On some computations of refined Donaldson-Thomas invariants

Alberto Cazzaniga

Lincoln College

University of Oxford

A thesis submitted for the degree of

Doctor of Philosophy

Trinity 2015

. . . La tesi è come il maiale,
non se ne butta via niente.
−Umberto Eco, *Come si fa una tesi di laurea.*

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Donaldson and Thomas defined Donaldson-Thomas (DT) invariants for moduli spaces of sheaves on proper Calabi-Yau threefolds in [DT, T]. The definition has been extended to more general moduli spaces of objects in 3-Calabi-Yau categories, and in the last few years there has been an increasing interest in studying refinements of DT invariants. This thesis is devoted to the computation of the generating series of refined DT invariants in some geometric examples.
In Chapter 2 we compute the generating series of refined DT invariants for the moduli space $\mathcal{M}_{n,r}$ of higher rank framed sheaves on $\mathbb{P}^3$, generalizing the result of [BBS] for the Hilbert scheme of n points on $\mathbb{C}^3$. We describe $\mathcal{M}_{n,r}$ as a global critical locus by means of the Beilinson spectral sequence, and compute the generating series using motivic wallcrossing. We interpret the result as a count of r-tuples of weighted 3D partitions in analogy with the refined topological vertex [IKV].
In Chapter 3 we study the refined DT invariants for the moduli space $P^r_{n,d}$ of higher rank (framed) stable pairs on the resolved conifold. Applying twice the Beilinson spectral sequence, we identify $P^r_{n,d}$ with the moduli space of PT-stable representations of the r-framed conifold quiver with superpotential, generalizing a result of [NN]. We compute a product expansion for the generating series of refined DT invariants of $P^r_{n,d}$, and we compare our results with the partition function counting $U(1)$-instantons on the minimal resolution of an A_{r-1} surface singularity.
In Chapter 4 we study the universal generating series of motivic DT invariants for deformations of graded 3-Calabi-Yau algebras. We consider Jacobi algebras which are derived equivalent to the resolved conifold and the resolution of a line of A_n-singularities, and investigate their marginal deformations. The universal series of motivic DT invariants are affected by the deformations, and their variation is interpreted in terms of virtual classes of some special representation varieties associated to the quiver.

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Contents

Introduction

Donaldson-Thomas (DT) invariants of Calabi-Yau threefolds were proposed by Donaldson and Thomas in [DT], and defined by Thomas in [T], as an analogue of the Casson invariant for real 3-dimensional manifolds, with the objective of producing a new numerical invariant of complex projective Calabi-Yau threefolds which is sensitive to their complex structure while being invariant under smooth deformations. Given a smooth complex projective Calabi-Yau threefold X with a fixed polarization L, the invariants are associated to the proper coarse moduli space $M_\alpha(X, L)$ of Gieseker-stable sheaves on X with trivial determinant and Chern character $\alpha \in \mathrm{H}^*(X, \mathbb{Q})$, under the assumption that stability coincides with semistability. More specifically, from the Calabi-Yau assumption the deformations $\mathrm{Ext}^1(E, E)$ of such a stable sheaf E on X are dual to the obstructions $\mathrm{Ext}^2(E, E)$, so that Thomas [T], following Behrend and Fantechi [BF1], constructs a perfect obstruction theory on $M_\alpha(X, L)$ which induces a 0-dimensional cycle $[M_\alpha(X, L)]_{\mathrm{vir}}$ whose degree $\mathrm{DT}_\alpha(X, L) \in \mathbb{Z}$ is the required invariant. The dependence on the polarization L has been interpreted in terms of wall-crossing by Joyce and Song [JS], who generalized the construction for the case of moduli spaces $M_\alpha^{ss}(X, L)$ of semistable sheaves by defining an invariant $\mathrm{DT}_\alpha^{ss}(X, L)$ which in general lies in $\mathbb{Q}$, but that is still invariant under smooth deformations of X.

A new approach to the study of DT invariants is due to Behrend [B], who interprets the numerical DT invariant as virtual Euler characteristic. In more detail, Behrend shows that for any scheme Z equipped with a perfect *symmetric* obstruction theory, there is an intrinsically defined constructible function ν_Z, called the Behrend function, such that when $Z = M_\alpha(X, L)$ is proper one has

$$\mathrm{DT}_\alpha(X, L) = \chi(Z, \nu_Z) = \sum_{k \in \mathbb{Z}} k \cdot \chi(\nu_Z^{-1}(k)),$$

where χ is the topological Euler characteristic. Behrend's insight gives a local approach to the theory, allowing more computational flexibility via stratification techniques, and extending the definition of DT invariants to non-proper settings, such as moduli spaces of sheaves on open Calabi-Yau threefolds.

In their work [DT], Donaldson and Thomas also observed that the moduli space of vector bundles on a Calabi-Yau threefold X is locally cut out as the critical locus of the Chern-Simons functional on an infinite-dimensional manifold. This led to the long standing conjecture that the moduli space of sheaves $M_\alpha(X, L)$ defined above is locally the critical locus of a regular function on a smooth scheme. Progress in this direction has been made by Joyce and Song in [JS], who proved by gauge theoretic arguments that this is true in the analytic category, and by Behrend in [B], who showed that the algebraic statement is true for any scheme with a perfect symmetric obstruction theory if one relaxes the requirement to Z being the zero locus of an *almost closed* 1-form. More recently, Brav, Bussi and Joyce showed in [BBJ] that if one considers a scheme Z which is the classical truncation of a derived scheme with a (-1)-shifted symplectic structure [PTVV], then Z arises locally as the critical locus of a regular function on a smooth scheme. This together with the construction of a (-1)-shifted symplectic structure on $M_\alpha(X, L)$ due to Pantev, Toën, Vaquié and Vezzosi [PTVV], shows that the natural local setting to study Donaldson-Thomas theory is

$$Z = \mathrm{Crit}(f\colon M \longrightarrow \mathbb{C}),$$

where M is a smooth scheme and f a regular function.

In the local setting, the work of Behrend suggested the existence of various refinements of Donaldson-Thomas theory. If $Z = \mathrm{Crit}(f)$ for a regular function f on a smooth scheme M, the value of the Behrend function ν_Z at a point z in Z is the value of the reduced Euler characteristic of the Milnor fiber of f at z, up to multiplication by $(-1)^{\dim M}$. This agrees with the Euler characteristic at z of the half Tate twist of Deligne's perverse sheaf of vanishing cycles $\Phi_f = \phi_f \mathbb{Q}(\frac{\dim M}{2})[\dim M]$, so that the sheaf Φ_f can be considered as a categorification of the numerical DT invariant. The cohomological DT invariant

$$\mathbb{H}^*_c(Z, \Phi_f)$$

admits a mixed Hodge structure, and one defines the weight polynomial

$$W(Z, \Phi_f) = \sum_n t^{\frac{n}{2}} \sum_i (-1)^i \dim_{\mathbb{Q}} \mathrm{Gr}^W_n \mathbb{H}^i_c(Z, \Phi_f)$$

to be the refined DT invariant. A global version of this invariant has been defined by Brav, Bussi, Dupont, Joyce and Szendrői in [BBDJS] for derived schemes with a (-1)-shifted symplectic structure and a choice of orientation [KS1] by gluing the local data. Another approach towards refining numerical DT invariants follows the insight of Kontsevich and Soibelman [KS1]. Since the DT invariant is a (weighted) Euler characteristic, it can be computed using cut and paste techniques, so that it is expected to admit a refinement in an enhanced version of the Grothendieck ring of varieties $K_0(\mathrm{Var}_{\mathbb{C}})$. When $Z = \mathrm{Crit}(f)$ as above, Kontsevich and Soibelman define the motivic DT invariant $[Z]_{\mathrm{vir}}$ of (Z, f) as the motivic class of vanishing cycles defined by Denef and Losev in [DL]. Also in this case, a construction due to Bussi, Joyce and Meinhardt [BJM] shows that, starting from a scheme with a (-1)-shifted symplectic structure and a choice of orientation, the local motivic DT invariants can be glued to obtain a global invariant.

The main objective of this thesis is to compute the generating series of refinements of the classical DT invariants in some geometric examples, related in different ways to recent developments in the field.

The first example we consider, which will be discussed in Chapter 2, consists of a higher rank refined generalization of the MNOP conjecture [MNOP] for the case of $\mathbb{C}^3$. In [MNOP], the authors conjectured that the generating series of numerical DT invariants of the Hilbert scheme of points on a quasi-projective Calabi-Yau threefold admits a product expansion $M(-q)^{\chi(X)}$, where

$$M(q) = \prod_{n\geq 1}(1-q^n)^{-n}$$

is the MacMahon function counting $3D$-partitions. The conjecture has been proven independently by Behrend and Fantechi [BF2], Levine and Pandharipande [LP] and Li [L], and it plays a prominent role in the work of Toda [To], where the author describes the relation between the generating series of DT invariants of Hilbert scheme of curves on Calabi-Yau threefolds and stable pairs invariants defined by Pandharipande and Thomas [PT1]. More recently, in [BBS] Behrend, Bryan and Szendrői computed a motivic refinement of the MNOP conjecture which involves a product of twisted MacMahon functions, and from [IKV] the result was interpreted as a weighted count of $3D$-partitions. We generalize the results in [BBS] for the higher rank case.

The moduli space $\mathcal{M}_{n,r}$ of higher rank sheaves on $\mathbb{C}^3$ we will be considering, which generalizes the case of the Hilbert scheme of points $\mathrm{Hilb}^n(\mathbb{C}^3)$, is defined by analogy with the algebraic incarnation of the moduli space of instantons on $\mathbb{R}^4$, as in Nakajima

[N]. Similarly to the very recent work [GKY], we will consider a projective toric compactification $\mathbb{C}^3 \subset \mathbb{P}^3$ obtained by adding an hyperplane p_∞. We define $\mathcal{M}_{n,r}$ to be the moduli functor whose geometric points parametrise isomorphism classes of torsion free sheaves F on $\mathbb{P}^3$ with a framing ϕ along p_∞, i.e. a fixed isomorphism $\phi\colon F|_{p_\infty} \longrightarrow \mathcal{O}_{p_\infty}^{\oplus r}$, and with Chern character $r - nh^3$. Section 2.1 is devoted to the proof of the following theorem, which shows that $\mathcal{M}_{n,r}$ arises as a global critical locus.

Theorem 0.1. *There is a smooth quasi-projective variety $U_{n,r}$ and a regular function*

$$\omega_{n,r}\colon U_{n,r} \longrightarrow \mathbb{C}$$

such that $\mathcal{M}_{n,r}$ is represented by $M_{n,r} = \mathrm{Crit}(\omega_{n,r})$.

This result is achieved by employing the techniques of the Beilinson spectral sequence for sheaves on $\mathbb{P}^3$ to show that $\mathcal{M}_{n,r}$ is represented by the moduli space of r-framed stable representations of the three loop quiver with superpotential. From Theorem 0.1 we can construct the topological coefficient system $\Phi_{\omega_{n,r}}$ on $M_{n,r}$, and define the refined DT invariant $W(M_{n,r}, \Phi_{\omega_{n,r}})$. We fix the rank r of the sheaves and study the generating series

$$\mathrm{DT}_{\mathbb{C}^3,r}(q) = 1 + \sum_{n\geq 1} W(M_{n,r}, \Phi_{\omega_{n,r}}) q^n,$$

obtaining the following theorem, which is the main result of Chapter 2.

Theorem 0.2. *The generating series $\mathrm{DT}_{\mathbb{C}^3,r}$ of refined DT invariants of the moduli space $M_{n,r}$ admits a product expansion*

$$\mathrm{DT}_{\mathbb{C}^3,r}(q) = \prod_{m=1}^{\infty} \prod_{k=0}^{rm-1} \left(1 - t^{2+k-\frac{rm}{2}}((-1)^r q)^m\right)^{-1}. \tag{1}$$

We give two different proofs of Theorem 0.2, and in both cases we pass through the generating series of motivic DT invariants and then consider the weight specialization. In Section 2.2.1, we compute the generating series by dimensional reduction closely following [BBS]. The approach of Section 2.2.2 is based on the techniques of motivic wallcrossing for framed objects developed by Mozgovoy in [Moz3], i.e. in this case we obtain the invariants for $\mathcal{M}_{n,r}$ from the universal series of the invariants of unframed representation of the three loop quiver in a critical chamber. Considering the numerical specialization $\mathrm{DT}_{\mathbb{C}^3,r}(q)\big|_{t^{\frac{1}{2}}=1}$ of the formula, we obtain the generating series of numerical DT invariants, which is given by the r-th power of $M((-1)^r q)$, as it was already observed by Oprea [O], and by Cirafici, Sinkovic and Szabo in [CSS] using localization techniques. In this sense, our formula (1) is also a refinement of the building block formula [GKY, Conjecture C] of Gholampour, Kool and Young for the case of $\mathbb{C}^3$ where one assumes the reflexive hull to be locally free (which is a trivial case for the purposes of the authors in [GKY]). We discuss in Section 2.2.3 how the series of refined DT-invariants for rank r sheaves factors as a product of twisted rank 1 contributions, and we describe the weight polynomial in terms of a count of weighted $3D$-partitions discussing asymptotics.

Chapter 3 is devoted to the study of refined invariants of the moduli space $\mathcal{P}^r_{n,d}(X)$ of higher rank framed stable pairs on a compactification of the resolved conifold singularity. Moduli spaces of stable pairs have been introduced by Pandharipande and Thomas in [PT1] in order to obtain better suited moduli spaces for the study of a correspondence with Gromov-Witten theory. They parametrise pairs (F, s) where F is a pure sheaf supported in dimension one on a Calabi-Yau threefold X, and $s\colon \mathcal{O}_X \longrightarrow F$ is a section which is surjective apart from a finite number of points. Nagao and Nakajima showed in [NN] that the moduli space $\mathcal{P}^1_{n,d}(X^o)$ of stable pairs on the resolved conifold singularity $X^o = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2})$ arises as a global proper critical locus of a regular function on a smooth scheme, by constructing an isomorphism between $\mathcal{P}^1_{n,d}(X^o)$ and the moduli space of PT-stable framed representations of the conifold quiver with superpotential. They use this description to compute the partition function of the numerical DT-invariants and to study its behavior under change of King stability condition. Morrison, Mozgovoy, Nagao and Szendrői refined in [MMNS] the results of Nagao and Nakajima, exhibiting a product expansion of the generating series of motivic DT invariants, and interpreted the result in terms of the formalism of the refined topological vertex [IKV]. In Chapter 3 we generalize this result to the case of higher rank framed stable pairs on the resolved conifold, and discuss its relation with invariants of surfaces.

In analogy with the case of $\mathbb{C}^3$, we consider the toric compactification $X^o \subset X = \mathrm{Proj}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2})$ obtained by adding a divisor D_∞. Following [Sh1] we study the moduli space of framed higher rank stable pairs $\mathcal{P}^r_{n,d}(X)$ whose geometric points are isomorphism classes of pairs (F, s) where F is a pure sheaf supported in dimension one away from D_∞, and $s\colon \mathcal{O}_X^{\oplus r} \longrightarrow F$ is a multi-section with cokernel supported on a finite number of points. We show in Sections 3.2 and 3.3 that $\mathcal{P}^r_{n,d}(X)$ arises as a global critical locus.

Theorem 0.3. *The moduli functor $\mathcal{P}^r_{n,d}(X)$ is represented by a proper scheme $P^r_{n,d}$, and it arises as the critical locus of a regular function*

$$\omega^r_{n,d}\colon U^r_{n,d} \longrightarrow \mathbb{C}$$

on a smooth quasi-projective variety $U^r_{n,d}$.

We achieve this result by applying twice the techniques of the Beilinson spectral sequence. Following a construction due to Diaconescu [Di2], we use the relative version of the Beilinson spectral sequence for sheaves on projective bundles, and we show that $P^r_{n,d}$ is isomorphic to the moduli space of asymptotically stable ADHM sheaves on $\mathbb{P}^1$ with framing rank r. The geometric points of this moduli space are given by the datum of a vector bundle E on $\mathbb{P}^1$ together with a decoration consisting of two sections $\phi_1, \phi_2 \in \mathrm{Hom}(E(1), E)$ satisfying an ADHM-type relation, and a framing given by a multi-section $\psi\colon \mathcal{O}^{\oplus r} \longrightarrow E$ which fulfills the requirements of asymptotic stability. We then employ the Beilinson spectral sequence for sheaves on $\mathbb{P}^1$ obtaining that $P^r_{n,d}$ is isomorphic to the moduli space of r-framed PT-stable representations of the conifold quiver with superpotential. This implies Theorem refteo2, giving a generalization of the results of Nagao and Nakajima for the Pandharipande-Thomas chamber. From Theorem 0.3 we can construct the topological coefficient system $\Phi_{\omega^r_{n,d}}$ and study the refined DT invariant $W(P^r_{n,d}, \Phi_{\omega^r_{n,d}})$. We fix the rank r, and we study the generating series of refined DT invariants

$$Z^r_X(s,T) = \sum_{n,d\geq 0} W(P^r_{n,d}, \Phi_{\omega^r_{n,d}}) s^n T^d,$$

obtaining the following product expansion, which gives the main result of Chapter 3.

Theorem 0.4. *The generating series Z^r_X of refined DT invariants of the moduli spaces of higher rank stable pairs admits a product expansion*

$$Z_{X,r}(s,T) = \prod_{n\geq 1} \prod_{j=0}^{rn-1} \left(1 - t^{\frac{rn}{2}-j-\frac{1}{2}}((-1)^r s)^n T\right).$$

Also in this case the formula is obtained in Section 3.4 using motivic wallcrossing for framed objects following [MMNS], and then considering the weight specialization. Since the moduli space $P^r_{n,d}$ is proper, the weight filtration of $\mathbb{H}^*(P^r_{n,d}, \Phi_{\omega^r_{n,d}})$ agrees with the degree filtration from [DMSS], so the product expansion that we obtain in Theorem 0.3 is actually computing the generating series of Poincaré polynomials of the critical cohomology. The Euler characteristic specialization $Z_{X,r}(s,T)\big|_{t^{\frac{1}{2}}=1}$ is given by

$$Z_{X,r}(s,T)\big|_{t^{\frac{1}{2}}=1} = \prod_{n\geq 1}(1-((-1)^r s)^n T)^{rn},$$

and it is obtained, up to the change of sign $(-1)^r s$ of the point-counting parameter s, as the r-th power of the generating series of numerical DT invariants of Pandharipande and Thomas stable pairs, as observed by Sheshmani in [Sh2] using torus localization techniques. In Section 3.4.1 we exhibit a factorization in terms of twisted rank 1 contributions in order to compare, as proposed by geometric engineering [KKV], the generating series Z_X^r with the partition function $Z_{A_{r-1},1}$ counting instantons on a resolved A_{r-1}-surface singularity [NY]. Despite observing a striking similarity in the factorizations of the two partition functions as the product of elementary contributions, we are unable to obtain a full correspondence, and we discuss how further research building on the recent work of Nekrasov and Okounkov [NO] might achieve a resolution of this discrepancy.

Chapter 4 is devoted to the study of the behavior of the universal generating series of motivic DT invariants of some graded Calabi-Yau algebras under graded deformations. From the work of Kontsevich [Ko] on homological mirror symmetry, the DT invariants of a Calabi-Yau threefold can be considered as invariants attached to the derived category of coherent sheaves on X. When X is a local Calabi-Yau threefold whose bounded derived category $\mathrm{D}^b(\mathrm{Coh}(X))$ is isomorphic to the derived category of modules over the path algebra $A(Q,W)$ of a quiver with superpotential, it is then natural to consider motivic DT invariants for the moduli spaces of finite-dimensional modules over $A(Q,W)$, as defined by Kontsevich and Soibelman in [KS1]. More precisely, using the fact that for such an algebra $A(Q,W)$ the trace of the superpotential W defines a regular function ω_d on the space $R_d(Q,W)$ of d-dimensional modules, one can define the virtual motive of the stack $\mathcal{M}_d(Q,W)$ of isomorphism classes of d-dimensional modules and study the universal series

$$U(Q,W) = \sum_d [\mathcal{M}_d(Q,W)]_{\mathrm{vir}} y^d.$$

By analogy with the classical picture, graded deformations of the 3-Calabi-Yau algebra $A(Q,W)$ play the same role as smooth families of Calabi-Yau threefolds, so it is interesting to investigate the behavior of the universal series $U(Q,W)$ under such deformations. We focus on the cases when the algebra $A(Q,W)$ is the noncommutative crepant resolution of the conifold singularity and of the resolution of a line of A_n-singularities. In particular, we concentrate on one parameter families of deformations of such algebras which are obtained from a marginal deformation W_q of the original superpotential W. We compute the universal generating series for the marginally deformed algebras in Theorems 4.6 and 4.16 via dimensional reduction of quivers with a cut [Mo1], expressing the series as a motivic exponential. The results show that the universal series of motivic DT invariants is far from being invariant under graded deformations of the 3-Calabi-Yau algebras, as observed also in [OU] and independently by Morrison. When the deformation parameter q is generic the generating series stays unchanged, and in Propositions 4.13 and 4.26 we show how the universal generating series can be understood in terms of the virtual motive of the stack of stable representations of the relevant quiver with q-deformed potential for some particular choice of King stability condition. While the universal series of motivic DT invariants of $A(Q, W_q)$ depends on the deformation parameter q, in all the examples that have been computed the Euler characteristic specialization is not affected by deformations [CMPS], so that it is natural to ask if Donaldson-Thomas invariants of graded 3-Calabi-Yau algebras are invariant under graded deformations for which the algebra is still Calabi-Yau.

Chapter 1

Some Preliminary Constructions and Results

1.1 Quivers with superpotential and moduli of their representations

In this section we recall some basic definitions and results for (framed) quivers (with superpotential) and moduli spaces of their representations, which are going to be used in the rest of this thesis. The presentation we propose is mostly inspired by the work of Bocklandt [Bo] and Ginzburg [Gi] for the definition and basic properties of representations of quivers with superpotentials, by the work of King [Ki1] for the construction of moduli spaces of quiver representations, and by the work of Reineke [R] for the discussion of the case of framed quiver representations. The examples where we link the geometry of crepant resolutions of some rational threefold singularities to that of moduli of representations of quivers with superpotential are due to the work of Van den Bergh on non-commutative crepant resolutions [VdB1].

1.1.1 Quivers with superpotential

A *quiver* Q is a directed graph, i.e. the datum $(V(Q), E(Q), h, t)$ of a set of vertices $V(Q)$, a set of arrows $E(Q)$ and two functions $h, t\colon E(Q) \longrightarrow V(Q)$ specifying the head and the tail of each arrow. We will consider only finite connected quivers, that is quivers whose sets of vertices and arrows are finite and whose underlying graph is connected.

Once a quiver Q is given, it is natural to define an associative $\mathbb{C}$-algebra A_Q called the *path algebra* of the quiver. If an arrow a_1 ends where another arrow a_2 starts we denote by $a_1 * a_2$ the path obtained by concatenation of the two arrows. The path algebra A_Q is spanned as a $\mathbb{C}$-vector space by paths p on Q obtained as consecutive concatenation of arrows, where we include the trivial paths e_i based at the vertices. The product on A_Q is defined by the $\mathbb{C}$-linear extension of concatenation of paths given by

$$(a_1 * \ldots * a_n) \cdot (a'_1 * \ldots * a'_m) = \begin{cases} a_1 * \ldots * a_n * a'_1 * \ldots * a'_m & \text{if } h(a_n) = t(a'_1); \\ 0 & \text{otherwise.} \end{cases}$$

This defines a structure of associative $\mathbb{C}$-algebra on A_Q with unit $\sum_{i \in V(Q)} e_i$.

Remark 1.1. The algebra A_Q is almost never commutative. Furthermore, A_Q is finite-dimensional if and only if the quiver Q does not contain oriented cycles or loops.

It is usually more interesting to study the whole category of modules over the path algebra of a quiver rather than the path algebra itself. This can be interpreted in terms of quiver representations.

A *representation of a quiver* Q is the assignment of a finite-dimensional vector space V_i to each vertex i and of a linear morphism $v_a : V_{t(a)} \longrightarrow V_{h(a)}$ to each arrow of Q. We call $\underline{d} = (\dim_{\mathbb{C}}(V_i))$ the *dimension vector* of the representation. A *morphism* between two representations $\{V_i, v_a\}$ and $\{V'_i, v'_a\}$ of Q is a collection of linear maps $\{\sigma_i : V_i \longrightarrow V'_i\}$ such that the diagrams

$$\begin{tikzcd} V_{t(a)} \arrow[r, "v_a"] \arrow[d, "\sigma_{t(a)}"'] & V_{h(a)} \arrow[d, "\sigma_{h(a)}"] \\ V'_{t(a)} \arrow[r, "v'_a"] & V'_{h(a)} \end{tikzcd}$$

commute.

Proposition 1.2. *The category* $\mathbf{mod}_{A_Q}$ *of finite-dimensional left* A_Q*-modules is equivalent to the category* $\mathbf{rep}_{\mathbb{C}}(Q)$ *of quiver representations of* Q*.*

Our interest in quiver representations arises from the possibility of constructing explicit moduli spaces which are analogous to moduli of bundles (or sheaves) on local geometries. In the case when one considers holomorphic bundles on a CY threefold the moduli space is cut out by the critical locus of the Chern-Simons functional. The idea of considering moduli of representations of quivers subject to relations coming from a superpotential arose in the physics literature, where it was used to construct a local finite-dimensional model of the CS functional.

We will denote by $\mathrm{Sup}(Q)$ the quotient of the path algebra A_Q by the commutator subspace $[A_Q, A_Q]$. The vector space $\mathrm{Sup}(Q)$ is generated by elements represented by cyclic monomials, i.e. elements $a_1 \cdot \ldots \cdot a_n$ satisfying $t(a_1) = h(a_n)$. Notice that $\mathrm{Sup}(Q)$ admits a grading defined by the length of the cyclic monomials; we will adopt the notation $\mathrm{Sup}(Q)_d$ for the graded part of degree d. For every arrow a in Q we can define a formal operation of differentiation $\partial_a \colon \mathrm{Sup}(Q) \longrightarrow A_Q$ as follows: given a cyclic monomial x define $\partial_a(x)$ to be zero if a does not appear in the monomial, and otherwise define it to be the path obtained by cyclically permuting x until a is in first position and then canceling it. The derivation ∂_a gives a well-defined linear map on $\mathrm{Sup}(Q)$, as it is independent of the cyclic ordering of the elements defining a cyclic monomial x.

Remark 1.3. The $\mathbb{C}$-linear extension of the function

$$\circlearrowleft \colon a_1 \ldots a_n \longmapsto \sum_{i=1}^{n} a_i \ldots a_n a_1 \ldots a_{i-1},$$

defined on the set of cyclic monomials, gives a morphism $\circlearrowleft \colon \mathrm{Sup}(Q) \longrightarrow A_Q$. For any arrow $a \in E(Q)$ there is a lift of ∂_a to a linear map $\widetilde{\partial}_a \colon A_Q \longrightarrow A_Q$ such that

$$\begin{tikzcd} \mathrm{Sup}(Q) \arrow[r, "\circlearrowleft"] \arrow[dr, "\partial_a"'] & A_Q \arrow[d, "\widetilde{\partial}_a"] \\ & A_Q \end{tikzcd}$$

is a commutative diagram.

A *superpotential* for Q is the choice of an element $W \in \mathrm{Sup}(Q)$. Consider the two sided ideal $I_W := \langle\langle \partial_a W : a \in E(Q) \rangle\rangle$ of A_Q. The *Jacobi algebra* of the quiver with superpotential (Q, W) is the quotient A_Q/I_W of the path algebra by the ideal of relations defined by the superpotential. We will denote the Jacobi algebra of (Q, W) by $A_{Q,W}$, or alternatively by $J(Q, W)$. One can define the category of representations of (Q, W) as the full subcategory of $\mathbf{rep}_{\mathbb{C}}(Q)$ whose objects are representations satisfying the relations in I_W. Similarly to the above one gets:

Proposition 1.4. *The category $\mathbf{mod}_{A_{Q,W}}$ of finite-dimensional left $A_{Q,W}$-modules is equivalent to the category $\mathbf{rep}_{\mathbb{C}}(Q, W)$ of quiver representations of (Q, W).*

Example 1.5. Let Q be a quiver with one vertex v_0 and three loops $\{a_i\}_{i \in I_3}$ based at that vertex, as indicated in Figure 1.1.

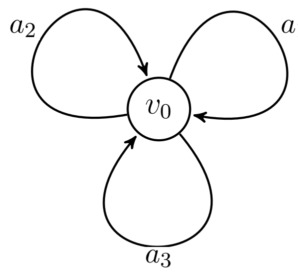

Figure 1.1: The three-loop quiver Q.

The path algebra A_Q is the ring of functions of the non-commutative $\mathbb{C}^3$, i.e. the free non-commutative algebra $\mathbb{C}\langle a_1, a_2, a_3\rangle$ with three generators. Consider now the superpotential $W = a_1a_2a_3 - a_1a_3a_2$. The ideal I_W is $\langle\langle [a_i, a_j]\rangle\rangle$ and so the path algebra $A_{Q,W}$ is isomorphic to $\mathbb{C}[a_1, a_2, a_3]$, the ring of functions of $\mathbb{C}^3$.

Example 1.6. Consider a quiver Q with two vertices $\{v_0, v_1\}$, two arrows $\{a_1, a_2\}$ from v_0 to v_1 and two arrows $\{b_1, b_2\}$ in the opposite direction. This is called the *conifold quiver* and is depicted in Figure 1.2.

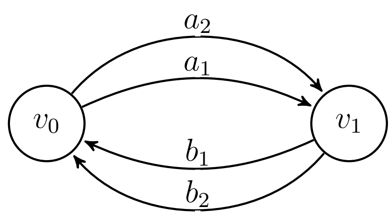

Figure 1.2: The conifold quiver Q.

Consider the semisimple algebra $S = \mathbb{C}[e_{v_0}, e_{v_1}]$ generated by the trivial paths at the vertices. The vector space $\langle a_i, b_i | i \in \{1, 2\}\rangle$ is a $S - S$ bimodule under the usual concatenation of paths. In this way the path algebra A_Q of the conifold quiver is the tensor algebra over S of this bimodule. If we consider the conifold superpotential $W = a_1b_1a_2b_2 - a_1b_2a_2b_1$ we obtain that the path algebra

$$A_{Q,W} = \frac{A_Q}{\langle\langle a_1b_ia_2 - a_2b_ia_1, b_1a_ib_2 - b_2a_ib_1\rangle\rangle}$$

is finite-dimensional over its center $Z(A_{Q,W}) \simeq \mathbb{C}[x,y,z,t]/(xt-yz)$, the coordinate ring of a three-dimensional nodal or conifold singularity, where

$$x = a_1b_1 + b_1a_1,\ y = a_1b_2 + b_2a_1,\ z = a_2b_1 + b_1a_2,\ t = a_2b_2 + b_2a_2.$$

In fact $A_{Q,W}$ is a non-commutative crepant resolution (NCCR) of the conifold singularity in the sense of Van den Bergh [VdB1, Definition 4.1].

Example 1.7. Consider the quiver Q_{n+1} with $n+1$ vertices labeled from 0 to n and with arrows depicted as in Figure 1.3. This is called the A_n-*quiver*.

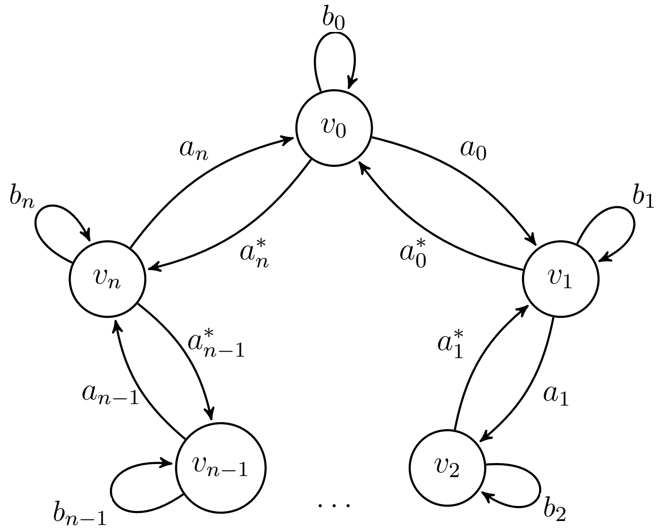

Figure 1.3: The A_n-quiver Q_{n+1}.

We are interested in the superpotential

$$W = \sum_{i=0}^{n} b_{i+1}a_i^*a_i - b_ia_ia_i^*, \tag{1.1}$$

where we consider the indices modulo $n+1$, so that in particular $b_{n+1} := b_0$. The center $Z(J(Q_{n+1},W))$ of the Jacobi algebra is isomorphic to $\mathbb{C}[x,y,z,t]/(xy-z^{n+1})$, the coordinate ring of a line of A_n-singularities, where

$$x = a_0 \dots a_n,\ y = a_n^* \dots a_0^*,\ z = \sum_{i=0}^{n} a_ia_i^*,\ t = \sum_{i=0}^{n} b_i.$$

Also in this case the superpotential algebra $A_{Q_{n+1},W}$ is a NCCR of the threefold singularity whose coordinate ring is determined by the center $Z(A_{Q_{n+1},W})$. We will see in Chapter 4 that Q_{n+1} is the McKay quiver for the embedding of $\mathbb{Z}/(n+1)\mathbb{Z}$ in $\mathrm{SL}(3,\mathbb{C})$ with weights $(1,-1,0)$.

Remark 1.8. Examples 1.6 and 1.7 are particular cases of a much more general theory initiated by Van den Bergh [VdB1]. For instance, every Gorenstein affine toric threefold admits a non-commutative crepant resolution [Bro, Theorem 8.6].

1.1.2 Moduli of quiver representations

We report here some definitions and results on moduli spaces of (semi)-stable quiver representations, developed in [Ki1] by King via Mumford's GIT theory of [MFK].

Given a quiver Q, fix an integer vector $\underline{d}$ in $\mathbb{N}^{V(Q)}$ and consider the affine variety

$$R_{\underline{d}}(Q) = \bigoplus_{a \in E(Q)} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{d_{t(a)}}, \mathbb{C}^{d_{h(a)}})$$

of quiver representations of Q of dimension vector $\underline{d}$. The algebraic group $G_{\underline{d}} = \prod_{i \in V(Q)} GL_{d_i}$ acts on $R_{\underline{d}}(Q)$ via $(g_i) \cdot v_a = g_{h(a)} v_a g_{t(a)}^{-1}$. The diagonal action of $\mathbb{C}^*$ leaves all the representations invariant, so that it is natural to consider the action of $PG_{\underline{d}} := \left(\prod_{i \in V(Q)} GL_{d_i}\right)/\mathbb{C}^*$. With this notation two $\underline{d}$-dimensional representations of Q are isomorphic if and only if they differ by an element of $PG_{\underline{d}}$.

In general the naïve quotient $R_{\underline{d}}(Q)/PG_{\underline{d}}$ is badly behaved, as it is often a non Hausdorff topological space (in the analytic topology).

The first idea to overcome this problem is to consider the classical Hilbert quotient

$$M_{\underline{d}}(Q) := \mathrm{Spec}(\mathbb{C}\left[R_{\underline{d}}(Q)\right]^{PG_{\underline{d}}}),$$

obtained from the ring of $PG_{\underline{d}}$-invariant functions. In this case the points of the quotient variety are in bijection with isomorphism classes of semisimple representations of the quiver, i.e. direct sums of representations which do not admit non-trivial subrepresentations. This can be rather unsatisfactory as it often leads to uninteresting moduli spaces; for instance, given a quiver with no cycles, the only simples are the one-dimensional representations based at the vertices, so that in this case the quotient is just a single point for each choice of dimension vector.

Mumford's GIT construction allows us to see this very last construction as a special case of a much more interesting class of moduli spaces. Consider a function $\theta \colon \mathbb{Z}^{V(Q)} \longrightarrow \mathbb{Z}$, and write θ_i for $\theta \cdot e_i$. If $\theta \cdot \underline{d} = 0$ then the character

$$\chi_\theta : (g_i) \longmapsto \prod_{i \in V(Q)} \det(g_i)^{\theta_i}$$

of $G_{\underline{d}}$ descends to a character of $PG_{\underline{d}}$. Denote by L_θ the $PG_{\underline{d}}$-equivariant line bundle on $R_{\underline{d}}(Q)$ associated to the character χ_θ, and let $H^0(R_{\underline{d}}(Q), L_\theta)$ be the vector space of its $PG_{\underline{d}}$-equivariant sections. We define the moduli space of χ_θ-semistable representations of Q of dimension vector $\underline{d}$ as

$$M^\theta_{\underline{d}}(Q) = \mathrm{Proj}\left(\bigoplus_{n\geq 0} H^0(R_{\underline{d}}(Q), L_{n\theta})\right).$$

It is well known that $M^\theta_{\underline{d}}(Q)$ is the quotient of the χ_θ-semistable locus

$$R^{\theta\text{-ss}}_{\underline{d}}(Q) = \{x \in R_{\underline{d}}(Q) \colon f(x) \neq 0 \text{ for some } f \in \bigoplus_{n>0} H^0(R_{\underline{d}}(Q), L_{n\theta})\}$$

by the following equivalence relation: $x \sim y$ if and only if the intersection of $\overline{PG_{\underline{d}} \cdot x}$ and $\overline{PG_{\underline{d}} \cdot y}$ in $R_{\underline{d}}(Q)^{\mathrm{ss}}$ is non-empty.

The moduli space $M^\theta_{\underline{d}}(Q)$ contains the stable locus $M^{\theta\text{-st}}_{\underline{d}}(Q)$ as an open subvariety, i.e. the orbit space of

$$R^{\theta\text{-st}}_{\underline{d}}(Q) = \left\{x \in R_{\underline{d}}(Q)^{\mathrm{ss}} \colon PG_{\underline{d}} \cdot x \text{ is closed and } \left|\mathrm{Stab}_{PG_{\underline{d}}}(x)\right| < \infty\right\}$$

with respect to the action of $PG_{\underline{d}}$.

Remark 1.9. The Hilbert quotient $M_{\underline{d}}(Q)$ coincides with the GIT quotient with respect to the trivial character. In particular, the Mumford GIT quotient $M^\theta_{\underline{d}}(Q)$ is relatively projective over the classical Hilbert GIT quotient $M_{\underline{d}}(Q)$.

King's main contribution in [Ki1] was to show that the notions of stability and semistability of Mumford's construction can be formulated purely in algebraic terms in the case of moduli of quiver representations. Let us discuss this now.

Given a dimension vector $\underline{d}$, we define the *slope* of $\underline{d}$ with respect to θ to be the rational number

$$\mu_\theta(\underline{d}) = \frac{\theta \cdot \underline{d}}{|\underline{d}|}, \tag{1.2}$$

where $|.|$ denotes the length of the integer vector. We will say that a $\underline{d}$-dimensional representation $(\underline{V}, \underline{v})$ of Q is *θ-semistable* if for every subrepresentation $(\underline{V}', \underline{v}')$ we have $\mu_\theta(\dim \underline{V}') \leq \mu_\theta(\dim \underline{V})$. Analogously, *$\theta$-stable representations* are θ-semistable representations for which the strict inequality holds whenever one considers a proper subrepresentation. Note that the notion of stability we consider here is presented differently from the one adopted in [Ki1], but they agree for representations satisfying $\theta \cdot \underline{d} = 0$.

It is a classical result that $\underline{\theta}$-semistable representations of a quiver Q of a given slope

μ form a full abelian subcategory of $\mathbf{mod}_{A_Q}$. From this it follows that θ-semistable representations of Q admit a *Jordan-Hölder filtration* by θ-semistables. In other words, under the correspondence of Proposition 1.2, given a θ-semistable A_Q-module M there is a proper increasing sequence of θ-semistable modules

$$0 = M_0 \subset \ldots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is θ-stable of slope $\mu(M)$. In particular, to each θ-semistable module M one can associate a graded θ-polystable object $\mathrm{gr}\,(M) = \oplus_i M_i/M_{i-1}$ that is unique up to permutation of the graded components. Two θ-semistable representations are *S-equivalent* if their associated θ-polystable objects are isomorphic.

Proposition 1.10 ([Ki1, Proposition 3.1,3.2])**.** *A point in $R_{\underline{d}}(Q)$ is χ_θ-semistable (resp. χ_θ-stable) if and only if the corresponding quiver representation is θ-semistable (resp. θ-stable). Furthermore, equivalence of χ_θ-semistable points coincides with S-equivalence of the corresponding semistable representations, so that the moduli space $M^\theta_{\underline{d}}(Q)$ parametrises θ-polystable objects of dimension vector $\underline{d}$.*

Remark 1.11. Note that when we use the word moduli space we are not distinguishing between fine and coarse moduli spaces. The stable locus of a quiver GIT construction with *primitive* dimension vector $\underline{d}$, i.e. a dimension vector such that $\gcd(d_i) = 1$, is a fine moduli space for the moduli problem classifying stable representations. If $\underline{d}$ is not primitive then it is only a coarse moduli space. The semistable quotient is just the coarse moduli space associated to the moduli problem classifying polystable representations up to isomorphism.

Now suppose we are given a quiver with superpotential (Q, W). By considering those representations which satisfy the relations given by the finitely-generated ideal I_W, we get a diagram of inclusions

$$\begin{array}{ccccc} R^{\theta\text{-st}}_{\underline{d}}(Q) & \subset & R^{\theta\text{-ss}}_{\underline{d}}(Q) & \subset & R_{\underline{d}}(Q) \\ \cup & & \cup & & \cup \\ R^{\theta\text{-st}}_{\underline{d}}(Q,W) & \subset & R^{\theta\text{-ss}}_{\underline{d}}(Q,W) & \subset & R_{\underline{d}}(Q,W), \end{array}$$

where the horizontal inclusions denote open embeddings while the vertical ones denote closed immersions.

Remark 1.12. The closed subscheme $R_{\underline{d}}(Q, W)$ of $R_{\underline{d}}(Q)$ is invariant under the action of $PG_{\underline{d}}$. Indeed, the formal derivation $\partial_a(x)$ of a cyclic monomial along an arrow a has head $t(a)$ and tail $h(a)$; so if (V_i, v_b) satisfies $\partial_a(W)(v_b) = 0$ then we have also $(g_i)\cdot\partial_a(W)(v_b) = 0$, as $(g_i)\cdot\partial_a(W)(v_b) = g_{h(a)}\partial_a(W)(v_b)g^{-1}_{t(a)}$.

Using Remark 1.12 we can construct the moduli scheme $M_{\underline{d}}(Q, W)$, and its stable and semistable versions $M_{\underline{d}}^{\theta\text{-(s)st}}(Q, W)$ as closed subschemes of the corresponding moduli spaces of quiver representations.

Remark 1.13. While the Mumford GIT construction in the case of quiver representations gives an integral scheme of finite type over $\mathbb{C}$, the moduli space of semistable representations of a quiver with superpotential (Q, W) can be a non-reduced and reducible scheme.

We conclude the section by explaining the existence of a chamber decomposition of the space of King's stability conditions for a quiver Q and a dimension vector $\underline{d}$. The equivariant Neron-Severi group $NS^{PG_{\underline{d}}}(R_{\underline{d}}(Q))$ of the affine variety of $\underline{d}$-dimensional representations of Q can be identified with the space of King's stability conditions $\{\theta \in \mathbb{Z}V(Q) \colon \theta \cdot \underline{d} = 0\}$. Since the multiplication of a stability condition by a strictly positive integer leaves the space of semistable representations of Q unchanged, we can consider the $\mathbb{Q}$-linearization $\mathrm{NS}_{\mathbb{Q}}^{PG_{\underline{d}}}(R_{\underline{d}}(Q))$. We divide $\mathrm{NS}_{\mathbb{Q}}^{PG_{\underline{d}}}(R_{\underline{d}}(Q))$ into regions called chambers so that choosing different stability conditions in the same chamber leaves the moduli space of stable representations unchanged. We adopt the notation $\underline{e} < \underline{d}$ for a non-zero dimension vector $\underline{e}$ such that $e_i \leq d_i$ for every vertex i and $e_i < d_i$ for at least one vertex i. Given a vector $\underline{e} < \underline{d}$, we let $H_{\underline{e}}$ be the hyperplane of $\mathbb{Q}$-linearized stability conditions vanishing on $\underline{e}$. This defines the region

$$H_{>\underline{e}} = \left\{\theta \in \mathrm{NS}_{\mathbb{Q}}^{PG_{\underline{d}}}(R_{\underline{d}}(Q)) \colon \theta \cdot \underline{e} > 0\right\},$$

which corresponds to choices of King's stabilities for which a stable representation of dimension vector $\underline{d}$ does not have subrepresentations of dimension $\underline{e}$. Two $\mathbb{Q}$-linearized stability conditions contained in a connected component of

$$\mathrm{NS}_{\mathbb{R}}^{PG_{\underline{d}}}(R_{\underline{d}}(Q)) \setminus \bigcup_{\underline{e}<\underline{d}} H_{\underline{e}}$$

give rise to the same moduli space of $\underline{d}$-dimensional stable representations of Q. In particular, any stability condition lying outside the system of walls $\{H_{\underline{e}}\}_{\underline{e}<\underline{d}}$ is *generic*, i.e. for such a choice stability coincides with semistability.

Notation 1.14. If a stability condition is generic and the dimension vector $\underline{d}$ is primitive we will denote by $M_{\underline{d}}^{\theta}(Q, W)$ the moduli space of stable representations, which in this case coincide with the semistable ones.

It will be useful to consider also the stacky versions of the spaces constructed above. We thus define the quotient Artin stack of representations of (Q, W) by

$$\mathcal{M}_d(Q, W) := [R_d(Q, W)/\operatorname{GL}_d],$$

and we denote by $\mathcal{M}_d^{\theta\text{-(s)st}}(Q,W)$ the Artin stack obtained as the global quotient of the space of (semi)stable representations by the group of changes of basis. Notice that the diagonal subgroup $\mathbb{C}^*$ acts trivially so that when θ is generic, $\mathcal{M}_d^{\theta}(Q,W)$ is obtained as $[M_d^{\theta}(Q,W)/\mathbb{C}^*]$, where the torus $\mathbb{C}^*$ acts trivially.

Example 1.15. Consider the quiver with superpotential (Q,W) of Example 1.5 and fix a dimension vector n. Then the representations of Q are triples (A_1,A_2,A_3) of $n\times n$ commuting matrices and GL_n acts by simultaneous conjugation. Thus the quotient $R_n(Q,W)/\mathrm{GL}_n$ is isomorphic to the Artin stack $\mathrm{Coh}_n(\mathbb{C}^3)$ of length n coherent sheaves on $\mathbb{C}^3$ supported on a finite set of points. Any such sheaf M is a $\mathbb{C}[x,y,z]$-module of dimension n, so by fixing an isomorphism $M\simeq\mathbb{C}^n$ we get a triple of commuting matrices (X,Y,Z). The condition for two such $\mathbb{C}[x,y,z]$-modules to be isomorphic is described exactly by the action of GL_n.

1.1.3 Framed quiver representations

The example we have in mind when approaching moduli of framed representations of a quiver is the Hilbert scheme of n points on $\mathbb{C}^3$.

Example 1.16. The Hilbert scheme of points $\mathrm{Hilb}^{[n]}(\mathbb{C}^3)$ is the moduli space parametrising closed zero-dimensional subschemes Z of $\mathbb{C}^3$ of length n, or equivalently the moduli space of ideal sheaves of $\mathbb{C}^3$ of colength n. It has been shown in [Sz2] that $\mathrm{Hilb}^{[n]}(\mathbb{C}^3)$ can be realized as the moduli space $\mathfrak{M}_n$ of cyclic modules with a generator over the path algebra of the quiver with superpotential (Q,W) defined in Example 1.5. More explicitly, $\mathfrak{M}_n$ parametrizes pairs (M,m) of n-dimensional $A_{Q,W}$-modules M generated by m, up to isomorphism preserving the generators. This can be formulated in terms of moduli of representations of an extended quiver $(\hat{Q}_1,W)$ with respect to a special stability condition.

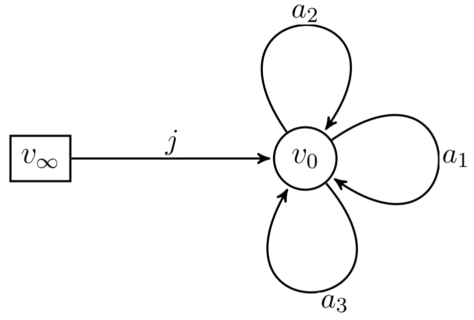

Figure 1.4: The extended quiver Q_1.

Let $(\hat{Q}_1, W)$ be the quiver with superpotential obtained from Q by adding a vertex v_∞, an arrow j from v_∞ to v_0 and leaving the superpotential unchanged. Consider a stability condition $\hat{\theta} = (\theta_\infty, \theta_0)$ such that $\theta_\infty + \theta_0 = 0$ and $\theta_\infty > 0$. Then a $(1,n)$-dimensional $\hat{\theta}$-semistable representation $(\{\hat{V}_i\}, \{\hat{\phi}_a\})$ of $(\hat{Q}_1, W)$ is a representation $\{V_0, \{\phi_a\}\}$ of (Q, W) together with an injective morphism $J\colon \mathbb{C} \longrightarrow V_0$ such that the only ϕ_{a_i}-invariant subspace of V_0 containing $\mathrm{Im}(J)$ is V_0 itself. In particular there are no strictly $\hat{\theta}$-semistable representations. It is clear then that $\mathfrak{M}_n$ is isomorphic to $M^{\hat{\theta}}_{(1,n)}(\hat{Q}_1, W)$, where we use the notation of 1.14.

Framed representations and framed modules have been widely discussed in the literature, each time with a slightly different flavor. We fix notation and then show how one can find the smooth models of [ER] and the cyclic representations of [Sz1] as special cases of framed representations.

Suppose we are given a quiver Q, a dimension vector $\underline{d}$, a vector θ in $\mathbb{Q}^{V(Q)}$, and a vector $\underline{n}$ in $\mathbb{N}^{V(Q)}$ distinct from $\underline{0}$. We construct a quiver $\hat{Q}_{\underline{n}}$ by adding a vertex v_∞ to $V(Q)$ and adding n_i arrows from v_∞ to the vertex v_i to $E(Q)$. Now we consider the dimension vector $\hat{\underline{d}} = (1, \underline{d})$ and the King stability condition $\hat{\theta} = (-\underline{d} \cdot \theta, \theta)$ on $\hat{Q}_n$. Note that θ in $\mathbb{Q}^{V(Q)}$ need not itself be a King stability condition on Q.

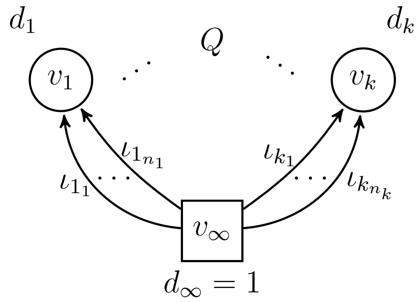

Figure 1.5: Framed quiver $\hat{Q}_{\underline{n}}$.

A *framed representation* of Q of type $\underline{n}$, or an $\underline{n}$-framed representation, is a representation of $\hat{Q}_{\underline{n}}$. If we are given a quiver with superpotential (Q, W) instead of just Q, a $\underline{n}$-framed representation of (Q, W) is a $\underline{n}$-framed representation of Q of type $\underline{n}$ satisfying the relations imposed by W. We say that a framed representation is θ-(semi)stable if it is $\hat{\theta}$-(semi)stable as an ordinary quiver representation. Notice that an $\underline{n}$-framed representation $\hat{V}$ of Q induces a representation of Q if one forgets the framing, and we will denote such a representation by V, omitting the $\hat{}$.

Remark 1.17. A framed representation $\hat{V}$ of Q is θ-semistable if and only if the following two conditions are satisfied

- every subrepresentation U with $U_\infty = 0$ satisfies $\dim(U) \cdot \theta \leq 0$;
- every subrepresentation U with $U_\infty = \mathbb{C}$ induces a quotient $\widetilde{V} = \hat{V}/U$ which is a usual representation of Q, and as such $\widetilde{V}$ has non-negative θ-slope, i.e. $\dim \widetilde{V} \cdot \theta \geq 0$.

We will denote by $M^{\theta-ss}_{\underline{d},\underline{n}}(Q)$ the moduli space of θ-semistable framed representations of Q of type $\underline{n}$. In the case of a quiver Q with superpotential W the variety $M^{\theta-ss}_{\underline{d},\underline{n}}(Q,W)$ is a closed subscheme of $M^{\theta-ss}_{\underline{d},\underline{n}}(Q)$. We define in analogy with the unframed case the moduli of stable representations, and we adopt the notation of 1.14 when there are no strictly semistable representations.

A special role is played by θ-semistable framed representations when

$$\theta = N\theta' - \underline{1}, \text{ for } \underline{1} = (1, \dots, 1) \tag{1.3}$$

where $\theta' \in \mathbb{N}^{V(Q)}$ is a King stability condition on Q for a dimension vector $\underline{d}$ and $N \geq |\underline{d}|$. Borrowing the notation of [ER], we will call V a smooth θ'-representation of Q if it is a θ-semistable framed representation of Q for a vector θ constructed as in equation (1.3).

Lemma 1.18 ([ER])**.** *Let θ' be a King stability condition on Q for a dimension vector $\underline{d}$ and let θ be as in equation (1.3). Then:*

1. *For every dimension vector $\underline{0} \neq \underline{e} \leq \underline{d}$ of Q we have $\hat{\theta} \cdot (0, \underline{e}) < 0$ if and only if $\hat{\theta} \cdot (0, \underline{e}) \leq 0$ if and only if $\theta' \cdot \underline{e} \leq 0$;*
2. *for every dimension vector $\hat{\underline{e}} < \hat{\underline{d}}$ the two inequalities $\hat{\theta} \cdot \hat{\underline{e}} < 0$ and $\hat{\theta} \cdot \hat{\underline{e}} \leq 0$ are equivalent, and this is true if and only if $\theta' \cdot \underline{e} < 0$.*

Proof. 1. We have $\hat{\theta} \cdot (0, \underline{e}) < 0$ if and only if $\theta' \cdot \underline{e} < |e|/N \in (0, 1)$, and since $\theta' \cdot \underline{e}$ is an integer this is equivalent to $\theta' \cdot \underline{e} \leq 0$. The same works for $\leq$ in place of strict inequalities.
2. We have $\hat{\theta} \cdot \hat{\underline{e}} < 0$ if and only if $\theta' \cdot \underline{e} < (|\underline{e}| - |\underline{d}|)/N \in (-1, 0)$, so this is equivalent to asking $\theta' \cdot \underline{e} < 0$. One gets the same result starting with a non-strict inequality. □

Corollary 1.19. *In the notation of equation (1.3), smooth θ'-representations of Q are automatically θ-stable as $\underline{n}$-framed representations of Q. In particular, the variety $M^{\theta}_{\underline{d},\underline{n}}(Q)$ is smooth and quasi-projective.*

Proof. This is a direct consequence of Lemma 1.18. □

The datum of a framed representation of type $\underline{n}$ is equivalent to that of a finite-dimensional A_Q-module M together with a choice of $|\underline{n}|$ elements $\{m_{i_j \in M}\}$, so that m_{i_j} is based at the vertex i. Rephrasing the results of Lemma 1.18, we see that $M^{\theta-ss}_{\underline{d},\underline{n}}$ parametrises pairs $(M, \{m_{i_j}\})$ where M is a θ-semistable module that does not admit any submodule M' containing the framing $\{m_{i_j}\}$ and satisfying $\theta \cdot \dim M' \geq \theta \cdot \dim M$. In other words, $M^{\theta-ss}_{\underline{d},\underline{n}}(Q)$ parametrises θ-semistable modules with a framing avoiding strictly semistable submodules. The case of quivers with superpotential works analogously, considering $A_{Q,W}$-modules instead of A_Q-modules.

Lemma 1.20. *Let $((Q, W), \underline{d})$ be a quiver with superpotential with fixed dimension vector $\underline{d}$, and let $\underline{n}$ be a framing type. Taking $\theta = -\underline{1}$ in $\mathbb{Z}^{V(Q)}$, $M^{\theta}_{\underline{d},\underline{n}}(Q, W)$ is the fine moduli space of $A_{Q,W}$-modules of dimension $\underline{d}$ with a choice of $|\underline{n}|$ generators at vertices prescribed by $\underline{n}$. For $|\underline{n}| = 1$, this is the moduli space of $A_{Q,W}$-cyclic modules with generator at a vertex which appeared in [Sz2].*

Proof. For $\underline{d}' < \underline{d}$ we have $\hat{\theta} \cdot \hat{\underline{d}'} = |\underline{d}| - |\underline{d}'| > 0$. Hence there are no submodules containing the set $\{m_{i_j}\}$ defined by the framing. There are no conditions imposed on the other submodules as $\theta \cdot \underline{e} < 0$ for any choice of non-zero $\underline{e}$. □

Example 1.21. Consider the quiver (Q, W) of Example 1.6 and framing vector $(1, 0)$ of Figure 1.6.

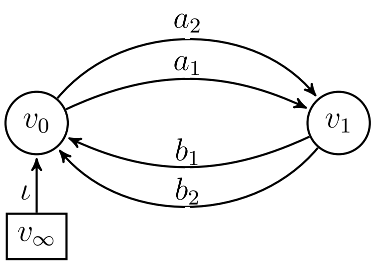

Figure 1.6: The framed conifold quiver Q_1.

Fix the dimension vector to be $\underline{1}$ so that W does not impose any relations. The chamber structure for the stability conditions on framed representations is determined by the equations

$$\theta_1 = 0, \ \theta_2 = 0, \ \theta_1 + \theta_2 = 0.$$

The diagonal line does not change the geometry of the moduli space, so the wall-crossing picture is given by the table

	$\theta_1 < 0$	$\theta_1 > 0$
$\theta_2 > 0$	$\mathcal{O}(-1) \oplus_{\mathbb{P}^1} \mathcal{O}(-1)$	$\emptyset$
$\theta_2 < 0$	$\mathcal{O}(-1) \oplus_{\mathbb{P}^1} \mathcal{O}(-1)$	$\mathcal{O}(-1) \oplus_{\mathbb{P}^1} \mathcal{O}(-1)^\dagger$

,

where the passage from the top left corner of the table to the bottom right corner is the celebrated Atiyah flop. Notice in particular that when the moduli space of framed representations is non-empty it is a commutative crepant resolution of the center of the Jacobi algebra of the unframed quiver.

Example 1.22. A completely analogous picture to the one in Example 1.21 works for the chamber structure for the quiver of Example 1.7 where we take $n = 0$, we choose the dimension vector to be $(1, 1)$ and the framing vector to be $(1, 0)$.

Remark 1.23. The two previous examples are again the shadow of a much more general theory. This was developed by Van den Bergh [VdB1], inspired by work of Bridgeland on threefold flops. Starting from the non-commutative crepant resolution of a Gorenstein toric affine three-dimensional variety one can reconstruct its commutative crepant resolutions via quiver GIT [Moz1].

1.2 Beilinson spectral sequence

In this section we discuss some consequences of results due to Beilinson and Orlov on the derived category of coherent sheaves on projective spaces [Be] and projective bundles [Or]. In particular, we describe how in these contexts one can obtain an approximation of a given sheaf F: more precisely, we identify the associated graded of a filtration of F as the limit of a spectral sequence whose first page can be explicitly described in terms of linear algebra. The canonical reference for this material is [OSS, Section 2.3], and our treatment follows it closely. The definitions of exceptional objects and exceptional collections are due to Bondal and Orlov, and a reference for this material is Huybrechts' book [Hu].

It is easy to prove that the Fourier-Mukai transform with kernel the structure sheaf of the diagonal is isomorphic to the identity functor on the derived category of a smooth projective variety. The Beilinson spectral sequence is a non-trivial consequence of this easy statement.

Let X be a smooth projective variety and consider the commutative diagram

$$\begin{tikzcd} & & X \arrow[dr, "a"] & \\ X \arrow[r, "\Delta"] & X \times X \arrow[ur, "p=\mathrm{pr}_1"] \arrow[dr, "q=\mathrm{pr}_2"'] & & \mathrm{pt}\,. \\ & & X \arrow[ur, "b"'] & \end{tikzcd} \tag{1.4}$$

We will say that a complex of coherent sheaves $U^\bullet$ on $X \times X$ is *decomposable* if each of its terms U^i is the tensor product of two coherent sheaves pulled back from the two factors, i.e. $U^i = p^*V^i \otimes q^*W^i$. In such a case we will adopt the notation $V^i \boxtimes W^i$, and we write

$$U^\bullet = [V^\bullet \boxtimes W^\bullet].$$

Assume that the structure sheaf $\Delta_*(\mathcal{O}_X)$ of the diagonal $\Delta(X)$ is quasi-isomorphic to a decomposable complex $U^\bullet = [V^\bullet \boxtimes W^\bullet]$. We consider the Fourier-Mukai transform

$$\mathcal{F}^{\mathcal{O}_{\Delta(X)}}_{X\to X} \colon \mathrm{D}^b(X) \longrightarrow \mathrm{D}^b(X)$$

sending a complex $F^\bullet$ to $\mathbb{R}q_*(p^*F^\bullet \otimes \mathcal{O}_{\Delta(X)})$, where all the functors we consider are derived. Given a sheaf F on X, from the general machinery of derived functors we have a spectral sequence

$$\mathrm{E}_1^{rs} = \mathbb{R}^s q_*((p^*F \otimes \mathcal{O}_{\Delta(X)})^r) \Longrightarrow \mathbb{R}^{s+r} q_*(p^*F \otimes \mathcal{O}_{\Delta(X)}).$$

We see immediately from the projection formula that

$$\mathbb{R}^{s+r} q_*(p^*F \otimes \mathcal{O}_{\Delta(X)}) = \begin{cases} F & \text{if } s+r=0; \\ 0 & \text{otherwise}\,. \end{cases}$$

Replacing $\mathcal{O}_{\Delta(X)}$ with the complex $U^\bullet$ we identify the E_1^{rs} page

$$\mathrm{E}_1^{rs} = \mathbb{R}^s q_*((F \otimes V^r) \boxtimes W^r),$$

and again by the projection formula we have

$$\mathrm{E}_1^{rs} = \mathrm{H}^s(X, F \otimes V^r) \otimes W^r.$$

In order to make this result geometric, one has to have a good understanding of the above decomposable complex. To be able to use the spectral sequence to understand sheaves on a certain variety, one would like the resolution $U^\bullet$ mentioned above to be as simple as possible. A possible requirement would be for the W^r's to be line bundles, so that the first page of the spectral sequence can be described in terms of linear maps between the cohomology groups and sections of line bundles. The most important example is the case of projective space $\mathbb{P}^n(\mathbb{C})$, as somehow many of the other explicit constructions descend from this. We provide a brief proof as it consists of a construction which will be used later.

Theorem 1.24 ([OSS, Theorem 3.1.3])**.** *Let F be a sheaf on $\mathbb{P}^n$ and denote by $\Omega^k_{\mathbb{P}^n}$ the k-th exterior product of the cotangent sheaf on $\mathbb{P}^n$. Then the spectral sequence*

$$\mathrm{E}_1^{rs} = \mathrm{H}^s(\mathbb{P}^n, F \otimes \Omega^{-r}_{\mathbb{P}^n}(-r)) \otimes \mathcal{O}_{\mathbb{P}^n}(r) \tag{1.5}$$

converges to

$$F^i = \begin{cases} F \ \textit{if}\ i = 0; \\ 0 \ \textit{otherwise}. \end{cases}$$

In other words, $\mathrm{E}_\infty^{rs} = 0$ if $r \neq -s$ and $\oplus_{s=0}^n \mathrm{E}_\infty^{-s,s}$ is the associated graded of a filtration of F. The analogue result holds for the spectral sequence

$$\mathrm{E}_1^{rs} = \mathrm{H}^s(\mathbb{P}^n, F \otimes \mathcal{O}_{\mathbb{P}^n}(-r)) \otimes \Omega^r_{\mathbb{P}^n}(r).$$

Proof. Consider the section s of $V = \mathcal{T}_{\mathbb{P}^n}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^n}(1)$ defined by

$$s([x],[y])\colon \lambda \cdot x \longmapsto \lambda \cdot x \mod \mathbb{C}y, \tag{1.6}$$

where we identify V with the bundle $\mathcal{H}om(q^*\mathcal{O}_{\mathbb{P}^n}(-1), p^*\mathcal{T}_{\mathbb{P}^n}(-1))$. Let e_s be evaluation on the section s, and let $\Lambda^k(e_s)$ be described locally by

$$(\phi_1 \wedge \ldots \wedge \phi_k) \mapsto \sum_{i=1}^k (-1)^i \phi_i(s)\, \phi_1 \wedge \ldots \wedge \widehat{\phi_i} \wedge \ldots \phi_k.$$

The complex

$$0 \longrightarrow \Omega^n(n) \boxtimes \mathcal{O}(-n) \xrightarrow{\wedge^n(e_s)} \Omega^{n-1}(n-1) \boxtimes \mathcal{O}(1-n) \longrightarrow \cdots$$

$$\cdots \longrightarrow \Omega^1(1) \boxtimes \mathcal{O}(-1) \xrightarrow{\quad e_s \quad} \mathcal{O} \longrightarrow 0$$

gives a decomposable resolution of the structure sheaf of the diagonal $\mathcal{O}_{\Delta(\mathbb{P}^n)}$ on $\mathbb{P}^n \times \mathbb{P}^n$. ∎

Remark 1.25. Consider diagram (1.4) with $X = \mathbb{P}^n$ and fix homogeneous coordinates $(w_0, \ldots, w_n)$ and $(z_0, \ldots, z_n)$ respectively on the first and second projective space in the product $\mathbb{P}^n \times \mathbb{P}^n$. Then the section s of equation (1.6) can be written explicitly in coordinates as

$$s = \sum_{i=0}^{n} e\left(\frac{\partial}{\partial w_i}\right) \boxtimes z_i, \tag{1.7}$$

where $\frac{\partial}{\partial w_i}$ are the sections of the twisted tangent sheaf $\mathcal{T}_{\mathbb{P}^n}(-1)$ defined by the dual of the Euler sequence.

Remark 1.26. The existence of a decomposable resolution of the diagonal is a very restrictive condition. For example, varieties with trivial canonical bundle do not have one. The requirement for W^r to be line bundles is even more restrictive. For instance, in the non-toric case, Grassmannians give a counterexample.

Recall that a collection $\{F_1, \ldots, F_n\}$ of objects in $D^b(X)$ is *exceptional* if

$$\mathrm{Ext}^k(F_i, F_j) = \begin{cases} \mathbb{C} & \text{if } l = 0, i = j; \\ 0 & \text{if } i > j \text{ or } l \neq 0, i = j. \end{cases} \tag{1.8}$$

It is said to be *strongly exceptional* if in addition

$$\mathrm{Ext}^k(F_i, F_j) = 0 \text{ if } k \neq 0, \text{ for all } i, j. \tag{1.9}$$

If the condition of being strongly exceptional for a collection $\{F_1, \ldots, F_n\}$ can be seen as an analogue of (semi-)orthogonality in linear algebra for $\mathrm{D}^b(X)$, the property of being full is parallel to the notion of a collection of elements in a vector space giving a generating set. Indeed, we say that a collection $\mathcal{F} = \{F_1, \ldots, F_n\}$ is *full* if the smallest full triangulated subcategory of $D^b(X)$ containing $\mathcal{F}$ is equivalent to $\mathrm{D}^b(X)$.

Corollary 1.27. *The collections*

$$\{\mathcal{O}_{\mathbb{P}^n}, \ldots, \mathcal{O}_{\mathbb{P}^n}(n)\}$$

and

$$\{\mathcal{O}_{\mathbb{P}^n}, \Omega^1_{\mathbb{P}^n}(1), \ldots, \Omega^n_{\mathbb{P}^n}(n)\}$$

are full strongly exceptional collections on $\mathbb{P}^n$.

Proof. The Beilinson theorem 1.24 proves that the collections are full. The fact they are strongly exceptional is an easy consequence of the Bott formula for the cohomology of the projective space with coefficients in $\Omega^p_{\mathbb{P}^n}(k)$ (Theorem 2.34 of Appendix 2.B). □

In the projective case, the notion of a tilting generator is closely related to that of a full strongly exceptional collection, but it has the advantage of generalizing to the quasi-projective setting.

Definition 1.28. Let Y be a smooth quasi-projective variety. We say that an object $\mathcal{E}$ in $\mathrm{D}^b(Y)$ is a *tilting generator* if

1. $\mathrm{Ext}^i(\mathcal{E},\mathcal{E}) = 0$ for all $i > 0$;
2. if $\mathrm{Ext}^i(\mathcal{E}, F) = 0$ for all i then F is quasi-isomorphic to the 0 object;
3. the algebra $A_T := \mathrm{End}_{D^b(X)}(\mathcal{E})$ has finite Hochschild dimension, that is there exists k big enough such that $\mathrm{HH}^k(A_T, M) = 0$ for any A_T bimodule M (see [W, Section 9.1] for a definition of Hochschild cohomology).

If $\mathcal{E}$ is concentrated in degree zero and locally free, then we say that $\mathcal{E}$ is a *tilting bundle* on X.

In fact, a full exceptional collection on a smooth projective variety defines a tilting generator.

Lemma 1.29 ([Ki2, Lemma 4.3])**.** *Let X be a smooth projective variety. Assume that $\{F_1, \dots, F_n\}$ is a full strongly exceptional collection in $D^b(X)$. Then*

$$\mathcal{E} = \bigoplus_{i=0}^{n} F_i \tag{1.10}$$

is a tilting generator.

Example 1.30. The vector bundles $\bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n}(i)$ and $\bigoplus_{p=0}^n \Omega^p_{\mathbb{P}^n}(p)$ are tilting bundles on $\mathbb{P}^n$.

The Beilinson spectral sequence admits a generalization for relative geometries fibered in projective spaces over smooth projective varieties. Given a vector bundle

$$\pi\colon V \longrightarrow Y$$

over a smooth projective variety, consider the associated projective bundle $\mathbb{P}(V)$ over Y. Recall that the construction of the associated projective bundle comes equipped with a relatively ample line bundle $\mathcal{O}_\pi(1)$, and one can construct the bundle of relative differentials Ω_π on $\mathbb{P}(V)$.

Theorem 1.31. *Given a sheaf F on $\mathbb{P}(V)$, there is a spectral sequence*

$$\mathrm{E}^1_{rs} = \pi^*\mathbb{R}^r\pi_*(F \otimes \Omega^r_\pi(r)) \otimes \mathcal{O}_\pi(-r)$$

converging to

$$F^i = \begin{cases} F & \text{if } i = 0; \\ 0 & \text{otherwise}. \end{cases}$$

Remark 1.32. An analogous description to Remark 1.25 holds for the relative setting in a case we will be particularly interested in. Assume that $X = \mathbb{P}^1$ and $V = \mathcal{O} \oplus \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)$. Denote by z_0 the section of $\mathcal{O}_\pi(1)$ corresponding to $(1,0,0)$ via the identification

$$\mathrm{H}^0(\mathbb{P}(V), \mathcal{O}_\pi(1)) = \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \oplus \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(-a_1)) \oplus \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(-a_2)); \tag{1.11}$$

and denote by z_i the sections of $\mathcal{O}_\pi(1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(a_i)$ corresponding respectively to $e_i = (0, \delta_{i1}, \delta_{i2})$, for $i \in \{1, 2\}$, where δ is the Kronecker delta. The dual of the relative Euler sequence

$$0 \longrightarrow \mathcal{O}_\pi(-1) \xrightarrow{\eta} \pi^*(V) \xrightarrow{\xi} \mathcal{T}_\pi(-1) \longrightarrow 0 \tag{1.12}$$

defines a section $\frac{\partial}{\partial z_0}$ of $\mathcal{T}_\pi(-1)$ and two sections $\frac{\partial}{\partial z_i}$ of $\mathcal{T}_\pi(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(-a_i)$. The Beilinson spectral sequence is recovered via the resolution of the structure sheaf of the relative diagonal of $\mathbb{P}(V) \times_{\mathbb{P}^1} \mathbb{P}(V)$ defined by the Koszul complex induced by the evaluation on the section $s = \sum z_i \boxtimes \frac{\partial}{\partial w_i}$, where the z_i's and w_i's refer to coordinates of the different projective bundles. In particular, if one identifies $\Omega_\pi(1)$ with the kernel of η^* using the Euler sequence, so that $\Omega^1_\pi(1)$ is identified with a subsheaf of the direct sum of line bundles $\pi^*(V^*)$, the evaluation map is given by

$$(z_0, z_1, z_2), \tag{1.13}$$

where there is a natural identification of $\mathcal{O}_\pi(-1) \boxtimes \pi^*(V)$ with $\mathcal{O}_\pi(-1) \otimes \pi^*(V) \boxtimes \mathcal{O}_{\mathbb{P}(V)}$. In the same way, the morphism $\Lambda^2(e_s)\colon \mathcal{O}_\pi(-2) \boxtimes \Omega^2_\pi(2) \longrightarrow \mathcal{O}_\pi(-1) \boxtimes \Omega^1_\pi(1)$ takes the form

$$(z_1 \boxtimes w_2 - z_2 \boxtimes w_1, z_2 \boxtimes w_0 - z_0 \boxtimes w_2, z_0 \boxtimes w_1 - z_1 \boxtimes w_0), \tag{1.14}$$

if one identifies $\Omega^2_\pi(2)$ with $\pi^*(\det(V^*)) \otimes \mathcal{O}_\pi(-1)$, and $\Omega_\pi(1)$ with a subsheaf of $\pi^*(V^*)$.

1.3 (Graded) Calabi-Yau algebras

Homological mirror symmetry (HMS), originally proposed by Kontsevich in its earliest formulation in [Ko], is one of the major motivations to study the bounded derived category of coherent sheaves of a Calabi-Yau threefold X, which we will denote by $\mathrm{D}^b(X)$. When X is assumed to be projective, the category $\mathrm{D}^b(X)$ is quite complicated. For example, one of the main tools used to understand the derived category of a variety is finding a tilting bundle, which allows us to realize the category as the derived category of modules over an algebra. However, when X is projective, Serre duality

$$\mathrm{Hom}_{D^b(X)}(E, F) \cong \mathrm{Hom}_{D^b(X)}(F, E[3])^* \tag{1.15}$$

implies no tilting bundle exists. Indeed, for any locally free sheaf V on X we have by Serre duality

$$\mathrm{Ext}^3(V, V) \cong \mathrm{Ext}^0(V, V)^* \neq 0. \tag{1.16}$$

Remark 1.33. From (1.16) it is clear that exceptional objects are not the right objects to consider when studying the derived category of projective Calabi-Yau varieties. To overcome this, Seidel and Thomas introduced spherical objects in [ST]. This thesis focuses primarily on local cases, so we refer to [Hu, Chapter 8] for a treatment of spherical objects and their properties.

The situation changes radically when considering local cases.

Example 1.34. Let $X = \{xy - wz = 0\}$ be the conifold singularity and let $f\colon Y \longrightarrow X$ be the crepant resolution obtained by blowing up along the plane $x = z = 0$. Then $\pi\colon Y \longrightarrow \mathbb{P}^1$ is a rank two vector bundle and $\mathcal{E} := \mathcal{O}_Y \oplus \pi^*(\mathcal{O}_{\mathbb{P}^1}(1))$ is a tilting bundle for Y [VdB1]. There is an equivalence of categories

$$\mathrm{D}^b(Y) \cong \mathrm{D}^b(\text{mod-}A_Y)$$

where A_Y is the endomorphism algebra $\mathrm{Hom}_{D^b(Y)}(\mathcal{E}, \mathcal{E})$.

Another remarkable source of examples is the derived McKay correspondence in dimension 3.

Theorem 1.35 ([BKR, Theorem 1.2])**.** *Let V be a 3-dimensional complex vector space, and let $G < \mathrm{SL}(V)$ be a finite subgroup. Denote by Y the G-Hilbert scheme of V, parametrising 0-dimensional G-invariant subschemes Z of V together with a fixed isomorphism between $\mathrm{H}^0(Z, \mathcal{O}_Z)$ and the regular representation $\mathbb{C}[G]$. Then $\pi\colon Y \longrightarrow \mathbb{C}^3/G$ is a crepant resolution and there is a derived equivalence*

$$\mathrm{D}^b(Y) \cong \mathrm{D}^b(\mathbb{C}[V] * G\text{-}mod),$$

where $\mathbb{C}[V] * G$ *is the skew group algebra, whose product is*

$$(s \otimes g) * (s' \otimes g') = s(g \cdot s) \otimes gg'.$$

Motivated by analogous results to Example 1.34 and Theorem 1.35, there has been increasing interest throughout the last decade in studying algebras A with $\mathrm{D}^b(A\text{-}mod)$ (denoted by $D(A)$ in the rest of the chapter) sharing the same behavior as the derived category of a non-compact Calabi-Yau threefold (or more generally of a non-compact Calabi-Yau n-fold). This led to the definition which follows, due to Ginzburg. Our approach to Calabi-Yau algebras follows the foundational paper of Ginzburg [Gi], but especially the work of Bocklandt [Bo] which focuses on the graded situation.

Consider a $\mathbb{C}$-algebra A and denote by A^e the A-bimodule $A \otimes_{\mathbb{C}} A^{op}$. The algebra $A \otimes_{\mathbb{C}} A$ admits two A^e-module structures: the outer structure is defined by $(a,b)x \otimes y = ax \otimes yb$ and the inner structure by $(a,b)x \otimes y = xb \otimes ay$. When A^e is regarded as a module over itself we consider it with respect to the outer structure. Given an A^e-module M, the vector space $\mathrm{Hom}_{A^e}(M, A^e)$ naturally acquires the structure of A^e-module via the inner structure.

Definition 1.36 ([Gi, Definition 3.2.3])**.** An algebra A is *Calabi-Yau of dimension* d (d-CY), for $d \geq 2$, if A admits a finite length resolution by finitely generated projective A^e-modules and

$$\mathrm{RHom}(A, A^e)[d] \cong A \text{ in } \mathrm{D}(A^e).$$

The following proposition draws an analogy between d-CY algebras and the derived category of d-CY varieties by matching the behavior of their Serre functors.

Proposition 1.37. *Let A be a d-*CY *algebra. Given X and Y in* $\mathrm{D}(A)$*, we have*

$$\mathrm{Ext}^i_{\mathrm{D}(A)}(X,Y) \cong \mathrm{Ext}^{d-i}_{\mathrm{D}(A)}(Y,X)^* \textit{ for } i \in \{0, \ldots, d\}. \tag{1.17}$$

The following proposition due to Ginzburg formalizes explicitly the relation between certain classes of open Calabi-Yau varieties and of Calabi-Yau algebras, essentially generalizing the observation of Van den Bergh discussed in Example 1.34.

Proposition 1.38 ([Gi, Proposition 3.3.1])**.** *Let X be a complex variety, projective over an affine variety. Assume that $\mathcal{E}$ is a tilting generator (cf. Definition 1.28) for X. Then:*

1. *the endomorphism algebra* $\mathrm{End}_{D^b(X)}(\mathcal{E})$ *is finitely generated as a module over its center* $\mathrm{Z}(\mathrm{End}_{D^b(X)}(\mathcal{E}))$*;*
2. *X is a Calabi-Yau variety of dimension d if and only if* $\mathrm{End}_{D^b(X)}(\mathcal{E})$ *is a d-*CY *algebra.*

Remark 1.39. Under the hypothesis of Proposition 1.38, denote by $A_{\mathcal{E}}$ the endomorphism algebra $\mathrm{End}_{D^b(X)}(\mathcal{E})$. Ginzburg observes in [Gi, Section 7.1] that there is an equivalence of categories

$$D^b(X) \cong D^b(\text{mod-}A_{\mathcal{E}}). \tag{1.18}$$

Given the importance of CY algebras in the study of homological properties of Calabi-Yau varieties, the question of constructing examples of such algebras has been of general interest. Physicists discovered that in certain cases Jacobi algebras $J(Q, W)$ (defined in Section 1.1) are Calabi-Yau. It was then conjectured and proven in certain cases by Ginzburg that, in the 3-dimensional case, *friendly* ([Gi, Definition 3.5.1]) CY algebras are superpotential algebras. In the case of graded quiver 3-CY algebras Bocklandt proved the conjecture without any friendliness assumption [Bo]. We restrict to this setting for the rest of the chapter. Notice that this case is somewhat special. For example Davison shows that there are examples of friendly 3-CY algebras which are not superpotential algebras [D].

Let A be the path algebra of a (finite connected) quiver, considered as a graded A^e-module with grading induced by the length of the paths. Denote by J the ideal $A_{>0}$. Let I be a finitely-generated graded two-sided ideal and assume that $I \subset J^2$.

Theorem 1.40 ([Bo, Theorem 3.1])**.** *If A/I is a graded* 3-CY *algebra, there is a homogeneous superpotential W in* $\mathrm{Sup}(Q)_{\geq 3}$ *such that* $A/I \cong J(Q, W)$*.*

Not all superpotential algebras are 3-CY. We will call a superpotential *good* if the algebra $J(Q, W)$ is 3-CY.

Example 1.41. Let Q be the one loop quiver. Consider the superpotential $W = a^n$ where a is the generator of $\mathbb{C}[Q]$. The algebra $J(Q, W)$ is not 3-CY as it has infinite global dimension as an A^e-module.

The property of a superpotential algebra $A = J(Q, W)$ being 3-CY can be described in terms of the exactness of a particular complex of graded projective A^e-modules constructed starting from (Q,W). Denote by S the semisimple module $A/A_{\geq 1}$. Given as S-bimodule T we can construct a projective A^e-module

$$F_T := A \otimes_S T \otimes_S A.$$

Given a quiver with superpotential (Q,W) consider the self-dual complex of projective bimodules

$$C^\bullet_{(Q,W)} \colon F_{\mathbb{C}\{i \circlearrowright W \colon i \in V(Q)\}} \xrightarrow{\delta_3} F_{\mathbb{C}\{\partial_a W \colon a \in E(Q)\}} \xrightarrow{\delta_2} F_{\mathbb{C}\{a \colon a \in E(Q)\}} \xrightarrow{\delta_1} F_S, \qquad (1.19)$$

where

- $\delta_1 \colon 1 \otimes a \otimes 1 \longmapsto a \otimes t(a) \otimes 1 - 1 \otimes h(a) \otimes a$;
- for $\partial_a W = a_1 \dots a_n$
$$\delta_2 \colon 1 \otimes a_1 \dots a_n \otimes 1 \longmapsto \sum_i a_1 \dots a_{i-1} \otimes a_i \otimes a_{i+1} \dots a_n;$$
- $\delta_3 \colon i \circlearrowright W \longrightarrow \sum_{a \in A_i} a \otimes \partial_a W \otimes 1 - 1 \otimes \partial_a W \otimes a$, where A_i is the set of cyclic paths in W passing through the vertex i.

Theorem 1.42 ([Bo, Theorem 4.3])**.** *Let (Q,W) be a quiver with superpotential. The Jacobi algebra $J(Q,W)$ is* 3-CY *if and only if the complex $C^\bullet_{(Q,W)}$ defines a projective bimodule resolution of $J(Q,W)$.*

Given a quiver Q, it is not an easy task in general to describe exactly which homogeneous superpotentials are good.

Remark 1.43. We mention a more general construction which associates a differential graded 3 Calabi-Yau algebra to any quiver with superpotential. Starting from a superpotential W for Q, Ginzburg defines in [Gi, Section 5.2] a differential graded algebra (dg-algebra) $\mathfrak{D}(Q,d_W)$ such that $\mathrm{H}^0_{d_W}(\mathfrak{D}(Q,d_W)) = J(Q,W)$. In analogy with 1.36, there is a definition of a Calabi-Yau dg-algebra 1.36, and with this definition it was proven later by Keller and Van den Bergh in [K] that all the dg-algebras $\mathfrak{D}(Q,d_W)$ defined from a quiver with superpotential are 3-CY. In particular, the complex (1.19) is a resolution of $J(Q,W)$ if and only if the dg-algebra $\mathfrak{D}(Q,d_W)$ has cohomology just in degree 0, and in this case the algebra $J(Q,W)$ is itself Calabi-Yau. In this thesis we restrict to the classical approach of Bocklandt and Ginzburg, avoiding the differential graded case. In a sense, graded Calabi-Yau algebras are the noncommutative analogue of local Calabi-Yau varieties, while enlarging our definition of Calabi-Yau algebra to the dg case would correspond to considering derived schemes. Since all of our constructions will be of a more classical nature, we prefer to restrict to the graded case.

The exactness of the complex in equation (1.19) can be checked on each graded piece, and it imposes an open condition for every graded part. We will say that a superpotential of degree d is *very general* if it lies outside a countable union of closed subvarieties of $\mathrm{Sup}_d(Q)$.

Corollary 1.44. *Let Q be a quiver admitting a good superpotential of degree d. Then the very general superpotential of degree d is good.*

Another remarkable consequence of Theorem 1.42 is the fact that the matrix-valued Hilbert series of graded quiver 3-CY algebras satisfy restrictive conditions. Given a quiver Q with n vertices and a homogeneous ideal of relations I the *matrix-valued Hilbert series* of $A = \mathbb{C}[Q]/I$ is

$$H_A(t) = \sum_{k\geq 0} h_k\, t^k \in \mathrm{Mat}_{n\times n}(\mathbb{Z})[[t]], \tag{1.20}$$

where $(h_k)_{ij} = \dim_{\mathbb{C}} e_i A_k e_j$, and we fix a numbering $\{1, \dots, n\}$ for the vertices of Q. The following observation due to Bocklandt implies that the matrix-valued Hilbert series of a graded quiver 3-CY algebra depends only on the adjacency matrix and the degree of the superpotential.

Corollary 1.45. *Let $A = J(Q, W)$ be a graded* 3-CY *algebra, and let d be the degree of W. If we denote by M the adjacency matrix of Q, we have*

$$H_A(t) = \frac{1}{\mathrm{Id} - (Mt - M^t t^{d-1} + \mathrm{Id}\, t^d)}, \tag{1.21}$$

where the equality is to be interpreted in the ring of matrix valued formal power series.

We can apply these results to the examples we considered at the beginning of the chapter.

Example 1.46. Let A_Y be the endomorphism algebra of the tilting bundle $\mathcal{E}$ of the crepant resolution Y of the conifold singularity constructed in Example 1.34. It is well known that A_Y is the algebra $J(Q, W)$ constructed from the conifold quiver depicted in Figure 1.2 with superpotential defined in Example 1.6. Proposition 1.38 implies that A_Y is 3-CY and so W is a good superpotential of degree 4. Hence, the very general superpotential of degree 4 for the conifold quiver is a good superpotential.

Example 1.47. Let G and V be as in the hypothesis of Theorem 1.35; assume in addition that G is abelian. Let $\mathrm{Irr}(G)$ be the set of isomorphism classes of irreducible representations of G, so that $V \cong \rho_1 \oplus \rho_2 \oplus \rho_3$ for some choice of ρ_i's in $\mathrm{Irr}(G)$. The McKay quiver Q_G is defined as follows:

- the set of vertices $V(Q_G)$ is $\mathrm{Irr}(G)$;
- there is an arrow $a_i^\rho : \rho\rho_i \longrightarrow \rho$ for all irreducible ρ and $i \in \{1,2,3\}$, where $\rho\rho_i := \rho \otimes \rho_i$.

Since $G < \mathrm{SL}(V)$, the determinant $\det(V)$ can be identified with the trivial representation. We can then define a superpotential W by setting

$$W = \sum_{\mathrm{Irr}(G)} (a_3^{\rho\rho_{12}} a_2^{\rho\rho_1} a_1^\rho - a_2^{\rho\rho_{13}} a_3^{\rho\rho_1} a_1^\rho),$$

where we denote by ρ_{ij} the representation $\rho_i \otimes \rho_j$. The algebra $J(Q_G, W)$ is isomorphic to the skew group algebra $\mathbb{C}[V] * G$ of Theorem 1.35 and is 3-CY [Gi, Theorem 4.4.6]. The non-abelian case, treated in [Gi, Section 4.4], is very similar, apart from the fact that one gets Morita equivalence with the skew group algebra rather than isomorphism. In particular, a very general superpotential W' of Q_G of degree 3 is good.

In [Bo, Section 5] Bocklandt explains how some techniques from the theory of Gröbner bases can be employed to identify good superpotentials. Denote by $\{a_1, \dots, a_n\}$ the set $E(Q)$ of the arrows of Q. We can extend the ordering

$$a_1 < \dots < a_n$$

on the set of arrows $E(Q)$ to a total ordering on the set $P(Q)_{\geq 1}$ of paths of nonzero length on Q by imposing

$$a_{i_1} \dots a_{i_l} < a_{j_1} \dots a_{j_m}$$

when $i_l < j_m$ or when $i_l = j_m$ and there is a k such that $a_{i_h} = a_{j_h}$ for $h < k$ and $a_{i_k} < a_{j_k}$. Given a linear combination of paths $\alpha = \sum_p a_p p$ in $\mathbb{C}[Q]$, we denote by $\mathrm{lt}(\alpha)$ the leading term of α, i.e. $a_{p'}p'$ where p' is maximal among the monomials appearing in α with non-zero coefficient.

Remark 1.48. If X an element in $\mathrm{Sup}(Q)$, we adopt the notation $\mathrm{lt}(X)$ for the leading term of its image $\circlearrowright X$ inside $\mathbb{C}[Q]$.

Let G be a set of elements in $\mathbb{C}[Q]$. We say that a triple of monomial terms (a, b, c) is an *ambiguity* for G if there exist $g_1, g_2 \in G$ such that

$$ab = \mathrm{lt}(g_1), bc = \mathrm{lt}(g_2).$$

Lemma 1.49 ([Bo, Lemma 5.1])**.** *Let Q be a quiver for which each vertex is the head and tail of at least two arrows. Assume that we fix an ordering for $E(Q)$. Suppose that W is an element of* $\mathrm{Sup}(Q)_d$ *such that*

1. *the leading terms of the relations $\partial_a W$ are distinct and the set of ambiguities is in bijective correspondence with $V(Q)$ so that*

$$\operatorname{amb}_v = (a, \operatorname{lt}(\partial_a W)b^{-1}, b) = (a, a^{-1}\operatorname{lt}(\partial_b W), b), \text{ and}$$
$$\operatorname{lt}(vWv) = a\operatorname{lt}(\partial_a W) = \operatorname{lt}(\partial_b W)b, \text{ and}$$

2. *for every vertex v there is an arrow a starting at v such that $\operatorname{lt}(\partial_b W)a = 0$ for all b in $E(Q)$.*

Then the complex $C^\bullet_{(Q,W)}$ is a resolution of $J(Q,W)$, i.e. W is a good superpotential.

Condition (1) ensures that the Poincaré polynomial of the Jacobi algebra $J(Q,W)$ is given by the expression in equation (1.21), and that $\{\partial_a W\colon a \in E(Q)\}$ is a Grobner basis for I_W.On the other hand, it follows from condition (2) that the last morphism in the complex $C^\bullet_{(Q,W)}$ of equation (1.42) is injective, and hence that $C^\bullet_W$ is a resolution of $J(Q,W)$.

Example 1.50. Let Q be the three-loop quiver of Figure 1.1. The Jacobi algebra $J(Q,W_q)$ defined by the marginally deformed potential $W_q = xyz - qxzy$ for $q \in \mathbb{C}^*$ is a 3-CY algebra. To prove this, we consider the ordering

$$x > y > z$$

on the set $E(Q)$ of arrows of Q. The leading terms of the relations induced by the potential are

$$\operatorname{lt}(\partial_x W_q) = yz,\ \operatorname{lt}(\partial_y W_q) = -qxz,\ \operatorname{lt}(\partial_z W_q) = xy.$$

Since none of the leading terms of the relations end in x, condition (2) of Lemma 1.49 is satisfied. There is just one ambiguity among the relations corresponding to the vertex 0:

$$(x, \operatorname{lt}(\partial_x W_q)z^{-1}, z) = (x, x^{-1}\operatorname{lt}(\partial_z W_q), z), \text{ and } x\partial_x W_q = \partial_z W_q z = \operatorname{lt}(0W0).$$

From Lemma 1.49, we obtain that the algebra $J(Q,W_q)$ is 3-CY.

1.4 Donaldson-Thomas invariants

1.4.1 A quick overview

This section is intended as motivation for the definition of the invariants of moduli spaces of sheaves and of moduli of representations of quivers that we will study in the following chapters. In the last few years, much progress has been made in the development of Donaldson-Thomas theory, with contributions from several groups of mathematicians, and we refer to [PT2] and [Sz4], and references therein, for two complementary surveys of the subject. We will instead present a summary of the main contributions which led to the definition of refined/motivic Donaldson-Thomas invariants in the local case, and which show why this setting is relevant for understanding the global picture.

Donaldson-Thomas invariants were conceived by S. K. Donaldson and R. P. Thomas in [DT] as an analogue for complex threefolds of the Casson invariant for real 3-dimensional manifolds. Following the work of Behrend and Fantechi on the intrinsic normal cone [BF1], Thomas constructed in [T] a perfect obstruction complex on proper moduli spaces of stable sheaves on Calabi-Yau varieties, and defined Donaldson-Thomas invariants as the result of the integration of the corresponding virtual fundamental class.

Theorem 1.51 (Thomas,[T, Corollary 3.39])**.** *Let (X, L) be a complex projective Calabi-Yau threefold (i.e. assume that $K_X \simeq \mathcal{O}_X$) and a fixed polarization L. Let $\mathcal{M}_X$ be the coarse moduli scheme of stable sheaves F on X with trivial determinant (i.e. $\bigwedge^r F \simeq \mathcal{O}_X$) and prescribed Hilbert polynomial, and assume that stability coincides with semistability. There exists a perfect symmetric deformation-obstruction complex on $\mathcal{M}_X$.*

Following [BF1], one can construct a 0-dimensional cycle

$$C \in A_0(\mathcal{M}_X),$$

coinciding with the top Chern class of $\mathcal{M}_X$ when the moduli space is smooth, depending just on the deformation-obstruction complex.

Definition 1.52. Given X and $\mathcal{M}_X$ as in the theorem above one defines the *Donaldson-Thomas invariant* of X as

$$\mathrm{DT}(\mathcal{M}_X) := C \cap [\mathcal{M}_X] = \int_{[\mathcal{M}_X]_{vir}} 1,$$

where we have adopted the common notation $[\mathcal{M}_X]_{vir}$ of virtual fundamental cycle for C.

Remark 1.53*.* One of the most important properties of the numerical invariant $\mathrm{DT}(\mathcal{M})$, proved by Thomas in [T], is that it is invariant under deformations of the underlying Calabi-Yau threefold X.

A natural question arising from the work of Thomas is whether the invariant obtained using the extra structure of a deformation-obstruction complex on the moduli space $\mathcal{M}_X$ does actually depend on this extra piece of information. This question was answered by K. Behrend in [B], where it was shown that the DT invariant defined above only depends on the scheme structure of the moduli space.

Theorem 1.54 (Behrend,[B]). *For any scheme Z, there is a canonical constructible function, called the Behrend function, denoted by*

$$\nu_Z\colon Z \longrightarrow \mathbb{Z},$$

such that when $Z = \mathcal{M}_X$ is a proper moduli space of stable sheaves on a CY 3*-fold as in Theorem 1.51, one has*

$$\chi(Z,\nu_Z) = \sum_{k\in\mathbb{Z}} k\chi(\nu_Z^{-1}(k)) = \int_{\mathcal{M}_X} \nu_Z d\chi \tag{1.22}$$

coinciding with $\mathrm{DT}(\mathcal{M}_X)$.

Remark 1.55. Note that $\chi(Z,\nu_Z)$ is well-defined without requiring any assumption of properness, so that one can take this as a generalization of Definition 1.52 for non-proper moduli spaces of sheaves over CY threefolds.

When a scheme Z is cut out as the critical locus of a regular function on a smooth scheme, one is able to obtain a very explicit description of the Behrend function, as explained for instance by Beherend and Fantechi in [BF2]. Let $f\colon M \longrightarrow \mathbb{C}$ be a regular function and let

$$Z := \mathrm{Crit}(f) = \mathrm{Z}(df)$$

be its scheme-theoretic critical locus. The deformations and obstructions of Z are controlled by the truncation $\tau_{\geq -1}(\mathbb{L}_Z^\bullet)$ of the cotangent complex $\mathbb{L}_Z^\bullet$, which in this simple case can be explicitly described by the complex of sheaves

$$\tau_{\geq -1}(\mathbb{L}_Z^\bullet) = [I/I^2 \longrightarrow \Omega_M|_Z],$$

where I is the ideal sheaf defining Z in M and $\Omega_M|_Z$ is the restriction of the sheaf of differentials to the critical locus. The morphism of complexes

$$\begin{tikzcd} \mathrm{E}^\bullet \arrow[d] & TM|_Z \arrow[r, "\partial^2 f"] \arrow[d, "\mathrm{ev}_{df}"'] & T^*M|_Z \arrow[d, "\mathrm{Id}"] \\ \tau_{\geq -1}(\mathbb{L}_Z^\bullet) & I/I^2 \arrow[r] & T^*M|_Z, \end{tikzcd} \tag{1.23}$$

where $\partial^2 f$ is induced by the Hessian of f, defines the perfect obstruction theory on Z. Indeed, $\mathrm{E}^\bullet$ is a complex of locally free sheaves on Z and the morphism of complexes (1.23) induces an isomorphism $\mathrm{H}^0(\mathrm{E}^\bullet) \simeq \mathrm{H}^0(\mathbb{L}_Z^\bullet)$ and a surjective morphism $\mathrm{H}^{-1}(\mathrm{E}^\bullet) \longrightarrow \mathrm{H}^{-1}(\mathbb{L}_Z^\bullet)$. The perfect obstruction theory is also symmetric, as there is an isomorphism $\mathrm{E}^\bullet \longrightarrow (\mathrm{E}^\bullet)^*[1]$ induced by the symmetry of the Hessian. Following the work of Behrend and Fantechi [BF1] one can define a virtual fundamental class $[Z]_{\mathrm{vir}} \in A(Z)$; from the symmetry of the obstruction theory it follows that $[Z]_{\mathrm{vir}}$ is zero-dimensional. A remarkable property of this local setting is that one is actually able to describe explicitly the value of the Behrend function ν_Z at a point of the critical locus. Given a point P in the critical locus we have

$$\nu_Z(P) = (-1)^{\dim M}(1 - \chi(MF_f(P))),$$

where $\mathrm{MF}_f(P)$ is the Milnor fiber of f at P, i.e. $\mathrm{MF}_f(P) = f^{-1}(t) \cap \mathrm{B}_\delta(P)$ where $0 < |t - f(P)| < \epsilon \ll \delta \ll 1$.

The reason we are particularly interested in the local case comes from the observation of Donaldson and Thomas that the moduli space of holomorphic bundles on a Calabi-Yau threefold is locally cut out as the critical locus of a holomorphic functional on an infinite-dimensional manifold [DT, Section 2]. Much work has been done and new mathematics has been developed to show that the same picture holds in the algebraic category. First of all, Behrend proved in [B] that a scheme Z (resp. a Deligne-Mumford stack) with a perfect symmetric obstruction theory is Zariski locally (resp. étale locally) the zero locus of an *almost closed* 1*-form* ω. More precisely, each point z in such a Z is contained in a Zariski (resp. etalé) open U which admits an embedding $U \hookrightarrow M$ in a smooth scheme M so that there is a 1-form $\omega \in \Omega^1_M$ for which scheme-theoretically $U = Z(w)$ and $d\omega \in I\Omega^2_M$. (Here we denote by I the ideal sheaf of U in M.) A natural question arose at this point: is it true that at least in the analytic topology the zero locus of an almost closed 1-form is locally the critical locus of a holomorphic function? Pandharipande and Thomas in [PT3] gave a negative answer to this question, exhibiting an example of an almost closed 1-form on $\mathbb{C}^2$ whose zero locus near the origin cannot be described as the critical locus of a holomorphic function. A different approach using gauge theoretic arguments was used by Joyce and Song to prove in [JS, Theorem 5.4]; they showed that, in the analytic topology, the moduli space of stable sheaves on a Calabi-Yau threefold is locally the critical locus of a holomorphic function, obtaining a finite-dimensional model of the Donaldson-Thomas construction. The authors also generalized this result to the case of semistable sheaves on Calabi-Yau threefolds, showing that analytically locally the stack of semistable sheaves admits an atlas $[S/G]$ such that S is locally the critical locus of a G-equivariant holomorphic function, where G is a complex reductive algebraic group. The next major step towards understanding the local structure of moduli spaces of stable sheaves on CY 3-folds is due to Pantev, Toën, Vaquié and Vezzosi [PTVV], who proved using derived algebraic geometry that such spaces admit a richer structure than a symmetric obstruction theory. They define (-1)-shifted symplectic structures on derived algebraic schemes, and they prove that the moduli space of stable sheaves $\mathcal{M}_X$ on a $3\,$CY-variety X admits such a structure. Following this, Brav, Bussi and Joyce proved in [BBJ] that the classical truncation $t_0(Z)$ of a derived scheme (Z, ω_Z) with a (-1)-shifted symplectic structure is Zariski locally the critical locus of a regular function on a smooth scheme, and this together with the results in [PTVV] implies that the moduli space $\mathcal{M}_X$ of stable sheaves on a Calabi-Yau threefold X is locally a critical locus. Further development for the case of (-1)-shifted symplectic structures on Artin stacks recently appeared in [BBBBJ].

Using the Behrend function in the analytic setting, or rather its version for Artin stacks locally of finite type over $\mathbb{C}$, Joyce and Song generalized in [JS] the definition of Donaldson-Thomas invariants to the case of moduli spaces of semistable sheaves on complex projective threefolds, and showed how the invariants depend on the choice of a stability condition by describing their behavior under wall crossing. In this setting the invariants remain again unchanged under deformation of the underlying Calabi-Yau threefold. In [JS] the authors also define the Behrend function over algebraically closed fields $\mathbb{K}$ of characteristic 0, and Bussi later proved in [Bu] that the definition of Donaldson-Thomas invariants can be extended to any algebraically closed field $\mathbb{K}$ of characteristic 0, by proving that the local description of the moduli spaces as critical loci still holds and showing that the Behrend identities are valid also in this context.

The analysis of the local case when $Z = \mathrm{Crit}(f\colon M \longrightarrow \mathbb{C})$ led Behrend to suggest a categorified version of the DT invariants, later formally defined and studied in the special case of the Hilbert scheme of four points on $\mathbb{C}^3$ by Dimca and Szendrői [DS].

Deligne defined in [De] the *complex of vanishing cycles* $\phi_f\mathbb{Q}$, a complex of sheaves of $\mathbb{Q}$-vector spaces with constructible cohomology supported on Z, whose pointwise

Euler characteristic

$$\chi(\phi_f)(P) := \sum_i (-1)^i \dim_{\mathbb{Q}} \mathcal{H}^i(Z, \phi_f \mathbb{Q})_P$$

satisfies $\chi(\phi_f)(P) = (-1)^{\dim M} \nu_Z(P)$. Taking hypercohomology with compact supports of a suitable shift by the dimension of M, one recovers the numerical DT invariant of (1.22):

$$\mathrm{DT}(Z) = \sum_i (-1)^i \dim_{\mathbb{Q}} \mathbb{H}^i_c(Z, \phi_f \mathbb{Q}[\dim M]).$$

At least in this local setting one could say that $\phi_f \mathbb{Q}[\dim M]$, called the *perverse sheaf of vanishing cycles*, gives a categorification of the numerical DT invariant, and that the vector space

$$\bigoplus_i \mathbb{H}^i_c(Z, \phi_f \mathbb{Q}[\dim M]) \tag{1.24}$$

gives a cohomological refinement of such. Starting from this description, other types of refinements of numerical Donaldson-Thomas invariants can be defined by considering the Poincaré polynomial or the weight polynomial of the hypercohomology (1.24). The existence of such refinements has been conjectured in the physics literature in some local cases [IKP, IKV]. Since moduli spaces of sheaves are locally the critical locus of a regular function on a smooth scheme, it is natural to ask whether it is possible to glue the local data $\phi_f \mathbb{Q}[\dim M]$ constructed around each point to obtain a global invariant. This is a very subtle question, and in order to perform the gluing it is necessary to consider a choice of *orientation* [KS1], which is an extra piece of information on the symmetric obstruction theory or on the (-1)-shifted symplectic structure. If $E^\bullet \longrightarrow \mathbb{L}^\bullet_Z$ is a symmetric obstruction theory on a scheme Z as described above, an orientation is a choice of a square root of the determinant bundle $\det(E^\bullet)$. It was proven by Brav, Bussi, Dupont, Joyce and Szendrői in [BBDJS] that, given a choice of orientation for the moduli space of sheaves on a Calabi-Yau threefold X, one can construct a global perverse sheaf $P^\bullet_{\mathcal{M}_X}$ by gluing the local data. This together with the result of Hua [H] on the existence of orientation data for moduli of stable sheaves on Calabi-Yau 3-folds, shows that there exists a categorification of the DT-invariant also in the global case, and hence that cohomological and refined invariants can be defined also in this setting. The authors also prove the analogous statement for derived schemes with a (-1)-shifted symplectic structure and a fixed choice of orientation [BBDJS, Corollary 6.11].

Remark 1.56. In the local case there exists a canonical choice of orientation. If X is the critical locus of a regular function f on a smooth scheme M and we consider the symmetric obstruction theory described in (1.23), we have

$$\det(E^\bullet) = \det(TM|_X)^{-1} \otimes \det(T^*M|_X) \cong K_M|_X^{\otimes 2}.$$

Thus, the choice of the square root K_M defines a canonical orientation on X.

There is another direction that one can take, following an insight of Kontsevich and Soibelman [KS1]. Since the DT invariant is an Euler characteristic type invariant, it can be computed using cut-and-paste techniques, i.e. by decomposing the variety into disjoint union of open and closed subvarieties. In the local setting $Z = \mathrm{Crit}(f\colon M \longrightarrow \mathbb{C})$, Denef and Losev constructed in [DL] the motivic class of vanishing cycles, a class

$$[\phi_f] \in \mathrm{K}^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-\frac{1}{2}}] \tag{1.25}$$

in a suitable enhancement of the Grothendieck ring of varieties. It was observed in [BBS] that the Euler characteristic of $-(-\mathbb{L}^{-\frac{1}{2}})^{\dim M}[\phi_f]$ recovers the classical DT invariant. Recent work of Bussi, Joyce and Meinhardt [BJM] uses the results of [BBJ] on the local structure of derived schemes with a (-1)-shifted symplectic structure admitting an orientation to generalize the definition to the global setting. In particular, in [BBJ] the authors define a motivic invariant for the moduli space of stable sheaves on a Calabi-Yau threefold with a choice of orientation, which refines the classical invariant of Donaldson and Thomas.

The following sections describe the construction of the class $-(-\mathbb{L}^{-\frac{1}{2}})^{\dim M}[\phi_f]$ in the local setting and some of its properties.

1.4.2 Grothendieck ring of varieties and its enhancements

The constructions we report in this subsection are the result of work by Denef and Losev [DL] and Loojienga [Lo], based on unpublished work of Kontsevich.

The *Grothendieck ring of varieties* $\mathrm{K}_0(\mathrm{Var}_{\mathbb{C}})$, introduced for the first time by Grothendieck in a letter to Serre, is the quotient of the free $\mathbb{Z}$-module generated by isomorphism classes of complex varieties (reduced separated schemes of finite type) by the scissor equivalence relation

$$[X] = [U] + [Y], \; U \subset X \text{ open and } Y = X \setminus U. \tag{1.26}$$

The product is induced by the Cartesian product of varieties. An element A of $\mathrm{K}_0(\mathrm{Var}_{\mathbb{C}})$ is said to be *effective* if $A = [X]$ for a variety X.

Remark 1.57. This ring is not sensitive to non-reduced scheme structure, so that we could have equivalently considered $\mathrm{K}_0(\mathrm{Sch}_{\mathbb{C}})$.

There are two properties of $\mathrm{K}_0(\mathrm{Var}_{\mathbb{C}})$ that will be used several times in this exposition:

- If $\pi\colon X \longrightarrow B$ is a Zariski locally trivial fibration with fiber F we have $[X] = [F][B]$;
- If $f\colon X \longrightarrow Y$ is a morphism that is a bijection at the level of geometric points, then $[X] = [Y]$.

One can refer for example to Bridgeland [Br, Lemma 2.5, 2.9] for a proof of these facts.

From Deligne's theory of mixed Hodge structures on compactly supported cohomology with coefficients in $\mathbb{Q}$, we get a morphism of rings

$$\mathrm{E}\colon\ \mathrm{K}_0(\mathrm{Var}_{\mathbb{C}}) \longrightarrow \mathbb{Z}[x, y]$$

defined on generators by the E-polynomial

$$\mathrm{E}([X]) = \sum_{p,q} x^p y^q \sum (-1)^i \dim \mathrm{H}^{p,q}(\mathrm{H}_c^i(X, \mathbb{Q})). \tag{1.27}$$

Forgetting the Hodge filtration, one is left with the finite increasing weight filtration

$$0 = W_0(\mathrm{H}_c^i(X, \mathbb{Q})) \subset W_1(\mathrm{H}_c^i(X, \mathbb{Q})) \ldots \subset W_n(\mathrm{H}_c^i(X, \mathbb{Q})) = \mathrm{H}_c^i(X, \mathbb{Q}).$$

Hence, one can consider the weight polynomial

$$\mathrm{W}\colon\ \mathrm{K}_0(\mathrm{Var}_{\mathbb{C}}) \longrightarrow \mathbb{Z}[t^{\frac{1}{2}}] \tag{1.28}$$

associating to a variety X the polynomial in half integer powers of t

$$\sum_n t^{\frac{n}{2}} \sum_i (-1)^i \dim \mathrm{Gr}_n^W \mathrm{H}_c^i(X, \mathbb{Q}),$$

where $\mathrm{Gr}_n^W \mathrm{H}_c^i(X, \mathbb{Q})$ is the n-th piece of the associated graded of the weight filtration. In particular, the weight polynomial can be obtained by formally setting $x = y = t^{\frac{1}{2}}$ in the E-polynomial. The further specialization $t^{\frac{1}{2}} = 1$ defines the Euler characteristic for cohomology with compact support

$$\chi\colon\ K_0(\mathrm{Var}_{\mathbb{C}}) \longrightarrow \mathbb{Z}. \tag{1.29}$$

As explained in [DL], in order to define the motivic class of vanishing cycles of a function as in (1.25) one needs to consider a certain enhancement of the Grothendieck ring of varieties.

Definition 1.58. Let $\hat{\mu} = \lim_{\leftarrow} \mu_n$ be the group of roots of unity. An action of $\hat{\mu}$ on a scheme X is *good* if the action factors through a finite quotient of $\hat{\mu}$ and every point x in X admits a $\hat{\mu}$-invariant affine open neighbourhood. An isomorphism between two schemes X and Y with a good $\hat{\mu}$-action is a $\hat{\mu}$-equivariant isomorphism of schemes $f\colon X \longrightarrow Y$.

We define $\widetilde{K}_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$ to be the quotient of the free $\mathbb{Z}$-module generated by isomorphism classes of varieties with a good $\hat{\mu}$-action by the analogue of the scissor equivalence relation of (1.26), where we allow only $\hat{\mu}$-invariant open sets U. Finally we consider the additive group $K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$ obtained as the quotient of $\widetilde{K}_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$ by the relation identifying the class $[V, \hat{\mu}]$ of a vector space with a linear $\hat{\mu}$-action with the class $[V]$ of V with the trivial $\hat{\mu}$-action. The group $K_0(\mathrm{Var}_{\mathbb{C}})$ sits inside $K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$ by considering the trivial action. One can endow $K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$ with a commutative ring structure that extends the one on $K_0(\mathrm{Var}_{\mathbb{C}})$; this involves the introduction of the product $*$ which involves a somewhat complicated construction introducing a Fermat curve. Since this will not be used in the following chapters we refer to [DL], [Lo] for the definition of the product.

Example 1.59. The ring $K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$ admits a square root of the class $\mathbb{L} = [\mathbb{A}^1]$ of the affine line, [DL, Lo]. The Thom-Sebastiani theorem of [DL] describes the motivic Milnor fiber of the sum of two functions $f = g_1 \boxplus g_2$, where $g_i\colon X_i \longrightarrow \mathbb{C}$ are two regular functions and $g_1 \boxplus g_2(x_1, x_2) = g_1(x_1) + g_2(x_2)$, in terms of the classes associated to the two functions g_1 and g_2:

$$[-\phi_f] = [-\phi_{g_1}][-\phi_{g_2}].$$

We consider the special case when $X_i = \mathbb{C}$ and $g_i(x_i) = x_i^2$. Studying the fiber over 0 of the embedded resolution $\mathrm{Bl}_0(\mathbb{C}^2)$ of the zero locus of f, it can be proven that $[\phi_f] = \mathbb{L}$. Since clearly $[\phi_{g_1}] = [\phi_{g_2}]$, it is natural to define $[\phi_{g_i}] := \mathbb{L}^{\frac{1}{2}}$. Following [DL] it is easy to show that $[\phi_{g_i}] = 1 - [\{p_1, p_2\}, \mu_2]$, where the generator of μ_2 acts on the two points interchanging them.

The second enhancement of the Grothendieck ring of varieties concerns the necessity of adding a formal inverse of the square root of the class $\mathbb{L}$ of the affine line. This procedure requires some care as it has been recently proven in [Bor] by L. Borisov that the class of the affine line is a zero divisor in $\mathrm{K}_0(\mathrm{Var}_{\mathbb{C}})$. Let $\mathrm{Tor}_{\mathbb{L}}$ be the ideal

$$\mathrm{Tor}_{\mathbb{L}} = \{A \in \mathrm{K}_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}}) : A * \mathbb{L} = 0\}.$$

We can invert the class of the square root of the affine line after we have killed the $\mathbb{L}$-torsion, obtaining

$$\mathcal{M}_{\mathbb{C}}^{\hat{\mu}} = \left(\mathrm{K}_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})/\,\mathrm{Tor}_{\mathbb{L}}\right)[\mathbb{L}^{-\frac{1}{2}}].$$

The invariants we will consider sit in a subring $\mathcal{M}_{\mathbb{C}}$ of $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$. Since $K_0(\mathrm{Var}_{\mathbb{C}})$ is included in $\mathrm{K}_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$, we can consider the subring $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{\frac{1}{2}}]$ of polynomials in half-integer powers of the class of the affine line. In particular, it follows from [DL] and [Lo] that the product of two elements in this subring is given by

$$\left(\sum_{i=0}^{n} a_i \mathbb{L}^{\frac{i}{2}}\right) * \left(\sum_{j=0}^{m} b_j \mathbb{L}^{\frac{j}{2}}\right) = \sum_{k=0}^{n+m} \sum_{i=0}^{k} a_i b_{k-i} \mathbb{L}^{\frac{k}{2}}, \tag{1.30}$$

where $a_i b_j$ denotes the usual product of classes in the Grothendieck ring of varieties. Proceeding exactly as above we localize with respect to the class $\mathbb{L}^{\frac{1}{2}}$ to obtain

$$\mathcal{M}_{\mathbb{C}} = \left(\mathrm{K}_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{\frac{1}{2}}]/\,\mathrm{Tor}_{\mathbb{L}}\right)[\mathbb{L}^{-\frac{1}{2}}], \tag{1.31}$$

where the product again corresponds to a rule completely analogous to the one of equation (1.30).

Remark 1.60. If A is in $\mathrm{Tor}_{\mathbb{L}}$ then $E(A) = 0$. Indeed, we have that E is a ring homomorphism, $E(\mathbb{L}) = xy$, and $\mathbb{Z}[x, y]$ is an integral domain.

From Remark 1.60 the E-polynomial, the weight polynomial, and the Euler characteristic extend to homomorphisms with source $\mathcal{M}_{\mathbb{C}}$ and target respectively $\mathbb{Z}[x^{\pm\frac{1}{2}}, y^{\pm\frac{1}{2}}]$, $\mathbb{Z}[t^{\pm\frac{1}{2}}]$ and $\mathbb{Z}$, by setting:

$$\begin{aligned} E &\colon \mathbb{L}^m \longmapsto (xy)^m, \\ W &\colon \mathbb{L}^m \longmapsto t^m, \\ \chi &\colon \mathbb{L}^m \longmapsto 1, \end{aligned}$$

for m a half integer.

The following remark explains some of the geometric reasons and conventions which make it necessary to define $\mathcal{M}_{\mathbb{C}}$ in order to construct a class whose Euler characteristic recovers the numerical DT invariant.

Remark 1.61. In order to be able to extend all these constructions to varieties locally cut out as the critical locus of a function on a smooth scheme one has to include the case of a smooth projective variety M endowed with the zero function f. In this case one has $[\phi_f] = -[M]$. Furthermore, the weight filtration of the cohomology coincides with the degree filtration, so that the weight polynomial is the Poincaré polynomial (with half integer exponents). From what we have seen above, we know that we have to consider the shift $\mathbb{Q}[\dim M]$ of the constant $\mathbb{Q}$-local system on M when we apply the functor of vanishing cycles, in order to preserve the correspondence with classical DT invariants, and to make the result a perverse sheaf. We take $\mathbb{P}^1$ as a simple example, and we refer to [Sz3, Appendix A] for a more detailed discussion on weights. In this case we have $\mathrm{H}^*(\mathbb{P}^1, \mathbb{Q}[1]) = \mathbb{Q}[1] \oplus \mathbb{Q}(-1)[-1]$ where $mathbbQ$ and $\mathbb{Q}(-1)$ are pure Tate Hodge structures of weight 0 and 2 respectively. To keep the property that weight and degree filtrations agree for the smooth projective case, it is thus convenient to tensor $\mathbb{Q}[\dim(X)]$ by a Tate twist $\mathbb{Q}\left(\dim(X)/2\right)$ of half the dimension of X, i.e. a formal object satisfying $\mathbb{Q}\left(\dim(X)/2\right)^{\otimes 2} = \mathbb{Q}(\dim X)$ [KS2, Section 3.4]. In particular, the Tate object $\mathbb{Q}(\frac{1}{2})$ can be seen as an analogue of the inverse $\mathbb{L}^{-\frac{1}{2}}$ of the square root of the Lefschetz motive in the motivic case. In the example of $\mathbb{P}^1$ we have $\mathrm{H}^*(\mathbb{P}^1, \mathbb{Q}(1/2)[1]) = \mathbb{Q}[1] \oplus \mathbb{Q}[-1]$, now with weights 1 and -1. Computing the weight polynomial with coefficient system $\mathbb{Q}(\frac{1}{2})[1]$ we obtain $t^{\frac{1}{2}} + t^{-\frac{1}{2}}$. In terms of motivic Milnor fibers this is translated into multiplication of the class $[\phi_f]$ by $-(-\mathbb{L}^{-\frac{1}{2}})^{\dim M}$, where again the meaning of the class $\mathbb{L}^{-\frac{1}{2}}$ becomes manifest by thinking about the Lefschetz motive $\mathbb{L}$ as the class corresponding to the Tate twist $\mathbb{Q}(-1)$ of weight 2, since we consider cohomology with compact support.

In order to perform wall-crossing we will need to be able to compute invariants on a degenerate wall. To do so we will also consider invariants in the Grothendieck ring $\mathrm{K}_0(\mathrm{St}_{\mathbb{C}})$ of stacks which are of finite type over $\mathbb{C}$ and have affine stabilizers at each point. It has been proven in [Br] that any stack of finite type over $\mathbb{C}$ with affine stabilizers is the image under a bijective morphism of a global quotient stack with stabilizer an affine group. This is the crucial fact that allows Bridgeland to prove that the Grothendieck ring $K_0(\mathrm{St}_{\mathbb{C}})$ can be obtained from $K_0(\mathrm{Var}_{\mathbb{C}})$ by inverting the class of linear algebraic groups $[\mathrm{GL}_d]$, where $[\mathrm{GL}_d]$ is a polynomial in the class of the affine line

$$[\mathrm{GL}_d] = \mathbb{L}^{\binom{d}{2}} \prod_{i=1}^{d} (\mathbb{L}^i - 1).$$

Again, in light of the result of [Bor], it is better to factor by the torsion ideal

$$\mathrm{Tor}_{\mathrm{GL}} = \{A \in \mathrm{K}_0(\mathrm{Var}_{\mathbb{C}}) : A \cdot [\mathrm{GL}_d] = 0\}$$

before localizing. We then have

$$\mathrm{K}_0(\mathrm{St}_{\mathbb{C}}) = (\mathrm{K}_0(\mathrm{Var}_{\mathbb{C}})/\operatorname{Tor}_{\mathrm{GL}})[[\mathrm{GL}_d]^{-1} : \ d \geq 1].$$

We will denote by $\mathcal{M}_{\mathbb{C}}^{st}$ the ring obtained by formally adding the square root of the class of the affine line $\mathbb{L}^{\frac{1}{2}}$, taking the quotient modulo the torsion ideal, and localizing with respect to $\mathbb{L}^{\frac{1}{2}}$.

Remark 1.62. The ring $\mathrm{K}_0(\mathrm{St}_{\mathbb{C}})$ can also be described as

$$\left(\mathrm{K}_0(\mathrm{Var}_{\mathbb{C}})/\operatorname{Tor}_{\mathbb{L},(1-\mathbb{L}^d)}\right)[\mathbb{L}^{-1}, (1-\mathbb{L}^d)^{-1} : \ d \geq 1].$$

In particular, it can be considered as a subring of $\mathcal{M}_{\mathbb{C}}[[\mathbb{L}^{-1}]]$ by identifying $(1 - \mathbb{L}^{-d})^{-1}$ with its power series expansion. Here the notation $\mathcal{M}_{\mathbb{C}}[[\mathbb{L}^{-1}]]$ indicates the dimensional completion of $\mathcal{M}_{\mathbb{C}}$ explained by Behrend and Dhillon in [BD, Section 2]. We recall its construction as it will be useful in the rest of the exposition. For each $n \geq 0$ let F^n be the subring of $\mathcal{M}_{\mathbb{C}}$ generated by elements

$$[X]\mathbb{L}^{-k}, \qquad \dim(X) - k \leq -n.$$

Every F^n gives an ideal in F^0, and we have a decreasing filtration

$$\ldots \subset F^2 \subset F^1 \subset F^0.$$

We denote by $\widehat{F^0}$ the completion of F^0 with respect to this filtration and define

$$\widetilde{\mathcal{M}}_{\mathbb{C}} = \mathcal{M}_{\mathbb{C}}[[\mathbb{L}^{-1}]] := \widehat{F_0} \otimes_{F^0} \mathcal{M}_{\mathbb{C}}. \tag{1.32}$$

We conclude this section with a brief description of the definition of the relative Grothendieck ring; we refer to [Lo, Section 4] for a more detailed presentation. In the case of varieties, we fix a variety S, and we let $K_0(\mathrm{Var}_S)$ be the quotient of the free $\mathbb{Z}$-module generated by isomorphism classes of varieties over S by the scissor relation for varieties over S. A multiplication on $K_0(\mathrm{Var}_S)$ is defined via the fiber product of varieties over S. The same procedure which led to the construction of $\mathcal{M}_{\mathbb{C}}$ starting from $\mathrm{K}_0(\mathrm{Var}_{\mathbb{C}})$ carries over to the relative setting, giving a ring $\mathcal{M}_S$. This generalizes to the construction of $\mathcal{M}_S^{st}$, the enhanced Grothendieck ring of stacks with affine stabilizers over a fixed stack S of finite type and with affine stabilizers.

1.4.3 Power structure on $\mathrm{K}_0(\mathrm{Var}_{\mathbb{C}})$ and $\widetilde{\mathcal{M}}_{\mathbb{C}}$

We recall the construction of a power structure on the Grothendieck ring of varieties; this will be extensively used in the following chapters, and in particular in Chapter 4 to compute motivic invariants of deformations of Calabi-Yau algebras. The power structure on $K_0(Var_{\mathbb{C}})$ was introduced by Gusein-Zade, Luengo and Melle-Hernandez in [GZLMH], and its geometrical interpretation due to Bryan and Morrison [BM, Section 2] can be used to reinterpret many computations of generating series of motivic DT invariants, as in for example [BBS, MMNS].

Definition 1.63. Let R be a ring. A *power structure* on R is an assignment

$$(1 + tR[[t]]) \times R \longrightarrow (1 + Rt[[t]]) \tag{1.33}$$

$$(A(t), M) \longrightarrow A(t)^M, \tag{1.34}$$

which satisfies the following:

(1) $A(t)^0 = 1$;

(2) $A(t)^1 = A(t)$;

(3) $A(t)^M A(t)^N = A(t)^{M+N}$;

(4) $A(t)^{M\cdot N} = (A(t)^M)^N$;

(5) $A(t)^M B(t)^M = (A(t)B(t))^M$;

(6) $(1 + t)^m = 1 + mt + O(t^2)$.

Gusein-Zade, Luengo and Melle-Hernandez proved in [GZLMH] that there is a unique power structure on the Grothendieck ring of varieties $K_0(\mathrm{Var}_{\mathbb{C}})$ such that for any variety X one has

$$(1 - t)^{-[X]} = \sum_{i=0}^{\infty} [\mathrm{Sym}^n(X)]t^n, \tag{1.35}$$

where $\mathrm{Sym}^n(X)$ is the n-th symmetric power of X. In [GZLMH] the authors also prove that the power structure is *effective*: given a collection of varieties $\{A_i\}_{i\in\mathbb{N}}$ and a variety $[B]$, then the series $A(t)^{[B]} = (\sum_i [A_i]t^i)^{[B]}$ has effective coefficients. In the case when A_0 is a point, the coefficients of the series $A(t)^{[B]}$ can be described explicitly:

$$(A(t)^{[B]})_{[t^n]} = \bigsqcup_{\lambda \vdash n} \left((B^{\lambda_i} \setminus \Delta \times A_i^{\lambda_i})/S_{\lambda_i}\right), \tag{1.36}$$

where $\lambda \vdash n$ denotes a partition of n, $\Delta \subset B^{\lambda_i}$ is the big diagonal

$$\Delta = \{(x_1, \dots, x_{\lambda_i}) \in B^{\lambda_i} \colon x_i \neq x_j \text{ for all } i, j\},$$

and S_{λ_i} acts by permuting simultaneously the factors in the product.

Remark 1.64. In [BM] the authors describe how to obtain a nice geometric interpretation of the power structure in the effective case. In the notation above, we define

$$\deg \colon \bigsqcup_{i=1}^{\infty} A_i \longrightarrow \mathbb{N}$$

as the function which sends a point x in A_i to i. Then the coefficient $(A(t)^{[M]})_{[t^n]}$ parametrises pairs (S, ϕ) where S is a subset of n elements of the geometric points of M and $\phi \colon S \longrightarrow \bigsqcup_{i=1}^{\infty} A_i$ satisfies $\sum_{x \in S} \deg(\phi(x)) = n$.

This power structure extends uniquely to a power structure on the ring $\widetilde{\mathcal{M}}_{\mathbb{C}}$ of (1.32), by requiring

$$(1-t)^{-\mathbb{L}^k[X]} = (1-\mathbb{L}^k t)^{-[X]} \tag{1.37}$$

for all half integers k.

Remark 1.65. In the case when k is a positive integer, the identity (1.37) is a consequence of a lemma due to B. Totaro [Gö, Lemma 4.4], which states that

$$[\mathrm{Sym}^n(\mathbb{C}^k \times X)] = \mathbb{L}^{nk}[\mathrm{Sym}^n(X)].$$

Remark 1.66. The extension of the power structure to $K_0(\mathrm{St}_{\mathbb{C}})$ (and the other enhancements) is still effective in the following sense: if $A(t)$ has coefficients which are all effective classes of stacks, B is a variety and $A(t)_{[t^0]} = 1$ then $(A(t))^{[B]}$ has coefficients which are effective classes of stacks. In particular, up to the fact that the A_i are stacks, the geometric description of Remark 1.64 is still valid.

We conclude the section by defining motivic exponentiation and describing one property which will be used in the subsequent chapters.

Definition 1.67. [GZLMH] The *motivic exponential* is the assignment

$$\mathrm{Exp} \colon \widetilde{\mathcal{M}}_{\mathbb{C}}[[t]] \longrightarrow 1 + \widetilde{\mathcal{M}}_{\mathbb{C}}[[t]] \tag{1.38}$$

defined by

$$\mathrm{Exp} \colon \sum_{k>0} [A_k] t^k \longmapsto \prod_{k>0} (1-t^k)^{-[A_k]}. \tag{1.39}$$

In particular, it follows directly from (1.37) that

$$\mathrm{Exp}\left(A(\mathbb{L}^k t)\right) = \mathrm{Exp}(A(t))\big|_{t \mapsto \mathbb{L}^k t} \tag{1.40}$$

for any half integer k.

1.4.4 DT invariants for quivers with superpotential (and a cut)

Let M be a smooth scheme and $f\colon M \longrightarrow \mathbb{C}$ a regular function. Following [BBS] we define the *motivic* DT *invariant* of $X{=}\mathrm{Crit}(f\colon M \longrightarrow \mathbb{C})$ to be the following multiple of the motive of vanishing cycles of (1.25)

$$[X]_{\mathrm{vir}} = -(-\mathbb{L}^{-\frac{1}{2}})^{\dim M}[\phi_f] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}. \tag{1.41}$$

We refer to the Laurent polynomial $W([X]_{\mathrm{vir}})$ with half integer exponents, defined in (1.28) as the *refined* DT *invariant* of X.

Remark 1.68. Both the motivic and the refined DT invariants we defined are invariants of X as the critical locus of f, so that in principle we should write $[X]_{\mathrm{vir}} = [(X,f)]_{\mathrm{vir}}$ and $\mathrm{DT}(X) = \mathrm{DT}(X,f)$. We omit the f in order to simplify the notation, unless it is not clear from the context which function cuts X out as a critical locus. In particular, if X is smooth we will write $[X]_{\mathrm{vir}}$ in place of $[(X,0)]_{\mathrm{vir}}$ to denote the class $-(-\mathbb{L})^{-\frac{\dim(X)}{2}}[X]$.

The class $[\phi_f]$ measures the difference between the class $[\psi_f]$ of nearby cycles, which can be described explicitly in terms of an embedded resolution, and the central fiber $[f^{-1}(0)]$. Theorem 1.75 of [BBS] explains that when f is equivariant with respect to a torus action with some good properties, then the class of nearby cycles is just the class of a generic fiber $[f^{-1}(1)]$.

Remark 1.69. It was proven independently by Navarro Aznar [Na, Proposition 15.11] and Saito [S] that the perverse sheaf $\phi_f\mathbb{Q}[\dim M]$ admits a mixed Hodge structure in the sense of Deligne. Furthermore,

$$W([X]_{\mathrm{vir}}) = \sum_n t^{\frac{n}{2}} \sum_i (-1)^i \dim_{\mathbb{Q}} \mathrm{Gr}_n^W \mathbb{H}_c^i(X, \phi_f\mathbb{Q}[m]\,(m/2)), \tag{1.42}$$

where m is the dimension of M and $\mathbb{Q}(m/2)$ is the Tate twist described in Remark 1.61. Refined DT invariants of the global critical locus X are the weight polynomials W for the critical cohomology of X.

Moduli spaces of representations of (framed) quivers with superpotential give examples of global critical loci. This has been known for quite a long time in the physics literature, and was proven by E. Segal [Se]. We sketch the proof here as it consists of a construction that we will use again.

Proposition 1.70. *Let θ be a generic stability parameter for a quiver with superpotential (Q, W). Then there is a regular function* $\operatorname{Tr} W$ *on the smooth variety $M_d^\theta(Q)$ such that*

$$\operatorname{Crit}(\operatorname{Tr} W) = M_d^\theta(Q, W).$$

Proof. The smoothness of $M_d^\theta(Q)$ is a consequence of the genericity of θ. Given a cyclic summand $c_1 \dots c_n$ in W, we can compute the trace of the linear map $v_{c_n} \dots v_{c_1}$ constructed starting from a representation (V, v) in $R_d^\theta(Q)$, thus obtaining a function

$$\widehat{\omega}_{\underline{d},\theta} \colon R_{\underline{d}}^\theta(Q) \longrightarrow \mathbb{C}. \tag{1.43}$$

Cyclicity of the trace shows that this induces a well-defined morphism on $R_d^\theta(Q)$ and that it descends to a function $\omega_{\underline{d},\theta} = \operatorname{Tr} W$ on the geometric quotient $M_d^\theta(Q)$. The vanishing of the derivatives of $\operatorname{Tr} W$ imposes the same relations as the ones given by formal derivations of the superpotential. □

Remark 1.71. Given a quiver with superpotential (Q, W), a framing vector $\underline{n}$, and θ in $\mathbb{Z}^{V(Q)}$ chosen as in (1.3), the moduli space $M_{\underline{d},\underline{n}}^\theta(Q, W)$ is a global critical locus on a smooth scheme.

There is a subset of the vector space $\operatorname{Sup}(Q)$ of superpotentials that give rise to critical loci admitting a torus actions, making the virtual class $[M_d^\theta(Q, W)]_{\mathrm{vir}}$ particularly well-behaved.

Definition 1.72. We say that (Q, W) admits a *cut* if there is a set of arrows I such that every cyclic monomial of W contains exactly one arrow in I. Equivalently, there is a $\mathbb{Z}$-grading g_I on the paths in Q defined on arrows by

$$g_I \colon a \longmapsto \begin{cases} 1 \text{ if } a \in I; \\ 0 \text{ otherwise,} \end{cases}$$

such that W is homogeneous of degree 1.

The following definition gives an important property that may be satisfied by a torus action on a variety. In particular, we will show that the torus action that one constructs on the moduli space of a quiver with superpotential admitting a cut satisfies this property.

Definition 1.73. [BBS] Let $f \colon M \longrightarrow \mathbb{C}$ be a regular function on a smooth quasi-projective variety and assume we have an action of a one-dimensional torus $\mathbb{C}^*$ on M. We say that f is *circle closed* if it is equivariant, i.e. $f(t \cdot m) = t f(m)$ for m in M and t in $\mathbb{C}^*$, and the limit $\lim_{t \to 0} t \cdot m$ exists for every $m \in M$.

Remark 1.74. Our definition of a circle closed action differs slightly from the notion of a circle compact action in [BBS]. Nevertheless it is proven in [Loc. Cit.] that the requirement of having a circle closed action is sufficient to obtain the characterization of $[\mathrm{Crit}(f)]_{\mathrm{vir}}$ that follows.

Theorem 1.75 ([BBS, Proposition 2.11])**.** *Let $f\colon M \longrightarrow \mathbb{C}$ be a regular function with a circle closed action. Then the class $[\mathrm{Crit}(f)]_{\mathrm{vir}}$ lies in $\mathcal{M}_{\mathbb{C}}$ and*

$$[\mathrm{Crit}(f)]_{\mathrm{vir}} = -(-\mathbb{L}^{-\frac{1}{2}})^{\dim M}([f^{-1}(1)] - [f^{-1}(0)]).$$

Lemma 1.76. *Let (Q, W) be a quiver with a superpotential admitting a cut and θ be a generic stability condition. Then f admits a circle closed torus action.*

Proof. Let I be a cut. Consider the action on $M_d^\theta(Q)$ induced by the action

$$\mathbb{C}^* \times R_d^\theta(Q) \longrightarrow R_d^\theta(Q), \quad (t, (v_a)) \longmapsto (t^{g_I(a)} v_a).$$

The action is circle compact with respect to the function $\mathrm{Tr}(W)$ defined in Theorem 1.70. First of all from the definition we have $\mathrm{Tr}\, W(t(v_a)) = t\, \mathrm{Tr}\, W(v_i)$, so that $\mathrm{Tr}\, W$ is equivariant. There is a projective morphism

$$\pi\colon M_d^\theta(Q, W) \longrightarrow M_d^0(Q, W)$$

to the affine variety $M_d^0(Q, W)$. It is clear that all limits $\lim_{t\to 0}(t(v_a))$ exist in $M_d^0(Q, W)$ as they are already well-defined in $R_d^0(Q)$. Since π is proper, the limits exist also in $M_d^\theta(Q, W)$. $\square$

Denote by $T\colon \mathbb{Z}^{V(Q)} \longrightarrow \mathbb{Z}$ the Tits form of Q defined by $T(\underline{d}) = \sum_{i\in V(Q)} d_i^2 - \sum_{a\colon i\to j} d_i d_j$. The following proposition describes the virtual class of the critical locus of $\mathrm{Tr}(W)$ for a particular class of dimension vectors including the case of quivers with framing, as defined in Section 1.1.3.

Proposition 1.77. *Let (Q, W) a quiver admitting a cut and θ a generic stability condition. Let $\underline{d}$ be a dimension vector with $d_i = 1$ for at least one vertex i. Then we have*

$$[\mathrm{Crit}(\omega_{\underline{d},\theta})]_{\mathrm{vir}} = -\,(-\mathbb{L})^{T(d)-1}\, \frac{[\widehat{\omega}_{\underline{d},\theta}^{-1}(0)] - [\widehat{\omega}_{\underline{d},\theta}^{-1}(1)]}{[PG_{\underline{d}}]}, \tag{1.44}$$

where $\widehat{\omega}\colon R_{\underline{d}}^\theta(Q, W) \longrightarrow \mathbb{C}$ is the lift of the function $\mathrm{Tr}(W)$ *of Theorem 1.70.*

Proof. The group $PG_{\underline{d}} = \prod \mathrm{GL}(d_i)/\mathbb{C}^*$ is isomorphic to a product of general linear groups, so that in particular it is a special algebraic group (i.e. every étale locally trivial fibration is automatically Zariski locally trivial). Hence we have that $R^\theta_{\underline{d}}(Q,W)$ is a Zariski locally trivial $PG_{\underline{d}}$-bundle. Since the lift $\hat{\omega}$ to $R^\theta_{\underline{d}}(Q,W)$ of $\mathrm{Tr}(W)$ is $PG_{\underline{d}}$-equivariant, we have that the preimages $\widehat{\omega}^{-1}(0)$ and $\widehat{\omega}^{-1}(1)$ are Zariski locally trivial $PG_{\underline{d}}$-bundles, so that $[\mathrm{Tr}(W)^{-1}(x)] = [\hat{\omega}^{-1}(x)]/[PG_{\underline{d}}]$ for $x \in \{0,1\}$. Now the result follows from dimension counting. $\square$

Remark 1.78. Under the hypotheses of Proposition 1.77 the refined DT invariant, obtained by taking the weight polynomial $W([\mathrm{Crit}(\omega_{\underline{d},\theta})]_{\mathrm{vir}})$, computes the weight polynomial of the hypercohomology group of $M^\theta_{\underline{d},\theta}(Q,W)$ with coefficient system $\phi_{\omega_{\underline{d}},\theta}$, i.e. the refined Donaldson-Thomas invariants of the moduli space of θ-stable quiver representation of (Q,W) with dimension $\underline{d}$.

If we assume that (Q,W), θ and $\underline{d}$ satisfy the hypotheses of Proposition 1.77, it is natural to generalize the definition of Donaldson-Thomas invariants to the case of the quotient stack $\mathcal{M}^\theta_{\underline{d}}(Q,W) = M^\theta_{\underline{d}}(Q,W)/\mathbb{C}^*$ as

$$[\mathcal{M}^\theta_{\underline{d}}(Q,W)]_{\mathrm{vir}} := \frac{[M^\theta_{\underline{d}}(Q,W)]_{\mathrm{vir}}}{[\mathbb{C}^*]_{\mathrm{vir}}}. \tag{1.45}$$

Then using Proposition 1.77 we immediately obtain the characterization

$$[\mathcal{M}^\theta_{\underline{d}}(Q,W)]_{\mathrm{vir}} = \frac{[\mathrm{Crit}(\widehat{\omega}_{\underline{d},\theta})]_{\mathrm{vir}}}{[G_{\underline{d}}]}. \tag{1.46}$$

We then generalize the definition of motivic Donaldson-Thomas invariants for the stack of representations $\mathcal{M}^\theta_{\underline{d}}(Q,W)$ to the case when $\underline{d}$ does not satisfy the hypotheses of Proposition 1.77 and θ is not necessarily generic, but where we still assume that the superpotential admits a cut. The motivic DT invariant of the stack of θ-semistable representations with dimension vector $\underline{d}$ is

$$[\mathcal{M}^\theta_{\underline{d}}(Q,W)]_{\mathrm{vir}} := -\left(-\mathbb{L}^{\frac{1}{2}}\right)^{T(q)} \frac{[\widehat{\omega}^{-1}_{\underline{d},\theta}(1)] - [\widehat{\omega}^{-1}_{\underline{d},\theta}(0)]}{[G_{\underline{d}}]_{vir}}. \tag{1.47}$$

Remark 1.79. When $\theta = \underline{0}$ then

$$[\mathcal{M}^{\underline{0}}_{\underline{d}}(Q,W)]_{\mathrm{vir}} = \frac{[\mathrm{Crit}(\omega_{\underline{d},\underline{0}})]_{\mathrm{vir}}}{[G_{\underline{d}}]_{\mathrm{vir}}}, \tag{1.48}$$

so that in this case the geometric meaning of the DT invariants is still related to the motivic vanishing cycle $[\phi_{\omega_{\underline{d},\underline{0}}}]$. Other stacky cases are not so easy to interpret from the geometric point of view. It has been proven for example in [Moz2, Remark

3.5] that there exists a quiver Q such that if we consider the trivial superpotential and a particular choice of stability condition θ, the function $\widehat{\omega}_{\underline{d},\theta}$ does not admit a $\mathbb{C}^*$-action for which it is circle closed. Hence we cannot apply Theorem 1.75 in this case. This demonstrates that the definition (1.47) is not entirely geometrically justified, but it will be useful when performing wall-crossing. Ben-Bessat, Brav, Bussi and Joyce define in [BBBBJ] motivic Donaldson-Thomas invariants for Artin stacks with a (-1)-shifted symplectic structure and orientation. It would be interesting to understand whether the ad hoc definition of equation (1.47) agrees with the result of their constructions.

In Chapters 2 and 3 we use Proposition 1.77 to give a geometric description of and techniques to compute virtual classes. In Chapter 4 we will consider DT invariants of some stacky moduli spaces of representations of quivers with superpotential, but we will concentrate on the case of the $\underline{0}$ stability condition so that the definition (1.47) is still geometric by Remark 1.79.

Chapter 2

Higher Rank Refined DT Invariants for $\mathbb{C}^3$

We consider the moduli problem $\mathscr{M}_{n,r}$ parametrising families of torsion free sheaves on the projective space $\mathbb{P}^3(\mathbb{C})$ with Chern character $(r, 0, 0, -n)$ together with a fixed isomorphism to the sheaf $\mathcal{O}_{p_\infty}^{\oplus r}$ at a given torus invariant plane p_∞. We prove that the moduli functor $\mathscr{M}_{n,r}$ is represented by a quasi-projective scheme $\mathcal{M}_{n,r}$ arising as the moduli space of stable representations of a framed quiver with superpotential. From Theorem 1.70 we obtain a description of $\mathcal{M}_{n,r}$ as the critical locus of a global regular function on a smooth scheme, thus generalizing the result for the rank one case that appeared for example in Szendrői [Sz2]. As explained in Section 1.4 this leads to the study of motivic/refined Donaldson-Thomas invariants of $\mathcal{M}_{n,r}$. We first give a direct computation of the generating function for the refined Donaldson Thomas invariants of $\mathcal{M}_{n,r}$ defined in (1.41) following Behrend, Bryan and Szendrői [BBS], and then we apply the techniques developed in Mozgovoy [Moz3] to obtain our expression via motivic wall-crossing. We obtain the product expansion

$$\sum_{n\geq 0}[\mathcal{M}_{n,r}]_{vir}q^n = \prod_{m\geq 1}\prod_{k=0}^{rm-1}(1-\mathbb{L}^{2+k-\frac{rm}{2}}((-1)^r q)^m)^{-1}$$

where $[\mathcal{M}_{n,r}]_{vir}$ lies in $\mathcal{M}_{\mathbb{C}}$ by Lemma 1.76. The numerical specialization of this formula computes the generating series for classical Donaldson-Thomas invariants, defined in (1.29) as the Euler characteristic of $\mathcal{M}_{n,r}$ weighted by the Behrend constructible function. Explicitly, we have

$$\sum_{n\geq 0}\mathrm{DT}(\mathcal{M}_{n,r})q^n = M((-1)^r q)^r,$$

where $M(q)$ is the MacMahon function

$$M(q) = \prod_{m\geq 1}(1-q^m)^{-m},$$

which counts three-dimensional (3D) partitions in the positive octant. This same generating series has already appeared in the literature in two different contexts. On one hand Cirafici, Sinkovic and Szabo proved in [CSS] that it corresponds to the partition function counting D0-D6 branes on $\mathbb{C}^3$ for the abelian gauge group $U(1)^n$. On the other hand, Oprea proves in [O] that $M((-1)^r q)^r$ corresponds to a virtual count of torus invariant (H,δ)-stable framed sheaves, for a given stability condition (H,δ) with $\delta = \delta(n,r)$ a "good" parameter, thus providing a generating series for $\mathrm{DT}(\mathcal{M}_{n,r}^{H,\delta})$. It follows from Appendix 2.A that the moduli spaces we consider coincide with the moduli spaces of Oprea as stability is automatically satisfied. Hence, our result consists of an alternative computation of the generating series of classical DT invariants, with our techniques having the advantage of computing more refined invariants for $\mathcal{M}_{n,r}$.
We conclude the chapter by showing that the weight specialization of the generating series $\sum_{n\geq 0}[\mathcal{M}_{n,r}]_{vir} q^n$ can be described as a weighted count of 3D partitions.

2.1 The geometric picture

Consider the projective space $\mathbb{P}^3(\mathbb{C})$ and fix once and for all a plane p_∞ cut out by the equation $z_0 = 0$ after a choice of homogeneous coordinates on $\mathbb{P}^3$.

Definition 2.1. Denote by $\mathcal{G}$ the framing datum $\left(p_\infty, \mathcal{O}_{p_\infty}^{\oplus r}\right)$. A *framed sheaf* on $\mathbb{P}^3$ with framing datum $\mathcal{G}$ is a pair (F,ϕ) where

- F is a torsion free sheaf on $\mathbb{P}^3$,
- $\phi : F|_{p_\infty} \longrightarrow \mathcal{O}_{p_\infty}^{\oplus r}$ is an isomorphism of sheaves.

A *morphism of framed sheaves* $\psi\colon (F,\phi) \longrightarrow (F',\phi')$ with framing type $\mathcal{G}$ is a morphism of sheaves $\psi\colon F \longrightarrow F'$ making the diagram

$$\begin{tikzcd} F|_{p_\infty} \arrow[rr, "\psi|_{p_\infty}"] \arrow[dr, "\phi"'] & & F'|_{p_\infty} \arrow[dl, "\phi'"] \\ & \mathcal{O}_{p_\infty} & \end{tikzcd}$$

commutative.

For a choice of two positive integers r and n we consider the moduli problem

$$\mathscr{M}_{n,r}\colon \text{AffSchemes} \longrightarrow \text{Set}^{op}$$

assigning to a Noetherian affine scheme of finite type S the family

$$\left\{ \begin{array}{c} \mathcal{F} \text{ torsion free sheaf on } \mathbb{P}^3 \times S \text{ flat over } S, \\ \Phi\colon \mathcal{F}_{p_\infty \times S} \longrightarrow \mathcal{O}_{p_\infty \times S} \text{ sheaf isomorphism,} \\ ch(\mathcal{F}_s) = r - nh^3, \text{ for each geometric point } s \text{ of } S \end{array} \right\} \Big/ \cong ,$$

where h is Poincaré dual to the class of a hyperplane in $\mathbb{P}^3$ and where isomorphisms are in the category of framed sheaves.

Remark 2.2. The functor $\mathscr{M}_{n,r}$ defines uniquely a functor $\mathscr{M}_{n,r}\colon$ Schemes $\longrightarrow$ Set^{op} by requiring the gluing conditions to be satisfied.

Remark 2.3. In the literature, for example in the pioneering work [HL] or in [O], one usually considers the problem of the representability of the moduli problem for (H, δ)-stable framed sheaves, where δ is a suitable polynomial, in order to avoid automorphisms. We prove in Appendix 2.A that framed sheaves on $\mathbb{P}^n$ are automatically stable for a suitable choice of (H, δ) and all $n \geq 3$.

2.1.1 A first glimpse of the quiver description

First of all we notice that the threefold situation is quite different from the surface case analyzed by Nakajima in [N] due to the Horrocks splitting principle, which says that for $n \geq 3$ a vector bundle on $\mathbb{P}^n$ which is trivial on a hyperplane is automatically trivial. For any r and n strictly positive integers, there are no locally free sheaves on $\mathbb{P}^3$ of cohomological type $(r, 0, 0, -n)$ restricting to the trivial sheaf at p_∞. In a sense, this is really a problem regarding torsion free sheaves rather then a partial compactification of a moduli problem for vector bundles, as it happens in the two dimensional case. Another crucial difference is given by the following observation due to Abe and Yoshinaga.

Theorem 2.4 ([AY, Theorem 0.2]). *Any reflexive sheaf on $\mathbb{P}^3$ splitting as $\oplus_{i=1}^n \mathcal{O}_{p_\infty}(a_i)^{\oplus r_i}$ on the plane p_∞ is isomorphic to $\oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^3}(a_i)^{\oplus r_i}$.*

The following corollary is an immediate consequence. The result will become evident also from the description we will give in the next section.

Corollary 2.5. *Any torsion free rank r framed sheaf F on $\mathbb{P}^3$ with framing $\mathcal{G}$ embeds in $\mathcal{O}_{\mathbb{P}^3}^{\oplus r}$.*

Proof. Embed F into its double dual $F^{\vee\vee}$. The double dual $F^{\vee\vee}$ is then a reflexive sheaf on $\mathbb{P}^3$ isomorphic along p_∞ to $\mathcal{O}_{p_\infty}^{\oplus r}$, hence it is the trivial vector bundle $\mathcal{O}_{\mathbb{P}^3}^{\oplus r}$. □

Given a framed sheaf (F, ϕ) of cohomological type $(r, 0, 0, -n)$, we have an exact sequence of sheaves

$$0 \longrightarrow F \longrightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus r} \longrightarrow G \longrightarrow 0$$

where G is a zero-dimensional sheaf of length n. Assuming that the inclusion $F \longrightarrow \mathcal{O}^{\oplus r}$ is compatible with the framing, we can associate to (F, ϕ) the n-dimensional $\mathbb{C}[z_1, z_2, z_3]$-module $\mathrm{H}^0(G)$ of dimension n together with a morphism $\psi\colon \mathbb{C}^r \longrightarrow \mathrm{H}^0(G)$. Note that we write $\mathbb{C}^r$ in place $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}^{\oplus r})$ to stress the fact that the framing given by the inclusion is fixing a basis for $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}^{\oplus r})$. The surjectivity of $\mathcal{O}_{\mathbb{P}^3}^{\oplus r} \longrightarrow G$ implies that the image of ψ generates $\mathrm{H}^0(G)$ as a $\mathbb{C}[z_1, z_2, z_3]$-module. In the language of Section 1.1, we say that we can associate to a framed sheaf (F, ϕ) a θ-stable r-framed representation V of the quiver with superpotential (Q, W) defined in Example 1.5. Explicitly V is a representation of a quiver with superpotential $(\hat{Q}_r, W)$:

- the set of vertices $V(\hat{Q}_r)$ is $\{\infty, 0\}$;
- the set of arrows $E(\hat{Q}_r)$ is $\{a_j, i_h\colon j \in \{1, \dots, 3\},\ h \in \{1, \dots, r\}\}$ with a_i's loops based at the vertex 0 and $i_h\colon \infty \longrightarrow 0$;
- the superpotential W is given by $a_1a_2a_3 - a_1a_3a_2$;
- the dimension vector $d = d(n)$ is $(1, n)$;
- the stability parameter θ is given by $(\theta_\infty, \theta_0)$ with $\theta_\infty + n\theta_0 = 0$ and $\theta_0 < 0$.

Recall that the two-sided ideal of relations associated to the superpotential is defined by

$$I_W = \langle\langle \partial_a W\colon\ a \in E(\hat{Q}_r)\rangle\rangle,$$

so that a representation $(\{A_i\}_{i\in\{1,\dots,3\}}, I)$ of $(\hat{Q}_r, W)$ has to satisfy the quadratic commutation relations

$$[A_i, A_j] = 0 \quad i, j \in \{1, \dots, 3\}.$$

Remark 2.6. It is easy to check that V is θ-stable if and only if it does not admit subrepresentations with dimension vector $(1, n')$ for $0 < n' < n$. As there are no $\mathbb{C}[z_1, z_2, z_3]-$invariant subspaces of $\mathrm{H}^0(G)$ containing the image of ψ this condition is clearly satisfied by the representation associated to the framed sheaf (F, ϕ).

2.1.2 Pointwise description of $\mathscr{M}_{n,r}$

This section is devoted to a rigorous proof that the naïve correspondence established in Section 2.1.1 is actually a bijection, using the language of monads. Similar results were obtained independently in [BdSHJ] for the rank one case and the overall construction is inspired by [N] and [CSS].

Definition 2.7. A *k-step linear monad* $C^\bullet$ on $\mathbb{P}^3$ is a complex of sheaves

$$V_{2-k}\otimes\mathcal{O}(2-k)\xrightarrow{\alpha_{2-k}}V_{3-k}\otimes\mathcal{O}(3-k)\xrightarrow{\alpha_{3-k}}\cdots\xrightarrow{\alpha_{-1}}V_0\otimes\mathcal{O}\xrightarrow{\alpha_0}V_1\otimes\mathcal{O}(1)$$

such that the V_i's are finite-dimensional complex vector spaces, and such that for each $l\neq 0$ the cohomology sheaf $\mathcal{H}^l(C^\bullet)$ is zero, where $V_i\otimes\mathcal{O}(i)$ is placed in degree i. We will call

$$(\dim(V_{2-k}),\ldots,\dim(V_1))$$

the *dimension vector* of the monad.

Morphisms of monads are just morphisms of complexes of sheaves, so that k-step linear monads become a full subcategory of $\mathrm{Kom}^b(\mathrm{Coh}(\mathbb{P}^3))$, the category of bounded complexes of coherent sheaves on $\mathbb{P}^3$.

Remark 2.8. It is clear that an isomorphism $\Phi\colon C^\bullet\longrightarrow C'^\bullet$ of k-step monads induces an isomorphism of sheaves $\mathcal{H}^0(C^\bullet)\cong\mathcal{H}^0(C'^\bullet)$.

Lemma 2.9. *Let F be a torsion free sheaf on $\mathbb{P}^3$ that is trivial on p_∞. Then we have for each $i\in\{0,2,3\}$*

$$\mathrm{H}^i(F(-j))=0,\quad \textit{for } j\in\{-1,-2\},\tag{2.1}$$

$$\mathrm{H}^i(F\otimes\Omega^1_{\mathbb{P}^3})=0,\tag{2.2}$$

$$\mathrm{H}^i(F\otimes\Omega^2_{\mathbb{P}^3}(1))=0.\tag{2.3}$$

Proof. This is a particular case of a more general vanishing statement for framed sheaves on $\mathbb{P}^n$ that we prove in Theorem 2.35 in Appendix 2.B. $\square$

Theorem 2.10. *Let F be a torsion free sheaf trivial on p_∞ with Chern character $(r,0,0,-n)$. Then F is the cohomology of a 4-step linear monad*

$$M_F:\quad V_{-2}\otimes\mathcal{O}(-2)\xrightarrow{\alpha_{-2}}V_{-1}\otimes\mathcal{O}(-1)\xrightarrow{\alpha_{-1}}V_0\otimes\mathcal{O}\xrightarrow{\alpha_0}V_1\otimes\mathcal{O}(1)$$

with dimension vector $(n,3n,3n+r,n)$.

Proof. Applying Beilinson's theorem (Theorem 1.24) to the twist $F(-1)$ we obtain a spectral sequence E_k^{pq} with E_1-sheet

$$\mathrm{E}_1^{pq} = \mathrm{H}^q(F \otimes \Omega_{\mathbb{P}^3}^{-p}(-p-1)) \otimes \mathcal{O}_{\mathbb{P}^3}(p)$$

converging to a graded object $\oplus \mathrm{E}_\infty^{-pp}$ associated to a filtration of $F(-1)$. From Lemma 2.9 we see that the only non-trivial part of the E_1 sheet is the line $\mathrm{E}_1^{\bullet 1}$. Hence the spectral sequence degenerates at the E_2 term, the complex $\mathrm{E}_1^{\bullet 1}$ has non-trivial cohomology just in degree -1, and $\mathcal{H}^{-1}(\mathrm{E}_1^{\cdot 1})$ is the trivial filtration of $F(-1)$. If we define

$$V_i := \mathrm{H}^1(F \otimes \Omega_{\mathbb{P}^3}^{1-i}(-i))$$

and we tensor the line $\mathrm{E}_1^{\bullet 1}$ by $\mathcal{O}_{\mathbb{P}^3}(1)$, we obtain a 4-step linear monad whose cohomology is F. By Lemma 2.9, the Riemann-Roch formula reads

$$\dim V_i = \left[\mathrm{ch}\left(F \otimes \Omega_{\mathbb{P}^3}^{1-i}(-i)\right) \mathrm{Td}(\mathbb{P}^3)\right]_3,$$

so that the verification of the dimensions of the V_i's follows easily, using the expression

$$\mathrm{Td}(\mathbb{P}^3) = 1 + 2h + \frac{11}{6}h^2 + h^3.$$

□

Corollary 2.11. *Suppose that F and F' are isomorphic torsion free sheaves on $\mathbb{P}^3$, both trivial on p_∞. Then the two corresponding 4-step linear monads M_F and $M_{F'}$ are isomorphic.*

Proof. The isomorphism $\phi\colon F \cong F'$ induces isomorphisms $\phi_i\colon V_i \cong V_i'$. The collection $\phi_i \otimes 1$ is then an isomorphism of monads. □

The description of Theorem 2.10 is not optimal. We show that up to isomorphism the monad can be put into a certain standard form. In particular, as $\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}(1))$ is isomorphic to $\oplus \mathbb{C} z_i$ we have

$$\alpha_{-2} = \sum_{i=0}^{3} a_i z_i, \ \ \alpha_{-1} = \sum_{i=0}^{3} b_i z_i, \ \ \text{and } \alpha_0 = \sum_{i=0}^{3} c_i z_i$$

where a_i, b_i and c_i are linear maps. With this notation the relations

$$b_i a_j + b_j a_i = 0, \ \ c_i b_j + c_j b_i = 0 \tag{2.4}$$

are satisfied for all i and j in $\{0, \ldots, 3\}$.

Fix a framed sheaf (F, ϕ) and its corresponding monad M_F. We denote by α_i^∞ the restriction $\alpha_i|_{p_\infty}$.

Lemma 2.12. *We have a decomposition* $V_0 = U \oplus W$*, where* W *is canonically identified with* $\mathrm{H}^0(F|_{p_\infty})$*, such that*

$$(c_1\ c_2\ c_3)^t \colon U \oplus \{0\} \longrightarrow V_1^{\oplus 3}$$

is an isomorphism.

Proof. Restricting the monad to p_∞, we get three exact sequences

$$0 \longrightarrow V_{-2} \otimes \mathcal{O}_{p_\infty}(-2) \xrightarrow{\alpha_{-2}^\infty} V_{-1} \otimes \mathcal{O}_{p_\infty}(-1) \longrightarrow \mathrm{Im}(\alpha_{-1}^\infty) \longrightarrow 0, \tag{2.5}$$

$$0 \longrightarrow \mathrm{Ker}(\alpha_0^\infty) \longrightarrow V_0 \otimes \mathcal{O}_{p_\infty} \xrightarrow{\alpha_0^\infty} V_1 \otimes \mathcal{O}_{p_\infty}(1) \longrightarrow 0, \tag{2.6}$$

$$0 \longrightarrow \mathrm{Im}(\alpha_{-1}^\infty) \longrightarrow \mathrm{Ker}(\alpha_0^\infty) \longrightarrow F|_{p_\infty} \longrightarrow 0. \tag{2.7}$$

From (2.5) we see that $\mathrm{H}^0(\mathrm{Im}(\alpha_{-1}^\infty)) = \mathrm{H}^1(\mathrm{Im}(\alpha_{-1}^\infty)) = 0$, so that we have an isomorphism $\mathrm{H}^0(\mathrm{Ker}(\alpha_0^\infty)) \cong \mathrm{H}^0(F|_{p_\infty})$. As F is trivial along p_∞, we have $\mathrm{H}^1(\mathrm{Ker}(\alpha_0^\infty)) = 0$. We thus obtain an exact sequence of vector spaces

$$0 \longrightarrow \mathrm{H}^0(\mathrm{Ker}(\alpha_0^\infty)) \longrightarrow V_0 \otimes \mathrm{H}^0(\mathcal{O}) \xrightarrow{\alpha_0^\infty} V_1 \otimes \mathrm{H}^0(\mathcal{O}(1)) \longrightarrow 0\,.$$

As $\mathrm{H}^0(\mathcal{O}(1))$ is isomorphic to $\oplus_{i=1}^3 \mathbb{C} z_i$, we can identify $V_1 \otimes \mathrm{H}^0(\mathcal{O}(1))$ with $V_1^{\oplus 3}$ and write α_0^∞ as $(c_1\ c_2\ c_3)^t$ where c_i is in $\mathrm{Hom}(V_0, V_1)$. In particular $(c_1\ c_2\ c_3)^t$ is surjective with kernel $W = \mathrm{H}^0(\mathrm{Ker}(\alpha_0^\infty))$. □

Lemma 2.13. *The morphism* $c_3 b_2 a_1$ *identifies* V_{-2} *and* V_1*. Under this identification the morphism*

$$(a_1, a_2, a_3) \colon V_1^{\oplus 3} \longrightarrow V_{-1}$$

is an isomorphism.

Proof. From Remark 1.25 we have

$$c_3 b_2 a_1 = \mathrm{H}^1\left(\mathrm{Id} \otimes \left(\frac{\partial}{\partial w_3} \wedge \frac{\partial}{\partial w_2} \wedge \frac{\partial}{\partial w_1}\right)\right) \colon\ \mathrm{H}^1(\mathbb{P}^3, F \otimes \Omega_{\mathbb{P}^3}(2)) \longrightarrow \mathrm{H}^1(\mathbb{P}^n, F(-1)). \tag{2.8}$$

Let ξ be the isomorphism between $\mathcal{O}(-2)$ and $\Omega_{\mathbb{P}^3}(2)$, and notice that we have

$$dy_i \left(\frac{\partial}{\partial w_i}\right) = \left(\frac{1}{w_0} dw_i - \frac{w_i}{w_0^2} dw_0\right)\left(\frac{\partial}{\partial w_i}\right) = \frac{1}{w_0}. \tag{2.9}$$

We get a commutative diagram

$$\begin{tikzcd}
F(-2) \arrow[d, "\xi"] \arrow[dr, "w_0"] & \\
F \otimes \Omega^3(2) \arrow[r, "\mathrm{Id} \otimes e\left(\frac{\partial}{\partial w_3} \wedge \frac{\partial}{\partial w_2} \wedge \frac{\partial}{\partial w_1}\right)"] & F(-1).
\end{tikzcd} \tag{2.10}$$

From the fact that F is isomorphic to $\mathcal{O}_{p_\infty}^{\oplus r}$ along p_∞, and thus in particular locally free in a neighborhood of p_∞, we have an exact sequence

$$0 \longrightarrow F(-2) \xrightarrow{w_0} F(-1) \longrightarrow \mathcal{O}_{p_\infty}(-1)^{\oplus r} \longrightarrow 0,$$

and from the vanishing of $\mathrm{H}^i(p_\infty, \mathcal{O}_{p_\infty}(-1))$ we immediately see that the map induced on the first cohomology groups is an isomorphism, thus proving that $c_3 b_2 a_1$ is an isomorphism.
Now suppose that $(v_1, v_2, v_3)^t$ is in the kernel of $(a_1\ a_2\ a_3)$. In particular $c_l b_k(\sum_i a_i v_i) = 0$ for all choices of l and k. From the relations (2.4) this is equivalent to asking that for each choice σ in Σ_3 we have

$$c_{\sigma(3)} b_{\sigma(2)} a_{\sigma(1)}(v_{\sigma(1)}) = 0.$$

This implies that $v_i = 0$ for all $i \in \{1, 2, 3\}$. □

From Lemma 2.12 we have that W is naturally identified with $\mathrm{H}^0(F|_{p_\infty})$. On the other hand, the choice of a framing for F induces a fixed isomorphism between $\mathrm{H}^0(F|_{p_\infty})$ and $\mathrm{H}^0(\mathcal{O}_{p_\infty}^{\oplus r})$. Thus the choice of a framing is equivalent to the choice of a basis for W. From now on, when we deal with this situation we write $\mathbb{C}^r$ in place of W to stress the fact that we are not considering the action of GL_r on the vector space W.

Theorem 2.14. *The monad of a framed sheaf (F, ϕ) is isomorphic to*

$$V_1 \otimes \mathcal{O}(-2) \xrightarrow{\alpha_{-2}} V_1^{\oplus 3} \otimes \mathcal{O}(-1) \xrightarrow{\alpha_{-1}} (V_1^{\oplus 3} \oplus \mathbb{C}^r) \otimes \mathcal{O} \xrightarrow{\alpha_0} V_1 \otimes \mathcal{O}(1) \tag{2.11}$$

with morphisms

$$\alpha_{-2} = \begin{pmatrix} A_1 z_0 + \mathrm{Id}\, z_1 \\ A_2 z_0 + \mathrm{Id}\, z_2 \\ A_3 z_0 + \mathrm{Id}\, z_3 \end{pmatrix}, \quad \alpha_{-1} = \begin{pmatrix} 0 & A_3 z_0 + \mathrm{Id}\, z_3 & -A_2 z_0 - \mathrm{Id}\, z_2 \\ -A_3 z_0 - \mathrm{Id}\, z_3 & 0 & A_1 z_0 + \mathrm{Id}\, z_1 \\ A_2 z_0 + \mathrm{Id}\, z_2 & -A_1 z_0 - \mathrm{Id}\, z_1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\alpha_0 = \begin{pmatrix} A_1 z_0 + \mathrm{Id}\, z_1 & A_2 z_0 + \mathrm{Id}\, z_2 & A_3 z_0 + \mathrm{Id}\, z_3 & I z_0 \end{pmatrix}. \tag{2.12}$$

In particular the linear maps A_i satisfy the commutation relation

$$[A_i, A_j] = 0, \quad \textit{for } i, j \textit{ in } I_3. \tag{2.13}$$

Proof. We use Lemmas 2.12 and 2.13 to identify V_{-2} with V_1, V_{-1} with $V_1^{\oplus 3}$ and V_0 with $V_1^{\oplus 3} \oplus \mathbb{C}^r$. Then it follows from the relations (2.4) that we can assume that

$$a_1 = \begin{pmatrix} \mathrm{Id}_V \\ 0 \\ 0 \end{pmatrix}, \; a_2 = \begin{pmatrix} 0 \\ \mathrm{Id}_V \\ 0 \end{pmatrix}, \; a_3 = \begin{pmatrix} 0 \\ 0 \\ \mathrm{Id}_V \end{pmatrix},$$

$$b_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id}_V \\ 0 & -\mathrm{Id}_V & 0 \\ 0 & J & K \end{pmatrix}, \; b_2 = \begin{pmatrix} 0 & 0 & -\mathrm{Id}_V \\ 0 & 0 & 0 \\ \mathrm{Id}_V & 0 & 0 \\ -J & 0 & L \end{pmatrix},$$

$$b_3 = \begin{pmatrix} 0 & \mathrm{Id}_V & 0 \\ -\mathrm{Id}_V & 0 & 0 \\ 0 & 0 & 0 \\ -K & -L & 0 \end{pmatrix},$$

$$c_1^t = \begin{pmatrix} \mathrm{Id}_V \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; c_2^t = \begin{pmatrix} 0 \\ \mathrm{Id}_V \\ 0 \\ 0 \end{pmatrix}, \; c_3^t = \begin{pmatrix} 0 \\ 0 \\ \mathrm{Id}_V \\ 0 \end{pmatrix}.$$

Furthermore, up to composing with the automorphism $(\mathrm{Id} \otimes 1, G \otimes 1, \mathrm{Id} \otimes 1)$, where G is the matrix

$$\begin{pmatrix} \mathrm{Id}_V & 0 & 0 & 0 \\ 0 & \mathrm{Id}_V & 0 & 0 \\ 0 & 0 & \mathrm{Id}_V & 0 \\ L & -K & J & Id_W \end{pmatrix},$$

we can assume that L, K and J are the zero matrices. Imposing the relations (2.4), we get a monad in standard form (2.12) and satisfying formula (2.13). □

Remark 2.15. Note that given a framed sheaf (F, ϕ), the datum of a representation $(\{A_i\}_{i \in I_3}, I)$ is determined only up to isomorphism of the monad. In other words, considering the action of GL_n given by

$$g \cdot (\{A_i\}, I) = \left(\{g A_i g^{-1}\}, gI\right),$$

we see that representations in the same GL_n-orbit define isomorphic framed sheaves.

Proposition 2.16. *Let A_1, A_2 and A_3 be endomorphisms of V_1 such that $[A_i, A_j] = 0$ and $I \colon \mathbb{C}^r \longrightarrow V_1$. The datum $(\{A_i\}, I)$ defines a monad as in Theorem 2.14 if and only if the following stability condition is satisfied: the only non-trivial subspace of V_1 that is A_i-invariant and that contains $\mathrm{Im}(I)$ is V_1 itself.*

Proof. Using the datum $(\{A_i\}, I)$ we can construct a complex as in Theorem 2.14. Let us begin by showing that α_{-2} is injective. It is clear that $\alpha|_{p_\infty}$ is injective as a morphism of vector bundles. Consider then a point $q = (1, z_1, z_2, z_3)$. As a morphism of vector bundles α_{-2} is not injective on the fiber $(V_1 \otimes \mathcal{O}(-2))_q$ if and only if there is a vector v in V such that $A_i v = z_i v$ for all i in I_3. As the A_i's have only a finite number of common eigenvalues we have that α_2 is not injective on the fibers only for a finite number of points in $\mathbb{P}^3$, hence it is injective as a morphism of sheaves.
We are left with proving the equivalence between the surjectivity of α_0 and the stability condition being satisfied. The surjectivity of α_0 on each fiber over p_∞ is automatic. So α_0 is a surjective morphism of sheaves if and only if $\eta_{\underline{z}} = (A_1 - z_1, A_2 - z_2, A_3 - z_3, I)$ is surjective for every (z_1, z_2, z_3), that is if and only if the transpose $\eta^t_{\underline{z}}$ is injective for all $\underline{z}$. Suppose there exists $\underline{z}$ such that $\eta^t_{\underline{z}}$ is not injective. Then there is $\phi \neq 0$ in $\mathrm{Ker}(\eta^t_{\underline{z}})$ and so $S = \mathrm{Ker}(\phi)$ contains $\mathrm{Im}(I)$ and is A_i-invariant, since $\phi(A_i v - z_i v) = 0$ for every choice of v in V and i in I_3. Conversely, suppose that a subspace S as in the statement exists. As the A_i's commute we have $\phi \neq 0$ in $S^\perp = \{\phi\colon \ \phi|_S = 0\}$ such that ϕ is a common eigenvector for the A^t_i with respective eigenvalues z_i, but then $\eta_{\underline{z}}$ is not surjective. □

Let (Q, W) be the three-loop quiver with the superpotential W defined in Example 1.5. As in Section 1.1, let $M^\theta_{n,r}(Q, W)$ be the fine moduli space parametrising flat families of θ-stable r-framed representations of Q for $\theta < 0$.

Theorem 2.17. *There is a bijection between the set of isomorphism classes of framed sheaves on $\mathbb{P}^3$ with Chern character $(r, 0, 0, -n)$ and the set of geometric points of the scheme $M^\theta_{n,r}(Q, W)$.*

Proof. By Theorem 2.14 we can associate to a framed sheaf (F, ϕ) an equivalence class of r-framed representations of Q satisfying the relations given by the superpotential W. Each representative is θ-stable by Proposition 2.16, so that we get a point of $M^\theta_{n,r}(Q, W)$. Conversely, starting from a r-framed θ-stable representation of (Q, W) we can construct a complex as in (2.12); the stability condition for the representation is equivalent to the requirement that the complex is a 4-step linear monad. The cohomology sheaf F has the correct Chern character by construction, and it comes equipped with a framing obtained by restricting the complex to the plane p_∞. The sheaf F thus obtained is torsion free as it is locally free along p_∞; furthermore, restricting to $\mathbb{P}^3 \setminus p_\infty$

$$\begin{tikzcd} V_1^{\oplus 3} \otimes \mathcal{O}_{\mathbb{C}^3} \arrow[r, "a_{-1}"] & \begin{matrix} V_1 \otimes \mathcal{O}_{\mathbb{C}^3} \\ \oplus \\ \mathbb{C}^r \otimes \mathcal{O}_{\mathbb{C}^3} \end{matrix} \arrow[r, "a_0"] \arrow[d, "j"] & V_1 \otimes \mathcal{O}_{\mathbb{C}^3} \\ & \mathbb{C}^r \otimes \mathcal{O}_{\mathbb{C}^3} & \end{tikzcd}$$

gives an inclusion of $F|_{\mathbb{C}^3}$ in $\mathbb{C}^r \otimes \mathcal{O}_{\mathbb{C}^3}$. It follows from Remark 2.8 and Corollary 2.11 that the construction we have exhibited is a bijection. □

2.1.3 Representability of $\mathscr{M}_{n,r}$

We use the relative version of Beilinson's Theorem (Theorem 1.31) to prove that the bijective correspondence obtained in Theorem (2.17) induces an equivalence of moduli functors.

Let S be a Noetherian scheme of finite type over $\mathbb{C}$ and consider the diagram

$$\begin{tikzcd} \mathbb{P}^3_S \arrow[r, "q"] \arrow[d, "p"] & \mathbb{P}^3 \arrow[d] \\ S \arrow[r] & \mathrm{pt} \end{tikzcd} .$$

We denote by $\mathcal{O}_p(1)$ and Ω^l_p respectively the relative twisting sheaf of $\mathbb{P}^3_S$ over S and the relative sheaf of l-forms of $\mathbb{P}^3_S$ over S. We will shorten $\Omega^l_p \otimes \mathcal{O}_p(k)$ to $\Omega^l_p(k)$. Given a sheaf F_1 on S and a sheaf F_2 on $\mathbb{P}^3$ we write $F_1 \boxtimes F_2$ for the tensor product $p^*F_1 \otimes q^*F_2$.

It will be useful to define a sheafified version of the definition of a linear monad.

Definition 2.18. A *k-step sheafified linear monad* is a complex of sheaves $\mathcal{C}^\bullet$ on $\mathbb{P}^3_S$ of the form

$$\mathcal{V}_{2-k} \boxtimes \mathcal{O}_{\mathbb{P}^3}(2-k) \longrightarrow \mathcal{V}_{3-k} \boxtimes \mathcal{O}_{\mathbb{P}^3}(3-k) \longrightarrow \cdots \longrightarrow \mathcal{V}_0 \boxtimes \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{V}_1 \boxtimes \mathcal{O}_{\mathbb{P}^3}(1) ,$$

where the $\mathcal{V}_j$'s are vector bundles on S, and such that for each $l \neq 0$ the cohomology sheaf $\mathcal{H}^l(\mathcal{C}^\bullet)$ is zero. The *dimension vector* of a sheafified monad is the collection of the ranks of the vector bundles $\mathcal{V}_i$.

We will use a relative version of Beilinson's theorem to realize families of framed sheaves on $\mathbb{P}^3$ as cohomologies of sheafified monads.

Theorem 2.19 ([OSS, Theorem 4.1.11]). *Let $\mathcal{F}$ be a torsion free sheaf on $\mathbb{P}^3_S$ flat over S. Then there is a spectral sequence E_r^{pq} of sheaves on $\mathbb{P}^3_S$ with E_1-term given by*

$$\mathrm{E}_1^{lk} = R^k p_*(\mathcal{F} \otimes \Omega_p^{-l}(-l)) \boxtimes \mathcal{O}_{\mathbb{P}^n}(l)$$

that converges to the associated graded $\bigoplus_l E_\infty^{-ll}$ of a filtration of $\mathcal{F}$.

We want $\mathcal{F}$ to be a flat family of framed sheaves on $\mathbb{P}^3$ parametrized by S. From the vanishings of Lemma 2.9 we get the following.

Theorem 2.20. *Let $\mathcal{F}$ be a family of framed sheaves on $\mathbb{P}^3$ with Chern character $(r, 0, 0, -n)$ over a base scheme S. Then $\mathcal{F}$ is the cohomology of a 4-step sheafified linear monad with dimension vector $(n, 3n, 3n + r, n)$.*

Proof. We apply Theorem 2.19 to $\mathcal{F} \otimes \mathcal{O}_q(-1)$. Since $\mathcal{F}$ is flat, for each point s we have

$$R^k p_*(\mathcal{F} \otimes \Omega_p^{-l}(-l-1))_s \cong \mathrm{H}^k(\mathbb{P}^3, \mathcal{F}_s \otimes \Omega_{\mathbb{P}^3}^{-l}(-l-1)).$$

We immediately see that the spectral sequence degenerates at the E_1-term, and hence $\mathcal{F}$ has cohomology only in degree 0. The computation of the rank of the monad is then just an application of the Riemann-Roch theorem as in Theorem 2.10. $\square$

This is all we need to prove the equivalence of functors.

Theorem 2.21. *The moduli functor $\mathscr{M}_{n,r}$ is representable, and it is isomorphic to $\mathcal{M}_{n,r} := M^\theta_{n,r}(Q, W)$ for $\theta < 0$.*

Proof. Given an affine scheme S and an element $\mathcal{F}$ of $\mathscr{M}_{n,r}(S)$ we obtain a 4-step sheafified linear monad $\mathcal{C}^\bullet_{\mathcal{F}}$. Restricting the monad to $S \times p_\infty$ we see that $\mathcal{C}^\bullet_{\mathcal{F}}$ is isomorphic to

$$(V \otimes \mathcal{O}_S) \boxtimes \mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{\alpha_{-2}} (V^{\oplus 3} \otimes \mathcal{O}_S) \boxtimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha_{-1}}$$

$$\xrightarrow{\alpha_{-1}} ((V^{\oplus 3} \oplus \mathbb{C}^r) \otimes \mathcal{O}_S) \boxtimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\alpha_0} (V \otimes \mathcal{O}_S) \boxtimes \mathcal{O}_{\mathbb{P}^3}(1)$$

where V is a n-dimensional vector space and where $\alpha_i|_{p_\infty}$ are defined by

$$\alpha_{-2}|_{p_\infty} = \sum_{i=1}^{3} (a_i \times 1_S) \boxtimes z_i, \quad \alpha_{-1}|_{p_\infty} = \sum_{i=1}^{3} (b_i \times 1_S) \boxtimes z_i, \quad \alpha_{-2}|_{p_\infty} = \sum_{i=1}^{3} (c_i \times 1_S) \boxtimes z_i$$

with a_i, b_i and c_i are as in the proof of Theorem 2.14. Imposing the conditions of a monad on the coordinate z_0 we obtain a flat family of r-framed θ-stable representations of Q of dimension n satisfying the relations in W. Conversely, let us consider a flat family of r-framed θ-stable representations of (Q, W) on an affine scheme S. We can construct a 4-step sheafified linear monad using a trivialization $(\{A_i(s)\}, I(s))$ of the family on S in the structural expressions of equation (2.12). This gives a family $\mathcal{F}$ of torsion free sheaves over S with the required Chern character, and the restriction of the monad to $S \times p_\infty$ defines a framing for $\mathcal{F}$. By Theorem 2.17, this defines an isomorphism of functors. □

2.2 The algebraic picture: invariants

Combining Theorem 2.21 and Theorem 1.70 we immediately obtain the following result.

Theorem 2.22. *The moduli scheme $\mathcal{M}_{n,r}$ is quasi-projective and arises as the global critical locus of a regular function on a smooth variety.*

Let us modify the notation from Section 1.1, and denote by $\omega_{n,r}$ the trace of the superpotential W defining the regular morphism on $M^\theta_{n,r}(Q)$ which cuts out the quasi-projective variety $M^\theta_{n,r}(Q, W)$.

We have then

$$[\mathcal{M}_{n,r}]_{vir} = -(-\mathbb{L}^{\frac{1}{2}})^{2n^2+rn}[\phi_{\omega_{n,r}}],$$

and by Lemma 1.76

$$[\phi_{\omega_{n,r}}] = [\omega^{-1}_{n,r}(1)] - [\omega^{-1}_{n,r}(0)].$$

Remark 2.23. In [BJM] Bussi, Joyce and Meinhardt explained that the motivic invariant of a moduli problem depends on considering it as a derived scheme together with a (-1)-shifted symplectic structure with orientation. To claim that our procedure is actually computing the motivic invariant of $\mathcal{M}_{n,r}$, we should construct on both $\mathcal{M}_{n,r}$ and $M^\theta_{n,r}(Q, W)$ a (-1)-shifted symplectic structure and prove that the scheme isomorphism we constructed can be promoted to an isomorphism preserving this structure. As far as the author is aware, this is not yet understood even for the case of the Hilbert scheme of points of $\mathbb{C}^3$, when $r = 1$. Nevertheless, the isomorphism we constructed defines naturally a coefficient system $\Phi_{n,r} := \phi_{\omega_{n,r}}\mathbb{Q}[m](\frac{m}{2})$ on $\mathcal{M}_{n,r}$, where $m = 2n^2 + rn$. The weight specialization of $[\mathcal{M}_{n,r}]_{\mathrm{vir}}$ computes the weight polynomial of $\mathcal{M}_{n,r}$ with respect to this coefficient system.

2.2.1 Direct computation of the generating series

We use the strategy of Behrend, Bryan and Szendrői [BBS] and Morrison [Mo1] to compute the generating series

$$\mathrm{DT}_{\mathbb{C}^3,r}(q) = 1 + \sum_{n\geq 1}[\mathcal{M}_{n,r}]_{vir}q^n$$

of rank r motivic degree zero DT invariants for $\mathbb{C}^3$ by relating it to the generating series of the class of the commuting variety. This is a very explicit version of the technique of dimensional reduction which we will use in Chapter 4. In this case, we use the fact that a_1 is a cut for (Q_r, W) to reduce the computation to a computation involving just the quiver obtained from Q_r removing the arrow a_1. The weight specialization gives the generating series of refined invariants.

We consider the commuting variety

$$C_n = \{(Y, Z) \in \mathrm{End}(\mathbb{C}^n)^{\oplus 2} \colon [X, Y] = 0\}, \tag{2.14}$$

and we denote by c_n its class in $\mathrm{K}(\mathrm{Var}_{\mathbb{C}})$. The quotient $C_n/\,\mathrm{GL}_n$ by the action of GL_n by change of basis gives the stack $\mathrm{Coh}_n(\mathbb{C}^2)$ of coherent sheaves on $\mathbb{C}^2$ supported on points and of length n. One has

$$[C_n/\,\mathrm{GL}_n] = \frac{c_n}{[\mathrm{GL}_n]}$$

where the classes can be considered in the ring $\widetilde{\mathcal{M}}_{\mathbb{C}}$ by Remark 1.62.

It was proven in [BM, Theorem 1'] that the generating series of the classes $[\mathrm{Coh}_n(\mathbb{C}^2)]$ admits a product formula

$$1 + \sum_{n\geq 1} \frac{c_n}{[\mathrm{GL}(n)]}q^n = \prod_{m=1}^{\infty}\prod_{k=1}^{\infty}(1 - \mathbb{L}^{2-k}q^m)^{-1}, \tag{2.15}$$

where both sides of the equation lie in $\widetilde{\mathcal{M}}_{\mathbb{C}}[[q]]$.

We recall the construction of $\mathcal{M}_{n,r}$ in order to fix some notation that will be useful in the following part of the section. Consider the affine space

$$V_{n,r} = \mathrm{End}(\mathbb{C}^n)^{\oplus 3} \times (\mathbb{C}^n)^{\oplus r}.$$

We have a GL_n-invariant function

$$\widetilde{\omega}_{n,r}\colon ((X,Y,Z,\{v_i\})) \longmapsto \mathrm{Tr}(X[Y,Z])$$

defined on $V_{n,r}$. Given a set of r vectors $\{v_i\}$ we will denote by $\mathrm{span}\langle X,Y,Z\rangle\{v_i\}$ the vector space generated by the images of the v_i's under all the possible monomials in X, Y and Z. For $j \in \{1,\dots,n\}$ we define

$$U_{n,r,j} = \{((X,Y,Z),\{v_i\}) \in V_{n,r}\colon \dim_{\mathbb{C}} \mathrm{span}\langle X,Y,Z\rangle\{v_i\} = j\}.$$

The restriction of $\widetilde{\omega}_{n,r}$ to $U_{n,r,j}$ will be denoted by $\widetilde{\omega}_{n,r,j}$. The variety $M^{\theta}_{n,r}(Q)$ is just the quotient of $U_{n,r,n}$ by the usual free action of GL_n on framed representations. The intersection of the critical locus of $\widetilde{\omega}_{n,r}$ with $U_{n,r,n}$ is a GL_n-invariant subscheme whose quotient by GL_n is $\mathcal{M}_{n,r}$.

Theorem 2.24. *The generating series* $\mathrm{DT}_{\mathbb{C}^3,r}$ *of rank r motivic DT invariants of* $\mathcal{M}_{n,r}$ *is given by*

$$\prod_{m=1}^{\infty}\prod_{k=0}^{rm-1}\left(1-\mathbb{L}^{2+k-\frac{rm}{2}}((-1)^r q)^m\right)^{-1}.$$

Proof. We begin by proving the equality

$$\mathbb{L}^{n^2+nr}c_n = [\widetilde{\omega}_{n,r}^{-1}(0)] - [\widetilde{\omega}_{n,r}^{-1}(1)]. \tag{2.16}$$

To start with, notice that we have an isomorphism

$$V_{n,r}\setminus \widetilde{\omega}_{n,r}^{-1}(0) \longrightarrow \mathbb{C}^* \times \widetilde{\omega}_{n,r}^{-1}(1),$$

defined by

$$V=(X,Y,Z,\underline{v}) \longmapsto (\widetilde{\omega}_{n,r}(V), \widetilde{\omega}_{n,r}(V)^{-1}X, Y, Z, \underline{v}).$$

From this it follows directly that

$$\mathbb{L}^{3n^2+nr} - \mathbb{L}[\widetilde{\omega}_{n,r}^{-1}(0)] = (1-\mathbb{L})([\widetilde{\omega}_{n,r}^{-1}(0)] - [\widetilde{\omega}_{n,r}^{-1}(1)]). \tag{2.17}$$

The variety $\mathrm{End}(\mathbb{C}^n)\times C_n \times (\mathbb{C}^n)^{\oplus r}$ is contained in $\widetilde{\omega}_{n,r}^{-1}(0)$. Its complement is an affine bundle in the Zariski topology with base $\mathrm{End}(\mathbb{C}^n)^2\setminus C_n$ and fibers of dimension n^2+nr-1, as each fiber is cut out by a single linear equation in an affine space. We then have

$$[\widetilde{\omega}_{n,r}^{-1}(0)] = \mathbb{L}^{nr}\left(\mathbb{L}^{n^2}c_n + \mathbb{L}^{n^2-1}\left(\mathbb{L}^{2n^2}-c_n\right)\right).$$

Substituting in (2.17) we get (2.16).

We now need to impose the stability condition. The affine space $V_{n,r}$ admits a stratification indexed by the dimension of the subspace generated by the framing, so that we have $[V_{n,r}] = \sum [V_{n,r,j}]$. In particular,

$$[\widetilde{\omega}_{n,r}^{-1}(0)] - [\widetilde{\omega}_{n,r}^{-1}(1)] = \sum_j ([\widetilde{\omega}_{n,r,j}^{-1}(0)] - [\widetilde{\omega}_{n,r,j}^{-1}(1)]). \tag{2.18}$$

For a given j we have a Zariski locally trivial fibration

$$\widetilde{\omega}_{n,r,j}^{-1}(1) \longrightarrow \mathrm{Gr}(j,n)$$

sending $(X,Y,Z,\{v_i\})$ to $\mathrm{span}\langle X,Y,Z\rangle\{v_i\}$. We compute the class of the fiber fixing a suitable basis. We can assume that the matrices X, Y and Z and the vectors v_i are decomposed into blocks of the form

$$\begin{pmatrix} X_1 & X_* \\ 0 & X_2 \end{pmatrix}, \begin{pmatrix} Y_1 & Y_* \\ 0 & Y_2 \end{pmatrix}, \begin{pmatrix} Z_1 & Z_* \\ 0 & Z_2 \end{pmatrix}, \begin{pmatrix} v_{i1} \\ 0 \end{pmatrix}.$$

The condition $\mathrm{Tr}(X[Y,Z]) = 1$ then becomes

$$\mathrm{Tr}(X_1[Y_1,Z_1]) + \mathrm{Tr}(X_2[Y_2,Z_2]) = 1.$$

We distinguish three cases. If $\mathrm{Tr}(X_1[Y_1,Z_1]) = 1$ then we have that the class of the fiber is

$$[\widetilde{\omega}_{j,r,j}^{-1}(1)][\widetilde{\omega}_{n-j,r}^{-1}(0)]\mathbb{L}^{(3j-r)(n-j)}.$$

If $\mathrm{Tr}(X_1[Y_1,Z_1]) = 0$ the class of the fiber is

$$[\widetilde{\omega}_{j,r,j}^{-1}(0)][\widetilde{\omega}_{n-j,r}^{-1}(1)]\mathbb{L}^{(3j-r)(n-j)}.$$

In the remaining case, $\mathrm{Tr}(X_1[Y_1,Z_1]) \neq 0,1$, we get

$$(\mathbb{L}-2)[\widetilde{\omega}_{j,r,j}^{-1}(1)][\widetilde{\omega}_{n-j,r}^{-1}(1)]\mathbb{L}^{(3j-r)(n-j)}.$$

Collecting the results we obtain

$$\begin{aligned}[\widetilde{\omega}_{n,r,j}^{-1}(1)] = \big([\widetilde{\omega}_{j,r,j}^{-1}(0)][[\widetilde{\omega}_{n-j,r}^{-1}(1)] + [\widetilde{\omega}_{j,r,j}^{-1}(1)][\widetilde{\omega}_{n-j,r}^{-1}(0)] + \\ +(\mathbb{L}-2)[\widetilde{\omega}_{j,r,j}^{-1}(1)][\widetilde{\omega}_{n-j,r}^{-1}(1)]\big)\,[\mathrm{Gr}(j,n)]\mathbb{L}^{(3j-r)(n-j)}.\end{aligned}$$

Reasoning in the same way, we get the expression

$$\big([\widetilde{\omega}_{j,r,j}^{-1}(0)][\widetilde{\omega}_{n-j,r}^{-1}(0)] + (\mathbb{L}-1)[\widetilde{\omega}_{j,r,j}^{-1}(1)][\widetilde{\omega}_{n-j,r}^{-1}(1)]\big)\,[\mathrm{Gr}(j,n)]\mathbb{L}^{(3j-r)(n-j)}$$

for the class $[\widetilde{\omega}_{n,r,j}^{-1}(0)]$. Taking the difference of the two classes and substituting (2.16) we have

$$[\widetilde{\omega}_{n,r,j}^{-1}(0)] - [\widetilde{\omega}_{n,r,j}^{-1}(1)] = c_{n-j}\left([\widetilde{\omega}_{j,r,j}^{-1}(0)] - [\widetilde{\omega}_{j,r,j}^{-1}(1)]\right)[\mathrm{Gr}(j,n)]\mathbb{L}^{(n+2j)(n-j)}.$$

Now recall that the class of the Grassmannian $\mathrm{Gr}(j,n)$ is

$$[\mathrm{Gr}(j,n)] = \frac{[\mathrm{GL}_n]}{[\mathrm{GL}_k][\mathrm{GL}_{n-k}]\mathbb{L}^{j(n-j)}}.$$

Substituting the result in formula (2.18) and dividing by $[\mathrm{GL}_n]$ we have

$$\mathbb{L}^{n^2+nr}\frac{c_n}{[GL_n]} = \sum_{j=0}^{n} -\frac{c_{n-j}}{[\mathrm{GL}_{n-j}]}\frac{[\varphi_{\widetilde{\omega}_{j,r,j}}]}{[\mathrm{GL}_k]}\mathbb{L}^{n^2-j^2},$$

where $[\varphi_{\widetilde{\omega}_{j,r,j}}]$ is the motivic class of vanishing cycles of $\widetilde{\omega}_{j,r,j}$. Next we notice that

$$-\frac{[\varphi_{\widetilde{\omega}_{j,r,j}}]}{[\mathrm{GL}_j]} = (-1)^{rj}\mathbb{L}^{j^2+jr}[\mathcal{M}_{j,r}]_{\mathrm{vir}}.$$

Multiplying both sides by $\mathbb{L}^{-\frac{nr}{2}}q^n$ and summing over $n \geq 0$ we find

$$1 + \sum_{n=1}^{\infty}\frac{c_n}{[\mathrm{GL}_n]}\left(\mathbb{L}^{\frac{r}{2}}q\right)^n = 1 + \sum_{n=1}^{\infty}\sum_{j=0}^{n}\left(\frac{c_{n-j}}{[\mathrm{GL}_{n-j}]}(\mathbb{L}^{-\frac{r}{2}}q)^{n-j}[\mathcal{M}_{j,r}]_{vir}((-1)^r q)^j\right)$$

Finally, using the expression (2.15), we have

$$\begin{aligned}\mathrm{DT}_{\mathbb{C}^3,r}((-1)^r q) &= \prod_{m\geq 1}\prod_{k\geq 1}\frac{\left(1-\mathbb{L}^{2-k+\frac{rm}{2}}q^m\right)^{-1}}{\left(1-\mathbb{L}^{2-k-\frac{rm}{2}}q^m\right)^{-1}}\\ &= \prod_{m\geq 1}\prod_{k=1}^{rm}\left(1-\mathbb{L}^{2-k+\frac{rm}{2}}q^m\right)^{-1}\\ &= \prod_{m\geq 1}\prod_{k=0}^{rm-1}\left(1-\mathbb{L}^{2+k-\frac{rm}{2}}q^m\right)^{-1}.\end{aligned}$$

□

2.2.2 Motivic invariants of $\mathcal{M}_{n,r}$ via wall-crossing

In this section we provide an alternative proof of Theorem 2.24. We compute the generating series $Z_{\mathbb{C}^3}(q)$ via motivic wall-crossing following the strategy of [MMNS]. This technique consists essentially of a simplification of [Moz3] where one has just two stability chambers, giving a neat interpretation in our case of the wall-crossing phenomenon.

Recall that given a quiver Q one defines the Euler-Ringel form on $\mathbb{Z}^{V(Q)}$ by

$$\chi(\underline{n}, \underline{n}') = \sum_{i \in V(Q)} n_i n_i' - \sum_{a \in E(Q)} n_{t(a)} n'_{h(a)},$$

and its antisymmetrized version by $\langle \underline{n}, \underline{n}' \rangle = \chi(\underline{n}, \underline{n}') - \chi(\underline{n}', \underline{n})$.

Following [KS1] we define the twisted motivic quantum torus associated to a quiver Q. This is a completion of the positive part of Kontsevich-Soibelman motivic quantum torus, defined by

$$\mathcal{T}_Q = \prod_{\underline{n} \in \mathbb{N}^{V(Q)}} \widetilde{\mathcal{M}}_{\mathbb{C}} y^{\underline{n}}, \tag{2.19}$$

where the multiplication of two formal variables $y^{\underline{n}}$ and $y^{\underline{n}'}$ is given by

$$y^{\underline{n}} \cdot y^{\underline{n}'} = \left(-\mathbb{L}^{\frac{1}{2}}\right)^{\langle \underline{n}, \underline{n}' \rangle} y^{\underline{n}+\underline{n}'}.$$

Remark 2.25. If Q is symmetric, the quantum torus is commutative.

Now consider the quiver with superpotential $(\hat{Q}_r, W)$ of Section 2.1. Recall that the choice of a μ_θ-stability consists in the choice of a vector $\theta = (\theta_\infty, \theta_0)$ in $\mathbb{R}^2$. In this case we have just one wall which is given by the line l defined by the equation $\theta_\infty = \theta_0$. We denote by $l^>$ the chamber where $\theta_\infty > \theta_0$ and by $l^<$ the other open chamber. The following lemma summarizes describes μ_θ-stability of framed representations of $\hat{Q}_r$.

Lemma 2.26. *Let $V = (X, Y, Z, \{v_i\})$ be a representation of $\hat{Q}_r$ with dimension vector $\underline{v}$ satisfying $v_\infty = 1$. Varying the stability parameter we get:*

- *for $\theta \in l^>$ the representation V is semistable if and only if it is stable if and only if* $\operatorname{span}\langle X, Y, Z\rangle\{v_i\} = V_0$*;*
- *if $\theta = \underline{0}$ then V is strictly semistable;*
- *if $\theta \in l^<$ then V is semistable if and only if it is stable if and only if $V_0 = 0$.*

It is easy to describe the Harder-Narasimhan filtration of a representation with respect to the choice of stability in the various chambers, and we refer to Figure 2.1 for a pictorial representation of the behavior of the HN-filtration when crossing the wall.

Lemma 2.27. *Let $V = (X, Y, Z, \{v_i\})$ be a representation of $\hat{Q}_r$ with dimension vector $\underline{v}$ such that $v_\infty = 1$. Then:*

- *if $\theta \in l^{>}$ then the HN filtration is $0 \subset W \subset V$ where W is the representation obtained restricting X, Y and Z to $\operatorname{span}\langle X, Y, Z\rangle\{v_i\}$. In this case we will denote θ by θ^+;*

- *if $\theta = \underline{0}$ then the HN filtration of V is trivial;*

- *if $\theta \in l^{<}$ then the HN filtration is $0 \subset W \subset V$ where W is the subrepresentation based at the vertex 0. In this case we will denote θ by θ^-.*

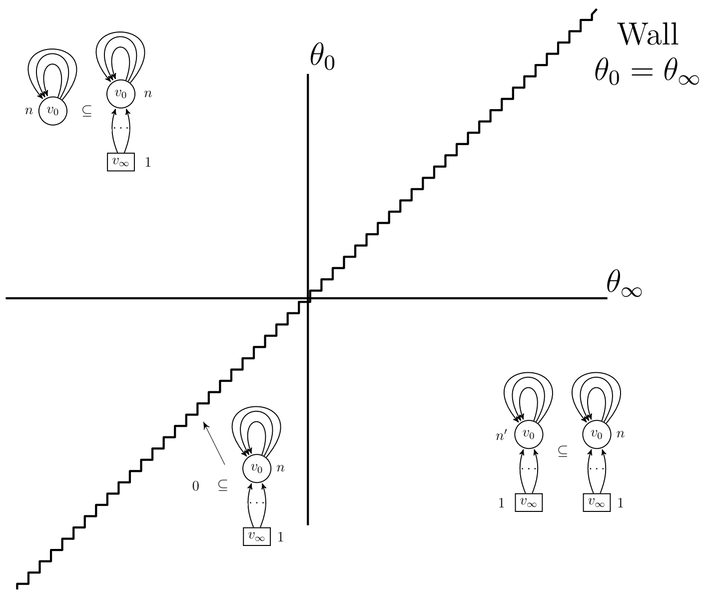

Figure 2.1: Change of HN-filtration when crossing the wall.

Given an integer n we will use the notation

$$A_n^{\pm} = [M_{n,r}^{\theta^{\pm}}(Q, W)]_{\mathrm{vir}}$$

for the motivic Donaldson-Thomas invariant of the moduli space of r-framed representations of (Q, W) with stability condition $\theta^{\pm}$. We then define

$$A_{U,n} = [M_{n,r}^{0}(Q, W)]_{\mathrm{vir}},$$

to be the virtual motive of the stack of framed representations with no stability condition imposed, and finally we define

$$B_{U,n} = [\mathcal{M}_{n}^{0}(Q, W)]_{\mathrm{vir}},$$

the universal virtual motive for unframed representations.

We define then the corresponding elements in the quantum torus:

$$A^{\pm} = \sum_n A_n^{\pm} y^{(1,n)};$$
$$A_U = \sum_n A_{U,n} y^{(1,n)};$$
$$B_U = \sum_n B_{U,n} y^{(0,n)}.$$

Remark 2.28. The definition of the product on $\mathcal{T}_{\hat{Q}_r}$ gives

$$y^{(0,n)} \cdot y^{(1,0)} = (-\mathbb{L}^{\frac{1}{2}})^{nr} y^{(1,n)}. \tag{2.20}$$

Notice also that

$$\mathcal{T}_{\hat{Q}_r} = \mathcal{T}_Q \oplus \bigoplus_d \mathcal{T}_Q \cdot y^{(d,0)},$$

and $\mathcal{T}_Q$ is a commutative subalgebra, so we can rewrite A^+ as

$$A^+ = Z_{\mathbb{C}^3,r}\left((-\mathbb{L}^{-\frac{1}{2}})^r y^{(0,1)}\right) \cdot y^{(1,0)}.$$

We combine powerful results proven by Mozgovoy in [Moz3], following in the case of quivers with a cut the original work of Kontsevich and Soibelman [KS1], with Lemma 2.27 together to get the following theorem.

Theorem 2.29. *We have the relations*

$$A_U = A^+ B_U = B_U A^- \tag{2.21}$$

in the twisted quantum torus $\mathcal{T}_{\hat{Q}_r}$.

Remark 2.30. Without going into too much detail, we give an idea of why this relation must hold. Let's concentrate on the "θ^+" equality. Given a representation V of the quiver $\hat{Q}_r$, we have a 1-step HN filtration with respect to θ^+ which is given by $0 \subset V^+ \subset V$, where V^+ is θ^+-stable. The quotient V/V^+ is an unframed representation of Q. Thus every representation of $\hat{Q}_r$ is an extension of a θ^+-stable representation by an unframed representation. This induces relations in the motivic Hall algebra which induce via the integration map a relation between the motivic DT invariants in the twisted quantum torus. We refer to [Moz3, Section 5] for a formal proof of this result.

Notice that there is only one θ^--stable representation of $\hat{Q}_r$ with framing rank 1, namely the only representation with dimension vector $(1,0)$. This immediately implies that $A^- = y^{(1,0)}$.

From Remark 2.28 and Theorem 2.29 we have:

$$Z_{\mathbb{C}^3}((-\mathbb{L}^{-\frac{1}{2}})^r y^{(0,1)}) \cdot y^{(1,0)} \cdot B_U(y^{(0,1)}) = B_U(y^{(0,1)}) \cdot y^{(1,0)},$$

and using again equation (2.20) we see that

$$Z_{\mathbb{C}^3}\left((-\mathbb{L}^{-\frac{1}{2}})^r y^{(0,1)}\right) B_U\left(\mathbb{L}^r y^{(0,1)}\right) \cdot y^{(1,0)} = B_U\left(y^{(0,1)}\right) \cdot y^{(1,0)}.$$

Writing $(-\mathbb{L}^{-\frac{1}{2}})^r y^{(0,1)} = q$, we have

$$Z_{\mathbb{C}^3}(q) = \frac{B_U\left((-\mathbb{L}^{\frac{1}{2}})^r q\right)}{B_U\left((-\mathbb{L}^{-\frac{1}{2}})^r q\right)}. \tag{2.22}$$

In this way the problem is reduced to the computation of the universal generating series B^U. Using (2.16) we can reduce the computation of B_U to the generating series for the stack of the commuting variety. We have then

$$B_U(t) = \prod_{m=1}^{\infty}\prod_{k=1}^{\infty}(1 - \mathbb{L}^{2-k} q^m)^{-1}.$$

Substituting in (2.22) we obtain the formula.

2.2.3 Combinatorial description of the invariants

The following purely combinatorial observation relates the generating series $DT_{\mathbb{C}^3,r}$ for the virtual Poincaré polynomial of the moduli spaces $\mathcal{M}_{n,r}$ to some counting of $3D$ partitions.

A *3D partition* π is the datum $\{\pi_{i,j}\}_{i,j}$ of some non-negative integers $\pi_{i,j}$ such that $\pi_{i,j} \geq \pi_{i+1,j}$, $\pi_{i,j} \geq \pi_{i,j+1}$ and such that the volume $|\pi| = \sum \pi_{i,j}$ is finite. The number $\pi_{i,j}$ can be interpreted as the number of boxes that are piled in position (i,j) in the tessellated positive quadrant, so that one can associate to a $3D$ partition some configuration of boxes in the positive octant of the 3-dimensional space. Given a partition π denote by $\omega_0(\pi)$ the number $\sum_i \pi_{i,i}$ of diagonal boxes, with $\omega_-(\pi)$ the number $\sum_{i>j} \pi_{i,j}$ of boxes below the diagonal and with $\omega_+(\pi)$ the number of boxes above the diagonal. Set

$$\widetilde{M}_r(q,t) := \prod_{m\geq 1}\prod_{k=1}^{rm}\left(1 - t^{4+rm-2k}q^m\right)^{-1},$$

so that $W(DT_{\mathbb{C}^3,r}) = \widetilde{M}_r((-1)^r q, t^{\frac{1}{2}})$, where $W(DT_{\mathbb{C}^3,r})$ is the generating series of the virtual weight polynomials of $\mathcal{M}_{n,r}$.

If we set $t = 1$ we obtain, as explained in Section 1.4, the classical DT invariants of $\mathcal{M}_{n,r}$. The series we obtain in this way is

$$1 + \sum_{n\geq 1} \mathrm{DT}(\mathcal{M}_{n,r})q^n = \prod_{n\geq 1}(1 - ((-1)^r q^n))^{-rn}. \tag{2.23}$$

Up to a formal change of coordinates $z = (-1)^r q$, this series is the r-th power of the MacMahon function $M(z) = \prod_{n\geq 1}(1-z^n)^{-n}$. By a celebrated theorem of MacMahon [M] the coefficient of z^n in $M(z)$ counts the number of 3D partitions of volume n.

We consider the action of a torus $T_\lambda = \mathbb{C}^*_\lambda$ on the moduli space $\mathcal{M}_{n,r}$. For a given λ in $\mathbb{C}^*$ let θ_λ be the automorphism of $\mathbb{P}^3$

$$\theta_\lambda \colon (z_0, z_1, z_2, z_3) \longmapsto (z_0, \lambda z_1, \lambda z_2, \lambda z_3),$$

and let $\xi_\lambda \colon \mathcal{O}_{p_\infty}^{\oplus r} \longrightarrow \mathcal{O}_{p_\infty}^{\oplus r}$ be the isomorphism rescaling sections by λ. We define an action of T_λ on $\mathcal{M}_{n,r}$ by

$$(F, \phi) \longmapsto (\theta_\lambda^*(F), \phi'), \tag{2.24}$$

where ϕ' is the composition

$$\phi' \colon \theta_\lambda^*(F)|_{p_\infty} \xrightarrow{\theta_\lambda^*(\phi)} \mathcal{O}_{p_\infty}^{\oplus r} \longrightarrow \mathcal{O}_{p_\infty}^{\oplus r} \xrightarrow{\xi_\lambda} \mathcal{O}_{p_\infty}^{\oplus r}.$$

The fixed points of this torus action are r-tuples of ideal sheaves $\oplus_{k=1}^r \mathcal{I}_k$ on $\mathbb{P}^3$ with the trivialization at p_∞ induced by the composition

$$\oplus_{k=1}^r \mathcal{I}_k \longrightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus r} \longrightarrow \mathcal{O}_{p_\infty}^{\oplus r}, \tag{2.25}$$

and such that the scheme defined by $\mathcal{I}_k$ is invariant by the action of $\mathbb{C}^*$ corresponding to rescaling the coordinate on $\mathbb{P}^3$. Thus, in analogy with the rank 1 case of the Hilbert scheme of points on $\mathbb{C}^3$, the classical DT invariant of $\mathcal{M}_{n,r}$ counts the number of torus invariant sheaves weighted by the coefficient $(-1)^{nr}$ which account for the value of the Behrend function of Theorem 1.54 at the torus invariant locus.

In [BBS] the authors prove that the generating series $DT_{\mathbb{C}^3,1}$ has the combinatorial description

$$\widetilde{M}_1(q,t)=\sum_{n\geq 0}\sum_{\pi\vdash n}t^{\sigma_n(\pi)}q^{|\pi|} \tag{2.26}$$

where the sum runs over the set P_n of 3D partitions of n and where

$$\sigma_n\colon P_n\longrightarrow\mathbb{Z},\quad \sigma_n=3\omega_0+\omega_+-\omega_-. \tag{2.27}$$

We can rewrite $\widetilde{M}_r(t,q)$ as

$$\prod_{i=1}^r\prod_{m\geq 0}\prod_{k=1}^m\left(1-t^{4+m-2k}(t^{r+1-2i}q)^m\right)^{-1},$$

and using equation (2.26) we have

$$\prod_{i=1}^r\sum_{\pi_i}t^{\sigma_{|\pi_i|}(\pi_i)+(r+1-2i)|\pi_i|}q^{|\pi_i|},$$

where each sum runs over the set of 3D partitions. Interchanging sum and product we get the final formulation

$$\sum_{\pi_1,\dots,\pi_r}\prod_{i=1}^r t^{\sigma_{|\pi_i|}(\pi_i)+(r+1-2i)|\pi_i|}q^{|\pi_i|}, \tag{2.28}$$

where the sum runs over r-tuple of partitions $\{\pi_i\}_{i\in\{1,\dots,r\}}$. We thus obtain the following combinatorial interpretation for the weight polynomial $W([\mathcal{M}_{n,r}]_{\mathrm{vir}})$: the alternating sum of the dimensions of the weight k part of the hypercohomology $\mathbb{H}^*(\mathcal{M}_{n,r},\Phi_{n,r})$ is the number of r-tuples of 3D partitions $\{\pi_i\}_{i\in\{1,\dots,r\}}$ such that $\sum_i|\pi_i|=n$, and such that

$$\sigma_{n,r}(\{\pi_i\}_i)=\sum_{i=1}^r\sigma_{|\pi_i|}(\pi_i)+(r+1-2i)|\pi_i|=k. \tag{2.29}$$

Remark 2.31. It has recently been proven by A. Morrison [Mo2], following some observations made by Hausel and Rodriguez-Villegas [HRV, Section 5], that the weight polynomials of $\mathcal{M}_{n,1}=\mathrm{Hilb}^n(\mathbb{C}^3)$ have Gaussian behavior for large n. More precisely, the renormalized function $\widetilde{\sigma}_n=n^{-\frac{2}{3}}\sigma_n\colon P_n\longrightarrow\mathbb{R}$ defines a random variable on the identically distributed probability space of 3D partitions of volume n. Morrison shows that as $n\to\infty$ the distribution of the random variable $\widetilde{\sigma}_n$ converges to a normal distribution with mean $\mu=3\xi(2)/(2\xi(3))^{\frac{2}{3}}$ and covariance $\sigma^2=1/(2\xi(3))^{\frac{1}{3}}$.

Several computer experiments show that the behavior of the weight polynomial of $\mathcal{M}_{n,r}$ is very similar for $r > 1$ to that observed by Hausel and Rodriguez-Villegas for the Hilbert scheme of points on $\mathbb{C}^3$. We conjecture that there exists a suitable renormalization $\widetilde{\sigma}_{n,r}$ of $\sigma_{n,r}$ which converges as $n \to \infty$ to a normal distribution.

2.A Stability of framed sheaves on $\mathbb{P}^m$

In this appendix we show that a framed sheaf on $\mathbb{P}^m$ which is trivial on a hyperplane is automatically stable in the sense of Huybrechts and Lehn [HL], and hence in particular has no non-trivial automorphisms.

Denote by p_∞ a fixed hyperplane in $\mathbb{P}^m$, and let H be the polarization defined by p_∞. Following Huybrechts and Lehn we will say that a pair (F, ϕ) is a *framed module* on $\mathbb{P}^m$ if $\phi\colon F \longrightarrow \mathcal{O}_{p_\infty}^{\oplus r}$ is not the zero map. Fix a rational polynomial with positive leading coefficient δ. A coherent module (F, ϕ) is $(H, \delta)-$*semi-stable* (or simply semi-stable if (H, δ) have been fixed) if for any framed subsheaf (F', ϕ')

$$r(P_{F'} - \delta) \leq r'(P_F - \delta), \tag{2.30}$$

and for any subsheaf F' contained in the kernel of the framing ϕ

$$rP_{F'} \leq r'(P_F - \delta). \tag{2.31}$$

Here we denote by P_E the Hilbert polynomial $\chi(E \otimes \mathcal{O}_{\mathbb{P}^m}(kH))$ of a coherent sheaf E on $\mathbb{P}^m$. In the usual way we define a coherent module to be stable when the inequalities (2.30) and (2.31) are strict for all proper subsheaves of F. A theorem [HL, Theorem 0.1] of Huybrechts and Lehn states that GIT allows us to construct scheme-theoretic moduli spaces of (semi)-stable coherent modules for more general choices of varieties and framings. We present here the version for $\mathbb{P}^m$ that will be of interest to us.

Theorem 2.32. *Fix a rational polynomial δ of degree at most n and with positive leading coefficient. There exists a projective scheme $M^{ss}_{n,r}(\mathbb{P}^m, \delta)$ which is the coarse moduli scheme for the moduli problem parametrising flat families of semi-stable coherent modules on $\mathbb{P}^m$ with Hilbert polynomial $rP_{\mathcal{O}_{\mathbb{P}^m}} - n$. The scheme $M^{ss}_{n,r}(\mathbb{P}^m, \delta)$ admits an open subset $M^{s}_{n,r}(\mathbb{P}^m, \delta)$ which is a fine moduli space for the moduli problem parametrising flat families of stable coherent modules.*

The next proposition proves that the generalization for $\mathbb{P}^m$ of the moduli functor $\mathcal{M}_{n,r}$ of Section 2.1 defines an open sub-functor of $M^s(\mathbb{P}^m, \delta)$ for a choice of $\delta = \delta(n, r)$ sufficiently large.

Proposition 2.33. *Let (F, ϕ) be a framed sheaf on $\mathbb{P}^m$, that is a pair consisting of a torsion free sheaf F on $\mathbb{P}^m$ and an isomorphism $\phi\colon F|_{p_\infty} \longrightarrow \mathcal{O}^{\oplus r}_{\mathbb{P}^m}$. Then (F, ϕ) is (H, δ)-stable for each choice of $\delta > (-1)^m(rk(F) - 1)\operatorname{ch}_m(F)$.*

Proof. Recall that F embeds into $\mathcal{O}_{\mathbb{P}^m}^{\oplus r}$, and since F is framed along p_∞ it has Chern character $r + (-1)^m n h^m$, where h is the class Poncaré dual to the cycle of the hyperplane p_∞. In particular, F has singularities on points. Since ϕ is an isomorphism, the condition (2.31) is automatically satisfied. It remains to prove that (2.30) holds. Suppose that (F', ϕ') is a framed subsheaf of (F, ϕ) of strictly smaller rank r'. Since we are assuming $\delta > (r-1)n$, we have that

$$P_{F'} \leq r'\chi(\mathcal{O}_{\mathbb{P}^m}(kH)) < \frac{r'}{r} P_F + \delta \frac{r - r'}{r} \delta.$$

We are left with the case when $rk(F') = rk(F)$. Denote by G and G' the quotients $\mathcal{O}_{\mathbb{P}^m}/F$ and $\mathcal{O}_{\mathbb{P}^m}/F'$. Both G and G' are supported on points and we have a surjection $G' \twoheadrightarrow G$. We have then $P_{F'} = rP_{\mathcal{O}_{\mathbb{P}^m}} - n'$, where $n' > n$, so that (2.30) is indeed satisfied. $\square$

2.B Some vanishing theorems for framed sheaves on $\mathbb{P}^m$

The following classical theorem due to R. Bott describes the dimensions of the sheaf cohomology of $\mathbb{P}^m$ with coefficient in various twists of exterior powers of the cotangent bundle $\Omega_{\mathbb{P}^m}$.

Theorem 2.34 ([OSS, Bott Formula])**.** *We have*

$$h^q\left(\mathbb{P}^m, \Omega^p_{\mathbb{P}^m}(k)\right) = \begin{cases} \binom{k+m-p}{k}\binom{k-1}{p} & \text{if } q = 0,\ k > p; \\ 1 & \text{if } k = 0,\ p = q; \\ \binom{-k+p}{-k}\binom{-k-1}{m-p} & \text{if } q = m, k < p - m; \\ 0 & \text{otherwise.} \end{cases} \tag{2.32}$$

Theorem 2.35. *Let F be a torsion free sheaf on $\mathbb{P}^m$ isomorphic to $\mathcal{O}_{\mathbb{P}^m}$ along a hyperplane p_∞, with $m \geq 2$. Then,*

$$\mathrm{H}^q(\mathbb{P}^m, F \otimes \Omega^p_{\mathbb{P}^m}(p-1)) = 0 \tag{2.33}$$

for $q \neq 1$ and for all p, where we define $\Omega^p_{\mathbb{P}^m}$ to be 0 for negative p.

Proof. Tensoring the short exact sequence of the hyperplane p_∞ by $F(k)$ we obtain

$$0 \longrightarrow F(k-1) \longrightarrow F \longrightarrow \mathcal{O}_{\mathbb{P}^{m-1}}(k) \longrightarrow 0.$$

Then we have:

$$\mathrm{H}^0(F(k-1)) \cong \mathrm{H}^0(F(k)) \qquad \text{for } k<0, \tag{2.34}$$

$$\mathrm{H}^q(F(k-1)) \cong \mathrm{H}^q(F(k)) \qquad \text{for } q \in \{2, \dots m-1\} \text{ and for all } k, \tag{2.35}$$

$$\mathrm{H}^m(F(k-1)) \cong \mathrm{H}^m(F(k)) \qquad \text{for } k \geq -m+1. \tag{2.36}$$

From Serre's vanishing theorem and equation (2.36) we immediately see that $\mathrm{H}^m(\mathbb{P}^m, F(k)) = 0$ for $k \geq -m$. Since F is torsion free there is an embedding $F \hookrightarrow \mathcal{L}$ for a certain direct sum of line bundles $\mathcal{L}$. Hence, by Serre's duality and Serre's vanishing theorem, we obtain starting from equation (2.34) that $\mathrm{H}^0(\mathbb{P}^m, F(k)) = 0$ for $k < 0$. From Serre's vanishing theorem and (2.35) we have that $\mathrm{H}^q(\mathbb{P}^m, F(k)) = 0$ for all k if we assume $q \in \{2, \dots, m-1\}$. Recall that for any $p \in \{0, \dots, m\}$ we have an exact sequence

$$0 \longrightarrow \Omega^p_{\mathbb{P}^m}(p) \longrightarrow \bigwedge^p \left(\mathcal{O}^{\oplus m+1}_{\mathbb{P}^m}\right) \longrightarrow \Omega^{p-1}_{\mathbb{P}^m}(p) \longrightarrow 0. \tag{2.37}$$

Tensoring this sequence by $F(k)$ implies immediately that from the isomorphisms (2.34)– (2.36) that we have just deduced

$$\mathrm{H}^0(F \otimes \Omega^p(k)) = 0 \text{ for } k < p; \tag{2.38}$$

$$\mathrm{H}^m(F \otimes \Omega^p(k)) = 0 \text{ for } k \geq p-m. \tag{2.39}$$

Rewriting part of Theorem 2.34 we have

$$\mathrm{H}^q(\mathbb{P}^m, F \otimes \Omega^n_{\mathbb{P}^m}(k)) = 0 \text{ for } q \in \{2, \dots, m-1\}. \tag{2.40}$$

Using equation (2.40) and the vanishing of the cohomology of the twists of the $F(k)$'s we see that

$$\mathrm{H}^q(\mathbb{P}^m, F \otimes \Omega^p_{\mathbb{P}^m}(k)) = 0 \text{ for } q \in \{2, \dots, m-2\}; \tag{2.41}$$

$$\mathrm{H}^q(\mathbb{P}^m, F \otimes \Omega^{m-1}_{\mathbb{P}^m}(k)) = 0 \text{ for } q = m-1,\ k \geq 0. \tag{2.42}$$

Proceeding by induction and using the long exact sequence in cohomology induced by (2.37), we obtain in particular

$$\mathrm{H}^q(\mathbb{P}^m, F \otimes \Omega^p_{\mathbb{P}^m}(k)) = 0 \text{ for } q \in \{2, \dots, m-1\}, k \geq 0. \tag{2.43}$$

Hence, all the cohomology vanishings listed in the statement of the theorem are satisfied. $\square$

Remark 2.36. Using Theorem (2.35) one can proceed in complete analogy with the $\mathbb{P}^3$ case to describe framed torsion free sheaves on $\mathbb{P}^m$ of cohomological type $r+(-1)^m nh^m$ in terms of representations of an r-framed quiver with one node and m loops satisfying commuting quadratic relations.

Chapter 3

Refined Invariants for Higher Rank Stable Pairs on the Conifold

We consider the moduli functor $\mathcal{P}^r_{n,d}(X)$ parametrising families of higher rank stable pairs $\mathcal{O}_X^{\oplus r} \xrightarrow{s} F$ on the toric projective bundle $X = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2})$ for a particular choice of limiting stability condition, studied by Sheshmani in [Sh1, Sh2]. We begin by using the relative Beilinson spectral sequence Theorem 1.31 for projective bundles to realize $\mathcal{P}^r_{n,d}(X)$ as a moduli problem $\mathcal{M}^{con}_{(d,n-d)}(\mathbb{P}^1, r)$ parametrising vector bundles on $\mathbb{P}^1$ with some extra decorations, following very closely Diaconescu's approach of [Di1]. More precisely, this reduction to $\mathbb{P}^1$ leads us to consider asymptotically stable ADHM sheaves on $\mathbb{P}^1$ with higher framing rank with respect to the choice of line bundles $L_i = \mathcal{O}_{\mathbb{P}^1}(1)$, where the ADHM relation in the decoration of the sheaves is a relative version of the classical ADHM relation for framed sheaves on $\mathbb{P}^2$ [ADHM]. The moduli scheme $M^{con}_{(d,n-d)}(\mathbb{P}^1, r)$ is isomorphic to the moduli space of PT-stable representations of the $(r, 0)$-framed conifold quiver with superpotential of Example 1.6, where PT-stability is a particular choice of a (limit) King stability condition. This generalizes to higher rank the description given for rank 1 by Nagao and Nakajima in [NN], and it gives an alternative proof for their results (only in the PT-chamber) for the rank 1 case. It follows that the moduli functor $\mathcal{P}^r_{n,d}(X)$ is represented by a proper scheme $P^r_{n,d}$ which is the critical locus of a regular function $\omega_{n,d,r}$ on a smooth quasi-projective variety. Using this description we compute the generating series of motivic/refined Donaldson-Thomas invariants employing the techniques developed in [MMNS]. If we denote by $Z_{X,r}(s,T)$ the generating series $\sum [P^r_{n,d}]_{\mathrm{vir}} s^n T^d$, we have the factorization

$$Z_{X,r}((-1)^r s, T) = \prod_{n\geq 1} \prod_{j=0}^{rn-1} \left(1 - \mathbb{L}^{\frac{rn}{2} - j - \frac{1}{2}} s^n T\right).$$

As the conifold superpotential admits a cut and the critical locus is proper, it follows from the work of Davison, Maulik, Schürmann and Szendrői and [DMSS, Corollary 3.1] that the cohomology with respect to the topological coefficient system $\Phi_{d,n,r} = \phi_{\omega_{d,n,r}}\mathbb{Q}[m]\left(\frac{m}{2}\right)$ is pure, where $m = n^2+(n-d)^2+nr$ is the dimension of the ambient space. It follows that the specialization $\mathbb{L}^{\frac{1}{2}} = t^{\frac{1}{2}}$ computes the generating series of the virtual Poincaré polynomials:

$$Z_{X,r}(s,T)\big|_{\mathbb{L}^{\frac{1}{2}}\mapsto t^{\frac{1}{2}}} = \sum_{n,d}\sum_{i}(-1)^{\frac{1}{2}}t^{\frac{i}{2}}\dim\mathbb{H}^i\left(P^r_{n,d},\Phi_{n,d,r}\right)s^nT^d.$$

The numerical specialization $\mathbb{L} = 1$ gives the generating series of the numerical Donaldson-Thomas invariants of higher rank stable pairs:

$$\sum_{n,d}\mathrm{DT}(P^r_{n,d}(X))T^ds^n = \prod_{n\geq 1}\left(1-((-1)^rs)^nT)\right)^r,$$

recovering the computations performed by Sheshmani in [Sh2] using virtual localization.

All of this carries through with minor modifications to the case when we replace X by the projective bundle $\mathbb{P}(\mathcal{O}^{\oplus 2}\oplus\mathcal{O}_{\mathbb{P}^1}(-2))$, extending the constructions to the case of the compactification of the line of resolved A_1-singularities. Since this example does not differ significantly from the one of resolved conifold, we relegate a brief discussion of higher rank pair invariants on X to the Appendix 3.A.

The main motivation for these constructions has been the desire to compare invariants of rank r stable pairs to invariants à la Nekrasov counting $U(1)$ instantons on the resolution of A_{r-1}-surface singularities, as suggested by geometric engineering and in [MO]. Unfortunately, our attempt does not seem to be very fruitful for these purposes, since despite the fact that the threefold and surface partition functions factorize in very similar ways, we are not able to find a suitable change of variables bringing one to the other. This is in line with the recent observations of Nekrasov and Okounkov [NO] that some extra interactions between the rank 1 contributions must be taken into account in order to realize a full correspondence between the two counting problems.

3.1 The moduli spaces

In this section we describe the two moduli spaces which are the main objects of interest of the current chapter, together with the moduli space of $(r,0)$-framed representations of the conifold quiver with superpotential of Example 1.6.

3.1.1 Higher rank stable pairs

The study of pairs consisting of a sheaf E on a smooth projective variety X together with some section $V \otimes \mathcal{O}_X \longrightarrow E$ is a relatively classical subject, and it was extensively studied by several authors beginning with Le Potier. More recently Pandharipande and Thomas [PT1] understood that the moduli spaces of these objects (for some particular choice of limit stability) are in a sense the right moduli spaces to be considered when studying the enumerative geometry of curves lying on Calabi-Yau threefolds. The key reason for preferring moduli of stable pairs rather than Hilbert scheme-like moduli spaces is that the former moduli problem eliminates the contributions coming from points which lie outside of the curves one is meant to count, thus ensuring properness of the moduli spaces also when the CY variety is quasi-projective. This makes their invariants well suited with respect to the (in part still conjectural) correspondence with Gromov-Witten invariants.

Denote by X a smooth projective threefold. A *rank r (framed) pair* on X is a pair (F, s) of a sheaf F supported in dimension 1 and a (multi-)section $s\colon \mathcal{O}_X^{\oplus r} \longrightarrow F$. We will consider morphisms between pairs preserving the source of the section, so that a morphism of pairs $\rho\colon (F, s) \longrightarrow (F', s')$ consists of a map $\rho\colon F \longrightarrow F'$ which sits in the commutative diagram

$$\begin{tikzcd}
\mathcal{O}_X^{\oplus r} \arrow[r, "s"] \arrow[d, "\mathrm{Id}"'] & F \arrow[d, "\rho"] \\
\mathcal{O}_X^{\oplus r} \arrow[r, "s'"] & F'.
\end{tikzcd}$$

Notation 3.1. In order to keep the notation as light as possible, we omit the adjective framed for higher rank stable pairs with morphisms as just described. Framing is crucial for all of our constructions, so for the rest of this chapter the reader should assume that we do not consider anything other then the framed case.

A pair (F, s) is said to be *stable* if F is pure and $\operatorname{coker}(s)$ is supported on a finite set of points.

Remark 3.2. As presented here the definition of stability for pairs might seem ad hoc. It has been shown by Pandharipande and Thomas [PT1, Lemma 1.3] that the condition of almost surjectivity of the section s arises naturally as the limit of a more familiar type of stability condition. The authors consider the limit of a family of stability conditions depending on the datum (H, δ) of a polarization H for X and a rational polynomial δ as the leading coefficient of δ goes to ∞, in a very similar way to what we have done in Appendix 2.A for framed sheaves on $\mathbb{P}^m$. We refer to their work for a careful explanation of this.

We will say that a rank r stable pair (F, s) on X is of *type* (n, β) when $\mathrm{ch}_2 F = \beta$, for some class $\beta \in H_2(X, \mathbb{Z})$, and we have $\chi(F) = n$, where χ here denotes the sheaf cohomology Euler characteristic.

Now let T be an affine scheme. A *family* (F_T, s_T) *of rank* r *stable pairs* of type (n, β) over T is the data of a coherent $\mathcal{O}_{X\times T}$-module F_T flat over T together with a multi-section $s_T \colon \mathcal{O}_{X\times T}^{\oplus r} \longrightarrow F_T$ such that for every geometric point $t \in T$ the restriction $(F_T|_t, s_T|_t)$ is a rank r stable pair of type (n, β) on $X \times \{t\}$. We say that two families F_T and F'_T over T are isomorphic if there is a morphism ρ_T of $\mathcal{O}_{X\times T}$-modules that makes the diagram

$$\begin{tikzcd}
\mathcal{O}_{X\times T}^{\oplus r} \arrow[r, "s_T"] \arrow[d, "\mathrm{Id}"'] & F_T \arrow[d, "\rho_T"] \\
\mathcal{O}_{X\times T}^{\oplus r} \arrow[r, "s'_T"] & F_{T'}
\end{tikzcd}$$

commutative. We denote by $\mathcal{P}^r_{n,\beta}$ the moduli functor AffSchemes $\longrightarrow$ Set^{op} which parametrises families of rank r stable pairs on X.

Sheshmani understood that considering moduli problems for higher rank stable pairs with framing, or in his terminology highly frozen stable pairs, one eliminates the automorphisms of the objects. Hence $\mathcal{P}^r_{n,\beta}$ is represented by a scheme.

Lemma 3.3 ([Sh2, Lemma 3.6])**.** *Suppose* $\rho \colon F \longrightarrow F$ *defines an automorphism of the stable pair* (F, s)*. Then* $\rho = \mathrm{Id}_F$*.*

Proof. There is a commutative diagram

$$\begin{tikzcd}
\mathcal{O}_X^{\oplus r} \arrow[r, "s"] \arrow[d, "\mathrm{Id}"] & \mathrm{Im}(s) \arrow[r, "\iota"] \arrow[d, "\mathrm{Id}"] & F \arrow[d, "\rho"] \\
\mathcal{O}_X^{\oplus r} \arrow[r, "s"] & \mathrm{Im}(s) \arrow[r, "\iota"] & F,
\end{tikzcd}$$

where ι is the inclusion map. Since s has cokernel Q supported on points, if we apply $\mathrm{Hom}(\,.\,, F)$ to the short exact sequence

$$0 \longrightarrow \mathrm{Im}(s) \longrightarrow F \longrightarrow Q \longrightarrow 0,$$

we obtain a short exact sequence

$$0 \longrightarrow \mathrm{Hom}(Q, F) \longrightarrow \mathrm{Hom}(F, F) \longrightarrow \mathrm{Hom}(\mathrm{Im}(s), F) \longrightarrow 0.$$

From the assumption of purity it follows that $\mathrm{Hom}(Q, F) = 0$ and so there is an isomorphism $\mathrm{Hom}(F, F) \cong \mathrm{Hom}(\mathrm{Im}(s), F)$, sending the morphism ρ to ι. But then $\rho = \mathrm{Id}$. □

This is one of the essential ingredients in the proof of the following theorem.

Theorem 3.4 ([Sh2, Theorem 3.7]). *The moduli space $\mathcal{P}^r_{n,\beta}$ is represented by a scheme $P^r_{n,\beta}$.*

As mentioned in the introduction of the chapter, we will be particularly interested in the case when X is the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2})$ over the projective line. The threefold X is a toric projective bundle which is a compactification of $X^o = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2})$ obtained by adding the divisor $\mathbb{P}(0 \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}) \cong \mathbb{P}^1 \times \mathbb{P}^1$ at ∞. We will denote the complementary divisor $X \setminus X^o$ by D_∞, its class in cohomology by H, and the cohomology class of a fiber by F. The rigid rational curve $b = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus 0^{\oplus 2})$ is disjoint from D_∞, and its class β in $H_2(X, \mathbb{Z})$ spans the entire kernel of $\cdot \cap H$.

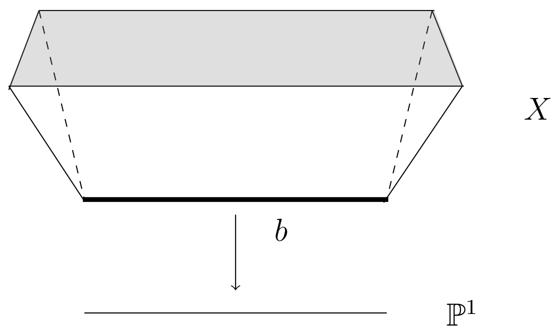

Figure 3.1: Compactification $X^0 \subset X$ (divisor D_∞ light grey, curve b bold).

We will denote by $P^r_{n,d}$ the proper moduli scheme representing the moduli problem for rank r stable pairs of type $(n, d\beta)$.

Remark 3.5. The condition that the support of F must be disjoint from D_∞ implies that the relevant geometry of the problem is happening on X^o, which is a local CY threefold. Similarly to the case of framed sheaves on $\mathbb{P}^3$, we compactify the CY geometry to a projective variety to use the classical techniques for studying sheaves on projective spaces or projective bundles.

3.1.2 Moduli of ADHM Sheaves

Moduli spaces of sheaves on complex algebraic varieties satisfying some ADHM type relations have been studied by several authors in different contexts. A particularly interesting case is when such a moduli space arises from a reduction process starting from a moduli space of sheaves on a local Calabi-Yau threefold fibered over a curve [Sz2]. A systematic approach to the subject has been carried through by Diaconescu during the last few years; see for instance [Di2]. This brief introduction to ADHM sheaves is strongly influenced by the mentioned above papers.

Let X be a complex projective variety, and let L_1 and L_2 be two line bundles on X. Denote by L_{12} the tensor product $L_1 \otimes L_2$. Let E_∞ be a coherent sheaf which will be referred to as the framing sheaf, in analogy with the original ADHM construction.

Definition 3.6. An *ADHM sheaf* on X is the datum $\mathcal{E} = (E, \phi_1, \phi_2, \varphi, \psi)$ of a coherent sheaf E on X and morphisms of sheaves as described in the diagram

$$\begin{tikzcd} & E \otimes L_2 \arrow[dr, "\phi_2"] & \\ E \otimes L_{12} \arrow[ur, "\phi_1 \otimes 1_{L_2}"] \arrow[dr, "\phi_2 \otimes 1_{L_1}"] \arrow[dd, "\varphi"'] & & E\,. \\ & E \otimes L_1 \arrow[ur, "\phi_1"] & \\ E_\infty \arrow[uurr, bend right, "\psi"'] & & \end{tikzcd}$$

These morphisms are required to satisfy the ADHM-like relation

$$[\phi_1, \phi_2] + \psi\varphi := \phi_1(\phi_2 \otimes 1_{L_1}) - \phi_2(\phi_1 \otimes 1_{L_2}) + \psi\varphi = 0. \tag{3.1}$$

Notation 3.7. We will adopt throughout the chapter the following terminology: if n and d are the rank and the degree of E, respectively, and r is the rank of the framing sheaf E_∞, we will say that $\mathcal{E}$ is a rank r ADHM sheaf of type (n, d).

Remark 3.8. The name ADHM sheaf is justified by the fact that, for each point x in X, equation (3.1) coincides with the classical ADHM relation that arises when describing $SU(r)$-instantons on $\mathbb{R}^4$, or equivalently torsion free sheaves F on $\mathbb{P}^2$ together with a fixed isomorphism $F|_{l_\infty} \cong \mathcal{O}_{l_\infty}^{\oplus r}$ along the line at infinity $l_\infty \cong \mathbb{P}^1$. Morally, one should really think of ADHM sheaves as the result which one expects when considering sheaves living on $\mathbb{P} = \mathbb{P}(\mathcal{O}_X \oplus L_1 \oplus L_2)$ with a framing along the divisor $\mathrm{D}_\infty \cong \mathbb{P}(0 \oplus L_1 \oplus L_2)$.

There are different possible definitions of morphisms of ADHM sheaves. Thinking about ADHM sheaves as the result of a reduction from the geometry of sheaves on the (compactification of the) total space of the rank two bundle $L_1 \oplus L_2 \longrightarrow X$, different choices of morphisms correspond to different requirements on the sheaves on the total space. Our choice is induced essentially from the necessity of avoiding non-trivial automorphisms in order to compute refined invariants.

Definition 3.9. A *morphism of ADHM sheaves* between $\mathcal{E}$ and $\mathcal{E}'$ is a morphism of sheaves $\theta \colon E \longrightarrow E'$ making the diagrams

$$\begin{tikzcd} E \otimes L_i \arrow[r, "\phi_1"] \arrow[d, "\theta \otimes 1_{L_i}"'] & E \arrow[d, "\theta"] \\ E' \otimes L_i \arrow[r, "\phi_i'"] & E' \end{tikzcd} \qquad \begin{tikzcd} E \otimes L_{12} \arrow[r, "\varphi"] \arrow[d, "\theta \otimes 1_{L_{12}}"'] & E_\infty \arrow[d, "\mathrm{Id}"] \\ E' \otimes L_{12} \arrow[r, "\varphi'"] & E_\infty \end{tikzcd} \qquad \begin{tikzcd} E_\infty \arrow[r, "\psi"] \arrow[d, "\mathrm{Id}"'] & E \arrow[d, "\theta"] \\ E_\infty \arrow[r, "\psi'"] & E' \end{tikzcd}$$

commute.

Remark 3.10. Notice that we do not consider ADHM subsheaves with framing different from E_∞. In the literature this choice of morphisms has appeared under the name of the Coulomb branch [CSS, Di2].

In order to obtain moduli spaces with reasonable geometric properties one has to introduce at this stage the choice of a stability condition. Again there is a dichotomy in the literature, depending on the choice of Definition 3.9. Our choice leads to the following definition of stability condition, as in [Di2].

Definition 3.11. An ADHM sheaf is said to be *asymptotically stable* if

- E is torsion free and $\mathrm{Im}(\psi) \neq 0$;
- there is no saturated subsheaf $E_0 \subset E$ containing $\mathrm{Im}(\psi)$ such that $\phi_i(E_0 \otimes L_i) \subset E_0$ and $\mathrm{Im}(\psi) \subset E_0$.

Remark 3.12. Again in this case, as explained in [Di1], the stability condition arises as the limit of δ-twisted μ-stabilities as the parameter δ goes to ∞.

The following technical lemma on destabilizing ADHM subsheaves will be useful in the following sections.

Lemma 3.13 ([Di2, Lemma 2.4])**.** *Let $\mathcal{E}$ be an ADHM sheaf. There exists a saturated subsheaf $E_0 \subset E$ such that $\phi_i(E_0(L_i)) \subset E_0$ and $\mathrm{Im}(\psi) \subset E_0$, and such that E_0 contains all the saturated subsheaves satisfying the same properties.*

We denote by $\mathscr{M}((X, E_\infty), n, d)$ the moduli functor parametrising flat families of stable ADHM sheaves on X with framing E_∞ and with underlying sheaf of rank n and degree d.

Theorem 3.14 ([Di2, Theorem 1.1])**.** *Assume that $r > 0$ and $n > 0$. The functor $\mathscr{M}((X, E_\infty), n, d)$ is represented by a quasi-projective variety $M((X, E_\infty), n, d)$.*

We restrict now to the case when $X \cong \mathbb{P}^1$, $E_\infty = \mathcal{O}_X^{\oplus r}$ and the bundle $L_1 \otimes L_2$ is isomorphic to $-K_X$. In this case we will denote the moduli scheme by $M_{(n,d)}(X, L_1, L_2, r)$. We further assume that the line bundles L_1 and L_2 have non-negative degree.

Lemma 3.15. *Let L_1, L_2 be line bundles on $\mathbb{P}^1$ with non-negative degree such that $L_{12} = -K_{\mathbb{P}^1}$. Then, the underlying sheaf E of an asymptotically stable ADHM sheaf $\mathcal{E}$ in $M_{(n,d)}(\mathbb{P}^1, L_1, L_2, r)$ has no direct summand with negative degree, and the morphism φ is identically zero.*

Proof. Since E is locally free on $\mathbb{P}^1$, it decomposes as the direct sum of line bundles. Let $E_{\geq 0}$ be the direct sum of all the summands in E with non-negative degree. Since $\deg(L_i) \geq 0$ we have that $\phi_i(E_{\geq 0} \otimes L_i) \subset E_{\geq 0}$. Also since each direct summand of $E_{\geq 0} \otimes L_{12}$ has strictly positive degree we have $\varphi_{E_{\geq 0} \otimes L_{12}} = 0$. Then

$$\mathcal{E}_{\geq 0} = (E_{\geq 0}, \phi_i|_{E_{\geq 0} \otimes L_i}, 0, \psi)$$

defines a saturated ADHM subsheaf $\mathcal{E}_{\geq 0}$ of $\mathcal{E}$ which contains $\mathrm{Im}(\psi)$. But then $E_{\geq 0} = E$ from stability. $\square$

The situation is more complicated if we drop the requirement that the L_i's have non-negative degree. In fact, it is Lemma 3.15 that allows us to employ the techniques of Section 1.2 to reduce the study of ADHM sheaves to a problem of linear algebra.

Remark 3.16. Considering $L_1 = \mathcal{O}_{\mathbb{P}^1}(-1)$, $L_2 = \mathcal{O}_{\mathbb{P}^1}(3)$ we run into trouble. Concretely, there are stable ADHM sheaves $\mathcal{E}$ with a direct summand with zero degree in E, and Lemma 3.15 does not hold for these sheaves. However, if we are required to restrict to the setting of ADHM sheaves without degree zero summands in E the same argument as in Lemma 3.15 works. Two interesting questions arise, which will not be addressed in the thesis, but might be interesting for future research:

- Can we describe explicitly the locus of ADHM sheaves with a degree zero summand? How big is it?

- Once we have a description of its complement in terms of quivers with superpotential, how do we incorporate the residual part? What role does it play when computing invariants?

Notation 3.17. In the rest of the chapter we will treat the case $L_1 = L_2 = \mathcal{O}_{\mathbb{P}^1}(1)$, and in this case we denote the moduli space of rank r asymptotically stable ADHM sheaves of type (n,d) by $M^{\mathrm{con}}_{(n,d)}(\mathbb{P}^1, r)$. In Appendix 3.A we will discuss the case $L_1 = \mathcal{O}_{\mathbb{P}^1}$, $L_2 = \mathcal{O}_{\mathbb{P}^1}(2)$, and the moduli spaces will be denoted by $M^{A_1}_{(n,d)}(\mathbb{P}^1, r)$.

3.2 From stable pairs on X to conifold-like ADHM sheaves

Let C be a smooth projective curve, let L_1 and L_2 be two line bundles on C, and let Y be the projective bundle $\mathbb{P}(\mathcal{O}_C \oplus L_1^\vee \oplus L_2^\vee)$. Denote by b the section $\mathbb{P}(\mathcal{O}_C \oplus 0 \oplus 0)$ and by β its class in $\mathrm{H}^2(Y, \mathbb{Z})$, and denote by D_∞ the divisor $\mathbb{P}(0 \oplus L_1 \oplus L_2)$. In [Di2, Theorem 1.9] Diaconescu proves that there is an isomorphism between the moduli space $\mathcal{P}^1_{n,d}(Y)$ of rank 1 stable pairs (F, s) of type $(d\beta, n)$ such that $\mathrm{Supp}(F)$ does not intersect D_∞, and the moduli space of asymptotically stable ADHM sheaves $M((C, \mathcal{O}_C), d, n + d(g(C) - 1))$ with respect to the choice of line bundles L_1 and L_2. Probably due to the fact that at the time there was not yet a good understanding of moduli of framed stable pairs and their enumerative interpretations, which later appeared in [Sh1], the author did not consider the higher rank case, but the constructions continue to work also in this setting, so that all of the credit for the results of this section goes to him. We report here the main parts of the construction including the higher rank case for the particular example of the resolved conifold, but one could replace $C = \mathbb{P}^1$ with a general smooth projective curve C and $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$ by the direct sum of two line bundles on C as above, in the same way as has been done in [Di2].

We switch to the dual notation $X = \mathrm{Proj}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2})$. Following the canonical notation for projective bundles in the algebro-geometric setting, we denote by π the projection to $\mathbb{P}^1$ and we denote by V the sheaf $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$. Let z_0 be the pull-back of the section $(1, 0, 0)$ of

$$\mathrm{H}^0(X, \mathcal{O}_X(1)) \cong \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \oplus \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))^{\oplus 2}, \tag{3.2}$$

and by z_1 and z_2 the pull-back of the sections $(1, 0)$ and $(0, 1)$ of

$$\mathrm{H}^0(X, \mathcal{O}_X(1) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(-1)) \cong \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})^{\oplus 2}. \tag{3.3}$$

The following lemma is a consequence of the relative Beilinson theorem for projective bundles (Theorem 1.31), combined with the observation of [Di2, Lemma 6.4] that the underlying sheaf F of a stable pair (F, s) is flat over $\mathbb{P}^1$, since it is pure and supported away from D_∞.

Lemma 3.18. *Let (F, s) be a stable pair of rank r. Then F admits a locally free resolution $\mathcal{F}^\bullet$*

$$0 \longrightarrow \mathcal{O}_X(-2) \otimes \pi^*(E \otimes \mathcal{O}_{\mathbb{P}^1}(2)) \xrightarrow{\alpha_{-2}} \mathcal{O}_X(-1) \otimes \pi^*(E \otimes \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}) \xrightarrow{\alpha_{-1}} \pi^* E \tag{3.4}$$

where $E = \mathrm{R}^0\pi_*(F)$, and the morphisms α_{-1}, α_{-2} are described by:

$$\alpha_{-2} = (z_2 - z_0 \otimes \pi^*\phi_2, z_0 \otimes \pi^*\phi_1 - z_1)^t\,, \tag{3.5}$$

and

$$\alpha_{-1} = (z_0 \otimes \pi^*\phi_1 - z_1, z_0 \otimes \pi^*\phi_2 - z_2)\,. \tag{3.6}$$

Furthermore, the morphisms ϕ_1 and ϕ_2 satisfy the ADHM-type identity

$$\phi_1\phi_2 - \phi_2\phi_1 = 0, \tag{3.7}$$

and the multi-section s induces a non trivial morphism $\psi\colon \mathcal{O}_{\mathbb{P}^1}^{\oplus r} \longrightarrow E$.

Proof. Since F is flat over $\mathbb{P}^1$ and the restriction F_{X_p} to the fiber of X over $p \in \mathbb{P}^1$ is supported on points we have

$$\mathrm{R}^i\pi_*(F) = 0,\ i \geq 1. \tag{3.8}$$

We denote by E the pushforward $\mathrm{R}\pi_*(F)$. From the vanishings (3.8) the relative Beilinson spectral sequence degenerates at the first page and the complex

$$0 \longrightarrow \mathcal{O}_X(-2) \otimes \pi^* \mathrm{R}^0\pi_*(F \otimes \Omega^2_\pi(2)) \xrightarrow{\beta_{-2}} \mathcal{O}_X(-1) \otimes \pi^* \mathrm{R}^0\pi_*(F \otimes \Omega^1_\pi(1)) \xrightarrow{\beta_{-1}} \pi^* E \tag{3.9}$$

has F in degree zero as the only non-zero cohomology. Since F is flat the sheaves $\mathrm{R}^0\pi_*(F \otimes \Omega^i_\pi(i))$ are locally free on $\mathbb{P}^1$ for all choices of i. Tensoring the Euler sequence by F and pushing forward we get an exact sequence

$$0 \longrightarrow \mathrm{R}^0\pi_*(F \otimes \Omega^1_\pi(1)) \longrightarrow \mathrm{R}^0\pi_*(F \otimes \pi^* V) \xrightarrow{\gamma} \mathrm{R}^0\, \pi_*(F(1)) \longrightarrow 0, \tag{3.10}$$

where γ is (ϕ_0, ϕ_1, ϕ_2) and where ϕ_i denotes $\mathrm{R}^0\pi_*(\mathrm{Id}_F \otimes z_i)$. Since F has support disjoint from D_∞ the morphism z_0 identifies F and its twist $F(1)$, and so ϕ_0 is an isomorphism. In this way we can identify $\mathrm{R}^0\pi_*(F\otimes\Omega^1_\pi(1))$ and $E\otimes\pi^*\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$. Identifying further $\Omega_\pi(2)$ with $\mathcal{O}_X(-1)\otimes\mathcal{O}_{\mathbb{P}^1}(2)$ we obtain a complex

$$\mathcal{O}_X(-2)\otimes\pi^*\,\mathrm{R}^0\pi_*(F(-1)\otimes\pi^*\mathcal{O}_{\mathbb{P}^1}(2))\xrightarrow{\alpha_{-2}}$$

$$\mathcal{O}_X(-1)\otimes\pi^*\,\mathrm{R}^0\pi_*(F\otimes\pi^*\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2})\xrightarrow{\alpha_{-1}}\pi^*\,\mathrm{R}^0\pi_*(F(1)) \tag{3.11}$$

whose only cohomology is F in degree 0. By Remark 1.32 the morphisms α_{-2} and α_{-1} can be written as

$$\begin{aligned}\alpha_{-1}&=(z_0\otimes\phi_1-z_1\otimes\phi_0, z_0\otimes\phi_2-z_2\otimes\phi_0)\\ \alpha_{-2}&=(z_2\otimes\phi_0-z_0\otimes\phi_2, z_0\otimes\phi_1-z_1\otimes\phi_0)^t,\end{aligned} \tag{3.12}$$

and from the fact that $\phi_0\colon\ \mathrm{R}^0\pi_*F\longrightarrow\mathrm{R}^0\pi_*F(1)$ is an isomorphism we get the desired complex. The fact that the ϕ_i's satisfy the ADHM-type identity (3.7) follows directly from the fact that 3.11 is a complex. From the functoriality of the Beilinson spectral sequence we get a non-zero morphism $\psi\colon\mathcal{O}^{\oplus r}\longrightarrow E$.

□

By Lemma 3.18 we can construct a conifold-like ADHM sheaf $\mathcal{E}=(E,\phi_1,\phi_2,\psi)$ starting from a pair (F,s). Conversely, starting with an ADHM sheaf on $\mathbb{P}^1$ one can pull back the datum to the projective bundle X, and construct a diagram of the form (3.4). The following two lemmas show that the ADHM sheaf constructed starting from a stable pair (F,s) is asymptotically stable, and that the complex associated to an ADHM sheaf naturally defines a stable pair.

Lemma 3.19. *The ADHM sheaf $\mathcal{E}$ constructed in Lemma 3.18 is asymptotically stable.*

Proof. We show that if $\mathcal{E}'\subset\mathcal{E}$ is a destabilizing subsheaf then the pair (F,s) from which E was constructed is not stable. The idea is to show that for a point p in $\mathbb{P}^1$ the restriction $s|_p$ of the multi-section to the fiber is not surjective at a generic point, using a result of Nakajima for framed sheaves on $\mathbb{P}^2$. Let p be a point in the base, and denote by X_p^o the subvariety of the fiber not intersecting D_∞, and let $\phi_{i,p}$ denote the restriction of the pullback $\pi^*\phi_i$ to X_p^o. By Lemma 3.18 the multi-section s is surjective on X_p^o if and only if the morphism

$$(E_p\oplus E_p\oplus\mathbb{C}^r)\otimes\mathcal{O}_{X_p^o}\xrightarrow{\alpha_2|_{X_p^o}\oplus\psi|_{X_p^o}}E_p\otimes\mathcal{O}_{X_p^o} \tag{3.13}$$

is surjective, where $\alpha_2|_{X_p^o}=(z_1-\phi_{1,p},z_2-\phi_{2,p})$. By [N, Lemma 2.7], this is equivalent to the fact that E_p does not admit a subspace $\widetilde{E}_p$ closed under $\phi_{i,p}$ and containing

$\mathrm{Im}(\phi_p)$. For a generic p in $\mathbb{P}^1$ the restriction of the ADHM subsheaf E' to p gives such a subspace. Hence the pair $\mathcal{O}^r \longrightarrow F$ is not generically surjective and we have a contradiction. $\square$

Lemma 3.20. *Let $\mathcal{E}$ be an asymptotically stable ADHM sheaf. The complex*

$$0 \longrightarrow \mathcal{O}_X(-2) \otimes \pi^*(E \otimes \mathcal{O}_{\mathbb{P}^1}(2)) \xrightarrow{\alpha_{-2}} \mathcal{O}_X(-1) \otimes \pi^*(E \otimes \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}) \xrightarrow{\alpha_{-1}} \pi^* E \tag{3.14}$$

has cohomology just in degree 0*, where*

$$\alpha_{-2} = (z_2 - z_0 \otimes \pi^*\phi_2, z_0 \otimes \pi^*\phi_1 - z_1)^t,$$

and

$$\alpha_{-1} = (z_0 \otimes \pi^*\phi_1 - z_1, z_0 \otimes \pi^*\phi_2 - z_2)$$

If we denote by F its cohomology in degree 0*, then $\pi^*(\psi)\colon \mathcal{O}_X^{\oplus r} \longrightarrow F$ is a stable pair of type* $(\deg(E) - rk(E), rk(E))$.

Proof. Suppose α_{-2} has non-trivial kernel. Then, $\mathrm{Ker}(\alpha_2)$ is torsion free, so that it is locally free outside a codimension 2 subvariety Z. Hence if we restrict to the fiber over a generic point p of $\mathbb{P}^1$ the morphism

$$E_p \otimes \mathcal{O}_{X_p} \xrightarrow{\alpha_2|_{X_p}} (E_p \oplus E_p) \otimes \mathcal{O}_{X_p} \tag{3.15}$$

is generically not injective. But this contradicts the fact that such a map can only have finitely many zeros by [N, Lemma 2.7]. Notice that since for a generic point p in $\mathbb{P}^1$ the restriction $\alpha_2|_{X_p}$ has only finitely many zeros, the cokernel $\mathrm{coker}(\alpha_2)$ is locally free in codimension 2. This together with the fact that $\mathrm{coker}(\alpha_2)$ is the quotient of two locally free sheaves implies that it is torsion free. Hence the cohomology $\mathrm{Ker}(\alpha_1)/\mathrm{Im}(\alpha_2)$ is torsion free. Since we have

$$\alpha_{-2} = (\alpha_{-2,1}, \alpha_{-2,2})^t = (-z_0 \otimes \phi_2 + z_2, -z_0 \otimes \phi_1 + z_1), \ \alpha_1 = (-\alpha_{-2,2}, \alpha_{-2,1}), \tag{3.16}$$

it is enough to prove that one of the two restrictions of $\alpha_{-2,2}$ and $\alpha_{-2,1}$ to a point y is an isomorphism in order to prove that the stalk $\mathrm{Ker}(\alpha_1)/\mathrm{Im}(\alpha_2)_y = 0$. But again from [N, Lemma 2.7] this gives a codimension 2 condition, so it follows that $\mathrm{Ker}(\alpha_1)/\mathrm{Im}(\alpha_2) = 0$ because it is torsion free.

Now we denote by F the cohomology in degree 0. We need to prove that $\pi^*(\psi)\colon \mathcal{O}^{\oplus r} \longrightarrow F$ is a stable pair. First of all notice that the support of F must be disjoint from

D_∞ since the sections z_1 and z_2 cannot vanish simultaneously on D_∞. Using the fact that (3.4) has F as its only cohomology and $\beta = H^2 - 2HF$, we see that the Chern character of F satisfies

$$\mathrm{ch}_0(F) = 0, \mathrm{ch}_1(F) = 0, \mathrm{ch}_2(F) = rk(E)\beta \tag{3.17}$$

where rk and deg denote the usual rank and degree of E, and $\chi(F) = \deg(E) + rk(E)$.

First we show that the cokernel of $s = \pi^*(\psi)$ is supported on points. Suppose that s is not surjective at a point x in X. Localizing at x and then dualizing this is equivalent to the fact that the morphism of vector spaces

$$E_p \otimes \mathbb{C}(x) \longrightarrow E_p(E_p \oplus E_p \oplus \mathbb{C}^r) \otimes \mathbb{C}(x), \tag{3.18}$$

is not injective, where E_p as usual denotes the fiber of E at $p = \pi(x)$. Again by [N, Lemma 2.7] this implies that there is a proper subspace E'_p which is closed under the restriction of the ϕ_i's to p and which contains $\mathrm{Im}(\psi_p)$. If we assume that the cokernel is not supported on points, such a subspace would exist for all p in $\mathbb{P}^1$, since the support of the cokernel is at least one-dimensional and does not intersect D_∞. But then the canonical saturated ADHM subsheaf $\mathcal{E}'$ of $\mathcal{E}$ defined in Lemma 3.13 has rank strictly smaller than the rank of $\mathcal{E}$ and so it is destabilizing. It remains to prove that F is pure, and from [Di2, Lemma 6.4] this is equivalent to proving that F is flat over $\mathbb{P}^1$. This is not affected by the change in the rank of the framing, so we refer to [Di2, Proposition 7.7]. □

Proposition 3.21. *The constructions of Lemma 3.18 and Lemma 3.20 define a bijection between $P_{n,d}(X)$ and $M_{d,n-d}(\mathbb{P}^1, r)$.*

Proof. Lemma 3.18 shows how to construct an asymptotically stable ADHM sheaf $\mathcal{E}$ starting from a stable pair (F, s). Suppose $\rho\colon (F, s) \longrightarrow (F', s')$ is an isomorphism of pairs. Since the Beilinson spectral sequence is functorial we get an isomorphism $\Phi(\rho)$ of the corresponding complexes (3.9), and this in turn gives an isomorphism $\widetilde{\Phi}(\rho)$ between the complexes $\mathcal{F}^\bullet$ and $\mathcal{F}'^\bullet$ which define $\mathcal{E}$ and $\mathcal{E}'$. This implies directly that $\mathrm{R}^0\pi_*(\rho)\colon E \longrightarrow E'$ induces the desired isomorphism of ADHM sheaves, since it satisfies the compatibility conditions because $\widetilde{\Phi}(\rho)$ is a morphism of complexes.

Lemma 3.20 shows how to construct a stable pair (F, s) from an ADHM sheaf. In particular, if $\xi\colon \mathcal{E} \longrightarrow \mathcal{E}'$ is an isomorphism of ADHM sheaves the pullback $\pi^*(\xi)$ defines an isomorphism of complexes (3.20). This induces an isomorphism between the cohomologies F and F'. The compatibility condition for the framing morphisms in Definition 3.9 implies that this is an isomorphism of stable pairs. The two constructions thus pass to the level of isomorphism classes, and it is not difficult to verify they are mutually inverse. □

We now want to prove a version of Proposition 3.21 that holds in families. Similarly to 2.21, it is based on a relative version of the Beilison theorem for projective bundles. Let X_T be the product $X \times T$ and let p_T be the projection onto T. Denote by $\mathcal{O}_{X_T}(j)$ the pullback $p_T^* \mathcal{O}_\pi(j)$ and define $\Omega_{X_T} := p_T^* \Omega_\pi$. Considering the diagram

$$\begin{tikzcd} X_T \times_{\mathbb{P}^1_T} X_T \arrow[r, "p_2"] \arrow[d, "p_1"'] & X_T \arrow[d, "\pi_T"] \\ X_T \arrow[r, "\pi_T"] & \mathbb{P}^1_T \end{tikzcd} \tag{3.19}$$

we have the following theorem.

Theorem 3.22 ([Di2, Lemma 7.1]). *Let F_T be a sheaf on X_T, and assume F_T is flat over T. Then there is a spectral sequence E_r^{pq} of sheaves on X_T with E_1-term given by*

$$\mathrm{E}_1^{lk} = \pi_T^* R^k \pi_{T*}(F_T \otimes \Omega_{X_T}^{-l}(-l)) \otimes \mathcal{O}_{X_T}(l)$$

that converges to the associated graded $\oplus_l E_\infty^{-ll}$ of a filtration of F_T.

As mentioned at the beginning of the section, the following theorem is implicit in the work of Diaconescu [Di2, Theorem 1.9].

Theorem 3.23. *The moduli space $P^r_{n,d}(X)$ is isomorphic to the moduli space of conifold-like ADHM sheaves $M^{con}_{(d,n-d)}(\mathbb{P}^1, r)$.*

Proof. Let T be an affine scheme and let (F_T, s_T) be a family of stable pairs over T. Since $\mathcal{F}_T$ is flat over T and F_t is flat over $\mathbb{P}^1$ for all $t \in T$, we have that $\mathcal{F}_T$ is flat over $\mathbb{P}^1 \times T$. Hence for each t in T we have

$$\mathrm{R}^i \pi_{T*}(F_T \otimes \Omega^{-j}_{\pi_T}(-j))\big|_{\mathbb{P}^1 \times \{t\}} = \mathrm{R}^i \pi_{\{t\}*}(F_{\{t\}} \otimes \Omega_{\pi_{\{t\}}}(-j)) = 0, \tag{3.20}$$

where $\pi_{\{t\}}$ the restriction of π_T to $X \times \{t\}$, because the restriction $(F_T|_{\{t\}}, s_T|_t)$ is a stable pair. By Theorem 3.22, if we denote $\mathrm{R}^0\, \pi_{T*}(F_T)$ by E_T, we get a complex of sheaves $\mathcal{F}_T^\bullet$

$$\mathcal{O}_{X_T}(-2) \otimes \pi^* \mathrm{R}^0 \pi_*(F \otimes \Omega^2_{X_T}(2)) \xrightarrow{\beta_{T-2}} \mathcal{O}_{X_T}(-1) \otimes \pi^* \mathrm{R}^0 \pi_{T*}(F \otimes \Omega^1_{X_T}(1)) \xrightarrow{\beta_{T-1}} \pi_T^* E_T \tag{3.21}$$

whose only cohomology is F_T in degree 0. Proceeding as in Lemma 3.18 but relative to T, we can construct a quasi-isomorphic complex

$$\mathcal{O}_{X_T}(-2) \otimes \pi^*(E_T \otimes \mathcal{O}_{\mathbb{P}^1}(2)) \xrightarrow{\alpha_{-2}} \mathcal{O}_{X_T}(-1) \otimes \pi^*(E_T \otimes \mathcal{O}_{\mathbb{P}^1_T}(1)^{\oplus 2}) \xrightarrow{\alpha_{-1}} \pi^* E_T, \tag{3.22}$$

where α_T^{-1} and α_T^{-2} are the relative version of (3.5) and (3.6). This constructs a flat family of asymptotically stable ADHM sheaves $(E_T, \phi_{T\,1}, \phi_{T\,2}, R^0\pi_{T\,*}(s_T))$, since the ADHM equation and the asymptotic stability can be checked pointwise. Conversely, if we start with a flat family of asymptotically stable ADHM sheaves on $\mathbb{P}^1_T$ we can pull-back the datum to obtain a complex of the form (3.22). It can be checked by restricting to each single fiber that the only cohomology F_T sits in degree zero. The pull-back $\pi_T^*(\psi_T)$ defines a multi-section s_T, and by Lemma 3.20 the restriction $(F_T|_t, s_T|_t)$ is a rank r stable pair. By Lemma 5.3 of [Di2] we obtain that F_T is flat over T so that (F_T, s_T) is a family of stable pairs. From Proposition 3.21 we see that the two constructions are inverse to each other. □

3.3 From conifold-like ADHM sheaves on curves to moduli of quiver representations

This section is devoted to the construction of a scheme-theoretic isomorphism between $M^{con}_{(n,d)}(\mathbb{P}^1, r)$ and the moduli space $M^{\theta}_{(n+d,d)}(Q_{(r,0)}, W)$ of θ-stable $(r, 0)$-framed representations of the conifold quiver Q with superpotential W depicted in Figure 3.2, where θ is a particular choice of stability condition appearing in the literature under the name of Pandharipande-Thomas stability [NN]. We proceed in analogy with Chapter 2, i.e. we first establish a bijection at the level of geometric points by means of the Beilinson spectral sequence (1.5), and then we prove that this extends to the level of families, thus giving an isomorphism of moduli spaces.

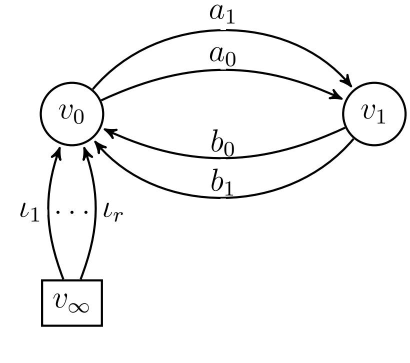

Figure 3.2: The framed conifold quiver $Q_{(r,0)}$.

The following lemma, due to K. Nagao and H. Nakajima [NN, Lemma 4.4], describes the chamber decomposition of the space of King stability conditions for the framed conifold quiver. Following the conventions of Chapter 1, it is important to recall that a vector θ in $\mathbb{R}^2$ defines a King stability condition for the framed quiver $Q_{(r,0)}$ with dimension vector $\underline{d}$ by taking $\hat{\theta} = \hat{\theta}(\underline{d}) = (-\theta \cdot \underline{d}, \theta)$.

Lemma 3.24. *Consider the set of walls $\mathcal{L}$ in $\mathbb{R}^2$ consisting of the lines depicted in Figure 3.3:*

- $L_m^+ = \{(\theta_0, \theta_1) \colon (m-1)\theta_0 + m\theta_1 = 0\}$*;*
- $L_0 = \{(\theta_0, \theta_1) \colon \theta_0 + \theta_1 = 0\}$*;*
- $L_m^- = \{(\theta_0, \theta_1) \colon m\theta_0 + (m-1)\theta_1 = 0\}$*;*

where m ranges through all the positive integers. Suppose that θ does not lie on one of the walls in $\mathcal{L}$. For any choice of dimension vector $\underline{d}$ the King stability condition $\hat{\theta}$ of $Q_{(r,0)}$ is generic.

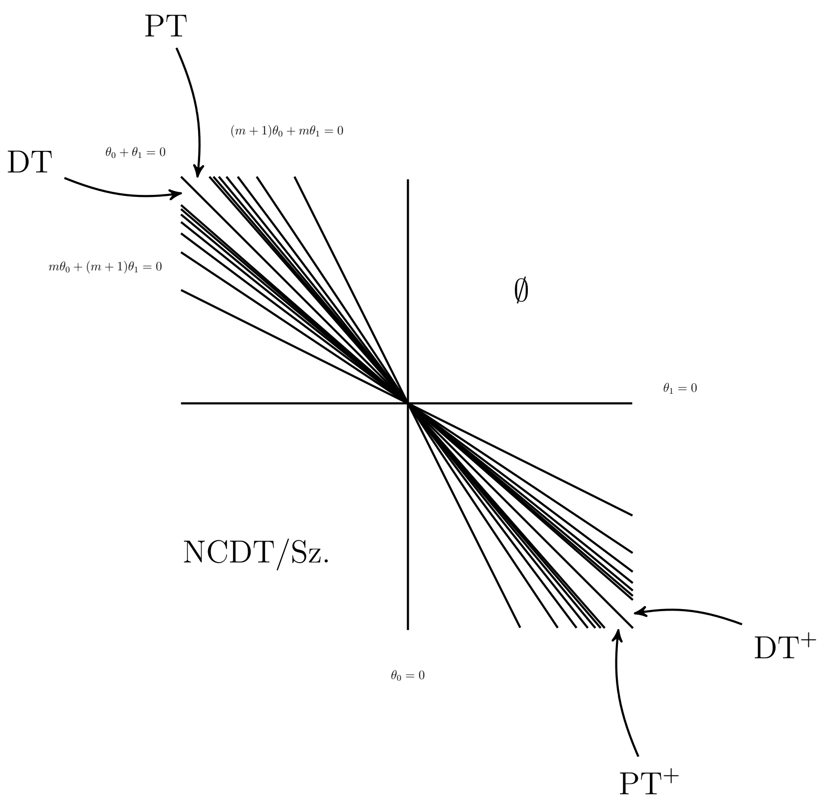

Figure 3.3: Chamber structure of the framed conifold quiver.

As mentioned above, we will be interested in one particular (limiting) stability condition which is referred to in the literature as PT stability. A framed representation V of $Q_{(r,0)}$ with dimension vector $(1,\underline{d})$ is said to be PT stable if any non trivial subrepresentation V' of dimension vector $(d'_\infty,\underline{d}')$ satisfies the inequality

$$\theta\cdot\underline{d}'-d'_\infty(\theta\cdot\underline{d})<0,$$

where θ is chosen such that

$$\theta_0<0,\ 0<\theta_0+\theta_1<\epsilon, \tag{3.23}$$

for a choice of $\epsilon=\epsilon(\underline{d})>0$ sufficiently small. To be more precise, for each fixed dimension vector $\underline{d}=(d_1,d_2)$ it is sufficient to consider θ lying in the chamber C_{d_1+1} consisting of the interior of the half cone spanned by the vectors $(-1,1)$ and $(-d_1,d_1+1)$. In this sense the PT stability can be understood as a limit of stability conditions approaching the wall given by the half line generated by the vector $(-1,1)$.

Remark 3.25. The notation PT, adopted from Nakajima and Nagao in [NN], is motivated by their proposition [NN, Proposition 2.11], where they prove that the moduli space of PT-stable representations of the framed conifold quiver is isomorphic to the moduli space of rank 1 stable pairs on the resolved conifold X^o, which was studied in the context of Donaldson-Thomas theory by Pandharipande and Thomas in [PT1].

In rest of the section we develop the constructions which will allow us to prove the following proposition.

Proposition 3.26. *The moduli space $M^{con}_{n,d}(\mathbb{P}^1,r)$ is isomorphic to $M^{\theta}_{(n+d,d)}(Q_{(r,0)},W)$. In particular, it is proper, and it arises as the global critical locus of a regular function $f_{n,d,r}\colon U^{con}_{n,d,r}\longrightarrow\mathbb{C}$ on the smooth quasi-projective variety $U^{con}_{n,d,r}$. That is, scheme-theoretically we have*

$$M^{con}_{n,d}(\mathbb{P}^1,r)=\{df_{n,d,r}=0\}\subset U^{con}_{n,d,r}.$$

Theorem 3.23 and Proposition 3.26 immediately imply the following theorem.

Theorem 3.27. *The moduli space $P^r_{n,d}$ is proper, and it arises as the scheme-theoretic critical locus of a regular function on a smooth quasi-projective variety.*

From conifold-like ADHM sheaves to quiver representations

Starting from an r-framed asymptotically stable conifold-like ADHM sheaf of type (n,d), we construct an $(r,0)$-framed PT-stable representation of the Jacobi algebra $J(Q,W)$ of the conifold quiver with superpotential defined in Example 1.6, with dimension vector $(n+d,d)$. In particular, we show that two isomorphic ADHM sheaves give rise to isomorphic representations if and only if they are isomorphic.

Let $\mathcal{E}=(E,\phi_1,\phi_2,\psi)$ be an asymptotically stable ADHM sheaf. Consider the diagram

$$\begin{tikzcd} & & \mathbb{P}^1 \arrow[dr] & \\ \mathbb{P}^1 \arrow[r, "\Delta"] & \mathbb{P}^1\times\mathbb{P}^1 \arrow[ur, "p"] \arrow[dr, "q"'] & & \mathrm{pt} \\ & & \mathbb{P}^1 \arrow[ur] & \end{tikzcd}$$

where the notation is consistent with that adopted in Section 1.2. We have seen that there is a locally free decomposable resolution of the pushforward of E along the diagonal

$$(E\otimes\Omega_{\mathbb{P}^1}(1))\boxtimes\mathcal{O}_{\mathbb{P}^1}(-1)\xrightarrow{\varphi}E\boxtimes\mathcal{O}\longrightarrow\Delta_*(E),$$

where φ is the evaluation on the section

$$\sum_i \mathrm{ev}\left(\frac{\partial}{\partial w_i}\right)\boxtimes z_i$$

for choices of homogeneous coordinates $(w_0:w_1)$ and $(z_0:z_1)$ on the first and second copy of $\mathbb{P}^1$ respectively. Under the identification $\Omega_{\mathbb{P}^1}(1)\cong\mathcal{O}_{\mathbb{P}^1}(-1)$ the morphism φ can be written explicitly as

$$-\left(\mathrm{Id}_E\otimes w_1\right)\boxtimes z_0+\left(\mathrm{Id}_E\otimes w_0\right)\boxtimes z_1. \tag{3.24}$$

Recall from Lemma 3.15 that E has no direct summand of negative degree. Thus, as a consequence of the convergence of the Beilinson spectral sequence on $\mathbb{P}^1$ there is a canonical two-step locally free resolution of E,

$$0\longrightarrow\mathrm{H}^0(E(-1))\otimes\mathcal{O}_{\mathbb{P}^1}(-1)\xrightarrow{\beta}\mathrm{H}^0(E)\otimes\mathcal{O}_{\mathbb{P}^1}\longrightarrow E\longrightarrow 0. \tag{3.25}$$

The morphism β is of the form $\beta=b_0z_0+b_1z_1$, where the b_i's are linear morphisms in $\mathrm{Hom}_{\mathbb{C}}\left(\mathrm{H}^0(E(-1)),\mathrm{H}^0(E)\right)$. From equation (3.24) it is clear that, up to sign, the

b_i's are the morphisms induced in cohomology by the multiplication by the section w_{i+1} of $\mathcal{O}_{\mathbb{P}^1}(1)$, where we read the indices modulo 2. Consider the twists $\phi_i(-1)$ of the morphisms ϕ_i in the ADHM datum, that is

$$\phi_i(-1) := \phi_i \otimes \mathrm{Id} \colon E(1) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow E \otimes \mathcal{O}_{\mathbb{P}^1}(-1).$$

The $\phi_i(-1)$'s define two linear morphisms

$$a_{i-1} \colon \mathrm{H}^0(E) \longrightarrow \mathrm{H}^0(E(-1))$$

by simply taking a_{i-1} to be the induced morphism $\mathrm{H}^0(\phi_i(-1))$ in cohomology. In the same way we obtain a framing

$$I \colon \mathbb{C}^r \longrightarrow \mathrm{H}^0(E),$$

where we denote $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}^{\oplus r})$ by $\mathbb{C}^r$ to stress the fact that morphisms between ADHM sheaves restrict to the identity at the level of framings, as in Definition 3.9.

Summarizing, we started from an ADHM sheaf $\mathcal{E}$ and we constructed:

- two vector spaces $V_0 = \mathrm{H}^0(E)$ and $V_1 = \mathrm{H}^0(E(-1))$ of dimension $n+d$ and d respectively;
- two morphisms $b_0, b_1 \colon V_1 \longrightarrow V_0$;
- two morphisms $a_0, a_1 \colon V_0 \longrightarrow V_1$;
- a framing $I \colon \mathbb{C}^r \longrightarrow V_0$, or equivalently r linear maps $\iota_j \colon \mathbb{C} \longrightarrow V_0$.

In other words, we have an $(r,0)$-framed representation V of the conifold quiver Q with dimension vector $(n+d,d)$. The following lemmas are devoted to showing that V sits in the locally closed subvariety $R^{\theta}_{(d+n,d)}(Q_{(r,0)}, W)$ of PT-stable representations of the conifold quiver with superpotential.

Lemma 3.28. *The representation V satisfies the relations*

$$a_0 b_j a_1 - a_1 b_j a_0, \ j \in \{0,1\}, \tag{3.26}$$

$$b_0 a_j b_1 - b_1 a_j b_0, \ j \in \{0,1\}. \tag{3.27}$$

Proof. We start with equation (3.26). We consider in particular the (-1)-twist of the ADHM relation

$$\phi_1(-1) \circ \phi_2 - \phi_2(-1) \circ \phi_1 = 0. \tag{3.28}$$

The functoriality of the Beilinson spectral sequence together with (3.28) gives a commutative diagram

$$\begin{tikzcd}
 & E(1) \arrow[rr, "\phi_1"] & & E \arrow[rr, "\phi_2(-1)"] & & E(-1) \\
E(1) \arrow[ur, "\mathrm{Id}"] \arrow[rr, "\phi_2"] & & E \arrow[rr, "\phi_1(-1)"] & & E(-1) \arrow[ur, "\mathrm{Id}"] & \\
 & F_1 \otimes \mathcal{O} \arrow[uu, "\mathrm{ev}"] \arrow[rr] & & V_0 \otimes \mathcal{O} \arrow[uu, "\mathrm{ev}"] \arrow[rr, "a_1 \otimes \mathrm{Id}"] & & V_1 \otimes \mathcal{O} \arrow[uu, "\mathrm{ev}"'] \\
F_1 \otimes \mathcal{O} \arrow[uu, "\mathrm{ev}"] \arrow[ur, "\mathrm{Id}"] \arrow[rr] & & V_0 \otimes \mathcal{O} \arrow[uu, "\mathrm{ev}"] \arrow[rr, "a_0 \otimes \mathrm{Id}"] & & V_1 \otimes \mathcal{O} \arrow[uu, "\mathrm{ev}"] \arrow[ur, "\mathrm{Id}"] & \\
 & V_0 \otimes \mathcal{O}(-1) \arrow[uu] \arrow[rr, "a_0 \otimes \mathrm{Id}"] & & V_1 \otimes \mathcal{O}(-1) \arrow[uu, "\beta"] \arrow[rr] & & F_{-2} \otimes \mathcal{O}(-1) \arrow[uu] \, , \\
V_0 \otimes \mathcal{O}(-1) \arrow[uu] \arrow[ur, "\mathrm{Id}"] \arrow[rr, "a_1\,\mathrm{Id}"] & & V_1 \otimes \mathcal{O}(-1) \arrow[uu, "\beta"] \arrow[rr] & & F_{-2} \otimes \mathcal{O}(-1) \arrow[uu] \arrow[ur, "\mathrm{Id}"] &
\end{tikzcd}$$

where we have denoted by F_1 and F_{-2} respectively the vector spaces $\mathrm{H}^0(E(1))$ and $\mathrm{H}^0(E(-2))$. The morphism of sheaves

$$(a_1 b_0 a_0 - a_0 b_0 a_1) z_0 + (a_1 b_1 a_0 - a_0 b_1 a_1) z_1$$

is identically 0, so the linear morphisms we have constructed must satisfy the relations of (3.26).
To prove that the representation V satisfies the relations of (3.27) it is enough to consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & E(-1) \arrow[r, "S(\alpha)"] & E^{\oplus 2} \arrow[r, "\alpha"] \arrow[d, "\phi_k \otimes \mathrm{Id}"'] & E(1) \arrow[r] \arrow[d, "\phi_k"] & 0 \\
 & & E(-1)^{\oplus 2} \arrow[r, "\alpha(-1)"] & E \, , &
\end{tikzcd}$$

where the first line is the syzygy resolution of $E(1)$ with $\alpha = \begin{pmatrix} w_0 & w_1 \end{pmatrix}$. Passing to the level of cohomology we get

$$V_1 \xrightarrow{\begin{pmatrix} b_0 & b_1 \end{pmatrix}^t} V_0^{\oplus 2} \xrightarrow{\mathrm{diag}(a_k)} V_1^{\oplus 2} \xrightarrow{\begin{pmatrix} b_1 & -b_0 \end{pmatrix}} V_0,$$

and since the composition $\alpha(-1) \circ (\phi_k \otimes \mathrm{Id}) \circ S(\alpha)$ is zero we get (3.27). □

Lemma 3.29. *The representation* V *is* PT-*stable.*

Proof. Assume first that there exists a destabilizing subrepresentation V' of V with $V'_\infty = 1$. The diagram

$$\begin{tikzcd} 0 \arrow[r] & V_1 \otimes \mathcal{O}(-1) \arrow[r] & V_0 \otimes \mathcal{O} \arrow[r] & E \arrow[r] & 0 \\ 0 \arrow[r] & V'_1 \otimes \mathcal{O}(-1) \arrow[r] \arrow[u, hook] & V'_0 \otimes \mathcal{O} \arrow[r] \arrow[u, hook] & E' \arrow[r] \arrow[u, hook] & 0 \end{tikzcd}$$

defines a locally free subsheaf E' of E. It is immediate from the definition of quiver subrepresentation that in fact E' defines an ADHM subsheaf of E containing $\mathrm{Im}(\psi)$. Furthermore, since we are assuming the subrepresentation V' of dimension $(1, d'_0, d'_1)$ to be destabilizing with respect to the PT-stability, we have

$$0 < \delta(d_1 - d'_1) \leq \theta_0((d'_0 - d'_1) - r),$$

for δ sufficiently small and positive. Hence the vector bundle E' must have rank strictly smaller than the rank of E. The saturation of E' in E is ϕ_i-invariant and defines a proper saturated subsheaf of E containing $\mathrm{Im}(\psi)$, thus contradicting the asymptotic stability.

We show now that a non-zero subrepresentation $V' \subset V$ with $V'_\infty = 0$ cannot be destabilizing with respect to PT-stability. The map $\beta = b_0 z_0 + b_1 z_1$ is injective on each fiber and defines a vector bundle E, so the restriction $\beta' = b'_0 z_0 + b'_1 z_1$ defines a non-trivial vector sub-bundle of E. This implies that the dimension vector of V' is of the form $(d'_1 + r', d'_1)$ for some $d_1 \geq 0$ and $r > 0$, and so $\theta \cdot \dim(V') < 0$. $\square$

In the following remarks, we discuss the behavior of the correspondence we just established with respect to isomorphisms. Together they prove that the representations defined starting from two ADHM sheaves are isomorphic if and only if the two ADHM sheaves are themselves isomorphic.

Remark 3.30. Starting from an isomorphism $\xi \colon \mathcal{E} \longrightarrow \mathcal{E}'$ of ADHM sheaves we can construct an isomorphism between the induced representations V and V'. From the functoriality of the Beilinson spectral sequence, ξ induces a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & V_1 \otimes \mathcal{O}(-1) \arrow[r, "\beta"] \arrow[d, "\mathrm{H}^0(\xi(-1))"] & V_0 \otimes \mathcal{O} \arrow[r] \arrow[d, "\mathrm{H}^0(\xi)"] & E \arrow[r] \arrow[d, "\xi"] & 0 \\ 0 \arrow[r] & V'_1 \otimes \mathcal{O}(-1) \arrow[r, "\beta'"] & V'_0 \otimes \mathcal{O} \arrow[r] & E' \arrow[r] & 0, \end{tikzcd} \tag{3.29}$$

so that we have two invertible morphisms $g_0 = \mathrm{H}^0(\xi)$ and $g_1 = \mathrm{H}^0(\xi(-1))$ respectively in $\mathrm{GL}(V_0)$ and $\mathrm{GL}(V_1)$, satisfying

$$b'_i = g_0 \circ b_i g_1^{-1}.$$

From Definition 3.9 we have $\xi(-1)\circ\phi_i = \phi'_i\circ\xi$, which in cohomology gives the relation

$$a'_i = g_1 \circ a_i \circ g_0^{-1}.$$

Since we also have also $\xi \circ \psi = \psi'$, it is clear that (Id, g_0, g_1) defines an isomorphism between the representations V and V'.

Remark 3.31. Suppose now that the two representations defined by the ADHM sheaves $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic. Denoting the two representations by V and V' and the isomorphism by (g_0, g_1) we obtain a morphism of sheaves $\xi\colon \mathcal{E} \longrightarrow \mathcal{E}'$ from the fact that an isomorphism between the Beilinson resolutions of E and E' induces an isomorphism of sheaves. Now it remains to check that the morphisms in the definitions of $\mathcal{E}$ and $\mathcal{E}'$ are compatible, but this is just a consequence of the fact that the morphisms of the corresponding Beilinson resolutions are compatible from the definition of isomorphism of quiver representations.

From quiver representations to conifold-like ADHM sheaves

We describe how to construct an asymptotically stable conifold-like ADHM sheaf of type $(d_0 - d_1, d_1)$ from the datum of a PT-stable r-framed representation V of the conifold quiver with dimension vector (d_0, d_1).

We construct a sheaf on $\mathbb{P}^1$ by reverse engineering the Beilinson theorem for sheaves on $\mathbb{P}^1$. Starting from the linear maps $b_i\colon V_1 \longrightarrow V_0$ of the given representation V, we consider the morphism of sheaves

$$\beta\colon V_1 \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow V_0 \otimes \mathcal{O}_{\mathbb{P}^1},\ \ \beta = b_0 z_0 + b_1 z_1.$$

Lemma 3.32. *The cokernel E of the morphism β is a locally free sheaf on $\mathbb{P}^1$ of rank $d_0 - d_1$ and degree d_1.*

Proof. We need to show that β is injective when considered as a morphism of vector bundles, i.e. that the restriction β_λ to the fiber over a point λ of $\mathbb{P}^1$ is an injective linear map. More precisely, we show that the non-injectivity of β_λ for $\lambda \in \mathbb{P}^1$ forces the representation V to be PT-unstable. Let K' be the kernel of β_λ, and assume $K' \neq 0$. Consider the $J(Q,W)$-module K generated by K', i.e. the subrepresentation K of V with underlying vector spaces

$$K_0 = \langle b_i K', b_i a_j b_k K', \ldots\rangle,\ \ K_1 = \langle K', a_i b_j K' \ldots\rangle$$

based respectively at the vertices 0 and 1 respectively. Using the relations imposed by the potential W we have

$$\beta_\lambda a_{i_1} b_{j_1} \dots a_{i_n} b_{j_n} v = b_{j_1} a_{i_1} \dots b_{j_n} a_{i_n} \beta_\lambda v,\ \ v \in V_1,$$

so that we have the equality $K_1 = K'$. It is then clear that $\dim K_0 \leq \dim K_1$, so that the subrepresentation K of V is destabilizing with respect to the PT stability. The rank and degree of E can be easily computed from the fact that E sits by construction in the exact sequence

$$0 \longrightarrow V_1 \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow V_0 \otimes \mathcal{O}_{\mathbb{P}^1} \longrightarrow E \longrightarrow 0. \tag{3.30}$$

□

We have thus constructed a candidate E for the underlying sheaf of an asymptotically stable conifold-like ADHM sheaf on $\mathbb{P}^1$. Notice in particular that it follows directly from the definition that E has no direct summand of negative degree, since from (3.30) one can easily deduce that $\mathrm{H}^1(E) = \mathrm{H}^1(E(-1)) = 0$. We now construct the ADHM morphisms $\phi_i \colon E(1) \longrightarrow E$. In order to do so, we will need to consider the syzygy resolution

$$0 \longrightarrow E(-1) \xrightarrow{\gamma = \begin{pmatrix} w_1 & -w_0 \end{pmatrix}^t} E^{\oplus 2} \xrightarrow{\begin{pmatrix} w_0 & w_1 \end{pmatrix}} E(1) \longrightarrow 0\ , \tag{3.31}$$

where (w_0, w_1) is a choice of coordinates on $\mathbb{P}^1$. We consider the diagram

$$\begin{tikzcd}
V_1 \otimes \mathcal{O}(-2) \arrow[r, "\begin{pmatrix} w_1 & -w_0 \end{pmatrix}^t"] \arrow[d, "\beta(-1)"'] & V_1^{\oplus 2} \otimes \mathcal{O}(-1) \arrow[d, "\mathrm{diag}(\beta)"] \arrow[dr, "\begin{pmatrix} a_k b_1 & a_k b_0 \end{pmatrix}"] & \\
V_0 \otimes \mathcal{O}(-1) \arrow[r, "\begin{pmatrix} w_1 & -w_0 \end{pmatrix}^t"] & V_0^{\oplus 2} \otimes \mathcal{O} \arrow[dr, "\begin{pmatrix} b_1 a_k \\ b_0 a_k \end{pmatrix}^t"'] & V_1 \otimes \mathcal{O}(-1) \arrow[d, "\beta"] \\
& & V_0 \otimes \mathcal{O},
\end{tikzcd} \tag{3.32}$$

where the left square is induced by the morphism γ in (3.31) and where the morphisms corresponding to vertical lines are defined as in equation (3.25). Since the linear maps a_i, b_j satisfy the relations induced by the superpotential W, diagram (3.32) is commutative.

Lemma 3.33. *The diagram (3.32) defines a morphism of sheaves $\phi_k\colon E(1) \longrightarrow E$ for $k \in \{0,1\}$.*

Proof. From the syzygy resolution (3.31), giving a morphism $\rho\colon E(1) \longrightarrow E$ is equivalent to defining a morphism of complexes $\rho\colon E_1^\bullet \longrightarrow \widetilde{E}_1^\bullet$ as in the diagram

$$\begin{tikzcd} E_1^\bullet\colon & E(-1) \arrow[r, "\gamma"] \arrow[d, "\rho_{-1}"'] & E^{\oplus 2} \arrow[d, "\rho_0"] \\ \widetilde{E}_1^\bullet\colon & 0 \arrow[r] & E. \end{tikzcd}$$

From the commutativity of (3.32) we can define a morphism $\rho\colon E_1^\bullet \longrightarrow \widetilde{E}_1^\bullet$, which in turn defines a morphism of sheaves $\phi_{k+1}\colon E(1) \longrightarrow E$. □

We need to prove that the ADHM morphisms ϕ_k satisfy the ADHM relation.

Lemma 3.34. *The morphisms ϕ_1 and ϕ_2 satisfy $\phi_1 \circ \phi_2(1) - \phi_2 \circ \phi_1(1) = 0$.*

Proof. From the definition of the ϕ_i's it suffices to check commutativity at the level of the syzygy resolutions. Define $\widetilde{\phi_k}\colon V_0^{\oplus 4} \longrightarrow V_0^{\oplus 2} \otimes \mathcal{O}$ as

$$\widetilde{\phi_k} = \begin{pmatrix} b_1 a_{k-1} & b_0 a_{k-1} & 0 & 0 \\ 0 & 0 & b_1 a_{k-1} & b_0 a_{k-1} \end{pmatrix} \otimes \mathrm{Id}\,.$$

The relation between the ϕ_i's can be recovered from

$$\begin{pmatrix} b_1 a_0 & b_0 a_0 \end{pmatrix} \otimes \mathrm{Id} \circ \widetilde{\phi_2} = \begin{pmatrix} b_1 a_1 & b_0 a_1 \end{pmatrix} \otimes \mathrm{Id} \circ \widetilde{\phi_1}. \tag{3.33}$$

□

We can deduce the asymptotic stability of the ADHM sheaf $\mathcal{E}$ from the PT-stability of the quiver representation.

Lemma 3.35. *The ADHM sheaf $\mathcal{E}$ is asymptotically stable.*

Proof. Assume that $\mathcal{E}$ admits an ADHM subsheaf $\mathcal{E}'$ contradicting asymptotic stability. By construction the representation V admits a subrepresentation V' with dimension vector $(n'+d', d')$, where n' is the rank of E' and d' is the degree. We have in particular that n' needs to be strictly smaller than n, and from the definition of PT-stability we have

$$\hat{\theta} \cdot \dim(V') = \theta_1(n - n') + \epsilon c,$$

for some integer c and $\epsilon > 0$ arbitrarily small. But then V' destabilizes V. □

We conclude with analogues of Remarks 3.30 and 3.31, i.e. by showing that two representations define isomorphic sheaves if and only if they are themselves isomorphic.

Remark 3.36. The construction preserves isomorphism classes. To show this it is enough to notice that two isomorphic representations V and V' define a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & V_1 \otimes \mathcal{O}(-1) \arrow[r, "\beta"] \arrow[d, "g_1\otimes \mathrm{Id}"] & V_0 \otimes \mathcal{O} \arrow[r] \arrow[d, "g_0\otimes \mathrm{Id}"] & E \arrow[r] \arrow[d, "\xi"] & 0 \\
0 \arrow[r] & V_1 \otimes \mathcal{O}(-1) \arrow[r, "\beta'"] & V_0 \otimes \mathcal{O} \arrow[r] & E' \arrow[r] & 0.
\end{tikzcd}$$

One can check that the commutativity conditions required in Definition 3.9 are satisfied using the relations $b_i' = g_0 b_i g_1^{-1}$ and $a_i' = g_1 a_i' g_0$.

Remark 3.37. Assume that two representations V and V' define isomorphic ADHM sheaves $\mathcal{E}$ and $\mathcal{E}'$. Recalling that (3.30) identifies the vector spaces in the representations with cohomology groups of E and E' and their twist by $\mathcal{O}_{\mathbb{P}^1}(-1)$, the isomorphism of sheaves $\xi\colon E \longrightarrow E'$ induces isomorphisms in cohomology $g_0 := \mathrm{H}^0(\xi)$ and $g_1 := \mathrm{H}^0(\xi \otimes \mathrm{Id}_{\mathcal{O}_{\mathbb{P}^1}(-1)})$ which fit in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & V_1 \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \arrow[r, "\beta"] \arrow[d, "g_1\otimes \mathrm{Id}"] & V_0 \otimes \mathcal{O} \arrow[r] \arrow[d, "g_0\otimes \mathrm{Id}"] & E \arrow[r] \arrow[d, "\xi"] & 0 \\
0 \arrow[r] & V_1' \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \arrow[r, "\beta'"] & V_0' \otimes \mathcal{O}_{\mathbb{P}^1} \arrow[r] & E' \arrow[r] & 0.
\end{tikzcd} \tag{3.34}$$

We just need to check that (g_0, g_1) defines a morphism of representations. The commutativity of (3.34) implies immediately that $b_i' = g_0 b_i g_1^{-1}$ for $i \in \{0,1\}$. The fact that $a_i' = g_1 a_i g_0^{-1}$ follows directly from the relation $\phi_k' \circ \xi = \xi \circ \phi_k$, once we look at the morphism induced in cohomology.

Moduli of conifold-like ADHM sheaves as a proper critical locus

Combining the results of the two previous subsections with the relative version of the Beilinson spectral sequence (Theorem 1.31) gives the proof of Proposition 3.26, which we report here.

Proposition 3.38. *There is a scheme-theoretic isomorphism between the moduli spaces* $M^{\mathrm{con}}_{(n,d)}(\mathbb{P}^1, r)$ *and* $M^{\theta}_{(n+d,d)}(Q_{(r,0)}, W)$.

Proof. Suppose we have a flat family $\mathcal{E}_S$ of ADHM sheaves over an affine scheme S, with underlying vector bundle E_S. Consider the diagram

$$\begin{tikzcd} \mathbb{P}^1_S \arrow[r, "q"] \arrow[d, "p"'] & \mathbb{P}^1 \arrow[d] \\ S \arrow[r] & \mathrm{pt}, \end{tikzcd}$$

so that we can adopt the notation of Section 2.1.3. By Theorem 2.19 there is a spectral sequence

$$\mathrm{E}_1^{rs} = \mathrm{R}^s\, p_*(E_S \otimes \Omega_p^{-r}(-r)) \boxtimes \mathcal{O}_{\mathbb{P}^1}(r)$$

whose associated graded converges to a filtration of E_S. Since E_S is flat over S we have by Grauert's theorem that

$$\mathrm{R}^s\, p_*(E_S \otimes \Omega_p^{-r}(-r))_s = \mathrm{H}^s(E_S|_s \otimes \Omega_{\mathbb{P}^1}^{-r}(-r)),$$

and as the fiber of E_S over a geometric point s has no direct summand of negative degree the spectral sequence degenerates at the E_1-term. Because S is affine, we get a two-step locally free resolution

$$0 \longrightarrow (V_1 \otimes \mathcal{O}_S) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{\beta_S} (V_0 \otimes \mathcal{O}_S) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow E_S \longrightarrow 0,$$

for some vector spaces V_0 and V_1 of dimension $n+d$ and d respectively. Notice also that $\beta_S = B_0 z_0 + B_1 z_1$, where B_0 and B_1 are two morphism of vector bundles from $V_1 \otimes \mathcal{O}_S$ to $V_0 \otimes \mathcal{O}_S$. The morphisms $\phi_k(S)$ in the definition of ADHM sheaf induce morphisms A_k in the opposite direction simply by taking the pushforward $\mathrm{R}^0 p_*$, and in the same way from the framing ψ_S we get a morphism $\mathcal{O}_S^{\oplus r} \longrightarrow V_0 \otimes \mathcal{O}_S$. We have thus constructed a flat family of representations of the conifold quiver $Q_{(r,0)}$ with superpotential. Since both the stability of the representations and the fact that they satisfy the relations induced by the superpotential W can be checked on geometric points, we can apply Lemma 3.28 and 3.29 to see that in fact we constructed a flat family of representations of $(Q_{(r,0)}, W)$ which are PT-stable.
Consider now a flat family of PT-stable representations of $(Q_{(r,0)}, W)$ with dimension $(1, d_0, d_1)$ on an affine scheme S. We construct E_S as the cokernel of the morphism

$$V_0 \otimes \mathcal{O}_S \boxtimes \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{B_0 \boxtimes z_0 + B_1 \boxtimes z_1} V_1 \otimes \mathcal{O}_S \boxtimes \mathcal{O}_{\mathbb{P}^1}\ .$$

Applying Lemma 3.32 for all the geometric points of S we obtain that E_S is locally free of rank $d_0 - d_1$ and degree d_1, and so also flat over S. A relative version of diagram (3.32) allows us to construct the morphisms ϕ_k for $k \in \{1, 2\}$, and one can check all the compatibilities of the morphisms and the asymptotic stability simply by looking at the pointwise constructions of the previous sections. In this way we obtain a flat family of asymptotically stable rank r ADHM sheaves on $\mathbb{P}^1$.

It is clear from Lemma 3.30 and 3.36 that this construction sends isomorphic families to isomorphic families. On the other hand, Lemma 3.31 and 3.37 show that the morphism of functors defined in this way is an isomorphism, thus concluding the proof. $\square$

3.4 Invariants

In this section we compute the generating series of motivic Donaldson-Thomas invariants for the moduli spaces $M^{\theta}_{\underline{d}}(Q_{(r,0)}, W)$ of PT-stable r-framed representations of the conifold quiver with superpotential. By Theorem 3.27, the specialization $\mathbb{L} = 1$ recovers the generating series of classical DT invariants of the moduli space of higher rank stable pairs on the resolution of the conifold singularity. As mentioned in Remark 2.23, it is not enough to have an isomorphism of schemes. In order to ensure that the expression we obtain for the quiver with superpotential is actually computing the generating series of motivic Donaldson-Thomas invariants for the higher rank stable pair moduli spaces, and in order to achieve this result we would need to enhance our construction to be an isomorphism of derived schemes with a (-1)-shifted symplectic structure and orientation, as explained in [BJM]. This is a very interesting question already for $r = 1$, but it will not be addressed in this thesis. Nevertheless, the isomorphism we constructed defines naturally a topological coefficient system $\Phi^r_{n,d} = \phi_{\omega^r_{n,d}}\left(\frac{k}{2}\right)[k]$ on $P^r_{(n,d)}$, where $k = 2nd - (d+n)^2 + nr$, and the specialization $\mathbb{L} = t^{\frac{1}{2}}$ computes the Poincaré polynomial of the hypercohomology of $P^r_{n,d}$ with respect to to $\Phi^r_{n,d}$. In this sense, the polynomial $[P^r_{n,d}]_{\mathrm{vir}}\big|_{\mathbb{L}^{\frac{1}{2}} \mapsto t^{\frac{1}{2}}}$ consists of a refinement of the classical DT invariant.

The techniques we use are inspired by the work of Morrison, Mozgovoy, Nagao and Szendröi [MMNS], and they consist of extending to generic framed quivers the procedure we have used in Section 2.2.2 for the framed 3-loop quiver with superpotential. The idea is to use the expression for the universal series of motivic invariants of the conifold quiver (Q, W) computed in [MMNS], as well as wall-crossing for framed objects, to obtain the invariants of $M^{\theta}_{\underline{d}}(Q_{(r,0)}, W)$.

We start by fixing some notation. Denote by α a dimension vector for the conifold quiver. We define Δ to be the union of

$$\Delta_+^{\mathrm{re}} = \{(i, i-1) \colon i \geq 1\} \cup \{(i-1, i) \colon i \geq 1\},$$

and

$$\Delta_+^{\mathrm{im}} = \{(i, i) \colon i \geq 1\}.$$

The universal series A_U of motivic Donaldson-Thomas invariants of (Q, W) lies in the Kontsevich-Soibelman quantum torus $\mathcal{T}_Q$ defined in (2.19), which is symmetric since the conifold quiver is symmetric. Using the fact that $I = \{a_1\}$ is a cut for (Q, W), the universal series

$$A_U = \sum_{\alpha} [\mathcal{M}_\alpha(Q, W)]_{\mathrm{vir}}\, y^\alpha \tag{3.35}$$

in the quantum torus $\mathcal{T}_Q$ has been computed in [MMNS, Theorem 2.1] via dimensional reduction:

$$A_U = \prod_{\alpha \in \Delta_+} A^\alpha,$$

where

$$A^\alpha = \prod_{j \geq 0} (1 - \mathbb{L}^{-j-\frac{1}{2}} y^\alpha), \quad \alpha \in \Delta_+^{\mathrm{re}}, \tag{3.36}$$

and

$$A^\alpha = \prod_{j \geq 0} (1 - \mathbb{L}^{-j} y^\alpha)(1 - \mathbb{L}^{-j+1} y^\alpha), \quad \alpha \in \Delta_+^{\mathrm{Im}}. \tag{3.37}$$

From now on we let ξ be a generic stability condition for Q, where here the word stability refers to the μ_ξ-stability of (1.2) and not necessarily to a King stability condition. Consider a representation V of (Q, W) and let $0 = V_0 \subset \ldots \subset V_n = V$ be its Harder-Narashiman filtration with respect to μ_ξ. We say that V has only positive (resp. negative) Harder-Narashiman factors if $\mu_\xi(V_i/V_{i-1}) > 0$ (resp. $\mu_\xi(V_i/V_{i-1}) < 0$) for all $i \in \{1, \ldots, n\}$. Let $\mathcal{M}_{\xi,\alpha}^{+}(Q, W)$ (resp. $\mathcal{M}_{\xi,\alpha}^{-}(Q, W)$) denote the stack of representations of (Q, W) which have only positive (resp. negative) Harder-Narasimhan factors. We let $A_{\xi,\alpha}^{\pm}$ be the virtual motive $[\mathcal{M}_{\xi,\alpha}^{\pm}(Q, W)]_{\mathrm{vir}}$ in $\widetilde{\mathcal{M}}_{\mathbb{C}}$ of Section 1.4, and we define the positive and negative generating series

$$A_\xi^{\pm} = \sum_{\alpha} A_{\xi,\alpha}^{\pm} y^\alpha. \tag{3.38}$$

The following lemma explains how the universal series factors depending on the positivity of the HN factors.

Lemma 3.39 ([MMNS, Lemma 2.6]). *We have a factorization*

$$A_\xi^\pm = \prod_{\pm\xi\cdot\alpha>0} A^\alpha \tag{3.39}$$

and the universal series can be expressed in terms of the positive and negative factors:

$$A_U = A_\xi^+ \cdot A_\xi^-. \tag{3.40}$$

Consider the extended quantum torus $\mathcal{T}_{\hat{Q}_r}$ where $\hat{Q}_r$ is the r-framed conifold quiver, and where the expression r-framed is intended as a shortening of $(r,0)$-framed. Let $y_\infty = y^{(\underline{0},1)}$. Note that

$$\mathcal{T}_{\hat{Q}_r} = \mathcal{T}_Q \oplus \bigoplus_d \widetilde{\mathcal{M}}_{\mathbb{C}}\; y_\infty^d, \tag{3.41}$$

where we identify y^α with $y^{(\alpha,0)}$. When $\alpha = (1,0)$ we will denote y^α by y_0, and we define similarly y_1. Since $\mathcal{T}_Q$ is commutative these two variables commute. From the definition of the product in Section 2.2.2 we have

$$y_\infty \cdot y^{(\alpha,0)} = \left(-\mathbb{L}^{\frac{1}{2}}\right)^{-r\alpha_0} y^{(\alpha,1)}, \tag{3.42}$$

where $\alpha = (\alpha_0, \alpha_1)$.

We define generating series in the extended quantum torus corresponding to invariants of framed sheaves:

$$\widetilde{A}_U = \sum_\alpha [\mathcal{M}^0_{\alpha,1}(\hat{Q}_r, W)]_{\mathrm{vir}} y^{(\alpha,1)}, \tag{3.43}$$

$$\widetilde{A}_\xi = \sum_\alpha [M^\xi_{\alpha,1}(\hat{Q}_r, W)]_{\mathrm{vir}} y^{(\alpha,1)}, \tag{3.44}$$

$$\widetilde{Z}_\xi = \sum_\alpha [M^\xi_{\alpha,1}(\hat{Q}_r, W)]_{\mathrm{vir}} y^{(\alpha,0)}. \tag{3.45}$$

Remark 3.40. It is worth recalling the notation we adopt in (3.43)-(3.45). The stack $\mathcal{M}^0_{\alpha,1}(\hat{Q}_r, W)$ parametrises r-framed representations of the conifold quiver with superpotential up to isomorphisms acting as change of basis at the vertices which are distinct from the framing vertex v_∞. The variety $M^\xi_{\alpha,1}(\hat{Q}_r, W)$ parametrises framed representations of (Q,W) which are stable with respect to the King stability condition induced by ξ up to isomorphisms acting as the identity on the vector space associated to the vertex of the framing.

In particular we have

$$y_\infty \cdot \widetilde{Z}_\xi((-\mathbb{L}^{\frac{1}{2}})^r y_0, y_1) = \widetilde{A}_\xi(y_0, y_1). \tag{3.46}$$

Recall that when $\hat{V}$ is a framed representation of Q we denote by V the representation obtained by forgetting the framing.

Lemma 3.41. *We have $\widetilde{A}_U = A_U \cdot y_\infty$.*

Proof. As we are considering the trivial stability condition, this follows from the fact that each framed representation $\hat{V}$ has a unique filtration

$$0 \subset V \subset \hat{V},$$

such that $\hat{V}/V$ is the simple representation based at the vertex ∞. □

Lemma 3.42. *Let $\hat{V}$ be an r-framed representation of Q. There is a filtration*

$$0 = \hat{V}^0 \subset \hat{V}^1 \subset \hat{V}^2 \subset \hat{V}^3 = \hat{V}$$

such that the quotients $\hat{U}^i = \hat{V}^i/\hat{V}^{i-1}$, when not zero, satisfy:

- *$\hat{U}^1_\infty = 0$ and each HN factor F of U^1 with respect to the stability condition ξ satisfies $\xi \cdot F > 0$;*
- *$\hat{U}^2_\infty = 1$ and $\hat{U}$ is a ξ-stable framed representation;*
- *$\hat{U}^3_\infty = 0$ and each HN factor F of U^3 with respect to the stability condition ξ satisfies $\xi \cdot F < 0$.*

Proof. For a given $\epsilon > 0$ sufficiently small, let μ_ϵ be the stability on $\hat{Q}_r$ defined by

$$\mu_\epsilon(\hat{V}) = \frac{\xi \cdot V}{\epsilon|V| + \hat{V}_\infty}.$$

Suppose we consider a representation $\hat{W}$ with $\hat{W}_\infty = 1$ and a subrepresentation $\hat{W}'$ with $\hat{W}'_\infty = 0$. We have $\xi \cdot \dim(\hat{W}') > 0$ if and only if $\mu_\epsilon(\hat{W}') > \mu_\epsilon(\hat{W})$. Consider the Harder-Narasimhan filtration of $\hat{V}$ with respect to μ_ϵ. Suppose the maximal μ_ϵ-destabilizing subrepresentation $\hat{V}^2$ of $\hat{V}$ has $\hat{V}^2_\infty = 1$. The quotient $\hat{U}^3$ has only negative HN-factors from the maximality of $\hat{V}^2$, and $\hat{V}^2$ is ξ-semistable. Indeed, if $\hat{U} \subset \hat{V}^2$ has $\hat{U}_\infty = 0$ then automatically $\xi \cdot U < 0$, and if $\hat{V}' \subset \hat{V}^2$ has $\hat{V}'_\infty = 1$ we have $(1 + \epsilon|V'|)\xi \cdot V^2 \geq (1 + \epsilon|V^2|)\xi \cdot V'$, so that $\xi \cdot V' - \xi \cdot V^2 < 0$. Suppose instead that the maximal μ_ϵ-destabilizing subrepresentation $\hat{U}$ of $\hat{V}$ has $\hat{U}_\infty = 0$. We see that $\hat{U}$ has only positive HN-factors with respect to ξ since for any subrepresentation $U' \subset U$, the condition $\mu_\epsilon(U') \leq \mu_\epsilon(\hat{U})$ together with $|\dim(U')| < |\dim(U)|$ implies that $\xi \cdot \dim(U') < \xi \cdot \dim(U)$. Then the maximal μ_ϵ-destabilizing subrepresentation $\hat{W}$ of $\hat{V}/\hat{U}$ necessarily has $\hat{W}_\infty = 1$. Denote by $\hat{V}^2$ its preimage in $\hat{V}$ under the projection. Then it is not difficult to check that $\hat{U} \subset \hat{W}' \subset \hat{V}^2$ is the desired sequence. □

Corollary 3.43 ([MMNS]). *There is a factorization*

$$\widetilde{A}_U = A_\xi^+ \widetilde{A}_\xi A_\xi^- . \tag{3.47}$$

Proof. This follows from the properties of the filtration of Lemma 3.42. □

Theorem 3.44. *We have*

$$Z_\xi(y_0, y_1) = \frac{A_\xi^-((-\mathbb{L}^{\frac{1}{2}})^r y_0, y_1)}{A_\xi^-((-\mathbb{L}^{\frac{1}{2}})^{-r} y_0, y_1)}. \tag{3.48}$$

Proof. From Lemma 3.39 and Corollary 3.43 we get

$$A_\xi^+ y_\infty A_\xi^-(\mathbb{L}^r y_0, y_1) = \widetilde{A}_U = A_\xi^+ \widetilde{A}_\xi A_\xi^-,$$

so that from equation (3.46) we obtain

$$y_\infty \cdot Z_\xi((-\mathbb{L}^{\frac{1}{2}})^r y_0, y_1) = y_\infty \cdot \frac{A_\xi^-(\mathbb{L}^r y_0, y_1)}{A_\xi^-(y_0, y_1)},$$

and changing variables we get the result. □

Corollary 3.45. *The generating series of motivic invariants of the moduli space of* PT*-stable r-framed representations of the conifold quiver with superpotential*

$$Z_{\mathrm{con}}^r(y_0, y_1) = \sum_{\underline{d}} \left[M_{\underline{d}}^\theta(Q_r, W) \right]_{\mathrm{vir}} y_0^{d_0} y_1^{d_1} \tag{3.49}$$

factorizes as

$$Z_{\mathrm{con}}^r((-1)^r y_0, y_1) = \prod_{m \geq 1} \prod_{j=0}^{rm-1} \left(1 - \mathbb{L}^{\frac{rm}{2} - j - \frac{1}{2}} y_0^m y_1^{m-1} \right). \tag{3.50}$$

Proof. It is enough to use equation (3.48) together with Lemma 3.39. □

As explained at the beginning of the section, it is natural to consider the hypercohomology of $P_{n,d}^r$ with respect to the coefficient system $\Phi_{n,d}^r$. The next theorem computes the generating series of the virtual Poincaré polynomials

$$p_{n,d}^r(t^{\pm\frac{1}{2}}) := \sum_i (-1)^i \dim_{\mathbb{Q}} \mathbb{H}^i(P_{n,d}^r, \Phi_{n,d}^r)\, t^{\frac{i}{2}} \tag{3.51}$$

Theorem 3.46. *The generating series of refined Donaldson-Thomas invariants of $P^r_{n,d}$ factorizes as*

$$Z^r_X\left(t^{\pm\frac{1}{2}}, y_0, y_1\right) = \sum p^r_{n,d}\left(t^{\pm\frac{1}{2}}\right) y_0^n y_1^d = \prod_{m\geq 1}\prod_{j=0}^{rm-1}\left(1 - t^{\frac{rm}{2}-j-\frac{1}{2}}((-1)^r y_0)^m y_1^{m-1}\right). \tag{3.52}$$

Proof. The specialization of (3.50) which sends $\mathbb{L}^{\frac{1}{2}} \mapsto t^{\frac{1}{2}}$ computes the weight polynomial of $P^r_{n,r}$ with respect to $\Phi^r_{n,d}$, as explained in 1.42. Since the moduli space $P^r_{n,d}$ is proper, we have that the weight filtration of the cohomology with coefficients in $\Phi^r_{n,d}$ coincides with the degree filtration, and so we obtain the desired result. $\square$

We rewrite now the series in term of the variables $s = q_0 q_1$ and $T = q_1^{-1}$, which we will call the *box counting* parameter and the *curve counting* parameter, following the terminology of the literature on the topological vertex. Factorizing in terms of rank 1 contributions we have:

$$Z^r_X\left(t^{\pm\frac{1}{2}}, (-1)^r s, T\right) = \prod_{l=0}^{r-1} Z_{\text{con},1}\left(t^{\pm\frac{1}{2}}, (-1)^r \mathbb{L}^{\frac{r-1}{2}-l} s, T\right), \tag{3.53}$$

so that it is evident that the rank r partition function consists of a multiple count of rank 1 contributions where the box counting parameter is twisted by a half power of the motive of the affine line. If we specialize further setting $\mathbb{L} \mapsto 1$, we obtain a product expansion for the partition function of classical DT invariants

$$\prod_{d\geq 1}(1 - ((-1)^r s)^d T)^{rd}, \tag{3.54}$$

which recovers a result of Sheshmani [Sh2]. In [Sh2] the author uses the non-geometric action of a torus $(\mathbb{C}^*)^r$ to reduce the computation of invariants for rank r stable pairs to r contributions of rank one. Notice that we cannot use localization techniques for the computation of refined invariants as it is not at all clear in general how the fixed points of the torus action contribute to the virtual Poincaré polynomial.

3.4.1 Comparison with Nekrasov partition function

As mentioned in the introduction to this chapter, the motivations for computing invariants of higher rank stable pairs on the conifold are two-fold. On one hand we have seen in (3.53) that the partition function of refined invariants of higher rank stable pairs factorizes into twisted elementary contributions, so that the generating series of refined invariants behaves similarly to the unrefined one but incorporates some extra twists whose nature is not entirely clear and which might lead to further interesting research. On the other hand, inspired by observations of Maulik and Okounkov in [MO] and of Günaydin, Neitzke, Pavlyk and Pioline [GNPP, Section 4], we want to compare higher rank invariants for the conifold to instanton type invariants on the resolution of an A_{r-1}-surface singularity.

This correspondence between invariants of surfaces and of threefolds has been investigated for some years in the physics literature and goes under the name of geometric engineering [KKV]. This predicts that the string partition function $Z^1_{X_{r_1}}$ of a certain threefold X_{r_1} can be computed starting from the partition function $Z^{r_1}_{Y_1}$ counting rank r_1 instantons on a surface Y_1. On the threefold side one considers the minimal resolution X_{r_1} of the orbifold quotient X/μ_{r_1} of the resolved conifold $X = \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ by the fiberwise action of μ_{r_1} with weights $(-1, 1)$. On the surface side, the variety Y_1 is just $\mathbb{C}^2$. The string partition function $Z^1_{X_{r_1}}$ can be taken to be the generating series of rank 1 Donaldson-Thomas invariants of stable pairs on X_{r_1} with support on the r_1 rational curves parallel to the base of the fibration $X_{r_1} \longrightarrow \mathbb{P}^1$. The instanton partition function $Z^r_{Y_1}$ has been computed by Nekrasov via equivariant integration on the moduli space $\mathfrak{M}_{n,r_2}$ of $SU(r_2)$-instantons on $\mathbb{R}^4$ of charge n, in order to understand the ill-defined expression

$$\int_{\mathfrak{M}_{n,r_2}} 1, \tag{3.55}$$

and it will be briefly described later on in the section. At the unrefined level, the correspondence has been checked using the topological vertex. For refined invariants the situation is much more complicated due to the lack of a mathematical formulation for the refined topological vertex, but already the $r_1 = 1$ case has given interesting results [Sz3].

The insight of [MO] and [GNPP] is that one should aim for a picture

$$\begin{tikzcd} X_{r_1} \arrow[d, dashed] & & Y_{r_2} \arrow[d, dashed] \\ Z^{r_2}_{X_{r_1}} & & Z^{r_1}_{Y_{r_2}} \arrow[ll, squiggly] \end{tikzcd} , \tag{3.56}$$

where Y_{r_2} is the resolution of an A_{r_2-1}-surface singularity. Here the arrow between the partition functions is representing the fact that the refined invariants of rank r_2 stable pairs on X_{r_1} could be computed starting from the surface side.

In the rest of this section we are going to assume that $r_1 = 1$, rename $r_2 = r$, and discuss how Theorem 3.46 fits into the picture.

It has been understood by Kronheimer and Nakajima [KN] that the moduli space $\mathfrak{M}^r_{n,1}$ of rank 1 instantons of charge n on Y_r is a hyperkähler variety isomorphic to $\mathrm{Hilb}^n(Y_r)$, the Hilbert scheme of n points on the resolution. As mentioned above the naïve integral $\int_{\mathfrak{M}^r_{n,1}} 1$ gives an ill-defined expression since the moduli spaces $\mathfrak{M}^r_{n,1}$ are non-compact and 1 is not a top-form. Nekrasov explained that if one appropriately interprets the integral in the equivariant setting, then it gives a count of charge n instantons on Y_r. In order to reduce to the equivariant setting we have to fix the action of a torus T on Y_r. Consider the usual action of $T = (\mathbb{C}^*)^2$ on $\mathbb{C}^2$ by dilation, and notice that the inclusion $\mu_r \subset (\mathbb{C}^*)^2$ induces an action of the quotient torus $T = (\mathbb{C}^*)^2/\mu_r$ on the quotient $\mathbb{C}^2/\mu_r$. This action has weights $(r, 0, 1)$ and $(0, r, 1)$ on the generators (u, v, w) of the ring of invariant functions $\mathbb{C}[u, v, w]/(uv - w^r)$. The action of T lifts to an action on the instanton moduli space $\mathfrak{M}^r_{n,1}$, so that the structure sheaf $\mathcal{O}_{\mathfrak{M}^r_{n,1}}$ defines a class in the T-equivariant K-theory of $\mathfrak{M}^r_{n,1}$. The class $\mathcal{O}_{\mathfrak{M}^r_{n,1}}$ replaces 1 in (3.55), and we just need to explain how to perform the equivariant integration of this class. Let $\pi^r_{n,1}$ be the morphism sending $\mathfrak{M}^r_{n,1}$ to a point. We have an action of T on each of the cohomology groups $\mathrm{R}^i\pi^r_{n,1\,*}\mathcal{O}_{\mathfrak{M}^r_{n,1}}$, so that the formal difference $\mathrm{R}\pi^r_{n,1\,*}\mathcal{O}_{\mathfrak{M}^r_{n,1}} = \sum_i(-1)^i\,\mathrm{R}\pi^r_{n,1\,*}\mathcal{O}_{\mathfrak{M}^r_{n,1}}$ defines a class in the T-equivariant K-theory of the point. The instanton partition function of Y_r is defined as

$$Z^1_{Y_r}(q_1, q_2) = \sum_n \Lambda^n\,\mathrm{char}_T(\mathrm{R}\pi^r_{n,1\,*}\mathcal{O}_{\mathfrak{M}^r_{n,1}}), \tag{3.57}$$

where (q_1, q_2) is a basis for the characters of the torus T, and where char_T computes the character of the virtual representation. Since all the T-weight spaces are finite-dimensional by a result of Nakajima and Yoshioka [NY, Section 4], this gives a well-defined formula.

We compute now a product expansion of the partition function of rank 1 instantons on Y_r. Denote by $\overline{\mathfrak{M}}^r_{n,1}$ the symmetric quotient $S^n(\mathbb{C}^2/\mu_r)$. The morphism $\pi^r_{n,1}$ factorizes through $\overline{\mathfrak{M}}^r_{n,1}$:

$$\begin{tikzcd} \mathfrak{M}^r_{n,1} \arrow[rr, "\sigma^r_n"] \arrow[dr, "\pi^r_{n,1}"'] & & \overline{\mathfrak{M}}^r_{n,1} \arrow[dl, "\widetilde{\pi}^r_{n,1}"] \\ & pt\ , & \end{tikzcd} \tag{3.58}$$

where σ_n^r is the composition of the Hilbert-Chow morphism with the n-th symmetric power of the projection $Y_r \longrightarrow \mathbb{C}^2/\mu_r$. Since the morphism σ_n^r is a projective symplectic resolution of singularities, we have that $\mathrm{R}^i\sigma^r_{n\,*}\mathcal{O}_{\mathfrak{M}^r_{n,1}}$ by the Grauert-Riemenschneider theorem. From the fact that $\overline{\mathfrak{M}}^r_{n,1}$ is an affine rational singularity we also get $\mathrm{H}^i(\overline{\mathfrak{M}}^r_{n,1}, \mathcal{O}_{\overline{\mathfrak{M}}^r_{n,1}}) = 0$, so that in fact $\mathrm{R}^i\pi^r_{n,1\,*}\mathcal{O}_{\mathfrak{M}^r_{n,1}} = 0$ for all $i > 0$ and $\mathrm{R}\pi^r_{n,1\,*}\mathcal{O}_{\mathfrak{M}^r_{n,1}}$ is a representation of T. From these observations we immediately get

$$\begin{aligned} Z^1_{Y_r}(q_1,q_2) &= \sum_n \Lambda^n \operatorname{char}_T \mathrm{H}^0\Big(\overline{\mathfrak{M}}^r_{n,1}, \mathcal{O}_{\overline{\mathfrak{M}}^r_{n,1}}\Big) \\ &= \sum_n \Lambda^n \operatorname{char}_T \left(\frac{\mathbb{C}[u_1,\dots,u_n,v_1,\dots,v_n,w_1,\dots,w_n]}{(u_iv_i - w_i^r \colon i \in \{1,\dots,r\})}\right)^{S_n} \\ &= \sum_n \Lambda^n \operatorname{char}_T S^n\left(\frac{\mathbb{C}[u,v,w]}{(uv-w^r)}\right) \\ &= \operatorname{char}_{T\times\mathbb{C}^*} S^*\left(\frac{\mathbb{C}[u,v,w]}{(uv-w^r)}\right) \\ &= \prod_{\substack{i_1\equiv i_2 (r)\\ i_1,i_2\geq 0}} \left(1 - q_1^{i_1}q_2^{i_2}\Lambda\right)^{-1}, \end{aligned} \tag{3.59}$$

where the last equality comes from the computation of the Hilbert series for the variety $\mathbb{C}^2/\mu_r$.

If we simplify further setting $r = 1$, the curly arrow in (3.56) has been explained by Szendrői in [Sz3].

Theorem 3.47. *Let $Z^1_X(t^{\frac{1}{2}}, s, T)$ be the generating series of refined invariants of rank 1 stable pairs moduli spaces on the resolved conifold X. Then if we set $q_1 = -t^{\frac{1}{2}}s, q_2 = -t^{-\frac{1}{2}}s$ and $\Lambda = -sT$, we have*

$$Z^1_X(s,T) = Z^1_{\mathbb{C}^2}(q_1,q_2,\Lambda)^{-1}. \tag{3.60}$$

This result is the key ingredient that together with the Lefschetz action on the critical cohomology allowed Szendrői to prove that the vector space $\oplus_{n,d}\mathbb{H}^*(P_{n,d},\Phi_{n,d})$ admits a $\mathrm{GL}_2 \times \mathbb{C}^*$-action making it isomorphic to the exterior algebra $\bigwedge \mathbb{C}[x,y]$.

When we have $r > 1$, we want to try to compare the generating series we computed in Theorem 3.46 with (3.59). In order to do so we notice that if we set $q_1 = -t^{\frac{1}{2}}s, q_2 = -t^{-\frac{1}{2}}s$ and $\Lambda = t^{\frac{r}{2}}s^rT$ the partition function for rank 1 instantons on Y_r factorizes as a twisted product of elementary contributions:

$$\begin{aligned}
Z^1_{Y_r}(q_1,q_2,\Lambda) &= \prod_{\substack{i_1\equiv i_2(r)\\ i_1,i_2\geq 0}} \left(1-q_1^{i_1}q_2^{i_2}\Lambda\right)^{-1} \\
&= \prod_{j=0}^{r}\prod_{k_1,k_2\geq 0}\left(1-t^{\frac{r(k_1-k_2)}{2}}s^{r(k_1+k_2+1)}s^{2j}T\right)^{-1} \\
&= \prod_{j=0}^{r-1}\prod_{m\geq 1}\prod_{k=0}^{m-1}\left(1-t^{\frac{rm}{2}-rj+\frac{r}{2}}s^{rm}(q)\right) \\
&= \prod_{j=0}^{r-1} Z_{\mathbb{C}^2}\left(\left(t^{\frac{1}{2}}s\right)^r,\left(t^{-\frac{1}{2}}s\right)^r, s^{2k+1}T\right).
\end{aligned} \tag{3.61}$$

We exhibit now another factorization of the series in Theorem 3.46 which is more suited for this comparison:

$$Z^r_X(t^{\pm\frac{1}{2}},s,T) = Z^1_X\left(t^{\pm\frac{r}{2}},s,t^{\frac{r-1}{2}-k}T\right). \tag{3.62}$$

The factorization of the two formulas into elementary contributions is extremely similar. The essential difference is in the nature of the twists: the twists of the $\mathbb{C}^2$ contributions on the instanton side are corrections which involve a rescaling of the curve counting parameter T by a power of the point counting parameter, while on the stable pairs side the corrections of the rank 1 contributions consist of a rescaling of the curve counting parameter by powers of the weight parameter t. This difference is not entirely clear, and it has been observed by Nekrasov and Okounkov that this discrepancy between the two formulas indicates a need for a less naïve approach to counting of higher rank objects. In particular, in [NO, Section 5.1] they define a class in the K-theory of the locus of stable pairs which are fixed by the non geometric action rescaling the framing which is sensitive to certain interactions between the various rank one components. They show that, up to some conjectural assumptions, this recovers some count of curves inside the 5-fold $X\times Y_r$, and that this should then recover fully the geometric engineering picture (3.56) for $r_1=1$ and $r_2=r$.

3.A Refined invariants for higher rank stable pairs on a line of resolved A_1-singularities

The same story we described in this chapter carries through without major modifications for the case when we consider the projective bundle $Y=\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 2}\oplus\mathcal{O}_{\mathbb{P}^1}(-2))$. The toric projective bundle Y is a compactification of the resolution of a line of A_1

singularities $V(xy - t^2) \subset \mathbb{C}^4$ obtained by adding the divisor $D_\infty = \mathbb{P}(0 \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$. Again in this case, the rational line b corresponding to the zero section of $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ defines a class $\beta \in \mathrm{H}^2(Y, \mathbb{Z})$ which spans the kernel of the cap product with D_∞.

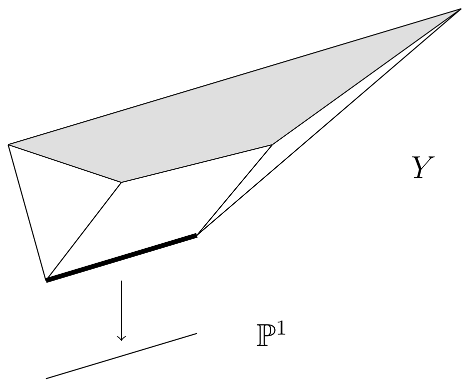

Figure 3.4: Compactification $Y^0 \subset Y$ (divisor D_∞ light grey, curve b bold).

We denote by $P^r_{n,d}(Y)$ the moduli space parametrising flat families of rank r stable pairs (F, s) with $\mathrm{ch}_2(F) = d\beta$, $\chi(F) = n$, and support disjoint from D_∞. Following the construction of [Di2] in the same way as in Section 3.2 we get an isomorphism between $P^r_{(n,d)}(Y)$ and the moduli space of rank r asymptotically stable ADHM sheaves $M^{A_1}_{(n-d,d)}(\mathbb{P}^1, r)$ defined in 3.17. The following proposition explains how we get a description of $P^r_{n,d}(Y)$ in terms of quiver representations.

Proposition 3.48. *The moduli space $P^r_{n,d}(Y)$ of rank r stable pairs is isomorphic to the moduli space $M^\theta_{(n,n-d)}(Q_{(r,0)}, W)$ of representations of the framed A_1-quiver with superpotential of Example 1.7, where θ is chosen in the* PT*-chamber as in (3.23).*

Proof. We need to prove there is an isomorphism

$$M^{A_1}_{(n-d,d)}(\mathbb{P}^1, r) \cong M^\theta_{(n,n-d)}(Q_{(r,0)}, W).$$

We explain how to construct a representation of $Q_{(r,0)}$ starting from a rank r asymptotically stable ADHM sheaf $\mathcal{E}$. Since $\mathrm{H}^1(E) = \mathrm{H}^1(E(-1)) = 0$ by Lemma 3.15, the Beilinson spectral sequence describing E on $\mathbb{P}^1$ degenerates at the E_1-sheet. It follows that we have a resolution

$$0 \longrightarrow \mathrm{H}^0(E(-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{a_0^* z_0 + a_1 z_1} \mathrm{H}^0(E) \otimes \mathcal{O}_{\mathbb{P}^1} \longrightarrow E \longrightarrow 0. \quad (3.63)$$

Hence we can construct:

- two vector spaces $V_0 = \mathrm{H}^0(E)$ and $V_1 = \mathrm{H}^0(E(-1))$;
- two morphisms $a_0^*, a_1 \colon V_1 \longrightarrow V_0$ from (3.63);
- two endomorphisms $b_0 \in \mathrm{End}(V_0)$ and $b_1 \in \mathrm{End}(V_1)$ induced respectively by Φ_1 and $\Phi_1 \otimes \mathrm{Id} \colon E(-1) \longrightarrow E(-1)$;
- two morphisms $-a_1^*, a_0 \colon V_0 \longrightarrow V_1$ that are induced by the composition

$$(\Phi_2 \otimes \mathrm{Id}) \circ \begin{pmatrix} w_1 & w_0 \end{pmatrix} : E^{\oplus 2} \longrightarrow E(-1);$$

- a morphism $\iota \colon \mathbb{C}^r \longrightarrow V_0$ from $\mathrm{H}^0(\psi)$.

The remaining steps of the proof are obtained in the same way as in Section 3.3. $\square$

As an immediate consequence we obtain that $P^r_{n,d}$ arises as the critical locus of a regular function $\omega^r_{n,d}$ on a smooth quasi-projective variety of dimension $4n(n-d)+nr$. Since $\{a_0^*, a_1^*\}$ is a cut for the superpotential W of the A_1-quiver, the class $[P^r_{n,d}]_{\mathrm{vir}}$ lies in the enhanced Grothendieck ring of varieties $\mathcal{M}_{\mathbb{C}}$ from Theorem 1.75. The techniques of Section 3.4 enable us to compute the generating series for $(r,0)$-framed representations of Q starting from the universal series of motivic Donaldson-Thomas invariants of Q without imposing any choice of stability condition. In particular, Bryan and Morrison in [BM] proved the formula

$$\begin{aligned} A_U(q) &= \sum_{\underline{d}} [\mathcal{M}_{\underline{d}}(Q,W)]_{\mathrm{vir}} \; q_0^{d_0} q_1^{d_1} \\ &= \prod_{m\geq 1} \prod_{k\geq 0} \left(1-\mathbb{L}^{-k} q^m\right)^{-1} \left(1-\mathbb{L}^{1-k} q^m\right)^{-1} \cdot \\ &\quad \cdot \prod_{m\geq 1} \prod_{k\geq 0} \left(1-\mathbb{L}^{-k} q^m q_0^{-1}\right)^{-1} \left(1-\mathbb{L}^{-k} q^m q_1^{-1}\right)^{-1}, \end{aligned}$$

where we adopt the notation $q := q_0 q_1$. From this together with Theorem 3.44 we have the following formula for the refined Donaldson-Thomas invariants of $P^r_{(n,d)}(Y)$.

Theorem 3.49. *The generating series $Z^r_{A_1}(q_0,q_1)$ of the virtual classes $[P^r_{(n,d)}(Y)]_{\mathrm{vir}}$ has a product expansion*

$$Z^r_{A_1}(q_0,q_1) = \prod_{m\geq 1} \prod_{k=0}^{rm-1} \left(1-\mathbb{L}^{\frac{rm}{2}-k}((-1)^r q_0 q_1)^m q_1^{-1}\right)^{-1}. \tag{3.64}$$

Proof. Denote by ξ the PT-stability conditions. From Theorem 3.44 it is enough to compute the generating series $A_\xi^-(q_0, q_1)$, which consists of the product of factors in the universal series A_U corresponding to vectors $\underline{d} = (d_0, d_1)$ satisfying $\xi \cdot \underline{d} < 0$. We have

$$A_\xi^-(q_0, q_1) = \prod_{m\geq 1}\prod_{k\geq 0} \left(1 - \mathbb{L}^{-k} q^m q_1^{-1}\right)^{-1},$$

so that

$$Z_{A_1}^r((-1)^r q_0, q_1) = \frac{\prod_{m\geq 1}\prod_{k\geq 0}\left(1 - \mathbb{L}^{\frac{rm}{2}-k} q^m q_1^{-1}\right)^{-1}}{\prod_{m\geq 1}\prod_{k\geq 0}\left(1 - \mathbb{L}^{-\frac{rm}{2}-k} q^m q_1^{-1}\right)^{-1}},$$

and simplifying we get 3.64. $\square$

In complete analogy with the conifold case we have a factorization of the generating series for refined invariants of rank r stable pairs as the product of r factors which consist of twists of the rank 1 contributions:

$$Z_{A_1}^r(q_0, q_1) = \prod_{j=0}^{r-1} Z_{A_1}^1\left((-1)^{r-1}\mathbb{L}^{\frac{rm-1}{2}-j} q_0, q_1\right). \tag{3.65}$$

Chapter 4

DT Invariants of Deformations of 3-Calabi-Yau Algebras

As explained in Chapter 1, the study of Donaldson-Thomas invariants of moduli spaces of sheaves on a 3-dimensional Calabi-Yau variety X gives information about the geometry of X. In the direction of homological mirror symmetry, Kontsevich and Soibelman proposed in [KS1] that one should really aim at computing Donaldson-Thomas invariants for 3-CY-categories, that is, triangulated categories with Serre functor behaving as for $\mathrm{D}^b(X)$ with X a Calabi-Yau threefold [KS1, Section 3.3]. Following the idea that, at least in many local cases such as in Proposition 1.38, the homological information of $\mathrm{D}^b(X)$ can be described by means of modules on a 3-CY algebra T_X, one can think of T_X as a non-commutative analogue of X. In the case when this algebra arises as the path algebra of quivers with superpotential, one can study its motivic Donaldson-Thomas invariants as defined by Kontsevich and Soibelman [KS1], and by Behrend, Bryan and Szendrői [BBS]. Continuing with the analogy, the universal generating series $A_U(T_X)$ of motivic Donaldson-Thomas invariants of T_X can be thought as the non-commutative counterpart of the generating series of motivic DT invariants for moduli of sheaves on X. It is then natural to investigate the behavior of $A_U(T_X)$ under deformations of the algebra T_X, in a similar way to that in which Thomas studied the invariance of the numerical DT invariant under deformations of the underlying Calabi-Yau variety in a smooth family. When doing so, we choose to restrict to the case when T_X is realized as the path algebra of a quiver with homogeneous superpotential with respect to the grading where all arrows have degree 1, and we consider only graded deformation of such algebras. The matrix-valued Hilbert series (cf. 1.20) remains constant under such deformations, so that our examples play the analogous role to families of projective Calabi-Yau threefolds in the classical picture of Thomas.

This chapter focuses on the study of the universal generating series of motivic DT invariants for graded deformations of 3-CY algebras in two examples: the conifold algebra of Example 1.46, and the non-commutative resolution of a line of A_n-singularities, which is a particular instance of the McKay correspondence of Example 1.47. We concentrate on a particular family of deformations which goes under the name of marginal deformations, showing that non-commutative motivic Donaldson-Thomas invariants are sensitive to the deformations of the algebra and discussing how the generating series is affected by these deformations. The first results are in line with the observations, made independently by Morrison [CMPS] and Okawa and Ueda [OU], that motivic Donaldson-Thomas invariants are not invariant under deformation. The second set of results describe the fact that generically the generating series of Donaldson-Thomas invariants is entirely determined by the virtual motive of the moduli stack of stable representations for some choice of stability condition. This last result differs quite drastically from the case of marginal deformations of $\mathbb{C}^3$ (cf. 4.A) where the universal series is determined by simple representations, and raises an interesting question regarding the role of these representations.

4.1 DT invariants of deformations of the conifold algebra

In this section we describe the moduli space of homogeneous deformations of the superpotential of the conifold algebra $J(Q, W)$, following [OU]. After discussing the connection between deformations of the superpotential W and deformations of $J(Q, W)$ as a graded 3-CY, we concentrate on a particular family of deformations and compute the universal series of motivic Donaldson-Thomas invariants for the deformed algebras. We then discuss the relationship between the motives appearing in the universal series and the geometry of representations of the conifold quiver with deformed superpotential with respect to some particular choice of stability condition.

4.1.1 Moduli of deformation of potentials

Let $J_0 = J(Q, W_0)$ be the conifold algebra of Example 1.6. It is clear from Theorem 1.40 that all the deformations of J_0 as a graded Calabi-Yau algebra can be obtained by deforming the superpotential in $\mathrm{Sup}(Q)_4$. Corollary 1.44 further implies that a very general deformation of W_0 inside $\mathrm{Sup}(Q)_4$ defines a deformation of J_0 as a graded

3-CY algebra. Not all the choices of W inside $\mathrm{Sup}_4(Q)$ give rise to a 3-CY algebra, though.

Remark 4.1. Consider the superpotential $W = a_1b_1a_1b_1$ and $J_W = J(Q,W)$. The dimension of the vector space $e_0(J_W)_3e_1$ is 7. This differs from the dimension predicted by Corollary 1.45 for a 3-CY algebra constructed from the conifold quiver and a superpotential of degree 4.

Following [OU] we study the (coarse) moduli space $\mathcal{M}_{pot}$ of superpotential algebras $J(Q,W)$ with W of degree 4 and we show that the complement of a quartic hypersurface in $\mathcal{M}_{pot}$ corresponds to genuine deformations of J_0 as a graded 3-CY algebra.

Let V_0 and V_1 be two complex vector spaces of dimension 2, and let V denote $V_0 \otimes V_1$. We regard V_i as the vector space of linear combinations of arrows from i to $i+1$ (with indices read mod 2). Let W be a homogeneous superpotential of Q of degree 4. Since it is defined only up to cyclic permutations we can assume without loss of generality that any cyclic monomial in W is represented by a path starting at 0. Furthermore, if we consider $v_i, v_i' \in V_i$ for $i \in \{0,1\}$, both $v_0v_1v_0'v_1'$ and $v_0'v_1'v_0v_1$ start at 0 and represent the same cyclic monomial. Hence, the degree 4 superpotential W of Q corresponds to an element in $U = \mathrm{Sym}^2(V_0 \otimes V_1)$. Consider the action of $\mathrm{G}_U = \mathrm{GL}(V_0) \times \mathrm{GL}(V_1)$ defined by

$$(g_0, g_1) \cdot ((v_0 \otimes v_1) \odot (v_0' \otimes v_1')) = (g_0(v_1) \otimes g_1(v_2)) \odot (g_0(v_0') \otimes g_1(v_1')), \tag{4.1}$$

where the symbol $\odot$ denotes the symmetric tensor of two vectors. Two superpotentials differing by the action of G_U define isomorphic algebras, so that we can define the moduli stack of superpotentials

$$\mathcal{M}_{pot} = [U/\,\mathrm{G}_U]\,.$$

We consider the character

$$\chi\colon G_U \longrightarrow \mathbb{C}^*,\ \chi\colon (g_0, g_1) \longmapsto \det(g_0g_1),$$

so that the associated GIT quotient of U by the action of G_U is

$$M_{pot} := U /\!\!/_{\chi} G_U = \mathrm{Proj}\left(\oplus_{n\geq 0}\, \mathbb{C}[\mathrm{Sym}^2(V_0 \otimes V_1)]^{GL(V_0)\times GL(V_1),\chi^n}\right).$$

The rest of the subsection explains the constructions necessary for the proof of the following proposition.

Proposition 4.2. *Consider the action of G_U on U of (4.1). Then*

1. *([OU, Theorem 1.1]) the coarse moduli space of potentials M_{pot} is isomorphic to the weighted projective space $\mathbb{P}(1,2,3,4)$;*

2. *given a generic superpotential W' in $Sup_4(Q)$, the algebra $A_{Q,W'}$ is* 3-CY*. Hence, there is a non-empty Zariski open subset of $\mathbb{P}(1,2,3,4)$ representing good superpotentials (cf. Sec. 1.3).*

First of all we notice that there is an isomorphism

$$M_{pot} = \operatorname{Proj}\left(\mathbb{C}[\operatorname{Sym}^2(V_0 \otimes V_1)]^{\operatorname{SL}(V_0)\times \operatorname{SL}(V_1)}\right),$$

where the grading puts $\operatorname{Sym}^i(U^*)$ in degree i. This GIT quotient can be described explicitly by means of the accidental isomorphism between $\operatorname{SL}(2)\times\operatorname{SL}(2)$ and $\operatorname{Spin}(4)$ induced by the isomorphism of root systems $A_1 \times A_1 \cong D_2$. The action of $\operatorname{SL}(V_i)$ on V_i preserves a generator ω_i of $\Lambda^2(V_i)$ for i in $\{0,1\}$. The non-degenerate symmetric bilinear form

$$\eta \in \operatorname{Sym}^2(V_0\otimes V_1)^*\colon\ \ \eta\colon (v_0^1\otimes v_1^1, v_0^2\otimes v_1^2) \longrightarrow \omega_0(v_0^1,v_0^2)\,\omega_1(v_1^1,v_1^2)$$

is preserved by the action of $\operatorname{SL}(V_0)\times\operatorname{SL}(V_1)$ defined in (4.1). We thus get a surjective morphism of Lie groups

$$\operatorname{SL}(V_0)\times\operatorname{SL}(V_1) \longrightarrow \operatorname{SO}(V_0\otimes V_1),$$

with kernel $\{\pm(\mathrm{Id},\mathrm{Id})\}$. Since $-(\mathrm{Id},\mathrm{Id})$ acts trivially on $\operatorname{Sym}^2(V_0\otimes V_1)$ we get obtain isomorphism

$$U/\!\!/_\chi G_U \cong \operatorname{Proj}\left(\mathbb{C}[\operatorname{Sym}^2(V)]^{\operatorname{SO}(V)}\right).$$

Lemma 4.3. *There is an isomorphism*

$$\mathbb{C}[\operatorname{Sym}^2(V)]^{\operatorname{SO}(V)} \simeq \mathbb{C}[z_1,\dots,z_4], \tag{4.2}$$

where z_i is the symmetric polynomial of degree i in the eigenvalues. We thus get

$$U/\!\!/_\chi G_U = \operatorname{Proj}(\mathbb{C}[z_1,\dots,z_4]) = \mathbb{P}(1,2,3,4),$$

Proof. Two symmetric complex matrices are similar if and only if they are orthogonally similar [G, Chap. 11, Sec. 2, Thm. 4] Hence, any symmetric complex matrix with distinct eigenvalues is orthogonally similar to a diagonal matrix. The discriminant Δ is a symmetric polynomial in the eigenvalues, so that we have

$$(\mathbb{C}[\mathrm{Sym}^2(V)][\Delta^{-1}])^{SO(V)} = \mathbb{C}[\mu_1, \mu_2, \mu_3, \mu_4]^{S_4}[\Delta^{-1}], \tag{4.3}$$

where the μ_i's are the eigenvalues of the symmetric non-singular matrix and the symmetric group S_4 permutes the eigenvalues. If we let $z_i = \sum_j \mu_j^i$ be the i-th symmetric polynomial in the eigenvalues μ_j we have then $\mathbb{C}[\mu_1, \mu_2, \mu_3, \mu_4]^{S_4}[\Delta^{-1}] \cong \mathbb{C}[z_1, z_2, z_3, z_4, \Delta^{-1}]$. We thus get

$$\mathbb{C}[\mathrm{Sym}^2(V)]^{\mathrm{SO}(V)} \cong (\mathbb{C}[\mathrm{Sym}^2(V)][\Delta^{-1}])^{\mathrm{SO}(V)} \cap \mathbb{C}[\mathrm{Sym}^2(V)] \cong \mathbb{C}[z_1, z_2, z_3, z_4]. \tag{4.4}$$

In other words the closure of the $\mathrm{SO}(V)$-orbit of any symmetric complex matrix contains a diagonal matrix. It is then clear that the unstable locus for the $\mathrm{SO}(V)$ action consists of the nilpotent matrices and that the GIT quotient is isomorphic to $\mathbb{P}(1, 2, 3, 4)$. □

From the general machinery of Geometric Invariant Theory [MFK] there is a morphism

$$\phi\colon U^{ss} \longrightarrow \mathbb{P}(1, 2, 3, 4)$$

which is a good categorical quotient.

In fact, we can ensure that the preimage of a Zariski open dense subset of $\mathbb{P}(1, 2, 3, 4)$ consists of superpotentials giving rise to Calabi-Yau algebras. We fix bases $\{a_1, a_2\}$ and $\{b_1, b_2\}$ for the vector spaces V_0 and V_1. Now consider then the homogeneous deformation of the conifold potential

$$W_{dc} = \lambda_0 a_1 b_1 a_2 b_2 - \lambda_1 a_1 b_2 a_2 b_1,$$

where the complex coefficients λ_i satisfy $\lambda_0\lambda_1 \neq 0$, and denote by $(Sup(Q)_4)_{<W}$ the subspace

$$(Sup(Q)_4)_{<W} = \{W'\colon\ \mathrm{lt}(0W'0) < \mathrm{lt}(0W0),\ \mathrm{lt}(1W'1) < \mathrm{lt}(1W1)\} \cup \{0\}$$

inside the vector space $Sup(Q)_4$ of degree 4 potentials.

Lemma 4.4. *Fix λ_0 and λ_1 such that $\lambda_0\lambda_1 \neq 0$. For any choice of a superpotential W in the affine subspace*

$$L = W_{dc} + (Sup(Q)_4)_{<W_0}$$

the algebra $J(Q, W)$ is 3-CY.

Proof. We consider the ordering

$$a_1 > a_2 > b_1 > b_2$$

on the set $E(Q)$ of arrows of the conifold quiver Q. The leading terms of the relations are

$$\begin{aligned}
\mathrm{lt}(\partial_{a_1} W) &= \lambda_0 b_1 a_2 b_2,\ \mathrm{lt}(\partial_{b_1} W) = \lambda_1 a_1 b_2 a_2, \\
\mathrm{lt}(\partial_{a_2} W) &= \lambda_1 b_1 a_1 b_2,\ \mathrm{lt}(\partial_{b_2} W) = \lambda_0 a_1 b_1 a_2.
\end{aligned}$$

Notice in particular that none of the leading terms ends in a_1 or b_1, so that condition (2) of Theorem 1.49 is satisfied. On the other hand, the only ambiguities are

$$\begin{aligned}
(a_1, \mathrm{lt}(\partial_{a_1} W) b_2^{-1}, b_2) &= (a_1, a_1^{-1} \mathrm{lt}(\partial_{b_2} W), b_2),\ a_1 \mathrm{lt}(\partial_{a_1} W) = \mathrm{lt}(\partial_{b_2} W) b_2 = \mathrm{lt}(0W0); \\
(b_1, \mathrm{lt}(\partial_{b_1} W) a_2^{-1}, a_2) &= (b_1, b_1^{-1} \mathrm{lt}(\partial_{a_2} W), a_2),\ b_1 \mathrm{lt}(\partial_{b_1} W) = \mathrm{lt}(\partial_{a_2} W) a_2 = \mathrm{lt}(1W1),
\end{aligned}$$

corresponding respectively to the vertices 0 and 1. It then follows from Theorem 1.49 that the algebra $J(Q, W)$ is 3-CY. □

The cyclic monomials

$$a_1 b_2 a_1 b_2,\ a_1 b_2 a_2 b_2,\ a_2 b_1 a_2 b_1,\ a_2 b_1 a_2 b_2,\ a_2 b_2 a_2 b_2$$

generate the linear subspace $(Sup(Q)_4)_{<W}$. We consider the basis

$$\{a_1 \otimes b_1,\ a_1 \otimes b_2,\ a_2 \otimes b_1,\ a_2 \otimes b_2\}$$

of $V_0 \otimes V_1$, so that any superpotential is represented by a symmetric 4×4 matrix. In these coordinates, a superpotential lying in the affine subspace L can be represented by a matrix of the form

$$\begin{pmatrix} 0 & 0 & 0 & \frac{1}{2}\lambda_0 \\ 0 & \alpha & -\frac{1}{2}\lambda_1 & \gamma \\ 0 & -\frac{1}{2}\lambda_1 & \beta & \delta \\ \frac{1}{2}\lambda_0 & \gamma & \delta & \epsilon \end{pmatrix}. \tag{4.5}$$

Hence, the generic superpotential in $Sup_4(Q)$ is good. Hence, there is a non-empty Zariski open subset of $\mathbb{P}(1, 2, 3, 4)$ representing equivalence classes of good superpotentials.

4.1.2 DT invariants of quantum/marginal deformations

We will now to concentrate on superpotentials in $\mathbb{P}(1,2,3,4)$ lying on the weighted projective line $l\colon \{z_1 = z_3 = 0\}$. In particular, we are only interested in potentials defining 3-CY algebras, so that we can exclude the point $p := (0,2,0,2)$. Thus, a potential in $l \setminus \{p\}$ admits a representative of the form

$$W_q = a_1b_1a_2b_2 - qa_1b_2a_2b_1, \tag{4.6}$$

where $q \in \mathbb{C}^*$, and it correspont to the point $(0, 2(1+q^2), 0, 2(1+q^4))$ on l.

Fix a dimension vector d for Q, and denote by $\mathcal{M}(J_{W_q}, d)$ the quotient stack

$$[\mathrm{Crit}(\omega_{q,d})/G_d]\,,$$

where $\omega_{q,d}$ is the regular function $\mathrm{Tr}(W_q)\colon R(Q,d) \longrightarrow \mathbb{C}$. The stack $\mathcal{M}(J_{W_q}, d)$ parametrises d-dimensional modules over the Jacobian algebra $J(Q, W_q)$. We will adopt the notation $\left[\mathcal{M}(J_{W_q}, d)\right]_{\mathrm{vir}}$ for the class

$$\frac{[\mathrm{Crit}(\omega_{q,d})]_{\mathrm{vir}}}{[G_d]_{\mathrm{vir}}}.$$

Recall that the virtual class of a regular function on a smooth scheme was defined in 1.41. Since the set $I = \{a_1\}$ is a cut for W_q, the class $\left[\mathcal{M}(J_{W_q}, d)\right]_{\mathrm{vir}}$ lives in the enhanced Grothendieck ring of varieties $\widetilde{\mathcal{M}}_{\mathbb{C}}$ of Section 1.4.2. We are interested in computing the universal series $A_{U,q}$ of motivic Donaldson-Thomas invariants for the quiver with superpotential (Q, W_q). This is the formal power series

$$A_{U,q} = 1 + \sum_{d \in \mathbb{N}^{V(Q)}} \left[\mathcal{M}(J_{W_q}, d)\right]_{\mathrm{vir}}\, y^d,$$

where we have introduced the multivariable notation $y^d = y_0^{d_0} y_1^{d_1}$. As we are in the case of a superpotential with a cut we can employ dimensional reduction, a procedure which has been analyzed by A. Morrison in [Mo1]. The idea is that it is possible to recover the motivic virtual class of the stack of modules over the Jacobi algebra from the class of the reduced variety

$$R(J_{W_q,I}, d) = \{(A_2, B_1, B_2) \in R(Q \setminus I, d)\colon B_1A_2B_2 - qB_2A_2B_1\}, \tag{4.7}$$

which forgets the matrices corresponding to arrows in the cut.

Theorem 4.5 ([Mo1, Proposition 5.1]). *There is an identity of generating series*

$$A_{U,q} = \sum_{d \in \mathbb{N}^{V(Q)}} \left(-\mathbb{L}^{\frac{1}{2}}\right)^{(d_0-d_1)^2} \frac{[R(J_{W_q,I}, d)]}{[G_d]}\, y^d.$$

The rest of the section will be devoted to the proof of the following theorem, which explicitly describes the universal series of motivic invariants of the marginally deformed conifold quiver with superpotential. We will compute the generating series by means of a stratification of the reduced variety $R(J_{W_q,I}, d)$ following the strategy of [MMNS, Section 2.2].

Theorem 4.6. *Let W_q be the deformed conifold potential of (4.6). The universal series of motivic Donaldson-Thomas invariants of (Q, W_q) factorizes as*

$$A_U(Q, W_q) = A_U(Q, W_1) \cdot \frac{\prod_{r \in \mathbb{N}} \operatorname{Exp}\left((\mathbb{L}-1)\frac{(y_0y_1)^r}{1-(y_0y_1)^r}\right)^{\delta_r(q)}}{\operatorname{Exp}\left((\mathbb{L}-1)\frac{y_0y_1}{1-y_0y_1}\right)}, \tag{4.8}$$

where $A_U(Q, W_1)$ is the universal series for the conifold quiver with undeformed superpotential, and where

$$\delta_r(q) = \begin{cases} 1 & \text{if } q \text{ is a primitive } r-th \text{ root of unity ,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.9}$$

Given a triple (A_2, B_1, B_2) in $R(J_{W_q,I}, d)$, consider the linear map

$$A_2 \oplus B_2 \in \operatorname{Hom}(V_0, V_1) \oplus \operatorname{Hom}(V_1, V_0) \subset \operatorname{End}(V),$$

where we denote by V the direct sum $V_0 \oplus V_1$ of dimension $d_0 + d_1$. The stratification of $R(J_{W_q,I}, d)$ is based on the following simple lemma about decomposing linear operators.

Lemma 4.7. *Let C be a linear endomorphism of a vector space W. There is a direct sum decomposition $W = W^I \oplus W^N$ such that*

- *$C = C^I \oplus C^N \in \operatorname{End}(W^I) \oplus \operatorname{End}(W^N)$, i.e. C is diagonal with respect to the decomposition $W = W^I \oplus W^N$;*
- *C^I is invertible;*
- *C^N is nilpotent.*

Proof. Let n be the dimension of W. Define two subspaces $W^N := \mathrm{Ker}(C^n)$ and $W^I = \mathrm{Im}(C^n)$. The subspaces W^I and W^N intersect trivially since $C^{2n}(w) = 0$ implies $C^n(w) = 0$. We thus obtain $W = W^I \oplus W^N$, and from the definition it is evident that C preserves both W^I and W^N. Invertibility of C^I and nilpotence of C^N are again direct consequences of the definition. □

Lemma 4.7 applied to $A_2 \oplus B_2$ defines decompositions $V_i = V_i^I \oplus V_i^N$, where the components are obtained by intersecting V_i with the invertible and the nilpotent subspaces of V. With respect to this decomposition we can write

$$\begin{aligned} A_2 &= A_2^I \oplus A_2^N \in \mathrm{Hom}(V_0^I, V_1^I) \oplus \mathrm{Hom}(V_0^N, V_1^N); \\ B_2 &= B_2^I \oplus B_2^N \in \mathrm{Hom}(V_1^I, V_0^I) \oplus \mathrm{Hom}(V_1^N, V_0^N), \end{aligned} \tag{4.10}$$

where A_2^I and B_2^I are required to be invertible. Clearly the subspaces V_0^I and V_1^I are of the same dimension. Notice also that both $A_2^N B_2^N$ and $B_2^N A_2^N$ are nilpotent endomorphisms. The following lemma ensures the compatibility of the matrix B_1 with respect to this decomposition.

Lemma 4.8. *Assume (A_2, B_1, B_2) lies in $R(J_{W_q,I}, d)$. Let $V_i = V_i^I \oplus V_i^N$ be the decomposition of (4.10) for $A_2 \oplus B_2$. Then we have*

$$B_1 = B_1^I \oplus B_1^N \in \mathrm{Hom}(V_1^I, V_0^I) \oplus \mathrm{Hom}(V_1^N, V_0^N).$$

Proof. First of all notice that B_1 maps V_1^N to V_0^N. Indeed, if $n \geq d_0$ and $v \in V_1^N$ we have $(B_2A_2)^n B_1(v) = q^{-n} B_1 (A_2B_2)^n(v) = 0$. Now take $n \geq \max(d_0, d_1)$. Writing the equality $B_1(A_2B_2)^n = q^n(B_2A_2)^n B_1$ in components we get

$$\begin{pmatrix} B_1^{II} & 0 \\ B_1^{NI} & B_1^{NN} \end{pmatrix} \begin{pmatrix} (A_2^I B_2^I)^n & 0 \\ 0 & 0 \end{pmatrix} = q^n \begin{pmatrix} (B_2^I A_2^I)^n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B_1^{II} & 0 \\ B_1^{NI} & B_1^{NN} \end{pmatrix},$$

and from the invertibility of $A_2^I B_2^I$ and $B_2^I A_2^I$ we see that $B_1^{NI} = 0$. □

Remark 4.9. This reasoning strongly depends on the assumption $q \neq 0$. Notice that this is the same condition for which the superpotential W_q actually defines a 3-CY algebra.

Remark 4.10. The notation $B_1 = B_1^I \oplus B_1^N$ is purely formal. It does not indicate any invertibility and nilpotence property of the components of B_1.

The reduced variety $R(J_{W_q,I}, d)$ admits a stratification $\{R_a(J_{W_q,I}, d)\}_a$ depending on the dimension $2a$ of the invertible subspace V^I of V. Define two representation varieties by

$$R_q^I(a) = \{(A_2, B_1, B_2) \in R(J_{W_q,I}, (a, a))\colon A_2 \oplus B_2 \text{ is invertible}\},$$

$$R_q^N(d) = \{(A_2, B_1, B_2) \in R(J_{W_q,I}, d) \colon A_2 \oplus B_2 \text{ is nilpotent}\},$$

and let $M(d, a)$ be the variety parametrising decompositions

$$\{V_0 = V_0^I \oplus V_0^N,\ V_1 = V_1^I \oplus V_1^N \colon \dim_{\mathbb{C}}(V_i^I) = a\}.$$

Notice that $M(d, a)$ is a Zariski open subset of the product of Grassmannians $\mathrm{Gr}(a, d_0) \times \mathrm{Gr}(a, d_1)$, and its class in $\mathrm{K}_0(\mathrm{Var}_{\mathbb{C}})$ is

$$[M(d, a)] = \frac{[\mathrm{GL}_{d_0}][\mathrm{GL}_{d_1}]}{[\mathrm{GL}_a]^2[\mathrm{GL}(d_0 - a)][\mathrm{GL}(d_1 - a)]}.$$

For every choice of a in $\{0, \dots, \min(d_0, d_1)\}$ the variety $M(a, d)$ admits a Zariski locally trivial tautological vector bundle $\mathcal{V}_0^I \oplus \mathcal{V}_0^N \oplus \mathcal{V}_1^I \oplus \mathcal{V}_0^N$. One can consider then the bundle of homomorphisms

$$\mathrm{Hom}(\mathcal{V}_0^\bullet, \mathcal{V}_1^\bullet) \oplus \mathrm{Hom}(\mathcal{V}_1^\bullet, \mathcal{V}_0^\bullet)^{\oplus 2}$$

for $\bullet \in \{I, N\}$, and it becomes evident that the deformed relation (4.7) defines a Zariski locally trivial fibration

$$\begin{tikzcd} R_q^I(a) \times R_q^N(d_0 - a, d_1 - a) \arrow[r, hook] & R_a(J_{W_q,I}, d) \arrow[d] \\ & \mathcal{M}(d, a). \end{tikzcd}$$

This gives the equality

$$\frac{[R_a(J_{W_q,I}, d)]}{[\mathrm{GL}_d]} = \frac{[R_q^I(a)]}{[\mathrm{GL}_a]^2} \frac{[R_q^N(d_0 - a, d_1 - a)]}{[\mathrm{GL}_{(d_0 - a, d_1 - a)}]}.$$

Setting

$$N_q(y_0, y_1) = \sum_d (-\mathbb{L}^{-\frac{1}{2}})^{(d_0 - d_1)^2} \frac{[R_q^N(d)]}{[\mathrm{GL}_d]} y^d,$$

$$I_q(y) = \sum_a \frac{[R_q^I(a)]}{[\mathrm{GL}_a]^2} y^a,$$

the universal series factors into invertible and nilpotent parts as

$$A_U(Q, W_q) = I_q(y_0 y_1) N_q(y_0, y_1).$$

We now compute the invertible and nilpotent contributions in the generating series by comparing them to the invertible and nilpotent parts in the undeformed generating series. The results are expressed in exponential form, and we refer to Section 1.4 for the definition and properties of motivic exponentiation.

Lemma 4.11. *The invertible part $I_q(y)$ factors as*

$$I_q(y) = \mathrm{Exp}\left(\frac{y}{1-y}\right) \cdot \prod_{r \in \mathbb{N}} \mathrm{Exp}\left((\mathbb{L}-1)\frac{y^r}{1-y^r}\right)^{\delta_r(q)}$$

for $\delta_r(q)$ as defined in 4.9.

Proof. We reduce the computation to the calculation of the generating series of the classes of the invertible part of the marginally deformed commuting varieties, i.e.

$$C_q^I(a) = \{(A,B) \in \mathrm{End}\,\mathbb{C}^a \colon AB - qBA = 0, B \text{ invertible}\}.$$

This has been computed by A. Morrison [CMPS], and we discuss the main ideas of the calculation in the Appendix 4.A to Chapter 4 for completeness. The map

$$R_q^I(a) \longrightarrow C_q^I(a), \quad (A_2, B_1, B_2) \longrightarrow (B_2^{-1}B_1, A_2B_2)$$

is a GL_a Zariski fiber bundle over $C_q^I(a)$, where the fiber over (X,Y) is

$$\mathrm{GL}_a(Y, X, \mathrm{Id}) = \{(Yg^{-1}, gX, g)\}.$$

We thus have

$$I_q(y_0y_1) = \sum_a \frac{[C_q^I(a)]}{[\mathrm{GL}_a]}\, y^a, \tag{4.11}$$

and the statement follows directly from Theorem 4.30. □

Lemma 4.12. *The nilpotent part of the generating series $N_q(y_0, y_1)$ is not affected by marginal deformations.*

Proof. Let us first set some combinatorial notation. Given a partition π of an integer n we will denote by $\pi(i)$ the number of parts of type i and by $l(\pi)$ the length of the partition. For a quadruple of partitions $(\pi_1, \pi_2, \pi_3, \pi_4)$, define sets of indices I_{π_k} for $1 \leq k \leq 4$ by

$$I_{\pi_k} = \{(i,j,h) \colon 1 \leq i \leq l(\pi_k), 1 \leq j \leq \pi_k(i), 1 \leq h \leq 2i\}$$

when $k \leq 2$ and

$$I_{\pi_k} = \{(i,j,k) \colon 1 \leq i \leq l(\pi_i), 1 \leq j \leq \pi_k(i),\ 1 \leq h \leq 2i-1\}$$

when $k \geq 3$.

Given an element (A_2, B_1, B_2) of $R_q^N(\alpha)$ there is a quadruple of partitions $(\pi_1, \dots, \pi_4)$ and a basis

$$\{\{a_h^{ij}\}_{I_{\pi_1}}, \{b_h^{ij}\}_{I_{\pi_2}}, \{c_h^{ij}\}_{I_{\pi_3}}, \{d_h^{ij}\}_{I_{\pi_4}}\} \tag{4.12}$$

of $V_0 \oplus V_1$ such that

$$V_0 \ni a_{2i}^{ij} \overset{A_2}{\mapsto} a_{2i-1}^{ij} \overset{B_2}{\mapsto} \dots \overset{A_2}{\mapsto} a_1^{ij}, \text{ s.t. } B_2(a_1^{ij}) = 0;$$
$$V_1 \ni b_{2i}^{ij} \overset{B_2}{\mapsto} b_{2i-1} \overset{A_2}{\mapsto} \dots \overset{B_2}{\mapsto} b_1, \text{ s.t. } A_2(b_1^{ij}) = 0;$$
$$V_0 \ni c_{2i-1}^{ij} \overset{A_2}{\mapsto} c_{2i-2}^{ij} \overset{B_2}{\mapsto} \dots \overset{B_2}{\mapsto} c_1^{ij}, \text{ s.t. } A_2(c_1^{ij}) = 0;$$
$$V_1 \ni d_{2i-1}^{ij} \overset{B_2}{\mapsto} d_{2i-2}^{ij} \overset{A_2}{\mapsto} \dots \overset{A_2}{\mapsto} d_1{}^{ij}, \text{ s.t. } B_2(d_1^{ij}) = 0.$$

In particular the π_i's contain all the combinatorial information necessary to describe the normal forms of both A_2B_2 and B_2A_2, which are respectively $(\pi_1, \pi_2, \pi_3 - \delta, \pi_4)$ and $(\pi_1, \pi_2, \pi_3, \pi_4 - \delta)$, where $\pi - \delta$ denotes $(\pi(1) - 1, \pi(2) - 1, \dots, \pi(n) - 1)$ for a partition π of length n. The nilpotent reduced variety $R_q^N(d)$ admits a stratification

$$R_q^N(d) = \bigsqcup_{\{\pi_i\} \in A_d} R_q(\{\pi_i\}),$$

where A_d is the set of quadruples of partitions satisfying

$$\{d_0 = \sum |\pi_i| - l(\pi_4), \quad d_1 = \sum |\pi_i| - l(\pi_3)\},$$

and where the stratum indexed by (π_i) corresponds to triples (A_2, B_1, B_2) with $A_2 \oplus B_2$ admitting a basis as in (4.12). Each stratum is a vector bundle

$$\begin{tikzcd}
F_q \arrow[r, hook] \arrow[d] & R_q(\{\pi_i\}) \arrow[d] \\
(A_2, B_2) \arrow[r, hook] & \{(A_2, B_2) \colon A_2 \oplus B_2 \sim N(\{\pi_i\})\}
\end{tikzcd}$$

with fiber $F_q = \{B_1 \colon B_1A_2B_2 - qB_2A_2B_1 = 0\}$ over pairs of matrices whose direct sum has a prescribed normal form. The base of the fibration is clearly independent on the deformation parameter q, and we claim the same if true for the fibers. Since $A_2 \oplus B_2$ is nilpotent, also the product B_2A_2 is nilpotent. So B_2A_2 is similar to qB_2A_2 via some change of basis, say given by P. Then

$$F_q \xrightarrow{P^{-1}\cdot\_} F_1$$

gives an isomorphism between the fibers. We are then reduced to the classical case considered in [MMNS, Section 2.2], so that

$$N_q(y_0, y_1) = N_1(y_0, y_1) = \operatorname{Exp}\left(\frac{2\mathbb{L}y_0y_1 - \mathbb{L}^{-\frac{1}{2}}(y_0 + y_1)}{\mathbb{L} - 1} \frac{1}{1 - y_0y_1} \right).$$

$\square$

Combining the results of the two lemmas, we obtain the result of Theorem 4.6.

Formula (4.8) implies an immediate but important geometric consequence. Since the generating series depends on the choice of the deformation parameter, we can conclude that motivic Donaldson-Thomas invariants of quiver with superpotential algebras are not invariant under graded deformation. This could provide motivation to investigate if the same assertion is true for motivic invariants of moduli spaces of sheaves with orientation, defined in [BJM], on Calabi-Yau threefolds.

4.1.3 Generating series and moduli of representations

Assuming that q is not a root of unity, the generating series has the form

$$A_{U,q} = \mathrm{Exp}\left(\frac{3\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}\frac{y_0y_1}{1-y_0y_1} - \frac{1}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}\frac{y_0+y_1}{1-y_0y_1}\right).$$

This result shows that there are some elementary classes in $\mathcal{M}_{\mathbb{C}}$ for low dimension vectors which determine via exponentiation the collection of motivic Donaldson-Thomas invariants. We investigate how these classes can be interpreted as virtual classes of the moduli space of the corresponding q-deformed variety of representations.

It is worth spending a few words on the undeformed case, as it will motivate our investigation of the deformed generating series. The universal series for the undeformed algebra is

$$A_{U,1} = \mathrm{Exp}\left(\frac{\mathbb{L}^{\frac{3}{2}} + \mathbb{L}^{\frac{1}{2}}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}\frac{y_0y_1}{1-y_0y_1} - \frac{1}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}\frac{y_0+y_1}{1-y_0y_1}\right).$$

Our first attempt is to interpret the motives in the generating series in terms of simple representations, in analogy with the case of the 3-loop quiver with marginally deformed superpotential 4.A. Define $\delta_0 = (1,0)$ and $\delta_1 = (0,1)$, and let δ be $\delta_0 + \delta_1$. We will denote by $M^s(J_1,d)$ (resp. $\mathcal{M}^s(J_1,d)$) the open subscheme (resp. substack) of $M(J_1,d)$ (resp. $\mathcal{M}(J_1,d)$) of simple modules. One can show the following:

- $M^s(J_1,\delta_i) = \{\mathrm{pt}\}$, so that $\mathcal{M}^s(J_q,\delta_i) = \{\mathrm{pt}\}/\mathbb{C}^*$ and
$$[\mathcal{M}^s(J_1,\delta_i)]_{vir} = 1/[\mathbb{C}^*]_{vir} = -(\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}})^{-1};$$
- $M^s(J_1,\delta) \cong \mathcal{O}_{\mathbb{P}^1}(-1,-1) \setminus Z$, where Z is the zero section, so that
$$[\mathcal{M}^s(J_1,\delta)]_{vir} = \frac{(\mathbb{L}^{\frac{1}{2}} + \mathbb{L}^{-\frac{1}{2}})(\mathbb{L}-1)}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}.$$

While the virtual motive of simple representations of dimension vector δ_i corresponds precisely to the coefficient of the δ_i term $y_i/(1-y_0y_1)$, this is not the case for dimension vector δ. If we consider the King stability condition $(1,-1)$ we have that $M^\theta(J_q,\delta)$ is the closure of the simple locus $M^s(J_q,\delta)$ and

$$\left[\mathcal{M}^\theta(J_1,\delta)\right]_{vir} = \frac{\mathbb{L}^{\frac{3}{2}}+\mathbb{L}^{\frac{1}{2}}}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}},$$

which gives the coefficient of $y^\delta/(1-y^\delta)$.

The necessity of considering also non-simple representations to understand the motives in the generating series becomes even more evident in the case of a deformed potential W_q for $q \neq 0,1$. For the dimension vectors δ_i the situation is completely analoguous to the undeformed case. In the dimension δ case, the moduli space $M^s(J_q,\delta)$ is cut out by the function $\omega_q^s = \mathrm{Tr}(W_q^s)$ on $M^s(J_q,\delta)$. It turns out that it is the disjoint union of 4 copies of $\mathbb{C}^*$, namely the coordinate axes minus the origin of the fibers over 0 and ∞ when we consider the inclusion $M^s(Q,\delta) \subset \mathcal{O}_{\mathbb{P}^1}(-1,-1)$. We compute the virtual class of the simple locus of representations of the Jacobian algebra. Using the description $M^s(Q,\delta) \subset \mathcal{O}_{\mathbb{P}^1}(-1,-1)$ we can describe ω_q^s locally on the two charts U_1 and U_2 which trivialize the bundle $\mathcal{O}_{\mathbb{P}^1}(-1,-1)$. On each of the charts our function can be expressed as

$$\omega_q^s \colon U_i \cap M^s(Q,\delta) \simeq \mathbb{C}^3 \setminus (\{\underline{0}\} \times \mathbb{C}) \longrightarrow \mathbb{C}$$
$$\omega_q^s \colon (x,y,z) \longrightarrow xyz.$$

Denoting by $\omega_q^s(i)$ the restriction of ω_q^s to $U_i \cap M^s(Q,\delta)$ we have

$$[(\omega_q^s(i))^{-1}(1)] - [(\omega_q^s(i))^{-1}(0)] = -2\mathbb{L}(\mathbb{L}-1),$$

and denoting by $\omega_q^s(12)$ the restriction to the overlap of the two charts we have

$$[(\omega_q^s(12))^{-1}(1)] - [(\omega_q^s(12))^{-1}(0)] = -(\mathbb{L}-1)^2.$$

We thus obtain

$$[\mathcal{M}^s(J_q,\delta)]_{vir} = \frac{3\mathbb{L}^{\frac{1}{2}}-2\mathbb{L}^{-\frac{1}{2}}-\mathbb{L}^{-\frac{3}{2}}}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}.$$

This agrees only birationally with the coefficient of $y^\delta/(1-y^\delta)$ in the argument of the exponential form of the universal series. Let us explain how replacing simplicity with King stability gives full agreement. First of all in this case it is not a sensible approach to consider the closure of the simple locus inside the moduli space of

θ-stable representations with dimension vector δ, as this subvariety does not arise as the critical locus of a function on $\mathcal{M}^\theta(Q,\delta)$. We proceed instead as follows: the function ω_q^s naturally extends to a function ω_q on all of $M^\theta(Q,\delta)$. The critical locus $M^\theta(J_q,\delta)$ of ω_q consists of the union of $M^s(J_q,\delta)$ and the zero section inside $M^\theta(Q,\delta)$ as depicted in Figure 4.1.

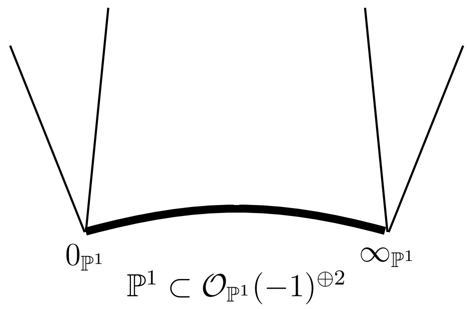

Figure 4.1: Critical locus of $Tr(W_q)$, $q \neq 1$ and dimension vector $(1,1)$.

The following proposition shows that in this way we recover the motive appearing as the coefficient of the $y^\delta/(1-y^\delta)$ in the exponential form of the universal series.

Proposition 4.13. *The virtual fundamental class of the moduli stack $\mathcal{M}^\theta(J_{\underline{q}},\delta)$ of θ-stable δ-dimensional representations is*

$$\frac{3\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}.$$

Proof. Since $\mathbb{C}^*$ acts trivially on $M^\theta(J_q,\delta)$ it suffices to compute $[M^\theta(J_q,\delta)]_{\mathrm{vir}}$. Notice that we can cover $M^\theta(J_q,\delta)$ with two charts U_1 and U_2 corresponding to the two charts of $\mathbb{P}^1$. The restriction of the function ω_q to each of these charts is

$$\omega_q \colon U_1 \cong \mathbb{C}^3 \longrightarrow \mathbb{C}, \quad \omega_q(x,y,z) = xyz.$$

Furthermore, $U_1 \cap U_2 \cong \mathbb{C}^2 \times \mathbb{C}^*$, where the coordinate z is different from 0. We then have

$$[M^\theta(J_q,\delta)]_{vir} = -(-\mathbb{L}^{-\frac{1}{2}})^3(-2(2\mathbb{L}^2 - \mathbb{L}) + (\mathbb{L}^2 - \mathbb{L})) = -3\mathbb{L}^{\frac{1}{2}} + \mathbb{L}^{-\frac{1}{2}}.$$

$\square$

The reason why we are forced to introduce the stability condition θ in order to describe the elementary virtual motives which determine the universal series is not entirely clear at this stage. It would be interesting to understand if the introduction of θ is related to motivic wall crossing, since in a sense the fundamental role played by the θ-stable representations in determining the universal series can be interpreted as the fact they are the building blocks out of which all the other representations are constructed.

4.2 DT invariants of deformations of noncommutative resolutions of $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_n$

Let Q be the McKay quiver of $G = \mathbb{Z}_n \subset \mathrm{SL}(3, \mathbb{C})$ with diagonal embedding with weights $(0, 1, -1)$. The McKay quiver is the same quiver we considered in Figure 1.2, so that we have

- n vertices $\{0, \ldots, n-1\}$;
- a loop b_i at each vertex;
- an arrow $a_i \colon i \longrightarrow i+1$ for $i \in \{0, \ldots, n-1\}$;
- an arrow $a_i^* \colon i+1 \longrightarrow i$ for $i \in \{0, \ldots, n-1\}$.

Notation 4.14. The indices appearing in the current section are to be understood modulo n. It will be mentioned explicitly when an index is to be taken in classical terms.

We have seen in Example 1.47 that a very generic degree 3 potential defines a 3-CY algebra, since the potential

$$W_{\underline{1}} := \sum_{i=0}^{n-1} b_{i+1} a_i^* a_i - b_i a_i a_i^*$$

defines an algebra $A(Q, W_{\underline{1}})$ isomorphic to the skew group algebra $\mathbb{C}[V] * G$. We concentrate in this section on the family of graded quantum/marginal deformations of the McKay potential $W_{\underline{1}}$ given by

$$W_{\underline{q}} = \sum_{i=0}^{n-1} q_i' b_{i+1} a_i^* a_i - q_i b_i a_i a_i^*, \tag{4.13}$$

where we assume $\prod_j q'_j q_j \neq 0$. Up to rescaling the a_i's we can assume that the coefficients q'_i are identically 1. Denoting by q the choice of an n-th root of the product of the q_i's, the change of variables

$$b'_0 = b_0,\ a'_{i-1} = q^{-i} q_0 \dots q_{i-1} a_{i-1},\ b'_i = q^i q_0^{-1} \dots q_{i-1}^{-1} b_i,\ a'_{n-1} = a_{n-1}$$

brings $W_{\underline{q}}$ into the form

$$W_q = \sum_{i=0}^{n-1} b_{i+1} a_i^* a_i - q b_i a_i a_i^*,$$

so that in fact the family of potentials we are considering corresponds to a one parameter deformation of the classical McKay algebra of G with embedding in $\mathrm{SL}(3, \mathbb{C})$ described above. Ginzburg observed in [Gi, Remark 4.5.3] that a q-deformation of the McKay superpotential defines a 3-CY algebra when q is assumed not to be a root of unity. The following lemma shows that it is enough to require that the deformation parameter does not vanish.

Lemma 4.15. *For any choice of potential W_q in the family*

$$\left\{ \sum_{i=0}^{n-1} b_{i+1} a_i^* - q b_i a_i a_i^* : q \in \mathbb{C}^* \right\}$$

the Jacobi algebra $J(Q, W)$ is 3-CY.

Proof. We consider the ordering

$$b_0 > a_0 > b_1 > a_1 > \dots > b_{n-1} > a_{n-1} > a_0^* > \dots > a_{n-1}^*$$

on the set $E(Q)$ of arrows of the A_n-quiver. The leading terms of the relations defined by the superpotential are

$$\mathrm{lt}(\partial_{a_i} W) = b_{i+1} a_i^*,\ \mathrm{lt}(\partial_{a_i^*} W) = -q b_i a_i,\ \mathrm{lt}(\partial_{b_i^*} W) = -q a_i a_{i^*}.$$

Since none of the leading terms ends in b_i the condition (2) of Theorem 1.49 is immediately satisfied. On the other hand, for each choice of i in $\{0, \dots, n-1\}$ there is only one ambiguity associated to the vertex i:

$$(b_i, \mathrm{lt}(\partial_{b_i} W)(a_i^*)^{-1}, a_i^*) = (b_i, b_i^{-1} \mathrm{lt}(\partial_{a_i^*} W), a_i^*),\ \text{and } b_i \,\mathrm{lt}(\partial_{b_i} W) = \mathrm{lt}(\partial_{a_i^*} W) a_i^* = \mathrm{lt}(iWi).$$

It follows then from Theorem 1.49 that the Jacobi algebra $J(Q, W)$ is 3-CY. $\square$

We compute the universal generating series of motivic Donaldson-Thomas invariants

$$A_{U,q} = 1 + \sum_{\underline{d}\in\mathbb{N}^{V(Q)}} \left[\mathcal{M}(J_{W_q}, d)\right]_{\mathrm{vir}}\ y^{\underline{d}},$$

where J_{W_q} is the Jacobi algebra $J(Q, W_q)$, and where we adopt the multi-index notation $y^{\underline{d}} = y_0^{d_1} \dots y_{n-1}^{d_{n-1}}$. Once again the result will be expressed in exponential form, and we will be able to interpret the motives in the argument of the exponential in terms of the virtual classes of representation spaces of J_{W_q}.

4.2.1 DT invariants of marginal/quantum deformations

The set of arrows $I = \{a_0^*, \dots, a_{n-1}^*\}$ is a cut for W_q. For a given dimension vector $\underline{d}$ we denote by V_i a d_i-dimensional vector space, and we define the reduced variety

$$R(J_{W_q,I}, \underline{d}) = \{A_i \in \mathrm{Hom}(V_i, V_{i+1}), B_i \in \mathrm{End}(V_i)\colon B_{i+1}A_i - qA_iB_i = 0\},$$

so that by [Mo1, Proposition 5.1]

$$A_{U,q} = \sum_{\underline{d}\in\mathbb{N}^{V(Q)}} \frac{[R(J_{W_q,I}, \underline{d})]}{[G_{\underline{d}}]}\, y^{\underline{d}}.$$

The remainder of this section will be devoted to the proof of the following theorem.

Theorem 4.16. *Let $A_U(Q, W_1)$ be the generating series for the undeformed case $q = 1$. Then*

$$A_U(Q, W_q) = A_U(Q, W_1) \cdot \frac{\prod_{r\in\mathbb{N}} \mathrm{Exp}\left((\mathbb{L}-1)\frac{y^{r\delta}}{1-y^{r\delta}}\right)^{\delta_r(q^n)}}{\mathrm{Exp}\left((\mathbb{L}-1)\frac{y^{\delta}}{1-y^{\delta}}\right)}, \tag{4.14}$$

where we set $\delta = (1, \dots, 1)$ and where

$$\delta_r(q^n) = \begin{cases} 1 & \text{if } q^n \text{ is a primitive } r\text{-th root of unity} \\ 0 & \text{otherwise} \end{cases}.$$

We rewrite the definition of the reduced variety in a more compact form. If we denote by V the direct sum $\oplus_i V_i$, the endomorphism $B = \oplus B_i$ is of block diagonal type (b.d.) and $A = \oplus A_i$ is of cyclic type (c.t.), so that

$$R(J_{W_q,I}, \underline{d}) = \{A, B \in \mathrm{End}(V)\colon BA - qAB,\ B \text{ b.d.}, A \text{ c.t.}\}. \tag{4.15}$$

By Lemma 4.7 there is a decomposition $V = V^I \oplus V^N$ such that A decomposes into the direct sum of its nilpotent and invertible part.

Lemma 4.17. *The endomorphism B is compatible with the decomposition $V = V^I \oplus V^N$.*

Proof. From the equality $BA = qAB$ we deduce that B preserves the nilpotent subspace V^N. Writing the equation $BA^{|\underline{d}|} = q^{|\underline{d}|}A^{|\underline{d}|}B$ with respect to the block decomposition we have

$$\begin{pmatrix} B^{II} & 0 \\ B^{IN} & B^{NN} \end{pmatrix} \begin{pmatrix} (A^I)^{|\underline{d}|} & 0 \\ 0 & 0 \end{pmatrix} = q^{|\underline{d}|} \begin{pmatrix} (A^I)^{|\underline{d}|} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B^{II} & 0 \\ B^{IN} & B^{NN} \end{pmatrix},$$

and from this we immediately see that $B^{IN} = 0$ for $q \neq 0$. $\square$

We can consider a further decomposition which combines both the original one of (4.15) and that of Lemma 4.17, so that we end up having $V_i = V_i^I \oplus V_i^N$ with $A_i = A_i^I \oplus A_i^N$, where A_i^I is invertible and the direct sum $\oplus A_i^N$ is nilpotent. Note that this implies that $\dim V_i^I = \dim V_j^I$ for any choice of i and j in $\mathbb{Z}_n$. Given $a \in \mathbb{N}$, denote by $a\delta \in \mathbb{N}^{V(Q)}$ the vector $(a, \ldots, a)$ as in Theorem 4.16. The variety $R(J_{W_q,I}, \underline{d})$ admits a stratification $\{R(J_{W_q,I}, \underline{d}, a\delta)\}_a$, where each stratum consists of representations of Q with dimension vector $\underline{d}$ which decompose into invertible and nilpotent part in such a way that $\dim(V_i^I) = a$. For each of these strata we have a Zariski locally trivial fibration

$$\begin{array}{ccc} R_q^I(a\delta) \times R_q^N(\underline{d} - a\delta) & \hookrightarrow & R(J_{W_q,I}, \underline{d}, a\delta) \\ & & \downarrow \\ & & \mathcal{M}(\underline{d}, a\delta), \end{array}$$

where $\mathcal{M}(\underline{d}, a\delta)$ is the space of decompositions $V_i = V_i^I \oplus V_i^N$ and where $R_q^I(a\delta)$ and $R_q^N(\underline{d} - a\delta)$ are respectively the spaces of the invertible and nilpotent part of the representation in the sense above. Proceeding in analogy with the conifold case we define

$$I_q(y) = \sum_{a \in \mathbb{N}} \frac{[R_q^I(a\delta)]}{[\mathrm{GL}_a]} y^a,$$

$$N_q(\underline{y}) = \sum_{\underline{d} \in \mathbb{N}^{V(Q)}} \frac{[R_q^N(\underline{d})]}{[\mathrm{GL}_{\underline{d}}]} y^{\underline{d}},$$

so that the universal series factors as $A_U(Q, W_q) = I_q(y_0 \ldots y_n) N_q(\underline{y})$. Again we compute separately the invertible and the nilpotent parts of the generating series and then combine the results.

Lemma 4.18. *The invertible part of the universal series has exponential form*

$$I_q(y) = \mathrm{Exp}\left(\frac{y}{1-y}\right) \prod_{r \in \mathbb{N}} \mathrm{Exp}\left((\mathbb{L}-1)\frac{y^r}{1-y^r}\right)^{\delta_r(q^n)},$$

where

$$\delta_r(q^n) = \begin{cases} 1 & \text{if } q^n \text{ is a primitive } r\text{-th root of unity;} \\ 0 & \text{otherwise.} \end{cases}.$$

Proof. Since the A_i's are invertible, we have the expression

$$B_i = q^i A_{i-1} \dots A_0 B_0 (A_{i-1} \dots A_0)^{-1}.$$

for $i \neq 1$, and we can use this to reduce the system of matrix equations to the single equation

$$B_0(A_{n-1} \dots A_0) - q^n (A_{n-1} \dots A_0) B_0 = 0.$$

Then we have that $R^I_q(a\delta)$ is a Zariski locally trivial fiber bundle over $C^I_{q^n}(a)$ with fiber GL_a^{n-1}, where $C^I_{q^n}(a)$ is the q^n-deformed invertible part of the commuting variety:

$$\{A \in \mathrm{GL}(V_0), B \in \mathrm{End}(V_0) \colon BA - q^n AB = 0\}.$$

From this it is clear that

$$I_q(y) = \sum_a \frac{[C^I_{q^n}(a)]}{[\mathrm{GL}_a]} y^a,$$

and so as in Lemma 4.11 we obtain the formula. □

Lemma 4.19. *The nilpotent contribution to the generating series is independent of the deformation.*

Proof. There is a collection $\{e_1, \dots, e_l\}$ such that $\{A^j e_i \colon A^j e_i \neq 0\}_{ij}$ is a basis of V. We say that e_i starts at a and ends at b if $e_i \in V_a$ and $A^k e_i \in V_b$, where k is the smallest integer such that $A^{k+1} e_i = 0$. We call the integer $\frac{k}{n} + 1$ the length of e_i. Next we define a collection of partitions $\{\pi(a,b)\}$ such that $\pi(a,b)_j$ is the number of vectors in $\{e_1, \dots, e_l\}$ starting at a, ending at b, and of length j. Denote by $J_{\{\pi(a,b)\}}$ the associated matrix in Jordan normal form. There is a block diagonal matrix P such that $PAP^{-1} = J_{\{\pi(a,b)\}}$. Note that the same initial set of vectors $\{e_1, \dots, e_l\}$ defines a basis for the matrix qA which bring it in the same Jordan normal form $J_{\{\pi(a,b)\}}$, so that we have a matrix P' of diagonal block type such that $P'qA(P')^{-1} = J_{\{\pi(a,b)\}}$. But then multiplication by $(P')^{-1}P$ defines an isomorphism between

$$\{B \colon BA - qAB = 0, B \text{ b. d. }\}$$

and

$$\{B\colon BA - AB = 0, B \text{ b. d.}\}.$$

Using the fact that there is a stratification $\{R_q^N(\underline{d} - a\delta, \pi)\}_{\{\pi(a,b)\}}$ of $R^N(\underline{d} - a\delta)$ indexed by the set of collections of partitions $\{\pi(a,b)\}$, together with the fact that each stratum is a vector bundle over the variety $N(\{\pi(a,b)\})$ of nilpotent matrices which are similar to $J_{\{\pi(a,b)\}}$, we can conclude the proof of the lemma. $\square$

From Lemma 4.12 and 4.19 we obtain a proof of Theorem 4.16.

The results obtained for the non-commutative crepant resolution of a line of A_n-singularities are in complete agreement with the results of section 4.1. The universal series of motivic invariants is affected by deformations of the potential.

4.2.2 Generating series and moduli of representations

We focus mainly on the space of representations with dimension vector $\delta = (1, \dots, 1)$, since only the coefficient of $\frac{y^\delta}{1-y^\delta}$ changes when considering a q-deformation of $J(Q, W_{\underline{1}})$. Let us start by discussing the undeformed case. Let δ_i be the dimension vector which takes value 1 on the vertex i and 0 otherwise. Define

$$\Delta := \{\delta_i + \ldots + \delta_{i+k}\colon i \in \{0, \dots, n-1\}, k \in \{0, \dots, n-2\}\},$$

where we read the indices modulo n, so that for example the vector $\delta_2 + \ldots + \delta_{n-1} + \delta_0$ lies in Δ. The universal series of motivic Donaldson-Thomas invariants of $J(Q, W_{\underline{1}})$, computed by Bryan and Morrison in [BM], reads

$$\operatorname{Exp}\left(\frac{\mathbb{L}^{\frac{3}{2}} + (n-1)\mathbb{L}^{\frac{1}{2}}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \frac{y^\delta}{1 - y^\delta} + \sum_{\alpha \in \Delta} \frac{\mathbb{L}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \frac{y^\alpha}{1 - y^\delta}\right). \tag{4.16}$$

While the virtual class of the stack $\mathcal{M}^s(J_{W_1}, \alpha)$ of simple representations with dimension vector α in Δ coincides with the coefficient of the term $y^\alpha/(1 - y^\delta)$, this is not the case for dimension vector δ. In the following example we show this phenomenon explicitly for the case $n = 2$; the argument can be easily generalized for all cyclic groups $\mathbb{Z}_n$.

Example 4.20. A representation (b_i, a_i, a_i^*) with dimension vector $(1, 1)$ is simple if and only if $(a_i, a_{i+1}^*) \neq \underline{0}$ for i in $\{0, 1\}$. We can realize $M^s(Q, \delta)$ as a subscheme of $M^\theta(Q, \delta)$ where $\theta = (-1, 1)$ is the stability condition giving the G-Hilbert scheme of

$\mathbb{C}^3$. The variety $M^\theta(Q,\delta)$ can be covered by two coordinate charts U_0 and U_1 with coordinates respectively

$$\begin{aligned} &y_{0,0}, y_{0,1}, x_{0,0}, x_{0,1}, x^*_{0,1}, \\ &y_{1,0}, y_{1,1}, x_{1,0}, x^*_{1,0}, x_{0,1}; \end{aligned} \tag{4.17}$$

and transition functions

$$y_{1,j} = y_{0,j}, x_{1,0} = x_{0,0}x_{0,1}, x^*_{1,0} = x^{-1}_{0,1}, x_{1,1} = x_{0,1}x^*_{0,1}.$$

The intersection $U_i^s = U_i \cap M^s(Q,\delta)$ is the complement of the 3-dimensional linear subspace cut out by $x_{i,i} = x_{i,i+1} = 0$. The function $\omega = \mathrm{Tr}(W_1)\colon M^s(Q,\delta) \longrightarrow \mathbb{C}$ can be expressed in coordinates as

$$\omega|_{U_i^s}(x_{i,j}) = (y_{i,i+1} - y_{i,i})(x_{i,i} - x_{i,i+1}x^*_{i,i+1})$$

on the open set U_i^s. We can thus compute

$$[\omega|^{-1}_{U_i^s}(1)] - [\omega|^{-1}_{U_i^s}(0)] = -\mathbb{L}^3(\mathbb{L}-1),$$

and on the double overlap

$$[\omega|^{-1}_{U_0^s\cap U_1^s}(1)] - [\omega|^{-1}_{U_0^s\cap U_1^s}(0)] = -\mathbb{L}^2(\mathbb{L}-1)^2.$$

The virtual class of the coarse moduli space of simple representations of J_{W_1} is

$$[M^s(J_{W_1},\delta)]_{vir} = -\mathbb{L}^{\frac{3}{2}} + \mathbb{L}^{-\frac{1}{2}},$$

which differs from the coefficient of $y^\delta/(1-y^\delta)$ by the class $\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}$.

Remark 4.21. This is a special instance of Theorem 3.4 in the paper [BJM] of Bussi, Joyce and Meinhardt. This theorem states that the virtual motive $[\mathrm{Crit}(f)]_{\mathrm{vir}}$ of the critical locus $X = \mathrm{Crit}(f)$ of a regular function $f\colon U \longrightarrow \mathbb{C}$ on a smooth scheme U depends only on the third thickening $X^{(3)}$ of X inside U and on the restriction $f^{(3)}$ of f to $X^{(3)}$. In our case, we have the following situation: we have a regular function $f\colon U \longrightarrow \mathbb{C}$ such that $X = \mathrm{Crit}(f)$ is smooth and the restriction of f to X is identically zero. Denote by $\widetilde{X}^3$ the third order thickening of X inside U and by $f^{(3)}$ the restriction of f to $\widetilde{X}^{(3)}$ as above. The third order thickening $X^{(3)}$ of X inside itself does not affect X and the restriction of f to X, so that we have a commutative diagram

$$\begin{tikzcd} X^{(3)} \arrow[r] \arrow[dr, "0"'] & \widetilde{X}^{(3)} \arrow[d, "f^3"] \\ & \mathbb{C}. \end{tikzcd}$$

It follows then from [BJM, Theorem 3.4] that $[(X,0)]_{\mathrm{vir}} = [\mathrm{Crit}(f)]_{\mathrm{vir}}$.

We thus proceed as in Subsection 4.1.3 extending the function ω^s to the locus of θ-stable representations of the McKay quiver, where θ is the King stability condition $(-(n-1), 1, \dots, 1)$ in the chamber which give rise to the G-Hilbert scheme.

Proposition 4.22. *The virtual class of the moduli space of θ-stable representations of the Jacobi algebra J_{W_1} is*

$$[M^\theta(J_{W_1}, \delta)]_{vir} = -\mathbb{L}^{\frac{3}{2}} - (n-1)\mathbb{L}^{\frac{1}{2}}.$$

Proof. We start describing an atlas $\{U_i\}_i$ for $M^\theta(Q, \delta)$ and describing how the function $\omega = \mathrm{Tr}(W_1)$ can be expressed locally in coordinates. We adopt the following notation: if $x_1, \dots, x_k$ are coordinates for $\mathbb{C}^k$, we denote by $(x_{i_1}, \dots, x_{i_s})$ the locus where all the x_{i_j}'s are not 0. If we denote by $\widetilde{U}$ the complement of $n(n+1)/2$ linear subspaces of codimension 2 in $\mathbb{C}^{2n}$

$$\bigcap_{j>i} \left(\mathbb{C}^{2n}_{\{a_i, a_i^*\}} \setminus V(a_i^*, a_j) \right),$$

a representation (a_i, a_i^*, b_i) of dimension δ is θ-stable only if it lies in

$$\mathbb{C}^n_{\{b_i\}} \times \widetilde{U}.$$

The open set $\widetilde{U}$ admits an open cover $\{\widetilde{U}_i\}$, where

$$\widetilde{U}_i = (\dots, a_{i-1}^*, a_{i+1}, \dots)$$

is the complement of a union of $n-1$ hyperplanes. If we identify the torus $T = \mathrm{GL}_\delta / \mathbb{C}^*$ with $(\mathbb{C}^*)^{n-1}$ so that the i-th copy of $\mathbb{C}^*$ acts with weight 1 on a_{i-1} and a_i^*, with weight -1 on a_i and a_{i-1}^*, and with weight 0 on the other variables, we get an explicit description

$$M^\theta(Q, \delta) \cong \frac{\mathbb{C}^n \times \widetilde{U}}{(\mathbb{C}^*)^{n-1}}.$$

The open sets $\widetilde{U}_i$ are T-invariant, and we can construct an atlas $\{U_i\}$ for $M^\theta(Q, \delta)$ taking $U_i := \mathbb{C}^n \times \widetilde{U}_i / T$. A system of coordinates for U_i is given by

$$\begin{gathered} b_{i,0}, \dots, b_{i,n-1}, \\ \dots, z_{i,i-1} = a_{i-1}a_{i-1}^*, z_{i,i} = \frac{\prod_{j>i} a_j}{\prod_{j<i} a_j^*} a_i, \\ z_{i,i}^* = \frac{\prod_{j<i} a_j^*}{\prod_{j>i} a_j} a_i^*, z_{i,i+1} = a_{i+1}a_{i+1}^*, \dots . \end{gathered} \tag{4.18}$$

Denote by ω_I the restriction of ω to the intersection of the U_i's whose indices lie in I. We have

$$\omega_{\{i\}} = (b_{i,i+1} - b_{i,i})z_{i,i}z_{i,i}^* + \sum_{j \neq i}(b_{i,j+1} - b_{i,j})z_i$$

in this system of coordinates. Up to another change of coordinates we can write locally

$$\omega_{\{i\}} = c_i(z_{i,i}z_{i,i}^* - z_{i,i+1}) + \sum_{j \neq i,i+1} c_{i,j}\widetilde{z}_{i,j}.$$

Using Thom-Sebastiani formula we can compute

$$[\omega_{\{i\}}^{-1}(1)] - [\omega_{\{i\}}^{-1}(0)] = -\mathbb{L}^{n+2}$$

and for all $i \in \{0, \dots n-2\}$

$$[\omega_{\{i,i+1\}}^{-1}(1)] - [\omega_{\{i,i+1\}}^{-1}(0)] = -\mathbb{L}^{n+2} + \mathbb{L}^{n+1}.$$

It is not difficult to prove, again using the Thom-Sebastiani theorem, that the analogous contribution for ω_I is trivial when I consists of a pair of non-consecutive indices (we do not consider $n-1$ and 0 to be consecutive here) or when I contains more then two indices. We thus obtain

$$[M^\theta(J_{W_1}, \delta)]_{vir} = -(-\mathbb{L}^{-\frac{1}{2}})^{2n+1}(-\mathbb{L}^{n+2} + (n-1)\mathbb{L}^{n+1}).$$

by a simple inclusion-exclusion argument. □

We now pass to the q-deformed case, and we assume that q is not a root of unity. As above we consider the function $\omega = \mathrm{Tr}(W_q)$ as a function over $M(Q, \delta)$, so that we can avoid any issue related to the commutativity of the various linear maps in a representation. We can write

$$W_q = \sum_{i=0}^{n-1} c_i a_i a_i^*.$$

after a change of variables. The following lemma describes the locus of simple representation of the Jacobi algebra J_{W_q}.

Lemma 4.23. *There is a (non-canonical) isomorphism*

$$M^s(J_{W_q}, \delta) \cong \mathbb{C}^* \sqcup \mathbb{C}^*.$$

Proof. Let $V = (a_i, a_i^*, b_i)$ be a representation of Q with dimension vector δ and let $c_i = b_{i+1} - qb_i$. The representation is simple if and only if

$$(a_i, a_j^*) \neq (0,0) \text{ for } i \neq j.$$

Considering $M^s(Q,\delta)$ inside $M^\theta(Q,\delta)$ as above, we have that the intersection between the open set U_i defined in Proposition 4.22 with the locus of simple representations is the complement of the $(n-i)(i+1)$ codimension 2 subspaces

$$x_{i,k} = x_{i,j} = 0, \text{ for } k \leq i, j \geq i+1,$$
$$x_{i,k} = x^*_{i,i} = 0, \text{ for } k \leq i.$$

On the other hand points in U_i corresponding to representations of J_{W_q} satisfying

$$c_{i,i}x_{i,i} = c_{i,i}x^*_{i,i} = x_{i,i}x^*_{i,i} = 0, \quad c_{i,j} = x_{i,j} = 0 \text{ for } j \neq i.$$

It is then clear that the locus of simple representations of the Jacobi algebra J_{W_q} does not intersect U_i for $i \neq 0, n-1$. Furthermore we have

$$U_0 \cap M^s(J_{W_q}) = \{x_{0,0} \neq 0, \ x^*_{0,0} = c_{0,j} = x_{0,k} = 0, \ k > 0 \text{ and for all } j\};$$
$$U_{n-1} \cap M^s(J_{W_q}) = \{x^*_{n-1,n-1} \neq 0, \ c_{n-1,j} = x_{n-1,j} = 0, \text{ for all } j\}.$$

□

The following example shows, in the case $G = \mathbb{Z}_2$, that the virtual motive of simple representations with dimension vector δ does not correspond to the coefficient of $y^\delta/(1-y^\delta)$ in the exponential of the q-deformed generating series.

Example 4.24. We proceed analogously to Example 4.20. We look at simple representations of the McKay quiver of G inside θ-stable representations for $\theta = (-1,1)$. On the open set U_i^s with coordinates as in (4.17), the function ω^s can be expressed as

$$y_{i,i+1}x_{i,i+1} + y_{i,i}x_{i,i}x^*_{i,i}.$$

The Thom-Sebastiani theorem together with an inclusion-exclusion argument gives

$$[(\omega^s_{\{i\}})^{-1}(1)] - [(\omega^s_{\{i\}})^{-1}(0)] = -\mathbb{L}^3 + \mathbb{L}^2,$$

and

$$[(\omega^s_{\{0,1\}})^{-1}(1)] - [(\omega^s_{\{0,1\}})^{-1}(0)] = 0.$$

In this way we obtain

$$[\mathcal{M}^s(J_{W_q},\delta)]_{vir} = 2\cdot\frac{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}.$$

This does not match, even at the birational level, the coefficient

$$\frac{3\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}$$

of the term $y^\delta/(1-y^\delta)$ in the exponential form of the universal series.

As in the conifold case, we now choose a stability condition θ and look at the critical locus of the extension of ω^s to the moduli space $M^\theta(J_{W_q},\delta)$. In particular, we take θ to be $(-(n-1),1,\dots,1)$, the stability condition that gives G-$Hilb(\mathbb{C}^3)$. The following propositions describe the geometry and the motivic virtual cycle of $M^\theta(J_{W_q},\delta)$.

Proposition 4.25. *The critical locus of the function*

$$\omega\colon M^\theta(J_{W_q},\delta)\longrightarrow\mathbb{C},\quad \omega=\mathrm{Tr}(W_q),$$

consists of the following components (see Figure 4.2 for an illustration):

- *a tree* Γ *of* $n-1$ *projective lines* $\mathbb{P}^1_{i,i+1}$ *with* $0_{\mathbb{P}^1_{i,i+1}}=\infty_{\mathbb{P}^1_{i+1,i+2}}$ *for* i *in* $\{1,\dots,n-2\}$*;*
- $n-2$ *disjoint affine lines, each intersecting* Γ *in one of its singular points;*
- *two pairs of affine lines meeting at a point which intersect respectively one of the two extremities* $0_{\mathbb{P}^1_{0,1}}$ *and* $\infty_{\mathbb{P}^1_{n-2,n-1}}$ *of* Γ*.*

Proof. We adopt the notation of Proposition 4.22, i.e. we consider $M^\theta(J_{W_q},\delta)$ inside $M^\theta(Q,\delta)$ and consider locally the coordinates defined in (4.18). Up to a further change of coordinates $c_i=b_{i+1}-qb_i$, we have

$$\omega_{\{i\}}=\ldots+c_{i,i-1}z_{i,i-1}+c_{i,i}z_{i,i}z^*_{i,i}+c_{i,i+1}z_{i,i+1}+\ldots. \tag{4.19}$$

In particular, the intersection of $\mathrm{Crit}(\omega)$ with U_i is the union of the three coordinate axes corresponding to the variables $c_{i,i}$, $z_{i,i}$ and $z^*_{i,i}$. The critical locus $\mathrm{Crit}(\omega)$ intersects the overlap $U_i\cap U_j$ of two charts only if i and j are consecutive integers (not read modulo n). The intersection of $\mathrm{Crit}(\omega)$ with $U_i\cap U_{i+1}$ consists of a pointed line $\mathbb{C}^*$ with coordinate $z^*_{i,i}=z^{-1}_{i+1,i+1}$, so that $\mathrm{Crit}(\omega_{\{i,i+1\}})$ consists of the union of a projective line $\mathbb{P}^1_{i,i+1}$ and two affine lines attached respectively at $0_{i,i+1}$ and $\infty_{i,i+1}$. This reasoning applied for $i\in\{0,\dots,n-2\}$ leads us to the required description of the critical locus. $\square$

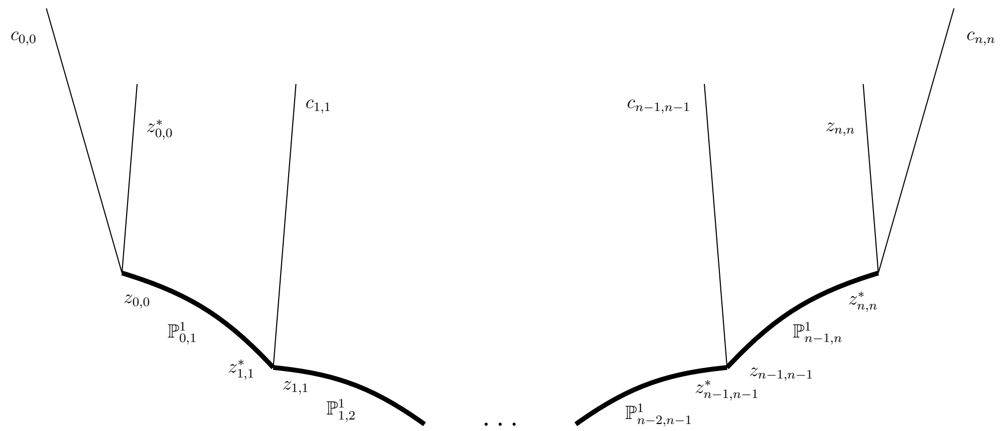

Figure 4.2: Critical locus of W_q for q not a root of unity.

Proposition 4.26. *The virtual class of the stacky quotient* $\mathrm{Crit}(\omega)/\mathbb{C}^*$ *is*

$$[\mathcal{M}^\theta(J_{W_q},\delta)]_{vir} = \frac{(n+1)\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}.$$

Proof. Since W_q admits a cut we have

$$[\mathrm{Crit}(\omega)]_{vir} = -(-\mathbb{L}^{-\frac{1}{2}})^{2n+1}([\omega^{-1}(1)] - [\omega^{-1}(0)]),$$

so we can combine the local description of ω from Proposition 4.25 and an inclusion-exclusion argument to compute $[\mathrm{Crit}(\omega)]_{vir}$. We have

$$\begin{aligned}[\omega^{-1}(1)] - [\omega^{-1}(0)] = &\sum_i([\omega^{-1}_{\{i\}}(1)] - [\omega^{-1}_{\{i\}}(0)]) - \sum_{i\neq j}([\omega^{-1}_{\{i,j\}}(1)] - [\omega^{-1}_{\{i,j\}}(0)]) + \\ &+ \sum_{i,j,k \text{ distinct}}([\omega^{-1}_{\{i,j,k\}}(1)] - [\omega^{-1}_{\{i,j,k\}}(0)]) - \dots .\end{aligned}$$

An easy argument involving the Thom-Sebastiani theorem implies that there is no contribution coming from the overlaps of charts with non-consecutive indices and from overlaps of multiplicity at least 3. We thus get

$$\begin{aligned}[\omega^{-1}(1)] - [\omega^{-1}(0)] = &\sum_{i=0}^{n-1}([\omega^{-1}_{\{i\}}(1)] - [\omega^{-1}_{\{i\}}(0)]) + \\ &\sum_{i=0}^{n-2}([\omega^{-1}_{\{i,i+1\}}(1)] - [\omega^{-1}_{\{i,i+1\}}(0)]).\end{aligned} \tag{4.20}$$

In light of the local form for $\omega_{\{i\}}$ of (4.19), we can compute the various contributions by means of the Thom-Sebastiani theorem. Recall that the class of the motivic vanishing cycles of the function

$$f_{i,j}\colon \mathbb{C}^2 \longrightarrow \mathbb{C}, \quad f_j(c_{i,j}, z_{i,j}) = c_{i,j}z_{i,j}$$

is $-\mathbb{L}$, and the class of the function

$$g_{i,i}\colon \mathbb{C}^3 \longrightarrow \mathbb{C} \quad g(c_{i,i}, z_{i,i}, z^*_{i,i}) = c_{i,i} z_{i,i} z^*_{i,i}$$

is $-2\mathbb{L}^2 + \mathbb{L}$. Thus we have

$$\sum_{i=0}^{n-1}([\omega^{-1}_{\{i\}}(1)] - [\omega^{-1}_{\{i\}}(0)]) = -n\mathbb{L}^n(2\mathbb{L} - 1). \tag{4.21}$$

Recall that the intersection $U_i \cap U_{i+1}$ can be expressed in the coordinates of U_i as the locus where $z^*_{i,i}$ does not vanish. Proceeding in an analogous way to the approach for $\omega_{\{i\}}$ we obtain

$$\sum_{i=0}^{n-2}([\omega^{-1}_{\{i,i+1\}}(1)] - [\omega^{-1}_{\{i,i+1\}}(0)]) = -(n-1)\mathbb{L}^n(\mathbb{L} - 1). \tag{4.22}$$

From equations (4.21) and (4.22) we conclude

$$[\mathrm{Crit}(\omega)]_{vir} = -(n+1)\mathbb{L}^{\frac{1}{2}} + \mathbb{L}^{-\frac{1}{2}}.$$

□

Similarly to the conifold case, the universal series of motivic Donaldson-Thomas invariants is determined by the non-commutative skeleton of the resolution of a line of A_n-singularities depicted in Figure 4.2. Again, it would be interesting to have a more intrinsic motivation for this phenomenon in order to clarify the role of θ-stable representations in the computation of the universal series.

4.A The deformed commuting variety

We describe in this appendix the computation of the generating series

$$A_q(y) = 1 + \sum_{n\geq 1} \frac{[C_q(n)]}{[\mathrm{GL}_n]} y^n,$$

where

$$C_q(n) = \{A, B \in \mathrm{End}(\mathbb{C}^n)\colon AB - qBA = 0\}$$

is the q-deformed commuting variety. The computation is a contribution of A. Morrison to [CMPS], and it is reported here for reasons of completeness. The technique used to obtain an exponential form for the universal series is similar to the one we employed in Sections 4.1 and 4.2, i.e. it consists of stratifying the space of q-commuting varieties based on the structure of the nilpotent and the invertible part of the matrices.

Recall we introduced the notation $C_q^I(n)$ for the variety of q-commuting matrices where the second matrix is invertible, and let $C_q^N(n)$ denote the analogous variety where we require the second matrix to be nilpotent. It will be also useful, for a given pair $*$ and $\circ$ of elements in $\{I, N\}$, to let $C_q^{*\circ}(n)$ be the variety of q-commuting matrices with nilpotency/invertibility specified respectively by the letter $*$ and $\circ$. We also introduce a similar notation for the generating series of the classes of the corresponding varieties:

$$A_q^{*\circ}(y) = 1 + \sum_{n\geq 1} \frac{[C_q^{*\circ}(n)]}{[\mathrm{GL}_n]} y^n.$$

Given a pair of matrices (A, B) in $C_q(n)$ there is a decomposition $\mathbb{C}^n = V^I \oplus V^N$ such that $B = B^I \oplus B^N$ with B^I invertible and B^N nilpotent, and such that $A = A^I \oplus A^N$ is also block diagonal. This reduces the problem to the study of the variety $C_q^*(n)$, where $*$ lies in $\{I, N\}$. Given a pair of q-commuting matrices in $C_q^*(n)$, one can further decompose V^* as $V^{*I} \oplus V^{*N}$ and obtain two pairs of q-commuting matrices (A^{*I}, B^{*N}), (A^{*N}, B^{*N}) in $C_q^{*I}(n)$ and $C_q^{*N}(n)$ respectively. We obtain in this way

$$A_q(y) = A_q^{NN}(y)A_q^{NI}(y)A_q^{IN}(y)A_q^{II}(y).$$

Proceeding exactly as in Lemma 4.12 (and 4.19) we obtain that if we require one of the two q-commuting varieties to be nilpotent the generating series is not affected by the deformation parameter. We have then

$$A_q^{NN}(y)A_q^{NI}(y)A_q^{IN}(y) = A_1^{NN}(y)A_1^{NI}(y)A_1^{IN}(y).$$

In particular, $A_1^{NI}(y)$ can be computed as follows. Consider the stratification $\{C_1^{NI}(\pi, n)\}$ of $C_1^{NI}(n)$ indexed by the set of partitions π of n which define the Jordan type of the second matrix. We have a Zariski trivial fibration

$$\begin{tikzcd}
\{A\colon AJ(\pi) - J(\pi)A = 0\} \arrow[r] & C_q^{NI}(\pi, n) \arrow[d] \\
& \{B\colon B \sim J(\pi)\},
\end{tikzcd}$$

where $J(\pi)$ is the standard Jordan form corresponding to the partition π. Computing the classes of the base and the fiber of the fibration we obtain

$$[C_q^{NI}(\pi)] = \frac{[\mathrm{GL}_n]}{[Stab_{\mathrm{GL}_n}(J(\pi))]}[Stab_{\mathrm{GL}_n}(J(\pi))] = [\mathrm{GL}_n],$$

so that

$$A_q^{NI}(y) = \mathrm{Exp}\left(\frac{y}{1-y}\right).$$

Using (1.37), we get

$$A_q^{NN}(y) = (A_q^{NI}(y))^{\frac{1}{\mathbb{L}-1}} = \mathrm{Exp}\left(\frac{1}{\mathbb{L}-1}\frac{y}{1-y}\right)$$

and finally we obtain the following expression for the product of the factors requiring at least one of the two matrices to be nilpotent:

$$A_q^{NN}(y)A_q^{NI}(y)A_q^{IN}(y) = \mathrm{Exp}\left(\frac{2\mathbb{L}-1}{\mathbb{L}-1}\frac{y}{1-y}\right).$$

We are then left with the factor $A_q^{II}(y)$. This contribution is genuinely affected by the deformation parameter.

Remark 4.27. In the case when q is not a root of unity the variety $C_q^{II}(n)$ is empty. Indeed, two q-commuting matrices A and B satisfy $\det(AB) = q^n \det(BA)$, and if they are invertible it follows that $q^n = 1$.

We can therefore restrict to the study of $C_q^{II}(n)$ when q is a primitive r-th root of unity and $n = rm$. First of all notice that the q-commutation relation implies that A is similar to $q^k A$ for all choices of k. If the Jordan normal block decomposition of A is

$$A \sim \bigoplus_{k=1}^{M} J(\mu_k, n_k),$$

the Jordan decomposition of $q^l A$ is

$$q^l A \sim J(q^l \mu_k, n_k).$$

We thus obtain

$$A \sim \bigoplus_{i=0}^{r-1} J_i,\ \ J_i = \bigoplus_{k=0}^{N} J(q^i \mu_k, n_k). \tag{4.23}$$

Remark 4.28. We will assume that J_i and J_k have no common eigenvalues for $i \neq k$. This can be done, without loss of generality, by replacing μ_l by $q^k \mu_l$ any time we have $\mu_i = q^k \mu_l$ changing only the ordering in the Jordan normal decomposition of A.

Lemma 4.29. *Suppose A has Jordan normal form as in (4.23) and assume that (A, B) lies in $C_q^I(n)$. If we write $B = (B_{ij})$ as a block matrix with respect to the decomposition induced by $A = \oplus J_i$ we have $B_{ij} = 0$ unless $q^{j-i+1} = 1$.*

Proof. In the block decomposition described in the lemma the equation $AB - qBA = 0$ can be rewritten in components as $J_iB_{ik} - qB_{ik}J_k = 0$. Recall that there is an invertible matrix P_l such that $P_lJ_lP_l^{-1} = q^lJ_0$, so that we can rewrite the equations as

$$q^iJ_0B' - q^{k+1}B'J_0,$$

where $B' = P_iB_{ik}P_k^{-1}$. We obtain that B' is 0 if $q^{j-i+1} \neq 1$. We can write $J_0 = D_0 + N_0$ with D_0 invertible and N_0 nilpotent. The equation can then be rewritten as

$$q^iD_0B' - q^{j+1}B'D_0 = q^{j+1}B'N_0 - q^iD_0B'.$$

Note that $(B')_{lm}$ gets rescaled by $q^i\mu_l - q^{j-i}\mu_m$, which is non-zero by Remark 4.28. On other hand, one easily see that the element in the bottom left corner of the right hand side of the equation is zero. Proceeding by induction it is not difficult to show that $B' = 0$. □

Theorem 4.30. *Let q be a primitive r-th root of unity. The generating series $A_q^{II}(y)$ can be expressed in exponential form as*

$$A_q^{II}(y) = \operatorname{Exp}\left((\mathbb{L}-1)\frac{y^r}{1-y^r}\right).$$

Proof. Consider the substack $C_1^{1,I}(n)$ of $C_1^{II}(n)$ defined by requiring that 1 be the only eigenvalue of the first matrix. From the definition of a power structure of Section 1.4.3 we have

$$A_1^{II}(y) = (A_1^{1I}(y))^{\mathbb{L}-1} = (A_1^{NN}(y)).$$

Similarly we can define the closed substack $C_q^{1I}(rm)$ of $C_q^{1I}(rm)$ consisting of q-commuting pairs where the first matrix has eigenvalues $1, q, \ldots, q^{r-1}$ all with the same multiplicity. Again using power structures we obtain

$$A_q^{II}(y) = A^{1I}(y^r)^{[\mathbb{C}^*/\mu_r]} = (A^{1I}(y^r))^{\mathbb{L}-1},$$

so that it is enough to prove

$$[C_q^{1I}(rm)/\operatorname{GL}_{rm}] = [C_1^{1I}(m)/\operatorname{GL}_m].$$

Let $\{C_1^{1I}(\pi, m)\}_\pi$ be the stratification of $C_1^{II}(m)$ indexed by the partition corresponding to the Jordan normal form of the matrix A. We have a Zariski locally trivial fibration

$$\begin{tikzcd} \{B\colon J_0B = BJ_0\} \arrow[r, hook] & C_1^{1I}(\pi, m) \arrow[d] \\ & \{A\colon A \sim J_0\}. \end{tikzcd}$$

The class of the base is given by

$$\frac{[\mathrm{GL}_m]}{[Stab_{\mathrm{GL}_m}(J_0)]},$$

and the fiber is isomorphic to the stabilizer group $Stab_{\mathrm{GL}_m}(J_0)$. We thus have

$$\left[\frac{C_1^{1I}(m)}{\mathrm{GL}_m}\right] = p(m),$$

where $p(n)$ is the number of partitions of the integer n. Consider now $C_q^{1I}(rm)$. It follows from Lemma 4.29 that this quotient stack admits a stratification $\{C_q^{1I}(\pi, rm)\}$ depending on the Jordan normal form of the matrix A. Denote by J_π the Jordan normal form of the matrix A described in (4.23). There is a Zariski locally trivial fibration

$$\begin{tikzcd} \{B\colon J_\pi B = qBJ_\pi\} \arrow[r, hook] & C_q^{1I}(\pi, rm) \arrow[d] \\ & \{A\colon A \sim J_\pi\}. \end{tikzcd} \tag{4.24}$$

We show that the class of the fiber is isomorphic to $[Stab_{Gl_n}(J_0)]^r$. By Lemma 4.29 the fiber is given by matrices B_{ik} with $q^{k+1-i} = 1$ satisfying

$$J_i B_{ik} = qB_{ik}J_k.$$

From the description of (4.23) there are matrices P_i such that $J_i = P_i J_0 P_i^{-1}$, so that the map

$$(B_{ij})_{ij} \longmapsto (B_{ij}P_j)_{ij}$$

defines an isomorphism between the fiber in (4.24) and r copies of the stabilizer $Stab_{\mathrm{GL}_m}(J_0)$. It remains to show that the class of the base is

$$\frac{[\mathrm{GL}_{rm}]}{[Stab_{\mathrm{GL}_{rm}}(J_\pi)]}.$$

In analogy with Lemma 4.29 write a matrix B in the stabilizer group $Stab_{\mathrm{GL}_{rm}}(J_\pi)$ in blocks B_{ik} where i and k vary between 0 and $r-1$. Then the block B_{ij} satisfies

$$J_i B_{ik} = B_{ik} J_k,$$

and using the fact that $J_k = q^k P_k J_0 P_k^{-1}$ this equation can be rewritten as

$$q^i J_0 P_i B_{ik} P_k^{-1} = q^j P_i B_{ik} P_k^{-1} J_0.$$

As in the last part of the proof of Lemma 4.29 we can prove that this implies $B_{ik} = 0$ whenever i is different from j. But then

$$(B_{ik}) \longmapsto (P_i B_{ik} P_k^{-1})$$

defines an isomorphism between $Stab_{\mathrm{GL}_{rm}}(J_\pi)$ and r copies of $Stab(\mathrm{GL}_m(J_0))$, and this completes the proof. $\square$

The computation of the generating series of the classes of the q-deformed commuting varieties together with Morrison's dimensional reduction [Mo1, Prop. 5.1] enable us to compute the generating series of motivic DT-invariants of the three loop quiver with deformed superpotential. We let Q be the three loop quiver of Example 1.5 and W_q to be the marginally deformed potential $W = a_1a_2a_3 - qa_1a_3a_2$.

Theorem 4.31 ([CMPS]). *If $q \in \mathrm{C}^*$ is a primitive rth root of unity, then*

$$U_{Q_1,W_q}(y) = \mathrm{Exp}\left(\frac{2\mathbb{L}-1}{\mathbb{L}-1}\frac{y}{1-y} + (\mathbb{L}-1)\frac{t^r}{1-y^r}\right).$$

Otherwise

$$U_{Q_1,W_q}(y) = \mathrm{Exp}\left(\frac{2\mathbb{L}-1}{\mathbb{L}-1}\frac{y}{1-y}\right). \tag{4.25}$$

Assume now that q is a generic deformation parameter. The motives appearing as the coefficients of $y/(1-y)$ in the generating series in terms of representations of (Q, W_q) admit a geometric interpretation in term of simple representations. Let $\mathcal{M}^s(J_q, 1)$ be the moduli stack of simple representations of (Q, W_q) with dimension vector 1. We can compute directly the virtual motive $[\mathcal{M}^s(J_q, 1)]_{vir}$:

$$[\mathcal{M}^s(J_q,1)]_{vir} = -\frac{\mathbb{L}^{-\frac{3}{2}}}{[\mathbb{C}^*]_{\mathrm{vir}}}\left([W_q^{-1}(1)] - [W_q^{-1}(0)]\right) = -\frac{-2\mathbb{L}^{\frac{1}{2}} + \mathbb{L}^{-\frac{1}{2}}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}},$$

and notice that this matches exactly the coefficient of $y/(1-y)$ inside the exponential of 4.25. The same phenomenon has been observed also for other more complicated homogeneous deformation of the potential [CMPS, Sec. 3].

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