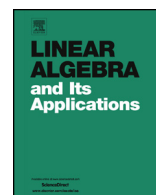




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Comparing the principal eigenvector of a hypergraph and its shadows

Gregory J. Clark^{*}, Felipe Thomaz, Andrew T. Stephen

Saïd Business School, University of Oxford, United Kingdom of Great Britain and Northern Ireland

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ABSTRACT

Graphs (i.e., networks) have become an integral tool for the representation and analysis of relational data. Advances in data gathering have led to multi-relational data sets which exhibit greater depth and scope. In certain cases, this data can be modeled using a hypergraph. However, in practice analysts typically reduce the dimensionality of the data (whether consciously or otherwise) to accommodate a traditional graph model. In recent years spectral hypergraph theory has emerged to study the eigenpairs of the adjacency hypermatrix of a uniform hypergraph. We show how analyzing multi-relational data, via a hypermatrix associated to the aforementioned hypergraph, can lead to conclusions different from those when the data is projected down to its co-occurrence matrix. To this end we consider how the principal eigenvector of a hypergraph and its shadow can vary in terms of their spectral rankings, Pearson/Spearman correlation coefficient, and Chebyshev distance. In particular, we provide an example of a uniform hypergraph where the most central vertex (à la eigencentrality) changes depending on the order of the associated matrix. To the best of our knowledge this is the first known hypergraph to exhibit this property. We further show that the aforementioned eigenvectors have a high

^{*} Corresponding author.

E-mail address: gregory.clark@sbs.ox.ac.uk (G.J. Clark).

Pearson correlation but are uncorrelated under the Spearman correlation coefficient.

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1. Introduction

There is a class of problems which seeks to quantify the importance of vertices (i.e., nodes) in a graph (i.e., network) according to some criterion. Centrality measures are typically employed in such cases. Examples of such measures include degree, betweenness, closeness, and eigenvector centrality. While the aforementioned notions of centrality are related they can vary in practice. This is particularly troublesome when two centrality measures identify different vertices as being ‘the most central’. Famously, the Krackhardt kite is an example of a graph where different vertices have the greatest degree, betweenness, and closeness centrality [1]. In a similar vein we construct a hypergraph whose most central vertex (by eigenvector centrality) changes depending on the order of the associated matrix (see Fig. 4). We further explore the extent to which the principal eigenvector of a hypergraph and its shadow (i.e., the multigraph formed from the co-occurrence matrix of the hypergraph) can vary according to their spectral ranking, Pearson/Spearman correlation coefficient, and Chebyshev distance.

In recent years, the principal eigenvector of the (normalized) adjacency hypermatrix of a k -uniform hypergraph (see [2–6]) has received increasing attention as a way to model multi-relational data which faithfully analogizes the graph case [7–12]. Despite these developments, the term ‘hypergraph’ has been historically employed in various contexts. For example, in [8] the authors define the adjacency *matrix* of a hypergraph to be $A = HWH^T - D$ where H is the $|V| \times |E|$ incidence matrix, W is a diagonal matrix of edge-weights, and D is the diagonal degree matrix. Note that A is precisely the co-occurrence matrix of H when W is the identity matrix.

This approach of using the co-occurrence matrix, or some variation thereof, as a stand-in for the adjacency hypermatrix has been the basis for a longstanding corpus of work. Generally speaking, such methods of *graph reduction* replace each hyperedge in a hypergraph with a traditional graph. A notable example is the *clique expansion* which replaces each hyperedge with a complete graph of the same size (other examples include the star expansion and Lawler expansion [13]). Because the term ‘expansion’ implies growth or extension we adopt the language of *shadow* from extremal hypergraph theory as it is indicative of information loss (see [14] for examples). The use of graph reduction techniques raises several important questions.

First, can a multi-relational system be faithfully modeled using a reduced graph? Clearly the answer to this question is context dependent and it has been explored in several areas. Sociologists have shown that modeling group effects, such as peer pressure and the adoption of social norms, cannot be reduced to pairwise interactions [15].

Further, diffusion processes on a hypergraph can account for inter-edge mediators but this property is lost under reduction [16–18]. Second, if we cannot faithfully model a multi-relational system with a reduced graph can we at least capture salient structural properties of the underlying system? There has been considerable work in network science showing that this is not the case. Notably the clique expansion of a hypergraph has been shown to misidentify important vertices when analyzing multi-relational data [19,20]. It is natural to wonder if the reduced graph is always inadequate for studying a hypergraph. Interestingly this is also not true. In [21] the authors characterize the conditions for which the hypergraph s - t cut problem is reducible to a graph s - t cut problem. Third, we consider the question of computational feasibility: what is the marginal cost of analyzing a hypergraph compared to its shadow? Computing the principal eigenvector of a hypergraph is considerably more difficult than computing the principal eigenvector of its shadow. However, there have been great strides in approximating the principal eigenvector of a hypergraph using either optimization [22,23] or iteration [24,25]. For exact computations one can consider an algebraic approach via the Lu-Man Method which was introduced in [26] and further developed in [27,28].

We now turn our attention to the question of novelty. That is, how ‘different’ is the principal eigenvector of a hypergraph from its shadow (i.e., the multigraph formed from its co-occurrence matrix)? We address this question in three ways. We first consider how different the spectral ranking of a hypergraph and its shadow can be. This is particularly important in practical contexts where achieving the top rank confers some sort of benefit (e.g., reputational benefits in a social system). To the best of the authors’ knowledge we provide the first known example of a hypergraph whose most central vertex changes depending on the order of the associated (hyper)matrix. This motivates the exploration of the Pearson and Spearman correlation between the aforementioned eigenvectors. We provide an example where the principal eigenvector of a hypergraph and its shadow are highly correlated by the Pearson correlation coefficient but are uncorrelated by the Spearman correlation coefficient. Finally, we provide an upper bound on the Chebyshev distance between the principal eigenvector of a hypergraph and its shadow and show that this bound is achieved asymptotically.

We begin by presenting the necessary background for our discussion in the following section. In Section 3 we describe a property of the principal eigenvector of a k -partite k -uniform hypergraph. We leverage this property to construct a hypergraph family (i.e., cauldron hypergraphs) in Section 4. In Section 5 we show that the most central vertex of this family depends on the order of the accompanying hypermatrix. We further explore how the principal eigenvectors of these hypergraphs and their shadows correlate in Section 6. In Section 7 we pivot to a structural approach and show how the loss of information incurred from depreciating data can lead to variations in the spectral ranking. Finally, in Section 8 we provide an upper bound on the Chebyshev distance between the aforementioned vectors. We further provide a family of hypergraphs which achieves this bound in the limit.

2. Preliminaries

A k -uniform hypergraph, abbreviated k -graph, is an ordered pair $H = ([n], E)$ where $E \subseteq \binom{[n]}{k}$. Throughout we will assume that all hypergraphs are uniform and we reserve the language of “hypergraph” specifically for k -graphs where $k > 2$. We maintain the notation of [4]. The (normalized) adjacency hypermatrix of a k -graph H is an order k and dimension n hypermatrix, denoted

$$\mathcal{A}(H) \in \mathbb{R}^{\overbrace{n \times n \times \cdots \times n}^{k \text{ times}}}$$

where

$$a_{i_1, i_2, \dots, i_k} = \frac{1}{(k-1)!} \begin{cases} 1 & : \{i_1, i_2, \dots, i_k\} \in E(H) \\ 0 & : \text{otherwise.} \end{cases}$$

Note that the order of a hypermatrix is the number of coordinates of each entry and the dimension of a hypermatrix is the number of possible indices for each coordinate. In this way $\mathcal{A}(H)$ is a collection of n^k elements $a_{i_1 i_2 \dots i_k} \in \mathbb{R}$ where $i_j \in [n]$.

Let H be a simple k -uniform hypergraph and $x \in \mathbb{C}^{|V|}$. For $e \in E$ we denote $x^e = \prod_{v \in e} x_v$. We say that (λ, x) is an eigenpair of H if it satisfies the eigenequations

$$\lambda x_i^{k-1} = \sum_{\substack{e \in E \\ i \in e}} x^{e \setminus i} \text{ for } i \in [n].$$

The hypergraph H defines a polynomial form

$$F_{\mathcal{A}(H)}(x) = k \cdot \sum_{e \in H} x^e. \quad (1)$$

A hypergraph $H = (V, E)$ is *connected* if for all $i, j \in V(H)$ there exists a path from i to j in H ; to be precise, there is a sequence of vertex pairs $((v_r, v_{r+1}))_{r=1}^s$ such that $v_1 = i, v_{s+1} = j$ and $\{v_t, v_{t+1}\} \subseteq e_t \in E(H)$. A directed graph $D = (V, A)$ is *strongly connected* if for any ordered pair of vertices (i, j) there is a directed path from i to j . Let H be a k -graph and define $D(H)$ to be the directed graph formed by replacing each hyperedge of H with a bi-directed complete graph of size k . The hypermatrix $\mathcal{A}(H)$ is said to be *weakly irreducible* if $D(H)$ is strongly connected [23,29]. Note that $\mathcal{A}(H)$ is weakly irreducible if and only if H is connected. We now state a version of the Perron-Frobenius Theorem using this language.

Theorem 1. (The Perron-Frobenius Theorem for Weakly Irreducible Nonnegative Tensors [23]) Let H be a connected k -graph. Then $\mathcal{A}(H)$ has a strictly positive eigenpair (λ, x) which is unique up to scaling.

We will make use of the following.

Lemma 2. ([4]) *Let H be a connected k -graph with principal eigenpair (λ, x) such that $\|x\|_k^k = 1$ then*

$$\lambda = \max_{y: \|y\|_k^k = 1} F_{A(H)}(y).$$

Henceforth we assume that the principal eigenvector of a k -graph is normalized so that its k -norm is 1.

The enterprise of this paper is to motivate the use of k -graphs to model k -relational data. We do so by comparing the principal eigenvector of the normalized adjacency hypermatrix of H with its co-occurrence matrix. We now make this notion precise.

A *multigraph* is a graph which permits multiple edges. More technically, a multigraph G is an ordered triple $G = (V, E, \mu)$ where V is the set of vertices, E a set of edges, and $\mu : E \rightarrow \mathbb{Z}_+$ is a multiplicity function. For convenience we will use exponents to denote the multiplicity of an edge in set notation. We say that $\mu_G(e)$ is the multiplicity of $e \in E(G)$ and suppress the subscript when the context is clear. Note that a simple graph can be thought of as a multigraph where the multiplicity function is simply the indicator function. The *adjacency matrix* of a multigraph $A(G)$ is the $|V| \times |V|$ matrix where $A(G)_{i,j} = \mu(ij)$ if $ij \in E(G)$ and is zero otherwise. We similarly define a multi-hypergraph and direct the interested reader to [30,31] for a definition of the adjacency matrix of a multi-hypergraph and to [23] for a Perron-Frobenius Theorem for multi-hypergraphs. While many of the results herein can be extended to multi-hypergraphs we are principally considered with the relationship between a (simple) hypergraph and its shadow. We now make the concept of a hypergraph shadow precise.

Definition 1. For a k -graph $H = (V, E)$ and $1 \leq s \leq k$, we define the s -shadow of H to be the multi-hypergraph formed by replacing each edge of H with a complete s -graph on k vertices. That is,

$$\partial^s(H) = \left(V, \left\{ \binom{e}{s} : e \in E(H) \right\}, \mu \right)$$

where $\mu(f) = |\{e \in E(H) : f \subseteq e\}|$.

The notation of ∂H is used for the (traditional) shadow operation which replaces each k -edge of a hypergraph with all possible $(k-1)$ -subedges (ignoring multiplicity). For clarity, we reserve the language of ‘shadow’ to mean 2-shadow as in Definition 1.

An example of a 3-graph and its 2-shadow is provided in Fig. 1. Note that when $s = 2$, $\partial^2(H)$ is a multigraph and when $s = 1$, $\partial^1(H)$ is a collection of disconnected loopy vertices. This motivates the following remark.

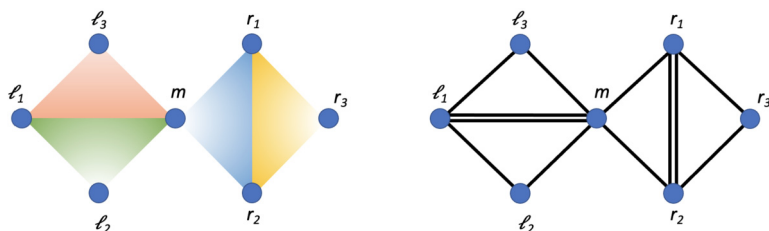


Fig. 1. A 3-graph $E(H) = \{[m, \ell_1, \ell_2], [m, \ell_1, \ell_3], [m, r_1, r_2], [r_1, r_2, r_3]\}$ where its edges are drawn as triangular faces on the left and its 2-shadow $E(\partial^2 H) = \{[m, \ell_1]^2, [m, \ell_2], [m, \ell_3], [m, r_1], [m, r_2], [\ell_1, \ell_2], [\ell_1, \ell_3], [r_1, r_2]^2, [r_1, r_3], [r_2, r_3]\}$ drawn on the right.

Remark 1. In order to facilitate conversation we abuse notation and refer to the normalized degree vector of H as the principal eigenvector of $\partial^1(H)$. As such we will write $x(\partial^1(H)) \equiv d(H)$ to mean the normalized degree vector of H as in Equation (2).

Remark 2. The adjacency matrix of $\partial^2(H)$ is the co-occurrence matrix of H .

We adopt the language of *spectral ranking* as in [32] so that ‘the most central vertices by eigenvector centrality’ have spectral rank 1. To this end we define the *vertex ranking of H according to its principal eigenvector x* to be

$$R(H) = (V_1, V_2, \dots, V_t) \text{ where } \bigsqcup_i V_i = V(H)$$

such that for $v \in V_i$ and $u \in V_j$ we have $x_v > x_u$ if $i < j$ and $x_v = x_u$ if $i = j$. We say that a hypergraph is *s-opaque* if its spectral ranking differs from that of its *s-shadow* (i.e., $R(H) \neq R(\partial^s H)$). We further define the *s-umbral index* of a hypergraph $u_s(H)$ to be the least index for which the spectral ranking of H and $\partial^s(H)$ differ. In the case when H is not *s-opaque* (i.e., $R(H) = R(\partial^s(H))$) we write $u_s(H) = 0$. We say that a hypergraph is *umbralific* if $R(\partial^s H)$ is distinct for all $1 \leq s \leq k$. Note that being umbralific is equivalent to the Spearman correlation coefficient of the principal eigenvector of $\partial^i(H)$ and $\partial^j(H)$, for $i \neq j$, being strictly less than 1. We denote the *Spearman correlation coefficient* as r_s and write r_p for the *Pearson correlation coefficient*.¹ To measure how uncorrelated the principal eigenvectors of a hypergraph and its shadows are we define the *shadow length* of a hypergraph as the geometric mean (i.e., the log-average) of 1 minus the Spearman correlation between all distinct pairs $\partial^i(H)$ and $\partial^j(H)$:

$$SL(H) = \left(\prod_{\{i,j\} \in \binom{[k]}{2}} 1 - r_s(x(\partial^i(H)), y(\partial^j(H))) \right)^{1/\binom{k}{2}}. \quad (2)$$

¹ Source code for eigenvector approximations and correlative analysis can be found at github.com/Ramsey003/Comparing-the-Principal-Eigenvector-of-a-Hypergraph-and-its-Shadows.

Table 1
Correlative statistics of various Hypergraph families. We consider the Pearson r_p and Spearman r_s correlation coefficients of y , the principal eigenvector of a hypergraph H ; x , the principal eigenvector of its shadow $\partial^2(H)$; and d , the degree vector of H . We further compute the shadow length of the hypergraph, $SL(H)$, via Equation (2). The hypergraph families considered include \mathcal{B}_t , the t -pleated bow tie hypergraph (Section 5, see Table 2 and Table 3); O_R , a modified octahedron (Section 7, see Table 4); and $\mathcal{S}(\eta, k)$, the k -uniform hyperstar with η edges (Section 8, see proof of Theorem 19). Approximations for y are computed as a constrained optimization problem via Lemma 2 using `scipy.optimize.minimize`. Approximations for x were computed using `networkx.eigenvector_centrality`. Notably, the principal eigenvector of \mathcal{B}_8 and its shadow are uncorrelated under the Spearman correlation coefficient.

| Hypergraph | $r_p(y, x)$ | $r_p(y, d)$ | $r_p(x, d)$ | $r_s(y, x)$ | $r_s(y, d)$ | $r_s(x, d)$ | $SL(H)$ |
|------------------------|-------------|-------------|-------------|-------------|-------------|-------------|---------|
| \mathcal{B}_8 | 0.99449 | 0.99596 | 0.99985 | 0.07116 | 0.72623 | 0.73439 | 0.40725 |
| \mathcal{B}_1 | 0.99549 | 0.98781 | 0.99229 | 0.88889 | 0.94281 | 0.94281 | 0.07136 |
| O_R | 0.99406 | 0.99139 | 0.99499 | 0.99070 | 0.91287 | 0.92144 | 0.03993 |
| $\mathcal{S}(\eta, k)$ | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 0.00000 |

Note that $SL(H) \geq 0$ and equality is achieved precisely when H is *not* umbralific. A summary of the shadow lengths of various hypergraph families explored in this paper is given in Table 1.

There is consensus among academics and practitioners that the degree vector and principal eigenvector of a *graph* are highly correlated [33,34]. As such, it is acceptable to use the degree vector of a graph as a proxy for its principal eigenvector in some empirical settings. We endeavor to find hypergraphs with large shadow length to challenge this assumption in general.

We make use of the following Lemma which is a restatement of Lemma 3.21 from [23] and is desideratum to the Collatz-Wielandt Theorem for hypermatrices.

Lemma 3. [23] *Let $\mathcal{A}(H)$ be the normalized adjacency hypermatrix of a k -graph (for $k \geq 2$) on n vertices. Let $x \in \mathbb{R}_{++}^n$ be strictly positive. Then*

$$\min_{i \in [n]} \frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}} \leq \rho(\mathcal{A}) \leq \max_{i \in [n]} \frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}}.$$

3. The principle eigenvector of a k -partite k -graph

In this section we consider hypergraph colorings and describe a spectral property of k -colorable k -graphs. We begin by presenting definitions related to graph coloring and adhere to the language of [35]. A *vertex coloring* of G is a map $c : V \rightarrow S$ such that $c(i) \neq c(j)$ when $ij \in E$. The *chromatic number* of G , denoted $\chi(G)$, is the smallest $|S|$ for which a coloring of G exists. A graph G is said to be *k -chromatic* if $\chi(G) = k$ and further G is said to be *k -colorable* if $\chi(G) \leq k$. A set of vertices $U \subseteq V$ is *independent* if $ij \notin E$ for all $i, j \in U$. A k -coloring of G is simply a partition of the vertex set of G into k independent sets and we refer to such independent sets as *color classes*. There are several notions of vertex colorings of hypergraphs but we consider the ‘strict’ definition which requires all vertices in an edge to be distinctly colored. That is to say,

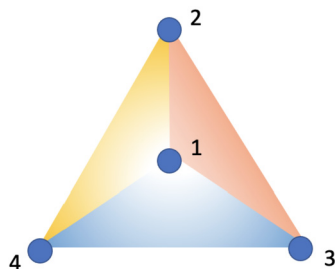


Fig. 2. The 3-uniform bottomless tetrahedron hypergraph whose edges $E(T) = \{[1, 2, 3], [1, 2, 4], [1, 3, 4]\}$ are drawn as monochromatic triangular faces. Note that $\{1\}$ forms an independent vertex cover.

a hypergraph is k -partite (or a k -cylinder) if there exists a coloring $c : V(H) \rightarrow [k]$ such that $c(i_1) \neq c(i_2) \neq \dots \neq c(i_t)$ for $i_1 i_2 \dots i_t \in E(H)$ [4].

The following elegant result, given in [36], provides insight into how vertices in a color class ‘compete’ for centrality in a bipartite graph.

Theorem 4. ([36]) *If S is an independent set of a connected graph G and x is the principal eigenvector of G , then*

$$\sum_{i \in S} x_i^2 \leq \frac{1}{2}.$$

Equality happens if and only if G is bipartite having S as one color class.

Interestingly the spectral characterization of bipartite graphs in Theorem 4 does not extend to k -partite hypergraphs. However, reframing Theorem 4 in terms of *independent vertex covers* admits a natural extension.

We say that $U \subseteq V(G)$ *covers* a (hyper)graph if every edge in the (hyper)graph has a vertex in U . An *independent vertex cover* is a vertex cover which is also independent. Thus an independent vertex cover has the property that every edge from the (hyper)graph contains *exactly* one vertex from the independent vertex cover. In graphs, having an independent vertex cover is equivalent to being bipartite since every edge has exactly one endpoint in both color classes. However, there exist k -graphs which have an independent vertex cover which are not k -partite for $k > 2$. For example, the bottomless tetrahedron

$$T = ([4], \{[1, 2, 3], [1, 2, 4], [1, 3, 4]\}),$$

is a 4-partite 3-graph which has an independent vertex cover, $\{1\}$ (a sketch is given in Fig. 2).

Lemma 5. *A graph has an independent vertex cover if and only if it is bipartite. However, a k -graph has an independent vertex cover if it is k -partite.*

Proof. Let G be a connected, nontrivial graph with an independent vertex cover, R . Let $B = V \setminus R$. Since G is nontrivial it has at least one edge. Whence R is independent there is at least one vertex not in R so that B is nonempty. We claim that $R \sqcup B$ induces a 2-coloring of G . Let $e = ij \in E(G)$. Since R is an independent vertex cover either $i \in U$ or $j \in U$, but not both. Without loss of generality, $i \in R$ and $j \in B$. Thus $R \sqcup B$ induces a 2-coloring of G as desired.

Instead suppose that G is bipartite with color classes $V = R \sqcup B$. Note that R (resp. B) is an independent vertex cover by definition: every edge has exactly one endpoint in R (resp. B). Similarly, color classes of a k -partite k -graph for an independent vertex cover. \square

We now show that Theorem 4 characterizes k -graphs with an independent vertex cover. Our proof is similar to that of Theorem 4. We include it as it succinctly highlights the mechanisms underpinning this phenomenon.

Theorem 6. *If S is an independent set of a connected k -graph H and y is the principal eigenvector of H , then*

$$\sum_{i \in S} y_i^k \leq \frac{1}{k}.$$

Equality occurs if and only if S is an independent vertex cover of H .

Proof. For each $i \in S$ we have

$$\rho y_i^{k-1} = \sum_{\substack{e \in E \\ i \in e}} y^{e \setminus i}.$$

Multiplying each equation by y_i and summing over $i \in S$ yields

$$\rho \sum_{i \in S} y_i^k = \sum_{i \in S} \sum_{i \in e} y^e.$$

Since S is independent and the entries of y are positive we have that

$$\sum_{i \in S} \sum_{i \in e} y^e \leq \frac{F_H(y)}{k} = \frac{\rho}{k}.$$

The desired inequality follows whence $\rho > 0$.

If S is an independent vertex cover then every edge contains exactly one vertex from S . We have

$$\sum_{i \in S} \sum_{i \in e} y^e = \sum_{e \in E} y^e = \frac{F_H(y)}{k}$$

where the first equality follows from the definition of an independent vertex cover (i.e., every edge contains exactly one vertex from S) and the second equality is immediate from Lemma 2. If instead equality holds then every edge e which does not have a vertex in S has $y_e = 0$ by the maximality of Lemma 2. Since y is strictly positive this cannot be the case, so every edge must have a vertex in S . It follows that S is a cover by definition. We assumed S is independent and thus S is an independent vertex cover. \square

We show that equality in Theorem 6 can hold for k -graphs which are not k -partite.

Lemma 7. *The principal eigenvector (λ, x) of the bottomless tetrahedron T satisfies*

$$\lambda = 3(2/3)^{2/3}, x = \langle 3^{-1/3}, 3^{-1/3}(2/3)^{1/3}, 3^{-1/3}(2/3)^{1/3}, 3^{-1/3}(2/3)^{1/3} \rangle.$$

In particular, T is a 4-partite 3-graph for which $x_1^3 = 1/3$.

Proof. Observe that $\|x\|_3 = 1$ and note that $x_2 = x_3 = x_4$ by automorphism. We verify the eigenequations

$$\lambda x_1^2 = 3(2/3)^{2/3} \cdot (3^{-1/3})^2 = 3(3^{-2/3} \cdot (2/3)^{2/3}) = 3x_2^2 = x_2x_3 + x_2x_4 + x_3x_4$$

and

$$\lambda x_2^2 = 3(2/3)^{2/3} \cdot (3^{-1/3}(2/3)^{1/3})^2 = 2(3^{1/3})(3^{-1/3}(2/3)^{1/3}) = 2x_1x_3 = x_1x_3 + x_1x_4.$$

The remaining eigenequations follow similarly. \square

Intuitively, Theorem 6 shows that vertices in an independent vertex cover ‘compete for centrality’ as in a zero-sum game. For simplicity we restrict our attention to color classes of k -partite k -graphs (which are independent vertex covers by Lemma 5). We do so because the shadow operation preserves the color classes of a hypergraph and thus the monochromatic vertices in a k -partite k -graph ‘compete for centrality’ in a stricter sense than they do in the shadow. In the following sections we apply Theorem 6 to leverage this spectral property of k -partite k -graphs.

4. Cauldron hypergraphs

Given a bipartite graph B with color classes $V(B) = C_1 \cup C_2$ and $t > 0$, we define the *cauldron 3-graph*, denoted $C(B, t)$, to be

$$E(C(B, t)) = \{[m, r_1, r_2]\} \cup \{[m, p, q] : pq \in E(B)\} \cup \{[r_1, r_2, v_i] : 0 \leq i \leq t\}.$$

A sketch of a cauldron graph is given in Fig. 3. Let (y, ρ) be the principal eigenpair of $C(B, t)$. Note that r_1 and r_2 are fixed under automorphism. Further, v_1, \dots, v_t are

similarly fixed under automorphism. It follows that $y(r_1) = y(r_2)$ and $y(v_i) = y(v_j)$. With this in mind we reserve the following notation to simplify our discussion.

Definition 2. Fix a Cauldron 3-graph $H = C(B, t)$ with the vertex labeling given in Fig. 3. To simplify our calculations we write (y, ρ) and (λ, x) to be the principal eigenpair of H and $\partial^2(H)$, respectively. Moreover we write (z, ζ) for the principal eigenpair of B . We denote

$$\alpha = y(m), \beta = y(r_1) = y(r_2), \gamma = y(v_i),$$

and

$$a = x(m), b = x(r_1) = x(r_2), c = x(v_i).$$

The following demonstrates Theorems 4 and 6.

Corollary 8. Let $H = C(B, t)$ where B has color classes $V(B) = C_1 \cup C_2$. Then

$$\sum_{v \in C_1} z_v^2 = \sum_{u \in C_2} z_u^2 \text{ and } \sum_{v \in C_1} y_v^3 = \sum_{u \in C_2} y_u^3.$$

However, $\sum_{v \in C_1} x_v^2$ and $\sum_{u \in C_2} x_u^2$ are not equal in general.

Proof. From Theorem 4 we have

$$1/2 = \sum_{u \in C_1} z_u^2 = \sum_{v \in C_2} z_v^2.$$

Moreover from Theorem 6 we equate

$$1/3 = \beta^3 + \sum_{u \in C_1} y_u^3 = \beta^3 + \sum_{v \in C_2} y_v^3$$

so that

$$\sum_{u \in C_1} y_u^3 = \sum_{v \in C_2} y_v^3.$$

Note that $\partial^2(H)$ is a 3-colorable 2-graph so that the inequality in Theorem 4 is strict. It is easy to see that $\sum_{v \in C_1} x_v^2$ and $\sum_{u \in C_2} x_u^2$ are not equal in general. \square

Our enterprise is to construct a hypergraph whose umbral index is 1. We do so by leveraging Corollary 8 to impose different restrictions on the principal eigenvector of a cauldron hypergraph and its shadow. We begin by proving the following spectral condition for an inversion in the spectral rankings $R(H)$ and $R(\partial^2(H))$ for m and $\{r_1, r_2\}$.

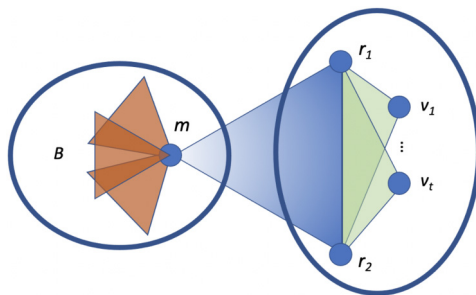


Fig. 3. Sketch of a cauldron graph whose edges are $E(C(B, t)) = \{[m, r_1, r_2]\} \cup \{[m, p, q] : pq \in E(B)\} \cup \{[r_1, r_2, v_i] : 0 \leq i \leq t\}$.

Theorem 9. Let H be a cauldron graph. Then $\alpha > \beta$ if and only if $(\rho - 1)\sqrt{\rho} > t$, or equivalently,

$$\rho > \frac{1}{3} \left(\frac{1}{2} \right)^{\frac{1}{3}} \left(27t^2 + 9\sqrt{9t^2 - \frac{4}{3}t - 2} - 2 \right)^{\frac{1}{3}} + \frac{2 \left(\frac{1}{2} \right)^{\frac{2}{3}}}{3 \left(27t^2 + 9\sqrt{9t^2 - \frac{4}{3}t - 2} \right)^{\frac{1}{3}}} + \frac{2}{3}.$$

Further $a > b$ precisely when $(\lambda^2 - 2\lambda)/(\lambda + 2) > t$, that is to say

$$\lambda > \frac{t + 2 + \sqrt{t^2 + 12t + 4}}{2}.$$

Proof. Let (y, ρ) and (λ, x) be the principal eigenvector of H and $\partial^2(H)$, respectively. For simplicity we write

$$\alpha = y(m), \beta = y(r_1) = y(r_2), \gamma = x(v_i).$$

We have

$$\begin{aligned} \rho\beta^2 &= \alpha\beta + t\beta\gamma \\ \rho\gamma^2 &= \beta^2. \end{aligned}$$

From the second eigenequation we have $\gamma = \beta/\sqrt{\rho}$ so that substitution into the first eigenequation yields

$$\frac{\alpha}{\beta} = \frac{\rho\sqrt{\rho} - t}{\sqrt{\rho}}.$$

It follows that $\alpha > \beta$ if and only if $(\rho - 1)\sqrt{\rho} > t$. Now consider $\partial^2(H)$ and let

$$a = x(m), b = x(r_1) = x(r_2), c = x(v_i).$$

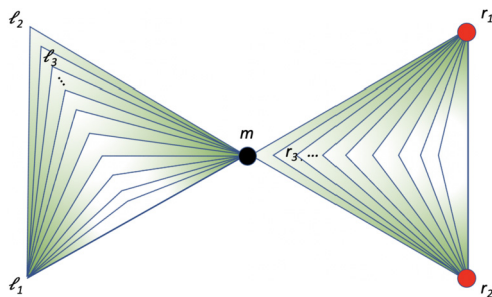


Fig. 4. The 8-pleated bow tie, \mathcal{B}_8 .

We have then that

$$\lambda b = a + (t+1)b + tc$$

$$\lambda c = 2b.$$

Similar to the previous case substitution, yields

$$\frac{a}{b} = \frac{\lambda^2 - \lambda(t+1) - 2t}{\lambda}.$$

Indeed $a > b$ precisely when $t < (\lambda^2 - 2\lambda)/(\lambda + 2)$ or equivalently

$$\lambda > \frac{t + 2 + \sqrt{t^2 + 12t + 4}}{2}. \quad \square$$

Our enterprise is to construct a hypergraph for which $(\rho - 1)\sqrt{\rho} < t$ but $(\lambda^2 - 2\lambda)/(\lambda + 2) > t$. In the following section we show the t -pleated bow tie graph $\mathcal{B}_t = (S_{t+1}, t)$ satisfies these conditions.

5. Pleated bow tie

We consider the t -pleated bow tie $\mathcal{B}_t = C(S_{t+1}, t)$ where S_t is the star with t rays. We make use of the following labeling

$$\mathcal{B}_t = \{[m, \ell_1, \ell_2], [m, r_1, r_2]\} \cup \left(\bigcup_{i=1}^t \{[m, \ell_1, \ell_{i+2}], [r_1, r_2, r_{i+2}]\} \right).$$

A drawing of \mathcal{B}_1 and $\partial^2 \mathcal{B}_1$ is given in Fig. 1 and a drawing of \mathcal{B}_8 is provided in Fig. 4. Our goal is to show that $u_2(\mathcal{B}_8) = 1$. We begin by showing that \mathcal{B}_8 is opaque with the aid of an approximation for the principal eigenvector found in Table 2.

Lemma 10. *We have*

$$4.690116 \leq \rho(\mathcal{B}_8) \leq 4.690119$$

Table 2

An approximation of the principal eigenvector $y(\mathcal{B}_8)$, the principal eigenvector $x(\partial^2(\mathcal{B}_8))$, and the degree vector $d(\mathcal{B}_8)$. Approximations for y are computed as a constrained optimization problem via Lemma 2 using `scipy.optimize.minimize`. Approximations for x were computed using `networkx.eigenvector_centrality`. Each approximation was rounded to seven decimal places. This suggests that the umbral index of \mathcal{B}_8 is 1 and further that \mathcal{B}_8 is umbralific. A proof of this statement can be found in Theorem 14.

| Vertex | $y(\mathcal{B}_8)$ | $x(\partial^2(\mathcal{B}_8))$ | $d(\mathcal{B}_8)$ |
|------------------------------------|--------------------|--------------------------------|---------------------|
| m | 0.5703211 | 0.5070057 | $0.\overline{185}$ |
| r_1, r_2 | 0.5725499 | 0.4554540 | $0.1\overline{6}$ |
| r_3, r_4, \dots, r_{10} | 0.2643757 | 0.0791819 | $0.0\overline{185}$ |
| ℓ_1 | 0.5261352 | 0.4625799 | $0.1\overline{6}$ |
| $\ell_2, \ell_3, \dots, \ell_{10}$ | 0.2529394 | 0.0842814 | $0.0\overline{185}$ |

and

$$11.503996 \leq \lambda(\partial^2(\mathcal{B}_8)) \leq 11.504157.$$

Proof. This follows from Table 2 and Lemma 3. \square

Lemma 11. \mathcal{B}_8 is 2-opaque; moreover, m and $\{r_1, r_2\}$ are inverted in the spectral ranking of \mathcal{B}_8 and $\partial^2(\mathcal{B}_8)$.

Proof. Let $t = 8$. From Lemma 10 we have

$$(\rho - 1)\sqrt{\rho} < 7.9990 < t.$$

Further $\lambda(\partial^2(\mathcal{B}_8)) \geq 11.503996$ hence

$$\lambda(\partial^2(\mathcal{B}_8)) \geq 11.503996 > \frac{t + \sqrt{t^2 + 12t + 4}}{2} \approx 10.403.$$

The conclusion follows from Theorem 9. \square

To prove that $u_2(\mathcal{B}_8) = 1$ we must prove that $R_1(\mathcal{B}_8) = \{r_1, r_2\}$ and $R_1(\partial^2(\mathcal{B}_8)) = \{m\}$.

Lemma 12. Fix t and let (ρ, y) be the principal eigenvector of \mathcal{B}_t . Then $R_1(\mathcal{B}_t) = \{r_1, r_2\}$ if and only if $(\rho - 1)\sqrt{\rho} < t$.

Proof. For simplicity we write

$$\alpha = y(m), \beta = y(r_1) = y(r_2), \gamma = y(r_i) \text{ for } i > 2, \delta = y(\ell_1) \text{ and } \varepsilon = y(\ell_j) \text{ for } j > 1.$$

As such, the eigenequations of \mathcal{B}_t are

$$\begin{cases} \rho\alpha^2 = \beta^2 + (t+1)\delta\varepsilon \\ \rho\beta^2 = \alpha\beta + t\beta\gamma \\ \rho\gamma^2 = \beta^2 \\ \rho\delta^2 = (t+1)\alpha\varepsilon \\ \rho\varepsilon^2 = \alpha\delta \end{cases}$$

We show that $\beta > \alpha > \gamma, \delta, \varepsilon$ if and only if $t < (\rho - 1)\sqrt{\rho}$. From Theorem 9 we have that $\beta > \alpha$.

The difference of the first and third eigenequation yields $\alpha > \gamma$. We now show $\alpha > \delta$. Rearranging the fourth eigenequation yields for ε and substituting into the fifth eigenequation yields

$$\left(\frac{\alpha}{\delta}\right)^3 = \frac{\rho^3}{(t+1)^2}.$$

Indeed, $\alpha > \delta$ if and only if $\rho^3 > (t+1)^2$. This inequality is satisfied when $(\rho - 1)\sqrt{\rho} > t$. As previously shown, $\delta^3 = (t+1)\varepsilon^3$ so that $\delta/\varepsilon > \sqrt[3]{t+1} > 1$ implying that $\delta > \varepsilon$. It follows that $\alpha > \varepsilon$ concluding the proof. \square

We now establish a similar characterization for $\partial^2\mathcal{B}_t$.

Lemma 13. Fix t and let (λ, x) be the principal eigenvector of $\partial^2\mathcal{B}_t$. We have that $R_1(\partial^2(\mathcal{B}_t)) = \{c\}$ if and only if $t < (2\lambda - \lambda^2)/(\lambda + 1)$ or equivalently

$$\lambda > \frac{t + 2 + \sqrt{t^2 + 12t + 4}}{2}.$$

Proof. For simplicity, we write

$$a = x(c), b = x(r_1) = x(r_2), c = x(r_i) \text{ for } i > 2, d = x(\ell_1), \text{ and } e = x(\ell_j) \text{ for } j > 1.$$

As such, the eigenequations of $\partial^2\mathcal{B}_t$ are

$$\begin{cases} \lambda a = 2b + (t+1)(d+e) \\ \lambda b = a + (t+1)b + tc \\ \lambda c = 2b \\ \lambda d = (t+1)(a+e) \\ \lambda e = a + d \end{cases}$$

We prove our claim by showing that $a > b, c, d, e$ when λ is sufficiently large. Appealing to Theorem 9 we find that $a > b$.

Taking the difference between the first and third eigenequations yields

$$\lambda(a - c) = (t+1)(d+e).$$

Whence $\lambda, t, d, e > 0$ we have that $a > c$.

Now consider the difference of the first and fourth eigenequations,

$$\lambda(a - d) = 2b + (t + 1)(d - a).$$

Suppose to the contrary that $a \leq d$. Then $\lambda(a - d) \leq 0$. Moreover, we have that $2b + (t + 1)(d - a) > 0$ since $b, t > 0$ and $d - a \geq 0$. This implies that $0 < 0$ which is a contradiction. It must be the case that $a > d$.

We conclude by showing $d > e$. Substituting the fifth eigenequation into the fourth yields

$$\lambda d = (t + 1)(\lambda e - d + e)$$

so that

$$d = e \left(\frac{(t + 1)(\lambda + 1)}{\lambda - (t + 1)} \right) > e. \quad \square$$

The following Theorem is immediate from Lemmas 11, 12, and 13.

Theorem 14. $u(\mathcal{B}_8) = 1$. That is, the spectral rank 1 vertices of \mathcal{B}_8 and its 2-shadow $\partial^2 \mathcal{B}_8$ are distinct.

We further make the following Conjecture.

Conjecture 1. We have $u(\mathcal{B}_t) = 1$ for $t \geq 8$.

Question 1. Do there exist k -graphs which are not k -partite that have an umbral index of 1 (cf., Theorem 6 and Lemma 7)? Moreover, what is the smallest such k -graph (in terms of vertices and/or edges) to have an umbral index of 1 for each k ?

6. Eigenvector correlation

In the previous section we showed that the most central vertex according to the principal eigenvector can vary under the shadow operation. It is natural to wonder to what extent this can occur. In this section we consider the Pearson and Spearman correlation coefficients of such vectors which we denote r_p and r_s . A summary of these correlations for various hypergraphs is provided in Table 1.

Our main result is that the principal eigenvectors of \mathcal{B}_8 and $\partial^2 \mathcal{B}_8$ are highly correlated under the Pearson correlation coefficient while they are uncorrelated under the Spearman correlation coefficient. We now prove that spectral ranking of \mathcal{B}_8 given in Table 2 is correct. We do so by providing a spectral condition for determining the spectral rank of \mathcal{B}_t .

Table 3

An approximation of the principal eigenvector $y(\mathcal{B}_1)$, the principal eigenvector $x(\partial^2(\mathcal{B}_1))$, and the degree vector $d(\mathcal{B}_1)$. Approximations for y are computed as a constrained optimization problem via Lemma 2 using `scipy.optimize.minimize`. Approximations for x were computed using `networkx.eigenvector_centrality`. Each approximation was rounded to seven decimal places. This suggests that the umbral index of \mathcal{B}_1 is 2 and further that \mathcal{B}_1 is umbralific.

| Vertex | $y(\mathcal{B}_1)$ | $x(\partial^2(\mathcal{B}_1))$ | $d(\mathcal{B}_1)$ |
|------------------|--------------------|--------------------------------|--------------------|
| m | 0.6431819 | 0.5591865 | 0.25 |
| r_1, r_2 | 0.5578859 | 0.4117513 | $0.1\bar{6}$ |
| r_3 | 0.4066793 | 0.2125560 | $0.08\bar{3}$ |
| ℓ_1 | 0.5425427 | 0.4189989 | $0.1\bar{6}$ |
| ℓ_2, ℓ_3 | 0.2524797 | 0.252481404 | $0.08\bar{3}$ |

Lemma 15. Fix t and let (ρ, y) be the principal eigenvector of \mathcal{B}_t . Then the spectral ranking of \mathcal{B}_t is

$$R(\mathcal{B}_t) = (\{r_1, r_2\}, m, \ell_1, \{r_3, \dots, r_{t+2}\}, \{\ell_2, \dots, \ell_{t+1}\})$$

if and only if $(\rho - 1)\sqrt{\rho} < t$ and $\rho^3 - (t + 1)^2 - \rho(t + 1)^{2/3} > 0$.

Proof. Assume our setup from Lemma 12. We will show $\beta > \alpha > \delta > \gamma > \varepsilon$ if and only if the aforementioned assumptions hold. In Lemma 12 we showed that, under these assumptions, $\beta > \alpha > \delta > \varepsilon$. It remains to be shown that $\delta > \gamma > \varepsilon$.

Substituting the third eigenequation into the second yields

$$\rho\alpha^2 = \rho\gamma^2 + (t + 1)\delta\varepsilon = \rho\gamma^2 + \frac{\alpha^2(t + 1)^2}{\rho^2}$$

so that

$$\left(\frac{\alpha}{\gamma}\right)^2 = \frac{\rho^3}{\rho^3 - (t + 1)^2}.$$

We have previously shown

$$\left(\frac{\alpha}{\delta}\right)^3 = \frac{\rho^3}{(t + 1)^2}$$

so that

$$\left(\frac{\gamma}{\delta}\right)^2 = \frac{\rho^3 - (t + 1)^2}{\rho(t + 1)^{4/3}}.$$

Indeed $\delta > \gamma$ if and only if

$$\rho(t + 1)^{4/3} + (t + 1)^2 - \rho^3 > 0$$

which is always true whence $t > (\rho - 1)\sqrt{\rho}$. Furthermore, since $\delta^3 = (t + 1)\varepsilon^3$ we have that

$$\left(\frac{\gamma}{\varepsilon}\right)^2 = \frac{\rho^3 - (t + 1)^2}{\rho(t + 1)^{2/3}}$$

so that $\gamma > \varepsilon$ when

$$\rho^3 - (t + 1)^2 - \rho(t + 1)^{2/3} > 0 \quad \square$$

Note that, by Lemma 10, \mathcal{B}_8 satisfies the assumptions of Lemma 15. Moreover, by Table 2 we have that the vertex ranking according to the degree vector of \mathcal{B}_8 as well as the principal eigenvector of \mathcal{B}_8 and $\partial^2\mathcal{B}_8$ are all distinct. As such we have the following.

Theorem 16. \mathcal{B}_8 is umbralific.

7. Modified octahedron

We previously considered a spectral condition for the deviation of the principal eigenvector of a hypergraph and its shadow. We now consider this question from a structural approach. Consider the modified octahedron given in Fig. 5. Let O_R denote the 3-graph formed by taking the red faces of the octahedron and the green edge. To be precise,

$$E(O_R) = \{[t, p, q], [t, r, s], [b, q, r], [b, p, s], [u, p, q]\}.$$

Similarly define O_B to be the 3-graph formed from the blue faces of the octahedron and the green edge,

$$E(O_B) = \{[t, p, r], [t, p, s], [b, p, q], [b, r, s], [u, p, q]\}.$$

An approximation of the principal eigenvectors of $O_R, O_B, \partial^2 O_R$, and $\partial^2 O_B$ is given in Table 4 and the corresponding spectral rankings are given in Table 5.

Definition 3. We say that two hypergraphs H_1 and H_2 are s -umbral mates if $H_1 \neq H_2$ but $\partial^s(H_1) = \partial^s(H_2)$.

Note that s -umbral mates demonstrate the loss of information incurred by using the co-occurrence matrix of a hypergraph instead of its adjacency hypermatrix. Observe that O_R and O_B are 2-umbral mates as $O_R \neq O_B$ and $\partial^2 O_R = \partial^2 O_B$. Moreover, t are b are identically situated in the 2-shadow while occupying distinct positions in O_R (and O_B). We thus expect the spectral ranking of the red/blue modified octahedron and its 2-shadow to differ.

Theorem 17. O_R and O_B are 2-opaque.

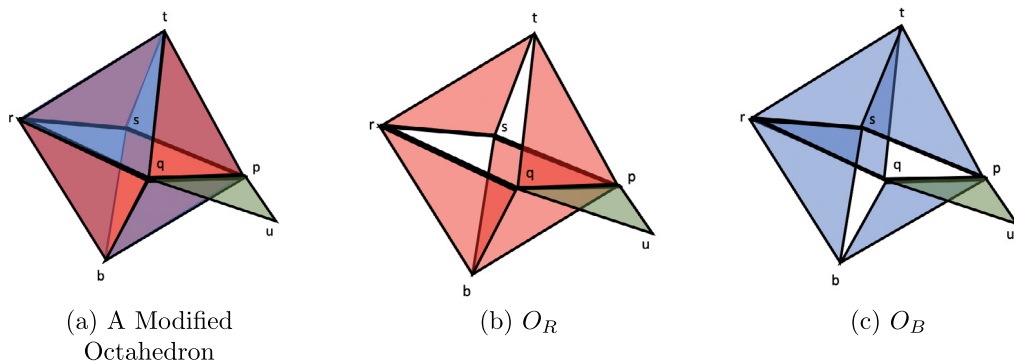


Fig. 5. A modified octahedron, O_R , and O_B , respectively.

Proof. Consider O_R . Let (ρ, y) and (λ, x) be the principal eigenvector of O_R and $\partial^2(O_R)$, respectively. We remark that $x_t = x_b$ by symmetry of $\partial^2(O_R)$. It remains to be shown that $y_t \neq y_b$. We will abuse notation and write v for y_v , the value of the principal eigenvector of the vertex v . By symmetry of O_R we have $p = q$ and $r = s$.

First suppose to the contrary that $q = r$. Taking the difference of their eigenequations yields

$$\rho q^2 - \rho r^2 = (rb + tq + uq) - (tr + qb)$$

so that $0 = uq$. It follows that $u = 0$ or $q = 0$. This cannot be the case as the principal eigenvector is strictly positive. Indeed $r \neq q$.

Now consider the difference of the eigenequations of t and b ,

$$\rho t^2 - \rho b^2 = rs + pq - (rq + ps) = r^2 + q^2 - 2rq = (r - q)^2.$$

Since $r \neq q$ we have that $(r - q)^2 > 0$ which implies that $t > b$. \square

The following is immediate from Table 5.

Theorem 18. O_R and O_B are umbralific.

8. Hyperstars and windmills

Finally we consider the Chebyshev distance to measure the extent to which the principal eigenvector of a hypergraph and its shadow can vary coordinate-wise.

Definition 4. The Chebyshev distance between two vectors $x, y \in \mathbb{R}^n$ is

$$D(y, x) = \max_i |x_i - y_i|.$$

Table 4

An approximation of the principal eigenvector $y(O_R)$ (resp. $y(O_B)$, the principal eigenvector $x(\partial^2(O_R))$, and the degree vector $d(O_R) = d(O_B)$. Approximations for y are computed as a constrained optimization problem via Lemma 2 using `scipy.optimize.minimize`. Approximations for x were computed using `networkx.eigenvector_centrality`. Each approximation was rounded to seven decimal places. A summary of the spectral ranking of O_R , O_B , and their shadows is given in Table 5.

| Vertex | $y(O_R)$ | $y(O_B)$ | $x(\partial^2(O_R)) = x(\partial^2(O_B))$ | $d(O_R) = d(O_B)$ |
|--------|-----------|-----------|---|-------------------|
| p | 0.5938351 | 0.5938351 | 0.4870931 | 0.2 |
| q | 0.5938351 | 0.5938351 | 0.4870931 | 0.2 |
| t | 0.5159682 | 0.5120581 | 0.3579397 | 0.1 $\bar{3}$ |
| b | 0.5120581 | 0.5159682 | 0.3579397 | 0.1 $\bar{3}$ |
| r | 0.4985602 | 0.4985602 | 0.3348463 | 0.1 $\bar{3}$ |
| s | 0.4985602 | 0.4985602 | 0.3348463 | 0.1 $\bar{3}$ |
| u | 0.3951650 | 0.3951650 | 0.2121199 | 0.0 $\bar{6}$ |

Table 5

The spectral ranking of $O_R, O_B, \partial^2 O_R$ (resp. $\partial^2 O_B$), and $\partial^1(O_R)$ (resp. $\partial^1(O_B)$) according to Table 4. This suggests the umbral index of O_R and O_B is 2 and further that they are umbralific.

| Rank | O_R | O_B | $\partial^2(O_R) = \partial^2(O_B)$ | $\partial^1(O_R) = \partial^1(O_B)$ |
|------|--------|--------|-------------------------------------|-------------------------------------|
| 1 | p, q | p, q | p, q | p, q |
| 2 | t | b | t, b | t, b, r, s |
| 3 | b | t | r, s, u | u |
| 4 | r, s | r, s | u | \emptyset |
| 5 | u | u | \emptyset | \emptyset |

We assumed that the principal eigenvector of a k -graph is normalized so that its k -norm is 1. This can be problematic when trying to interpret the Chebyshev distance between the principal eigenvector of two (hyper)graphs of different uniformity. For example, consider the principal eigenvector of a single k -edge, $E_k = \{[k], \{[k]\}\}$, which is $y = (k^{-1/k})_{i=1}^k$. The 2-shadow of a single k -edge, $\partial^2(E_k) = K_k$, is the complete graph whose principle eigenvector is $x = (k^{-1/2})_{i=1}^k$. From a practical standpoint, each vertex in E_k and $\partial^2(E_k)$ occupies an identical position in the network so the projection of the single-edge hypergraph down to the complete graph does not lose any information. As such, we would expect $D(y, x)$ to be zero but this is clearly not the case. To account for this we measure the coordinate-wise distance between a particular weighting of the vectors which we introduce below.

A *weighting* of a k -graph H is a map $w : V(H) \rightarrow \mathbb{R}_{\geq 0}$ so that $\sum_{v \in V} w(v) = 1$ [37]. We consider the k -optimal weighting $w_k(x) = x^k$ where k is the uniformity of the (hyper)graph [14,22,38]. To simplify notation we write $w_H(x) = w_{\text{rank}(H)}(x)$.

Thus we consider the Chebyshev distance between the k -optimal weighting of the principal eigenvector of a hypergraph and its shadow when they are both of unit length under the 1-norm:

$$D(w_H(y), w_{\partial^2 H}(x)) = \max_v |y_v^k - x_v^2| \text{ where } \|y\|_k = 1 = \|x\|_2.$$

Consider now the k -uniform hyperstar $\mathcal{S}(\eta, k)$ which consists of η k -edges all sharing a common vertex. That is,

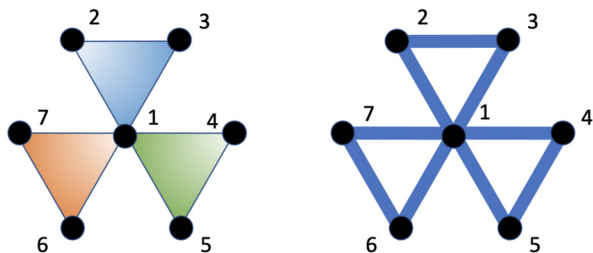


Fig. 6. $\mathcal{S}(3, 3)$ where $E(\mathcal{S}(3, 3)) = \{[1, 2, 3], [1, 4, 5], [1, 6, 7]\}$ and its shadow, $W(3, 2)$, where $E(W(3, 2)) = \{[1, 2], [1, 3], [1, 4], [1, 5], [1, 6], [1, 7], [2, 3], [4, 5], [6, 7]\}$, respectively.

$$E(\mathcal{S}(\eta, k)) = \{[1, (i-1)(k-1) + 2, \dots, (i-1)(k-1) + k] : 1 \leq i \leq \eta\}.$$

The 2-shadow of a k -star is a windmill graph (also known as a fan or friendship graph). In [39], the author determined the spectrum and network properties of windmill graphs. Adhering to their notation, the windmill graph $W(\eta, k)$ consists of η copies of the complete graph K_k joined at a single vertex. We provide a drawing of $\mathcal{S}(3, 3)$ and $W(3, 2)$ in Fig. 6. With this notation we have $\partial^2(\mathcal{S}(\eta, k)) = W(\eta, k-1)$.

Theorem 19. *Let y and x be the principal eigenvectors of a k -graph H and its 2-shadow $\partial^2(H)$, respectively. Then the Chebyshev distance between the rank-optimal weighting of the principal eigenvectors is*

$$D(w_H(y), w_{\partial^2(H)}(x)) = \max_v |y_v^k - x_v^2| \leq 1/2.$$

Moreover, for

$$\Delta_k := \max_{H \in \mathcal{H}(k)} D(w_H(y), w_{\partial^2 H}(x)),$$

where the maximum is taken over all connected k -graphs, we have

$$\lim_{k \rightarrow \infty} \Delta_k = 1/2.$$

Proof. From Theorem 6 we have $\max y_v^k \leq 1/k$ and $\max x_v^2 \leq 1/2$ so that $D(w_H(y), w_{\partial^2(H)}(x)) \leq 1/2$. Let (λ, x) be the principal eigenpair of $W(\eta, k)$. From [39] we have

$$\lambda = \frac{k-1}{2} + \sqrt{\left(\frac{k-1}{2}\right)^2 + \eta k}$$

where $\lambda x_1 = 1 - x_1$ and $x_i = (1 - x_1)/(\eta k)$ for $i > 1$. Solving for x_1 and normalizing such that $\|x\|_2 = 1$ yields

$$x_1 = \frac{2}{\zeta \sqrt{\frac{(\frac{2}{\zeta} - 1)^2}{k\eta} + \frac{4}{\zeta^2}}} \text{ for } \zeta = k + \sqrt{(k-1)^2 + 4k\eta} + 1.$$

For fixed k , $\lim_{\eta \rightarrow \infty} x_1^2 = 1/2$. Now consider the k -star $\mathcal{S}(n, k)$ and observe that vertex-1 forms an independent vertex cover. Because $\{1\}$ is an independent vertex cover we have from Theorem 6 that $y_1^k = 1/k$. We have shown $\lim_k \Delta_k = 1/2$ as desired. \square

Conjecture 2. $\Delta_k = 1/2 + o(1)$ for all $k \geq 3$.

Declaration of competing interest

There is no competing interest.

Data availability

No data was used for the research described in the article.

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