# Strominger′s System on non-Kähler Hermitian Manifolds

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*To my parents and brothers.*

*All conditioned phenomena*
*Are like dreams, illusions, bubbles, and shadows;*
*Like dew drops and a lightning flash:*
*Thus should be contemplated.*

*- Vajracchedikā Prajñāpāramitā Sūtra*

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# Abstract

In this thesis, we investigate the Strominger system on non-Kähler manifolds.

We will present a natural generalization of the Strominger system for non-Kähler hermitian manifolds $M$ with $c_1(M) = 0$. These manifolds are more general than balanced hermitian manifolds with holomorphically trivial canonical bundles. We will then consider explicit examples when $M$ can be realized as a principal torus fibration over a Kähler surface $S$. We will solve the Strominger system on such construction which also includes manifolds of topology $(k-1)(S^2 \times S^4)\#k(S^3 \times S^3)$.

We will investigate the anomaly cancellation condition on the principal torus fibration $M$. The anomaly cancellation condition reduces to a complex Monge-Ampère-type PDE, and we will prove existence of solution following Yau's proof of the Calabi-conjecture [Yau78], and Fu and Yau's analysis [FY08].

Finally, we will discuss the physical aspects of our work. We will discuss the Strominger system using $\alpha'$-expansion and present a solution up to $(\alpha')^1$-order. In the $\alpha'$-expansion approach on a principal torus fibration, we will show that solving the anomaly cancellation condition in topology is necessary and sufficient to solving it analytically. We will discuss the potential problems with $\alpha'$-expansion approach and consider the full Strominger system with the Hull connection. We will show that the $\alpha'$-expansion does not correctly capture the behaviour of the solution even up to $(\alpha')^1$-order and should be used with caution.

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# Chapter 1

# Introduction

In this thesis we investigate Strominger's system on non-Kähler hermitian manifolds. This system of differential equations originates from compactifying the low energy limit of the $10$-dimensional heterotic string theory on a compact $6$-dimensional Riemannian manifold $M$ requiring that the effective field theory on the $4$-dimensional Minkowski space to have $N = 1$ supersymmetry.

It has been well established that the low energy behaviour of heterotic string theory is correctly approximated by $N = 1$ $10$-dimensional heterotic supergravity [BBS07, Pol05, MS10]. The bosonic part of the supergravity action in the string frame up to the first order in $\alpha'$, is

$$\begin{aligned} S = -\frac{1}{2\kappa_{10}^2} \int_{M_{10}} e^{-2\hat{\phi}} \left( \hat{R}_S * 1 - 4d\hat{\phi} \wedge * d\hat{\phi} + \frac{1}{2} \hat{H} \wedge * \hat{H} \right) \\ + \alpha' \frac{e^{-\hat{\phi}}}{4} \mathrm{tr} \left( \hat{F} \wedge * \hat{F} - \hat{R} \wedge * \hat{R} \right) + O(\alpha'^2), \end{aligned}$$

where $\hat{R}_S$ is the Ricci scalar, $\hat{\phi}$ is the dilaton, $\hat{H}$ is a $3$-form, $\hat{F}$ is the curvature of the vector bundle, and $\hat{R}$ is the curvature $2$-form. It has been shown that the equations of motion of this action can be solved by requiring that the background is supersymmetric and satisfies an anomaly cancellation condition [Iva10, MS10].

Strominger [Str86, Str90] found that the internal Riemannian manifold $(M, g)$ should admit a complex structure $J$, and a hermitian form $\omega$ with a non-vanishing holomorphic $(3, 0)$-form $\Omega$. Additionally $(M, J, g, \omega)$ should be accompanied by a holomorphic vector bundle $(V, H)$ and the following system of equations should be satisfied:

$$d^{*}\omega = Jd\log ||\Omega||_{g}, \tag{1.1}$$

$$F \wedge \omega^{2} = 0, \text{ and } F^{2,0} = F^{0,2} = 0, \tag{1.2}$$

$$i\partial\bar{\partial}\omega = \frac{\alpha'}{4}\mathrm{tr}(R \wedge R - F \wedge F). \tag{1.3}$$

Note that there was a minus sign error in Equation (1.1) of [Str86] which was later corrected by Strominger in [Str90]. Strominger didn't specify which connection to use when computing $\mathrm{tr}R \wedge R$, later Hull [Hul86] has shown this should be the Hull connection.

A simple solution of the Strominger system can be constructed when $(M, J, g, \omega)$ is Calabi-Yau. Since $c_1(M) = 0$, we can use the Calabi-Yau theorem and find a Kähler form $\omega$ whose metric $g$ is Ricci-flat. We can solve the Strominger system with $(M, J, g, \omega)$ and $(V = T^{1,0}M, g)$. This is called the standard embedding in string theory.

The virtue of the Strominger system is that it fits in well with Miles Reid's fantasy. Reid considered complex 3-manifolds with a trivial canonical bundle; we will call such manifolds **Reid manifolds**. He [Rei87] conjectured that these manifolds fit into one universal moduli space which is connected via birational transformations and smoothings. This conjecture has been tested [CGH90, CGGK96] for a large number of Kähler Reid manifolds, i.e. Calabi-Yau manifolds.

Strominger system provides the means to study Reid's fantasy via metric geometry even in the non-Kähler case. Li and Yau [LY05] showed that the first condition

Equation (1.1) of the Strominger system is equivalent to

$$d(||\Omega||_\omega \, \omega \wedge \omega) = 0.$$

Thus in the non-Kähler case, the Calabi-Yau metrics are generalized to conformally balanced metrics.

Balanced metrics are more flexible than Kähler metrics. The existence of balanced metrics is preserved under birational transformations [AB95], and under conifold smoothings [FLY08]. If the manifold satisfies the $\partial\bar{\partial}$-lemma, then the balanced condition is preserved under small deformations of the complex structure [Wu06]. However, unlike Calabi-Yau metrics, uniqueness is not guaranteed for balanced metrics. This is where the holomorphic vector bundle $(V, H)$ can help by introducing extra restrictions from Equation (1.2), (1.3).

Regarding the vector bundle condition, Equation (1.2), Chuan [Chu10] considered the conifold transition and the family of balanced metrics studied in [FLY08]. He has shown that one can find a holomorphic vector bundle for each balanced metric that fits into a family of hermitian-Yang-Mills vector bundles throughout the conifold transition. Thus the vector bundle condition can be carried over along the conifold transition.

Given a holomorphic vector bundle $(V, H)$ which satisfies Equation (1.2), there is an extra restriction on the balanced metric via the anomaly cancellation condition, Equation (1.3). Due to the non-linearity of the anomaly cancellation condition, solving the equation is a challenge. However, in the explicit construction of principal torus bundles over a K3 surface [GP04], Fu and Yau [FY08] proved the existence of solutions to the anomaly cancellation condition using the Chern connection for $R$ in Equation (1.3).

Meanwhile, D. Grantcharov, G. Grantcharov, and Poon [GGP08] constructed

Calabi-Yau with torsion structures on principal torus bundles $M$ of [GP04] which are more general than those considered in [FY08]. They found examples of non-Kähler manifolds $M$ that have vanishing first Chern class $c_1(M) = 0$ but do not necessarily admit holomorphically trivial canonical bundles. A natural question to ask is whether the Strominger system can be generalized to non-Kähler manifolds with $c_1(M) = 0$ which do not necessarily have a holomorphic $(3,0)$-form.

In this thesis, we investigate Strominger system on a non-Kähler hermitian manifold $(M, J, g, \omega)$ with $c_1(M) = 0$. We derive a generalization of the conformally balanced condition and discuss the existence of solutions when $M$ is a principal torus bundle constructed in [GP04]. We show that the Strominger system solved with the Chern connection in Equation (1.3) has a solution on such $M$ using the continuity method. Finally, we investigate the Strominger system with the Hull connection and discuss the implication of our work in the context of heterotic string theory.

## 1.1 New results and outline

In **Chapter 2**, we review some results from complex geometry, and analysis. Often the general theorems will be tailored to our needs.

In **Chapter 3**, we consider the Strominger system on a hermitian manifold $(M, J, g, \omega, \Omega)$ with a hermitian vector bundle $(V, H)$. Here $\Omega$ is a nowhere vanishing holomorphic $(3,0)$-form. We will show that the conformally balanced condition $d(||\Omega||_\omega \, \omega \wedge \omega) = 0$ is equivalent to

$$\mathcal{R}^B = 0,$$

where $\mathcal{R}^B$ is the Ricci form of the Bismut connection when $\Omega$ exists. Note that $\mathcal{R}^B = 0$ implies that $c_1(M) = 0$. However, the converse is not true: $c_1(M) = 0$

is not sufficient to guarantee existence of solution to $\mathcal{R}^B = 0$ on $M$ [FG03]. We will investigate hermitian structures $(M, J, g, \omega)$ such that $c_1(M) = 0$ and $\mathcal{R}^B = 0$. In addition to these conditions, if $(M, J)$ admits a holomorphically trivial canonical bundle, then $\mathcal{R}^B = 0$ will imply that the conformally balanced condition is satisfied. Thus the hermitian structure $(M, J, g, \omega)$ that we consider in this thesis includes the structure imposed by the Strominger system.

We focus on the explicit construction of [GP04]. The non-Kähler manifold $M$ constructed in [GP04] is a torus fibration over a Kähler surface $S$:

$$\begin{tikzcd}
M \arrow[d] & T^2 \arrow[l] \\
S &
\end{tikzcd}$$

The torus fibration is determined by two $(1,1)$-forms $\frac{\beta^1}{2\pi}, \frac{\beta^2}{2\pi} \in H^2(S, \mathbb{Z})$. This construction will allow us to reduce the Strominger system on $M$ to differential equations on $S$. We present new solutions to the Strominger system.

1. Conformally balanced condition
    (a) We prove that $(S, g_S)$ should have zero or positive Ricci scalar curvature. By the Enriques-Kodaira classification this implies that $S$ can only be $T^4$, K3, rational, or ruled surface.
    (b) We show that if $(S, g_S)$ has zero scalar curvature then $(1,1)$-forms $\beta^i$s are anti-self-dual and vice versa. This is the case when our general construction reduces to that of [FY08].
    (c) We show that if $(S, g_S)$ has positive scalar curvature, then the conformally balanced condition can be solved with a Hodge dual condition $\beta^1 = *\beta^2$.
    (d) If $(S, g_S)$ has positive scalar curvature, we show that we can solve the

conformally balanced condition with self-dual $\beta^1$ and anti-self-dual $\beta^2$.

   (e) With the combination of Hodge dual solutions and self-dual/anti-self-dual solutions, we can obtain the non-Kähler manifold $(k-1)(S^2 \times S^4)\#k(S^3 \times S^3)$ [GGP08]. In particular, when $k = 2$, it is known that the manifold admits complex structures $J$ such that the canonical bundle is holomorphically trivial [FP09]. Thus this manifold is a good candidate for heterotic string compactifications.

2. Vector bundle condition

   (a) We discuss a generalization of the standard embedding when $(M, \omega)$ is non-Kähler and show that if $(T^{1,0}M, h)$ admits a hermitian-Yang-Mills connection, then $(V = T^{1,0}M, h)$ is a solution of the vector bundle condition.

   (b) We show that this variation of the standard embedding fails for the principal torus fibration $M$ we are considering; that is we prove that $(T^{1,0}M, h)$ cannot be a solution to the vector bundle conditions when $S$ is not K3 or $T^4$.

3. Anomaly cancellation condition

   (a) We will show that the anomaly cancellation condition reduces to the following PDE:

$$i\partial\bar{\partial}e^u \wedge \omega_S - i\frac{\alpha'}{2}\partial\bar{\partial}u \wedge \lambda^2\omega_S - i\frac{\alpha'}{2}\partial\bar{\partial}(e^{-u}\rho) - \frac{\alpha'}{2}\partial\bar{\partial}u \wedge \partial\bar{\partial}u + \mu\frac{\omega_S^2}{2} = 0.$$

   Note that $\lambda^2$ is the scalar curvature of $S$ and this equation correctly reduces to the anomaly cancellation condition of [FY08] when $\lambda^2 = 0$.

   (b) In general, $\rho$ in the above PDE is a complicated $(1,1)$-form. We will present an explicit expression for $\rho$ for the Hodge dual solution 1.(c) and the self-dual/anti-self-dual solution of the conformally balanced condition 1.(d). The explicit expression will be important when proving existence of solution of the PDE in general.

(c) We correct a result in [FY08] and show that the Strominger system can be solved when $S = T^4$. This means that the Iwasawa manifold can be considered as a solution of the system.

In **Chapter 4**, we explore the analytic properties of the anomaly cancellation condition PDE and prove existence of a solution. The proof is valid for cases when the vector bundle condition is solved on $S$ and the conformally balanced condition is solved either via the Hodge dual condition or the self-dual/anti-self-dual condition. In particular, the existence implies that the Strominger system has a solution on $(k-1)(S^2 \times S^4)\#k(S^3 \times S^3)$ for $k \geq 2$, since the conformally balanced condition can be solved with Hodge dual solution and self-dual/anti-self-dual solution [GGP08].

We prove the existence of a solution by writing the PDE in the following form:

$$\begin{aligned}
&2\alpha i \det g' \cdot g'^{\bar{j}i} \frac{\partial^2}{\partial z^i \partial z^{\bar{j}}} \left( \frac{\partial u_t}{\partial z^k} \right) \\
&\quad = - \det g' g'^{\bar{j}i} \frac{\partial}{\partial z^k} \left( e^{u_t} i g_{i\bar{j}} - \alpha t \lambda^2 i g_{i\bar{j}} + t\alpha e^{-u_t} \rho_{i\bar{j}} \right) + \frac{\partial}{\partial z^k} \left( \det g \cdot F_{t,u_t,du_t} \right),
\end{aligned}$$

where we have introduced $t \in [0, 1]$, the parameter on which we use the continuity method. Define a set $T$ by

$$T := \{t \in [0,1];\ \text{the PDE has a solution for } t\}.$$

When $t = 0$, we have a trivial solution $u = \mathrm{constant}$, thus $0 \in T$. When $t = 1$, the PDE becomes the anomaly cancellation condition and we would like to show that $1 \in T$. The idea is to show that $T$ is open and closed in $[0, 1]$, and since $T$ is not empty $T = [0, 1]$.

1. To show that $T$ is open, we use the implicit function theorem.
2. To show that $T$ is closed, we use a priori estimates following Yau's proof of the Calabi conjecture [Yau78] and Fu and Yau's analysis [FY08]. Assuming $u \in C^{5,\alpha}(M)$, the key estimates we derive are the following:
   (a) $u$ is bounded from below and above.
   (b) The first order derivatives of $u$ are bounded and the PDE is uniformly elliptic.
   (c) The second order derivatives of $u$ are bounded.
   (d) The third order derivatives of $u$ are bounded.

   We prove $T$ is closed by showing that it contains its limit points. This will be proved using Schauder estimates, bootstrap procedure and the Rellich-Kondrachov theorem.

In **Chapter 5**, we investigate the Strominger system with the Hull connection in Equation (1.3). While mathematically the Chern connection is the most natural connection to consider, Hull [Hul86] derived the physical connection that string theory demands. We investigate how the Strominger system behaves using the Hull connection.

1. We first consider $\alpha'$-expansion on the principal torus fibration.
   (a) We emphasize that $(\alpha')^0$-order of the Strominger system implies that $\omega$ is Kähler at $0$th-order [IP01, GMW04, FT11].
   (b) We solve the conformally balanced condition up to $(\alpha')^1$-order and deduce $(S, g_S)$ must be Kähler Ricci-flat, thus a Calabi-Yau.

   (c) We solve the vector bundle condition up to $(\alpha')^1$-order when $V = \pi^*E$ and $E \to S$ is a stable bundle. The condition reduces to

$$\mathrm{tr}_{\omega_S} F = 0.$$

   (d) We solve the anomaly cancellation condition up to $(\alpha')^1$-order. We prove that the necessary and sufficient condition for the anomaly cancellation condition to have an analytic solution is $p_1(M) = p_1(V)$. Thus in this case solving the anomaly cancellation condition topologically is equivalent to solving it analytically.

   (e) We discuss the problems of the $\alpha'$-expansion, and the possible complications when the solution of the Strominger system with respect to the Hull connection is not analytic.

2. We consider the exact solution of the Strominger system with respect to the Hull connection.

   (a) We derive curvature identities between Chern, Bismut, and Hull connections.

   (b) We express the anomaly cancellation in terms of the Hull connection:

$$i\partial\bar{\partial}\omega = \frac{\alpha'}{4}\left(\mathrm{tr}R^H \wedge R^H - \mathrm{tr}F \wedge F\right),$$

   and show that $\mathrm{tr}R^H \wedge R^H$ has $(3,1) + (1,3)$-components in general. We will derive a sufficient condition such that $\left(\mathrm{tr}R^H \wedge R^H\right)^{(3,1)} = 0$, and also present conditions when $\left(\mathrm{tr}R^H \wedge R^H\right)^{(2,2)}$ simplifies.

   (c) We apply the anomaly cancellation condition to the principal torus bundle and prove that a solution $u_C$ of the Strominger system with respect to the Chern connection converges to a solution $u_H$ of the system with respect to the Hull connection. The explicit computation of the Hull connection to show convergence is rather complicated, the details are left in **Appendix A**.

(d) By investigating the behaviour of $u_C \to u_H$, we show that not all of the exact solutions to the Strominger system is obtained through the $\alpha'$-expansion. Conversely, it is not clear whether all of the $\alpha'$-expansion solution can be realized as an analytic approximation of a full solution.

In **Chapter 6**, we will discuss further work that can be done to improve the existing results of this thesis. In **Appendix B**, we present some numerical solutions to the anomaly cancellation condition that is work in progress.

# Chapter 2

# Background

In this chapter we review some general aspects of complex geometry, analysis, and heterotic string theory. Most of the contents in this chapter is based upon [GH94], [Huy05] and [Mor07]. We set conventions that will be used throughout this thesis.

## 2.1 Complexification

We recall that a **complex vector bundle** $V$ over a manifold $M$ is a vector bundle whose fibres $V_p$ are complex vector spaces for $p \in M$. At each fibre $V_p$ , we introduce a hermitian inner product $H_p$ that is a complex-valued bilinear form which is linear in its first argument and anti-linear in its second, that is

$$H_p(u,v) = \overline{H_p(v,u)}\,,\ H_p(\lambda u, v) = \lambda H_p(u,v)\,,\ H_p(u,\lambda v) = \bar{\lambda} H_p(u,v)\,,$$

where $u, v \in V$. If the hermitian product $H_p$ can be patched together and can be realized as a section of $(V \otimes \overline{V})^*$, then we call it a **hermitian metric** $H \in \Gamma(V \otimes \overline{V})^*$. We call a complex vector bundle a **hermitian vector bundle** if it is equipped with a hermitian metric and denote it as $(V, H)$. When the manifold $M$ admits an integrable complex structure $J$, then we have a notion of holomorphicity. A complex vector bundle $V$ over $(M, J)$ is called a **holomorphic vector bundle** if it admits a trivialization with holomorphic transition functions.

In this thesis we only deal with bundles over a complex manifold $(M, J)$. Consider a complex manifold $(M, J)$ with a riemannian metric $\tilde{g}$, we can always construct a metric $g$ from $\tilde{g}$ as in the following:

$$g(\cdot,\cdot) := \frac{1}{2}\left[\tilde{g}(J\cdot, J\cdot) + \tilde{g}(\cdot,\cdot)\right].$$

Such metric $g$ can be expressed in holomorphic coordinates as

$$g = g_{i\bar{j}} dz^i \otimes dz^{\bar{j}} + g_{\bar{j}i} d\bar{z}^{\bar{j}} \otimes dz^i,$$

where $i, j, k$ are holomorphic indices and $h := g_{i\bar{j}} dz^i \otimes dz^{\bar{j}}$ can be thought of the hermitian metric on $T^{1,0}M$ and $\bar{h}$ the hermitian metric on $T^{0,1}M$. It is also common to call $g$ a **hermitian metric** of $(M, J)$, and we will use such terminology if it doesn't cause any confusion. We define the **hermitian form** as

$$\omega(\cdot,\cdot) := g(J\cdot,\cdot).$$

We call a hermitian form **Kähler** if it is closed $d\omega = 0$.

## 2.2 Connections

A **connection** $\nabla$ on a holomorphic vector bundle $V \to M$ is a map

$$\nabla : \Gamma(V) \to \Gamma(T^*M \otimes V)$$

satisfying the Leibniz rule,

$$\nabla(fs) = df \otimes s + f\nabla s\,,$$

for any $f \in C^\infty(M)$ and $s \in \Gamma(V)$. We extend the connection to act on $V$-valued $p$-forms $\alpha$ by imposing the Leibniz rule,

$$\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^p \alpha \wedge \nabla s\,,$$

for all $\alpha \in \Omega^p M$ and $s \in \Gamma(V)$. When $(M, J)$ is a complex manifold the 1-form part of $\nabla$ can be decomposed naturally: $\nabla = \nabla^{1,0} \oplus \nabla^{0,1}$. The $\nabla^{0,1}$ part of the connection is closely related to $\bar\partial$ on $M$; $\nabla^{0,1}(f \cdot s) = \bar\partial f \cdot s + f\nabla^{0,1}s$. When $\nabla^{0,1} = \bar\partial$ on a holomorphic section $s$, we call the connection **compatible with the holomorphic structure**.

Using the connection $\nabla$ we can define the curvature $F$ of $\nabla$ as follows. The operator $\nabla^2$ is linear over $C^\infty(M)$:

$$\begin{aligned}\nabla^2(fs) &= \nabla(df \otimes s + f\nabla s) = -df \otimes \nabla s + df \otimes \nabla s + f\nabla^2 s \\ &= f\nabla^2 s\,.\end{aligned}$$

Thus $\nabla^2$ is a bundle map $V \to \bigwedge^2(T^*M) \otimes V$ and it is a section of $\bigwedge^2(T^*M) \otimes \mathrm{End}\,(V)$. This $\mathrm{End}\,(V)$-valued 2-form $F$ is called the **curvature** of the connection $\nabla$.

Using a local basis frame $(e_1, ..., e_\gamma)$ for an open patch $U \subset V$, we can express a connection $\nabla$ in terms of a 1-form $A$ which takes values in $\mathrm{End}\,(V)$

$$\nabla e_\alpha = A_\alpha{}^\beta e_\beta\,.$$

If the connection is compatible with the holomorphic structure then

$$\nabla^{0,1}(fe_\alpha) = \bar{\partial} f \cdot e_\alpha + A^{0,1}{}_\alpha{}^\beta e_\beta = \bar{\partial} f \cdot e_\alpha,$$

therefore $A^{0,1} = 0$ and the connection 1-form $A$ only has $(1,0)$-components.

The curvature 2-form can be computed using this frame,

$$\nabla^2 e_\alpha = \nabla\left(A_\alpha{}^\beta e_\beta\right) = \left(dA_\alpha{}^\gamma - A_\alpha{}^\beta \wedge A_\beta{}^\gamma\right) e_\gamma.$$

This is the Cartan structure equations for the curvature matrix with respect to the frame $e$,

$$F = dA - A \wedge A.$$

Note that we have a minus sign due to the fact that we defined the connection as a section of $T^*M \otimes \mathrm{End}\,(V)$.

We denote the index structure of the curvature form as $F_{ab\alpha}{}^\beta$, where $a$, $b$ are general indices (not necessary holomorphic) on $M$ and $\alpha$, $\beta$ are general indices (not necessary holomorphic) on the vector bundle $V$. We can take the trace of $F$ with respect to $\alpha$, $\beta$. We define the **Ricci form** as

$$\mathcal{R} := i\mathrm{tr}_V F = \frac{i}{2} F_{ab\alpha}{}^\alpha dz^a \wedge dz^b\,.$$

Now we review important connections that will be used through out this thesis.

### 2.2.1 Hermitian connection

The hermitian connection $D$ will be the primary connection we use in this thesis as it is the unique connection on a holomorphic hermitian vector bundle $(V, H)$ that is compatible with the hermitian structure. The conditions we impose are the following,

1. Hermitian metric preservation: $DH = 0 \Leftrightarrow dH_{i\bar{j}} = H(De_i, e_j) + H(e_i, De_j)$.
2. Holomorphic structure compatibility: $\nabla^{0,1} = \bar{\partial}$.

**Proposition 1.** *If $(V, H)$ is a hermitian vector bundle over a complex manifold $(M, J)$, then there is a unique connection $D$ called the* ***hermitian connection*** *that is compatible with both the hermitian metric $H$ and the complex structure $J$.*

*In holomorphic coordinates, D is given by,*

$$A = \partial H \cdot H^{-1}.$$

*Proof.* Let $(e_1, ..., e_k)$ be a holomorphic frame for $E$ and put $H_{i\bar{j}} = \langle e_i, e_j \rangle$. Since $D$ is compatible with the holomorphic structure, $De_i$ is of type $(1, 0)$. By the metric compatibility condition, we obtain,

$$dH_{i\bar{j}} = \langle De_i, e_j \rangle + \langle e_i, De_j \rangle = A_i{}^k H_{k\bar{j}} + \bar{A}_{\bar{j}}{}^{\bar{k}} H_{i\bar{k}} \,.$$

On right hand side, the first term is a $(1, 0)$-form and the second term is a $(0, 1)$-form. Therefore we have,

$$\partial H = AH, \qquad \bar{\partial}\bar{H} = \bar{A}\bar{H}.$$

$A = \partial H \cdot H^{-1}$ is the unique solution to both of these equations. $\square$

The curvature form of the hermitian connection can be obtained as the following.

**Proposition 2.** *The curvature form of a hermitian connection $D$ on $(V, H)$ is a $(1, 1)$-form.*

*In a holomorphic frame, it is given by*

$$F = \bar{\partial}(\partial H \cdot H^{-1})\,.$$

*Proof.* This is a result of direct computation.

$$\begin{aligned} F = dA - A \wedge A &= d(\partial H \cdot H^{-1}) - \partial H \cdot H^{-1} \wedge \partial H \cdot H^{-1} \\ &= \bar{\partial}(\partial H) \cdot H^{-1} - \partial H \wedge dH^{-1} + \partial H \wedge \partial H^{-1}, \end{aligned}$$

where we have used $H^{-1} \cdot \partial H = -\partial H^{-1} \cdot H$. Expanding $d$ and simplifying we obtain

$$F = \bar{\partial}(\partial H) \cdot H^{-1} - \partial H \wedge \bar{\partial} H^{-1} = \bar{\partial}(\partial H \cdot H^{-1}).$$

This completes the proof. □

Using the hermitian connection we can derive a local expression of the Ricci form of the hermitian connection:

$$\mathcal{R}^{\text{herm}} = i\mathrm{tr}_V \bar{\partial}\left(\partial H \cdot H^{-1}\right) = -i\partial\bar{\partial} \log \det H.$$

Note that $\det H$ is a tensor density and this is a purely local expression. Using this expression we can express the first Chern class as

$$c_1(V, H) = \left[\frac{i}{2\pi}\mathrm{tr}_V F\right] = -\frac{i}{2\pi}\partial\bar{\partial} \log \det H.$$

Consider the curvature form $F$ of a hermitian connection. We can take the trace of $F$ with respect to the hermitian form $\omega$ of the hermitian manifold $(M, J, g)$ in the following sense:

$$(\mathrm{tr}_\omega F)_\alpha{}^\beta := i\omega^{i\bar{j}} F_{i\bar{j}\alpha}{}^\beta = g^{i\bar{j}} F_{i\bar{j}\alpha}{}^\beta.$$

Note that $\mathrm{tr}_\omega F \in \mathrm{End}\,(V)$. We call a hermitian connection **hermitian-Yang-Mills** if it satisfies

$$\mathrm{tr}_\omega F = \mu \mathbb{1},$$

where $\mu$ is some constant.

We can find $\mu$ using the definition of a degree of a vector bundle. A **degree** of a holomorphic vector bundle $V$ over $M$ is defined as

$$\deg(V) = \int_M c_1(V) \wedge *\omega\,,$$

where $\omega$ is the hermitian form of a the Gauduchon metric [Gau77] i.e. $\omega$ satisfies $\partial\bar{\partial}(*\omega) = 0$. Kähler metrics are trivially Gauduchon. Note that for Gauduchon metrics the degree is well defined. This is because changing the hermitian metric on $V$ will change the first Chern class by $c_1(V, h_2) - c_1(V, h_1) = \partial\bar{\partial} f$ where $f = -i/2\pi \log(\det h_2/\det h_1)$ is a globally defined function on $M$. The integration will remove the extra term (Stokes' theorem). We take the Ricci form as a representative of the Chern class and obtain:

$$\begin{aligned}\deg(V) &= \int \frac{\mathcal{R}}{2\pi} \wedge *\omega = \int \frac{i}{2\pi} \omega^{i\bar{j}} F_{i\bar{j}\alpha}{}^{\alpha}\,\mathrm{vol} \\ &= \frac{1}{2\pi} \mu\, \mathrm{rank}(V)\, \mathrm{vol}(M).\end{aligned}$$

Therefore,

$$\mu = \frac{2\pi}{\mathrm{vol}(M)} \frac{\deg(V)}{\mathrm{rank}(V)},$$

in particular when $c_1(V) = 0$, then we obtain $\mu = 0$.

### 2.2.2 Canonical connections

On a hermitian manifold $(M, J, g)$, it is often more useful to talk about connections that preserve the hermitian metric $g$ and complex structure $J$, i.e.

1. Hermitian metric preservation: $\nabla g = 0 \Leftrightarrow dg_{i\bar{j}} = g(\nabla e_i, \bar{e}_{\bar{j}}) + g(e_i, \nabla \bar{e}_{\bar{j}})$.
2. Complex structure preservation: $\nabla J = 0$.

The preservation condition of the complex structure can be understood using a projection operator $P := (\mathbb{1} - iJ)/2$. Note that $P(e_{\bar{j}}, \cdot) = 0$ and $P(v, e^i) = v^i$, thus $P$ projects $v$ to its holomorphic component. Now we take the covariant derivative of $P(e_{\bar{j}}, e^i)$ and obtain

$$0 = \nabla P(e_{\bar{j}}, e^i) = P(\nabla e_{\bar{j}}, e^i),$$

where we have used $P(e_{\bar{j}}, \cdot) = 0$. This implies that $\nabla e_{\bar{j}} \in \Gamma(T^*M \otimes T^{0,1}M)$. Using this method we can show that the complex structure preservation condition is equivalent to imposing

$$\nabla \in \Gamma\left(T^*M \otimes \mathrm{End}\left(T^{1,0}M\right)\right) \oplus \Gamma\left(T^*M \otimes \mathrm{End}\left(T^{0,1}M\right)\right).$$

If we consider the connection components ${\Gamma_{ab}}^c$ of $\nabla$, this is condition implies that the only non-vanishing components are ${\Gamma_{ai}}^j$ and ${\Gamma_{a\bar{i}}}^{\bar{j}}$. This is significantly weaker than the holomorphic structure compatibility condition; holomorphic structure compatibility condition implies that the connection components are of pure type, i.e. ${\Gamma_{ki}}^j$ and ${\Gamma_{\bar{k}\bar{i}}}^{\bar{j}}$. A hermitian manifold in fact admits a family of connections that preserve the hermitian metric $g$, and complex structure $J$.

**Proposition 3** ([Gau97]). *Denote $A, B, C \in \Gamma(TM)$. Any hermitian manifold $(M, J, g, \omega)$ admits a one-parameter family of connections $t \in \mathbb{R}$ that preserve the hermitian metric $g$*

*and the complex structure $J$:*

$$g(\nabla^t_A B, C) = g(\nabla^{LC}_A B, C) + \frac{t-1}{4}\,(d^c\omega)\,(A, B, C) + \frac{t+1}{4}\,(d^c\omega)\,(A, JB, JC),$$

*where $d^c = i(\bar\partial - \partial)$ and $\nabla^{LC}$ is the Levi-Civita connection. These connections are called the* ***canonical connections****.*

Note that when the metric is Kähler we have $d^c\omega = 0$, and the canonical connections degenerate into the Levi-Civita connection. Thus in the Kähler case, the Levi-Civita connection is the natural connection to consider. Among the canonical connections in the non-Kähler case, two connections stand out. The connection with $t = 1$ is called the Chern connection and the one with $t = -1$ is called the Bismut connection. We will discuss these connection in detail in the coming sections.

For canonical connections, there are some components of the Riemann tensor that vanish. In the convention we have used for the curvature tensor, the Riemann tensor can be expressed in local coordinates as follows

$$R_{abc}{}^{d} = \partial_a \Gamma_{bc}{}^{d} - \partial_b \Gamma_{ac}{}^{d} - \Gamma_{ac}{}^{e}\Gamma_{be}{}^{d} + \Gamma_{bc}{}^{e}\Gamma_{ae}{}^{d}.$$

Since the canonical connections preserve the complex structure $\Gamma_{\cdot i}{}^{\bar j} = \Gamma_{\cdot \bar j}{}^{i} = 0$. This can be used to show that $R_{abi}{}^{\bar j} = R_{ab\bar j}{}^{i} = 0$.

### 2.2.3 The Levi-Civita connection on Kähler manifolds

We review some identities related to the Levi-Civita connection on Kähler manifolds. The Levi-Civita connection preserves the complex structure in the Kähler case, and using the symmetries $R_{abcd} = R_{cdab}$ of the Riemann tensor with respect to the Levi-Civita connection $\nabla^{LC}$ we obtain

$$\begin{aligned} &R_{abcd}\, dz^a \otimes dz^b \otimes dz^c \otimes dz^d \\ &\qquad = R_{i\bar{j}k\bar{l}}\, dz^i \otimes d\bar{z}^{\bar{j}} \otimes dz^k \otimes d\bar{z}^{\bar{l}} + R_{i\bar{j}\bar{l}k}\, dz^i \otimes d\bar{z}^{\bar{j}} \otimes d\bar{z}^{\bar{l}} \otimes dz^k \\ &\qquad\quad + R_{\bar{j}ik\bar{l}}\, d\bar{z}^{\bar{j}} \otimes dz^i \otimes dz^k \otimes d\bar{z}^{\bar{l}} + R_{\bar{j}i\bar{l}k}\, d\bar{z}^{\bar{j}} \otimes dz^i \otimes d\bar{z}^{\bar{l}} \otimes dz^k. \end{aligned}$$

We define the **Ricci form** of $(T^{\mathbb{C}}M, \nabla^{LC})$ to be that of $(T^{1,0}M, \nabla^{LC})$ as defined before:

$$\mathcal{R} := iR_{i\bar{j}k\bar{l}} g^{k\bar{l}} dz^i \wedge dz^{\bar{j}}.$$

On $T^{\mathbb{C}}M$ the **Ricci curvature tensor** is defined as

$$R_{ab} = -R_{acb}{}^{c}.$$

By using the decomposition and symmetry of the Riemann tensor we can see that $R_{ij} = R_{\bar{i}\bar{j}} = 0$ and $R_{i\bar{j}} = R_{\bar{j}i}$. Also because the Ricci tensor is a $(1,1)$-type, we have $R(J\cdot, J\cdot) = R(\cdot,\cdot)$. These are the exact same symmetries as the hermitian metric, and we define a $(1,1)$-form similarly to what we did for the hermitian form

$$\tilde{\mathcal{R}} = R(J\cdot, \cdot),$$

where $R$ is the Ricci tensor. Using the Bianchi identity $R_{i\bar{j}k\bar{l}} + R_{\bar{j}ki\bar{l}} + R_{ki\bar{j}\bar{l}} = 0$ with $R_{ki\bar{j}\bar{l}} = 0$, we can see that $\tilde{\mathcal{R}}$ is in fact the Ricci form defined before:

$$\begin{aligned} \mathcal{R} &= iR_{i\bar{j}k\bar{l}} g^{k\bar{l}} dz^i \wedge d\bar{z}^{\bar{j}} = -iR_{\bar{j}ki}{}^{k} dz^{\bar{j}} \wedge dz^i \\ &= \tilde{\mathcal{R}}. \end{aligned}$$

Thus on a Kähler manifold the components of the Ricci tensor and those of the Ricci form are the same. This is not true in general for non-Kähler manifolds.

Another property that is useful on Kähler manifolds is the fact that the **Ricci scalar curvature** $g^{ab}R_{ab}$ and the **hermitian scalar curvature** $\langle \omega, \mathcal{R} \rangle$ are related by a factor:

$$\langle \omega, \mathcal{R} \rangle = \frac{1}{2} g^{ab} R_{ab}.$$

### 2.2.4 The Chern connection

We have defined the Chern connection $\nabla^C$ from the canonical connections with $t = 1$. i.e.

$$g(\nabla^C_A B, C) = g(\nabla^C_A B, C) + \frac{1}{2}(d^c\omega)(A, JB, JC),$$

where $A, B, C \in \Gamma(TM)$. In this section, we will give an alternative equivalent definition that will be useful as well.

The complexified tangent bundle $T^{\mathbb{C}}M$ has the following natural decomposition,

$$(T^{\mathbb{C}}M, g) = (T^{1,0}M, h) \oplus (T^{0,1}M, \bar{h}),$$

where $g = h_{i\bar{j}} e^i \otimes e^{\bar{j}} + h_{\bar{j}i} e^{\bar{j}} \otimes e^i$. $(T^{1,0}M, h)$ is a holomorphic hermitian vector bundle over $(M, J)$ and thus we can define a hermitian connection $\nabla^h$ on it as before. On $(T^{0,1}M, \bar{h})$ we take the complex conjugate of the connection $\nabla^h$ and obtain $\bar{\nabla}^{\bar{h}}$. The connection on $T^{\mathbb{C}}M$ defined by $\nabla^C := \nabla^h \oplus \bar{\nabla}^{\bar{h}}$ is called the **Chern connection**. We state this definition formally in the following proposition.

**Proposition 4.** *A complex manifold $M$ has a unique connection $\nabla^C$ called the **Chern connection** on $T^{\mathbb{C}}M$ that satisfies the following conditions,*

1. *it preserves the metric $g$: $\nabla^C g = 0$,*

2. *it is compatible with the holomorphic structure of $T^{1,0}M$:*

   *$(\nabla^C)^{0,1} = \bar{\partial}$ on holomorphic sections of $T^{1,0}M$.*

3. *it is compatible with the anti-holomorphic structure of* $T^{0,1}M$*:*

   $(\nabla^C)^{1,0} = \partial$ *on anti-holomorphic sections of* $T^{0,1}M$*.*

*Denoting the hermitian part of the metric as* $h$ *in a holomorphic frame, the connection* $1$*-form matrix can be written as the following,*

$$A = \begin{pmatrix} \partial h h^{-1} & 0 \\ 0 & \bar{\partial} h^t h^{-t} \end{pmatrix}.$$

*Proof.* This proposition is a direct result of constructing the Chern connection on $(T^{\mathbb{C}}M, g)$ from the hermitian connection $(T^{1,0}M, h)$ as described before. □

Note that this definition gives the same connection when taking $t = 1$ of the canonical connections. Often the Chern connection is defined as the connection that preserves the metric $g$, complex structure $J$, and it is pure in its indices in the physics literature. The condition that the connection should be pure in its indices is essentially the holomorphic structure compatibility condition, and implies that the connection preserves the complex structure $\nabla J = 0$.

We can compute the curvature of the Chern connection directly.

**Proposition 5.** *The curvature* $2$*-form of the Chern connection in a holomorphic frame can be expressed as the following,*

$$R^C = \begin{pmatrix} \bar{\partial}(\partial h h^{-1}) & 0 \\ 0 & \partial(\bar{\partial} h^t h^{-t}) \end{pmatrix}.$$

*Proof.* The curvature form of the Chern connection can be derived using the curvature expression we have derived for the hermitian connection using $(T^{\mathbb{C}}M, g) = (T^{1,0}, h) \oplus (T^{0,1}, \bar{h})$. □

The decomposition $T^{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M$ simplifies computation when dealing with canonical connections. We can simply consider a canonical connection as $\Gamma(T^*M \otimes \operatorname{End} T^{1,0})$ and take the complex conjugate to extend it to $\Gamma(T^*M \otimes \operatorname{End} T^{0,1})$. We will use this technique when deriving connection identities in the following sections.

Here we use this trick to derive an expression for the torsion of the Chern connection $\nabla^C$.

**Remark 6.** The Chern connection $\nabla^C$ on $T^{1,0}M$ is given by

$$g(\nabla^C_A Y, \bar{Z}) = g(\nabla^{LC}_A Y, \bar{Z}) + \frac{1}{2} g(d^c\omega(A, Y), \bar{Z}).$$

The torsion of the Chern connection on $T^{1,0}M$ can be expressed by

$$T^C(A, Y, \bar{Z}) = g(T^C(A, Y), \bar{Z}) = g(d^c\omega(A, Y), \bar{Z}).$$

This has close resemblance with the Hull connection which we will define in the following sections.

**Remark 7.** Note that the expression for the torsion tensor $T^C|_{T^{1,0}M} = d^c\omega$ is only valid on $T^{1,0}M$. If we consider the full Chern connection on $T^{\mathbb{C}}M$, the torsion fails to be skew-symmetric because the term involving $d^c\omega$ is twisted by $J$ in general, i.e. $(\nabla^C_X Y)_\flat := (\nabla^{LC}_X Y)_\flat + \frac{1}{2}(d^c\omega)(X, JY, J\cdot)$.

### 2.2.5 The Bismut connection

Bismut [Bis89] first introduced a connection with totally skew-symmetric torsion on $T^{\mathbb{C}}M$ to prove a local index formula. We have defined the Bismut connection via the canonical connection with

$$g(\nabla^B_A B, C) = g(\nabla^{LC}_A B, C) - \frac{1}{2}(d^c\omega)(A, B, C).$$

Such connection also plays an important role in string theory. We can give an equivalent definition as the following.

**Proposition 8.** *A complex manifold $M$ has a unique connection $\nabla^B$ called the* ***Bismut connection*** *on $T^{\mathbb{C}}M$ that satisfies the following conditions,*

1. *it preserves the metric: $\nabla^B g = 0$,*
2. *it preserves the complex structure: $\nabla^B J = 0$,*
3. *the torsion contracted with the metric, i.e $T^B(X, Y, Z) = g(T^B(X, Y), Z)$, is totally skew-symmetric.*

*The torsion $T^B$ of $\nabla^B$ can be expressed via the hermitian form $\omega$,*

$$T^B = -d^c\omega.$$

*Proof.* This is a corollary of Proposition 3 with $t = -1$. □

### 2.2.6 The Hull connection

Strominger system is used to find suitable backgrounds for heterotic string theory. The connection that enters Strominger system has been worked out by Hull [Hul86], and it turns out that the connection is not a canonical connection. However the Hull connection is closely related to the Bismut connection.

**Definition 9.** On a hermitian manifold $(M, J, g)$, we define **Hull connection** $\nabla^H$ by

$$g(\nabla^H_X Y, Z) = g(\nabla^{LC}_X Y, Z) + \frac{1}{2}(d^c\omega)(X, Y, Z).$$

The Hull connection satisfies the following conditions.

1. It preserves the metric: $\nabla^H g = 0$.

2. It does not preserve the complex structure: $\nabla^H J \neq 0$.
3. The torsion contracted with the metric, i.e. $T^H(X, Y, Z) = g(T^H(X, Y), Z)$ is totally anti-symmetric and is given by

   $$T^H = d^c\omega,$$

   which is the torsion of the Bismut connection with opposite sign.

## 2.3 Lefschetz decomposition

We briefly review Lefschetz decomposition in complex 2 dimensions. The result will be used extensively in the following chapters.

First we define the **Hodge $*$-operator** by

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \, \mathrm{vol},$$

where $\langle \alpha, \beta \rangle = \alpha^{a_{i_1} \cdots a_{i_n}} \beta_{a_{i_1} \cdots a_{i_n}} / n!$ and the indices are raised using the metric $g$. The **Lefschetz operator** $L$ on $\alpha$ is defined by

$$L\alpha = \omega \wedge \alpha,$$

and the **dual Lefschetz operator** $\Lambda$ is defined by

$$\langle \Lambda\alpha, \beta \rangle = \langle \alpha, L\beta \rangle \text{ for all } \beta \in \bigwedge^* T^*M.$$

Note that when $\beta = 1$, we have

$$\Lambda\alpha = \langle \alpha, \omega \rangle.$$

We will often interchange the two notations. With the dual Lefschetz operator, we can define primitive forms. A differential form $\alpha$ is **primitive** if $\Lambda\alpha = 0$, and we denote the linear space of $k$-th order primitive forms as $P^k$.

We are ready to state the Lefschetz decomposition.

**Proposition 10** ([Huy05]). *There exists a direct sum decomposition of $k$-th order differential forms:*

$$\bigwedge^k T^*M = \bigoplus_{i\geq 0} L^i(P^{k-2i}).$$

*This is called the* ***Lefschetz decomposition***.

We will be particularly interested in when $k = 2$,

$$\begin{aligned}\bigwedge^2 T^*M &= L^1(P^0) \oplus L^0(P^2) \\ &= (C^\infty(M)\cdot\omega) \oplus (P^{2,0} \oplus P^{1,1} \oplus P^{0,2}).\end{aligned}$$

In Proposition 1.2.31 of [Huy05], the following formula has been derived: Suppose $\dim_{\mathbb{C}}(M) = n$ and $\alpha \in P^k$, then the following formula holds

$$*L^j\alpha = (-1)^{\frac{k(k+1)}{2}} \frac{j!}{(n-k-j)!} L^{n-k-j}(-J)\circ\alpha,$$

where $(-J)\circ\alpha = \alpha(J\bullet, \cdots, J\bullet)$. Using this formula we can show that $*\omega = \omega$ in complex 2-dimensions. Thus the primitive condition $\langle\omega,\alpha\rangle = \Lambda\alpha = 0$ is equivalent to $\omega\wedge\alpha = 0$. Denote the space of self-dual forms $\bigwedge^2_+$ and anti-self-dual forms as $\bigwedge^2_-$. Using the Lefschetz decomposition and the formula above we obtain,

$$\bigwedge^2_+ \cong P^{2,0} \oplus (C^\infty(M)\cdot\omega) \oplus P^{0,2}, \qquad \bigwedge^2_- \cong P^{1,1}.$$

We will often use the fact that the only self-dual $(1,1)$-form is a multiple of $\omega$, and all primitive $(1,1)$-forms are anti-self-dual.

## 2.4 Banach spaces

We introduce some results of analysis which we will use when showing that the Strominger system has a solution. The results are taken from [Joy07, Eva10, Kry96, GT01], and often stated for special cases that we are interested in.

### 2.4.1 $C^k$ spaces and Hölder spaces

Let $(M, J, g)$ be a hermitian manifold. For an integer $k \geq 0$, we define $C^k(M)$ as the space of functions $f$ whose $k$-th derivative is continuous and bounded. We define a norm on this space by

$$||f||_{C^k} := \sum_{j=0}^{k} \sup_M (\nabla'^j f),$$

where $\nabla'$ is the Chern connection. $C^k$ is a Banach space with this norm.

When discussing elliptic PDEs, $C^k$ spaces are not ideal to work with. Hölder spaces $C^{k,\alpha}$ are the correct spaces to consider. Heuristically, $C^{k,\alpha}$ can be thought of a space of functions that are "$k.\alpha$-times" differentiable. First let us define $C^{0,\alpha}$. Suppose $|x - y|$ be the distance between $x, y \in M$ calculated using $\nabla'$, and $r(g)$ be the injectivity radius of $g$. Let $\alpha \in (0, 1)$ , then $f \in C^\infty(M)$ is called **Hölder continuous with exponent** $\alpha$ if the following is finite

$$[f]_\alpha = \sup_{\substack{x \neq y \\ |x-y| < r(g)}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

We denote the set of functions $f$ which are Hölder continuous with exponent $\alpha$ as $C^{0,\alpha}(M)$. Any Hölder continuous function is continuous thus on a compact manifold $M$ thus $C^{0,\alpha}(M)$ is consisted of bounded functions. We take the norm on $C^{0,\alpha}(M)$ to be $||f||_{C^{0,\alpha}} = ||f||_{C^0} + [f]_\alpha$. Hölder spaces are Banach spaces with this norm.

Higher order $C^{k,\alpha}$ are defined analogously. If $f \in C^k(M)$ and $\nabla^k f$ is Hölder continuous with exponent $\alpha$, we say that $f \in C^{k,\alpha}(M)$. The norm is given by $||f||_{C^{k,\alpha}(M)} = ||f||_{C^k} + [\nabla^k f]_\alpha$.

Hölder continuity can be thought of fractional differentiation as mentioned before. An easy way to see this is to consider $f \in C^1(M)$:

$$\begin{aligned}[f]_\alpha &= \sup \frac{|f(x)-f(y)|^\alpha}{|x-y|^\alpha}|f(x)-f(y)|^{1-\alpha} \\ &\leq ||\nabla f||^\alpha_{C^0} \left(2||f||_{C^0}\right)^{1-\alpha},\end{aligned}$$

where we have used the mean value theorem. Thus Hölder continuous functions can be thought of "$0.\alpha$"-differentiable functions.

### 2.4.2 Lebesque spaces and Sobolev spaces

For $p \geq 1$, we define the **Lebesque space** $L^p(M)$ to be the set of locally integrable functions $f$ on $M$ which have finite norm:

$$||f||_{L^p} = \left(\int_M |f|^p \cdot \mathrm{vol}\right)^{1/p}.$$

A well known inequality regarding Lebesque spaces is the Hölder inequality: Suppose $p, q, r \geq 1$ and $1/r = 1/p + 1/q$. If $f_1 \in L^p(M), f_2 \in L^q(M)$ then $f_1 f_2 \in L^r(M)$ and

$$||f_1 f_2||_{L^r} \leq ||f_1||_{L^p}||f_2||_{L^q}.$$

We will often use this inequality in the form such that $f_2 = 1$ and the volume of the compact manifold is normalized to 1.

Sobolev spaces are defined analogously as Lebesque spaces. We define a **Sobolev space** $L^p_k(M)$ to be a set of functions $f \in L^p(M)$ such that $f$ is $k$-times weakly differentiable and the following **Sobolev norm** on $L^p_k(M)$ is finite

$$||f||_{L^p_k} = \left( \sum_{j=0}^{k} \int_M |\nabla'^j f|^p \cdot \mathrm{vol} \right)^{1/p},$$

where $\nabla'$ is the Chern connection and $|\nabla'^j f|^p$ is computed using the metric $g$. Sobolev spaces are Banach spaces with respect to this norm.

## 2.5 Embedding theorems

Embedding theorems will play an important role in proving the existence of solution. They will be used to derive a priori estimates, and show that a sequence has a convergent point.

First we state the implicit function theorem for Banach spaces. This theorem will be used in the continuity method when proving the set of solutions is open.

**Theorem 11** (Implicit function theorem)**.** *Suppose $X, Y, Z$ are Banach spaces and $U, V$ are open neighbourhoods around $0$ of $X, Y$ respectively. Suppose there exists a function $F \in C^k$ such that maps the origin of $X \times Y$ to $Z$, that is*

$$\begin{aligned} F : X \times Y &\to Z \\ (0,0) &\mapsto 0. \end{aligned}$$

*If the linear map $dF|_X : Y \to Z$ at $(0,0)$ is one-to-one and onto as vector spaces, then there exists a connected neighbourhood $U'$ of $0$ in $X$ and a unique map $G \in C^k$ such that*

$$\begin{aligned} G : U' &\to V \\ 0 &\mapsto 0, \end{aligned}$$

*and $F(x, G(x)) = 0$ for all $x \in U'$.*

We will also use embedding theorems to find a priori estimates and limit points of sequences. The following is a famous theorem due to Sobolev. The presentation is based on [Aub98, Joy07].

**Theorem 12** (Sobolev embedding theorem)**.** *Suppose $M$ is a real $n$-dimensional manifold. Let $k, l$ be integers such that $k \geq l \geq 0$; $q, r$ real numbers such that $q, r \geq 1$; and $\alpha \in (0, 1)$.*

1. *If $q^{-1} \leq r^{-1} + (k-l)/n$, then $L^q_k(M)$ can be continuously embedded in $L^r_l(M)$.*
2. *If $q^{-1} \leq (k-l-\alpha)/n$, then $L^q_k(M)$ can be continuously embedded in $C^{l,\alpha}(M)$.*

We will denote continuous embeddings by $L^q_k(M) \hookrightarrow L^r_l(M)$. With slightly stronger conditions some Sobolev spaces can be compactly embedded into others. Here **compactly embedded** $X \hookrightarrow Y$ means to be continuously embedded and every sequence in a bounded set in $X$ has a subsequence that converges in $Y$.

**Theorem 13** (Rellich-Kondrachov theorem)**.** *Suppose $M$ is a real $n$-dimensional riemannian manifold. Let $k, l$ be integers with $k \geq l \geq 0$; $q, r$ be real numbers with $q, r \geq 1$; and $\alpha \in (0, 1)$.*

1. *If $q^{-1} < r^{-1} + (k-l)/n$, then the embedding $L^q_k(M) \hookrightarrow L^r_l(M)$ is compact.*
2. *If $q^{-1} < (k-l-\alpha)/n$, then the embedding $L^q_k(M) \hookrightarrow C^{l,\alpha}(M)$ is compact. Also $C^{k,\alpha}(M) \hookrightarrow C^k(M)$ is compact.*

The compact embedding $C^{5,\alpha}(M) \hookrightarrow C^5(M)$ will play a key role when showing the set of solutions is closed using the continuity method.

## 2.6 Maximum principle and Schauder estimates

We are interested in uniformly elliptic operators $L$ of order 2. For such operators, we can use the maximum principle.

**Theorem 14** (Weak maximum principle). *Let $L$ be elliptic in a bounded domain $\Omega$. Suppose*

$$Lu \geq 0 \text{ in } \Omega,$$

*with $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, then the maximum of $u$ in $\bar{\Omega}$ is obtained on $\partial\Omega$. In other words,*

$$\sup_{\bar{\Omega}}(u) = \sup_{\partial\Omega}(u).$$

This maximum principle will be used in the following way in this thesis: We will work on a compact manifold $M$ and construct a non-constant function $u \in C^2(M)$ and an elliptic operator $L$. Since $M$ is a compact domain, $u$ will be bounded and achieve its maximum at a point $p$. Take the bounded domain $\Omega$ such that $p \in \Omega$. By the maximum principle we obtain $Lu(p) \leq 0$. This will be the key idea we use in finding 2nd order and 3rd order a priori estimates of our elliptic PDE.

Another tool we will use in finding a priori estimates will be Schauder's estimates.

**Theorem 15** (Schauder interior estimates). *Let $B_1, B_2$ be the balls of radius $1, 2$ in $\mathbb{R}^n$, and let $u \in C^2(B_2)$ be a solution*

$$Lu = a^{ij}D_{ij}u + b^i D_i u + cu = f,$$

*where $f \in C^{0,\alpha}(B_2)$ and the coefficients satisfy the following conditions.*

1. *Uniformly elliptic condition: $a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2$ for all $x \in B_, \xi \in \mathbb{R}^n$, where $\lambda$ is a positive constant.*
2. *Hölder continuous condition: $a^{ij}, b^j, c \in C^{0,\alpha}(B_2)$.*

*Let us denote a positive constant $\Lambda$ such that*

$$||a^{ij}||_{C^{0,\alpha}}, ||b^i||_{C^{0,\alpha}}, ||c||_{C^{0,\alpha}} \leq \Lambda.$$

*Then we have*

$$u|_{B_1} \in C^{2,\alpha}(B_1) \text{ and } ||u|_{B_1}||_{C^{2,\alpha}} \leq C(||f||_{C^{0,\alpha}} + ||u||_{C^0}),$$

*where* $C$ *is a constant that depends on* $n$*,* $\alpha$*,* $\lambda$*, and* $\Lambda$*.*

*More generally, if the coefficients satisfy the conditions with* $C^{0,\alpha}(B_2)$ *replaced by* $C^{l,\alpha}(B_2)$ *where* $l \geq 0$ *is a integer, then*

$$u|_{B_1} \in C^{l+2,\alpha}(B_1) \text{ and } ||u|_{B_1}||_{C^{l+2,\alpha}} \leq C(||f||_{C^{l,\alpha}} + ||u||_{C^0}),$$

*where* $C$ *is a constant that depends on* $n$*,* $\alpha$*,* $\lambda$*, and* $\Lambda$*.*

In our case, we are working on a compact manifold $M$. Denote the finite set of indices as $I$. Using compactness, we can cover the manifold with finite open cover $\{X_i; i \in I\}$ and $\{Y_i; i \in I\}$, where for each $i$ we have $X_i \subset Y_i$ and $(X_i, Y_i)$ is diffeomorphic to $(B_1, B_2)$. Since $(X_i, Y_i)$ is diffeomorphic to $(B_1, B_2)$ we can use the Schauder interior estimates for the open covering $X_i$ and obtain a global estimates on $M$. Thus we can replace $B_1$ and $B_2$ with $M$ and state the result. This will be used in the continuity method.

## 2.7 Conventions

We summarize the conventions we use in this work.

### 2.7.1 Indices

**General index on $V$** $\alpha$, $\beta$, $\gamma$.

**General index on $M$** $a$, $b$, $c$, $d$.

**Complex index on $M$** $i$, $\bar{j}$, $k$, $\bar{l}$, $m$, $\bar{n}$, $p$, $\bar{q}$, $r$, $\bar{s}$, $t$, $\bar{u}$.

### 2.7.2 Metric, complex structure, fundamental forms

**The original metric on $S$** $g_S$ or $g$.

**The new metric $g'$ on $S$** $g'_{i\bar{j}} := \left(e^u - \alpha\lambda^2 + \frac{\alpha\lambda^2}{2}e^{-u}\right) g_{i\bar{j}} + 2\alpha u_{i\bar{j}}$.

**Complex structure** $J\varphi = (-1)^r \varphi \circ J$, where $\varphi$ is a $r$-form.

**Kähler form $\omega_S = \omega$ on $S$** $\omega_S := i g_{i\bar{j}} dz^i \wedge dz^{\bar{j}}$.

**Hermitian form $\omega'$** $\omega' := \left(e^u - \alpha\lambda^2 + \frac{\alpha\lambda^2}{2}e^{-u}\right) \omega_S + 2\alpha i \partial\bar{\partial} u$.

**Hermitian form $\omega_u$ on $M$** $\omega_u = e^u \omega_S + \frac{i}{2}\theta \wedge \bar{\theta}$.

**Hodge dual $*$** $\alpha \wedge *\beta = \langle \alpha, \beta \rangle \cdot \mathrm{vol}$.

### 2.7.3 Differentiation, differential forms

**$u_i$** $u_i := \partial_i u = \partial u / \partial z^i$.

**$\triangle\psi$** $\triangle\psi := g^{i\bar{j}} \partial_i \bar{\partial}_{\bar{j}} \psi$, or $\Delta\psi \frac{\omega_S^2}{2} := i\partial\bar{\partial}\psi \wedge \omega_S$.

**$\nabla u \cdot \nabla v$** $\nabla u \cdot \nabla v := \frac{1}{2} g^{i\bar{j}} (u_i v_{\bar{j}} + u_{\bar{j}} v_i) = \frac{1}{2}(i\partial u \wedge \bar{\partial} v - i\bar{\partial} u \wedge \partial v) \wedge \omega_S$.

**$dz^{a_1 a_2 \cdots a_n}$** $dz^{a_1} \wedge dz^{a_2} \wedge \cdots \wedge dz^{a_n}$.

**$d^c$** $d^c = i(\bar{\partial} - \partial)$ equivalently $d^c = (-1)^r J d J$ on a $r$-form.

### 2.7.4 Connections, and curvature

**$V$** Holomorphic hermitian vector bundle over $M$.

**$\nabla$ or $\nabla^{LC}$** Levi-Civita connection.

**$\nabla'$ or $\nabla^C$** Chern connection.

**$\nabla^B$** Bismut connection.

$\boldsymbol{\nabla^H}$ Hull connection.

$\boldsymbol{D}$ Hermitian connection on $TV$.

$\boldsymbol{R}$ Curvature of a connection on $TM$.

$\boldsymbol{F}$ Curvature of $D$ on $TV$.

### 2.7.5 Constants, and functions

$\boldsymbol{\alpha}$ $\alpha := \alpha'/2$.

$\boldsymbol{C}$**,** $\boldsymbol{c}$**,** $\boldsymbol{\kappa}$ Generic positive constants.

$\boldsymbol{\epsilon}$ Constants that are free to choose.

$\boldsymbol{f}$**,** $\boldsymbol{\tilde{f}}$**,** $\boldsymbol{h}$**,** $\boldsymbol{\tilde{h}}$ Positive bounded smooth functions.

# Chapter 3

# The Strominger system

Strominger [Str86] investigated heterotic superstring backgrounds with $N = 1$ supersymmetry. We consider a hermitian 3-fold $(M, J, g, \omega)$ which admits a holomorphic $(3, 0)$-form $\Omega$. In addition to the hermitian manifold, we have a holomorphic hermitian vector bundle $(V, H)$. The conditions Strominger derived are:

$$\begin{aligned} d^*\omega &= Jd\log \|\Omega\|_g, \\ F \wedge \omega^2 &= 0, \text{ and } F^{2,0} = F^{0,2} = 0, \\ i\partial\bar{\partial}\omega &= \frac{\alpha'}{4}\mathrm{tr}(R \wedge R - F \wedge F), \end{aligned}$$

where $R$ denotes the curvature form on $TM$ and $F$ is the curvature form of the hermitian connection with respect to $H$. We will call the first equation the conformally balanced condition, the second conditions the vector bundle conditions, and the last equation the anomaly cancellation condition.

In Strominger's original work, the author did not specify which connection should be used in computing $R$. Hull [Hul86] worked out the connection $\nabla^H$ that string theory requires. It turns out that the connection $\nabla^H$ is not one of the canonical connections. This makes the Hull connection $\nabla^H$ rather awkward to work with. In this work we will mainly use the Chern connection when computing $R$. In Chapter 5 we will discuss the Strominger system with the Hull connection and address the implication of this work in terms of string theory.

## 3.1 Construction of the non-Kähler manifold

We are interested in a non-Kähler manifold $M$ which can be realized as a torus fibration over a Kähler surface $S$. Goldstein and Prokushkin [GP04] constructed such $M$ as a principal bundle over a hermitian manifold $(S, g_S, \omega_S)$ where $S$ is not necessarily Kähler.

**Theorem 16** ([GP04]). *Let $\beta^1$ and $\beta^2$ be closed $2$-forms on a hermitian $2$-fold $(S, g_S, \omega_S)$ such that $\beta^1/2\pi,\ \beta^2/2\pi \in H^2(S, \mathbb{Z})$. Then there is a $3$-fold $M$ which is a $T^2$ principal bundle over $S$.*

$$\begin{tikzcd}
M \arrow[d] & T^2 \arrow[l] \\
S &
\end{tikzcd}$$

*Furthermore:*

1. *If $\beta^1 + i\beta^2$ has no component in $\bigwedge^{0,2} T^*S$, then the complex structure is integrable.*
2. *Denote the local coordinates on $T^2$ as $x$ and $y$; and define $\alpha^1$ and $\alpha^2$ using the Poincaré lemma $d\alpha^i := \beta^i$. Then $\gamma^1, \gamma^2$ defined as $\gamma^1 := dx + \alpha^1$ and $\gamma^2 := dy + \alpha^2$ give a connection on the $T^2$ fibres.*

   *Furthermore the following $(1,0)$-form is well-defined*

$$\theta := dx + \alpha^1 + i(dy + \alpha^2),$$

*and a metric on $M$ has the following form*

$$g = g_S + (dx + \alpha^1)^2 + (dy + \alpha^2)^2.$$

3. *If either $\beta^1$ or $\beta^2$ is not exact, then $M$ is a non-Kähler manifold.*

Goldstein and Prokushkin also studied the topology of $M$ and showed that

1. The Betti numbers satisfy the following:

$$h^{1,0}(M) = h^{1,0}(S) \qquad\qquad h^{0,1}(M) = h^{0,1}(S) + 1.$$

In particular $h^{0,1}(M) = h^{1,0}(M) + 1$.

2. The Hodge numbers satisfy the following:

$$\begin{aligned} b_1(M) &= b_1(S) + 1, \qquad & b_2(M) &= b_2(S) - 1 \qquad & &\text{when } \beta^2 = n\beta^1 \\ b_1(M) &= b_1(S), \qquad & b_2(M) &= b_2(S) - 2 \qquad & &\text{when } \beta^2 \neq n\beta^1. \end{aligned}$$

3. The Euler number is zero: $\chi(M) = 0$.

A topological observation that will be useful later will be the relation between the first Pontryagin classes $p_1(S)$ of $S$ and $p_1(M)$ of $M$.

**Proposition 17.** *If $M$ is a principal torus fibration over $S$, then $p_1(M) = \pi^* p_1(S)$.*

*Proof.* $M$ and $S$ fit into a short exact sequence as follows

$$0 \to T^2 \hookrightarrow M \xrightarrow{\pi} S \to 0,$$

therefore the Chern class satisfies $c(M) = c(S) \wedge c(T^2)$. From this expression we obtain,

$$c_1(M) = c_1(S) + c_1(T^2) \qquad c_2(M) = c_2(S) + c_1(S) \wedge c_1(T^2) + c_2(T^2).$$

Using the definition of the first Pontryagin class $p_1(M) := c_1(M)^2/2 - c_2(M)$, we obtain $p_1(M) = p_1(S) + p_1(T^2)$. Since $p_1(T^2) = 0$, we obtain $p_1(M) = p_1(S)$. $\square$

## 3.2 Conformally balanced condition

We start by investigating the properties of the conformally balanced condition. The following proposition shows that the conformally balanced condition is equivalent to various conditions. Note that the following proposition holds for a general $m$-fold.

**Proposition 18.** *Let $(M, J, g, \omega)$ be a compact hermitian $m$-fold with non-vanishing holomorphic $(m, 0)$-form $\Omega$. Denote the Ricci form with respect to the Bismut connection as $\mathcal{R}^B_M$. Then the following conditions are equivalent:*

1. $d^*\omega = Jd \log ||\Omega||_g$,
2. $\mathcal{R}^B_M = 0$,
3. $d(||\Omega||_\omega \, \omega^{m-1}) = 0$.

*Proof.* We prove this by showing $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$.

- $(1) \Rightarrow (2)$ : The Ricci form of the Bismut connection $\mathcal{R}^B$ and the Ricci form of the Chern connection $\mathcal{R}^C$ satisfies the following equation,

$$\mathcal{R}^B = \mathcal{R}^C - dd^*\omega.$$

Locally, $\mathcal{R}^C = -i\partial\bar{\partial}\log\det(g_{i\bar{j}})$, and by using $d^*\omega = Jd\log||\Omega||_g$, we can show that $\mathcal{R}^B = 0$ by the following computation,

$$\begin{aligned}\mathcal{R}^B &= -i\partial\bar{\partial}\log\left[\det(g_{i\bar{j}})||\Omega||_g^2\right]\\ &= -i\,m!\,\partial\bar{\partial}\log\left(|f|^2\right)\\ &= 0,\end{aligned}$$

where $f$ a holomorphic function defined as $f := \Omega_{12\cdots m}$ and we have used $||\Omega||_g^2 = \Omega_{i_1\cdots i_m}\bar{\Omega}_{\bar{j}_1\cdots\bar{j}_m}g^{i_1\bar{j}_1}\cdots g^{i_m\bar{j}_m} = m!|f|^2\det(g^{i\bar{j}})$.

- $(2) \Rightarrow (3)$ : We use the local form of $\mathcal{R}^B = 0$ and obtain,

$$\begin{aligned}0 &= dd^*\omega + i\partial\bar{\partial}\log\det(g_{i\bar{j}})\\ &= d(d^*\omega - Jd\log||\Omega||_g), \qquad (3.1)\end{aligned}$$

where we have used $\log\det g_{i\bar{j}} = -2\log||\Omega||_g + \log m! + \log|f|^2$ from the definition of $||\Omega||_g$.

Note that if we define $\widetilde{\omega} := ||\Omega||^{1/(m-1)}\omega$, then $\tilde{*}\widetilde{\omega} = \widetilde{\omega}^{m-1}/(m-1)!$ and $d^{\tilde{*}}\widetilde{\omega} = 0$ is equivalent to $d(||\Omega||_g\,\omega^{m-1}) = 0$.

Since $M$ is compact, $d^{\tilde{*}}\widetilde{\omega} = 0$ is equivalent to $dd^{\tilde{*}}\widetilde{\omega} = 0$. We compute $dd^{\tilde{*}}\widetilde{\omega}$:

$$\begin{aligned}dd^{\tilde{*}}\widetilde{\omega} &= -d\,\tilde{*}\,\frac{d||\Omega||_g\wedge\omega^{m-1} + ||\Omega||_g\,d\omega^{m-1}}{(m-1)!}\\ &= -d\,\tilde{*}\,||\Omega||_g\left(\frac{d\log||\Omega||_g\wedge\omega^{m-1}}{(m-1)!} + d*\omega\right),\end{aligned}$$

where we have used $\tilde{*}\widetilde{\omega} = \widetilde{\omega}^{m-1}/(m-1)!$ in the first line. If $\alpha$ is a real $(2m-1)$-

form then $\tilde{*}\alpha = ||\Omega||_g^{-1} * \alpha$. Therefore, we have

$$\begin{aligned} dd^{\tilde{*}}\widetilde{\omega} &= dd^*\omega - d * \frac{d\log||\Omega||_g \wedge \omega^{m-1}}{(m-1)!} \\ &= dd^*\omega - dJd\log||\Omega||_g \\ &= 0, \end{aligned}$$

where we have used $*(df \wedge \omega^{m-1}) = (m-1)!Jdf$ [Huy05, Proposition 1.2.31] and Equation (3.1). Thus, $\mathcal{R}^B = 0$ implies $d(||\Omega||_\omega\, \omega^{m-1}) = 0$.

- (3) $\Rightarrow$ (1) : [LY05] Let $f$ be a real positive function, then $d(f\omega^{m-1}) = df \wedge \omega^{m-1} + fd\omega^{m-1}$. Applying Hodge star $*$ and using $*(df \wedge \omega^{m-1}) = (m-1)!Jdf$, we obtain,

$$\begin{aligned} *d(f\omega^{m-1}) &= -(m-1)!fd^*\omega + (m-1)!Jdf, \\ &= (m-1)!f(-d^*\omega + Jd\ln f). \end{aligned}$$

If we take $f = ||\Omega||_\omega$, we conclude $d(||\Omega||_\omega\ \omega^{m-1} = 0$ if and only if $d^*\omega = Jd\ln||\Omega||_\omega$.

□

If $(M, J, g, \omega)$ admits a top $(m, 0)$-form $\Omega$, then $c_1(M) = 0$. In the Kähler case the converse is true: if $c_1(M) = 0$ then $M$ admits a top holomorphic $(m, 0)$-form $\Omega$. The proof normally involves applying the Calabi-Yau theorem to find a Ricci flat metric.

However, in the non-Kähler case, Calabi-Yau theorem does not apply and as a result $c_1(M) = 0$ does not imply existence of a top holomorphic $(m, 0)$-form. Such counter examples can be found on Hopf surfaces, and Enriques surfaces [FG04, FP09].

In complex 3-dimensions, the following remark give a sufficient condition for the existence of a holomorphic $(3,0)$-form.

**Remark 19.** Consider a complex 3-dimensional manifold $M$. If $h^{0,1}(M) = 0$, then $M$ admits a holomorphic $(3,0)$-form.

*Discussion.* Since we are on a complex manifold, we have a $(3,0)$-form $\Psi$ which is not necessarily holomorphic. We take a $\bar{\partial}$ derivative of $\Psi$ and obtain:

$$\bar{\partial}\Psi = \alpha \wedge \Psi,$$

where $\alpha$ is a $(0,1)$-form. By applying $\bar{\partial}$ once more we obtain,

$$0 = \bar{\partial}\alpha \wedge \Psi.$$

Since $\Psi$ is a nowhere vanishing $(3,0)$-form and $\bar{\partial}\alpha$ is a $(0,2)$-form, the last equation implies that $\bar{\partial}\alpha = 0$, thus $\alpha \in H^{0,1}(M)$. Because $h^{0,1}(M) = 0$, we can write $\alpha = \bar{\partial} f$ for some function $f \in C^\infty(M)$. Integrating $\bar{\partial}\Psi = \bar{\partial} f \wedge \Psi$, we obtain

$$\bar{\partial}(e^{-f}\Psi) = 0,$$

thus $\Omega := e^{-f}\Psi$ is a holomorphic $(3,0)$-form. $\square$

We will solve the conformally balanced condition by solving $\mathcal{R}^B_M = 0$ in complex 3-dimensions. An advantage of $\mathcal{R}^B_M = 0$ over $d\left(||\Omega||_\omega\, \omega \wedge \omega\right) = 0$ is that it can be studied even on hermitian manifolds that do not admit holomorphically trivial canonical bundles. This provides a natural generalization of the Strominger system on manifolds with $c_1(M) = 0$. Non-Kähler manifolds $M$ with $c_1(M) = 0$ have been studied in [GGP08, FP11].

We would like to investigate the conformally balanced condition over $M$. Consider a principal torus bundle $M$ constructed in Theorem 16. D. Grantcharov, G.

Grantcharov, and Poon derived the relation between the Ricci forms.

**Proposition 20** ([GGP08])**.** *Let $\mathcal{R}^B_M$ and $\mathcal{R}^B_S$ be the Ricci forms of the Bismut connection on $M$ and $S$ respectively. We have the following relation,*

$$\mathcal{R}^B_M = \pi^*\mathcal{R}^B_S - \sum_{l=1}^{2} d\left(\langle\omega_S, \beta^l\rangle\alpha^l\right).$$

In what follows we will be interested in manifolds with hermitian forms $\omega_u := e^u\pi^*\omega_S + \frac{i}{2}\theta\wedge\bar{\theta}$ such that the conformally balanced condition is satisfied. We will find the Ricci form $\mathcal{R}^B_{Mu}$ of $\omega_u$ in terms of the Ricci form $\mathcal{R}^B_S$.

In our situation we have a conformal scaling $e^u$ on $S$. We investigate how the Ricci form on $S$ changes under the conformal scaling $\omega^u_S = e^u\omega_S$.

**Lemma 21.** *For the conformal scaling $\omega^u_S = e^u\omega_S$ on $S$, the curvature tensor $R^C_{Su}$ of the Chern connection scales as the following,*

$$R^C_{Su} = \bar{\partial}\partial u\cdot \mathbb{1}_S + R^C_S.$$

*Furthermore, the Ricci form $\mathcal{R}^C := i\mathrm{tr}R^C$ scales as,*

$$\mathcal{R}^C_{Su} = 2i\bar{\partial}\partial u + \mathcal{R}^C_S.$$

*Proof.* This is a result of direct computation,

$$\begin{aligned}
R^C_{Su} &= \bar{\partial}\left[\partial(e^u H_S)\cdot e^{-u}H_S^{-1}\right],\\
&= \bar{\partial}\left[(e^u\partial u H_S + e^u\partial H_S)e^{-u}H_S^{-1}\right]\\
&= \bar{\partial}\partial u\cdot\mathbb{1}_S + R^C_S.
\end{aligned}$$

This completes the proof. ☐

We are ready to prove the main result of this section. The following proposition reduces the conformally balanced condition on $M$ to a condition on $S$. Furthermore it gives a family of conformally balanced metrics on $M$.

**Proposition 22.** *Let $\mathcal{R}^B_{Mu}$ be the Ricci form of the Bismut connection for $M$ with hermitian form $\omega_u := e^u\pi^*\omega_S + \frac{i}{2}\theta \wedge \bar{\theta}$. If $\beta^l$'s are harmonic $(1,1)$-forms, then we can write $\mathcal{R}^B_{Mu}$ in terms of the Ricci form on $S$, and $\beta^l$s:*

$$\mathcal{R}^B_{Mu} = \pi^* \left( \mathcal{R}_S - \sum_{l=1}^{2} \langle \omega_S, \beta^l \rangle \beta^l \right).$$

*Therefore, if $\mathcal{R}_S = \sum_{l=1}^{2} \langle \omega_S, \beta^l \rangle \beta^l$, then $\mathcal{R}^B_{Mu} = 0$ for the family of hermitian forms $\omega_u := e^u\pi^*\omega_S + \frac{i}{2}\theta \wedge \bar{\theta}$.*

In other words, this proposition shows that if $\omega_0$ satisfies the conformally balanced condition, then $\omega_u$ also satisfies the condition for all $u$.

*Proof of Proposition 22.* Recall the result of Proposition 20:

$$\mathcal{R}^B_{Mu} = \pi^*\mathcal{R}^B_{Su} - \sum_{l=1}^{2} d\left( \langle \omega_S, \beta^l \rangle \alpha^l \right).$$

First we compute $\mathcal{R}^B_{Su}$ by using $\mathcal{R}^B = \mathcal{R}^C - dd^*\omega$,

$$\begin{aligned}
\mathcal{R}^B_{Su} &= \mathcal{R}^C_{Su} - dd^{*_{Su}}\omega^u_S \\
&= 2i\bar{\partial}\partial u + \mathcal{R}^C_S + d(*_{S_u}(du \wedge \omega^u_S)) \\
&= 2i\bar{\partial}\partial u + \mathcal{R}^C_S + d(Jdu) \\
&= \mathcal{R}^C_S,
\end{aligned}$$

where we have used $*(df \wedge \omega) = Jdf$ [Huy05, Proposition 1.2.31]. We substitute this expression into Proposition 20 and obtain,

$$\mathcal{R}^B_{Mu} = \pi^*\mathcal{R}^C_S - \sum_{l=1}^{2} d\left(\langle\omega_S, \beta^l\rangle\alpha^l\right). \tag{3.2}$$

Now we use the fact that $S$ is Kähler. For Kähler metrics, all the canonical connections are the same, we can therefore denote the Ricci form has $\mathcal{R}_S$ without ambiguity. Since $\beta^l$ is a harmonic $(1,1)$-form and the metric is Kähler, its trace is constant [Bes08, 2.33]. Therefore we have $d(\langle\omega_S, \beta^l\rangle_{\omega_S}\alpha^l) = \langle\omega_S, \beta^l\rangle_{\omega_S}\beta^l$. Substituting this expression into Equation (3.2) we obtain the result:

$$\mathcal{R}^B_{Mu} = \pi^*\left(\mathcal{R}_S - \sum_{l=1}^{2}\langle\omega_S, \beta^l\rangle\beta^l\right).$$

This completes the proof. □

## 3.3 Computation of $R^C_{Mu}$

In this section, we compute $R^C_{Mu}$. The results will be used in the vector bundle condition and in the anomaly cancellation condition.

We compute $R^C_{Mu}$ using the Chern connection and reduce it down to an expression on $S$. To carry out the computation of the curvature 2-forms with respect to the Chern connection we construct holomorphic frames. Let $\{z^i = x^i + iy^i;\ i = 1, 2\}$ be a local coordinate in $S$. By construction $\{\pi^* dz^i\}$ are holomorphic $(1,0)$-forms on $M$. On the torus fibre direction recall that we have a $(1,0)$-from $\theta = dx + \pi^*\alpha^1 + i(dy + \pi^*\alpha^2)$, where by definition $\beta^i = d\alpha^i$. We construct a holomorphic $(1,0)$-form $\theta^0$ from $\theta$ by subtracting the anti-holomorphic part. Since $\beta^i$'s are harmonic on a Kähler metric $g_S$, we can use the $\bar{\partial}$-Poincaré lemma and express $\beta^1 = \bar{\partial}\xi$ and $\beta^2 = \bar{\partial}\zeta$ locally. Consider $\theta^0 := \theta - \pi^*(\xi + i\zeta)$, then $\theta^0$ is a $(1,0)$-form by construction and it is holomorphic. Holomorphicity can be checked explicitly,

$$\begin{aligned}\bar{\partial}\theta^0 &= \bar{\partial}\theta - \pi^*(\bar{\partial}\xi + i\bar{\partial}\zeta) \\ &= d\theta - \pi^*(\beta^1 + i\beta^2) = 0,\end{aligned}$$

where we have used $\partial\theta = 0$ since $d\theta$ is a $(1,1)$-form. To simplify our notation, we define $\phi := \xi + i\zeta$ locally and express $\theta^0 = \theta - \pi^*\phi$.

Now we have $\{\pi^* dz^i, \theta^0\}$ as our holomorphic basis $(1,0)$-forms. We seek for holomorphic $(1,0)$-vector fields dual to the basis we have constructed. We take $U_0 := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ which is dual to $\theta^0$ and orthogonal to $\{\pi^* dz^i\}$. We find $\{U_i\}$ by a horizontal lifting of $\{\pi^* dz^i\}$ that satisfies $\theta^0(U_i) = 0$. Since $\theta^0(\partial/\partial z^i) = \frac{\partial}{\partial z^i} \lrcorner [\alpha^1 + i\alpha^2 - \phi]$, the following expression for $U_i$ is dual to $\pi^* dz^i$ and orthogonal to $\theta^0$,

$$U_i = \frac{\partial}{\partial z^i} - \left(\frac{\partial}{\partial z^i} \lrcorner [(\alpha^1 + i\alpha^2) - \phi]\right) U_0.$$

To summarize, so far we have constructed a holomorphic $(1,0)$-form frame $\{\pi^* dz^i, \theta^0\}$ and a holomorphic $(1,0)$-vector frame $\{U_i, U_0\}$ which are dual to each other. We would like to express the metric $g$ in the holomorphic frame. The easiest way to achieve this is by applying $U_i$ in $g$. For example, if we denote the hermitian part of $g$ in this holomorphic frame as $h$, we can find its components in the following way,

$$\begin{aligned}h_{1\bar{1}} &= g_M(U_1, \overline{U}_{\bar{1}}) \\ &= \pi^* g_S(U_1, \overline{U}_{\bar{1}}) + \frac{\theta(U_1) \otimes \bar{\theta}(U_{\bar{1}})}{2}.\end{aligned}$$

$\theta(U_1)$ can be computed as in the following,

$$\begin{aligned}\theta(U_i) &= (\theta^0 + \pi^*\phi)(U_i) = \pi^*\phi(U_i) \\ &= \phi_i.\end{aligned}$$

Using this expression we finally obtain,

$$h_{1\bar{1}} = g_{1\bar{1}} + \frac{|\phi_1|^2}{2}.$$

Similarly, we can work out the other components, and in this basis the hermitian part of the metric takes the following form,

$$\begin{aligned}h &= \begin{pmatrix} g_{1\bar{1}} + \frac{|\phi_1|^2}{2} & g_{1\bar{2}} + \frac{\phi_1\bar{\phi}_2}{2} & \frac{\phi_1}{2} \\ g_{2\bar{1}} + \frac{\phi_2\phi_1}{2} & g_{2\bar{2}} + \frac{|\phi_2|^2}{2} & \frac{\phi_2}{2} \\ \frac{\bar{\phi}_1}{2} & \frac{\bar{\phi}_2}{2} & 1 \end{pmatrix} \\ &= \begin{pmatrix} g + BB^* & B \\ B^* & 1 \end{pmatrix},\end{aligned}$$

where we introduced $B := \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}/2$. It is straightforward to check that the inverse metric is given by

$$h^{-1} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ -B^*g^{-1} & 1 + B^*g^{-1}B \end{pmatrix}.$$

Using $h$ and $h^{-1}$ we can compute the curvature 2-form $R_M$.

**Proposition 23** ([FY08]). *The curvature 2-form $R_M$ on $M$ can be expressed as*

$$R_M^C = \bar{\partial}(\partial h \cdot h^{-1}) = \begin{pmatrix} R_{1\bar{1}} & R_{1\bar{2}} \\ R_{2\bar{1}} & R_{2\bar{2}} \end{pmatrix},$$

*where*

$$
\begin{aligned}
R_{1\bar{1}} &= R_S + \bar{\partial}\left(B\partial B^* \cdot g^{-1}\right), \\
R_{1\bar{2}} &= -R_S B + \partial g \cdot g^{-1} \wedge \bar{\partial} B - \bar{\partial}\left(B\partial B^* \cdot g^{-1} B\right), \\
R_{2\bar{1}} &= \bar{\partial}\left(\partial B^* \cdot g^{-1}\right), \\
R_{2\bar{2}} &= -\bar{\partial}\left(\partial B^* \cdot g^{-1} B\right).
\end{aligned}
$$

## 3.4 Vector bundle condition

We would like to find a holomorphic vector bundle $V$ that admits a hermitian-Yang-Mills connection and

$$\mathrm{tr}_{\omega_u} F = 0.$$

A natural holomorphic vector bundle to consider would be $T^{1,0}M$. In string theory taking $V = T^{1,0}M$ is known as the **standard embedding**.

If the $(M, J, g, \omega_u)$ satisfies the conformally balanced condition and $(T^{1,0}M, H)$ admits a hermitian-Yang-Mills connection, then we can show that $(T^{1,0}M, H)$ satisfies the vector bundle condition. The following proposition can be thought of a non-Kähler version of the standard embedding in string theory.

**Proposition 24.** *If $(M, J, g, \omega)$ is a hermitian manifold with $c_1(M) = 0$, and admits a hermitian-Yang-Mills connection on its holomorphic tangent bundle $(T^{1,0}M, g)$, then the curvature form $R$ with respect to the Chern connection satisfies the vector bundle condition:*

$$R \wedge *\omega = 0, \text{ and } R^{2,0} = R^{0,2} = 0.$$

*Proof.* Since $T^{1,0}M$ admits a hermitian-Yang-Mills connection, we have

$$R \wedge *\omega = \mu \cdot \mathbb{1},$$

where $\mu = \frac{2\pi}{\mathrm{vol}(M)} \frac{\deg(V)}{\mathrm{rank}(V)}$. The vanishing first Chern class $c_1(T^{1,0}M) = 0$ implies that that the degree of the vector bundle $T^{1,0}M$ is zero:

$$\deg(T^{1,0}M) = \int c_1(T^{1,0}M) \wedge *\omega = 0,$$

therefore $R \wedge *\omega = 0$. □

Note that in particular, if $(M, J)$ admits a nowhere vanishing top holomorphic $(3, 0)$-form $\Omega$, then $c_1(M) = 0$, and we can apply this proposition.

A natural question to ask is: when does $(T^{1,0}M, H)$ admit a hermitian-Yang-Mills connection? In the Kähler case $(T^{1,0}M, H)$ admits a hermitian-Yang-Mills connection when it is stable by the Donaldson-Uhlenbeck-Yau theorem [Don85, UY86]. In the non-Kähler case, if $(T^{1,0}M, H)$ is stable then $T^{1,0}M$ admits a hermitian-Yang-Mills connection by the Li-Yau theorem [LY87]. Note that in the non-Kähler case, the stability of $(T^{1,0}M, H)$ is not a necessary condition for $(T^{1,0}M, H)$ to admit a hermitian-Yang-Mills connection. However, it seems stability of $(T^{1,0}M, H)$ is sufficient in either cases. Finding a class of manifolds with stable $(T^{1,0}M, H)$ is a challenge in itself.

Consider when $(M, J, g, \omega)$ is Kähler and we would like to find a hermitian structure $(T^{1,0}M, H)$ which is stable. If the hermitian structure of $(T^{1,0}M, H)$ is induced from a Kähler structure $(M, J, g, \omega)$, then we have the following remark.

**Remark 25.** If the hermitian structure of $(T^{1,0}M, H)$ is induced from a Kähler structure $(M, J, g, \omega)$, then $(T^{1,0}M, H)$ is stable if and only if $(M, J, g, \omega)$ is Kähler-Einstein.

*Discussion.* When $(M, J, g, \omega)$ is Kähler and the hermitian structure on $(T^{1,0}M, H)$ is induced from the Kähler structure i.e. $H = g$, then the Levi-Civita connection on $TM$ becomes the hermitian connection on $(T^{1,0}M, g)$. We can see that the hermitian-Yang-Mills connection on $(T^{1,0}M, g)$ is equivalent to $(M, J, g, \omega)$ being

Kähler-Einstein:

$$\mathrm{tr}_\omega R = g^{i\bar{j}} R_{i\bar{j}k}{}^{m} = -g^{i\bar{j}} R_{\bar{j}ki}{}^{m},$$

where we have used the Bianchi identity. Thus

$$\mathrm{tr}_\omega R = \mu \cdot \mathbb{1} \Leftrightarrow R_{k\bar{l}} = \mu g_{k\bar{l}}.$$

By the Donaldson-Uhlenbeck-Yau theorem this is equivalent to $(T^{1,0}M, g)$ being stable with respect to $\omega$. □

However, even when a manifold does not admit a Kähler-Einstein metric, there could still exist $(T^{1,0}M, H)$ that admits a hermitian-Yang-Mills connection. From the remark above, in this case $(T^{1,0}M, H)$ will induce hermitian metric on $M$ which cannot be Kähler.

In our construction $(M, J, g_u, \omega_u)$, we show that $(T^{1,0}M, g_u)$ does not admit a hermitian-Yang-Mills connection.

**Corollary 26.** *Suppose $(M, J, g_u, \omega_u)$ satisfies the conformally balanced condition and $S$ has a positive scalar curvature. Then $(T^{1,0}M, g_u)$ does not admit a hermitian-Yang-Mills connection.*

*Proof.* We use proof by contradiction. Since $(M, J, g_u, \omega_u)$ satisfies the conformally balanced condition, we have $c_1(M) = 0$. This implies that if $T^{1,0}M$ admits a hermitian-Yang-Mills connection, it should satisfy $\mathrm{tr}_{\omega_u} R^C_M = 0$. Denote the curvature form $R^C_M$ as

$$R^C_M = \begin{pmatrix} R_{1\bar{1}} & R_{1\bar{2}} \\ R_{2\bar{1}} & R_{2\bar{2}} \end{pmatrix}.$$

Then $\mathrm{tr}_{\omega_u} R^C_M = 0$ implies $R_{1\bar{1}} \wedge \omega_u^2 = R_{2\bar{2}} \wedge \omega_u^2 = 0$. Meanwhile by Fu and Yau's computation (Proposition 23), we know that $R_M^C$ is a $(1,1)$-form on $S$. Therefore,

$$0 = R_{1\bar{1}} \wedge \omega_u^2 = R_{1\bar{1}} \wedge e^{2u}\omega_S \wedge i\theta \wedge \bar{\theta}$$

is equivalent to $R_{1\bar{1}} \wedge \omega_S = 0$ and similarly $R_{2\bar{2}} \wedge \omega_u^2 = 0$ is equivalent to $R_{2\bar{2}} \wedge \omega_S = 0$. We take the trace of these expressions and obtain:

$$\begin{aligned} \mathrm{tr} R_{1\bar{1}} \wedge \omega_S &= \mathrm{tr}\left[R_S + \bar{\partial}\left(B\partial B^* \cdot g^{-1}\right)\right] \wedge \omega_S = 0, \\ \mathrm{tr} R_{2\bar{2}} \wedge \omega_S &= \mathrm{tr}\left[-\bar{\partial}\left(\partial B^* \cdot g^{-1} B\right)\right] \wedge \omega_S = 0. \end{aligned}$$

Adding the two equations and using the cyclic property of the trace, we obtain

$$\mathrm{tr} R_S \wedge \omega_S = 0,$$

which is a contradiction to our assumption that $S$ has positive scalar curvature, i.e. $\mathrm{tr} R_S \wedge \omega_S \neq 0$. □

This implies that we cannot use the standard embedding $(V = T^{1,0}M, g_u)$ to solve the vector bundle condition in our case. This is one striking difference between non-Kähler manifolds and Calabi-Yau manifolds.

We can construct a vector bundle $V$ equipped with a hermitian metric $H$ following [FY08].

**Lemma 27** ([FY08]). *Let $(E, H)$ be a stable vector bundle over the base space $S$ with degree $0$ with respect to the Kähler metric $\omega_S$. Then $V = \pi^* E$ is a vector bundle over $M$ with degree $0$ with respect to the metric $\omega_u$. Furthermore, $\pi^* H$ is the hermitian-Yang-Mills metric on $V$.*

*Proof.* By the Donaldson-Uhlenbeck-Yau theorem, there exists a hermitian Yang-

Mills metric $H$ on $E$ with curvature $F_H$ that satisfies

$$F_H \wedge \omega_S = 0,$$

since $\deg(E) = 0$. We look into the pull-back $V = \pi^* E$ with the metric $\pi^* H$. The curvature $\pi^* F_H$ of $V$ satisfies,

$$\pi^* F_H \wedge \omega_u^2 = \pi^*(F_H \wedge \omega_S) \wedge (\pi^*(e^{2u}\omega_S) + \pi^* e^u \theta \wedge \bar{\theta}) = 0.$$

Thus $(V, \pi^* H)$ is a vector bundle with degree $0$ with respect to $\omega_u$. Furthermore, $\pi^* H$ is the hermitian-Yang-Mills metric on $V$. $\square$

K. Becker, M. Becker, Fu, Tseng, and Yau found the most general form of a vector bundle that satisfies the vector bundle condition. They prove the following Proposition.

**Proposition 28** ([BBF$^+$06]). *Let $(V, H)$ be a hermitian-Yang-Mills vector bundle over $(M, \omega_u)$ with gauge group $SU(r)$. If $(M, \omega_u, V, H)$ is a solution to the Strominger's system, then $V$ is of the following form,*

$$V = \pi^* E \otimes L,$$

*where $(E, H)$ is a hermitian-Yang-Mills vector bundle over $S$ and $L$ is a flat line bundle over $M$.*

The authors prove this proposition by considering the most general gauge field on $V$ and use the hermitian-Yang-Mills condition.

**Corollary 29.** *Let $(V, H)$ be a hermitian-Yang-Mills vector bundle over $(M, \omega_u)$ with gauge group $SU(r)$. Then the hermitian-Yang-Mills connection over $V$ is unique up to scaling.*

*Proof.* By Proposition 28, $V = \pi^* E \otimes L$. This implies the curvature can be written in the following form,

$$F = F_E \otimes \mathbb{1} + \mathbb{1} \otimes F_L.$$

We impose the hermitian-Yang-Mills condition on each term independently.

1. $F_E \otimes \mathbb{1}$**:** When imposing the hermitian-Yang-Mills condition on $F_E$ we obtain,

   $$\begin{aligned} F_E \wedge \omega_u^2 &= F_E \wedge \left( e^u \omega_S + \frac{i}{2} \theta \wedge \bar{\theta} \right)^2, \\ &= F_E \wedge e^u \omega_S \wedge i\theta \wedge \bar{\theta} = 0, \end{aligned}$$

   since $F_E$ is a pull-back of a two form on $S$. Because $\theta \wedge \bar{\theta}$ are forms on the fibre, we obtain $F_E \wedge \omega_S = 0$. This is the hermitian-Yang-Mills condition on $E$ and by the Donaldson-Uhlenbeck-Yau theorem, the connection on $E$ is unique.

2. $\mathbb{1} \otimes F_L$**:** Since the line bundle is constructed over the $T^2$ fibration of $M$, the connection on the line bundle would be of the following form

   $$A_L = p(dx + \alpha^1) + q(dy + \alpha^2),$$

   where $p, q$ are constants. The hermitian-Yang-Mills connection gives

   $$F_L \wedge \omega_u^2 = (p\beta^1 + q\beta^2) \wedge \left( e^u \omega_S + \frac{i}{2} \theta \wedge \bar{\theta} \right)^2 = 0,$$

   which implies $p\langle \beta^1, \omega_S \rangle + q\langle \beta^2, \omega_S \rangle = 0$. This expression determines $A_L$ uniquely up to scaling.

This completes the proof. □

## 3.5 Anomaly cancellation condition

In this section, we focus on the anomaly cancellation condition,

$$i\partial\bar{\partial}\omega_u = \frac{\alpha'}{4}(\mathrm{tr}R_M^u \wedge R_M^u - \mathrm{tr}F_H \wedge F_H).$$

### 3.5.1 Computation of $\mathrm{tr}R_M^u \wedge R_M^u$

Fu and Yau compute $\mathrm{tr}R_M^u \wedge R_M^u$:

**Proposition 30** ([FY08]). *Consider the curvature 2-form $R_M^u$ with respect to the Chern connection. Then we have the following equation*

$$\mathrm{tr}R_M^u \wedge R_M^u = \pi^*(\mathrm{tr}R_S^u \wedge R_S^u + 2i\partial\bar{\partial}(e^{-u}\rho)),$$

*where $\rho := -i\mathrm{tr}\{\bar{\partial}B \wedge \partial B^* \cdot g^{-1}\}$ is a well-defined 2-form.*

The computation is straight forward yet somewhat long. The details can be found in the original paper.

We can express $R_S^u$ in terms of $R_S$ and express $\mathrm{tr}R_M^u \wedge R_M^u$ entirely in terms of forms on $S$.

**Lemma 31.** $\mathrm{tr}R_M^u \wedge R_M^u$ *can be expressed in terms of the pull-back of a form on $S$ as follows*

$$\mathrm{tr}R_M^u \wedge R_M^u = \pi^*\left(2\bar{\partial}\partial u \wedge \bar{\partial}\partial u + 2\bar{\partial}\partial u \wedge \mathrm{tr}R_S + \mathrm{tr}R_S \wedge R_S + 2i\partial\bar{\partial}(e^{-u}\rho)\right).$$

*Proof.* We use the result of Proposition 30:

$$\mathrm{tr}R_M^u \wedge R_M^u = \pi^*(\mathrm{tr}R_S^u \wedge R_S^u + 2i\partial\bar{\partial}(e^{-u}\rho)).$$

By lemma 21, $\mathrm{tr}R_S^u \wedge R_S^u = 2\bar{\partial}\partial u \wedge \bar{\partial}\partial u + 2\bar{\partial}\partial u \wedge \mathrm{tr}R_S + \mathrm{tr}R_S \wedge R_S$. Substituting this expression into the expression above we derive the lemma. □

Now we compute $i\partial\bar{\partial}\omega_u$.

### 3.5.2 Computation of $i\partial\bar{\partial}\omega_u$

We express $i\partial\bar{\partial}\omega_u$ in terms of the base space.

**Lemma 32.** *$i\partial\bar{\partial}\omega_u$ can be expressed in terms of the Kähler form $\omega_S$ on the base space and the $\beta^l$s as,*

$$i\partial\bar{\partial}\omega_u = i\partial\bar{\partial}e^u \wedge \omega_S - \frac{1}{2}(\beta^1 \wedge \beta^1 + \beta^2 \wedge \beta^2).$$

*Proof.* We carry out the calculations explicitly,

$$\begin{aligned} i\partial\bar{\partial}\omega_u &= i\partial\bar{\partial}\left(e^u\omega_S + \frac{i}{2}\theta \wedge \bar{\theta}\right) \\ &= i\partial\bar{\partial}e^u \wedge \omega_S - \frac{1}{2}(\beta^1 \wedge \beta^1 + \beta^2 \wedge \beta^2) \end{aligned}$$

In the second line, we have used $d\theta = \beta^1 + i\beta^2$ and $\bar{\partial}\theta = 0$ since $d\theta$ is a $(1,1)$-form. $\square$

### 3.5.3 Anomaly cancellation condition as a PDE on $S$

In the previous sections we have proved that a stable holomorphic vector bundle $E$ with an hermitian-Yang-Mills metric $H$ over $S$ can be pulled back to $M$ and furthermore $(V = \pi^*E, \pi^*H)$ is hermitian-Yang-Mills. We also discussed a hermitian-Yang-Mills vector bundle $(V, H)$ when $V = T^{1,0}S \oplus L$. In both of the constructions, $F$ is a $(1,1)$-form on $S$. We will only consider such examples when $F \in \Gamma(\Omega^{1,1}S)$. We reduce the anomaly cancellation condition to a PDE on $S$.

**Proposition 33.** *Consider the hermitian manifold $(M, \omega_u)$ with the vector bundle $(V, H)$ such that $F \in \Gamma(\Omega^{1,1}S)$. The anomaly cancellation condition reduces to a PDE of $u$ on $S$*

*of the following form*

$$
\begin{aligned}
&i\partial\bar{\partial}e^{u}\wedge\omega_S - i\frac{\alpha'}{2}\partial\bar{\partial}u\wedge\mathcal{R}_S - i\frac{\alpha'}{2}\partial\bar{\partial}(e^{-u}\rho) - \frac{\alpha'}{2}\partial\bar{\partial}u\wedge\partial\bar{\partial}u \\
&= \frac{\alpha'}{4}\mathrm{tr}R_S\wedge R_S - \frac{\alpha'}{4}\mathrm{tr}F\wedge F + \frac{1}{2}(\beta^1\wedge\beta^1+\beta^2\wedge\beta^2).
\end{aligned}
$$

*Proof.* We use the expression derived on $i\partial\bar{\partial}u$ and $\mathrm{tr}R^u_M\wedge R^u_M$. The anomaly cancellation condition becomes

$$
\begin{aligned}
&i\partial\bar{\partial}u\wedge\omega_S - \frac{1}{2}(\beta^1\wedge\beta^1+\beta^2\wedge\beta^2) \\
&= \frac{\alpha'}{4}\left(2\partial\bar{\partial}u\wedge\partial\bar{\partial}u + 2\partial\bar{\partial}u\wedge\mathcal{R}_S + \mathrm{tr}R_S\wedge R_S + 2i\partial\bar{\partial}(e^{-u}\rho)\right) - \frac{\alpha'}{4}\mathrm{tr}F\wedge F,
\end{aligned}
$$

where we have used the definition $\mathcal{R}_S := i\mathrm{tr}R_S$. Simplifying this expression we obtain the result of this proposition. $\square$

## 3.6 Reduction of the Strominger system

Suppose we have solved the vector bundle condition using a pull-back $V=\pi^*E$ of a stable bundle $E$ over $S$. Applying Proposition 22 and Proposition 33, the Strominger system on $(M,J,g_u,\omega_u)$ reduces to two equations on the base space $S$:

1. Conformally balanced condition:

$$
\mathcal{R}_S = \sum_{l=1}^{2}\langle\omega_S,\beta^l\rangle\beta^l,
$$

   where $\beta^l$s are harmonic $(1,1)$-forms.

2. Vector bundle and the anomaly cancellation condition:

$$
\begin{aligned}
&i\partial\bar{\partial}e^{u}\wedge\omega_S - i\frac{\alpha'}{2}\partial\bar{\partial}u\wedge\mathcal{R}_S - i\frac{\alpha'}{2}\partial\bar{\partial}(e^{-u}\rho) - \frac{\alpha'}{2}\partial\bar{\partial}u\wedge\partial\bar{\partial}u\\
&= \frac{\alpha'}{4}\mathrm{tr}R_S\wedge R_S - \frac{\alpha'}{4}\mathrm{tr}F\wedge F + \frac{1}{2}\left(\beta^1\wedge\beta^1 + \beta^2\wedge\beta^2\right).
\end{aligned}
$$

Throughout this work we will commonly refer to the last equation as 'the PDE of $u$' or simply 'the PDE'. In Chapter 4, we will prove that this PDE always has a solution $u$ under some assumptions. Therefore if we can find a solution to the conformally balanced condition and vector bundle condition, then a solution of the Strominger system always exists under some mild assumptions.

### 3.6.1 Solving the conformally balanced condition

We investigate solutions of the conformally balanced condition in this section. The conformally balanced condition gives restriction on the base space $S$.

**Proposition 34.** *If $(M, J, g_u, \omega_u)$ satisfies the conformally balanced condition then the metric on the base $S$ must have zero or positive Ricci scalar curvature.*

*Proof.* The metric $g$ satisfies the conformally balanced condition, thus we have the following equation,

$$\mathcal{R}_S = \sum_{l=1}^{2}\langle\omega_S, \beta^l\rangle\beta^l.$$

We contract this with $\omega_S$ and obtain,

$$\langle\omega_S, \mathcal{R}_S\rangle = \sum_{l=1}^{2}\left(\langle\omega_S, \beta^l\rangle\right)^2 \geq 0,$$

since $\omega_S$ and $\beta^l$'s are real forms. □

This proposition implies that either $S$ should be $T^4$, K3 or have a positive Ricci scalar curvature. By the Enriques-Kodaira classification, positive Ricci scalar curvature implies that $S$ is either a rational or a ruled surface [LeB91, Sun97]. The following conjecture could be used to find examples of $S$.

**Conjecture 35.** *[LeB91] Let $S$ be a complex 2-dimensional compact Kähler surface. Then the following are equivalent.*

1. *$S$ admits a riemannian metric with positive Ricci scalar curvature.*
2. *$S$ admits a Kähler metric with positive Ricci scalar curvature.*
3. *$S$ is either rational or ruled.*

Currently the missing part of the proof is $3 \Rightarrow 2$. In our case we are interested in $2 \Rightarrow 3$ which has been proven. Using this result we come to conclude the following.

**Corollary 36.** *If $(M, J, g_u, \omega_u, \Omega)$ satisfies the conformally balanced condition, then $S$ must be either $T^4$, K3, ruled or rational surface.*

#### 3.6.1.1 Primitive solutions

The simplest solution to the conformally balanced condition is a solution with Ricci scalar curvature $0$. The following proposition investigates this case.

**Proposition 37.** *Suppose that $(M, J, g_u, \omega_u)$ satisfies the conformally balanced condition. $S$ is $T^4$ or a K3 surface if and only if $\beta^l$'s are primitive.*

*Proof.* If $\beta^l$'s are primitive then $\langle \omega_S, \beta^l \rangle = 0$, therefore $\mathcal{R}_S = 0$. Since $S$ is Kähler this implies that $S$ is $T^4$ or K3.

Conversely, suppose $S$ is $T^4$ or K3 then we have

$$0 = \mathcal{R}_S = \sum_l \langle \omega_S, \beta^l \rangle \beta^l.$$

Therefore $\beta^l$'s should be linearly dependent or primitive. If they are primitive we are done. Suppose $\beta^l$'s are linearly dependent where $\beta^2 = \kappa \beta^1$. Using this expression we obtain

$$0 = \langle \omega_S, \beta^1 \rangle \beta^1 + \kappa^2 \langle \omega_S, \beta^1 \rangle \beta^1,$$

which implies that $\langle \omega_S, \beta^1 \rangle = 0$. Therefore, $\beta^l$'s should be primitive. $\square$

This is precisely the setting in Goldstein, Prokushkin [GP04] and Fu, Yau [FY08]. We would like to relax the condition so that $S$ need not be a surface with Ricci flat metric.

#### 3.6.1.2 Hodge dual solutions

In this thesis, we generalize the work in the literature by considering the case when $\beta^1$ and $\beta^2$ are Hodge dual to each other:

$$*_g \beta^1 = \beta^2.$$

This condition will allow $S$ with $c_1(S) \neq 0$, and generate new examples of $(M, \omega_u)$. First we investigate the structures on $S$.

**Proposition 38.** *If $S$ is a 2-dimensional Kähler surface, and $\beta^l$ are closed real $(1,1)$-forms that satisfy $\beta^1 = *_g \beta^2$ then*

1. *$\beta^1$, $\beta^2$ are harmonic,*
2. *$\langle \beta^1, \omega_S \rangle = \langle \beta^2, \omega_S \rangle = \lambda$,*
3. *$\beta^1 + \beta^2 = \lambda \omega_S$,*

*where $\lambda$ is a constant (which can be zero).*

*Furthermore, if $(M, \omega_u)$ satisfies the conformally balanced condition then $\mathcal{R}_S = \lambda^2 \omega_S$; that is $(S, \omega_S)$ is Kähler-Einstein with constant positive curvature.*

*Proof.* In the following proofs we will use the fact that in real $4$-dimensions the Lefschetz decomposition theorem [Huy05, Proposition 1.2.31] tells us that for $(1,1)$-forms (1) the anti-self-dual condition is equivalent to the primitive condition, and (2) all self-dual $2$-forms are multiple of $\omega_S$.

1. $\beta^1$ and $\beta^2$ are harmonic because $\beta^l$'s are closed and Hodge dual to each other,

   $$d\beta^2 = d * \beta^1 = 0.$$

   Thus $d\beta^1 = d^*\beta^1 = 0$ and similarly $d\beta^2 = d^*\beta^2 = 0$.

2. Note that the $(1,1)$-form $*(\beta^1 - \beta^2)$ is anti-self-dual and therefore primitive;

   $$\langle \beta^1 - \beta^2, \omega_S \rangle = 0.$$

   This implies that $\langle \beta^1, \omega_S \rangle = \langle \beta^2, \omega_S \rangle = \lambda$. Because $\beta^l$'s are harmonic $(1,1)$-forms, $\lambda$ is a constant [Bes08, 2.33].

3. On the other hand, $\beta^1 + \beta^2$ is self-dual and from the Lefschetz decomposition $\beta^1 + \beta^2$ must be a multiple of $\omega_S$. Suppose that $\beta^1 + \beta^2 = f\omega_S$ for some function $f$. Then by contracting this with $\omega_S$ we obtain $f = \lambda$, therefore $\beta^1 + \beta^2 = \lambda\omega_S$.

Finally applying the facts we have derived before and assuming $(M, \omega_u)$ satisfies the conformally balanced condition we obtain

$$\begin{aligned} \mathcal{R}_S &= \langle \beta^1, \omega_S \rangle \beta^1 + \langle \beta^2, \omega_S \rangle \beta^2 = \lambda(\beta^1 + \beta^2) \\ &= \lambda^2 \omega_S. \end{aligned}$$

This completes the proof. □

Note that the Hodge dual condition $\beta^1 = *\beta^2$ reduces to the primitive condition when $\beta^1 = -\beta^2$. Thus part of the Fu and Yau's primitive solutions are included in the Hodge dual case.

Conversely, if $S$ is a Kähler-Einstein manifold with constant positive Ricci scalar curvature $4\lambda^2$, then we can find $M$ such that its hermitian form $\omega_u$ satisfies the conformally balanced condition. We can show this by choosing $\beta^1 = \beta^2 = \lambda^2\omega_S/4$. By assumption $\mathcal{R}_S = \lambda^2\omega_S$, meanwhile $\langle\beta^1, \omega_S\rangle\beta^1 + \langle\beta^2, \omega_S\rangle\beta^2 = \lambda^2\omega_S$. Therefore the conformally balanced condition is satisfied. Note that $\beta^l$'s are automatically harmonic, because $\omega_S$ is Kähler.

#### 3.6.1.3 Self-dual and anti-self-dual solution

We can also solve the conformally balanced condition by assuming $\beta^1$ is self-dual and $\beta^2$ is anti-self-dual. By the Lefschetz decomposition theorem $\beta^1 \in L(C^\infty(M))$, and we write $\beta^1 = \frac{f}{\sqrt{2}}\omega_S$.

If we would like $(M, \omega_u)$ to satisfy the conformally balanced condition on $M$, then we must have

$$\mathcal{R}_S = f^2\omega_S,$$

which gives a restriction on $S$. We consider $S$ that admits a positive scalar curvature. By the Yamabe problem, we can conformally scale the metric on $S$ so that it has constant positive scalar curvature $\lambda^2$. Therefore we will consider $(S, g_S, \omega_S)$ that is Kähler-Einstein with positive scalar curvature $\lambda^2$, and $\beta^1 = \frac{\lambda}{\sqrt{2}}\omega_S$, $\beta^2$ anti-self-dual.

We will focus mostly on the Hodge dual solutions, however we will generalize most of the results found in the Hodge dual case to this case as well. Especially, the existence of solution proved in Chapter 4 can be applied to this case as well.

#### 3.6.1.4 Other Solutions

Apart from primitive and Hodge dual solutions, in principle it is possible to construct other solutions of the conformally balanced condition. $S$ should be ruled or rational thus perhaps del Pezzo and Hirzebruch surfaces would give examples as well. However, it seems more complicated in these cases to prove the existence of a solution $u$.

### 3.6.2 Solving the anomaly cancellation condition

Recall the anomaly cancellation condition as a PDE of $u$:

$$\begin{aligned} &i\partial\bar{\partial}e^u \wedge \omega_S - i\frac{\alpha'}{2}\partial\bar{\partial}u \wedge \mathcal{R}_S - i\frac{\alpha'}{2}\partial\bar{\partial}(e^{-u}\rho) - \frac{\alpha'}{2}\partial\bar{\partial}u \wedge \partial\bar{\partial}u \\ &= \frac{\alpha'}{4}\mathrm{tr}R_S \wedge R_S - \frac{\alpha'}{4}\mathrm{tr}F \wedge F + \frac{1}{2}(\beta^1 \wedge \beta^1 + \beta^2 \wedge \beta^2). \end{aligned}$$

Since $\omega_S$ and $\mathcal{R}_S$ are closed the left hand side can be written as an exact form. Integrating over $S$ we obtain

$$\int_S \frac{\beta^1}{2\pi} \wedge \frac{\beta^1}{2\pi} + \int_S \frac{\beta^2}{2\pi} \wedge \frac{\beta^2}{2\pi} = \alpha' \left(-c_2(M) + c_2(V) - \frac{1}{2}c_1(V)^2\right), \tag{3.3}$$

where we have used $-\mathrm{tr}F\wedge F/(8\pi^2) = p_1(V) = c_1(V)^2/2 - c_2(V)$. The left hand side of this equation is the self-intersection numbers of $\beta^1/2\pi$ and $\beta^2/2\pi$. By construction, $\beta^i/2\pi$ are the Chern classes of a line bundle over $S$, thus the self-intersection numbers are integer valued. We consider $V = \pi^* E$ that is hermitian-Yang-Mills. Such hermitian-Yang-Mills vector bundles over Kähler surfaces are well studied, and we can find $E$ such that the above condition is satisfied. Suppose $E$ is stable bundle over $S$ that satisfies Equation (3.3) then we can write

$$\frac{\alpha'}{4}\mathrm{tr}R_S \wedge R_S - \frac{\alpha'}{4}\mathrm{tr}F \wedge F + \frac{1}{2}(\beta^1 \wedge \beta^1 + \beta^2 \wedge \beta^2) = -\mu\frac{\omega_S^2}{2},$$

where $\mu$ is some smooth function such that

$$\int_S \mu \frac{\omega_S^2}{2} = 0.$$

Fu and Yau showed that on $T^4$ there is no solution to the PDE of $u$. However their proof has a mistake in it. We prove the following lemma and investigate the constraints when $S = T^4$.

**Lemma 39.** *Suppose* $(V, H)$ *is a hermitian vector bundle over a hermitian* $n$*-fold* $(M, \omega)$*. If* $F \wedge *\omega = 0$*, then*

$$||F||^2 \cdot \text{vol} := \text{tr} F \wedge *F = -\text{tr} F \wedge F \wedge \frac{\omega^{n-2}}{(n-2)!}.$$

*Proof.* Since $F \wedge *\omega = 0$, the curvature 2-form is primitive. We can use Lefschetz decomposition and obtain the following

$$*F = -\frac{1}{(n-2)!}\omega^{n-2} \wedge F.$$

Thus the lemma follows. □

**Remark 40.** Suppose we want to find a solution of the Strominger system of the form $(M, \omega_u, V, H)$ when $S = T^4$. Fu and Yau originally proved that $S$ cannot be $T^4$. However, there is a mistake in the original argument. In fact, the work in [FY08] implies there exists a solution. We present below a corrected version of their argument. We wedge the PDE with $\omega_u$ and integrate over $S$ to obtain

$$\begin{aligned}\int_S \left( i\partial\bar{\partial} e^u \wedge \omega_S - i\frac{\alpha'}{2}\partial\bar{\partial} u \wedge \mathcal{R}_S - i\frac{\alpha'}{2}\partial\bar{\partial}(e^{-u}\rho) - \frac{\alpha'}{2}\partial\bar{\partial} u \wedge \partial\bar{\partial} u \right) \wedge \omega_u \\ = \int_S \left( -\frac{\alpha'}{4}\text{tr} F \wedge F + \frac{1}{2}(\beta^1 \wedge \beta^1 + \beta^2 \wedge \beta^2) \right) \wedge \omega_u,\end{aligned}$$

where we have used $R_S = 0$ for $S = T^4$. Using Stokes theorem, we show that the left hand side is zero:

$$
\begin{aligned}
&\int_S \left( i\partial\bar{\partial}e^u \wedge \omega_S - i\frac{\alpha'}{2}\partial\bar{\partial}u \wedge \mathcal{R}_S - i\frac{\alpha'}{2}\partial\bar{\partial}(e^{-u}\rho) - \frac{\alpha'}{2}\partial\bar{\partial}u \wedge \partial\bar{\partial}u \right) \wedge \omega_u \\
&\quad = -\frac{1}{2}\int_S \left( d^c e^u \wedge \omega_S - \frac{\alpha'}{2}d^c u \wedge \mathcal{R}_S - \frac{\alpha'}{2}d^c(e^{-u}\rho) + i\frac{\alpha'}{2}d^c u \wedge \partial\bar{\partial}u \right) \wedge d\omega_u = 0,
\end{aligned}
$$

since $d\omega_u \in \Gamma\left(\Omega^3 S \oplus (\Omega^2 S \wedge \Omega M)\right)$ and the other terms are in the section of $\Omega^3 S$. Meanwhile, since $\beta^i$ are anti-self dual and primitive we have

$$\beta^1 \wedge \beta^i = -\beta^i \wedge *\beta^i = -||\beta^i||^2 \cdot \text{vol}.$$

Also by Lemma 39, we have

$$-\text{tr}F \wedge F \wedge \omega_u = ||F|| \cdot \frac{\omega_u^3}{3}.$$

Therefore, the right hand side can be written as

$$\int_S \left( \frac{\alpha'}{4}||F||^2 - \frac{1}{2}(||\beta^1||^2 + ||\beta^2||^2) \right) \frac{\omega_S^2}{2} \wedge \omega_u = 0.$$

Notice the minus sign difference with the original arguments of [FY08]. By choosing $F$ and $\beta^i$s, this equation can be satisfied.

Suppose it is possible to find $V$ such that we can write the anomaly cancellation condition as

$$i\partial\bar{\partial}e^u \wedge \omega_S - i\frac{\alpha'}{2}\partial\bar{\partial}u \wedge \mathcal{R}_S - i\frac{\alpha'}{2}\partial\bar{\partial}(e^{-u}\rho) - \frac{\alpha'}{2}\partial\bar{\partial}u \wedge \partial\bar{\partial}u + \mu\frac{\omega_S^2}{2} = 0.$$

In the next section, we reduce this PDE into a simpler form by looking into special cases and generate some examples.

#### 3.6.2.1 Primitive case: Ricci-flat

In the case when $\beta^1$, and $\beta^2$ are primitive $(1,1)$-forms, $S$ should be Ricci-flat and the PDE reduces to

$$i\partial\bar{\partial}e^u \wedge \omega_S - i\frac{\alpha'}{2}\partial\bar{\partial}(e^{-u}\rho) - \frac{\alpha'}{2}\partial\bar{\partial}u \wedge \partial\bar{\partial}u + \mu\frac{\omega_S^2}{2} = 0.$$

This PDE has been extensively studied in [FY08]. In this thesis we study when $S$ is not Ricci flat.

#### 3.6.2.2 Non-Ricci flat case

In general $S$ need not be Ricci flat. However, we have shown that $S$ should be positive Ricci scalar curvature $2\lambda^2$. Using Lefschetz decomposition this implies that

$$\mathcal{R}_S = \lambda^2\omega_S + \varphi,$$

where $\varphi$ is a primitive $(1,1)$-form.

The form $\rho$ can be simplified under certain conditions. The following expression is useful when studying $\rho$.

**Proposition 41.** *The $(1,1)$-form $\rho$ is well-defined and can be expressed as the following,*

$$\rho = i\bar{\sigma}_{k\bar{j}}g^{\bar{j}i}\sigma_{\bar{l}i}dz^k \wedge dz^{\bar{l}} + \frac{\lambda^2}{2}\omega_S + \lambda(\beta^1 + \beta^2 - \lambda\omega_S),$$

*where $\sigma = \beta^1 + i\beta^2 - \frac{\lambda}{2}(i+1)\omega_S$.*

*Proof.* Recall that locally we have,

$$\beta^1 = \bar{\partial}\xi, \qquad \xi = \xi_1 dz^1 + \xi_2 dz^2,$$

$$\beta^2 = \bar{\partial}\zeta, \qquad \zeta = \zeta_1 dz^1 + \zeta_2 dz^2,$$

$$\phi = \xi + i\zeta, \qquad B = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},\ B^* = \begin{pmatrix} \bar{\phi}_1 & \bar{\phi}_2 \end{pmatrix}.$$

We start from the definition of $\rho$

$$\begin{aligned} \rho &= i\partial B^* \cdot g^{-1} \wedge \bar{\partial} B \\ &= i\partial_k \bar{\phi}_{\bar{j}} g^{\bar{j}i} \bar{\partial}_{\bar{l}} \phi_i dz^k \wedge d\bar{z}^{\bar{l}}. \end{aligned}$$

Meanwhile we can express the components of $\bar{\partial}\phi$ and $\partial\phi$ in terms of $\sigma$ and $\omega_S$.

$$\begin{aligned} \partial_k \bar{\phi}_{\bar{j}} &= \bar{\sigma}_{k\bar{j}} + \frac{\lambda}{2}(-i+1)\omega_{k\bar{j}} \\ \bar{\partial}_{\bar{l}} \phi_i &= \sigma_{\bar{l}i} + \frac{\lambda}{2}(i+1)\omega_{\bar{l}i}. \end{aligned}$$

Using these expressions we obtain

$$\begin{aligned} \rho &= i\bar{\sigma}_{k\bar{j}} g^{\bar{j}i} \sigma_{\bar{l}i} dz^k \wedge d\bar{z}^{\bar{l}} + \frac{\lambda^2}{2}\omega_S + \frac{\lambda}{2}(1-i)\sigma + \frac{\lambda}{2}(1+i)\bar{\sigma} \\ &= i\bar{\sigma}_{k\bar{j}} g^{\bar{j}i} \sigma_{\bar{l}i} dz^k \wedge d\bar{z}^{\bar{l}} + \frac{\lambda^2}{2}\omega_S + \lambda(\beta^1 + \beta^2 - \lambda\omega_S), \end{aligned}$$

where we have used

$$\frac{\lambda}{2}(1-i)\sigma + \frac{\lambda}{2}(1+i)\bar{\sigma} = \lambda(\beta^1 + \beta^2 - \lambda\omega_S).$$

This completes the proof. □

#### 3.6.2.3 Hodge dual case

When $*_g\beta^1 = \beta^2$, by Proposition 38 $\mathcal{R}_S$ reduces to the following

$$\mathcal{R}_S = \lambda^2\omega_S.$$

**Lemma 42.** *If $\beta^1 = *\beta^2$ then, $\rho$ reduces to the following*

$$\rho = i\bar{\sigma}_{k\bar{j}}g^{\bar{j}i}\sigma_{l\bar{i}}dz^k \wedge d\bar{z}^{\bar{l}} + \frac{\lambda^2}{2}\omega_S.$$

*Proof.* This is direct result of Proposition 41 and using $\beta^1 + \beta^2 = \lambda\omega_S$ from Proposition 38. ☐

The following lemma will be useful latter when we analyse the Strominger's system for the simplest cases.

**Lemma 43.** *If $\beta^1 = *\beta^2$ and $\beta^1 = n\beta^2$, then $\rho$ reduces as the following,*

$$\rho = \frac{\lambda^2}{2}\omega_S.$$

*Proof.* Since $\beta^1 = *\beta^2$ and $\beta^1 = n\beta^2$, we have $\beta^1 = \beta^2$. Applying Lemma 38, we have $\beta^1 = \beta^2 = \lambda\omega_S/2$.

Now we apply this fact to $\sigma$,

$$\begin{aligned}\sigma &= (1-i)\frac{\lambda}{2}\omega_S + \frac{\lambda}{2}(i-1)\omega_S,\\ &= 0,\end{aligned}$$

thus $\rho = \lambda^2\omega_S/2$. ☐

To summarize, in the Hodge dual case the PDE of $u$ simplifies to

$$i\partial\bar{\partial}e^{u}\wedge\omega_S - i\frac{\alpha'}{2}\partial\bar{\partial}u\wedge\lambda^2\omega_S - i\frac{\alpha'}{2}\partial\bar{\partial}(e^{-u}\rho) - \frac{\alpha'}{2}\partial\bar{\partial}u\wedge\partial\bar{\partial}u + \mu\frac{\omega_S^2}{2} = 0,$$

and furthermore when $\beta^1 = \beta^2$ it simplifies further to

$$i\partial\bar{\partial}e^{u}\wedge\omega_S - i\frac{\alpha'}{2}\partial\bar{\partial}u\wedge\lambda^2\omega_S - i\frac{\alpha'}{2}\partial\bar{\partial}e^{-u}\wedge\frac{\lambda^2}{2}\omega_S - \frac{\alpha'}{2}\partial\bar{\partial}u\wedge\partial\bar{\partial}u + \mu\frac{\omega_S^2}{2} = 0.$$

#### 3.6.2.4 Self-dual/anti-self-dual case

We also consider the case when $(S, g_S, \omega_S)$ is Kähler-Einstein with constant positive hermitian curvature $\lambda^2$

$$\mathcal{R}_S = \lambda^2\omega_S.$$

The following is a solution of the conformally balanced condition: $\beta^1 = \frac{\lambda}{2}\omega_S$, $\beta^2$ is anti-self-dual. In this case, the $\rho$ can be derived as in the following.

**Lemma 44.** *When $\beta^1 = \frac{\lambda}{2}\omega_S$ and $\beta^2$ is anti-self-dual,*

$$\rho = i(\beta^2)_{k\bar{j}}g^{\bar{j}i}(\beta^2)_{\bar{l}i} + \frac{\lambda^2}{2}\omega_S.$$

*Proof.* We use Proposition 41:

$$\begin{aligned}\rho &= i\left({\beta^1}_{k\bar{j}} - i{\beta^2}_{k\bar{j}}\right)g^{\bar{j}i}\left({\beta^1}_{\bar{l}i} + i{\beta^2}_{\bar{l}i}\right)\\ &= \frac{\lambda^2}{2}\omega_S + i{\beta^2}_{k\bar{j}}g^{\bar{j}i}{\beta^2}_{\bar{l}i} - \frac{\lambda}{\sqrt{2}}i\beta^2_{k\bar{l}} - \frac{\lambda}{\sqrt{2}}i{\beta^2}_{\bar{l}k}.\end{aligned}$$

Using the anti-symmetry property of $\beta^2$ we obtain the result. □

Note the similarity of Lemma 44 and Lemma 42. This will allow us to apply the existence proof of Chapter 4 to both Hodge dual case and self-dual/anti-self-dual case. This is of an interest because this type of construction will lead to $(k-1)(S^2 \times$

$S^4)\#k(S^3 \times S^3)$ [GGP08], and the result of Chapter 4 will guarantee a solution to the anomaly cancellation condition on $(k-1)(S^2 \times S^4)\#k(S^3 \times S^3)$ on $(M, \omega_u, V, H)$. We will discuss this further in later chapters.

## 3.7 Summary and future directions

We have discussed the Strominger system in general settings. In the following chapter we will be considering the Strominger system on $(M, \omega_u, V = \pi^* E, H)$ with a stable bundle $E$ over $S$. The conformally balanced condition will be solved with the Hodge dual solution or with the self-dual/anti-self-dual solution. Recall in this case the Strominger system with respect to the Chern connection reduces to

1. Conformally balanced condition:

$$\mathcal{R}_S = \sum_{i=1}^{2} \langle \omega_S, \beta^i \rangle \beta^i,$$

   with $\beta^i \in H^2(S, \mathbb{Z})$.

2. Vector bundle condition is satisfied by $V = \pi^* E$ and $E$ is stable over $S$.

3. Anomaly cancellation condition reduces to the following PDE:

$$i\partial\bar{\partial}e^u \wedge \omega_S - i\frac{\alpha'}{2}\partial\bar{\partial}u \wedge \lambda^2\omega_S - i\frac{\alpha'}{2}\partial\bar{\partial}(e^{-u}\rho) - \frac{\alpha'}{2}\partial\bar{\partial}u \wedge \partial\bar{\partial}u + \mu\frac{\omega_S^2}{2} = 0,$$

   when the following condition is satisfied

$$Q\left(\frac{\beta^1}{2\pi}\right) + Q\left(\frac{\beta^2}{2\pi}\right) = \alpha'(-p_1(S) + p_1(E)), \tag{3.4}$$

   where $Q(\rho_1, \rho_2) := \int_S \rho_1 \wedge \rho_2$ gives the intersection number.

In Chapter 4, we will show that the PDE always has a solution under some mild curvature assumptions. This implies that if we solve the conformally balanced condition, and find a stable vector bundle $E$ over $S$ that satisfies Equation (3.4), then the Strominger system has a solution with respect to the Chern connection. Note that Equation (3.4) can be thought of as solving the anomaly cancellation in topology on $S$, and therefore, in this case solving Equation (3.4) is sufficient and necessary for existence of an analytic solution.

# Chapter 4

# Existence Theorem

Consider the construction $(M, \omega_u, V = \pi^* E, H)$ discussed in the Chapter 3 with the Hodge dual condition $\beta^1 = *\beta^2$ or the self-dual $\beta^1 = \frac{\lambda}{2}\omega_S$ anti-self-dual $\beta^2$ condition. We have shown that the Strominger system reduces to the following:

1. Conformally balanced condition:

$$\mathcal{R}_S = \sum_{i=1}^{2} \langle \omega_S, \beta^i \rangle \beta^i,$$

   with $\beta^i \in H^2(S, \mathbb{Z})$.

2. Vector bundle condition is satisfied by assumption.
3. Anomaly cancellation condition reduces to the following PDE:

$$i\partial\bar{\partial} e^u \wedge \omega_S - i\frac{\alpha'}{2}\partial\bar{\partial} u \wedge \lambda^2 \omega_S - i\frac{\alpha'}{2}\partial\bar{\partial}(e^{-u}\rho) - \frac{\alpha'}{2}\partial\bar{\partial} u \wedge \partial\bar{\partial} u + \mu\frac{\omega_S^2}{2} = 0,$$

   when the following condition is satisfied

$$Q\left(\frac{\beta^1}{2\pi}\right) + Q\left(\frac{\beta^2}{2\pi}\right) = \alpha'(-p_1(S) + p_1(E)), \tag{4.1}$$

where $Q$ gives the self-interaction number.

In this chapter, we will show that the PDE has a solution.

In the proof of existence, we will use the continuity method first pioneered by Yau [Yau78]. We will follow the proof outlined in [Yau78, Aub70, Joy07, FY08]. The PDE is closely related to [FY08], but due to $S$ having positive scalar curvature we have a extra term. We derive estimates needed to prove existence and improve the proof of [FY08].

The fact that the anomaly cancellation condition has a solution for the Hodge dual, and self-dual/anti-self-dual solutions will imply that there exists a solution on $(k-1)(S^2 \times S^4)\#k(S^3 \times S^3)$ [GGP08]. This will be of a particular interest when $k = 2$, as it is known that $(S^2 \times S^4)\#2(S^3 \times S^3)$ admits a holomorphically trivial canonical bundle [FP09], thus it is a prime candidate for heterotic string compactification.

## 4.1 Main theorem

In the following theorem, we claim that the PDE we are interested in has a solution, and furthermore defines a new hermitian metric on $S$.

**Theorem 45.** *When $\lambda^2\alpha' < 4$, the following differential equation for $u$,*

$$i\partial\bar{\partial}e^u \wedge \omega_S - i\frac{\alpha'}{2}\partial\bar{\partial}u \wedge \lambda^2\omega_S - i\frac{\alpha'}{2}\partial\bar{\partial}(e^{-u}\rho) - \frac{\alpha'}{2}\partial\bar{\partial}u \wedge \partial\bar{\partial}u + \mu\frac{\omega_S^2}{2} = 0,$$

*has a smooth solution such that*

$$\omega' = e^u\omega_S - \frac{\alpha'}{2}\lambda^2\omega_S + \frac{\alpha'}{2}e^{-u}\rho + \alpha' i\partial\bar{\partial}u$$

*defines a hermitian metric on $S$.*

*Proof.* First we introduce $\alpha = \alpha'/2$ to simplify our notation.

**Continuity method:** We will prove Theorem 45 using the continuity method. We introduce a parameter $t$:

$$i\partial\bar{\partial}e^u \wedge \omega_S - i\alpha\lambda^2 t\partial\bar{\partial}u \wedge \omega_S - i\alpha t\partial\bar{\partial}(e^{-u}\rho) - \alpha\partial\bar{\partial}u \wedge \partial\bar{\partial}u + \mu t\frac{\omega_S^2}{2} = 0 \tag{4.2}$$

When $t = 0$, this equation has a solution $u = \text{constant}$, we want to show that it also has a solution $u$ for $t = 1$.

**Setup:** We impose the following conditions on the solution:

- Normalization:

$$\left(\int_S e^{-4u}\frac{\omega_S^2}{2}\right)^{\frac{1}{4}} = A, \quad \int_S \frac{\omega_S^2}{2} = 1. \tag{4.3}$$

- Elliptic condition:

$$\omega' = e^u\omega_S - \alpha t\lambda^2\omega_S + t\alpha e^{-u}\rho + 2i\alpha\partial\bar{\partial}u > 0. \tag{4.4}$$

The normalization is introduced to compute the estimates later.

Denote the space of functions whose $k$-derivatives are Hölder continuous with exponent $0 < \alpha_0 < 1$ as $C^{k,\alpha_0}(S)$. Consider the following sets,

- $B_A$ is a set of $u \in C^{5,\alpha_0}(S)$ that satisfies the normalization condition (4.3).
- $T$ is a set of $t$ such that the differential equation (4.2) has a solution $u$ which satisfies the the normalization condition (4.3) and the elliptic condition (4.4).

Note that $0 \in T$ with $u = -\ln A$, and also $T \subset [0, 1]$. If we can show that $T$ is open and closed in $[0, 1]$ then this implies $1 \in T$ and the PDE has a solution.

**Open:** Consider a pair $(t, u) \in [0, 1] \times B_A$. We define a natural map $\widetilde{K}$ by the following

$$\widetilde{K}(t,u) := *_{\omega_S}\left(i\partial\bar{\partial}e^u \wedge \omega_S - i\alpha t\lambda^2\partial\bar{\partial}u \wedge \omega_S - i\alpha t\partial\bar{\partial}(e^{-u}\rho) - \alpha\partial\bar{\partial}u \wedge \partial\bar{\partial}u + \mu t\frac{\omega_S^2}{2}\right).$$

It is straight forward to verify that $\int_S \widetilde{K}(t,u) = 0$, as $\int_S \mu = 0$ and $S$ has no boundaries. Thus it is natural to introduce the following set $C_0^{3,\alpha_0}(S) = \{\psi \in C^{3,\alpha_0}; \int_S \psi = 0\}$ and we can view the map $\widetilde{K}$ as $\widetilde{K} : [0,1] \times B_A \to C_0^{3,\alpha_0}(S)$.

Now consider a pair $(t_0, u_{t_0})$ where $t_0 \in T$ and $u_{t_0} \in B_A$ is a solution to the PDE with $t = t_0$. If we can show that $\widetilde{K}$ maps an open neighbourhood $U_0$ of $(t_0, u_{t_0}) \in [0,1] \times B_A$ to an open neighbourhood $V$ of $0 \in C_0^{3,\alpha_0}(S)$ then this will imply that $T$ is open. This is because $\widetilde{K}$ will map an arbitrary element $(t, u_t) \in U_0$ to $\psi \in C_0^{3,\alpha_0}(S)$, and this will imply that

$$i\partial\bar{\partial}e^{u_t} \wedge \omega_S - i\alpha t\lambda^2\partial\bar{\partial}u_t \wedge \omega_S - i\alpha t\partial\bar{\partial}(e^{-u_t}\rho) - \alpha\partial\bar{\partial}u_t \wedge \partial\bar{\partial}u_t + \mu' t\frac{\omega_S^2}{2} = 0,$$

where $\mu' = \mu - \psi$. Thus $(t, u_t)$ is also a solution, implying $T$ is open.

We show that $\widetilde{K}$ is a homeomorphism using the implicit function theorem. For $(t_0, u_{t_0}) \in \ker\widetilde{K}$, we write the linearisation of $\widetilde{K}$ as $K := d\widetilde{K}$. Explicitly $K(\phi)$ is defined as

$$K(\phi) := *_{\omega_S}\left[i\partial\bar{\partial}(e^{u_{t_0}}\phi) \wedge \omega_S - i\alpha t_0\lambda^2\partial\bar{\partial}\phi \wedge \omega_S + i\alpha t_0\partial\bar{\partial}(e^{-u_{t_0}}\phi\rho) - 2\alpha\partial\bar{\partial}u_{t_0} \wedge \partial\bar{\partial}\phi\right].$$

Note that the principle part of $*_{\omega_S}K$ is

$$i\partial\bar{\partial}\phi \wedge (e^{u_{t_0}}\omega_S - \alpha t_0\lambda^2\omega_s + t_0\alpha e^{-u_{t_0}}\rho + 2\alpha i\partial\bar{\partial}u_{t_0}),$$

and by the elliptic condition it follows that $K$ is elliptic. We use the fact that $K$ is elliptic with Lemma 46 to show that $K$ is invertible. By the implicit function theorem this implies that $\widetilde{K}$ is homeomorphic at $u_{t_0}$.

- **One-to-one:** By Lemma 46, $\phi \in \ker K$ is a no where zero function that has constant sign. Therefore $\ker K \cap T_{u_{t_0}} B_A = 0$ and $K$ is one-to-one.
- **Onto:** We show that $K$ is onto by realizing for any $\psi \in C_0^{3,\alpha_0}(S)$ there exists a weak solution $\phi_1$ of $K(\phi_1) = \psi$ because $\psi \perp \ker K^*$ by Lemma 46. According to Schauder theory, $\phi_1 \in C^{5,\alpha_0}(S)$ when $\psi \in C_0^{3,\alpha_0}(S)$. If we take $c_0 = -\frac{\int e^{-4u_{t_0}}\phi_1}{\int e^{-4u_{t_0}}\phi_0}$, then $\phi_1 + c_0\phi_0 \in T_{u_{t_0}} B_A$ and $K(\phi_1 + c_0\phi_0) = \psi$, thus $K$ is onto.

Since $d\widetilde{K}_{(t_0,u_{t_0})}$ is a bijection, $\widetilde{K}$ is a homeomorphism and this implies $T$ is open.

**Closed:** We show that $T$ is closed by showing that it contains its limit points. More explicitly, suppose $(t_s)$ is a sequence in $T$ that converges to $\lim_{s\to\infty} t_s = t'$.

Consider the PDE in the determinant form $\det g'/\det g = F_{t,u_t,du_t}$. This form will be discussed in detail in Lemma 47. Here $F_{t,u_t,du_t}$ is a smooth function of $t$, $u_t$, and the first order derivatives of $u_t$. We take a derivative with respect to $\partial_k$ and obtain

$$\begin{aligned}
&2\alpha i \det\left(e^{u_t} i g_{i\bar{j}} - \alpha t \lambda^2 i g_{i\bar{j}} + t\alpha e^{-u_t}\rho_{i\bar{j}} + 2\alpha i u_{t,i\bar{j}}\right) g'^{\bar{j}i} \frac{\partial^2}{\partial z^i \partial z^{\bar{j}}} \left(\frac{\partial u_t}{\partial z^k}\right) \\
&\quad = -\det g' g'^{\bar{j}i} \frac{\partial}{\partial z^k}\left(e^{u_t} i g_{i\bar{j}} - \alpha t \lambda^2 i g_{i\bar{j}} + t\alpha e^{-u_t}\rho_{i\bar{j}}\right) + \frac{\partial}{\partial z^k}\left(\det g \cdot F_{t,u_t,du_t}\right).
\end{aligned}$$

We consider this PDE with $t_s$ in place of $t$ and use the bootstrap argument. In Proposition 63 and Proposition 91, we show that if $u \in C^4(S)$ then $g'_{i\bar{j}}$ is positive and bounded away from zero. This implies that the PDE above is uniformly elliptic. In Proposition 74 and Proposition 91, we show that if $u \in C^5(S)$, then there are a priori estimates of $u_{i\bar{j}k}$, this implies that the coefficients of the PDE above are at least $C^{0,\alpha}(S)$ for any $\alpha \in (0,1)$. Using Schauder estimates discussed in Chapter 2 we obtain $C^{2,\alpha}(S)$ a priori estimates on $\partial u/\partial z^k$. This tells us that the coefficients are now in $C^{1,\alpha}(S)$. Repeating this process we obtain that $||u_{t_s}||_{C^{5,\alpha}} \leq C$ for some constant $C$.

Since the sequence $u_{t_s}$ is bounded in $C^{5,\alpha}(S)$ the Rellich-Kondrachov theorem implies that the sequence $u_{t_s}$ should lie in a compact subset of $C^5(S)$. Therefore there should exists a subsequence $u_{t_c}$ that converges to $u$ in $C^5(S)$.

Because $u_{t_c}$ converges to $u$ in $C^5(S)$, we can take the limit of the PDE above with respect to this subsequence $\lim_{c\to\infty} t_c \to t'$ and this tells us that $u$ will be a solution of the PDE with $t = t'$. Since $u \in C^5(S)$, we can repeat the bootstrap procedure and conclude $u \in C^{5,\alpha}(S)$. Therefore, $T$ contains its limit points and it is closed.

**Regularity:** We have showed that the PDE has a solution $u \in C^{5,\alpha}(S)$. By repeating the bootstrap process we can see that this in fact implies that $u \in C^{\infty}(S)$. Therefore, $g'$ is a smooth hermitian metric on $S$. □

In the following sections, we give a detailed proof of lemmas and theorems that are required to prove Theorem 45. We use the notation established in the proof of Theorem 45 throughout this section.

## 4.2 To show $T$ is open.

The following lemma is used to prove that $T$ is open.

**Lemma 46.** *The operator $K$ defined in Theorem 45 has the following properties,*

- $\ker K^* = \mathbb{R}$,
- $\ker K = \{\mathbb{R}\phi_0; \phi_0$ *is a no where zero function that has constant sign*$\}$.

*Proof.* We would like to use Corollary (7.29) of [LT95] that states properties of an elliptic operator $\triangle'$. Define $\omega'_{t_0}$ to be the principal part of $*_{\omega_S} K$:

$$i\partial\bar{\partial}\phi \wedge \omega'_{t_0} := i\partial\bar{\partial}\phi \wedge (e^{u_{t_0}}\omega_S - \alpha t_0 \lambda^2 \omega_S + t_0 \alpha e^{-u_{t_0}}\rho + 2i\alpha\partial\bar{\partial}u_{t_0}).$$

Then we define an elliptic operator $\triangle'$ associated with $K$ in the following way,

$$\triangle' := i\Lambda_{\omega'_{t_0}}\partial\bar{\partial},$$

where $\Lambda_{\omega'_{t_0}}$ is the dual Lefschetz operator. $\triangle'$ is an elliptic operator because $\omega'_{t_0} > 0$. $\triangle'$ is smooth because we are assuming that $u_{t_0}$ is a solution of the PDE and $\rho$, $\mu$ are smooth.

We use the definition of a laplacian $i\partial\bar{\partial}\psi \wedge \omega'_{t_0} = \triangle'(\psi)\frac{\omega'^2_{t_0}}{2}$ to establish $\ker \triangle'^* = \ker K$,

$$\begin{aligned}
&\int K^*(\psi)\phi\frac{\omega_S^2}{2} \\
&\quad = \int \psi\left[i\partial\bar{\partial}(e^{u_{t_0}}\phi) \wedge \omega_S - i\alpha t_0 \lambda^2 \partial\bar{\partial}\phi \wedge \omega_S + i\alpha t\partial\bar{\partial}(e^{-u_{t_0}}\phi\rho) - 2\partial\bar{\partial}u_{t_0} \wedge \partial\bar{\partial}\phi\right] \\
&\quad = \int \phi i\bar{\partial}\partial\psi \wedge (e^{u_{t_0}}\omega_S - \alpha t_0\lambda^2\omega_S + t_0\alpha e^{-u_{t_0}}\rho + 2i\alpha\partial\bar{\partial}u_{t_0}) \\
&\quad = -\int \triangle'(\psi)\phi\frac{\omega'^2_{t_0}}{2}.
\end{aligned}$$

Therefore $\ker \triangle'^* = \ker K$.

Corollary (7.29) of [LT95] states that every function $f \in \ker \triangle'^*$ is nowhere zero and has a constant sign; and $\ker \triangle' = \mathbb{R}$. Therefore we obtain this lemma. □

## 4.3 Simplification of the PDE and an Overview

The PDE can be written in several equivalent ways. Depending on the context, we will choose the most convenient form of the PDE. In the following lemma we introduce the first equivalent formulation.

**Lemma 47.** *The PDE,*

$$i\partial\bar{\partial}e^u \wedge \omega_S - i\alpha\lambda^2 t\partial\bar{\partial}u \wedge \omega_S - i\alpha t\partial\bar{\partial}(e^{-u}\rho) - \alpha\partial\bar{\partial}u \wedge \partial\bar{\partial}u + \mu t\frac{\omega_S^2}{2} = 0,$$

*is equivalent to*

$$\frac{\det g'}{\det g} = F_{t,u_t,du_t},$$

*where $g'$ is defined by $\omega' = e^u\omega_S - \alpha\lambda^2 t\omega_S + \alpha t e^{-u}\rho + 2\alpha i\partial\bar{\partial}u$, and $F_{t,u_t,du_t}$ only depends on $t$ and up to the first order differentiation of $u_t$.*

*Proof.* The easiest way to show this is to use the following identity,

$$\frac{\omega'^2}{2} = \frac{\det g'}{\det g}\frac{\omega^2}{2}.$$

We compute $\omega'^2$ explicitly and obtain:

$$\begin{aligned}\omega'^2 &= (e^u - \alpha\lambda^2 t)^2\omega_S^2 + \alpha^2 t^2 e^{-2u}\rho^2 - 4\alpha^2\partial\bar{\partial}u \wedge \partial\bar{\partial}u \\ &\quad + 2\alpha e^{-u}(e^u - \alpha\lambda^2 t)\omega_S \wedge \rho t + 4\alpha^2 e^{-u} t\rho \wedge i\partial\bar{\partial}u + 4\alpha(e^u - \alpha\lambda^2)\omega_S \wedge i\partial\bar{\partial}u.\end{aligned}$$

By using the PDE and contracting with $\omega^2/2$, and we get the following expression:

$$\begin{aligned}\frac{\det g'}{\det g} &= (e^u - \alpha\lambda^2 t)^2 + \frac{\alpha^2}{2}e^{-2u}\left\langle t^2\rho^2, \frac{\omega_S^2}{2}\right\rangle + \alpha(1 - \alpha\lambda^2 t e^{-u})\left\langle t\rho \wedge \omega_S, \frac{\omega_S^2}{2}\right\rangle \\ &\quad - 2\alpha e^u\left\langle i\partial u \wedge \bar{\partial}u \wedge \omega_S, \frac{\omega_S^2}{2}\right\rangle + 2\alpha^2 e^u\left\langle i\partial u \wedge \bar{\partial}u \wedge t\rho, \frac{\omega_S^2}{2}\right\rangle \\ &\quad + 2\alpha^2 e^{-u}\left\langle i\bar{\partial}u \wedge t\partial\rho, \frac{\omega_S^2}{2}\right\rangle - 2\alpha^2 e^{-u}\left\langle i\partial u \wedge t\bar{\partial}\rho, \frac{\omega_S^2}{2}\right\rangle \\ &\quad + 2\alpha^2 e^{-u}\left\langle i\partial\bar{\partial}\rho, \frac{\omega_S^2}{2}\right\rangle - 2\alpha\mu t \\ &:= F_{t,u_t,du_t}.\end{aligned}$$

This proves the lemma. $\square$

If $\beta^1 = n\beta^2$, then $\rho$ simplifies and we can write the PDE explicitly in the following way.

**Lemma 48.** *If $\beta^1 = n\beta^2$, the PDE simplifies and can be written in the following three equivalent forms:*

- *In differential forms:*

$$i\partial\bar{\partial}\left(e^u - \alpha\lambda^2 tu - \frac{\alpha\lambda^2 t}{2}e^{-u}\right) \wedge \omega_S - \alpha\partial\bar{\partial}u \wedge \partial\bar{\partial}u + \mu\frac{\omega_S^2}{2} = 0. \tag{4.5}$$

- *In a complex Monge-Ampère-type equation:*

$$\frac{\det g'}{\det g} = F_{t,u_t,du_t}.$$

- *In components:*

$$g^{\bar{j}i}\partial_i\partial_{\bar{j}}\left(e^u - \alpha\lambda^2 tu - \frac{\alpha\lambda^2 t}{2}e^{-u}\right) + 2\alpha\frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}} + \mu = 0. \tag{4.6}$$

*Proof.* When $\beta^1 = n\beta^2$, we use the fact that $\rho$ reduces to $\rho = \frac{\lambda^2}{2}\omega_S$. We conclude that the PDE reduces to Equation (4.5).

Equation (4.6) can be derived by using the following identity,

$$\left\langle \partial\bar{\partial}u \wedge \partial\bar{\partial}u, \frac{\omega_S^2}{2} \right\rangle = -2\frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}},$$

and the definition of the laplacian,

$$i\partial\bar{\partial}f \wedge \omega_S = \triangle f\,\frac{\omega_S^2}{2}.$$

Using these identities in Equation (4.5) we obtain Equation (4.6). $\square$

When $\beta^1 = n\beta^2$ we will commonly refer to the PDE as the ′simplified PDE′ or just ′the PDE′ if it doesn′t cause any confusion. We first consider the simplified PDE and show it has a solution. Later we will extend the result to the general PDE.

## 4.4 Zeroth order estimate

In this section, we obtain a lower bound and an upper bound of $u$ subjected to the following conditions

- Normalization:

$$\left(\int_S e^{-4u}\frac{\omega_S^2}{2}\right)^{\frac{1}{4}} = A, \quad \int_S \frac{\omega_S^2}{2} = 1, \quad (4.3)$$

- Elliptic condition:

$$\omega' = e^u\omega_S - \alpha\lambda^2\omega_S + \alpha\frac{\lambda^2}{2}e^{-u}\omega_S + 2i\alpha\partial\bar{\partial}u > 0. \quad (4.4)$$

Naively, we can try to write the simplified PDE (4.5) in terms of the simplified PDE of Fu and Yau [FY08]:

$$i\partial\bar{\partial}\left(e^u - \alpha f e^{-u}\right)\wedge\omega_S - \alpha\partial\bar{\partial}u\wedge\partial\bar{\partial}u + \mu\frac{\omega_S^2}{2} = 0,$$

with $f = \lambda^2 e^u u + \lambda^2 t/2$. However this approach is not fruitful as the elliptic condition (4.4) does not translate directly into the elliptic condition of [FY08]. We work out the bounds of $u$ following the ideas of [FY08].

First we establish some conventions that will be used. We replace $\lambda^2 t$ by $\lambda^2$ and $\mu t$ by $\mu$ to simplify the notation. A subscript on a function will denote its partial derivative, i.e. $u_i := \partial_i u$. Also we introduce the following shorthand notation, $|\nabla u|^2 = g^{i\bar{j}}u_i u_{\bar{j}}$ and $\triangle u = g^{i\bar{j}}u_{i\bar{j}}$.

We show that $u$ is bounded by using integration by parts.

**Lemma 49.** *Suppose $\omega'$ satisfies the elliptic condition $\omega' > 0$, then for any $k \in \mathbb{R}^+$,*

$$\int i\partial\bar{\partial}(e^{-ku}) \wedge \omega' \geq -k \int e^{-ku} i\partial\bar{\partial}u \wedge \omega'. \tag{4.7}$$

*Equivalently, for $p \in \mathbb{R}^+$ we have,*

$$\int i\partial\bar{\partial}(e^{pu}) \wedge \omega' \geq p \int e^{pu} i\partial\bar{\partial}u \wedge \omega'. \tag{4.8}$$

*Proof.* Recall the definition of the laplacian:

$$\triangle\psi \frac{\omega_S^2}{2} := i\partial\bar{\partial}\psi \wedge \omega_S = g^{i\bar{j}}\partial_i\bar{\partial}_{\bar{j}}\psi \frac{\omega_S^2}{2}.$$

Since $\omega' > 0$, it can be associated to a metric $g'$. The inequality comes from the following computation,

$$\begin{aligned} \int i\partial\bar{\partial}(e^{-ku}) \wedge \omega' &= k^2 \int e^{-ku} g'^{i\bar{j}} u_i u_{\bar{j}} \frac{\omega'^2}{2} - k \int e^{-ku} g'^{i\bar{j}} u_{i\bar{j}} \frac{\omega'^2}{2}, \\ &\geq -k \int e^{-ku} g'^{i\bar{j}} u_{i\bar{j}} \frac{\omega'^2}{2}, \\ &= -k \int e^{-ku} i\partial\bar{\partial}u \wedge \omega'. \end{aligned}$$

The second inequality is obtained by replacing $k$ with $-p$. This completes the proof. □

### 4.4.1 Estimate of inf u

We show that $u$ is bounded by below using Inequality (4.7). The following lemma is obtained by substituting $\omega'$ in the right hand side of Inequality (4.7) and using integration by parts,

**Lemma 50.** *For a solution of the PDE under the elliptic condition $\omega' > 0$, we have the following inequality for $|\nabla u|$,*

$$k \int \left(e^{-u}\right)^{k-1} |\nabla u|^2 \leq C \int \left(e^{-u}\right)^{k+1} + C \int \left(e^{-u}\right)^{k},$$

*where $C$ is a constant depending on $\alpha'\lambda^2$, and $\mu$.*

*Proof.* First we substitute $\omega'$ on the right hand side of Inequality (4.7)

$$\begin{aligned}
-k \int e^{-ku} i\partial\bar{\partial}u \wedge \omega' =& - k \int e^{-ku} i\partial\bar{\partial}u \wedge \left(e^{u}\omega_S - \alpha\lambda^2\omega_S + e^{-u}\alpha\frac{\lambda^2}{2}\omega_S\right) \\
&+ 2ik \int e^{-ku} \left(\partial\bar{\partial}e^{u} - \alpha\lambda^2\partial\bar{\partial}u - \alpha\frac{\lambda^2}{2}\partial\bar{\partial}e^{-u}\right) \wedge \omega_S \\
&+ 2k \int e^{-ku}\mu,
\end{aligned}$$

where we have used the PDE (Equation (4.5)). This simplifies further using the Leibniz rule,

$$\begin{aligned}
-k \int e^{-ku} i\partial\bar{\partial}u \wedge \omega' =& k \int e^{-ku} \left(e^{u} - \alpha\lambda^2 + \alpha\frac{\lambda^2}{2}e^{-u}\right) i\partial\bar{\partial}u \wedge \omega_S \\
&+ 2k \int e^{-ku} \left(e^{u} - \alpha\frac{\lambda^2}{2}e^{-u}\right) i\partial u \wedge \bar{\partial}u \wedge \omega_S + 2k \int e^{-ku}\mu.
\end{aligned}$$

Meanwhile the left hand side can be computed similarly,

$$\begin{aligned}
\int i\partial\bar{\partial}(e^{-ku}) \wedge \omega' &= \int \left(e^{u} - \alpha\lambda^2 + \alpha\frac{\lambda^2}{2}e^{-u}\right) \triangle(e^{-ku}), \\
&= \int e^{-ku} \left(e^{u} - \alpha\lambda^2 + \alpha\frac{\lambda^2}{2}e^{-u}\right) (k^2|\nabla u|^2 - k\triangle u).
\end{aligned}$$

Combining the two expressions, we obtain the following,

$$\begin{aligned}
&k^2\int e^{-ku}\left(e^u-\alpha\lambda^2+\alpha\frac{\lambda^2}{2}e^{-u}\right)|\nabla u|^2 \qquad (4.9)\\
&\quad\geq 2k\int e^{-ku}\left(e^u-\alpha\lambda^2+\alpha\frac{\lambda^2}{2}e^{-u}\right)\triangle u+2k\int e^{-ku}\left(e^u-\alpha\frac{\lambda^2}{2}e^{-u}\right)|\nabla u|^2\\
&\qquad+2k\int e^{-ku}\mu.
\end{aligned}$$

By integrating by parts on $2\int e^{-ku}(e^u-\alpha\lambda^2+\alpha\frac{\lambda^2}{2}e^{-u})\triangle u$, we conclude

$$\begin{aligned}
&2\int e^{-ku}\left(e^u-\alpha\lambda^2+\alpha\frac{\lambda^2}{2}e^{-u}\right)\triangle u\\
&\quad=2(k-1)\int e^{-(k-1)u}|\nabla u|^2-2\alpha\lambda^2k\int e^{-ku}|\nabla u|^2+2(k+1)\int\alpha\frac{\lambda^2}{2}e^{-(k+1)u}|\nabla u|^2.
\end{aligned}$$

We substitute this expression into Inequality (4.9) and simplify,

$$\begin{aligned}
-2\int e^{-ku}\mu &\geq k\int e^{-(k-1)u}|\nabla u|^2+k\int\alpha\frac{\lambda^2}{2}e^{-(k+1)u}|\nabla u|^2-\alpha\lambda^2k\int e^{-ku}|\nabla u|^2\\
&=k\left(1-\alpha\frac{\lambda^2}{2}\right)\int e^{-(k-1)u}|\nabla u|^2+\frac{\lambda^2}{2}k\alpha\int\left(e^{-u/2}-e^{u/2}\right)^2e^{-ku}|\nabla u|^2\\
&\geq kC^{-1}\int e^{-(k-1)u}|\nabla u|^2.
\end{aligned}$$

The last inequality follows because the second term is positive and we assume $4-\alpha'\lambda^2>0$, thus $C$ is a constant depending on $\alpha'\lambda^2$. We use $C$ to denote a generic constant from now. This inequality implies that,

$$k\int e^{-(k-1)u}|\nabla u|^2\leq C\int e^{-ku},$$

where $C$ now depends on $\alpha\lambda^2$ and $\mu$. We weaken this inequality, because in general when $\beta^1\neq\beta^2$, we do not obtain such strong inequality. The above inequality can be weaken to,

$$k\int e^{-(k-1)u}|\nabla u|^2\leq C\int e^{-ku-u}+C\int e^{-ku}.$$

This completes the proof. $\square$

Using the Sobolev inequality and the Hölder inequality we can obtain the following inequality from the lemma above.

**Lemma 51.** *For a solution $u$ of the PDE under the elliptic condition $\omega' > 0$ and normalization $\int_S \frac{\omega_S^2}{2} = 1$, the following inequality holds,*

$$\left(\int (e^{-u})^{2k}\right)^{\frac{1}{2}} \leq C\left(\int (e^{-u})^{k+2}\right)^{\frac{k}{k+2}} + Ck\left(\int (e^{-u})^{k+2}\right)^{\frac{k+1}{k+2}} + Ck\int (e^{-u})^{k+2}, \quad (4.10)$$

*where $C$ depends on $\alpha'\lambda^2, \mu$ and the Sobolev constant of $g$.*

*Proof.* The Sobolev embedding theorem (Theorem 12) states that the following embedding is continuous,

$$L_k^p(\mathbb{R}^n) \subseteq L_l^q(\mathbb{R}^n)$$

when $\frac{1}{q} = \frac{1}{p} - \frac{k-l}{n}$. Consider the Sobolev inequality for $n = 4, l = 1, k = 0, q = 2$, and $p = 4$. This leads to the following inequality,

$$\left[\int \left(e^{-\frac{k}{2}u}\right)^4\right]^{1/4} \leq C_S\left[\left(\int \left(e^{-\frac{k}{2}u}\right)^2\right)^{1/2} + \left(\int \left|\nabla e^{-\frac{k}{2}u}\right|^2\right)^{1/2}\right],$$

where $C_S$ is the Sobolev constant. By squaring, and using Lemma 50 with the inequality of arithmetic and geometric means, we derive the following,

$$\begin{aligned}\left(\int (e^{-u})^{2k}\right)^{1/2} &\leq C\left[\int \left(e^{-\frac{k}{2}u}\right)^2 + \int \left|\nabla e^{-\frac{k}{2}u}\right|^2\right] \\ &\leq C\left[\int e^{-ku} + k\int \left(e^{-u}\right)^{k+2} + k\int \left(e^{-u}\right)^{k+1}\right],\end{aligned}$$

where $C$ depends on $\alpha'\lambda^2$, $\mu$, and $C_S$. Now using Hölder's inequality, we finally obtain

$$\left(\int (e^{-u})^{2k}\right)^{\frac{1}{2}} \leq C\left(\int (e^{-u})^{k+2}\right)^{\frac{k}{k+2}} + Ck\left(\int (e^{-u})^{k+2}\right)^{\frac{k+1}{k+2}} + Ck\int (e^{-u})^{k+2}.$$

This completes the proof. $\square$

To obtain $\inf u$, the strategy we use is to reiterate Inequality (4.10). It is easy to see that when $k = 2$, Inequality (4.10) cannot be iterated any more. This gives a natural motivation to define the normalization of the following integral,

$$A := \left(\int e^{-4u}\right)^{1/4} < 1.$$

We finally state the following proposition concerning $\inf u$.

**Proposition 52.** *If $u$ is a solution of the PDE under the elliptic condition $\omega' > 0$, and normalizations $A < 1$, $\int \frac{\omega_S^2}{2} = 1$, then we have,*

$$\inf_S u \geq -\log(C_1 A),$$

*where $C_1$ is a constant that depends on $\alpha'\lambda^2, \mu$ and the Sobolev constant of $g$.*

*Proof.* We use Inequality (4.10) and reiterate it to obtain the limit. Regarding the integral there are two cases, (1) when $\int (e^{-u})^k \leq 1$ for all $k \geq 4$ and (2) when there exists some $k_0$ such that $\int (e^{-u})^{k_0} > 1$. Hölder inequality tells us that if $\int (e^{-u})^{k_0} > 1$ then $\int (e^{-u})^k > 1$ for all $k \geq k_0$. We first focus on the first case as it is simpler and then deal with the second one.

1. When for all $k \geq 4$, $\int (e^{-u})^k \leq 1$, Inequality (4.10) implies

$$\left(\int (e^{-u})^{2k}\right)^{1/2} \leq Ck\left(\int \left(e^{-u}\right)^{k-2}\left(e^{-u}\right)^{4}\right)^{\frac{k}{k+2}}$$
$$\leq Ck\left[\left(\int (e^{-u})^{k}\right)^{\frac{k-2}{k}}\left(\int (e^{-u})^{2k}\right)^{\frac{2}{k}}\right]^{\frac{k}{k+2}},$$

   by the Hölder inequality. We rearrange the terms and obtain

$$\int \left(e^{-u}\right)^{2k} \leq Ck^{2\cdot\frac{k+2}{k-2}}\left(\int \left(e^{-u}\right)^{k}\right)^{2}$$
$$\leq Ck^{2}\left(\int \left(e^{-u}\right)^{k}\right)^{2}.$$

   By taking $k = 2^\beta$, and the power $1/2^{\beta+1}$ on both sides we have

$$\left(\int \left(e^{-u}\right)^{2^{\beta+1}}\right)^{\frac{1}{2^{\beta+1}}} \leq C(2^{\beta})^{\frac{1}{2^{\beta+1}}}\left(\int (e^{-u})^{2^{\beta}}\right)^{\frac{1}{2^{\beta}}}$$
$$\leq \cdots$$
$$\leq C(2^{\beta})^{\frac{1}{2^{\beta+1}}}(2^{\beta-1})^{\frac{1}{2^{\beta}}}\cdots(2^{2})^{\frac{1}{2^{3}}}\left(\int \left(e^{-u}\right)^{4}\right)^{\frac{1}{4}}.$$

   When taking the limit $\beta \to \infty$, we can check the constant converges to some $C_1$ by the ratio test. Therefore,

$$||e^{-u}||_{\infty} \leq C_1 A,$$

   where $C_1$ depends on $\alpha'\lambda^2, \mu$ and the Sobolev constant of $\omega_S$.

2. Suppose $\int (e^{-u})^k > 1$ for $k \geq k_0$. The strategy is to reduce $k$ until $k < k_0$ so that $\int (e^{-u})^k \leq 1$. Once this is achieved we can use the results described above.

Suppose we start from $k \geq k_0$ and $\int (e^{-u})^k > 1$. Inequality (4.10) implies,

$$\begin{aligned}\left(\int (e^{-u})^{2k}\right)^{1/2} &\leq Ck \int (e^{-u})^{k+2}, \\ &\leq Ck \left(\int (e^{-u})^k\right)^{\frac{k-2}{k}} \left(\int (e^{-u})^{2k}\right)^{\frac{2}{k}},\end{aligned}$$

by the Hölder inequality. By taking $k = 2^\beta$ and iterating this inequality we obtain

$$\begin{aligned}\left(\int \left(e^{-u}\right)^{2^{\beta+1}}\right)^{\frac{1}{2^{\beta+1}-4}} &\leq C(2^\beta)^{\frac{1}{2^{\beta+1}-4}} \left(\int \left(e^{-u}\right)^{2^\beta}\right)^{\frac{1}{2^\beta-4}} \\ &\leq \cdots \\ &\leq C(2^\beta)^{\frac{1}{2^{\beta+1}-4}} (2^{\beta-1})^{\frac{1}{2^\beta-4}} \cdots (2^{\beta_0})^{\frac{1}{2^{\beta_0+1}-4}} \left(\int \left(e^{-u}\right)^{2^{\beta_0}}\right)^{\frac{1}{2^{\beta_0}-4}},\end{aligned}$$

where $\beta_0$ is the first integer so that $2^{\beta_0} < k_0$, i.e. $\int (e^{-u})^{2^\beta_0} < 1$, and $C$ is a generic constant that depends on $\alpha'\lambda^2$, $\mu$, the Sobolev constant of $g$, $k$ and $k_0$. When we reach $\beta_0$, we proceed on with the method described in the first case, to reduce the power of $e^{-u}$ on the right hand side. This will result in

$$\left(\int \left(e^{-u}\right)^{2^{\beta+1}}\right)^{\frac{1}{2^{\beta+1}}} \leq CA^{\frac{2^{\beta_0}}{2^{\beta_0}-4} \cdot \frac{2^{\beta+1}-4}{2^{\beta+1}}},$$

where $C$ now also depends on $\beta$.

As we did in the first case we take $\beta \to \infty$, it can be shown that $C$ converges to a constant using the ratio test, and since we assume $A < 1$, this implies that

$$||e^{-u}||_\infty \leq C_1 A,$$

where we defined $C_1$ and used the same notation as before.

Therefore we obtain $\inf_S u \geq -\log(C_1 A)$ in both cases. □

### 4.4.2 Estimate of sup u

We follow the same procedure with $k$ replaced with $-p$. Inequality (4.9) becomes,

$$\begin{aligned}
&p^2\int e^{pu}\left(e^u-\alpha\lambda^2+\frac{\alpha\lambda^2}{2}e^{-u}\right)|\nabla u|^2\\
&\geq -2p\int\left(e^u-\alpha\lambda^2+\frac{\alpha\lambda^2}{2}e^{-u}\right)\triangle u-2p\int e^{pu}\left(e^u-\frac{\alpha\lambda^2}{2}e^{-u}\right)|\nabla u|^2\\
&\quad -2p\int e^{pu}\mu.
\end{aligned}$$

Using integration by parts, this inequality reduces to,

$$2\int e^{pu}\mu\geq p\left(1-\frac{\alpha\lambda^2}{2}\right)\int e^{pu+u}|\nabla u|^2+p\frac{\alpha\lambda^2}{2}\int\left(e^{u/2}-e^{-u/2}\right)^2 e^{pu}|\nabla u|^2,$$

which implies

$$p\int e^{pu+u}|\nabla u|^2\leq C\int e^{pu}\mu, \tag{4.11}$$

where $C$ depends on $\alpha\lambda^2$. An interesting case is when $t=0$. Recall that $\mu$ is a shorthand notation for $\mu t$. Thus when $t=0$, we obtain $p\int e^{pu+u}|\nabla u|^2=0$ and therefore $|\nabla u|^2\equiv 0$. We summarize this in the following remark.

**Remark 53.** When $t=0$, the PDE,

$$i\partial\bar{\partial}e^u\wedge\omega_S-i\alpha t\partial\bar{\partial}u\wedge\lambda^2\omega_S-i\alpha t\partial\bar{\partial}e^{-u}\wedge\frac{\lambda^2}{2}\omega_S-\alpha\partial\bar{\partial}u\wedge\partial\bar{\partial}u+\mu t\frac{\omega_S^2}{2}=0$$

has a unique constant solution $u=-\ln A$ which satisfies the normalization conditions

$$\left(\int_S e^{-4u}\frac{\omega_S^2}{2}\right)^{\frac{1}{4}}=A,\quad \int_S\frac{\omega_S^2}{2}=1.$$

**Proposition 54.** *Suppose $u$ is a solution to the PDE under the elliptic condition $\omega'>0$,* *and normalizations* $A < 1$, $\int \frac{\omega_S^2}{2} = 1$. *If we choose* $A$ *so that* $C_1 A < 1$*, then we have*

$$\sup u \leq C_2,$$

*where* $C_2$ *is a constant that depends on* $\alpha\lambda^2$*,* $\mu$*, the Sobolev constant of* $g$*, and* $A$*.*

*Proof.* The proof is similar to the proof of Proposition 52. We use the Sobolev inequality as in Lemma 52 to obtain the following

$$\left(e^{2pu}\right)^{1/2} \leq C_S^2 \left[\int e^{pu} + \int \frac{p^2}{4} e^{pu} |\nabla u|^2\right],$$

where $C_S$ is the Sobolev constant. From Equation 4.11, and Proposition 52 we obtain

$$p \int e^{pu} |\nabla u|^2 \leq C \int e^{pu}, \tag{4.12}$$

where we have chosen $A$ small so that $e^u > (C_1 A)^{-1} > 1$. By using this expression in the Sobolev inequality we obtain

$$\begin{aligned} \left(\int e^{2^{\beta+1} u}\right)^{\frac{1}{2^{\beta+1}}} &\leq C (2^\beta)^{\frac{1}{2^\beta}} \left(\int e^{2^\beta u}\right)^{\frac{1}{2^\beta}} \\ &\leq \cdots \\ &\leq C \int e^u, \end{aligned}$$

where $C$ now depends on $\alpha\lambda^2$, $\mu$, $C_S$, and $\beta$. By sending $\beta \to \infty$, and noting that $C$ converges to some constant we obtain

$$||e^u||_\infty \leq C \int e^u.$$

Note that we have transformed a question of the upper bound of $e^u$ to a question of the upper bound of the average of $e^u$, therefore it is natural to consider the

Poincaré's inequality.

$$\begin{aligned}\int e^{2u} - \left(\int e^u\right)^2 = \int \left(e^u - \int e^u\right)^2 &\leq C \int \nabla e^u \\ &\leq C \int e^u,\end{aligned}$$

where the last inequality comes from Inequality (4.12).

Now we focus on the function $e^u$. From the normalization condition $(\int e^{-4u})^{1/4} = A$, we can expect $S$ can be divided into two regions:

$$U_1 := \left\{x \in S; e^u(x) \leq \frac{2}{A}\right\}, \qquad U_2 := \left\{x \in S; e^{u(x)} > \frac{2}{A}\right\}.$$

Note that neither $U_1$ nor $U_2$ is empty; if so it will lead to contradiction. Bear in mind that $U_2$ is the region we are interested in. We use Young's inequality $2\int_{U_1} e^u \int_{U_2} e^u \leq \frac{1}{\epsilon}\left(\int_{U_1} e^u\right)^2 + \epsilon\left(\int_{U_2} e^u\right)^2$ in the following inequality

$$\begin{aligned}\left(\int_S e^u\right)^2 &\leq \left(1+\frac{1}{\epsilon}\right)\left(\int_{U_1} e^u\right)^2 + (1+\epsilon)\left(\int_{U_2} e^u\right)^2 \\ &\leq \left(1+\frac{1}{\epsilon}\right) \operatorname{vol}(U_1) \int_{U_2} e^{2u} + (1+\epsilon) \operatorname{vol}(U_2) \int_{U_2} e^{2u} \\ &\leq \left(1+\frac{1}{\epsilon}\right) \frac{4}{A^2} + (1+\epsilon) \operatorname{vol}(U_2) \int_S e^{2u},\end{aligned}$$

where we have used the Hölder inequality in the second line. Now we use the Poincaré inequality we have derived before, and simplify the expression to obtain

$$[1-(1+\epsilon)\operatorname{vol}(U_2)]\left(\int e^u\right)^2 - C\int e^u - \left(1+\frac{1}{\epsilon}\right)\frac{4}{A^2} \leq 0.$$

We will show that $\operatorname{vol}(U_2) < 1$ which allows us to pick $\epsilon$ small enough so that

$[1-(1+\epsilon)\,\mathrm{vol}(U_2)] < 1$. This in turn will imply that $\int e^u$ is bounded.

$$\begin{aligned}
A^4 &= \int_{U_1} e^{-4u} + \int_{U_2} e^{-4u} \\
&\leq \sup e^{-4u}\,\mathrm{vol}(U_1) + \left(\frac{A}{2}\right)^4 \mathrm{vol}(U_2) \\
&= \left(-\sup e^{-4u} + \frac{A^4}{16}\right)\mathrm{vol}(U_2) + \sup e^{-4u}.
\end{aligned}$$

As a result, we obtain

$$\mathrm{vol}(U_2) \leq 1 - \frac{\frac{15}{16}A^4}{\sup e^{-4u} - \frac{A^4}{16}} < 1,$$

where we have used $\sup e^{-4u} > A^4/16$; otherwise it contradicts the normalization condition. Therefore we can pick a small $\epsilon$ which implies that $\int e^u$ is bounded. Therefore $e^u$ is bounded as well. The inequality derived above shows that the bound depends on $A$ as well as $\alpha\lambda^2$, $C_S$, and $\mu$. This completes the proof. $\square$

## 4.5 Estimates of the determinant and normal coordinates

### 4.5.1 Estimates of the determinant

In this section we find the upper and lower bounds of the determinant $F$

$$F := \frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}},$$

and introduce some normal coordinates that will be useful for estimates later on. We will need $F$ to be lower bounded to show that the PDE can be realized as a uniformly elliptic PDE. Most of the work in this section is primarily based on

[FY08] and [Aub70, Yau78].

First we derive $F$ explicitly. From the definition of $\omega'$:

$$\begin{aligned}\omega'^2 &= \left(e^u - \alpha\lambda^2 + \frac{\alpha\lambda^2}{2}e^{-u}\right)^2 \omega_S^2 + 4i\alpha\partial\bar{\partial}u \wedge \left(e^u - \alpha\lambda^2 + \frac{\alpha\lambda^2}{2}e^{-u}\right)\omega_S \\ &\quad - 4\alpha^2\partial\bar{\partial}u \wedge \partial\bar{\partial}u, \\ &= \left(e^u - \alpha\lambda^2 + \frac{\alpha\lambda^2}{2}e^{-u}\right)^2 \omega_S^2 + 2\alpha\left(e^u - \alpha\lambda^2 + \frac{\alpha\lambda^2}{2}e^{-u}\right)\triangle u \cdot \omega_S^2 \\ &\quad - 4\alpha\left[\left(i\partial\bar{\partial}e^u - i\alpha\lambda^2\partial\bar{\partial}u - i\frac{\alpha\lambda^2}{2}\partial\bar{\partial}e^{-u}\right)\wedge\omega_S + \mu\frac{\omega_S^2}{2}\right],\end{aligned}$$

where we have used the definition of the laplacian and the PDE. We simplify this by computing further,

$$\omega'^2 = \left(e^u - \alpha\lambda^2 + \frac{\alpha\lambda^2}{2}e^{-u}\right)^2 \omega_S^2 - 2\alpha\left(e^u - \frac{\alpha\lambda^2}{2}e^{-u}\right)|\nabla u|^2\omega_S^2 - 2\alpha\mu\omega_S^2.$$

From $\frac{\omega'^2}{2} = \frac{\det(g'_{i\bar{j}})}{\det(g_{i\bar{j}})}\frac{\omega_S^2}{2}$, we obtain,

$$F(t,q) = \left(e^u - \alpha\lambda^2 t + \frac{\alpha\lambda^2 t}{2}e^{-u}\right)^2 - 2\alpha\left(e^u - \frac{\alpha\lambda^2 t}{2}e^{-u}\right)|\nabla u|^2 - 2\alpha\mu t.$$

We would like to show that $F$ is bounded. First, we find an upper bound for $|\nabla u|^2$.

**Lemma 55.** *If $\alpha\lambda^2 < 2$ and $A$ is small enough such that $C_1 A < 1$, then we have the following upper bound for $|\nabla u|^2$,*

$$|\nabla u|^2 \leq \frac{1+C}{\alpha}e^u \leq C_3$$

*where $C$ is a constant that depends on $\alpha$, $\lambda^2$ and $\mu$. $C_3$ is a constant that depends on $\alpha$, $\lambda^2$, $\mu$, and $A$.*

*Proof.* We use the result that $e^{-\inf u} \leq C_1 A$ and assume $C_1 A < 1$. By the elliptic condition we also have $F > 0$, and thus

$$
\begin{aligned}
0 &< e^{-2u} F \\
&= 1 - 2\alpha e^{-u}\lambda^2 t - 2\alpha e^{-u}|\nabla u|^2 + \alpha^2\lambda^2 t e^{-3u}|\nabla u|^2 \\
&\quad + e^{-2u}\left(\alpha\lambda^2 t + \alpha^2\lambda^4 t^2 - 2\alpha\mu t - \alpha^2\lambda^4 t^2 e^{-u} + \frac{\alpha^2\lambda^4}{4} t^2 e^{-2u}\right).
\end{aligned}
$$

We define $C := \alpha\lambda^2 + \alpha^2\lambda^4 + 2\alpha \sup|\mu| + \alpha^4\lambda^4/4$, then we can write the last inequality as,

$$
0 < e^{-2u}F \leq 1 - 2\alpha e^{-u}\left(1 - \frac{\alpha\lambda^2}{2}(C_1 A)^2\right)|\nabla u|^2 + (C_1 A)^2 C.
$$

Since $1 - \alpha\lambda^2 (C_1 A)^2/2 \geq 1/2$, we obtain the upper bound,

$$
|\nabla u|^2 \leq \frac{1 + (C_1 A)^2 C}{\alpha} e^u \leq \frac{1 + C}{\alpha} e^u.
$$

This completes the proof. □

Since we have shown that $u$ is bounded in the previous section, this Lemma implies the following corollary.

**Corollary 56.** *If* $\alpha\lambda^2 < 2$ *and* $C_1 A < 1$, *then* $F$ *is bounded from above.*

*Proof.* We have shown that $u$ is bounded and by the above Lemma $|\nabla u|$ is upper bounded. From the definition of $F$ one can see that this implies that $F$ is bounded from above as well. □

We would like to show that $F$ is uniformly bounded $e^{-2u}F(t, \cdot) > \kappa$ for any constant $\kappa \in (0, 1)$. Intuitively, this make sense because,

$$
\frac{1}{C_1 A} \leq e^u,
$$

and if we choose $A$ to be small then $\det g' \sim e^{2u}$ will be large. Thus for a given constant $\kappa \in (0,1)$, we expect there exists $A$ such that $e^{-2u}F > \kappa$. Proving this fact, however, tuns out to be quite technical. We will use Fu and Yau's result.

To use Fu and Yau's result, we compare our problem with theirs. In [FY08], the PDE and the elliptic condition are:

1. PDE of Fu and Yau,

$$\triangle(e^u - \alpha t f e^{-u}) + 2\alpha \frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}} + \mu t = 0,$$

2. Elliptic condition of Fu and Yau,

$$(e^u + \alpha t f e^{-u})\omega_S + 2\alpha i\partial\bar{\partial}u > 0,$$

where $f$ is a positive function. If we take $f = \lambda^2 u e^u + \lambda^2/2$, then we see that the PDE of Fu and Yau becomes the PDE we are interested in. To use Fu and Yau's result, we need to make sure that the elliptic condition of Fu and Yau is satisfied with this choice of $f$ as well. With this choice of $f$ Fu and Yau's elliptic condition becomes

$$\left(e^u + \alpha\lambda^2 t u + \frac{\alpha\lambda^2 t}{2} e^{-u}\right)\omega_S + 2\alpha i\partial\bar{\partial}u > 0.$$

The elliptic condition we impose implies Fu and Yau's elliptic condition when $C_1 A < 1$:

$$\begin{aligned} 0 &< \left(e^u - \alpha\lambda^2 t + \frac{\alpha\lambda^2 t}{2} e^{-u}\right)\omega_S + 2\alpha i\partial\bar{\partial}u \\ &\leq \left(e^u + \alpha\lambda^2 t u + \frac{\alpha\lambda^2 t}{2} e^{-u}\right)\omega_S + 2\alpha i\partial\bar{\partial}u, \end{aligned}$$

when $C_1 A < 1$ because $0 < -\log(C_1 A) \leq u$. Therefore, we can use Fu and Yau's result in our case as well.

We state the following result from [FY08].

**Proposition 57** ([FY08]). *Let $t \in T$ and $u$ be a solution of the PDE under the following conditions,*

1. *Elliptic condition:*

$$\omega' = \left(e^u - \alpha\lambda^2 t + \frac{\alpha\lambda^2 t}{2}e^{-u}\right)\omega_S + 2\alpha i\partial\bar{\partial}u > 0.$$

2. *Normalization conditions:*

$$\left(\int_S e^{-4u}\right)^{1/4} = A < 1, \quad \int_S \frac{\omega_S^2}{2} = 1.$$

*If $C_1 A < 1$, then given any constant $\epsilon \in (0,1)$, we can find a small $A$ such that $F$ is uniformly lower bounded by*

$$F(t,\cdot) > \epsilon e^{2u} \geq \epsilon(C_1 A)^{-2}.$$

Fu and Yau prove this proposition using proof by contradiction.

### 4.5.2 Normal coordinates

Let us introduce normal coordinates around a point $p$ such that,

$$g = |dz^1|^2 + |dz^2|^2,$$

where our convention here is $|dz^i|^2 = dz^i \otimes d\bar{z}^{\bar{i}} + d\bar{z}^{\bar{i}} \otimes dz^i$. In this coordinates $g'$ becomes the following form,

$$\begin{aligned} g' = &\left(e^u - \alpha\lambda^2 + \frac{\alpha\lambda^2}{2}e^{-u} + 2\alpha u_{1\bar{1}}\right)|dz^1|^2 + 2\alpha u_{1\bar{2}}(dz^1 \otimes d\bar{z}^{\bar{2}} + d\bar{z}^{\bar{2}} \otimes dz^1) \\ &+ \left(e^u - \alpha\lambda^2 + \frac{\alpha\lambda^2}{2}e^{-u} + 2\alpha u_{2\bar{2}}\right)|dz^2|^2 + 2\alpha u_{2\bar{1}}(dz^2 \otimes d\bar{z}^{\bar{1}} + d\bar{z}^{\bar{1}} \otimes dz^2). \end{aligned}$$

Since $T_p^{(1,0)}M$ is isomorphic to $\mathbb{C}^2$, we can think of $g_{i\bar{j}}$ and $g'_{i\bar{j}}$ as invertible hermitian $2 \times 2$ matrices. It is clear that $g_{i\bar{j}}$ and $g'_{i\bar{j}}$ commute, thus they are simultaneously diagonalizable. We summarize this in the following lemma.

**Lemma 58.** *We can always find a normal coordinate with respect to $g$ at $p$, such that $g$ and $g'$ has the following form,*

$$g = |dz^1|^2 + |dz^2|^2,$$

*and*

$$\begin{aligned} g' = &\left(e^u - \alpha\lambda^2 + \frac{\alpha\lambda^2}{2}e^{-u} + 2\alpha u_{1\bar{1}}\right)|dz^1|^2 \\ &+ \left(e^u - \alpha\lambda^2 + \frac{\alpha\lambda^2}{2}e^{-u} + 2\alpha u_{2\bar{2}}\right)|dz^2|^2. \end{aligned}$$

Another coordinate system that is useful is a normal coordinate with $u_2 = u_{\bar{2}} = 0$. This coordinate system can be found by taking a normal coordinate system with respect to $g$ and rotating it.

**Lemma 59.** *We can always find a normal coordinate with respect to $g$ at $p$, such that*

$$g = |dz^1|^2 + |dz^2|^2,$$

*and*

$$u_2 = u_{\bar{2}} = 0, \quad u_1 = u_{\bar{1}} \geq 0.$$

*Proof.* This normal coordinate can be obtained by an orthonormal rotation of a normal coordinate at $p$ to adapt the system such that $\partial u/\partial x^1 \geq 0$ and $\partial u/\partial y^1 = \partial u/\partial x^2 = \partial u/\partial y^2 = 0$. □

## 4.6 Overview on second and third order estimates

In the following sections we will find a priori estimates of the second order derivatives and third order derivatives of $u$. The general strategy we use is to start from a function $G$ that has free parameters $\lambda$. Since we are working on a compact manifold we can find a point $q$ where $G$ will take its maximum. At that point we will have $\triangle G(q) \leq 0$. By choosing $\lambda$ we can find second order and third order estimates.

When computing $\triangle G(q) \leq 0$ one key idea is to use the PDE. The PDE we are interested in is a second order PDE, and this will allow use to write second order differentiation in terms of lower order differentiations. Thus by using it we can reduce the order of $\triangle G$ to lower order derivatives.

Suppose we want to find the estimates of second order derivatives of $u$. A natural function to consider is $G = g^{i\bar{j}} g'_{i\bar{j}}$. When computing $\triangle G$ this will increase the second order derivatives to forth order derivatives which makes the problem worse. However, by using the PDE we reduce the forth order derivatives into third order. We can use the PDE once more to reduce the third order derivatives in terms of second order derivatives, and through this process $\triangle G(q) \leq 0$ will lead to an inequality of second order derivatives. There are some technical differences, but this general approach also applies to third order a priori estimates.

Finally, one can ask how to construct functions $G$ such that this approach works. The starting point comes from [Cal58, Aub70, Yau78]. In the original work, the authors consider the following functions for the a priori estimates:

$$G = \frac{g'_{1\bar{1}} g'_{2\bar{2}}}{2} e^{-\lambda u} \qquad \text{second order}$$

$$G = g^{i\bar{l}} g^{\bar{j}m} g^{k\bar{n}} u_{i\bar{j}k} u_{\bar{l}m\bar{n}} + \lambda \triangle u \qquad \text{third order}$$

In our case these functions are not enough to find bounds we are seeking for. This is because our PDE has first order derivatives as well as second order derivatives.

This introduces extra terms when reducing the higher order differentials into lower order ones. Generally, these extra terms prevent us from finding desired bounds and we need to modify the functions $G$.

Suppose we want to find a function $G$ such that it can be used to find second order estimates. The strategy is to add an extra term on to $G$ such that $\triangle G$ behaves nicely, naively we are adding extra terms on $G$ so that we can *control* the highest derivative term. When finding second order a priori estimates $\lambda|\nabla u|^2$ is a natural candidate to consider. This is because we have already found estimates up to first order thus we *know* how $\lambda|\nabla u|^2$ behaves. When it comes to third order estimates, we can use a similar approach. But we have more candidates we can consider: $u$, $|\nabla u|^2$, $\triangle u$, $|\nabla_{ik} u|^2$, and the combination of these terms.

In the following sections we use these ideas outlined here to find the a priori estimates of the derivatives of $u$. While the ideas are simple, the computation turns out to be quite complicated.

## 4.7 Second order estimate

In this section we find the second order a priori estimate of $u \in C^4(M)$. This will be used to show that the PDE is uniformly bounded in the following sense; take the PDE in the complex Monge-Ampère form and differentiate it to obtain

$$\det g' g'^{i\bar{j}} \alpha \frac{\partial^2}{\partial z^i \partial z^{\bar{j}}} \frac{\partial u}{\partial z^k} = \partial_k (F \det g) - \partial_k \left[ \left( e^u - \alpha\lambda^2 + \frac{\alpha\lambda^2}{2} e^{-u} \right) g_{i\bar{j}} \right].$$

If we can show that $g'_{i\bar{j}}$ is bounded, this implies the above PDE of $u_k$ is uniformly elliptic.

In the previous sections we have shown that $F$ is bounded. By the inequality of arithmetic and geometric means, in normal coordinates we have

$$e^u - \alpha\lambda^2 + \frac{\alpha\lambda^2}{2}e^{-u} + \alpha\triangle u \geq F^{1/2} > \kappa^{1/2}(C_1 A)^{-1} > 0.$$

In this section, we will show that $L(u) := e^u - \alpha\lambda^2 + \alpha\lambda^2 e^{-u}/2 + \alpha\triangle u$ has an upper bound. Then with the fact that $F$ is bounded this will imply that $g'_{1\bar{1}} + g'_{2\bar{2}}$ and $g'_{1\bar{1}}g'_{2\bar{2}}$ are bounded. This in turn implies $g'_{i\bar{i}} = e^u - \alpha\lambda^2 + \alpha\lambda^2 e^{-u}/2 + 2\alpha u_{i\bar{i}}$ is bounded as well.

We use the techniques of [Aub70, Yau78, Joy07, FY08] to find an upper bound for $L(u)$. We will consider a point $q_1$ where

$$G_1 := \log L(u) - \lambda_1 u + \lambda_2 |\nabla u|^2$$

reaches its maximum. If we denote the laplacian with respect to the Chern connection of $g'$ as $\triangle'$, then at $q_1$ we have

$$\triangle' G_1(q_1) \leq 0.$$

By choosing $\lambda_1$ and $\lambda_2$ carefully, this will give us an inequality at $q_1$ that will be used to show that $L(u)$ is bounded from above on $S$. Note that the function $G_1$ which we maximize has an extra term compared to [Aub70, Yau78, Joy07]. The role of this term will be discussed after Lemma 62.

First we find the forth order derivative of $u$ and reduce the order of derivatives using the PDE. This will be our main line of attack: reduce higher order derivatives using the PDE and find the estimates.

**Lemma 60.** *We can find the forth order derivative of $u$:*

$$2\alpha g'^{i\bar j}(\nabla_{i\bar j k\bar l}u) = -\, g'^{\bar j i} g_{i\bar j}\nabla_{k\bar l}\left(e^u + e^{-u}\frac{\alpha\lambda^2}{2}\right) - \frac{\nabla_k F\nabla_{\bar l}F}{F^2} + \frac{\nabla_{k\bar l}F}{F}$$
$$+\, g'^{\bar j m} g'^{\bar n i}\nabla_{\bar l} g'_{m\bar n}\cdot\nabla_{\bar k} g'_{i\bar j} + 2\alpha g'^{\bar j i}\left(R_{k\bar l\bar j}{}^{\bar m}\partial_{i\bar m}u - R_{i\bar j\bar l}{}^{\bar m}\partial_{k\bar m}u\right).$$

*Proof.* We take the $\log$ function of the definition of $F$ and differentiate it with respect to $\nabla_{\bar l}$:

$$g'^{\bar j i}\nabla_{\bar l} g'_{i\bar j} = F^{-1}\nabla_{\bar l}F,$$

note that we have used the fact that $\nabla$ is the Levi-Civita connection of $g$. We take another derivative with respect to $\nabla_k$ and reorganize the terms:

$$g'^{\bar j i}\nabla_{k\bar l} g'_{i\bar j} = g'^{\bar j m} g'^{\bar n i}\nabla_k g'_{m\bar n}\cdot\nabla_{\bar l} g'_{i\bar j} - F^{-2}\nabla_k F\nabla_{\bar l}F + F^{-1}\nabla_{k\bar l}F,$$

where we have used $\nabla_k g'^{\bar j i} = -g'^{\bar j m} g'^{\bar n i}\nabla_k g'_{m\bar n}$. By using the definition $g'_{i\bar j} = (e^u - \alpha\lambda^2 + e^{-u}\alpha\lambda^2/2)g_{i\bar j} + 2\alpha u_{i\bar j}$, we obtain

$$2\alpha g'^{\bar j i}\nabla_{k\bar l i\bar j}u = g'^{\bar j m} g'^{\bar n i}\nabla_k g'_{m\bar n}\cdot\nabla_{\bar l} g'_{i\bar j} - g'^{\bar j i} g_{i\bar j}\nabla_{k\bar l}\left(e^u + e^{-u}\frac{\alpha\lambda^2}{2}\right) - \frac{\nabla_k F\nabla_{\bar l}F}{F^2} + \frac{\nabla_{k\bar l}F}{F}.$$

We can change the order of derivatives on the left hand side using

$$\nabla_{k\bar l i\bar j}u = \nabla_{i\bar j k\bar l}u - R_{k\bar l\bar j}{}^{\bar m}\partial_{i\bar m}u + R_{i\bar j\bar l}{}^{\bar m}\partial_{k\bar m}u,$$

and obtain the result of this lemma. □

The Chern connection $\nabla'$ with respect to $g'$, and the Levi-Civita connection $\nabla$ with respect to $g$ are closely related. When we act the laplacian $\triangle'$ with respect to the Chern connection on a function $\psi$, we have the following:

$$\begin{aligned}\triangle'\psi = g'^{\bar{j}i}\nabla'_{i\bar{j}}\psi &= g'^{\bar{j}i}(\partial_i\nabla'_{\bar{j}}\psi - \Gamma'^{\ a}_{i\bar{j}}\nabla'_a\psi)\\ &= g'^{i\bar{j}}\partial_{i\bar{j}}\psi = g'^{\bar{j}i}\nabla_{i\bar{j}}\psi.\end{aligned}$$

We use this property in the following lemma which is used to find $\triangle'\triangle u$, and $\triangle' G_1$.

**Corollary 61.** *We have the following equation:*

$$\begin{aligned}\triangle'(\triangle u) = \frac{1}{2\alpha}\Bigg[ &- g'^{i\bar{j}}g_{i\bar{j}}\triangle\left(e^u + e^{-u}\frac{\alpha\lambda^2}{2}\right) + \frac{\triangle F}{F} - \frac{|\nabla F|^2}{F^2} + 2\alpha H\\ &+ g'^{i\bar{n}}g'^{m\bar{j}}g^{k\bar{l}}\nabla_k g'_{m\bar{n}}\cdot\nabla_{\bar{l}}g'_{i\bar{j}}\Bigg],\end{aligned}$$

*where* $H := g'^{\bar{j}i}g^{\bar{k}l}\left(R_{k\bar{l}\bar{j}}{}^{\bar{m}}\partial_{i\bar{m}}u - R_{i\bar{j}\bar{l}}{}^{\bar{m}}\partial_{k\bar{m}}u\right)$.

*Proof.* Contract the results of the lemma above with $g^{k\bar{l}}$, then we obtain the following corollary. ☐

Using this corollary and some additional estimates, we can derive the estimate of $LF\triangle' G_1$ in the following lemma.

**Lemma 62.** *When* $\alpha\lambda^2 < 1$ *and* $C_1A < 1$, *we have the following estimate:*

$$\begin{aligned}LF\triangle' G_1 \geq &\left[\frac{\lambda_1}{\alpha C_1 A} - C(\lambda_2^2 + \lambda_2 + 1)\right]L^2 - C(\lambda_1 + \lambda_2 + 1)L\\ &+ \left(\frac{\kappa\lambda_2}{2(C_1A)^2} - C\right)\Gamma - C\sqrt{\Gamma} - C,\end{aligned}$$

*where* $\Gamma := g^{i\bar{j}}g^{k\bar{l}}\nabla_{ik}u\nabla_{\bar{j}\bar{l}}u$ *and* $C$ *is a constant that depends on* $\alpha$, $\lambda^2$, $\mu$, $A$, *and both the Sobolev constant and the curvature bound of* $g$.

The proof of this lemma is rather technical and we leave it for the next section. From this lemma we can see why we need the extra $\lambda_2|\nabla u|^2$ term in $G_1$. This is to force the highest order in $\Gamma$ to have a positive leading term. Fix $C$ and take $\lambda_2$ such that

$$\frac{\kappa\lambda_2}{2(C_1A)^2} - C > 0.$$

Then by completing square and since at $q_1$ we have $LF\triangle' G_1 < 0$, we obtain

$$0 \geq \left[\frac{\lambda_1}{\alpha C_1 A} - C(\lambda_2^2 + \lambda_2 + 1)\right] L^2 - C(\lambda_1 + \lambda_2 + 1)L - C.$$

Now choose $\lambda_1$ such that

$$\frac{\lambda_1}{\alpha C_1 A} - C(\lambda_2^2 + \lambda_2 + 1) > 0.$$

This implies that $L$ is bounded from above at $q_1$.

From the definition of $G_1$, we see that $G_1$ is bounded because $L$, $u$ and $|\nabla u|$ are also bounded. By assumption $G_1$ takes its maximum value at $q_1$ therefore we have

$$\log L - \lambda_1 u + \lambda_2 |\nabla u|^2 \leq G_1(q_1),$$

thus $\log L \leq G_1(q_1) + \lambda_1 \sup u$ and $L$ is bounded on $S$. Since both $F$ and $L$ are bounded on $S$, this implies that $g'_{i\bar{j}}$ is bounded as well. We summarize this in the following proposition.

**Proposition 63.** *Let $t \in T$ and $u \in C^4(S)$ be a solution of the PDE under the following conditions,*

1. *Elliptic condition:*

$$\omega' = \left(e^u - \alpha\lambda^2 t + \frac{\alpha\lambda^2 t}{2} e^{-u}\right)\omega_S + 2\alpha i\partial\bar{\partial}u > 0.$$

2. *Normalization conditions:*

$$\left(\int_S e^{-4u}\right)^{1/4} = A < 1, \quad \int_S \frac{\omega_S^2}{2} = 1.$$

*If $C_1 A < 1$ and $\alpha\lambda^2 < 2$, then $e^u - \alpha\lambda^2 t + e^{-u}\alpha\lambda^2 t/2 + 2\alpha\triangle u$ has an upper bound depending on $\alpha$, $\mu$, $\lambda^2$, the Sobolev constant of $g$, the curvature bound of $g$, and $A$.*

*Furthermore, with the result of Proposition 57, this implies that $g'_{i\bar{j}}$ is bounded from above and below uniformly and positively away from zero.*

As discussed in the beginning of this section, this proposition implies that the PDE is uniformly elliptic.

### 4.7.1 Estimates for Lemma 62

In this section we derive the estimates in Lemma 62. First we compute $\triangle' G_1$ in the following lemma,

**Lemma 64.** *$LF\triangle' G_1$ satisfies the following inequality:*

$$\begin{aligned} LF\triangle' G_1 \geq & -\lambda_1 LF\triangle' u + \lambda_2 LF\triangle' |\nabla u|^2 - L\triangle\left(e^u + e^{-u}\frac{\alpha\lambda^2}{2}\right) + \frac{HF}{2} \\ & + \frac{1}{2}\left(\triangle F - \frac{|\nabla F|^2}{F}\right) + F\triangle'\left(e^u + \frac{\alpha\lambda^2}{2}e^{-u}\right) - CL(u)^2, \end{aligned} \tag{4.13}$$

*where $C$ is a constant that depends on $\alpha$, $\lambda^2$, $\mu$, $A$, and Sobolev constant of $g$.*

*Proof.* We compute $\triangle' G_1$ directly:

$$\triangle' G_1 = -L^{-2}|\nabla' L|^2_{g'} + L^{-1}\triangle' L - \lambda_1\triangle' u + \lambda_2\triangle'|\nabla u|^2.$$

We now compute term by term in normal coordinates.

In normal coordinates that diagonalizes $g'$, the first term satisfies the following inequality

$$
\begin{aligned}
L^{-2}|\nabla' L|^2_{g'} &= L^{-2}\sum_i \frac{g'^{i\bar i}}{4}\left|\sum_j \frac{\partial_i g'_{j\bar j}}{\sqrt{g'_{j\bar j}}}\sqrt{g'_{j\bar j}}\right|^2 \\
&\le L^{-2}\sum_i \frac{g'^{i\bar i}}{4}\sum_j \frac{\partial_i g'_{j\bar j}}{\sqrt{g'_{j\bar j}}}\frac{\partial_{\bar i} g'_{j\bar j}}{\sqrt{g'_{j\bar j}}}\sum_k g'_{k\bar k} \\
&\le L^{-1}\sum_{i,j}\frac{g'^{i\bar i}g'^{j\bar j}}{2}\partial_i g'_{j\bar j}\partial_{\bar i} g'_{j\bar j},
\end{aligned}
$$

where we have used the Cauchy-Schwarz inequality. Note that $\partial_i g'_{j\bar j} = \partial_j g'_{i\bar j} + \left(e^u + \frac{\alpha\lambda^2}{2}e^{-u}\right)_i$ and similarly $\partial_{\bar i} g'_{j\bar j} = \partial_{\bar j} g'_{j\bar i} + (e^u + \frac{\alpha\lambda^2}{2}e^{-u})_{\bar i}$. We use these equations and compute:

$$
\begin{aligned}
2L^{-1}|\nabla' L|^2_{g'} \le{}& \sum_{i\neq k} g'^{i\bar i}g'^{k\bar k}\left(\partial_k g'_{i\bar k} + \left(e^u + \frac{\alpha\lambda^2}{2}e^{-u}\right)_i\right)\left(\partial_{\bar k} g'_{i\bar k} + \left(e^u + \frac{\alpha\lambda^2}{2}e^{-u}\right)_{\bar i}\right) \\
&+ \sum_i (g'^{i\bar i})^2\partial_i g'_{i\bar i}\partial_{\bar i} g'_{i\bar i} \\
\le{}& \sum_{i,k} g'^{i\bar i}g'^{k\bar k}\partial_k g'_{i\bar k}\partial_{\bar k} g'_{k\bar i} + \sum_{i\neq k} g'^{i\bar i}g'^{k\bar k}\left(e^u + \frac{\alpha\lambda^2}{2}e^{-u}\right)_i\partial_{\bar i} g'_{k\bar k} \\
&+ \sum_{i\neq k} g'^{i\bar i}g'^{k\bar k}\left(e^u + \frac{\alpha\lambda^2}{2}e^{-u}\right)_{\bar i}\partial_i g'_{k\bar k},
\end{aligned}
$$

where we have added $2\sum_{i\neq k} g'^{i\bar i}g'^{k\bar k}\left(e^u + \alpha\lambda^2 e^{-u}/2\right)_i\left(e^u + \alpha\lambda^2 e^{-u}/2\right)_{\bar i}$ which is positive and rearranged the index on the derivative of $g'$. Using the Cauchy-Schwartz inequality on the last two terms we obtain

$$
\begin{aligned}
2L(u)^{-1}|\nabla' L|^2_{g'} \le{}& \sum_{i,k} g'^{i\bar i}g'^{k\bar k}\partial_k g'_{i\bar k}\partial_{\bar k} g'_{k\bar i} + \sum_{i\neq k}\left(g'^{k\bar k}\right)^2\partial_{\bar i} g'_{k\bar k}\partial_i g'_{k\bar k} \\
&+ \sum_i (g'^{i\bar i})^2\left(e^u + \frac{\alpha\lambda^2}{2}e^{-u}\right)_i\left(e^u + \frac{\alpha\lambda^2}{2}e^{-u}\right)_{\bar i} \\
\le{}& \sum_{i,j,k} g'^{i\bar i}g'^{k\bar k}\partial_j g'_{i\bar k}\partial_{\bar j} g'_{k\bar i} + C\frac{L(u)^2}{F},
\end{aligned}
$$

where we have used the fact that $u$ and the first order derivatives of $u$ are bounded, $\sum_i (g'^{i\bar{i}})^2 \leq 4L^2/F^2$, and $F$ is lower bounded. $C$ is a constant that depends on $u$, the upper bound of $\nabla u$ and the lower bound of $F$.

Using Lemma 61 it is straightforward to derive the following expression for the second term,

$$\begin{aligned} L^{-1}\triangle' L = L^{-1}\Bigg[&\triangle'\left(e^u + \frac{\alpha\lambda^2}{2}e^{-u}\right) \\ &+ \frac{1}{2}\Bigg\{ - g'^{\bar{j}i} g_{i\bar{j}} \triangle \left(e^u + e^{-u}\frac{\alpha\lambda^2}{2}\right) + \frac{\triangle F}{F} - \frac{|\nabla F|^2}{F^2} + H \\ &\qquad + g'^{i\bar{n}} g'^{\bar{j}m} g^{\bar{k}l} \nabla_l g'_{m\bar{n}} \nabla_{\bar{k}} g'_{i\bar{j}} \Bigg\}\Bigg]. \end{aligned}$$

We substitute these expressions into $\triangle' G_1$ and use $F g'^{\bar{j}i} g_{i\bar{j}} = 2L(u)$ in normal coordinates to obtain the result of this lemma. □

We estimate Inequality (4.13) term-by-term using normal coordinates that will simplify the estimates. The first term is estimated in the following lemma.

**Lemma 65.** *If* $\alpha\lambda^2 < 2$ *and* $C_1 A < 1$, *then the first term on the right hand side of Inequality* (4.13), *has the following lower bound:*

$$-\lambda_1 L F \triangle' u \geq -\lambda_1 C L + \frac{\lambda_1}{\alpha C_1 A} L^2,$$

*where* $C$ *is a constant that depends on* $\alpha$, $\lambda^2$, $\mu$, *Sobolev constant of* $g$, *and* $A$.

*Proof.* Using the definitions, we compute the following

$$\begin{aligned} -\lambda_1 L F g'^{i\bar{j}} u_{i\bar{j}} &= -\frac{\lambda_1 L F}{2\alpha} g'^{i\bar{j}} \left[ g'_{i\bar{j}} - \left(e^u - \alpha\lambda^2 + \frac{\alpha\lambda^2 e^{-u}}{2}\right) g_{i\bar{j}} \right] \\ &= -\frac{\lambda_1 L F}{\alpha} + \frac{\lambda_1}{\alpha}\left(e^u - \alpha\lambda^2 + e^{-u}\frac{\alpha\lambda^2}{2}\right) L^2, \end{aligned}$$

where we used $F g'^{i\bar{j}} g_{i\bar{j}}/2 = L$. We have shown $|\nabla u|$ is bounded above, and this implies that $F$ has some upper bound $C$. Using this we obtain the following lower bound:

$$-\lambda_1 LF\triangle' u \geq -\lambda_1 CL + \frac{\lambda_1}{\alpha C_1 A}L^2.$$

This completes the proof. $\square$

The second term is addressed in the following lemma.

**Lemma 66.** *If $\alpha\lambda^2 < 2$ and $C_1 A < 1$, then the second term on the right hand side of Inequality* (4.13)*, has the following lower bound:*

$$\lambda_2 LF\triangle'|\nabla u|^2 \geq \left(\frac{\kappa\lambda_2}{2(C_1A)^2} - 1\right)\Gamma - C(\lambda_2^2 + \lambda_2)L^2 - C\lambda_2 L,$$

*where $C$ is a constant that depends on $\alpha$, $\lambda^2$, $\mu$, Sobolev constant of $g$, and $A$.*

*Proof.* By direct computation, we can expand the second term:

$$\lambda_2 LFg'^{i\bar j}\partial_i\partial_{\bar j}(u_k u_{\bar k}) = \lambda_2 LFg'^{i\bar j}u_{\bar j k}u_{i\bar k} + \lambda_2 LFg'^{i\bar j}(u_{i\bar j k}u_{\bar k} + u_k u_{i\bar j\bar k}) + \lambda_2 LFg'^{i\bar j}u_{ik}u_{\bar i\bar k}.$$

The first term can be bounded using the definition of $g'$ and $L$:

$$\lambda_2 LFg^{i\bar j}u_{\bar j k}u_{i\bar k} \geq -2\lambda_2 LFg'^{i\bar j}g'_{\bar j k}\left(e^u - \alpha\lambda^2 + e^{-u}\frac{\alpha\lambda^2}{2}\right)g_{i\bar k} \geq -C\lambda_2 L, \tag{4.14}$$

where $C$ is a generic constant that depends on the bounds of $u$.

Using the PDE we obtain the following expression for the second term

$$\begin{aligned}
&\lambda_2 LFg'^{i\bar j}(u_{i\bar j k}u_{\bar k} + u_k u_{i\bar j\bar k})\\
&\quad = -2\lambda_2 L\bigg[\left(e^u + \frac{\alpha\lambda^2}{2}e^{-u}\right)|\nabla u|^4 + \left(e^u - \frac{\alpha\lambda^2}{2}e^{-u}\right)|\nabla u|^2\triangle u\\
&\qquad\qquad + \left(e^u - \frac{\alpha\lambda^2}{2}e^{-u}\right)\nabla u\circ\nabla|\nabla u|^2 + \nabla u\circ\nabla\mu\bigg]\\
&\quad \geq -\lambda_2 LC|\nabla|\nabla u|^2| - \lambda_2 CL^2 - \lambda_2 CL,
\end{aligned}$$

now $C$ depends on the bound of $u$, $\nabla u$, and $\mu$. We can obtain the bound on $|\nabla|\nabla u|^2|$ by using a normal coordinate that also satisfies $u_2 = u_{\bar 2} = 0$.

$$\begin{aligned}
|\nabla|\nabla u|^2|^2 &= (u_{i1}u_{\bar i 1} + u_{i1}u_{\bar i \bar 1} + u_{i\bar 1}u_{\bar i 1} + u_{i\bar 1}u_{\bar i \bar 1})|\nabla u|^2 \\
&\le 2(u_{i1}u_{\bar i \bar 1} + u_{\bar i 1}u_{i\bar 1})|\nabla u|^2
\end{aligned}$$

Here we observe that

$$\begin{aligned}
u_{\bar i 1}u_{i\bar 1} &\le u_{\bar i j}u_{i\bar j} = (u_{1\bar 1} + u_{2\bar 2})^2 - 2\det u_{i\bar j} \\
&\le (\triangle u)^2 - \frac{2}{\alpha}\left[\triangle\left(e^u - \alpha\lambda^2 u - \frac{\alpha\lambda^2}{2}e^{-u}\right) + \mu\right] \\
&\le (\triangle u)^2 - \frac{2}{\alpha}\left(e^u - \frac{\alpha\lambda^2}{2}e^{-u}\right)|\nabla u|^2 - \frac{2}{\alpha}\left(e^u - \alpha\lambda^2 + \frac{\alpha\lambda^2}{2}e^{-u}\right)\triangle u + C \\
&\le \left(\frac{L}{\alpha}\right)^2 - CL + C \le \frac{L^2}{\alpha^2} + C,
\end{aligned}$$

where we have used the PDE and the fact that $C_1 A < 1$. Using this inequality and defining $\Gamma := g^{i\bar j}g^{k\bar l}\nabla_{ik}u\nabla_{\bar j\bar l}u$, we can find a bound for $|\nabla|\nabla u|^2|$:

$$|\nabla|\nabla u|^2| \le \sqrt{2}|\nabla u|[\Gamma^{1/2} + L/\alpha] + C.$$

Combining this expression with the inequality for the second term, we finally obtain the following bound:

$$\begin{aligned}
\lambda_2 L F g'^{i\bar j}(u_{i\bar j k}u_{\bar k} + u_k u_{i\bar j \bar k}) &\ge -\lambda_2 C L\sqrt{\Gamma} - \lambda_2 C L^2 - \lambda_2 C L \\
&= (\sqrt{\Gamma} - \lambda_2 C L)^2 - \Gamma - C(\lambda_2^2 + \lambda_2)L^2 - \lambda_2 C L \\
&\ge -\Gamma - C(\lambda_2^2 + \lambda_2)L^2 - \lambda_2 C L. \qquad (4.15)
\end{aligned}$$

We can find a bound for the last term from a straight forward computation. In normal coordinates, we have

$$\begin{aligned}\lambda_2 F L g'^{i\bar{j}} u_{ik} u_{\bar{j}\bar{k}} &= \lambda_2 \frac{F}{2}\left[\left(1+\frac{g'_{2\bar{2}}}{g'_{1\bar{1}}}\right) u_{1k} u_{\bar{1}\bar{k}} + \left(\frac{g'_{1\bar{1}}}{g'_{2\bar{2}}}+1\right) u_{2k} u_{\bar{2}\bar{k}}\right] \\ &\geq \lambda_2 \frac{F}{2}\Gamma \geq \frac{\kappa\lambda}{2(C_1 A)^2}\Gamma \end{aligned} \tag{4.16}$$

Combining Inequality (4.14), (4.15), and (4.16), we obtain the bound of this lemma.

□

It is easy to find estimates for the third term.

**Lemma 67.** *If $\alpha\lambda^2 < 2$ and $C_1 A < 1$, then the third term of the right hand side of Lemma* (4.13) *has the following lower bound:*

$$-L\triangle\left(e^u + e^{-u}\frac{\alpha\lambda^2}{2}\right) \geq -CL^2 - CL,$$

*where $C$ is a constant that depends on $\alpha$, $\lambda^2$, $\mu$, the Sobolev constant of $g$, and $A$.*

*Proof.* This estimate can be derived by simple computation:

$$\begin{aligned}-L\triangle\left(e^u + e^{-u}\frac{\alpha\lambda^2}{2}\right) &= -L\left[\left(e^u + e^{-u}\frac{\alpha\lambda^2}{2}\right)|\nabla u|^2 + \left(e^u - e^{-u}\frac{\alpha\lambda^2}{2}\right)\triangle u\right] \\ &\geq -CL^2 - CL,\end{aligned}$$

where $C$ depends on the bound of $u$ and $\nabla u$ and we have used $e^u - e^{-u}\alpha\lambda^2/2 > 0$. □

The forth term involves the Riemann tensor. We assume $g$ is a given metric that is $C^\infty$.

**Lemma 68.** *If $\alpha\lambda^2 < 2$ and $C_1 A < 1$, then the forth term of the right hand side of*

*Inequality* (4.13) *has the following lower bound:*

$$\frac{HF}{2} \geq -CL^2 - C,$$

*where $C$ is a constant that depends on $\alpha$, $\lambda^2$, $\mu$, the Sobolev constant of $g$, $A$, and the curvature bounds of $g$.*

*Proof.* We use normal coordinates that diagonalizes $u_{i\bar{j}}$ and expand $g'$ in terms of $g$:

$$\begin{aligned}
\frac{HF}{2} &= \alpha \left( e^u - \alpha\lambda^2 + e^{-u}\frac{\alpha\lambda^2}{2} \right) \left[ R_{k\bar{k}\bar{i}}{}^{\bar{m}} \partial_{i\bar{m}} u - R_{i\bar{i}\bar{k}}{}^{\bar{m}} \partial_{k\bar{m}} u \right] \\
&\quad + 2\alpha^2 \left( R_{k\bar{k}\bar{1}}{}^{\bar{1}} u_{1\bar{1}} u_{2\bar{2}} + R_{k\bar{k}\bar{2}}{}^{\bar{2}} u_{2\bar{2}} u_{1\bar{1}} - R_{1\bar{1}\bar{k}}{}^{\bar{k}} u_{k\bar{k}} u_{2\bar{2}} - R_{2\bar{2}\bar{k}}{}^{\bar{k}} u_{k\bar{k}} u_{1\bar{1}} \right) \\
&= 2\alpha^2 \left( R_{1\bar{1}\bar{2}}{}^{\bar{2}} + R_{2\bar{2}\bar{1}}{}^{\bar{1}} \right) u_{1\bar{1}} u_{2\bar{2}} - 2\alpha^2 R_{1\bar{1}\bar{2}}{}^{\bar{2}} \left( u_{2\bar{2}} \right)^2 - 2\alpha^2 R_{2\bar{2}\bar{1}}{}^{\bar{1}} \left( u_{1\bar{1}} \right)^2 \\
&\geq -4\alpha^2 \max \left| R_{i\bar{j}\bar{k}}{}^{\bar{l}} \right| u_{1\bar{1}} u_{2\bar{2}} - 2\alpha^2 \max \left| R_{i\bar{j}\bar{k}}{}^{\bar{l}} \right| \sum \left( u_{i\bar{i}} \right)^2
\end{aligned}$$

By completing the square and using the definition of $L$ we finally obtain the following estimate:

$$\begin{aligned}
\frac{HF}{2} &\geq -C(\triangle u)^2 = -\frac{C}{\alpha^2} \left[ L - \left( e^u - \alpha\lambda^2 + \alpha\lambda^2 \frac{e^{-u}}{2} \right) \right]^2 \\
&\geq -CL^2 - C,
\end{aligned}$$

where $C$ depends on $\alpha$, the bound of $u$ and the curvature bound of $g$. $\square$

The estimate for the fifth term is somewhat long. We first estimate $\nabla|\nabla u|^2$ and $\triangle|\nabla u|^2$ which will be used in finding an estimate for the fifth term and also in the third order estimates of $u$.

**Lemma 69.** *If $\alpha\lambda^2 < 2$ and $C_1 A < 1$, then there exists a constant $C$ that depends on $\alpha\lambda^2$,*

*$\mu$, the Sobolev constant of $g$, and $A$ which satisfies*

$$|\nabla|\nabla u|^2| \le \sqrt{2}\left(\frac{L}{\alpha} + \sqrt{\Gamma} + C\right)|\nabla u|.$$

*Furthermore, if $G_1$ takes its maximum value at $q_1$, then we have*

$$\triangle|\nabla u|^2 \le (C\lambda_2^2 + C\lambda_2 + C)L^2 + 2\Gamma + C(\lambda_1 + \lambda_2)L + C.$$

*Proof.* We expand the expression and use normal coordinates described in Lemma 59:

$$\begin{aligned}
|\nabla|\nabla u|^2|^2 &= \sum_i (u_{i1}u_{\bar 1} + u_1 u_{i\bar 1})(u_{1\bar i}u_{\bar 1} + u_1 u_{\bar i\bar 1}) \\
&= \sum_i (u_{i1}u_{1\bar i} + u_{i1}u_{\bar i\bar 1} + u_{i\bar 1}u_{1\bar i} + u_{i\bar 1}u_{\bar i\bar 1})|\nabla u|^2 \\
&\le 2\left(\sum_i u_{1\bar i}u_{\bar 1 i} + \Gamma\right)|\nabla u|^2,
\end{aligned}$$

where we have used the Cauchy-Schwarz inequality. Meanwhile we have

$$\begin{aligned}
u_{i\bar 1}u_{\bar i 1} &\le u_{i\bar j}u_{\bar i j} = (\triangle u)^2 - 2\det u_{i\bar j} \\
&\le \frac{L^2}{\alpha^2} - \left(e^u - \alpha\lambda^2 + \frac{\alpha\lambda^2}{2}e^{-u}\right)\frac{L}{\alpha^2} + C \\
&\le \frac{L^2}{\alpha^2} + C,
\end{aligned}$$

where we have used the PDE and $C$ is a generic constant the depends on the bound of $u$. Therefore we finally get

$$|\nabla|\nabla u|^2|^2 \le 2\left(\frac{L^2}{\alpha^2} + \Gamma + C\right)|\nabla u|^2.$$

Now we focus on $\triangle|\nabla u|^2$. Using the normal coordinates that diagonalizes $g_{i\bar j}$, we can find $\triangle|\nabla u|^2$ at $q_1$:

$$\begin{aligned}\triangle|\nabla u|^2 &= 2\nabla\triangle u\circ\nabla u+\Gamma+(\triangle u)^2-2\det u_{i\bar{j}}\\ &\le 2\nabla\triangle u\circ\nabla u+\Gamma+\frac{L^2}{\alpha^2}-\frac{L}{\alpha^2}\left(e^u-\alpha\lambda^2+\frac{\alpha\lambda^2}{2}e^{-u}\right)+C\\ &\le 2\nabla\triangle u\circ\nabla u+\Gamma+\frac{L^2}{\alpha^2}+C,\end{aligned}$$

where we have used the PDE. Since $G_1$ takes its maximum value at $q_1$, we have $\nabla G_1(q_1)=0$. This implies

$$\nabla\triangle u=\frac{1}{\alpha}\left[L\lambda_1\nabla u-L\lambda_2\nabla|\nabla u|^2-\left(e^u-\frac{\alpha\lambda^2}{2}e^{-u}\right)\right],$$

at $q_1$. We use this to obtain the following

$$\begin{aligned}\triangle|\nabla u|^2 &\le \frac{2}{\alpha}L\lambda_1|\nabla u|^2+\frac{2}{\alpha}L\lambda_2|\nabla|\nabla u|^2||\nabla u|+\Gamma+\frac{L^2}{\alpha^2}+C\\ &\le\left(\frac{1}{\alpha^2}+\frac{2\sqrt{2}}{\alpha^2}|\nabla u|^2\lambda_2\right)L^2+2C\lambda_2L\sqrt{\Gamma}+\Gamma+\lambda_1CL+\lambda_2CL+C\\ &\le(C+C\lambda_2)L^2+C\lambda_2^2L^2+2\Gamma+C(\lambda_1+\lambda_2)L+C,\end{aligned}$$

which concludes the proof. ☐

We also need the estimates for the derivatives of $F$ to find a bound for the fifth term.

**Lemma 70.** *If $\alpha\lambda^2<2$ and $C_1A<1$, then we have the following estimate:*

$$|\nabla F|\le C\left(L+\sqrt{\Gamma}+1\right).$$

*Furthermore, if $G_1$ take its maximum value at $q_1$, then we have*

$$|\triangle F|\le C|\triangle|\nabla u|^2|+C|\nabla|\nabla u|^2|+CL+C.$$

*where $C$ is a constant that depends on $\alpha$, $\lambda^2$, $\mu$, the Sobolev constant of $g$, and $A$.*

*Proof.* Take a derivative of the definition of $F$:

$$\begin{aligned}\nabla F &= 2\left(e^u - \alpha\lambda^2 + \frac{\alpha\lambda^2}{2}e^{-u}\right)\left(e^u - \frac{\alpha\lambda^2}{2}e^{-u}\right)\nabla u \\ &\quad - 2\alpha\left(e^u + \frac{\alpha\lambda^2}{2}e^{-u}\right)\nabla u|\nabla u|^2 - 2\alpha\left(e^u - \frac{\alpha\lambda^2}{2}e^{-u}\right)\nabla|\nabla u|^2 - 2\alpha\nabla\mu.\end{aligned}$$

Therefore, we have $|\nabla F| \leq C|\nabla|\nabla u|^2| + C$. Meanwhile, the upper bound for $|\nabla|\nabla u|^2|$ can be read from Lemma 69. Thus we obtain,

$$|\nabla F| \leq C\left(L + \sqrt{\Gamma} + 1\right).$$

We take another derivative and contract it with the metric.

$$\begin{aligned}\triangle F &= 2\left(e^u - \frac{\alpha\lambda^2}{2}e^{-u}\right)^2|\nabla u|^2 + 2\left(e^u - \alpha\lambda^2 + \frac{\alpha\lambda^2}{2}e^{-u}\right)\left(e^u + \frac{\alpha\lambda^2}{2}e^{-u}\right)|\nabla u|^2 \\ &\quad + 2\left(e^u - \alpha\lambda^2 + \frac{\alpha\lambda^2}{2}e^{-u}\right)\left(e^u - \frac{\alpha\lambda^2}{2}e^{-u}\right)\triangle u - 2\alpha\left(e^u - \frac{\alpha\lambda^2}{2}e^{-u}\right)|\nabla u|^4 \\ &\quad - 2\alpha\left(e^u + \frac{\alpha\lambda^2}{2}e^{-u}\right)\triangle u|\nabla u|^2 - 4\alpha\left(e^u + \frac{\alpha\lambda^2}{2}e^{-u}\right)\nabla u \circ \nabla|\nabla u|^2 \\ &\quad - 2\alpha\left(e^u - \frac{\alpha\lambda^2}{2}e^{-u}\right)\triangle|\nabla u|^2 - 2\alpha\triangle\mu.\end{aligned}$$

Therefore we have,

$$\begin{aligned}|\triangle F| &\leq C|\triangle|\nabla u|^2| + C|\nabla|\nabla u|^2| + CL + C \\ &\leq C(\lambda_2^2 + \lambda_2^2 + 1)L^2 + 2C\Gamma + C(\lambda_1 + \lambda_2)L + C(L + \sqrt{\Gamma}) + CL + C,\end{aligned}$$

where we have used Lemma 69. Combining the two results for $|\nabla F|$ and $\triangle F$ we get this lemma. □

Now we are ready to find the estimate of the fifth term.

**Lemma 71.** *If $\alpha\lambda^2 < 2$ and $C_1 A < 1$, then the fifth term of the right hand side of Inequality* (4.13) *has the following lower bound at $q_1$:*

$$\frac{1}{2}\left(\triangle F - \frac{|\nabla F|^2}{F}\right) \geq -C(\lambda_2^2 + \lambda_2 + 1)L^2 - C(\lambda_1 + \lambda_2 + 1)L - C\Gamma - C\sqrt{\Gamma} - C,$$

*where $C$ is a constant that depends on $\alpha$, $\lambda^2$, $\mu$, the Sobolev constant of $g$, and $A$.*

*Proof.* Using Lemma 70, we have

$$\frac{1}{2}\left(\triangle F - \frac{|\nabla F|^2}{F}\right) \geq -C|\triangle|\nabla u|^2| - C|\nabla|\nabla u|^2| - CL - C - C\sqrt{\Gamma}.$$

Using Lemma 69 we obtain the result of this lemma. □

Finally we find an estimate for the last term.

**Lemma 72.** *If $\alpha\lambda^2 < 2$ and $C_1 A < 1$, then the sixth term of Inequality* (4.13) *has the following lower bound:*

$$F\triangle'\left(e^u + \frac{\alpha\lambda^2}{2}e^{-u}\right) \geq -CL - C,$$

*where $C$ is a constant that depends on $\alpha\lambda^2$, $\mu$, the Sobolev constant of $g$, and $A$.*

*Proof.* We expand the laplacian and obtain

$$\begin{aligned} F\triangle'\left(e^u + \frac{\alpha\lambda^2}{2}e^{-u}\right) &\geq \left(e^u - \frac{\alpha\lambda^2}{2}e^{-u}\right) F g'^{i\bar{j}} u_{i\bar{j}} \\ &= \left(e^u - \frac{\alpha\lambda^2}{2}e^{-u}\right)(g'_{1\bar{1}}u_{2\bar{2}} + g'_{2\bar{2}}u_{1\bar{1}}), \end{aligned}$$

where we have used normal coordinates that diagonalize $g'$ as well. Expressing $g'$

in terms of $g$ and using the PDE we obtain

$$\begin{aligned}
F\triangle'\left(e^u+\frac{\alpha\lambda^2}{2}e^{-u}\right)\geq\left(e^u-\frac{\alpha\lambda^2}{2}e^{-u}\right)\Bigg[&\left(e^u-\alpha\lambda^2+\frac{\alpha\lambda^2}{2}e^{-u}\right)\triangle u\\
&-2\left\{\triangle\left(e^u-\alpha\lambda^2-\frac{\alpha\lambda^2}{2}e^{-u}\right)+\mu\right\}\Bigg]\\
=-\left(e^u-\frac{\alpha\lambda^2}{2}e^{-u}\right)&\left(e^u+\alpha\lambda^2+\frac{\alpha\lambda^2}{2}e^{-u}\right)\triangle u-C,
\end{aligned}$$

therefore we obtain the result of this lemma. □

Now we are ready to prove Lemma 62.

*Proof of Lemma 62.* Using the estimates derived before, from Inequality (4.13) we obtain the following lower bound

$$\begin{aligned}
LF\triangle'G_1\geq&-\lambda_1LF\triangle'u+\lambda_2LF\triangle'|\nabla u|^2-L\triangle\left(e^u+e^{-u}\frac{\alpha\lambda^2}{2}\right)+\frac{HF}{2}\\
&+\frac{1}{2}\left(\triangle F-\frac{|\nabla F|^2}{F}\right)+F\triangle'\left(e^u+\frac{\alpha\lambda^2}{2}e^{-u}\right)-CL(u)^2\\
\geq&\left[\frac{\lambda_1}{\alpha C_1A}-C(\lambda_2^2+\lambda_2+1)\right]L^2-C(\lambda_1+\lambda_2+1)L\\
&+\left(\frac{\kappa\lambda_2}{2(C_1A)^2}-C\right)\Gamma-C\sqrt{\Gamma}-C.
\end{aligned}$$

This completes the proof. □

## 4.8 Third order estimates

In this section we estimate third order derivatives of $u\in C^5(M)$ following [Aub70, Yau78, FY08]. First we introduce some definitions for this section.

**Definition 73.** The following notations will be used in this section:

$$\begin{aligned}
\Gamma &:= g^{i\bar{j}} g^{k\bar{l}} \nabla_{ik} u \nabla_{\bar{j}\bar{l}} u, \\
\Theta &:= g'^{i\bar{l}} g'^{\bar{j}m} g'^{k\bar{n}} \nabla_{i\bar{j}k} u \nabla_{\bar{l}m\bar{n}} u, \\
\Xi &:= g'^{i\bar{j}} g'^{k\bar{l}} g'^{m\bar{n}} \nabla_{ikm} u \nabla_{\bar{j}\bar{l}\bar{n}} u, \\
\Phi &:= g'^{i\bar{j}} g'^{k\bar{l}} g'^{m\bar{n}} g'^{p\bar{q}} \nabla_{ik\bar{n}p} u \nabla_{\bar{j}\bar{l}m\bar{q}} u, \\
\Psi &:= g'^{i\bar{j}} g'^{k\bar{l}} g'^{m\bar{n}} g'^{p\bar{q}} \nabla_{i\bar{l}m\bar{q}} u \nabla_{\bar{j}k\bar{n}p} u.
\end{aligned}$$

We use the same technique applied in finding the second order estimates to find the third order estimates. Calabi [Cal58] originally suggested the function $S = \Theta + \lambda \triangle \phi$ to find the third order estimate in the Aubin-Calabi-Yau theorem. In our case, the function we consider is

$$G_2 := \triangle u \cdot \Theta + \lambda_3 \Theta + \lambda_4 \triangle u \cdot \Gamma + \lambda_5 |\nabla u|^2 \Gamma + \lambda_6 \Gamma.$$

A similar function has been used in [FY08]. The following lemma suggests why we need $G_2$ instead of $S$.

**Lemma 85.** *Suppose $\alpha\lambda^2 < 2$ and $C_1 A < 1$. If $G_2$ takes its maximum at $q_2$, then we have the following estimate for $\triangle'\Theta$ at $q_2$:*

$$\triangle'\Theta \geq \left[2c - \frac{2c}{\delta} - \epsilon\right] \Theta^2 - C\Gamma\Theta - C\Gamma^2 - C\Xi - C$$

*where $\epsilon$, $0 < \delta < 1$ are positive constants that we are free to choose, and $C$ is a constant that depends on $\epsilon$, $\alpha$, $\lambda^2$, $g$, $\mu$, and $A$.*

From this lemma, we can see why the original function $S$ suggested by Calabi does not work in our case. Suppose we take Calabi's function $S$, and $S$ takes its maximum at $q$. Then we have

$$\begin{aligned}0 &\geq \triangle' S = \triangle'\Theta + \lambda\triangle'\triangle u \\ &\geq \left[2c - \frac{2c}{\delta} - \epsilon\right]\Theta^2 - C\Gamma\Theta - C\Gamma^2 - C\Xi - C + \lambda(\kappa\Theta + C\Gamma + C),\end{aligned}$$

where we have used estimates of $\triangle'\triangle u$ that we will derive later. We can see that the leading order terms of this inequality are negative, thus we cannot find a bound on the terms.

On the contrary, we can pick $\lambda_i$ of the extra terms introduced in $G_2$ such that all the highest order terms in $G_2$ are positive. The additional terms introduced are natural extensions of $S$ in the following sense: the goal of $G_2$ is to find the estimates of third order derivatives of $u$, therefore we should construct $G_2$ with derivatives equal or lower than the third order derivative: $\Theta$, $\Gamma$, $u$, $\nabla u$, and $\triangle u$. We already have estimates of $u$, $\nabla u$, and $\triangle u$. Thus these terms can be treated as *0th order terms*. This leads to constructing $G_2$ as it is.

Using the estimates derived in subsequent sections we can find an expression for $\triangle' G_2$.

**Lemma 89.** *Suppose $\alpha\lambda^2 < 2$, $C_1 A < 1$, and $\lambda_3$ is a large constant such that $\lambda_3 + \triangle u > 0$. If $G_2$ takes its maximum at $q_2$, then we have the following inequality at $q_2$:*

$$\begin{aligned}\triangle' G_2 \geq &\left[\kappa + (\lambda_3 + \triangle u)\left(2c - \frac{2c}{\delta} - \epsilon\right) - \frac{C}{\lambda_3 + \triangle u}\right]\Theta^2 \\ &+ \left[\kappa\lambda_4 - C - \frac{C}{\triangle u + \lambda_3}(\lambda_4 + \lambda_5 + \lambda_6) - C(\lambda_3 + \triangle u)\right]\Gamma\Theta \\ &+ \left[\kappa\lambda_5 - C\lambda_4 - \frac{C}{\triangle u + \lambda_3}\lambda_5 - C(\lambda_3 + \triangle u)\right]\Gamma^2 \\ &+ \left[\kappa\lambda_6 - C\lambda_4 - C\lambda_5 - C - \frac{C}{\triangle u + \lambda_3}(\lambda_4 + \lambda_5 + \lambda_6) - C(\lambda_3 + \triangle u)\right]\Xi \\ &- C\Theta - C\Gamma - C,\end{aligned}$$

*where* $\epsilon$, $0 < \delta < 1$ *are positive smooth constants that we are free to choose;* $c$, $\kappa$ *are constants that depends on* $\alpha$, $\lambda^2$, $\mu$, *the Sobolev constant of* $g$, *and the curvature bound of* $g$*;* $C$ *is a constant that depends on* $\epsilon$, $\alpha$, $\lambda^2$, $\mu$, *the Sobolev constant of* $g$, *and the curvature bound of* $g$*.*

Now we choose functions $\epsilon$, $\delta$ such that the coefficient of $\Theta^2$ is positive to simplify this result. Since $0 < \delta < 1$ and $\lambda_3 + \triangle u > 0$, we can find a $\epsilon$, $\delta$ such that

$$2c - \frac{2c}{\delta} \geq -\frac{\kappa/4}{\lambda_3 + \triangle u}, \quad -\epsilon \geq -\frac{\kappa/4}{\lambda_3 + \triangle u}.$$

Therefore when $\lambda_3$ is a large positive constant, the $\Theta^2$ term has a positive coefficient, i.e.

$$\left(\kappa - \frac{C}{\lambda_3 + \triangle u}\right)\Theta^2 \geq \kappa\Theta^2.$$

So far, we have only assumed $\lambda_3$ is a large positive constant that is large such that $\lambda_3 + \triangle u > 0$. We fix $\lambda_3$, $\lambda_4$, $\lambda_5$, $\lambda_6$ by requiring the coefficients of $\Gamma\Theta$, $\Gamma^2$, $\Xi$ to be positive. This will give three inequalities, and since we have four $\lambda_i$'s this can be thought of an underdetermined problem. Naively, we can expect a solution and we show that this is indeed the case. First we note that there is a general pattern among the coefficients, and we exploit this on $\Gamma^2$ coefficient as well:

$$\begin{aligned}&\left[\kappa\lambda_5 - C\lambda_4 - \frac{C}{\lambda_3 + \triangle u}\lambda_5 - C(\lambda_3 + \triangle u)\right]\Gamma^2 \\ &\quad \geq \left[\kappa\lambda_5 - C\lambda_4 - \frac{C}{\lambda_3 + \triangle u}(\lambda_4 + \lambda_5 + \lambda_6) - C(\lambda_3 + \triangle u)\right]\Gamma^2.\end{aligned}$$

Note that for an arbitrary positive constants $C_4$, $C_5$, $C_6$, and given $\lambda_3$, we can always find $\lambda_4$, $\lambda_5$, $\lambda_6$ that satisfy

$$\begin{aligned}\kappa\lambda_4 - C - C(\lambda_3 + \sup \triangle u) &= C_4\\ \kappa\lambda_5 - C\lambda_4 - C(\lambda_3 + \sup \triangle u) &= C_5\\ \kappa\lambda_6 - C\lambda_4 - C\lambda_5 - C - C(\lambda_3 + \sup \triangle u) &= C_6.\end{aligned}$$

With the choice of $\lambda_4$, $\lambda_5$, $\lambda_6$ and the new coefficient of $\Gamma^2$, the following condition is sufficient for all of the coefficients of $\Gamma\Theta$, $\Gamma^2$, $\Xi$ to be positive

$$C_i - \frac{C}{\triangle u + \lambda_3}(\lambda_4 + \lambda_5 + \lambda_6) \geq 0 \text{ for } i = 4, 5, 6.$$

This can always be satisfied for large $C_i$ and $\lambda_3$. To show this we write $\lambda_4$ in terms of $C_4$ and $\lambda_3$:

$$-\frac{C}{\triangle u + \lambda_3}\lambda_4 \geq -\frac{C}{\kappa} - \frac{C}{\triangle u + \lambda_3}\left(\frac{1}{\kappa} + \frac{C_4}{\kappa}\right),$$

where we have used the fact that $\triangle u$ is bounded. Using this inequality and the definitions of $\lambda_5$ and $\lambda_6$ we can derive

$$\begin{aligned}&-\frac{C}{\triangle u + \lambda_3}\lambda_5 \geq -\frac{C}{\kappa} - \frac{C}{\kappa^2} - \frac{C}{\triangle u + \lambda_3}\left(\frac{1}{\kappa^2} + \frac{C_4}{\kappa^2} + \frac{C_5}{\kappa}\right),\\ &-\frac{C}{\triangle u + \lambda_3}\lambda_6 \geq -\frac{C}{\kappa} - \frac{C}{\kappa^2} - \frac{C}{\kappa^3} - \frac{C}{\triangle u + \lambda_3}\left(\frac{1}{\kappa} + \frac{1}{\kappa^2} + \frac{1}{\kappa^3} + \frac{C_4}{\kappa^2} + \frac{C_4}{\kappa^3} + \frac{C_5}{\kappa^2} + \frac{C_6}{\kappa}\right).\end{aligned}$$

Therefore we obtain,

$$\begin{aligned}&C_i - \frac{C}{\triangle u + \lambda_3}(\lambda_4 + \lambda_5 + \lambda_6)\\ &\quad \geq C_i - C\left[\frac{1}{\kappa} + \frac{1}{\kappa^2} + \frac{1}{\kappa^3}\right]\\ &\qquad - \frac{C}{\lambda_3 + \triangle u}\left[\frac{1}{\kappa} + \frac{1}{\kappa^2} + \frac{1}{\kappa^3} + \frac{C_4}{\kappa} + \frac{C_4}{\kappa^2} + \frac{C_4}{\kappa^3} + \frac{C_5}{\kappa} + \frac{C_5}{\kappa^2} + \frac{C_6}{\kappa}\right]. \qquad (4.17)\end{aligned}$$

Now we fix the generic constant $C$ and take $C_i$'s such that the following is satisfied

$$C_i - C\left[\frac{1}{\kappa} + \frac{1}{\kappa^2} + \frac{1}{\kappa^3}\right] > 0.$$

From Equation (4.17), we can see that for such $C_i$'s we can take $\lambda_3$ large such that all the coefficients of $\Gamma\Theta$, $\Theta^2$, $\Xi$ are positive.

Using the result of this process, we obtain

$$\triangle' G_2 \geq \kappa\Theta^2 + \kappa\Gamma\Theta + \kappa\Gamma^2 - C\Theta - C\Gamma - C,$$

where we are using $\kappa$ and $C$ in a generic sense. We are assuming $G_2$ obtains its maximum at $q_2$, i.e. $\triangle' G_2(q_2) \leq 0$. This implies that $\Theta$ and $\Gamma$ are bounded at $q_2$. Therefore $G_2$ is bounded at $q_2$, and we have

$$(\lambda_3 + \triangle u)\Theta + (\lambda_4 \triangle u + \lambda_5 |\nabla u|^2 + \lambda_6)\Gamma \leq G_2(q_2).$$

Hence $\Theta \leq G_2(q_2)/(\lambda_3 + \triangle u)$, which gives an upper bound for the third derivatives of $u$. We summarize this in the following proposition.

**Proposition 74.** *Let $t \in T$ and $u \in C^5(S)$ be a solution of the PDE under the following conditions,*

1. *Elliptic condition:*

$$\omega' = \left(e^u - \alpha\lambda^2 t + \frac{\alpha\lambda^2 t}{2} e^{-u}\right)\omega_S + 2\alpha i\partial\bar{\partial}u > 0.$$

2. *Normalization conditions:*

$$\left(\int_S e^{-4u}\right)^{1/4} = A < 1, \quad \int_S \frac{\omega_S^2}{2} = 1.$$

*If $C_1A < 1$ and $\alpha\lambda^2 < 2$, then $\nabla_{i\bar{j}k}u$ has an upper bound that depends on $\alpha$, $\lambda^2$, $\mu$, $A$, the Sobolev constant of $g$, and the curvature bound of $g$.*

### 4.8.1 Some basic estimates

First we present some basic estimates of derivatives of $u$ which will serve as building blocks for estimates of $\triangle' G_2$.

**Lemma 75.** *If $\alpha\lambda^2 < 2$ and $C_1A < 1$, then we have the following estimates:*

1. $\kappa\Gamma - C\sqrt{\Theta} - C \leq \triangle'|\nabla u|^2 \leq C(\sqrt{\Theta} + \Gamma + 1)$, *where $\kappa$ is a constant that depends on the bounds of $g$, $u$, and $\triangle u$. $C$ is a constant that depends on the bounds of $g$, $u$, $\nabla u$, and the curvature bound of $g$.*

2. $|\nabla_{ik}|\nabla u|^2| \leq C(\sqrt{\Xi} + \sqrt{\Gamma} + \sqrt{\Theta})$, *where $C$ is a constant that depends on the bounds of $\nabla u$, and $\triangle u$.*

3. $\triangle'(\triangle u) \geq \kappa\Theta - C\Gamma - C$, *where $C$ is a constant that depends on the bounds of $g$, $u$, $\nabla u$, and $\triangle u$.*

4. $|\nabla|\nabla u|^2| \leq C(\sqrt{\Gamma} + 1)$, *where $C$ is a constant that depends on the bounds of $g$, and $\nabla u$.*

5. $|\nabla \triangle u| \leq C\sqrt{\Theta}$, *where $C$ is a constant that depends on the bounds of $g$.*

6. $|\nabla\triangle|\nabla u|^2| \leq C(\sqrt{\Phi} + \sqrt{\Theta} + \sqrt{\Xi\Gamma} + \sqrt{\Gamma\Theta} + \sqrt{\Gamma} + \sqrt{\Psi} + 1)$, *where $C$ is a constant that depends on the bounds of $g$, $\nabla u$, $\triangle u$, the Riemann curvature $R$ of $g$, and $\nabla R$.*

7. $|\nabla(\nabla u \circ \nabla|\nabla u|^2)| \leq C(\Gamma + \sqrt{\Gamma} + \sqrt{\Theta} + \sqrt{\Xi} + 1)$, *where $C$ is a constant that depends on the bounds of $g$, $\nabla u$, and $\triangle u$.*

*Proof.* Expanding the definition of $\triangle'|\nabla u|^2$, the first estimate is obtained by

$$\begin{aligned}\triangle'|\nabla u|^2 &= g^{i\bar{j}}g'^{k\bar{l}}(\nabla_{k\bar{l}i}u\nabla_{\bar{j}}u + \nabla_{\bar{l}i}u\nabla_{k\bar{j}}u + \nabla_{ki}u\nabla_{\bar{l}\bar{j}}u + \nabla_i u\nabla_{k\bar{l}\bar{j}}u)\\ &\geq g'^{k\bar{l}}g^{i\bar{j}}(\nabla_{k\bar{l}i}u\nabla_{\bar{j}}u + \nabla_i u\nabla_{k\bar{l}\bar{j}}u + \nabla_{ki}u\nabla_{\bar{l}\bar{j}}u)\\ &\geq \kappa\Gamma - C\sqrt{\Theta} - C,\end{aligned}$$

where we have used $\nabla_{k\bar{l}\bar{j}}u = \nabla_{\bar{l}k\bar{j}}u - R_{k\bar{j}\bar{l}}{}^{\bar{n}}\nabla_{\bar{n}}u$. $\kappa$ is a constant that depends on the bounds of $g$, $u$, and $\triangle u$. $C$ is a constant that depends on the bounds of $g$, $u$, $\nabla u$, and the curvature bound of $g$. Similarly the upper bound can be read off from the first line:

$$\triangle'|\nabla u|^2 \leq C(\Gamma + \sqrt{\Theta} + 1).$$

The second estimate can be obtain using similar methods.

$$\begin{aligned}\nabla_{kl}\left(g^{i\bar{j}}\nabla_i u\nabla_{\bar{j}}u\right) &= g^{i\bar{j}}\left(\nabla_{kli}u\nabla_{\bar{j}}u + \nabla_{li}u\nabla_{k\bar{j}}u + \nabla_{ki}u\nabla_{l\bar{j}}u + \nabla_i u\nabla_{kl\bar{j}}u\right)\\ &\leq C\left(\sqrt{\Xi} + \sqrt{\Gamma} + \sqrt{\Theta}\right),\end{aligned}$$

where $C$ is a constant that depends on the bounds of $g$, $\nabla u$, and $\triangle u$.

The third estimate can be found starting from the computation of $\triangle'(\triangle u)$ obtained in the previous sections (Corollary 61).

$$\begin{aligned}\triangle'(\triangle u) &\geq -C - C|\triangle F| - C|\nabla F|^2 + 2\alpha g'^{i\bar{n}}g'^{m\bar{j}}g^{k\bar{l}}\nabla_{k\bar{n}m}u\nabla_{\bar{l}i\bar{j}}u\\ &\geq \kappa\Theta - C(\Gamma + 1),\end{aligned}$$

where we have used the result of the next lemma (Lemma 76). $\kappa$ is a constant that depends on the bounds of $g$, $u$, and $\triangle u$. $C$ is a constant that depends on the bounds of $g$, $u$, $\nabla u$, $\triangle u$, and the curvature bound of $g$.

The forth estimate can be derived as the following:

$$\begin{aligned} |\nabla_k |\nabla u|^2| &= |g^{i\bar{j}}(\nabla_{ki}u\nabla_{\bar{j}}u + \nabla_i u \nabla_{k\bar{j}}u)| \\ &\leq C(\sqrt{\Gamma}+1), \end{aligned}$$

where $C$ is a constant that depends on the bounds of $g$, and $\nabla u$.

The fifth estimate is straightforward computation:

$$|\nabla_k \triangle u| = |g^{i\bar{j}}\nabla_{ki\bar{j}}u| \leq C\sqrt{\Theta},$$

where $C$ is a constant that depends on the bounds of $g$.

The sixth estimate is obtain from the following equation:

$$\begin{aligned} \nabla_m \triangle |\nabla u|^2 = g^{k\bar{l}} g^{i\bar{j}} \big( &\nabla_{mk\bar{l}i}u\nabla_{\bar{j}}u + \nabla_{k\bar{l}i}u\nabla_{m\bar{j}}u + \nabla_{m\bar{l}i}u\nabla_{k\bar{j}}u + \nabla_{\bar{l}i}u\nabla_{mk\bar{j}}u \\ &+ \nabla_{mki}u\nabla_{\bar{l}\bar{j}}u + \nabla_{ki}u\nabla_{m\bar{l}\bar{j}}u + \nabla_{mi}u\nabla_{k\bar{l}\bar{j}}u + \nabla_i u\nabla_{mk\bar{l}\bar{j}}u \big). \end{aligned}$$

Therefore, we can estimate

$$|\nabla \triangle |\nabla u|^2| \leq C\left[\sqrt{\Phi} + \sqrt{\Theta} + \sqrt{\Xi\Gamma} + \sqrt{\Gamma}(\sqrt{\Theta}+1) + \sqrt{\Psi} + 1\right],$$

where we have used $\nabla_{m\bar{l}\bar{j}}u = \nabla_{\bar{j}m\bar{l}}u - R_{m\bar{j}\bar{l}}{}^{\bar{n}}\nabla_{\bar{n}}u$. $C$ is a constant that depends on the bounds of $g$, $\nabla u$, $\triangle u$, the Riemann curvature $R$ of $g$, and $\nabla R$.

The seventh estimate comes from the following computation:

$$\begin{aligned} \nabla_m(\nabla u \circ \nabla |\nabla u|^2) = \frac{g^{i\bar{j}}}{2}\big( &\nabla_{mi}u\nabla_{\bar{j}}|\nabla u|^2 + \nabla_i u\nabla_{m\bar{j}}|\nabla u|^2 \\ &+ \nabla_{m\bar{j}}u\nabla_i|\nabla u|^2 + \nabla_{\bar{j}}u\nabla_{mi}|\nabla u|^2\big). \end{aligned}$$

Therefore

$$|\nabla(\nabla u \circ \nabla|\nabla u|^2)| \leq C\sqrt{\Gamma}|\nabla|\nabla u|^2| + C|\triangle|\nabla u|^2| + C|\nabla|\nabla u|^2| + C|\nabla_{mi}|\nabla u|^2|,$$

where $C$ is a constant that depends on the bounds of $g$, $\nabla u$, and $\triangle u$. Using the estimates derived before we get the result of this lemma.

This completes the proof. $\square$

We can also derive estimates for the derivative of $F$.

**Lemma 76.** *If $\alpha\lambda^2 < 2$ and $C_1 A < 1$, then we have the following estimates:*

1. $|\nabla F| \leq C(\sqrt{\Gamma} + 1)$, *where $C$ is a constant that depends on the bounds of $u$, $\nabla u$, and $\triangle u$.*
2. $|\triangle F| \leq C(\Gamma + \sqrt{\Gamma} + \sqrt{\Theta} + 1)$, *and* $|\nabla_{ik} F| \leq C(1 + \sqrt{\Gamma} + \sqrt{\Theta} + \sqrt{\Xi})$,
3. $|\nabla\triangle F| \leq C(1 + \sqrt{\Gamma} + \sqrt{\Theta} + \sqrt{\Xi} + \sqrt{\Phi} + \sqrt{\Psi} + \Gamma + \sqrt{\Xi\Gamma} + \sqrt{\Gamma\Theta})$,

*Proof.* The first estimate is a direct result of Lemma 70, since now we have an estimate of $L$.

The second estimate can be obtained from Lemma 70 as well:

$$\begin{aligned} |\triangle F| &\leq C|\nabla|\nabla u|^2| + |\triangle|\nabla u|^2| + C \\ &\leq C(\sqrt{\Gamma} + 1) + C(\sqrt{\Theta} + \Gamma + 1), \end{aligned}$$

where we have used the result of Lemma 75 and the fact that now we have an estimate for $L$. $\nabla F$ can be computed

$$\begin{aligned} \nabla F = {} & 2\left(e^u - \alpha\lambda^2 + \frac{\alpha\lambda^2}{2}e^{-u}\right)\left(e^u - \frac{\alpha\lambda^2}{2}e^{-u}\right)\nabla u \\ & - 2\alpha\left(e^u + \frac{\alpha\lambda^2}{2}e^{-u}\right)\nabla u|\nabla u|^2 - 2\alpha\left(e^u - \frac{\alpha\lambda^2}{2}e^{-u}\right)\nabla|\nabla u|^2 - 2\alpha\nabla\mu. \end{aligned}$$

From this expression we obtain:

$$\begin{aligned}
|\nabla_{ik}F| &\leq C|\nabla_{ik}u| + C|\nabla|\nabla u|^2| + C|\nabla_{ik}|\nabla u|^2| + C \\
&\leq C\sqrt{\Gamma} + C\left(\sqrt{\Gamma}+1\right) + C\left(\sqrt{\Xi}+\sqrt{\Gamma}+\sqrt{\Theta}\right) + C,
\end{aligned}$$

where we have used Lemma 75. $C$ is a constant that depends on the bounds of $u$, $\nabla u$, and $\mu$.

The third estimate comes from taking the derivative of $\triangle F$:

$$\begin{aligned}
\triangle F = {} & 2\left(e^u - \frac{\alpha\lambda^2}{2}e^{-u}\right)^2 |\nabla u|^2 + 2\left(e^u - \alpha\lambda^2 + \frac{\alpha\lambda^2}{2}e^{-u}\right)\left(e^u + \frac{\alpha\lambda^2}{2}e^{-u}\right)|\nabla u|^2 \\
&+ 2\left(e^u - \alpha\lambda^2 + \frac{\alpha\lambda^2}{2}e^{-u}\right)\left(e^u - \frac{\alpha\lambda^2}{2}e^{-u}\right)\triangle u - 2\alpha\left(e^u - \frac{\alpha\lambda^2}{2}e^{-u}\right)|\nabla u|^4 \\
&- 2\alpha\left(e^u + \frac{\alpha\lambda^2}{2}e^{-u}\right)\triangle u|\nabla u|^2 - 4\alpha\left(e^u + \frac{\alpha\lambda^2}{2}e^{-u}\right)\nabla u \circ \nabla|\nabla u|^2 \\
&- 2\alpha\left(e^u - \frac{\alpha\lambda^2}{2}e^{-u}\right)\triangle|\nabla u|^2 - 2\alpha\triangle\mu.
\end{aligned}$$

Therefore we obtain,

$$\begin{aligned}
|\nabla\triangle F| &\leq C\left[1 + |\nabla|\nabla u|^2| + |\nabla\triangle u| + |\nabla(\nabla u \circ \nabla|\nabla u|^2)| + |\nabla\triangle|\nabla u|^2| + |\triangle|\nabla u|^2|\right], \\
&\leq C\Big(1 + \sqrt{\Gamma} + 1 + \sqrt{\Theta} + \Gamma + \sqrt{\Gamma} + \sqrt{\Theta} + \sqrt{\Xi} + 1 + \sqrt{\Phi} + \sqrt{\Theta} \\
&\qquad + \sqrt{\Xi\Gamma} + \sqrt{\Gamma\Theta} + \sqrt{\Gamma} + \sqrt{\Psi} + 1 + \sqrt{\Theta} + \Gamma + 1\Big),
\end{aligned}$$

where we have used the estimates derived in Lemma 75. This completes the proof.

□

We conclude this section with the following lemma.

**Lemma 77.** *We have the following estimates for* $\Gamma$*:*

$$|\nabla'\Gamma| \leq C\left(\sqrt{\Xi\Gamma} + \sqrt{\Gamma\Theta} + \sqrt{\Gamma}\right),$$
$$\triangle'\Gamma \geq \kappa\Xi + \kappa\Theta - \tilde{\epsilon}\Phi - C\Gamma - C,$$

*where* $\epsilon$ *is a positive constant that we are free to choose;* $C$ *is a constant that depends on* $\epsilon$*,* $g$ *and its curvature bound; and* $\kappa$ *is a constant that depends on the bounds of* $g$*,* $u$*, and* $\triangle u$*.*

*Proof.* We take a derivative of $\Gamma$:

$$\nabla_m\Gamma = g^{i\bar{j}}g^{k\bar{l}}(\nabla_{mik}u\nabla_{\bar{j}\bar{l}}u + \nabla_{ik}u\nabla_{m\bar{j}\bar{l}}u).$$

Therefore,

$$|\nabla\Gamma| \leq C\left(\sqrt{\Xi\Gamma} + \sqrt{\Gamma}(\sqrt{\Theta} + C)\right),$$

where we have used $\nabla_{m\bar{j}\bar{l}}u = \nabla_{\bar{j}m\bar{l}}u - R_{m\bar{l}\bar{j}}{}^{\bar{n}}\nabla_{\bar{n}}u$. $C$ is a constant that depends on $\nabla u$ and $R$.

We take another derivative on the expression we have obtained above and contract it with $g'^{m\bar{n}}$

$$\begin{aligned}
\triangle'\Gamma &= g^{i\bar{j}}g^{k\bar{l}}g'^{m\bar{n}} \\
&\qquad \times \big[(\nabla_{mi\bar{n}k}u - \nabla_m R_{\bar{n}ki}{}^{p}\nabla_p u - R_{\bar{n}ki}{}^{p}\nabla_{mp}u - R_{\bar{n}mi}{}^{p}\nabla_{pk}u - R_{\bar{n}mk}{}^{p}\nabla_{ip}u)\nabla_{\bar{j}\bar{l}}u \\
&\qquad\quad + \nabla_{mik}u\nabla_{\bar{n}\bar{j}\bar{l}}u + (\nabla_{i\bar{n}k}u - R_{\bar{n}ik}{}^{p}\nabla_p u)(\nabla_{\bar{j}m\bar{l}}u - R_{\bar{j}m\bar{l}}{}^{\bar{q}}\nabla_{\bar{q}}u) \\
&\qquad\quad + \nabla_{ik}u(\nabla_{\bar{n}\bar{j}m\bar{l}}u - \nabla_{\bar{n}}R_{\bar{j}m\bar{l}}{}^{\bar{q}}\nabla_{\bar{q}}u - R_{\bar{j}m\bar{l}}{}^{\bar{q}}\nabla_{\bar{n}\bar{q}}u - R_{m\bar{n}\bar{j}}{}^{\bar{q}}\nabla_{\bar{q}\bar{l}}u - R_{m\bar{n}\bar{l}}{}^{\bar{q}}\nabla_{\bar{j}\bar{q}}u)\big] \\
&\geq \kappa\Xi + \kappa\Theta - C\sqrt{\Phi\Gamma} - C\Gamma - C\sqrt{\Gamma} - C\sqrt{\Theta} - C,
\end{aligned}$$

where $\kappa$ is a constant that depends on the bounds of $g$, $u$, and $\triangle u$. $C$ is a constant that depends on the bound of $u$, $\nabla u$, and $R$.

By noting that

$$-C\sqrt{\Phi\Gamma} = \left(\sqrt{\tilde{\epsilon}\Phi} - \frac{C}{\sqrt{\tilde{\epsilon}}}\sqrt{\Gamma}\right)^2 - \tilde{\epsilon}\Phi - C\Gamma$$

for any positive constant $\epsilon$ we obtain this lemma. □

### 4.8.2 Estimates for $\triangle'\Theta$

In this section, we derive estimates for $\triangle'\Theta$ (Lemma 84) following [Yau78, Aub70].

**Lemma 78.** $\triangle'\Theta$ *satisfies the following equation:*

$$\begin{aligned}
\triangle'\Theta = \Big[ & 2g'^{i\bar{u}}g'^{\bar{s}t}g'^{\bar{l}r}g'^{\bar{j}m}g'^{k\bar{n}} + 2g'^{i\bar{s}}g'^{\bar{l}t}g'^{r\bar{u}}g'^{\bar{j}m}g'^{k\bar{n}} + 2g'^{i\bar{s}}g'^{\bar{l}r}g'^{\bar{j}t}g'^{m\bar{u}}g'^{k\bar{n}} \\
& + 2g'^{i\bar{s}}g'^{\bar{l}r}g'^{\bar{j}m}g'^{k\bar{u}}g'^{\bar{n}t} + g'^{i\bar{u}}g'^{\bar{l}t}g'^{\bar{j}r}g'^{m\bar{s}}g'^{k\bar{n}} + g'^{i\bar{l}}g'^{\bar{j}t}g'^{r\bar{u}}g'^{m\bar{s}}g'^{k\bar{n}} \\
& + g'^{i\bar{l}}g'^{\bar{j}r}g'^{m\bar{u}}g'^{\bar{s}t}g'^{k\bar{n}} + g'^{i\bar{l}}g'^{\bar{j}r}g'^{m\bar{s}}g'^{k\bar{u}}g'^{\bar{n}t}\Big]g'^{p\bar{q}} \\
& \times \nabla_{i\bar{j}k}u\nabla_{\bar{l}m\bar{n}}u\nabla_p g'_{t\bar{u}}\nabla_{\bar{q}}g'_{r\bar{s}} && \textit{(first class)} \\
- & \left[2g'^{i\bar{s}}g'^{\bar{l}r}g'^{\bar{j}m} + g'^{i\bar{l}}g'^{\bar{j}r}g'^{m\bar{s}}\right] g'^{k\bar{n}}g'^{p\bar{q}} \\
& \times \left[\nabla_p g'_{r\bar{s}}\nabla_{i\bar{j}k}u\nabla_{\bar{q}\bar{l}m\bar{n}}u + \nabla_{\bar{q}}g'_{r\bar{s}}\nabla_{pi\bar{j}k}u\nabla_{\bar{l}m\bar{n}}u\right] && \textit{(second class)} \\
- & \left[2g'^{i\bar{s}}g'^{\bar{l}r}g'^{\bar{j}m} + g'^{i\bar{l}}g'^{\bar{j}r}g'^{m\bar{s}}\right]g'^{k\bar{n}}g'^{p\bar{q}} \\
& \times \left[\nabla_{\bar{q}}g'_{r\bar{s}}\nabla_{i\bar{j}k}u\nabla_{p\bar{l}m\bar{n}}u + \nabla_p g'_{r\bar{s}}\nabla_{\bar{q}i\bar{j}k}u\nabla_{\bar{l}m\bar{n}}u\right] && \textit{(third class)} \\
- & \left[2g'^{i\bar{s}}g'^{\bar{l}r}g'^{\bar{j}m} + g'^{i\bar{l}}g'^{\bar{j}r}g'^{m\bar{s}}\right]g'^{k\bar{n}}g'^{p\bar{q}}\nabla_{p\bar{q}}g'_{r\bar{s}}\nabla_{i\bar{j}k}u\nabla_{\bar{l}m\bar{n}}u && \textit{(forth class)} \\
+ & g'^{i\bar{l}}g'^{\bar{j}m}g'^{k\bar{n}}g'^{p\bar{q}}\left[\nabla_{p\bar{q}i\bar{j}k}u\nabla_{\bar{l}m\bar{n}}u + \nabla_{i\bar{j}k}u\nabla_{p\bar{q}\bar{l}m\bar{n}}u\right] && \textit{(fifth class)} \\
+ & g'^{i\bar{l}}g'^{\bar{j}m}g'^{k\bar{n}}g'^{p\bar{q}}\left[\nabla_{\bar{q}i\bar{j}k}u\nabla_{p\bar{l}m\bar{n}}u + \nabla_{pi\bar{j}k}u\nabla_{\bar{q}\bar{l}m\bar{n}}u\right] && \textit{(sixth class)} \\
- & C\Theta.
\end{aligned}$$

*Proof.* This is a result of direct computation. □

We find the estimates class-by-class.

**Lemma 79.** *If $\alpha\lambda^2 < 2$ and $C_1 A < 1$, then the first class has the following lower bound:*

$$\begin{aligned}\text{(first class)} \geq \Big[& 2g'^{i\bar{u}}g'^{\bar{s}t}g'^{\bar{l}r}g'^{\bar{j}m}g'^{k\bar{n}} + 2g'^{i\bar{s}}g'^{\bar{l}t}g'^{r\bar{u}}g'^{\bar{j}m}g'^{k\bar{n}} + 2g'^{i\bar{s}}g'^{\bar{l}r}g'^{\bar{j}t}g'^{m\bar{u}}g'^{k\bar{n}} \\ &+ 2g'^{i\bar{s}}g'^{\bar{l}r}g'^{\bar{j}m}g'^{k\bar{u}}g'^{\bar{n}t} + g'^{i\bar{u}}g'^{\bar{l}t}g'^{\bar{j}r}g'^{m\bar{s}}g'^{k\bar{n}} + g'^{i\bar{l}}g'^{\bar{j}t}g'^{r\bar{u}}g'^{m\bar{s}}g'^{k\bar{n}} \\ &+ g'^{i\bar{l}}g'^{\bar{j}r}g'^{m\bar{u}}g'^{\bar{s}t}g'^{k\bar{n}} + g'^{i\bar{l}}g'^{\bar{j}r}g'^{m\bar{s}}g'^{k\bar{u}}g'^{\bar{n}t}\Big]g'^{p\bar{q}} \\ &\times 4\alpha^2\nabla_{i\bar{j}k}u\nabla_{\bar{l}m\bar{n}}u\nabla_{p\bar{u}t}u\nabla_{\bar{q}r\bar{s}}u - \epsilon\Theta^2 - C\Theta,\end{aligned}$$

*where we have the freedom to choose $\epsilon$, and $C$ is a constant that depends on $\epsilon$, the bounds of $g$, $u$, $\nabla u$, and $\triangle u$.*

*Proof.* We express $g'$ in terms of $g$ and find the lower bounds. We do the computation for one term and the rest follows accordingly.

$$\begin{aligned}& 2g'^{i\bar{u}}g'^{\bar{s}t}g'^{\bar{l}r}g'^{\bar{j}m}g'^{k\bar{n}}g'^{p\bar{q}}\nabla_{i\bar{j}k}u\nabla_{\bar{l}m\bar{n}}u\nabla_p g'_{t\bar{u}}\nabla_{\bar{q}}g'_{r\bar{s}} \\ &= 2g'^{i\bar{u}}g'^{\bar{s}t}g'^{\bar{l}r}g'^{\bar{j}m}g'^{k\bar{n}}g'^{p\bar{q}}\nabla_{i\bar{j}k}u\nabla_{\bar{l}m\bar{n}}u \\ &\quad\times\Bigg[4\alpha^2\nabla_{p\bar{u}t}u\nabla_{\bar{q}r\bar{s}}u + \left(e^u - \frac{\alpha\lambda^2 e^{-u}}{2}\right)^2 g_{t\bar{u}}g_{r\bar{s}}\nabla_p u\nabla_{\bar{q}}u \\ &\qquad + 2\alpha\nabla_{p\bar{u}t}u\left(e^u - \frac{\alpha\lambda^2}{2}e^{-u}\right)g_{r\bar{s}}\nabla_{\bar{q}}u + 2\alpha\nabla_{\bar{q}r\bar{s}}u\left(e^u - \frac{\alpha\lambda^2}{2}e^{-u}\right)g_{t\bar{u}}\nabla_p u\Bigg] \\ &\geq 8\alpha^2 g'^{i\bar{u}}g'^{\bar{s}t}g'^{\bar{l}r}g'^{\bar{j}m}g'^{k\bar{n}}g'^{p\bar{q}}\nabla_{i\bar{j}k}u\nabla_{\bar{l}m\bar{n}}u\nabla_{p\bar{u}t}u\nabla_{\bar{q}r\bar{s}}u - C\Theta^{3/2} - C\Theta,\end{aligned}$$

where $C$ is a constant that depends on the bounds of $g$, $u$, $\nabla u$, and $\triangle u$. After similar computation for other terms and splitting $\Theta^{3/2}$, we obtain this lemma. □

The second class can be estimated similarly.

**Lemma 80.** *If $\alpha\lambda^2 < 2$ and $C_1 A < 1$, then the second class has the following lower bound:*

$$\begin{aligned}\text{(second class)} \geq -&\left[2g'^{i\bar{s}}g'^{\bar{l}r}g'^{\bar{j}m} + g'^{i\bar{l}}g'^{\bar{j}r}g'^{m\bar{s}}\right]g'^{k\bar{n}}g'^{p\bar{q}} \\ &\times 2\alpha\Big[\nabla_{p\bar{s}r}u\nabla_{i\bar{j}k}u\nabla_{\bar{q}\bar{l}m\bar{n}}u + \nabla_{\bar{q}r\bar{s}}u\nabla_{pi\bar{j}k}u\nabla_{\bar{l}m\bar{n}}u\Big] - \tilde{\epsilon}\Phi - C\Theta,\end{aligned}$$

*where $\tilde{\epsilon}$ is a positive constant that we have the freedom to choose, and $C$ is a constant that depends on $\tilde{\epsilon}$, the bounds of $f$, $g$, $u$, $\nabla u$, and $\triangle u$.*

*Proof.* A typical term in the second class is:

$$\begin{aligned}
&-2g'^{i\bar{s}}g'^{\bar{l}r}g'^{\bar{j}m}g'^{k\bar{n}}g'^{p\bar{q}}\left[\nabla_p g'_{r\bar{s}}\nabla_{i\bar{j}k}u\nabla_{\bar{q}\bar{l}m\bar{n}}u+\nabla_{\bar{q}}g'_{r\bar{s}}\nabla_{pi\bar{j}k}u\nabla_{\bar{l}m\bar{n}}u\right]\\
&\quad=-2g'^{i\bar{s}}g'^{\bar{l}r}g'^{\bar{j}m}g'^{k\bar{n}}g'^{p\bar{q}}\\
&\qquad\times\Bigg[2\alpha\nabla_{p\bar{s}r}u\nabla_{i\bar{j}k}u\nabla_{\bar{q}\bar{l}m\bar{n}}u+2\alpha\nabla_{\bar{q}r\bar{s}}u\nabla_{pi\bar{j}k}u\nabla_{\bar{l}m\bar{n}}u\\
&\qquad\qquad+\left(e^u-\frac{\alpha\lambda^2}{2}e^{-u}\right)g_{r\bar{s}}\left(\nabla_p u\nabla_{i\bar{j}k}u\nabla_{\bar{q}\bar{l}m\bar{n}}u+\nabla_{\bar{q}}u\nabla_{pi\bar{j}k}u\nabla_{\bar{l}m\bar{n}}u\right)\Bigg]\\
&\quad\geq-4\alpha g'^{i\bar{s}}g'^{\bar{l}r}g'^{\bar{j}m}g'^{k\bar{n}}g'^{p\bar{q}}\left[\nabla_{p\bar{s}r}u\nabla_{i\bar{j}k}u\nabla_{\bar{q}\bar{l}m\bar{n}}u+\nabla_{\bar{q}r\bar{s}}u\nabla_{pi\bar{j}k}u\nabla_{\bar{l}m\bar{n}}u\right]-C\sqrt{\Theta\Phi}
\end{aligned}$$

We note that

$$-C\sqrt{\Theta\Phi}=\left(\sqrt{\tilde{\epsilon}}\sqrt{\Phi}-\frac{C}{\sqrt{\tilde{\epsilon}}}\sqrt{\Theta}\right)^2-\tilde{\epsilon}\Phi-\frac{C}{\tilde{\epsilon}}\sqrt{\Theta}\geq-\tilde{\epsilon}\Phi-C\Theta.$$

We can do the same estimate for the other term and obtain the result of this lemma.

□

The third class is estimated in the following lemma.

**Lemma 81.** *If $\alpha\lambda^2<2$ and $C_1A<1$, then the third class has the following lower bound:*

$$\begin{aligned}
\textit{(third class)}\geq&-\left[2g'^{i\bar{s}}g'^{\bar{l}r}g'^{\bar{j}m}+g'^{i\bar{l}}g'^{\bar{j}r}g'^{m\bar{s}}\right]g'^{k\bar{n}}g'^{p\bar{q}}\\
&\times2\alpha\left[\nabla_{\bar{q}r\bar{s}}u\nabla_{i\bar{j}k}u\nabla_{p\bar{l}m\bar{n}}u+\nabla_{p\bar{s}r}u\nabla_{\bar{q}i\bar{j}k}u\nabla_{\bar{l}m\bar{n}}u\right]-\tilde{\epsilon}\Psi-C\Theta,
\end{aligned}$$

*where we have the freedom to choose a positive constant $\tilde{\epsilon}$, and $C$ is a constant that depends on $\tilde{\epsilon}$, the bounds of $g$, $u$, and $\nabla u$.*

*Proof.* A generic term in the third class can be estimated as the following:

$$\begin{aligned}
&-2g'^{i\bar{s}}g'^{\bar{l}r}g'^{\bar{j}m}g'^{k\bar{n}}g'^{p\bar{q}} \\
&\quad\times\Bigg[2\alpha\left(\nabla_{\bar{q}r\bar{s}}u\nabla_{i\bar{j}k}u\nabla_{p\bar{l}m\bar{n}}u+\nabla_{p\bar{s}r}u\nabla_{\bar{q}i\bar{j}k}u\nabla_{\bar{l}m\bar{n}}u\right) \\
&\qquad+\left(e^{u}-\frac{\alpha\lambda^{2}}{2}e^{-u}\right)g_{r\bar{s}}\left(\nabla_{\bar{q}}u\nabla_{p\bar{l}m\bar{n}}u\nabla_{i\bar{j}k}u+\nabla_{p}u\nabla_{\bar{q}i\bar{j}k}u\nabla_{\bar{l}m\bar{n}}u\right)\Bigg] \\
&\quad\geq-4\alpha g'^{i\bar{s}}g'^{\bar{l}r}g'^{\bar{j}m}g'^{k\bar{n}}g'^{p\bar{q}}\left(\nabla_{\bar{q}r\bar{s}}u\nabla_{i\bar{j}k}u\nabla_{p\bar{l}m\bar{n}}u+\nabla_{p\bar{s}r}u\nabla_{\bar{q}i\bar{j}k}u\nabla_{\bar{l}m\bar{n}}u\right)-C\sqrt{\Theta\Psi},
\end{aligned}$$

where $C$ depends on the bounds of $g$, $u$, and $\nabla u$.

By using $-C\sqrt{\Theta\Psi}\geq-\tilde{\epsilon}\Psi-C\Theta$ and repeating this process for the other term, we obtain the result of this lemma. □

To find the estimates of the forth class we use the PDE to reduce the other derivatives and use estimates of $\nabla F$ and $\triangle F$ that we have derived before.

**Lemma 82.** *If $\alpha\lambda^2<2$ and $C_1A<1$, then the forth class has the following lower bound:*

$$\begin{aligned}
\textit{(forth class)}\geq&-\left[2g'^{i\bar{s}}g'^{\bar{l}r}g'^{\bar{j}m}+g'^{i\bar{l}}g'^{\bar{j}r}g'^{m\bar{s}}\right]g'^{k\bar{n}}g'^{\bar{w}t}g'^{\bar{u}v}4\alpha^{2}\nabla_{r\bar{u}t}u\nabla_{\bar{s}v\bar{w}}u\nabla_{i\bar{j}k}u\nabla_{\bar{l}m\bar{n}}u \\
&-\epsilon\Theta^{2}-C\Theta\left(\Gamma+1\right),
\end{aligned}$$

*where we have the freedom to choose a bounded positive constant $\epsilon$ and $C$ is a constant that depends on $\epsilon$, the bounds of $g$, $u$, $\nabla u$, and $\triangle u$.*

*Proof.* Expand $g'$ in a typical term in the forth class then we get

$$\begin{aligned}
&-2g'^{i\bar{s}}g'^{\bar{l}r}g'^{\bar{j}m}g'^{k\bar{n}}g'^{p\bar{q}}\nabla_{p\bar{q}}g'_{r\bar{s}}\nabla_{i\bar{j}k}u\nabla_{\bar{l}m\bar{n}}u \\
&\quad\geq-2g'^{i\bar{s}}g'^{\bar{l}r}g'^{\bar{j}m}g'^{k\bar{n}}\left(2\alpha g'^{p\bar{q}}\nabla_{p\bar{q}r\bar{s}}u\right)\nabla_{i\bar{j}k}u\nabla_{\bar{l}m\bar{n}}u-C\Theta \\
&\quad\geq-2g'^{i\bar{s}}g'^{\bar{l}r}g'^{\bar{j}m}g'^{k\bar{n}}\nabla_{i\bar{j}k}u\nabla_{\bar{l}m\bar{n}}u\left[-\frac{\nabla_{\bar{s}}F\nabla_{r}F}{F^{2}}+\frac{\nabla_{r\bar{s}}F}{F}+g'^{\bar{w}t}g'^{\bar{u}v}\nabla_{r}g'_{t\bar{u}}\nabla_{\bar{s}}g'_{v\bar{w}}\right] \\
&\qquad-C\Theta
\end{aligned}$$

where $C$ depends on the bounds of $g$, $u$, and $\nabla u$. In this inequality, we have used Lemma 60 to reduce the forth order derivative into lower order terms. Expanding $g'$ again we obtain:

$$\begin{aligned}
&-2g'^{i\bar{s}}g'^{\bar{l}r}g'^{\bar{j}m}g'^{k\bar{n}}g'^{p\bar{q}}\nabla_{p\bar{q}}g'_{r\bar{s}}\nabla_{i\bar{j}k}u\nabla_{\bar{l}m\bar{n}}u \\
&\quad \geq -2g'^{i\bar{s}}g'^{\bar{l}r}g'^{\bar{j}m}g'^{k\bar{n}}g'^{\bar{w}t}g'^{\bar{u}v}4\alpha^2\nabla_{r\bar{u}t}u\nabla_{\bar{s}v\bar{w}}u\nabla_{i\bar{j}k}u\nabla_{\bar{l}m\bar{n}}u \\
&\qquad - C\Theta\left(|\nabla F|^2 + |\triangle F|\right) - C\Theta \\
&\quad \geq -2g'^{i\bar{s}}g'^{\bar{l}r}g'^{\bar{j}m}g'^{k\bar{n}}g'^{\bar{w}t}g'^{\bar{u}v}4\alpha^2\nabla_{r\bar{u}t}u\nabla_{\bar{s}v\bar{w}}u\nabla_{i\bar{j}k}u\nabla_{\bar{l}m\bar{n}}u \\
&\qquad - C\Theta\left(\Gamma + \sqrt{\Gamma} + 1 + \sqrt{\Theta}\right) - C\Theta,
\end{aligned}$$

where we have used Lemma 75. Realizing $-C\Theta^{3/2} \geq -\epsilon\Theta^2 - C\Theta$, and following the same process for the other term and obtain the result of this lemma. □

We use the PDE and the estimates of $\nabla F$ and $\triangle F$ for the fifth class as well.

**Lemma 83.** *If $\alpha\lambda^2 < 2$ and $C_1 A < 1$, then the fifth class has the following lower bound:*

$$\begin{aligned}
\textit{(fifth class)} \geq\; & g'^{i\bar{l}}g'^{\bar{j}m}g'^{k\bar{n}}g'^{p\bar{s}}g'^{\bar{q}r}2\alpha\left[\nabla_{i\bar{s}r}u\nabla_{p\bar{q}k\bar{j}}u\nabla_{\bar{l}m\bar{n}}u + \nabla_{\bar{l}r\bar{s}}u\nabla_{p\bar{q}m\bar{n}}u\nabla_{i\bar{j}k}u\right] \\
& + g'^{i\bar{l}}g'^{\bar{j}m}g'^{k\bar{n}}g'^{\bar{q}r}g'^{p\bar{s}}2\alpha\big[(\nabla_{i\bar{j}r\bar{s}}u\nabla_{k\bar{q}p}u + \nabla_{\bar{j}r\bar{s}}u\nabla_{ik\bar{q}p}u)\nabla_{\bar{l}m\bar{n}}u \\
& \qquad\qquad + (\nabla_{\bar{l}m\bar{s}r}u\nabla_{\bar{n}p\bar{q}}u + \nabla_{m\bar{s}r}u\nabla_{\bar{l}\bar{n}p\bar{q}}u)\nabla_{i\bar{j}k}u\big] \\
& - g'^{i\bar{l}}g'^{\bar{j}m}g'^{k\bar{n}}(g'^{\bar{q}t}g'^{r\bar{u}}g'^{p\bar{s}} + g'^{\bar{q}r}g'^{p\bar{u}}g'^{\bar{s}t})4\alpha^2 \\
& \quad \times \left[\nabla_{i\bar{u}t}u\nabla_{\bar{j}r\bar{s}}u\nabla_{k\bar{q}p}u\nabla_{\bar{l}m\bar{n}}u + \nabla_{\bar{l}t\bar{u}}u\nabla_{m\bar{s}r}u\nabla_{\bar{n}p\bar{q}}u\nabla_{i\bar{j}k}u\right] \\
& - \tilde{\epsilon}\Phi - \tilde{\epsilon}\Psi - \epsilon\Theta^2 - C\Theta - C\Xi - C\Gamma^2 - C\Gamma\Theta - C
\end{aligned}$$

*where we have the freedom to choose positive constant $\epsilon$ and $\tilde{\epsilon}$. $C$ is a constant that depends on $\epsilon$, $\tilde{\epsilon}$, the bounds of $g$, $u$, $\nabla u$, and the curvature bound of $g$.*

*Proof.* We differentiate the result of Lemma 60 and obtain:

$$\begin{aligned}
2\alpha g'^{p\bar{q}}\nabla_{p\bar{q}i\bar{j}k}u = &-\nabla_i\left[g'^{p\bar{q}}g_{p\bar{q}}\nabla_{k\bar{j}}\left(e^u + e^{-u}\frac{\alpha\lambda^2}{2}\right)\right] \\
&+ 2\alpha g'^{p\bar{q}}\left(R_{i\bar{q}\bar{l}}{}^{\bar{n}}\nabla_{pk\bar{n}}u - R_{\bar{q}pk}{}^{m}\nabla_{pm\bar{j}}u - R_{pik}{}^{m}\nabla_{\bar{q}m\bar{j}}u\right) \\
&+ 2\alpha\nabla_i\left[g'^{p\bar{q}}(R_{k\bar{j}\bar{q}}{}^{\bar{n}}\nabla_{p\bar{n}}u - R_{p\bar{q}\bar{j}}{}^{\bar{n}}\nabla_{k\bar{n}}u)\right] \\
&+ 2\alpha g'^{p\bar{n}}g'^{\bar{q}m}\nabla_i g'_{m\bar{n}}\nabla_{p\bar{q}k\bar{j}}u + \nabla_i\left(g'^{\bar{q}m}g'^{p\bar{n}}\nabla_{\bar{j}}g'_{m\bar{n}}\nabla_k g'_{p\bar{q}}\right) \\
&+ \frac{\nabla_{i\bar{j}k}F}{F} - \frac{\nabla_{i\bar{j}}F\nabla_k F + \nabla_{\bar{j}k}F\nabla_i F + \nabla_{ik}F\nabla_{\bar{j}}F}{F^2} + 2\frac{\nabla_i F\nabla_{\bar{j}}F\nabla_k F}{F^3}.
\end{aligned}$$

Using this expression, we can derive an estimate on a typical term of the fifth class:

$$\begin{aligned}
&2\alpha g'^{i\bar{l}}g'^{\bar{j}m}g'^{k\bar{n}}g'^{p\bar{q}}\nabla_{p\bar{q}i\bar{j}k}u\nabla_{\bar{l}m\bar{n}}u \\
&\geq 2\alpha g'^{i\bar{l}}g'^{\bar{j}m}g'^{k\bar{n}}g'^{p\bar{s}}g'^{\bar{q}r}\nabla_i g'_{r\bar{s}}\nabla_{p\bar{q}k\bar{j}}u\nabla_{\bar{l}m\bar{n}}u - C\Theta \\
&\quad + g'^{i\bar{l}}g'^{\bar{j}m}g'^{k\bar{n}}\nabla_{\bar{l}m\bar{n}}u\Bigg[\nabla_i(g'^{\bar{q}r}g'^{p\bar{s}}\nabla_{\bar{j}}g'_{r\bar{s}}\nabla_k g'_{p\bar{q}}) + \frac{\nabla_{i\bar{j}k}F}{F} + 2\frac{\nabla_i F\nabla_{\bar{j}}F\nabla_k F}{F^3} \\
&\qquad - \frac{\nabla_{i\bar{j}}F\nabla_k F + \nabla_{\bar{j}k}F\nabla_i F + \nabla_{ik}F\nabla_{\bar{j}}F}{F^2}\Bigg],
\end{aligned}$$

where $C$ is a constant that depends on the curvature bound of $g$. The first term can be estimated as the following by expanding $g'$:

$$\begin{aligned}
&2\alpha g'^{i\bar{l}}g'^{\bar{j}m}g'^{k\bar{n}}g'^{p\bar{s}}g'^{\bar{q}r}\nabla_i g'_{r\bar{s}}\nabla_{p\bar{q}k\bar{j}}u\nabla_{\bar{l}m\bar{n}}u \\
&\geq 4\alpha^2 g'^{i\bar{l}}g'^{\bar{j}m}g'^{k\bar{n}}g'^{p\bar{s}}g'^{\bar{q}r}\nabla_{i\bar{s}r}u\nabla_{p\bar{q}k\bar{j}}u\nabla_{\bar{l}m\bar{n}}u - C\sqrt{\Psi}\Theta,
\end{aligned}$$

where $C$ now also depends on the bounds of $g$, $u$, and $\nabla u$. The estimate for the second term can be obtained by expanding $g'$ with estimates $|\nabla_i g'_{t\bar{u}}| \leq C(1+\sqrt{\Theta})$,

$|\nabla_{i\bar{j}} g'_{r\bar{s}}| \leq C(1 + \sqrt{\Theta} + \sqrt{\Psi})$, and $|\nabla_{ik} g'_{p\bar{q}}| \leq C(1 + \sqrt{\Theta} + \sqrt{\Phi})$:

$$\begin{aligned}
& g'^{i\bar{l}} g'^{\bar{j}m} g'^{k\bar{n}} \nabla_{\bar{l}m\bar{n}} u \nabla_i (g'^{\bar{q}r} g'^{p\bar{s}} \nabla_{\bar{j}} g'_{r\bar{s}} \nabla_k g'_{p\bar{q}}) \\
& \geq + g'^{i\bar{l}} g'^{\bar{j}m} g'^{k\bar{n}} g'^{\bar{q}r} g'^{p\bar{s}} 4\alpha^2 \left[ \nabla_{i\bar{j}r\bar{s}} u \nabla_{k\bar{q}p} u + \nabla_{\bar{j}r\bar{s}} u \nabla_{ik\bar{q}p} u \right] \nabla_{\bar{l}m\bar{n}} u \\
& \quad - g'^{i\bar{l}} g'^{\bar{j}m} g'^{k\bar{n}} \left( g'^{\bar{q}t} g'^{r\bar{u}} g'^{p\bar{s}} + g'^{\bar{q}r} g'^{p\bar{u}} g'^{\bar{s}t} \right) 8\alpha^3 \nabla_{i\bar{u}t} u \nabla_{\bar{j}r\bar{s}} u \nabla_{k\bar{q}p} u \nabla_{\bar{l}m\bar{n}} u \\
& \quad - C \left( \Theta^{3/2} + \Theta + \sqrt{\Theta} + \sqrt{\Psi\Theta} + \sqrt{\Phi\Theta} \right).
\end{aligned}$$

The terms involving $F$ can be estimated using Lemma 76. We finally obtain the following estimate:

$$\begin{aligned}
& g'^{i\bar{l}} g'^{\bar{j}m} g'^{k\bar{n}} g'^{p\bar{q}} \nabla_{p\bar{q}i\bar{j}k} u \nabla_{\bar{l}m\bar{n}} u \\
& \geq g'^{i\bar{l}} g'^{\bar{j}m} g'^{k\bar{n}} g'^{p\bar{s}} g'^{\bar{q}r} 2\alpha \nabla_{i\bar{s}r} u \nabla_{p\bar{q}k\bar{j}} u \nabla_{\bar{l}m\bar{n}} u \\
& \quad + g'^{i\bar{l}} g'^{\bar{j}m} g'^{k\bar{n}} g'^{\bar{q}r} g'^{p\bar{s}} 2\alpha \left[ \nabla_{i\bar{j}r\bar{s}} u \nabla_{k\bar{q}p} u + \nabla_{\bar{j}r\bar{s}} u \nabla_{ik\bar{q}p} u \right] \nabla_{\bar{l}m\bar{n}} u \\
& \quad - g'^{i\bar{l}} g'^{\bar{j}m} g'^{k\bar{n}} \left( g'^{\bar{q}t} g'^{r\bar{u}} g'^{p\bar{s}} + g'^{\bar{q}r} g'^{p\bar{u}} g'^{\bar{s}t} \right) 4\alpha^2 \nabla_{i\bar{u}t} u \nabla_{\bar{j}r\bar{s}} u \nabla_{k\bar{q}p} u \nabla_{\bar{l}m\bar{n}} u \\
& \quad - C\sqrt{\Theta} - C\Theta - C\Theta^{3/2} - C\sqrt{\Psi\Theta} - C\sqrt{\Phi\Theta} - C\sqrt{\Gamma\Theta} \\
& \quad - C\sqrt{\Xi\Theta} - C\Gamma\sqrt{\Theta} - C\sqrt{\Xi\Gamma\Theta} - C\sqrt{\Gamma}\Theta - C\sqrt{\Theta}\Gamma^{3/2},
\end{aligned}$$

where we have used Lemma 76. The other term can be estimated similarly and by splitting terms we obtain the result. □

Collecting all the estimates we have derived, we prove the following lemma.

**Lemma 84.** *Suppose $\alpha\lambda^2 < 2$ and $C_1 A < 1$. If $G_2$ takes its maximum at $q_2$, then we have* *the following estimate for $\triangle'\Theta$ at $q_2$:*

$$\begin{aligned}\triangle'\Theta \geq g'^{i\bar{i}}g'^{j\bar{j}}g'^{k\bar{k}}g'^{p\bar{p}}\Bigg[&\left|\nabla_{\bar{p}i\bar{j}k}u - 2\alpha g'^{l\bar{l}}\nabla_{\bar{p}l\bar{j}}u\nabla_{i\bar{l}k}u\right|^2\\ &+\left|\nabla_{pi\bar{j}k}u - 2\alpha g'^{l\bar{l}}\left(\nabla_{p\bar{j}l}u\nabla_{i\bar{l}k}u + \nabla_{p\bar{l}i}u\nabla_{l\bar{j}k}u\right)\right|^2\Bigg]\\ -\tilde{\epsilon}\Phi - \tilde{\epsilon}\Psi - \epsilon\Theta^2 - C\Gamma\Theta - C\Gamma^2 - C\Xi - C,&\end{aligned}$$

*where $\epsilon$, $\tilde{\epsilon}$ are positive constants that we are free to choose, and $C$ is a constant that depends on $\epsilon$, $\tilde{\epsilon}$, $\alpha$, $\lambda^2$, $g$, $\mu$, and $A$.*

*Proof.* We use the estimates we have derived and use normal coordinate that diagonalize $g'$ to obtain Lemma 84. □

We would like to construct $G_2$ such that $\triangle' G_2(q_2) \leq 0$ gives us a bound on $\Theta$. Suppose $G_2$ only have $\Theta$, then from the previous lemma we immediately observe that the leading order of $\Phi$, and $\Psi$ are negative. By introducing positive constant $\delta$, we can make the leading orders positive.

$$\begin{aligned}\triangle'\Theta \geq\ & g'^{i\bar{i}}g'^{j\bar{j}}g'^{k\bar{k}}g'^{p\bar{p}}\\ &\times\Bigg[\left|\sqrt{\delta}\nabla_{\bar{p}i\bar{j}k}u - \frac{2\alpha}{\sqrt{\delta}}g'^{l\bar{l}}\nabla_{\bar{p}l\bar{j}}u\nabla_{i\bar{l}k}u\right|^2\\ &\quad+\left|\sqrt{\delta}\nabla_{pi\bar{j}k}u - \frac{2\alpha}{\sqrt{\delta}}g'^{l\bar{l}}\left(\nabla_{p\bar{j}l}u\nabla_{i\bar{l}k}u + \nabla_{p\bar{l}i}u\nabla_{l\bar{j}k}u\right)\right|^2\Bigg]\\ &+(1-\delta-\tilde{\epsilon})\Psi + (1-\delta-\tilde{\epsilon})\Phi + \left[2c - \frac{2c}{\delta} - \epsilon\right]\Theta^2\\ &- C\Gamma\Theta - C\Gamma^2 - C\Xi - C. \qquad (4.18)\end{aligned}$$

If $0 < \delta < 1$, then we can always find a small $\tilde{\epsilon}$ such that $1-\delta-\tilde{\epsilon} > 0$. Therefore we obtain the following lemma.

**Lemma 85.** *Suppose $\alpha\lambda^2 < 2$ and $C_1 A < 1$. If $G_2$ takes its maximum at $q_2$, then we have* *the following estimate for $\triangle'\Theta$ at $q_2$:*

$$\triangle'\Theta \geq (1-\delta-\tilde{\epsilon})\Phi + \left[2c - \frac{2c}{\delta} - \epsilon\right]\Theta^2 - C\Gamma\Theta - C\Gamma^2 - C\Xi - C,$$

*where $\epsilon$, $\tilde{\epsilon}$, $\delta$ are positive constants we are free to choose, and $C$ is a constant that depends on $\epsilon$, $\tilde{\epsilon}$, $\alpha$, $\lambda^2$, $g$, $\mu$, and $A$.*

### 4.8.3 Estimates of $\triangle' G_2$

In this section we give the details of the estimates of $\triangle' G_2$:

$$\triangle' G_2 = \triangle'(\triangle u \cdot \Theta) + \lambda_3 \triangle'\Theta + \lambda_4 \triangle'(\triangle u \cdot \Gamma) + \lambda_5 \triangle'(|\nabla u|^2 \Gamma) + \lambda_6 \triangle'\Gamma.$$

**Lemma 86.** *At $q_2$ we have*

$$\begin{aligned}\triangle'(\triangle u \cdot \Theta) \geq {} & \triangle u \cdot \triangle'\Theta + \kappa\Theta^2 - C\Theta\Gamma - C\Theta \\ & - \frac{C}{\triangle u + \lambda_3}\Big[\Theta^2 + \lambda_4\Gamma\Theta + \lambda_5\left(\Gamma\Theta + \Gamma^2 + \Gamma + \Theta\right) \\ & \qquad + (\lambda_4 + \lambda_5 + \lambda_6)(\Xi + \Gamma + \Theta + \Gamma\Theta)\Big],\end{aligned}$$

*where $C$ is a constant that depends on the bounds of $g$, $\nabla u$, $\triangle u$, and the curvature bound of $g$.*

*Proof.* At $q_2$ we are assuming $G_2$ reaches its maximum thus we have $\nabla' G_2 = 0$. This can be used to find an expression for $\nabla'\Theta$ as in the following:

$$\nabla'\Theta = -\frac{1}{\triangle u + \lambda_3}\left(\Theta\nabla'\triangle u + \lambda_4\nabla'(\triangle u \cdot \Gamma) + \lambda_5\nabla'(|\nabla u|^2\Gamma) + \lambda_6\nabla'\Gamma\right).$$

From this expression we can estimate $|\nabla'\Theta|$:

$$\begin{aligned}
|\nabla'\Theta| &\leq \frac{C}{\triangle u+\lambda_3}\left(\Theta|\nabla'\triangle u| + \left[\lambda_4|\nabla'\triangle u| + \lambda_5|\nabla'|\nabla u|^2|\right]\Gamma + (\lambda_4+\lambda_5+\lambda_6)|\nabla'\Gamma|\right)\\
&\leq \frac{C}{\triangle u+\lambda_3}\Big[\Theta^{3/2} + \lambda_4\Gamma\sqrt{\Theta} + \lambda_5\left(\Gamma^{3/2}+\sqrt{\Gamma}\right)\\
&\qquad +(\lambda_4+\lambda_5+\lambda_6)\left(\sqrt{\Xi\Gamma}+\sqrt{\Gamma\Theta}+\sqrt{\Gamma}\right)\Big],
\end{aligned}$$

where we have used Lemma 75, and 77. $C$ is a constant that depends on the bounds of $g$, $\nabla u$, $\triangle u$, and the curvature bound of $g$.

By expanding the laplacian and using the above expression with Lemma 75, we obtain

$$\begin{aligned}
\triangle'(\triangle u\cdot\Theta) &\geq \triangle u\cdot\triangle'\Theta + \Theta\triangle'\triangle u - C|\nabla'\triangle u||\nabla'\Theta|,\\
&\geq \triangle u\cdot\triangle'\Theta + \kappa\Theta^2 - C\Theta\Gamma - C\Theta\\
&\quad -\frac{C}{1+\lambda_3}\Big[\Theta^2 + \lambda_4\Gamma\Theta + \lambda_5\left(\Gamma^{3/2}\sqrt{\Theta}+\sqrt{\Gamma\Theta}\right)\\
&\qquad + (\lambda_4+\lambda_5+\lambda_6)\left(\sqrt{\Xi\Gamma\Theta}+\sqrt{\Gamma}\Theta+\sqrt{\Gamma\Theta}\right)\Big].
\end{aligned}$$

We split $-C\sqrt{\Xi\Gamma\Theta}$ in the following way,

$$-C\sqrt{\Xi\Gamma\Theta} = \left(\sqrt{\Xi}-C\sqrt{\Gamma\Theta}\right)^2 - \Xi - C\Gamma\Theta.$$

By splitting other terms similarly we obtain this lemma. □

**Lemma 87.** *We have the following lower bound:*

$$\triangle'(|\nabla u|^2\Gamma) \geq \kappa\Gamma^2 - \epsilon\Phi - C\Gamma - C\Xi - C\Theta - C$$

*where $\epsilon$ is a positive constant we are free to choose, and $C$ is a constant that depends on $\epsilon$, the bounds of $f$, $g$, $u$, $\nabla u$, and $\triangle u$.*

*Proof.* We expand the laplacian and use Lemma 75 and 77.

$$\begin{aligned}\triangle'(|\nabla u|^2\Gamma) &= \triangle'|\nabla u|^2 \cdot \Gamma + 2\nabla'|\nabla u|^2 \circ \nabla'\Gamma + |\nabla u|^2\triangle'\Gamma \\ &\geq (\kappa\Gamma - C\sqrt{\Theta} - C)\Gamma - C(\sqrt{\Gamma}+1)(\sqrt{\Xi\Gamma} + \sqrt{\Gamma\Theta} + \sqrt{\Gamma}) \\ &\quad + C(\kappa\Xi + \kappa\Theta - \epsilon\Phi - C\Gamma - C),\end{aligned}$$

where $\tilde{\epsilon}$ is a positive constant, and $C$ depends on the bounds of $\epsilon$, $g$, $u$, $\nabla u$, and $\triangle u$. We use the following identity:

$$-C\Gamma\sqrt{\Xi} = \left(\Gamma\delta - \frac{C}{\delta}\sqrt{\Xi}\right)^2 - \Gamma^2\delta^2 - \frac{C^2}{\delta^2}\Xi.$$

By choosing $\delta$ small enough so that $\kappa - \delta^2 > 0$, and using similar identities to simplify we obtain the result of this lemma. ☐

**Lemma 88.** *We have the following lower bound:*

$$\triangle'(\triangle u \cdot \Gamma) \geq \kappa\Theta\Gamma - C\Gamma^2 - \epsilon\Phi - C\Xi - C\Theta - C,$$

*where $\epsilon$ is a positive constant we are free to choose; and $C$ is a constant that depends on $\epsilon$, the bounds of $g$ and curvature bounds of $g$.*

*Proof.* We observe

$$\begin{aligned}\triangle'(\triangle u \cdot \Gamma) &\geq \triangle'(\triangle u) \cdot \Gamma - C|\nabla'\triangle u||\nabla'\Gamma| + \triangle u \cdot \triangle'\Gamma \\ &\geq (\kappa\Theta - C\sqrt{\Theta} - C)\Gamma - C\sqrt{\Theta}\left(\sqrt{\Xi\Gamma} + \sqrt{\Gamma\Theta} + \sqrt{\Gamma}\right) \\ &\quad + \triangle u \cdot (\kappa\Xi + \kappa\Theta - \epsilon\Phi - C\Gamma - C),\end{aligned}$$

where we have used Lemma 75 and 77. $C$ is a constant that depends on the bounds of $g$ and the curvature bounds of $g$.

By splitting terms in the following way

$$\kappa\Theta\Gamma - C\sqrt{\Theta}\Gamma - C\sqrt{\Gamma}\Theta \geq \kappa\Theta\Gamma - C\Gamma - C\Theta,$$

we obtain the result of this lemma. □

Gathering all the estimates of this section, we obtain the following

$$\begin{aligned}
\triangle' G_2 \geq\, & (\lambda_3 + \triangle u)\triangle'\Theta + \left(\kappa - \frac{C}{\triangle u + \lambda_3}\right)\Theta^2 \\
& + \left[\kappa\lambda_4 - C - \frac{C}{\triangle u + \lambda_3}(\lambda_4 + \lambda_5 + \lambda_6)\right]\Gamma\Theta \\
& + \left[\kappa\lambda_5 - C\lambda_4 - \frac{C}{\triangle u + \lambda_3}\lambda_5\right]\Gamma^2 \\
& + \left[\kappa\lambda_6 - C\lambda_4 - C\lambda_5 - \frac{C}{\triangle u + \lambda_3}(\lambda_4 + \lambda_5 + \lambda_6)\right]\Xi \\
& - \epsilon\Phi - C\Theta - C\Gamma - C.
\end{aligned}$$

By using the estimates on $\triangle'\Theta$ (Lemma 85), we obtain the following inequality

$$\begin{aligned}
\triangle' G_2 \geq\, & \left[(\lambda_3 + \triangle u)(1 - \delta - \tilde{\epsilon}) - \epsilon\right]\Phi \\
& + \left[\kappa - (\lambda_3 + \triangle u)\left(2c - \frac{2c}{\delta} - \epsilon\right) - \frac{C}{\lambda_3 + \triangle u}\right]\Theta^2 \\
& + \left[\kappa\lambda_4 - C - \frac{C}{\triangle u + \lambda_3}(\lambda_4 + \lambda_5 + \lambda_6) - C(\lambda_3 + \triangle u)\right]\Gamma\Theta \\
& + \left[\kappa\lambda_5 - C\lambda_4 - \frac{C}{\triangle u + \lambda_3}\lambda_5 - C(\lambda_3 + \triangle u)\right]\Gamma^2 \\
& + \left[\kappa\lambda_6 - C\lambda_4 - C\lambda_5 - C - \frac{C}{\triangle u + \lambda_3}(\lambda_4 + \lambda_5 + \lambda_6) - C(\lambda_3 + \triangle u)\right]\Xi \\
& - C\Theta - C\Gamma - C.
\end{aligned}$$

For small $0 < \delta < 1$ we can always find $\tilde{\epsilon}$ such that $1 - \delta - \tilde{\epsilon} > 0$, and large $\lambda_3$ we can always find $\epsilon$ small such that $(\lambda_3 + \triangle u)(1 - \delta - \tilde{\epsilon}) - \epsilon > 0$. This gives us the main result of this section.

**Lemma 89.** *Suppose* $\alpha\lambda^2 < 2$*,* $C_1 A < 1$*, and* $\lambda_3 + \triangle u > 0$*. If* $G_2$ *takes its maximum at* $q_2$*, then we have the following inequality at* $q_2$*:*

$$\begin{aligned}\triangle' G_2 \geq & \left[\kappa + (\lambda_3 + \triangle u)\left(2c - \frac{2c}{\delta} - \epsilon\right) - \frac{C}{\lambda_3 + \triangle u}\right]\Theta^2 \\ & + \left[\kappa\lambda_4 - C - \frac{C}{\triangle u + \lambda_3}(\lambda_4 + \lambda_5 + \lambda_6) - C(\lambda_3 + \triangle u)\right]\Gamma\Theta \\ & + \left[\kappa\lambda_5 - C\lambda_4 - \frac{C}{\triangle u + \lambda_3}\lambda_5 - C(\lambda_3 + \triangle u)\right]\Gamma^2 \\ & + \left[\kappa\lambda_6 - C\lambda_4 - C\lambda_5 - C - \frac{C}{\triangle u + \lambda_3}(\lambda_4 + \lambda_5 + \lambda_6) - C(\lambda_3 + \triangle u)\right]\Xi \\ & - C\Theta - C\Gamma - C,\end{aligned}$$

*where* $0 < \epsilon, \delta < 1$ *are positive constants that we are free to choose;* $c$*,* $\kappa$ *are constants that depends on* $\alpha$*,* $\lambda^2$*,* $\mu$*, the Sobolev constant of* $g$*, and the curvature bound of* $g$*;* $C$ *is a constant that depends on* $\epsilon$*,* $\alpha$*,* $\lambda^2$*,* $\mu$*, the Sobolev constant of* $g$*, and the curvature bound of* $g$*.*

## 4.9 The general case

In the previous sections we have shown that the PDE has a solution when they are Hodge dual to each other $\beta^1 = *\beta^2$ and $\beta^1 = \beta^2$. In this section, we prove that the PDE has a solution for Hodge dual solutions $\beta^1 = *\beta^2$ in general, and also when $\beta^1$ is self-dual and $\beta^2$ is anti-self-dual.

The proof of each case is similar. We will prove the Hodge dual case first and show that the same technique can be applied for the self-dual/anti-self-dual case.

### 4.9.1 Hodge dual case

We look into the Hodge dual case when $\beta^1 \neq \beta^2$. Recall the PDE:

$$i\partial\bar{\partial}e^u \wedge \omega_S - i\alpha t\lambda^2 \partial\bar{\partial}u \wedge \omega_S - i\alpha t\partial\bar{\partial}(e^{-u}\rho) - \alpha\partial\bar{\partial}u \wedge \partial\bar{\partial}u + \mu t\frac{\omega_S^2}{2} = 0.$$

We use the same simplified notation as in Section 4.4, by renaming $t\lambda^2$ to $\lambda^2$, $t\mu$ to $\mu$, and $t\rho$ to $\rho$. We prove $u$ is lower bounded in the general case following a similar procedure discussed in the previous sections.

The general Hodge dual case differs from the simple case of $\beta^1 = \beta^2$ mainly in terms of $\rho$. The following lemma applies for $\rho$ in the general case when $\beta^1 = *\beta^2$.

**Lemma 90.** *The* $(1,1)$*-form* $\rho$ *is real and satisfies the following inequality*

$$i\partial u \wedge \bar{\partial}u \wedge \rho \geq \frac{\lambda^2}{2}|\nabla u|^2 \frac{\omega_S^2}{2}.$$

*Proof.* Using the definition of $\rho$

$$\rho = i\partial_k \bar{\phi}_{\bar{j}} g^{\bar{j}i} \bar{\partial}_{\bar{l}} \phi_i dz^k \wedge d\bar{z}^{\bar{l}}.$$

We can check that it is real explicitly,

$$\begin{aligned}\bar{\rho} &= -i\bar{\partial}_{\bar{k}} \phi_j g^{j\bar{i}} \partial_l \bar{\phi}_{\bar{i}} d\bar{z}^{\bar{l}} \wedge dz^k \\ &= \rho.\end{aligned}$$

In the previous sections, we have shown that

$$\rho = i\bar{\sigma}_{k\bar{j}} g^{\bar{j}i} \sigma_{\bar{l}i} dz^k \wedge dz^{\bar{l}} + \frac{\lambda^2}{2}\omega_S,$$

when $\beta^1 = *\beta^2$ (Proposition 41). Hence to prove the inequality we show that the term involving $\sigma$ is positive. For simplicity we introduce the following notation

$$\varphi = i\bar{\sigma}_{k\bar{j}}g^{\bar{j}i}\sigma_{\bar{l}i}dz^k \wedge dz^{\bar{l}}.$$

$i\partial u \wedge \bar{\partial} u \wedge \varphi$ can be computed using coordinates

$$\begin{aligned} i\partial u \wedge \bar{\partial} u \wedge \varphi &= i(u_1 u_{\bar{1}}\varphi_{2\bar{2}} - u_1 u_{\bar{2}}\varphi_{2\bar{1}} - u_2 u_{\bar{1}}\varphi_{1\bar{2}} + u_2 u_{\bar{2}}\varphi_{1\bar{1}})dz^{1\bar{1}2\bar{2}} \\ &= -\frac{i}{\det g_{i\bar{j}}} u_{\bar{j}}(\operatorname{adj}\varphi)^{\bar{j}i}u_i \frac{\omega_S^2}{2}, \end{aligned}$$

where we have used $\mathrm{vol} = -\det g_{i\bar{j}}dz^{1\bar{1}2\bar{2}}$ and $\operatorname{adj}\varphi$ denotes the adjugate matrix of $\varphi$ when the components of $\varphi$ are treated as a $2 \times 2$ matrix, i.e. $\operatorname{adj}\varphi \cdot \varphi = \det \varphi \mathbb{1}$. Meanwhile when treating $\varphi$ as a matrix, it can be expressed as

$$\varphi = iMg^{-1}M^*,$$

where $M$ is a matrix with components $M_{k\bar{j}} = \bar{\sigma}_{k\bar{j}}$. Therefore,

$$i\partial u \wedge \bar{\partial} u \wedge \varphi = \frac{1}{\det g_{i\bar{j}}} \left(u^* \cdot \operatorname{adj} M^* \cdot \operatorname{adj} g^{-1} \cdot \operatorname{adj} M \cdot u\right) \frac{\omega_S^2}{2} \geq 0,$$

where $u := (u_1\ u_2)^T$. □

Now we can find the lower bound of $u$ for the general Hodge dual case using this lemma.

**Proposition 91.** *Suppose $\alpha\lambda^2 < 2$. If $u$ is a solution of the general PDE under the elliptic condition $\omega' > 0$, and normalizations $A < 1$, $\int \frac{\omega_S^2}{2} = 1$, then we have,*

$$\inf_S u \geq -\log(C_1 A),$$

*where $C_1$ is a constant that depends on $\alpha'\lambda^2, \mu$ and the Sobolev constant of $\omega_S$.*

*Proof.* We use Lemma 4.7:

$$\int i\partial\bar{\partial}(e^{-ku}) \wedge \omega' \geq -ik \int e^{-ku} \partial\bar{\partial} u \wedge \omega'.$$

We compute the right hand side of this inequality:

$$\begin{aligned}
&-ik \int e^{-ku} \partial\bar{\partial} u \wedge (e^u \omega_S - \alpha\lambda^2 \omega_S + \alpha e^{-u}\rho + 2i\alpha\partial\bar{\partial} u) \\
&\quad = -ik \int e^{-ku} \partial\bar{\partial} u \wedge (e^u \omega_S - \alpha\lambda^2 \omega_S + \alpha e^{-u}\rho) \\
&\qquad + ik \int e^{-ku} 2 \left(\partial\bar{\partial} e^u \wedge \omega_S - \alpha\partial\bar{\partial} u \wedge \lambda^2 \omega_S - \alpha t \partial\bar{\partial}(e^{-u}\rho)\right) + \int e^{-ku} 2k\mu \\
&\quad = k \int e^{-ku}(e^u - \alpha\lambda^2)\triangle u + 2k \int e^{-ku}\mu + 2k \int e^{-ku+u} |\nabla u|^2 \\
&\qquad + ik \int e^{-ku-u} \alpha(\partial\bar{\partial} u \wedge \rho - 2\partial u \wedge \bar{\partial} u \wedge \rho - 2\bar{\partial} u \wedge \partial\rho + 2\partial u \wedge \bar{\partial}\rho - 2\partial\bar{\partial}\rho)
\end{aligned}$$

where we have used the PDE and the Leibniz rule. On the other hand, the left hand side of the inequality can be computed using the chain rule:

$$\begin{aligned}
&i \int \partial\bar{\partial} e^{-ku} \wedge (e^u \omega_S - \alpha\lambda^2 \omega_S + \alpha e^{-u}\rho) \\
&\quad = k^2 \int e^{-ku}(e^u - \alpha\lambda^2)|\nabla u|^2 - k \int e^{-ku}(e^u - \alpha\lambda^2)\triangle u \\
&\qquad + i\alpha \int e^{-ku-u} \rho \wedge (k^2 \partial u \wedge \bar{\partial} u - k\partial\bar{\partial} u).
\end{aligned}$$

We substitute these two expressions into the inequality and obtain:

$$\begin{aligned}
&k \int e^{-ku}(e^u - \alpha\lambda^2)|\nabla u|^2 + k\alpha \int e^{-ku-u} i\partial u \wedge \bar{\partial} u \wedge \rho \\
&\quad \geq 2 \int e^{-ku}(e^u - \alpha\lambda^2)\triangle u + 2k \int e^{-ku+u}|\nabla u|^2 + 2k \int e^{-ku}\mu \\
&\qquad + 2\alpha \int e^{-ku-u} i(\partial\bar{\partial} u \wedge \rho - \bar{\partial} u \wedge \partial\rho + \partial u \wedge \bar{\partial}\rho - \partial u \wedge \bar{\partial} u \wedge \rho - \partial\bar{\partial}\rho).
\end{aligned}$$

We use the following expression by Stokes′ theorem.

$$\begin{aligned}0 &= \int \partial\bar{\partial}(e^{-ku-u}\rho)\\&= \int e^{-ku-u}\big[(k+1)^2\partial u\wedge\bar{\partial}u\wedge\rho + (-k-1)\partial\bar{\partial}u\wedge\rho - (-k-1)\bar{\partial}u\wedge\partial\rho\\&\qquad\qquad + (-k-1)\partial u\wedge\bar{\partial}\rho + \partial\bar{\partial}\rho\big],\end{aligned}$$

and substitute this into the inequality:

$$\begin{aligned}&k\int e^{-ku}(e^u-\alpha\lambda^2)|\nabla u|^2 + k\alpha\int e^{-ku-u} i\partial u\wedge\bar{\partial}u\wedge\rho\\&\geq 2k\int e^{-ku}(e^u-\alpha\lambda^2)|\nabla u|^2 + 2\int e^{-ku}\mu\\&\quad + 2\alpha\left(\frac{1}{k+1}-1\right)\int e^{-ku-u} i\partial\bar{\partial}\rho + 2\alpha k\int e^{-ku-u} i\partial u\wedge\bar{\partial}u\wedge\rho,\end{aligned}$$

where we have used integration by parts on the term including $\triangle u$. We simplify and obtain

$$\begin{aligned}&2\alpha\left(1-\frac{1}{k+1}\right)\int e^{-ku-u} i\partial\bar{\partial}\rho - 2\int e^{-ku}\mu\\&\quad\geq k\int e^{-ku}(e^u-\alpha\lambda^2)|\nabla u|^2 + k\alpha\int e^{-ku-u} i\partial u\wedge\bar{\partial}u\wedge\rho\\&\quad\geq k\int e^{-ku}(e^u-\alpha\lambda^2)|\nabla u|^2 + k\alpha\int e^{-ku-u}\frac{\lambda^2}{2}|\nabla u|^2,\end{aligned}$$

where we have used Lemma 90.

Therefore we conclude:

$$\begin{aligned}&2i\alpha\left(1-\frac{1}{k+1}\right)\int e^{-ku-u}\partial\bar{\partial}\rho - 2\int e^{-ku}\mu\\&\geq k\left(1-\frac{\lambda^2}{2}\alpha\right)\int e^{-ku+u}|\nabla u|^2 + \frac{\lambda^2}{2}\alpha k\int e^{-ku}\left(e^{-u/2}-e^{u/2}\right)^2|\nabla u|^2\\&\geq k\left(1-\frac{\lambda^2}{2}\alpha\right)\int e^{-ku+u}|\nabla u|^2,\end{aligned}$$

which implies

$$k\int e^{-(k-1)u}|\nabla u|^2 \leq C\int e^{-ku-u} + C\int e^{-ku}.$$

We follow the discussion in the previous sections on the lower bound of $u$ and obtain an estimate $\inf u \geq -\ln(C_1 A)$. $\square$

$A$ gives the normalization of the solution $u$ which we are free to choose. We can set $A$ small enough so that $e^u$ becomes large enough to control the extra term $e^{-u}|\mathrm{tr}(\bar{\partial}B\wedge\partial B^*\cdot g)|$. Thus the analysis for the special case still holds for the general case.

### 4.9.2 Self-dual/anti-self-dual case

The proof for the self-dual $\beta^1$ and anti-self-dual $\beta^2$ case is similar to the Hodge dual case. Note that for this case the $(1,1)$-form $\rho$ becomes

$$\rho = i(\beta^2)_{k\bar{j}}g^{\bar{j}i}(\beta^2)_{\bar{l}i} + \frac{\lambda^2}{2}\omega_S.$$

Note that $\rho$ is of the same form as in the Hodge dual case with $\sigma = \beta^2$. Therefore, Lemma 90 still holds for the self-dual/anti-self-dual case and the existence proof follows from the exact same procedure as in the Hodge dual case.

### 4.9.3 $(k-1)(S^2\times S^4)\#k(S^3\times S^3)$

The fact that the anomaly cancellation condition admits a solution in the Hodge dual case and also in the self-dual/anti-self-dual case implies that the PDE has a solution when the Strominger system is considered on the connected sum of $(k-1)(S^2\times S^4)\#k(S^3\times S^3)$. This is because the $\beta^1$, $\beta^2$ chosen to construct $(k-1)(S^2\times S^4)\#k(S^3\times S^3)$ is either Hodge dual or self-dual/anti-self-dual [GGP08]. From [GGP08] Chapter 4, $(k-1)(S^2\times S^4)\#k(S^3\times S^3)$ is constructed by the following choice of $\beta^1$ and $\beta^2$.

1. For $k = 1$, $\beta^1 = *\beta^2$. Thus the PDE has a solution by the proof of Hodge dual case.
2. For $k = 2$, $\beta^1 = \frac{\lambda}{2}\omega_S$ and $\beta^2$ is anti-self-dual. Thus the PDE has a solution by the proof of self-dual/anti-self-dual case.
3. For $3 \leq k \leq 8$, $\beta^1 = \frac{\lambda}{2}\omega_S$ and $\beta^2$ is anti-self-dual. Thus the PDE has a solution by the proof of self-dual/anti-self-dual case.
4. For $k \geq 9$, the construction is such that $\langle \omega_S, \beta^1 \rangle = \langle \omega_S, \beta^2 \rangle = 2$ and $\langle \omega_S, \beta^1 + \beta^2 \rangle > 0$. The first condition implies that $\beta^1 - \beta^2$ is primitive and anti-self-dual (Lefschetz decomposition in real $4$-dimensions). Meanwhile, the second condition implies that $\beta^1 + \beta^2 \in L(C^\infty(M))$. Therefore,

   $$\beta^1 = \frac{\lambda}{\sqrt{2}}\omega_S + \varphi, \quad \beta^2 = \frac{\lambda}{\sqrt{2}}\omega_S - \varphi,$$

   where $\varphi$ is a primitive form and the normalization is chosen to match that of [GGP08]. Therefore $\beta^1 = *\beta^2$, and the PDE has a solution in this case as well.

In summary, the anomaly cancellation condition has a solution on $(k-1)(S^2 \times S^4)\#k(S^3 \times S^3)$ for all $k \in \mathbb{N}$.

# Chapter 5

# Heterotic String Theory Aspects

## 5.1 Statement of main results

In this chapter, we study the Strominger system in the context of heterotic string theory. We investigate two approaches: the $\alpha'$-expansion and the exact solutions of the Strominger system with the Hull connection.

In the $\alpha'$-expansion approach, we prove the following proposition.

**Proposition 95.** *Consider* $(M, \omega_u, \Omega, V, H)$ *as a solution of the Strominger system up to* $(\alpha)'$*-order. The anomaly cancellation condition has a solution up to* $(\alpha')^1$*-order if and only if* $\mathrm{tr} F \wedge F \in \Gamma(\bigwedge^{2,2} T^*S)$ *and* $p_1(M) = p_1(V)$.

The condition $p_1(M) = p_1(V)$ is commonly called solving the anomaly cancellation condition in topology. This proposition states that the topological solution is necessary and sufficient to guarantee existence of an analytical solution in $\alpha'$-expansion for the principal torus construction.

In fact, we derive necessary and sufficient conditions for existence of solution up to $(\alpha')^1$-order.

**Proposition 96.** *Consider* $(M, \omega_u, \Omega, V = \pi^* E, H)$ *as a* $\alpha'$*-expansion solution of the Strominger system up to* $(\alpha')^1$*-order. The following is sufficient and necessary for existence* *of solution up to $(\alpha')^1$-order.*

1. *The conformally balanced condition has a solution up to $(\alpha')^1$-order if and only if $\omega_{u(0)}$ is Kähler and Ricci-flat.*
2. *The vector bundle condition has a solution up to $(\alpha')^1$-order if and only if $\mathrm{tr}_{\omega_S} F = 0$.*
3. *The anomaly cancellation condition has a solution up to $(\alpha')^1$-order if and only if $p_1(V) = p_1(S)$.*

The $\alpha'$-expansion is problematic if the solution of the Strominger system is not analytic to begin with. One limitation of the $\alpha'$-expansion approach is that at $0$-th order, any hermitian manifold $(M, J, g, \omega)$ that satisfies the conformally balanced condition should be Kähler.

We discuss exact solutions of the Strominger system with respect to the Hull connection. Particularly, we focus on the anomaly cancellation condition with respect to the Hull connection. We show that the Hull connection introduces additional terms compared to the Chern connection and these terms introduce extra conditions in the anomaly cancellation condition.

We will show that in general $\mathrm{tr} R^H \wedge R^H$ has a $(3,1)$-component and if $(\mathrm{tr} R^H \wedge R^H)^{(3,1)} = 0$, then $dT^H \in L^2(C^\infty(M))$. Here $L^2(C^\infty(M))$ denotes the Lefschetz decomposition.

Finally, we investigate the Strominger system with the Hull connection in our case $(M, \omega_u, V, H)$. We will prove the following proposition.

**Proposition 106.** *Consider $(M, \omega_u, V, H)$. Denote a solution of the Strominger system with respect to the Chern connection as $u_C$. Then we have the following estimate,*

$$\left|\left| i\partial\bar{\partial}\omega_{u_C} - \frac{\alpha'}{4}\left[ (\mathrm{tr} R^H \wedge R^H)(u_C) - \mathrm{tr} F \wedge F \right] \right|\right| \leq C(A^6 + A^5 + A^4 + A^3 + A^2),$$

*where $||\varphi|| := |\langle \varphi, \omega_S^2/2 \rangle|$ for $\varphi \in L^2_{\omega_S}(C^\infty(M))$, and $A = \left(\int e^{-4u_C}\right)^{1/4}$. $C$ is a constant* *that depends on* $\langle \beta^1 \wedge \beta^1/2 + \beta^2 \wedge \beta^2/2, \omega_S^2/2\rangle$, $\alpha'\lambda^2$, $\mu$, *the Sobolev constant of* $g_S$, *and* $C_1$ *of Chapter 4.*

*Therefore when* $A \to 0$, $u_C$ *can be considered as a solution of the anomaly cancellation condition with respect to the Hull connection.*

Using this proposition we show that the $\alpha'$-expansion fails to capture the behaviour of exact solutions of the Strominger system with respect to the Hull connection even up to $\alpha'$-order. This strongly suggests that the solution of the Strominger system with respect to the Hull connection is not analytic. Thus the $\alpha'$-expansion approach should be used with caution.

## 5.2 Strominger system in $\alpha'$-expansion

We introduce $\alpha'$-expansion to the Strominger system and solve it by $\alpha'$-order. This approach can be valid if the solution is analytic, however if the solution is not analytic to start with this approach may not give a correct solution even in $(\alpha')^0$-th order. We discuss potential problems with $\alpha'$-expansion in the next section. In this section, we show that the $\alpha'$-expansion approach is rather restrictive and show that it only allows solutions with $S$ being a Calabi-Yau manifold.

First, we introduce the objects that have $\alpha'$-expansions. We assume that $u$ and the connection 1-forms $\alpha^1$, $\alpha^2$ of the fibre have $\alpha'$-expansions:

$$u = u_{(0)} + \alpha' \cdot u_{(1)} + O(\alpha'^2),$$

$$\alpha^i = \alpha^i_{(0)} + \alpha' \cdot \alpha^i_{(0)} + O(\alpha'^2).$$

The $\alpha'$ expansion will manifest itself to $\omega_u$ as well:

$$\begin{aligned}\omega_u &= \omega_{(0)} + \alpha' \cdot \omega_{u(1)} + O(\alpha'^2)\\ &= e^{u_{(0)}}\left(1 + \alpha' \cdot u_{(1)}\right)\omega_S + \frac{i}{2}\theta \wedge \bar{\theta} + O(\alpha'^2),\end{aligned}$$

where $\theta = dz + \left(\alpha^1_{(0)} + i\alpha^2_{(0)}\right) + \alpha' \cdot \left(\alpha^1_{(1)} + i\alpha^2_{(1)}\right) + O(\alpha'^2)$. In this section, we will often write $\omega$ for $\omega_u$ to simplify the notation. With these expansions in mind we solve the Strominger system.

### 5.2.1 $(\alpha')^0$-order

#### 5.2.1.1 Conformally balanced condition and anomaly cancellation condition

The conformally balanced condition in the 0-th order of $\alpha'$ gives us the following equation:

$$d\left(||\Omega||_{(0)}\, \omega_{(0)} \wedge \omega_{(0)}\right) = 0 + O(\alpha').$$

Thus $\omega_{(0)}$ should be conformally balanced.

On the other hand, the 0-th order expansion of the anomaly cancellation condition gives us the following equation:

$$i\partial\bar{\partial}\omega_{(0)} = 0 + O(\alpha').$$

Therefore, $\omega_{(0)}$ is astheno-Kähler; recall that $\omega$ is called **astheno-Kähler**, if $i\partial\bar{\partial}\omega^{n-2} = 0$ in complex $n$-dimensions. In complex 3-dimensions, this condition is sometimes called strong Kähler with torsion. In general this condition does not imply Kähler. However when the metric is conformally balanced the astheno-Kähler condition implies that the metric is Kähler.

**Proposition 92** ([IP01, GMW04, FT11])**.** *Consider a compact complex $n$-manifold $(M, J)$.*

*Suppose the hermitian form is astheno-Kähler* $i\partial\bar{\partial}(\omega^{n-2}) = 0$ *and conformally balanced* $d(e^{-2\phi}{*\omega}) = 0$*, where* $e^{-2\phi}$ *is some conformal scaling. Then* $\omega$ *is Kähler* $d\omega = 0$*.*

*Proof.* The general proof can be found in the original papers. We will present a short proof on $3$-complex dimensions that closely resembles the argument in [IP01, GMW04]. We assume that the metric is conformally balanced:

$$d\left(e^{-2\phi}\omega \wedge \omega\right) = 2e^{-2\phi}\omega \wedge (d\omega - d\phi \wedge \omega) = 0.$$

We define a $(2,1)+(1,2)$-form $W_3 := d\omega - d\phi \wedge \omega$. By the conformally balanced condition we have $W_3 \wedge \omega = 0$ which implies that $W_3$ is primitive.

We compute $*(d^c\omega)$ using the Lefschetz decomposition:

$$*(d^c\omega) = -\omega \wedge d\phi + W_3 = d\omega - 2d\phi \wedge \omega = e^{2\phi} d\left(e^{-2\phi}\omega\right).$$

Using this expression, we can show that $d^c\omega = 0$ by the following computation:

$$||e^{-\phi}d^c\omega||^2 = \int e^{-2\phi} d^c\omega \wedge *(d^c\omega) = \int -d(d^c\omega) \wedge e^{-2\phi}\omega = 0,$$

since $\omega$ is astheno-Kähler and we are focusing on $n = 3$. □

From this proposition, we conclude that $\omega_{(0)}$ is Kähler $d\omega_{(0)} = 0$.

**Remark 93.** When solving the Strominger system via $(\alpha')$-expansion, the $0$th-order hermitian form should be always Kähler. In [MS10], the authors argue the Chern connection at $0$th-order is necessary Kähler and this is true in general. Thus the Chern connection can be used to solve the Strominger system via $\alpha'$-expansion. In fact, in $(\alpha')^0$-order $\omega_{(0)}$ is Kähler therefore Chern connection, Bismut connection, and Hull connection all reduce to the Levi-Civita connection, and we are free to choose any connection we like.

The Kähler condition is a very strong condition, and simplifies the expression on the expansion of $\omega$. Using the expansion we obtain

$$\begin{aligned} d\omega_{(0)} = e^{u_{(0)}} du_{(0)} \wedge \omega_S + & \frac{i}{2}(\beta^1_{(0)} + i\beta^2_{(0)}) \wedge \left(d\bar{z} + (\alpha^1_{(0)} - i\alpha^2_{(0)})\right) \\ & - \frac{i}{2}\left(dz + (\alpha^1_{(0)} + i\alpha^2_{(0)})\right) \wedge (\beta^1_{(0)} - i\beta^2_{(0)}). \end{aligned}$$

From this we conclude $(\beta^1_{(0)} + i\beta^2_{(0)}) \wedge d\bar{z} = 0$. This implies $\beta^1_{(0)} = \beta^2_{(0)} = 0$, because $\beta^1, \beta^2 \in H^2(S, 2\pi\mathbb{Z})$. Also we conclude that $\alpha^1_{(0)}, \alpha^2_{(0)} \in H^1(S, 2\pi\mathbb{Z})$. Since now we have $\beta^1_{(0)} = \beta^2_{(0)} = 0$, this gives us $du_{(0)} = 0$ therefore $u_{(0)}$ is a constant.

#### 5.2.1.2 Vector bundle condition

In $(\alpha')^0$-th order the only condition on the vector bundle is

$$\mathrm{tr}_{\omega_{(0)}} F = 0.$$

Often the holomorphic tangent bundle $T^{1,0}M$ with the Ricci-flat metric $g$ is considered as the vector bundle $V$ to satisfy this condition. This is called the **standard embedding**.

In [BBF+06], the authors show that the most general holomorphic bundle with $\mathrm{SU}(n)$-structure is of a form $V = \pi^* E \otimes L$, where $E$ is a holomorphic vector over $S$ and $L$ is a line bundle over the fibres. If we take $V = \pi^* E$ then we obtain

$$\mathrm{tr}_{\omega_{(0)}} F = e^{-u_{(0)}} \mathrm{tr}_{\omega_S} F = 0,$$

which is a condition on $S$.

In summary, the conformally balanced condition and the anomaly cancellation condition up to 0th-order implies that

1. $\omega_{(0)}$ is Kähler with $c_1(M) = 0$.

2. $u_0$ is a constant.

3. $\beta^i$'s have no $0$th-order terms.

4. If $V = \pi^* E$, then $\mathrm{tr}_{\omega_S} F = 0$.

## 5.2.2 $(\alpha')^1$-order

### 5.2.2.1 Conformally balanced condition

We use $\mathcal{R}^B = 0$ formulation of the conformally balanced condition. In the previous chapters, we have shown that this is equivalent to

$$\mathcal{R}_S = \langle \omega_S, \beta^1 \rangle \beta^1 + \langle \omega_S, \beta^2 \rangle \beta^2.$$

Since $\beta^i = \alpha' \cdot \beta^i_{(1)} + O(\alpha'^2)$, we conclude $\mathcal{R}_S = 0 + O(\alpha'^2)$. Therefore, $\omega_{u(0)}$ should be Ricci-flat.

### 5.2.2.2 Anomaly cancellation condition

The anomaly cancellation condition in $(\alpha')^1$-order is

$$i\partial\bar{\partial}\omega_{(1)} = \frac{1}{4}\left(\mathrm{tr} R_{(0)} \wedge R_{(0)} - \mathrm{tr} F \wedge F\right).$$

Hull [Hul86] has derived that the connection in $R_{(0)}$ should be the Hull connection. However, we have shown that $\omega_{(0)}$ is Kähler in the previous sections, thus the Hull connection reduces to the Levi-Civita connection. In fact, all of the canonical connections reduce to the Levi-Civita connection as well so we can use the results of the previous chapters on the Chern connection in this condition. Using the fact that $u_{(0)}$ is a constant and the derivation of the previous chapters, we conclude

$$\mathrm{tr} R_{(0)} \wedge R_{(0)} = \pi^*(\mathrm{tr} R_S \wedge R_S - 2e^{-u_{(0)}} i\partial\bar{\partial}\rho).$$

On the other hand, we have

$$i\partial\bar{\partial}\omega_{(1)} = e^{u_{(0)}} i\partial\bar{\partial}u_1 \wedge \omega_S,$$

where we have used the fact that $\beta^1_{(0)} = \beta^2_{(0)} = 0$.

Therefore, the anomaly cancellation condition becomes

$$\pi^*\left(e^{u_{(0)}} i\partial\bar{\partial}u_1 \wedge \omega_S + 2e^{-u_{(0)}} i\partial\bar{\partial}\rho\right) = \frac{\alpha'}{4}\left[\pi^*\left(\mathrm{tr}R_S \wedge R_S\right) - \mathrm{tr}F \wedge F\right].$$

From this expressions we obtain the following remark.

**Remark 94.** $\mathrm{tr}F \wedge F \in \Gamma(\bigwedge^{2,2} T^*S)$ is a necessary condition for the Strominger system to have a solution on $(M, \omega_u, V, H)$.

If we construct $V$ by $V = \pi^* E$, then $\mathrm{tr}F \wedge F \in \Gamma(\bigwedge^{2,2} T^*S)$ automatically. In fact, if $\mathrm{tr}F \wedge F \in \Gamma(\bigwedge^{2,2} T^*S)$, then we have a necessary and sufficient condition on the existence of solution.

**Proposition 95.** *Consider $(M, \omega_u, \Omega, V, H)$ as a solution of the Strominger system up to $(\alpha)'$-order. The anomaly cancellation condition has a solution up to $(\alpha')^1$-order if and only if $\mathrm{tr}F \wedge F \in \Gamma(\bigwedge^{2,2} T^*S)$ and $p_1(M) = p_1(V)$.*

*Proof.* The first part of the statement comes from the remark above. We assume $\mathrm{tr}F \wedge F$ only has components in $\Omega^{2,2}S$ and write the anomaly cancellation condition as the following,

$$i\partial\bar{\partial}u_{(1)} \wedge \omega_S = \frac{e^{-u_{(0)}}}{4}\left(\mathrm{tr}R_S \wedge R_S - \mathrm{tr}F \wedge F\right) - 2e^{-2u_{(0)}} i\partial\bar{\partial}\rho.$$

Note that all of the forms are top forms on $S$. Thus the right hand side can be expressed as a function $f$ multiplied by the volume form $\omega_S^2/2$. We use the definition of the Laplacian and write the anomaly cancellation condition as

$$\triangle u_{(1)}\frac{\omega_S^2}{2} = f\,\frac{\omega_S^2}{2},$$

where $f := \left\langle \frac{e^{-u_{(0)}}}{4}\left(\mathrm{tr}R_S\wedge R_S - \mathrm{tr}F\wedge F\right) - 2e^{-2u_{(0)}}i\partial\bar{\partial}\rho, \frac{\omega_S^2}{2}\right\rangle$. This is the Poisson equation on a compact complex manifold. It is well known [Aub98, Theorem 4.7] that the necessary and sufficient condition for existence of solution is $\int_S f = 0$; in other words the $(\alpha')^1$-order equation has a solution if and only if

$$\int \frac{e^{-u_{(0)}}}{4}\left(\mathrm{tr}R_S\wedge R_S - \mathrm{tr}F\wedge F\right) - 2e^{-2u_{(0)}}i\partial\bar{\partial}\rho = 0,$$

which is equivalent to $p_1(V) = p_1(S) = p_1(M)$. □

#### 5.2.2.3 Vector bundle condition

If $V = \pi^*E$, and the vector bundle condition is satisfied in $(\alpha')^0$-order then it is satisfied automatically in $(\alpha')^1$-order. This can be easily seen from

$$\omega_{u(1)} = e^{u_{(0)}}u_{(1)}\omega_S + \frac{i}{2}\theta_{(0)}\wedge\bar{\theta}_{(1)} + \frac{i}{2}\theta_{(1)}\wedge\bar{\theta}_{(0)},$$

and since $F \in \Gamma(\bigwedge^{1,1}T^*S)$ we have

$$\mathrm{tr}_{\omega_{u(1)}}F = e^{u_{(0)}}u_{(1)}\mathrm{tr}_{\omega_S}F = u_{(1)}\mathrm{tr}_{\omega_{u(0)}}F.$$

Therefore if $V = \pi^*E$ the solution of the vector bundle condition in the 0th-order is sufficient and necessary for the 1st-order solution.

### 5.2.3 Summary of $\alpha'$-expansion solutions.

In summary, we have proved the following proposition.

**Proposition 96.** *Consider $(M, \omega_u, \Omega, V = \pi^* E, H)$ as a $\alpha'$-expansion solution of the Strominger system up to $(\alpha')^1$-order. The following is sufficient and necessary for existence of solution up to $(\alpha')^1$-order.*

1. *The conformally balanced condition has a solution up to $(\alpha')^1$-order if and only if $\omega_{u(0)}$ is Kähler and Ricci-flat.*
2. *The vector bundle condition has a solution up to $(\alpha')^1$-order if and only if $\mathrm{tr}_{\omega_S} F = 0$.*
3. *The anomaly cancellation condition has a solution up to $(\alpha')^1$-order if and only if $p_1(V) = p_1(S)$.*

However, the $\alpha'$-expansion approach has a potential problem. Noticeably, if the general solution of the Strominger system is not analytic, we run into problems immediately. We discuss the problems with $\alpha'$-expansion in the next section in detail.

## 5.3 Comments on the $\alpha'$-expansion

The $\alpha'$-expansion approach to the Strominger system is rather incomplete. A general solution to the Strominger system with the torus fibration construction we are considering allows $S$ to be a Kähler surfaces other than a Calabi-Yau surface. However once we impose the $\alpha'$-expansion, $S$ is forced to be Calabi-Yau. This behaviour occurs because of three main reasons:

1. When we expand functions in the power of $\alpha'$, we are assuming which functions *will* have $\alpha'$-corrections and *how*. This is a quite strong condition which we are imposing extra to the Strominger system. In terms of the PDE, this affects what the principal symbol is for each $\alpha'$-order. Explicitly the general

PDE we are trying to solve is

$$i\partial\bar{\partial}e^u \wedge \omega_S - i\frac{\alpha'}{2}\partial\bar{\partial}u \wedge \lambda^2\omega_S - i\frac{\alpha'}{2}\partial\bar{\partial}(e^{-u}\rho) - \frac{\alpha'}{2}\partial\bar{\partial}u \wedge \partial\bar{\partial}u + \mu\frac{\omega_S^2}{2} = 0.$$

We easily see that this PDE has a non-linear term $\partial\bar{\partial}u \wedge \partial\bar{\partial}u$ in the principal symbol. However, once we impose the $\alpha'$-expansions, it modifies the principal symbol into a linear operator. In this case the PDE becomes

$$i\partial\bar{\partial}e^u \wedge \omega_S + \mu\frac{\omega_S^2}{2} = 0,$$

and the principal symbol becomes that of the Laplacian. The behaviour of the two principal symbols are different and thus we have different solutions as a result.

2. The $\alpha'$-expansion approach is problematic when the solution is not analytic. In the previous chapters, we have shown that the solution $u$ to the Strominger system is smooth, i.e. $u \in C^\infty(M)$. The family of smooth functions is larger than the family of analytic functions, thus assuming the solution is analytic and imposing the $\alpha'$-expansion will limit the number of solutions obtained through the approach.

3. Another problem with the $\alpha'$-expansion is that there is no guarantee that the sum $\sum_i(\alpha')^i u_{(i)}$ of the $\alpha'$-solution will converge to a solution $u$ of the PDE. This is a similar situation to first order deformations and obstructions; it is not clear which $\alpha'$-solution can be realized as a solution of the Strominger system. A local existence theorem for the Cauchy problem with an analytical solution has been addressed in the Cauchy-Kowalevski theorem. It is less clear whether a similar theorem holds in the Strominger system, and in more general the equation of motion of heterotic string theory.

## 5.4 Hull connection

We have discussed the limitations of $\alpha'$-expansion in the previous chapter. It may be more natural to consider the full Strominger system with Hull connection. We investigate properties of the Hull connection in this section, and show that the Hull connection is closely related to the Chern connection.

### 5.4.1 Connection identities

Chern connection, Bismut connection and Hull connection are closely related. In this section we derive some identities that shows the similarity of these connections. For $A, B, C, D \in \Gamma(TM)$, the Riemann tensor is defined as

$$R(A,B)C := \nabla_A \nabla_B C - \nabla_B \nabla_A C - \nabla_{[A,B]} C,$$

and the covariant Riemann tensor is defined as

$$R(A,B,C,D) := g(R(A,B)C, D).$$

Bismut connection and Hull connection have totally skew-symmetric torsion, i.e. if we use the metric $g$ to lower the indices of the torsion, we obtain a $3$-form:

$$T(X,Y,Z) := g(T(X,Y), Z).$$

We derive connection and curvature identities with our convention. Not all of the identities are new, similar identites on the Bismut connection and Chern connection can be found in [Bis89, IP01]. The identities regarding the Hull connection are new.

Consider a connection $\nabla^T$ with torsion $T$ which is of the following form,

$$\nabla^T := \nabla^{LC} + \frac{1}{2}T,$$

where $\nabla^{LC}$ is the Levi-Civita connection. We have the following relation between the Riemann tensor of connection $\nabla^T$ and the Levi-Civita connection $\nabla^{LC}$.

**Proposition 97.** *Suppose $\nabla^T$ is a connection with totally skew-symmetric torsion $T$. Then the Riemann tensor $R^T$ with respect to $\nabla^T$ and the Riemann tensor $R^{LC}$ with respect to the Levi-Civita connection $\nabla^{LC}$ are related by*

$$\begin{aligned} R^T(A,B,C,D) = R^{LC}&(A,B,C,D) \\ &- \frac{1}{4}\left[g(T(A,D),T(B,C)) - g(T(B,D),T(A,C))\right] \\ &+ \frac{1}{2}\left[\left(\nabla^{LC}_A T\right)(B,C,D) - \left(\nabla^{LC}_B T\right)(A,C,D)\right]. \end{aligned}$$

*Proof.* We start from the definition of the Riemann tensor and substitute the relation between $\nabla^T$ and $\nabla^{LC}$

$$\begin{aligned} R^T(A,B)C &= \nabla^T_A\left(\nabla^{LC}_B C + \frac{1}{2}T(B,C)\right) - \nabla^T_B\left(\nabla^{LC}_A C + \frac{1}{2}T(A,C)\right) \\ &\quad - \nabla^{LC}_{[A,B]}C - T([A,B],C) \\ &= R^{LC}(A,B)C + \frac{1}{2}\left(\nabla^{LC}_A[T(B,C)] - \nabla^{LC}_B[T(A,C)]\right) \\ &\quad + \frac{1}{4}\left[T(A,T(B,C)) - T(B,T(A,C))\right] \\ &\quad + \frac{1}{2}\left(T(A,\nabla^{LC}_B C) - T(B,\nabla^{LC}_A C) - T([A,B],C)\right). \end{aligned}$$

Now we use the fact that the torsion of the Levi-Civita connection vanishes, i.e. $\nabla^{LC}_A B - \nabla^{LC}_B A - [A,B] = 0$ and contract the expression with the metric. After some rearrangement and using the fact that $T$ is totally skew-symmetric, we obtain the following

$$
\begin{aligned}
&R^{T}(A,B,C,D)\\
&=R^{LC}(A,B,C,D)-\frac{1}{4}\left[g(T(A,D),T(B,C))-g(T(B,D),T(A,C))\right]\\
&\quad+\frac{1}{2}\left(\nabla^{LC}_{A}[T(B,C,D)]-T(B,C,\nabla^{LC}_{A}D)-\nabla^{LC}_{B}[T(A,C,D)]+T(A,C,\nabla^{LC}_{B}D)\right)\\
&\quad+\frac{1}{2}\left[-T(\nabla^{LC}_{A}B,C,D)-T(B,\nabla^{LC}_{A}C,D)+T(\nabla^{LC}_{B}A,C,D)+T(A,\nabla^{LC}_{B}C,D)\right].
\end{aligned}
$$

Using the Leibniz rule on $\nabla^{LC}_{A}[T(B,C,D)]$ we obtain the proposition. □

By definition, Bismut connection and Hull connection have totally skew-symmetric torsion with opposite sign. Using the proposition above, we obtain the following corollary.

**Corollary 98** ([Fer10]). *Denote the Riemann tensor of the Bismut connection as $R^{B}$ and that of the Hull connection as $R^{H}$, then the Riemann tensor of each connection satisfies the following identity*

$$
R^{H}(A,B,C,D)=R^{B}(C,D,A,B)+\frac{1}{2}dT^{H}(A,B,C,D).
$$

*Proof.* We use Proposition 97:

$$
\begin{aligned}
R^{H}(A,B,C,D)&=R^{B}(C,D,A,B)+\frac{1}{2}\left[\left(\nabla^{LC}_{A}T^{H}\right)(B,C,D)-\left(\nabla^{LC}_{B}T^{H}\right)(A,C,D)\right]\\
&\quad+\frac{1}{2}\left[\left(\nabla^{LC}_{C}T^{H}\right)(D,A,B)-\left(\nabla^{LC}_{D}T^{H}\right)(C,A,B)\right]\\
&=R^{-}(C,D,A,B)+\frac{1}{2}dT^{H}(A,B,C,D),
\end{aligned}
$$

where we have used the definition of $d\alpha(A,B,C,D):=\sum_{[A,B,C,D]}(\nabla^{LC}_{A}\alpha)(B,C,D)$ for arbitrary 2-form $\alpha$, where $\sum_{[A,B]}$ denote the anti-symmetric sum, for example $\sum_{[A,B]}\alpha(A,B):=\alpha(A,B)-\alpha(B,A)$. □

Meanwhile, on $T^{1,0}M$ we have the following observation.

**Corollary 99.** *Consider* $A, B \in \Gamma(TM)$ *and* $X, Y, Z, W \in \Gamma(T^{1,0}M)$. *On* $T^{1,0}M$, *the Hull connection* $\nabla^H$ *and the Chern connection* $\nabla^C$ *are identical, i.e.*

$$g(\nabla^H_A Z, \overline{W}) = g(\nabla^C_A Z, \overline{W}).$$

*Furthermore, on* $T^{1,0}M$ *the Riemann curvature tensors are identical, i.e.*

$$R^H(A, B, Z, \overline{W}) = R^C(A, B, Z, \overline{W}),$$

*which implies that* $R^H(X, Y, Z, \overline{W}) = R^H(\overline{X}, \overline{Y}, Z, \overline{W}) = 0$.

*Proof.* The connection identity and the curvature identity is true because if we restrict the definition of the Chern connection to $T^{1,0}M$ we obtain the following,

$$g(\nabla^C_B Z, \overline{W}) = g(\nabla^{LC}_B Z, \overline{W}) + \frac{1}{2} g(d^c\omega(B, Z), \overline{W}),$$

which is the same expression for the Hull connection restricted to $T^{1,0}M$. The vanishing components come from the fact that the curvature 2-form of the Chern connection is of type $(1,1)$. □

From the above corollaries, we can deduce properties of the covariant curvature tensor $R^H$ of the Hull connection. We will refer to the **component type** of the Riemann tensor by the type of the vector fields contracted to it, that is we will refer to $(1,1)(1,1)$-component of $R^H$ by $R^H(X, \overline{Y}, Z, \overline{W})$ for $X, Y, Z, W \in \Gamma(T^{1,0}M)$.

**Corollary 100.** *Denote* $X, Y, Z, W \in \Gamma(TM)$ *and the covariant Riemann curvature tensor of the Hull connection as* $R^H$. *The non-vanishing components of* $R^H$ *are the following and its complex conjugate.*

1. $(1,1)(1,1)$*-component:* $R^H(X, \overline{Y}, Z, \overline{W}) = R^C(X, \overline{Y}, Z, \overline{W})$.

2. $(1,1)(2,0)$*-component:* $R^H(X,\overline{Y},Z,W)=R^B(Z,W,X,\overline{Y})$.

3. $(0,2)(2,0)$*-component:* $R^H(\overline{X},\overline{Y},Z,W)=\frac{1}{2}dT^H(\overline{X},\overline{Y},Z,W)$.

*Proof.* We have the following identity

$$R^H(A,B,C,D)=R^B(C,D,A,B)+\frac{1}{2}dT^H(A,B,C,D),$$

with $T^H=d^c\omega$ thus $dT^H\in\Omega^{(2,2)}M$. Also the Bismut connection preserves the complex structure, therefore $R^B\in\Gamma\left((\Omega^2M\otimes\operatorname{End}T^{(1,0)}M)\oplus(\Omega^2M\otimes\operatorname{End}T^{(0,1)}M)\right)$. This implies that $R^B$ is of a type $(\cdot,\cdot)(1,1)$.

1. $(2,0)(2,0)$-component vanishes. We show this using the fact that $R^B$ is of type $(\cdot,\cdot)(1,1)$ and $dT^H$ is a $(2,2)$-form. Then $R^H(X,Y,Z,W)=R^B(Z,W,X,Y)+\frac{1}{2}dT^H(X,Y,Z,W)=0$.

2. $(1,1)(1,1)$-component is identical to the Chern connection because we have shown that $R^C=R^H$ on $T^{1,0}M$.

3. $(1,1)(2,0)$-component identity follows from Corollary 98, and since $dT^H$ is a $(2,2)$-form.

4. $(0,2)(0,2)$-component identity follows from Corollary 98, and the fact that $R^B$ is of a type $(\cdot,\cdot)(1,1)$.

□

Proposition 97 and Corollary 98 can be applied on $T^{1,0}M$ with the Chern connection and Bismut connection. We obtain the following corollary.

**Corollary 101.** *For* $X,Y,Z,W\in\Gamma\left(T^{(1,0)}M\right)$*, we have the following identities for the covariant Riemann tensors.*

1. $R^B(X,\overline{Y},Z,\overline{W})=R^C(Z,\overline{W},X,\overline{Y})+\frac{1}{2}dT^B(X,\overline{Y},Z,\overline{W})$.

2. $R^B(X,Y,Z,\overline{W}) = (\nabla^{LC}_X T^B)(Y,Z,\overline{W}) - (\nabla^{LC}_Y T^B)(X,Z,\overline{W})$.

*Proof.* The first identity is equivalent to Corollary 98 restricted to $T^{1,0}M$. The second identity comes from Proposition 97 with $T^B = -T^C$ on $T^{1,0}M$. $\square$

### 5.4.2 $\mathrm{tr}R^H \wedge R^H$

In general to find a physical solution, the Hull connection should be used to solve the anomaly cancellation condition,

$$i\partial\bar{\partial}\omega = \frac{\alpha'}{4}(\mathrm{tr}R^H \wedge R^H - \mathrm{tr}F \wedge F).$$

In this section, we focus on $\mathrm{tr}R^H \wedge R^H$.

#### 5.4.2.1 $(\mathrm{tr}R^H \wedge R^H)^{(3,1)}$

First we note that a necessary condition for the anomaly cancellation condition to have a solution is that $\mathrm{tr}R^H \wedge R^H$ is a $(2,2)$-form. However $\mathrm{tr}R^H \wedge R^H$ has a $(3,1)+(1,3)$-component in general. This can be seen from the curvature identities we have derived before,

1. $(1,1)(1,1)$-component: $R^H(X,\overline{Y},Z,\overline{W}) = R^C(X,\overline{Y},Z,\overline{W})$.
2. $(1,1)(2,0)$-component: $R^H(X,\overline{Y},Z,W) = R^B(Z,W,X,\overline{Y})$.
3. $(0,2)(2,0)$-component: $R^H(\overline{X},\overline{Y},Z,W) = \frac{1}{2}dT^H(\overline{X},\overline{Y},Z,W)$.

Because the $(\cdot,\cdot)(2,0)$-component has $(1,1)(2,0)$ and $(0,2)(2,0)$-type, this implies that $\mathrm{tr}R^H \wedge R^H$ will have a $(3,1)+(1,3)$ components. If we denote the basis for a unitary frame as $e_i, \bar{e}_{\bar{j}}$, then the $(3,1)$-component is of the following form.

$$\begin{aligned}(\mathrm{tr}R^H \wedge R^H)^{(3,1)}(X,Y,Z,\overline{W}) &= \frac{1}{4!}\sum_{\substack{i,j\\ [X,Y,Z,\overline{W}]}} R^H(X,\overline{W},e_i,e_j)R^H(Y,Z,\bar{e}_{\bar{j}},\bar{e}_{\bar{i}})\\ &= \frac{1}{4!}\sum_{\substack{i,j\\ [X,Y,Z,\overline{W}]}} R^B(e_i,e_j,X,\overline{W})\cdot\frac{1}{2}dT^H(\bar{e}_{\bar{j}},\bar{e}_{\bar{i}},Z,W),\end{aligned}$$

where $\sum_{[X,Y,Z,\overline{W}]}$ denotes the anti-symmetrized sum. $dT^H$ is a $(2,2)$-form. By the Lefschetz decomposition, we have

$$dT^H \in L^2(C^\infty(M)) \oplus L^1(P^{1,1}(M)) \oplus P^{2,2}(M),$$

where $L$ is the Lefschetz operator and $P$ is the space of primitive forms. If $dT^H$ only has components in $L^2(C^\infty(M))$ then $(3,1)$-component of $\mathrm{tr}R^H \wedge R^H$ vanishes.

**Lemma 102.** *If $dT^H$ only has components in $L^2(C^\infty(M))$, then $(3,1)$-component of $\mathrm{tr}R^H \wedge R^H$ is zero.*

*Proof.* Denote the basis for a unitary frame as $e_i, \bar{e}_{\bar{j}}$ and suppose $dT^H = f\omega\wedge\omega$. Then we have

$$\begin{aligned}4!dT^H(Y,Z,\bar{e}_{\bar{j}},\bar{e}_{\bar{i}}) &= 8f\left[\omega(Y,Z)\omega(\bar{e}_{\bar{j}},\bar{e}_{\bar{i}}) + \omega(Y,\bar{e}_{\bar{j}})\omega(\bar{e}_{\bar{i}},Z) + \omega(Y,\bar{e}_{\bar{i}})\omega(Z,\bar{e}_{\bar{j}})\right]\\ &= 8f\left[g(\bar{e}_{\bar{j}},Y)g(\bar{e}_{\bar{i}},Z) - g(\bar{e}_{\bar{i}},Y)g(\bar{e}_{\bar{j}},Z)\right].\end{aligned}$$

Using this expression and the curvature identities, we can express the $(3,1)$-component of $\mathrm{tr}R^H \wedge R^H$ as the following

$$\begin{aligned}(\mathrm{tr}R^H \wedge R^H)^{(3,1)}(X,Y,Z,\overline{W}) &= \sum_{\substack{i,j\\ [X,Y,Z,\overline{W}]}} R^B(e_i,e_j,X,\overline{W})\cdot\frac{1}{2}dT^H(Y,Z,\bar{e}_{\bar{j}},\bar{e}_{\bar{i}})\\ = \frac{1}{6}f\sum_{\substack{i,j\\ [X,Y,Z,\overline{W}]}} &\left[(\nabla^{LC}_{e_i}T^B)(e_j,X,\overline{W}) - (\nabla^{LC}_{e_j}T^B)(e_i,X,\overline{W})\right]\times\left[Y_{\bar{j}}Z_{\bar{i}} - Y_{\bar{i}}Z_{\bar{j}}\right]\end{aligned}$$

Since we are using a unitary frame, we have $Y_{\bar{j}} = Y^j$ component-wise. Therefore, the above expression can be simplified

$$\begin{aligned}(\mathrm{tr}R^H \wedge R^H)^{(3,1)}(X, Y, Z, \overline{W}) &= \frac{2}{3} f \sum_{\substack{i,j \\ [X,Y,Z,\overline{W}]}} (\nabla^{LC}_Z T^B)(Y, X, \overline{W}) \\ &= \frac{1}{36} f dT^B(Z, Y, X, \overline{W}) = 0,\end{aligned}$$

since $dT^B$ is a $(2,2)$-form. □

It is less clear whether $dT^H \in L^2(C^\infty(M))$ is a necessary condition for the $(3,1)$-component of $\mathrm{tr}R^H \wedge R^H$ to vanish. When there are components in other spaces, for example $dT \in L^1(P^{1,1}(M))$, the $(3,1)$-component may not vanish. Explicitly, if $dT = f\omega \wedge \varphi$ where $\varphi$ is a primitive form we obtain,

$$(\mathrm{tr}R^H \wedge R^H)^{(3,1)}(X, Y, Z, \overline{W}) \sim \sum_{[X,Y,Z,\overline{W}]} R^B(\varphi(X), Y, Z, \overline{W}),$$

where the musical isomorphism is understood implicitly. It will be interesting to find a necessary condition for the existence of a solution to $(\mathrm{tr}R^H \wedge R^H)^{(3,1)} = 0$.

#### 5.4.2.2 $(\mathrm{tr}R^H \wedge R^H)^{(2,2)}$

The $(2,2)$-component of $\mathrm{tr}R^H \wedge R^H$ can be derived as the following

$$\begin{aligned}&(\mathrm{tr}R^H \wedge R^H)^{(2,2)}(X, \overline{Y}, Z, \overline{W}) \\ &= (\mathrm{tr}R^C \wedge R^C)(X, \overline{Y}, Z, \overline{W}) - \frac{1}{4!} \sum_{\substack{i,j \\ [X,Y,Z,\overline{W}]}} R^B(e_i, e_j, X, \overline{Y}) R^B(\bar{e}_{\bar{j}}, \bar{e}_{\bar{i}}, \overline{W}, Z) \\ &\quad - \frac{1}{4 \times 4!} \sum_{\substack{i,j \\ [X,Y,Z,\overline{W}]}} dT^H(X, Z, \bar{e}_{\bar{i}}, \bar{e}_{\bar{j}}) \cdot dT^H(\overline{Y}, \overline{W}, e_j, e_i)\end{aligned}$$

The last term can be simplified if $dT^H \in L^2(C^\infty(M))$.

**Lemma 103.** *If* $\dim_{\mathbb{C}} M = n$ *and* $dT^H \in L^2(C^\infty(M))$ *then*

$$\sum_{\substack{i,j \\ [X,Y,Z,\overline{W}]}} dT^H(X,Z,\bar{e}_{\bar{i}},\bar{e}_{\bar{j}}) \cdot dT^H(\overline{Y},\overline{W},e_j,e_i) = 8\sqrt{\frac{2\langle dT^H, dT^H\rangle}{n(n-1)}} \cdot dT^H(X,Z,\overline{W},\overline{Y}).$$

*Proof.* The proof is similar to the proof of Lemma 102. We write $dT^H = f\omega \wedge \omega$ and use the computation in Lemma 102:

$$dT^H(\overline{Y},\overline{W},e_j,e_i) = \frac{1}{3} f \left[Y_j W_i - Y_i W_j\right].$$

Therefore,

$$\begin{aligned}\sum_{\substack{i,j \\ [X,Y,Z,\overline{W}]}} dT^H(X,Z,\bar{e}_{\bar{i}},\bar{e}_{\bar{j}}) \cdot dT^H(\overline{Y},\overline{W},e_j,e_i) &= \frac{2}{3} f \sum_{\substack{i,j \\ [X,Y,Z,\overline{W}]}} dT(X,Z,\bar{e}_{\bar{i}},\bar{e}_{\bar{j}}) Y_j W_i \\ &= 16 f dT^H(X,Z,\overline{W},\overline{Y})\end{aligned}$$

Meanwhile,

$$dT^H \wedge * dT^H = f^2 \omega^2 \wedge * \omega^2 = 2 f^2 n(n-1)\frac{\omega^n}{n!},$$

where we have used $*\omega^2/2 = \omega^{n-2}/(n-2)!$ in $n$-complex dimensions. Therefore we have

$$\langle dT^H, dT^H\rangle = 2f^2 n(n-1).$$

Substituting this expression for $F$ we obtain this formula. □

## 5.5 Anomaly cancellation condition on $(M, \omega_u, V, H)$

In this section we explore the anomaly cancellation condition with the Hull connection

$$i\partial\bar{\partial}\omega_u = \frac{\alpha'}{4}\left(\mathrm{tr} R^H \wedge R^H - \mathrm{tr} F \wedge F\right),$$

in our setting $(M, J, g_u, \omega_u, V, H)$. We will apply the general results derived in the previous section to our particular case. We focus on $\mathrm{tr} R^H \wedge R^H$.

### 5.5.1 $(\mathrm{tr} R^H \wedge R^H)^{(3,1)}$

In this section we show that the $(3, 1)$-component of $\mathrm{tr} R^H \wedge R^H$ vanishes automatically. First we observe that

$$dT^H = 2i\partial\bar{\partial}\omega_u = 2i\partial\bar{\partial}e^u \wedge \omega_S - \beta^1 \wedge \beta^1 - \beta^2 \wedge \beta^2,$$

and by construction all of the forms are on $S$. Thus it should be a multiple of the top form on $S$. Therefore, we have

$$dT^H = f\frac{\omega_S^2}{2},$$

where $f := 2\triangle e^u - \langle \beta^1 \wedge \beta^1 + \beta^2 \wedge \beta^2, \omega_S^2/2\rangle$. Now we use Lemma 102 and prove $(\mathrm{tr} R^H \wedge R^H)^{(3,1)} = 0$.

**Proposition 104.** *Consider $(M, J, g_u, \omega_u, V, H)$ as a solution of the Strominger system with respect to the Hull connection. Then*

$$(\mathrm{tr} R^H \wedge R^H)^{(3,1)} = 0.$$

*Proof.* We compute this directly,

$$\begin{aligned}(\mathrm{tr} R^H \wedge R^H)^{(3,1)}(X, Y, Z, \overline{W}) &= \frac{1}{4!} \sum_{\substack{i,j \\ [X,Y,Z,\overline{W}]}} R^H(X, \overline{W}, e_i, e_j) R^H(Y, Z, \bar{e}_{\bar{j}}, \bar{e}_{\bar{i}}) \\ &= \frac{1}{4!} \sum_{\substack{i,j \\ [X,Y,Z,\overline{W}]}} R^B(e_i, e_j, X, \overline{W}) \cdot \frac{1}{2} dT^H(\bar{e}_{\bar{j}}, \bar{e}_{\bar{i}}, Z, W),\end{aligned}$$

from the connection identities derived in the previous sections. We use the fact that the metric is scaled by

$$g_u = e^u g_S + \frac{1}{2}\theta \odot \bar{\theta},$$

where $\theta \odot \bar{\theta} := \theta \otimes \bar{\theta} + \bar{\theta} \otimes \theta$. Thus the unitary frame $\{e^i\}$ on $M$ is related to the unitary frame $\{\tilde{e}^i\}$ on $S$ by

$$e^i = e^u \tilde{e}^i \text{ and } e_i = e^{-u}\tilde{e}_i.$$

Therefore

$$dT^H(\bar{e}_{\bar{j}}, \bar{e}_{\bar{j}}, Z, W) = e^{-2u} dT^H(\bar{\tilde{e}}_{\bar{j}}, \bar{\tilde{e}}_{\bar{i}}, Z, W) = e^{-2u} f \frac{\omega_S^2}{2}(\bar{\tilde{e}}_{\bar{j}}, \bar{\tilde{e}}_{\bar{i}}, Z, W).$$

Since $Z, W \in \Gamma(TS)$, we can apply the computation of Lemma 102, and we obtain the result of this lemma. $\square$

### 5.5.2 $(\mathrm{tr} R^H \wedge R^H)^{(2,2)}$

From the derivations in the previous section, we know that the $(2,2)$-component of $\mathrm{tr} R^H \wedge R^H$ has three types of contributions

$$\begin{aligned}
&(\mathrm{tr} R^H \wedge R^H)^{(2,2)}(X, \overline{Y}, Z, \overline{W}) \\
&\quad = (\mathrm{tr} R^C \wedge R^C)(X, \overline{Y}, Z, \overline{W}) - \frac{1}{4!} \sum_{\substack{i,j \\ [X,Y,Z,\overline{W}]}} R^B(e_i, e_j, X, \overline{Y}) R^B(\bar{e}_{\bar{j}}, \bar{e}_{\bar{i}}, \overline{W}, Z) \\
&\quad - \frac{1}{4 \times 4!} \sum_{\substack{i,j \\ [X,\overline{Y},Z,\overline{W}]}} dT^H(X, Z, \bar{e}_{\bar{i}}, \bar{e}_{\bar{j}}) \cdot dT^H(\overline{Y}, \overline{W}, e_j, e_i).
\end{aligned}$$

$\mathrm{tr} R^C \wedge R^C$ has been investigated in detail in the previous chapters. We look into the other two terms.

**Lemma 105.** *Consider $(M, \omega_u, V, H)$ as a solution to the Strominger system with respect* *to the Hull connection. Then we have*

$$\sum_{\substack{i,j\\ [X,Y,Z,\overline{W}]}} dT^H(X,Z,\bar{e}_{\bar{i}},\bar{e}_{\bar{j}})\cdot dT^H(\overline{Y},\overline{W},e_j,e_i) = 16e^{-4u}\left(\triangle e^u - B\right)dT^H,$$

*where* $B := \langle \beta^1\wedge\beta^1/2 + \beta^2\wedge\beta^2/2, \omega_S^2/2\rangle$.

*Proof.* We will use Lemma 103. We denote the unitary frame of $\omega_u$ as $\{e^i\}$ and that of $\omega_S$ as $\{\tilde{e}^i\}$ then since $\omega_u = e^u\omega_S + \frac{i}{2}\theta\wedge\bar{\theta}$, we have

$$e^i = e^u\tilde{e}^i \text{ and } e_i = e^{-u}\tilde{e}_i.$$

Therefore,

$$\begin{aligned}&\sum_{\substack{i,j\\ [X,Y,Z,\overline{W}]}} dT^H(X,Z,\bar{e}_{\bar{i}},\bar{e}_{\bar{j}})\cdot dT^H(\overline{Y},\overline{W},e_j,e_i)\\ &\qquad = \sum_{\substack{i,j\\ [X,Y,Z,\overline{W}]}} e^{-4u}dT^H(X,Z,\bar{\tilde{e}}_{\bar{i}},\bar{\tilde{e}}_{\bar{j}})\cdot dT^H(\overline{Y},\overline{W},\tilde{e}_j,\tilde{e}_i).\end{aligned}$$

Now $dT^H\in\Gamma(\Omega^{2,2}S)$ thus $dT^H = f\omega_S\wedge\omega_S$. We compute $dT^H$:

$$dT^H = 2i\partial\bar{\partial}e^u\wedge\omega_S - \beta^1\wedge\beta^1 - \beta^2\wedge\beta^2 = (2\triangle e^u - 2B)\frac{\omega_S^2}{2},$$

where $B := \langle\beta^1\wedge\beta^1/2 + \beta^2\wedge\beta^2/2, \omega_S^2/2\rangle$. Now we can use Lemma 103 with $n = 2$, which gives the result of this lemma. □

The term

$$\frac{1}{4!}\sum_{\substack{i,j\\ [X,Y,Z,\overline{W}]}} R^B(e_i,e_j,X,\overline{Y})R^B(\bar{e}_{\bar{j}},\bar{e}_{\bar{i}},\overline{W},Z)$$

is trickier than the other terms to write explicitly. We have done an explicit computation in Appendix A, and showed that this term is non-vanishing only when

$X, Y, Z, W \in T^{1,0}S$, and we have the following expression

$$\begin{aligned}
&\frac{1}{4!}\sum_{\substack{i,j\\ [X,\overline{Y},Z,\overline{W}]}} R^B(e_i,e_j,X,\overline{Y})R^B(\bar{e}_{\bar{j}},\bar{e}_{\bar{i}},\overline{W},Z)\\
&\sim \left[e^{-2u}\left(|\nabla u|+e^{-u}+1\right)^2+e^{-4u}\left(|\nabla^2 u|+e^{-u}|\nabla u|+e^{-u}\right)^2\right]\\
&\qquad \times \text{(terms on } S \text{ independent of } u).
\end{aligned}$$

### 5.5.3 Anomaly cancellation condition with respect to the Chern and Hull connection

Now we are ready to prove the main proposition of this chapter.

**Proposition 106.** *Consider $(M, \omega_u, V, H)$. Denote a solution of the Strominger system with respect to the Chern connection as $u_C$. Then we have the following estimate,*

$$\left|\left| i\partial\bar{\partial}\omega_{u_C} - \frac{\alpha'}{4}\left[(\mathrm{tr}R^H\wedge R^H)(u_C) - \mathrm{tr}F\wedge F\right]\right|\right| \le C(A^6+A^5+A^4+A^3+A^2),$$

*where $||\varphi|| := |\langle \varphi, \omega_S^2/2\rangle|$ for $\varphi \in L^2_{\omega_S}(C^\infty(M))$, and $A = \left(\int e^{-4u_C}\right)^{1/4}$. $C$ is a constant that depends on $\langle \beta^1\wedge\beta^1/2+\beta^2\wedge\beta^2/2, \omega_S^2/2\rangle$, $\alpha'\lambda^2$, $\mu$, the Sobolev constant of $g_S$, and $C_1$ of Chapter 4.*

*Therefore when $A \to 0$, $u_C$ can be considered as a solution of the anomaly cancellation condition with respect to the Hull connection.*

*Proof.* Consider the anomaly cancellation condition with respect to the Hull connection then we have

$$\begin{aligned}
0 &= i\partial\bar{\partial}\omega_u - \frac{\alpha'}{4}\left(\mathrm{tr}R^H\wedge R^H - \mathrm{tr}F\wedge F\right)\\
&= i\partial\bar{\partial}\omega_u - \frac{\alpha'}{4}\left(\mathrm{tr}R^C\wedge R^C - \mathrm{tr}F\wedge F\right)\\
&\quad + \frac{\alpha'}{3}e^{-4u}\left(\triangle e^u - B\right)i\partial\bar{\partial}\omega_u + \sum_{i,j} R^B(e_i,e_j,\cdot,\cdot)\wedge R^B(\bar{e}_{\bar{j}},\bar{e}_{\bar{i}},\cdot,\cdot).
\end{aligned}$$

We have proven existence of solution $u_C$ of the Strominger system with respect to the Chern connection in Theorem 45. With this solution $u_C$ the above equation becomes

$$\begin{aligned} & i\partial\bar{\partial}\omega_{u_C} - \frac{\alpha'}{4}\left[(\mathrm{tr}R^H \wedge R^H)(u_C) - \mathrm{tr}F \wedge F\right] \\ & = \frac{\alpha'}{3} e^{-4u_C}\left(\triangle e^{u_C} - B\right)^2 \frac{\omega_S^2}{2} + \sum_{i,j} R^B(e_i, e_j, \cdot, \cdot) \wedge R^B(\bar{e_{\bar{j}}}, \bar{e_{\bar{i}}}, \cdot, \cdot), \end{aligned} \tag{5.1}$$

where $B := \langle \beta^1 \wedge \beta^1/2 + \beta^2 \wedge \beta^2/2, \omega_S^2/2\rangle$, and we have used the fact that $i\partial\bar{\partial}\omega_{u_C} = (\triangle e^{u_C} - B)\omega_S^2/2$. Recall that in Chapter 4, we have derived the following estimate

$$e^{-u_C} \leq C_1 A,$$

where $A := \left(\int_S e^{-4u_C}\right)^{1/4}$. We use this to find an estimate of the right hand side of Equation (5.1).

**Estimates of $e^{-4u_C}\left(\triangle e^{u_C} - B\right)^2$.**

We expand the Laplacian

$$e^{-4u_C}(\triangle e^{u_C} - B)^2 = (e^{-u_C}|\nabla u_C|^2 + e^{-u_C}\triangle u_C - 2e^{-2u_C}B)^2,$$

and find estimates of the individual terms.

1. $e^{-u_C}|\nabla u_C|^2$: From Lemma 50 with $k = 2$, we have

$$\begin{aligned} \int e^{-u_C}|\nabla u_C|^2 &\leq C\int e^{-3u_C} + C\int e^{-2u_C} \\ &\leq C\left(\int e^{-4u_C}\right)^{3/4} + C\left(\int e^{-4u_C}\right)^{2/4}, \end{aligned}$$

   where we have used the Hölder inequality in the second line and $C$ is a positive constant that depends on $\alpha'\lambda^2$, and $\mu$. Since we have normalized the metric $g_S$ to volume 1, we have the following

$$\sup\left(e^{-u_C}|\nabla u_C|^2\right) \leq CA^3 + CA^2.$$

2. $e^{-u_C}\triangle u_C$: Consider the Sobolev embedding theorem of $L_2^1(S) \hookrightarrow L_0^2(S)$:

$$\int e^{-u} + \sum_a \int e^{-u}|\partial_a u| + \sum_{a,b} \int e^{-u}|\partial_a\partial_b u| \leq C_S \left(\int (e^{-u})^2\right)^{1/2} \leq C_S \left(\int (e^{-u})^4\right)^{1/4},$$

   Therefore, we obtain

$$\sup(|e^{-u_C}\triangle u_C|) \leq C_S A,$$

   where we have used the fact that $\mathrm{vol}(S) = 1$.

3. $-2e^{-2u_C}B$: Using the fact that $B$ is a bounded function on $S$ by definition, we obtain the following inequality

$$\sup\left(-2e^{-2u_C}B\right) \leq CA^2,$$

   where $C$ is a constant that depends on $B$ and $C_1$ of Chapter 4.

Using the estimates derived above, we finally conclude

$$e^{-4u_C}(\triangle e^{u_C} - B)^2 \leq C(A^3 + A^2 + A)^2, \tag{5.2}$$

where $C$ is a constant that depends on $B$, $\alpha'\lambda^2$, $\mu$, Sobolev constant of $g_S$ and $C_1$.

**Estimates of $\sum_{i,j} R^B(e_i, e_j, \cdot, \cdot) \wedge R^B(\bar{e}_{\bar{j}}, \bar{e}_{\bar{i}}, \cdot, \cdot)$.** In the Appendix, we have shown that

$$\begin{aligned}
&\frac{1}{4!} \sum_{\substack{i,j \\ [X,\overline{Y},Z,\overline{W}]}} R^B(e_i, e_j, X, \overline{Y}) R^B(\bar{e}_{\bar{j}}, \bar{e}_{\bar{i}}, \overline{W}, Z) \\
&\quad \sim \left[ e^{-2u} \left( |\nabla u| + e^{-u} + 1 \right)^2 + e^{-4u} \left( |\nabla^2 u| + e^{-u} |\nabla u| + e^{-u} \right)^2 \right] \\
&\qquad \times \text{(terms on } S \text{ independent of } u\text{)}.
\end{aligned}$$

Using the Sobolev embedding $L^1_2(S) \hookrightarrow L^2_0(S)$ we have discussed above, we can simplify the terms with $u = u_C$

$$\begin{aligned}
&\left| \frac{1}{4!} \sum_{\substack{i,j \\ [X,\overline{Y},Z,\overline{W}]}} R^B(e_i, e_j, X, \overline{Y}) R^B(\bar{e}_{\bar{j}}, \bar{e}_{\bar{i}}, \overline{W}, Z) \right| \\
&\quad \leq C \left[ e^{-2u_C} \left( |\nabla u_C| + e^{-u_C} + 1 \right)^2 + e^{-4u_C} \left( |\nabla^2 u_C| + A + e^{-u_C} \right)^2 \right] \\
&\quad \leq C \Big[ e^{-2u_C} \left( |\nabla u_C|^2 + 2|\nabla u_C| + 1 + A^2 + 4A \right) \\
&\qquad\quad + e^{-4u_C} \left( |\nabla^2 u_C|^2 + 2A |\nabla^2 u_C| + 4A^2 + 2A \right) \Big],
\end{aligned}$$

where $C$ is a constant that depends on the Sobolev constant and the terms on $S$. We use the Sobolev embedding once again, and obtain the following,

$$\left\| \left| \frac{1}{4!} \sum_{\substack{i,j \\ [X,\overline{Y},Z,\overline{W}]}} R^B(e_i, e_j, \cdot, \cdot) R^B(\bar{e}_{\bar{j}}, \bar{e}_{\bar{i}}, \cdot, \cdot) \right| \right\| \leq C \left[ A^6 + A^5 + A^4 + A^3 + A^2 \right]. \tag{5.3}$$

Using Inequality (5.2) and (5.3) in Equation (5.1), we obtain the result of this proposition. □

We can use this proposition to show potential problems with the $\alpha'$-expansion approach. Let us take $A = \alpha'/(5C) \leq 1$, then we have the following from Proposition 106

$$\left|\left|i\partial\bar{\partial}\omega_{u_C} - \frac{\alpha'}{4}\left[(\mathrm{tr}R^H \wedge R^H)(u_C) - \mathrm{tr}F \wedge F\right]\right|\right| \le (\alpha')^2.$$

Therefore, with this choice of normalization $A$, the solution $u_C$ of the Strominger system with respect to the Chern connection becomes an exact solution of the Strominger system up to $\alpha'$ order with respect to the Hull connection. As discussed in Chapter 3 and 4, there exists $\omega_{u_C}$ over Kähler surfaces $S$ that are not Calabi-Yau. This is contrary to the findings when using $\alpha'$-expansion that restricts $S$ to a Calabi-Yau surface. This suggests that not all of the exact solutions of the Strominger system with respect to the Hull connection can be approximated using the $\alpha'$-expansion. This result is expected; in Chapter 4, we have shown that the solution $u_C$ is smooth, however the $\alpha'$-expansion approach assumes $u_C$ is analytic. Since the set of smooth functions is larger than that of analytic functions, it is expected that the $\alpha'$-expansion approach will leave out a large number of smooth solutions.

We can ask the converse: when can an $\alpha'$-expansion solution be realized as a exact solution of the Strominger system? The $\alpha'$-expansion approach is built on the assumption that the exact solution is analytic and thus can be expanded in the following form:

$$\omega := \omega_0 + \alpha'\omega_1 + (\alpha')^2\omega_2 + \cdots .$$

Often $\omega$ is truncated $\omega = \omega_0 + \alpha'\omega_1 + O(\alpha')$, and the equations are solved in $\alpha'$ order-to-order. This approach has the following problems.

1. There is no convergence proof of the sum $\omega = \omega_0 + \alpha'\omega_1 + (\alpha')^2\omega_2 + \cdots$.
2. Even if the expansion does converge, there is no guarantee that it will converge to an exact solution of the PDEs we are considering.

Note that the Cauchy-Kowalevski theorem addresses similar issues for Cauchy initial value problems locally. It will be interesting to see if a similar theorem can be established for the anomaly cancellation condition.

# Chapter 6

# Conclusions and outlook

In this thesis we have investigated the Strominger system on a non-Kähler hermitian manifold in detail.

In Chapter 3, we have solved the conformally balanced condition, and the vector bundle condition when $M$ is a principal torus bundle over a Kähler surface $S$. The general results correctly reproduce special cases considered in [GP04] and [FY08].

In Chapter 4, we have shown that the anomaly cancellation condition has a solution using the continuity method. The result can be applied to various non-Kähler manifolds, and in particular to $(k-1)(S^2 \times S^4)\#k(S^3 \times S^3)$. When $k = 2$, it is known that $(S^2 \times S^4)\#2(S^3 \times S^3)$ admits complex structures $J$ with holomorphically trivial canonical bundles. Thus it is a good candidate for heterotic string compactification.

In Chapter 5, we investigated the Strominger system in $\alpha'$-expansion context and also by using the Hull connection. We have shown that the $\alpha'$ -expansion does not reproduce the solution of the system with respect to the Hull connection and thus should be used with caution.

The work of this thesis can be improved further. In Chapter 3, we show that

Iwasawa manifolds can be solutions of the Strominger system since $S = T^4$ in this case. We could explicitly work out the Strominger system with the Hull connection in detail and understand the solution better.

In the previous chapters, we have shown that the Strominger system has a solution on $(k-1)(S^2 \times S^4)\#k(S^3 \times S^3)$. When $k = 2$ in particular it is known that $(S^2 \times S^4)\#2(S^3 \times S^3)$ admits infinitely many complex structures that have holomorphically trivial canonical bundles [FP09]. It will be interesting to see if the particular complex structure we use in this thesis is one of these complex structures.

In this thesis, we have shown that the $\alpha'$-expansion does not correctly capture the behaviour of the solution even up to $(\alpha')^1$-order. We could improve our understanding of the $\alpha'$-expansion by investigating when the exact solution of the Strominger system is analytic and by establishing a similar result to the Cauchy-Kowalevski theorem. Ideally, we should try to to discuss the $\alpha'$-expansion in higher order corrections along the lines of [WW87].

A particularly interesting non-Kähler hermitian manifold to consider might be $(M, J, g, \omega)$ that admits a conformally balanced metric and and also $(T^{1,0}M, g)$ is stable. In such cases, we can take $(V = T^{1,0}M, g)$ and reduce the Strominger system to

$$i\partial\bar{\partial}\omega = \frac{\alpha'}{4}\left(\mathrm{tr}R^H \wedge R^H - \mathrm{tr}R^C \wedge R^C\right).$$

This can be thought as a modified version of the standard embedding. Using the derivation of the curvature identities, it is possible to simplify this expression and gain insight of the behaviour of the non-Kähler solution. This equation will show the effects of the Hull connection components that do not preserve the complex structure.

We could also provide numerical solutions to demonstrate an example explicitly. This is still work in progress, and we have included some preliminary numerical results in Appendix B.

Finally, the estimates in Chapter 4 can be improved. Note that we assume $u \in C^{5,\alpha}(M)$ and then use the following chain of uniform bounds independent of $t$ to show that $T$ is closed:

$$||\partial^2 u||_{C^{0,\alpha}(M)} \leq C \to ||\partial^2 u||_{C^{1,\alpha}(M)} \leq C \to ||\partial^2 u||_{C^{2,\alpha}(M)} \leq C \to ||\partial^2 u||_{C^{3,\alpha}(M)} \leq C,$$

where $C$ is some generic constant. This procedure seems convoluted since we start from a strong assumption $u \in C^{5,\alpha}(M)$ and then show a somewhat weaker estimate $||\partial^2 u||_{C^{0,\alpha}(M)} \leq C$. Ideally, we would like to start from a weaker assumption and obtain a stronger estimate. It has been pointed out to the author by Professor Jan Kristensen that it might be possible to present a more elegant and simple proof using viscosity theory.

# Appendix A

# Computation of connection components

In this section, we compute the Hull connection and the Bismut connection explicitly. Since the Hull connection is not a canonical connection, using holomorphic coordinates does not simplify the computation unlike the case with the Chern connection. In this section we use a unitary basis and Cartan's structure equations to derive the results. The results will be used to show that the solution of the Strominger system with respect to the Chern connection converges to a solution to the system with respect to the Hull connection in Chapter 5.

## A.1 Setup and conventions

We establish the notation of this appendix. First of all, we will use a unitary frame $\{\theta^1, \bar{\theta}^{\bar{1}}, \theta^2, \bar{\theta}^{\bar{2}}, \theta, \bar{\theta}\}$ so the metric can be expressed in the following form,

$$\begin{aligned} g &= \frac{1}{2}(\theta^1 \odot \bar{\theta}^{\bar{1}} + \theta^2 \odot \bar{\theta}^{\bar{2}} + \theta \odot \bar{\theta}), \\ &= \frac{e^{2\phi}}{2}(\tilde{\theta}^1 \odot \bar{\tilde{\theta}}^{\bar{1}} + \tilde{\theta}^2 \odot \bar{\tilde{\theta}}^{\bar{2}}) + \frac{1}{2}(\theta \odot \bar{\theta}) \end{aligned}$$

where $\odot$ is the symmetric product such that $\theta^1 \odot \theta^2 = \theta^1 \otimes \theta^2 + \theta^2 \otimes \theta^1$, $2\phi = u$, and $\{\tilde{\theta}^a\}$ are the unitary basis 1-forms on the base before the conformal scaling.

We introduce the following conventions for this section.

- Index $\mu, \nu, \rho, \sigma, \tau$ for general index, i.e. $\theta^p \in \{\theta^1, ..., \bar{\theta}^{\bar{2}}, \theta, \bar{\theta}\}$. Index $i, j, k$ for holomorphic indices i.e. $\theta^i \in \{\theta^1, \theta^2, \theta\}$ and vice versa for $\bar{i}, \bar{j}, \bar{k}$.

- Index $\alpha, \beta, \gamma$ for index on the base $S$, i.e. $\theta^a \in \{\theta^1, ..., \bar{\theta}^{\bar{2}}\}$. And index $a, b, c$ for the holomorphic indices on the base $S$, i.e. $\theta^a \in \{\theta^1, \theta^2\}$ and vice versa for $\bar{a}, \bar{b}, \bar{c}$ indices.

- We will use the following convention: $\theta^0 := \theta$ and its complex conjugate $\bar{\theta}^{\bar{0}} = \bar{\theta}$.

We use the following convention for the Cartan structure equations:

$$T^\mu = d\theta^\mu + \gamma^\mu{}_\nu \wedge \theta^\nu, \tag{A.1}$$

$$R^\mu{}_\nu = d\gamma^\mu{}_\nu + \gamma^\mu{}_\rho \wedge \gamma^\rho{}_\nu, \tag{A.2}$$

where the $\gamma^\mu{}_\nu$ are connection one forms that are related to the connection components, $\gamma^\mu{}_\nu := \Gamma_{\rho\nu}{}^\mu \theta^\rho$.

- $\nabla^{LC}$ denotes the Levi-Civita connection, $\nabla^-$ the Bismut connection, and $\nabla^+$ the Hull connection. In this section only, we use $\nabla$ to denote a metric preserving connection with totally skew-symmetric $(2,1) + (1,2)$-torsion, i.e. $\nabla = \nabla^\pm$.

- $\tilde{\Gamma}$ denotes the Christoffel symbols of the Levi-Civita connection, and $\Gamma$ denotes the connection components of $\nabla$. $\Gamma^+$ and $\Gamma^-$ are the connection components of $\nabla^+$ and $\nabla^-$ respectively.

## A.2 Properties of the connection

In this section, we assume that a general connection $\nabla$ preserves the metric and has torsion which is a $(1,2)+(2,1)$-form using the metric isomorphism. The Hull connection and the Bismut connection are examples of such connection $\nabla$.

### A.2.1 Preservation of the metric

Since $\nabla$ preserves the metric, this gives constraints on the connection components. For a unitary frame, the metric preservation condition is,

$$\begin{aligned} \nabla_\mu g_{\nu\rho} &= \partial_\mu g_{\nu\rho} - \Gamma_{\mu\nu}{}^{\sigma} g_{\sigma\rho} - \Gamma_{\mu\rho}{}^{\sigma} g_{\nu\sigma}, \\ 0 &= -\Gamma_{\mu\nu\rho} - \Gamma_{\mu\rho\nu}. \end{aligned}$$

Thus the connection components are antisymmetric on the last two indices. Using this property, it is straightforward to derive the following identity,

$$\Gamma_{\mu\nu\rho} = \Gamma_{[\mu\nu]\rho} - \Gamma_{[\nu\rho]\mu} + \Gamma_{[\rho\mu]\nu}. \tag{A.3}$$

### A.2.2 Some vanishing components

We show that connection components with pure lowered indices vanish:

$$\Gamma_{ijk} = \Gamma_{\bar{i}\bar{j}\bar{k}} = 0.$$

In case of the Bismut connection, this can be easily seen from the complex structure preservation condition $\nabla^- J = 0$ that $\Gamma^{-}_{\mu i}{}^{\bar{j}} = \Gamma^{-}_{\mu\bar{j}}{}^{i} = 0$. Thus the above identities follows. We show that it is true for both Hull and Bismut connections.

**Lemma 107.** *Suppose $\nabla$ is metric preserving connection with skew-symmetric torsion* *that is a* $(2,1)+(1,2)$*-form. Then the following connection components* $\Gamma$ *vanish*

$$\Gamma_{ijk} = \Gamma_{\overline{i}\overline{j}\bar{k}} = 0. \tag{A.4}$$

*Proof.* Since $\nabla$ preserves the hermitian metric $g$, we have

$$0 = \nabla_X\left[g(Y,Z)\right] = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

where $X, Y, Z \in \Gamma(T^{1,0}M)$ and we have used the fact that $g$ is hermitian. Now we follow the arguments of [Tag11]:

$$\begin{aligned} g(\nabla_X Y, Z) = \frac{1}{2}\big[&g(\nabla_X Y, Z) - g(\nabla_Y X, Z) - g(T(X,Y),Z) \\ &+ g(\nabla_Z Y, X) - g(\nabla_Y Z, X) - g(T(Z,Y),X) \\ &+ g(\nabla_Z X, Y) - g(\nabla_X Z, Y) - g(T(Z,X),Y)\big], \end{aligned}$$

where we have used the fact that the skew-symmetric torsion is a $(2,1)+(1,2)$-form, therefore $T(X,Y) \in \Gamma(T^{1,0}M)$. We therefore obtain

$$\begin{aligned} g(\nabla_X Y, Z) &= \frac{1}{2}\left[g([X,Y],Z) + g([Z,Y],X) + g([Z,X],Y)\right] \\ &= 0, \end{aligned}$$

because the complex structure is integrable, i.e. $[X,Y] \in \Gamma(T^{1,0}M)$ (Frobenius theorem). Since $g(\nabla_X Y, Z) = 0$ we obtain

$$\Gamma_{ijk} = \Gamma_{\overline{i}\overline{j}\bar{k}} = 0.$$

□

### A.2.3 Computation of the connection components

When computing the connection components explicitly, it is convenient to introduce the following $(1,2)$-tensor $\vartheta$ which is antisymmetric on its covariant indices $\frac{1}{2}\vartheta_{\rho\nu}{}^{\mu}\theta^{\rho}\wedge\theta^{\nu} := d\theta^{\mu}$. Using this notation the first Cartan structure equation becomes,

$$\frac{1}{2}T_{\rho\nu}{}^{\mu}\theta^{\rho}\wedge\theta^{\nu} = \frac{1}{2}\vartheta_{\rho\nu}{}^{\mu}\theta^{\rho}\wedge\theta^{\nu} + \Gamma_{\rho\nu}{}^{\mu}\theta^{\rho}\wedge\theta^{\nu}.$$

This gives an expression of $\Gamma_{[\mu\nu]\rho}$ in terms of the torsion $T_{\mu\nu\rho}$ and $\vartheta_{\mu\nu\rho}$. Using Equation (A.3) and the fact that $T$ is a 3-form we obtain,

$$\Gamma_{\rho\nu\mu} = \frac{1}{2}\left(T_{\rho\nu\mu} - \vartheta_{\rho\nu\mu} + \vartheta_{\nu\mu\rho} - \vartheta_{\mu\rho\nu}\right). \tag{A.5}$$

Because $\nabla$ preserves the metric and we are using a unitary frame, not all of $\Gamma_{\mu\nu\rho}$ are independent. We have shown that $\Gamma$ is antisymmetric in the last two indices, i.e. $\Gamma_{\mu\nu\rho} = -\Gamma_{\mu\rho\nu}$, and using the fact that $\Gamma$ is a real object only the following components are independent,

1. $\Gamma_{00\bar{0}}$, $\Gamma_{00\bar{\alpha}}$, $\Gamma_{0\bar{0}\alpha}$, $\Gamma_{0\bar{0}\bar{\alpha}}$, $\Gamma_{0\alpha\bar{\beta}}$, $\Gamma_{0\bar{\alpha}\bar{\beta}}$.

2. $\Gamma_{\alpha 0\bar{0}}$, $\Gamma_{\alpha 0\bar{\beta}}$, $\Gamma_{\alpha\bar{0}\beta}$, $\Gamma_{\alpha\bar{0}\bar{\beta}}$, $\Gamma_{\alpha\beta\bar{\gamma}}$, $\Gamma_{\alpha\bar{\beta}\bar{\gamma}}$.

To compute the components explicitly, we take specific values for $\mu$ in the first Cartan structure equation:

$$T^{\mu} = d\theta^{\mu} + \gamma^{\mu}{}_{\nu}\wedge\theta^{\nu},$$

and derive the connection components for $\nabla$,

1. When $\mu = \alpha$, we can express $\Gamma$ in terms of $\tilde{\Gamma}$ which denotes the connection

components for the Levi-Civita connection on $S$,

$$\begin{aligned} T^{\alpha} &= d(e^{\phi}\tilde{\theta}^{\alpha}) + \gamma^{\alpha}{}_{\nu} \wedge \theta^{\nu}, \\ &= d\phi \wedge \theta^{\alpha} - e^{-\phi}\tilde{\Gamma}_{\gamma\beta}{}^{\alpha}\theta^{\gamma} \wedge \theta^{\beta} + \Gamma_{\rho\nu}{}^{\alpha}\theta^{\rho} \wedge \theta^{\nu}, \\ \frac{1}{2}T_{\rho\nu}{}^{\alpha}\theta^{\rho} \wedge \theta^{\nu} &= \partial_{\gamma}\phi\delta^{\alpha}_{\beta}\theta^{\gamma} \wedge \theta^{\beta} - e^{-\phi}\tilde{\Gamma}_{\gamma\beta}{}^{\alpha}\theta^{\gamma} \wedge \theta^{\beta} + \Gamma_{\rho\nu}{}^{\alpha}\theta^{\rho} \wedge \theta^{\nu}. \end{aligned}$$

   (a) $0\bar{0}$ component: $T_{0\bar{0}}{}^{\alpha} = 2\Gamma_{[0\bar{0}]}{}^{\alpha}$, and $\vartheta_{0\bar{0}}{}^{\alpha} = 0$.

   (b) $0\beta$ component: $T_{0\beta}{}^{\alpha} = 2\Gamma_{[0\beta]}{}^{\alpha}$, and $\vartheta_{0\beta}{}^{\alpha} = 0$.

   (c) $\gamma\beta$ component:
   $T_{\gamma\beta}{}^{\alpha} = 2(\partial_{[\gamma}\phi\delta^{\alpha}_{\beta]} - e^{-\phi}\tilde{\Gamma}_{[\gamma\beta]}{}^{\alpha} + \Gamma_{[\gamma\beta]}{}^{\alpha})$, and $\vartheta_{\gamma\beta}{}^{\alpha} = 2(\partial_{[\gamma}\phi\delta^{\alpha}_{\beta]} - e^{-\phi}\tilde{\Gamma}_{[\gamma\beta]}{}^{\alpha})$.

2. When $\mu = 0$, the first Cartan structure equation becomes,

$$\frac{1}{2}T_{\rho\nu}{}^{0}\theta^{\rho} \wedge \theta^{\nu} = \beta^{1} + i\beta^{2} + \Gamma_{\rho\nu}{}^{0}\theta^{\rho} \wedge \theta^{\nu}.$$

   (a) $0\bar{0}$ component: $T_{0\bar{0}}{}^{0} = 2\Gamma_{[0\bar{0}]}{}^{0}$, and $\vartheta_{0\bar{0}}{}^{0} = 0$.

   (b) $0\beta$ component: $T_{0\beta}{}^{0} = 2\Gamma_{[0\beta]}{}^{0}$, and $\vartheta_{0\beta}{}^{0} = 0$.

   (c) $\bar{0}\beta$ component: $T_{\bar{0}\beta}{}^{0} = 2\Gamma_{[\bar{0}\beta]}{}^{0}$, and $\vartheta_{\bar{0}\beta}{}^{0} = 0$.

   (d) $\alpha\beta$ component: $T_{\alpha\beta}{}^{0} = (\beta^{1} + i\beta^{2})_{\alpha\beta} + 2\Gamma_{[\alpha\beta]}{}^{0}$, and $\vartheta_{\alpha\beta}{}^{0} = (\beta^{1} + i\beta^{2})_{\alpha\beta}$.

Now using Equation (A.5), we can compute the independent components listed above. A straightforward computation reveals:

1. Vanishing components:

$$\Gamma_{00\bar{0}} = \Gamma_{00\bar{a}} = 0.$$

2. Purely torsional terms:

$$\Gamma_{0\bar{0}a} = \frac{1}{2}T_{0\bar{0}a}, \quad \Gamma_{0\bar{0}\bar{a}} = \frac{1}{2}T_{0\bar{0}\bar{a}}, \quad \Gamma_{0\bar{a}\bar{b}} = \frac{1}{2}T_{0\bar{a}\bar{b}}, \quad \Gamma_{a0\bar{0}} = \frac{1}{2}T_{a0\bar{0}}, \quad \Gamma_{a\bar{0}b} = \frac{1}{2}T_{a\bar{0}b}.$$

3. $\beta^i$ terms:

$$\Gamma_{0a\bar{b}} = \frac{1}{2}(T_{0a\bar{b}} + \vartheta_{a\bar{b}0}), \qquad \Gamma_{a0\bar{b}} = \frac{1}{2}(T_{a0\bar{b}} + \vartheta_{a\bar{b}0}), \qquad \Gamma_{a\bar{0}\bar{b}} = \frac{1}{2}(T_{a\bar{0}\bar{b}} + \vartheta_{a\bar{b}\bar{0}}),$$
$$\Gamma_{a\bar{b}0} = \frac{1}{2}(T_{a\bar{b}0} - \vartheta_{a\bar{b}0}), \qquad \Gamma_{a\bar{b}\bar{0}} = \frac{1}{2}(T_{a\bar{b}\bar{0}} - \vartheta_{a\bar{b}\bar{0}}).$$

4. Base terms:

$$\Gamma_{ab\bar{c}} = e^{-\phi}\tilde{\Gamma}_{ab\bar{c}} + \delta_{a\bar{c}}\partial_b\phi + \frac{1}{2}T_{ab\bar{c}},$$
$$\Gamma_{a\bar{b}\bar{c}} = e^{-\phi}\tilde{\Gamma}_{a\bar{b}\bar{c}} - 2\delta_{a[\bar{b}}\partial_{\bar{c}]}\phi + \frac{1}{2}T_{a\bar{b}\bar{c}}.$$

Note that the Kronecker $\delta$'s are defined with respect to the covariant and contravariant indices, for instance $\delta_1^1 = 1$. When the index is lowered this translates into $\delta_{1\bar{1}} = \frac{1}{2}$.

## A.3 The Bismut and Hull connection

So far we have only assumed that $\nabla$ is a metric preserving connection with $(1,2)+(2,1)$-form torsion. Now we consider the Bismut connection $\nabla^-$ and the Hull connection $\nabla^+$.

The Bismut connection $\nabla^-$ is the unique metric preserving and complex structure preserving connection with skew symmetric torsion. The Bismut connection can be expressed in terms of the Levi-Civita connection $\nabla^{LC}$ in the following way,

$$\nabla^- = \nabla^{LC} + \frac{1}{2}T = \nabla^{LC} - \frac{1}{2}d^c\omega.$$

Note that we are using the metric isomorphism implicitly. The Hull connection is defined as,

$$\nabla^+ = \nabla^{LC} + \frac{1}{2}d^c\omega.$$

It can be checked that $\nabla^+$ preserves the metric and has skew-symmetric torsion by construction. However the Hull's connection no longer preserves the complex structure. The torsion of the Bismut connection $T^-$ and that of the Hull connection $T^+$, are given from the above definitions as $T^\pm = \pm d^c\omega$. To keep our notation as simple as possible, we often drop ± and express a generic connection which can be either the Bismut or the Hull connection as $\nabla = \nabla^\pm$ and similarly also for the torsion as $T = T^\pm$.

We use the explicit expression for $d^c\omega$ to simplify the connection components further,

$$\begin{aligned} d^c\omega &= 2e^{2\phi} d^c\phi \wedge \omega_S - \frac{1}{2}(\bar{\partial}\theta \wedge \bar{\theta} + \theta \wedge \partial\bar{\theta}), \\ &= i d^c\phi \wedge (\theta^1 \wedge \bar{\theta}^{\bar{1}} + \theta^2 \wedge \bar{\theta}^{\bar{2}}) - \frac{1}{2}[(\beta^1 + i\beta^2) \wedge \bar{\theta} + \theta \wedge (\beta^1 - i\beta^2)]. \end{aligned}$$

From this expression we obtain the components of the torsion,

1. Vanishing torsion components:

$$T_{a0\bar{0}} = T_{\bar{a}0\bar{0}} = T_{0ab} = T_{0\bar{a}\bar{b}} = 0.$$

2. $\beta^i$ terms:

$$T^\pm_{a\bar{b}\bar{0}} = \mp\frac{1}{2}(\beta^1 + i\beta^2)_{a\bar{b}}, \qquad T^\pm_{a\bar{b}0} = \mp\frac{1}{2}(\beta^1 - i\beta^2)_{a\bar{b}}.$$

3. Torsion coming from the base:

$$\begin{aligned} T^\pm_{12\bar{1}} &= \mp\partial_2\phi, & T^\pm_{12\bar{2}} &= \pm\partial_1\phi, \\ T^\pm_{1\bar{1}\bar{2}} &= \mp\bar{\partial}_{\bar{2}}\phi, & T^\pm_{2\bar{1}\bar{2}} &= \pm\bar{\partial}_{\bar{1}}\phi. \end{aligned}$$

Thus for the Bismut and Hull connections the torsional connection components vanish. We summarize the result in the following list.

1. Vanishing components:

$$\Gamma_{00\bar{0}} = \Gamma_{00\bar{a}} = \Gamma_{0\bar{0}a} = \Gamma_{0\bar{0}\bar{a}} = \Gamma_{0\bar{a}\bar{b}} = \Gamma_{a0\bar{0}} = \Gamma_{a\bar{0}b} = 0.$$

2. $\beta^i$ terms:

$$\Gamma^{\pm}_{0a\bar{b}} = \begin{cases} 0 \\ \frac{1}{2}(\beta^1 - i\beta^2)_{a\bar{b}} \end{cases}, \quad \Gamma^{\pm}_{a0\bar{b}} = \begin{cases} \frac{1}{2}(\beta^1 - i\beta^2)_{a\bar{b}} \\ 0 \end{cases}, \quad \Gamma^{\pm}_{a\bar{0}\bar{b}} = \begin{cases} \frac{1}{2}(\beta^1 + i\beta^2)_{a\bar{b}} \\ 0 \end{cases}.$$

3. Base terms:

$$\Gamma^{\pm}_{11\bar{1}} = e^{-\phi}\tilde{\Gamma}_{11\bar{1}} + \frac{1}{2}\partial_1\phi, \quad \Gamma^{\pm}_{12\bar{1}} = e^{-\phi}\tilde{\Gamma}_{12\bar{1}} + \begin{cases} 0 \\ \partial_2\phi \end{cases}, \quad \Gamma^{\pm}_{21\bar{1}} = e^{-\phi}\tilde{\Gamma}_{21\bar{1}} \pm \frac{1}{2}\partial_2\phi,$$

$$\Gamma^{\pm}_{22\bar{2}} = e^{-\phi}\tilde{\Gamma}_{22\bar{2}} + \frac{1}{2}\partial_2\phi, \quad \Gamma^{\pm}_{21\bar{2}} = e^{-\phi}\tilde{\Gamma}_{21\bar{2}} + \begin{cases} 0 \\ \partial_1\phi \end{cases}, \quad \Gamma^{\pm}_{12\bar{2}} = e^{-\phi}\tilde{\Gamma}_{12\bar{2}} \pm \frac{1}{2}\partial_1\phi,$$

$$\Gamma^{\pm}_{22\bar{1}} = e^{-\phi}\tilde{\Gamma}_{22\bar{1}}, \quad \Gamma^{\pm}_{1\bar{1}\bar{2}} = e^{-\phi}\tilde{\Gamma}_{1\bar{1}\bar{2}} + \begin{cases} -\bar{\partial}_{\bar{2}}\phi \\ 0 \end{cases},$$

$$\Gamma^{\pm}_{11\bar{2}} = e^{-\phi}\tilde{\Gamma}_{11\bar{2}}, \quad \Gamma^{\pm}_{2\bar{1}\bar{2}} = e^{-\phi}\tilde{\Gamma}_{2\bar{1}\bar{2}} + \begin{cases} \bar{\partial}_{\bar{1}}\phi \\ 0 \end{cases}.$$

## A.4 The Riemann curvature tensor

Using Cartan structure equations, the general form of the Riemann curvature tensor can be derived:

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \partial_\mu \Gamma_{\nu\rho\sigma} - \partial_\nu \Gamma_{\mu\rho\sigma} - \Gamma_{\mu\rho}{}^{\tau}\Gamma_{\nu\tau\sigma} + \Gamma_{\nu\rho}{}^{\tau}\Gamma_{\mu\tau\sigma} \\ &\quad + \left(T_{\mu\nu}{}^{\tau} - \Gamma_{\mu\nu}{}^{\tau} + \Gamma_{\nu\mu}{}^{\tau}\right)\Gamma_{\tau\rho\sigma}. \end{aligned}$$

The Riemann curvature tensor in this form can be thought as a section of $\bigwedge^2 T^*M \otimes \bigwedge^2 T^*M$. In this perspective we will refer to types of the Riemann tensor as the type of the two 2-forms. For instance, $R_{i\bar{j}lm}$ will be referred to a $(1,1)(2,0)$-type. In our setting, we have five independent types: $(2,0)(2,0)$, $(2,0)(1,1)$, $(2,0)(0,2)$, $(1,1)(2,0)$, and $(1,1)(1,1)$. However because the pure types of the connection components vanish, the $(2,0)(2,0)$-type of the Riemann tensor vanish. Also the Riemann tensors simplify because we have a set of vanishing connection components, and none of the connection components depend on $0$ nor $\bar{0}$ direction, i.e. $\partial_0\Gamma = \partial_{\bar{0}}\Gamma = 0$.

An arbitrary component of the Riemann tensor can be read from the expression above and the connection component we have derived before. We are interested in the following term in Chapter 5:

$$\frac{1}{4!} \sum_{\substack{i,j \\ [X,\overline{Y},Z,\overline{W}]}} R^B(e_i, e_j, X, \overline{Y}) R^B(\bar{e}_{\bar{j}}, \bar{e}_{\bar{i}}, \overline{W}, Z).$$

We first note that when $X, Y \notin \Gamma(TS)$, then $R^B(e_i, e_j, X, \overline{Y}) = 0$. This is because the connection components of the Bismut connection vanishes when it takes values on the fibre directions, i.e. $\Gamma^-_{a0b} = \Gamma^-_{a\bar{0}b} = \Gamma^-_{a0\bar{0}} = 0$.

Since the unitary basis is related by $\theta^a = e^u\tilde{\theta}^a$ and $e_a = e^{-u}\bar{e}_a$, while $\theta^0$ and $e_0$ are invariant. We obtain two types of Riemann tensor behaviour:

1. $R^B(e_0, e_a, X, \overline{Y}) = e^{-u} R^B(e_0, \tilde{e}_{\tilde{a}}, X, \overline{Y})$.

2. $R^B(e_a, e_b, X, \overline{Y}) = e^{-2u} R^B(\tilde{e}_{\tilde{a}}, \tilde{e}_{\tilde{b}}, X, \overline{Y})$.

We investigate the expression of each Riemann tensor.

**Lemma 108.** *Consider the Bismut connection $\nabla^-$ on $(M, \omega_u)$. Then the Riemann tensor $R^B$ of the Bismut connection has the following form*

$$|R^B(e_0, e_a, X, \overline{Y})| \sim \left[ |\nabla_a u| + e^{-u} + 1 \right] \times \textit{(terms on } S \textit{ independent of } u\textit{)}.$$

*Proof.* We take $\mu = 0$, $\nu = a$, $\rho = b$, $\sigma = \bar{c}$ for the general expression for $R^B$ and obtain

$$R^B_{0ab\bar{c}} = -\partial_a \Gamma_{0b\bar{c}} - \Gamma_{0b}{}^d \Gamma_{ad\bar{c}} + \Gamma_{ab}{}^d \Gamma_{0d\bar{c}} + (T_{0a}{}^d - \Gamma_{0a}{}^d)\Gamma_{db\bar{c}},$$

where we have omitted vanishing components. Using the connection components we derived before, we obtain

$$R^B_{0ab\bar{c}} = -\frac{1}{2}\partial_a(\beta^1 - i\beta^2)_{b\bar{c}} - \frac{1}{2}\Gamma^{-}_{a\bar{c}}{}^d(\beta^1 - i\beta^2)_{db} + \frac{1}{2}\Gamma^{-}_{ab}{}^d(\beta^1 - i\beta^2)_{d\bar{c}} - (\beta^1 - i\beta^2)_a{}^d \Gamma^{-}_{db\bar{c}}.$$

We are interested in term containing $u$, substituting $\Gamma^{-}_{a\bar{c}}{}^d$ terms we obtain,

$$R_{0ab\bar{c}} \sim [|\nabla_a u| + e^{-u} + 1] \times \text{(base terms)}.$$

Thus we obtain the result of this lemma. $\square$

We have a similar result for $R^B(e_a, e_b, X, \overline{Y})$.

**Lemma 109.** *Consider the Bismut connection $\nabla^-$ on $(M, \omega_u)$. Then the Riemann tensor $R^B$ of the Bismut connection has the following form*

$$|R^B(e_a, e_b, X, \overline{Y})| \sim \left[ |\nabla_a \nabla_b u| + e^{-u}|\nabla u| + e^{-u} \right] \times \textit{(terms on } S \textit{ independent of } u\textit{)}.$$

*Proof.* We take $\mu = a$, $\nu = b$, $\rho = c$, $\sigma = \bar{d}$ in the general expression for $R^B$, and follow the proof of the previous lemma. Using the general expression, $R^B$ becomes

$$R^B_{abc\bar{d}} = \partial\Gamma^-_{bcd} - \partial_b\Gamma^-_{acd} - \Gamma^{-\ e}_{ac}\Gamma^-_{be\bar{d}} + \Gamma^{-\ e}_{bc}\Gamma^-_{ae\bar{d}} + (T_{ab}{}^{e} - \Gamma^{-\ e}_{ab} + \Gamma^{-\ e}_{ba})T_{ec\bar{d}}.$$

From the connection components we have derived, we see that except from the second order derivatives, the terms have $e^{-u}$ factor. Thus we have

$$R^B_{abc\bar{d}} \sim \left[\nabla_a\nabla_b u + e^{-u}\nabla u + e^{-u}\right] \times \text{(base terms)}.$$

Therefore obtain the result of this lemma. □

Finally we conclude that the behaviour of $\mathrm{tr}R^B \wedge R^B$.

**Lemma 110.** *Consider the Bismut connection $\nabla^-$ on $(M, \omega_u)$. Then*

$$\begin{aligned}&\frac{1}{4!}\sum_{\substack{i,j\\ [X,\overline{Y},Z,\overline{W}]}} R^B(e_i, e_j, X, \overline{Y})R^B(\bar{e}_{\bar{j}}, \bar{e}_{\bar{i}}, \overline{W}, Z)\\ &\quad\sim \left[e^{-2u}\left(|\nabla u| + e^{-u} + 1\right)^2 + e^{-4u}\left(|\nabla^2 u| + e^{-u}|\nabla u| + e^{-u}\right)^2\right]\\ &\qquad\times \textit{(terms on } S \textit{ independent of } u\textit{).}\end{aligned}$$

*Proof.* The computation done for the connection components are in the unitary frame $\theta^a$ which is related to the unitary frame on $S$ by $\theta^a = e^u\tilde{\theta}^a$. Since $u$ is a function on $S$ we express the result in terms of the unitary frame $\tilde{e}_{\tilde{a}}$ on $S$:

1. $R^B(e_0, e_a, X, \overline{Y}) = e^{-u}R^B(e_0, \tilde{e}_{\tilde{a}}, X, \overline{Y})$.
2. $R^B(e_a, e_b, X, \overline{Y}) = e^{-2u}R^B(\tilde{e}_{\tilde{a}}, \tilde{e}_{\tilde{b}}, X, \overline{Y})$.

By squaring these expression and taking the trace we get the result of this lemma.

□

# Appendix B

# Local Numerical Examples

In this appendix we reduce the PDE

$$i\partial\bar{\partial}e^{u}\wedge\omega_S - i\frac{\alpha'}{2}\partial\bar{\partial}u\wedge\lambda^2\omega_S - i\frac{\alpha'}{2}\partial\bar{\partial}(e^{-u}\rho) - \frac{\alpha'}{2}\partial\bar{\partial}u\wedge\partial\bar{\partial}u + \mu\frac{\omega_S^2}{2} = 0,$$

into an ordinary differential equation by considering cases when $(S, \omega_S)$ is a cohomogeneity one metric. The work here is still work in progress.

We assume $u$ only depends on the direction that is invariant under the action of the isometry group of $g_S$. This reduces the PDE into an ODE. We solve the resulting ODE numerically on a given patch on $S$ for some simple examples.

We have solved the ODE numerically using Mathematica and Maple. We only present the graph of each numerical solution to illustrate the behaviour of the solution. Typically, $u(s)$ is a function of a variable $s$ which takes values in $s \in [0, \infty)$. We map this interval to $[0, \pi/2)$ using $s = \tan t$. The plots are done with respect to $t$.

In each examples, the initial conditions are chosen such that the numerical solution is bounded. In Chapter 4, we have proved that $u$ is bounded given that $u$ has sufficiently large initial value (Section 4.4, Chapter 4). We choose $u$ sufficiently large such that this condition is satisfied.

## B.1 Simplification of $\rho$

First we look into cases when we can simplify $\rho$. In general $\rho$ is a complicated $(1,1)$-form. However by Corollary 43, when $\beta^1 = n\beta^2$ for some integer $n$, we can express $\rho$ as

$$\rho = \frac{\lambda^2}{2}\omega_S.$$

In this case, the PDE simplifies to,

$$i\partial\bar{\partial}\left(e^u - \frac{\alpha'\lambda^2}{2}u - \frac{\alpha'\lambda^2}{4}e^{-u}\right) \wedge \omega_S - \frac{\alpha'}{2}\partial\bar{\partial}u \wedge \partial\bar{\partial}u + \mu\frac{\omega_S^2}{2} = 0. \tag{B.1}$$

## B.2 Simplification of $\omega_S$

We simplify the PDE further by taking special cases of $(S, \omega_S)$. We consider the Gibbons-Pope metric given in [GP78] and Gauntlett-Martelli-Sparks-Yau metric given in [GMSY07].

## B.3 Metrics of [GP78]

We explore the PDE on $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{CP}^2$ with Gibbons and Pope's [GP78] metric, which is the standard Kähler-Einstein metric written in cohomogeneity one form. The metrics have isometries that leave one "radial" direction $r$ invariant. We will assume $u = u(r)$ and reduce the PDE into an ODE.

### B.3.1 $\mathbb{CP}^1 \times \mathbb{CP}^1$

In [GP78], the authors study the following generic metric

$$g_S = d\chi^2 + A(\chi)(d\psi + \epsilon\cos\theta d\phi)^2 + B(\chi)(d\theta^2 + \sin^2\theta d\phi^2).$$

They show that for specific $A(\chi)$ and $B(\chi)$ this generic metric becomes metrics on different manifolds. For $\mathbb{CP}^1 \times \mathbb{CP}^1$, the authors propose $\epsilon = 0, A = \sin^2(\chi\sqrt{\Lambda})/\Lambda$ and $B = 1/\Lambda$, where $\Lambda$ is a constant. Then the metric becomes

$$g_S = d\chi^2 + \frac{\sin^2(\chi\sqrt{\Lambda})}{\sqrt{\Lambda}}d\psi^2 + \frac{1}{\Lambda}(d\theta^2 + \sin^2\theta d\phi^2),$$

where $0 \leq \chi \leq \frac{\pi}{\sqrt{\Lambda}}$, $0 \leq \theta \leq \pi$, and $0 \leq \psi, \phi \leq 2\pi$.

This metric can be obtained from the standard Fubini-Study metric using stereographic projection. Denote the inhomogeneous coordinate for $\mathbb{CP}^1 \times \mathbb{CP}^1$ as $(z_1, z_2)$. If we take the following stereographic projection,

$$z_1 = \cot(\chi\sqrt{\Lambda})e^{i\psi}, \qquad z_2 = \cot\left(\frac{\theta}{2}\right)e^{i\phi},$$

then the standard Fubini-Study metric on $\mathbb{CP}^1 \times \mathbb{CP}^1$ takes the form of the metric above with the choice of $\epsilon$, $A$, $B$.

If we think of $\chi$ as the radial direction as in [GP78] and assume $u = u(\chi)$, we can reduce the PDE into a ODE of $\chi$. Note that this assumption implies that we are only considering the effects of one of the $\mathbb{CP}^1$ and ignoring the contribution of the other. This is acceptable as $\mathbb{CP}^1 \times \mathbb{CP}^1$ is a trivial fibration. We carry out the numerical analysis on a given patch.

- **Assumptions:** We assume $u = u(\chi)$. We take $\beta^1 = \beta^2 = \frac{\lambda}{2}\omega_S$ with a trivial vector bundle $V$, i.e. $F_H = 0$.
- **Conformally Balanced Condition:** Explicit computation shows that $\mathcal{R}_S = 2\omega_S$. The conformally balanced condition gives an condition on $\lambda^2$. From Proposition 22,

  $$0 = \mathcal{R}^B_{Mu} = \pi^*(2\omega_S - \lambda^2\omega_S),$$

  thus $\lambda^2 = 2$.

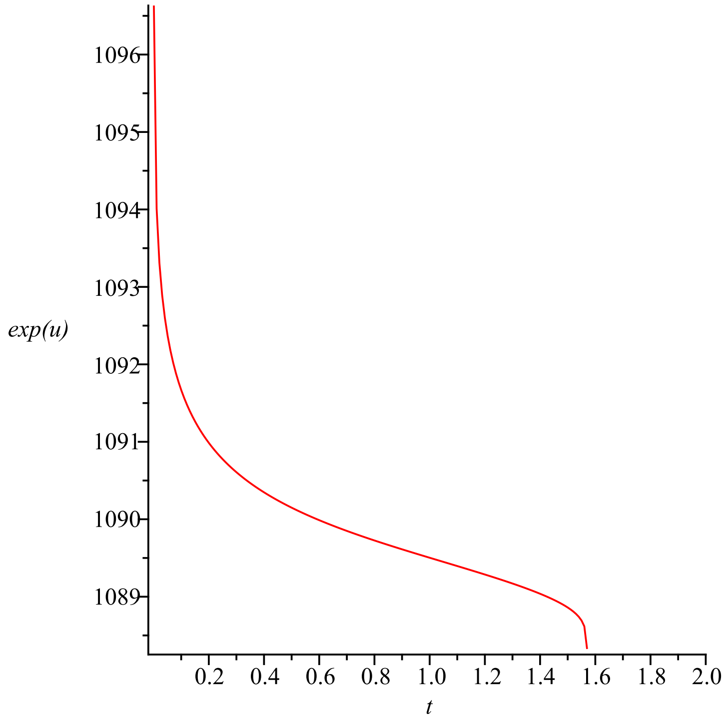

Figure B.1: Numerical solution on $\mathbb{CP}^1 \times \mathbb{CP}^1$

- **Computation of $\mu$:** Since we are on $\mathbb{CP}^1 \times \mathbb{CP}^1$, due to dimensional reasons we obtain $\mathrm{tr} R \wedge R = 0$. We can read $\mu$ off from the following definition,

  $$-\mu \frac{\omega_S^2}{2} = \frac{\alpha'}{4} \mathrm{tr} R \wedge R - \frac{\alpha'}{4} \mathrm{tr} F_H \wedge F_H + \langle \beta^1, \beta^2 \rangle \frac{\omega_S^2}{2}, \\ = \frac{\lambda^2}{2} \frac{\omega_S^2}{2}.$$

  Thus, $\mu = -1$.

- **ODE:** Now the PDE reduces to the following ODE,

  $$\left( e^u - \frac{e^{-u}}{8} \right) (1+s)^2 (u')^2 s + \left( e^u - \frac{1}{4} + \frac{e^{-u}}{8} \right) (1+s)^2 (su'' + u') - 1 = 0,$$

  where we have taken $\alpha' = 1/4$ for numerical computations. Also we have defined $s := \cot^2(\chi\sqrt{\Lambda})$.

- **Numerical solution:** We have solved the non-linear ODE with the following initial values $u(0) = 7$ and $u'(0) = -1$. We have chosen the initial values so that $u$ is bounded. We have used a reparametrization $s = \tan(t)$ to illustrate the behaviour of the solution on $[0, \pi/2)$. Note that the scaling factor $\exp(u)$ does not have a singularity and it is bounded by the results of Chapter 4 (Figure B.1).

### B.3.2 $\mathbb{CP}^2$

The next example we consider is $\mathbb{CP}^2$. The Fubini-Study metric on $\mathbb{CP}^2$ in inhomogeneous coordinates $z_i$ is given as,

$$(g_{i\bar{j}}) = \frac{1}{2(1+|\mathbf{z}|^2)^2} \begin{pmatrix} 1+|z_2|^2 & -z_2\bar{z}_1 \\ -z_1\bar{z}_2 & 1+|z_1|^2, \end{pmatrix}$$

where $|\mathbf{z}|^2 := |z_1|^2 + |z_2|^2$.

$\mathrm{SU}(3)$ acts on $\mathbb{CP}^2$ on the homogeneous coordinates in the standard way and an arbitrary $\mathrm{U}(2)$ matrix can be embedded into $\mathrm{SU}(3)$ as in the following,

$$\begin{pmatrix} \mathrm{U}(2) & 0 \\ 0 & [\det \mathrm{U}(2)]^{-1} \end{pmatrix}.$$

It can be checked that this embedding leaves $r^2 := |z_1|^2 + |z_2|^2$ invariant. Following [GP78], we parametrize $z_1$ and $z_2$ using Euler angles,

$$z_1 = r\cos\frac{\theta}{2}\exp\left(i\frac{\psi+\phi}{2}\right), \qquad z_2 = r\sin\frac{\theta}{2}\exp\left(i\frac{\psi-\phi}{2}\right), \tag{B.2}$$

with the coordinates having the following values,

$$0 \leq \theta \leq \pi, \qquad 0 \leq \phi \leq 2\pi,$$

$$0 \leq \psi \leq 4\pi, \qquad 0 \leq r \leq \infty.$$

In these coordinates, the Fubini-Study metric becomes,

$$g = \frac{1}{(1+r^2)^2} dr^2 + \frac{r^2}{4(1+r^2)} \left(\sigma_1^2 + \sigma_2^2\right) + \frac{r^2}{4(1+r^2)^2} \sigma_3^2, \tag{B.3}$$

where $\sigma_i$'s are the left-invariant one forms on $\mathrm{SU}(2)$, explicitly,

$$\sigma_1 = \cos\psi d\theta + \sin\theta \sin\psi d\phi,$$

$$\sigma_2 = -\sin\psi d\theta + \sin\theta \cos\psi d\phi,$$

$$\sigma_3 = \cos\theta d\phi + d\psi.$$

The symmetry of the Fubini-Study metric becomes clear in this form. We assume $u$ only depends on $r^2$ and solve the PDE numerically.

- **Assumptions:** We take $\beta^1 = \beta^2 = \frac{\lambda}{2}\omega$ and take $V$ to be a trivial vector bundle, i.e. $F_H = 0$.

- **Computation:** It can derived that $\mathcal{R}_S = 3\,\omega_S$ thus from the conformally balanced condition $\lambda^2 = 3$. We take $\alpha' = 1/6$ for the numerical computations. The function $\mu$ can be read off from the following definition,

  $$-\mu \frac{\omega_S^2}{2} = \frac{\alpha'}{4} \mathrm{tr} R \wedge R - \frac{\alpha'}{4} \mathrm{tr} F_H \wedge F_H + \langle \beta^1, \beta^2 \rangle \frac{\omega_S^2}{2}.$$

  Explicit calculations show that,

  $$\mathrm{tr} R \wedge R = -\frac{3}{2} \frac{\omega_S^2}{2},$$

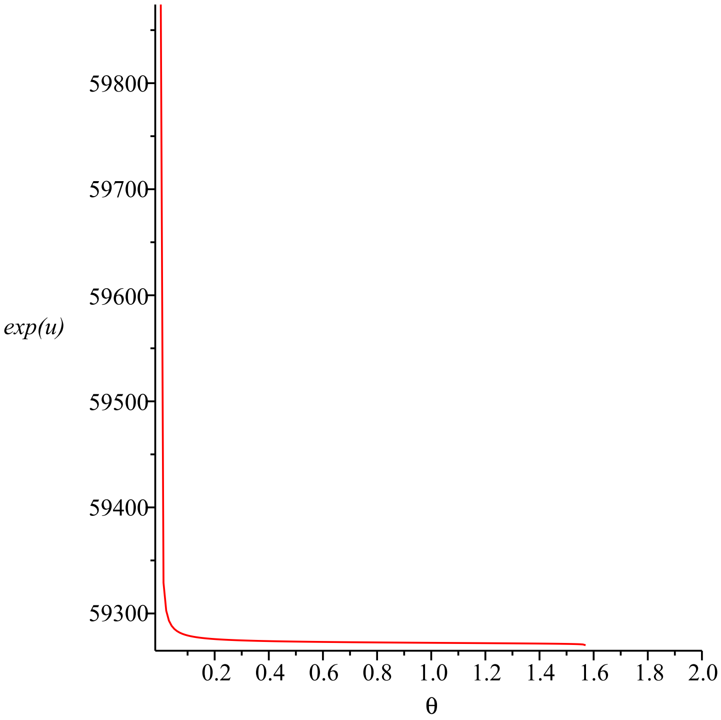

Figure B.2: Numerical solution on $\mathbb{CP}^2$

giving us $\mu = -23/16$.

- **ODE:** Now the PDE reduces into the following ODE,

$$\begin{aligned}\left(e^u - \frac{e^u}{8}\right)(u')^2\, s(1+s)^2 \\ + \left(e^u - \frac{1}{4} + \frac{e^{-u}}{8}\right)\left[s(1+s)^2u'' + (1+s)(2+s)u'\right] \\ + \frac{1}{3}(1+s)^3u'(su''+u') - \frac{23}{32} = 0,\end{aligned}$$

  where $s := r^2$, $s \in [0,\infty)$ and $u' := \frac{du(s)}{ds}$.

- **Numerical solution:** The numeric solution has been computed with $u(0) = 11$ and $u'(0) = -10$. The initial condition was chosen so that $u$ is bounded. We have used $s^2 = \tan(t)$ to plot the scaling $\exp(u)$ as a function of $t \in [0, \pi/2)$. $u$ is smooth and it is bounded (Section 4.4, Chapter 4).

## B.4 Complex structures and cohomogeneity one metrics

By a further coordinate transformation $r = \tan(t)$ with $0 \leq t < \pi/2$, the standard Fubini-Study Metric (B.3) of $\mathbb{CP}^2$ in real coordinates reads,

$$g = dt^2 + \frac{\sin^2 t}{4}(\sigma_1^2 + \sigma_2^2) + \frac{\cos^2 t \sin^2 t}{4}\sigma_3^2. \tag{B.4}$$

A natural generalization of this metric is of the following form,

$$g_S = dt^2 + a^2(t)\sigma_1^2 + b^2(t)\sigma_2^2 + c^2(t)\sigma_3^2.$$

This type of metric has been studied thoroughly in [GMSY07] and we apply this metric to our PDE.

Since the PDE is in complex coordinates, we need to introduce a complex structure on this metric. We take the following basis $(1, 0)$-forms which gives us the complex structure,

$$\theta^1 = dt + ic(t)\sigma_3, \qquad \theta^2 = a(t)\sigma_1 + ib(t)\sigma_2, \tag{B.5}$$

Frobenius theorem states that if the $(0, 2)$-part of $d\theta^1$ and $d\theta^2$ vanish then the complex structure is integrable. In this case the vanishing condition is the following:

$$a^2 - b^2 + (a\dot{b} - \dot{a}b)c = 0,$$

where the dot indicates derivatives with respect to $t$. The Metric (B.4) satisfy this condition trivially as $a = b$.

The metric in [GMSY07] also satisfies this integrable condition. This can be seen from the following ODEs derived in [GMSY07],

$$\begin{aligned}\dot{a}(t) &= -\frac{1}{2bc}(-a^2+b^2+c^2),\\ \dot{b}(t) &= -\frac{1}{2ca}(a^2-b^2+c^2),\\ \dot{c}(t) &= -\frac{1}{2ab}(a^2+b^2-c^2)+\Lambda ab.\end{aligned}$$

In the new $\theta^i$ coordinates we have,

$$\omega_S = \frac{i}{2}\left(\theta^1 \wedge \bar{\theta}^{\bar{1}} + \theta^2 \wedge \bar{\theta}^{\bar{2}}\right).$$

We assume $u$ depends only on the radial coordinate $t$, i.e. $u = u(t)$ and compute $\partial u$, $\bar{\partial} u$ and $\partial\bar{\partial} u$ in $\theta$ coordinates:

$$\partial u = \frac{\dot{u}}{2}\,\theta^1, \qquad \bar{\partial} u = \frac{\dot{u}}{2}\,\bar{\theta}^{\bar{1}}, \qquad \partial\bar{\partial} u = \frac{1}{4}\left(\frac{\dot{c}}{c}\dot{u} + \ddot{u}\right)\theta^1 \wedge \bar{\theta}^{\bar{1}} - \frac{c\dot{u}}{4ab}\,\theta^2 \wedge \bar{\theta}^{\bar{2}}.$$

We can substitute these expressions into the simplified PDE (B.1) and reduce the PDE into an ODE. The process of reducing the PDE to an ODE is somewhat long. We have used Maple to find the explicit ODE when necessary. We carry out numerical computations for three cases: the $\mathbb{CP}^2$ metric given in [GP78], the $\mathbb{CP}^1 \times \mathbb{CP}^1$ and the $\mathbb{CP}^2$ metric given in [GMSY07].

### B.4.1 $\mathbb{CP}^2$: revisited

We use the complex structure given in (B.5) and compute for the $\mathbb{CP}^2$ metric given in [GP78], with $a = b = \sin^2 t/4$, $c = \cos^2 t \sin^2 t/4$. Note that the complex structure is different from the one given in Equation (B.2). We have solved the resulting ODE numerically with initial conditions $u(0) = 2$ and $u(0) = -0.1$ (Figure B.3). These initial values were chosen so that $u$ is bounded (Section 4.4, Chapter 4).

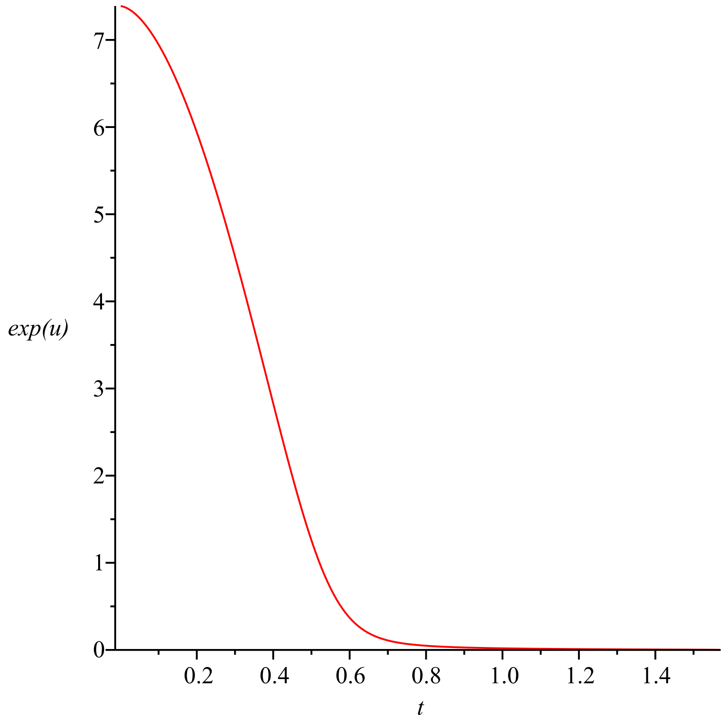

Figure B.3: Numerical solution on $\mathbb{CP}^2$ with the complex structure (B.5)

### B.4.2 $\mathbb{CP}^1 \times \mathbb{CP}^1$ of [GMSY07]

The $\mathbb{CP}^1 \times \mathbb{CP}^1$ metric proposed in [GMSY07] is the following,

$$g_S = dt^2 + \frac{\cos^2(\sqrt{3}t)}{3}\sigma_1^2 + \frac{1}{3}\sigma_2^2 + \frac{\sin^2(\sqrt{3}t)}{3}\sigma_3^2,$$

with $0 \le t \le \pi/(2\sqrt{3})$.

- **Assumptions:** We use the same setting as the $\mathbb{CP}^1 \times \mathbb{CP}^1$ computation before. We take $\beta^1 = \beta^2 = \sqrt{2}\omega_S$ with a trivial vector bundle $V$, i.e., $F_H = 0$. Also $\mu = -1$ stays the same as it does not depend on the complex structure.

- **Numerical solution:** We have solved the ODE numerically with the following initial conditions: $u(0) = 5$ and $u'(0) = -10$. The scaling factor $\exp(u)$ on $\omega_S$ is bounded.

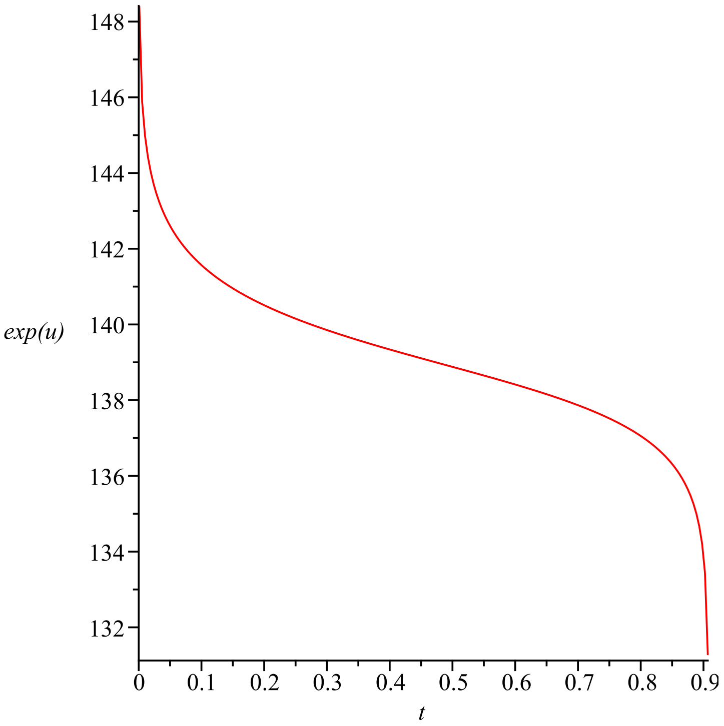

Figure B.4: Numerical solution on $\mathbb{CP}^1 \times \mathbb{CP}^1$ with the complex structure (B.5)

### B.4.3 $\mathbb{CP}^2$ of [GMSY07]

The $\mathbb{CP}^2$ metric given in [GMSY07] is the following,

$$g_S = dt^2 + \cos^2\left(t + \frac{\pi}{4}\right)\sigma_1^2 + \sin^2\left(t + \frac{\pi}{4}\right)\sigma_2^2 + \sin^2(2t)\sigma_3^2,$$

with $0 \leq t \leq \pi/4$.

- **Assumptions:** We use the same setting as the $\mathbb{CP}^2$ computation before with $\beta^1 = \beta^2 = \frac{\sqrt{3}}{2}\omega_S$ and $\mu = -23/16$.

- **Numerical solution:** The numerical solution for $u(0) = 5$ and $u'(0) = -10$ has been plotted in Figure B.5.

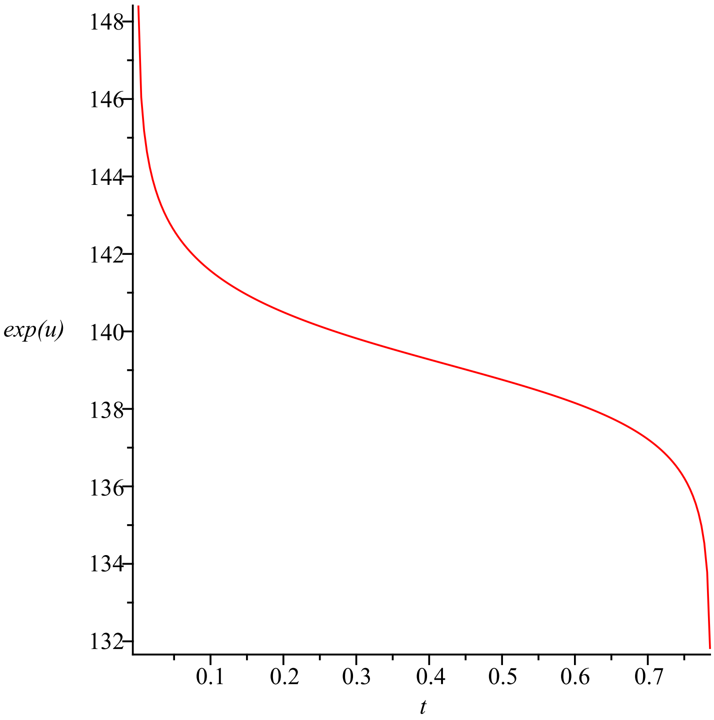

Figure B.5: Numerical solution on $\mathbb{CP}^2$ with the complex structure (B.5)

## B.5 $S^3 \times S^3 \to \mathbb{CP}^1 \times \mathbb{CP}^1$

We consider the base space $\mathbb{CP}^1 \times \mathbb{CP}^1$ with the standard Fubini-Study metric,

$$g_S = \frac{dz_1 \otimes d\bar{z}_{\bar{1}} + d\bar{z}_{\bar{1}} \otimes dz_1}{(1+|z_1|^2)^2} + \frac{dz_2 \otimes d\bar{z}_{\bar{2}} + d\bar{z}_{\bar{2}} \otimes dz_2}{(1+|z_2|^2)^2}.$$

We take $\beta^l$'s as the following,

$$\beta^1 = \sqrt{2}\frac{dz_1 \wedge d\bar{z}_{\bar{1}}}{(1+|z_1|^2)^2}, \qquad \beta^2 = \sqrt{2}\frac{dz_2 \wedge d\bar{z}_{\bar{2}}}{(1+|z_2|^2)^2}.$$

This corresponds to having $S^3 \times S^3$ as the entire space over $\mathbb{CP}^1 \times \mathbb{CP}^1$ [GGP08]. In this setting since $\beta^1 \neq \beta^2$, we cannot use Corollary 43. We use Proposition 41 to obtain $\rho$ in general,

$$\rho = (2-2\lambda)\beta^1 + \frac{\lambda^2}{2}\omega_S.$$

Since $\beta^1$ is harmonic and $\omega_S$ is Kähler, $\rho$ is also harmonic respect to $\Delta_{\bar\partial}$. Using this fact, we can simplify the PDE in the following form,

$$i\partial\bar\partial e^u \wedge \omega_S - i\frac{\alpha'}{2}\partial\bar\partial u \wedge \lambda^2\omega_S - i\frac{\alpha'}{2}\partial\bar\partial e^{-u} \wedge \rho - \frac{\alpha'}{2}\partial\bar\partial u \wedge \partial\bar\partial u + \mu\frac{\omega_S^2}{2} = 0.$$

We assume $u = u(s)$ where $s := |z_1|^2$, and reduce the PDE into an ODE. A straightforward yet somewhat long calculation produces the following ODE,

$$\frac{1}{2}\left(-\frac{\alpha'}{2}e^{-u}\lambda^2 + 2e^u\right)(u')^2 s + \frac{1}{2}\left(\frac{\alpha'}{2}e^{-u}\lambda^2 - 2\frac{\alpha'}{2}\lambda^2 + 2e^u\right)(u''\cdot s + u') + \frac{\mu}{(1+s)^2} = 0,$$

where the prime derivative is a shorthand notation for $d/ds$.

- **Assumptions:** We take $\beta^1 = \sqrt{2}\frac{dz_1\wedge d\bar z_{\bar 1}}{(1+|z_1|^2)^2}$, $\beta^2 = \sqrt{2}\frac{dz_2\wedge d\bar z_{\bar 2}}{(1+|z_2|^2)^2}$ with a trivial vector bundle $V$, i.e. $F_H = 0$.

- **Conformally Balanced Condition:** Explicit computation shows that $\mathcal{R}_S = 2\omega_S$. The conformally balanced condition (cf. Proposition 22)

$$0 = \mathcal{R}^B_{Mu} = \pi^*(2\omega_S - \lambda^2\omega_S),$$

  gives $\lambda^2 = 2$.

- **Computation of $\mu$:** Since we are on $\mathbb{CP}^1 \times \mathbb{CP}^1$, we get $\mathrm{tr}R \wedge R = 0$. We can read $\mu$ off from the following definition,

$$\begin{aligned} -\mu\frac{\omega_S^2}{2} &= \frac{\alpha'}{4}\mathrm{tr}R\wedge R - \frac{\alpha'}{4}\mathrm{tr}F_H \wedge F_H + \langle\beta^1, \beta^2\rangle\frac{\omega_S^2}{2}, \\ &= \frac{\lambda^2}{2}\frac{\omega_S^2}{2}. \end{aligned}$$

  Thus, $\mu = -1$.

- **ODE:** Now the PDE reduces to the following ODE,

$$\left(e^{u}-\frac{e^{-u}}{8}\right)(1+s)^{2}(u')^{2}s+\left(e^{u}-\frac{1}{4}+\frac{e^{-u}}{8}\right)(1+s)^{2}(su''+u')-1=0,$$

  where $s := \cot(\chi\sqrt{\Lambda})^2$ and we have taken $\alpha' = 1/4$ for numerical computations.

- **Numerical solution:** We have solved the non-linear ODE with the following initial values $u(0) = 7$ and $u'(0) = -1$. We have used a reparametrization $s = \tan(t)$ to illustrate the behaviour in an interval $t \in [0, \pi/2)$. Note that the scaling factor $\exp(u)$ has no singularity (Figure B.1) and it is bounded by the results in Chapter 4.

## B.6 Comments

Some numerical plots seem to suggest that $\exp(u)$ develops a singularity. However, by the analytic proof of Chapter 4, all of the solutions are smooth. Closer investigation on the numerical data with higher resolution, confirms that all the solutions are smooth and bounded.

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