velocity.

1.3.2. Decoupling method and cancellations. Ω (or \mathfrak{w}) apparently verifies a good transport equation (1.6). As such, it is regarded as the relatively stationary (or transported) part of ∂v , which does not disperse as a free wave. We start with taking away the transported part from the velocity and derive a better equation for the dispersive part.

To this end, we introduce a decomposition for the velocity

$$v^i = v_+^i + \eta^i, \quad (1.20)$$

where v_+ is a vector-valued wave function, satisfying a wave equation system; and η is the part of the velocity determined by the vorticity. $\square_{\mathbf{g}}\eta$ is supposed to cancel the vorticity derivative on the right hand side of (1.7). v_+ and ϱ will be treated as wave functions, for which we apply the energy argument together with establishing the Strichartz estimate, since they verify better wave equations than (1.13). To control η , we will rely on elliptic estimates and the transport equation for the vorticity.

More precisely, we define the vector field η^i by

$$\Lambda^2 \eta^i := (I - \Delta_e) \eta^i = \text{curl } \mathfrak{w}^i, \quad (1.21)$$

where Δ_e is the Laplace-Beltrami operator of the Euclidean metric. By direct computation,

$$\text{curl } \mathfrak{w}^n = e^\varrho \text{curl } \Omega^n + \epsilon^{nij} \mathfrak{w}_j \partial_i \varrho.$$

Hence, we can substitute (1.21) to the right hand side of (1.7) to derive

$$\square_{\mathbf{g}} v^i = c^2 \Delta_e \eta^i - c^2 \eta^i + c^2 \mathfrak{w}_n \partial_m \varrho \epsilon^{imn} + \mathcal{Q}^i.$$

Since the induced metric $g_{ij} = c^{-2} \delta_{ij}$ is conformally flat, there holds

$$\Delta_g(\eta^i) = c^2(\Delta_e(\eta^i) - \partial^j(\log c) \partial_j(\eta^i)).$$

Hence in view of (1.20), we obtain the wave equation for v_+

$$\begin{aligned} \square_{\mathbf{g}} v_+^i &= -\mathbf{T}\mathbf{T}v_+^i + \text{Tr}k\mathbf{T}v_+^i + \Delta_g v_+^i \\ &= \mathbf{T}\mathbf{T}\eta^i - \text{Tr}k\mathbf{T}\eta^i - c^2(\eta^i - \partial^j \log c \partial_j(\eta^i)) + c^2 \mathfrak{w}_n \partial_m \varrho \epsilon^{imn} + \mathcal{Q}^i, \end{aligned} \quad (1.22)$$

⁹This estimate is as important if adopting the alternative approach in [34] to prove Strichartz estimate.

which can be recast for short as

$$\square_{\mathbf{g}} v_+^i = \mathbf{T}\mathbf{T}\eta^i - \text{Tr}k\mathbf{T}\eta^i + \tilde{\mathcal{Q}}^i - c^2\eta^i, \quad (1.23)$$

with the quadratic term

$$\tilde{\mathcal{Q}}^i = \mathcal{Q}^i + c^2(\mathbf{w}_n \partial_m \varrho \epsilon^{imn} + \partial^j(\log c) \partial_j(\eta^i)).$$

Such reduction transforms the higher order linear term (1.21), which appears on the right hand side of the equation (1.7), to the term $\mathbf{T}\mathbf{T}\eta$. The latter is a linear higher order term. Due to the complexity and difficulty in controlling the term $\mathbf{T}\mathbf{T}\eta$ and its derivatives, we cancel this term in all the applications of the wave equation (1.23). As the consequence, merely the spatial derivatives of $\mathbf{T}\eta$ are involved in the analysis of this article, which can be well controlled by the derivative bound of vorticity via elliptic estimates.

Due to the term $\mathbf{T}\mathbf{T}\eta$ in (1.23), we consider the following equation for wave functions v_+ and ϱ , which better represents the structure of (1.23),

$$\square_{\mathbf{g}} \Psi = W + \mathbf{T}Y - \text{Tr}kY, \quad (1.24)$$

where the pair of functions (Ψ, Y) is either $(v_+, \mathbf{T}\eta)$ or $(\varrho, 0)$ with the corresponding error terms contained in W .

We rely on (1.24) to derive energy and flux control and also strichartz estimates for $\partial\Psi$. In all the applications of (1.24), we manage to cancel the term $\mathbf{T}Y$.

- For energy estimates and the flux estimates along null cones, this is done by constructing the energy densities with the help of the first order system (2.12) which is equivalent to (1.24), and constructing the modified current (2.23) which originally was introduced in [42, Section 3.2]. Combining the elliptic estimates for η , we can obtain the total energy of ϱ, v . (See Corollary 3.3).
- To bound the Strichartz norm $\|\partial\Psi\|_{L_t^2 L_x^\infty}$ with $\Psi = v_+, \varrho$, instead of treating the full right hand side of (1.24) as the inhomogeneous term of (1.15), we adapt the method of linearization in [42, Section 4.2] to cancel $\mathbf{T}Y$ by the Duhamel's principle. (See (4.5).)

We complete the above propagation of energy and flux and the reduction to Strichartz estimates by using merely the bound of $\|\Omega(t)\|_{H_x^{s'-\frac{1}{2}}}$. This solves the difficulty (a) in Section 1.3.1, with the regularity consistent with Conjecture 1.

1.3.3. Trilinear structure for the propagation of the vorticity. Due to (c) in Section 1.3.1, we have to control the higher order norm $\|\Omega(t)\|_{H_x^{s'}}, 2 < s' < s$ for $t \leq T$, even though the energy control of v, ϱ and the linearization only rely on the bound of $\|\Omega\|_{H_x^{s'-\frac{1}{2}}}$. Therefore we need to solve the difficulty in (b).

To this end, we first note that to bound $\|\Omega(t)\|_{H_x^{s'}}$, it suffices to bound $\|\text{curl } \mathfrak{C}(t)\|_{H_x^\alpha}$ with $0 \leq \alpha \leq s' - 2, 0 < t \leq T$, in the same manner as in (1.19) by applying elliptic estimates for the Hodge operator $(\text{div}, \text{curl})$.

To control $\|\text{curl } \mathfrak{C}\|_{L_x^2}$, by applying (3.4) to $F = G = \text{curl } \mathfrak{C}$, we then integrate the following identity in $t' \in [0, t]$,

$$\partial_t \int_{\mathbb{R}^3} |\text{curl } \mathfrak{C}|^2 dx = \int_{\mathbb{R}^3} (2\mathbf{T} \text{curl } \mathfrak{C} \cdot \text{curl } \mathfrak{C} - \text{Tr } \overset{\circ}{k} |\text{curl } \mathfrak{C}|^2) dx. \quad (1.25)$$

Recall $\text{Tr } \overset{\circ}{k} = -\text{div } v$ and the following symbolic formula obtained from (3.1)

$$\mathbf{T} \text{curl } \mathfrak{C} - \text{curl } \mathbf{T}\mathfrak{C} = \partial v \partial \mathfrak{C}. \quad (1.26)$$

Since $\|\partial v\|_{L_t^1[0,T]L_x^\infty}$ is expected to be bounded, the right hand side of (1.25) can be controlled by using elliptic estimates and Gronwall's inequality, except the highest order term

$$\mathcal{I} = \int_0^t \int_{\mathbb{R}^3} \operatorname{curl} \mathbf{T} \mathfrak{C} \operatorname{curl} \mathfrak{C} dx dt'.$$

Instead of substituting the symbolic formula (1.17) to $\mathbf{T} \mathfrak{C}$, we observe trilinear structures in \mathcal{I} by deriving the precise formula in (3.13), with the main terms listed below ¹⁰

$$\mathcal{I} = \int_0^t \int_{\Sigma_{t'}} e^{-\varrho} \{ \partial^n \partial_m v^j \partial_n \Omega_j + \partial_j \mathbf{T} \varrho \partial_m \Omega^j \} \operatorname{curl} \mathfrak{C}^m dx dt' + \dots$$

See the complete formula in (3.15). Since we do not have a bound on $\|\partial \Omega\|_{L_t^1 L_x^\infty([0,T] \times \mathbb{R}^3)}$, we will estimate the integral using integration by parts.

For the first term in \mathcal{I} , by using $\partial_m(\operatorname{curl} \mathfrak{C})^m = 0$ we calculate the integrand

$$\begin{aligned} e^{-\varrho} \partial^n \partial_m v^j \partial_n \Omega_j \operatorname{curl} \mathfrak{C}^m &= \partial_m (\partial^n v^j \partial_n \Omega_j \operatorname{curl} \mathfrak{C}^m e^{-\varrho}) - e^{-\varrho} \partial^n v^j \partial_m \partial_n \Omega_j \operatorname{curl} \mathfrak{C}^m \\ &\quad - \partial_m (e^{-\varrho}) \partial^n v^j \partial_n \Omega_j \operatorname{curl} \mathfrak{C}^m. \end{aligned}$$

The first term vanishes due to integration by parts on $\Sigma_{t'}$, and the other terms can be treated by using the elliptic estimate, Sobolev embedding and Gronwall's inequality.

For the second term in the integrand of \mathcal{I} , note

$$\begin{aligned} e^{-\varrho} \partial_j \mathbf{T} \varrho \partial_m \Omega^j \operatorname{curl} \mathfrak{C}^m &= \mathbf{T} (e^{-\varrho} \partial_j \varrho \partial_m \Omega^j \operatorname{curl} \mathfrak{C}^m) - \partial_j \varrho \mathbf{T} (e^{-\varrho} \partial_m \Omega^j \operatorname{curl} \mathfrak{C}^m) \\ &\quad + e^{-\varrho} [\partial_j, \mathbf{T}] \varrho \partial_m \Omega^j \operatorname{curl} \mathfrak{C}^m. \end{aligned}$$

We integrate by parts in the spacetime to treat the first term on the right hand side with the help of (3.4), which only generates lower order terms. For the second term, we can substitute the transport equations of $\partial \Omega$ and $\operatorname{curl} \mathfrak{C}$ to save derivatives. The commutator term is of lower order. See the proof of (3.5) for the full detail.

To control $\|\operatorname{curl} \mathfrak{C}\|_{H_x^\alpha}$ with $\alpha = s' - 2$, $0 < t \leq T$, besides entailing the above set of integration by parts, for the estimate of the highest order, we have an additional issue coming from the high-low interaction for the product in the first term below

$$\int_0^t \|\partial v \partial \mathfrak{C}(t')\|_{H_x^{s'-2}} \|\operatorname{curl} \mathfrak{C}(t')\|_{H_x^{s'-2}} dt', \quad (1.27)$$

which comes from the commutator term in (1.26). Note the norm $\|\partial v\|_{L_t^2 B_{\infty,2,x}^{s'-2}}$ ¹¹ can be bounded by establishing Strichartz estimate for v_+ and the elliptic estimate for η . Thus (1.27) can be treated by using Gronwall's inequality and the elliptic estimate. The s - s' energy hierarchy for v , ϱ and \mathfrak{w} helps crucially to complete the estimate of (1.27), since by our approach we can not control $\|\partial v\|_{L_t^2 B_{\infty,2,x}^{s-2}}$, even with smoother vorticity data.

1.3.4. Fundamental structures for the causal geometry of the acoustical spacetime. At last, we consider the difficulty from (c) in Section 1.3.1, that is to control the acoustical null cones, with the focus on bounding $\operatorname{tr} \chi$, the most important quantity in causal geometry. The regularity from the general derivative of the metric is by no means sufficient for the purpose even in the previous works for quasilinear wave equations.

Since the null area expansion $\operatorname{tr} \chi$ verifies the Raychaudhuri equation (7.12), i.e.

$$L \operatorname{tr} \chi + \frac{1}{2} (\operatorname{tr} \chi)^2 = -\mathbf{R}_{LL} + l.o.t.,$$

¹⁰ ∂^n here is $\delta^{nl} \partial_l$ instead of n -th order derivative. We will clarify whenever such confusion may occur.

¹¹The definition of the Besov norm can be found at the end of Section 3.1.

the strategy is to gain regularity by taking advantage of the structure of the Ricci component \mathbf{R}_{LL} , which works particularly well in Einstein vacuum spacetime due to the vanishing \mathbf{Ric} .

For the acoustical spacetime, we derive in (7.30) that

$$\mathbf{R}_{LL} = L(\Xi_L) - e^\varrho \delta_{ij} \mathbf{N}^j \operatorname{curl} \Omega^i + l.o.t., \quad (1.28)$$

where the one form $\Xi_\mu = \Gamma_{\alpha\beta}^\eta(\mathbf{g}) \mathbf{g}^{\alpha\beta} \mathbf{g}_{\eta\mu}$, $\Gamma(\mathbf{g})$ is the Christoffel symbol of \mathbf{g} (see (7.26) and (7.27)), and \mathbf{N} is the outward unit normal of $S_{t,u} := \Sigma_t \cap C_u$. The basic analysis on null hypersurfaces relies on the energy fluxes of v, ϱ and the vorticity controlled in Section 6. We can control the double-curl flux of vorticity $\|\mu^{s'-2} P_\mu \operatorname{curl} \mathfrak{C}\|_{L^2(C_u)}^2 + \|\operatorname{curl} \mathfrak{C}\|_{L^2(C_u)}^2$ ¹² by energy method and the trilinear estimates, nevertheless, can not bound $\partial \mathfrak{C}$ at the same level, since there lacks the trilinear structure if using $\mathbf{T} \partial \mathfrak{C}$ to propagate the energy, and there is neither a proper Hodge systems for $\mathfrak{C} = e^{-\varrho} \operatorname{curl} \Omega$ available on $S_{t,u}$. Applying the trace inequality and using the full energy bound of $\|\mathfrak{C}(t)\|_{H_x^{s'-1}}$ on Σ_t lead to a loss of $\frac{1}{2}$ -derivative due to the restriction to $S_{t,u}$. We then lose the bound on $\|\operatorname{curl} \Omega\|_{L^\infty(S_{t,u})}$ by a $\frac{1}{2}$ -derivative.

For quasilinear wave equations (1.12), one can gain regularity for $\operatorname{tr} \chi + \Xi_L$ as in [17, 18, 34, 45] due to (1.28) and the absence of vorticity. Because of the weak regularity, the derivative of the term $\Xi_L = \Xi_\mu L^\mu$ caused the main difficulty for proving the sharp local well-posedness for the solution of (1.12) via the geometric approach devised in [17] and [18]. The difficulty was solved in [45] by introducing the geometric normalization via the conformal change of the spacetime metric and bounding the conformal energy by an un-canonical energy method. The issue from Ξ_L remains the same for the compressible Euler equations, for which we adopt the method in [45].

For the compressible Euler flow, the rough $\operatorname{curl} \Omega$ also leads to a serious difficulty in bounding $\|\operatorname{tr} \chi - \frac{2}{\tilde{r}}\|_{L_t^2 L_x^\infty(u \geq 0)}$ in (8.9). Here $\tilde{r} = t - u$, and the estimate is crucial for proving the Strichartz estimate in Theorem 4.3. The region of the norm is, roughly, the domain of influence of a unit ball with the time-span nearly the large frequency λ fixed in Theorem 4.3. In such region, $0 \leq u \leq t$, with $u = 0$ on the boundary of the domain of influence and $u = t$ along the time-axis. (We refer to Section 5.1 for the set-up of the acoustical null cones C_u .)

We write below the transport equation for $z = \operatorname{tr} \chi + \Xi_L - \frac{2}{\tilde{r}}$ in (8.42),

$$Lz + \frac{2z}{t-u} = e^\varrho \operatorname{curl} \Omega_{\mathbf{N}} + \frac{2}{r} (\Xi_L - k_{\mathbf{N}\mathbf{N}}) + \cdots,$$

with nonlinear terms omitted. The term $\operatorname{curl} \Omega$ on the right hand side is merely bounded in $H^{\frac{1}{2}+}(S_{t,u})$, which is short of $\frac{1}{2}$ -spatial-derivative on $S_{t,u}$ to give the bound of $\|z\|_{L^\infty(S_{t,u})}$ directly. As shown in (7.34), $\operatorname{curl} \Omega_i \mathbf{N}^i = \epsilon^{AB} \nabla_A \Omega_B$ where ∇ denotes the Levi-Civita connection on $S_{t,u}$ with respect to the induced metric of the acoustical metric \mathbf{g} , $\{e_A\}_{A=1}^2$ forms the orthonormal basis and ϵ^{AB} is the volume form on $S_{t,u}$. Since this is an angular derivative, we can not decompose it into $LF + E$ with controllable functions F and E .

To solve the difficulty, we uncover fundamental structures in Section 7 on the angular derivatives of \mathbf{R}_{LL} and the component of the second fundamental form $k_{\mathbf{N}\mathbf{N}}$, in the acoustical spacetime.

Note the L^∞ bound on $\tilde{r}^{\frac{1}{2}} z$ can be obtained by using the Sobolev inequality

$$|\tilde{r}^{\frac{1}{2}} z| \lesssim \|\tilde{r}^{\frac{1}{2}-\frac{2}{p}} (\tilde{r} \nabla)^{(\leq 1)} z\|_{L^p(S_{t,u})},^{13} \quad 0 < 1 - \frac{2}{p} < s' - 2. \quad (1.29)$$

¹²We refer to (2.2) for the definition of the Littlewood-Paley projector.

¹³Let $\mathfrak{P}^{(i)}$ represent applying the operator \mathfrak{P} i times, and $\mathfrak{P}^{(0)}$ be the identity operator. We set $\mathfrak{P}^{(\leq m)} = \sum_{i=0}^m \mathfrak{P}^{(i)}$.

To bound the right hand side, we consider the transport equation of ∇z by differentiating (8.42). It is crucial to observe the following trace decomposition derived in (7.31),

$$\nabla(e^\varrho \operatorname{curl} \Omega_{\mathbf{N}}) = \nabla_L(e^\varrho \operatorname{curl} \Omega)_A + e^\varrho e_{A_i} \Pi^{ij} \epsilon_{jm}{}^l (\operatorname{curl}^2 \Omega)_l \mathbf{N}^m + l.o.t.,$$

with $\Pi^{ij} = \mathbf{g}^{ij} + \mathbf{T}^i \mathbf{T}^j - \mathbf{N}^i \mathbf{N}^j$, since this quantity can not be directly bounded by the double-curl flux of vorticity. This leads to

$$\nabla_L \nabla z + \frac{3}{t-u} \nabla z = \nabla_L(e^\varrho \operatorname{curl} \Omega)_A + e^\varrho e_{A_i} \Pi^{ij} \epsilon_{jm}{}^l (\operatorname{curl}^2 \Omega)_l \mathbf{N}^m + \frac{2}{\tilde{r}} \nabla(\Xi_L - k_{\mathbf{N}\mathbf{N}}) + \dots \quad (1.30)$$

Combining the first terms on both sides, we derive the transport equation for $\nabla_A z - (e^\varrho \operatorname{curl} \Omega)_A$ in (8.45) and obtain the following estimate by integrating along the null cone C_u ,

$$\begin{aligned} \|\tilde{r}(\nabla_A z - e^\varrho (\operatorname{curl} \Omega)_A)\|_{L_\omega^p(S_{t,u})} &\lesssim \tilde{r}^{-1} \int_{t_{\min}}^t \|\tilde{r} \nabla(\Xi_L - k_{\mathbf{N}\mathbf{N}})\|_{L_\omega^p} dt' \\ &+ \tilde{r}^{-1} \int_{t_{\min}}^t \|\tilde{r}^2 \operatorname{curl}^2 \Omega\|_{L_\omega^p} dt' + \dots, \end{aligned} \quad (1.31)$$

where $t_{\min} = \max(u, 0)$, $\omega \in \mathbb{S}^2$ on $S_{t,u}$ is the pull-back spherical coordinate via the null geodesic flow, (see the construction in Section 5,) and we omitted the terms of initial data and nonlinear terms. We can control $\|\tilde{r}^{\frac{3}{2}}(\nabla_A z - e^\varrho (\operatorname{curl} \Omega)_A)\|_{L_\omega^p(S_{t,u})}$ by bounding the right hand side of the inequality with the flux control in Section 6 and its consequences in Proposition 8.6. The desired bound of ∇z in (1.29) follows since the other term $\tilde{r} e^\varrho (\operatorname{curl} \Omega)_A$ can be easily bounded in L_ω^p . Thus we can obtain the pointwise bound on $\tilde{r}^{\frac{1}{2}} z$.

However the bound of $\|z\|_{L_t^2 L_x^\infty(u \geq 0)}$ does not follow from the pointwise estimate of $\tilde{r}^{\frac{1}{2}} z$, since $\tilde{r} = t - u$ vanishes at the time-axis. In view of (1.29), to bound $\|z\|_{L_t^2 L_x^\infty(u \geq 0)}$, we need the bound of $\|\sup_{0 \leq u \leq t} \|\tilde{r} \nabla z\|_{L_\omega^p}\|_{L_t^2}$. The term $\frac{2}{\tilde{r}} \nabla(\Xi_L - k_{\mathbf{N}\mathbf{N}})$ in (1.30) is the obstruction for controlling the above bound since we need the bound of $\|\tilde{r} \nabla(\Xi_L - k_{\mathbf{N}\mathbf{N}})\|_{L_t^2 L_u^\infty L_\omega^p(C_u)}$ in view of (1.31), much stronger than the available bound on flux. To solve this issue, we derive a trace decomposition in Proposition 7.7 that

$$k_{\mathbf{N}\mathbf{N}} - \frac{1}{2} \Xi_L = -\frac{1}{2} (L(\log c + \varrho) + 2L(v)_{\mathbf{N}}). \quad (1.32)$$

Based on the above identity, and the equation $L \log \mathbf{b} = -k_{\mathbf{N}\mathbf{N}}$ in (7.11), we consider the normalized quantity $\mathscr{Y} = \mathbf{b}(\operatorname{tr} \chi + \Xi_L) - \frac{2}{\tilde{r}}$ and note $z = \mathbf{b}^{-1} \mathscr{Y} + 2 \frac{\mathbf{b}^{-1} - 1}{\tilde{r}}$, where the last term and \mathbf{b}^{-1} are easier to control. It boils down to deriving the bound of $\|\mathscr{Y}\|_{L_t^2 L_x^\infty(u \geq 0)}$. Note the transport equation of $\nabla \mathscr{Y}$ takes the form,

$$\begin{aligned} \nabla_L(\nabla_A \mathscr{Y} - \mathbf{b} e^\varrho (\operatorname{curl} \Omega)_A) + \frac{2}{\tilde{r}} (\nabla_A \mathscr{Y} - \mathbf{b} e^\varrho (\operatorname{curl} \Omega)_A) \\ = -\frac{2}{\tilde{r}} \nabla(\Xi_L - 2k_{\mathbf{N}\mathbf{N}}) + \mathbf{b} e^\varrho e_{A_i} \Pi^{ij} \epsilon_{jm}{}^l (\operatorname{curl}^2 \Omega)_l \mathbf{N}^m + \dots \quad (1.33) \end{aligned}$$

For the first term on the right hand side, we have a good trace decomposition by using (1.32). Due to the decomposition, this term can be canceled by slightly normalizing the quantity $\nabla_A \mathscr{Y} - \mathbf{b} e^\varrho (\operatorname{curl} \Omega)_A$ on the left hand side of (1.33). Thus we can control the estimates of \mathscr{Y} and z . (See details in the proof of Proposition 8.19.) The estimate of $\operatorname{tr} \chi - \frac{2}{\tilde{r}}$ follows from the bound of $\|\Xi_L\|_{L_t^2 L_x^\infty}$ by the bootstrap assumption (2.1) and the estimate of z .

1.4. Organization of the proof. The proof of Theorem 1.1 consists of three main building blocks: energy and flux propagation, linearization and reduction for Strichartz estimates, and the control of the causal geometry of the acoustical spacetime. Uniqueness can follow from a similar energy argument. We omit the details to avoid repetition.

The energy propagation includes the propagations of v, ϱ and the vorticity from the Cauchy data, completed in Section 2 and 3 under the main bootstrap assumption (2.1) in $\mathbb{R}^3 \times [0, T]$. In Section 2, the energy estimates for v_+, ϱ up to the order of H_x^2 can be bounded together with bounding the L_x^p norms of $\partial\Omega, \partial\mathbf{w}, \partial^2\eta$, with $p > 3$ from the initial data, summarized in Corollary 2.11. To bound the highest order energy $\|\partial v(t), \partial\varrho(t)\|_{H_x^{s-1}}$, Proposition 2.15 shows that it relies on the norms of $\|\text{curl}\Omega(t)\|_{H_x^{s-2}}$ and $\|\Omega(0)\|_{H_x^{\frac{3}{2}+}}$. We then bound the stronger norm $\|\Omega\|_{C_t^0[0, T]H_x^2}$ in (3.5) in Section 3.1 by performing the trilinear estimates. This completes the highest order energy control on v and ϱ in Corollary 3.3. We then use this result to bound the highest order energy for the vorticity in Proposition 3.5.

To prove (2.1), as in [45], we take the scheme initiated in [17] (see also [18, 20, 42]). In Section 4-5 we obtain the Strichartz estimates in Theorem 4.1 based on Theorem 5.3, which proves (2.1). The proof consists of a series of reductions. We first prove Theorem 4.1 by applying the dyadic Strichartz estimates in Theorem 4.3 to the Littlewood-Paley pieces $P_\lambda\partial v_+$ and $P_\lambda\partial\varrho$, with large λ . By running the \mathcal{TT}^* argument,¹⁴ we then reduce the proof of Theorem 4.10, which is the rescaled statement of Theorem 4.3, to the decay estimate in Theorem 4.12. By proper localizations in the physical space, it is then reduced to the proof of Proposition 4.13 and then further reduced to controlling the conformal energy in Theorem 5.3.

To prove Theorem 5.3, we run the multiplier approach in [45, Section 7] which includes a conformal change of the spacetime metric for normalizing the causal geometry. To obtain the necessary geometric control, we carry out geometric analysis of the acoustical null cones in Section 8 and Section 9, with preliminaries given in Section 5-7. In Section 5, we give the geometric set-up for the acoustical null cones. In Section 6, we control derivatives of the metric and vorticity along the null cones via the energy argument in the ambient acoustical spacetime, upto the highest order, with the help of the trace estimates and elliptic estimates for the derivatives of η . In Section 7, we derive the important trace decompositions for $\nabla(\mathbf{R}_{LL})$ and the second fundamental form in Proposition 7.5 and Proposition 7.7, and also provide the structure equations of the null tetrad and the necessary decompositions of Riemann curvature. In Section 8, by using the results in Proposition 7.5 and Proposition 7.7, we prove Proposition 8.2 simultaneously with other estimates therein. With the help of them, in Section 9, we control the renormalized mass aspect function and the derivatives of the conformal factor used for the geometric normalization in Proposition 9.4 and Proposition 9.5. Thus we complete the full set of geometric estimates for running the proof in [45, Section 7]. This completes the proof of Theorem 5.3.

2. ENERGY ESTIMATES FOR WAVE FUNCTIONS (v_+, ϱ)

2.1. Bootstrap assumptions. For the fixed number $2 < s < \frac{5}{2}$, we fix

$$0 < \epsilon_0 < \frac{s-2}{5}, \quad s' - 2 = \delta_0 = \epsilon_0^2.$$

We make the following bootstrap assumption in a spacetime slab $[0, T] \times \mathbb{R}^3$ with $T > 0$ that

$$\|\partial\varrho, \partial v_+, \partial v\|_{L_t^2 L_x^\infty}^2 + \sum_{\lambda \geq 2} \lambda^{2\delta_0} \|P_\lambda(\partial\varrho, \partial v_+)\|_{L_t^2 L_x^\infty}^2 \leq 1, \quad (2.1)$$

¹⁴See [42, Section 4] and [45, Section 3 and Section 9] for the \mathcal{TT}^* argument and also for more details of the reduction.

where P_λ is the Littlewood-Paley projector with frequency $\lambda = 2^k$ defined for any function f by

$$P_\lambda f(x) = f_\lambda(x) = \int_{\mathbb{R}^3} e^{ix \cdot \xi} \beta(\lambda^{-1} \xi) \hat{f}(\xi) d\xi \quad (2.2)$$

with β a smooth function supported in the shell $\{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$ satisfying $\sum_{k \in \mathbb{Z}} \beta(2^k \xi) = 1$ for $\xi \neq 0$. We refer to [35] for detailed properties of Littlewood-Paley decompositions.

We also make an auxiliary bootstrap assumption that

$$\|\partial \varrho\|_{L_t^2 L_x^3([0, T] \times \mathbb{R}^3)} \leq 1, \quad (2.3)$$

which will be improved in (2.68) to

$$\|\partial \varrho\|_{L_t^2 L_x^3([0, T] \times \mathbb{R}^3)} \leq CT^{\frac{1}{2}}.$$

(2.1) will be improved to

$$\|\partial v_+, \partial \varrho, \partial v\|_{L_t^2 L_x^\infty}^2 + \sum_{\lambda \geq 2} \lambda^{2\delta_0} \|P_\lambda(\partial \varrho, \partial v_+)\|_{L_t^2 L_x^\infty}^2 \leq CT^{2\gamma_1}, \quad (2.4)$$

where $0 < \gamma_1 \leq \epsilon_0$. The C 's are universal constants in the above two estimates. With $0 < T < 1$ sufficiently small, we can have $\max(CT^{\frac{1}{2}}, CT^{2\gamma_1}) < 1$. The improvement will be achieved in Section 4 by establishing the dyadic Strichartz estimate in Theorem 4.3 for (1.15), and by a delicate linearization from (1.24) of $\Psi = (v_+, \varrho)$ to (1.15).

To prove the Strichartz estimate (2.4), we need the full energy control for v, ϱ in Corollary 3.3. To complement the energy estimates on v_+, ϱ by using wave equations, the estimates for η will be achieved by elliptic estimates combined with the energy estimates for $\text{curl}^{(\leq 2)} \Omega$, completed in Section 3.

Since any point $(x, t) \in \mathbb{R}^3 \times [0, T]$ can be reached by following the integral curve of \mathbf{T} , denoted by $x(t)$, from the initial slice

$$c(x(t), t)/c(x(0), 0) - 1 = c(x(0), 0)^{-1} \int_0^t c'(\varrho) \mathbf{T} \varrho; \quad \varrho(x(t), t) = \varrho(x(0), 0) + \int_0^t \mathbf{T} \varrho. \quad (2.5)$$

By (1.10) at $t = 0$, $|\varrho| \leq C_1$. By the evolution of ϱ and (2.1), we can obtain for all $t \in [0, T]$, $|\varrho| \leq C_1 + 1$. Since c, c' are smooth functions about ϱ , $|c', c| \lesssim 1$. By using (2.1)

$$\left| \frac{c(x(t), t)}{c(x(0), 0)} - 1 \right| \lesssim c(x(0), 0)^{-1} T^{\frac{1}{2}} \leq C c_0^{-1} T^{\frac{1}{2}} < \frac{1}{2}$$

as long as we fix $0 < T^{\frac{1}{2}} < \frac{1}{2} c_0 C^{-1}$, where C is a universal constant. This leads to $\frac{1}{2} < c(x(t), t)/c(x(0), 0) < \frac{3}{2}$ and thus for $t \in [0, T]$,

$$\frac{1}{2} c_0 \leq \frac{1}{2} c(x(0), 0) \leq c \leq \frac{3}{2} c(x(0), 0) \leq C_0, \quad (2.6)$$

where C_0 is a universal constant.

This fact together with the fact that $|c', c''| \lesssim 1$ in $\mathbb{R}^3 \times [0, T]$ will be frequently used in this paper. The metric $g_{ij} = c^{-2} \delta_{ij}$ is thus always a conformally flat Riemannian metric on Σ_t for $0 < t \leq T$.

Similar to (2.5), we can also obtain the bound on v due to (2.1) and the C_1 bound in (1.10). Thus we will frequently use $|\varrho, v| \lesssim 1$, and

$$\|C(f)\|_{L_x^\infty} \lesssim 1, \quad \|\partial(C(f))\|_{L_x^\infty} \lesssim \|\partial f\|_{L_x^\infty} \quad (2.7)$$

with $f = \varrho$ or v , where $C(y)$ is a smooth function of y .

2.1.1. *The second fundamental form.* Next we derive the formula for the second fundamental form.

Proposition 2.1. *Define the second fundamental form for \mathbf{T} on Σ_t as $k_{ij} := -\frac{1}{2}\mathcal{L}_{\mathbf{T}}g_{ij}$. Let $g^{ij}k_{ij} = \text{Tr}k$, and $\hat{k}_{ij} = k_{ij} - \frac{1}{3}\text{Tr}kg_{ij}$. There hold*

$$k_{ij} = -\frac{1}{2}c^{-2}(-2\mathbf{T}(\log c)\delta_{ij} + \partial_i v_j + \partial_j v_i) \quad (2.8)$$

$$\text{Tr}k = 3\mathbf{T}\log c - \text{div}v. \quad (2.9)$$

Proof. By the fact that

$$[\mathbf{T}, \partial_i] = -\partial_i v^a \partial_a, \quad (2.10)$$

we can compute by definition that

$$\begin{aligned} k_{ij} &= -\frac{1}{2}\mathcal{L}_{\mathbf{T}}g_{ij} = -\frac{1}{2}(\mathbf{T}g_{ij} - g([\mathbf{T}, \partial_i], \partial_j) - g(\partial_i, [\mathbf{T}, \partial_j])) \\ &= -\frac{1}{2}(\mathbf{T}(c^{-2})\delta_{ij} + \partial_i v^a g_{aj} + \partial_j v^a g_{ia}). \end{aligned}$$

This gives (2.8). Taking trace implies (2.9) immediately. \square

2.2. Preliminaries for energy estimates. To begin with, we give the uniform method to treat the equations (1.23) and (1.8) without involving the term $\mathbf{T}\mathbf{T}\eta$ in analysis.

2.2.1. *Reduction to the first order system.* We note the main equations (1.23) and (1.8) take the form of

$$\square_{\mathbf{g}}\Psi = W + \mathbf{T}Y - \text{Tr}kY, \quad (2.11)$$

which can be written as the first order equation system for a pair of functions (U, V)

$$\begin{cases} \mathbf{T}U = V + F_U \\ \mathbf{T}V = \Delta_g U + F_V + \text{Tr}kV \end{cases} \quad (2.12)$$

with

$$U = \Psi, \quad F_U = -Y, \quad F_V = -W. \quad (2.13)$$

(2.11) for $\Psi = v_+$ can be written as (2.12) with

$$U = v_+^i, \quad V = \mathbf{T}v^i, \quad F_U = -\mathbf{T}\eta^i, \quad F_V = -\tilde{\mathcal{Q}}^i + c^2\eta^i; \quad (2.14)$$

(2.11) for $\Psi = \varrho$ can be written as (2.12) with

$$U = \varrho, \quad V = \mathbf{T}\varrho, \quad F_U = 0, \quad F_V = -\mathcal{Q}^0. \quad (2.15)$$

We denote U_{v_+} and U_{ϱ} the functions U listed above for $\Psi = v_+$ or ϱ respectively. The same convention applies to V and the errors F_U and F_V . We set the vector-valued function $\mathbf{U} = (U_{v_+}, U_{\varrho})$ and $\mathbf{V} = (V_{v_+}, V_{\varrho})$ to unify both cases. In the same manner

$$F_{\mathbf{U}} = (F_{U_{v_+}}, 0), \quad F_{\mathbf{V}} = (F_{V_{v_+}}, F_{V_{\varrho}}).$$

By abuse of notation, when using U and V unless specified, we mean the components in the corresponding vectors \mathbf{U} and \mathbf{V} . Similarly, F_U and F_V may represent the components in $F_{\mathbf{U}}$ or $F_{\mathbf{V}}$ respectively. We can directly write in view of (2.14) and (2.15) that

$$F_{\mathbf{U}} = (-\mathbf{T}\eta, 0). \quad (2.16)$$

Consolidating the quadratic forms \mathcal{Q}^0 in (1.8) and $\tilde{\mathcal{Q}}^i$ in (1.23), we write both components of $F_{\mathbf{V}}$ as

$$F_V = (C(\varrho) + 1) \cdot \mathcal{Z} + c^2(\eta + \mathbf{w} \cdot \partial\varrho + \partial(\log c) \cdot \partial\eta), \quad (2.17)$$

where $C(\varrho)$ denotes some smooth functions of ϱ , which may vary when multiplied to each term in

$$\mathcal{L} = (\partial v)^2 + \partial \varrho \cdot (\partial v + \partial \varrho). \quad (2.18)$$

Here we use ∂f to denote the component of total derivative $(\partial f, \mathbf{T}f)$ for a function f .

To obtain energy estimate, it is crucial not to apply \mathbf{T} derivative to (2.11) since $\mathbf{T}^2 Y$ may not be under control by the given regularity of the data. Hence we will only apply spatial differentiation to (2.12) instead. By using (2.12), we can keep track of the error terms produced by applying the spatial derivatives or Littlewood-Paley projector P_λ with $\lambda > 1$ to (2.11).

To be more precise, by using (2.10) we differentiate (2.12) to obtain the equation system for $(U_i^{(1)}, V_i^{(1)}) = (\partial_i U, \partial_i V)$

$$\begin{cases} \mathbf{T}U^{(1)} = V^{(1)} + F_{U^{(1)}} \\ \mathbf{T}V^{(1)} = \Delta_g U^{(1)} + F_{V^{(1)}} + \text{Tr}k V^{(1)}, \end{cases} \quad (2.19)$$

where

$$\begin{aligned} F_{U^{(1)}} &= \partial_i F_U - \partial_i v^m \partial_m U \\ F_{V^{(1)}} &= \partial_i (c^2) \Delta_e U - \partial_i v^m \partial_m V - \partial_i (c^2 \partial^m (\log c)) \partial_m U + V \partial_i \text{Tr}k + \partial_i F_V. \end{aligned} \quad (2.20)$$

For calculating the above error terms, we used the commutator formula for the scalar function $f = U$,

$$\partial_i \Delta_g f - \Delta_g (\partial_i f) = \partial_i (c^2) \Delta_e f - \partial_i (c^2 \partial^j (\log c)) \partial_j f.$$

For (U, V) satisfying (2.12), by applying the Littlewood-Paley projection P_μ with $\mu > 1$ to (2.12), we obtain (2.12) for the pair of functions $(U_\mu, V_\mu) := (P_\mu U, P_\mu V)$ with F_{U_μ} and F_{V_μ} given by

$$\begin{cases} F_{U_\mu} = -[P_\mu, v^m] \partial_m U + P_\mu F_U, \\ F_{V_\mu} = [P_\mu, c^2] \Delta_e U + P_\mu F_V + [P_\mu, \text{Tr}k] V - [P_\mu, c^2 \partial_l (\log c)] \partial^l U - [P_\mu, v^m] \partial_m V. \end{cases} \quad (2.21)$$

Define the energy for (U, V) satisfying (2.12) by

$$\mathcal{E}(t) = \mathcal{E}[U](t) := \frac{1}{2} \int_{\Sigma_t} (g^{ij} \partial_i U \partial_j U + |V|^2) d\mu_g, \quad (2.22)$$

which may also be denoted by $\mathcal{E}_U(t)$. Here $d\mu_g$ is the area element of (Σ_t, g) . With $U^{(0)} = U$, $V^{(0)} = V$, we define

$$\begin{aligned} \mathcal{E}^{(m)}(t) &= \mathcal{E}[U^{(m)}](t), \quad m = 0, 1; \quad \mathcal{E}^{(\leq 1)}(t) = \sum_{m=0}^1 \mathcal{E}[U^{(m)}](t), \\ \mathcal{E}_\mu^{(1)}(t) &= \mathcal{E}[U_\mu^{(1)}](t). \end{aligned}$$

For the component of a vector-valued function such as v_+^i , we denote the sum of the energy of all components of U^i by the same notation whenever no confusion occurs.

The low order energy control can be undertaken for v and ϱ directly by using (1.7) and (1.8) under the assumption of (2.1). This will be given in (2.35) and in Corollary 2.8.

For the higher order energy, we will construct the modified current for the equation (2.11), which cancels the term $\mathbf{T}Y$ on the right hand side. Only the spatial derivative of Y will be involved in our analysis. This enables us to control energy (and H^s -energy flux in Section 6) for v_+ and ϱ by (2.11) from initial data merely by using the norm of $\|\text{curl} \Omega\|_{H_x^{s' - \frac{3}{2}}}$ and (2.1).

2.2.2. *The modified current.* We construct the modified energy current of scalar functions U ¹⁵ satisfying the equation (2.11), or identically, (2.12) with (2.13),

$$\mathcal{P}[U]_\mu = -F_U \mathbf{D}_\mu U + Q[U]_{\mu\nu} \mathbf{T}^\nu + \frac{1}{2} F_U^2 \mathbf{D}_\mu t, \quad (2.23)$$

where $Q_{\alpha\beta} := Q[f]_{\alpha\beta}$ is the standard energy momentum tensor for scalar functions f

$$Q[f]_{\alpha\beta} = \partial_\alpha f \partial_\beta f - \frac{1}{2} \mathbf{g}_{\alpha\beta} \mathbf{D}^\mu f \mathbf{D}_\mu f \quad (2.24)$$

with \mathbf{D} the Levi-Civita connection of \mathbf{g} and indices lifted or lowered by \mathbf{g} . Applying the divergence theorem to \mathcal{P}_μ in the spacetime region $\bigcup_{0 \leq t' \leq t} \Sigma_t$ yields

$$\int_{\Sigma_t} \mathcal{P}_\mu \mathbf{T}^\mu = \int_{\Sigma_0} \mathcal{P}_\mu \mathbf{T}^\mu - \int_0^t \int_{\Sigma_{t'}} \mathbf{D}^\mu \mathcal{P}_\mu, \quad (2.25)$$

where we hide the standard volume element on Σ_t and $\bigcup_{0 \leq t' \leq t} \Sigma_{t'}$ which are $d\mu_g$ and $d\mu_g dt'$. $d\mu_g$ is always comparable to $d\mu_e$ due to (2.6).

Now we show

$$\int_{\Sigma_t} \mathbf{T}^\mu \mathcal{P}_\mu = \mathcal{E}(t), \quad (2.26)$$

$$\mathbf{D}^\mu \mathcal{P}_\mu = -F_V \cdot V - \mathbf{D}^i F_U \mathbf{D}_i U - (k^{ij} - \frac{1}{2} \text{Tr} k c^2 \delta^{ji}) \mathbf{D}_i U \mathbf{D}_j U - \frac{1}{2} \text{Tr} k V^2. \quad (2.27)$$

Indeed, we compute

$$\begin{aligned} \mathbf{T}^\mu \mathcal{P}_\mu &= -F_U \mathbf{T} U + Q_{\mu\nu} \mathbf{T}^\mu \mathbf{T}^\nu + \frac{1}{2} F_U^2 \mathbf{T}(t) \\ &= \frac{1}{2} (\mathbf{T} U \mathbf{T} U + \mathbf{D}^i U \mathbf{D}_i U) + \frac{1}{2} F_U^2 - F_U \mathbf{T} U \\ &= \frac{1}{2} (|V|^2 + c^2 \delta^{ij} \partial_i U \partial_j U). \end{aligned}$$

This is identical to the integrand of (2.22). Thus (2.26) is proved.

Since (2.12) implies $V = -F_U + \mathbf{T} U$, we have from the definition of \mathcal{P}_μ that

$$\begin{aligned} \mathbf{D}^\mu \mathcal{P}_\mu &= -\mathbf{D}^\mu F_U \mathbf{D}_\mu U - F_U \square_{\mathbf{g}} U + \mathbf{D}^\mu Q_{\mu\nu} \mathbf{T}^\nu + \frac{1}{2} Q_{\mu\nu}^{(\mathbf{T})} \pi^{\mu\nu} \\ &\quad + F_U \mathbf{D}^\mu F_U \mathbf{D}_\mu t + \frac{1}{2} F_U^2 \square_{\mathbf{g}} t \\ &= \mathbf{T} F_U \mathbf{T} U - \mathbf{D}^i F_U \mathbf{D}_i U - F_U \square_{\mathbf{g}} U + \square_{\mathbf{g}} U \mathbf{D}_{\mathbf{T}} U + \frac{1}{2} Q_{\mu\nu}^{(\mathbf{T})} \pi^{\mu\nu} \\ &\quad - F_U \mathbf{T} F_U + \frac{1}{2} \text{Tr} k F_U^2 \\ &= (-F_U + \mathbf{T} U) (\square_{\mathbf{g}} U + \mathbf{T} F_U) - \mathbf{D}^i F_U \mathbf{D}_i U + \frac{1}{2} Q_{\mu\nu}^{(\mathbf{T})} \pi^{\mu\nu} + \frac{1}{2} \text{Tr} k F_U^2. \end{aligned}$$

Here $^{(\mathbf{T})} \pi := \mathcal{L}_{\mathbf{T}} \mathbf{g}$ is the deformation tensor of \mathbf{T} . In view of (2.11) and (2.13), we then obtain

$$\mathbf{D}^\mu \mathcal{P}_\mu = (-F_V + \text{Tr} k F_U) V - \mathbf{D}^i F_U \mathbf{D}_i U + \frac{1}{2} Q_{\mu\nu}^{(\mathbf{T})} \pi^{\mu\nu} + \frac{1}{2} \text{Tr} k F_U^2. \quad (2.28)$$

¹⁵This is adapted from the one introduced in [42, Section 3.2].

Note ${}^{(\mathbf{T})}\pi_{ij} = -2k_{ij}$ is the non-trivial part of ${}^{(\mathbf{T})}\pi$. By using the first equation in (2.12) we have

$$\begin{aligned} -\frac{1}{2}Q_{\mu\nu}{}^{(\mathbf{T})}\pi^{\mu\nu} &= Q_{ij}k^{ij} = k^{ij}(\mathbf{D}_i U \mathbf{D}_j U - \frac{1}{2}\mathbf{g}_{ij}\mathbf{D}^\alpha U \mathbf{D}_\alpha U) \\ &= k^{ij}\mathbf{D}_i U \mathbf{D}_j U - \frac{1}{2}\text{Tr}k \mathbf{D}^i U \mathbf{D}_i U + \frac{1}{2}\text{Tr}k((F_U)^2 + 2VF_U + V^2). \end{aligned}$$

Substituting the above identity to (2.28) implies (2.27) in view of (2.11).

Lemma 2.2 (Fundamental inequalities for higher order energies). *If $\|\partial v\|_{L_t^1 L_x^\infty([0,T]\times\mathbb{R}^3)} \lesssim 1$, there hold with $\alpha > 0$ that*

$$\mathcal{E}(t)^{\frac{1}{2}} \lesssim \mathcal{E}(0)^{\frac{1}{2}} + \int_0^t (\|\partial F_U\|_{L_x^2} + \|F_V\|_{L_x^2}) dt', \quad (2.29)$$

$$\mathcal{E}^{(1)}(t)^{\frac{1}{2}} \lesssim \mathcal{E}^{(1)}(0)^{\frac{1}{2}} + \int_0^t (\|\partial F_{U^{(1)}}\|_{L_x^2} + \|F_{V^{(1)}}\|_{L_x^2}) dt', \quad (2.30)$$

$$\|\lambda^\alpha \mathcal{E}_\lambda^{(1)}(t)^{\frac{1}{2}}\|_{l_\lambda^2} \lesssim \|\lambda^\alpha \mathcal{E}_\lambda^{(1)}(0)^{\frac{1}{2}}\|_{l_\lambda^2} + \int_0^t (\|\lambda^\alpha \partial F_{U_\lambda^{(1)}}\|_{l_\lambda^2 L_x^2} + \|\lambda^\alpha F_{V_\lambda^{(1)}}\|_{l_\lambda^2 L_x^2}) dt'. \quad (2.31)$$

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Remark 2.3. Since $\int_{\Sigma_t} |\nabla f|_g^2 d\mu_g \approx \int_{\Sigma_t} |\partial f|^2 d\mu_e$, with $d\mu_e$ the volume element of the Euclidean metric, we will not distinguish the metric used for \dot{H}_1 norm. And the assumption $\|\partial v\|_{L_t^1 L_x^\infty([0,T]\times\mathbb{R}^3)} \lesssim 1$ is a consequence of (2.1).

Proof. We first derive from (2.27) that

$$\int_0^t \int_{\Sigma_{t'}} |\mathbf{D}^\alpha \mathcal{P}_\alpha| \lesssim \int_0^t \{(\|F_V(t')\|_{L_x^2} + \|\partial F_U(t')\|_{L_x^2}) \mathcal{E}^{\frac{1}{2}}(t') + \|k(t')\|_{L_x^\infty} \mathcal{E}(t')\} dt'. \quad (2.32)$$

Note $|c', c^{-1}| \lesssim 1$ due to (2.6) and (2.7). We can bound $|\mathbf{T} \log c| \lesssim |\mathbf{T} \varrho|$. Hence using (2.8) and the first equation in (1.4), we have

$$|k| \lesssim |\partial v|. \quad (2.33)$$

In view of (2.33), substituting (2.32) and (2.26) into (2.25) gives

$$\mathcal{E}(t) \leq \mathcal{E}(0) + \int_0^t \{(\|F_V(t')\|_{L_x^2} + \|\partial F_U(t')\|_{L_x^2}) \mathcal{E}^{\frac{1}{2}}(t') + \|\partial v(t')\|_{L_x^\infty} \mathcal{E}(t')\} dt'.$$

(2.29) follows immediately by using Gronwall's inequality and the bound $\|\partial v\|_{L_t^1 L_x^\infty([0,T]\times\mathbb{R}^3)} \lesssim 1$.

Applying (2.29) to $(U^{(1)}, V^{(1)})$ with error terms verifying (2.20), we can obtain (2.30). Applying (2.29) to $(U_\lambda^{(1)}, V_\lambda^{(1)})$ with error terms (2.21) substituted by $(U^{(1)}, V^{(1)})$ gives (2.31). \square

Corollary 2.4. *Under the assumption of (2.1), there hold the energy estimates for $l = 0, 1$,*

$$\mathcal{E}^{(l)}(t)^{\frac{1}{2}} \lesssim \mathcal{E}^{(l)}(0)^{\frac{1}{2}} + \int_0^t \{ \|\partial \partial^{(l)} F_U, \partial^{(l)} F_V\|_{L_x^2} + l(\|\partial v\|_{H_x^1} + \|\partial \varrho\|_{H_x^1})(\|\partial U\|_{L_x^\infty} + \|V\|_{L_x^\infty}) \} dt'. \quad (2.34)$$

¹⁶In this paper, for any f , $\|f\|_{l_\lambda^2} := (\sum_{\lambda>1} |f_\lambda|^2)^{\frac{1}{2}}$ with λ dyadic.

Proof. The case of $l = 0$ in (2.34) is (2.29). We only need to consider the first order estimate, for which we need to bound the integrand of the right hand side of (2.30). In view of (2.20), we compute that

$$\begin{aligned} \|\partial F_{U^{(1)}}\|_{L_x^2} &\lesssim \|\partial^2 F_U\|_{L_x^2} + \|\partial v\|_{L_x^\infty} \|\partial^2 U\|_{L_x^2} + \|\partial^2 v\|_{L_x^2} \|\partial U\|_{L_x^\infty}, \\ \|F_{V^{(1)}}\|_{L_x^2} &\lesssim \|\partial \varrho\|_{L_x^\infty} \|\Delta_e U\|_{L_x^2} + \|\partial v\|_{L_x^\infty} \|\partial V\|_{L_x^2} + \|\partial \text{Tr} k\|_{L_x^2} \|V\|_{L_x^\infty} \\ &\quad + \|\partial^2(c^2)\|_{L_x^2} \|\partial U\|_{L_x^\infty} + \|\partial F_V\|_{L_x^2} \\ &\lesssim \|\partial \varrho, \partial v\|_{L_x^\infty} \mathcal{E}^{(1)}(t)^{\frac{1}{2}} + (\|V\|_{L_x^\infty} + \|\partial U\|_{L_x^\infty}) (\|\partial^2 c\|_{L_x^2} + \|\partial \text{Tr} k\|_{L_x^2}) + \|\partial F_V\|_{L_x^2}. \end{aligned}$$

By using (2.9) and the first equation in (1.4), $|c, c', c'', c^{-1}| \lesssim 1$, and Sobolev embedding, we bound

$$\|\partial \text{Tr} k\|_{L_x^2} + \|\partial^2(c^2)\|_{L_x^2} \lesssim \|\partial v\|_{H_x^1} + \|\partial \varrho\|_{H_x^1}.$$

Combining the above two estimates, we can obtain (2.34). \square

The purpose of introducing the modified current is mainly to control the higher order energy. For the low order estimates, we adopt the standard method.

Proposition 2.5 (0-order energy). *Under the assumption (2.1), there holds*

$$\|\partial v, \partial \varrho\|_{L^2(\Sigma_t)} \lesssim \|\partial v, \partial \varrho\|_{L^2(\Sigma_0)} + \int_0^t \|\text{curl} \Omega(t')\|_{L_x^2} dt'. \quad (2.35)$$

Proof. To prove (2.35), we recall the standard energy approach. By applying the divergence theorem to the energy current $\mathcal{P}_\alpha^{(\mathbf{T})}[f] = Q[f]_{\alpha\beta} \mathbf{T}^\beta$, we can obtain the energy identity

$$\int Q_{\alpha\beta} \mathbf{T}^\alpha \mathbf{T}^\beta(t) d\mu_g - \int Q_{\alpha\beta} \mathbf{T}^\alpha \mathbf{T}^\beta(0) d\mu_g = - \int_{[0,t] \times \mathbb{R}^3} \left(\square_{\mathbf{g}} f \mathbf{D}_{\mathbf{T}} f + \frac{1}{2} (\mathbf{T}) \pi_{\alpha\beta} Q^{\alpha\beta} \right). \quad (2.36)$$

For any smooth scalar function, since

$$Q[f]_{\mathbf{T}\mathbf{T}} = \frac{1}{2} ((\mathbf{T}f)^2 + c^2 \delta^{ij} \partial_i f \partial_j f),$$

there holds for $0 \leq t \leq T$

$$\int_{\Sigma_t} Q[f]_{\mathbf{T}\mathbf{T}} d\mu_g \approx \int_{\Sigma_t} (|\partial f|^2 + |\mathbf{T}f|^2) d\mu_e = \|\partial f(t)\|_{L_x^2}^2, \quad (2.37)$$

where the constants in the symbol \approx depend on c_0 and C_0 .

Since the non-trivial component of the deformation tensor $(\mathbf{T})\pi_{ij} = -2k_{ij}$, we use (2.33) to bound $|Q^{\alpha\beta}(\mathbf{T})\pi_{\alpha\beta}| \lesssim |\partial v| |\partial f|^2$. Due to (2.1),

$$\|\partial f(t)\|_{L_x^2} \lesssim \|\partial f(0)\|_{L_x^2} + \int_0^t \|\square_{\mathbf{g}} f(t')\|_{L_x^2} dt'. \quad (2.38)$$

Applying (2.38) to (1.7) for v and to (1.8) for ϱ , and using (1.4) lead to

$$\begin{aligned} &\|\partial v(t)\|_{L_x^2} + \|\partial \varrho(t)\|_{L_x^2} \\ &\lesssim \|\partial v(0)\|_{L_x^2} + \|\partial \varrho(0)\|_{L_x^2} + \int_0^t (\|\square_{\mathbf{g}} v\|_{L_x^2} + \|\square_{\mathbf{g}} \varrho\|_{L_x^2}) dt' \\ &\lesssim \|\partial v(0)\|_{L_x^2} + \|\partial \varrho(0)\|_{L_x^2} + \int_0^t \{ (|\partial \varrho| + |\partial v|)^2 \|_{L_x^2} + \|\partial \varrho \Omega\|_{L_x^2} + \|\text{curl} \Omega(t')\|_{L_x^2} \} dt' \\ &\lesssim \|\partial v(0)\|_{L_x^2} + \|\partial \varrho(0)\|_{L_x^2} + \int_0^t \|\partial \varrho, \partial v\|_{L_x^\infty} \|\partial \varrho, \partial v, \Omega\|_{L_x^2} + \|\text{curl} \Omega\|_{L_x^2} \} dt'. \end{aligned}$$

Since $|\Omega| \lesssim |\partial v|$, we can incorporate this term as part of ∂v in the last line. The consequence drops out by using Gronwall's inequality and (2.1). \square

In order to carry out energy estimate for the dyadic pairs (U_μ, V_μ) , we need to derive a series of product estimates and commutator estimates with Littlewood-Paley theory. Since the estimates are not limited to the applications in the energy estimates, we provide them in Section 10.

Now we give the energy inequality of the highest order.

Proposition 2.6. *With $0 < \alpha < 1$, under the assumption of (2.1), there holds*

$$\begin{aligned} \|\mu^\alpha \mathcal{E}_\mu^{(1)}(t)^{\frac{1}{2}}\|_{l_\mu^2 L_x^2} &\lesssim \|\mu^\alpha \mathcal{E}_\mu^{(1)}(0)^{\frac{1}{2}}\|_{l_\mu^2 L_x^2} + T^{\frac{1}{2}} \sup_{0 \leq t' \leq t} \mathcal{E}^{(\leq 1)}(t')^{\frac{1}{2}} \\ &\quad + \int_0^t (\|\partial v, \partial(c^2), \text{Trk}\|_{H_x^{1+\alpha}} \|\partial U, V\|_{L_x^\infty} + \|\partial(\partial F_U, F_V)\|_{\dot{H}_x^\alpha}) dt'. \end{aligned}$$

Proof. In view of (2.31), we control the integrand with the help of (2.21). Recall that

$$\begin{aligned} F_{U_\mu^{(1)}} &= -[P_\mu, v^m] \partial_m U^{(1)} + P_\mu F_{U^{(1)}} \\ F_{V_\mu^{(1)}} &= [P_\mu, c^2] \Delta_e U^{(1)} + P_\mu F_{V^{(1)}} + [P_\mu, \text{Trk}] V^{(1)} - \frac{1}{2} [P_\mu, \partial(c^2)] \partial U^{(1)} - [P_\mu, v^m] \partial_m V^{(1)}. \end{aligned}$$

To treat the commutators, we will employ commutator estimates provided in Section 10. By applying (10.4) to $(F, G) = (v, U^{(1)})$, we can bound

$$\begin{aligned} \|\mu^\alpha \partial F_{U_\mu^{(1)}}\|_{l_\mu^2 L_x^2} &\lesssim \|\mu^\alpha \partial [P_\mu, v] \partial U^{(1)}\|_{l_\mu^2 L_x^2} + \|\mu^\alpha \partial P_\mu F_{U^{(1)}}\|_{l_\mu^2 L_x^2} \\ &\lesssim \|\mu^\alpha \partial P_\mu F_{U^{(1)}}\|_{l_\mu^2 L_x^2} + \|\partial v\|_{L_x^\infty} \|\partial U^{(1)}\|_{H_x^\alpha} + \|\partial v\|_{H_x^{1+\alpha}} \|U^{(1)}\|_{L_x^\infty}. \end{aligned} \quad (2.39)$$

To estimate $\|\mu^\alpha F_{V_\mu^{(1)}}\|_{l_\mu^2 L_x^2}$, we first apply (10.2) to $(F, G) = (c^2, \partial U^{(1)})$ and $(v, V^{(1)})$ to derive

$$\|\mu^\alpha [P_\mu, c^2] \Delta_e U^{(1)}\|_{l_\mu^2 L_x^2} + \|\mu^\alpha [P_\mu, v] \partial V^{(1)}\|_{l_\mu^2 L_x^2} \lesssim \|\partial \varrho\|_{L_x^\infty} \|\partial U^{(1)}\|_{H_x^\alpha} + \|\partial v\|_{L_x^\infty} \|V^{(1)}\|_{H_x^\alpha}.$$

Note $V^{(1)} = \partial V$. Applying (10.3) to $(F, G) = (\partial(c^2), U^{(1)})$ and (Trk, V) yields

$$\begin{aligned} \|\mu^\alpha [P_\mu, \partial(c^2)] \partial U^{(1)}\|_{l_\mu^2 L_x^2} + \|\mu^\alpha [P_\mu, \text{Trk}] V^{(1)}\|_{l_\mu^2 L_x^2} \\ \lesssim \|\partial \varrho, \text{Trk}\|_{L_x^\infty} \|\partial U^{(1)}, V^{(1)}\|_{H_x^\alpha} + \|\partial(c^2), \text{Trk}\|_{H_x^{1+\alpha}} \|U^{(1)}, V\|_{L_x^\infty}. \end{aligned}$$

Combining the above two estimates, the formula of $F_{V_\mu^{(1)}}$ and also using (2.33), we have

$$\begin{aligned} \|\mu^\alpha F_{V_\mu^{(1)}}\|_{l_\mu^2 L_x^2} &\lesssim \|\mu^\alpha P_\mu F_{V^{(1)}}\|_{l_\mu^2 L_x^2} + \|\partial \varrho, \partial v\|_{L_x^\infty} \|\partial U^{(1)}, V^{(1)}\|_{H_x^\alpha} \\ &\quad + \|\partial(c^2), \text{Trk}\|_{H_x^{1+\alpha}} \|U^{(1)}, V\|_{L_x^\infty}. \end{aligned} \quad (2.40)$$

It remains to bound the first terms on the right hand side of (2.39) and (2.40). In view of (2.20), we apply (10.8) to $(F, G) = (\partial v, \partial U)$ to derive

$$\begin{aligned} \|\mu^\alpha P_\mu \partial F_{U^{(1)}}\|_{l_\mu^2 L_x^2} &\lesssim \|\partial^2 F_U\|_{\dot{H}_x^\alpha} + \|\mu^\alpha P_\mu \partial(\partial v \cdot \partial U)\|_{l_\mu^2 L_x^2} \\ &\lesssim \|\partial^2 F_U\|_{\dot{H}_x^\alpha} + \|\partial^2 v\|_{H_x^\alpha} \|\partial U\|_{L_x^\infty} + \|\partial v\|_{L_x^\infty} \|\partial^2 U\|_{H_x^\alpha}. \end{aligned}$$

Applying (10.7) to $(F, G) = (V, \text{Trk}), (\partial U, \partial(c^2)), (\partial v, V)$ and $(\partial(c^2), \partial U)$ yields

$$\begin{aligned} \|\mu^\alpha P_\mu F_{V^{(1)}}\|_{l_\mu^2 L_x^2} &\lesssim \|\partial v, \text{Trk}\|_{H_x^{1+\alpha}} \|V\|_{L_x^\infty} + \|\partial(c^2)\|_{H_x^{1+\alpha}} \|\partial U\|_{L_x^\infty} \\ &\quad + \|\partial v, \partial \varrho\|_{L_x^\infty} \|V, \partial U\|_{H_x^{1+\alpha}} + \|\partial F_V\|_{\dot{H}_x^\alpha}, \end{aligned}$$

where we also used (2.33) to bound $|\text{Trk}| \lesssim |\partial v|$.

We summarize the above estimates as

$$\begin{aligned} & \|\mu^\alpha P_\mu \partial F_{U^{(1)}}\|_{l_\mu^2 L_x^2} + \|\mu^\alpha P_\mu F_{V^{(1)}}\|_{l_\mu^2 L_x^2} \\ & \lesssim \|\partial v, \partial(c^2), \text{Tr}k\|_{H^{1+\alpha}} \|V, \partial U\|_{L_x^\infty} + \|\partial v, \partial \varrho\|_{L_x^\infty} \|V, \partial U\|_{H_x^{1+\alpha}} + \|\partial^2 F_U\|_{\dot{H}_x^\alpha} + \|\partial F_V\|_{\dot{H}_x^\alpha}. \end{aligned} \quad (2.41)$$

Substituting the inequality to (2.31) implies that for $0 < \alpha < 1$,

$$\begin{aligned} \|\mu^\alpha \mathcal{E}_\mu(t)^{\frac{1}{2}}\|_{l_\mu^2 L_x^2} & \lesssim \|\mu^\alpha \mathcal{E}_\mu^{\frac{1}{2}}(0)\|_{l_\mu^2 L_x^2} + \int_0^t \|\partial v, \partial \varrho\|_{L_x^\infty} (\|\mu^\alpha \mathcal{E}_\mu^{(1)}(t')^{\frac{1}{2}}\|_{l_\mu^2} + \mathcal{E}^{(\leq 1)}(t')^{\frac{1}{2}}) dt' \\ & \quad + \int_0^t (\|\partial v, \partial(c^2), \text{Tr}k\|_{H_x^{1+\alpha}} \|\partial U, V\|_{L_x^\infty} + \|\partial(\partial F_U, F_V)\|_{\dot{H}_x^\alpha}) dt'. \end{aligned}$$

Proposition 2.6 follows by applying the Gronwall's inequality with the help of $\|\partial v, \partial \varrho\|_{L_t^1 L_x^\infty} \lesssim 1$ due to (2.1). \square

The main task will be to control $\|\partial(\partial F_U, F_V)\|_{L_t^1 H_x^\alpha}$ with $0 \leq \alpha \leq s-2$ for both $U = v_+$ and $U = \varrho$. In particular for $U = v_+$, due to (2.14), we need to provide estimates for $\partial \eta$. This will be carried out in the following subsection.

2.3. Preliminary Estimates for Ω and η . We first rely on the definition of η in (1.21) and the equation (1.6) to prove the following estimates for the vorticity and η .

Lemma 2.7. (1) For any $p \geq 2$,

$$\|\Omega(t)\|_{L_x^p} \lesssim \|\Omega(0)\|_{L_x^p} \lesssim 1. \quad (2.42)$$

(2) Let $0 < \epsilon \leq s-2$. For any $2 \leq p \leq \frac{3}{1-\epsilon}$, there hold

$$\|\partial \Omega, \partial \mathfrak{w}\|_{L_x^p} \lesssim \|\partial \varrho\|_{L_x^p} + 1, \quad (2.43)$$

$$\|\text{curl} \Omega, \mathfrak{C}\|_{L_x^p} \lesssim 1, \quad (2.44)$$

$$\|\partial^2 \eta\|_{L_x^p} \lesssim \|\partial \varrho\|_{L_x^p} + 1. \quad (2.45)$$

Substituting the estimate (2.44) into (2.35) implies the lowest order energy estimate,

Corollary 2.8.

$$\|\partial \varrho, \partial v\|_{L^2(\Sigma_t)} \lesssim \|\partial \varrho, \partial v\|_{L^2(\Sigma_0)} + T \lesssim 1.$$

Proof of Lemma 2.7. The first inequality in (2.42) can be obtained by integrating (1.6) with the help of the bound $\|\partial v\|_{L_t^1 L_x^\infty} \lesssim 1$ due to (2.1). The second one is due to Sobolev embedding $\|\Omega(0)\|_{L_x^p} \lesssim \|\Omega(0)\|_{H_x^{\frac{3}{2}+}} \lesssim 1$ for all $p > 2$.

Similarly, by integrating (1.9), for $2 \leq p \leq \frac{3}{1-\epsilon}$

$$\|\mathfrak{C}(t)\|_{L_x^p} \lesssim \|\mathfrak{C}(0)\|_{L_x^p} + \int_0^t \|\partial v\|_{L_x^\infty} \|\partial \Omega\|_{L_x^p} dt. \quad (2.46)$$

Recall from (1.5) and the definition of \mathfrak{C} that

$$\text{div} \Omega = -\Omega^a \partial_a \varrho, \quad \text{curl} \Omega = e^\varrho \mathfrak{C}.$$

The L^p estimate for the above Hodge system gives

$$\|\partial \Omega\|_{L_x^p} \lesssim \|\Omega \cdot \partial \varrho\|_{L_x^p} + \|\mathfrak{C}\|_{L_x^p}.$$

Substituting the above estimate into (2.46) implies

$$\|\mathfrak{C}(t)\|_{L_x^p} \lesssim \|\mathfrak{C}(0)\|_{L_x^p} + \int_0^t \|\partial v\|_{L_x^\infty} (\|\Omega \cdot \partial \varrho\|_{L_x^p} + \|\mathfrak{C}\|_{L_x^p}) dt'.$$

By using (2.42) for the estimate of $\|\Omega\|_{L_x^p}$ and applying (2.1) for $\|\partial v, \partial \varrho\|_{L_t^2 L_x^\infty} \lesssim 1$, we can obtain

$$\|\mathfrak{C}(t)\|_{L_x^p} \lesssim 1.$$

This gives the estimate of (2.44) and we can bound

$$\|\partial \Omega\|_{L_x^p} \lesssim \|\Omega\|_{L_x^\infty} \|\partial \varrho\|_{L_x^p} + 1 \lesssim \|\partial \varrho\|_{L_x^p} + 1$$

which is the first estimate in (2.43).

It is straightforward to compute,

$$\operatorname{curl} \mathfrak{w}_i = \operatorname{curl} (\Omega e^\varrho)_i = \epsilon_i^{mn} (\partial_m \Omega_n + \Omega_n \partial_m \varrho) e^\varrho.$$

We hence have obtained the Hodge system

$$\operatorname{div} \mathfrak{w} = 0, \quad \operatorname{curl} \mathfrak{w}_n = e^\varrho ((\operatorname{curl} \Omega)_n + \epsilon_n^{ij} \Omega_j \partial_i \varrho). \quad (2.47)$$

It follows by the L^p estimate for the above Hodge system, (2.44) and (2.42) that

$$\|\partial \mathfrak{w}\|_{L_x^p} \lesssim \|\mathfrak{C}\|_{L_x^p} + \|\partial \varrho\|_{L_x^p} \|\Omega\|_{L_x^\infty} \lesssim 1 + \|\partial \varrho\|_{L_x^p}, \quad (2.48)$$

which is the second estimate of (2.43).

Also in view of (2.47), (2.44) and (2.42), we have

$$\|\partial^2 \eta\|_{L_x^p} \lesssim \|\partial^2 \Lambda^{-2} (\operatorname{curl} \mathfrak{w})\|_{L_x^p} \lesssim \|\Omega \cdot \partial \varrho\|_{L_x^p} + \|\operatorname{curl} \Omega\|_{L_x^p} \lesssim \|\partial \varrho\|_{L_x^p} + 1.$$

Here we recall that Λ^{-2} is the inverse operator of $\Lambda^2 = I - \Delta_e$. This gives (2.45). The proof of Lemma 2.7 is complete. \square

Next, we give more estimates on η .

Proposition 2.9. *There hold the following estimates for $t \in [0, T]$,*

$$\|\eta\|_{H_x^2} \lesssim 1, \quad \|\eta\|_{L_x^\infty} \lesssim 1, \quad (2.49)$$

$$\|\mathbf{T}\eta\|_{H_x^1} \lesssim 1 + l(\|\partial \eta\|_{L_x^\infty} + \|\partial v\|_{H_x^1}(\|\partial \varrho\|_{L_x^3} + 1)), \quad l = 0, 1 \quad (2.50)$$

$$\|\partial \mathbf{T}\eta\|_{H_x^1} \lesssim \|\partial v\|_{H_x^1}(\|\partial \eta\|_{L_x^\infty} + \|\partial \varrho\|_{L_x^3} + 1). \quad (2.51)$$

Proof. The first estimate in (2.49) is obtained by using (2.45) and Corollary 2.8, the second one follows immediately by Sobolev embedding.

Consider the estimate of $\mathbf{T}\eta$. We first derive the symbolic formula

$$\mathbf{T}(\operatorname{curl} \mathfrak{w})_n = \partial(\partial v \Omega e^\varrho) + \mathfrak{C} e^{2\varrho} \partial v + \partial v \cdot \partial \varrho \Omega e^\varrho. \quad (2.52)$$

Indeed, by using the second identity in (2.47), (1.6), (1.9) and the first equation in (1.4) that

$$\begin{aligned} \mathbf{T}(\operatorname{curl} \mathfrak{w})_n &= \mathbf{T}(\mathfrak{C}_n e^{2\varrho}) + \mathbf{T}(\Omega_j \epsilon_n^{ij} \partial_i (e^\varrho)) \\ &= \mathbf{T} \mathfrak{C}_n e^{2\varrho} + \mathfrak{C}_n \mathbf{T}(e^{2\varrho}) + \mathbf{T} \Omega_j \epsilon_n^{ij} \partial_i (e^\varrho) + \Omega_j \epsilon_n^{ij} \mathbf{T} \partial_i (e^\varrho) \\ &= (\mathbf{T} \mathfrak{C}_n + 2 \mathfrak{C}_n \mathbf{T} \varrho) e^{2\varrho} + \Omega^a \partial_a v_j \epsilon_n^{ij} \partial_i (e^\varrho) + \Omega_j \epsilon_n^{ij} (\partial_i \mathbf{T}(e^\varrho) + [\mathbf{T}, \partial_i] e^\varrho) \\ &= -2 \partial_b (\Omega_j \partial_a v^j \epsilon_n^{ab} e^\varrho) + \partial_a v_n \operatorname{curl} \Omega^a e^\varrho + 2 \mathfrak{C}_n e^{2\varrho} \mathbf{T} \varrho + \partial_i (\Omega_j \epsilon_n^{ij} \mathbf{T}(e^\varrho)) \\ &\quad - (\operatorname{curl} \Omega)_n \mathbf{T}(e^\varrho) + \Omega \partial v \partial e^\varrho \\ &= -2 \partial_b (\Omega_j \partial_a v^j \epsilon_n^{ab} e^\varrho) + \partial_i (\Omega_j \epsilon_n^{ij} \mathbf{T}(e^\varrho)) + \mathfrak{C}_n e^{2\varrho} \partial v + \Omega \partial v \partial e^\varrho, \end{aligned}$$

where we used (2.10) and also have written the higher order terms on the right hand side into divergence form. By using the first equation in (1.4) again, we can obtain (2.52).

We next show the m -order derivative estimates

$$\|\partial^m \Lambda^{-2} \mathbf{T} \operatorname{curl} \mathfrak{w}\|_{H_x^1} \lesssim m \|\partial v\|_{H_x^1} \|\partial \varrho\|_{L_x^3} + \|\partial v\|_{H_x^1}^{\max(m-1, 0)}, \quad m = 0, 1, 2, \quad (2.53)$$

by using the following standard estimates for scalar functions F ,

$$\|\Lambda^{-1}F\|_{L_x^2} \lesssim \|F\|_{L_x^{\frac{6}{5}}}, \quad \|\Lambda^{-2}F\|_{L_x^2} \lesssim \|F\|_{L_x^1}, \quad (2.54)$$

which follows directly from the duality argument, Sobolev embedding and L^2 estimates for the operator $\partial\Lambda^{-1}$. The constants in the inequalities are the universal Sobolev constants.

By using (2.54), in view of (2.52), we derive by using (2.42), (2.44), Corollary 2.8 and Sobolev embedding that

$$\begin{aligned} \|\Lambda^{-2}\mathbf{T} \operatorname{curl} \mathbf{w}\|_{L_x^2} &\lesssim \|\Lambda^{-2}\partial(\partial v \Omega e^\varrho)\|_{L_x^2} + \|\Lambda^{-2}(\partial v(e^\varrho \mathfrak{C} + \partial \varrho \Omega)e^\varrho)\|_{L_x^2} \\ &\lesssim \|\partial v \cdot \Omega\|_{L_x^{\frac{6}{5}}} + \|\partial v e^\varrho(\mathfrak{C} e^\varrho + \partial \varrho \Omega)\|_{L_x^1} \\ &\lesssim \|\partial v\|_{L_x^2} \|\Omega\|_{L_x^3} + \|\partial v\|_{L_x^2} (\|\partial \varrho\|_{L_x^2} \|\Omega\|_{L_x^\infty} + \|\mathfrak{C}\|_{L_x^2}) \lesssim 1; \end{aligned}$$

and

$$\begin{aligned} \|\partial\Lambda^{-2}\mathbf{T} \operatorname{curl} \mathbf{w}\|_{L_x^2} &\lesssim \|\partial v\|_{L_x^2} \|\Omega\|_{L_x^\infty} + \|\Lambda^{-1}(\partial v(e^\varrho \mathfrak{C} + \partial \varrho \Omega)e^\varrho)\|_{L_x^2} \\ &\lesssim 1 + \|\partial v(e^\varrho \mathfrak{C} + \partial \varrho \Omega)e^\varrho\|_{L_x^{\frac{6}{5}}} \\ &\lesssim 1 + \|\mathfrak{C}\|_{L_x^3} \|\partial v\|_{L_x^2} + \|\Omega\|_{L_x^\infty} \|\partial v\|_{L_x^6} \|\partial \varrho\|_{L_x^3} \\ &\lesssim 1 + \|\partial v\|_{H_x^1} \|\partial \varrho\|_{L_x^3}. \end{aligned}$$

Similarly, also using (2.43)

$$\begin{aligned} \|\partial^2\Lambda^{-2}\mathbf{T} \operatorname{curl} \mathbf{w}\|_{L_x^2} &\lesssim \|\partial(\partial v \Omega e^\varrho)\|_{L_x^2} + \|\partial v(\mathfrak{C} e^\varrho + \Omega \partial \varrho) e^\varrho\|_{L_x^2} \\ &\lesssim \|\partial^2 v\|_{L_x^2} \|\Omega\|_{L_x^\infty} + \|\partial v\|_{L_x^6} \|\mathfrak{C}, \partial \Omega\|_{L_x^3} + \|\Omega\|_{L_x^\infty} \|\partial v\|_{L_x^6} \|\partial \varrho\|_{L_x^3} \\ &\lesssim \|\partial v\|_{H_x^1} (\|\partial \varrho\|_{L_x^3} + 1). \end{aligned}$$

Therefore (2.53) is proved.

On the other hand by the definition of η in (1.21)

$$\mathbf{T}(I - \Delta_e)\eta = \mathbf{T} \operatorname{curl} \mathbf{w}$$

which implies

$$\mathbf{T}\eta = (I - \Delta_e)^{-1}(-[\mathbf{T}, I - \Delta_e]\eta + \mathbf{T} \operatorname{curl} \mathbf{w}).$$

By using (2.10), there holds symbolically that

$$[\mathbf{T}, \Delta_e]\eta = \partial(\partial v \partial \eta) + \partial v \cdot \partial^2 \eta.$$

Thus,

$$\mathbf{T}\eta = \Lambda^{-2}(\partial(\partial v \partial \eta) + \partial v \partial^2 \eta + \mathbf{T} \operatorname{curl} \mathbf{w}). \quad (2.55)$$

By using (2.54), we first give the base order estimate for $\mathbf{T}\eta$ with the help of the above identity,

$$\begin{aligned} \|\mathbf{T}\eta\|_{L_x^2} &\leq \|\Lambda^{-1}(\partial v \partial \eta)\|_{L_x^2} + \|\Lambda^{-2}(\partial v \partial^2 \eta)\|_{L_x^2} + \|\Lambda^{-2}\mathbf{T} \operatorname{curl} \mathbf{w}\|_{L_x^2} \\ &\lesssim \|\partial v \partial \eta\|_{L_x^{\frac{6}{5}}} + \|\partial v \cdot \partial^2 \eta\|_{L_x^1} + \|\Lambda^{-2}\mathbf{T} \operatorname{curl} \mathbf{w}\|_{L_x^2} \\ &\lesssim \|\partial v\|_{L_x^2} (\|\partial \eta\|_{L_x^3} + \|\partial^2 \eta\|_{L_x^2}) + \|\Lambda^{-2}\mathbf{T} \operatorname{curl} \mathbf{w}\|_{L_x^2} \\ &\lesssim 1 + \|\Lambda^{-2}\mathbf{T} \operatorname{curl} \mathbf{w}\|_{L_x^2}, \end{aligned}$$

where we employed Corollary 2.8, Sobolev embedding and the first estimate in (2.49).

For higher order derivatives, in view of (2.55), using (2.49), Corollary 2.8, Sobolev embedding and the first estimate in (2.54), we derive

$$\begin{aligned}
\|\partial \mathbf{T} \eta\|_{L_x^2} &\lesssim \|\partial v \cdot \partial \eta\|_{L_x^2} + \|\Lambda^{-1}(\partial v \cdot \partial^2 \eta)\|_{L_x^2} + \|\partial \Lambda^{-2} \mathbf{T} \operatorname{curl} \mathbf{w}\|_{L_x^2} \\
&\lesssim \|\partial v\|_{L_x^2} \|\partial \eta\|_{L_x^\infty} + \|\partial v \cdot \partial^2 \eta\|_{L_x^{\frac{6}{5}}} + \|\partial \Lambda^{-2} \mathbf{T} \operatorname{curl} \mathbf{w}\|_{L_x^2} \\
&\lesssim \|\partial v\|_{L_x^2} \|\partial \eta\|_{L_x^\infty} + \|\partial v\|_{L_x^3} \|\partial^2 \eta\|_{L_x^2} + \|\partial \Lambda^{-2} \mathbf{T} \operatorname{curl} \mathbf{w}\|_{L_x^2} \\
&\lesssim \|\partial \eta\|_{L_x^\infty} + \|\partial v\|_{L_x^3} + \|\Lambda^{-2} \mathbf{T} \operatorname{curl} \mathbf{w}\|_{H_x^1}; \\
\|\partial^2 \mathbf{T} \eta\|_{L_x^2} &\lesssim \|\partial(\partial v \partial \eta)\|_{L_x^2} + \|\partial v \cdot \partial^2 \eta\|_{L_x^2} + \|\partial \Lambda^{-2} \mathbf{T} \operatorname{curl} \mathbf{w}\|_{H_x^1} \\
&\lesssim \|\partial^2 v\|_{L_x^2} \|\partial \eta\|_{L_x^\infty} + \|\partial v\|_{L_x^6} \|\partial^2 \eta\|_{L_x^3} + \|\partial \Lambda^{-2} \mathbf{T} \operatorname{curl} \mathbf{w}\|_{H_x^1} \\
&\lesssim \|\partial v\|_{H_x^1} (\|\partial \eta\|_{L_x^\infty} + \|\partial \varrho\|_{L_x^3} + 1) + \|\partial \Lambda^{-2} \mathbf{T} \operatorname{curl} \mathbf{w}\|_{H_x^1},
\end{aligned}$$

where we also used (2.45) to derive the last line.

Note by using Corollary 2.8 and Sobolev embedding

$$\|\partial v\|_{L_x^3} \lesssim \|\partial v\|_{H_x^1}^{\frac{1}{2}} \|\partial v\|_{L_x^2}^{\frac{1}{2}} + \|\partial v\|_{L_x^2} \lesssim \|\partial v\|_{H_x^1}^{\frac{1}{2}} + 1.$$

Applying (2.53) to the above inequalities leads to

$$\begin{aligned}
\|\mathbf{T} \eta\|_{H_x^l} &\lesssim 1 + l(\|\partial \eta\|_{L_x^\infty} + \|\partial v\|_{H_x^1}(\|\partial \varrho\|_{L_x^3} + 1)), \quad l = 0, 1 \\
\|\partial \mathbf{T} \eta\|_{H_x^1} &\lesssim \|\partial v\|_{H_x^1}(\|\partial \eta\|_{L_x^\infty} + \|\partial \varrho\|_{L_x^3} + 1).
\end{aligned}$$

These are (2.50) and (2.51). □

2.4. Energy estimates for v_+ and ϱ . We will provide the energy estimates for v_+ and ϱ in this subsection. To distinguish the energies with (U, V, F_U, F_V) defined in (2.14) and (2.15), we denote the two sets of energies by $\mathcal{E}_{v_+}^{(l)}(t)$ and $\mathcal{E}_\varrho^{(l)}(t)$ respectively.

2.4.1. Lower order energy estimates. We give the first order energy estimate, where the 0-order energy estimate in Corollary 2.8 will be frequently used.

Proposition 2.10 (First order energies). *For (U, V, F_U, F_V) given in (2.14) and (2.15), there hold*

(0)

$$\|F_U\|_{H_x^1} \lesssim 1 + \|\partial \eta\|_{L_x^\infty} + \|\partial v\|_{H_x^1}(\|\partial \varrho\|_{L_x^3} + 1); \quad (2.56)$$

$$\|\mathcal{Z}\|_{L_x^2} + \|F_V\|_{L_x^2} \lesssim \|\partial v, \partial \varrho\|_{L_x^\infty} + 1 + \|\partial \varrho\|_{L_x^3} \quad (2.57)$$

and in particular $F_U \equiv 0$ for $U = \varrho$.

(1)

$$\|\partial^2 F_U\|_{L_x^2} \lesssim (1 + \|\partial v_+\|_{H_x^1})(\|\partial \eta\|_{L_x^\infty} + \|\partial \varrho\|_{L_x^3} + 1) \quad (2.58)$$

$$\|\partial F_V\|_{L_x^2} \lesssim (\|\partial v, \partial \varrho, \partial \eta\|_{L_x^\infty} + 1)(\|\partial \varrho\|_{L_x^3} + 1)(\|\partial \varrho\|_{H_x^1} + \|\partial^2 v_+\|_{L_x^2} + 1). \quad (2.59)$$

(2)

$$\mathcal{E}_{v_+}^{(\leq 1)}(t)^{\frac{1}{2}} + \mathcal{E}_\varrho^{(\leq 1)}(t)^{\frac{1}{2}} \lesssim \mathcal{E}_{v_+}^{(\leq 1)}(0)^{\frac{1}{2}} + \mathcal{E}_\varrho^{(\leq 1)}(0)^{\frac{1}{2}} + 1 \lesssim 1. \quad (2.60)$$

Proof. We first give the 0-order error estimates in (0). Recall $F_U = (-\mathbf{T}\eta, 0)$, (2.56) follows from (2.50). Recall from (2.17) and Corollary 2.8, using the first estimate in (2.49), (2.42) and Sobolev embedding we bound

$$\begin{aligned} \|\mathcal{Z}\|_{L_x^2} + \|F_V\|_{L_x^2} &\lesssim (\|\partial v\|_{L_x^2} + \|\partial\varrho\|_{L_x^2})(\|\partial v\|_{L_x^\infty} + \|\partial\varrho\|_{L_x^\infty}) + \|\eta, (\Omega, \partial\eta) \cdot \partial\varrho\|_{L_x^2} \\ &\lesssim \|\partial v, \partial\varrho\|_{L_x^\infty} + \|\eta\|_{L_x^2} + \|\Omega\|_{L_x^\infty} \|\partial\varrho\|_{L_x^2} + \|\partial\eta\|_{L_x^6} \|\partial\varrho\|_{L_x^3} \\ &\lesssim \|\partial v, \partial\varrho\|_{L_x^\infty} + 1 + \|\partial\varrho\|_{L_x^3}. \end{aligned}$$

This gives (2.57).

The estimate of (2.58) follows from (2.51), $\|\partial^2 F_U\|_{L_x^2} = \|\partial^2 \mathbf{T}\eta\|_{L_x^2}$, and the derivative estimates

$$\|\partial^m v\|_{L_x^2} \lesssim \|\partial^m \eta\|_{L_x^2} + \|\partial^m v_+\|_{L_x^2} \lesssim \|\partial^m v_+\|_{L_x^2} + 1, \quad m = 0, 1, 2, \quad (2.61)$$

which is derived by using (2.49).

Next we consider $\|F_V\|_{H_x^1}$. In view of (2.17),

$$\partial F_V = \partial((C(\varrho) + 1) \cdot \mathcal{Z}) + I \quad (2.62)$$

with

$$I = \partial(c^2(\eta + \mathfrak{w} \cdot \partial\varrho + \partial(\log c) \cdot \partial\eta))$$

and \mathcal{Z} given in (2.18). Expanding $\partial\mathcal{Z}$ gives

$$\partial\mathcal{Z} = (\partial\varrho + \partial v) \cdot (\partial\partial\varrho + \partial\partial v). \quad (2.63)$$

It follows by using (1.4) and (2.61)

$$\begin{aligned} \|\partial\mathcal{Z}\|_{L_x^2} &\lesssim \|\partial\varrho, \partial v\|_{L_x^\infty} \cdot (\|\partial\partial\varrho, \partial^2 v_+, \partial\mathbf{T}v\|_{L_x^2} + 1) \\ &\lesssim \|\partial\varrho, \partial v\|_{L_x^\infty} (\|\partial\varrho\|_{H_x^1} + \|\partial^2 v_+\|_{L_x^2} + \|\partial\varrho\|^2_{L_x^2} + 1) \\ &\lesssim \|\partial\varrho, \partial v\|_{L_x^\infty} (\|\partial\varrho\|_{H_x^1} + \|\partial^2 v_+\|_{L_x^2} + 1), \end{aligned} \quad (2.64)$$

where we have used Sobolev embedding on \mathbb{R}^3 and Corollary 2.8 to bound

$$\|\partial\varrho\|_{L_x^4} \lesssim \|\partial\varrho\|_{H_x^1}^{\frac{3}{4}} \|\partial\varrho\|_{L_x^2}^{\frac{1}{4}} \lesssim \|\partial\varrho\|_{H_x^1} + 1. \quad (2.65)$$

The estimate for the term I in (2.62) is performed term by term as follows:

$$\|\partial(c^2\eta)\|_{L_x^2} \lesssim \|\partial\varrho\|_{L_x^2} \|\eta\|_{L_x^\infty} + \|\partial\eta\|_{L_x^2} \lesssim 1$$

where we used (2.49) and Corollary 2.8.

$$\begin{aligned} \|\partial(c^2\mathfrak{w}\partial\varrho)\|_{L_x^2} &\lesssim \|\partial\varrho\|_{L_x^6} \|\partial\mathfrak{w}\|_{L_x^3} + \|\mathfrak{w}\|_{L_x^\infty} \|\partial\varrho\|^2_{L_x^2} \\ &\lesssim \|\partial\varrho\|_{H_x^1} (\|\partial\varrho\|_{L_x^3} + 1) \end{aligned}$$

where we used (2.42) and (2.43) to bound \mathfrak{w} , used (2.65) and Sobolev embedding to bound norms of $\partial\varrho$. By using the first estimate in (2.49) and (2.65), we can obtain

$$\begin{aligned} \|\partial(\partial(c^2)\partial\eta)\|_{L_x^2} &\lesssim \|\partial^2(c^2)\|_{L_x^2} \|\partial\eta\|_{L_x^\infty} + \|\partial^2\eta\|_{L_x^2} \|\partial(c^2)\|_{L_x^\infty} \\ &\lesssim (\|\partial\varrho\|_{H_x^1} + 1) \|\partial\eta\|_{L_x^\infty} + \|\partial\varrho\|_{L_x^\infty}. \end{aligned}$$

Combining the above three estimates gives

$$\|I\|_{L_x^2} \lesssim (\|\partial\varrho\|_{H_x^1} + 1) (\|\partial\eta\|_{L_x^\infty} + \|\partial\varrho\|_{L_x^3} + 1) + \|\partial\varrho\|_{L_x^\infty}.$$

In view of (2.62), substituting (2.64), $\|\mathcal{Z}\|_{L_x^6} \lesssim \|\partial v, \partial \varrho\|_{L_x^\infty} (\|\partial \varrho\|_{H_x^1} + \|\partial^2 v_+\|_{L_x^2} + 1)$ and the estimate for $\|I\|_{L_x^2}$ leads to

$$\begin{aligned} \|\partial F_V\|_{L_x^2} &\lesssim \|\partial \mathcal{Z}\|_{L_x^2} + \|\partial C(\varrho)\|_{L_x^3} \|\mathcal{Z}\|_{L_x^6} + \|I\|_{L_x^2} \\ &\lesssim (\|\partial v, \partial \varrho, \partial \eta\|_{L_x^\infty} + 1)(\|\partial \varrho\|_{L_x^3} + 1)(\|\partial \varrho\|_{H_x^1} + \|\partial^2 v_+\|_{L_x^2} + 1). \end{aligned}$$

This gives (2.59).

We now rewrite the error estimates (2.56)-(2.59) in view of and $\partial \eta = \partial v - \partial v_+$ as

$$\|F_U\|_{H_x^2} + \|F_V\|_{H_x^1} \lesssim (\|\partial v, \partial \varrho, \partial v_+\|_{L_x^\infty} + 1)(\|\partial \varrho\|_{L_x^3} + 1)(\mathcal{E}_{v_+}^{(\leq 1)}(t)^{\frac{1}{2}} + \mathcal{E}_\varrho^{(\leq 1)}(t)^{\frac{1}{2}} + 1). \quad (2.66)$$

Substituting the above estimates to (2.34) and also applying (2.61) imply

$$\begin{aligned} &\sum_{l=0}^1 (\mathcal{E}_{v_+}^{(l)}(t)^{\frac{1}{2}} + \mathcal{E}_\varrho^{(l)}(t)^{\frac{1}{2}}) \\ &\lesssim \int_0^t (\|\partial v, \partial v_+, \partial \varrho\|_{L_x^\infty} + 1)(\|\partial \varrho\|_{L_x^3} + 1)(\mathcal{E}_\varrho^{(\leq 1)}(t)^{\frac{1}{2}} + \mathcal{E}_{v_+}^{(\leq 1)}(t)^{\frac{1}{2}} + 1) \\ &\quad + \sum_{l=0}^1 (\mathcal{E}_{v_+}^{(l)}(0)^{\frac{1}{2}} + \mathcal{E}_\varrho^{(l)}(0)^{\frac{1}{2}}). \end{aligned} \quad (2.67)$$

Similar to (2.61), there holds the derivative estimate

$$\|\partial^m v_+\|_{L_x^2} \lesssim \|\partial^m \eta\|_{L_x^2} + \|\partial^m v\|_{L_x^2} \lesssim \|\partial^m v\|_{L_x^2} + 1, \quad m = 0, 1, 2.$$

Therefore when $t = 0$, also using (1.4), $V_{v_+} = \mathbf{T}v$ in (2.14) and Sobolev embedding $\|\partial \varrho\|_{L_x^4} \lesssim \|\partial \varrho\|_{H_x^1}$, we have

$$\mathcal{E}_{v_+}^{(l)}(0)^{\frac{1}{2}} \lesssim \|\partial v(0)\|_{H_x^l} + \|\partial \varrho(0)\|_{H_x^l}, \quad l = 0, 1.$$

Similarly in view of (2.15) and the first equation in (1.4), we derive

$$\mathcal{E}_\varrho^{(l)}(0)^{\frac{1}{2}} \lesssim \|\partial \varrho(0)\|_{H_x^l} + \|\partial v(0)\|_{H_x^l}, \quad l = 0, 1.$$

Thus we have the comparison result for the initial data that

$$\mathcal{E}_{v_+}^{(l)}(0) + \mathcal{E}_\varrho^{(l)}(0) \lesssim 1, \quad l = 0, 1.$$

Using (2.1) and (2.3) and applying Gronwall's inequality to (2.67), (2.60) can be proved. \square

Improvement on (2.3). As a direct consequence of (2.60), by Sobolev embedding, we obtain $\|\partial \varrho\|_{L_x^3} \lesssim \|\partial \varrho\|_{H_x^1} \lesssim 1$. Hence

$$\|\partial \varrho\|_{L_x^3} \lesssim 1, \quad \|\partial \varrho\|_{L_t^2 L_x^3} \lesssim T^{\frac{1}{2}}. \quad (2.68)$$

Thus the bootstrap assumption (2.3) is improved.

We summarize some important estimates below for future reference.

Corollary 2.11. *Let $0 < \epsilon \leq s - 2$. There hold*

$$\|\partial v, \partial v_+, \partial \varrho\|_{H_x^1} + \|\partial C(\varrho)\|_{H_x^1} \lesssim 1, \quad (2.69)$$

$$\|\mathfrak{w}, \Omega\|_{H_x^1} \lesssim 1, \quad \|\partial \Omega, \partial \mathfrak{w}, \partial^2 \eta\|_{L_x^p} \lesssim 1, \quad (2.70)$$

$$\|\text{Ric}(g)\|_{L_x^2} \lesssim 1, \quad (2.71)$$

where $2 \leq p \leq \frac{3}{1-\epsilon}$ and $C(y)$ are smooth functions; $\text{Ric}(g)$ is the Ricci curvature of the induced metric g . And there holds the following error estimates

$$\|F_U\|_{H_x^2} + \|F_V\|_{H_x^1} \lesssim \|\partial v, \partial \varrho, \partial v_+\|_{L_x^\infty} + 1. \quad (2.72)$$

Proof. The first set of estimates in (2.69) is a consequence of Corollary 2.8, (2.60) and (2.61). The second estimate follows by using the first set of estimates and the smoothness of C'' and Sobolev embedding. Substituting (2.68) and (2.60) into (2.66) implies (2.72). It follows from (2.69) and Sobolev embedding that $\|\partial\varrho\|_{L_x^p} \lesssim 1$ if $2 \leq p \leq \frac{3}{1-\epsilon}$. The derivative estimates in (2.70) can then be derived by combining this estimate, (2.43) and (2.45). The L_x^2 estimates for \mathfrak{w} and Ω have been obtained in (2.42). Note $\text{Ric}(g) = g(\partial^2 g + \partial g \partial g)$. (2.71) is a consequence of the derivative estimate of ϱ in (2.69) together with Sobolev embedding, (2.6) and $|\varrho| \leq C$. \square

Using [1] or [29, Theorem 5.4], there exists a constant $d_0 > 0$ depending only on the constant bounds in (2.6) and the bound of $\|\text{Ric}\|_{L_x^2}$, which is the uniform lower bound of radius of injectivity on Σ_t for $t \in [0, T]$. The comparison between the Euclidean and geodesic ball on Σ_t will be used in Section 4.2. We assume T satisfies $0 < T \leq \min(d_0, 1)$ throughout the paper.

Due to Corollary 2.11 and the smoothness of $C(y)$, there hold $\|C(\varrho), C'(\varrho)\|_{L_x^\infty} + \|\partial\varrho\|_{H_x^1} \lesssim 1$. We can derive

Lemma 2.12. *Let $0 < \alpha \leq \frac{1}{2}$ be fixed. For $C(\varrho)$ a smooth function of ϱ , there hold for scalar functions f that*

$$\|\Lambda^\alpha(C(\varrho)f)\|_{L_x^2} \lesssim \|\Lambda^\alpha f\|_{L_x^2} + \|f\|_{L_x^2}, \quad (2.73)$$

$$\|\Lambda^{\frac{1}{2}+\alpha}(C(\varrho)f)\|_{L_x^2} \lesssim \|\Lambda^{\frac{1}{2}+\alpha} f\|_{L_x^2} + \|f\|_{L_x^2}. \quad (2.74)$$

Proof. For (2.73), we apply (10.13) to $(F, G) = (C(\varrho), f)$ to obtain

$$\|\Lambda^\alpha(C(\varrho)f)\|_{L_x^2} \leq \|C(\varrho)\|_{L_x^\infty} \|\Lambda^\alpha f\|_{L_x^2} + \|C(\varrho)\|_{B_{\infty,2}^\alpha} \|f\|_{L_x^2}. \quad (2.75)$$

Note that due to Bernstein inequality for the dyadic frequency $\mu > 1$,

$$\|\mu^\alpha P_\mu C(\varrho)\|_{L_x^\infty} \lesssim \|\mu^{\alpha+\frac{1}{2}} P_\mu \partial C(\varrho)\|_{L_x^2}.$$

Thus for all $0 \leq \alpha \leq \frac{1}{2}$

$$\|\mu^\alpha P_\mu C(\varrho)\|_{l_\mu^2 L_x^\infty} \lesssim \|\partial C(\varrho)\|_{H_x^1} \lesssim \|\partial\varrho\|_{H_x^1} \lesssim 1.$$

Here we recall $\|a_\mu\|_{l_\mu^2}$ is the usual l^2 norm for the number sequence of $a_\mu, \mu > 1$. Combining this inequality with (2.75) implies (2.73).

For (2.74) and $0 < \alpha < \frac{1}{2}$, we first derive

$$\mu^{\frac{1}{2}+\alpha} \|P_\mu(C(\varrho)f)\|_{L_x^2} \leq \mu^{\frac{1}{2}+\alpha} \|C(\varrho)P_\mu f\|_{L_x^2} + \mu^{\frac{1}{2}+\alpha} \|[P_\mu, C(\varrho)]f\|_{L_x^2}.$$

For the second term on the right hand side, we apply (10.16) to $(F, G) = (C(\varrho), f)$ to derive

$$\|\mu^{\frac{1}{2}+\alpha} [P_\mu, C(\varrho)]f\|_{l_\mu^2 L_x^2} \lesssim \|\partial C(\varrho)\|_{H_x^1} \|\Lambda^\alpha f\|_{L_x^2} \lesssim \|\Lambda^\alpha f\|_{L_x^2},$$

where we used (2.69). Combining the above two inequalities gives (2.74) for the case $0 < \alpha < \frac{1}{2}$.

If $\alpha = \frac{1}{2}$, we can directly obtain (2.74) by using (2.69) and Sobolev embedding. \square

Proposition 2.13. *Let $0 < \epsilon \leq s - 2$. There hold the following estimates*

$$\|\partial\eta\|_{L^\infty} + \|\partial^2\eta\|_{L_x^p} \lesssim 1, \quad 2 \leq p \leq \frac{3}{1-\epsilon} \quad (2.76)$$

$$\|\partial\Omega\|_{\dot{H}_x^\alpha} \lesssim \|\operatorname{curl}\Omega\|_{\dot{H}_x^\alpha} + 1, \quad 0 < \alpha \leq \frac{1}{2} + \epsilon \quad (2.77)$$

$$\|\operatorname{curl}\mathfrak{w}\|_{H_x^\alpha} \lesssim \|\operatorname{curl}\Omega\|_{H_x^\alpha} + 1, \quad 0 < \alpha \leq \frac{1}{2} + \epsilon \quad (2.78)$$

$$\|\eta\|_{H_x^{2+\alpha}} \lesssim \|\operatorname{curl}\Omega\|_{H_x^\alpha} + 1, \quad 0 < \alpha \leq \frac{1}{2} + \epsilon \quad (2.79)$$

$$\|\mathbf{T}\eta\|_{H_x^2} \lesssim 1, \quad (2.80)$$

$$\|\mathbf{T}\eta\|_{L_x^\infty} \lesssim 1, \quad (2.81)$$

$$\|\mathbf{T}\eta\|_{H_x^{2+\epsilon}} \lesssim \|\partial v, \partial\varrho\|_{H_x^{1+\epsilon}} + (\|\partial v\|_{L_x^\infty} + 1)(\|\operatorname{curl}\Omega\|_{H_x^\epsilon} + 1). \quad (2.82)$$

Remark 2.14. The above results hold the same if $0 < \epsilon \leq s' - 2$ with $2 < s' < s$ and the constants in the bounds depend merely on $\|\Omega(0)\|_{H^{s'-\frac{1}{2}}}$, (2.1) and the bound on $\|\partial v, \partial\varrho\|_{H_x^1}$.

Proof. The L_x^p estimate in (2.76) has been included in (2.70). The estimate of $\|\partial\eta\|_{L_x^\infty}$ is a consequence of the L_x^p estimate and (2.49) by using Sobolev embedding.

Next we consider (2.77). Note that by using (1.5)

$$\begin{aligned} \int_{\Sigma_t} \mu^{2\alpha} |\partial P_\mu \Omega|^2 &= \int_{\Sigma_t} \mu^{2\alpha} \{|\operatorname{curl} P_\mu \Omega|^2 + |\operatorname{div} P_\mu \Omega|^2\} dx \\ &= \int_{\Sigma_t} \mu^{2\alpha} \{|P_\mu \operatorname{curl} \Omega|^2 + |P_\mu \operatorname{div} \Omega|^2\} dx \\ &= \int_{\Sigma_t} \mu^{2\alpha} \{|P_\mu \operatorname{curl} \Omega|^2 + |P_\mu (\Omega \partial \varrho)|^2\} dx. \end{aligned}$$

Thus,

$$\|\mu^\alpha \partial P_\mu \Omega\|_{l_\mu^2 L_x^2}^2 = \|\mu^\alpha P_\mu \operatorname{curl} \Omega\|_{l_\mu^2 L_x^2}^2 + \|\mu^\alpha P_\mu (\Omega \partial \varrho)\|_{l_\mu^2 L_x^2}^2. \quad (2.83)$$

We consider the second term on the right hand side of (2.83) by applying (10.12) to $(F, G) = (\Omega, \partial \varrho)$, which gives

$$\|\mu^\alpha P_\mu (\Omega \partial \varrho)\|_{l_\mu^2 L_x^2} \lesssim \|\Omega\|_{\dot{H}_x^{\frac{1}{2}+\alpha}} \|\partial \varrho\|_{H_x^1} + \|\Omega\|_{L_x^\infty} \|\partial \varrho\|_{\dot{H}_x^\alpha}, \quad 0 < \alpha \leq \frac{1}{2} + \epsilon. \quad (2.84)$$

If $\alpha \leq \frac{1}{2}$, using (2.70) and (2.42), the right hand side is bounded by $\|\partial \varrho\|_{H_x^1}$. In this case, by using (2.69), we can obtain

$$\|\partial\Omega\|_{\dot{H}_x^\alpha} \lesssim \|\operatorname{curl}\Omega\|_{\dot{H}_x^\alpha} + \|\partial\varrho\|_{H_x^1} \lesssim \|\operatorname{curl}\Omega\|_{\dot{H}_x^\alpha} + 1.$$

(2.77) is proved. If $\alpha > \frac{1}{2}$, note that with $\theta = \frac{\alpha - \frac{1}{2}}{\alpha}$ and using (2.70),

$$\|\Omega\|_{\dot{H}_x^{\frac{1}{2}+\alpha}} \leq \|\Omega\|_{\dot{H}_x^{1+\alpha}}^\theta \|\Omega\|_{\dot{H}_x^1}^{1-\theta} \lesssim \|\Omega\|_{\dot{H}_x^{1+\alpha}}^\theta.$$

Combining the above estimate with (2.83) and (2.84), by using Young's inequality, we can obtain (2.77) for the case $\frac{1}{2} < \alpha \leq \frac{1}{2} + \epsilon$.

Recall from the second equation of (2.47), applying Lemma 2.12 and (2.84)

$$\begin{aligned} \|\operatorname{curl}\mathfrak{w}\|_{H_x^\alpha} &\lesssim \|\operatorname{curl}\Omega\|_{H_x^\alpha} + \|\Omega \cdot \partial \varrho\|_{H_x^\alpha}, \quad 0 < \alpha \leq \frac{1}{2} + \epsilon \\ &\lesssim \|\operatorname{curl}\Omega\|_{H_x^\alpha} + \|\Omega\|_{\dot{H}_x^{\frac{1}{2}+\alpha}} + 1 \\ &\lesssim \|\operatorname{curl}\Omega\|_{H_x^\alpha} + 1, \end{aligned}$$

where we also used (2.42), (2.77) and (2.69) to derive the last two lines. Thus (2.78) is proved. (2.79) is a direct consequence of (2.78) and the definition of η in (1.21).

(2.80) can be derived by substituting the first estimate in (2.76), (2.69) and (2.68) into the estimates (2.50) and (2.51). (2.81) is its consequence due to Sobolev embedding.

We next consider (2.82) in view of (2.55),

$$\|\partial^2 \mathbf{T}\eta\|_{H_x^\epsilon} \leq I + J + K,$$

where

$$I = \|\partial(\partial v \cdot \partial \eta)\|_{H_x^\epsilon}, \quad J = \|\partial v \cdot \partial^2 \eta\|_{H_x^\epsilon}, \quad K = \|\mathbf{T} \operatorname{curl} \mathbf{w}\|_{H_x^\epsilon}.$$

For the first term I , applying (10.8) to $F = \partial v$, $G = \partial \eta$ and $\alpha = \epsilon$ gives

$$I \lesssim \|\partial v\|_{H_x^{1+\epsilon}} \|\partial \eta\|_{L_x^\infty} + \|\partial v\|_{L_x^\infty} \|\partial \eta\|_{H_x^{\epsilon+1}}.$$

For the second term J , we apply (10.7) to $F = \partial v$, $G = \partial \eta$ and $\alpha = \epsilon$ to derive

$$J \lesssim \|\partial v\|_{H_x^{1+\epsilon}} \|\partial \eta\|_{L_x^\infty} + \|\partial v\|_{L_x^\infty} \|\partial \eta\|_{H_x^{\epsilon+1}}.$$

Also by using (2.76) and (2.79), we can conclude that

$$I + J \lesssim \|\partial v\|_{H_x^{1+\epsilon}} + \|\partial v\|_{L_x^\infty} (\|\operatorname{curl} \Omega\|_{H_x^\epsilon} + 1).$$

For K , in view of (2.52), we apply (2.73) with $\alpha = \epsilon$ and $C(y) = e^y$ to derive

$$\begin{aligned} \|\mathbf{T} \operatorname{curl} \mathbf{w}\|_{H_x^\epsilon} &\lesssim \|\partial(\partial v \Omega e^\varrho)\|_{\dot{H}_x^\epsilon} + \|\mathfrak{C} e^{2\varrho} \partial v\|_{H_x^\epsilon} + \|\partial v \cdot \partial(e^\varrho) \Omega\|_{H_x^\epsilon} \\ &\lesssim \|\partial(\partial v \Omega)\|_{H_x^\epsilon} + \|\operatorname{curl} \Omega \cdot \partial v\|_{H_x^\epsilon} + \|\partial v \cdot \partial \varrho \Omega\|_{H_x^\epsilon}. \end{aligned} \quad (2.85)$$

For the first term on the right hand side, applying (10.8) to $(F, G) = (\partial v, \Omega)$ and $\alpha = \epsilon$, also using (2.77) with $\alpha = \epsilon$ and (2.42) we obtain

$$\begin{aligned} \|\partial(\partial v \Omega)\|_{H_x^\epsilon} &\lesssim \|\partial v\|_{H_x^{1+\epsilon}} \|\Omega\|_{L_x^\infty} + \|\partial v\|_{L_x^\infty} (\|\operatorname{curl} \Omega\|_{H_x^\epsilon} + 1) \\ &\lesssim \|\partial v\|_{H_x^{1+\epsilon}} + \|\partial v\|_{L_x^\infty} (\|\operatorname{curl} \Omega\|_{H_x^\epsilon} + 1). \end{aligned}$$

For the second term on the right of (2.85), we apply (10.7) with $(F, G) = (\partial v, \Omega)$ with $\alpha = \epsilon$, also using (2.77) and (2.42), to have

$$\begin{aligned} \|\Lambda^\epsilon(\operatorname{curl} \Omega \cdot \partial v)\|_{L_x^2} &\lesssim \|\partial v\|_{H_x^{1+\epsilon}} \|\Omega\|_{L_x^\infty} + \|\partial v\|_{L_x^\infty} \|\partial \Omega\|_{H_x^\epsilon} \\ &\lesssim \|\partial v\|_{H_x^{1+\epsilon}} + \|\partial v\|_{L_x^\infty} (\|\operatorname{curl} \Omega\|_{H_x^\epsilon} + 1). \end{aligned}$$

For the third term on the right of (2.85), we apply (10.15) to $(G_1, G_2, G_3) = (\partial v, \partial \varrho, \Omega)$ that

$$\begin{aligned} \|\Lambda^\epsilon(\partial v \cdot \partial \varrho \cdot \Omega)\|_{L_x^2} &\lesssim \|\Lambda^\epsilon \partial v\|_{H_x^1} \|\partial v\|_{H_x^1} \|\Omega\|_{H_x^1} + \|\Lambda^\epsilon \Omega\|_{H_x^1} \|\partial v\|_{H_x^1} \|\partial \varrho\|_{H_x^1} \\ &\quad + \|\Lambda^\epsilon \partial \varrho\|_{H_x^1} \|\partial v\|_{H_x^1} \|\Omega\|_{H_x^1}. \end{aligned}$$

Since $\|\Omega\|_{H_x^1} \lesssim 1$ in (2.70), also using (2.69) and (2.77),

$$\|\Lambda^\epsilon(\partial v \cdot \partial \varrho \cdot \Omega)\|_{L_x^2} \lesssim \|\Lambda^\epsilon(\partial v, \partial \varrho)\|_{H_x^1} + \|\Lambda^\epsilon \operatorname{curl} \Omega\|_{L_x^2} + 1.$$

Summing up the above estimates for three terms and also using (2.69) imply

$$K = \|\mathbf{T} \operatorname{curl} \mathbf{w}\|_{H_x^\epsilon} \lesssim \|\Lambda^\epsilon(\partial v, \partial \varrho)\|_{H_x^1} + (\|\partial v\|_{L_x^\infty} + 1)(\|\operatorname{curl} \Omega\|_{H_x^\epsilon} + 1).$$

Combining the above estimate with the estimate for $I + J$ gives (2.82). \square

2.4.2. *Highest order energy estimates for v_+ and ϱ .* We will control the highest order energy for v_+ and ϱ , by giving the control of F_U and F_V at the highest order, and using Proposition 2.6.

Proposition 2.15. *Let $0 < \epsilon \leq s - 2$.*

(1) *For (U, V, F_U, F_V) in (2.14) and (2.15), there hold*

$$\|\partial^2 F_U\|_{H_x^\epsilon} \lesssim \|\partial v, \partial \varrho\|_{H_x^{1+\epsilon}} + (\|\partial v\|_{L_x^\infty} + 1)(\|\operatorname{curl} \Omega\|_{H_x^\epsilon} + 1), \quad (2.86)$$

$$\|\partial F_V\|_{H_x^\epsilon} \lesssim (\|\partial(\partial v, \partial \varrho)\|_{\dot{H}_x^\epsilon} + \|\operatorname{curl} \Omega\|_{H_x^\epsilon} + 1)(\|\partial v, \partial \varrho\|_{L_x^\infty} + 1). \quad (2.87)$$

(2) *Denote $\mathcal{E}_\mu^{(1)}(t) := \mathcal{E}_{v_+, \mu}^{(1)}(t) + \mathcal{E}_{\varrho, \mu}^{(1)}(t)$ for short. There holds the following energy estimate*

$$\begin{aligned} \|\mu^\epsilon \mathcal{E}_\mu^{(1)}(t)\|_{L_\mu^2}^{\frac{1}{2}} &\lesssim 1 + \int_0^t \|\operatorname{curl} \Omega\|_{H_x^\epsilon} (\|\partial \varrho, \partial v_+\|_{L_x^\infty} + 1) dt' \\ &\quad + \|\mu^\epsilon \mathcal{E}_\mu^{(1)}(0)\|_{L_\mu^2}^{\frac{1}{2}}. \end{aligned} \quad (2.88)$$

To prove the above result, we need a preliminary comparison result.

Lemma 2.16. *Let $0 < \epsilon \leq s - 2$.*

$$\|\partial \partial v\|_{\dot{H}_x^\epsilon} \lesssim \left(\sum_{\mu > 1} \mu^{2\epsilon} \mathcal{E}_\mu^{(1)}(t) \right)^{\frac{1}{2}} + \|\operatorname{curl} \Omega\|_{H_x^\epsilon} + 1, \quad (2.89)$$

$$\|\partial \partial \varrho, \partial \operatorname{Tr} k, \partial^2(c^2)\|_{\dot{H}_x^\epsilon} \lesssim \left(\sum_{\mu > 1} \mu^{2\epsilon} \mathcal{E}_{\varrho, \mu}^{(1)}(t) \right)^{\frac{1}{2}} + 1. \quad (2.90)$$

Proof. Due to (2.79) and $v = v_+ + \eta$, we can obtain

$$\|\partial^2 v\|_{\dot{H}_x^\epsilon} \lesssim \|\partial^2 v_+\|_{\dot{H}_x^\epsilon} + \|\operatorname{curl} \Omega\|_{H_x^\epsilon} + 1.$$

Since $\mathbf{T}v = -c^2 \partial \varrho$ in (1.4), $\partial \mathbf{T}v = c^2(c^{-1}c'(\varrho) \partial \varrho \partial \varrho + \partial^2 \varrho)$. We apply Lemma 2.12 to derive

$$\|c^2 \partial^2 \varrho\|_{H_x^\epsilon} \lesssim \|\partial^2 \varrho\|_{H_x^\epsilon}.$$

By using (10.14) with $F = G = \partial \varrho$, (2.69) and Lemma 2.12, we have

$$\|C(\varrho) \partial \varrho \partial \varrho\|_{H_x^\epsilon} \lesssim \|\partial \varrho\|^2_{H_x^1} \lesssim \|\partial \varrho\|_{H_x^1}^2 \lesssim 1. \quad (2.91)$$

Combining the above two estimates yields

$$\|\partial \mathbf{T}v\|_{\dot{H}_x^\epsilon} \lesssim \|\partial^2 \varrho\|_{H_x^\epsilon} + 1.$$

Hence (2.89) is proved.

The estimate in (2.90) for $\partial \partial \varrho$ follows by definition. In view of (2.9) and (1.4)

$$\partial \operatorname{Tr} k = 3\partial((\log c)' \mathbf{T} \varrho) + \partial \mathbf{T} \varrho = C(\varrho)(\partial \partial \varrho + \partial \varrho \partial \varrho),$$

where $C(\varrho)$ represents several smooth functions of ϱ . $\partial^2(c^2)$ can also be written in the same symbolic form. Applying Lemma 2.12 and (2.91) leads to

$$\|\partial \operatorname{Tr} k, \partial^2(c^2)\|_{\dot{H}_x^\epsilon} \lesssim \|\partial \partial \varrho\|_{H_x^\epsilon} + 1.$$

Thus (2.90) is proved. \square

Proof of Proposition 2.15. Note $F_U = 0$ in (2.15), (2.86) holds trivially in this case. Recall the formula of F_U from (2.14) and $\|\partial^2 F_U\|_{H_x^\epsilon} = \|\partial^2 \mathbf{T} \eta\|_{H_x^\epsilon}$. (2.86) follows immediately by using (2.82).

Recall the definition of F_V in (2.17) and $\mathbf{w} = \Omega e^e$. We symbolically write

$$\begin{aligned} \partial F_V &= \partial(\mathcal{Z}(C(\varrho) + 1)) + \partial(c^2(\eta + \mathbf{w} \cdot \partial \varrho + \partial \log c \partial \eta)) \\ &= (\partial \mathcal{Z} + \partial(\eta + \Omega \cdot \partial \varrho + \partial \varrho \partial \eta))C(\varrho) + \partial \mathcal{Z} + (\mathcal{Z} + \eta + \Omega \cdot \partial \varrho + \partial \varrho \partial \eta) \partial C(\varrho), \end{aligned}$$

where $C(\varrho)$ are smooth functions of ϱ and may vary when distributed to different factors.

Since in (2.59) we have obtained the estimate for $\|\partial F_V\|_{L_x^2}$, we only consider the highest order estimate. Applying (2.73) twice gives

$$\begin{aligned}\|\partial F_V\|_{\dot{H}_x^\epsilon} &\lesssim \|\partial\eta\|_{H_x^\epsilon} + \|\partial\mathcal{Z}\|_{H_x^\epsilon} + \|\partial(\Omega \cdot \partial\varrho + \partial\eta \cdot \partial\varrho)\|_{H_x^\epsilon} + \|(\mathcal{Z} + \eta + \Omega\partial\varrho + \partial\eta\partial\varrho)\partial C(\varrho)\|_{H_x^\epsilon} \\ &\lesssim 1 + \|\partial\mathcal{Z}\|_{H_x^\epsilon} + \|\partial(\Omega \cdot \partial\varrho + \partial\eta \cdot \partial\varrho)\|_{H_x^\epsilon} + \|(\mathcal{Z} + \eta + \Omega\partial\varrho + \partial\eta\partial\varrho)\partial\varrho\|_{H_x^\epsilon},\end{aligned}$$

where we also employed the first estimate in (2.49). For the lower order term, applying (10.14) to $(F, G) = (\eta, \partial\varrho)$ yields

$$\begin{aligned}\|\eta \cdot \partial\varrho\|_{H_x^\epsilon} &\lesssim \|\eta\|_{H_x^{\frac{1}{2}+\epsilon}} \|\partial\varrho\|_{H_x^1} + \|\partial\varrho\|_{H_x^{\frac{1}{2}+\epsilon}} \|\eta\|_{H_x^1} \\ &\lesssim \|\eta\|_{H_x^1} \|\partial\varrho\|_{H_x^1} \lesssim 1,\end{aligned}$$

where we used (2.69) and (2.49).

Next we estimate the quadratic terms. Recall (2.63) for the form of $\partial\mathcal{Z}$. We apply (10.8) with

$$\alpha = \epsilon, F = \partial v, \partial\varrho; G = \partial v, \partial\varrho, \Omega, \partial\eta$$

to derive

$$\begin{aligned}\|\partial\mathcal{Z}\|_{H_x^\epsilon} + \|\partial(\Omega \cdot \partial\varrho + \partial\eta \cdot \partial\varrho)\|_{H_x^\epsilon} \\ \lesssim \|\partial(\partial v, \partial\varrho)\|_{H_x^\epsilon} \|\partial v, \partial\varrho, \Omega, \partial\eta\|_{L_x^\infty} + \|\partial v, \partial\varrho\|_{L_x^\infty} \|\partial(\partial v, \partial\varrho, \Omega, \partial\eta)\|_{H_x^\epsilon}.\end{aligned}$$

By using (2.77) and (2.79),

$$\|\partial\Omega\|_{H_x^\epsilon} + \|\partial^2\eta\|_{H_x^\epsilon} \lesssim \|\operatorname{curl}\Omega\|_{H_x^\epsilon} + 1. \quad (2.92)$$

Using the above estimate, (2.42) and (2.76), we obtain

$$\begin{aligned}\|\partial\mathcal{Z}\|_{H_x^\epsilon} + \|\partial(\Omega \cdot \partial\varrho + \partial\eta \cdot \partial\varrho)\|_{H_x^\epsilon} \\ \lesssim (\|\partial(\partial v, \partial\varrho)\|_{H_x^\epsilon} + \|\operatorname{curl}\Omega\|_{H_x^\epsilon} + 1)(\|\partial v, \partial\varrho\|_{L_x^\infty} + 1).\end{aligned}$$

For the cubic terms, we apply (10.15) to $G_1 = \partial v, \partial\varrho, \Omega, \partial\eta$, $G_2 = \partial v, \partial\varrho$ and $G_3 = \partial\varrho$.

$$\begin{aligned}\|(\mathcal{Z} + \Omega\partial\varrho + \partial\eta\partial\varrho)\partial\varrho\|_{H_x^\epsilon} &\lesssim \|\partial v, \partial\varrho, \Omega, \partial\eta\|_{H_x^{1+\epsilon}} \|\partial v, \partial\varrho\|_{H_x^1} \|\partial\varrho\|_{H_x^1} \\ &\quad + \|\partial v, \partial\varrho, \Omega, \partial\eta\|_{H_x^1} (\|\partial v, \partial\varrho\|_{H_x^{1+\epsilon}} \|\partial\varrho\|_{H_x^1} + \|\partial v, \partial\varrho\|_{H_x^1} \|\partial\varrho\|_{H_x^{1+\epsilon}}).\end{aligned}$$

Using (2.92), (2.49) and Corollary 2.11, we derive

$$\|(\mathcal{Z} + \Omega\partial\varrho + \partial\eta\partial\varrho)\partial\varrho\|_{H_x^\epsilon} \lesssim \|\partial(\partial v, \partial\varrho)\|_{H_x^\epsilon} + \|\operatorname{curl}\Omega\|_{H_x^\epsilon} + 1.$$

Summing up the estimates for the lower order, quadratic and cubic terms yields (2.87).

We hence combine the estimates of (2.86) and (2.87) with (2.89) to conclude

$$\|\partial(\partial F_U, F_V)\|_{H_x^\epsilon} \lesssim \left(\sum_{\mu>1} \mu^{2\epsilon} \mathcal{E}_\mu^{(1)}(t)\right)^{\frac{1}{2}} + \|\operatorname{curl}\Omega\|_{H_x^\epsilon} + 1)(\|\partial\varrho, \partial v\|_{L_x^\infty} + 1). \quad (2.93)$$

To prove (2.88), we will substitute (2.86) and (2.87) to the right hand side of the inequality in Proposition 2.6. Recall from Lemma 2.16 that

$$\|\partial^2(c^2), \partial\partial v, \partial\partial\varrho, \partial(\operatorname{Tr}k)\|_{H_x^\epsilon} \lesssim \left(\sum_{\mu>1} \mu^{2\epsilon} \mathcal{E}_\mu^{(1)}(t)\right)^{\frac{1}{2}} + \|\operatorname{curl}\Omega\|_{H_x^\epsilon} + 1.$$

Moreover, by (1.4) and the first estimate in (2.76), for both the cases of (U, V) in (2.14) and (2.15)

$$\|\partial v\|_{L_x^\infty} + \|\partial U, V\|_{L_x^\infty} \lesssim \|\partial\varrho, \partial v_+\|_{L_x^\infty} + 1. \quad (2.94)$$

Hence by using (2.60), (2.93) and (2.94), we derive

$$\begin{aligned}
& \|\mu^\epsilon \mathcal{E}_\mu^{(1)}(t)^{\frac{1}{2}}\|_{l_\mu^2 L_x^2} \\
& \lesssim \|\mu^\epsilon \mathcal{E}_\mu^{(1)}(0)^{\frac{1}{2}}\|_{l_\mu^2 L_x^2} + T^{\frac{1}{2}} \sup_{0 \leq t' \leq t} \mathcal{E}^{(\leq 1)}(t')^{\frac{1}{2}} \\
& + \int_0^t (\|\partial v, \partial(c^2), \text{Tr}k\|_{H_x^{1+\epsilon}} \|\partial U, V\|_{L_x^\infty} + \|\partial(\partial F_U, F_V)\|_{H_x^\epsilon}) dt' \\
& \lesssim \|\mu^\epsilon \mathcal{E}_\mu^{(1)}(0)^{\frac{1}{2}}\|_{l_\mu^2 L_x^2} + T^{\frac{1}{2}} + \int_0^t (\|\mu^\epsilon \mathcal{E}_\mu^{(1)}(t')^{\frac{1}{2}}\|_{l_\mu^2} + \|\text{curl} \Omega\|_{H_x^\epsilon} + 1)(\|\partial \varrho, \partial v_+\|_{L_x^\infty} + 1) dt'.
\end{aligned}$$

(2.88) can be obtained by Gronwall's inequality and using (2.1). \square

3. $H^{2+\delta}$ ENERGY ESTIMATES FOR VORTICITY

Let $0 \leq \delta \leq s' - 2$. The purpose of this section is to derive the bound of $\|\Omega\|_{H^{2+\delta}(\Sigma_t)}$ for all $0 < t \leq T$. Neither ∂v nor Ω is sufficiently smooth for providing such bound by using the transport equation if following the standard product estimates under the bootstrap assumption (see (1.18)). Applying the curl-operator to (1.9) followed by pairing the resulting equation with a curl-structure leads to a series of crucial cancellations including the div-curl structure in space and integration by part in spacetime, with the help of the Hodge system

$$\text{div} v = -\mathbf{T}\varrho, \quad \text{curl} v = \Omega e^\varrho$$

and the transport equations (1.6) and (1.9).

Recall from (2.88) that we need to obtain the bound of $\|\text{curl} \Omega(t)\|_{H_x^{s-2}}$ for all $0 < t \leq T$ to obtain the highest order energy estimates for the wave functions (v_+, ϱ) . Under our assumption on the initial vorticity, we will bound $\|\mathfrak{C}(t)\|_{H_x^1}$ norm to close the energy estimate in (2.88). We then further obtain the bound for $\|\mathfrak{C}(t)\|_{H_x^{1+\delta}}$ with the help of the bound on the highest order energy for (v_+, ϱ) and (2.1). For both estimates on $\mathfrak{C}(t)$, we heavily rely on the the particular structure in the curl-equation of (1.9).

We first give the precise formulae of $\mathbf{T} \text{curl} F$ for vector fields F .

Proposition 3.1. *There hold for Σ_t -tangent vector fields F that*¹⁷

$$\mathbf{T} \text{curl} F^m - \mathbf{T}\varrho \text{curl} F^m = \partial_n v^m \text{curl} F^n - \epsilon^{mni} \partial_n v_j \partial_i F^j + \text{curl} \mathbf{T} F^m, \quad (3.1)$$

and identically,

$$\mathbf{T}(e^{-\varrho} \text{curl} F^m) = e^{-\varrho} \partial_n v^m \text{curl} F^n - \epsilon^{mni} \partial_n v_j \partial_i F^j e^{-\varrho} + \text{curl} \mathbf{T} F^m e^{-\varrho}. \quad (3.2)$$

Let $(\partial v_j \wedge \partial F^j)^m = \epsilon^{mni} \partial_n v_j \partial_i F^j$. There holds

$$\begin{aligned}
& \text{curl}(e^{-\varrho} \partial v_j \wedge \partial F^j)_m \\
& = e^{-\varrho} (\partial^n \partial_m v_j \partial_n F^j + (\text{curl}^2 v_j + \partial_j \mathbf{T}\varrho) \partial_m F^j + \partial_m v^j (-\text{curl}^2 F_j + \partial_j(\text{div} F)) \\
& - \partial_n v^j \partial^n \partial_m F_j - \epsilon_m^{ni} \epsilon_i^{ab} \partial_a v^j \partial_b F_j \partial_n \varrho).
\end{aligned} \quad (3.3)$$

Proof. In view of (2.10)

$$[\mathbf{T}, \partial_j] F^i = -\partial_j v^l \partial_l F^i,$$

¹⁷We use the Euclidean metric δ_{ij} to lift and lower the indices of tensor fields.

we can derive

$$\begin{aligned}\epsilon^{mj}{}_i[\mathbf{T}, \partial_j]F^i &= \epsilon^m{}_i{}^j \partial_j v^l \partial_l F^i \\ &= \epsilon^{mij} \partial_j v^l (\partial_i F_l + \epsilon_{li}{}^n \text{curl } F_n) \\ &= \epsilon^{mij} \partial_j v^l \partial_i F_l + \partial_n v^m \text{curl } F^n - \text{div } v \text{curl } F^m,\end{aligned}$$

where we calculated for $Z_n = \text{curl } F_n$ that

$$\epsilon^{mj}{}_i \epsilon_l{}^{ni} \partial_j v^l Z_n = (\delta_l^m \delta_j^n - \delta_l^j \delta^{mn}) \partial_j v^l Z_n = \partial^n v^m Z_n - \text{div } v Z^m$$

to obtain the last line. (3.1) follows by applying (1.4) to $\text{div } v$.

(3.2) is a consequence of multiplying (3.1) by $e^{-\varrho}$.

Next we prove (3.3). We first directly compute $\epsilon^{iab} \epsilon_{mni} = \delta_m^a \delta_n^b - \delta_n^a \delta_m^b$. With its help, we derive

$$\begin{aligned}\epsilon_{mn}{}^i \partial^n (e^{-\varrho} \partial v_j \wedge \partial F^j)_i &= \epsilon_{mn}{}^i \partial^n (\epsilon_i{}^{ab} \partial_a v^j \partial_b F_j e^{-\varrho}) \\ &= e^{-\varrho} \{ (\delta_m^a \delta_n^b - \delta_n^a \delta_m^b) \partial^n (\partial_a v^j \partial_b F_j) - \epsilon_{mn}{}^i \epsilon_i{}^{ab} \partial^n \varrho \partial_a v^j \partial_b F_j \} \\ &= e^{-\varrho} \{ (\delta_m^a \delta_n^b - \delta_n^a \delta_m^b) (\partial^n \partial_a v^j \partial_b F_j + \partial_a v^j \partial^n \partial_b F_j) - \epsilon_{mn}{}^i \epsilon_i{}^{ab} \partial^n \varrho \partial_a v^j \partial_b F_j \} \\ &= e^{-\varrho} (\partial^n \partial_m v^j \partial_n F_j - \Delta_e v^j \partial_m F_j + \partial_m v^j \Delta_e F_j - \partial_n v^j \partial^n \partial_m F_j - \epsilon_{mn}{}^i \epsilon_i{}^{ab} \partial^n \varrho \partial_a v^j \partial_b F_j).\end{aligned}$$

Noting for a vector-valued function G there holds

$$\Delta_e(G^j) = -(\text{curl}^2 G)^j + \partial_j(\text{div } G)$$

and using the first equation in (1.4) when $G = v$, we can obtain (3.3) from the above identity. \square

Proof of (1.9). (1.9) is a direct consequence of applying (3.2) to $F = \Omega$. Indeed,

$$\mathbf{T}(e^{-\varrho} \text{curl } \Omega^m) = \partial_n v^m (e^{-\varrho} \text{curl } \Omega^n) - \epsilon^{mni} \partial_n v_j \partial_i \Omega^j e^{-\varrho} + \text{curl } \mathbf{T} \Omega^m e^{-\varrho}.$$

By using (1.6), (1.5) and $\Omega = \mathbf{w} e^{-\varrho}$, we calculate

$$\begin{aligned}\text{curl}(\mathbf{T} \Omega^m) &= \epsilon^{mij} \partial_i (\mathbf{T} \Omega_j) = \epsilon^{mij} \partial_i (\Omega^a \partial_a v_j) \\ &= \epsilon^{mij} \partial_i \Omega^a \partial_a v_j + \partial_a \mathbf{w}^m \Omega^a = \epsilon^{mij} \partial_i \Omega^a (\partial_j v_a + \epsilon_{aj}{}^l \mathbf{w}_l) + \partial_a \mathbf{w}^m \Omega^a \\ &= \epsilon^{mij} \partial_i \Omega^a \partial_j v_a - (\delta_a^m \delta^{il} - \delta^{ml} \delta_a^i) \mathbf{w}_l \partial_i \Omega^a + \partial_a \mathbf{w}^m \Omega^a \\ &= \epsilon^{mij} \partial_i \Omega^a \partial_j v_a - (\partial_i \Omega^m \mathbf{w}^i - \mathbf{w}^m \partial_a \Omega^a) + \partial_a \mathbf{w}^m \Omega^a \\ &= \epsilon^{mij} \partial_i \Omega^a \partial_j v_a - (\partial_i \mathbf{w}^m e^{-\varrho} \mathbf{w}^i + \mathbf{w}^m \partial_i (e^{-\varrho}) \mathbf{w}^i + \mathbf{w}^m \Omega^a \partial_a \varrho) + \partial_a \mathbf{w}^m \Omega^a \\ &= \epsilon^{mij} \partial_i \Omega^a \partial_j v_a.\end{aligned}$$

Combining the above two calculations gives (1.9). \square

Recall that for any vector fields Z tangent to Σ_t and any scalar function f there holds $\int_{\Sigma_t} \mathcal{L}_Z(f d\mu_e) = \int_{\Sigma_t} \text{div}(f Z) d\mu_e = 0$. To derive the H^2 energy of vorticity, we derive the energy formula for Σ_t -tangent vector fields F, G

$$\begin{aligned}\partial_t \int_{\Sigma_t} \langle F, G \rangle_e d\mu_e &= \int_{\Sigma_t} (\langle F, \mathbf{T} G \rangle_e + \langle G, \mathbf{T} F \rangle_e) d\mu_e + \int_{\Sigma_t} \langle F, G \rangle_e (\partial_t + \mathcal{L}_v) d\mu_e \\ &= \int_{\Sigma_t} (\langle F, \mathbf{T} G \rangle_e + \langle G, \mathbf{T} F \rangle_e) d\mu_e + \int_{\Sigma_t} -\langle F, G \rangle_e \text{Tr } \overset{\circ}{k} d\mu_e,\end{aligned}\tag{3.4}$$

where we used $\mathcal{L}_{\mathbf{T}} d\mu_e = -\delta^{ij} \overset{\circ}{k}_{ij} d\mu_e = -\text{Tr } \overset{\circ}{k} d\mu_e$, and $\langle \cdot, \cdot \rangle_e$ means the inner product is taken by using the Euclidean metric.

3.1. H^2 estimate for vorticity. We first derive the energy estimate for vorticity in H_x^2 , which is sufficient to complete the energy estimate for the wave function (v_+, ϱ) given in (2.88).

Proposition 3.2 (H^2 bound of vorticity). *There holds for $0 < t \leq T$ that*

$$\|\operatorname{curl} \mathfrak{C}\|_{L^2(\Sigma_t)} + \|\partial \mathfrak{C}\|_{L^2(\Sigma_t)} + \|\partial^2 \Omega\|_{L^2(\Sigma_t)} + \|\partial^2 \mathfrak{w}\|_{L^2(\Sigma_t)} \lesssim 1. \quad (3.5)$$

Proof. We first reduce the proof of (3.5) to showing the first estimate therein.

By $\mathfrak{C}^i = e^{-\varrho} \operatorname{curl} \Omega^i$, we derive

$$\operatorname{div} \mathfrak{C} = \partial_i (e^{-\varrho}) \operatorname{curl} \Omega^i.$$

By the elliptic estimate for hodge system, Sobolev embedding on \mathbb{R}^3 , (2.69) and (2.44), we can obtain that

$$\begin{aligned} \|\partial \mathfrak{C}\|_{L^2(\Sigma_t)} &\lesssim \|\partial(e^{-\varrho})\|_{L_x^6} \|\operatorname{curl} \Omega\|_{L^3(\Sigma_t)} + \|\operatorname{curl} \mathfrak{C}\|_{L^2(\Sigma_t)} \\ &\lesssim \|\partial \varrho\|_{L_x^6} + \|\operatorname{curl} \mathfrak{C}\|_{L^2(\Sigma_t)} \lesssim \|\operatorname{curl} \mathfrak{C}\|_{L^2(\Sigma_t)} + 1. \end{aligned}$$

Moreover, by the elliptic estimate and the formula

$$\Delta_e \Omega^j = -\operatorname{curl}^2 \Omega^j + \partial^j \operatorname{div} \Omega,$$

we derive by using (2.69), (2.70), (2.42), (1.5) and the Sobolev embedding that

$$\begin{aligned} \|\partial^2 \Omega\|_{L^2(\Sigma_t)} &\lesssim \|\operatorname{curl}(e^\varrho \mathfrak{C}^i)\|_{L^2(\Sigma_t)} + \|\partial(\Omega \partial \varrho)\|_{L^2(\Sigma_t)} + \|\partial \varrho \cdot \operatorname{curl} \Omega\|_{L^2(\Sigma_t)} \\ &\lesssim \|\operatorname{curl} \mathfrak{C}\|_{L^2(\Sigma_t)} + \|\partial \varrho\|_{L^6(\Sigma_t)} \|\mathfrak{C}, \partial \Omega\|_{L^3(\Sigma_t)} + \|\Omega\|_{L_x^\infty} \|\partial^2 \varrho\|_{L^2(\Sigma_t)} \\ &\lesssim \|\operatorname{curl} \mathfrak{C}\|_{L^2(\Sigma_t)} + 1. \end{aligned} \quad (3.6)$$

Next, we recall that $\mathfrak{w} = \Omega e^\varrho$, which gives

$$\begin{aligned} \partial^2 \mathfrak{w} &= \partial^2 (\Omega e^\varrho) = e^\varrho (\partial^2 \Omega + \partial \Omega \partial \varrho) + \partial(e^\varrho \partial \varrho) \Omega \\ &= e^\varrho \{\partial^2 \Omega + \partial \Omega \partial \varrho + \Omega((\partial \varrho)^2 + \partial^2 \varrho)\}. \end{aligned} \quad (3.7)$$

Note that by using (2.70) and (2.69) we can derive by using Sobolev embedding

$$\|\partial \Omega \cdot (\partial \varrho, \partial v)\|_{L_x^2} + \|\Omega(\partial \varrho)^2\|_{L_x^2} \lesssim \|\partial \Omega\|_{L_x^3} \|\partial \varrho, \partial v\|_{L_x^6} + \|\Omega\|_{H_x^1} \|\partial \varrho\|_{H_x^1}^2 \lesssim 1. \quad (3.8)$$

Hence taking L_x^2 norm of the expression for $\partial^2 \mathfrak{w}$ yields

$$\begin{aligned} \|\partial^2 \mathfrak{w}\|_{L_x^2} &\lesssim \|\partial^2 \Omega\|_{L_x^2} + \|\partial \Omega \partial \varrho\|_{L_x^2} + \|\Omega(\partial \varrho)^2\|_{L_x^2} + \|\Omega \partial^2 \varrho\|_{L_x^2} \\ &\lesssim \|\partial^2 \Omega\|_{L_x^2} + \|\Omega\|_{L_x^\infty} \|\partial^2 \varrho\|_{L_x^2} + 1 \lesssim \|\partial^2 \Omega\|_{L_x^2} + 1, \end{aligned}$$

where we used (2.69) and (2.42). Combined with (3.6), we have

$$\|\partial^2 \mathfrak{w}\|_{L_x^2} \lesssim \|\operatorname{curl} \mathfrak{C}\|_{L_x^2} + 1,$$

as desired. For future reference, we can similarly obtain

$$\|\operatorname{curl} \mathfrak{C}\|_{L_x^2} \lesssim \|\partial^2 \mathfrak{w}\|_{L_x^2} + 1. \quad (3.9)$$

Therefore

$$\|\partial \mathfrak{C}\|_{L^2(\Sigma_t)} + \|\partial^2 \Omega\|_{L^2(\Sigma_t)} + \|\partial^2 \mathfrak{w}\|_{L^2(\Sigma_t)} \lesssim \|\operatorname{curl} \mathfrak{C}\|_{L^2(\Sigma_t)} + 1. \quad (3.10)$$

To prove (3.5), it suffices to consider the first estimate.

Now we apply (3.4) to $F = G = \operatorname{curl} \mathfrak{C}$. By using Gronwall's inequality, $\operatorname{Tr} \overset{\circ}{k} = -\operatorname{div} v = \mathbf{T} \varrho$ and (2.1),

$$\int_{\Sigma_t} |\operatorname{curl} \mathfrak{C}|^2 d\mu_e \lesssim \|\operatorname{curl} \mathfrak{C}\|_{L^2(\Sigma_0)}^2 + \left| \int_0^t \int_{\Sigma_{t'}} \langle \operatorname{curl} \mathfrak{C}, 2\mathbf{T} \operatorname{curl} \mathfrak{C} - \mathbf{T} \varrho \operatorname{curl} \mathfrak{C} \rangle_e d\mu_e dt' \right|. \quad (3.11)$$

To treat the last term on the right hand side, which will be denoted as I , we apply (3.1) to $F = \mathfrak{C}$, followed with using (3.10) and the first equation in (1.4) to derive

$$\begin{aligned} I &\lesssim \left| \int_0^t \int_{\Sigma_{t'}} \langle \operatorname{curl} \mathfrak{C}, \operatorname{curl} \mathbf{T}\mathfrak{C} \rangle_e d\mu_e dt' \right| + \int_0^t \|\partial v\|_{L_x^\infty} \|\partial \mathfrak{C}\|_{L_x^2} \|\operatorname{curl} \mathfrak{C}\|_{L_x^2} dt' \\ &\lesssim \left| \int_0^t \int_{\Sigma_{t'}} \langle \operatorname{curl} \mathfrak{C}, \operatorname{curl} \mathbf{T}\mathfrak{C} \rangle_e d\mu_e dt' \right| + \int_0^t \|\operatorname{curl} \mathfrak{C}\|_{L_x^2} (1 + \|\operatorname{curl} \mathfrak{C}\|_{L_x^2}) \|\partial v\|_{L_x^\infty} dt'. \end{aligned}$$

The second term on the right will be treated by Gronwall's inequality and using (2.1). We focus on the first term on the right hand side, denoted by $|I^1|$ with

$$I^1 = \int_0^t \int_{\Sigma_{t'}} \langle \operatorname{curl} \mathfrak{C}, \operatorname{curl} \mathbf{T}\mathfrak{C} \rangle_e d\mu_e dt'. \quad (3.12)$$

By using (1.9), we compute

$$\operatorname{curl} \mathbf{T}\mathfrak{C}_m = -2 \operatorname{curl} (e^{-\varrho} \partial v_j \wedge \partial \Omega^j)_m + \operatorname{curl} (\partial_a v \mathfrak{C}^a)_m. \quad (3.13)$$

Note that

$$\begin{aligned} \operatorname{curl} (\partial_a v \mathfrak{C}^a)_m &= \epsilon_{mni} \partial^n (\partial_a v^i \operatorname{curl} \Omega^a e^{-\varrho}) \\ &= \epsilon_{mni} \partial^n \partial_a v^i \mathfrak{C}^a + \epsilon_{mni} \partial_a v^i \partial^n \mathfrak{C}^a \\ &= \partial_a \mathfrak{w}_m \mathfrak{C}^a + \epsilon_{mni} \partial_a v^i \partial^n \mathfrak{C}^a. \end{aligned} \quad (3.14)$$

Thus by using (3.10) and (2.70), we can directly bound

$$\begin{aligned} \|\operatorname{curl} (\partial_a v \mathfrak{C}^a)\|_{L^2(\Sigma_t)} &\lesssim \|\partial \mathfrak{w}\|_{L_x^3} \|\mathfrak{C}\|_{L_x^6} + \|\partial_a v\|_{L_x^\infty} \|\partial \mathfrak{C}\|_{L_x^2} \\ &\lesssim \|\mathfrak{C}\|_{H_x^1} (\|\partial \mathfrak{w}\|_{L_x^3} + \|\partial v\|_{L_x^\infty}) \lesssim (\|\operatorname{curl} \mathfrak{C}\|_{L_x^2} + 1) (\|\partial v\|_{L_x^\infty} + 1). \end{aligned}$$

Hence the second term in (3.13) has been treated.

It remains to bound the L_x^2 -norm of the first term on the right hand side of (3.13). For this purpose, we apply (3.3) to $F = \Omega$ and observe that the first term on the right takes the form of $\partial^2 v \cdot \partial_j \Omega$, which is only expected to be in $L_x^{\frac{3}{2}}$ instead of the favourable L_x^2 . To solve this difficulty, we derive carefully the integral below,

$$\begin{aligned} I_+^1 &= \int_0^t \int_{\Sigma_{t'}} \operatorname{curl} (e^{-\varrho} \partial v_j \wedge \partial \Omega^j)_m \operatorname{curl} \mathfrak{C}^m d\mu_e dt' \\ &= \int_0^t \int_{\Sigma_{t'}} \{ e^{-\varrho} (\partial^n \partial_m v^j \partial_n \Omega_j + (\partial_j \mathbf{T}\varrho + \operatorname{curl}^2 v_j) \partial_m \Omega^j + \partial_m v^j (-\operatorname{curl}^2 \Omega_j + \partial_j (\operatorname{div} \Omega)) \\ &\quad - \partial_n v^j \partial^n \partial_m \Omega_j - \epsilon_{mni} \epsilon^{iab} \partial_a v^j \partial_b \Omega_j \partial^n \varrho) \} \operatorname{curl} \mathfrak{C}^m d\mu_e dt'. \end{aligned} \quad (3.15)$$

The above integral can be decomposed into two parts. For the following part, denoted by $I_{+,1}^1$, we will employ integration by parts in \mathbb{R}^3 and in spacetime,

$$I_{+,1}^1 = \int_0^t \int_{\Sigma_{t'}} e^{-\varrho} (\partial^n \partial_m v^j \partial_n \Omega_j + \partial_j \mathbf{T}\varrho \partial_m \Omega^j) \operatorname{curl} \mathfrak{C}^m d\mu_e dt', \quad (3.16)$$

while the remaining terms are denoted by $I_{+,2}^1$. Using the second equation in (2.47), (2.42) and (3.8), we carry out the direct estimate below,

$$\begin{aligned} |I_{+,2}^1| &\lesssim \int_0^t \int_{\Sigma_{t'}} (|\operatorname{curl} \mathfrak{w}| |\partial \Omega| + |\partial v| (|\partial^2 \Omega| + |\partial \Omega \cdot \partial \varrho|)) |\operatorname{curl} \mathfrak{C}| d\mu_e dt' \\ &\lesssim \int_0^t \|\operatorname{curl} \mathfrak{C}\|_{L_x^2} \{ \|\partial \Omega| (|\partial \Omega| + |\Omega \partial \varrho|)\|_{L_x^2} + \|\partial v\|_{L_x^\infty} (\|\partial^2 \Omega\|_{L_x^2} + \|\partial \Omega \cdot \partial \varrho\|_{L_x^2}) \} \\ &\lesssim \int_0^t \|\operatorname{curl} \mathfrak{C}\|_{L_x^2} (\|\partial \Omega\|^2_{L_x^2} + 1 + \|\partial v\|_{L_x^\infty} (\|\partial^2 \Omega\|_{L_x^2} + 1)) dt'. \end{aligned}$$

By using Sobolev embedding, (2.70) and (3.6)

$$\|\partial \Omega\|_{L_x^2}^2 \lesssim \|\partial \Omega\|_{H_x^1} \|\partial \Omega\|_{L_x^3} \lesssim \|\operatorname{curl} \mathfrak{C}\|_{L_x^2} + 1.$$

Substituting the above estimate in the estimate for $|I_{+,2}^1|$, also using (3.6) again, we can obtain

$$|I_{+,2}^1| \lesssim \int_0^t \|\operatorname{curl} \mathfrak{C}\|_{L_x^2} (\|\operatorname{curl} \mathfrak{C}\|_{L_x^2} + 1) (1 + \|\partial v\|_{L_x^\infty}) dt',$$

which can be further treated by Gronwall's inequality.

For the first term in (3.16), we observe the div-curl structure and perform integration by parts on Σ_t ,

$$\begin{aligned} I_{+,1,1}^1 &= \int_0^t \int_{\Sigma_{t'}} e^{-\varrho} \partial^n \partial_m v^j \partial_n \Omega_j \operatorname{curl} \mathfrak{C}^m d\mu_e dt' \\ &= \int_0^t \int_{\Sigma_{t'}} \{ \partial_m (e^{-\varrho} \partial^n v^j \partial_n \Omega_j \operatorname{curl} \mathfrak{C}^m) - \partial_m (e^{-\varrho} \partial_n \Omega_j \operatorname{curl} \mathfrak{C}^m) \partial^n v^j \} d\mu_e dt'. \end{aligned} \quad (3.17)$$

The first term is a boundary term, denoted by $I_{+,1,1,b}^1$ which vanishes identically. Due to $\partial_m (\operatorname{curl} \mathfrak{C})^m = 0$, we obtain

$$\begin{aligned} I_{+,1,1}^1 &= - \int_0^t \int_{\Sigma_{t'}} \partial_m (e^{-\varrho} \partial^n \Omega_j \operatorname{curl} \mathfrak{C}^m) \partial_n v^j d\mu_e dt' \\ &= - \int_0^t \int_{\Sigma_{t'}} (\partial_m (e^{-\varrho}) \partial_n \Omega_j + e^{-\varrho} \partial_m \partial_n \Omega_j) \operatorname{curl} \mathfrak{C}^m \partial^n v^j d\mu_e dt'. \end{aligned}$$

We then use (3.6) and the first estimate in (3.8) to derive

$$\begin{aligned} |I_{+,1,1}^1| &\lesssim \int_0^t (1 + \|\partial^2 \Omega\|_{L_x^2}) \|\operatorname{curl} \mathfrak{C}\|_{L_x^2} \|\partial v\|_{L_x^\infty} dt' \\ &\lesssim \int_0^t (1 + \|\operatorname{curl} \mathfrak{C}\|_{L_x^2}) \|\operatorname{curl} \mathfrak{C}\|_{L_x^2} \|\partial v\|_{L_x^\infty} dt'. \end{aligned}$$

The above term can be then treated by Gronwall's inequality.

Next, we consider the second term of $I_{+,1}^1$ in (3.16), which is denoted by $I_{+,1,2}^1$ below. In view of (3.4), we integrate by parts in spacetime to obtain

$$\begin{aligned}
I_{+,1,2}^1 &= \int_0^t \int_{\Sigma_{t'}} \partial_j \mathbf{T} \varrho \partial_m \Omega^j e^{-\varrho} \operatorname{curl} \mathfrak{C}^m d\mu_e dt' \\
&= \int_0^t \int_{\Sigma_{t'}} ([\partial_j, \mathbf{T}] \varrho + \mathbf{T} \partial_j \varrho) \partial_m \Omega^j e^{-\varrho} \operatorname{curl} \mathfrak{C}^m d\mu_e dt' \\
&= \int_0^t \int_{\Sigma_{t'}} (\partial_j v^a \partial_a \varrho + \operatorname{Tr} \overset{\circ}{k} \partial_j \varrho) \partial_m \Omega^j e^{-\varrho} \operatorname{curl} \mathfrak{C}^m d\mu_e dt' \\
&\quad + \int_0^t \partial_t \int_{\Sigma_{t'}} \partial_j \varrho \partial_m \Omega^j e^{-\varrho} \operatorname{curl} \mathfrak{C}^m d\mu_e dt' - \int_0^t \int_{\Sigma_{t'}} \partial_j \varrho \mathbf{T} (\partial_m \Omega^j e^{-\varrho} \operatorname{curl} \mathfrak{C}^m) d\mu_e dt',
\end{aligned} \tag{3.18}$$

where we used (2.10) to treat the commutator.

By using (3.8) and (2.1), we control the first term on the right hand side

$$\begin{aligned}
|I_{+,1,2,1}^1| &= \left| \int_0^t \int_{\Sigma_{t'}} (\partial_j v^a \partial_a \varrho + \operatorname{Tr} \overset{\circ}{k} \partial_j \varrho) \partial_m \Omega^j e^{-\varrho} \operatorname{curl} \mathfrak{C}^m d\mu_e dt' \right| \\
&\lesssim \int_0^t \|\partial v\|_{L_x^\infty} \|\partial \varrho \cdot \partial \Omega\|_{L_x^2} \|\operatorname{curl} \mathfrak{C}\|_{L_x^2} dt' \\
&\lesssim \int_0^t \|\partial v\|_{L_x^\infty} \|\operatorname{curl} \mathfrak{C}\|_{L_x^2} dt' \lesssim \sup_{0 \leq t' \leq t} \|\operatorname{curl} \mathfrak{C}\|_{L^2(\Sigma_{t'})}.
\end{aligned}$$

Similarly, we bound the boundary term by

$$|I_{+,1,2,2}^1| \lesssim \sup_{0 \leq t' \leq t} \{\|\partial \varrho \cdot \partial \Omega\|_{L_x^2} \|\operatorname{curl} \mathfrak{C}\|_{L_x^2}\} \lesssim \sup_{0 \leq t' \leq t} \|\operatorname{curl} \mathfrak{C}\|_{L^2(\Sigma_{t'})}.$$

For the last term in $I_{+,1,2}^1$, which is denoted by $I_{+,1,2,3}^1$, we first show the following preliminary estimates

$$\|\mathbf{T}(\partial \Omega \cdot (e^{-\varrho} + 1))\|_{L_x^2} \lesssim 1. \tag{3.19}$$

Indeed, we compute by using (1.6), (1.4) and (2.10) that

$$\mathbf{T} \partial \Omega = \partial \Omega \partial v + \Omega \partial^2 v, \quad \mathbf{T}(\partial \Omega e^{-\varrho}) = e^{-\varrho}(\partial \Omega \partial v + \Omega \partial^2 v). \tag{3.20}$$

Therefore, it follows by using (3.8), (2.42) and (2.69) that

$$\|\mathbf{T}(\partial \Omega (e^{-\varrho} + 1))\|_{L_x^2} \lesssim \|\partial \Omega \cdot \partial v\|_{L_x^2} + \|\Omega\|_{L_x^\infty} \|\partial^2 v\|_{L_x^2} \lesssim 1.$$

This gives (3.19).

In view of (1.9), we can symbolically write

$$\mathbf{T} \mathfrak{C} = e^{-\varrho} \partial v \cdot \partial \Omega; \quad \operatorname{curl} \mathbf{T} \mathfrak{C} = \partial(e^{-\varrho} \partial v \cdot \partial \Omega). \tag{3.21}$$

Using the above symbolic formulas, (2.10) and $\mathfrak{C} = e^{-\varrho} \operatorname{curl} \Omega$, we derive

$$\begin{aligned}
I_{+,1,2,3}^1 &= \int_0^t \int_{\Sigma_{t'}} \{ \partial_j \varrho \partial_m \Omega^j e^{-\varrho} (\operatorname{curl} \mathbf{T} \mathfrak{C}^m + [\mathbf{T}, \operatorname{curl}] \mathfrak{C}^m) + \partial_j \varrho \mathbf{T}(\partial_m \Omega^j e^{-\varrho}) \operatorname{curl} \mathfrak{C}^m \} d\mu_e dt' \\
&= \int_0^t \int_{\Sigma_{t'}} \partial \varrho \{ \partial \Omega e^{-\varrho} \partial(e^{-\varrho} \partial v \cdot \partial \Omega) + \partial v \partial \mathfrak{C} \partial \Omega e^{-\varrho} + \mathbf{T}(\partial \Omega e^{-\varrho}) \operatorname{curl} \mathfrak{C} \} d\mu_e dt' \\
&= \int_0^t \int_{\Sigma_{t'}} \{ \partial \varrho \partial \Omega e^{-2\varrho} (\partial^2 v \partial \Omega + \partial v \partial \varrho \partial \Omega + \partial v \partial^2 \Omega) + \partial \varrho \mathbf{T}(\partial \Omega e^{-\varrho}) \operatorname{curl} \mathfrak{C} \} d\mu_e dt'.
\end{aligned}$$

By using (3.19), we can estimate the last term in the last line,

$$\|\partial \varrho \mathbf{T}(\partial \Omega e^{-\varrho}) \operatorname{curl} \mathfrak{C}\|_{L_x^1} \lesssim \|\partial \varrho\|_{L_x^\infty} \|\operatorname{curl} \mathfrak{C}\|_{L_x^2}.$$

For the remaining terms in the last line, we employ Sobolev embedding, (2.69) and (2.70) to derive

$$\begin{aligned} & \|\partial_j \varrho \partial \Omega e^{-2\varrho} (\partial^2 v \partial \Omega + \partial v \partial \varrho \partial \Omega + \partial v \partial^2 \Omega)\|_{L_x^1} \\ & \lesssim \|\partial \varrho\|_{L_x^6} (\|\partial^2 v\|_{L_x^2} \|\partial \Omega\|_{L_x^6}^2 + \|\partial \Omega\|_{L_x^6} \|\partial \Omega\|_{L_x^3} \|\partial v\|_{L_x^6} \|\partial \varrho\|_{L_x^6} + \|\partial^2 \Omega\|_{L_x^2} \|\partial \Omega\|_{L_x^6} \|\partial v\|_{L_x^6}) \\ & \lesssim (\|\operatorname{curl} \mathfrak{C}\|_{L_x^2} + 1)(\|\operatorname{curl} \mathfrak{C}\|_{L_x^2} + 1), \end{aligned}$$

where we used Sobolev embedding and (3.6) to treat $\|\partial \Omega\|_{L_x^6} + \|\partial^2 \Omega\|_{L_x^2}$ in the last step.

Combining the estimates for the two parts gives

$$|I_{+,1,2,3}^1| \lesssim \int_0^t (\|\operatorname{curl} \mathfrak{C}\|_{L_x^2} + 1)(\|\partial \varrho\|_{L_x^\infty} + \|\operatorname{curl} \mathfrak{C}\|_{L_x^2} + 1) dt',$$

which then can be treated by Gronwall's inequality.

We now summarize the above calculations as

$$|I| + |I^1| \lesssim \int_0^t (\|\operatorname{curl} \mathfrak{C}\|_{L^2(\Sigma_{t'})} + 1)^2 (\|\partial v, \partial \varrho\|_{L_x^\infty} + 1) dt' + \sup_{0 \leq t' \leq t} \|\operatorname{curl} \mathfrak{C}\|_{L^2(\Sigma_{t'})}. \quad (3.22)$$

Substituting the above estimate in (3.11) yields

$$\begin{aligned} \int_{\Sigma_t} |\operatorname{curl} \mathfrak{C}|^2 d\mu_e & \lesssim \|\operatorname{curl} \mathfrak{C}\|_{L^2(\Sigma_0)}^2 + \int_0^t (\|\operatorname{curl} \mathfrak{C}\|_{L_x^2} + 1)^2 (\|\partial v, \partial \varrho\|_{L_x^\infty} + 1) dt' \\ & \quad + \sup_{0 \leq t' \leq t} \|\operatorname{curl} \mathfrak{C}\|_{L^2(\Sigma_{t'})}, \end{aligned}$$

which implies the first estimate in (3.5) by using Gronwall's inequality and using (3.9) to bound $\|\operatorname{curl} \mathfrak{C}\|_{L^2(\Sigma_0)} \lesssim 1$. \square

Corollary 3.3. *For $0 \leq \epsilon \leq s - 2$, there hold*

$$\|\Omega\|_{H_x^2} + \|\operatorname{curl} \Omega\|_{H_x^1} \lesssim 1 \quad (3.23)$$

$$\|\partial v, \partial \varrho, \operatorname{Tr} k, \partial(c^2)\|_{H_x^{1+\epsilon}} \lesssim 1, \quad \|\mu^\epsilon \mathcal{E}_\mu^{(1)}(t)^{\frac{1}{2}}\|_{l_\mu^2} \lesssim 1 \quad (3.24)$$

$$\|\mathbf{T}^2 v, \mathbf{T}^2 \varrho\|_{H_x^\epsilon} \lesssim 1. \quad (3.25)$$

Proof. (3.23) is a consequence of (3.5) and (2.70). The second estimate in (3.24) is derived by substituting (3.5) and (2.1) to (2.88) provided that $\|\mu^\epsilon \mathcal{E}_\mu^{(1)}(0)^{\frac{1}{2}}\|_{l_\mu^2} \lesssim 1$. Indeed to see this, we repeat the proof in Lemma 2.16 at $t = 0$ to obtain

$$\|\partial^2 v_+\|_{H_x^\epsilon} \lesssim \|\partial^2 v\|_{H_x^\epsilon} + \|\operatorname{curl} \Omega\|_{H_x^\epsilon} + 1, \quad \|\partial \mathbf{T} v\|_{\dot{H}^\epsilon} \lesssim \|\partial^2 \varrho\|_{H_x^\epsilon} + 1.$$

Moreover similar to (2.78) we can derive $\|\operatorname{curl} \Omega\|_{H_x^\epsilon} \lesssim \|\operatorname{curl} \mathfrak{w}\|_{H_x^\epsilon} + 1$, which also holds at $t = 0$. Combining the above estimates yields $\|\mu^\epsilon \mathcal{E}_{v_+, \mu}^{(1)}(0)^{\frac{1}{2}}\|_{l_\mu^2} \lesssim 1$. The estimate for $\|\mu^\epsilon \mathcal{E}_{\varrho, \mu}^{(1)}(0)^{\frac{1}{2}}\|_{l_\mu^2} \lesssim 1$ follows directly by the data assumption.

The $\dot{H}^{1+\epsilon}$ bound in the first estimate in (3.24) follows by using the second estimate with the help of Lemma 2.16 and (3.23), while the lower order bound can be obtained by (2.69).

To show (3.25), we first derive by using (1.4) and (2.10) symbolically that

$$\begin{aligned} \mathbf{T}^2(v) &= \mathbf{T}(-c^2 \partial \varrho) = c c' \mathbf{T} \varrho \partial \varrho + c^2 \partial v \partial \varrho + c^2 \partial \mathbf{T} \varrho \\ \mathbf{T}^2 \varrho &= \mathbf{T}(\partial v) = \partial \mathbf{T} v + (\partial v)^2. \end{aligned}$$

Hence, symbolically, we can write

$$\mathbf{T}^2 v, \mathbf{T}^2 \varrho = C(\varrho)(\partial(v, \varrho))^2 + C(\varrho)\partial\mathbf{T}(\varrho, v).$$

By using (2.73) and (10.14)

$$\begin{aligned} \|\mathbf{T}^2 v, \mathbf{T}^2 \varrho\|_{H_x^\epsilon} &\lesssim \|(\partial(v, \varrho))^2\|_{H_x^\epsilon} + \|\partial\mathbf{T}(\varrho, v)\|_{H_x^\epsilon} \\ &\lesssim \|\partial(v, \varrho)\|_{H_x^{\epsilon+\frac{1}{2}}} \|\partial(v, \varrho)\|_{H_x^1} + \|\partial\mathbf{T}(\varrho, v)\|_{H_x^\epsilon}. \end{aligned}$$

Therefore $\|\mathbf{T}^2 v, \mathbf{T}^2 \varrho\|_{H_x^\epsilon} \lesssim 1$ follows by using (3.24). \square

Let $0 < \alpha < 1$ be fixed. Define for a function f the following Besov norm

$$\|f\|_{B_{\infty,2,x}^\alpha} = \|f\|_{L_x^\infty} + \|\mu^\alpha P_\mu f\|_{l_\mu^2 L_x^\infty}.$$

Corollary 3.4. *Let $0 < \delta \leq s' - 2$. There hold the following estimates*

$$\begin{aligned} \|\partial v\|_{B_{\infty,2,x}^\delta} &\lesssim \|\partial v_+\|_{B_{\infty,2,x}^\delta} + 1 \\ \|\partial v\|_{L_x^\infty} &\lesssim \|\partial v_+\|_{L_x^\infty} + 1. \end{aligned}$$

Proof. By using $v = v_+ + \eta$, Bernstein inequality, the estimate for η in (2.79) and (3.5)

$$\begin{aligned} \|\partial v\|_{B_{\infty,2,x}^\delta} &\lesssim \|\partial v_+\|_{B_{\infty,2,x}^\delta} + \|\operatorname{curl} \Omega\|_{H_x^{\frac{1}{2}+\delta}} + 1 \\ &\lesssim \|\partial v_+\|_{B_{\infty,2,x}^\delta} + 1. \end{aligned}$$

Similarly, by using (2.80),

$$\begin{aligned} \|\mathbf{T}v\|_{B_{\infty,2,x}^\delta} &\lesssim \|\mathbf{T}\eta\|_{B_{\infty,2,x}^\delta} + \|\mathbf{T}v_+\|_{B_{\infty,2,x}^\delta} \\ &\lesssim \|\mathbf{T}\eta\|_{H^{\frac{3}{2}+\delta}} + \|\mathbf{T}v_+\|_{B_{\infty,2,x}^\delta} \\ &\lesssim 1 + \|\mathbf{T}v_+\|_{B_{\infty,2,x}^\delta}. \end{aligned}$$

The L_x^∞ estimate can be derived in the same way. \square

3.2. $H^{2+\delta}$ bound for vorticity. Let $0 < \delta \leq s' - 2$ be fixed. In this subsection, we derive the highest order energy bound for the vorticity. Such bound for the vorticity, together with the flux on $\operatorname{curl} \mathfrak{C}$ which will be bounded in Section 6.2, is used particularly for controlling the geometry of the acoustical null cones.

Proposition 3.5 (Highest-order energy control for vorticity). *Let $0 < \delta \leq s' - 2$. There holds the following estimate*

$$\|\operatorname{curl} \mathfrak{C}\|_{H^\delta(\Sigma_t)} + \|\partial \mathfrak{C}\|_{H^\delta(\Sigma_t)} + \|\partial^2 \mathfrak{w}\|_{H^\delta(\Sigma_t)} + \|\partial^2 \Omega\|_{H^\delta(\Sigma_t)} \lesssim 1. \quad (3.26)$$

Since the L_x^2 estimates have been proved in (3.5), we only need to prove the highest order estimates in the above result. We first prove two sets of preliminary estimates.

Lemma 3.6. *Let $0 < \delta \leq s' - 2$ and $C(y)$ be any smooth function. There hold*

$$\|\mu^\delta [\mathbf{T}, P_\mu \operatorname{curl}] F^m\|_{l_\mu^2 L_x^2} \lesssim \|\partial v\|_{B_{\infty,2,x}^\delta} \|\partial F\|_{L_x^2} + \|\partial v\|_{L_x^\infty} \|\partial F\|_{H_x^\delta}, \quad (3.27)$$

$$\|C(\varrho)F\|_{H_x^{1+\delta}} \lesssim \|F\|_{H_x^{1+\delta}}. \quad (3.28)$$

Proof. In view of (3.1) and using (1.4) to treat $\mathbf{T}\varrho$ therein, symbolically, we can write

$$[\mathbf{T}, P_\mu \operatorname{curl}] F^m = [\mathbf{T}, P_\mu] \operatorname{curl} F^m + P_\mu (\partial v \cdot \partial F). \quad (3.29)$$

Since

$$[P_\mu, \mathbf{T}] \operatorname{curl} F^m = [P_\mu, v] \partial \operatorname{curl} F^m,$$

by applying (10.2) to $F = v$ and $G = \operatorname{curl} F$, we derive

$$\|\mu^\delta[P_\mu, \mathbf{T}] \operatorname{curl} F\|_{l_\mu^2 L_x^2} \lesssim \|\partial v\|_{L_x^\infty} \|\operatorname{curl} F\|_{H_x^\delta}. \quad (3.30)$$

For the other term in (3.29), we apply (10.13) with $F = \partial v$ and $G = \partial F$ to have

$$\|\partial v \cdot \partial F\|_{H_x^\delta} \lesssim \|\partial v\|_{B_{\infty,2,x}^\delta} \|\partial F\|_{L_x^2} + \|\partial v\|_{L_x^\infty} \|\partial F\|_{H_x^\delta}.$$

We combine the above estimate with (3.30) to derive (3.27).

Next we prove (3.28). It suffices to consider the highest order estimate. We apply (2.73) to $\alpha = \delta$, $C(\varrho)$ and $f = \partial F$; and apply (10.14) to $\alpha = \delta$, F and $G = \partial(C(\varrho))$ to derive

$$\begin{aligned} \|\Lambda^\delta \partial(FC(\varrho))\|_{L_x^2} &\lesssim \|\Lambda^\delta(\partial FC(\varrho))\|_{L_x^2} + \|\Lambda^\delta(F\partial(C(\varrho)))\|_{L_x^2} \\ &\lesssim \|\partial F\|_{H_x^\delta} + \|F\|_{H_x^{\frac{1}{2}+\delta}} \|\partial(C(\varrho))\|_{H_x^1} + \|F\|_{H_x^1} \|\partial(C(\varrho))\|_{H_x^{\frac{1}{2}+\delta}} \\ &\lesssim \|\partial F\|_{H_x^\delta} + \|F\|_{H_x^1}, \end{aligned}$$

where we used $\|\partial\varrho, \partial(C(\varrho))\|_{H_x^1} \lesssim 1$ in (2.69). Thus (3.28) is proved. \square

Next we provide a comparison result.

Lemma 3.7. *Let $0 < \delta \leq s' - 2$. There hold the following estimates,*

$$\|\partial\mathfrak{C}\|_{H_x^\delta} + \|\partial^2\Omega\|_{H_x^\delta} + \|\partial^2\mathfrak{w}\|_{H_x^\delta} \lesssim \|\operatorname{curl}\mathfrak{C}\|_{\dot{H}_x^\delta} + 1. \quad (3.31)$$

Proof. We first show the following preliminary estimates

$$\|(e^{\pm\varrho} + 1)\partial\varrho \cdot (\mathfrak{C} + \partial\Omega)\|_{H_x^\delta} \lesssim 1, \quad (3.32)$$

$$\|\Omega \cdot \partial^2\varrho\|_{H_x^\delta} + \|\partial(\Omega \cdot \partial\varrho)\|_{H_x^\delta} \lesssim 1, \quad (3.33)$$

$$\|(e^{\pm\varrho} + 1)(\partial\varrho)^2\Omega\|_{H_x^\delta} \lesssim 1. \quad (3.34)$$

Indeed, applying (10.14) to $F = \partial\varrho$ and $G = \mathfrak{C} + \partial\Omega$, and using the result $\|\partial\varrho, \mathfrak{C}, \partial\Omega\|_{H_x^1} \lesssim 1$ which is from (2.69) and (3.5), we have

$$\begin{aligned} \|(e^{\pm\varrho} + 1)\partial\varrho \cdot (\mathfrak{C} + \partial\Omega)\|_{H_x^\delta} &\lesssim \|\partial\varrho \cdot (\mathfrak{C} + \partial\Omega)\|_{H_x^\delta} \\ &\lesssim \|\partial\varrho\|_{H_x^1} (\|\mathfrak{C}\|_{H_x^1} + 1) \lesssim 1, \end{aligned}$$

where for the first inequality we applied Lemma 2.12. This gives (3.32).

By using (10.13), (3.23), (2.42) and (3.24),

$$\begin{aligned} \|\Omega \cdot \partial^2\varrho\|_{H_x^\delta} &\lesssim \|\Omega\|_{B_{\infty,2,x}^\delta} \|\partial^2\varrho\|_{L_x^2} + \|\Omega\|_{L_x^\infty} \|\partial^2\varrho\|_{H_x^\delta} \\ &\lesssim \|\Omega\|_{H_x^{\frac{3}{2}+\delta}} \|\partial^2\varrho\|_{L_x^2} + \|\Omega\|_{L_x^\infty} \|\partial^2\varrho\|_{H_x^\delta} \lesssim 1. \end{aligned}$$

Due to $\partial(\Omega\partial\varrho) = \Omega\partial^2\varrho + \partial\Omega\partial\varrho$, combining the above estimate with (3.32) yields the second estimate in (3.33).

For (3.34), we first use (2.73), and then apply (10.15) with $G_1 = G_2 = \partial\varrho$ and $G_3 = \Omega$ to derive

$$\begin{aligned} \|(e^{\pm\varrho} + 1)(\partial\varrho)^2\Omega\|_{H_x^\delta} &\lesssim \|(\partial\varrho)^2\Omega\|_{H_x^\delta} \lesssim \|\partial\varrho\|_{H_x^1}^2 \|\Omega\|_{H_x^{1+\delta}} + \|\Omega\|_{H_x^1} \|\partial\varrho\|_{H_x^{1+\delta}} \|\partial\varrho\|_{H_x^1} \\ &\lesssim 1, \end{aligned}$$

where we used (3.24) and (3.23).

Now we consider the estimate in (3.31). By using the equation

$$\operatorname{div} \mathfrak{C} = -\partial_i \varrho \mathfrak{C}^i \quad (3.35)$$

and the elliptic estimate for the div-curl system, we derive

$$\|\partial\mathfrak{C}\|_{H_x^\delta} \lesssim \|\partial_i \varrho \cdot \mathfrak{C}^i\|_{H_x^\delta} + \|\operatorname{curl} \mathfrak{C}\|_{H_x^\delta}. \quad (3.36)$$

Applying (3.32) to the first term gives the first estimate in (3.31).

For the second estimate in (3.31), similar to (3.6), we derive

$$\begin{aligned}\|\partial^2 \Omega\|_{H_x^\delta} &\lesssim \|\operatorname{curl}^2 \Omega\|_{H_x^\delta} + \|\partial(\Omega \cdot \partial \varrho)\|_{H_x^\delta} \\ &\lesssim \|\operatorname{curl}(e^\varrho \mathfrak{C})\|_{H_x^\delta} + \|\partial(\Omega \cdot \partial \varrho)\|_{H_x^\delta} \\ &\lesssim \|e^\varrho \operatorname{curl} \mathfrak{C}\|_{H_x^\delta} + \|e^\varrho \partial \varrho \cdot \mathfrak{C}\|_{H_x^\delta} + \|\partial(\Omega \cdot \partial \varrho)\|_{H_x^\delta}.\end{aligned}\quad (3.37)$$

Applying Lemma 2.12 to the first term, (3.32) to the second term and (3.33) to the third term leads to

$$\|\partial^2 \Omega\|_{H_x^\delta} \lesssim \|\operatorname{curl} \mathfrak{C}\|_{H_x^\delta} + 1. \quad (3.38)$$

Finally, we estimate $\|\partial^2 \mathfrak{w}\|_{H_x^\delta}$ in view of the symbolic formula (3.7) for $\partial^2 \mathfrak{w}$

$$\begin{aligned}\|\partial^2 \mathfrak{w}\|_{H_x^\delta} &\lesssim \|e^\varrho \{\partial^2 \Omega + \partial \Omega \partial \varrho + \Omega((\partial \varrho)^2 + \partial^2 \varrho)\}\|_{H_x^\delta} \\ &\lesssim \|\partial^2 \Omega\|_{H_x^\delta} + \|\partial \Omega \partial \varrho\|_{H_x^\delta} + \|\Omega \partial^2 \varrho\|_{H_x^\delta} + \|\Omega(\partial \varrho)^2\|_{H_x^\delta} \\ &\lesssim \|\partial^2 \Omega\|_{H_x^\delta} + 1 \lesssim \|\operatorname{curl} \mathfrak{C}\|_{H_x^\delta} + 1,\end{aligned}$$

where we used (2.73), (3.32)-(3.34) and the estimate (3.38). Using the L_x^2 bound in (3.5), (3.31) is proved. \square

For estimates in the fractional Sobolev spaces, we recall the standard trichotomy of the Littlewood-Paley theory. For smooth functions F and G ,

$$P_\mu(F \cdot G) = P_\mu[F \cdot G]_{HL} + P_\mu[F \cdot G]_{LH} + P_\mu[F \cdot G]_{HH} \quad (3.39)$$

where ¹⁸

$$\begin{aligned}P_\mu[F \cdot G]_{HL} &= P_\mu(P_\mu F P_{\leq \mu} G), \\ P_\mu[F \cdot G]_{LH} &= P_\mu(P_{\leq \mu} F P_\mu G), \\ P_\mu[F \cdot G]_{HH} &= P_\mu\left(\sum_{\lambda > \mu} P_\lambda F P_\lambda G\right).\end{aligned}$$

Lemma 3.8. *Let $0 < \delta \leq s' - 2$ and let $\mathcal{I}_\mu = \mu^{2\delta} \int_0^t \int_{\Sigma_{t'}} P_\mu(\operatorname{curl} \mathbf{T} \mathfrak{C}^m) P_\mu \operatorname{curl} \mathfrak{C}_m d\mu_e dt'$. There holds the following estimate,*

$$\begin{aligned}\sum_{\mu > 1} |\mathcal{I}_\mu| &\lesssim \int_0^t (\|\operatorname{curl} \mathfrak{C}\|_{\dot{H}_x^\delta} + 1) \{ \|\partial v\|_{B_{\infty,2,x}^\delta} + \|\partial \varrho\|_{L_x^\infty} + (1 + \|\partial v\|_{L_x^\infty}) (\|\operatorname{curl} \mathfrak{C}\|_{\dot{H}_x^\delta} + 1) \} dt' \\ &\quad + \sup_{0 \leq t' \leq t} \|\Lambda^\delta \operatorname{curl} \mathfrak{C}(t')\|_{L_x^2}.\end{aligned}$$

Proof. In view of (3.13),

$$\begin{aligned}\mathcal{I}_\mu &= \mu^{2\delta} \int_0^t \int_{\Sigma_{t'}} (-2P_\mu \operatorname{curl}(e^{-\varrho} \partial v_j \wedge \partial \Omega^j)_m + P_\mu \operatorname{curl}(\partial_a v \mathfrak{C}^a)_m) \cdot P_\mu \operatorname{curl} \mathfrak{C}^m d\mu_e \\ &= \mathcal{I}_\mu^1 + \mathcal{I}_\mu^2.\end{aligned}\quad (3.40)$$

For the term \mathcal{I}_μ^2 , we recall the formula in (3.14). We first bound the second term in (3.14) by applying (10.13) to $F = \partial v$ and $G = \partial \mathfrak{C}$,

$$\begin{aligned}\|\mu^\delta P_\mu(\partial v \cdot \partial \mathfrak{C})\|_{l_\mu^2 L_x^2} &\lesssim \|\partial v\|_{B_{\infty,2,x}^\delta} \|\partial \mathfrak{C}\|_{L_x^2} + \|\partial \mathfrak{C}\|_{H_x^\delta} \|\partial v\|_{L_x^\infty} \\ &\lesssim \|\partial v\|_{B_{\infty,2,x}^\delta} + (\|\operatorname{curl} \mathfrak{C}\|_{\dot{H}_x^\delta} + 1) \|\partial v\|_{L_x^\infty},\end{aligned}$$

¹⁸The more precise form of the trichotomy decomposition can be found in [33]. The decomposition is written schematically for simplicity without altering analysis.

where we used (3.5) and (3.31) to derive the last line.

For the first term in (3.14), we apply (10.14) to $F = \partial \mathfrak{w}$ and $G = \mathfrak{C}$, and also use (3.5) to derive

$$\|\mu^\delta P_\mu(\partial \mathfrak{w} \cdot \mathfrak{C})\|_{l_\mu^2 L_x^2} \lesssim \|\partial \mathfrak{w}\|_{H_x^{\frac{1}{2}+\delta}} \|\mathfrak{C}\|_{H_x^1} + \|\partial \mathfrak{w}\|_{H_x^1} \|\mathfrak{C}\|_{H_x^{\frac{1}{2}+\delta}} \lesssim 1.$$

Combining the above two estimates implies

$$\sum_{\mu > 1} |\mathcal{I}_\mu^2| \lesssim \int_0^t \|\operatorname{curl} \mathfrak{C}\|_{\dot{H}_x^\delta} (\|\partial v\|_{B_{\infty,2,x}^\delta} + (\|\operatorname{curl} \mathfrak{C}\|_{\dot{H}_x^\delta} + 1) \|\partial v\|_{L_x^\infty} + 1) dt'. \quad (3.41)$$

For the hard term \mathcal{I}_μ^1 ¹⁹, applying (3.3) to $F = \Omega$, similar to (3.15), we write

$$\begin{aligned} \mathcal{I}_\mu^1 &= \mu^{2\delta} \int_0^t \int_{\Sigma'_t} \operatorname{curl} P_\mu(e^{-\varrho} \partial v_j \wedge \partial \Omega^j)_m P_\mu \operatorname{curl} \mathfrak{C}^m d\mu_e dt' \\ &= \mu^{2\delta} \int_0^t \int_{\Sigma'_t} P_\mu \{e^{-\varrho} (\partial^n \partial_m v^j \partial_n \Omega_j + (\partial_j \mathbf{T} \varrho + \operatorname{curl}^2 v_j) \partial_m \Omega^j + \partial_m v^j (-\operatorname{curl}^2 \Omega_j + \partial_j (\operatorname{div} \Omega)) \\ &\quad - \partial_n v^j \partial^n \partial_m \Omega_j - \epsilon_m^{ni} \epsilon_i^{ab} \partial_a v^j \partial_b \Omega_j \partial_n \varrho)\} P_\mu \operatorname{curl} \mathfrak{C}^m d\mu_e dt'. \end{aligned}$$

Let us write the hard term below, which will be treated by integration by parts

$$\mathcal{I}_{+,\mu}^1 = \mu^{2\delta} \int_0^t \int_{\Sigma'_t} P_\mu \{e^{-\varrho} (\partial^n \partial_m v^j \partial_n \Omega_j + \partial_j \mathbf{T} \varrho \partial_m \Omega^j)\} P_\mu \operatorname{curl} \mathfrak{C}^m d\mu_e dt'. \quad (3.42)$$

The remaining terms are denoted by $\mathcal{I}_{-,\mu}^1$

$$\mathcal{I}_{-,\mu}^1 = \mu^{2\delta} \int_0^t \int_{\Sigma'_t} P_\mu (e^{-\varrho} \mathcal{A}_m) P_\mu \operatorname{curl} \mathfrak{C}^m d\mu_e dt'$$

where

$$\begin{aligned} \mathcal{A}_m &= \operatorname{curl}^2 v_j \partial_m \Omega^j + \partial_m v^j (-\operatorname{curl}^2 \Omega_j + \partial_j (\operatorname{div} \Omega)) - \partial_n v^j \partial^n \partial_m \Omega_j - \epsilon_m^{ni} \epsilon_i^{ab} \partial_a v^j \partial_b \Omega_j \partial_n \varrho \\ &= \operatorname{curl} \mathfrak{w} \partial \Omega + \partial v \partial^2 \Omega + \partial v \cdot \partial \Omega \cdot \partial \varrho \\ &= e^\varrho (\operatorname{curl} \Omega + \Omega \partial \varrho) \partial \Omega + \partial v \partial^2 \Omega + \partial v \cdot \partial \Omega \cdot \partial \varrho, \end{aligned}$$

and we used the second equation in (2.47) to derive the last line. To control this term, we first directly obtain

$$\sum_{\mu > 1} |\mathcal{I}_{-,\mu}^1| \lesssim \int_0^t \|e^{-\varrho} \mathcal{A}\|_{\dot{H}_x^\delta} \|\operatorname{curl} \mathfrak{C}\|_{\dot{H}_x^\delta} dt'. \quad (3.43)$$

We then estimate by using Lemma 2.12 that

$$\|e^{-\varrho} \mathcal{A}\|_{\dot{H}_x^\delta} \lesssim \|\operatorname{curl} \Omega \cdot \partial \Omega\|_{\dot{H}_x^\delta} + \|\Omega \partial \varrho \partial \Omega\|_{H_x^\delta} + \|\partial v \partial^2 \Omega\|_{H_x^\delta} + \|\partial v \cdot \partial \Omega \cdot \partial \varrho\|_{H_x^\delta}.$$

By using (10.14) and (3.5)

$$\|\partial \Omega \cdot \partial \Omega\|_{\dot{H}_x^\delta} \lesssim \|\partial \Omega\|_{H_x^{\frac{1}{2}+\delta}} \|\partial \Omega\|_{H_x^1} \lesssim \|\partial \Omega\|_{H_x^1}^2 \lesssim 1.$$

By using (10.13), (3.5) and (3.31), we can obtain

$$\begin{aligned} \|\partial v \partial^2 \Omega\|_{H_x^\delta} &\lesssim \|\partial v\|_{B_{\infty,2,x}^\delta} \|\partial^2 \Omega\|_{L_x^2} + \|\partial v\|_{L_x^\infty} \|\partial^2 \Omega\|_{H_x^\delta} \\ &\lesssim \|\partial v\|_{B_{\infty,2,x}^\delta} + \|\partial v\|_{L_x^\infty} (\|\operatorname{curl} \mathfrak{C}\|_{\dot{H}_x^\delta} + 1). \end{aligned}$$

¹⁹We dropped the constant -2 which does not influence the analysis.

For the cubic terms, we employ (10.15) to obtain

$$\begin{aligned} \|(\Omega + \partial v + \partial \varrho) \cdot \partial \Omega \cdot \partial \varrho\|_{H_x^\delta} &\lesssim (\|\Lambda^\delta \Omega\|_{H_x^1} + \|\Lambda^\delta(\partial v, \partial \varrho)\|_{H_x^1}) \|\partial \Omega\|_{H_x^1} \|\partial \varrho\|_{H_x^1} \\ &\quad + \|\Omega, \partial v, \partial \varrho\|_{H_x^1} (\|\partial \Omega\|_{H_x^1} \|\Lambda^\delta \partial \varrho\|_{H_x^1} + \|\Lambda^\delta \partial \Omega\|_{H_x^1} \|\partial \varrho\|_{H_x^1}) \\ &\lesssim \|\Lambda^\delta \partial \Omega\|_{H_x^1} + 1 \lesssim \|\operatorname{curl} \mathfrak{C}\|_{\dot{H}_x^\delta} + 1, \end{aligned} \quad (3.44)$$

where we employed Corollary 3.3, (3.5) and (3.31) to get the last line.

Substituting the above three estimates to (3.43) yields

$$\sum_{\mu > 1} |\mathcal{I}_{-, \mu}^1| \lesssim \int_0^t \|\operatorname{curl} \mathfrak{C}\|_{\dot{H}_x^\delta} \{ \|\partial v\|_{B_{\infty, 2, x}^\delta} + (1 + \|\partial v\|_{L_x^\infty}) (\|\operatorname{curl} \mathfrak{C}\|_{\dot{H}_x^\delta} + 1) \} dt'. \quad (3.45)$$

Next we consider the term of (3.42). We first separate the two terms below, for which we will perform integration by parts in different ways,

$$\begin{aligned} \mathcal{J}_\mu &= \mu^{2\delta} \int_0^t \int_{\Sigma_{t'}} P_\mu [\partial^n \partial_m v^j \cdot \partial_n \Omega_j e^{-\varrho}] P_\mu \operatorname{curl} \mathfrak{C}^m d\mu_e dt', \\ \mathcal{K}_\mu &= \mu^{2\delta} \int_0^t \int_{\Sigma_{t'}} P_\mu [\partial_j \mathbf{T} \varrho \cdot \partial_m \Omega^j e^{-\varrho}] P_\mu \operatorname{curl} \mathfrak{C}^m d\mu_e dt'. \end{aligned} \quad (3.46)$$

For \mathcal{J}_μ , using $\partial^m P_\mu \operatorname{curl} \mathfrak{C}_m = 0$, we derive

$$\begin{aligned} \mathcal{J}_\mu &= \mu^{2\delta} \int_0^t \int_{\Sigma_{t'}} P_\mu [\partial_m (\partial^n v^j \cdot \partial_n \Omega_j e^{-\varrho}) - \partial^n v^j \partial_m (\partial_n \Omega_j e^{-\varrho})] P_\mu \operatorname{curl} \mathfrak{C}^m d\mu_e dt' \\ &= \mu^{2\delta} \int_0^t \int_{\Sigma_{t'}} \{ \partial_m (P_\mu (\partial^n v^j \cdot \partial_n \Omega_j e^{-\varrho}) P_\mu \operatorname{curl} \mathfrak{C}^m) \\ &\quad - P_\mu [\partial^n v^j \partial_m (\partial_n \Omega_j \cdot e^{-\varrho})] P_\mu \operatorname{curl} \mathfrak{C}^m \} d\mu_e dt'. \end{aligned}$$

Again, since the boundary term vanishes, we only need to treat the last line. By trichotomy in (3.39), we decompose $\mathcal{J}_\mu = \mathcal{J}_{\mu, HL} + \mathcal{J}_{\mu, LH} + \mathcal{J}_{\mu, HH}$,

$$\begin{aligned} \mathcal{J}_{\mu, HL} &= \mu^{2\delta} \int_0^t \int_{\Sigma_{t'}} P_\mu [\partial^n v_\mu^j \cdot P_{\leq \mu} \partial_m (e^{-\varrho} \partial_n \Omega_j)] P_\mu \operatorname{curl} \mathfrak{C}^m d\mu_e dt', \\ \mathcal{J}_{\mu, LH} &= \mu^{2\delta} \int_0^t \int_{\Sigma_{t'}} P_\mu [P_{\leq \mu} \partial^n v^j \cdot P_\mu \partial_m (e^{-\varrho} \partial_n \Omega_j)] P_\mu \operatorname{curl} \mathfrak{C}^m d\mu_e dt', \\ \mathcal{J}_{\mu, HH} &= \mu^{2\delta} \int_0^t \int_{\Sigma_{t'}} \sum_{\lambda \geq \mu} P_\mu [P_\lambda (\partial^n v^j) \cdot P_\lambda \partial_m (e^{-\varrho} \partial_n \Omega_j)] P_\mu \operatorname{curl} \mathfrak{C}^m d\mu_e dt', \end{aligned}$$

where we neglected the $-$ sign, since it does not influence the estimate; and $F_\mu^i = P_\mu(F^i)$. By using Hölder inequality and Cauchy-Schwarz inequality, we derive

$$\begin{aligned} \sum_{\mu > 1} |\mathcal{J}_{\mu, HL}| &\lesssim \int_0^t \|\partial v\|_{B_{\infty, 2, x}^\delta} \|\partial(e^{-\varrho} \partial \Omega)\|_{L_x^2} \|\operatorname{curl} \mathfrak{C}\|_{\dot{H}_x^\delta} dt'; \\ \sum_{\mu > 1} (|\mathcal{J}_{\mu, LH}| + |\mathcal{J}_{\mu, HH}|) &\lesssim \int_0^t \|\partial v\|_{L_x^\infty} \|\partial(e^{-\varrho} \partial \Omega)\|_{\dot{H}_x^\delta} \|\operatorname{curl} \mathfrak{C}\|_{\dot{H}_x^\delta} dt'. \end{aligned} \quad (3.47)$$

By using (3.28) with $F = \partial \Omega$ and (3.31), we have

$$\|\partial(e^{-\varrho} \partial \Omega)\|_{\dot{H}_x^\delta} \lesssim \|\operatorname{curl} \mathfrak{C}\|_{\dot{H}_x^\delta} + 1.$$

Note the lower order estimate holds due to (2.69), the first estimate in (3.8) and (3.5)

$$\|\partial(e^{-\varrho}\partial\Omega)\|_{L_x^2} \lesssim \|\operatorname{curl} \mathfrak{C}\|_{L_x^2} + 1 \lesssim 1.$$

Substituting the above two estimates to (3.47) yields

$$\sum_{\mu>1} |\mathcal{J}_\mu| \lesssim \int_0^t \{ \|\partial v\|_{B_{\infty,2,x}^\delta} + \|\partial v\|_{L_x^\infty} (\|\operatorname{curl} \mathfrak{C}\|_{\dot{H}_x^\delta} + 1) \} \|\operatorname{curl} \mathfrak{C}\|_{\dot{H}_x^\delta} dt'. \quad (3.48)$$

Next we treat \mathcal{K}_μ in (3.46) by performing the spacetime integration by parts by virtue of (3.4).

$$\begin{aligned} \mathcal{K}_\mu &= \mu^{2\delta} \int_0^t \int_{\Sigma_{t'}} P_\mu (\partial_j \mathbf{T} \varrho \partial_m \Omega^j e^{-\varrho}) P_\mu \operatorname{curl} \mathfrak{C}^m d\mu_e dt' \\ &= \mu^{2\delta} \int_0^t \int_{\Sigma_{t'}} P_\mu ([\partial_j, \mathbf{T}] \varrho + \mathbf{T} \partial_j \varrho) \partial_m \Omega^j e^{-\varrho} P_\mu \operatorname{curl} \mathfrak{C}^m d\mu_e dt' \\ &= \mu^{2\delta} \int_0^t \int_{\Sigma_{t'}} P_\mu [\mathbf{T} (\partial_j \varrho \partial_m \Omega^j e^{-\varrho}) - \partial_j \varrho \mathbf{T} (\partial_m \Omega^j e^{-\varrho}) + \partial_j v^n \partial_n \varrho \partial_m \Omega^j e^{-\varrho}] P_\mu \operatorname{curl} \mathfrak{C}^m d\mu_e dt' \\ &= \mu^{2\delta} \int_0^t \{ \mathbf{T} P_\mu (\partial_j \varrho \partial_m \Omega^j e^{-\varrho}) + [P_\mu, \mathbf{T}] (\partial_j \varrho \partial_m \Omega^j e^{-\varrho}) + P_\mu [-\partial_j \varrho \mathbf{T} (\partial_m \Omega^j e^{-\varrho}) \\ &\quad + \partial_j v^n \partial_n \varrho \partial_m \Omega^j e^{-\varrho}] \} P_\mu \operatorname{curl} \mathfrak{C}^m d\mu_e dt' \\ &= \mu^{2\delta} \int_0^t \int_{\Sigma_{t'}} (\mathbf{T} \{ P_\mu (\partial_j \varrho \partial_m \Omega^j e^{-\varrho}) \cdot P_\mu \operatorname{curl} \mathfrak{C}^m \} - P_\mu (\partial_j \varrho \partial_m \Omega^j e^{-\varrho}) \mathbf{T} P_\mu \operatorname{curl} \mathfrak{C}^m) d\mu_e dt' \\ &\quad + \mu^{2\delta} \int_0^t \int_{\Sigma_{t'}} \{ [P_\mu, \mathbf{T}] (\partial_j \varrho \partial_m \Omega^j e^{-\varrho}) + P_\mu [-\partial_j \varrho \mathbf{T} (\partial_m \Omega^j e^{-\varrho}) + \partial_j v^n \partial_n \varrho \partial_m \Omega^j e^{-\varrho}] \} P_\mu \operatorname{curl} \mathfrak{C}^m d\mu_e dt' \\ &= \mu^{2\delta} \{ \int_0^t \int_{\Sigma_{t'}} \operatorname{Tr} \overset{\circ}{k} P_\mu (\partial_j \varrho \partial_m \Omega^j e^{-\varrho}) P_\mu \operatorname{curl} \mathfrak{C}^m d\mu_e dt' + \int_0^t \partial_t \int_{\Sigma_{t'}} P_\mu (\partial_j \varrho \partial_m \Omega^j e^{-\varrho}) P_\mu \operatorname{curl} \mathfrak{C}^m d\mu_e dt' \} \\ &\quad + \mu^{2\delta} \int_0^t \int_{\Sigma_{t'}} \{ [P_\mu, \mathbf{T}] (\partial_j \varrho \partial_m \Omega^j e^{-\varrho}) + P_\mu (-\partial_j \varrho \mathbf{T} (\partial_m \Omega^j e^{-\varrho}) + \partial_j v^n \partial_n \varrho \partial_m \Omega^j e^{-\varrho}) \} P_\mu \operatorname{curl} \mathfrak{C}^m d\mu_e dt' \\ &\quad - \mu^{2\delta} \int_0^t \int_{\Sigma_{t'}} P_\mu (\partial_j \varrho \partial_m \Omega^j e^{-\varrho}) \mathbf{T} P_\mu \operatorname{curl} \mathfrak{C}^m d\mu_e dt'. \end{aligned}$$

Let us denote the last line by \mathcal{K}_μ^- and the remaining four terms together by \mathcal{K}_μ^+ . We further decompose \mathcal{K}_μ^- as follows

$$\begin{aligned} \mathcal{K}_\mu^- &= \mu^{2\delta} \int_0^t \int_{\Sigma_{t'}} P_\mu (\partial_j \varrho \partial_m \Omega^j e^{-\varrho}) \mathbf{T} P_\mu \operatorname{curl} \mathfrak{C}^m d\mu_e dt' \\ &= \mu^{2\delta} \int_0^t \int_{\Sigma_{t'}} P_\mu (\partial_j \varrho \partial_m \Omega^j e^{-\varrho}) ([\mathbf{T}, P_\mu \operatorname{curl}] \mathfrak{C}^m + P_\mu \operatorname{curl} \mathbf{T} \mathfrak{C}^m) d\mu_e dt' \\ &= \mathcal{K}_{1,\mu}^- + \mathcal{K}_{2,\mu}^-, \end{aligned}$$

where the $-$ sign is dropped without influencing the result. We can bound the second term by using (3.21) and the finite band property ([22, Page 2])

$$\begin{aligned} |\mathcal{K}_{2,\mu}^-| &\leq \left| \int_0^t \int_{\Sigma_{t'}} \mu^{2\delta} P_\mu (\partial \varrho \partial \Omega e^{-\varrho}) P_\mu \partial(e^{-\varrho} \partial v \cdot \partial \Omega) d\mu_e dt \right| \\ &\lesssim \mu^{1+2\delta} \int_0^t \|P_\mu (\partial \varrho e^{-\varrho} \cdot \partial \Omega)\|_{L_x^2} \|P_\mu (e^{-\varrho} \partial v \partial \Omega)\|_{L_x^2} dt'. \end{aligned}$$

By using (2.74)

$$\begin{aligned} \text{Er} &:= \|\Lambda^{\frac{1}{2}+\delta}(\partial\varrho e^{-\varrho} \cdot \partial\Omega)\|_{L_x^2} + \|\Lambda^{\frac{1}{2}+\delta}(e^{-\varrho}\partial v \cdot \partial\Omega)\|_{L_x^2} \\ &\lesssim \|\partial\varrho\partial\Omega\|_{H_x^{\frac{1}{2}+\delta}} + \|\partial v\partial\Omega\|_{H_x^{\frac{1}{2}+\delta}}. \end{aligned}$$

The L^2 norms are bounded by a universal constant due to (3.8). For the $\dot{H}^{\frac{1}{2}+\delta}$ norms, applying (10.14) to $(F, G) = (\partial\varrho, \partial\Omega)$ and $(F, G) = (\partial v, \partial\Omega)$ leads to

$$\begin{aligned} \text{Er} &\lesssim \|\partial\varrho, \partial v\|_{H_x^{1+\delta}} \|\partial\Omega\|_{H_x^1} + \|\partial\varrho, \partial v\|_{H_x^1} \|\partial\Omega\|_{H_x^{1+\delta}} + 1 \\ &\lesssim 1 + \|\partial\Omega\|_{H_x^{1+\delta}} \lesssim \|\text{curl } \mathfrak{C}\|_{\dot{H}_x^\delta} + 1, \end{aligned}$$

where we used Corollary 3.3 and (3.31) to derive the last line. We thus conclude that

$$\begin{aligned} \sum_{\mu>1} |\mathcal{K}_{2,\mu}^-| &\lesssim \int_0^t \|\Lambda^{\frac{1}{2}+\delta}(e^{-\varrho}\partial\varrho\partial\Omega)\|_{L_x^2} \|\Lambda^{\frac{1}{2}+\delta}(e^{-\varrho}\partial v \cdot \partial\Omega)\|_{L_x^2} dt' \\ &\lesssim \int_0^t (\|\text{curl } \mathfrak{C}\|_{\dot{H}_x^\delta} + 1)^2 dt'. \end{aligned}$$

For the other term, we first derive

$$\sum_{\mu>1} |\mathcal{K}_{1,\mu}^-| \lesssim \int_0^t \|\Lambda^\delta(e^{-\varrho}\partial\varrho\partial\Omega)\|_{L_x^2} \|\mu^\delta[\mathbf{T}, P_\mu \text{curl}] \mathfrak{C}\|_{l_\mu^2 L_x^2} dt'.$$

Due to (3.32)

$$\|e^{-\varrho}\partial\varrho\partial\Omega\|_{H_x^\delta} \lesssim 1. \quad (3.49)$$

Combining the above estimate with (3.27) for $F = \mathfrak{C}$, (3.5) and (3.31) yields

$$\begin{aligned} \sum_{\mu>1} |\mathcal{K}_{1,\mu}^-| &\lesssim \int_0^t (\|\partial v\|_{B_{\infty,2,x}^\delta} \|\partial\mathfrak{C}\|_{L_x^2} + \|\partial v\|_{L_x^\infty} \|\partial\mathfrak{C}\|_{H_x^\delta}) dt' \\ &\lesssim \int_0^t \{\|\partial v\|_{B_{\infty,2,x}^\delta} + \|\partial v\|_{L_x^\infty} (\|\text{curl } \mathfrak{C}\|_{\dot{H}_x^\delta} + 1)\} dt'. \end{aligned}$$

Hence

$$\sum_{\mu>1} |\mathcal{K}_\mu^-| \lesssim \int_0^t \{\|\partial v\|_{B_{\infty,2,x}^\delta} + (\|\text{curl } \mathfrak{C}\|_{\dot{H}_x^\delta} + 1 + \|\partial v\|_{L_x^\infty}) (\|\text{curl } \mathfrak{C}\|_{\dot{H}_x^\delta} + 1)\} dt'. \quad (3.50)$$

We go back to consider \mathcal{K}_μ^+ , which has four terms, denoted by $\mathcal{K}_{i,\mu}^+$ with $i = 1, \dots, 4$. By using (3.20), we symbolically rewrite the term below

$$\begin{aligned} \mathcal{K}_{1,\mu}^+ &= \int_0^t \int_{\Sigma_{t'}} \mu^{2\delta} P_\mu (-\partial_j \varrho \mathbf{T} (\partial_m \Omega^j e^{-\varrho}) + \partial_j v^n \partial_n \varrho \partial_m \Omega^j e^{-\varrho}) P_\mu \text{curl } \mathfrak{C}^m d\mu_e dt' \\ &= \int_0^t \int_{\Sigma_{t'}} \mu^{2\delta} P_\mu \{(\partial\varrho(\partial v \partial\Omega + \Omega \partial^2 v) e^{-\varrho})\} P_\mu \text{curl } \mathfrak{C} d\mu_e dt'. \end{aligned}$$

We apply (10.7) to $F = \partial\varrho \cdot \Omega$ and $G = \partial v$ following with using (3.33) and Corollary 3.3 to derive

$$\begin{aligned} \|\partial\varrho \cdot \Omega \cdot \partial^2 v\|_{\dot{H}_x^\delta} &\lesssim \|\partial\varrho\Omega\|_{L_x^\infty} \|\partial^2 v\|_{H_x^\delta} + \|\partial v\|_{L_x^\infty} \|\partial\varrho \cdot \Omega\|_{H_x^{1+\delta}} \\ &\lesssim \|\partial\varrho\|_{L_x^\infty} + \|\partial v\|_{L_x^\infty}. \end{aligned}$$

Thus it follows by using Lemma 2.12 and (3.44) that

$$\begin{aligned} \sum_{\mu>1} |\mathcal{K}_{1,\mu}^+| &\lesssim \int_0^t \|\partial_\varrho(\partial v \partial \Omega + \Omega \partial^2 v)\|_{H_x^\delta} \|\Lambda^\delta \operatorname{curl} \mathfrak{C}\|_{L_x^2} dt' \\ &\lesssim \int_0^t (\|\partial_\varrho\|_{L_x^\infty} + \|\partial v\|_{L_x^\infty} + \|\operatorname{curl} \mathfrak{C}\|_{\dot{H}_x^\delta} + 1) \|\operatorname{curl} \mathfrak{C}\|_{\dot{H}_x^\delta} dt'. \end{aligned}$$

For the term

$$\mathcal{K}_{2,\mu}^+ = \mu^{2\delta} \int_0^t \int_{\Sigma_{t'}} \operatorname{Tr} \overset{\circ}{k} P_\mu (\partial_\varrho \partial \Omega e^{-\varrho}) P_\mu \operatorname{curl} \mathfrak{C} d\mu_e dt'$$

by using (3.49) and $\operatorname{Tr} \overset{\circ}{k} = -\operatorname{div} v$ we estimate

$$\begin{aligned} \sum_{\mu>1} |\mathcal{K}_{2,\mu}^+| &\lesssim \int_0^t \|\operatorname{Tr} \overset{\circ}{k}\|_{L_x^\infty} \|\Lambda^\delta (\partial_\varrho \partial \Omega e^{-\varrho})\|_{L_x^2} \|\Lambda^\delta \operatorname{curl} \mathfrak{C}\|_{L_x^2} dt' \\ &\lesssim \int_0^t \|\partial v\|_{L_x^\infty} \|\Lambda^\delta \operatorname{curl} \mathfrak{C}\|_{L_x^2} dt'. \end{aligned}$$

For the term

$$\mathcal{K}_{3,\mu}^+ = \mu^{2\delta} \int_0^t \int_{\Sigma_{t'}} [P_\mu, \mathbf{T}] (\partial_\varrho \partial \Omega e^{-\varrho}) P_\mu \operatorname{curl} \mathfrak{C} d\mu_e dt'$$

we recall that $[P_\mu, \mathbf{T}] = [P_\mu, v] \partial$ and apply (10.2) to $F = v$ and $G = \partial_\varrho \partial \Omega e^{-\varrho}$ with $\alpha = \delta$,

$$\begin{aligned} \sum_{\mu>1} |\mathcal{K}_{3,\mu}^+| &\lesssim \int_0^t \|\mu^\delta [P_\mu, v] \partial (\partial_\varrho \partial \Omega e^{-\varrho})\|_{l_\mu^2 L_x^2} \|\Lambda^\delta \operatorname{curl} \mathfrak{C}\|_{L_x^2} dt' \\ &\lesssim \int_0^t \|\partial v\|_{L_x^\infty} \|\partial_\varrho \partial \Omega e^{-\varrho}\|_{H_x^\delta} \|\Lambda^\delta \operatorname{curl} \mathfrak{C}\|_{L_x^2} dt' \\ &\lesssim \int_0^t \|\partial v\|_{L_x^\infty} \|\Lambda^\delta \operatorname{curl} \mathfrak{C}\|_{L_x^2} dt', \end{aligned}$$

where we used (3.49) to obtain the last line.

For the boundary term,

$$\mathcal{K}_{4,\mu}^+ = \mu^{2\delta} \int_0^t \partial_t \int_{\Sigma_{t'}} P_\mu (\partial_\varrho \partial \Omega e^{-\varrho}) P_\mu \operatorname{curl} \mathfrak{C} d\mu_e dt',$$

by using (3.49) again, we derive

$$\begin{aligned} \sum_{\mu>1} |\mathcal{K}_{4,\mu}^+| &\lesssim \sup_{0 \leq t' \leq t} \|\Lambda^\delta (\partial_\varrho \partial \Omega e^{-\varrho})\|_{L_x^2} \|\Lambda^\delta \operatorname{curl} \mathfrak{C}\|_{L_x^2} \\ &\lesssim \sup_{0 \leq t' \leq t} \|\Lambda^\delta \operatorname{curl} \mathfrak{C}\|_{L_x^2}(t'). \end{aligned}$$

Hence we summarize the above estimates for the terms in \mathcal{K}_μ^+ , and combine the estimate of (3.50) to conclude

$$\begin{aligned} \sum_{\mu>1} |\mathcal{K}_\mu| &\lesssim \int_0^t (\|\partial v, \partial_\varrho\|_{L_x^\infty} + \|\Lambda^\delta \operatorname{curl} \mathfrak{C}\|_{L_x^2} + 1) (\|\Lambda^\delta \operatorname{curl} \mathfrak{C}\|_{L_x^2} + 1) dt' \\ &\quad + \sup_{0 \leq t' \leq t} \|\Lambda^\delta \operatorname{curl} \mathfrak{C}\|_{L_x^2}(t') + \int_0^t \|\partial v\|_{B_{\infty,2,x}^\delta} dt'. \end{aligned}$$

Combining the above estimate with (3.48) and (3.45) implies the result in Lemma 3.8. \square

Proof of Proposition 3.5. We apply (3.4) to $F = G = P_\mu \operatorname{curl} \mathfrak{C}$ to obtain

$$\begin{aligned}
& \sum_{\mu > 1} \mu^{2\delta} \int_{\Sigma_t} |P_\mu \operatorname{curl} \mathfrak{C}|^2 d\mu_e \\
& \lesssim \|\Lambda^\delta \operatorname{curl} \mathfrak{C}(0)\|_{L_x^2}^2 + \sum_{\mu > 1} \left| \int_0^t \mu^{2\delta} \mathbf{T} P_\mu \operatorname{curl} \mathfrak{C} \cdot P_\mu \operatorname{curl} \mathfrak{C} d\mu_e dt' \right| \\
& \quad + \int_0^t \|\partial v\|_{L_x^\infty} \|\Lambda^\delta \operatorname{curl} \mathfrak{C}\|_{L_x^2}^2 dt' \\
& \lesssim \|\Lambda^\delta \operatorname{curl} \mathfrak{C}(0)\|_{L_x^2}^2 + \sum_{\mu > 1} \left| \int_0^t \mu^{2\delta} ([\mathbf{T}, P_\mu \operatorname{curl}] + P_\mu \operatorname{curl} \mathbf{T}) \mathfrak{C} \cdot P_\mu \operatorname{curl} \mathfrak{C} d\mu_e dt' \right| \\
& \quad + \int_0^t \|\partial v\|_{L_x^\infty} \|\Lambda^\delta \operatorname{curl} \mathfrak{C}\|_{L_x^2}^2 dt' \\
& \lesssim \|\Lambda^\delta \operatorname{curl} \mathfrak{C}(0)\|_{L_x^2}^2 + \sum_{\mu > 1} \left| \int_0^t \mu^{2\delta} [\mathbf{T}, P_\mu \operatorname{curl}] \mathfrak{C} \cdot P_\mu \operatorname{curl} \mathfrak{C} d\mu_e dt' \right| \\
& \quad + \int_0^t \|\partial v\|_{L_x^\infty} \|\Lambda^\delta \operatorname{curl} \mathfrak{C}\|_{L_x^2}^2 dt' + \sum_{\mu > 1} |\mathcal{I}_\mu| \\
& \lesssim \int_0^t (\|\operatorname{curl} \mathfrak{C}\|_{\dot{H}_x^\delta} + 1) \{ \|\partial v\|_{B_{\infty,2,x}^\delta} + \|\partial \varrho\|_{L_x^\infty} + (1 + \|\partial v\|_{L_x^\infty}) (\|\operatorname{curl} \mathfrak{C}\|_{\dot{H}_x^\delta} + 1) \} dt' \\
& \quad + \sup_{0 \leq t' \leq t} \|\Lambda^\delta \operatorname{curl} \mathfrak{C}\|_{L_x^2}(t') + \|\Lambda^\delta \operatorname{curl} \mathfrak{C}(0)\|_{L_x^2}^2,
\end{aligned}$$

where we used $\operatorname{Tr} \overset{\circ}{k} = -\operatorname{div} v$, and applied (3.27) to $F = \mathfrak{C}$ together with (3.31), and Lemma 3.8 to obtain the last inequality. Similar to (3.31), $\|\Lambda^\delta \operatorname{curl} \mathfrak{C}(0)\|_{L_x^2} \lesssim \|\partial^2 \mathfrak{w}(0)\|_{H_x^\delta} + 1 \lesssim 1$. Proposition 3.5 then follows by using Gronwall's inequality and using (3.31). \square

4. REDUCTION TO THE STRICHARTZ ESTIMATES FOR THE LINEARIZED WAVE EQUATION

Due to Corollary 3.4, the main task from now on is to improve the bootstrap assumption in (2.1) by establishing Strichartz estimate for the wave function $\Psi = (v_+, \varrho)$, which is restated below.

Theorem 4.1 (Main estimate). *Let $s > 2$ and let $\Psi = (v_+, \varrho)$ be the pair of solutions of (1.23) and (1.8). If Ψ satisfies (2.1), then there holds with a number $8\delta_0 < \gamma_0 < s - 2$*

$$\|\partial \Psi\|_{L_{[0,T]}^2 L_x^\infty}^2 + \sum_{\lambda \geq 2} \lambda^{2\gamma_0} \|\bar{P}_\lambda \partial \Psi\|_{L_{[0,T]}^2 L_x^\infty}^2 \lesssim T^{2\gamma_1} \quad (4.1)$$

where \bar{P}_λ denote the Littlewood-Paley projections with $\sum_\lambda \bar{P}_\lambda = \operatorname{Id}$ in $L^2(\mathbb{R}^3)$ and $0 < \gamma_1 \leq \epsilon_0$.

Remark 4.2. Different from [42, Section 4] and [45, Section 3], \mathbf{T} -derivative estimates in (4.1) will be derived by using (1.4) and the spatial derivative estimate in (4.1) due to a potential issue from commutations.

We first reduce the proof of Theorem 4.1 to the proof of Strichartz estimates on small time intervals. Let λ be a fixed large dyadic number and let $0 < \epsilon_0 < \frac{s-2}{5}$ and $\delta_0 = \epsilon_0^2$ be as fixed in Section 2. By using the bootstrap assumption (2.1), we can partition $[0, T]$ into disjoint union of

sub-intervals $I_k := [t_{k-1}, t_k]$ of total number $\lesssim \lambda^{8\epsilon_0}$ with the properties that $|I_k| \leq \lambda^{-8\epsilon_0} T$ and

$$\|\partial\Psi\|_{L_{I_k}^2 L_x^\infty}^2 + \sum_{\mu \geq 2} \mu^{2\delta_0} \|\bar{P}_\mu \partial\Psi\|_{L_{I_k}^2 L_x^\infty}^2 \lesssim \lambda^{-8\epsilon_0}. \quad (4.2)$$

The total number of intervals I_k depends on λ , which is denoted by k_λ .

Let \mathcal{H} be the space of pairs of functions $\vartheta = (\vartheta_0, \vartheta_1)$, with the norm

$$\|\vartheta\|_{\mathcal{H}}^2 = \|\partial\vartheta_0\|_{L_x^2}^2 + \|\vartheta_1 + v^i \partial_i \vartheta_0\|_{L_x^2}^2.$$

For the pair of functions $f[t] := (f(t), \partial_t f(t))$,

$$\|f[t]\|_{\mathcal{H}}^2 = \|\partial f(t)\|_{L_x^2}^2 + \|\mathbf{T}f(t)\|_{L_x^2}^2.$$

On each time interval I_k we will show the following dyadic Strichartz estimate.

Theorem 4.3 (Dyadic Strichartz estimates for the linearized wave equation). *Fix $\lambda \geq \Lambda$ with Λ a large constant. Let \mathbf{g} be the acoustical metric given in (1.3), and ψ be a solution of*

$$\square_{\mathbf{g}} \psi = 0 \quad (4.3)$$

on the time interval I_k . Then for any $q > 2$ sufficiently close to 2 there holds

$$\|P_\lambda \partial\psi\|_{L_{I_k}^q L_x^\infty} \lesssim \lambda^{\frac{3}{2} - \frac{1}{q}} \|\psi[t_k]\|_{\mathcal{H}},$$

where $\partial = \partial, \mathbf{T}$.

4.1. Proof of Theorem 4.1 assuming Theorem 4.3. By the reproducing property, we can write $\bar{P}_\mu = P_\mu^2$ with P_μ the Littlewood-Paley projection associated to a different symbol. We now recall that the solution U of (2.11) verifies the first order system (2.12) with correspondence given in (2.13). We also obtained that the corresponding $(U_\mu, V_\mu) = (P_\mu U, P_\mu V)$ verifies (2.12) with F_{U_μ} and F_{V_μ} given in (2.21). Recast the equation for U_μ into a form of (2.11) by using (2.13), we have

$$\square_{\mathbf{g}} P_\mu U = -\mathbf{T}F_{U_\mu} + \text{Tr}k F_{U_\mu} - F_{V_\mu}.$$

We will apply the above calculation to U being either of the function in Ψ according to (2.14) and (2.15). The first term on the right hand side will be shortly transformed into the lower order term $-F_{U_\mu}$, incorporated into the functions of initial data in Duhamel's principle.

To be precise, we define $\mathcal{W}(t, s)$ to be the operator defined on \mathcal{H} such that, for each $\vartheta := (\vartheta_0, \vartheta_1) \in \mathcal{H}$ on Σ_s , $\phi(t, s, x) = \mathcal{W}(t, s)(\vartheta)$ is the unique solution of the initial value problem such that

$$\square_{\mathbf{g}} \phi = 0 \quad (4.4)$$

with

$$\phi(t; s, x) = \vartheta_0, \quad \partial_t \phi(t; s, x) = \vartheta_1, \quad \text{at } t = s.$$

Then, by an adaption from [42, Section 4], we derive the representation formula by the Duhamel's principle for $t \in I_k = [t_{k-1}, t_k]$ that

$$\begin{aligned} P_\mu U(t) &= \mathcal{W}(t, t_{k-1}) (P_\mu U(t_{k-1}), \partial_t P_\mu U(t_{k-1}) - F_{U_\mu}(t_{k-1})) \\ &\quad + \int_{t_{k-1}}^t \{\mathcal{W}(t, s)(0, -R_\mu(s)) + \mathcal{W}(t, s)(F_{U_\mu}(s), 0)\} ds, \end{aligned} \quad (4.5)$$

where

$$R_\mu = -F_{V_\mu} + v^i \partial_i F_{U_\mu}. \quad (4.6)$$

Now we apply P_μ to the both sides and take the spatial derivative.

$$\begin{aligned} P_\mu^2 \partial_m U(t) &= \int_{t_{k-1}}^t \left\{ \partial_m P_\mu \mathcal{W}(t, s)(0, -R_\mu(s)) + \partial_m P_\mu \mathcal{W}(t, s)(F_{U_\mu}(s), 0) \right\} ds \\ &\quad + \partial_m P_\mu \mathcal{W}(t, t_{k-1}) (P_\mu U(t_{k-1}), \partial_t P_\mu U(t_{k-1}) - F_{U_\mu}(t_{k-1})). \end{aligned} \quad (4.7)$$

By using Theorem 4.3, we have for any one-parameter family of data $\vartheta(s) := (\vartheta_0(s), \vartheta_1(s)) \in \mathcal{H}$ with $s \in I_k := [t_{k-1}, t_k]$ that

$$\|P_\mu \partial \mathcal{W}(t, s)(\vartheta(s))\|_{L_{[s, t_k]}^q L_x^\infty} \lesssim \mu^{\frac{3}{2} - \frac{1}{q}} \|\vartheta(s)\|_{\mathcal{H}}.$$

In view of the Minkowski inequality we then obtain

$$\begin{aligned} \left\| \int_{t_{k-1}}^t P_\mu \partial \mathcal{W}(t, s)(\vartheta(s)) ds \right\|_{L_{I_k}^2 L_x^\infty} &\lesssim \int_{t_{k-1}}^{t_k} \|P_\mu \partial \mathcal{W}(t, s)(\vartheta(s))\|_{L_{[s, t_k]}^2 L_x^\infty} ds \\ &\lesssim |I_k|^{\frac{1}{2} - \frac{1}{q}} \mu^{\frac{3}{2} - \frac{1}{q}} \int_{I_k} \|\vartheta(s)\|_{\mathcal{H}} ds. \end{aligned}$$

Since $|I_k| \lesssim T\mu^{-8\epsilon_0}$, it follows that

$$\left\| \int_{t_{k-1}}^t P_\mu \partial \mathcal{W}(t, s)(\vartheta(s)) ds \right\|_{L_{I_k}^2 L_x^\infty} \lesssim T^{\frac{1}{2} - \frac{1}{q}} \mu^{(\frac{1}{2} - \frac{1}{q})(1 - 8\epsilon_0)} \int_{I_k} \mu \|\vartheta(s)\|_{\mathcal{H}} ds.$$

Applying the above inequality to (4.7) gives, with $\delta_2 := (\frac{1}{2} - \frac{1}{q})(1 - 8\epsilon_0)$, that

$$\|\bar{P}_\mu \partial_m U\|_{L_{I_k}^2 L_x^\infty} \lesssim T^{\frac{1}{2} - \frac{1}{q}} \mu^{\delta_2} \|\mu(F_{U_\mu}, -R_\mu)\|_{L_{I_k}^1 \mathcal{H}} + T^{\frac{1}{2} - \frac{1}{q}} B_\mu(t_{k-1}), \quad (4.8)$$

where

$$B_\mu(t) := \mu^{\delta_2} \|\mu(P_\mu U(t), \partial_t P_\mu U(t) - F_{U_\mu}(t))\|_{\mathcal{H}}.$$

In the following we will control the right hand side of (4.8). Positive indices ϵ_0, q, δ are chosen such that $4\epsilon_0 + \delta_2 + \delta < s - 2$, and $\delta_2 + \delta < 4\epsilon_0$.

Remark 4.4. For convenience, we choose $\frac{1}{2} - \frac{1}{q} = \epsilon_0$. Hence $0 < \delta < \min(s - 2 - 4\epsilon_0 - \epsilon_0(1 - 8\epsilon_0), 4\epsilon_0 - \epsilon_0(1 - 8\epsilon_0))$. Since $s - 2 > 5\epsilon_0 > 0$, this allows us to achieve $\delta > 8\epsilon_0^2 = 8\delta_0$ in the range.

4.1.1. *Estimates for R_μ, F_{U_μ} .* To treat the first term on the right of (4.8), we note

$$\begin{aligned} \|(F_{U_\mu}(s), -R_\mu(s))\|_{\mathcal{H}}^2 &= \|(-R_\mu + v^m \partial_m F_{U_\mu})(s)\|_{L_x^2}^2 + \|\partial F_{U_\mu}(s)\|_{L_x^2}^2 \\ &= \|F_{V_\mu}(s)\|_{L_x^2}^2 + \|\partial F_{U_\mu}(s)\|_{L_x^2}^2. \end{aligned} \quad (4.9)$$

Lemma 4.5. *For any $\delta_1 > \delta > 0$ satisfying $b := \delta_2 + \delta_1 < 4\epsilon_0$, there holds*

$$\left(\sum_{\mu > \Lambda} \sum_{k=1}^{\kappa_\mu} \|\mu^{1+\delta_2+\delta}(F_{U_\mu}(s), -R_\mu(s))\|_{L_{I_k}^1 \mathcal{H}}^2 \right)^{\frac{1}{2}} \lesssim 1. \quad (4.10)$$

Proof. Recall from the definitions of (2.21),

$$\begin{aligned} F_{V_\mu} &= [P_\mu, c^2] \Delta_e U - [P_\mu, v^m] \partial_m V + P_\mu F_V + [P_\mu, \text{Tr} k] V - [P_\mu, c^2 \partial_t (\log c)] \partial^l U, \\ \partial_i F_{U_\mu} &= -[P_\mu, v^m] \partial_i \partial_m U - [P_\mu, \partial_i v^m] \partial_m U + \partial_i P_\mu F_U. \end{aligned}$$

These terms will be divided into two types and thus treated differently,

$$\begin{aligned}\mathcal{I}_1 &:= \sum_{\mu > \Lambda} \sum_{k=1}^{\kappa_\mu} \|\mu^{1+\delta_2+\delta} ([P_\mu, c^2] \Delta_e U, [P_\mu, v] \partial V, [P_\mu, v] \partial^2 U)\|_{L_{I_k}^1 L_x^2}^2, \\ \mathcal{I}_2 &:= \sum_{\mu > \Lambda} \sum_{k=1}^{\kappa_\mu} \|\mu^{1+\delta_2+\delta} \check{R}_\mu\|_{L_{I_k}^1 L_x^2}^2,\end{aligned}$$

where symbolically

$$\check{R}_\mu = [P_\mu, \text{Tr}k]V + [P_\mu, \partial_i v^m] \partial_m U + [P_\mu, \partial_t(c^2)] \partial^l U + P_\mu F_V + \partial P_\mu F_U.$$

It suffices to show that

$$\mathcal{I}_1^{\frac{1}{2}} \lesssim \|\partial^2 U, \partial V\|_{L_I^\infty L_x^2}, \quad (4.11)$$

$$\mathcal{I}_2^{\frac{1}{2}} \lesssim \|\partial v_+, \partial \varrho\|_{L_I^1 L_x^\infty} + \|\mu^{1+b} \partial P_\mu F_U\|_{L_I^1 l_\mu^2 L_x^2} + \|\mu^{1+b} P_\mu F_V\|_{L_I^1 l_\mu^2 L_x^2}. \quad (4.12)$$

By (4.2) and Corollary 3.4, we have $\|\partial \varrho, \partial v\|_{L_{I_k}^1 L_x^\infty} \lesssim \mu^{-8\epsilon_0} T^{\frac{1}{2}}$. We can apply (10.1) to obtain

$$\begin{aligned}\|\mu^{1+\delta+\delta_2} ([P_\mu, c^2] \Delta_e U, [P_\mu, v] \partial V, [P_\mu, v] \partial^2 U)\|_{L_{I_k}^1 L_x^2} \\ \lesssim \mu^{\delta+\delta_2} \|\partial \varrho, \partial v\|_{L_{I_k}^1 L_x^\infty} \|\partial^2 U, \partial V\|_{L_t^\infty L_x^2} \\ \lesssim \mu^{\delta+\delta_2-8\epsilon_0} \|\partial^2 U, \partial V\|_{L_t^\infty L_x^2}.\end{aligned}$$

Recall also that $\kappa_\mu \lesssim \mu^{8\epsilon_0}$. We can obtain

$$\sum_{k=1}^{\kappa_\mu} \|\mu^{1+\delta+\delta_2} ([P_\mu, c^2] \Delta_e U, [P_\mu, v] \partial V, [P_\mu, v] \partial^2 U)\|_{L_{I_k}^1 L_x^2}^2 \leq C \mu^{2(\delta+\delta_2-4\epsilon_0)} \|\partial^2 U, \partial V\|_{L_t^\infty L_x^2}^2.$$

Since $0 < \delta < \delta_1$ and $b := \delta_2 + \delta_1 < 4\epsilon_0$, we have

$$\mathcal{I}_1 \lesssim \Lambda^{2(b-4\epsilon_0)} \|\partial^2 U, \partial V\|_{L_t^\infty L_x^2}^2 \lesssim \|\partial^2 U, \partial V\|_{L_t^\infty L_x^2}^2,$$

which gives (4.11).

Next we prove (4.12). Since $0 < \delta < \delta_1$, we observe that for any function a_μ there holds

$$\begin{aligned}\sum_{\mu > \Lambda} \sum_{k=1}^{\kappa_\mu} \|\mu^\delta a_\mu\|_{L_{I_k}^1 L_x^2}^2 &\leq \sum_{\mu > \Lambda} \|\mu^\delta a_\mu\|_{L_I^1 L_x^2}^2 \leq \left(\int_I \sum_{\mu > \Lambda} \|\mu^\delta a_\mu\|_{L_x^2} \right)^2 \\ &\lesssim \left(\int_I \|\mu^{\delta_1} a_\mu\|_{l_\mu^2 L_x^2} \right)^2.\end{aligned} \quad (4.13)$$

The first three terms in \check{R}_μ can be treated directly by using (4.13). We first note that

$$[P_\mu, f]G = P_\mu(fG) - fP_\mu G.$$

Applying (10.6) to the first term yields

$$\|\mu^{1+b} [P_\mu, f]G\|_{l_\mu^2 L_x^2} \lesssim \|f\|_{L_x^\infty} \|G\|_{H_x^{b+1}} + \|f\|_{H_x^{1+b}} \|G\|_{L_x^\infty}.$$

With the help of the above estimate and (4.13),

$$\begin{aligned}\|\mu^{1+b} ([P_\mu, \text{Tr}k]V, [P_\mu, \partial v] \partial U, [P_\mu, \partial_t(c^2)] \partial^l U)\|_{l_\mu^2 L_x^2} \\ \lesssim \|\partial v, \text{Tr}k, \partial(c^2)\|_{L_x^\infty} \|V, \partial U\|_{H_x^{b+1}} + \|\partial v, \text{Tr}k, \partial(c^2)\|_{H_x^{1+b}} \|V, \partial U\|_{L_x^\infty}.\end{aligned}$$

Using (2.94) and taking L_I^1 norm gives

$$\begin{aligned} & \|\mu^{1+b}([P_\mu, \text{Tr}k]V, [P_\mu, \partial v]\partial U, [P_\mu, \partial_l(c^2)]\partial^l U)\|_{l_\mu^2 L_x^2} \\ & \lesssim \|\partial \varrho, \partial v_+\|_{L_I^1 L_x^\infty} \|V, \partial U, \partial v, \text{Tr}k, \partial(c^2)\|_{L_t^\infty H_x^{1+b}}. \end{aligned}$$

(4.12) then follows by applying Corollary 3.3.

Thus

$$\begin{aligned} \mathcal{I}_1^{\frac{1}{2}} + \mathcal{I}_2^{\frac{1}{2}} & \lesssim \|\partial v_+, \partial \varrho\|_{L_I^1 L_x^\infty} + \|\mu^{1+b} P_\mu F_U\|_{L_I^1 l_\mu^2 H_x^1} \\ & \quad + \|\mu^{1+b} P_\mu F_V\|_{L_I^1 l_\mu^2 L_x^2} + \|\partial^2 U, \partial V\|_{L_I^\infty L_x^2}. \end{aligned}$$

We now combine Corollary 3.3, (2.86), (2.87) and Corollary 3.4 to obtain

$$\|\mu^{1+b} P_\mu F_U\|_{L_I^1 l_\mu^2 H_x^1} + \|\mu^{1+b} P_\mu F_V\|_{L_I^1 l_\mu^2 L_x^2} \lesssim \|\partial v_+, \partial \varrho\|_{L_I^1 L_x^\infty} + 1.$$

Using the above estimate, $\|\partial v_+, \partial \varrho\|_{L_I^1 L_x^\infty} \lesssim 1$ due to (2.1), and (2.60), we can conclude (4.10). \square

4.1.2. *Estimate for $B_\mu(t_{k-1})$.* Recall (2.12) with U being $P_\mu U$. By the definition of \mathcal{H} , we derive $B_\mu(t)^2 = \mu^{2\delta_2+2}(\|\mathbf{T}P_\mu U(t) - F_{U_\mu}(t)\|_{L_x^2}^2 + \|\partial P_\mu U(t)\|_{L_x^2}^2) = \mu^{2\delta_2+2}(\|P_\mu V(t)\|_{L_x^2}^2 + \|\partial P_\mu U(t)\|_{L_x^2}^2)$. (4.14)

Hence, by using (3.24), we can obtain directly

Lemma 4.6. *If $4\epsilon_0 + \delta + \delta_2 < s - 2$, there holds*

$$\sum_{\mu > \Lambda} \sum_{k=1}^{\kappa_\mu} \mu^{2\delta} B_\mu(t_{k-1})^2 \lesssim 1.$$

Indeed since $\kappa_\mu \lesssim \mu^{8\epsilon_0}$, we can derive in view of the energy bound in (3.24) that

$$\begin{aligned} \sum_{\mu > \Lambda} \sum_{k=1}^{\kappa_\mu} \mu^{2\delta} B_\mu(t_{k-1})^2 & \lesssim \sum_{\mu > \Lambda} \mu^{2\delta+2\delta_2+8\epsilon_0} \sup_{t \in I} \mathcal{E}_\mu^{(1)}(t) \\ & \lesssim \sum_{\mu > \Lambda} \mu^{2\delta+2\delta_2+8\epsilon_0-2(s-2)} \cdot \sup_{t \in I} \sup_{\mu > \Lambda} \mu^{2(s-2)} \mathcal{E}_\mu^{(1)}(t) \lesssim 1. \end{aligned}$$

In view of (4.8), Lemma 4.5, Lemma 4.6 and writing

$$\sum_{\mu > \Lambda} \|\mu^\delta P_\mu \partial_m U\|_{L_I^2 L_x^\infty}^2 = \sum_{\mu > \Lambda} \sum_{k=1}^{\kappa_\mu} \|\mu^\delta P_\mu \partial_m U\|_{L_{I_k}^2 L_x^\infty}^2,$$

we can obtain the following result.

Proposition 4.7. *For any $q > 2$ sufficiently close to 2 and any $\delta > 0$ sufficiently small such that $4\epsilon_0 + \delta_2 + \delta < s - 2$, where $\delta_2 := (\frac{1}{2} - \frac{1}{q})(1 - 8\epsilon_0)$, and $\delta_2 + \delta < 4\epsilon_0$, for (U, V) in (2.14) and (2.15) satisfying (2.12) there holds*

$$\sum_{\mu > \Lambda} \|\mu^\delta P_\mu \partial U\|_{L_I^2 L_x^\infty}^2 \lesssim T^{1-\frac{2}{q}}.$$

Therefore for $\Psi = (v_+, \varrho)$, we have obtained

$$\|\partial \Psi\|_{L_I^2 L_x^\infty}^2 + \sum_{\mu > \Lambda} \|\mu^\delta P_\mu \partial \Psi\|_{L_I^2 L_x^\infty}^2 \leq CT^{1-\frac{2}{q}}.$$

We need the following result for obtaining the control on $\mathbf{T}\Psi$.

Lemma 4.8. *There holds the following commutator estimate for scalar functions G and f ,*

$$\|[P_\mu, G]\partial f\|_{L_x^\infty} \lesssim \mu^{-\frac{1}{2}} \|\partial G\|_{L_x^\infty} \|\partial^2 f\|_{L_x^2}, \quad \mu > 1.$$

Indeed, from (10.1), (10.5) and using Bernstein inequality, we derive

$$\begin{aligned} \|[P_\mu, G]\partial_m f\|_{L_x^\infty} &\lesssim \|[P_\mu, G]\partial f_{\leq \mu}\|_{L_x^\infty} + \|P_\mu(\sum_{\lambda > \mu} G_\lambda \partial f_\lambda)\|_{L_x^\infty} \\ &\lesssim \mu^{-1} \|\partial G\|_{L_x^\infty} \|\partial f_{\leq \mu}\|_{L_x^\infty} + \sum_{\lambda > \mu} \lambda^{-1} \|\partial G_\lambda\|_{L_x^\infty} \|\partial f_\lambda\|_{L_x^\infty} \\ &\lesssim \|\partial G\|_{L_x^\infty} \left\{ \mu^{-1} \sum_{l \leq \mu} \|l^{\frac{3}{2}} \partial f_l\|_{L_x^2} + \mu^{-\frac{1}{2}} \sum_{\lambda > \mu} \left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}} \|\lambda P_\lambda \partial f\|_{L_x^2} \right\} \\ &\lesssim \mu^{-\frac{1}{2}} \|\partial G\|_{L_x^\infty} \|\partial^2 f\|_{L_x^2}. \end{aligned}$$

This implies Lemma 4.8.

Corollary 4.9. *With the same choices of q, δ_2, δ as in Proposition 4.7, for $\Psi = (v_+, \varrho)$, there hold*

$$\|\mathbf{T}\Psi\|_{L_I^2 L_x^\infty}^2 + \sum_{\mu > \Lambda} \|\mu^\delta \bar{P}_\mu \mathbf{T}\Psi\|_{L_I^2 L_x^\infty}^2 \leq CT^{1-\frac{2}{q}}.$$

Proof. By using (1.4) and (1.20), we can derive

$$\begin{aligned} \mathbf{T}v_+ &= \mathbf{T}v - \mathbf{T}\eta = -c^2 \partial \varrho - \mathbf{T}\eta, \\ \mathbf{T}\varrho &= -\operatorname{div} v = -(\partial v_+ + \partial \eta). \end{aligned}$$

By using (2.76), (2.80) and Sobolev embedding,

$$\begin{aligned} \|\mathbf{T}v_+\|_{L^\infty} &\lesssim \|\partial \varrho\|_{L_x^\infty} + \|\mathbf{T}\eta\|_{L_x^\infty} \lesssim \|\partial \varrho\|_{L_x^\infty} + 1; \\ \|\mathbf{T}\varrho\|_{L_x^\infty} &\lesssim \|\partial v_+\|_{L_x^\infty} + \|\partial \eta\|_{L_x^\infty} \lesssim \|\partial v_+\|_{L_x^\infty} + 1. \end{aligned}$$

Hence the $L_I^2 L_x^\infty$ estimates in Corollary 4.9 follows immediately as a consequence of the $L_I^2 L_x^\infty$ estimates in Proposition 4.7.

By Sobolev embedding and (2.80), we derive

$$\|\mu^\delta \bar{P}_\mu \mathbf{T}\eta\|_{L_\mu^2 L_x^\infty} \lesssim \|\mathbf{T}\eta\|_{H_x^2} \lesssim 1$$

Applying Lemma 4.8 to $G = c^2$, together with using (2.69) implies

$$\|\mu^\delta \bar{P}_\mu (c^2 \partial \varrho)\|_{L_\mu^2 L_I^2 L_x^\infty} \lesssim \|\mu^\delta \bar{P}_\mu \partial \varrho\|_{L_I^2 L_\mu^2 L_x^\infty} + \|\partial \varrho\|_{L_I^2 L_x^\infty}.$$

These two estimates imply $\sum_{\mu > \Lambda} \|\mu^\delta \bar{P}_\mu \mathbf{T}v_+\|_{L_I^2 L_x^\infty}^2 \leq CT^{1-\frac{2}{q}}$, by using the estimate of $\partial \varrho$ in Proposition 4.7.

Note by Berntein inequality, (2.79) and (3.23), we derive

$$\|\mu^\delta \bar{P}_\mu \partial \eta\|_{L_\mu^2 L_x^\infty} \lesssim \|\mu^{\delta+\frac{3}{2}} \bar{P}_\mu \partial \eta\|_{L_\mu^2 L_x^2} \lesssim \|\operatorname{curl} \Omega\|_{H_x^{\frac{1}{2}+\delta}} + 1 \lesssim 1.$$

Combining the above estimate with the estimate of ∂v_+ in Proposition 4.7 gives

$$\|\mu^\delta \bar{P}_\mu \mathbf{T}\varrho\|_{L_I^2 L_\mu^2 L_x^\infty} \lesssim T^{\frac{1}{2}} + \|\mu^\delta \bar{P}_\mu \partial v_+\|_{L_I^2 L_\mu^2 L_x^\infty} \lesssim T^{\frac{1}{2}-\frac{1}{q}}.$$

Hence the proof of Corollary 4.9 is complete. \square

Rename the choice of δ, q in Remark 4.4 by $\gamma_0 = \delta$, and $\gamma_1 = \frac{1}{2} - \frac{1}{q}$. The proof of Theorem 4.1 is complete.

4.2. Prove Theorem 4.3 using dispersive estimate. In order to prove Theorem 4.3 on each spacetime slab $I_k \times \mathbb{R}^3$, we consider the coordinate change $(t, x) \rightarrow (\lambda(t - t_k), \lambda x)$. The interval I_k becomes $I_* = [0, \tau_*]$ with $\tau_* \leq \lambda^{1-8\epsilon_0} T$. Under the rescaled coordinates the function $\Phi = (\varrho, v)$ and the metric component \mathbf{g} in (1.3) become

$$\Phi(\lambda^{-1}t + t_k, \lambda^{-1}x) \quad \text{and} \quad \mathbf{g}(\Phi(\lambda^{-1}t + t_k, \lambda^{-1}x)) \quad (4.15)$$

which are still denoted as Φ and \mathbf{g} . In view of (4.2), Corollary 3.4 and $|I_k| \leq \lambda^{-8\epsilon_0} T$, we have

$$\|\partial \mathbf{g}\|_{L_{I_*}^1 L_x^\infty} \lesssim \lambda^{-8\epsilon_0}.$$

Therefore, to prove Theorem 4.3, it is equivalent to show the following Strichartz estimate on I_* with respect to Littlewood-Paley projection P on the frequency domain $\{1/2 \leq |\xi| \leq 2\}$. Here we fix the convention that $P = P_1$, which is the Littlewood-Paley projection P_λ with $\lambda = 1$.

Theorem 4.10. *If there is a large number Λ such that for $\lambda \geq \Lambda$, on the time interval $I_* = [0, \tau_*]$, there holds*

$$\|\partial v_+, \partial \varrho\|_{L_{I_*}^2 L_x^\infty}^2 + \lambda^{2\delta_0} \sum_{\mu \geq 2} \mu^{2\delta_0} \|\bar{P}_\mu \partial(v_+, \varrho)\|_{L_{I_*}^2 L_x^\infty}^2 \lesssim \lambda^{-1-8\epsilon_0} \quad (4.16)$$

then for any solution ψ of $\square_{\mathbf{g}} \psi = 0$ on the time interval I_* and $q > 2$ sufficiently close to 2, there holds

$$\|P \partial \psi\|_{L_{I_*}^q L_x^\infty} \lesssim \|\psi[0]\|_{\mathcal{H}}. \quad (4.17)$$

By Bernstein inequality and the following lemma, (4.17) holds on any subinterval of I_* with length comparable to 1. Our task is to show (4.17) holds on I_* with the constant bound independent of the frequency λ .

Lemma 4.11. *Under the bootstrap assumption (2.1), there holds, for any solution ψ of (4.19), the standard energy estimate*

$$\|\psi[t]\|_{\mathcal{H}} \lesssim \|\psi[0]\|_{\mathcal{H}}. \quad (4.18)$$

The above result follows directly from (2.38). The proof of Theorem 4.10 crucially relies on the following decay estimate.

Theorem 4.12 (Decay estimate). *Let $0 < \epsilon_0 < \frac{s-2}{5}$ be a fixed number. There exists a large number Λ such that for any $\lambda \geq \Lambda$ and any solution ψ of the equation*

$$\square_{\mathbf{g}} \psi = 0 \quad (4.19)$$

on the time interval $I_* = [0, \tau_*]$ with $\tau_* \leq \lambda^{1-8\epsilon_0} T$, there is a function $\mathfrak{d}(t)$ satisfying

$$\|\mathfrak{d}\|_{L^{\frac{q}{2}}} \lesssim 1, \text{ for } q > 2 \text{ sufficiently close to } 2 \quad (4.20)$$

such that for any $1 \leq t_0 \leq t \leq \tau_*$ there holds

$$\|P \mathbf{T} \psi(t)\|_{L_x^\infty} \leq \left(\frac{1}{(1 + |t - t_0|)^{\frac{2}{q}}} + \mathfrak{d}(t) \right) \left(\sum_{m=0}^3 \|\partial^m \psi(t_0)\|_{L_x^1} + \sum_{m=0}^2 \|\partial^m \mathbf{T} \psi(t_0)\|_{L_x^1} \right). \quad (4.21)$$

Assuming Theorem 4.12, we can prove Theorem 4.10 by running the $\mathcal{T}\mathcal{T}^*$ argument in $[1, \tau_* - 1]$. See [42, Section 4] and [45, Section 9].

To prove Theorem 4.12, we carry out a further localization of the solution for (4.19) in physical space in $I_* \times \mathbb{R}^3$ with $I_* = [0, \tau_*]$ under the rescaled coordinates, where $\tau_* \leq \lambda^{1-8\epsilon_0} T$. For each t denote by $g(t)$ or g the induced Riemannian metric of (4.15) on $\Sigma_t = \{t\} \times \mathbb{R}^3$. Given $d > 0$ and $\mathbf{p} \in \Sigma_t$ we use $B_d(\mathbf{p})$ and $B_d(\mathbf{p}, g)$ to denote the Euclidean ball and the geodesic ball on Σ_t with respect to g . We can find $R > 0$ such that

$$B_R(\mathbf{p}) \subset B_{\frac{1}{2}}(\mathbf{p}, g(t)), \quad \forall \mathbf{p} \in \Sigma_t \text{ and } 0 \leq t \leq \tau_*. \quad (4.22)$$

This is achievable due to the ellipticity condition from (2.6). We take a sequence of Euclidean balls $\{B_J\}$ with radius R such that their union covers \mathbb{R}^3 and any ball in this collection intersect at most 20 other balls. Let $\{\mathcal{X}_J\}$ be a partition of unity subordinate to the cover $\{B_J\}$. We may assume that $\sum_{m=1}^3 |\partial^m \mathcal{X}_J|_{L_x^\infty} \leq C_1$ holds uniformly in J . By using this partition of unity and a standard argument we can reduce the proof of Theorem 4.12 to establishing the following dispersive estimate for the solution of $\square_{\mathbf{g}} \psi = 0$ with $\psi[t_0] := (\psi(t_0), \partial_t \psi(t_0))$ supported on an Euclidean ball of radius R .

Proposition 4.13. *There is a large constant Λ such that for any $\lambda \geq \Lambda$ and any solution ψ of*

$$\square_{\mathbf{g}} \psi = 0$$

on the time interval $[0, \tau_]$ with $\tau_* \leq \lambda^{1-8\epsilon_0} T$ and with $\psi[t_0]$ supported in the Euclidean ball B_R of radius R , there exists a function $\mathfrak{d}(t)$ satisfying*

$$\|\mathfrak{d}\|_{L^{\frac{q}{2}}[0, \tau_*]} \lesssim 1 \quad \text{for } q > 2 \text{ sufficiently close to } 2 \quad (4.23)$$

such that for all $1 \leq t_0 \leq t \leq \tau_$,*

$$\|P\mathbf{T}\psi(t)\|_{L_x^\infty} \lesssim \left(\frac{1}{(1 + |t - t_0|)^{\frac{2}{q}}} + \mathfrak{d}(t) \right) (\|\psi[t_0]\|_{\mathcal{H}} + \|\psi(t_0)\|_{L^2}). \quad (4.24)$$

Proof of Theorem 4.12 assuming Proposition 4.13. By using the partition of unity $\{\mathcal{X}_J\}$ we decompose $\psi = \sum_J \psi_J$, with ψ_J the solution of $\square_{\mathbf{g}} \psi_J = 0$ satisfying the initial conditions

$$\psi_J(t_0) = \mathcal{X}_J \psi(t_0), \quad \partial_t \psi_J(t_0) = \mathcal{X}_J \partial_t \psi(t_0).$$

By using (4.24) in Proposition 4.13, we have

$$\|P\mathbf{T}\psi_J(t)\|_{L_x^\infty} \lesssim \left(\frac{1}{(1 + |t - t_0|)^{\frac{2}{q}}} + \mathfrak{d}(t) \right) (\|\psi_J[t_0]\|_{\mathcal{H}} + \|\psi_J(t_0)\|_{L^2}).$$

In view of Lemma 4.11 and the Sobolev embedding $W^{2,1} \hookrightarrow L^2$ in \mathbb{R}^3 , we obtain

$$\|P\mathbf{T}\psi_J(t)\|_{L_x^\infty} \lesssim \left(\frac{1}{(1 + |t - t_0|)^{\frac{2}{q}}} + \mathfrak{d}(t) \right) \left(\sum_{m=0}^3 \|\partial^m \psi_J(t_0)\|_{L_x^1} + \sum_{m=0}^2 \|\partial^m \mathbf{T}\psi_J(t_0)\|_{L_x^1} \right).$$

Summing over J and using the fact that any ball B_J intersects with at most 20 other balls, we can conclude the desired estimate. \square

5. REDUCTION TO THE BOUNDEDNESS OF CONFORMAL ENERGY

The main task now is to prove Proposition 4.13, which will be reduced further to controlling conformal energy. To define the conformal energy, we set up a foliation of the acoustical spacetime by outgoing acoustical null cones, which covers the domain of influence of $B_{\frac{1}{2}}(\mathbf{p}, g(t_0))$. Without loss of generality, it suffices to fix $t_0 = 1$, since otherwise we can repeat our construction and proof in a subinterval of $[0, \tau_*]$.

5.1. Construction of foliations of acoustical null cones. Let \mathbf{p} be the center of B_R in Proposition 4.13 at $t = t_0$. We denote by Γ^+ the time axis passing through \mathbf{p} which is defined to be the integral curve of the forward unit normal \mathbf{T} with $\Gamma^+(t_{\mathbf{p}}) = \mathbf{p}$. We similarly extend the integral curve of $-\mathbf{T}$ from $t = t_{\mathbf{p}}$ till $t = 0$, which is still denoted as Γ^+ by abuse of notation, and denote the point $\Gamma^+(0) = \mathbf{o}$. We will only consider the segment of $\Gamma^+(t)$ with $t \in [0, \tau_*]$.

Define the optical function u to be the solution of the Eikonal equation

$$\mathbf{g}^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0. \quad (5.1)$$

We refer to the level sets of the optical function u , denoted by C_u , as the acoustical null cones. The global optical function u is constructed as follows.

Let p be any point on Γ^+ , and $L_\omega, \omega \in \mathbb{S}^2$ be the family of null vectors in $\mathcal{T}_p\mathcal{M}$. For each $\omega \in \mathbb{S}^2$, define the vector field L' to be the generator of the null geodesic Υ_ω in (\mathcal{M}, g) by

$$\mathbf{D}_{L'}L' = 0, \quad \frac{d}{ds}\Upsilon_\omega(s) = L', \quad L'(s) = 1, \quad (5.2)$$

and when $s = 0$, $L' = L_\omega$. The null vector L_ω can be decomposed as $L_\omega = \mathbf{T} + \mathbf{N}_\omega$, where $\mathbf{N}_\omega, \omega \in \mathbb{S}^2$ is the family of the unit vectors in $\mathcal{T}_p\Sigma_{t_p}$. We set $u = t$ at $p \in \Gamma^+$. The ruled hypersurface formed by $\{\Upsilon_\omega, \omega \in \mathbb{S}^2\}$ is the level set of u , which is denoted by C_u . This immediately yields $L'(u) = 0$. One can directly check via (5.2) and Eikonal equation that $L' = -\mathbf{D}u$, and clearly

$$\mathbf{T}u = 1, \quad \text{on } \Gamma^+(t_p). \quad (5.3)$$

As such, the optical function u has been defined in the causal future of \mathbf{o} , denoted by \mathcal{D}^+ , with the level sets C_u being the outgoing null (acoustical) cones with vertex on Γ^+ at $t = u$. With $S_{t,u} = C_u \cap \Sigma_t$, which is a smooth surface diffeomorphic to \mathbb{S}^2 , we denote the two solid cones

$$\mathcal{D}_0^+ = \bigcup_{\{t \in [t_0, \tau_*], 0 \leq u \leq t\}} S_{t,u} \quad \text{and} \quad \mathcal{D}^+ = \bigcup_{\{t \in [0, \tau_*], 0 \leq u \leq t\}} S_{t,u}.$$

Next we extend the foliation of spacetime by the null cones to a neighbourhood of \mathcal{D}^+ in $\bigcup_{t \in [0, \tau_*]} \Sigma_t$. Recall that $\tau_* \leq \lambda^{1-8\epsilon_0}T$ and $0 < T \leq d_0$, where $d_0 > 0$ is the radius of injectivity on Σ_t for $t \leq [0, T]$. Let $\mathbf{v}_* = \frac{4}{5}\tau_*$. We can guarantee that there is a neighbourhood \mathcal{O} of \mathbf{o} contained in the geodesic ball $B_{\tau_*}(\mathbf{o}, g)$ of radius τ_* on $\{t = 0\}$ such that it can be foliated by the level sets $S_{\mathbf{v}}$ of a function \mathbf{v} taking all values in $[0, \mathbf{v}_*]$ with $\mathbf{v}(\mathbf{o}) = 0$ and with each $S_{\mathbf{v}}$, for $\mathbf{v} > 0$, diffeomorphic to \mathbb{S}^2 ; see Proposition 5.1 shortly for various important properties. Let $a^{-1} = |\nabla \mathbf{v}|_g$ be the lapse function on $\{t = 0\}$ and we know $\lim_{\mathbf{v} \rightarrow 0} a = 1$ in Proposition 5.1. Then, in \mathcal{O} the metric g can be written as

$$ds^2 = a^2 d\mathbf{v}^2 + \gamma_{AB} d\omega^A d\omega^B, \quad (5.4)$$

where ω^A , $A = 1, 2$, denote the angular variables on \mathbb{S}^2 and γ is the induced metric on $S_{\mathbf{v}}$. At $t = 0$, we denote by $\mathbf{N} = \mathbf{N}(\mathbf{v}, \omega)$ the outward unit normal of the foliation of $S_{\mathbf{v}}$ and we also note that $\mathbf{N} \rightarrow \mathbf{N}_\omega$ as $\mathbf{v} \rightarrow 0$ for $\omega \in \mathbb{S}^2$.

With the initial datum at $S_{\mathbf{v}}$ given by

$$L' = a^{-1}(\mathbf{T} + \mathbf{N}), \quad 0 < \mathbf{v} \leq \mathbf{v}_*, \quad t = 0, \quad (5.5)$$

we define L', Υ_ω to be the vector field and the null geodesic satisfying (5.2). By setting $u = -\mathbf{v}$, the level set of u , denoted by C_u , is the ruled hypersurface of the null geodesics $\{\Upsilon_\omega, \omega \in \mathbb{S}^2\}$, and thus $L'(u) = 0$. And we can check $L' = -\mathbf{D}u$.

This gives an extension of u satisfying (5.1) in the causal future of $\bigcup_{0 \leq \mathbf{v} \leq \mathbf{v}_*} S_{\mathbf{v}}$, denoted by $\widetilde{\mathcal{D}^+}$. It is foliated by the null cones C_u , initiating from $S_{\mathbf{v}}$ with $u = -\mathbf{v}$ at $t = 0$, and those initiating from the time axis Γ^+ with $u = t$. We still denote $S_{t,u} = C_u \cap \Sigma_t$ for $-\mathbf{v}_* \leq u < 0$ and $t \in [0, \tau_*]$. $\widetilde{\mathcal{D}^+}$ can be written as

$$\widetilde{\mathcal{D}^+} = \bigcup_{\{t \in [0, \tau_*], -\mathbf{v}_* \leq u \leq t\}} S_{t,u}.$$

Define in $\widetilde{\mathcal{D}^+}$ for all $0 < t \leq \tau_*$ that

$$\mathbf{b}^{-1} := \mathbf{T}u = -\langle L', \mathbf{T} \rangle.$$

By (5.5) and (5.3), we can see the initial and the boundary values of \mathbf{b} verify $\mathbf{b} = a$ at $t = 0$ and $\mathbf{b} = 1$ on $\Gamma^+(t)$.

Next, with ∇ the shorthand notation for ∇_g , we directly compute

$$\nabla u = \mathbf{D}u + \langle \mathbf{D}u, \mathbf{T} \rangle \mathbf{T} = -L' + \mathbf{b}^{-1} \mathbf{T}$$

and thus $|\nabla u|_g = \mathbf{b}^{-1}$. Let \mathbf{N} be the outward unit normal of the radial foliation $S_{t,u}$ with $-\mathbf{v}_* \leq u \leq t$ on Σ_t . By the above calculation

$$\mathbf{N} = \mathbf{b}L' - \mathbf{T}.$$

Moreover, also using $L' = -\mathbf{D}u$,

$$-\mathbf{N}(u) = \mathbf{b}^{-1} = \mathbf{T}u = |\nabla u|.$$

We can directly see that

$$\mathbf{N} \rightarrow \mathbf{N}_\omega(\Gamma^+(t)), \quad \mathbf{b} \rightarrow 1 \text{ as } u \rightarrow t.$$

In Lemma 8.4, we will show that $|\mathbf{b} - 1| \leq \frac{1}{4}$. Assuming this property, we can prove as in [45, Section 4] that

$$B_{\frac{1}{2}}(\mathbf{p}, g(t_0)) \subset \mathcal{D}_0^+ \cap \Sigma_{t_0},$$

which, together with (4.22), implies $B_R(\mathbf{p}) \subset \mathcal{D}_0^+ \cap \Sigma_{t_0}$.

For convenience in $\widetilde{\mathcal{D}^+}$, we introduce the pair of null frames

$$L = \mathbf{b}L' = \mathbf{T} + \mathbf{N}, \quad \underline{L} = \mathbf{T} - \mathbf{N}, \quad (5.6)$$

and define the projection tensor as

$$\Pi^{\mu\nu} = \mathbf{g}^{\mu\nu} + \mathbf{T}^\mu \mathbf{T}^\nu - \mathbf{N}^\mu \mathbf{N}^\nu \quad (5.7)$$

or identically

$$\Pi^{\mu\nu} = \mathbf{g}^{\mu\nu} + \frac{1}{2}(L^\mu \underline{L}^\nu + \underline{L}^\mu L^\nu).$$

Since \mathbf{N} is the unit normal to $S_{t,u}$ in Σ_t , $\Pi_{\mu\nu}$ gives the induced metric on $S_{t,u}$, which will be denoted as γ , with ∇ the corresponding Levi-Civita connection on $S_{t,u}$. We can write down the line element of the metric g as

$$ds^2 = \mathbf{b}^2 du^2 + \gamma_{AB} d\omega^A d\omega^B. \quad (5.8)$$

Let $\{e_A, e_B\}$ with $A, B = 1, 2$ be the orthonormal basis of the tangent bundle on $S_{t,u}$. We now introduce the connection coefficient on C_u by using the null pair $e_4 = L$, $e_3 = \underline{L}$. The null second fundamental forms χ and $\underline{\chi}$, the torsion ζ , and the Ricci coefficient $\underline{\zeta}$ of the foliation $S_{t,u}$ are defined by

$$\begin{aligned} \chi_{AB} &= \mathbf{g}(\mathbf{D}_A e_4, e_B), & \underline{\chi}_{AB} &= \mathbf{g}(\mathbf{D}_A e_3, e_B), \\ \zeta_A &= \frac{1}{2} \mathbf{g}(\mathbf{D}_3 e_4, e_A), & \underline{\zeta}_A &= \frac{1}{2} \mathbf{g}(\mathbf{D}_4 e_3, e_A). \end{aligned} \quad (5.9)$$

We denote by $\text{tr}\chi$ and $\hat{\chi}$ the trace and traceless part of χ taken by the metric γ , and apply the same convention to $\underline{\chi}$.

On each $C_u \cap \widetilde{\mathcal{D}^+}$, with $t_{\min} = \max\{u, 0\}$, we write $\Upsilon(t, \omega) := \Upsilon(s(t, \omega), \omega)$ for $t_{\min} \leq t \leq \tau_*$ by change of parameter. The pull-back coordinates (t, ω^1, ω^2) induced by the null geodesic flow $\Upsilon(t, \omega), \omega \in \mathbb{S}^2$ along C_u together with the function u define a system of coordinates on $\widetilde{\mathcal{D}^+}$. Indeed, the short-time well-posedness of the null geodesic family $\{\Upsilon(s(t), \omega, u)\}$, for all $\omega \in \mathbb{S}^2$, $-\mathbf{v}_* \leq u \leq t$ follows from the standard ODE theory. The semi-global behaviour throughout $\widetilde{\mathcal{D}^+}$, or more precisely, the global diffeomorphism of the null exponential map, $\Upsilon : [t_{\min}, \tau_*] \times \mathbb{S}^2 \rightarrow C_u \cap \widetilde{\mathcal{D}^+}$ for all C_u contained in $\widetilde{\mathcal{D}^+}$, can be guaranteed by running the same continuity argument for proving [40, Theorem 1.2], based on (8.4), Proposition 8.3, (8.18) and (2.1).

Hence for each $p \in \widetilde{\mathcal{D}^+}$, there exists a unique triplet (t, ω, u) such that $p = \Upsilon(t, \omega, u)$. For the case that $t = 0$ in the triplet, in view of (5.4), we note on each $S_{\mathbf{v}}$, $\{\partial_{\omega^1}, \partial_{\omega^2}\}$ is the pair of pull-back coordinate frame by the diffeomorphism $\mathbb{S}^2 \rightarrow S_{\mathbf{v}}$ with $0 < \mathbf{v} \leq \mathbf{v}_*$. (See more details in the proof in [44, Section 10] for Proposition 5.1.)

For $t > 0$, we introduce the transport coordinate on C_u by

$$\frac{d}{dt}x^\mu(\Upsilon(t, \omega)) = L^\mu(t, \omega), \quad t > 0 \quad (5.10)$$

and adopt the pull-back coordinate frame $\{\partial_{\omega^A}, A = 1, 2\}$ on $S_{t,u}$ defined by the diffeomorphism $\Upsilon(t, \cdot, u) : \mathbb{S}^2 \rightarrow S_{t,u}, t > 0$. Along the cone C_u , $L = \partial_t^{20}$ together with the the pull-back coordinate frame $\{\partial_{\omega^1}, \partial_{\omega^2}\}$ forms a set of coordinate frame on C_u . We can derive

$$\begin{aligned} \frac{d}{dt}\gamma(\partial_{\omega^A}, \partial_{\omega^B}) &= \gamma(\mathbf{D}_L \partial_{\omega^A}, \partial_{\omega^B}) + \gamma(\partial_{\omega^A}, \mathbf{D}_L \partial_{\omega^B}) \\ &= \gamma(\mathbf{D}_{\partial_{\omega^A}} L, \partial_{\omega^B}) + \gamma(\partial_{\omega^A}, \mathbf{D}_{\partial_{\omega^B}} L) \\ &= 2\chi(\partial_{\omega^A}, \partial_{\omega^B}). \end{aligned} \quad (5.11)$$

The second fundamental form of $S_{t,u}$ in Σ_t for $0 \leq t \leq \tau_*$ is given by

$$\theta(X, Y) = \langle \nabla_X \mathbf{N}, Y \rangle \quad (5.12)$$

for any vector fields X, Y tangent to $S_{t,u}$. The trace of θ is defined by $\text{tr}\theta = \gamma^{AB}\theta_{AB}$, and the traceless part of θ is denoted by $\hat{\theta}$.

Let $v_t = \frac{\sqrt{|\gamma|}}{\sqrt{|\gamma^{(0)}|}}$, where $\gamma^{(0)}$ is the canonic round metric on \mathbb{S}^2 . Since v_t is defined on $S_{t,u}$, we may write it as $v_{t,u}$ in particular when u is also varying. By $\mathcal{L}_L \gamma = 2\chi$,

$$L(v_t) = v_t \text{tr}\chi. \quad (5.13)$$

We define $S_{t,u}$ -tangent tensor field F if F verifies $i_L F = 0$ and $i_{\underline{L}} F = 0$. For such a tensor field, $|F|$ is the norm of F under the induced metric γ . We will use the two norms

$$\|F\|_{L_x^q(S_{t,u})}^q = \int_{S_{t,u}} |F|^q d\mu_\gamma \quad \text{and} \quad \|F\|_{L_\omega^q(S_{t,u})}^q = \int_{\mathbb{S}^2} |F|^q(\omega) d\mu_{\mathbb{S}^2}.$$

For $S_{t,u}$ -tangent tensor field F on C_u , we introduce the mixed norms

$$\|F\|_{L_\omega^q L_t^\infty(C_u)}^q = \int_{\mathbb{S}^2} \sup_{\Upsilon_\omega} |F|^q d\mu_{\mathbb{S}^2} \quad \text{and} \quad \|F\|_{L_x^q L_t^\infty(C_u)}^q = \int_{\mathbb{S}^2} \sup_{t \in \Upsilon_\omega} (v_t |F|^q) d\mu_{\mathbb{S}^2}$$

and

$$\|F\|_{L_t^p L_x^q(C_u)}^p = \int_{t_1}^{t_2} \left(\int_{\mathbb{S}^2} |F|^q v_{t'} d\mu_{\mathbb{S}^2} \right)^{\frac{p}{q}} dt'.$$

For tensor fields F defined on $\Sigma_t \cap \widetilde{\mathcal{D}^+}$ we use the norms

$$\|F\|_{L_u^p L_x^q}^p = \int_{u_1}^{u_2} \left(\int_{\mathbb{S}^2} |F|^q v_{t,u'} d\mu_{\mathbb{S}^2} \right)^{\frac{p}{q}} du', \quad \|F\|_{L_x^q L_u^\infty} = \left(\int_{\mathbb{S}^2} \left(\sup_u (v_t |F|^q) \right) (\omega) d\mu_{\mathbb{S}^2} \right)^{\frac{1}{q}}.$$

We may write $d\mu_{\mathbb{S}^2}$ as $d\omega$ for convenience. The ranges t_1, t_2 and u_1 and u_2 are determined by the integral regions.

Finally, we recall that the existence of the \mathbf{v} -foliation with the desired properties in a neighbourhood of \mathbf{o} on $\{t = 0\}$ is guaranteed by the following result. The proof of the following result depends on the Ricci curvature of the induced metric g on Σ_t , which is of the same regularity level as in [45]. See the proof in [44, Section 10].

²⁰This is the partial differentiation with u and ω fixed, instead of the ∂_t in the cartesian frame.

Proposition 5.1. *On $\{t = 0\}$ there exists a function \mathbf{v} with $0 \leq \mathbf{v} \leq \mathbf{v}_* = \frac{4}{5}\tau_*$ such that each level set $S_{\mathbf{v}}$ is diffeomorphic to \mathbb{S}^2 and*

$$\mathrm{tr} \theta + k_{\mathbf{NN}} = \frac{2}{a\mathbf{v}} + \mathrm{Tr} k - \Xi_4, \quad a(\mathbf{o}) = 1. \quad (5.14)$$

Let $\gamma^{(0)}$ be the canonical round metric on \mathbb{S}^2 and γ be the induced metric of g on $S_{\mathbf{v}}$. Let $\overset{\circ}{\gamma} = \mathbf{v}^{-2}\gamma$ and $\tilde{\gamma} = \mathbf{v}^2\gamma^{(0)}$. Then on $\cup_{0 \leq \mathbf{v} \leq \mathbf{v}_*} S_{\mathbf{v}}$ there hold

$$|a - 1| \lesssim \lambda^{-4\epsilon_0} < \frac{1}{4}, \quad \|\mathbf{v}^{\frac{1}{2} - \frac{2}{q_*}}(\hat{\theta}, \nabla \log a)\|_{L^{q_*}(S_{\mathbf{v}})} \lesssim \lambda^{-\frac{1}{2}}, \quad (5.15)$$

$$\|\nabla \log a\|_{L_{\mathbf{v}}^2 L_{S_{\mathbf{v}}}^\infty} + \|\hat{\chi}\|_{L_{\mathbf{v}}^2 L_{S_{\mathbf{v}}}^\infty} \lesssim \lambda^{-\frac{1}{2}}, \quad (5.16)$$

$$|\overset{\circ}{\gamma} - \gamma^{(0)}| + \|\partial_\omega(\overset{\circ}{\gamma} - \gamma^{(0)})\|_{L^{q_*}(S_{\mathbf{v}})} \lesssim \lambda^{-4\epsilon_0}, \quad (5.17)$$

$$\|\mathbf{v}^{\frac{1}{2} - \frac{2}{q_*}} \nabla(\log \sqrt{|\gamma|} - \log \sqrt{|\tilde{\gamma}|})\|_{L^{q_*}(S_{\mathbf{v}})} \lesssim \lambda^{-\frac{1}{2}}, \quad (5.18)$$

where $0 < 1 - \frac{2}{q_*} < s - 2$ and for scalar functions f , $\|f\|_{L_{\omega}^{q_*}(S_{\mathbf{v}})}^{q_*} := \int_{\mathbb{S}^2} |f|^{q_*}(\omega) d\mu_{\mathbb{S}^2}$. Moreover

$$|a - 1| \lesssim \lambda^{-4\epsilon_0}, \quad \|\mathbf{v}^{-\frac{1}{2}}(a - 1)\|_{L^\infty} \lesssim \lambda^{-\frac{1}{2}}, \quad \frac{\sqrt{|\gamma|}}{\sqrt{|\gamma^{(0)}|}} \approx \mathbf{v}^2 \quad (5.19)$$

and there holds the inclusion

$$\cup_{0 \leq \mathbf{v} \leq \mathbf{v}_*} S_{\mathbf{v}} \subset B_{\tau_*}(\mathbf{o}). \quad (5.20)$$

5.2. Reduction from the dyadic Strichartz estimates to the boundedness of conformal energy. In order to give the definition of our conformal energy, we take two smooth nonnegative cut-off functions $\underline{\varpi}$ and ϖ depending only on two variables t, u ; for $t > 0$ they are defined in a manner such that

$$\underline{\varpi} = \begin{cases} 1 & \text{on } 0 \leq u \leq t \\ 0 & \text{on } u \leq -\frac{t}{4}, \end{cases} \quad \text{and} \quad \varpi = \begin{cases} 1 & \text{on } 0 \leq \frac{u}{t} \leq \frac{1}{2} \\ 0 & \text{if } \frac{u}{t} \geq \frac{3}{4} \text{ or } u \leq -\frac{t}{4}. \end{cases}$$

We may define ϖ and $\underline{\varpi}$ such that they coincide in the region $\cup_{\{t \in [t_0, \tau_*], -\frac{t}{4} < u \leq 0\}} S_{t,u}$.

Definition 5.2. *For any scalar function ψ vanishing outside \mathcal{D}^+ , we define the conformal energy $\mathcal{C}[\psi]$ of ψ by*

$$\mathcal{C}[\psi](t) = \mathcal{C}[\psi]^{(i)}(t) + \mathcal{C}[\psi]^{(e)}(t),$$

where

$$\begin{aligned} \mathcal{C}[\psi]^{(i)}(t) &= \int_{\Sigma_t} (\underline{\varpi} - \varpi) t^2 (|\mathbf{D}\psi|^2 + |\tilde{r}^{-1}\psi|^2) d\mu_g, \\ \mathcal{C}[\psi]^{(e)}(t) &= \int_{\Sigma_t} \varpi (\tilde{r}^2 |\mathbf{D}_L \psi|^2 + \tilde{r}^2 |\nabla \psi|^2 + |\psi|^2) d\mu_g. \end{aligned}$$

We will prove the following boundedness theorem for the conformal energy in the rest of this paper, combined with [45, Section 7].

Theorem 5.3 (Boundedness theorem). *Let (4.16) hold. Let ψ be any solution of $\square_{\mathbf{g}} \psi = 0$ on $I_* = [0, \tau_*]$ with $\psi[t_0]$ supported in $B_R \subset \mathcal{D}^+ \cap \Sigma_{t_0}$. Then, for $t \in [t_0, \tau_*]$, the conformal energy of ψ satisfies the estimate*

$$\mathcal{C}[\psi](t) \lesssim (1+t)^{2\epsilon} \left(\|\psi[t_0]\|_{\mathcal{H}}^2 + \|\psi(t_0)\|_{L^2(\Sigma)}^2 \right),$$

where the fixed constant $\epsilon > 0$ can be arbitrarily small.

Under the assumption (4.16), to show Theorem 5.3 implies Proposition 4.13, we refer to ²¹ [45, Section 4.1], for which we need the following results on $\widetilde{\mathcal{D}^+}$,

$$\|\varpi(\hat{\chi}, \nabla \log \mathbf{b}, \text{tr} \chi - \frac{2}{\tilde{r}})\|_{(L^{\frac{q}{2}}[0, \tau_*] L_x^\infty \cap \widetilde{\mathcal{D}^+})} \leq C \lambda^{\frac{2}{q}-1-4\epsilon_0(\frac{4}{q}-1)}, \quad (5.21)$$

where $\tilde{r} = t - u$, $q > 2$ and is sufficiently close to 2;

$$|\mathbf{b} - 1| \leq \frac{1}{2}, \quad \|\text{tr} \theta - \frac{2}{\tilde{r}}\|_{L^3(\Sigma_t \cap \widetilde{\mathcal{D}^+})} \leq C, \quad (5.22)$$

and the estimates

$$C^{-1} \gamma^{(0)}(X, X) \leq \tilde{r}^{-2} \gamma(X, X) \leq C \gamma^{(0)}(X, X), \quad v_t / \tilde{r}^2 \approx 1, \quad (5.23)$$

where X is any $S_{t,u}$ tangent vector field with $S_{t,u}$ contained in $\widetilde{\mathcal{D}^+}$ and $C > 0$ is a universal constant.

The assumptions (5.22) and (5.23) are used to prove the scaling-invariant inequalities in Lemma 5.5 and Proposition 5.4 which are involved in proving Proposition 4.13. The first assumption in (5.22) implies \mathbf{b} can be regarded as a positive constant away from zero. (5.23) ensures that the area element v_t and \tilde{r}^2 are comparable. Thus for any tensor F on $S_{t,u}$ and $1 \leq q < \infty$ we have

$$\|F\|_{L^q(S_{t,u})} \approx \|\tilde{r}^{\frac{2}{q}} F\|_{L_\omega^q(S_{t,u})}.$$

With the series of reductions, the proof of the main theorem, Theorem 1.1, has been reduced to the proof of Theorem 5.3.

The proofs of Theorem 5.3 had always been the most important part of the series of works on rough solutions for quasilinear wave equations [18, 20, 42, 45]. In particular, due to the optimal regularity assumption on the data in [45], the proof of the boundedness theorem in [45] is completely different from the previous works and the result has a growth with time, i.e. $(t+1)^{2\epsilon}$. The reason of such harmless loss is due to the weak regularity of the spacetime metric and the null hypersurfaces therein. The proof in [45, Section 7] contains two major ingredients: one is to use the conformal method to normalize the null cones; the other is to adapt the hierarchic approach in [11] to the rough spacetime to obtain the weighted energy flux together with the weighted energy, so as to compensate the weak control in particular on the normalized mass aspect function (see (9.8)). The analysis in the proof is hard to be relaxed further.

In our situation, the appearance of the rough vorticity derivative in (1.7) lowers the regularity of the Ricci curvature significantly, which makes it much harder to control the null hypersurfaces of the spacetime time. Our task is to gain the complete set of the geometric control required in [45, Section 7] by utilizing the geometric structures of the acoustical spacetime derived in Section 7. The first set of estimates will be achieved in Proposition 8.2 and Proposition 8.3, which will complete the proof of (5.21)-(5.23), (see Remark 8.5). We will control the normalized mass aspect function and the conformal factor for applying the conformal method in Section 9.

Next, we provide a set of analytic tools under the original coordinates in the region of $\widetilde{\mathcal{D}^+}$, for which, we rely on bootstrap assumptions (5.22) and (5.23).

Proposition 5.4. *Under the assumption (5.23), there hold the following Sobolev inequalities*

(i) *For $2 \leq q < \infty$ and any $S_{t,u}$ -tangent tensor F , there hold*

$$\|\tilde{r}^{1-2/q} F\|_{L^q(S_{t,u})} \lesssim \|\tilde{r} \nabla F\|_{L^2(S_{t,u})}^{1-2/q} \|F\|_{L^2(S_{t,u})}^{2/q} + \|F\|_{L^2(S_{t,u})}, \quad (5.24)$$

and

$$\tilde{r}^{\frac{1}{2}} \|F\|_{L^\infty(S_{t,u})} \lesssim \|\tilde{r} \nabla F\|_{L^4(S_{t,u})} + \|F\|_{L^4(S_{t,u})}. \quad (5.25)$$

²¹See also [42, Section 4.3].

(ii) For any $\delta \in (0, 1)$, any $q \in (2, \infty)$ and any scalar function f there holds

$$\sup_{S_{t,u}} |f| \lesssim \tilde{r}^{\frac{2\delta(q-2)}{2q+\delta(q-2)}} \left(\int_{S_{t,u}} (|\nabla f|^2 + \tilde{r}^{-2}|f|^2) \right)^{\frac{1}{2} - \frac{\delta q}{2q+\delta(q-2)}} \left(\int_{S_{t,u}} (|\nabla f|^q + \tilde{r}^{-q}|f|^q) \right)^{\frac{2\delta}{2q+\delta(q-2)}}.$$

We refer to [10, Chapter 3] and [18, 21, 42, 45] for the above inequalities. We will also need a collection of trace inequalities for future reference.

Lemma 5.5. *Under the assumptions (5.22) and (5.23) there hold on $S_{t,u}$ for scalar functions F the following trace inequalities*

$$\begin{aligned} \int_{S_{t,u}} |F|^2 &\lesssim (\|F\|_{\dot{H}^1(\Sigma_t \cap \{u' \geq u\})} + \|F\|_{L^6(\Sigma_t \cap \{u' \geq u\})}) \|F\|_{L^2(\Sigma_t \cap \{u' \geq u\})}, \\ \|F\|_{L^4(S_{t,u})} + \|\tilde{r}^{-\frac{1}{2}} F\|_{L^2(S_{t,u})} &\lesssim \|F\|_{\dot{H}^1(\Sigma_t \cap \{u' \geq u\})} + \|F\|_{L^6(\Sigma_t \cap \{u' \geq u\})}, \end{aligned}$$

where $-\mathbf{v}_* \leq u \leq t$.

The above result will be always used together with Sobolev embedding on Σ_t .

Proof. This results can be obtained by slightly adapting the original proof in [41, Section 7.2]. \square

We note that in $\widetilde{\mathcal{D}^+}$, $0 \leq \tilde{r} \leq \frac{9}{5}\tau_*$. Thus, back to the coordinate before rescaling, there holds $0 \leq \tilde{r} \lesssim \lambda^{-8\epsilon_0} T \lesssim T$. This fact will be constantly used in the rest of this section and Section 6.

Lemma 5.6 (Dyadic trace inequality). *Let $0 < \alpha < \frac{1}{2}$. Under the assumptions of (5.22) and (5.23), there hold the following estimates for scalar functions F*

$$\|\mu^\alpha [P_\mu, v] F\|_{l_\mu^2 L^2(S_{t,u})} \lesssim \|\partial v\|_{H_x^1} \|F\|_{H^\alpha(\Sigma_t)}, \quad (5.26)$$

$$\|\mu^\alpha P_\mu F\|_{l_\mu^2 L^2(S_{t,u})} \lesssim \|F\|_{H^{\frac{1}{2}+\alpha}(\Sigma_t)}, \quad (5.27)$$

$$\|F\|_{L^2(S_{t,u})} \lesssim \|F\|_{H^{\frac{1}{2}+\alpha}(\Sigma_t)}, \quad (5.28)$$

where P_μ is a Littlewood-Paley projector with the smooth symbol supported in a dyadic shell $\{C^{-1} < |\xi| < C, \xi \in \mathbb{R}^3\}$.

Proof. (5.28) is the standard trace inequality.

To prove (5.26), we apply Lemma 5.5 and Lemma 10.4 to obtain

$$\begin{aligned} \|\mu^\alpha [P_\mu, v] F\|_{L^2(S_{t,u})} &\lesssim \|\mu^{\alpha-\frac{1}{2}} [P_\mu, v] F\|_{H^1(\Sigma_t)}^{\frac{1}{2}} \|\mu^{\alpha+\frac{1}{2}} [P_\mu, v] F\|_{L^2(\Sigma_t)}^{\frac{1}{2}} \\ &\lesssim \|\partial v\|_{L_x^6} \left(\sum_{\lambda \leq \mu} \left(\frac{\lambda}{\mu} \right)^{\frac{1}{2}-\alpha} + \sum_{\lambda > \mu} \left(\frac{\mu}{\lambda} \right)^{\frac{1}{2}+\alpha} \right) \|\lambda^\alpha F_\lambda\|_{L_x^2}. \end{aligned}$$

Since $\|\partial v\|_{L_x^6} \lesssim \|\partial v\|_{H_x^1} \lesssim 1$ due to (2.69), taking l_μ^2 norm gives (5.26).

By using Lemma 5.5 and the finite band property,

$$\|P_\mu F\|_{L^2(S_{t,u})}^2 \lesssim \|P_\mu F\|_{H^1(\Sigma_t)} \|P_\mu F\|_{L^2(\Sigma_t)} \lesssim \|\mu^{\frac{1}{2}} P_\mu F\|_{L^2(\Sigma_t)}^2.$$

Multiplying the above inequality by $\mu^{2\alpha}$, followed with taking l_μ^2 norms on both sides, gives (5.27). \square

Lemma 5.7. *Let $s - 2 \geq \delta > 1 - \frac{2}{p}$. Under the assumptions (5.22) and (5.23), there hold for scalar functions F ,*

$$\|\tilde{r}F\|_{L_t^2 L_\omega^p(C_u)} \lesssim \|\mu^\delta \tilde{P}_\mu F\|_{l_\mu^2 L^2(C_u)} + T^{\frac{1}{2}} \sup_{0 \leq t \leq T} \|F\|_{L^2(\Sigma_t)}, \quad (5.29)$$

$$\|\tilde{r}F\|_{L_u^2 L_\omega^p(u \geq u_0)} \lesssim \|\Lambda^\delta F\|_{L^2(\Sigma_t)} + \|F\|_{L^2(\Sigma_t)}, \quad (5.30)$$

$$\tilde{r}^{\frac{1}{2}} \|F\|_{L_\omega^{2p}(S_{t,u})} \lesssim \|\Lambda^\delta F\|_{H^1(\Sigma_t)} + \|F\|_{H^1(\Sigma_t)}, \quad (5.31)$$

where \tilde{P}_μ is a Littlewood-Paley projector in \mathbb{R}^3 under the original coordinates, which may have slightly different smooth symbol from either \bar{P}_μ or P_μ .

Proof. For the Littlewood-Paley projectors $\sum_\lambda \bar{P}_\lambda = Id$, in view of the reproducing property $\bar{P}_\lambda = P_\lambda^2$, we can decompose $F = \sum_{0 < \mu \leq 1} P_\mu^2 F + \sum_{\mu > 1} P_\mu^2 F$.

If $\mu > 1$, by using (5.24)

$$\begin{aligned} \|\tilde{r}^{1-2/p} P_\mu^2 F\|_{L^p(S_{t,u})} &\lesssim \|\tilde{r} \nabla P_\mu^2 F\|_{L^2(S_{t,u})}^{1-2/p} \|P_\mu^2 F\|_{L^2(S_{t,u})}^{2/p} + \|P_\mu^2 F\|_{L^2(S_{t,u})} \\ &\lesssim \mu^{1-\frac{2}{p}} \|\tilde{r} \tilde{P}_\mu P_\mu F\|_{L^2(S_{t,u})}^{1-\frac{2}{p}} \|P_\mu^2 F\|_{L^2(S_{t,u})}^{\frac{2}{p}} + \|P_\mu^2 F\|_{L^2(S_{t,u})} \\ &\lesssim ((\mu \tilde{r})^{1-\frac{2}{p}} + 1) \|\tilde{P}_\mu F\|_{L^2(S_{t,u})} \lesssim (\mu^{1-\frac{2}{p}} + 1) \|\tilde{P}_\mu F\|_{L^2(S_{t,u})}, \end{aligned}$$

where we used $|\nabla f| \lesssim |\partial f|$. In the last line above, we have regarded both $P_\mu^2 F, \tilde{P}_\mu P_\mu F$ as $\tilde{P}_\mu F$, which are Littlewood-Paley projection associated to some smooth symbols. For the lower frequency term, applying Lemma 5.5 leads to

$$\|\tilde{r}^{1-\frac{2}{p}} P_{\leq 1}^2 F\|_{L^p(S_{t,u})} \lesssim \|P_{\leq 1}^2 F\|_{H^1(\Sigma_t)} \lesssim \|F\|_{L^2(\Sigma_t)}.$$

Therefore we have obtained for $\delta > 1 - \frac{2}{p}$

$$\|\tilde{r}^{1-\frac{2}{p}} F\|_{L^p(S_{t,u})} \lesssim \|\mu^\delta \tilde{P}_\mu F\|_{l_\mu^2 L^2(S_{t,u})} + \|F\|_{L^2(\Sigma_t)}.$$

Integrating the inequality along C_u with L_t^2 gives (5.29). Integrating in u from t to u_0 gives (5.30).

To prove (5.31), we first derive for $\mu > 1$

$$\|P_\mu^2 F\|_{L_\omega^{2p}} \lesssim \|P_\mu^2 F\|_{L_\omega^4}^{\frac{2}{p}} \|P_\mu^2 F\|_{L_\omega^\infty}^{1-\frac{2}{p}}.$$

Thus by using (5.25)

$$\begin{aligned} \tilde{r}^{\frac{1}{2}} \|P_\mu^2 F\|_{L_x^\infty} &\lesssim \tilde{r} \|\nabla P_\mu^2 F\|_{L^4(S_{t,u})} + \|P_\mu^2 F\|_{L^4(S_{t,u})} \\ &\lesssim \tilde{r} \mu \|\tilde{P}_\mu P_\mu F\|_{L^4(S_{t,u})} + \|P_\mu^2 F\|_{L^4(S_{t,u})}. \end{aligned}$$

We then apply the L^4 estimate in Lemma 5.5 to derive

$$\begin{aligned} \tilde{r}^{\frac{1}{2}} \|P_\mu^2 F\|_{L_\omega^{2p}} &\lesssim \|P_\mu^2 F\|_{H_x^1}^{\frac{2}{p}} (\tilde{r} \mu \|\tilde{P}_\mu P_\mu F\|_{L^4(S_{t,u})} + \|P_\mu^2 F\|_{L^4(S_{t,u})})^{1-\frac{2}{p}} \\ &\lesssim \|P_\mu^2 F\|_{H_x^1}^{\frac{2}{p}} (\mu \|\tilde{P}_\mu F\|_{H_x^1} + \|P_\mu^2 F\|_{H_x^1})^{1-\frac{2}{p}} \\ &\lesssim \|\tilde{P}_\mu F\|_{H_x^1} (\mu^{1-\frac{2}{p}} + 1). \end{aligned} \quad (5.32)$$

For $F_{\leq 1} = \sum_{0 < \mu \leq 1} P_\mu^2 F$, by the same procedure, we can obtain

$$\tilde{r}^{\frac{1}{2}} \|F_{\leq 1}\|_{L_\omega^{2p}} \lesssim \|F\|_{H_x^1}.$$

We sum the estimate (5.32) for $\mu > 1$, then combine the result with the above estimate to obtain (5.31) with $\delta > 1 - \frac{2}{p}$. \square

6. CONTROL OF FLUX

In order to understand the analytic property of the acoustical null cones, we will control the energy fluxes for derivatives of v , ϱ , and for $\text{curl } \mathfrak{C}$ along null cones.

6.1. Flux for ∂v and $\partial \varrho$. We define the flux of a function U on the part of $t_{\min} \leq t' \leq t$ on null cone C_u by

$$\mathcal{F}[U](C_u) = \int_{C_u \cap \{t_{\min} \leq t' \leq t\}} (|LU|^2 + |\nabla U|^2)$$

where C_u on the left hand side is a short-hand notation for $C_u \cap \{t_{\min} \leq t' \leq t\}$, and we hide the volume element $d\mu_\gamma dt'$ on C_u for convenience.

To control the flux of (v, ϱ) along null cones C_u , we apply the divergence theorem to \mathcal{P}_μ in (2.23) and (2.11) in the spacetime region $\widetilde{\mathcal{D}^+} \cap \{u' \geq u\} \cap \{t_{\min} \leq t' \leq t\}$, where $t_{\min} = \max\{u, 0\}$. This leads to

$$\begin{aligned} & \int_{C_u \cap \{t_{\min} \leq t' \leq t\}} L^\mu \mathcal{P}_\mu \\ &= \int_{\Sigma_t \cap \{u' \geq u\}} \mathcal{P}_\mu \mathbf{T}^\mu - \int_{\Sigma_{t_{\min}} \cap \{u' \geq u\}} \mathcal{P}_\mu \mathbf{T}^\mu + \int_{\widetilde{\mathcal{D}^+} \cap \{u' \geq u\} \cap \{t_{\min} \leq t' \leq t\}} \mathbf{D}^\mu \mathcal{P}_\mu. \end{aligned} \quad (6.1)$$

The volume element $d\mu_g$ on Σ_t is always comparable to $d\mu_e$ and has been omitted in the above. In particular if $t_{\min} = u$, then $\Sigma_{t_{\min}} \cap \{u' \geq u\}$ is only the point $\Gamma^+(u)$. In this situation, the corresponding integral vanishes.

We now compute

$$\begin{aligned} L^\mu \mathcal{P}_\mu &= -F_U LU + Q_{\mu\nu} \mathbf{T}^\nu L^\mu + \frac{1}{2} F_U^2 L^\mu \mathbf{D}_\mu t \\ &= -F_U LU + \frac{1}{2} ((LU)^2 + (\nabla U)^2) + \frac{1}{2} F_U^2 L(t) \\ &= \frac{1}{2} ((LU - F_U)^2 + |\nabla U|^2), \end{aligned}$$

where we used $Q(L, \mathbf{T})[f] = \frac{1}{2}((Lf)^2 + |\nabla f|^2)$. Substituting this identity to (6.1) implies the following result.

Lemma 6.1 (Fundamental estimate for flux). *There holds on $\widetilde{\mathcal{D}^+}$ for (U, V, F_U, F_V) satisfying (2.12),*

$$\mathcal{F}[U](C_u) \lesssim \int_{C_u} |F_U|^2 + \mathcal{E}[U](t) + \mathcal{E}[U](t_{\min}) + \left| \int_{\widetilde{\mathcal{D}^+} \cap \{u' \geq u\} \cap \{t_{\min} \leq t' \leq t\}} \mathbf{D}^\mu \mathcal{P}_\mu \right|, \quad (6.2)$$

where the integrand of the last term can be found in (2.27), and the term $\mathcal{E}[U](t_{\min})$ vanishes if $u > 0$.

In Section 8, we need the flux control for the metric components \mathbf{g} . Since $v = v_+ + \eta$, we will apply Lemma 6.1 to wave functions (v_+, ϱ) , and use the trace inequalities and elliptic estimates to control derivatives of η .

Proposition 6.2 (H^2 flux for (v, ϱ)). *Under the assumptions (5.22) and (5.23) on $\widetilde{\mathcal{D}^+}$, for the density ϱ and the component of velocity v^i , there holds*

$$\mathcal{F}[\partial \varrho](C_u) + \mathcal{F}[\partial v](C_u) \lesssim 1.$$

Proof. We will use the equation (2.19) with (2.14), (2.15), and recall $(U_i^{(1)}, V_i^{(1)}) = (\partial_i U, \partial_i V)$. Since $(U_i^{(1)}, V_i^{(1)})$ involves merely the spatial derivatives of (U, V) , we will obtain the flux of time derivatives by using (1.4). Since $U = v_+$ or ϱ in (2.19), to recover the full control on v , we will employ the trace inequality to control $\|L\partial\eta\|_{L^2(C_u)}$ and $\|\nabla\partial\eta\|_{L^2(C_u)}$. Note that $\|F_U\|_{L^2(C_u)}$ appears on the right hand side of (6.2). Since it also contains the terms of η , we will treat such term by virtue of trace inequalities. To this end, we first show

$$\|L(\partial\eta)\|_{L^2(S_{t,u})} + \|\nabla(\partial\eta)\|_{L^2(S_{t,u})} + \|F_{U^{(1)}}\|_{L^2(S_{t,u})} \lesssim 1, \quad (6.3)$$

which immediately implies

$$\|L(\partial\eta)\|_{L^2(C_u)} + \|\nabla(\partial\eta)\|_{L^2(C_u)} + \|F_{U^{(1)}}\|_{L^2(C_u)} \lesssim T^{\frac{1}{2}}. \quad (6.4)$$

$L = \mathbf{T} + \mathbf{N}$ in (5.6) will be frequently used in the proof. To see the first estimate in (6.3), we derive by using (5.28) that

$$\begin{aligned} \|L\partial\eta\|_{L^2(S_{t,u})} &\lesssim \|\mathbf{N}\partial\eta\|_{L^2(S_{t,u})} + \|\mathbf{T}\partial\eta\|_{L^2(S_{t,u})} \\ &\lesssim \|\partial^2\eta\|_{H^{\frac{1}{2}+}(\Sigma_t)} + \|\mathbf{T}\partial\eta\|_{H^{\frac{1}{2}+}(\Sigma_t)}. \end{aligned}$$

For the second term, by (2.10), $\mathbf{T}\partial\eta = \partial\mathbf{T}\eta - \partial v^m \partial_m \eta$. By using (2.80), Sobolev inequality, (2.76) and (2.69), we derive

$$\begin{aligned} \|\mathbf{T}\partial\eta\|_{H_x^{\frac{1}{2}+}} &\lesssim \|\partial\mathbf{T}\eta\|_{H_x^1} + \|\partial v \cdot \partial\eta\|_{H_x^1} \\ &\lesssim 1 + \|\partial^2 v \partial\eta\|_{L_x^2} + \|\partial v \partial^2 \eta\|_{L_x^2} + \|\partial v \cdot \partial\eta\|_{L_x^2} \\ &\lesssim 1 + \|\partial v\|_{H_x^1} \|\partial\eta\|_{L_x^\infty} + \|\partial^2 \eta\|_{L_x^3} \|\partial v\|_{L_x^6} \\ &\lesssim \|\partial v\|_{H_x^1} + 1 \lesssim 1. \end{aligned}$$

For the first term, by using (2.79) and (3.23),

$$\|\partial^2 \eta\|_{H^{\frac{1}{2}+}(\Sigma_t)} \lesssim \|\operatorname{curl} \Omega\|_{H_x^{\frac{1}{2}+}} \lesssim \|\operatorname{curl} \Omega\|_{H_x^1} \lesssim 1. \quad (6.5)$$

Hence,

$$\|L(\partial\eta)\|_{L^2(S_{t,u})} \lesssim 1. \quad (6.6)$$

In view of (5.28), (6.5) also implies

$$\|\nabla(\partial\eta)\|_{L^2(S_{t,u})} \lesssim \|\partial^2 \eta\|_{L^2(S_{t,u})} \lesssim \|\partial^2 \eta\|_{H^{\frac{1}{2}+}(\Sigma_t)} \lesssim 1.$$

Thus the first two estimates in (6.3) are proved.

For the last estimate in (6.3), we recall from (2.20),

$$\begin{aligned} \|F_{U^{(1)}}\|_{L^2(S_{t,u})} &\lesssim \|\partial F_U\|_{L^2(S_{t,u})} + \|\partial v \cdot \partial U\|_{L^2(S_{t,u})} \\ &\lesssim \|\partial\mathbf{T}\eta\|_{L^2(S_{t,u})} + \|\partial v\|_{L^4(S_{t,u})} \|\partial U\|_{L^4(S_{t,u})}. \end{aligned}$$

By using (5.28) and Lemma 5.5, the energy estimates (2.60), (2.69), and (2.80), we have

$$\|F_{U^{(1)}}\|_{L^2(S_{t,u})} \lesssim \|\partial\mathbf{T}\eta\|_{H_x^{\frac{1}{2}+}} + \|\partial v\|_{H_x^1} \|\partial U\|_{H_x^1} \lesssim 1,$$

as desired in (6.3). Thus the proof of (6.3) is completed.

Next we apply (6.2) to $U^{(1)}, V^{(1)}, F_{U^{(1)}}, F_{V^{(1)}}$. Similar to (2.32)

$$\|\mathbf{D}^\alpha \mathcal{P}_\alpha\|_{L^1(\widetilde{\mathcal{D}^+})} \lesssim (\|F_{V^{(1)}}\|_{L_t^1 L_x^2} + \|\partial F_{U^{(1)}}\|_{L_t^1 L_x^2}) \sup_{t' \leq t} \mathcal{E}_U^{(1)}(t')^{\frac{1}{2}} + \|k\|_{L_t^1 L_x^\infty} \sup_{t' \leq t} \mathcal{E}_U^{(1)}(t').$$

Recall from (2.20) and the calculation in Corollary 2.4

$$\begin{aligned}
\|\partial F_{U^{(1)}}\|_{L_x^2} &\lesssim \|\partial^2 v\|_{L_x^2} \|\partial U\|_{L_x^\infty} + \|\partial v\|_{L_x^\infty} \|\partial^2 U\|_{L_x^2} + \|\partial^2 F_U\|_{L_x^2} \\
&\lesssim \|\partial U, \partial v\|_{L_x^\infty} + \|\partial^2 F_U\|_{L_x^2}, \\
\|F_{V^{(1)}}\|_{L_x^2} &\lesssim \|\partial \varrho, \partial v\|_{L_x^\infty} \mathcal{E}_U^{(1)}(t)^{\frac{1}{2}} + \|V, \partial U\|_{L_x^\infty} (\|\partial v\|_{H_x^1} + \|\partial \varrho\|_{H_x^1}) + \|\partial F_V\|_{L_x^2} \\
&\lesssim \|\partial \varrho, \partial v, V, \partial U\|_{L_x^\infty} + \|\partial F_V\|_{L_x^2},
\end{aligned}$$

where we also used (2.60) and (2.69). Substituting (2.72) into the above inequalities yields

$$\|\partial F_{U^{(1)}}\|_{L_x^2} + \|F_{V^{(1)}}\|_{L_x^2} \lesssim \|\partial v, \partial \varrho, \partial v_+\|_{L_x^\infty} + 1$$

for $U = v_+$ and $U = \varrho$.

By (2.33), $\|k\|_{L_t^1 L_x^\infty} \lesssim T^{\frac{1}{2}}$ by (2.1). By using the boundedness of energy in (2.60), we summarize the above calculations and derive with the help of (2.1) that

$$\|\mathbf{D}^\alpha \mathcal{P}_\alpha\|_{L^1(\widetilde{\mathcal{D}^+})} \lesssim \|\partial v, \partial \varrho, \partial v_+\|_{L_t^1 L_x^\infty(\widetilde{\mathcal{D}^+})} + T \lesssim T^{\frac{1}{2}}. \quad (6.7)$$

Substituting (6.7), the last estimate in (6.4) and the boundedness of energy (2.60) to (6.2) yields

$$\|L\partial U, \nabla \partial U\|_{L^2(C_u)} \lesssim 1, \text{ for } U = v_+, \varrho.$$

Using the first two estimates in (6.4), $v = v_+ + \eta$, and the first equation in (1.4), we can conclude

$$\|L\partial(v, \varrho), \nabla \partial(v, \varrho), L\mathbf{T}\varrho, \nabla \mathbf{T}\varrho\|_{L^2(C_u)} \lesssim 1. \quad (6.8)$$

We further apply the second equation in (1.4)

$$|L\mathbf{T}v| + |\nabla \mathbf{T}v| \lesssim |L(c^2 \partial \varrho)| + |\nabla(c^2 \partial \varrho)| \lesssim |L\partial \varrho| + |\nabla \partial \varrho| + |\partial \varrho|^2. \quad (6.9)$$

By using Lemma 5.5 and (2.69), we can bound

$$\|\partial \varrho\|_{L^2(C_u)}^2 \lesssim \sup_{t' \leq t} \|\partial \varrho\|_{L^4(S_{t', u})}^2 \cdot T^{\frac{1}{2}} \lesssim \|\partial \varrho\|_{L_t^\infty H_x^1}^2 T^{\frac{1}{2}} \lesssim T^{\frac{1}{2}}.$$

Hence by using (6.8) we have

$$\|L\mathbf{T}v, \nabla \mathbf{T}v\|_{L^2(C_u)} \lesssim 1.$$

Thus the proof of Proposition 6.2 is complete. \square

Proposition 6.3 ($H^{2+\epsilon}$ flux for (v, ϱ)). *Let $0 < \epsilon \leq s - 2$. Under the assumption (5.22) and (5.23) on $\widetilde{\mathcal{D}^+}$, there holds*

$$\|\mu^\epsilon \mathcal{F}^{\frac{1}{2}}[P_\mu \partial v](C_u)\|_{l_\mu^2} + \|\mu^\epsilon \mathcal{F}^{\frac{1}{2}}[P_\mu \partial \varrho](C_u)\|_{l_\mu^2} + \|\mu^\epsilon \mathcal{F}^{\frac{1}{2}}[P_\mu \partial v_+](C_u)\|_{l_\mu^2} \lesssim 1.$$

Remark 6.4. To avoid unnecessary technical baggage, we will not derive the flux control for $\mathbf{T}v$, since only the weaker control in (8.27) is required in Section 8. By using the second equation of (1.4), $\mathbf{T}v = -c^2 \partial \varrho$, we can directly get the bound for $\mathbf{T}v$ in (8.27) by using the above result with the help of the trace inequality and energy estimates. See more details in the proof of (8.27) in Section 8.

Proof. We apply (6.2) to $(U_\mu^{(1)}, V_\mu^{(1)}) = (P_\mu U^{(1)}, P_\mu V^{(1)})$ with

$$(U^{(1)}, V^{(1)}) = (\partial v_+, \partial \mathbf{T}v), \quad (U^{(1)}, V^{(1)}) = (\partial \varrho, \partial \mathbf{T}\varrho)$$

to obtain

$$\begin{aligned} \sum_{\mu>1} \mu^{2\epsilon} \mathcal{F}[U_\mu^{(1)}](C_u) &\lesssim \sum_{\mu>1} \left\{ \int_{C_u \cap \{t' \leq t\}} \mu^{2\epsilon} |F_{U_\mu^{(1)}}|^2 + \mu^{2\epsilon} (\mathcal{E}[U_\mu^{(1)}](t) + \mathcal{E}[U_\mu^{(1)}](t_{\min})) \right\} \\ &\quad + \sum_{\mu>1} \mu^{2\epsilon} \left| \int_{\widetilde{\mathcal{D}^+} \cap \{u' \geq u\} \cap \{t_{\min} \leq t' \leq t\}} \mathbf{D}^\alpha \mathcal{P}_\alpha[U_\mu^{(1)}] \right|. \end{aligned} \quad (6.10)$$

Substituting $(U_\mu^{(1)}, V_\mu^{(1)})$ into (2.32) implies

$$\begin{aligned} \sum_{\mu>1} \mu^{2\epsilon} \|\mathbf{D}^\alpha \mathcal{P}_\alpha[U_\mu^{(1)}]\|_{L^1(\widetilde{\mathcal{D}^+} \cap \{u' \geq u\} \cap \{t_{\min} \leq t' \leq t\})} &\lesssim \int_0^t \|\mu^\epsilon \mathcal{E}[U_\mu^{(1)}]^{\frac{1}{2}}(t')\|_{l_\mu^2}^2 \\ &\quad \cdot \{\|\mu^\epsilon \mathcal{E}[U_\mu^{(1)}]^{\frac{1}{2}}(t')\|_{l_\mu^2} \|k(t')\|_{L_x^\infty} + \|\mu^\epsilon F_{V_\mu^{(1)}}(t')\|_{l_\mu^2 L_x^2} + \|\mu^\epsilon \partial F_{U_\mu^{(1)}}(t')\|_{l_\mu^2 L_x^2}\} dt', \end{aligned} \quad (6.11)$$

where the formulas of $F_{U_\mu^{(1)}}$ and $F_{V_\mu^{(1)}}$ can be found in (2.21). To control the right hand side, we will rely on the estimates of $F_{U_\mu^{(1)}}$, $F_{V_\mu^{(1)}}$ in the proof of Proposition 2.6 and the results of Corollary 3.3.

On Σ_t with $0 < t \leq T$, we derive from (2.39) and (2.41) that

$$\begin{aligned} &\|\mu^\epsilon \partial F_{U_\mu^{(1)}}\|_{l_\mu^2 L_x^2} + \|\mu^\epsilon F_{V_\mu^{(1)}}\|_{l_\mu^2 L_x^2} \\ &\lesssim \|\partial v, \partial(c^2), \text{Tr}k\|_{H^{1+\epsilon}} \|V, \partial U\|_{L_x^\infty} + \|\partial v, \partial \varrho\|_{L_x^\infty} \|V^{(1)}, \partial U\|_{H_x^{1+\epsilon}} + \|\partial^2 F_U\|_{H_x^\epsilon} + \|\partial F_V\|_{\dot{H}_x^\epsilon} \\ &\lesssim \|\partial v, \partial \varrho, \partial v_+\|_{L_x^\infty} \|\partial U, V^{(1)}, \partial v, \partial(c^2), \text{Tr}k\|_{H_x^{1+\epsilon}} + \|\partial^2 F_U\|_{H_x^\epsilon} + \|\partial F_V\|_{\dot{H}_x^\epsilon} \\ &\lesssim (\mathcal{E}^{(\leq 1)}(t))^{\frac{1}{2}} + \|\mu^\epsilon \mathcal{E}_\mu^{(1)}(t)^{\frac{1}{2}}\|_{l_\mu^2} + \|\text{curl } \Omega\|_{H_x^\epsilon} + 1)(\|\partial v_+, \partial v, \partial \varrho\|_{L_x^\infty} + 1) \\ &\lesssim \|\partial v_+, \partial v, \partial \varrho\|_{L_x^\infty} + 1, \end{aligned} \quad (6.12)$$

where we employed (2.86) and (2.87) together with Lemma 2.16 to derive the line of (6.12), and used Corollary 3.3 and (3.5) to derive the last line. We then substitute the above estimate to (6.11). By using (2.1) and Corollary 3.3, we can conclude that

$$\sum_{\mu>1} \mu^{2\epsilon} \|\mathbf{D}^\alpha \mathcal{P}_\alpha[U_\mu^{(1)}]\|_{L^1(\widetilde{\mathcal{D}^+})} \lesssim T^{\frac{1}{2}}. \quad (6.13)$$

Next, we bound $\|\mu^\epsilon F_{U_\mu^{(1)}}\|_{l_\mu^2 L^2(C_u)}$ in (6.10). Using (2.20) and (2.21) directly implies

$$\begin{aligned} \|\mu^\epsilon F_{U_\mu^{(1)}}\|_{l_\mu^2 L^2(C_u)} &\lesssim \|\mu^\epsilon [P_\mu, v] \partial U^{(1)}\|_{l_\mu^2 L^2(C_u)} + \|\mu^\epsilon P_\mu F_{U^{(1)}}\|_{l_\mu^2 L^2(C_u)} \\ &\lesssim T^{\frac{1}{2}} (\|\partial v\|_{L_t^\infty H_x^1} \|\partial U^{(1)}\|_{L_t^\infty H_x^\epsilon} + \|\Lambda^{\frac{1}{2}+\epsilon} F_{U^{(1)}}\|_{L_t^\infty L_x^2}), \end{aligned} \quad (6.14)$$

where we applied (5.26) to $\partial U^{(1)}$, and (5.27) to $F_{U^{(1)}}$ to obtain the last inequality. For the second term on the right hand side, we recall (2.16) and use (10.14) with $\alpha = \frac{1}{2} + \epsilon$ to obtain

$$\begin{aligned} \|\Lambda^{\frac{1}{2}+\epsilon} F_{U^{(1)}}\|_{L_x^2} &\lesssim \|\Lambda^{\frac{1}{2}+\epsilon} \partial F_U\|_{L_x^2} + \|\Lambda^{\frac{1}{2}+\epsilon} (\partial v \partial U)\|_{L_x^2} \\ &\lesssim \|\Lambda^{\frac{1}{2}+\epsilon} \partial \mathbf{T} \eta\|_{L_x^2} + \|\partial v\|_{H_x^{1+\epsilon}} \|\partial U\|_{H_x^1} + \|\partial U\|_{H_x^{1+\epsilon}} \|\partial v\|_{H_x^1} \\ &\lesssim \|\partial \mathbf{T} \eta\|_{H_x^1} + \|\partial U\|_{H_x^{1+\epsilon}} \lesssim 1, \end{aligned}$$

where we used (2.80) and Corollary 3.3 to derive the last inequality.

Substituting the above estimate to (6.14) and using Corollary 3.3 to treat the first term in the last line of (6.14) imply

$$\|\mu^\epsilon F_{U_\mu^{(1)}}\|_{l_\mu^2 L^2(C_u)} \lesssim T^{\frac{1}{2}}.$$

This controls the first term on the right hand side of (6.10). Substituting this estimate and (6.13) into (6.10) and using Corollary 3.3 imply

$$\|\mu^\epsilon \mathcal{F}^{\frac{1}{2}}[U_\mu^{(1)}](C_u)\|_{l_\mu^2}^2 \lesssim 1 + \sup_{t' \leq t} \sum_{\mu > 1} \mu^{2\epsilon} \mathcal{E}_\mu^{(1)}(t') \lesssim 1. \quad (6.15)$$

This together with Proposition 6.2 immediately gives the control of the flux for $(\partial v_+, \partial \varrho)$ to the highest order. Similar to the H^2 -case, in view of $v = v_+ + \eta$, we need to show

$$\|\mu^\epsilon (LP_\mu \partial \eta, \nabla P_\mu \partial \eta)\|_{l_\mu^2 L^2(C_u)} \lesssim T^{\frac{1}{2}} \lesssim 1. \quad (6.16)$$

For simplicity, we denote by $S = S_{t',u}$ with $t_{\min} \leq t' \leq t$. We first apply (5.27) to $F = \partial^2 \eta$. By using (2.79) and (3.5),

$$\|\Lambda^\alpha \partial^2 \eta\|_{L_t^\infty L_x^2} \lesssim 1, \quad 0 < \alpha \leq \frac{1}{2} + \epsilon. \quad (6.17)$$

Hence, by using (5.27)

$$\|\mu^\epsilon \nabla P_\mu \partial \eta\|_{l_\mu^2 L^2(S)} \lesssim \|\mu^\epsilon \partial P_\mu \partial \eta\|_{l_\mu^2 L^2(S)} \lesssim \|\Lambda^{\frac{1}{2}+\epsilon} \partial^2 \eta\|_{L_t^\infty L_x^2} \lesssim 1. \quad (6.18)$$

For the first estimate in (6.16), we derive

$$\begin{aligned} \|\mu^\epsilon LP_\mu \partial \eta\|_{l_\mu^2 L^2(S)} &\lesssim \|\mu^\epsilon \mathbf{N}^i \partial_i P_\mu \partial \eta\|_{l_\mu^2 L^2(S)} + \|\mu^\epsilon \mathbf{T} P_\mu \partial \eta\|_{l_\mu^2 L^2(S)} \\ &\lesssim \|\mu^\epsilon P_\mu \partial^2 \eta\|_{l_\mu^2 L^2(S)} + \|\mu^\epsilon [\mathbf{T}, P_\mu] \partial \eta\|_{l_\mu^2 L^2(S)} + \|\mu^\epsilon P_\mu \mathbf{T} \partial \eta\|_{l_\mu^2 L^2(S)}. \end{aligned}$$

The estimate of the first term on the right hand side of the last line is already included in (6.18). It suffices to show

$$\|\mu^\epsilon [\mathbf{T}, P_\mu] \partial \eta\|_{l_\mu^2 L^2(S)} + \|\mu^\epsilon P_\mu \mathbf{T} \partial \eta\|_{l_\mu^2 L^2(S)} \lesssim 1. \quad (6.19)$$

For the first estimate, we apply (5.26) to obtain

$$\|\mu^\epsilon [P_\mu, v] \partial^2 \eta\|_{l_\mu^2 L^2(S)} \lesssim \|\partial v\|_{H_x^1} \|\partial^2 \eta\|_{H_x^\epsilon} \lesssim 1,$$

where we used (6.17) and (2.69) to get the last bound. For the other term, we first use (6.17), (2.49), (10.14), Corollary 3.3 to bound

$$\begin{aligned} \|\Lambda^{\frac{1}{2}+\epsilon} [\mathbf{T}, \partial] \eta\|_{L_x^2} &\leq \|\Lambda^{\frac{1}{2}+\epsilon} (\partial v \cdot \partial \eta)\|_{L_x^2} \\ &\lesssim \|\partial v\|_{H_x^{1+\epsilon}} \|\partial \eta\|_{H_x^1} + \|\partial v\|_{H_x^1} \|\partial \eta\|_{H_x^{1+\epsilon}} \\ &\lesssim 1. \end{aligned}$$

Next we use (5.27) with $\alpha = \epsilon$, (2.80) together with the above estimate to derive

$$\|\mu^\epsilon P_\mu \mathbf{T} \partial \eta\|_{l_\mu^2 L^2(S)} \lesssim \|\Lambda^{\frac{1}{2}+\epsilon} ([\mathbf{T}, \partial] + \partial \mathbf{T}) \eta\|_{L_t^\infty L_x^2} \lesssim 1.$$

Thus (6.19) is proved. We have completed the proof of

$$\|\mu^\epsilon LP_\mu \partial \eta\|_{l_\mu^2 L^2(C_u)} \lesssim T^{\frac{1}{2}}.$$

The proof of (6.16) is thus completed.

Combining (6.15) with (6.16), and applying $v = v_+ + \eta$ imply

$$\|\mu^\epsilon \mathcal{F}^{\frac{1}{2}}[v_\mu^{(1)}](C_u)\|_{l_\mu^2} \lesssim \|\mu^\epsilon \mathcal{F}^{\frac{1}{2}}[P_\mu v_+^{(1)}](C_u)\|_{l_\mu^2} + \|\mu^\epsilon (\nabla_L P_\mu \partial \eta, \nabla P_\mu \partial \eta)\|_{l_\mu^2 L^2(C_u)} \lesssim 1.$$

Using $\mathbf{T} \varrho = -\operatorname{div} v$ in (1.4), the above estimate together with (6.15) leads to

$$\|\mu^\epsilon \mathcal{F}^{\frac{1}{2}}[P_\mu \partial v, P_\mu \partial v_+, P_\mu \partial \varrho](C_u)\|_{l_\mu^2} \lesssim 1.$$

Thus Proposition 6.3 is proved. \square

6.2. Flux of $\text{curl } \mathfrak{C}$. In this subsection, we derive the flux control of $\text{curl } \mathfrak{C}$ up to the highest order along C_u , which will be crucial for Section 8.

Along C_u , as a consequence of Proposition 3.5, and the trace inequalities (5.27) and (5.31), we can bound

$$\|\mu^{\delta+\frac{1}{2}} P_\mu \partial \Omega\|_{L_\mu^2 L^2(S_{t,u})} + \tilde{r}^{\frac{1}{2}} \|\partial \Omega\|_{L_\omega^{2p}(S_{t,u})} \lesssim 1, \quad (6.20)$$

with $0 \leq 1 - \frac{2}{p} < \delta \leq s' - 2$. Nevertheless, the analysis in Section 8 requires us to gain a $\frac{1}{2}$ derivative more than the above bounds. The goal is achieved in this subsection merely for $\text{curl } \mathfrak{C}$, whereas the full control for $\partial^2 \Omega$ may not actually hold under our assumption on the initial data.

We will introduce an energy argument identical to Section 3, with the flux obtained as the boundary term on C_u . Recall that the energy bound of $\text{curl } \mathfrak{C}$ in Section 3 is derived by taking advantage of the trilinear structure due to the pairing of $\mathbf{T} \text{curl } \mathfrak{C} \cdot \text{curl } \mathfrak{C}$, with the help of a series of integration by parts. We lose the structure if propagate the general $\partial \mathfrak{C}$ directly. The elliptic estimates for the Hodge system on Σ_t allows us to obtain the H_x^δ bound for $\partial^2 \Omega, \partial \mathfrak{C}, \partial^2 \mathfrak{w}$ whence $\|\text{curl } \mathfrak{C}\|_{H_x^\delta}$ is bounded. (See (3.5) and Proposition 3.5.) However there is no reasonable Hodge system on $S_{t,u}$ for \mathfrak{C} , thus we can not control $\partial \mathfrak{C}$ by the bound of $\text{curl } \mathfrak{C}$ in $L^2(S_{t,u})$ without loss. In Section 8, we will use merely the highest-order flux of $\text{curl } \mathfrak{C}$ to obtain the needed regularity of null cones by uncovering a series of geometric structures of the acoustical spacetime.

We first give the fundamental inequality for bounding the energy flux of $\text{curl } \mathfrak{C}$.

Lemma 6.5. *Let $-v_* \leq u_0 \leq \tau_*$ be fixed and F and G be one-tensor fields. There holds on $\widetilde{\mathcal{D}^+}$ that*

$$\begin{aligned} & \left| \int_{C_{u_0} \cap \{t_{\min} \leq t' \leq t\}} c^3 \langle F, G \rangle_e d\mu_\gamma dt + \int_{\widetilde{\mathcal{D}^+} \cap \{u \geq u_0\} \cap \{t_{\min} \leq t' \leq t\}} (\langle \mathbf{T} F, G \rangle_e + \langle \mathbf{T} G, F \rangle_e) d\mu_e dt \right| \\ & \lesssim \int_0^t \|\partial v(t')\|_{L_x^\infty} \int_{\Sigma_{t'}} |\langle F, G \rangle| d\mu_e dt' \\ & + \left| \int_{\Sigma_t} \langle F, G \rangle_e d\mu_g \right| + \left| \int_{\Sigma_{t_{\min}}} \langle F, G \rangle_e d\mu_g \right|, \end{aligned} \quad (6.21)$$

where $t_{\min}(u_0) = \max(u_0, 0)$, and thus the last term vanishes if $u_0 \geq 0$.

Proof. Let $\mathcal{V}^\mu = \langle F, G \rangle_e \mathbf{T}^\mu$. By applying the divergence theorem, we have

$$\begin{aligned} & \int_{C_{u_0} \cap \{t' \leq t\}} \mathcal{V}^\mu \mathbf{D}_\mu u b d\mu_\gamma dt' + \int_{\Sigma_{t_{\min}}} \mathcal{V}^\mu \mathbf{D}_\mu t - \int_{\Sigma_t} \mathcal{V}^\mu \mathbf{D}_\mu t \\ & = - \int_{\widetilde{\mathcal{D}^+} \cap \{u \geq u_0\} \cap \{t_{\min} \leq t' \leq t\}} \mathbf{D}_\mu \mathcal{V}^\mu d\mu_g dt. \end{aligned} \quad (6.22)$$

With $\bar{\Pi}_\mu^\nu = \delta_\mu^\nu + \mathbf{T}^\nu \mathbf{T}_\mu$, which is a Σ_t -tangent tensor since the contraction of the tensor with \mathbf{T} vanishes,

$$e_{i\nu} \mathbf{D}_i (\mathcal{V}^\mu \bar{\Pi}_\mu^\nu) = e_{i\nu} (\mathbf{D}_i \mathcal{V}^\mu \bar{\Pi}_\mu^\nu + \mathcal{V}^\mu \mathbf{D}_i \bar{\Pi}_\mu^\nu) = \mathbf{D}_i \mathcal{V}^j \bar{\Pi}_j^i + e_{i\nu} \mathcal{V}^\mu \mathbf{D}_i (\mathbf{T}_\mu \mathbf{T}^\nu),$$

where we decomposed Σ_t -tangent tensor fields, with indices lifted or lowered by g , with respect to $e_i = c^{-1} \partial_i$, $i = 1, 2, 3$ which form the orthonormal basis in (Σ_t, g) . And \mathbf{D} denotes the covariant derivative in the spacetime.

By the definition of \mathcal{V} , $\mathcal{V}^\mu \bar{\Pi}_\mu^\nu = 0$. Thus we can derive from the above identity that

$$\mathbf{D}_i \mathcal{V}^i = \langle \mathcal{V}, \mathbf{T} \rangle \text{Tr} k.$$

Using the above identity and the facts that $\langle \mathcal{V}, \mathbf{D}u \rangle = \langle F, G \rangle_e \mathbf{b}^{-1}$, $\langle \mathcal{V}, \mathbf{D}t \rangle = \langle F, G \rangle_e$, we compute

$$\begin{aligned} \mathbf{D}_\mu \mathcal{V}^\mu &= -\mathbf{T} \langle \mathcal{V}, \mathbf{T} \rangle + \mathbf{D}_i \mathcal{V}^i = \mathbf{T}(\langle F, G \rangle_e) - \langle F, G \rangle_e \text{Tr}k \\ &= \langle \mathbf{T}F, G \rangle_e + \langle \mathbf{T}G, F \rangle_e - \langle F, G \rangle_e \text{Tr}k. \end{aligned}$$

Substituting the above identity to (6.22) yields

$$\begin{aligned} \int_{C_{u_0} \cap \{t' \leq t\}} \mathbf{b}^{-1} \langle F, G \rangle_e \mathbf{b} d\mu_\gamma dt' &= \int_{\Sigma_t} \langle F, G \rangle_e d\mu_g - \int_{\Sigma_{t_{\min}}} \langle F, G \rangle_e d\mu_g \\ &\quad - \int_{\widetilde{\mathcal{D}^+} \cap \{u \geq u_0\} \cap \{t_{\min} \leq t' \leq t\}} \{\langle \mathbf{T}F, G \rangle_e + \langle F, \mathbf{T}G \rangle_e - \langle F, G \rangle_e \text{Tr}k\} d\mu_g dt'. \end{aligned} \quad (6.23)$$

Since $d\mu_g = c^{-3} d\mu_e$, we can replace F by $c^{\frac{3}{2}} F$ and G by $c^{\frac{3}{2}} G$ in the above identity. This implies

$$\begin{aligned} \int_{C_{u_0} \cap \{t' \leq t\}} c^3 \langle F, G \rangle_e d\mu_\gamma dt &= \int_{\Sigma_t} \langle F, G \rangle_e d\mu_e - \int_{\Sigma_{t_{\min}}} \langle F, G \rangle_e d\mu_e \\ &\quad - \int_{\widetilde{\mathcal{D}^+} \cap \{u \geq u_0\} \cap \{t_{\min} \leq t' \leq t\}} \{\langle \mathbf{T}F, G \rangle_e + \langle F, \mathbf{T}G \rangle_e - \langle F, G \rangle_e (\text{Tr}k - 3\mathbf{T}(\log c))\} d\mu_e dt. \end{aligned}$$

By (2.9), $\text{Tr}k - 3\mathbf{T}(\log c) = -\text{div } v$. (6.21) follows by substituting this formula to the above calculation. \square

Next, we prove the main result of this subsection.

Proposition 6.6. *Under the assumptions (5.22) and (5.23), there hold on $C_u \cap \widetilde{\mathcal{D}^+}$ that,*

$$\| \text{curl} \mathfrak{C} \|_{L^2(C_u)} \lesssim 1, \quad (6.24)$$

$$\| \mu^\delta P_\mu \text{curl} \mathfrak{C} \|_{l_\mu^2 L^2(C_u)} \lesssim 1, \quad 0 \leq \delta \leq s' - 2, \quad (6.25)$$

$$\| \tilde{r} \text{curl}^2 \Omega \|_{L_t^2 L_\omega^p(C_u)} \lesssim 1, \quad 0 \leq 1 - \frac{2}{p} < s' - 2. \quad (6.26)$$

Proof. Firstly, it is direct to compute $\epsilon_{mij} \partial^m \mathfrak{C}^i = e^{-\varrho} \epsilon_{mij} \partial^m (\text{curl } \Omega)^i - \epsilon_{mij} \partial^m \varrho \mathfrak{C}^i$. By using (5.31) with $s' - 2 \geq \delta > 1 - \frac{2}{p}$, we derive

$$\begin{aligned} \| \tilde{r} \text{curl}^2 \Omega \|_{L_t^2 L_\omega^p(C_u)} &\lesssim \| \tilde{r} \partial \varrho \cdot \mathfrak{C} \|_{L_t^2 L_\omega^p(C_u)} + \| \tilde{r} \text{curl} \mathfrak{C} \|_{L_t^2 L_\omega^p(C_u)} \\ &\lesssim \| \tilde{r}^{\frac{1}{2}} \partial \varrho \|_{L_t^2 L_\omega^{2p}(C_u)} \| \tilde{r}^{\frac{1}{2}} \mathfrak{C} \|_{L_t^\infty L_\omega^{2p}(C_u)} + \| \tilde{r} \text{curl} \mathfrak{C} \|_{L_t^2 L_\omega^p(C_u)} \\ &\lesssim T^{\frac{1}{2}} \| \partial \varrho \|_{L_t^2 L_x^\infty} \| \Lambda^\delta \mathfrak{C} \|_{L_t^\infty H_x^1} + \| \tilde{r} \text{curl} \mathfrak{C} \|_{L_t^2 L_\omega^p(C_u)} \\ &\lesssim T^{\frac{1}{2}} + \| \tilde{r} \text{curl} \mathfrak{C} \|_{L_t^2 L_\omega^p(C_u)}, \end{aligned}$$

where we used (2.1) and (3.26) to obtain the last line.

To derive (6.26), we apply (5.29) to $F = \text{curl}^2 \Omega$ by assuming (6.25) to derive

$$\| \tilde{r} \text{curl} \mathfrak{C} \|_{L_t^2 L_\omega^p(C_u)} \lesssim \| \mu^\delta \tilde{P}_\mu \text{curl} \mathfrak{C} \|_{l_\mu^2 L^2(C_u)} + \| \text{curl} \mathfrak{C} \|_{L_t^\infty L_x^2} \lesssim 1,$$

where we also used (3.5).

Next we prove (6.24) and (6.25). The analysis overlaps with Section 3, except that all the boundary terms along C_u were absent therein.

To prove (6.24), we apply (6.21) to $F = G = \text{curl } \mathfrak{C}$. Consider the following integral on $\widetilde{\mathcal{D}^+} \cap \{u \geq u_0\} \cap \{t_{\min} \leq t' \leq t\}$, and the range of t' will be hidden for short,

$$\begin{aligned} \int_{\widetilde{\mathcal{D}^+} \cap \{u \geq u_0\}} \langle \mathbf{T} \text{curl } \mathfrak{C}, \text{curl } \mathfrak{C} \rangle_e d\mu_e dt' &= \int_{\widetilde{\mathcal{D}^+} \cap \{u \geq u_0\}} \langle \text{curl } \mathbf{T} \mathfrak{C}, \text{curl } \mathfrak{C} \rangle_e d\mu_e dt' \\ &+ \int_{\widetilde{\mathcal{D}^+} \cap \{u \geq u_0\}} \langle [\mathbf{T}, \text{curl}] \mathfrak{C}, \text{curl } \mathfrak{C} \rangle_e d\mu_e dt'. \end{aligned} \quad (6.27)$$

By using (3.1) and the first equation in (1.4), the last line can be bounded by

$$\int_0^t \|\partial v\|_{L_x^\infty} \|\partial \mathfrak{C}\|_{L_x^2}^2 dt' \lesssim 1$$

with the help of (3.5) and (2.1). Thus

$$\left| \int_{\widetilde{\mathcal{D}^+} \cap \{u \geq u_0\}} \langle \mathbf{T} \text{curl } \mathfrak{C}, \text{curl } \mathfrak{C} \rangle_e d\mu_e dt' \right| \lesssim \left| \int_{\widetilde{\mathcal{D}^+} \cap \{u \geq u_0\}} \langle \text{curl } \mathbf{T} \mathfrak{C}, \text{curl } \mathfrak{C} \rangle_e d\mu_e dt' \right| + 1. \quad (6.28)$$

The first term on the right hand side of (6.28) has the same integrand as the term (3.12). The only difference is that the integral is on the spacetime domain enclosed by the boundary $\{t' = t_{\min}\}$, $\{t' = t\}$ and $\{u = u_0\}$. Therefore whenever undertaking integration by parts, we need to keep track of the additional boundary terms along C_{u_0} compared with the treatment for (3.12). The boundary terms on C_u arisen from the integration by parts to give (3.17) and (3.18) are denoted by $I_{+,1,1,b}^1$ and $I_{+,1,2,b}^1$ respectively. Let us compute $I_{+,1,1,b}^1$ first.

$$\begin{aligned} I_{+,1,1,b}^1 &= \int_{t_{\min}}^t \int_{\Sigma_{t'} \cap \{u \geq u_0\}} \partial_m (e^{-\varrho} \partial^n v_j \partial_n \Omega^j \text{curl } \mathfrak{C}^m) d\mu_e dt' \\ &= \int_{t_{\min}}^t \int_{\Sigma_{t'} \cap \{u \geq u_0\}} \{ \text{div}_g (e^{-\varrho} \partial^n v^j \partial_n \Omega_j \text{curl } \mathfrak{C}) \\ &\quad + 3 \partial_m \log c \cdot e^{-\varrho} \partial^n v^j \partial_n \Omega_j \text{curl } \mathfrak{C}^m \} c^3 c^{-3} d\mu_e dt' \\ &= \int_{t_{\min}}^t \int_{\Sigma_{t'} \cap \{u \geq u_0\}} \text{div}_g (c^3 e^{-\varrho} \partial^n v^j \partial_n \Omega_j \text{curl } \mathfrak{C}) d\mu_g dt', \end{aligned} \quad (6.29)$$

where we calculated $\text{div}_g X = \text{div}_e X - 3X(\log c)$ for Σ_t -tangent vector field X , since $g_{ij} = c^{-2} \delta_{ij}$.

$$I_{+,1,1,b}^1 = \int_{C_{u_0}} c^3 e^{-\varrho} \partial^n v^j \partial_n \Omega_j \text{curl } \mathfrak{C}^m \mathbf{N}^l g_{ml} d\mu_\gamma dt'.$$

By using Lemma 5.5 and (3.23), there holds on $\widetilde{\mathcal{D}^+}$

$$\|\partial \Omega\|_{L^4(S_{t,u})} + \|\partial \Omega\|_{L^2(S_{t,u})} \lesssim 1.$$

Thus by using the above inequality and (2.1), we have

$$|I_{+,1,1,b}^1| \lesssim \|\partial v\|_{L_t^2 L_x^\infty} \sup_{t_{\min} \leq t' \leq t} \|\partial \Omega\|_{L^2(S_{t',u_0})} \|\text{curl } \mathfrak{C}\|_{L^2(C_{u_0})} \lesssim \|\text{curl } \mathfrak{C}\|_{L^2(C_{u_0})}.$$

For the term in $I_{+,1,2,b}^1$, when integrated in $\widetilde{\mathcal{D}^+} \cap \{u \geq u_0\} \cap \{t_{\min} \leq t' \leq t\}$, the term

$$\int_{t_{\min}}^t \partial_t \int_{\Sigma_{t'} \cap \{u \geq u_0\}} \partial_j \varrho \partial^m \Omega^j e^{-\varrho} \text{curl } \mathfrak{C}_m d\mu_e dt'$$

contributes the additional boundary term

$$I_{+,1,2,b}^1 = \int_{C_{u_0}} c^3 e^{-\varrho} \partial_j \varrho \partial^m \Omega^j \text{curl } \mathfrak{C}_m d\mu_\gamma dt'.$$

Alternatively, if we consider $I_{+,1,2}^1$ on $\widetilde{\mathcal{D}^+} \cap \{u \geq u_0\}$ by applying (6.23) to $F = c^{\frac{3}{2}} \partial_j \varrho$ and $G = c^{\frac{3}{2}} \partial_m \Omega^j e^{-\varrho} \operatorname{curl} \mathfrak{C}^m$, we can get the same additional boundary term as above. Similar to the estimate for $I_{+,1,1,b}^1$,

$$\begin{aligned} |I_{+,1,2,b}^1| &\lesssim \|\partial \varrho\|_{L_t^2 L_x^\infty} \sup_{t_{\min} \leq t' \leq t} \|\partial \Omega\|_{L^2(S_{t',u_0})} \|\operatorname{curl} \mathfrak{C}\|_{L^2(C_{u_0})} \\ &\lesssim \|\operatorname{curl} \mathfrak{C}\|_{L^2(C_{u_0})}. \end{aligned}$$

The control of the rest of the terms can be found in the estimate for $|I^1|$ in (3.22). Therefore, from (6.28) and the above boundary estimates, we conclude

$$\begin{aligned} & \left| \int_{\widetilde{\mathcal{D}^+} \cap \{u \geq u_0\}} \langle \mathbf{T} \operatorname{curl} \mathfrak{C}, \operatorname{curl} \mathfrak{C} \rangle_e d\mu_e dt \right| \\ & \lesssim |I_{+,1,1,b}^1| + |I_{+,1,2,b}^1| + 1 + \int_0^t (\|\operatorname{curl} \mathfrak{C}\|_{L^2(\Sigma_t)} + 1)^2 (\|\partial v, \partial \varrho\|_{L_x^\infty} + 1) + \sup_{0 \leq t' \leq t} \|\operatorname{curl} \mathfrak{C}\|_{L^2(\Sigma_{t'})} \\ & \lesssim \|\operatorname{curl} \mathfrak{C}\|_{L^2(C_{u_0})} + 1, \end{aligned}$$

where we used $\|\partial v, \partial \varrho\|_{L_t^1 L_x^\infty} \lesssim T^{\frac{1}{2}}$ and the estimate (3.5) to obtain the last line.

Thus in view of (6.21), we have

$$\|\operatorname{curl} \mathfrak{C}\|_{L^2(C_{u_0})}^2 \lesssim 1 + \|\operatorname{curl} \mathfrak{C}\|_{L^2(C_{u_0})},$$

which gives $\|\operatorname{curl} \mathfrak{C}\|_{L^2(C_{u_0})} \lesssim 1$. This shows (6.24).

To prove (6.25), we apply (6.21) to $F = G = P_\mu \operatorname{curl} \mathfrak{C}_i$ to bound

$$\begin{aligned} \sum_{\mu > 1} \left| \int_{C_{u_0}} c^3 \mu^{2\delta} |P_\mu \operatorname{curl} \mathfrak{C}|_e^2 d\mu_\gamma dt \right| &\lesssim \sum_{\mu > 1} \left| \int_{\widetilde{\mathcal{D}^+} \cap \{u \geq u_0\}} \mu^{2\delta} \langle \mathbf{T} P_\mu \operatorname{curl} \mathfrak{C}, P_\mu \operatorname{curl} \mathfrak{C} \rangle_e d\mu_e dt \right| \\ &\quad + \sup_{t' \leq t} \int_{\Sigma_{t'}} \sum_{\mu > 1} \mu^{2\delta} |P_\mu \operatorname{curl} \mathfrak{C}|^2 d\mu_e, \end{aligned} \quad (6.30)$$

where we also used (2.1) for deriving the term in the last line.

For the first term on the right hand side,

$$\begin{aligned} \mathcal{B}_\mu &= \int_{\widetilde{\mathcal{D}^+} \cap \{u \geq u_0\}} \mu^{2\delta} \langle \mathbf{T} P_\mu \operatorname{curl} \mathfrak{C}, P_\mu \operatorname{curl} \mathfrak{C} \rangle_e d\mu_e dt' \\ &= \int_{\widetilde{\mathcal{D}^+} \cap \{u \geq u_0\}} \mu^{2\delta} \langle [\mathbf{T}, P_\mu \operatorname{curl}] \mathfrak{C} + \operatorname{curl} P_\mu \mathbf{T} \mathfrak{C}, P_\mu \operatorname{curl} \mathfrak{C} \rangle_e d\mu_e dt'. \end{aligned}$$

Then

$$\begin{aligned} \sum_{\mu > 1} |\mathcal{B}_\mu| &\lesssim \sum_{\mu > 1} \int_0^t \|\mu^\delta [\mathbf{T}, P_\mu \operatorname{curl}] \mathfrak{C}\|_{l_\mu^2 L_x^2} \|\mu^\delta P_\mu \operatorname{curl} \mathfrak{C}\|_{l_\mu^2 L_x^2} \\ &\quad + \sum_{\mu > 1} \left| \int_{\widetilde{\mathcal{D}^+} \cap \{u \geq u_0\}} \mu^{2\delta} \langle \operatorname{curl} P_\mu \mathbf{T} \mathfrak{C}, P_\mu \operatorname{curl} \mathfrak{C} \rangle_e d\mu_e dt' \right|. \end{aligned} \quad (6.31)$$

The first term on the right can be bounded by using (3.27), (3.31) and (2.1),

$$\begin{aligned} \sum_{\mu > 1} \int_0^t \|\mu^\delta [\mathbf{T}, P_\mu \operatorname{curl}] \mathfrak{C}\|_{l_\mu^2 L_x^2} \|\mu^\delta P_\mu \operatorname{curl} \mathfrak{C}\|_{l_\mu^2 L_x^2} \\ \lesssim \int_0^t (\|\partial v\|_{B_{\infty,2,x}^\delta} + \|\partial v\|_{L_x^\infty}) \|\partial \mathfrak{C}\|_{H_x^\delta}^2 dt' \lesssim 1. \end{aligned} \quad (6.32)$$

The last line of (6.31) has the same integrand as $\sum_{\mu>1} |\mathcal{I}_\mu|$ with \mathcal{I}_μ defined in Lemma 3.8. We will repeat the procedure in the proof of Lemma 3.8. Due to the integral region is changed to $\widetilde{\mathcal{D}^+} \cap \{u \geq u_0\}$, the only difference is to control the two additional boundary terms on C_u generated by the integration by parts in (3.46), which are detailed below

$$\begin{aligned}\mathcal{J}_{\mu,b} &= \mu^{2\delta} \int_{t_{\min}}^t \int_{\Sigma_{t'} \cap \{u \geq u_0\}} \partial_m (P_\mu (\partial_n v^j \cdot \partial^n \Omega_j e^{-\varrho}) P_\mu \operatorname{curl} \mathfrak{E}^m) d\mu_e dt', \\ \mathcal{K}_{\mu,b} &= \int_{C_{u_0}} c^3 \mu^{2\delta} P_\mu (\partial_j \varrho \partial_m \Omega^j e^{-\varrho}) P_\mu \operatorname{curl} \mathfrak{E}^m d\mu_\gamma dt',\end{aligned}$$

where the second term is contributed by \mathcal{K}_μ in (3.46).

For the first term, in the exactly same calculation as in (6.29), we carry out integration by parts on $\Sigma_{t'} \cap \{u \geq u_0\}$,

$$\begin{aligned}\mathcal{J}_{\mu,b} &= \int_{\widetilde{\mathcal{D}^+} \cap \{u \geq u_0\}} \mu^{2\delta} \operatorname{div}_g (c^3 P_\mu (\partial_n v^j \cdot \partial^n \Omega_j e^{-\varrho}) P_\mu \operatorname{curl} \mathfrak{E}) d\mu_g dt' \\ &= \mu^{2\delta} \int_{C_{u_0}} c^3 P_\mu (\partial_n v^j \cdot \partial^n \Omega_j e^{-\varrho}) P_\mu \operatorname{curl} \mathfrak{E}^m \mathbf{N}^n c^{-2} \delta_{mn} d\mu_\gamma dt' .\end{aligned}$$

Hence

$$\sum_{\mu>1} (|\mathcal{J}_{\mu,b}| + |\mathcal{K}_{\mu,b}|) \lesssim \|\mu^\delta P_\mu \operatorname{curl} \mathfrak{E}\|_{l_\mu^2 L^2(C_{u_0})} \|\mu^\delta P_\mu ((\partial v, \partial \varrho) \cdot \partial \Omega e^{-\varrho})\|_{l_\mu^2 L^2(C_{u_0})}. \quad (6.33)$$

We then use (5.27) combined with (10.14) to treat the last term

$$\begin{aligned}\|\mu^\delta P_\mu ((\partial v, \partial \varrho) \cdot \partial \Omega e^{-\varrho})\|_{l_\mu^2 L^2(C_{u_0})} &\lesssim T^{\frac{1}{2}} \|\Lambda^{\frac{1}{2}+\delta} ((\partial v, \partial \varrho) \cdot \partial \Omega e^{-\varrho})\|_{L_t^\infty L_x^2} \\ &\lesssim T^{\frac{1}{2}} (\|\Lambda^{1+\delta} (\partial v, \partial \varrho)\|_{L_t^\infty L_x^2} \|\partial \Omega e^{-\varrho}\|_{L_t^\infty H_x^1} \\ &\quad + \|\partial v, \partial \varrho\|_{L_t^\infty H_x^1} \|\Lambda^{1+\delta} (\partial \Omega e^{-\varrho})\|_{L_t^\infty L_x^2}).\end{aligned} \quad (6.34)$$

Note that for the last term in the above inequality, we can apply (3.28) to $F = \partial \Omega$ to bound

$$\|\Lambda^{1+\delta} (\partial \Omega e^{-\varrho})\|_{L_t^\infty L_x^2} \lesssim \|\partial \Omega\|_{H_x^{1+\delta}}.$$

Substituting the above inequality to (6.34) gives

$$\begin{aligned}&\|\mu^\delta P_\mu ((\partial v, \partial \varrho) \cdot \partial \Omega e^{-\varrho})\|_{l_\mu^2 L^2(C_u)} \\ &\lesssim T^{\frac{1}{2}} (\|\Lambda^{1+\delta} (\partial v, \partial \varrho)\|_{L_t^\infty L_x^2} \|\partial \Omega e^{-\varrho}\|_{L_t^\infty H_x^1} + \|\partial v, \partial \varrho\|_{L_t^\infty H_x^1} \|\partial \Omega\|_{L_t^\infty H_x^{1+\delta}}) \\ &\lesssim 1,\end{aligned}$$

where we employed Corollary 3.3, (3.8) and (3.26).

Hence combining the above inequality with (6.33), we conclude

$$\sum_{\mu>1} (|\mathcal{J}_{\mu,b}| + |\mathcal{K}_{\mu,b}|) \lesssim \|\mu^\delta P_\mu \operatorname{curl} \mathfrak{E}\|_{l_\mu^2 L^2(C_{u_0})}. \quad (6.35)$$

Note the change of the integral region of \mathcal{I}_μ in (3.40) to $\widetilde{\mathcal{D}^+} \cap \{u \geq u_0\}$ only requires us to provide the above two additional estimates. The remaining estimate for the term is carried out exactly as in Lemma 3.8, and controlled by the same bound. Then by the estimate in Lemma 3.8, in

view of (6.31), (6.32) and using (3.31), we derive

$$\begin{aligned}
\sum_{\mu>1} |\mathcal{B}_\mu| &\lesssim \sum_{\mu>1} (|\mathcal{J}_{\mu,b}| + |\mathcal{K}_{\mu,b}|) + 1 + \int_0^t (\|\operatorname{curl} \mathfrak{C}\|_{\dot{H}_x^\delta} + 1) \{ \|\partial v\|_{B_{\infty,2,x}^\delta} + \|\partial \varrho\|_{L_x^\infty} \\
&\quad + (1 + \|\partial v\|_{L_x^\infty}) (\|\operatorname{curl} \mathfrak{C}\|_{\dot{H}_x^\delta} + 1) \} dt' + \sup_{0 \leq t' \leq t} \|\Lambda^\delta \operatorname{curl} \mathfrak{C}\|_{L_x^2}(t') \\
&\lesssim 1 + \|\mu^\delta P_\mu \operatorname{curl} \mathfrak{C}\|_{l_\mu^2 L^2(C_{u_0})},
\end{aligned}$$

where we used (6.35), Corollary 3.4, (2.1) and (3.26) in the above.

Recall from (6.30), we can conclude by using (3.26) that

$$\begin{aligned}
\sum_{\mu>1} \left| \int_{C_{u_0}} c^3 \mu^{2\delta} |P_\mu \operatorname{curl} \mathfrak{C}|_e^2 d\mu_\gamma dt \right| \\
\lesssim 1 + \|\mu^\delta P_\mu \operatorname{curl} \mathfrak{C}\|_{l_\mu^2 L^2(C_{u_0})} + \sup_{t' \leq t} \int_{\Sigma_{t'}} \sum_{\mu>1} \mu^{2\delta} |P_\mu \operatorname{curl} \mathfrak{C}|^2 d\mu_e \\
\lesssim 1 + \|\mu^\delta P_\mu \operatorname{curl} \mathfrak{C}\|_{l_\mu^2 L^2(C_{u_0})}.
\end{aligned}$$

This implies (6.25). Hence the proof of Proposition 6.6 is complete. \square

7. FUNDAMENTAL STRUCTURES FOR THE CAUSAL GEOMETRY OF THE ACOUSTICAL SPACETIME

The analysis of the causal geometry is mainly on controlling the derivatives of the optical function u . The first derivative of u is already encoded in the null tetrad. The Hessian of the optical function u defined in (5.1) is usually decomposed into the connection coefficients (5.9) of the null tetrad $\{L, \underline{L}, e_A, A = 1, 2\}$. Among them, it is important to control the null second fundamental form χ and the torsion ζ . $\operatorname{tr} \chi$ is usually controlled by using the Raychaudhuri equation (see (7.12)), which contains the Ricci component \mathbf{R}_{LL} ; while for ζ and $\hat{\chi}$, one may rely on the Hodge system on $S_{t,u}$ (see (7.17), (7.18) and (7.16)), due to the limited regularity on the Riemann curvature. The estimates on them are coupled together via a bootstrap argument. The control of $\operatorname{tr} \chi$ plays a more fundamental role since it is crucial for guaranteeing the coordinate system by t, u and $\omega \in \mathbb{S}^2$ to be well-defined in $\widetilde{\mathcal{D}^+}$, as explained in Section 5. To control $\operatorname{tr} \chi$, it typically relies on the specific structure in \mathbf{R}_{LL} to gain better regularity over the general control from the spacetime metric, c.f. [18]-[21] and [34, 43, 45].

Due to the decoupling method and a series of cancellations, in our work the regularity of the general spacetime metric is maintained at the same level as in [45], under our assumption of the data. The main defect caused by the rough vorticity derivative actually occurs on the acoustical null cone, since $\operatorname{curl} \Omega$ is the main uncontrollable term in \mathbf{R}_{LL} , appearing in the Raychaudhuri equation, which at the first glance would fail the estimate on $\operatorname{tr} \chi$, and collapse any further analysis on the cone.

In order to gain the sufficient regularity for proving Theorem 5.3 as well as showing (5.21)-(5.23), we have to investigate at a much more delicate level on the specific structure of the acoustical metric. Here we uncover two fundamental structures: one is given in (7.31) on $\nabla(\operatorname{curl} \Omega_{\mathbf{N}})$; the other is given in (7.40) on $k_{\mathbf{NN}} - \frac{1}{2} \Xi_L$, where Ξ is a one-form defined in (7.26). Both of the structures will play crucial roles for the analysis of $\operatorname{tr} \chi$ in Section 8.

We start with recalling the basic calculations by virtue of the null tetrad $\{L, \underline{L}, e_A, A = 1, 2\}$, which appeared in [18, 45] for the spacetime therein.

Proposition 7.1.

$$\mathbf{D}_A e_4 = \chi_{AB} e_B - k_{AN} e_4 \quad \mathbf{D}_A e_3 = \underline{\chi}_{AB} e_B + k_{AN} e_3 \quad (7.1)$$

$$\mathbf{D}_4 e_4 = -k_{NN} e_4 \quad \mathbf{D}_4 e_3 = 2\underline{\zeta}_A e_A + k_{NN} e_3 \quad (7.2)$$

$$\mathbf{D}_3 e_4 = 2\zeta_A e_A + k_{NN} e_4 \quad \mathbf{D}_4 e_A = \nabla_L e_A + \underline{\zeta}_A e_4 \quad (7.3)$$

$$\mathbf{D}_B e_A = \nabla_B e_A + \frac{1}{2} \chi_{AB} e_3 + \frac{1}{2} \underline{\chi}_{AB} e_4 \quad \mathbf{D}_3 e_3 = (-2\zeta_A + 2k_{NA}) e_A - k_{NN} e_3 \quad (7.4)$$

$$\chi_{AB} = \theta_{AB} - k_{AB}, \quad \underline{\zeta}^A = -k_{\mathbf{N}}^A, \quad \zeta^A = \nabla^A \log \mathbf{b} + k_{\mathbf{N}}^A, \quad (7.5)$$

where $e_4 = L$ and $e_3 = \underline{L}$.

As a direct consequence of (7.1)-(7.4), the following decomposition holds under the null tetrad.

Corollary 7.2. *Let $h = \frac{1}{2} \text{tr} \chi$ and $\underline{h} = \frac{1}{2} \text{tr} \underline{\chi}$. For a scalar function f , there holds*

$$\square_{\mathbf{g}} f = \Delta f - L \underline{L} f - (h - k_{NN}) \underline{L} f - \underline{h} L f + 2\underline{\zeta}^A \nabla_A f. \quad (7.6)$$

7.0.1. *Commutation formulas.* We recall the following commutation relations used in [45, Section 5] (see also in [18, 20]).

Proposition 7.3. (1) *There holds for the scalar functions f that*

$$[L, \mathbf{T}] f = \frac{1}{2} [L, \underline{L}] f = (\underline{\zeta}^A - \zeta^A) \nabla_A f - k_{NN} \mathbf{N} f. \quad (7.7)$$

(2) *There holds for $S_{t,u}$ -tangent m -covariant tensor fields U_A that*

$$\begin{aligned} & \nabla_L \nabla_B U_A - \nabla_B \nabla_L U_A \\ &= -\chi_{BC} \cdot \nabla_C U_A + \sum_i (\chi_{A_i B} \underline{\zeta}_C - \chi_{BC} \underline{\zeta}_{A_i} + \mathbf{R}_{A_i C 4 B}) U_{A_1 \dots \check{C} \dots A_m} \end{aligned} \quad (7.8)$$

and for any scalar function f there holds

$$[L, \nabla_A] f = -\chi_{AB} \nabla_B f. \quad (7.9)$$

Consequently, for any scalar function f there holds

$$L \Delta f + \text{tr} \chi \Delta f = \Delta L f - 2\hat{\chi} \cdot \nabla^2 f - \nabla_A \chi_{AC} \nabla_C f + (\text{tr} \chi \underline{\zeta}_C - \chi_{AC} \underline{\zeta}_A - \delta^{AB} \mathbf{R}_{CA 4 B}) \nabla_C f. \quad (7.10)$$

7.0.2. *Null Structure Equations.* We will rely heavily on the following structure equations for analysing the connection coefficients on null hypersurfaces C_u in $\widetilde{\mathcal{D}^+}$ (see [10, Chapter 7], [18] and [45, Section 5]):

Proposition 7.4 (Transport equations and Hodge systems for connection coefficients).

$$L \mathbf{b} = -\mathbf{b} k_{NN}, \quad (7.11)$$

$$L \text{tr} \chi + \frac{1}{2} (\text{tr} \chi)^2 = -|\hat{\chi}|^2 - k_{NN} \text{tr} \chi - \mathbf{R}_{44}, \quad (7.12)$$

$$\nabla_L \hat{\chi}_{AB} + \frac{1}{2} \text{tr} \chi \hat{\chi}_{AB} = -k_{NN} \hat{\chi}_{AB} - (\mathbf{R}_{4A 4 B} - \frac{1}{2} \mathbf{R}_{44} \delta_{AB}), \quad (7.13)$$

$$L \text{tr} \underline{\chi} + \frac{1}{2} \text{tr} \chi \text{tr} \underline{\chi} = 2 \hat{\chi} \cdot \underline{\chi} + k_{NN} \text{tr} \underline{\chi} - \hat{\chi} \cdot \hat{\chi} + 2|\underline{\zeta}|^2 + \delta^{AB} \mathbf{R}_{A 3 4 B}, \quad (7.14)$$

$$\nabla_L \zeta + \frac{1}{2} \text{tr} \chi \zeta = -(k_{BN} + \zeta_B) \hat{\chi}_{AB} - \frac{1}{2} \text{tr} \chi k_{AN} - \frac{1}{2} \mathbf{R}_{A 4 4 3}, \quad (7.15)$$

$$(d\dot{h}v\hat{\chi})_A + \hat{\chi}_{AB} \cdot k_{B\mathbf{N}} = \frac{1}{2}(\nabla \text{tr}\chi + k_{A\mathbf{N}}\text{tr}\chi) + \mathbf{R}_{B4BA}, \quad (7.16)$$

$$d\dot{h}v\zeta = \frac{1}{2}(\mu - k_{\mathbf{N}\mathbf{N}}\text{tr}\chi - 2|\zeta|^2 - |\hat{\chi}|^2 - 2k_{AB}\hat{\chi}_{AB}) - \frac{1}{2}\delta^{AB}\mathbf{R}_{A43B}, \quad (7.17)$$

$$c\dot{h}rl\zeta = -\frac{1}{2}\hat{\chi} \wedge \hat{\chi} + \frac{1}{2}\epsilon^{AB}\mathbf{R}_{B43A}, \quad (7.18)$$

$$\begin{aligned} \nabla_{\underline{L}}\hat{\chi}_{AB} + \frac{1}{2}\text{tr}\chi\hat{\chi}_{AB} &= -\frac{1}{2}\text{tr}\chi\hat{\chi}_{AB} + 2\nabla_A\zeta_B - d\dot{h}v\zeta\delta_{AB} + k_{\mathbf{N}\mathbf{N}}\hat{\chi}_{AB} \\ &\quad + (2\zeta_A\zeta_B - |\zeta|^2\delta_{AB}) + \mathbf{R}_{A43B} - \frac{1}{2}\delta^{CD}\mathbf{R}_{C43D}\delta_{AB}, \end{aligned} \quad (7.19)$$

where the mass aspect function $\mu := \underline{L}\text{tr}\chi + \frac{1}{2}\text{tr}\chi\text{tr}\chi$; and for an $S_{t,u}$ -tangent tensor field F , $\nabla_L F := L^\mu \mathbf{D}_\mu F$, with \mathbf{D} the covariant derivative of $(\mathcal{M}, \mathbf{g})$.

There holds the following Hodge system on $S_{t,u}$ contained in $\widetilde{\mathcal{D}^+}$

$$d\dot{h}v(\zeta - \underline{\zeta}) = |\underline{\zeta}|^2 - |\zeta|^2 - (\underline{\zeta} - \zeta)\nabla\varphi, \quad c\dot{h}rl(\zeta - \underline{\zeta}) = -2c\dot{h}rl\underline{\zeta} \quad (7.20)$$

with $\varphi = \log \sqrt{|\gamma|} - \log \sqrt{|\tilde{\gamma}|}$ with $\tilde{\gamma} = (t - u)^2\gamma^{(0)}$.

The schematic form of (7.20) was used in [45, Section 5] crucially to provide the control of ζ , since the regularity of null cones in [45] is much weaker than the previous works. The explicit form in (7.20) shows clearly the relation of ζ and $\underline{\zeta}$ which simplifies the control on ζ in Section 8.

Proof. The majority of the above equations has appeared for a couple of times in literature. We only prove the new formula (7.20).

By using the last two equations in (7.5), we have $\zeta + \underline{\zeta} = \nabla \log \mathbf{b}$. Since the right hand side of this identity vanishes after taking $c\dot{h}rl$, the $c\dot{h}rl$ equation follows as a direct consequence.

Note $L(t - u) = 1$ and $\underline{L}(t - u) = 1 - 2\mathbf{b}^{-1}$. To show the first equation, we use the definition $\varphi = \log \sqrt{|\gamma|} - \log \sqrt{|\tilde{\gamma}|}$ to compute

$$L\varphi = \text{tr}\chi - \frac{2}{\tilde{r}}, \quad \underline{L}\varphi = \text{tr}\chi + (2\mathbf{b}^{-1} - 1)\frac{2}{\tilde{r}}, \quad \mathbf{N}\varphi = \text{tr}\theta - \frac{2\mathbf{b}^{-1}}{\tilde{r}}, \quad (7.21)$$

where the last one can be derived by the first two by using $2\mathbf{N} = L - \underline{L}$. Hence, due to (7.7),

$$[L, \underline{L}]\varphi = 2(\underline{\zeta}_A - \zeta_A)\nabla_A\varphi - 2k_{\mathbf{N}\mathbf{N}}(\text{tr}\theta - \frac{2\mathbf{b}^{-1}}{\tilde{r}}).$$

Hence in view of (7.21), we can derive

$$L\text{tr}\chi - \underline{L}\text{tr}\chi + L((2\mathbf{b}^{-1} - 1)\frac{2}{\tilde{r}}) + \underline{L}(\frac{2}{\tilde{r}}) = -2k_{\mathbf{N}\mathbf{N}}(\text{tr}\theta - \frac{2\mathbf{b}^{-1}}{\tilde{r}}) + 2(\underline{\zeta} - \zeta) \cdot \nabla\varphi. \quad (7.22)$$

Since we can directly check

$$L((2\mathbf{b}^{-1} - 1)\frac{2}{\tilde{r}}) + \underline{L}(\frac{2}{\tilde{r}}) = 4\tilde{r}^{-1}L(\mathbf{b}^{-1}) = 4\tilde{r}^{-1}\mathbf{b}^{-1}k_{\mathbf{N}\mathbf{N}},$$

substituting the identity to (7.22) leads to a cancellation with the right hand side,

$$L\text{tr}\chi - \underline{L}\text{tr}\chi = -2k_{\mathbf{N}\mathbf{N}}\text{tr}\theta + 2(\underline{\zeta} - \zeta) \cdot \nabla\varphi. \quad (7.23)$$

Next, we substitute (7.14) and (7.17) to the left hand side of the above identity. Using

$$\underline{\chi} - \chi = -2\theta, \quad \underline{\chi}_{AB} + \chi_{AB} = -2k_{AB} \quad (7.24)$$

we can obtain

$$2d\dot{h}v(\underline{\zeta} - \zeta) - 2\text{tr}\theta k_{\mathbf{N}\mathbf{N}} + 2(|\underline{\zeta}|^2 - |\zeta|^2) = L\text{tr}\chi - \underline{L}\text{tr}\chi = -2k_{\mathbf{N}\mathbf{N}}\text{tr}\theta + 2(\underline{\zeta} - \zeta) \cdot \nabla\varphi.$$

This implies the first equation in (7.20). \square

7.0.3. *Structures of Ricci curvature in the acoustical spacetime.* As seen in (5.21), we need to provide control on L_x^∞ norm of $\text{tr}\chi - \frac{2}{r}$. To achieve this end, the fundamental structure uncovered in [17] and [18] gives an important decomposition for \mathbf{R}_{LL} , based on the following formula of Ricci component under the cartesian coordinates,

$$\mathbf{R}_{\alpha\beta} = -\frac{1}{2}\square_{\mathbf{g}}(\mathbf{g}_{\alpha\beta}) + \frac{1}{2}(\mathbf{D}_\alpha\Xi_\beta + \mathbf{D}_\beta\Xi_\alpha) + S_{\alpha\beta}, \quad (7.25)$$

where Ξ is a 1-form defined by

$$\Xi_\gamma = (\Gamma_{\alpha\beta}^\eta - \hat{\Gamma}_{\alpha\beta}^\eta)\mathbf{g}^{\alpha\beta}\mathbf{g}_{\gamma\eta}, \quad (7.26)$$

with $\hat{\Gamma}$ being the Christoffel symbol of a smooth reference metric $\hat{\mathbf{g}}$. For convenience, $\hat{\mathbf{g}}$ is chosen to be the Minkowski metric \mathbf{m} . Under this choice, the symmetric two-tensor field $S_{\alpha\beta}$ is quadratic in $\partial\mathbf{g}$. Since the Christoffel symbol verifies

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}(\partial_\beta\mathbf{g}_{\alpha\gamma} + \partial_\alpha\mathbf{g}_{\beta\gamma} - \partial_\gamma\mathbf{g}_{\alpha\beta}),$$

we can directly compute

$$\Xi_\gamma = \mathbf{g}^{\alpha\beta}(\partial_\alpha\mathbf{g}_{\beta\gamma} - \frac{1}{2}\partial_\gamma\mathbf{g}_{\alpha\beta}). \quad (7.27)$$

Now by adopting the decomposition of $\mathbf{R}_{\alpha\beta}$ in (7.25), we will show in (7.30) that there holds for the component of Ricci curvature \mathbf{R}_{44} of the acoustical metric that

$$\mathbf{R}_{44} = L(\Xi_L) - e^\varrho\delta_{ij}\mathbf{N}^j\text{curl}\Omega^i + \mathcal{Q}(\partial\mathbf{g}, \partial\mathbf{g}),$$

where $\Xi_L = \Xi_\mu L^\mu$, (alternatively, $\Xi_L = \Xi_4$) and the last term is quadratic in $\partial\mathbf{g}$. For the angular derivative of $\text{curl}\Omega^i\mathbf{N}^j\delta_{ij}$, we will obtain a trace decomposition.

Recall from (1.3) that under the Cartesian coordinate frame $\partial_t = \partial_0, \partial_i, i = 1, 2, 3$

$$\begin{aligned} \mathbf{g}_{00} &= -1 + c^{-2}|v|^2, & \mathbf{g}_{0i} &= -c^{-2}v_i, & \mathbf{g}_{ij} &= c^{-2}\delta_{ij} \\ \mathbf{g}^{00} &= -1, & \mathbf{g}^{0i} &= -v^i & \mathbf{g}^{ij} &= c^2\delta^{ij} - v^iv^j. \end{aligned} \quad (7.28)$$

We denote by $\tilde{\pi} = f(\mathbf{g})\partial\mathbf{g}$ with f a smooth function, and $\pi = \tilde{\pi} \cdot X$, that is the contraction of $\tilde{\pi}$ with the metric \mathbf{g} by the tensor fields L, \underline{L} or Π in (5.7) denoted in general by X .

We first prove the following decompositions by direct calculations.

Proposition 7.5 (Decompositions of Ricci components).

$$\mathbf{R}_{34} = -c^{-2}\mathbf{D}^\alpha(v^i)\mathbf{D}_\alpha(v^j)\delta_{ij} + \frac{1}{2}c^2\square_{\mathbf{g}}(c^{-2}) + \frac{1}{2}(\mathbf{D}_L\Xi_{\underline{L}} + \mathbf{D}_{\underline{L}}\Xi_L) + S_{\mathbf{T}\mathbf{T}} - S_{\mathbf{N}\mathbf{N}}, \quad (7.29)$$

$$\begin{aligned} \mathbf{R}_{44} &= -\exp\varrho\mathbf{N}^j\text{curl}\Omega_j + \delta^{ij}c^{-2}\mathbf{N}^j\mathcal{Q}^i - \frac{1}{2}c^2\square_{\mathbf{g}}(c^{-2}) + \mathbf{D}_L\Xi_L + S_{44} \\ &\quad - c^{-2}\mathbf{D}^\alpha(v^i)\mathbf{D}_\alpha(v^j)\delta_{ij} + 2\delta_{ij}\mathbf{N}^j\mathbf{D}^\alpha(c^{-2})\mathbf{D}_\alpha(v^i), \end{aligned} \quad (7.30)$$

$$\Pi^{ij}\partial_j(\mathbf{N}^m\text{curl}\Omega_m) = \Pi_l^i\mathbf{D}_L(\Pi^{jl}\text{curl}\Omega_j) + \Pi^{ij}\epsilon_{j\mathbf{m}}^l\text{curl}^2\Omega_l\mathbf{N}^m + \partial\Omega \cdot (\chi + \pi) \cdot X, \quad (7.31)$$

$$\Pi^{ij}\mathbf{R}_{ij} = -c^2\square_{\mathbf{g}}(c^{-2}) + \nabla_A\Xi^A + \text{tr}\theta\Xi_{\mathbf{N}} + \pi \cdot \pi, \quad (7.32)$$

where Π is defined in (5.7), $\cdot X$ means contracted by the combination of null vector fields L, \underline{L} , or Π .

Remark 7.6. Note that for a Σ_t -tangent tensor F , denoting by ∇ the Levi-Civita connection of g for short,

$$\nabla_A F_B = \nabla_A F_B + \theta_{AB}F_{\mathbf{N}}. \quad (7.33)$$

Thus, by using ${}^{(g)}\Gamma$ to represent the Christoffel symbol of the Riemannian metric g , we can derive

$$\begin{aligned}\mathbf{N}^i \operatorname{curl} F_i &= \mathbf{N}^i \epsilon_i^{mn} \partial_m (F_n) = \mathbf{N}^i \epsilon_i^{AB} (\nabla_A F_B + {}^{(g)}\Gamma_{mn}^l F_l e_A^m e_B^n) \\ &= \epsilon^{AB} (\nabla_A F_B + \theta_{AB} F_N + {}^{(g)}\Gamma_{mn}^l F_l e_A^m e_B^n) = \epsilon^{AB} \nabla_A F_B,\end{aligned}\quad (7.34)$$

where other terms are cancelled since ϵ^{AB} is the volume form of $(S_{t,u}, \gamma)$, anti-symmetric about $A, B = 1, 2$.

Therefore the first term on the right of (7.30) is

$$\exp \varrho \mathbf{N}^i \operatorname{curl} \Omega_i = \exp \varrho \epsilon^{AB} \nabla_A \Omega_B, \quad (7.35)$$

which does not directly take the form of $\nabla_L P + E$ with the scalar functions P and E verifying good estimates.

Proof of Proposition 7.5. We first compute by using (7.25)

$$\begin{aligned}\mathbf{R}_{34} &= \mathbf{R}_{\mathbf{T}\mathbf{T}} - \mathbf{R}_{\mathbf{N}\mathbf{N}} \\ &= -\frac{1}{2}(\mathbf{T}^\alpha \mathbf{T}^\beta - \mathbf{N}^\alpha \mathbf{N}^\beta) \square_{\mathbf{g}} \mathbf{g}_{\alpha\beta} + \frac{1}{2}(\mathbf{D}_L \Xi_L + \mathbf{D}_L \Xi_L) + S_{\mathbf{T}\mathbf{T}} - S_{\mathbf{N}\mathbf{N}}.\end{aligned}$$

In view of (7.28), we compute

$$\begin{aligned}\mathbf{T}^\alpha \mathbf{T}^\beta \square_{\mathbf{g}} \mathbf{g}_{\alpha\beta} &= \mathbf{T}^0 \mathbf{T}^0 \square_{\mathbf{g}} \mathbf{g}_{00} + 2\mathbf{T}^0 \mathbf{T}^i \square_{\mathbf{g}} \mathbf{g}_{0i} + \mathbf{T}^i \mathbf{T}^j \square_{\mathbf{g}} \mathbf{g}_{ij} \\ &= \square_{\mathbf{g}}(-1 + c^{-2}|v|^2) - 2v^i \square_{\mathbf{g}}(c^{-2}v_i) + v^i v^j \delta_{ij} \square_{\mathbf{g}}(c^{-2}) \\ &= 2c^{-2} \mathbf{D}^\alpha(v^i) \mathbf{D}_\alpha(v^j) \delta_{ij};\end{aligned}$$

and

$$\mathbf{N}^\alpha \mathbf{N}^\beta \square_{\mathbf{g}} \mathbf{g}_{\alpha\beta} = \mathbf{N}^i \mathbf{N}^j \square_{\mathbf{g}} \mathbf{g}_{ij} = \mathbf{N}^i \mathbf{N}^j \delta_{ij} \square_{\mathbf{g}}(c^{-2}) = c^2 \square_{\mathbf{g}}(c^{-2}).$$

Thus we can obtain (7.29).

Next we calculate \mathbf{R}_{44} . Noting that $L = \mathbf{T} + \mathbf{N}$ gives $L^i = \mathbf{T}^i + \mathbf{N}^i = v^i + \mathbf{N}^i$, we have

$$\begin{aligned}L^\alpha L^\beta \square_{\mathbf{g}} \mathbf{g}_{\alpha\beta} &= L^0 L^0 \square_{\mathbf{g}} \mathbf{g}_{00} + 2L^0 L^i \square_{\mathbf{g}} \mathbf{g}_{0i} + L^i L^j \square_{\mathbf{g}} \mathbf{g}_{ij} \\ &= \square_{\mathbf{g}} \mathbf{g}_{00} + 2(v^i + \mathbf{N}^i) \square_{\mathbf{g}} \mathbf{g}_{0i} + (v^i + \mathbf{N}^i)(v^j + \mathbf{N}^j) \square_{\mathbf{g}} \mathbf{g}_{ij} \\ &= \square_{\mathbf{g}}(-1 + c^{-2}|v|^2) + 2(v^i + \mathbf{N}^i) \square_{\mathbf{g}}(-c^{-2}v_i) + |v + \mathbf{N}|_e^2 \square_{\mathbf{g}}(c^{-2}) \\ &= \square_{\mathbf{g}}(c^{-2}|v|^2) - 2v^i \square_{\mathbf{g}}(c^{-2}v_i) - 2\mathbf{N}^i \square_{\mathbf{g}}(c^{-2}v_i) + |v + \mathbf{N}|_e^2 \square_{\mathbf{g}}(c^{-2}) \\ &= -2\delta_{ij} \mathbf{N}^j \square_{\mathbf{g}}(c^{-2}v^i) + \delta_{ij}(2v^i \mathbf{N}^j + \mathbf{N}^i \mathbf{N}^j) \square_{\mathbf{g}}(c^{-2}) + 2c^{-2} \mathbf{D}^\alpha(v^i) \mathbf{D}_\alpha(v^j) \delta_{ij} \\ &= -2\delta_{ij} c^{-2} \mathbf{N}^j \square_{\mathbf{g}} v^i + c^2 \square_{\mathbf{g}}(c^{-2}) + 2c^{-2} \mathbf{D}^\alpha(v^i) \mathbf{D}_\alpha(v^j) \delta_{ij} - 4\delta_{ij} \mathbf{N}^j \mathbf{D}^\alpha(c^{-2}) \mathbf{D}_\alpha(v^i).\end{aligned}$$

Substituting (1.7) to the first term on the right hand side yields

$$\begin{aligned}L^\alpha L^\beta \square_{\mathbf{g}} \mathbf{g}_{\alpha\beta} &= 2 \exp \varrho \delta_{ij} \mathbf{N}^j \operatorname{curl} \Omega^i - 2\delta_{ij} c^{-2} \mathbf{N}^j \mathcal{Q}^i \\ &\quad + c^2 \square_{\mathbf{g}}(c^{-2}) + 2c^{-2} \mathbf{D}^\alpha(v^i) \mathbf{D}_\alpha(v^j) \delta_{ij} - 4\delta_{ij} \mathbf{N}^j \mathbf{D}^\alpha(c^{-2}) \mathbf{D}_\alpha(v^i).\end{aligned}$$

(7.30) follows by substituting the above identity to the formula (7.25) for \mathbf{R}_{LL} .

Now we prove (7.31).

$$\begin{aligned}\Pi^{ij} \partial_j (\mathbf{N}^m \operatorname{curl} \Omega_m) &= \Pi^{ij} (\partial_j \operatorname{curl} \Omega_m \mathbf{N}^m + \operatorname{curl} \Omega_m \partial_j \mathbf{N}^m) \\ &= \Pi^{ij} (\partial_m (\operatorname{curl} \Omega)_j \mathbf{N}^m) + \Pi^{ij} \epsilon_{jm}^l \operatorname{curl}^2 \Omega_l \mathbf{N}^m + \Pi^{ij} \operatorname{curl} \Omega_m \partial_j \mathbf{N}^m \\ &= \Pi^{ij} (L(\operatorname{curl} \Omega_j) - \mathbf{T}(\operatorname{curl} \Omega_j)) + \Pi^{ij} \epsilon_{jm}^l \operatorname{curl}^2 \Omega_l \mathbf{N}^m + \Pi^{ij} \operatorname{curl} \Omega_m \partial_j \mathbf{N}^m.\end{aligned}$$

By using (1.9) and the first equation in (1.4), we have

$$\mathbf{T} \operatorname{curl} \Omega_j = \mathbf{T} \varrho \operatorname{curl} \Omega_j + \partial v \partial \Omega = \partial v \partial \Omega. \quad (7.36)$$

For the other term,

$$\begin{aligned} \Pi_l^i \Pi^{jl} L(\operatorname{curl} \Omega)_j &= \Pi_l^i \Pi^{jl} (\mathbf{D}_L(\operatorname{curl} \Omega)_j + {}^{(\mathbf{g})} \Gamma \cdot \operatorname{curl} \Omega) \\ &= \Pi_l^i \mathbf{D}_L(\Pi^{jl} \operatorname{curl} \Omega_j) - \Pi_l^i \operatorname{curl} \Omega_j \mathbf{D}_L \Pi^{jl} + \Pi_l^i {}^{(\mathbf{g})} \Gamma \cdot \operatorname{curl} \Omega \\ &= \Pi_l^i \mathbf{D}_L(\Pi^{jl} \operatorname{curl} \Omega_j) + \operatorname{curl} \Omega_{\mathbf{N}} k_{\mathbf{N}j} \Pi^{ij} + \pi \cdot \operatorname{curl} \Omega. \end{aligned} \quad (7.37)$$

For deriving the last line, by using (7.2) and (7.5) we computed

$$\begin{aligned} \Pi_l^i \operatorname{curl} \Omega_j \mathbf{D}_L \Pi^{jl} &= \Pi_\nu^i \operatorname{curl} \Omega_\mu (\mathbf{D}_L \mathbf{T}^\mu \mathbf{T}^\nu + \mathbf{T}^\mu \mathbf{D}_L \mathbf{T}^\nu - \mathbf{D}_L \mathbf{N}^\mu \mathbf{N}^\nu - \mathbf{N}^\mu \mathbf{D}_L \mathbf{N}^\nu) \\ &= -\Pi_\nu^i \operatorname{curl} \Omega_l \mathbf{N}^l \mathbf{D}_L \mathbf{N}^\nu = -\operatorname{curl} \Omega_{\mathbf{N}} k_{\mathbf{N}j} \Pi^{ij}. \end{aligned}$$

Also using $\Pi^{ij} \partial_j \mathbf{N}^m = \Pi^{ij} (\nabla_j \mathbf{N}^m - {}^{(g)} \Gamma \cdot \mathbf{N}) = (\chi + \pi) \cdot X$, (7.31) can then be derived by combining (7.37) with (7.36).

At last we prove (7.32) by using (7.25) and (7.33),

$$\begin{aligned} \Pi^{ij} \mathbf{R}_{ij} &= -\frac{1}{2} \Pi^{ij} \square_{\mathbf{g}} \mathbf{g}_{ij} + \Pi^{ij} \mathbf{D}_i \Xi_j + \Pi^{ij} S_{ij} \\ &= -\frac{1}{2} \Pi^{ij} c^{-2} \delta_{ij} c^2 \square_{\mathbf{g}}(c^{-2}) + \nabla_A \Xi^A + \operatorname{tr} \theta \Xi_{\mathbf{N}} + \pi \cdot \pi \\ &= -c^2 \square_{\mathbf{g}}(c^{-2}) + \nabla_A \Xi^A + \operatorname{tr} \theta \Xi_{\mathbf{N}} + \pi \cdot \pi, \end{aligned}$$

which gives (7.32). \square

Besides the structure of Ricci curvature, we give an important cancellation between $k_{\mathbf{N}\mathbf{N}}$ and $\frac{1}{2} \Xi_4$.

Proposition 7.7.

$$\Xi_\mu \mathbf{T}^\mu = \operatorname{Tr} k, \quad (7.38)$$

$$\Xi_j = \partial_j (\log c - \varrho), \quad (7.39)$$

$$k_{\mathbf{N}\mathbf{N}} = \frac{1}{2} (\Xi_L - L(\log c + \varrho) - 2L(v)_{\mathbf{N}}), \quad (7.40)$$

where we denote, for any vector field Y , $Y(v)_{\mathbf{N}} = Y(v^i) \mathbf{N}^j \mathbf{g}_{ij}$.

Proof. We first compute $\Xi_\gamma \mathbf{T}^\gamma$ by using (7.27).

$$\begin{aligned} \Xi_{\mathbf{T}} &= \mathbf{g}^{\alpha\beta} (\partial_\alpha \mathbf{g}_{\beta\gamma} - \frac{1}{2} \partial_\gamma \mathbf{g}_{\alpha\beta}) \mathbf{T}^\gamma \\ &= \mathbf{g}^{\alpha\beta} \partial_\alpha \mathbf{g}_{\beta 0} \mathbf{T}^0 + \mathbf{g}^{\alpha\beta} \partial_\alpha \mathbf{g}_{\beta i} \mathbf{T}^i - \frac{1}{2} \mathbf{g}^{\alpha\beta} \mathbf{T}(\mathbf{g}_{\alpha\beta}). \end{aligned}$$

The last term on the right hand side can be computed as follows

$$\begin{aligned} \mathbf{g}^{\alpha\beta} \mathbf{T}(\mathbf{g}_{\alpha\beta}) &= 2(-v^i) \mathbf{T}(-c^{-2} v_i) + (c^2 \delta^{ij} - v^i v^j) \mathbf{T}(c^{-2} \delta_{ij}) - \mathbf{T}(-1 + c^{-2} |v|^2) \\ &= 2v^i \mathbf{T}(c^{-2} v_i) + c^2 \delta^{ij} \mathbf{T}(c^{-2}) \delta_{ij} - |v|^2 \mathbf{T}(c^{-2}) - \mathbf{T}(c^{-2} |v|^2) \\ &= -6 \mathbf{T} \log c. \end{aligned}$$

Now we compute the remaining terms

$$\begin{aligned}
& \mathbf{g}^{\alpha\beta} \partial_\alpha \mathbf{g}_{\beta 0} \mathbf{T}^0 + \mathbf{g}^{\alpha\beta} \partial_\alpha \mathbf{g}_{\beta i} \mathbf{T}^i \\
&= \mathbf{g}^{00} \partial_0 \mathbf{g}_{00} + \mathbf{g}^{0i} (\partial_0 \mathbf{g}_{i0} + \partial_i \mathbf{g}_{00}) + \mathbf{g}^{ij} \partial_i \mathbf{g}_{j0} \\
&+ \mathbf{g}^{00} \partial_0 \mathbf{g}_{0i} v^i + \mathbf{g}^{0j} (\partial_0 \mathbf{g}_{ij} + \partial_j \mathbf{g}_{0i}) v^i + \mathbf{g}^{lj} \partial_l \mathbf{g}_{ji} v^i \\
&= -\partial_0(-1 + c^{-2}|v|^2) - v^i (\partial_0(-c^{-2}v_i) + \partial_i(-1 + c^{-2}|v|^2)) + (c^2 \delta^{ij} - v^i v^j) \partial_i(-c^{-2}v_j) \\
&+ (-1) \partial_0(-c^{-2}v_i) v^i + (-v^j) v^i (\partial_0(c^{-2} \delta_{ij}) + \partial_j(-c^{-2}v_i)) + (c^2 \delta^{lj} - v^l v^j) \partial_l(c^{-2} \delta_{ij}) v^i \\
&= -\mathbf{T}(c^{-2}|v|^2) + v^i \mathbf{T}(c^{-2}v_i) + c^2 \delta^{ij} \partial_i(-c^{-2}v_j) + \mathbf{T}(c^{-2}v_i) v^i - v^j \mathbf{T}(c^{-2} \delta_{ij}) v^i \\
&+ c^2 \delta^{lj} \partial_l(c^{-2} \delta_{ij}) v^i \\
&= -\operatorname{div} v.
\end{aligned}$$

Combining the above two calculations yields (7.38) in view of (2.9).

We now consider (7.39).

$$\begin{aligned}
\mathbf{g}^{\alpha\beta} \partial_\alpha \mathbf{g}_{\beta j} &= \mathbf{g}^{00} \partial_0 \mathbf{g}_{0j} + \mathbf{g}^{0i} \partial_0 \mathbf{g}_{ij} + \mathbf{g}^{i0} \partial_i \mathbf{g}_{0j} + \mathbf{g}^{il} \partial_i \mathbf{g}_{lj} \\
&= -\partial_0(-c^{-2}v_j) + (-v^i) [\partial_0(c^{-2} \delta_{ij}) + \partial_i(-c^{-2}v_j)] + \mathbf{g}^{il} \partial_i(c^{-2} \delta_{lj}) \\
&= \partial_0(c^{-2}v_j) - v^i [\partial_0(c^{-2}) \delta_{ij} - \partial_i(c^{-2}v_j)] + (c^2 \delta^{il} - v^j v^l) \partial_i(c^{-2}) \delta_{lj} \\
&= c^{-2} \partial_0 v_j + v^i \partial_i(c^{-2}v_j) + (c^2 \delta^{il} - v^i v^l) \partial_i(c^{-2}) \delta_{lj} \\
&= c^{-2} \mathbf{T} v_j - 2 \partial_j \log c,
\end{aligned}$$

$$\begin{aligned}
\mathbf{g}^{\alpha\beta} \partial_j \mathbf{g}_{\alpha\beta} &= 2 \mathbf{g}^{0i} \partial_j \mathbf{g}_{0i} + \mathbf{g}^{00} \partial_j \mathbf{g}_{00} + \mathbf{g}^{il} \partial_j \mathbf{g}_{il} \\
&= -2v^i \partial_j(-c^{-2}v_i) - \partial_j(-1 + c^{-2}|v|^2) + (c^2 \delta^{il} - v^i v^l) \partial_j \mathbf{g}_{il} \\
&= 2v^i \partial_j(c^{-2}v_i) - \partial_j(c^{-2}|v|^2) + (c^2 \delta^{il} - v^i v^l) \partial_j(c^{-2} \delta_{il}) \\
&= -6 \partial_j \log c.
\end{aligned}$$

Combining the above calculations with (7.27) implies

$$\Xi_j = \mathbf{g}^{\alpha\beta} (\partial_\alpha \mathbf{g}_{\beta j} - \frac{1}{2} \partial_j \mathbf{g}_{\alpha\beta}) = c^{-2} \mathbf{T} v_j + \partial_j \log c$$

and the second equation in (1.4) gives (7.39).

From (7.38) and (7.39) we have

$$\Xi_\mu L^\mu = \operatorname{Tr} k + \mathbf{N}(\log c - \varrho). \quad (7.41)$$

Combining (7.41) with (2.9) yields

$$\Xi_L = 2 \mathbf{T}(\log c + \varrho) + L(\log c - \varrho). \quad (7.42)$$

We derive by using (2.8) that

$$k_{\mathbf{N}\mathbf{N}} = -c^{-2} \delta_{ij} \mathbf{N}(v^i) \mathbf{N}^j + \mathbf{T} \log c = -\mathbf{g}_{ij} \mathbf{N}(v^i) \mathbf{N}^j + \mathbf{T} \log c.$$

By using the second equation in (1.4) and (7.42), we have

$$\begin{aligned}
k_{\mathbf{N}\mathbf{N}} &= \mathbf{T} \log c - \mathbf{N} \varrho - L(v)_{\mathbf{N}} = \mathbf{T}(\log c + \varrho) - L \varrho - L(v)_{\mathbf{N}} \\
&= \frac{1}{2} (\Xi_L - L(\log c + \varrho)) - L(v)_{\mathbf{N}}.
\end{aligned}$$

This gives (7.40). The proof of Proposition 7.7 is complete. \square

Finally, we recall a result from the previous works [21, 43], and [45, Lemma 5.12].

Lemma 7.8 (Decomposition of Riemann curvature). *We denote by $\tilde{\pi} = f(\mathbf{g})\partial\mathbf{g}$ with f a smooth function, and $\pi = \tilde{\pi} \cdot X$. Let $\mathbf{A} = \tilde{\chi}, \text{tr}\chi - \frac{2}{r}, \pi$, and $\mathbf{E} = \mathbf{A} \cdot \pi + \text{tr}\chi \cdot \pi$.*

(i) *Let $\mathcal{D}_* = (\nabla, \nabla_L)$. There hold*

$$\mathbf{R}_{4A4B}, \mathbf{R}_{A443}, \mathbf{R}_{44}, \mathbf{R}_{4A} = \mathcal{D}_*\pi + \mathbf{E}.$$

(ii) *There exist scalar π , 1-form \mathbf{E} and $S_{t,u}$ tangent 2-vector π_{AB} such that*

$$\delta^{AB}\mathbf{R}_{CA4B} = \nabla_C\pi + \nabla^B\pi_{CB} + \mathbf{E}_C \quad \text{and} \quad \mathbf{R}_{CA4B} = \nabla\pi + \mathbf{E}.^{22}$$

(iii) *There exists 1-form π and scalar \mathbf{E} that $\mathbf{R}_{ABAB} = d\pi + \mathbf{E}$.*

(iv) *There exist 1-forms π and scalar functions \mathbf{E} such that*

$$\delta^{AB}\mathbf{R}_{B43A} = d\pi + \mathbf{E}, \quad \epsilon^{AB}\mathbf{R}_{A43B} = \text{curl}\pi + \mathbf{E}.$$

We recall the argument in [21, Proposition 4.1]. There holds under the coordinate frame $e_\alpha, e_\beta, e_\gamma, e_\delta$ in $(\mathcal{M}, \mathbf{g})$ the following decomposition,

$$\mathbf{R}_{\alpha\beta\gamma\delta} = \mathbf{D}_\alpha \overset{\circ}{\pi}_{\beta\delta\gamma} + \mathbf{D}_\beta \overset{\circ}{\pi}_{\alpha\gamma\delta} - \mathbf{D}_\alpha \overset{\circ}{\pi}_{\beta\gamma\delta} - \mathbf{D}_\beta \overset{\circ}{\pi}_{\delta\alpha\gamma} + E_{\alpha\beta\gamma\delta}$$

with $E = \mathbf{g} \cdot \tilde{\pi} \cdot \tilde{\pi}$ and $\overset{\circ}{\pi}_{\alpha\beta\gamma} = \partial_\gamma \mathbf{g}_{\alpha\beta}$. We contract the above identity by the null tetrad, and use Proposition 7.1 for the covariant derivatives on L, \underline{L}, Π . This gives the results in (i)-(iii).

The proof of (iv) needs a minor change due to the change of the spacetime metric. We compute with the help of Bianchi identity that

$$\delta^{AB}\mathbf{R}_{B43A} = \delta^{AB}(\mathbf{R}_{AB} - \delta^{CD}\mathbf{R}_{ACBD}), \quad \epsilon^{AB}\mathbf{R}_{AB43} = -2\epsilon^{AB}\mathbf{R}_{A43B}.$$

For $\delta^{AB}\mathbf{R}_{AB}$ we use $\delta^{AB}\mathbf{R}_{AB} = \nabla_A \Xi_A + \mathbf{E}$ which follows from (7.32) together with (1.8). We then can obtain (iv) by using (iii) and the above calculations.

8. CAUSAL GEOMETRY OF THE ACOUSTICAL SPACETIME

In this section, we establish a set of crucial estimates on connection coefficients in $\widetilde{\mathcal{D}^+}$ set up in Section 5 under the rescaled coordinates, that is $(t, x) \rightarrow (\lambda(t - t_k), \lambda x)$ as done in (4.15). Here $\lambda \geq \Lambda > 1$ with Λ sufficiently large and fixed. Recall that $\widetilde{\mathcal{D}^+}$ is contained in $I_* \times \mathbb{R}^3$ with $I_* = [0, \tau_*]$ and $\tau_* \leq \lambda^{1-8\epsilon_0}T$. For the rescaled components of the metric \mathbf{g} , we denote by $\tilde{\pi}$ the collection of terms taking the form of $\tilde{f}(\varrho, v)\partial(\varrho, v)$, with \tilde{f} smooth functions of its variables. In view of (7.28) this extends the class of $\tilde{\pi}$ in Lemma 7.8. According to (4.2) and Corollary 3.4, $\tilde{\pi}$ verifies

$$\|\tilde{\pi}\|_{L_t^2 L_x^\infty(I \times \mathbb{R}^3)} + \lambda^{\delta_0} \left(\sum_{\mu \geq 2} \mu^{2\delta_0} \|P_\mu \tilde{\pi}\|_{L_t^2 L_x^\infty(I \times \mathbb{R}^3)}^2 \right)^{\frac{1}{2}} \lesssim \lambda^{-1/2-4\epsilon_0}, \quad (8.1)$$

where $\delta_0 = s' - 2$, and P_μ is the Littlewood-Paley projection in (4.2). To derive the last inequality in (8.1) we combined (4.2) with applying Lemma 4.8 to $(f, G) = (\partial(\varrho, v), \tilde{f})$ and using (2.69), followed with rescaling. In the following sections we will work under the condition (8.1).

We fix the convention that

$$\begin{aligned} \tilde{r} &= t - u, & \widetilde{\text{tr}\chi} &= \text{tr}\chi + \Xi_4, & \mathfrak{U} &= \frac{\mathbf{b}^{-1} - 1}{\tilde{r}}, \\ z &= \widetilde{\text{tr}\chi} - \frac{2}{t - u}, & \mathscr{Y} &= \mathbf{b}(\widetilde{\text{tr}\chi} - \frac{2}{\mathbf{b}(t - u)}) = \mathbf{b}(z - 2\mathfrak{U}). \end{aligned} \quad (8.2)$$

Similar to [45, Lemma 5.1], we have the following results for the initial data along the null cones for the geometric quantities.

²²Here for π and \mathbf{E} we refer to their general symbolic definitions.

Lemma 8.1. *Let $t_{\min} = \max\{u, 0\}$.*

(i) *On any null cone C_u initiating from a point on the time axis Γ^+ at $t = u \geq 0$, there hold*

$$\tilde{r}^{\frac{3}{2}} \nabla z, \tilde{r}^2 \mu \rightarrow 0 \text{ as } t \rightarrow u, \quad \lim_{t \rightarrow u} \|\hat{\chi}, \zeta, \underline{\zeta}, \pi, \text{curl} \Omega, z, \frac{\mathbf{b}-1}{\tilde{r}}, \tilde{r} \nabla \mathcal{Y}\|_{L^\infty(S_{t,u})} < \infty.$$

Along any null cone C_u in $\widetilde{\mathcal{D}^+}$ with $u < 0$,

$$\mathcal{Y} \rightarrow 0 \text{ as } t \rightarrow t_{\min}.$$

(ii) *Let $\overset{\circ}{\gamma} := (t-u)^{-2} \gamma$ be the rescaled metric on $S_{t,u}$ and let $\gamma^{(0)}$ denote the canonical metric on \mathbb{S}^2 . Then, relative to the pull-back coordinates by the null geodesic flow $\Upsilon(t, \cdot, u) : \mathbb{S}^2 \rightarrow S_{t,u}$, there hold for $u \geq 0$ that*

$$\lim_{t \rightarrow u} \overset{\circ}{\gamma}_{ab} = \gamma_{ab}^{(0)}, \quad \lim_{t \rightarrow u} \partial_c \overset{\circ}{\gamma}_{ab} = \partial_c \gamma_{ab}^{(0)}, \quad (8.3)$$

where $a, b, c = 1, 2$; and there holds the estimate of (5.17) if $u < 0$.

(iii) *On $\bigcup_{\mathbf{v} \in (0, \mathbf{v}_*]} S_{\mathbf{v}}$ there hold $\mathbf{b} - a \rightarrow 0$, $|\mathbf{v}z| \lesssim \lambda^{-4\epsilon_0}$ and $\|\mathbf{v}^{\frac{3}{2}} \nabla z\|_{L^\infty_\omega L^p_\omega} + \|\mathbf{v}^{\frac{1}{2}} z\|_{L^\infty} \lesssim \lambda^{-\frac{1}{2}}$,*

where $0 < 1 - \frac{2}{p} < s - 2$, and $\|F\|_{L^\infty_\omega L^p_\omega} = \sup_{\mathbf{v} \in (0, \mathbf{v}_]} \left(\int_{S_{\mathbf{v}}} |F|^p d\omega \right)^{1/p}$ for any tensor field F .*

When $u \geq 0$, the proof is based on the local expansion of the geometric quantities at the vertex of the cone C_u . The items (i) and (ii) in Lemma 8.1 can be found from [38, 39] and [43, Section 2], if $u \geq 0$. If $u < 0$, the results are based on Proposition 5.1. The item (iii) also follows from Proposition 5.1.

Now we state the main results of this section. We may hide the range for u, t in \mathcal{D}^+ or $\widetilde{\mathcal{D}^+}$ for short and refer to Section 5.1 for their definitions.

Proposition 8.2. *Let p be a fixed number satisfying $0 < 1 - \frac{2}{p} < s' - 2$. Let $\mathcal{D}_* = (\nabla, \nabla_L)$. Under the assumption (4.2), there hold on $\widetilde{\mathcal{D}^+} \subset [0, \tau_*] \times \Sigma$ the estimates,*

$$\tilde{r} \widetilde{tr\chi} \approx 1, \|\tilde{r}^{\frac{1}{2}} z\|_{L^\infty(\widetilde{\mathcal{D}^+})} \lesssim \lambda^{-\frac{1}{2}} \quad (8.4)$$

$$\|\tilde{r}^{\frac{3}{2}} \nabla z\|_{L^\infty_t L^\infty_\omega L^p_\omega(\widetilde{\mathcal{D}^+})} \lesssim \lambda^{-\frac{1}{2}} \quad (8.5)$$

$$\|\tilde{r} \nabla z\|_{L^2_t L^p_\omega(C_u \cap \widetilde{\mathcal{D}^+})} + \|\tilde{r} \nabla \hat{\chi}\|_{L^2_t L^p_\omega(C_u \cap \widetilde{\mathcal{D}^+})} \lesssim \lambda^{-\frac{1}{2}} \quad (8.6)$$

$$\|z, \hat{\chi}, tr\chi - \frac{2}{\tilde{r}}, \zeta\|_{L^2_t L^q_x(\widetilde{\mathcal{D}^+})} \lesssim \lambda^{\frac{2}{q}-1-4\epsilon_0(\frac{4}{q}-1)}, \quad 2 < q < 4 \quad (8.7)$$

$$\left\| \frac{\mathbf{b}^{-1} - 1}{\tilde{r}} \right\|_{L^2_t L^\infty_\omega(C_u \cap \widetilde{\mathcal{D}^+})} + \left\| \frac{\mathbf{b}^{-1} - 1}{\tilde{r}^{\frac{1}{2}}} \right\|_{L^{2p}_\omega(C_u \cap \widetilde{\mathcal{D}^+})} + \|\tilde{r} \mathcal{D}_* \left(\frac{\mathbf{b}^{-1} - 1}{\tilde{r}} \right)\|_{L^2_t L^p_\omega(C_u \cap \widetilde{\mathcal{D}^+})} \lesssim \lambda^{-\frac{1}{2}} \quad (8.8)$$

and there holds in \mathcal{D}^+ (that is where $0 \leq u \leq t \leq \tau_$),*

$$\|z, \hat{\chi}, \zeta, tr\chi - \frac{2}{\tilde{r}}, \frac{\mathbf{b}^{-1} - 1}{\tilde{r}}\|_{L^2_t L^\infty_x} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}. \quad (8.9)$$

Proposition 8.3. *Let p be as fixed in Proposition 8.2. On the null cone C_u contained in $\widetilde{\mathcal{D}^+}$, there hold*

$$\|z\|_{L^2_t L^\infty_\omega(C_u \cap \widetilde{\mathcal{D}^+})} + \|\hat{\chi}\|_{L^2_t L^\infty_\omega(C_u \cap \widetilde{\mathcal{D}^+})} + \|\zeta\|_{L^2_t L^\infty_\omega(C_u \cap \widetilde{\mathcal{D}^+})} \lesssim \lambda^{-\frac{1}{2}}, \quad (8.10)$$

$$\|\partial_\omega(\overset{\circ}{\gamma} - \gamma^{(0)})\|_{L^p_\omega L^\infty_t(C_u \cap \widetilde{\mathcal{D}^+})} \lesssim \lambda^{-4\epsilon_0}, \quad \|\overset{\circ}{\gamma} - \gamma^{(0)}\|_{L^\infty} \lesssim \lambda^{-4\epsilon_0}, \quad (8.11)$$

where $\overset{\circ}{\gamma} = (t-u)^{-2} \gamma$.

The proof of the above results relies on a bootstrap argument. For p as fixed in Proposition 8.2, we make the bootstrap assumptions on any C_u contained in $\widetilde{\mathcal{D}^+}$,

$$\|\hat{\chi}\|_{L_t^2 L_\omega^\infty(C_u)} + \|z\|_{L_t^2 L_\omega^\infty(C_u)} + \|\zeta\|_{L_t^2 L_\omega^\infty(C_u)} \leq \lambda^{-\frac{1}{2} + \epsilon_0}, \quad (8.12)$$

$$\|\hat{\chi}\|_{L_t^2 L_\omega^\infty(C_u \cap \mathcal{D}^+)} + \|z\|_{L_t^2 L_\omega^\infty(C_u \cap \mathcal{D}^+)} + \|\zeta\|_{L_t^2 L_\omega^\infty(C_u \cap \mathcal{D}^+)} \leq \lambda^{-\frac{1}{2}}, \quad (8.13)$$

$$\|\partial_\omega(\overset{\circ}{\gamma} - \gamma^{(0)})\|_{L_t^\infty L_\omega^p} \leq \lambda^{-\epsilon_0}, \quad \|\overset{\circ}{\gamma} - \gamma^{(0)}\|_{L^\infty} \leq \lambda^{-\epsilon_0}. \quad (8.14)$$

We also assume that on any $S_{t,u} \subset \widetilde{\mathcal{D}^+}$, there hold

$$\|\mathrm{tr}\theta - \frac{2}{\tilde{r}}\|_{L^3(\Sigma_t \cap \widetilde{\mathcal{D}^+})} \leq 1, \quad \|(\lambda\tilde{r})^{\frac{1}{2}}(\nabla\mathbf{b}, \hat{\chi})\|_{L_\omega^q} \leq \lambda^{2\epsilon_0}, \quad (8.15)$$

$$|\mathbf{b} - 1| \leq \frac{1}{2}, \quad (8.16)$$

where $0 < 1 - \frac{2}{q} < s - 2$ is fixed.

(8.12) will be improved to (8.10), (8.13) will be improved in the estimates (8.9) at the end of this section. (8.14) will be improved to (8.11). (See Remark 8.17.)

By repeating the proof in [45, Lemma 5.4] with the help of the transport equations (7.11), (5.13), and the data in Lemma 8.1, we can derive as a direct consequence of the estimate of z in (8.12), (8.1) and (8.16) the following result

Lemma 8.4. *On $\widetilde{\mathcal{D}^+}$ there hold*

$$v_t \approx (t - u)^2, \quad (8.17)$$

$$|\mathbf{b} - 1| \lesssim \lambda^{-4\epsilon_0} < \frac{1}{4}. \quad (8.18)$$

Remark 8.5. In view of (8.1) and (7.5), (5.21) can be proved by (8.7). The first assumption in (5.23) and the second assumption in (5.22) are incorporated in (8.14) and (8.15). The first assumption in (5.22) and the second assumption in (5.23) are proved in Lemma 8.4.

Next we recall important inequalities for carrying out analysis. Using Lemma 8.4 and the second assumption in (8.14), we can show that on $\widetilde{\mathcal{D}^+}$ there hold the following Sobolev inequalities:

- For any $S_{t,u}$ -tangent tensor field F (including scalar field) and $2 < q < \infty$ there hold (see [10, 22, 18])

$$\|F\|_{L_\omega^q(S_{t,u})} \lesssim \|\tilde{r}\nabla F\|_{L_\omega^2(S_{t,u})}^{1-\frac{2}{q}} \|F\|_{L_\omega^2(S_{t,u})}^{\frac{2}{q}} + \|F\|_{L_\omega^2(S_{t,u})}, \quad (8.19)$$

$$\|F\|_{L_\omega^\infty(S_{t,u})} \lesssim \|r\nabla F\|_{L_\omega^q(S_{t,u})} + \|F\|_{L_\omega^2(S_{t,u})}, \quad (8.20)$$

which have appeared in Proposition 5.4 in a slightly different form.

- For $q \geq 2$, there holds for any $S_{t,u}$ -tangent tensor field F (including scalar field) on C_u contained in $\widetilde{\mathcal{D}^+}$

$$\|\tilde{r}^{\frac{1}{2}-\frac{1}{q}} F\|_{L_x^{2q} L_t^\infty(C_u)}^2 \lesssim \left\| \lim_{t \rightarrow t_{\min}} \tilde{r}^{\frac{1}{2}} F(t) \right\|_{L_\omega^{2q}}^2 + \|F\|_{L_\omega^\infty L_t^2} \left(\|\tilde{r}\nabla_L F\|_{L_\omega^q L_t^2} + \|F\|_{L_\omega^q L_t^2} \right). \quad (8.21)$$

See [43, Lemma 2.13], and the proof in [40, Section 8].

Hardy-Littlewood maximal function. For a scalar function $f(t)$ defined on $[0, \tau_*]$, its Hardy-Littlewood maximal function is defined by

$$\mathcal{M}(f)(t) = \sup_{0 \leq t' \leq \tau_*} \frac{1}{|t - t'|} \int_{t'}^t |f(\tau)| d\tau.$$

It is well-known that for any $1 < q < \infty$ there holds

$$\|\mathcal{M}(f)\|_{L_t^q} \lesssim \|f\|_{L_t^q}. \quad (8.22)$$

Using Lemma 8.4²³ and the second assumption in (8.14), we can obtain the following control along the null cones.

Proposition 8.6 (L^p Control of the flux). *Under the assumption (8.15), there hold for $0 \leq 1 - \frac{2}{q} < s' - 2$ on $\widetilde{\mathcal{D}^+}$ the following inequalities*

$$\|curl^2 \Omega\|_{L^2(C_u \cap \widetilde{\mathcal{D}^+})} + \|\tilde{r}^{1-\frac{2}{q}} curl^2 \Omega\|_{L_t^2 L_x^q(C_u \cap \widetilde{\mathcal{D}^+})} \lesssim \lambda^{-\frac{3}{2}} \quad (8.23)$$

$$\|\tilde{r}^{\frac{1}{2}} \partial \Omega\|_{L_\omega^{2q}(\Sigma_t \cap \widetilde{\mathcal{D}^+})} \lesssim \lambda^{-\frac{3}{2}}; \quad (8.24)$$

and with $\mathcal{D}_* = (\nabla, \nabla_L)$ and $0 \leq 1 - \frac{2}{p} < s - 2$ that

$$\|\tilde{r}^{1-\frac{2}{p}} \partial \tilde{\pi}\|_{L_u^2 L_x^p(\Sigma_t \cap \widetilde{\mathcal{D}^+})} \lesssim \lambda^{-\frac{1}{2}} \quad (8.25)$$

$$\|\tilde{\pi}\|_{L_u^2 L_\omega^p(\Sigma_t \cap \widetilde{\mathcal{D}^+})} + \|\tilde{r}^{\frac{1}{2}} \tilde{\pi}\|_{L^\infty L_\omega^{2p}(\Sigma_t \cap \widetilde{\mathcal{D}^+})} \lesssim \lambda^{-\frac{1}{2}} \quad (8.26)$$

$$\|\mathcal{D}_* \tilde{\pi}\|_{L^2(C_u \cap \widetilde{\mathcal{D}^+})} + \|\tilde{r}^{1-\frac{2}{p}} \mathcal{D}_* \tilde{\pi}\|_{L_t^2 L_x^p(C_u \cap \widetilde{\mathcal{D}^+})} \lesssim \lambda^{-\frac{1}{2}} \quad (8.27)$$

$$\|\tilde{r}(\nabla \pi, \nabla_L \pi), \pi\|_{L_t^2 L_\omega^p(C_u \cap \widetilde{\mathcal{D}^+})} \lesssim \lambda^{-\frac{1}{2}}, \quad (8.28)$$

where $\pi = \tilde{\pi} \cdot X$ with X being tensor fields L, \underline{L}, Π or the products of them.

For (8.25)-(8.28), by using Corollary 3.3, (5.30), (5.31), Proposition 6.2 and Proposition 6.3, for the case that $\tilde{\pi} = \partial \Phi$ with $\Phi = v, \varrho$, the proof of the above results is similar to [45, Lemma 5.5]; and the results for the general form of $\tilde{\pi}$ follow as the consequences as in [45, Proposition 5.6]. The proof for (8.27) and (8.28) in [45, Lemma 5.5] is based on the proof in [43, Proposition 2.6] under the assumption (8.15). This assumption will be proved by showing Proposition 8.11 shortly.

Recall that in Proposition 6.3, we did not provide the control on the flux of $\mathbf{T}v$ upto the highest order. This slightly influences the proof of (8.27). The terms of vorticity in (8.23) and (8.24) did not appear in the previous works on quasilinear wave equations. Thus we will focus on proving (8.23), (8.24), and (8.27) for the case $\tilde{\pi} = \partial(v, \varrho)$. The proof of (8.28) can be found in [45, Lemma 5.7] under the assumption (8.15), and using the estimates (8.27) and (8.1).

Proof. The estimates in (8.23) and (8.24) are consequences of (6.26) and the second estimate in (6.20) after rescaling.

To prove (8.27), we recall from [43, Proposition 2.5, Proposition 2.6] that

$$\begin{aligned} \|\tilde{r}^{1-\frac{2}{p}}(\nabla f, Lf)\|_{L_t^2 L_x^p(C_u)} &\lesssim \mathcal{F}^{\frac{1}{2}}[f](C_u) + \sum_{l>1} l^{1-\frac{2}{p}} \mathcal{F}^{\frac{1}{2}}[P_l f](C_u) \\ &\quad + \|f\|_{H^{s-1}(\Sigma_{\tau_*})} + \|f\|_{H^{s-1}(\Sigma_{t_{\min}})}. \end{aligned} \quad (8.29)$$

If the cone C_u is initiated from Γ^+ , the last term of the last line actually vanishes.

We can apply the above inequality to $f = \partial v, \partial \varrho$ to obtain for $0 \leq 1 - \frac{2}{p} < s - 2$

$$\|\tilde{r}^{1-\frac{2}{p}}(\nabla f, Lf)\|_{L_t^2 L_x^p(C_u)} \lesssim \lambda^{-\frac{1}{2}}, \quad (8.30)$$

which is due to Proposition 6.2, Proposition 6.3 and Corollary 3.3, followed with rescaling.

²³From now on, we will frequently use Lemma 8.4 without explicit mention.

To complete the proof of (8.27), we need to obtain the same control for $f = \mathbf{T}v$. From (6.9), we bound

$$\|\tilde{r}(L\mathbf{T}v, \nabla\mathbf{T}v)\|_{L_t^2 L_\omega^p(C_u \cap \widetilde{\mathcal{D}^+})} \lesssim \|\tilde{r}(L\partial\varrho, \nabla\partial\varrho, (\partial\varrho)^2)\|_{L_t^2 L_\omega^p(C_u \cap \widetilde{\mathcal{D}^+})}.$$

The first two terms on the right hand side can be bounded by (8.30). For the quadratic term, we use the second inequality in (8.26) to derive

$$\|\tilde{r}(\partial\varrho)^2\|_{L_t^2 L_\omega^p(C_u \cap \widetilde{\mathcal{D}^+})} \lesssim \|\tilde{r}^{\frac{1}{2}}\partial\varrho\|_{L_t^\infty L_\omega^p(C_u)}^2 \tau_*^{\frac{1}{2}} \lesssim \lambda^{-1+\frac{1}{2}-4\epsilon_0} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}.$$

Hence, we conclude

$$\|\tilde{r}(L\mathbf{T}v, \nabla\mathbf{T}v)\|_{L_t^2 L_\omega^p(C_u \cap \widetilde{\mathcal{D}^+})} \lesssim \lambda^{-\frac{1}{2}}.$$

The proof of (8.27) is completed. \square

The transport lemma. We will use transport equations to control the connection coefficients. Note for any fixed point $p \in \widetilde{\mathcal{D}^+}$, there exists a unique null geodesic through the point such that $p = \Upsilon(t, u, \omega)$, which is either initiated from $S_{0,-u}$ at the slice of $\{t = 0\}$ or from the vertex $t = u$ at the time axis Γ^+ if $u \geq 0$. We may regard $p = (t, u, \omega)$ for short. The following result can be derived in view of (5.13) and (8.17) and will be frequently used.

Lemma 8.7 (The transport lemma). *For C_u contained in $\widetilde{\mathcal{D}^+}$ let $t_{\min} = \max\{u, 0\}$. For any $S_{t,u}$ -tangent tensor field F satisfying*

$$\nabla_L F + \frac{m}{2} \text{tr}\chi F = W$$

with a constant m , there holds

$$v_t^{\frac{m}{2}} F(t) = \lim_{\tau \rightarrow t_{\min}} v_\tau^{\frac{m}{2}} F(\tau) + \int_{t_{\min}}^t v_{t'}^{\frac{m}{2}} W dt'.$$

Similarly, for the transport equation

$$\nabla_L F + \frac{m}{t-u} F = G \cdot F + W$$

with a constant m , if $\|G\|_{L_\omega^\infty L_t^1} \leq C$, then there holds

$$\tilde{r}^m |F(t)| \lesssim \lim_{\tau \rightarrow t_{\min}} (\tau - u)^m |F(\tau)| + \int_{t_{\min}}^t (t' - u)^m |W| dt'.$$

The same result holds when $\frac{2}{t-u}$ in the transport equation is replaced by $\text{tr}\chi$. The above integrals are taken along the null geodesic $\Upsilon(\cdot, u, \omega)$ on C_u .

We will also employ the Codazzi equations on the spheres $S_{t,u}$ for which we recall the following elliptic estimates, which hold under the assumption (8.14).

Lemma 8.8. *Let \mathcal{D}_1 be the operator that maps an $S_{t,u}$ -tangent 1-form F to $(d\sharp F, c\sharp\text{rl}F)$; and let \mathcal{D}_2 be the operator that sends an $S_{t,u}$ -tangent symmetric 2-tensor field F to $d\sharp v F$. Let \mathcal{D} denote either \mathcal{D}_1 or \mathcal{D}_2 and let $p > 2$ be the number in (8.14). Then for $2 \leq q \leq p$ there holds*

$$\|\nabla F\|_{L^q(S_{t,u})} + \|\tilde{r}^{-1} F\|_{L^q(S_{t,u})} \lesssim \|\mathcal{D}F\|_{L^q(S_{t,u})}$$

for any $S_{t,u}$ -tangent tensor F in the domain of \mathcal{D} .

It follows from the above result, (8.19), (8.20) and the duality argument that

Proposition 8.9. *Let F be a covariant symmetric traceless 2-tensor satisfying the Hodge system*

$$d\sharp v F = \nabla G + e \quad \text{on } S_{t,u} \quad (8.31)$$

for some scalar function G and 1-form e . For $2 < q < \infty$ and $\frac{1}{q'} = \frac{1}{2} + \frac{1}{q}$ there hold

$$\|F\|_{L^q(S_{t,u})} \lesssim \|G\|_{L^q(S_{t,u})} + \|e\|_{L^{q'}(S_{t,u})}; \quad (8.32)$$

and

$$\|F\|_{L^\infty(S_{t,u})} \lesssim \tilde{r}^{1-\frac{2}{q}} (\|\nabla G\|_{L^q(S_{t,u})} + \|e\|_{L^q(S_{t,u})}). \quad (8.33)$$

Similarly, for the Hodge system

$$\begin{cases} d\sharp v F = \nabla \cdot G_1 + e_1, \\ c\sharp \text{rl} F = \nabla \cdot G_2 + e_2, \end{cases} \quad (8.34)$$

with 1-forms $G = (G_1, G_2)$ and scalar functions $e = (e_1, e_2)$, there hold (8.32) and (8.33) for any $q > 2$.

Proposition 8.10. *Let F and G be $S_{t,u}$ -tangent tensor fields of suitable type satisfying (8.31) or (8.34) with certain term e . Suppose G is a projection of a tensor field \tilde{G} to tangent space of $S_{t,u}$ by $\Pi_\mu^{\mu'} \tilde{G}_{\mu' \dots}$ or takes the form of $\mathbf{N}^\mu \tilde{G}_{\mu \dots}, L^\mu \tilde{G}_{\mu \dots}$. Under the assumption (8.15), for $q > 2$, $1 \leq c < \infty$ and $\delta > 0$ sufficiently close to 0, there holds*

$$\|F\|_{L^\infty(S_{t,u})} \lesssim \|\mu^\delta P_\mu \tilde{G}\|_{l_\mu^c L^\infty(S_{t,u})} + \|\tilde{G}\|_{L^\infty(S_{t,u})} + \tilde{r}^{1-\frac{2}{q}} \|e\|_{L^q(S_{t,u})}. \quad (8.35)$$

Here \tilde{G} is regarded as its components under the coordinate frame $\partial_t, \partial_1, \partial_2, \partial_3$.

The proof of the above result can be found in [43, Section 5].

A sketch for the proof of Proposition 8.2 and Proposition 8.3. In comparison with the analysis in [45, Section 5], due to the rough $\text{curl} \Omega$, we have to carry out normalizations by using (7.30), (7.31) and (7.40) in order to obtain the estimates of z and ∇z . The quantity \mathcal{V} in (8.2) is introduced for this purpose. We also simplify the proof of the ζ -estimate by using (7.20) and Proposition 8.10. It is used in controlling $\nabla \mathcal{V}, \nabla z$ and also $z, \hat{\chi}$.

(1) We derive structure equations (8.41)-(8.46) with the help of (7.30), (7.31) and Lemma 7.8, then achieve a set of preliminary estimates in Proposition 8.11 by using transport equations, which gives the estimates for \mathcal{U} in Proposition 8.2 and improves the auxiliary assumption in (8.15).

(2) In Section 8.2, by using (8.45), we obtain the bounds of ∇z in (8.5) and (8.6), and with the help of the Codazzi equation (8.44) the bound of $\nabla \hat{\chi}$ in (8.6). Once the L^p bound in (8.5) is obtained, we have the pointwise control on $\tilde{r}^{\frac{1}{2}} z$ in (8.4) by Sobolev embedding. In Section 8.3, we obtain the sharp bound on ζ with the help of the Hodge system (7.20). By using (8.6), also in view of the equation (5.11) for propagating the metric component of γ and the Sobolev embedding (8.20), the proof of Proposition 8.3 is completed.

(3) In the region where $u \approx t$, the bound on $\|z\|_{L_t^2 L^\infty(\mathcal{D}^+)}$ can not follow from the pointwise control of $\tilde{r}^{\frac{1}{2}} z$, since \tilde{r} can be close to 0 in such region. This step is completed in Proposition 8.19 in Section 8.4. To remove the potential singularity, we derive the transport equation (8.87) for $\nabla \mathcal{V}$, from which we see the potential singularity arises exactly due to $\nabla(k_{\mathbf{NN}} - \frac{1}{2}\Xi_4)$. We then derive the trace decomposition of $\nabla(k_{\mathbf{NN}} - \frac{1}{2}\Xi_4)$ by using (7.40), and use the structure to remove the singular term. With the help of the normalized transport equation (8.88), we can obtain the L_ω^p bound on $\tilde{r}\nabla \mathcal{V}$, and the desired bounds on $\nabla z, z$ and $\text{tr} \chi - \frac{2}{\tilde{r}}$ follow as the consequences. With the bound on ∇z , we achieve the strong control on $\hat{\chi}$ in (8.7) and (8.9) by applying Proposition 8.9 and Proposition 8.10 to (7.16). The proof of Proposition 8.2 hence can be completed.

8.1. The preliminary estimates on $\hat{\chi}$, z and ζ . The goal of this subsection is to show the following preliminary estimates.

Proposition 8.11 (L^p and L^{2p} estimates on $\widetilde{\mathcal{D}^+}$). *Let $0 \leq 1 - \frac{2}{p} < s - 2$, and let $\mathcal{U} = \frac{\mathbf{b}^{-1}-1}{\tilde{r}}$. There hold in $\widetilde{\mathcal{D}^+}$ the following estimates*

$$\|\mathcal{U}\|_{L_t^2 L_\omega^\infty(C_u)} + \|\tilde{r} \mathcal{D}_* \mathcal{U}\|_{L_t^2 L_\omega^p(C_u)} + \|\tilde{r}^{\frac{1}{2}} \nabla \log \mathbf{b}\|_{L_\omega^p(S_{t,u})} + \|\tilde{r}^{\frac{1}{2}} \mathcal{U}\|_{L_\omega^{2p} L_t^\infty(C_u)} \lesssim \lambda^{-\frac{1}{2}}, \quad (8.36)$$

$$\|\mathcal{U}\|_{L_t^2 L^\infty(\mathcal{D}^+)} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}, \quad \|\mathcal{U}\|_{L_t^{\frac{q}{2}} L^\infty(\widetilde{\mathcal{D}^+})} \lesssim \lambda^{\frac{2}{q}-1-4\epsilon_0(\frac{4}{q}-1)}, \quad 2 < q < 4. \quad (8.37)$$

Let $\mathbf{A} = \hat{\chi}, \zeta, z, \pi$. There hold in $\widetilde{\mathcal{D}^+}$ that

$$\|\mathbf{A}, \tilde{r} \nabla_L \mathbf{A}\|_{L_t^2 L_\omega^p(C_u)} \lesssim \lambda^{-\frac{1}{2}} \quad (8.38)$$

$$\|\tilde{r}^{\frac{1}{2}} \mathbf{A}\|_{L_\omega^p(S_{t,u})} \lesssim \lambda^{-\frac{1}{2}} \quad (8.39)$$

$$\|\tilde{r}^{\frac{1}{2}} \mathbf{A}\|_{L_\omega^{2p} L_t^\infty(C_u \cap \mathcal{D}^+)} + \|\tilde{r}^{\frac{1}{2}} \pi\|_{L_\omega^{2p}(S_{t,u})} + \|\tilde{r}^{\frac{1}{2}} \mathcal{U}, \tilde{r}^{\frac{1}{2}} z\|_{L_\omega^{2p} L_t^\infty(C_u)} \lesssim \lambda^{-\frac{1}{2}}. \quad (8.40)$$

Remark 8.12. Note due to $\zeta = \nabla \log \mathbf{b} + k_{\mathbf{NN}}$ in (7.5), (8.39) improves the second estimates in (8.15). In view of (7.5), $\text{tr}\theta - \frac{2}{r} = z + \pi$. Hence using $\tilde{r} \lesssim \lambda^{1-8\epsilon_0} T$ and (8.40) implies $\|\pi, z\|_{L^3(\Sigma_t \cap \widetilde{\mathcal{D}^+})} \lesssim \lambda^{-4\epsilon_0}$. The assumption on $\text{tr}\theta$ in (8.15) can be improved provided that $\lambda \geq \Lambda$ with Λ sufficiently large.

To prove the above proposition, we first derive some symbolic null structure equations for ease of analysis.

Lemma 8.13. *Let \mathbf{A} denote terms $\hat{\chi}, z, \pi$. There hold the following structure equations,*

$$\nabla_L \nabla \log \mathbf{b} + \frac{1}{2} \text{tr}\chi \nabla \log \mathbf{b} = -\hat{\chi} \cdot \nabla \log \mathbf{b} - \nabla(k_{\mathbf{NN}}) \quad (8.41)$$

$$Lz + \frac{2z}{t-u} = -\frac{1}{2} \Xi_4^2 + (\Xi_4 - k_{\mathbf{NN}}) \widetilde{\text{tr}\chi} - |\hat{\chi}|^2 - \frac{1}{2} z^2 + \pi \cdot \pi + e^\theta \text{curl} \Omega_{\mathbf{N}} \quad (8.42)$$

$$\nabla_L \hat{\chi} + \frac{1}{2} \text{tr}\chi \hat{\chi} = \pi \cdot \mathbf{A} + \tilde{r}^{-1} \pi + (\nabla, \nabla_L) \pi + e^\theta \text{curl} \Omega_{\mathbf{N}} \quad (8.43)$$

$$dkv \hat{\chi} = \frac{1}{2} \nabla z + \nabla \pi + \frac{\pi}{t-u} + \mathbf{A} \cdot \pi \quad (8.44)$$

$$\nabla_L (\nabla z - e^\theta \text{curl} \Omega)_A + \frac{3}{t-u} (\nabla z - e^\theta \text{curl} \Omega)_A \quad (8.45)$$

$$= \mathbf{A} \cdot \nabla z + \frac{1}{(t-u)} (\nabla_A (\Xi_4 - k_{\mathbf{NN}}) + e^\theta (\text{curl} \Omega)_A) + (z, \pi) \cdot \nabla \pi \\ + e^\theta (e_{Ai} \Pi^{ij} \epsilon_{jm}^l (\text{curl}^2 \Omega)_l \mathbf{N}^m + X \cdot (\chi + \pi) \cdot \partial \Omega) + \nabla \hat{\chi} \cdot \hat{\chi},$$

where $\cdot X$ represents the contractions with L, \underline{L}, Π .

Hence with \mathfrak{A} an element of $\hat{\chi}, z, \nabla \log \mathbf{b}$, there holds the symbolic formula

$$\nabla_L \mathfrak{A} + \frac{m}{\tilde{r}} \mathfrak{A} = \mathbf{A} \cdot \mathbf{A} + \mathcal{D}_* \pi + \tilde{r}^{-1} \pi + e^\theta \text{curl} \Omega_{\mathbf{N}}, \quad m = 1, 2, \quad (8.46)$$

where on the right hand side $\mathbf{A} = z, \hat{\chi}, \zeta, \pi$, and all the possible terms appeared in the collection of the transport equations for \mathfrak{A} are included. $m = 1$ if $\mathfrak{A} = \hat{\chi}, \nabla \log \mathbf{b}$ and $m = 2$ if $\mathfrak{A} = z$.

Proof. (8.41) can be directly obtained by using (7.11) and (7.8).

Combining (7.30) with the equations (1.7), (1.8), and $\mathbf{D}_L L = -k_{\mathbf{NN}} L$ in (7.2), we can obtain

$$\mathbf{R}_{44} = -e^\theta \text{curl} \Omega_{\mathbf{N}} + \pi \cdot \pi + L(\Xi_4). \quad (8.47)$$

In view of the definition of $\widetilde{\text{tr}}\chi$, substituting (8.47) into (7.12) yields

$$L\widetilde{\text{tr}}\chi + \frac{1}{2}(\widetilde{\text{tr}}\chi)^2 = (\Xi_4 - k_{\mathbf{NN}})\widetilde{\text{tr}}\chi - \frac{1}{2}\Xi_4^2 - |\hat{\chi}|^2 + \pi \cdot \pi + e^e \text{curl } \Omega_{\mathbf{N}}, \quad (8.48)$$

which gives (8.42) by using the definition of z . (8.43) can be obtained by the substitutions of (8.47) and Lemma 7.8 (i) into (7.13). The proof of (8.44) can be obtained by substituting Lemma 7.8 (ii) into (7.16).

To derive (8.45), we directly take the covariant derivative on (8.42) and use the commutation formula (7.8) to obtain

$$\nabla_L \nabla z + \frac{3}{t-u} \nabla z = -\hat{\chi} \cdot \nabla z + \frac{1}{2}(\Xi_4 - z) \nabla z + \nabla G, \quad (8.49)$$

where G denotes the right hand side of (8.42). With the help of (7.31), we derive

$$\begin{aligned} & \nabla_A (\exp \varrho \text{curl } \Omega_{\mathbf{N}}) \\ &= \nabla_L (e^e \text{curl } \Omega)_A + e^e (e_{Ai} \Pi^{ij} \epsilon_{jm}^l (\text{curl }^2 \Omega)_l \mathbf{N}^m + (\pi + \chi) \cdot \partial \Omega \cdot X). \end{aligned} \quad (8.50)$$

Note that the first term on the right is of the type $\partial \text{curl } \Omega$, which is the second order derivative of Ω . There is no direct bound for this term along C_u . Therefore we renormalize the equation (8.49) by subtracting $(\nabla_L + \frac{3}{t-u})(e^e \text{curl } \Omega)_A$ from both sides. This gives (8.45). \square

Proof of (8.36) and (8.37). Recall from Lemma 8.1, $\lim_{t \rightarrow t_{\min}} (\mathbf{b} - a) = 0$ with $a = 1$ if $u \geq 0$ and the function a in Proposition 5.1 if $u < 0$. By using (7.11), we have

$$\frac{\mathbf{b}^{-1} - a^{-1}}{\tilde{r}} = \frac{1}{\tilde{r}} \int_{t_{\min}}^t L(\mathbf{b}^{-1}) dt' = \frac{1}{\tilde{r}} \int_{t_{\min}}^t \mathbf{b}^{-1} k_{\mathbf{NN}} dt'. \quad (8.51)$$

Since $a = 1$ if $u \geq 0$, and for $u \leq 0$, we derive in view of $\frac{\mathbf{b}^{-1} - 1}{\tilde{r}} = \frac{\mathbf{b}^{-1} - a^{-1}}{\tilde{r}} + \frac{a^{-1} - 1}{\tilde{r}}$,

$$\frac{\mathbf{b}^{-1} - 1}{\tilde{r}} = \frac{1}{\tilde{r}} \int_{t_{\min}}^t \mathbf{b}^{-1} k_{\mathbf{NN}} dt' + \frac{a^{-1} - 1}{\tilde{r}}. \quad (8.52)$$

For both cases above, by using (8.18) and (5.19),

$$\begin{aligned} \left\| \frac{\mathbf{b}^{-1} - 1}{\tilde{r}} \right\|_{L_t^2 L_\omega^\infty(C_u \cap \widetilde{\mathcal{D}^+})} &\lesssim \|k\|_{L_t^2 L_\omega^\infty(C_u \cap \widetilde{\mathcal{D}^+})} + \left\| \frac{1 - a^{-1}}{\mathbf{v}^{\frac{1}{2}}} \mathbf{v}^{\frac{1}{2}} \tilde{r}^{-1} \right\|_{L_t^2 L_\omega^\infty(C_u \cap \widetilde{\mathcal{D}^+})} \\ &\lesssim \lambda^{-\frac{1}{2}} + \lambda^{-\frac{1}{2}} \|\mathbf{v}^{\frac{1}{2}}(t + \mathbf{v})^{-1}\|_{L_t^2}, \end{aligned}$$

where we used (8.1) for the bound of k , and the last term vanishes unless $\mathbf{v} = -u$, for $u < 0$. Hence by direct calculation,

$$\left\| \frac{\mathbf{b}^{-1} - 1}{\tilde{r}} \right\|_{L_t^2 L_\omega^\infty(C_u \cap \widetilde{\mathcal{D}^+})} \lesssim \lambda^{-\frac{1}{2}}.$$

Next, when $u \geq 0$, by using (8.1) and (8.22)

$$\begin{aligned} \left\| \frac{\mathbf{b}^{-1} - 1}{\tilde{r}} \right\|_{L_t^2 L^\infty(\mathcal{D}^+)} &\lesssim \|\tilde{r}^{-1} \int_u^t \|k_{\mathbf{NN}}\|_{L_\omega^\infty} \|L_t^2 L_u^\infty(\mathcal{D}^+)\| \lesssim \|\mathcal{M}(\|k_{\mathbf{NN}}\|_{L_\omega^\infty})\|_{L_t^2} \\ &\lesssim \|k_{\mathbf{NN}}\|_{L_t^2 L^\infty(\mathcal{D}^+)} \lesssim \lambda^{-\frac{1}{2} - 4\epsilon_0}; \end{aligned}$$

similar to the above estimate, if $u \leq 0$, noting that $\sup_{0 \leq \mathbf{v} \leq \mathbf{v}_*} \mathbf{v}^{\frac{1}{2}} \tilde{r}^{-1} \lesssim t^{-\frac{1}{2}}$, we can derive with the help of (8.52), (5.19) and (8.1) that

$$\begin{aligned} \left\| \frac{\mathbf{b}^{-1} - 1}{\tilde{r}} \right\|_{L_t^{\frac{q}{2}} L^\infty(\widetilde{\mathcal{D}^+})} &\lesssim \|k_{\mathbf{NN}}\|_{L_t^{\frac{q}{2}} L^\infty(\widetilde{\mathcal{D}^+})} + \left\| \frac{1 - a^{-1}}{\mathbf{v}^{\frac{1}{2}}} \mathbf{v}^{\frac{1}{2}} \tilde{r}^{-1} \right\|_{L_t^{\frac{q}{2}} L^\infty(\widetilde{\mathcal{D}^+})} \\ &\lesssim \|k\|_{L_t^{\frac{q}{2}} L^\infty(\widetilde{\mathcal{D}^+})} + \lambda^{\frac{2}{q} - 1 - 4\epsilon_0(\frac{4}{q} - 1)} \\ &\lesssim \lambda^{\frac{2}{q} - 1 - 4\epsilon_0(\frac{4}{q} - 1)}. \end{aligned}$$

Next we prove the derivative estimate in (8.36). By using (7.11),

$$\tilde{r}L\tilde{U} = L(\mathbf{b}^{-1}) - \tilde{r}^{-1}(\mathbf{b}^{-1} - 1) = \mathbf{b}^{-1}k_{\mathbf{NN}} - \tilde{U},$$

on $C_u \cap \widetilde{\mathcal{D}^+}$, we derive by using (8.1) and the first estimate of (8.36) which has been proved above

$$\|\tilde{r}L\tilde{U}\|_{L_t^2 L_\omega^p} \lesssim \|k_{\mathbf{NN}}\|_{L_t^2 L_\omega^p} + \|\tilde{U}\|_{L_t^2 L_\omega^p} \lesssim \lambda^{-\frac{1}{2}}.$$

Recall that by using the first estimate in (8.12), $\|\hat{\chi}\|_{L_\omega^\infty L_t^1} \lesssim \lambda^{-3\epsilon_0}$. Hence we can apply Lemma 8.7 to (8.41), by using (i) in Lemma 8.1 for $\nabla \mathbf{b} = \mathbf{b}(\zeta + \underline{\zeta})$ when $u \geq 0$. By using (8.28), this leads to

$$\|\nabla \log \mathbf{b}\|_{L_t^2 L_\omega^p} + \tilde{r}^{\frac{1}{2}} \|\nabla \log \mathbf{b}\|_{L_\omega^p} \lesssim \|\tilde{r} \nabla k_{\mathbf{NN}}\|_{L_t^2 L_\omega^p} \lesssim \lambda^{-\frac{1}{2}}.$$

Similarly, if $u \leq 0$, we use the initial condition in (5.15) to derive

$$\tilde{r} \|\nabla \log \mathbf{b}\|_{L_\omega^p} \lesssim \int_{t_{\min}}^t \|\tilde{r} \nabla k_{\mathbf{NN}}\|_{L_\omega^p} dt' + \lim_{t \rightarrow 0} \tilde{r} \|\nabla \log \mathbf{b}\|_{L_\omega^p}.$$

Noting that at $t = 0$, (5.15) implies $\mathbf{v}^{\frac{1}{2}} \|\mathbf{v}^{\frac{1}{2}} \nabla \log a\|_{L_\omega^p} \lesssim \lambda^{-\frac{1}{2}} \mathbf{v}^{\frac{1}{2}}$. Thus if \mathbf{v} is fixed,

$$\|\tilde{r}^{-1} \mathbf{v}^{\frac{1}{2}} \|\mathbf{v}^{\frac{1}{2}} \nabla \log a\|_{L_\omega^p} \|_{L_t^2} \lesssim \lambda^{-\frac{1}{2}},$$

and

$$\tilde{r}^{-\frac{1}{2}} \mathbf{v}^{\frac{1}{2}} \|\mathbf{v}^{\frac{1}{2}} \nabla \log a\|_{L_\omega^p} \lesssim \lambda^{-\frac{1}{2}}.$$

We then conclude by using (8.28) on $C_u \cap \widetilde{\mathcal{D}^+}$

$$\tilde{r}^{\frac{1}{2}} \|\nabla \log \mathbf{b}\|_{L_\omega^p} + \|\nabla \log \mathbf{b}\|_{L_t^2 L_\omega^p} \lesssim \lambda^{-\frac{1}{2}} + \|\tilde{r} \nabla k_{\mathbf{NN}}\|_{L_t^2 L_\omega^p} \lesssim \lambda^{-\frac{1}{2}}.$$

Thus the second and the third estimates in (8.36) are proved. The last one follows by applying (8.21) with the help of the first two estimates in (8.36), Lemma 8.1 (i) and the second estimate in (5.19). The proofs for (8.36) and (8.37) are therefore complete. \square

Proof of (8.38)-(8.40). Note (8.38), (8.39) and the second estimate in (8.40) hold for π , which is a consequence of (8.28) and the last estimate of (8.26). To prove (8.38)-(8.40), it remains to consider $\mathbf{A} = z, \hat{\chi}, \zeta$.

We first note from (8.12) and (8.1)

$$\|\mathbf{A}\|_{L_t^2 L_\omega^\infty(C_u \cap \widetilde{\mathcal{D}^+})} \lesssim \lambda^{-\frac{1}{2} + \epsilon_0}. \quad (8.53)$$

Since we have proved in (8.36) that for $0 \leq 1 - \frac{2}{p} < s - 2$ there holds on $\widetilde{\mathcal{D}^+}$ that

$$\tilde{r}^{\frac{1}{2}} \|\nabla \log \mathbf{b}\|_{L_\omega^p(S_{t,u})} + \|\nabla \log \mathbf{b}\|_{L_t^2 L_\omega^p(C_u)} \lesssim \lambda^{-\frac{1}{2}}. \quad (8.54)$$

In view of (8.41), using the above estimates, (8.28) and (8.53) implies

$$\begin{aligned} \|\tilde{r}\nabla_L \nabla \log \mathbf{b}\|_{L_t^2 L_\omega^p(C_u \cap \widetilde{\mathcal{D}^+})} &\leq \|(\tilde{r}\mathbf{A} + 1)\nabla \log \mathbf{b}\|_{L_t^2 L_\omega^p(C_u)} + \|\tilde{r}\nabla \pi\|_{L_t^2 L_\omega^p(C_u)} \\ &\leq \|\tilde{r}^{\frac{1}{2}}\mathbf{A}\|_{L_t^2 L_\omega^\infty} \|\tilde{r}^{\frac{1}{2}}\nabla \log \mathbf{b}\|_{L_t^\infty L_\omega^p} + \|\nabla \log \mathbf{b}\|_{L_t^2 L_\omega^p(C_u)} + \lambda^{-\frac{1}{2}} \\ &\lesssim \lambda^{-\frac{1}{2}-3\epsilon_0} + \lambda^{-\frac{1}{2}}. \end{aligned}$$

Thus, it follows by using the last identity in (7.5) and (8.28) that

$$\|\tilde{r}\nabla_L \zeta\|_{L_t^2 L_\omega^p(C_u \cap \widetilde{\mathcal{D}^+})} \lesssim \|\nabla_L \pi\|_{L_t^2 L_\omega^p(C_u)} + \|\tilde{r}\nabla_L \nabla \log \mathbf{b}\|_{L_t^2 L_\omega^p(C_u)} \lesssim \lambda^{-\frac{1}{2}}.$$

Similarly, by using (7.5), (8.54), the estimates for π in (8.38) and (8.39), we have on $\widetilde{\mathcal{D}^+}$

$$\begin{aligned} \|\zeta\|_{L_t^2 L_\omega^p(C_u)} &\lesssim \|\nabla \log \mathbf{b}\|_{L_t^2 L_\omega^p} + \|\pi\|_{L_t^2 L_\omega^p} \lesssim \lambda^{-\frac{1}{2}}, \\ \|\tilde{r}^{\frac{1}{2}}\zeta\|_{L_\omega^p(S_{t,u})} &\lesssim \|\tilde{r}^{\frac{1}{2}}\pi\|_{L_\omega^p} + \|\tilde{r}^{\frac{1}{2}}\nabla \log \mathbf{b}\|_{L_\omega^p} \lesssim \lambda^{-\frac{1}{2}}. \end{aligned}$$

Thus (8.38) and (8.39) hold for ζ .

By applying Lemma 8.7 to (8.46) for both $\mathfrak{A} = \hat{\chi}$ and z with $m = 1$ or 2 , we derive

$$\tilde{r}^m |\mathfrak{A}(t)| \lesssim \lim_{\tau \rightarrow t_{\min}} |(\tau - u)^m \mathfrak{A}(\tau)| + \int_{t_{\min}}^t \tilde{r}^m (|\mathbf{A} \cdot \mathbf{A}| + |\mathcal{D}_* \pi| + |\tilde{r}^{-1} \pi| + |\operatorname{curl} \Omega|) dt'. \quad (8.55)$$

If $u \geq 0$, for the data of $\mathfrak{A} = z, \hat{\chi}$, we apply the result (i) in Lemma 8.1, the first term on the right of (8.55) vanishes.

If $u < 0$, for $\mathfrak{A} = \hat{\chi}$, we note that due to (5.15), (7.5) and applying the second estimate in (8.26) for π ,

$$\|\mathbf{v}^{\frac{1}{2}-\frac{2}{p}} \hat{\chi}\|_{L^p(S_{\mathbf{v}})} \lesssim \lambda^{-\frac{1}{2}}, \quad 0 \leq 1 - \frac{2}{p} < s - 2;$$

for $\mathfrak{A} = z$, we apply (iii) in Lemma 8.1. Thus, for $\mathbf{v} > 0$ fixed, since $\tilde{r} = t + \mathbf{v}$,

$$\begin{aligned} \|\tilde{r}^{-m+\frac{1}{2}} \mathbf{v}^m \mathfrak{A}(0, \mathbf{v}, \omega)\|_{L_t^\infty L_\omega^p} &\lesssim \|\mathbf{v}^{\frac{1}{2}} \mathfrak{A}(0, \mathbf{v}, \omega)\|_{L_\omega^p} \lesssim \lambda^{-\frac{1}{2}}, \\ \|\tilde{r}^{-m} \mathbf{v}^m \mathfrak{A}(0, \mathbf{v}, \omega)\|_{L_t^2 L_\omega^p} &\leq \|\tilde{r}^{-1} \mathbf{v} \mathfrak{A}(0, \mathbf{v}, \omega)\|_{L_t^2 L_\omega^p} \lesssim \mathbf{v}^{\frac{1}{2}} \|\mathfrak{A}(0, \mathbf{v}, \omega)\|_{L_\omega^p} \lesssim \lambda^{-\frac{1}{2}}. \end{aligned} \quad (8.56)$$

Therefore by using (8.55), in both cases, for $\mathfrak{A} = \hat{\chi}, z$,

$$\begin{aligned} \|\tilde{r}^{\frac{1}{2}} \mathfrak{A}\|_{L_\omega^p} &\lesssim \lambda^{-\frac{1}{2}} + \|\tilde{r}^{-\frac{1}{2}} \int_{t_{\min}}^t \tilde{r} (|\mathcal{D}_* \pi| + |\mathbf{A} \cdot \mathbf{A}| + |\tilde{r}^{-1} \pi| + |\operatorname{curl} \Omega|) dt'\|_{L_\omega^p} \\ \|\mathfrak{A}\|_{L_t^2 L_\omega^p} &\lesssim \lambda^{-\frac{1}{2}} + \|\tilde{r}^{-1} \int_{t_{\min}}^t \tilde{r} (|\mathcal{D}_* \pi| + |\mathbf{A} \cdot \mathbf{A}| + |\tilde{r}^{-1} \pi| + |\operatorname{curl} \Omega|) dt'\|_{L_t^2 L_\omega^p}, \end{aligned}$$

which lead to

$$\begin{aligned} \|\tilde{r}^{\frac{1}{2}} \mathfrak{A}\|_{L_\omega^p(S_{t,u})} + \|\mathfrak{A}\|_{L_t^2 L_\omega^p(C_u)} &\lesssim \lambda^{-\frac{1}{2}} + \|\tilde{r} \mathcal{D}_* \pi\|_{L_t^2 L_\omega^p(C_u)} + \|\pi\|_{L_t^2 L_\omega^p(C_u)} \\ &\quad + \|\tilde{r} \operatorname{curl} \Omega\|_{L_t^2 L_\omega^p(C_u)} + \|\tilde{r} \mathbf{A} \cdot \mathbf{A}\|_{L_t^2 L_\omega^p(C_u)}. \end{aligned} \quad (8.57)$$

We apply (8.28) and (8.24) to derive

$$\|\tilde{r} \mathcal{D}_* \pi\|_{L_t^2 L_\omega^p(C_u \cap \widetilde{\mathcal{D}^+})} + \|\pi\|_{L_t^2 L_\omega^p(C_u \cap \widetilde{\mathcal{D}^+})} + \|\tilde{r} \operatorname{curl} \Omega\|_{L_t^2 L_\omega^p(C_u \cap \widetilde{\mathcal{D}^+})} \lesssim \lambda^{-\frac{1}{2}}. \quad (8.58)$$

Thus by using (8.53), we derive on $\widetilde{\mathcal{D}^+}$

$$\begin{aligned} \|\tilde{r}^{\frac{1}{2}}\mathfrak{A}\|_{L_\omega^p(S_{t,u})} + \|\mathfrak{A}\|_{L_t^2 L_\omega^p(C_u \cap \widetilde{\mathcal{D}^+})} &\lesssim \lambda^{-\frac{1}{2}} + \|\tilde{r}\mathbf{A} \cdot \mathbf{A}\|_{L_t^2 L_\omega^p(C_u)} \\ &\lesssim \lambda^{-\frac{1}{2}} + \|\mathbf{A}\|_{L_t^2 L_\omega^\infty} \|\tilde{r}\mathbf{A}\|_{L_t^\infty L_\omega^p} \\ &\lesssim \lambda^{-\frac{1}{2}} + \lambda^{-\frac{1}{2}+\epsilon_0} \tau_*^{\frac{1}{2}} (\|\tilde{r}^{\frac{1}{2}}\hat{\chi}\|_{L_t^\infty L_\omega^p} + \|\tilde{r}^{\frac{1}{2}}z\|_{L_\omega^p} + \lambda^{-\frac{1}{2}}), \end{aligned} \quad (8.59)$$

where we used the proved estimate of (8.39) for $\mathbf{A} = \zeta, \pi$. Note $\tau_*^{\frac{1}{2}}\lambda^{-\frac{1}{2}+\epsilon_0} \lesssim \lambda^{-3\epsilon_0}$, which can be sufficiently small with $\lambda \geq \Lambda$ for sufficiently large Λ . Therefore, we can conclude for $S_{t,u}$ contained in $\widetilde{\mathcal{D}^+}$ that

$$\|\tilde{r}^{\frac{1}{2}}\hat{\chi}\|_{L_\omega^p} + \|\tilde{r}^{\frac{1}{2}}z\|_{L_\omega^p} \lesssim \lambda^{-\frac{1}{2}}$$

which completes the proof of (8.39). Substituting the above estimate to (8.59) yields the bound $\|\mathfrak{A}\|_{L_t^2 L_\omega^p} \lesssim \lambda^{-\frac{1}{2}}$. Thus the first estimate of (8.38) is completed.

Note that we have shown by using (8.39) and (8.53)

$$\|\tilde{r}\mathbf{A} \cdot \mathbf{A}\|_{L_t^2 L_\omega^p(C_u \cap \widetilde{\mathcal{D}^+})} \lesssim \lambda^{-\frac{1}{2}-3\epsilon_0}. \quad (8.60)$$

Combining this estimate with (8.58) and (8.46), also using the first estimate in (8.38), we can obtain the last estimate in (8.38) for $\mathbf{A} = \hat{\chi}, z$. Since other cases have been proved, the estimate of (8.38) is also proved.

Applying (8.21) with the help of (8.13), (8.1), Lemma 8.1 (i), (8.38) and Minkowski inequality, we can obtain in \mathcal{D}^+ that

$$\|\tilde{r}^{\frac{1}{2}}\mathbf{A}\|_{L_\omega^{2p} L^\infty(C_u)}^2 \lesssim \|\mathbf{A}\|_{L_\omega^\infty L_t^2(C_u)} (\|\tilde{r}\nabla \mathbf{A}\|_{L_\omega^p L_t^2(C_u)} + \|\mathbf{A}\|_{L_\omega^p L_t^2(C_u)}) \lesssim \lambda^{-1}. \quad (8.61)$$

We postpone the proof for the last pair of estimates in (8.40) to the end of the proof of (8.6), which will be given in Section 8.2. \square

8.2. Control of ∇z , $\nabla \hat{\chi}$ and $\tilde{r}^{\frac{1}{2}}|z|$. In this subsection, we give the estimates of ∇z and $\nabla \hat{\chi}$ in (8.5) and (8.6), and obtain the bound of $|z|$ in (8.4) as a consequence.

Since the right hand side of (8.42) is not bounded in $L_\omega^\infty L_t^1(C_u \cap \widetilde{\mathcal{D}^+})$, the pointwise estimate for z does not directly follow. There hold in view of the Sobolev embedding (8.20),

$$\begin{aligned} |\tilde{r}^{\frac{1}{2}}z(t, u, \omega)| &\lesssim \|\tilde{r}^{\frac{1}{2}}(\tilde{r}\nabla)^{(\leq 1)}z\|_{L_\omega^p(S_{t,u})}, \\ \|z\|_{L_t^2 L^\infty(\mathcal{D}^+)} &\lesssim \|(\tilde{r}\nabla)^{(\leq 1)}z\|_{L_t^2 L_u^\infty L_\omega^p(\mathcal{D}^+)}, \end{aligned} \quad (8.62)$$

where $p > 2$.

It is natural to consider the bounds on the right hand sides of the above two inequalities. Nevertheless, the estimate of $\|(\tilde{r}\nabla)^{(\leq 1)}z\|_{L_t^2 L_u^\infty L_\omega^p(\mathcal{D}^+)}$ is hardly to be obtained directly. Therefore, as the first step, we give the bound of $\tilde{r}^{\frac{1}{2}}|z|$ via the first inequality. In the second step, we will derive the bound for $\|z\|_{L_t^2 L^\infty(\mathcal{D}^+)}$ by carrying out a further normalization on $\text{tr}\chi$ in Section 8.4.

We first derive the estimates in (8.5) and (8.6), which are recast below.

Proposition 8.14. *For $0 \leq 1 - \frac{2}{p} < s' - 2$, there holds*

$$\|\tilde{r}^{\frac{3}{2}}\nabla z\|_{L_t^\infty L_u^\infty L_\omega^p(\widetilde{\mathcal{D}^+})} + \|\tilde{r}(\nabla(\hat{\chi}, z))\|_{L_t^2 L_\omega^p(C_u \cap \widetilde{\mathcal{D}^+})} \lesssim \lambda^{-\frac{1}{2}}. \quad (8.63)$$

Proof. By using (8.45), according to (8.53), we employ Lemma 8.7 to derive that

$$\begin{aligned} \tilde{r}^3 |\nabla_A z - e^\varrho(\operatorname{curl} \Omega)_A| &\lesssim \lim_{\tau \rightarrow t_{\min}} |(\tau - u)^3 (\nabla z - e^\varrho \operatorname{curl} \Omega)_A(\tau)| \\ &\quad + \int_{t_{\min}}^t (\tilde{r}^2 |\nabla \pi| + \tilde{r}^3 |\nabla \hat{\chi} \cdot \hat{\chi}| + \tilde{r}^3 |(z, \pi) \cdot \nabla \pi|) \\ &\quad + \int_{t_{\min}}^t \tilde{r}^3 (|\partial \Omega \cdot (\mathbf{A} + \tilde{r}^{-1})| + |\operatorname{curl}^2 \Omega|). \end{aligned} \quad (8.64)$$

Hence, we bound on $\widetilde{\mathcal{D}^+}$

$$\begin{aligned} &\|\tilde{r}(\nabla_A z - e^\varrho(\operatorname{curl} \Omega)_A)\|_{L_\omega^p(S_{t,u})} \\ &\lesssim \tilde{r}^{-2} \lim_{\tau \rightarrow t_{\min}} (\tau - u)^3 (|\nabla z| + |\operatorname{curl} \Omega|) \|_{L_\omega^p} + \tilde{r}^{-1} \int_{t_{\min}}^t \|\tilde{r} \nabla \pi\|_{L_\omega^p} dt' \\ &\quad + \tilde{r}^{-1} \int_{t_{\min}}^t \|\tilde{r} \partial \Omega, \tilde{r}^2 \operatorname{curl}^2 \Omega\|_{L_\omega^p} dt' + \|\tilde{r} \nabla \hat{\chi}, \tilde{r} \nabla \pi, \tilde{r} \partial \Omega\|_{L_t^2 L_\omega^p} \|\mathbf{A}\|_{L_t^2 L_\omega^\infty}. \end{aligned}$$

Due to $\tilde{r} \lesssim \tau_*$, by (8.24), there holds on $\widetilde{\mathcal{D}^+}$

$$\|\tilde{r} \partial \Omega\|_{L_\omega^p(S_{t,u})} \lesssim \lambda^{-1-4\epsilon_0}. \quad (8.65)$$

By (8.23) and (8.65), we obtain along $C_u \cap \widetilde{\mathcal{D}^+}$

$$\tilde{r}^{-1} \int_{t_{\min}}^t \|\tilde{r} \partial \Omega, \tilde{r}^2 \operatorname{curl}^2 \Omega\|_{L_\omega^p} dt' \lesssim \lambda^{-1-4\epsilon_0},$$

and by using (8.28) and (8.65)

$$\|\tilde{r} \nabla \pi, \tilde{r} \partial \Omega\|_{L_t^2 L_\omega^p(C_u \cap \widetilde{\mathcal{D}^+})} \lesssim \lambda^{-\frac{1}{2}}.$$

Hence by using the above three estimates, in view of Lemma 8.1 (i) and (iii), (8.53) and (8.28), we then obtain

$$\begin{aligned} \|\tilde{r} \nabla z\|_{L_\omega^p(S_{t,u})} &\lesssim \tilde{r}^{-2} |\min(u, 0)|^{\frac{3}{2}} \lambda^{-\frac{1}{2}} + \tilde{r}^{-1} \int_{t_{\min}}^t \|\tilde{r} \nabla \pi\|_{L_\omega^p} dt' \\ &\quad + \lambda^{-1+\epsilon_0} + \lambda^{-\frac{1}{2}+\epsilon_0} \|\tilde{r} \nabla \hat{\chi}\|_{L_t^2 L_\omega^p(C_u)}, \end{aligned} \quad (8.66)$$

which implies

$$\begin{aligned} \|\tilde{r} \nabla z\|_{L_t^2 L_\omega^p(C_u \cap \widetilde{\mathcal{D}^+})} &\lesssim \lambda^{-\frac{1}{2}} + \|\tilde{r} \nabla \pi\|_{L_t^2 L_\omega^p(C_u \cap \widetilde{\mathcal{D}^+})} + \lambda^{-3\epsilon_0} \|\tilde{r} \nabla \hat{\chi}\|_{L_t^2 L_\omega^p(C_u \cap \widetilde{\mathcal{D}^+})} \\ &\lesssim \lambda^{-\frac{1}{2}} + \lambda^{-3\epsilon_0} \|\tilde{r} \nabla \hat{\chi}\|_{L_t^2 L_\omega^p(C_u \cap \widetilde{\mathcal{D}^+})}, \end{aligned} \quad (8.67)$$

where we used (8.22) and (8.28).

By using (8.44) and Lemma 8.8,

$$\|\tilde{r} \nabla \hat{\chi}\|_{L_\omega^p} + \|\hat{\chi}\|_{L_\omega^p} \lesssim \|\tilde{r}(\nabla \pi, \tilde{r}^{-1} \pi, \nabla z)\|_{L_\omega^p} + \|\tilde{r} \mathbf{A} \cdot \pi\|_{L_\omega^p}. \quad (8.68)$$

By using (8.60), taking L_t^2 norm on (8.68) along C_u in $\widetilde{\mathcal{D}^+}$ we obtain

$$\|\tilde{r} \nabla \hat{\chi}\|_{L_t^2 L_\omega^p(C_u)} + \|\hat{\chi}\|_{L_t^2 L_\omega^p(C_u)} \lesssim \lambda^{-\frac{1}{2}} + \|\tilde{r} \nabla z\|_{L_t^2 L_\omega^p(C_u)},$$

where we used (8.28). As long as $\lambda \geq \Lambda$ with Λ sufficiently large, substituting the above estimate to the last term of (8.67) gives

$$\|\tilde{r} \nabla \hat{\chi}, \tilde{r} \nabla z\|_{L_t^2 L_\omega^p(C_u \cap \widetilde{\mathcal{D}^+})} \lesssim \lambda^{-\frac{1}{2}}.$$

With the first estimate in the above, we can use (8.66) and (8.28) to bound on $\widetilde{\mathcal{D}^+}$

$$\|\tilde{r}^{\frac{3}{2}}\nabla z\|_{L_\omega^p(S_{t,u})} \lesssim \lambda^{-\frac{1}{2}} + \|\tilde{r}\nabla\pi\|_{L_t^2L_\omega^p(C_u)} + \tau_*^{\frac{1}{2}}\lambda^{-\frac{1}{2}+\epsilon_0}\|\tilde{r}\nabla\hat{\chi}\|_{L_t^2L_\omega^p(C_u)} \lesssim \lambda^{-\frac{1}{2}}.$$

Thus we completed the proof of (8.63), which are the estimates for $\nabla\hat{\chi}, \nabla z$ in (8.5) and (8.6). \square

As the direct consequence of the second estimates in (8.63), by using (8.38) and (8.20),

$$\|z, \hat{\chi}\|_{L_t^2L_\omega^\infty(C_u \cap \widetilde{\mathcal{D}^+})} \lesssim \lambda^{-\frac{1}{2}}. \quad (8.69)$$

It remains to derive the ζ estimate for improving (8.12).

We now complete the proof of the last pair of estimates in (8.40) for $0 \leq 1 - \frac{2}{p} < s - 2$. For the estimate $\|\tilde{r}^{\frac{1}{2}}\mathcal{Z}\|_{L_\omega^{2p}(\widetilde{\mathcal{D}^+})}$, using (8.2), (8.69) and the first estimate in (8.36) we bound

$$\|\mathcal{Z}\|_{L_t^2L_\omega^\infty(C_u \cap \widetilde{\mathcal{D}^+})} \lesssim \|z\|_{L_t^2L_\omega^\infty(C_u \cap \widetilde{\mathcal{D}^+})} + \|\mathcal{U}\|_{L_t^2L_\omega^\infty(C_u \cap \widetilde{\mathcal{D}^+})} \lesssim \lambda^{-\frac{1}{2}}. \quad (8.70)$$

By using (7.11), (8.1), (8.36), (8.38) and (8.39), we obtain on $C_u \cap \widetilde{\mathcal{D}^+}$ that

$$\begin{aligned} \|\tilde{r}L\mathcal{Z}\|_{L_t^2L_\omega^p} &\lesssim \|\tilde{r}\nabla_L \mathbf{b}(z - 2\mathcal{U})\|_{L_t^2L_\omega^p} + \|\tilde{r}\nabla_L z\|_{L_t^2L_\omega^p} + \|\tilde{r}\nabla_L \mathcal{U}\|_{L_t^2L_\omega^p} \\ &\lesssim \|\pi\|_{L_t^2L_\omega^\infty} \|\tilde{r}(z, \mathcal{U})\|_{L_\omega^p} + \|\tilde{r}\nabla_L \mathbf{A}, \tilde{r}\nabla_L \mathcal{U}\|_{L_t^2L_\omega^p} \lesssim \lambda^{-\frac{1}{2}}. \end{aligned}$$

By virtue of the above two inequalities and (8.21), noting $\lim_{t \rightarrow \min} \tilde{r}^{\frac{1}{2}}|\mathcal{Z}| = 0$ by using Lemma 8.1 (i), we obtain in $\widetilde{\mathcal{D}^+}$ that

$$\|\tilde{r}^{\frac{1}{2}}\mathcal{Z}\|_{L_\omega^{2p}L_t^\infty(C_u)} \lesssim \lambda^{-\frac{1}{2}}.$$

This implies $\|\tilde{r}^{\frac{1}{2}}z\|_{L_\omega^{2p}L_t^\infty(C_u)} \lesssim \lambda^{-\frac{1}{2}}$ in view of (8.2) and the last estimate in (8.36). The proof of the last pair of estimates in (8.40) is completed.

(8.5) implies the second estimate in (8.4) in view of (8.62) and (8.39) for z . The first estimate of (8.4) follows as a consequence.

From (8.4), in the region $u \leq 0$, we can derive

$$|z| \lesssim (t + \mathbf{v})^{-\frac{1}{2}}\lambda^{-\frac{1}{2}}$$

and thus if $u \leq \frac{5t}{6}$

$$|z| \lesssim t^{-\frac{1}{2}}\lambda^{-\frac{1}{2}}. \quad (8.71)$$

Hence, the estimate for z in (8.7) holds in the region where $u \leq \frac{5t}{6}$, whereas in the region that $u \geq \frac{5t}{6}$ for the z estimates in (8.7) and (8.9), we need to seek for a different approach. This is achieved in Section 8.4 by considering the transport equations of \mathcal{Z} and $\nabla\mathcal{Z}$. Such analysis will be based on the estimates of ζ in (8.7) and (8.9).

Before proceeding to the control of ζ , we give a consequence of (8.6).

Lemma 8.15. *Let $\varphi := \log \sqrt{|\gamma|} - \log \sqrt{|\bar{\gamma}|}$ on $S_{t,u}$ and $0 \leq 1 - \frac{2}{p} < s' - 2$. There hold on $\widetilde{\mathcal{D}^+}$ the estimates*

$$\|\tilde{r}^{\frac{1}{2}}\nabla\varphi\|_{L_t^\infty L_\omega^p(C_u)} + \|\nabla\varphi\|_{L_t^2L_\omega^p(C_u)} \lesssim \lambda^{-\frac{1}{2}}.$$

Proof. By using (7.8) and $L\varphi = \text{tr}\chi - \frac{2}{t-u}$ we derive that

$$\nabla_L \nabla\varphi + \frac{1}{2}\text{tr}\chi \nabla\varphi = -\hat{\chi} \cdot \nabla\varphi + \nabla(\text{tr}\chi - \frac{2}{t-u}).$$

By using (8.53), with $m = 1$ we apply Lemma 8.7 to obtain

$$\tilde{r}|\nabla\varphi| \lesssim \lim_{\tau \rightarrow t_{\min}} (\tau - u)|\nabla\varphi|(\tau) + \int_{t_{\min}}^t \tilde{r}|\nabla(\text{tr}\chi - \frac{2}{\tilde{r}})|dt'.$$

For null cones C_u with $u \geq 0$ we use (8.3) to see the limit on the right hand side vanishes; and for null cones C_u with $u < 0$ we use (5.18). Hence for $S_{t,u}$ and C_u contained in $\widetilde{\mathcal{D}^+}$,

$$\begin{aligned} \|\tilde{r}^{\frac{1}{2}} \tilde{\nabla} \varphi\|_{L_\omega^p(S_{t,u})} &\lesssim \|\tilde{r}^{-\frac{1}{2}} \lim_{\tau \rightarrow t_{\min}} (\tau - u) |\tilde{\nabla} \varphi|(\tau)\|_{L_\omega^p(S_{t,u})} + \|\tilde{r} \tilde{\nabla}(\text{tr} \chi - \frac{2}{\tilde{r}})\|_{L_t^2 L_\omega^p(C_u \cap \widetilde{\mathcal{D}^+})}, \\ \|\tilde{\nabla} \varphi\|_{L_t^2 L_\omega^p(C_u)} &\lesssim \|\tilde{r}^{-1} \lim_{\tau \rightarrow t_{\min}} (\tau - u) |\tilde{\nabla} \varphi|(\tau)\|_{L_t^2 L_\omega^p(C_u)} + \|\tilde{r} \tilde{\nabla}(\text{tr} \chi - \frac{2}{\tilde{r}})\|_{L_t^2 L_\omega^p(C_u \cap \widetilde{\mathcal{D}^+})}. \end{aligned}$$

Similar to (8.56), we can bound the first terms on the right hand side by $\lambda^{-\frac{1}{2}}$. Consequently,

$$\|\tilde{r}^{\frac{1}{2}} \tilde{\nabla} \varphi\|_{L_\omega^p(S_{t,u})} + \|\tilde{\nabla} \varphi\|_{L_t^2 L_\omega^p(C_u)} \lesssim \lambda^{-\frac{1}{2}} + \|\tilde{r} \tilde{\nabla}(\text{tr} \chi - \frac{2}{\tilde{r}})\|_{L_t^2 L_\omega^p(C_u \cap \widetilde{\mathcal{D}^+})}.$$

Since $\text{tr} \chi - \frac{2}{t-u} = z - \Xi_4$, we may use (8.6) and (8.28) to obtain $\|\tilde{r} \tilde{\nabla}(\text{tr} \chi - \frac{2}{\tilde{r}})\|_{L_t^2 L_\omega^p(C_u)} \lesssim \lambda^{-\frac{1}{2}}$. Therefore

$$\|\tilde{r}^{\frac{1}{2}} \tilde{\nabla} \varphi\|_{L_\omega^p(S_{t,u})} + \|\tilde{\nabla} \varphi\|_{L_t^2 L_\omega^p(C_u)} \lesssim \lambda^{-\frac{1}{2}}.$$

Hence the proof of Lemma 8.15 is complete. \square

8.3. Estimates of ζ .

Proposition 8.16. *Let $0 \leq 1 - \frac{2}{p} < s' - 2$. There holds for $C_u \cap \widetilde{\mathcal{D}^+}$ that*

$$\|\tilde{r} \tilde{\nabla} \zeta\|_{L_t^2 L_\omega^p(C_u)} + \|\zeta\|_{L_t^2 L_\omega^p(C_u)} \lesssim \lambda^{-\frac{1}{2}}. \quad (8.72)$$

Proof. By using Lemma 8.8, (7.20), (8.40) and (8.20) on $S = S_{t,u}$,

$$\begin{aligned} \|\tilde{r} \tilde{\nabla}(\zeta - \underline{\zeta})\|_{L_\omega^p(S)} + \|\zeta - \underline{\zeta}\|_{L_\omega^p(S)} &\lesssim \|\zeta - \underline{\zeta}\|_{L^\infty(S)} \|\tilde{r} \tilde{\nabla} \varphi\|_{L_\omega^p(S)} + \|\tilde{r} |\mathbf{A}|^2\|_{L_\omega^p(S)} + \|\tilde{r} \tilde{\nabla} \underline{\zeta}\|_{L_\omega^p(S)} \\ &\lesssim \|(\tilde{r} \tilde{\nabla})^{(\leq 1)}(\zeta - \underline{\zeta})\|_{L_\omega^p(S)} \|\tilde{r} \tilde{\nabla} \varphi\|_{L_\omega^p(S)} + \|\tilde{r} |\mathbf{A}|^2\|_{L_\omega^p(S)} + \|\tilde{r} \tilde{\nabla} \underline{\zeta}\|_{L_\omega^p(S)}. \end{aligned}$$

Thus by using the first estimate in Lemma 8.15, (8.60) and (8.28), we can obtain

$$\|\tilde{r} \tilde{\nabla}(\zeta - \underline{\zeta})\|_{L_t^2 L_\omega^p(C_u)} + \|\zeta - \underline{\zeta}\|_{L_t^2 L_\omega^p(C_u)} \lesssim \lambda^{-\frac{1}{2}},$$

which implies (8.72) by applying (8.28) again to $\underline{\zeta} = -k_{\text{AN}}$. \square

Now apply (8.20) to ζ , together with using (8.72). Combining with (8.69), this leads to the estimate for any C_u contained in $\widetilde{\mathcal{D}^+}$,

$$\|\hat{\chi}, z, \zeta\|_{L_t^2 L_\omega^\infty(C_u)} \lesssim \lambda^{-\frac{1}{2}},$$

which gives (8.10) and improves (8.12).

Remark 8.17. As a consequence of the estimates of $\hat{\chi}$ and z in (8.10) together with (8.6), (8.11) can be proved by using the transport equation (5.11) and its angular derivative. (See the proof in [45, Section 5.5.2].) Thus the proof of Proposition 8.3 is complete.

Next we provide the control of ζ in (8.7) and (8.9). The following result is actually stronger than stated therein, and will be crucially used in the proof of estimates of $z, \hat{\chi}$ in (8.7) and (8.9).

Proposition 8.18. *There holds on $\widetilde{\mathcal{D}^+}$ that*

$$\|\zeta\|_{L_t^2 L_x^\infty(\widetilde{\mathcal{D}^+})} \lesssim \lambda^{-\frac{1}{2} - 4\epsilon_0}. \quad (8.73)$$

Proof. Applying Proposition 8.10 to (7.20) we have

$$\|\zeta\|_{L^\infty(S_{t,u})} \lesssim \|\mu^\delta P_\mu \tilde{\pi}\|_{L^\infty(S_{t,u})} + \|\tilde{\pi}\|_{L^\infty(S_{t,u})} + \tilde{r}^{1-\frac{2}{p}} \|(\zeta - \underline{\zeta}) \cdot \nabla \varphi, \zeta \cdot \zeta, \underline{\zeta} \cdot \underline{\zeta}\|_{L^p(S_{t,u})} \quad (8.74)$$

with $0 < \delta < s' - 2$ sufficiently small and $0 < 1 - \frac{2}{p} < s' - 2$.

By using the first estimate in Lemma 8.15, (8.1) and (8.39)

$$\begin{aligned} \|\tilde{r}(|\zeta|^2 + |\underline{\zeta}|^2 + |(\zeta - \underline{\zeta}) \cdot \nabla \varphi|)\|_{L_t^2 L_u^\infty L_\omega^p(\widetilde{\mathcal{D}^+})} &\lesssim \|\tilde{r} \nabla \varphi, \tilde{r} \zeta, \tilde{r} \underline{\zeta}\|_{L^\infty L_\omega^p(\widetilde{\mathcal{D}^+})} \|\zeta, \underline{\zeta}\|_{L_t^2 L_x^\infty(\widetilde{\mathcal{D}^+})} \\ &\lesssim \lambda^{-4\epsilon_0} (\|\zeta\|_{L_t^2 L_x^\infty(\widetilde{\mathcal{D}^+})} + \|\underline{\zeta}\|_{L_t^2 L_x^\infty(\widetilde{\mathcal{D}^+})}) \\ &\lesssim \lambda^{-4\epsilon_0} \|\zeta\|_{L_t^2 L_x^\infty(\widetilde{\mathcal{D}^+})} + \lambda^{-\frac{1}{2}-8\epsilon_0}. \end{aligned}$$

Taking $L_t^2 L_u^\infty$ norm on (8.74) followed with substituting the above inequality into the resulting inequality and using (8.1), we can obtain (8.73). \square

8.4. Improved estimates of z and $\hat{\chi}$. In order to complete the sets of estimates (8.9) and (8.7), due to $\text{tr}\chi - \frac{2}{r} = z + \pi$, it only remains to provide the estimates of z and $\hat{\chi}$ therein. We first derive the improved estimates for z below, for which we construct the quantity \mathcal{Z} whose angular derivative exhibits favourable structures. With the help of the control of $\nabla \mathcal{Z}$, we can derive the desired estimates of ∇z by comparison, and then for $\hat{\chi}$ by applying estimates (8.33) and (8.35) to (8.44).

Proposition 8.19. *Let $0 \leq 1 - \frac{2}{p} < s' - 2$. There hold the following estimates,*

$$\|z\|_{L_t^2 L_u^\infty L_\omega^p(\mathcal{D}^+)} + \|\tilde{r} \nabla \mathcal{Z}, \tilde{r} \nabla z\|_{L_t^2 L_u^\infty L_\omega^p(\mathcal{D}^+)} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0} \quad (8.75)$$

$$\|z\|_{L_t^2 L_x^\infty(\mathcal{D}^+)} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0} \quad (8.76)$$

$$\|\tilde{r} \nabla z\|_{L_t^{\frac{q}{2}} L_u^\infty L_\omega^p(\widetilde{\mathcal{D}^+})} \lesssim \lambda^{\frac{2}{q}-1-4\epsilon_0(\frac{4}{q}-1)}, \quad 2 < q < 4 \quad (8.77)$$

$$\|\hat{\chi}\|_{L_t^2 L_x^\infty(\mathcal{D}^+)} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0} \quad (8.78)$$

$$\|z, \hat{\chi}\|_{L_t^{\frac{q}{2}} L_x^\infty(\widetilde{\mathcal{D}^+})} \lesssim \lambda^{\frac{2}{q}-1-4\epsilon_0(\frac{4}{q}-1)}, \quad 2 < q < 4. \quad (8.79)$$

Proof. We first prove (8.75). By integrating the transport equation (8.42) along null geodesics and also using (8.4), we have from Lemma 8.7 that

$$\begin{aligned} \tilde{r}^2 |z(t)| &\lesssim \left| \lim_{\tau \rightarrow t_{\min}} (\tau - u)^2 z(\tau) \right| \\ &\quad + \left| \int_{t_{\min}}^t \tilde{r}^2 (|\mathbf{A} \cdot \mathbf{A}| + |\tilde{r}^{-1} \pi| + |\text{curl} \Omega|) dt' \right|. \end{aligned} \quad (8.80)$$

For null cones C_u with $u \geq 0$, the limit term vanishes due to Lemma 8.1 (i). Thus we can bound

$$|z(t)| \lesssim \tilde{r}^{-1} \int_{t_{\min}}^t \tilde{r} (|\mathbf{A} \cdot \mathbf{A}| + |\text{curl} \Omega| + \tilde{r}^{-1} |\pi|) dt'.$$

Note that by using (8.40)

$$\|\tilde{r} \mathbf{A} \cdot \mathbf{A}\|_{L_\omega^p} \lesssim \|\tilde{r}^{\frac{1}{2}} \mathbf{A}\|_{L_t^\infty L_\omega^{2p}}^2 \lesssim \lambda^{-1}.$$

Using the above estimate and (8.65), taking L_ω^p norm of z gives

$$\begin{aligned} \|z\|_{L_\omega^p} &\lesssim \|\tilde{r} \text{curl} \Omega\|_{L_t^\infty L_\omega^p} + \|\tilde{r} \mathbf{A} \cdot \mathbf{A}\|_{L_t^\infty L_\omega^p} + \tilde{r}^{-1} \int_u^t \|\pi\|_{L_\omega^p} dt' \\ &\lesssim \lambda^{-1} + \tilde{r}^{-1} \int_u^t \|\pi\|_{L_\omega^p} dt'. \end{aligned} \quad (8.81)$$

It follows by using (8.22) and (8.1) that

$$\left(\int_0^{\tau_*} \sup_{0 \leq u' \leq t} \|z\|_{L_\omega^p(S_{t,u'})}^2 dt\right)^{\frac{1}{2}} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0} + \|\pi\|_{L_t^2 L^\infty(\mathcal{D}^+)} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}.$$

Thus we obtained the first estimate in (8.75).

Recall from the line of (8.64), the first term of $\nabla\pi$ on the right hand side is a higher order linear term, which comes from differentiating $(\Xi_4 - k_{\mathbf{NN}})\widetilde{\text{tr}}\chi$ in (8.42). It becomes a singular term in particular in the region allowing \tilde{r} to be close to 0, since $\nabla\pi$ is not sufficiently smooth. This is the main hurdle for us to estimate $\|\tilde{r}\nabla z\|_{L_t^2 L_\omega^\infty L_\omega^p(\mathcal{D}^+)}$. Our strategy is to construct the quantity \mathcal{V} , whose major part is z . We will see in (8.88) that the transport equation of $\nabla\mathcal{V}$ does not contain such higher order linear term, which solves the issue of the potential singularity.

We recall from (7.40) that

$$\Xi_4 - 2k_{\mathbf{NN}} = L(\log c + \varrho) + 2c^{-2}L(v^i)\mathbf{N}^j\delta_{ij}.$$

By using (7.8),

$$\begin{aligned} \nabla(\Xi_4 - 2k_{\mathbf{NN}}) &= \nabla L(\log c + \varrho) + 2\nabla(L(v^i)\mathbf{N}^j c^{-2}\delta_{ij}) \\ &= \nabla_L \nabla(\log c + \varrho) + 2\nabla_L \nabla(v^i)\mathbf{N}^j c^{-2}\delta_{ij} + \chi \cdot \nabla(\log c + \varrho) \\ &\quad + 2\chi \cdot \nabla(v^i)\mathbf{N}^j c^{-2}\delta_{ij} + 2L(v^i)\nabla(\mathbf{N}^j g_{ij}) \\ &= \nabla_L \{\nabla(\log c + \varrho) + 2\nabla(v^i)\mathbf{N}^j c^{-2}\delta_{ij}\} - 2\nabla_L(\mathbf{N}^j c^{-2}\delta_{ij})\nabla(v^i) \\ &\quad + \chi \cdot (\nabla(\log c + \varrho) + 2\nabla(v^i)\mathbf{N}^j c^{-2}\delta_{ij}) + 2L(v^i)\nabla(\mathbf{N}^j g_{ij}). \end{aligned}$$

By using Proposition 7.1, we can derive symbolically

$$\nabla_L(\mathbf{N}^j c^{-2}\delta_{ij})\nabla(v^i), L(v^i)\nabla(\mathbf{N}^j g_{ij}) = \pi \cdot \pi + \chi \cdot \pi, \quad (8.82)$$

where $\pi \cdot \mathbf{g}$ has been regarded as a term of π . Indeed, from (7.2) and $2\mathbf{N} = L - \underline{L}$, we compute

$$2\langle \mathbf{D}_L \mathbf{N}, e_A \rangle = \langle \mathbf{D}_L L, e_A \rangle - \langle \mathbf{D}_L \underline{L}, e_A \rangle = -2\zeta_A = 2k_{\mathbf{AN}}.$$

Noting $\langle \nabla_L \mathbf{N}, \mathbf{N} \rangle = 0$, also by denoting $(\mathbf{g})\Gamma \cdot X$ as π , we can write $\nabla_L(\mathbf{N}^j c^{-2}\delta_{ij})\nabla(v^i) = \pi \cdot \pi \cdot \mathbf{g}$.

The symbolic formula for the other term in (8.82) can be obtained by noting $\nabla_A \mathbf{N}^i = \theta_{AB} e_B^i = (\chi_{AB} + k_{AB})e_B^i$ due to the first identity in (7.5).

For convenience, we denote $\pi_1 = \nabla(\log c + \varrho) + 2\nabla(v^i)\mathbf{N}^j c^{-2}\delta_{ij}$, which can be symbolically regarded as π . Thus

$$\nabla(\Xi_4 - 2k_{\mathbf{NN}}) = \nabla_L(\pi_1) + \chi \cdot \pi + \pi \cdot \pi. \quad (8.83)$$

In view of

$$\chi_{AB} = \frac{1}{2}\delta_{AB}\text{tr}\chi + \hat{\chi}_{AB} = \frac{1}{2}\delta_{AB}(\widetilde{\text{tr}}\chi - \Xi_4) + \hat{\chi}_{AB}, \quad (8.84)$$

and in view of the definition that $z = \widetilde{\text{tr}}\chi - \frac{2}{\tilde{r}}$, we derive from (8.83) the symbolic form

$$\nabla(\Xi_4 - 2k_{\mathbf{NN}}) = \nabla_L(\pi_1) + (\mathbf{A} + \tilde{r}^{-1}) \cdot \pi. \quad (8.85)$$

On the other hand, we multiply (8.48) by \mathbf{b} and apply (7.11)

$$L(\mathbf{b}\widetilde{\text{tr}}\chi) + \frac{1}{2}\mathbf{b}(\widetilde{\text{tr}}\chi)^2 = \frac{2}{\tilde{r}}(\Xi_4 - 2k_{\mathbf{NN}}) + (\Xi_4 - 2k_{\mathbf{NN}})(\mathbf{b}\widetilde{\text{tr}}\chi - \frac{2}{\tilde{r}}) + \mathbf{b} \cdot G,$$

with

$$G = e^\varrho \text{curl} \Omega_{\mathbf{N}} - |\hat{\chi}|^2 + \pi \cdot \pi, \quad (8.86)$$

where the last term of G is a symbolic representation.

We differentiate the above transport equation with the help of (7.8)

$$\begin{aligned} \nabla_L \nabla(\mathbf{btr}\chi) + \widetilde{\text{tr}\chi} \nabla(\mathbf{btr}\chi) &= -\chi \cdot \nabla(\mathbf{btr}\chi) + \frac{1}{2} \nabla \mathbf{b}(\widetilde{\text{tr}\chi})^2 + \frac{2}{\tilde{r}} \nabla(\Xi_4 - 2k_{\mathbf{NN}}) \\ &\quad + \nabla\{(\Xi_4 - 2k_{\mathbf{NN}})(\mathbf{btr}\chi - \frac{2}{\tilde{r}}) + \mathbf{b} \cdot G\}. \end{aligned} \quad (8.87)$$

By using (8.84)

$$\begin{aligned} \nabla_L \nabla(\mathbf{btr}\chi) + \frac{3}{2} \widetilde{\text{tr}\chi} \nabla(\mathbf{btr}\chi) - \frac{2}{\tilde{r}} \nabla(\Xi_4 - 2k_{\mathbf{NN}}) - \mathbf{b} \nabla G \\ = (-\hat{\chi} + \pi) \nabla(\mathbf{btr}\chi) + \nabla \pi(\mathbf{btr}\chi - \frac{2}{\tilde{r}}) + \nabla \mathbf{b}(G + (\widetilde{\text{tr}\chi})^2), \end{aligned}$$

where the right hand side is a symbolic expression. In view of $z = \widetilde{\text{tr}\chi} - \frac{2}{\tilde{r}}$, applying (8.85) to the above identity gives

$$\begin{aligned} \nabla_L (\nabla(\mathbf{btr}\chi) - \frac{2}{\tilde{r}} \pi_1) + \frac{3}{2} \widetilde{\text{tr}\chi} (\nabla(\mathbf{btr}\chi) - \frac{2}{\tilde{r}} \pi_1) \\ = \mathbf{A}(\nabla(\mathbf{btr}\chi) - \frac{2}{\tilde{r}} \pi_1) + \tilde{r}^{-1} \pi(\mathbf{A} + \tilde{r}^{-1}) + \nabla \pi(\mathbf{btr}\chi - \frac{2}{\tilde{r}}) \\ + \nabla \mathbf{b}(G + (\widetilde{\text{tr}\chi})^2) + \mathbf{b} \nabla G. \end{aligned}$$

With $\tilde{G} = -|\hat{\chi}|^2 + \pi \cdot \pi$, we recast $G = e^\varrho \text{curl } \Omega_{\mathbf{N}} + \tilde{G}$ in view of (8.50). Hence

$$\nabla G = \nabla(e^\varrho \text{curl } \Omega_{\mathbf{N}}) + \nabla \tilde{G}.$$

Substituting the trace decomposition (8.50) to the first term on the right hand side yields

$$\begin{aligned} \nabla_L (\nabla_A \mathcal{Y} - \mathbf{b} e^\varrho (\text{curl } \Omega)_A - \frac{2}{\tilde{r}} \pi_1) + \frac{3}{2} \widetilde{\text{tr}\chi} (\nabla_A \mathcal{Y} - \mathbf{b} e^\varrho (\text{curl } \Omega)_A - \frac{2}{\tilde{r}} \pi_1) \\ = \mathbf{A}(\nabla \mathcal{Y} - \mathbf{b} e^\varrho (\text{curl } \Omega)_A - \frac{2}{\tilde{r}} \pi_1) + \mathbf{b}(\tilde{r}^{-1} + \mathbf{A}) e^\varrho \text{curl } \Omega + \nabla \pi \cdot \mathcal{Y} \\ + \tilde{r}^{-1} \pi(\mathbf{A} + \tilde{r}^{-1}) + \nabla \mathbf{b}(\tilde{G} + (\widetilde{\text{tr}\chi})^2) + \mathbf{b} \nabla \tilde{G} \\ + \mathbf{b} e^\varrho (e_{A_i} \Pi^{ij} \epsilon_{jm}{}^l (\text{curl }^2 \Omega)_l \mathbf{N}^m + (\pi + \chi) \partial \Omega \cdot X), \end{aligned} \quad (8.88)$$

where we also used (7.11) and (7.5).

We note in general for $C_u \cap \bar{\mathcal{D}}^+$, combining (8.10) and (8.1) gives

$$\|\mathbf{A}\|_{L_t^2 L_\omega^\infty(C_u)} \lesssim \lambda^{-\frac{1}{2}}. \quad (8.89)$$

By using Lemma 8.1 (i), when $u \geq 0$, $\tilde{r}^{\frac{3}{2}} \nabla \mathcal{Y} \rightarrow 0$, $\tilde{r} \pi_1 \rightarrow 0$ and $\tilde{r} |\text{curl } \Omega| \rightarrow 0$ as $t \rightarrow u$. Due to the fact that $\|\mathbf{A}\|_{L_\omega^\infty L_t^1} \lesssim 1$ derived from (8.89), we apply Lemma 8.7 to (8.88) to derive

$$\begin{aligned} \tilde{r}^3 |\nabla \mathcal{Y} - \mathbf{b} e^\varrho (\text{curl } \Omega)_A - \frac{2}{\tilde{r}} \pi_1| \\ \lesssim \int_{t_{\min}}^t \tilde{r}^3 \left| \nabla \pi \cdot \mathcal{Y} + \tilde{r}^{-1} \pi(\mathbf{A} + \tilde{r}^{-1}) + \nabla \mathbf{b}(\tilde{G} + (\widetilde{\text{tr}\chi})^2) + \mathbf{b} \nabla \tilde{G} \right| dt' \\ + \int_{t_{\min}}^t \tilde{r}^3 \mathbf{b} e^\varrho |e_{A_i} \Pi^{ij} \epsilon_{jm}{}^l (\text{curl }^2 \Omega)_l \mathbf{N}^m + (\tilde{r}^{-1} + \mathbf{A}) e^\varrho \text{curl } \Omega + (\pi + \chi) \partial \Omega \cdot X| dt', \end{aligned}$$

where $t_{\min} = u$, since $u \geq 0$.

By using $|\mathbf{b} - 1| \leq \frac{1}{2}$ and (8.4), $\zeta = \nabla \log \mathbf{b} + \pi$ and $z = \widetilde{\text{tr}}\chi - \frac{2}{\tilde{r}}$, we can derive symbolically,

$$\begin{aligned} & \tilde{r}^3 |\nabla \mathcal{Y} - \mathbf{b}e^\varrho(\text{curl } \Omega)_A - \frac{2}{\tilde{r}}\pi_1| \\ & \lesssim \int_{t_{\min}}^t \tilde{r}^3 \left(|\nabla \pi \cdot \mathcal{Y}| + \tilde{r}^{-1} |\mathbf{A} \cdot \mathbf{A}| + (|\zeta| + |\pi|) \cdot (|\tilde{G}| + \tilde{r}^{-2}) + |\nabla \tilde{G}| \right) dt' \\ & + \int_{t_{\min}}^t \tilde{r}^3 (|\text{curl}^2 \Omega| + \tilde{r}^{-1} |\partial \Omega| + |\mathbf{A} \cdot \partial \Omega|) dt'. \end{aligned}$$

Hence

$$\begin{aligned} & \tilde{r} \|\nabla \mathcal{Y} - \mathbf{b}e^\varrho(\text{curl } \Omega)_A - \frac{2}{\tilde{r}}\pi_1\|_{L_\omega^p} \\ & \lesssim \int_{t_{\min}}^t \tilde{r} (\|\nabla \pi \cdot \mathcal{Y}\|_{L_\omega^p} + \|\text{curl}^2 \Omega\|_{L_\omega^p}) dt' + \tilde{r}^{-1} \int_{t_{\min}}^t (\|\zeta\|_{L_\omega^p} + \|\pi\|_{L_\omega^p} + \|\tilde{r} \partial \Omega\|_{L_\omega^p}) dt' \\ & + \int_{t_{\min}}^t \tilde{r} \left(\|\mathbf{A}(\tilde{r}^{-1} |\mathbf{A}| + |\tilde{G}|)\|_{L_\omega^p} + \|\nabla \tilde{G}\|_{L_\omega^p} + \|\mathbf{A} \cdot \partial \Omega\|_{L_\omega^p} \right). \end{aligned} \quad (8.90)$$

Since $|\nabla \tilde{G}| \lesssim |\nabla \hat{\chi}| \cdot |\hat{\chi}| + |\nabla \pi| |\pi|$, we use (8.89), (8.65), the second estimate in (8.6) and (8.28) to obtain

$$\begin{aligned} & \|\tilde{r} \nabla \tilde{G}\|_{L_t^1 L_\omega^p(C_u)} + \|\tilde{r} \mathbf{A} \cdot \partial \Omega\|_{L_t^1 L_\omega^p(C_u)} \\ & \lesssim \|\mathbf{A}\|_{L_t^2 L_\omega^\infty(C_u)} \|\tilde{r}(\nabla \hat{\chi}, \nabla \pi, \partial \Omega)\|_{L_t^2 L_\omega^p(C_u)} \lesssim \lambda^{-1}; \end{aligned} \quad (8.91)$$

and for the lower order terms since $\tilde{G} = \mathbf{A} \cdot \mathbf{A}$,

$$\begin{aligned} & \|\tilde{r} \mathbf{A} \cdot \mathbf{A} \cdot (\mathbf{A} + \tilde{r}^{-1})\|_{L_t^1 L_\omega^p(C_u)} \\ & \lesssim \|\mathbf{A} \cdot \mathbf{A}\|_{L_t^1 L_\omega^p(C_u)} + \|\mathbf{A}\|_{L_t^2 L_\omega^\infty(C_u)} \|\tilde{r} \mathbf{A} \cdot \mathbf{A}\|_{L_t^2 L_\omega^p(C_u)} \\ & \lesssim \|\mathbf{A}\|_{L_t^2 L_\omega^\infty(C_u)} (\|\mathbf{A}\|_{L_t^2 L_\omega^p(C_u)} + \|\tilde{r} \mathbf{A} \cdot \mathbf{A}\|_{L_t^2 L_\omega^p(C_u)}) \lesssim \lambda^{-1}, \end{aligned} \quad (8.92)$$

where we used (8.89), (8.38) and (8.40).

Using (8.28) and (8.70) implies

$$\|\tilde{r} \nabla \pi \cdot \mathcal{Y}\|_{L_t^1 L_\omega^p(C_u)} \lesssim \|\mathcal{Y}\|_{L_t^2 L_\omega^\infty(C_u)} \|\tilde{r} \nabla \pi\|_{L_t^2 L_\omega^p(C_u)} \lesssim \lambda^{-1}.$$

By using the above estimate together with (8.91), (8.92) and the second estimate in (8.23), we can bound on $S_{t,u}$ contained in \mathcal{D}^+ ,

$$\tilde{r} \|\nabla \mathcal{Y} - \mathbf{b}e^\varrho(\text{curl } \Omega)_A - \frac{2}{\tilde{r}}\pi_1\|_{L_\omega^p} \lesssim \lambda^{-1} + \tilde{r}^{-1} \int_{t_{\min}}^t (\|\zeta\|_{L_\omega^p} + \|\pi\|_{L_\omega^p}) dt'.$$

It then follows by using (8.22), (8.1) and (8.73) that

$$\begin{aligned} & \|\tilde{r}(\nabla \mathcal{Y} - \mathbf{b}e^\varrho(\text{curl } \Omega)_A - \frac{2}{\tilde{r}}\pi_1)\|_{L_t^2 L_u^\infty L_\omega^p(\mathcal{D}^+)} \lesssim \|\zeta, \pi\|_{L_t^2 L_x^\infty(\mathcal{D}^+)} + \lambda^{-\frac{1}{2}-4\epsilon_0} \\ & \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}. \end{aligned}$$

Using (8.1) and (8.65), we conclude

$$\|\tilde{r} \nabla \mathcal{Y}\|_{L_t^2 L_u^\infty L_\omega^p(\mathcal{D}^+)} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0},$$

as desired in the second estimates of (8.75).

By using (8.2), $\nabla z = \mathbf{b}^{-1}\nabla\mathcal{Z} - \nabla \log \mathbf{b}(z - 2\mathcal{U}) + 2\nabla\mathcal{U}$; and by using (7.5) $\nabla \log \mathbf{b} = \zeta + \pi$. Hence using (8.16) and (8.36) we derive

$$\begin{aligned}\tilde{r}\|\nabla z\|_{L_\omega^p} &\lesssim \tilde{r}\|\nabla\mathcal{Z}\|_{L_\omega^p} + \|\tilde{r}^{\frac{1}{2}}\nabla \log \mathbf{b}\|_{L_\omega^p}\|\tilde{r}^{\frac{1}{2}}(|z| + |\mathcal{U}|)\|_{L_\omega^\infty} + \|\nabla\mathbf{b}\|_{L_\omega^\infty} \\ &\lesssim \tilde{r}\|\nabla\mathcal{Z}\|_{L_\omega^p} + \lambda^{-\frac{1}{2}}\|\tilde{r}^{\frac{1}{2}}(|z| + |\mathcal{U}|)\|_{L_\omega^\infty} + \|\zeta + \pi\|_{L_\omega^\infty}.\end{aligned}$$

Taking the $L_t^2 L_u^\infty$ norm in \mathcal{D}^+ on both sides of the above inequality, with the help of (8.4), (8.37), (8.73), (8.1), and the estimates of $\nabla\mathcal{Z}$ in (8.75), we obtain

$$\|\tilde{r}\nabla z\|_{L_t^2 L_u^\infty L_\omega^p(\mathcal{D}^+)} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0} + \lambda^{-4\epsilon_0}\|\mathcal{U}\|_{L_t^2 L^\infty(\mathcal{D}^+)} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}$$

as desired in (8.75). In the region where $-\mathbf{v}_* \leq u \leq \frac{5t}{6}$, using (8.5) implies

$$\|\tilde{r}\nabla z\|_{L_\omega^p} \lesssim t^{-\frac{1}{2}}\|\tilde{r}^{\frac{3}{2}}\nabla z\|_{L_\omega^p(S)} \lesssim \lambda^{-\frac{1}{2}}t^{-\frac{1}{2}}.$$

Taking $L_t^{\frac{q}{2}} L_u^\infty$ norm in $\widetilde{\mathcal{D}^+}$, we can conclude (8.77).

Using (8.75) and (8.20) we can derive (8.76). For the estimate of z in (8.79), it remains to consider the region $u < 0$, which can be derived immediately by integrating (8.71).

Next, consider the estimate for $\hat{\chi}$ in (8.78). In view of (8.44), applying (8.33) and Proposition 8.10 yields

$$\begin{aligned}\|\hat{\chi}\|_{L^\infty(S_{t,u})} &\lesssim \|\mathcal{D}_2^{-1}(\nabla z)\|_{L^\infty(S_{t,u})} + \|\mathcal{D}_2^{-1}(\nabla\pi + \tilde{r}^{-1}\pi + \mathbf{A} \cdot \pi)\|_{L^\infty(S_{t,u})} \\ &\lesssim \|\tilde{r}^{1-\frac{2}{p}}\nabla z\|_{L^p(S_{t,u})} + \|\mu^{0+}P_\mu\tilde{\pi}\|_{l_\mu^2 L^\infty(S_{t,u})} + \|\tilde{r}^{1-\frac{2}{p}}(\mathbf{A} \cdot \pi, \tilde{r}^{-1}\pi)\|_{L^p(S_{t,u})} \\ &\lesssim \|\tilde{r}^{1-\frac{2}{p}}\nabla z\|_{L^p(S_{t,u})} + \|\mu^{0+}P_\mu\tilde{\pi}\|_{l_\mu^2 L^\infty(S_{t,u})} + (1 + \lambda^{-4\epsilon_0})\|\pi\|_{L^\infty(S_{t,u})},\end{aligned}$$

where $0 < 1 - \frac{2}{p} < s' - 2$, for the last inequality we used (8.39). By virtue of (8.1) and the estimate for ∇z in (8.75), we have

$$\|\hat{\chi}\|_{L_t^2 L_x^\infty(\mathcal{D}^+)} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0};$$

and by using (8.1) and (8.77) with $2 < q < 4$, we have

$$\|\hat{\chi}\|_{L_t^{\frac{q}{2}} L_x^\infty(\widetilde{\mathcal{D}^+})} \lesssim \lambda^{\frac{2}{q}-1-4\epsilon_0(\frac{4}{q}-1)}.$$

We therefore have obtained the estimates for $\hat{\chi}$ in (8.78) and (8.79). Thus the proof of Proposition 8.19 is completed. \square

In summary, (8.73) is stronger than the estimates of ζ stated in (8.9) and (8.7), the estimates (8.76), (8.78) and (8.79) are included in (8.9) and (8.7). Since $\text{tr}\chi - \frac{2}{\tilde{r}} = z + \pi$, using (8.1) and combining the estimates for z in (8.9) and (8.7), the estimates of $\text{tr}\chi - \frac{2}{\tilde{r}}$ therein can be obtained immediately. Thus, the proof of Proposition 8.2 is completed.

9. REGULARITY OF THE CONFORMAL METRIC AND THE MASS ASPECT FUNCTION

Let us set in \mathcal{D}^+

$$L\sigma = \frac{1}{2}\Xi_L, \quad \sigma(\Gamma^+) = 0 \tag{9.1}$$

where Γ^+ is the time axis. In order to prove Theorem 5.3, we need to carry out the conformal change of metric in the spacetime region \mathcal{D}^+ by introducing the metric $\tilde{\mathbf{g}} = e^{2\sigma}\mathbf{g}$. The conformal method was introduced in [45] to treat the term $-\Xi_L$ in $\text{tr}\chi - \widetilde{\text{tr}\chi}$. It is in particular crucial for solving the significant difficulty caused by the weak regularity on $\nabla\text{tr}\chi$ and μ due to the rough data in [45] for the equation (1.12). The rough Ξ_L derivative causes the same hurdle in the general acoustical spacetime as for the irrotational case. Therefore, in this section, we provide the control on σ and μ required for proving Theorem 5.3. This theorem is the main building

block to complete the dispersive estimate as indicated in Section 5.2. By the completion of the section, we will be able to achieve the complete set of estimates for the geometric quantities required by reproducing the proof of Theorem 5.3 in [45, Section 7].

In this section, instead of bounding $\tilde{\mu}$ directly as in [45], we introduce a further normalization on the mass aspect function in (9.9) to cope with the issue of the rough vorticity derivative, with the help of the transport equation of $\text{curl } \Omega$ in (1.9). It turns out that the influence of the rough vorticity can be reduced to a lower order, which can be seen from Proposition 9.3 and the resulting estimates in Proposition 9.4.

We recall the preliminary estimates, which can be obtained in the exact same way as in [45, Lemma 6.1].

Lemma 9.1. *Let $0 \leq 1 - \frac{2}{p} < s - 2$. Within \mathcal{D}^+ , there hold*

$$\begin{aligned} \|\tilde{r}^{\frac{1}{2}} L\sigma\|_{L_{\omega}^{2p}(C_u)} + \|r^{\frac{1}{2} - \frac{2}{p}} \nabla \sigma\|_{L_t^p L^\infty(C_u)} + \|\nabla \sigma\|_{L_t^2 L^p(C_u)} &\lesssim \lambda^{-\frac{1}{2}} \\ \|\sigma\|_{L^\infty} &\lesssim \lambda^{-8\epsilon_0}, \quad \|\tilde{r}^{-\frac{1}{2}} \sigma\|_{L^\infty} \lesssim \lambda^{-\frac{1}{2} - 4\epsilon_0}. \end{aligned} \quad (9.2)$$

The above estimates of $\nabla \sigma$ are much weaker for our purpose, the improved estimate on $\nabla \sigma$ will be achieved with the help of the estimate of a normalized mass aspect function $\tilde{\mu}$ (see (9.8) for the definition). Let us first recall the null transport equation for the mass aspect function μ from [45, (6.12)].

Lemma 9.2.

$$\begin{aligned} L\mu + \text{tr}\chi\mu &= R(\mu) - k_{\mathbf{NN}} L\text{tr}\chi + 2(\zeta_A - \zeta_A) \nabla_A \text{tr}\chi + \frac{1}{2} (\text{tr}\chi \hat{\chi} \cdot \hat{\chi} + \text{tr}\chi |\hat{\chi}|^2) \\ &\quad + \text{tr}\chi \left(d\sharp v \zeta + |\zeta|^2 + \frac{1}{2} \delta^{AB} \mathbf{R}_{A34B} + L(k_{\mathbf{NN}}) - (d\sharp v \pi + \mathbf{E}) \right) \\ &\quad - 2\hat{\chi}_{AB} (2\nabla_A \zeta_B + k_{\mathbf{NN}} \hat{\chi}_{AB} + 2\zeta_A \zeta_B + \mathbf{R}_{A43B}), \end{aligned} \quad (9.3)$$

where $\mathbf{E} = \mathbf{A} \cdot \pi + \text{tr}\chi \cdot \pi$, and

$$R(\mu) := -\underline{L}\mathbf{R}_{44} - \text{tr}\chi \mathbf{R}_{34} - \frac{1}{2} \text{tr}\chi \mathbf{R}_{44}.$$

For deriving the above equation, we used (7.12), (7.14), (7.7) and (7.19).

By virtue of (7.29), (7.30), Proposition 7.1 and the fact that $S = \pi \cdot \pi$, we can derive

$$\begin{aligned} \mathbf{R}_{44} &= L(\Xi_L) + k_{\mathbf{NN}} \Xi_L + \pi \cdot \pi - e^g \delta_{ij} \mathbf{N}^j \text{curl } \Omega^i, \\ \mathbf{R}_{34} &= \frac{1}{2} (\underline{L}(\Xi_L) + L(\Xi_L)) + \Xi \cdot (\zeta + \underline{\zeta}) + k_{\mathbf{NN}} \cdot \Xi + \pi \cdot \pi, \\ -\underline{L}\mathbf{R}_{44} &= -\underline{L}L(\Xi_L) - \underline{L}(k_{\mathbf{NN}}) \Xi_L + k_{\mathbf{NN}} \cdot (\zeta + k) \cdot \Xi + k \cdot \mathbf{D}\Xi \\ &\quad + \delta_{ij} \underline{L}(e^g \mathbf{N}^j \text{curl } \Omega^i) + \underline{L}(\pi \cdot \pi), \end{aligned}$$

for which we used (1.8), (7.2) and (7.3) to simplify the right hand sides of the above formulas.

Hence

$$\begin{aligned} R(\mu) &= -\underline{L}L(\Xi_L) - \frac{1}{2} \text{tr}\chi \underline{L}(\Xi_L) - \frac{1}{2} \text{tr}\chi L(\Xi_L) - \frac{1}{2} \text{tr}\chi L(\Xi_L) \\ &\quad + \delta_{ij} \underline{L}(e^g \mathbf{N}^j \text{curl } \Omega^i) + \frac{1}{2} e^g \text{tr}\chi \mathbf{N}^i \text{curl } \Omega^j + \text{tr}\chi \pi \cdot \pi + \mathbf{A} \cdot \mathbf{D}\tilde{\pi} + \mathbf{A}^3. \end{aligned} \quad (9.4)$$

The first term of the last line can be written as

$$\delta_{ij} \underline{L}(e^g \mathbf{N}^j \text{curl } \Omega^i) = \delta_{ij} (-L + 2\mathbf{T})(e^g \mathbf{N}^j \text{curl } \Omega^i). \quad (9.5)$$

Meanwhile in view of (7.10), (7.16) and (9.1), we have

$$\begin{aligned} L\mathring{\Delta}\sigma + \text{tr}\chi\mathring{\Delta}\sigma &= \frac{1}{2}\mathring{\Delta}(\Xi_4) - 2\hat{\chi}_{AC}\mathring{\nabla}_A\mathring{\nabla}_C\sigma - \mathring{\nabla}_A\text{tr}\chi\mathring{\nabla}_A\sigma - 2\delta^{AB}\mathbf{R}_{CA4B} \cdot \mathring{\nabla}_C\sigma \\ &\quad - \left(\frac{1}{2}k_{\mathbf{AN}}\text{tr}\chi - \hat{\chi}_{AB}k_{B\mathbf{N}} \right) \mathring{\nabla}_A\sigma - \chi_{AB}\zeta_A\mathring{\nabla}_B\sigma. \end{aligned} \quad (9.6)$$

Applying (7.6) to Ξ_4 gives the null decomposition of $\square_{\mathbf{g}}(\Xi_4)$

$$\square_{\mathbf{g}}(\Xi_4) = -L\mathring{L}(\Xi_4) + \mathring{\Delta}(\Xi_4) - \frac{1}{2}\text{tr}\chi\mathring{L}(\Xi_4) - \frac{1}{2}\text{tr}\chi\mathring{L}(\Xi_4) + 2\zeta^A\mathring{\nabla}_A(\Xi_4) + k_{\mathbf{NN}}\mathring{L}(\Xi_4). \quad (9.7)$$

Apart from the terms contributed by vorticity, we observe that the leading terms of $R(\mu)$ and the right hand side of (9.6) contain all the second order terms of (9.7). As in [45, Section 6] we can derive a transport equation for the renormalized mass aspect function

$$\check{\mu} = 2\mathring{\Delta}\sigma + \mu - \text{tr}\chi k_{\mathbf{NN}} + \frac{1}{2}\text{tr}\chi\mathring{\Xi}_L. \quad (9.8)$$

To further cancel the higher order terms of vorticity in $R(\mu)$, we will derive the transport equation with the help of the decomposition of (9.5) for the following quantity

$$\tilde{\mu} = \check{\mu} + \delta_{ij}e^\varrho\mathbf{N}^i\text{curl}\Omega^j. \quad (9.9)$$

Similar to the calculation in [45, (6.15) and Lemma 6.2] for $\check{\mu}$, by using (9.3)-(9.7) and (7.12) we derive for $\tilde{\mu}$ that

$$\begin{aligned} L\tilde{\mu} + \text{tr}\chi\tilde{\mu} &= \square_{\mathbf{g}}(\Xi_4) - 2 \left(2\delta^{AB}\mathbf{R}_{CA4B} + \frac{1}{2}k_{\mathbf{AN}}\text{tr}\chi - \hat{\chi} \cdot k + \chi \cdot \zeta \right) \mathring{\nabla}\sigma \\ &\quad + 2(\zeta - \tilde{\zeta})\mathring{\nabla}\text{tr}\chi - 4\hat{\chi} \cdot \mathring{\nabla}\tilde{\zeta} - 2\hat{\chi}_{AB}(k_{\mathbf{NN}}\hat{\chi}_{AB} + 2\zeta_A\zeta_B + \mathbf{R}_{A43B}) \\ &\quad + \text{tr}\chi \left(\text{div}\zeta + |\zeta|^2 + \frac{1}{2}\delta^{AB}\mathbf{R}_{A34B} \right) + \frac{1}{2}(\text{tr}\chi\hat{\chi} \cdot \hat{\chi} + \text{tr}\chi|\hat{\chi}|^2) \\ &\quad + 2k_{\mathbf{NN}}(|\hat{\chi}|^2 + k_{\mathbf{NN}}\text{tr}\chi + \mathbf{R}_{44}) + \mathbf{A} \cdot (\mathbf{D}\tilde{\pi} + \mathbf{E}) + \text{tr}\chi(\text{div}\pi + \mathbf{E}) \\ &\quad + 2\delta_{ij}\mathbf{T}(e^\varrho\text{curl}\Omega^i\mathbf{N}^j) + (\text{tr}\chi + \frac{1}{2}\text{tr}\chi)\delta_{ij}e^\varrho\mathbf{N}^i\text{curl}\Omega^j, \end{aligned} \quad (9.10)$$

where $\tilde{\zeta} = \mathring{\nabla}\sigma + \zeta$ and $\mathbf{E} = \mathbf{A} \cdot \mathbf{A} + \text{tr}\chi \cdot \mathbf{A}$.

Next we simplify the above equation in two steps.

Step 1. Recall (1.9) with $\mathfrak{C} = e^{-\varrho}\text{curl}\Omega$. Also using the first equation in (1.4), we derive

$$\mathbf{T}(\text{curl}\Omega^i) = \partial v \partial \Omega. \quad (9.12)$$

By using Proposition 7.1 and (7.7), we can compute $\mathbf{D}_{\mathbf{T}}\mathbf{N} = \frac{1}{4}(\mathbf{D}_L L - \mathbf{D}_{\mathring{L}}\mathring{L} + [\mathring{L}, L]) = \zeta_A e_A$. Therefore,

$$\mathbf{T}(\mathbf{N}^i) = \mathbf{D}_{\mathbf{T}}\mathbf{N} + \pi = \zeta_A e_A^i + \pi.$$

Denote the line of (9.11) by \mathcal{J} for which we derive

$$\begin{aligned} \mathcal{J} &= e^\varrho(\mathbf{T}\varrho\mathbf{N}^i\text{curl}\Omega^j\delta_{ij} + (\zeta + \pi)\text{curl}\Omega + \partial v \cdot \partial\Omega \cdot \mathbf{N}^i) + (\text{tr}\chi + \frac{1}{2}\text{tr}\chi)\delta_{ij}e^\varrho\mathbf{N}^i\text{curl}\Omega^j \\ &= e^\varrho\mathbf{A}\partial\Omega \cdot X + \tilde{r}^{-1}\delta_{ij}e^\varrho\mathbf{N}^i\text{curl}\Omega^j. \end{aligned} \quad (9.13)$$

Step 2. We next compute the term $\square_{\mathbf{g}}(\Xi_L)$ by using (7.42),

$$\square_{\mathbf{g}}(\Xi_L) = 2\square_{\mathbf{g}}(\mathbf{T}(\log c + \varrho)) + \square_{\mathbf{g}}(L(\log c - \varrho)). \quad (9.14)$$

Note that there holds for scalar functions ϕ the following commutation formula,

$$\begin{aligned} [\square_{\mathbf{g}}, \mathbf{T}]\phi &= -\mathbf{T}\text{Tr}k\mathbf{T}\phi + [\Delta_g, \mathbf{T}]\phi \\ &= -\mathbf{T}\text{Tr}k\mathbf{T}\phi + \nabla_g(k\nabla_g\phi) + k \cdot \nabla_g^2\phi + \mathbf{R}^{\mu i}_{\mathbf{T}i}\mathbf{D}_\mu\phi \\ &= \mathbf{g}(\partial\tilde{\pi} \cdot \partial\phi + \partial^2\phi \cdot \tilde{\pi}). \end{aligned} \quad (9.15)$$

Combining the equation (1.8) with the above commutation formula for $\phi = \log c + \varrho$ gives

$$\square_{\mathbf{g}}\mathbf{T}(\log c + \varrho) = f(\varrho)\partial\tilde{\pi} \cdot \tilde{\pi} + f(\varrho) \cdot \tilde{\pi}^3, \quad (9.16)$$

where $f(\varrho)$ represents smooth functions of ϱ , and we used the fact that $\tilde{\pi} \cdot \mathbf{g}$ still can be denoted by $\tilde{\pi}$.

Next we employ the null decomposition of the operator $\square_{\mathbf{g}}$ in (7.6), (7.7), (7.9) and (7.10) to compute for scalar functions ϕ

$$\begin{aligned} [\square_{\mathbf{g}}, L]\phi &= L([L, \underline{L}]\phi) + [\underline{\Delta}, L]\phi + \frac{1}{2}L\text{tr}\chi L\phi + 2\underline{\zeta} \cdot [\nabla, L]\phi - 2\nabla_L\underline{\zeta} \cdot \nabla\phi \\ &\quad + (k_{\mathbf{NN}} - \frac{1}{2}\text{tr}\chi)[\underline{L}, L]\phi - \nabla_L(k_{\mathbf{NN}} - \frac{1}{2}\text{tr}\chi)\underline{L}\phi \\ &= \nabla_L(2(\underline{\zeta} - \zeta) \cdot \nabla\phi - 2k_{\mathbf{NN}}\mathbf{N}\phi) + \text{tr}\chi\underline{\Delta}\phi + 2\hat{\chi} \cdot \nabla^2\phi + \text{d}\not{v}\chi_C\nabla_C\phi \\ &\quad - (\text{tr}\chi\underline{\zeta} - \chi_{AC}\underline{\zeta}_A - \delta^{AB}\mathbf{R}_{CA4B})\nabla_C\phi + \frac{1}{2}L\text{tr}\chi L\phi + 2\underline{\zeta} \cdot \chi \cdot \nabla\phi - 2\nabla_L\underline{\zeta} \cdot \nabla\phi \\ &\quad + (k_{\mathbf{NN}} - \frac{1}{2}\text{tr}\chi)(2(\zeta - \underline{\zeta}) \cdot \nabla\phi + 2k_{\mathbf{NN}}\mathbf{N}\phi) - L(k_{\mathbf{NN}} - \frac{1}{2}\text{tr}\chi)\underline{L}\phi. \end{aligned} \quad (9.17)$$

We now claim that

$$L\text{tr}\chi = -2\tilde{r}^{-2} + \mathcal{D}_*\pi + \mathbf{E}, \quad (9.18)$$

$$L\text{tr}\underline{\chi} = \text{d}\not{v}\pi + \nabla\pi + \mathbf{E} + 2\tilde{r}^{-2}, \quad (9.19)$$

$$\nabla_L\underline{\zeta} = \text{tr}\chi\mathbf{A} + \mathcal{D}_*\pi + \mathbf{E}, \quad (9.20)$$

$$\text{d}\not{v}\chi = \nabla\text{tr}\chi + \nabla\pi + \mathbf{E}, \quad (9.21)$$

where $\mathbf{E} = \mathbf{A} \cdot \mathbf{A} + \text{tr}\chi \cdot \mathbf{A}$ and $\mathbf{A} = \hat{\chi}, z, \pi, \zeta$.

To see (9.18), we recast (7.12) by using Lemma 7.8 (i),

$$L\text{tr}\chi = -\frac{1}{2}(z + \pi + \frac{2}{\tilde{r}})\text{tr}\chi + \mathcal{D}_*\pi + \mathbf{E} = -\tilde{r}^{-1}\text{tr}\chi + \mathcal{D}_*\pi + \mathbf{E}.$$

Note

$$\tilde{r}^{-1}\text{tr}\chi = (\tilde{r}^{-1} - \frac{1}{2}\text{tr}\chi)(\text{tr}\chi - \frac{2}{\tilde{r}}) + \frac{2}{\tilde{r}^2} + \frac{1}{2}\text{tr}\chi(\text{tr}\chi - \frac{2}{\tilde{r}}) = 2\tilde{r}^{-2} + \mathbf{E}. \quad (9.22)$$

Combining the above two identities, (9.18) follows as a consequence.

By using (7.24), we can derive

$$\text{tr}\chi\text{tr}\chi = -\text{tr}\chi^2 - 2\text{tr}k \cdot \text{tr}\chi = -(\text{tr}\chi - \frac{2}{\tilde{r}} + \frac{2}{\tilde{r}})\text{tr}\chi + \mathbf{E} = -\frac{2}{\tilde{r}}\text{tr}\chi + \mathbf{E}.$$

Combining the above identity with (9.22), (9.19) can be obtained in view of (7.14) and the decomposition of curvature in Lemma 7.8 (iv). (9.20) can be obtained in view of (7.15) and Lemma 7.8 (i). (9.21) can be obtained in view of (7.16) and Lemma 7.8 (ii).

Now we set $\phi = \log c - \varrho$ in (9.17), then substitute the symbolic formulas (9.18)-(9.21) to (9.17). Also by using Lemma 7.8 (ii), we can derive

$$[\square_{\mathbf{g}}, L](\log c - \varrho) = \tilde{r}^{-1} \text{d}\not{v}\pi + \tilde{r}^{-2}\pi + (\nabla\text{tr}\chi, \mathcal{D}_*\pi, \mathbf{E}) \cdot \mathbf{A}. \quad (9.23)$$

By using (1.8)

$$L\Box_{\mathbf{g}}(\log c - \varrho) = f(\varrho)\mathcal{D}_*\tilde{\pi} \cdot \tilde{\pi} + f(\varrho) \cdot (\tilde{\pi})^3.$$

Combining the above two identities yields

$$\Box_{\mathbf{g}}(L(\log c - \varrho)) = f(\varrho)\mathcal{D}_*\tilde{\pi} \cdot \tilde{\pi} + f(\varrho) \cdot (\tilde{\pi})^3 + \tilde{r}^{-1} \mathfrak{d}\mathfrak{I}v \pi + \tilde{r}^{-2} \pi + (\nabla \text{tr} \chi, \mathcal{D}_*\pi, \mathbf{E}) \cdot \mathbf{A}.$$

Now combining the above identity with (9.16) in view of (9.14) gives

$$\Box_{\mathbf{g}}(\Xi_L) = f(\varrho)\mathbf{D}\tilde{\pi} \cdot \tilde{\pi} + f(\varrho) \cdot (\tilde{\pi})^3 + \tilde{r}^{-1} \mathfrak{d}\mathfrak{I}v \pi + \tilde{r}^{-2} \pi + (\nabla \text{tr} \chi, \mathbf{D}\pi, \mathbf{E}) \cdot \mathbf{A}. \quad (9.24)$$

Since we will carry out L^p estimate instead of the derivative estimates of the right hand side of the above equation, we can drop the smooth function $f(\varrho)$ since $|f(\varrho)| \lesssim 1$.

Proposition 9.3. *For $\tilde{\mu}$ defined in (9.9), there holds the transport equation*

$$\begin{aligned} (L\tilde{\mu} + \text{tr} \chi \tilde{\mu}) - (\tilde{r}^{-1}(\mathfrak{d}\mathfrak{I}v \pi + \epsilon^{AB} \nabla_A \mathfrak{w}_B) + \tilde{r}^{-2} \pi) &= \hat{\chi} \cdot \nabla \tilde{\zeta} + \nabla \sigma \cdot (\mathbf{E} + \nabla \pi + \nabla \text{tr} \chi) \\ &\quad + \mathbf{A} \cdot (\nabla \text{tr} \chi, \mathbf{E}, \mathbf{D}\tilde{\pi}, e^e \partial \Omega), \end{aligned} \quad (9.25)$$

where $\mathbf{A} = \hat{\chi}, \zeta, \pi, z$, $\mathbf{E} = \mathbf{A} \cdot \mathbf{A} + \text{tr} \chi \cdot \mathbf{A}$, and “ X ” on the right hand side has been omitted since X are all bounded frames²⁴ and the equation will not be further differentiated.

Proof. Note due to Proposition 7.1, $\mathbf{D}\pi = \mathbf{D}\tilde{\pi} \cdot X + \mathbf{E}$. Hence substituting (9.24) and (9.13) to (9.11), also using Lemma 7.8, we can conclude

$$\begin{aligned} (L\tilde{\mu} + \text{tr} \chi \tilde{\mu}) - (\tilde{r}^{-1} \mathfrak{d}\mathfrak{I}v \pi + \tilde{r}^{-2} \pi) - \tilde{r}^{-1} \delta_{ij} \mathbf{N}^i e^e \text{curl} \Omega^j &= \hat{\chi} \cdot \nabla \tilde{\zeta} + \nabla \sigma \cdot (\mathbf{E} + \nabla \pi + \nabla \text{tr} \chi) \\ &\quad + \mathbf{A} \cdot (\nabla \text{tr} \chi, \mathbf{E}, \mathbf{D}\tilde{\pi}) + e^e \partial \Omega \mathbf{A}. \end{aligned} \quad (9.26)$$

Now we recast (7.35) as

$$\begin{aligned} e^e \mathbf{N}^i \text{curl} \Omega_i &= \epsilon^{AB} \nabla_A (e^e \Omega_B) - e^e \epsilon^{AB} \nabla_A \varrho \Omega_B = \epsilon^{AB} \nabla_A \mathfrak{w}_B - \epsilon^{AB} \nabla_A \varrho \mathfrak{w}_B \\ &= \epsilon^{AB} \nabla_A \mathfrak{w}_B + \pi \cdot \pi, \end{aligned} \quad (9.27)$$

where we used the definition of \mathfrak{w} for giving the symbolic form of the second term. Using the above formula for $\tilde{r}^{-1} e^e \mathbf{N}^i \text{curl} \Omega_i$ and since we can regard $\tilde{r}^{-1} \pi \cdot \pi = \mathbf{E} \cdot \mathbf{A}$, (9.25) follows as a consequence of the above calculation and (9.26). \square

Let $\tilde{\zeta} = \zeta + \nabla \sigma$. We recall the Hodge operator \mathcal{D}_1 which sends an $S_{t,u}$ -tangent tensor F to $(\mathfrak{d}\mathfrak{I}v F, \text{curl} F)$. Thus we can write $\nabla \tilde{\zeta} = \nabla \mathcal{D}_1^{-1}(\mathfrak{d}\mathfrak{I}v \tilde{\zeta}, \text{curl} \tilde{\zeta})$ and use the Hodge system

$$\begin{aligned} \mathfrak{d}\mathfrak{I}v \tilde{\zeta} &= \frac{1}{2}(\tilde{\mu} - 2|\zeta|^2 - |\hat{\chi}|^2 - 2k_{AB} \hat{\chi}_{AB}) + \mathfrak{d}\mathfrak{I}v \pi_2 + \mathbf{E} \\ &= \frac{1}{2}(\tilde{\mu} - \delta_{ij} e^e \mathbf{N}^i \text{curl} \Omega^j) + \mathfrak{d}\mathfrak{I}v \pi_2 + \mathbf{E}, \end{aligned} \quad (9.28)$$

$$\text{curl} \tilde{\zeta} = \frac{1}{2} \epsilon^{AB} k_{AC} \hat{\chi}_{CB} + \frac{1}{2} \epsilon^{AB} \mathbf{R}_{B43A} = \text{curl} \pi_3 + \mathbf{E}, \quad (9.29)$$

which are directly derived from (7.17) and (7.18) together with the curvature decomposition Lemma 7.8 (iv), where π_2 and π_3 are 1-forms of type π .

By using Proposition 9.3 and the Hodge system (9.28) and (9.29), we will prove

Proposition 9.4. *For any p satisfying $0 \leq 1 - \frac{2}{p} < s' - 2$ there hold*

$$\|\nabla \sigma\|_{L_u^2 L_t^2 L_\omega^\infty(\mathcal{D}^+)} + \|\tilde{r} \tilde{\mu}, \tilde{r} \tilde{\mu}, \tilde{r} \nabla \tilde{\zeta}\|_{L_u^2 L_t^2 L_\omega^p(\mathcal{D}^+)} \lesssim \lambda^{-4\epsilon_0}, \quad (9.30)$$

$$\|\tilde{r}^{\frac{3}{2}} \tilde{\mu}, \tilde{r}^{\frac{3}{2}} \tilde{\mu}\|_{L_u^2 L_t^\infty L_\omega^p(\mathcal{D}^+)} \lesssim \lambda^{-4\epsilon_0}. \quad (9.31)$$

²⁴This means $|X^\mu| \lesssim 1$ where $X = X^\mu \partial_\mu$ is the decomposition relative to the Cartesian coordinates.

Proof. The core estimates in the above are the estimates of $\tilde{\mu}$. We first give the necessary error estimates in the sequel.

Recall that $\mathbf{E} = \mathbf{A} \cdot \mathbf{A} + \text{tr}\chi \cdot \mathbf{A}$ and $\text{tr}\chi = \widetilde{\text{tr}\chi} - \Xi_4$. By using (8.4), (8.38) and (8.40),

$$\|\tilde{r}\mathbf{E}\|_{L_t^2 L_\omega^p(C_u \cap \mathcal{D}^+)} \leq \|\tilde{r}\widetilde{\text{tr}\chi}\mathbf{A}\|_{L_t^2 L_\omega^p(C_u \cap \mathcal{D}^+)} + \|\tilde{r}\mathbf{A} \cdot \mathbf{A}\|_{L_t^2 L_\omega^p(C_u \cap \mathcal{D}^+)} \lesssim \lambda^{-\frac{1}{2}}.$$

Moreover, by the following consequence of (8.9),

$$\|\mathbf{A}\|_{L_t^2 L_x^\infty(\mathcal{D}^+)} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}, \quad (9.32)$$

also by using (8.4) and (8.40), it is easily seen that

$$\|\tilde{r}\mathbf{E}\|_{L_t^2 L_u^\infty L_\omega^p(\mathcal{D}^+)} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}, \quad \|\tilde{r}^{\frac{3}{2}}\mathbf{E}\|_{L_u^2 L_t^\infty L_\omega^p(\mathcal{D}^+)} \lesssim \lambda^{-4\epsilon_0}. \quad (9.33)$$

Using $\text{tr}\chi = z - \Xi_4 + \frac{2}{\tilde{r}}$ again, also using the estimates in (8.6), (8.28) and (8.65), we summarize the obtained estimates on $C_u \cap \mathcal{D}^+$

$$\|\tilde{r}(\nabla \text{tr}\chi, \mathbf{E}, \nabla \pi, e^\ell \partial \Omega)\|_{L_t^2 L_\omega^p(C_u)} \lesssim \lambda^{-\frac{1}{2}}. \quad (9.34)$$

It follows from the above estimates and (8.25) that

$$\|\tilde{r}(\nabla \text{tr}\chi, \mathbf{D}\tilde{\pi}, e^\ell \partial \Omega, \mathbf{E})\|_{L_u^2 L_t^2 L_\omega^p(\mathcal{D}^+)} \lesssim \lambda^{-4\epsilon_0}. \quad (9.35)$$

By making use of (9.35), (9.28), (9.29), and Lemma 8.8 we obtain that

$$\begin{aligned} \|\tilde{r}\tilde{\nabla}\tilde{\zeta}\|_{L_u^2 L_t^2 L_\omega^p(\mathcal{D}^+)} &\lesssim \|\tilde{r}\tilde{\mu}\|_{L_u^2 L_t^2 L_\omega^p(\mathcal{D}^+)} + \|\tilde{r}(\mathbf{E} + \nabla \pi + e^\ell \partial \Omega)\|_{L_u^2 L_t^2 L_\omega^p(\mathcal{D}^+)} \\ &\lesssim \lambda^{-4\epsilon_0} + \|\tilde{r}\tilde{\mu}\|_{L_u^2 L_t^2 L_\omega^p(\mathcal{D}^+)}. \end{aligned} \quad (9.36)$$

Using the above estimate, in view of $\tilde{\nabla}\sigma = \tilde{\zeta} - \zeta$, (8.20), the first estimate in (8.72) and the last estimate in the line of (9.2) we have

$$\|\tilde{\nabla}\sigma\|_{L_u^2 L_t^2 L_\omega^\infty(\mathcal{D}^+)} \lesssim \|\tilde{r}\tilde{\mu}\|_{L_u^2 L_t^2 L_\omega^p(\mathcal{D}^+)} + \lambda^{-4\epsilon_0}. \quad (9.37)$$

It boils down to controlling $\tilde{\mu}$. Note that $\tilde{\mu}$ verifies the transport equation (9.25). By using Lemma 8.1 (i), $\lim_{t \rightarrow u} |\tilde{r}^2 \tilde{\mu}| = 0$. Thus, by Lemma 8.7 and (8.17), we have

$$|\tilde{r}^2 \tilde{\mu}| \lesssim \left| \int_u^t \tilde{r}^2 |L\tilde{\mu} + \text{tr}\chi \tilde{\mu}| dt' \right|. \quad (9.38)$$

We first compute

$$\left\| \frac{1}{\tilde{r}} \int_u^t \tilde{r}^2 \tilde{\nabla}\sigma (\nabla \pi + \nabla \text{tr}\chi + \mathbf{E}) \right\|_{L_u^2 L_t^\infty L_\omega^p(\mathcal{D}^+)} \lesssim \|\tilde{\nabla}\sigma\|_{L_u^2 L_t^2 L_x^\infty(\mathcal{D}^+)} \|\tilde{r}\nabla \pi, \tilde{r}\nabla \text{tr}\chi, \tilde{r}\mathbf{E}\|_{L_u^\infty L_t^2 L_\omega^p(\mathcal{D}^+)}.$$

Substituting (9.34) to the right hand side above implies

$$\left\| \frac{1}{\tilde{r}} \int_u^t \tilde{r}^2 \tilde{\nabla}\sigma (\nabla \pi + \nabla \text{tr}\chi + \mathbf{E}) \right\|_{L_u^2 L_t^\infty L_\omega^p(\mathcal{D}^+)} \lesssim \lambda^{-\frac{1}{2}} \|\tilde{\nabla}\sigma\|_{L_u^2 L_t^2 L_\omega^\infty(\mathcal{D}^+)}.$$

By using (8.89) and (9.35),

$$\begin{aligned} &\left\| \frac{1}{\tilde{r}} \int_u^t \tilde{r}^2 \mathbf{A} \cdot (\nabla \text{tr}\chi, \mathbf{D}\tilde{\pi}, e^\ell \partial \Omega, \mathbf{E}) \right\|_{L_u^2 L_t^\infty L_\omega^p(\mathcal{D}^+)} \\ &\lesssim \|\mathbf{A}\|_{L_u^\infty L_t^2 L_\omega^\infty(\mathcal{D}^+)} \|\tilde{r}(\nabla \text{tr}\chi, \mathbf{D}\tilde{\pi}, e^\ell \partial \Omega, \mathbf{E})\|_{L_u^2 L_t^2 L_\omega^p(\mathcal{D}^+)} \\ &\lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}. \end{aligned}$$

For the term $\hat{\chi} \cdot \nabla \tilde{\zeta}$, we use (9.36) and (8.89) to derive that

$$\begin{aligned} \left\| \frac{1}{\tilde{r}} \int_u^t \tilde{r}^2 \hat{\chi} \cdot \nabla \tilde{\zeta} dt' \right\|_{L_u^2 L_t^\infty L_\omega^p(\mathcal{D}^+)} &\lesssim \|\hat{\chi}\|_{L_u^\infty L_t^2 L_\omega^\infty(\mathcal{D}^+)} \|\tilde{r} \nabla \tilde{\zeta}\|_{L_u^2 L_t^2 L_\omega^p(\mathcal{D}^+)} \\ &\lesssim (\lambda^{-4\epsilon_0} + \|\tilde{r} \tilde{\mu}\|_{L_u^2 L_t^2 L_\omega^p(\mathcal{D}^+)}) \lambda^{-\frac{1}{2}}. \end{aligned}$$

Finally, for the second part on the left hand side of (9.25), we use (8.22) and (8.28) to derive that

$$\left\| \tilde{r}^{-1} \int_u^t (\tilde{r}' \mathfrak{d}v \pi, \pi, \tilde{r}' \epsilon^{AB} \nabla_A \mathfrak{w}_B) \right\|_{L_t^2 L_\omega^p(C_u \cap \mathcal{D}^+)} \lesssim \|\tilde{r} \nabla \pi, \pi\|_{L_t^2 L_\omega^p(C_u \cap \mathcal{D}^+)} \lesssim \lambda^{-\frac{1}{2}},$$

where \mathfrak{w} has been treated symbolically as ∂v above.

Now we divide (9.38) by \tilde{r} and use the above estimates to derive that

$$\|\tilde{r} \tilde{\mu}\|_{L_u^2 L_t^2 L_\omega^p(\mathcal{D}^+)} \lesssim \lambda^{-4\epsilon_0} \|\tilde{r} \tilde{\mu}\|_{L_u^2 L_t^2 L_\omega^p(\mathcal{D}^+)} + \lambda^{-4\epsilon_0} \|\nabla \sigma\|_{L_u^2 L_t^2 L_\omega^\infty(\mathcal{D}^+)} + \lambda^{-4\epsilon_0},$$

which in view of (9.37) gives

$$\|\tilde{r} \tilde{\mu}\|_{L_u^2 L_t^2 L_\omega^p(\mathcal{D}^+)} \lesssim \lambda^{-4\epsilon_0}. \quad (9.39)$$

Substituting the above estimate to (9.37) and (9.36) gives the first and the last two estimates in (9.30).

Next, we may divide (9.38) by $\tilde{r}^{\frac{1}{2}}$ and employ the similar argument as above to derive

$$\|\tilde{r}^{\frac{3}{2}} \tilde{\mu}\|_{L_u^2 L_t^\infty L_\omega^p(\mathcal{D}^+)} \lesssim \lambda^{-\frac{1}{2}}.$$

We thus have obtained the estimates for $\tilde{\mu}$ in (9.31).

At last we consider the estimates for $\tilde{\mu}$. In view of (9.9), using (8.65), on any $S_{t,u}$ contained in \mathcal{D}^+ we bound

$$\|\tilde{r} \tilde{\mu}\|_{L_\omega^p} \lesssim \|\tilde{r} \tilde{\mu}\|_{L_\omega^p} + \|\tilde{r} e^{\varrho} \delta_{ij} \mathbf{N}^i \text{curl} \Omega^j\|_{L_\omega^p} \lesssim \|\tilde{r} \tilde{\mu}\|_{L_\omega^p} + \lambda^{-1}.$$

The estimates of $\tilde{\mu}$ in (9.30) and (9.31) will follow from integrating the above inequality by using the estimates of $\tilde{\mu}$ therein. The proof is thus complete. \square

Proposition 9.5. $\nabla \sigma$ can be decomposed as

$$\nabla \sigma = \mathbf{A} + \mathbf{A}^\dagger + \mu^\dagger,$$

where \mathbf{A}^\dagger and μ^\dagger are 1-forms satisfying the estimates

$$\|\mathbf{A}^\dagger\|_{L_t^2 L_x^\infty(\mathcal{D}^+)} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0} \quad \text{and} \quad \|\mu^\dagger\|_{L_u^2 L^\infty(\mathcal{D}^+)} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}.$$

Proof. We introduce an $S_{t,u}$ -tangent 1-tensor field μ , defined on each $S_{t,u}$ in \mathcal{D}^+ by

$$\mathfrak{d}v \mu = \frac{1}{2}(\tilde{\mu} - \bar{\mu}), \quad \text{curl} \mu = 0, \quad (9.40)$$

where $\bar{f} := \frac{1}{|S_{t,u}|} \int_{S_{t,u}} f d\mu_\gamma$.

By using Lemma 8.8, (9.40), (9.30) and (8.20), we derive with $0 \leq 1 - \frac{2}{p} < s' - 2$

$$\|\tilde{r} \nabla \mu, \mu\|_{L_t^2 L_u^2 L_\omega^p} + \|\mu\|_{L_t^2 L_u^2 L_x^\infty} \lesssim \lambda^{-4\epsilon_0}. \quad (9.41)$$

In order to represent $\nabla \sigma$, we first derive in view of (9.28) and (9.29),

$$\zeta + \nabla \sigma - \mu = \mathcal{D}_1^{-1} \left(\mathfrak{d}v \tilde{\zeta} - \frac{1}{2}(\tilde{\mu} - \bar{\mu}), \text{curl} \tilde{\zeta} \right).$$

We next derive the transport equation of μ for representing μ .

By direct calculation, we have

$$L\bar{f} = \overline{Lf} + \overline{\text{tr}\chi f - \overline{\text{tr}\chi} f}.$$

Denote by G the right hand side of (9.25) in Proposition 9.3, and note that $\overline{\tilde{r}^{-1}\nabla_A\pi_A} = 0$ and $\overline{\tilde{r}^{-1}\epsilon^{AB}\nabla_A\mathfrak{w}_B} = 0$. We thus can obtain

$$\begin{aligned} & L(\tilde{\mu} - \bar{\mu}) + \text{tr}\chi(\tilde{\mu} - \bar{\mu}) \\ &= -(\text{tr}\chi - \overline{\text{tr}\chi})\bar{\mu} + G - \bar{G} + \tilde{r}^{-1}(\text{div}\pi + \epsilon^{AB}\nabla_A\mathfrak{w}_B) + \tilde{r}^{-2}(\pi - \bar{\pi}). \end{aligned}$$

By using (7.8) and Lemma 7.8 (ii), similar to [45, Proposition 6.4], we have

$$\begin{aligned} \text{div}(\nabla_L\mu + \frac{1}{2}\text{tr}\chi\mu) &= G_1 + \frac{1}{2}(\tilde{r}^{-1}(\text{div}\pi + \epsilon^{AB}\nabla_A\mathfrak{w}_B) + \tilde{r}^{-2}(\pi - \bar{\pi})), \\ \text{curl}(\nabla_L\mu + \frac{1}{2}\text{tr}\chi\mu) &= G_2, \end{aligned} \tag{9.42}$$

where

$$\begin{aligned} G_1 &= \frac{1}{2}\nabla\text{tr}\chi \cdot \mu + \hat{\chi} \cdot \nabla\mu + \frac{1}{2}(G - \bar{G}) + \mu \cdot (\nabla\pi + \mathbf{E}) - \frac{1}{2}(\text{tr}\chi - \overline{\text{tr}\chi})\bar{\mu}, \\ G_2 &= \hat{\chi} \cdot \nabla\mu + \frac{1}{2}\nabla\text{tr}\chi \cdot \mu + \mu \cdot (\nabla\pi + \mathbf{E}). \end{aligned}$$

Consequently

$$\nabla_L\mu + \frac{1}{2}\text{tr}\chi\mu = \frac{1}{2}\mathcal{D}_1^{-1}(\tilde{r}^{-1}(\text{div}\pi + \epsilon^{AB}\nabla_A\mathfrak{w}_B) + \tilde{r}^{-2}(\pi - \bar{\pi}), 0) + \mathcal{D}_1^{-1}(G_1, G_2), \tag{9.43}$$

which together with Lemma 8.7 implies that

$$\mu = v_t^{-\frac{1}{2}} \int_u^t v_{t'}^{\frac{1}{2}} \mathcal{D}_1^{-1}(G_1, G_2) dt' + v_t^{-\frac{1}{2}} \int_u^t v_{t'}^{\frac{1}{2}} \mathcal{D}_1^{-1}(\tilde{r}^{-1}(\text{div}\pi + \epsilon^{AB}\nabla_A\mathfrak{w}_B) + \tilde{r}^{-2}(\pi - \bar{\pi}), 0) dt'.$$

Therefore $\nabla\sigma = \mathbf{A} + \mu^\dagger + \mathbf{A}^\dagger$ with $\mathbf{A} = -\zeta$ and

$$\begin{aligned} \mu^\dagger &= v_t^{-\frac{1}{2}} \int_u^t v_{t'}^{\frac{1}{2}} \mathcal{D}_1^{-1}(G_1, G_2) dt', \\ \mathbf{A}^\dagger &= v_t^{-\frac{1}{2}} \int_u^t v_{t'}^{\frac{1}{2}} \mathcal{D}_1^{-1}(\tilde{r}^{-1}(\text{div}\pi + \epsilon^{AB}\nabla_A\mathfrak{w}_B) + \tilde{r}^{-2}(\pi - \bar{\pi}), 0) dt' \\ &\quad + \mathcal{D}_1^{-1}\left(\text{div}\tilde{\zeta} - \frac{1}{2}(\tilde{\mu} - \bar{\mu}), \text{curl}\tilde{\zeta}\right). \end{aligned}$$

Note the first line on the right hand side of \mathbf{A}^\dagger term contains vorticity. The control of the last term in \mathbf{A}^\dagger and the term G_1 also involves the estimate of $\partial\Omega$. Now we show

$$\|\tilde{r}(G_1, G_2)\|_{L_u^2 L_t^1 L_\omega^p(\mathcal{D}^+)} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}. \tag{9.44}$$

To prove the above estimate, we first treat the last term in G_1 symbolically by

$$(\text{tr}\chi - \overline{\text{tr}\chi})\bar{\mu} = \mathbf{A} \cdot \tilde{\mu},$$

where we have ignored the operator of taking average.

Hence, we schematically recast the terms of G_1 and G_2 as

$$G_1, G_2 = (\mu, \nabla\sigma) \cdot (\widetilde{\nabla\text{tr}\chi}, \widetilde{\nabla\pi}, \mathbf{E}) + \hat{\chi} \cdot (\widetilde{\nabla\zeta}, \widetilde{\nabla\mu}) + \mathbf{A} \cdot (\widetilde{\nabla\text{tr}\chi}, \mathbf{E}, \mathbf{D}\tilde{\pi}, e^g\partial\Omega^j, \tilde{\mu}).$$

By using (9.41), (8.63), (8.89), (9.34), (9.35) and (9.30), we derive

$$\begin{aligned} \|\tilde{r}(G_1, G_2)\|_{L_u^2 L_t^1 L_\omega^p(\mathcal{D}^+)} &\lesssim \|\not{A}, \not{V}\sigma\|_{L_u^2 L_t^2 L_\omega^\infty} \|\tilde{r}(\not{V}\widetilde{\text{tr}\chi}, \not{V}\pi, \mathbf{E})\|_{L_u^\infty L_t^2 L_\omega^p} \\ &\quad + \|\mathbf{A}\|_{L_u^\infty L_t^2 L_\omega^\infty} \|\tilde{r}(\not{V}\tilde{\zeta}, \not{V}\not{A}, \not{V}\widetilde{\text{tr}\chi}, \mathbf{E}, \mathbf{D}\tilde{\pi}, \tilde{\mu}, e^e \partial\Omega)\|_{L_u^2 L_t^2 L_\omega^p} \\ &\lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}, \end{aligned}$$

as desired in (9.44). As its consequence, in view of (8.17) and Proposition 8.9, we can obtain $\|\mu^\dagger\|_{L_u^2 L^\infty} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}$.

Note that by using (7.17) and Lemma 7.8 (iv), $\bar{\mu} = \bar{\mathbf{E}}$. Hence, by the definition of $\tilde{\mu}$ in (9.9),

$$\tilde{\mu} = \bar{\mathbf{E}} + \overline{\delta_{ij} e^e \mathbf{N}^i \text{curl } \Omega^j}.$$

Thus by using (8.35) with $0 < c < s' - 2$, (9.28) and (9.29)

$$\begin{aligned} \|\mathcal{D}_1^{-1} \left(\text{div } \tilde{\zeta} - \frac{1}{2}(\tilde{\mu} - \bar{\mu}), \text{curl } \tilde{\zeta} \right)\|_{L_t^2 L_u^\infty L_\omega^\infty} &\lesssim \|\tilde{r}\mathbf{E}\|_{L_t^2 L_u^\infty L_\omega^p} + \|\tilde{r}\delta_{ij} e^e \mathbf{N}^i \text{curl } \Omega^j\|_{L_t^2 L_u^\infty L_\omega^p} + \|\ell^c P_\ell \tilde{\pi}\|_{L_t^2 L_u^2 L_\omega^\infty} + \|\tilde{\pi}\|_{L_t^2 L_\omega^\infty} \\ &\lesssim \|\tilde{r}\mathbf{E}\|_{L_t^2 L_u^\infty L_\omega^p} + \|\ell^c P_\ell \tilde{\pi}\|_{L_t^2 L_u^2 L_\omega^\infty} + \|\tilde{\pi}\|_{L_t^2 L_\omega^\infty} + \|\tilde{r}\partial\Omega\|_{L_t^2 L_u^\infty L_\omega^p} \\ &\lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}, \end{aligned}$$

where we used (9.33), (8.1) and (8.65) to derive the last inequality.

Finally, by using (8.35) with $0 < c < s' - 2$, (8.1) and (8.17)

$$\begin{aligned} \|v_t^{-\frac{1}{2}} \int_u^t v_t^{\frac{1}{2}} \mathcal{D}_1^{-1}(\tilde{r}^{-1}(\text{div } \pi + \epsilon^{AB} \not{V}_A \mathbf{w}_B) + \tilde{r}^{-2}(\pi - \bar{\pi}), 0) dt'\|_{L_t^2 L_x^\infty} &\lesssim \|\ell^c P_\ell \tilde{\pi}\|_{L_t^2 L_u^2 L_\omega^\infty} + \|\tilde{\pi}\|_{L_t^2 L_\omega^\infty} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}, \end{aligned}$$

where we regarded $\mathbf{w} = \partial v = \tilde{\pi}$. Therefore the proof of Proposition 9.5 is complete. \square

10. APPENDIX

In this section, we rely on the trichotomy (3.39) of the Littlewood-Paley projections to derive commutator estimates and product estimates, which are the basic analytic tools to treat the analysis in the fractional Sobolev spaces.

Lemma 10.1. (1) For smooth scalar functions F and G , $1 \leq p, q, r \leq \infty$ satisfying $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$,

$$\|[P_\mu, F]G\|_{L_x^q} \lesssim \mu^{-1} \|\partial F\|_{L_x^p} \|G\|_{L_x^r}. \quad (10.1)$$

(2) For smooth scalar functions F and G , with $0 < \alpha < 1$,

$$\|\mu^\alpha [P_\mu, F] \partial G\|_{l_\mu^2 L_x^2} \lesssim \|\partial F\|_{L_x^\infty} \|G\|_{H_x^\alpha}, \quad (10.2)$$

$$\|\mu^\alpha [P_\mu, F] \partial G\|_{l_\mu^2 L_x^2} \lesssim \|F\|_{L_x^\infty} \|\partial G\|_{H_x^\alpha} + \|F\|_{H_x^{1+\alpha}} \|G\|_{L_x^\infty}, \quad (10.3)$$

$$\|\mu^\alpha \partial [P_\mu, F] \partial G\|_{l_\mu^2 L_x^2} \lesssim \|\partial F\|_{L_x^\infty} \|\partial G\|_{H_x^\alpha} + \|\partial F\|_{H_x^{\alpha+1}} \|G\|_{L_x^\infty}. \quad (10.4)$$

Proof. We recall from [42, (6.195)] that for smooth scalar functions f and W ,

$$[P_\mu, f]W = [P_\mu, f]W_{\leq \mu} + \sum_{\lambda > \mu} P_\mu(f_\lambda W_\lambda). \quad (10.5)$$

We consider (10.2) by applying (10.5) to $(f, W) = (F, \partial G)$.

$$\|\mu^\alpha [P_\mu, F](\partial G)_{\leq \mu}\|_{L_x^2} \lesssim \|\partial F\|_{L_x^\infty} \sum_{l \leq \mu} \left(\frac{l}{\mu}\right)^{1-\alpha} \|l^\alpha G_l\|_{L_x^2}.$$

Taking l_μ^2 norm gives

$$\|\mu^\alpha [P_\mu, F](\partial G)_{\leq \mu}\|_{l_\mu^2 L_x^2} \lesssim \|\partial F\|_{L_x^\infty} \|G\|_{H_x^\alpha}.$$

For the high-high interaction term in (10.5), by using the finite band property

$$\|\mu^\alpha \sum_{\lambda > \mu} P_\mu(F_\lambda(\partial G)_\lambda)\|_{L_x^2} \lesssim \sum_{\lambda > \mu} \left(\frac{\mu}{\lambda}\right)^\alpha \|(\partial F)_\lambda\|_{L_x^\infty} \|\lambda^\alpha G_\lambda\|_{L_x^2}.$$

Taking l_μ^2 norm implies

$$\|\mu^\alpha \sum_{\lambda > \mu} P_\mu(F_\lambda(\partial G)_\lambda)\|_{l_\mu^2 L_x^2} \lesssim \|\partial F\|_{L_x^\infty} \|G\|_{H_x^\alpha}.$$

(10.2) follows by combining the estimates for both parts.

For (10.3), by applying (10.5) to $(f, W) = (F, \partial G)$, we treat the first term by using the trichotomy and orthogonality property of the Littlewood-Paley projection,

$$[P_\mu, F](\partial G)_{\leq \mu} = P_\mu(F_\mu(\partial G)_{\leq \mu}) - F P_\mu(\partial G).$$

For this term, we compute

$$\|\mu^\alpha [P_\mu, F](\partial G)_{\leq \mu}\|_{l_\mu^2 L_x^2} \lesssim \|G\|_{L^\infty} \|F\|_{H_x^{\alpha+1}} + \|F\|_{L_x^\infty} \|G\|_{H_x^{\alpha+1}}.$$

The high-high interaction term can be controlled by

$$\mu^\alpha \left\| \sum_{\lambda > \mu} P_\mu(F_\lambda(\partial G)_\lambda) \right\|_{L_x^2} \lesssim \sum_{\lambda > \mu} \left(\frac{\mu}{\lambda}\right)^\alpha \|F_\lambda\|_{L^\infty} \lambda^{1+\alpha} \|G_\lambda\|_{L_x^2}.$$

Thus

$$\|\mu^\alpha \sum_{\lambda > \mu} P_\mu(F_\lambda(\partial G)_\lambda)\|_{l_\mu^2 L_x^2} \lesssim \|F\|_{L^\infty} \|G\|_{H_x^{\alpha+1}}.$$

(10.3) follows by combining the two terms.

For (10.4), we first note that

$$\partial[P_\mu, F]\partial G = [P_\mu, \partial F]\partial G + [P_\mu, F]\partial^2 G.$$

Applying (10.2) with (F, G) replaced by $(F, \partial G)$ to the second term, (10.3) with (F, G) replaced by $(\partial F, G)$ for the first term, (10.4) follows immediately. \square

Next we give the first set of the product estimates.

Lemma 10.2. *For $\alpha > 0$, there hold for scalar functions F and G that*

$$\|F \cdot G\|_{\dot{H}_x^\alpha} \lesssim \|F\|_{L_x^\infty} \|G\|_{\dot{H}_x^\alpha} + \|F\|_{\dot{H}_x^\alpha} \|G\|_{L_x^\infty} \quad (10.6)$$

$$\|F \cdot \partial G\|_{\dot{H}_x^\alpha} \lesssim \|F\|_{\dot{H}_x^{1+\alpha}} \|G\|_{L_x^\infty} + \|F\|_{L_x^\infty} \|\partial G\|_{H_x^\alpha} \quad (10.7)$$

$$\|\partial(F \cdot G)\|_{\dot{H}_x^\alpha} \lesssim \|\partial F\|_{H_x^\alpha} \|G\|_{L_x^\infty} + \|F\|_{L_x^\infty} \|\partial G\|_{H_x^\alpha} \quad (10.8)$$

$$\|F \cdot \partial^2 G\|_{\dot{H}_x^\alpha} \lesssim \|F\|_{\dot{H}_x^{2+\alpha}} \|G\|_{L_x^\infty} + \|F\|_{L_x^\infty} \|\partial^2 G\|_{H_x^\alpha} \quad (10.9)$$

$$\|\partial(F \cdot G)\|_{\dot{H}_x^\alpha} \lesssim \|F\|_{\dot{H}_x^{1+\alpha}} \|G\|_{L_x^\infty} + \|F\|_{H_x^1} \|G\|_{\dot{H}_x^{\frac{3}{2}+\alpha}} \quad (10.10)$$

$$\|F \cdot \partial G\|_{\dot{H}_x^\alpha} \lesssim \|F\|_{\dot{H}_x^{1+\alpha}} \|G\|_{L_x^\infty} + \|F\|_{H_x^1} \|G\|_{\dot{H}_x^{\frac{3}{2}+\alpha}} \quad (10.11)$$

$$\|F \cdot G\|_{\dot{H}_x^\alpha} \lesssim \|F\|_{\dot{H}_x^{\frac{1}{2}+\alpha}} \|G\|_{H_x^1} + \|F\|_{L_x^\infty} \|G\|_{\dot{H}_x^\alpha}, \quad (10.12)$$

where \dot{H}^α denotes the Sobolev norm of H^α with the L^2 norm excluded.

Proof. Now we prove (10.6). By trichotomy,

$$\begin{aligned}\mu^\alpha P_\mu(F \cdot G) &= \mu^\alpha P_\mu[F \cdot G]_{HL} + \mu^\alpha P_\mu[F \cdot G]_{LH} + \mu^\alpha P_\mu[F \cdot G]_{HH} \\ &= I_\mu + J_\mu + K_\mu.\end{aligned}$$

For the three terms, by using Hölder's inequality, we can compute

$$\begin{aligned}\|I_\mu\|_{l_\mu^2 L_x^2} &\lesssim \|F\|_{\dot{H}_x^\alpha} \|G\|_{L_x^\infty}; \quad \|J_\mu\|_{l_\mu^2 L_x^2} \lesssim \|F\|_{L_x^\infty} \|G\|_{\dot{H}_x^\alpha} \\ \|K_\mu\|_{l_\mu^2 L_x^2} &\lesssim \left\| \sum_{\lambda > \mu} \left(\frac{\mu}{\lambda}\right)^\alpha \|F_\lambda\|_{L_x^\infty} \|\lambda^\alpha G_\lambda\|_{L_x^2} \right\|_{l_\mu^2} \lesssim \|F\|_{L_x^\infty} \|G\|_{\dot{H}_x^\alpha}.\end{aligned}$$

This gives (10.6).

Next we prove (10.7). Again by using trichotomy,

$$\begin{aligned}\mu^\alpha P_\mu(F \cdot \partial G) &= \mu^\alpha P_\mu[F \cdot \partial G]_{HL} + \mu^\alpha P_\mu[F \cdot \partial G]_{LH} + \mu^\alpha P_\mu[F \cdot \partial G]_{HH} \\ &= I_\mu + J_\mu + K_\mu.\end{aligned}$$

By the finite band property,

$$\begin{aligned}\|I_\mu\|_{L_x^2} &\lesssim \|\mu^{1+\alpha} F_\mu\|_{L_x^2} \sum_{l \leq \mu} \frac{l}{\mu} \|G_l\|_{L_x^\infty} \\ \|J_\mu\|_{L_x^2} &\lesssim \|\mu^\alpha \partial G_\mu\|_{L_x^2} \|F_{\leq \mu}\|_{L_x^\infty} \lesssim \|\mu^\alpha \partial G_\mu\|_{L_x^2} \|F\|_{L_x^\infty} \\ \|K_\mu\|_{L_x^2} &\lesssim \sum_{\lambda > \mu} \left(\frac{\mu}{\lambda}\right)^\alpha \|\lambda^\alpha \partial G_\lambda\|_{L_x^2} \|F_\lambda\|_{L_x^\infty}.\end{aligned}$$

Summing the above terms in terms of l_μ^2 , we can conclude (10.7). (10.9) can be similarly proved.

Next we consider (10.10) by using trichotomy. For simplicity, we set $\mathcal{I}_\mu = \mu^{1+\alpha} \|P_\mu(F \cdot G)\|_{L_x^2}$.

$$\mathcal{I}_\mu \lesssim \mu^{1+\alpha} (\|P_\mu[F \cdot G]_{HL}\|_{L_x^2} + \|P_\mu[F \cdot G]_{LH}\|_{L_x^2} + \|P_\mu[F \cdot G]_{HH}\|_{L_x^2}).$$

By using the finite band property and Bernstein inequality, we obtain

$$\begin{aligned}\mu^{1+\alpha} \|P_\mu[F \cdot G]_{HL}\|_{L_x^2} &\lesssim \mu^{1+\alpha} \|P_\mu F\|_{L_x^2} \|G_{\leq \mu}\|_{L_x^\infty} \\ \mu^{1+\alpha} \|P_\mu[F \cdot G]_{LH}\|_{L_x^2} &\lesssim \|P_{\leq \mu} F\|_{L_x^6} \|P_\mu G\|_{L_x^3} \mu^{1+\alpha} \\ &\lesssim \|F\|_{H_x^1} \|P_\mu G\|_{\dot{H}_x^{\frac{3}{2}+\alpha}} \\ \mu^{1+\alpha} \|P_\mu[F \cdot G]_{HH}\|_{L_x^2} &\lesssim \sum_{\lambda > \mu} \|P_\lambda F\|_{L_x^6} \left(\frac{\mu}{\lambda}\right)^{1+\alpha} \|\lambda^{\frac{3}{2}+\alpha} P_\lambda G\|_{L_x^2},\end{aligned}$$

which implies

$$\|\mathcal{I}_\mu\|_{l_\mu^2} \lesssim \|F\|_{H_x^{1+\alpha}} \|G\|_{L_x^\infty} + \|F\|_{H_x^1} \|G\|_{\dot{H}_x^{\frac{3}{2}+\alpha}}$$

as desired.

Next we prove (10.11). Let $\mathcal{J}_\mu = \mu^\alpha \|P_\mu(F \cdot \partial G)\|_{L_x^2}$. In view of the trichotomy in (3.39), we estimate by using the finite band property and Bernstein inequality that

$$\begin{aligned} \mu^\alpha \|P_\mu[F \cdot \partial G]_{HL}\|_{L_x^2} &\lesssim \sum_{\lambda < \mu} \frac{\lambda}{\mu} \|\mu^{1+\alpha} P_\mu F\|_{L_x^2} \|P_\lambda G\|_{L_x^\infty}, \\ \mu^\alpha \|P_\mu[F \cdot \partial G]_{LH}\|_{L_x^2} &\lesssim \|P_{\leq \mu} F\|_{L_x^6} \mu^\alpha \|P_\mu \partial G\|_{L_x^3} \lesssim \|F\|_{L_x^6} \|\mu^{\alpha+\frac{1}{2}} P_\mu \partial G\|_{L_x^2}, \\ \mu^\alpha \|P_\mu[F \cdot \partial G]_{HH}\|_{L_x^2} &\lesssim \mu^\alpha \sum_{\lambda > \mu} \|P_\lambda F\|_{L_x^6} \|P_\lambda \partial G\|_{L_x^3} \\ &\lesssim \sum_{\lambda > \mu} \left(\frac{\mu}{\lambda}\right)^\alpha \|P_\lambda F\|_{H_x^1} \|\lambda^{\frac{1}{2}+\alpha} P_\lambda \partial G\|_{L_x^2}. \end{aligned}$$

Taking l_μ^2 norm for the above three terms implies

$$\|\mathcal{J}_\mu\|_{l_\mu^2} \lesssim \|F\|_{H_x^{1+\alpha}} \|G\|_{L_x^\infty} + \|F\|_{H_x^1} \|\partial G\|_{H_x^{\frac{1}{2}+\alpha}}$$

as desired.

At last we consider (10.12). We estimate the terms in (3.39) as follows

$$\begin{aligned} \|\mu^\alpha P_\mu[F \cdot G]_{HL}\|_{L_x^2} &\lesssim \mu^\alpha \|F_\mu\|_{L_x^2} \sum_{l \leq \mu} l^{\frac{3}{2}} \|P_l G\|_{L_x^2} \\ &\lesssim \mu^{\alpha+\frac{1}{2}} \|F_\mu\|_{L_x^2} \sum_{l \leq \mu} \left(\frac{l}{\mu}\right)^{\frac{1}{2}} \|l P_l G\|_{L_x^2}, \\ \|\mu^\alpha P_\mu[F \cdot G]_{LH}\|_{L_x^2} &\lesssim \|F_{\leq \mu}\|_{L_x^\infty} \mu^\alpha \|G_\mu\|_{L_x^2}, \\ \|\mu^\alpha P_\mu[F \cdot G]_{HH}\|_{L_x^2} &\lesssim \|F\|_{L_x^\infty} \sum_{\lambda > \mu} \left(\frac{\mu}{\lambda}\right)^\alpha \|\lambda^\alpha G_\lambda\|_{L_x^2}. \end{aligned}$$

Summing up the three inequalities implies

$$\|\mu^\alpha P_\mu(FG)\|_{l_\mu^2 L_x^2} \lesssim \|F\|_{\dot{H}^{\frac{1}{2}+\alpha}} \|G\|_{H_x^1} + \|F\|_{L_x^\infty} \|G\|_{\dot{H}_x^\alpha}.$$

The proof of (10.12) is completed. \square

Lemma 10.3. *Let $0 < \alpha < 1$ be fixed.*

$$\|\Lambda^\alpha(F \cdot G)\|_{L_x^2} \lesssim \|F\|_{B_{\infty,2,x}^\alpha} \|G\|_{L_x^2} + \|F\|_{L_x^\infty} \|G\|_{\dot{H}_x^\alpha}, \quad (10.13)$$

$$\|\Lambda^\alpha(F \cdot G)\|_{L_x^2} \lesssim \|F\|_{H_x^{\frac{1}{2}+\alpha}} \|G\|_{H_x^1} + \|G\|_{H_x^{\frac{1}{2}+\alpha}} \|F\|_{H_x^1}, \quad (10.14)$$

$$\|\Lambda^\alpha(G_1 G_2 G_3)\|_{L_x^2} \lesssim \sum_{j=1}^3 (\|\Lambda^\alpha G_j\|_{H_x^1} \Pi_{l \neq j} \|G_l\|_{H_x^1}). \quad (10.15)$$

Proof. (10.14) and (10.15) are [42, (6.188) and Lemma 18] respectively.

By using the trichotomy in (3.39)

$$\begin{aligned} \mu^\alpha \|P_\mu[F \cdot G]_{HL}\|_{L_x^2} &\lesssim \mu^\alpha \|F_\mu\|_{L_x^\infty} \|G_{\leq \mu}\|_{L_x^2} \lesssim \mu^\alpha \|F_\mu\|_{L_x^\infty} \|G\|_{L_x^2}, \\ \mu^\alpha \|P_\mu[F \cdot G]_{LH}\|_{L_x^2} &\lesssim \mu^\alpha \|F_{\leq \mu}\|_{L_x^\infty} \|P_\mu G\|_{L_x^2} \lesssim \|F\|_{L_x^\infty} \mu^\alpha \|P_\mu G\|_{L_x^2}, \\ \mu^\alpha \|P_\mu[F \cdot G]_{HH}\|_{L_x^2} &\lesssim \sum_{\lambda \geq \mu} \left(\frac{\mu}{\lambda}\right)^\alpha \|F_\lambda\|_{L_x^\infty} \|\lambda^\alpha P_\lambda G\|_{L_x^2}. \end{aligned}$$

(10.13) follows by taking l_μ^2 norms on the above inequalities. \square

Lemma 10.4. *For $0 < \alpha < 1/2$ there hold*

$$\begin{aligned}\mu^{-\frac{1}{2}+\alpha}\|\partial[P_\mu, F]G\|_{L_x^2} &\lesssim \|\partial F\|_{L_x^6}\left(\sum_{\lambda\leq\mu}\left(\frac{\lambda}{\mu}\right)^{1/2-\alpha}\|\lambda^\alpha G_\lambda\|_{L_x^2}+\sum_{\lambda>\mu}\left(\frac{\mu}{\lambda}\right)^{1/2+\alpha}\|\lambda^\alpha G_\lambda\|_{L_x^2}\right) \\ \mu^{\frac{1}{2}+\alpha}\|[P_\mu, F]G\|_{L_x^2} &\lesssim \|\partial F\|_{L_x^6}\left(\sum_{\lambda\leq\mu}\left(\frac{\lambda}{\mu}\right)^{1/2-\alpha}\|\lambda^\alpha G_\lambda\|_{L_x^2}+\sum_{\lambda\geq\mu}\left(\frac{\mu}{\lambda}\right)^{1+\alpha}\|\lambda^\alpha G_\lambda\|_{L_x^2}\right).\end{aligned}\tag{10.16}$$

This is [42, Lemma 23].

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REFERENCES

- [1] Anderson, M. T., *Cheeger-Gromov theory and applications to general relativity*. In *The Einstein equations and the large scale behavior of gravitational fields*, pages 347–377. Birkhäuser, Basel, 2004.
- [2] Bahouri, H. and Chemin, J.Y., *Équations d’ondes quasilineaires et effet dispersif*, Internat. Math. Res. Notices 1999, no. 21, 1141–1178.
- [3] Bahouri, H. and Chemin, J.Y., *Équations d’ondes quasilineaires et estimation de Strichartz*, Amer. J. Math., 121 (1999), 1337–1377.
- [4] Bourgain, J. and Li, D., *Strong Ill-Posedness of the 3D Incompressible Euler Equation in Borderline Spaces*, International Mathematics Research Notices, rnz158, <https://doi.org/10.1093/imrn/rnz158>, 110 pages, 2019.
- [5] Bressan, A. *Hyperbolic Systems of Conservation Laws. The One Dimensional Cauchy Problem*. Oxford University Press, 2000.
- [6] Bressan, A. *Hyperbolic Systems of Conservation Laws in One Space Dimension* ICM 2002, Vol. I, 159–178
- [7] P. Brenner. *The Cauchy problem for symmetric hyperbolic systems in L_p* . Math. Scand., 19:27–37, 1966.
- [8] P. Brenner. *The Cauchy problem for systems in L_p and $L_{p,\alpha}$* . Ark. Mat., 11:75–101, 1973.
- [9] Christodoulou, D., *The Formation of Shocks in 3-Dimensional Fluids*, EMS Monographs in Mathematics, European Mathematical Society, Zürich, 2007. viii+992 pp.
- [10] Christodoulou, D. and Klainerman, S., *The Global Nonlinear Stability of Minkowski Space*, Princeton Mathematical Series 41, 1993.
- [11] Dafermos, M. and Rodnianski, I., *A new physical-space approach to decay for the wave equation with applications to black hole spacetimes*, XVIth International Congress on Mathematical Physics 2010, 421–432, World Sci. Publ., Hackensack, NJ.
- [12] Disconzi, M., Luo, C., Mazzone, G. and Speck, J. *Rough sound waves in 3D compressible euler flow with vorticity.*, arXiv:1909.02550v1, 100 pages.
- [13] Glimm, J. *Solutions in the large for nonlinear hyperbolic systems of equations*. Comm. Pure Appl. Math., 18:697–715, 1965.
- [14] Hughes, T., Kato, T. and Marsden, J. E., *Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear electrodynamics and general relativity*, Arch. Rat. Mech. Anal., 63(1977), 273–294.
- [15] Kato, T., *The Cauchy problem for quasi-linear symmetric hyperbolic systems*. Arch. Rational Mech. Anal. 58(1975), no. 3, 181–205. doi:10.1007/BF00280740
- [16] Kato, T. and Ponce, G., *Commutator estimates and the Euler and Navier-Stokes equations*. Comm. Pure Appl. Math. 41(7), 891–907 (1988)
- [17] Klainerman, S., *A commuting vectorfield approach to Strichartz type inequalities and applications to quasi-linear wave equations*, Int. Math. Res. Notices 2001, No 5, 221–274.
- [18] Klainerman, S. and Rodnianski, I., *Improved local well-posedness for quasilinear wave equations in dimension three*, Duke Math. J., 117 (2003), 1–124.
- [19] Klainerman, S. and Rodnianski, I. *Ricci defects of microlocalized Einstein metrics*. J. Hyperbolic Differ. Equ. 1 (2004), no. 1, 85–113.

- [20] Klainerman, S. and Rodnianski, I., *Rough solutions of the Einstein vacuum equations*, Ann. Math. 161 (2005), 1143–1193.
- [21] Klainerman, S. and Rodnianski, I., *Causal structure of microlocalized rough Einstein metrics*, Ann. Math., 161 (2005), 1195–1243.
- [22] Klainerman, S. and Rodnianski, I., *A geometric Littlewood-Paley theory*, Geom. Funct. Anal., 16 (2006), 126–163.
- [23] Klainerman, S., Rodnianski, I. and Szeftel, J., *The bounded L^2 curvature conjecture*, Invent. Math. 202 (2015), no. 1, 91–216.
- [24] Krieger J.; Lührmann J. Concentration compactness for the critical Maxwell-Klein-Gordon equation. *Ann. PDE.* 1 (2015), no. 1, Art. 5, 208 pp.
- [25] Lindblad, H. *Counterexamples to local existence for quasilinear wave equations*. Am. J. Math. 118(1), 1–16 (1996)
- [26] Luk, J and Speck, J., *The hidden null structure of the compressible Euler equations and a prelude to applications*, arXiv:1610.00743v1, 2016
- [27] Majda, A., *Compressible fluid flow and systems of conservation laws in several space variables*. Applied Mathematical Sciences, 53. Springer, New York, 1984.
- [28] Oh S; Tataru D. Global well-posedness and scattering of the (4+1)-dimensional Maxwell-Klein-Gordon equation. *Invent. Math.* 205 (2016), no. 3, 781–877.
- [29] Petersen, P., *Convergence theorems in Riemannian geometry*. In Comparison geometry (Berkeley, CA, 1993–94), volume 30 of Math. Sci. Res. Inst. Publ., pages 167–202. Cambridge Univ. Press, Cambridge, 1997.
- [30] Smith, H. F., *A parametrix construction for wave equations with $C^{1,1}$ coefficients*. Ann. Inst. Fourier (Grenoble), 48 (1998), no. 3, 797–835.
- [31] Sterbenz, J., and Tataru, D. *Regularity of Wave-Maps in Dimension $2 + 1$* Commun. Math. Phys. (2010) 298: 231.
- [32] Tao, T., *Global regularity of wave maps. II. Small energy in two dimensions*. Commun. Math. Phys. 224(2), 443–544 (2001)
- [33] Tao, T., *Product estimates, multilinear estimates*. <https://www.math.ucla.edu/~tao/254a.1.01w/>
- [34] Smith, H. F. and Tataru, D., *Sharp local well-posedness results for the nonlinear wave equation*, Ann. Math., 162 (2005), 291–366.
- [35] Stein, E. M., *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, With the assistance of Timothy S. Murphy, Princeton Mathematical Series 43, Monographs in Harmonic Analysis III, Princeton University Press, Princeton, NJ, 1993.
- [36] Tataru, D., *Strichartz estimates for operators with nonsmooth coefficients and the nonlinear wave equation*, Amer. J. Math. 122 (2000), 349–376.
- [37] Tataru, D., *Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients III*, J. Amer. Math. Soc., 15 (2002), 419–442.
- [38] Wang, Q., *Causal geometry of Einstein-vacuum spacetimes*, Ph.D thesis of Princeton University, 2006.
- [39] Wang, Q., *On the geometry of null cones in Einstein-vacuum spacetimes*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 26 (2009), no. 1, 285–328.
- [40] Wang, Q., *Improved Breakdown Criterion for Einstein Vacuum Equations in CMC Gauge*, Comm. Pure Appl. Math., Vol. LXV, 21–76 (2012).
- [41] Wang, Q., *Improved Breakdown Criterion for Einstein Vacuum Equations in CMC Gauge*, arXiv:1004.2938, 52 pages.
- [42] Wang, Q., *Rough solutions of Einstein vacuum equations in CMCSH gauges*, Communications in Mathematical Physics, 328 (2014), Issue 3, 1275–1340.
- [43] Wang, Q., *Causal geometry of rough Einstein CMCSH spacetime*, Journal of Hyperbolic Differential Equations, 11 (2014), No. 3, 563–601.
- [44] Wang, Q., *A geometric approach for sharp local well-posedness of quasilinear wave equations*, arXiv:1408.3780v1, 2014, 94 pages.
- [45] Wang, Q., *A geometric approach for sharp local well-posedness of quasilinear wave equations*, Annals of PDE, 3 (2017), no. 1, 108 pages.

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Indices

| | |
|---|---|
| v | velocity of the compressible Euler flow |
| ϱ | (normalized) density function |
| p | pressure function |
| c | sound speed |
| v_+ | wave function part of v |
| \mathbf{w} | vorticity |
| Ω | normalized vorticity |
| Σ_t | level set of the time function t |
| \mathbf{T} | time-like unit normal of Σ_t |
| $\mathbf{g}, \mathbf{g}^{-1}$ | acoustical metric and its inverse metric |
| g | induced metric of \mathbf{g} on Σ_t |
| e | Euclidean metric on \mathbb{R}^3 |
| k | second fundamental form of Σ_t |
| $\text{Tr}k$ | trace of k taken by the induced metric g |
| $(X)\pi$ | deformation tensor of the vector field X with respect to \mathbf{g} |
| ∂ | derivative of space variables |
| ∂ | ∂ or \mathbf{T} |
| \mathbf{D} | Levi-Civita connection of \mathbf{g} |
| ∇ or ∇_g | Levi-Civita connection of g on Σ_t |
| div | divergence operator in (\mathbb{R}^3, e) |
| curl | curl operator in (\mathbb{R}^3, e) |
| $\square_{\mathbf{g}}, \Delta_e$, and Δ_g | Laplace-Beltrami operators of \mathbf{g} , of the Euclidean metric e in \mathbb{R}^3 and of the metric g respectively |
| $\mathcal{E}_U^{(m)}(t)$ | m -order-energy of the function U on Σ_t |
| $\mathbf{R}_{\mu\nu}$ | component of Ricci tensor in the acoustical spacetime |
| $\mathbf{R}_{\mu\nu\gamma\delta}$ | component of the Riemann curvature tensor in the acoustical spacetime |
| $\text{Ric}(g)$ and \mathbf{Ric} | Ricci tensor of g and \mathbf{g} respectively |
| P_λ | Littlewood-Paley projection of the dyadic frequency λ |
| $P_{\leq\lambda}, P_{>\lambda}$ | sums of Littlewood-Paley projections with frequency no more than λ , and greater than λ respectively |
| L and \underline{L} | null and conjugate null vector fields |
| u | optical function |
| C_u | level set of u which is a null hypersurface |
| $S_{t,u}$ | intersection of C_u and Σ_t |
| γ | induced metric of \mathbf{g} on $S_{t,u}$ |
| $d\mu_g$ and $d\mu_\gamma$ | standard area elements on (Σ_t, g) and $(S_{t,u}, \gamma)$ respectively |
| \mathbf{N} | outward unit normal of the radial foliation $S_{t,u} \subset \Sigma_t$ |
| θ | second fundamental form on $S_{t,u} \subset \Sigma_t$ defined by the normal vector field \mathbf{N} |
| \tilde{r} | $t - u$ |
| \mathbf{b} | null lapse function |

| | |
|--|---|
| χ and $\text{tr}\chi$ | null second fundamental form and its trace contracted by γ |
| $\underline{\chi}$ and $\text{tr}\underline{\chi}$ | conjugate null second fundamental form and its trace contracted by γ |
| ∇ | Levi-Civita connection of γ on $S_{t,u}$ |
| Δ | Laplace-Beltrami operator of the metric γ on $S_{t,u}$ |
| div | divergence operator of the connection ∇ |
| curl | curl operator of the connection of ∇ |
| $\zeta, \underline{\zeta}$ | torsion tensor fields on null cones |
| μ | mass aspect function |