Excursion Sets of Planar Gaussian Fields

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This thesis is dedicated to my Mum for reading me 'Tootles the Taxi' and countless other books.

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Abstract

Gaussian fields are widely used for modelling spatial phenomena in disciplines such as cosmology, medical imaging and quantum mechanics. Geometric properties of the excursion sets of such fields are well studied, as they provide important test statistics in these areas. There is also theoretical interest in the large scale topological behaviour of excursion sets of Gaussian fields, due to recent results suggesting a connection with discrete percolation models. In this thesis we study the geometry of planar Gaussian fields with a focus on one particular quantity; the number of excursion/level set components in a large domain, which is of interest both for statistical testing and in relation to percolation.

Chapter 1 introduces the topic of excursion/level sets of Gaussian fields. This includes motivation for their study, an outline of some mathematical techniques for their analysis and a summary of the relevant literature.

In Chapter 2 we consider the mean number of connected components of a Gaussian excursion set or level set in a large domain. Nazarov and Sodin have shown that this quantity satisfies a law of large numbers as the size of the domain increases. The value of the corresponding limit, at different levels, can be viewed as a functional which describes the asymptotic density of excursion set components. We derive a new integral representation of this functional, which provides an alternative way of analysing it. The representation is based on a deterministic decomposition of excursion set components in terms of critical points, which is analogous to the Poincaré-Hopf theorem for the Euler characteristic. We also derive some basic properties of the excursion set functional, including symmetries, upper/lower bounds and continuity properties. Finally, we explicitly compute this functional for a simple class of fields, which poses some interesting questions about the general properties of the functional.

In Chapter 3, we extend the analysis of our integral representations to show that the mean functionals are continuously differentiable for a wide class of fields. This follows from a more general result, in which we derive differentiability of the functionals from the decay of the probability of 'four-arm events' for the field conditioned to have a saddle point at the origin. We discuss how this compares to related results for percolation models. For some fields, including two important examples known

as the Random Plane Wave and the Bargmann-Fock field, we also derive stochastic monotonicity of the conditioned field, which proves that the mean functionals are monotone, with respect to the level, on certain regions.

In Chapter 4 we give the most significant result of this thesis: a lower bound on the variance of the number of Gaussian excursion/level set components in a large domain. More precisely, we prove a lower bound on the order of the fluctuations of these variables, for certain classes of field and certain levels, which implies the corresponding variance bound. These bounds are conjectured to be optimal for generic fields, based on the behaviour of other geometric quantities and also on the corresponding properties of discrete percolation models. The bounds are intuitively driven by 'level shifts' of the field and are proven by comparing the number of components at different levels, which relies heavily on the results from Chapters 2 and 3.

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Chapter 1

Introduction

1.1 Basic setting

We begin with a quick description of the main objects of study for this thesis: excursion sets of Gaussian fields. For readers unfamiliar with this material, a more detailed account can be found in Appendix A.

For some $d \in \mathbb{N}$, let $f : \mathbb{R}^d \to \mathbb{R}$ be a Gaussian random field, that is, f is a random process indexed by $\mathbb{R}^d$ such that for any $m \in \mathbb{N}$ and any $x_1, \ldots, x_m \in \mathbb{R}^d$, $(f(x_1), \ldots, f(x_m))$ is a Gaussian vector. The finite dimensional distributions of the field are completely specified by its mean and covariance functions $\mu : \mathbb{R}^d \to \mathbb{R}$ and $\kappa : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined by

$$\mu(x) = \mathbb{E}(f(x)) \qquad \text{and} \qquad \kappa(x, y) = \mathrm{Cov}(f(x), f(y)).$$

The covariance function is positive definite and symmetric. We say that f is stationary if its distribution is invariant under translations of $\mathbb{R}^d$. Under this assumption, which we now make, the covariance of f may be expressed as

$$K(x - y) = \kappa(x, y)$$

for a positive definite function $K : \mathbb{R}^d \to \mathbb{R}$. We also now assume that f is normalised to have mean zero and variance one, so that $\mu(x) = 0$ and $K(0) = \mathbb{E}(f(x)^2) = 1$ for all $x \in \mathbb{R}^d$. This means that the distribution of f is entirely specified by K. If K is continuous (which will be true throughout our analysis), then Bochner's theorem states that it is the Fourier transform of some probability measure ν:

$$K(x) = \int_{\mathbb{R}^d} e^{it \cdot x} \, d\nu(t).$$

The measure ν is known as the spectral measure of the field and must be Hermitian; that is, $\nu(A) = \nu(-A)$ for all Borel sets A. If ν has a density ρ with respect to

Lebesgue measure, we refer to this as the spectral density. Since the spectral measure determines the covariance function, it also determines the distribution of f. We say that f is isotropic if its distribution is invariant under rotations. In this case both the covariance function and the spectral measure will also be invariant under rotations, and, in particular, the covariance function can be expressed as

$$K(x) = k\left(|x|^2\right)$$

for some function $k : [0, \infty) \to \mathbb{R}$ (where $|\cdot|$ denotes the Euclidean norm). Results for Gaussian fields are often stated in terms of the covariance function or equivalently in terms of the spectral measure.

For this thesis, we are interested in the geometry of the (upper-)excursion sets and level sets of f, defined, respectively, as

$$\left\{x \in \mathbb{R}^d \;\middle|\; f(x) \geq \ell\right\} \qquad \text{and} \qquad \left\{x \in \mathbb{R}^d \;\middle|\; f(x) = \ell\right\} \qquad \text{for } \ell \in \mathbb{R}.$$

To simplify notation, these sets will be denoted by $\{f \geq \ell\}$ and $\{f = \ell\}$ respectively. We will only consider upper-excursion sets, because by symmetry of the Gaussian distribution, $\{f \leq \ell\}$ has the same distribution as $\{f \geq -\ell\}$. Motivated by applications (to be described in the next section), we will consider the case where these sets, and the fields generating them, are somewhat regular. For example, if f is continuously differentiable; then for fixed ℓ the level set $\{f = \ell\}$ will consist of simple curves almost surely. By Kolmogorov's theorem (Theorem A.1.2), smoothness assumptions on the field can be justified by smoothness of the covariance function K. We will always assume that f is at least twice continuously differentiable, which will allow us to apply certain Morse theoretic arguments. This assumption also ensures that any reasonable topological events which we may be interested in are measurable.

Various geometric properties of Gaussian excursion/level sets are relevant for applications (and we will give examples of these momentarily), however our primary object of study in Chapters 2–4 is the number of connected components of an excursion set (or level set) contained in a large domain. We will also restrict to fields defined on the plane. Explicitly, for a given field f and any $R > 0$, $\ell \in \mathbb{R}$ we define

$$N_{ES}(f, R, \ell) = N_{ES}(R, \ell)$$

to be the number of connected components of $\{f \geq \ell\}$ which intersect $B(R)$ (the ball of radius R centred at the origin) but not $\mathbb{R}^2 \backslash B(R)$ and we define

$$N_{LS}(f, R, \ell) = N_{LS}(R, \ell)$$

in the corresponding way for $\{f = \ell\}$. We are then interested in the statistics of these quantities as R tends to infinity. In the remainder of this chapter, we hope to justify this apparently narrow focus, and convince the reader of the value of understanding the behaviour of these variables.

1.2 Motivation

1.2.1 Direct applications

Smooth Gaussian fields are used in many areas of science to model spatial phenomena, and as a result, studying their geometry has wide ranging implications. Here we will describe a few of the most relevant such applications.

Cosmological theories predict that the Cosmic Microwave Background Radiation observed on Earth can be well modelled as a realisation of a stationary, isotropic Gaussian field on the two-dimensional sphere [41]. This prediction broadly agrees with experimental evidence, however there are some discrepancies which have important implications for the underlying physical theories. Topological and geometric quantities provide a useful way of testing the Gaussianity prediction and can also shed light on particular discrepancies between the theory and observations. A classical way of doing this, is to compare the expected Euler characteristic of the excursion set of a random field (which is explicitly computable in the Gaussian case and in some non-Gaussian cases) with the observed normalised Euler characteristic of the background radiation at a range of different levels (see, for example, [38]). Some recent work [87] suggests that performing the same comparison for the number of excursion set components can give a better indication of discrepancies with the Gaussian theory. The mean of this quantity is unknown, so must be estimated by simulation. Analytical results for the large scale statistics of two dimensional Gaussian excursion sets would therefore likely improve this testing procedure. There would also be cosmological applications of more general results in this direction: the cosmic web can be modelled as a Gaussian field on $\mathbb{R}^3$ and its excursion sets can be analysed in similar ways [88].

Excursion sets of Gaussian fields are also important for medical imaging. As an example, we consider Positron Emission Tomography, which can be used to understand brain activity. The output of this procedure may be a set of voxels on a discrete approximation of the brain, which is a combination of signals, representing brain activity, and ambient noise. The medical community is naturally interested in knowing whether particular treatments result in a significant change in brain activity, and, if so, in which areas. A common statistical approach to this problem is to treat the data as a realisation of some Gaussian field on a three-dimensional manifold and use the maximum of the field as a test statistic [108]. Since the distribution of the maximum is unknown, it is replaced with the distribution of the Euler characteristic at high levels. This is justified by the 'Euler characteristic heuristic' which states that at high levels ℓ, any excursion set above ℓ is very likely to have a single, simply connected component which will have an Euler characteristic of one. Therefore the probability that the maximum of the field exceeds this level is approximately the expected Euler characteristic of the excursion set above this level (which is known). This method is considered superior to the naive approach of performing multiple tests on the discrete values of the voxels, as it takes advantage of the natural continuity of the problem and avoids difficulties associated with multiple testing for a large number of random variables. Similar approaches can be applied to more complicated types of data, such as that resulting from functional Magnetic Resonance Imaging (fMRI), Electroencephalography (EEG) or Magnetoencephalography (MEG) [52].

Yet another application of Gaussian fields can be found in quantum mechanics. Let Ω be a connected, compact subset of a Riemannian 2-manifold. The quantum billiards model is the set of solutions $\phi \in C^2(\Omega)$ to the Helmholtz equation

$$\Delta\phi + \lambda\phi = 0 \quad \text{on } \Omega$$

where Δ is the Laplacian operator, $\lambda \geq 0$ and boundary conditions are imposed if necessary. Equivalently, the solution is an eigenfunction of the Laplacian. This model describes the wavefunction of a particle travelling through Ω with specular reflections at the boundary and energy λ. The zero level sets, or nodal sets, of such functions have been well studied in the physics literature (see [47] for an overview). In part, this is because of evidence that properties of the nodal set (such as the number of connected components for different values of λ) can help distinguish between chaotic and integrable dynamics for the quantum billiard model [16]. There is also some interest in the excursion sets of these eigenfunctions at high levels, as these give the areas in which a quantum particle has the highest probability of being found [109, Chapter 1].

In a seminal paper [13], Berry conjectured that eigenfunctions of the Laplacian defined on a 'generic' Riemannian manifold (in any dimension greater than one) should exhibit universal behaviour. In particular, he argued that an eigenfunction with eigenvalue λ should be well modelled by a random, isotropic wave with wavenumber $\sqrt{\lambda}$, and that this wave should have a Gaussian distribution. This became known as

Berry's Random Wave Model, and is useful (if true) as it allows one to analyse eigenfunctions of the Laplacian using probabilistic tools. Physicists have found a variety of evidence in favour of Berry's conjecture, although some discrepancies remain (see [27] and references therein).

An important example of the Random Wave Model occurs on the plane. One can define a random isotropic superposition of m monochromatic plane waves as

$$f_m(x) := \sqrt{\frac{2}{m}} \sum_{j=1}^{m} \cos(\theta_j \cdot x + \phi_j)$$

where the θ_j are uniformly distributed on the unit circle, the ϕ_j are uniformly distributed on $[0, 2\pi]$ and all variables are independent. Taking the number of waves m to infinity, it can be shown that this converges in distribution to a mean zero Gaussian field with covariance

$$K(x) = \mathbb{E}(f(0)f(x)) = J_0(|x|)$$

where J_0 is the zero-th Bessel function. It can also be shown that, almost surely, realisations of this field are eigenfunctions of the Laplacian with eigenvalue -1 (this can be generalised to an arbitrary negative eigenvalue by a rescaling of $\mathbb{R}^2$). This field is known as the Random Plane Wave, and is an important object of study in the literature on excursion sets of Gaussian fields due to its role in Berry's conjecture.

1.2.2 Mathematical motivations

The study of level sets of Gaussian fields (particularly nodal sets) can also be motivated mathematically. In many circumstances, one may be interested in the behaviour of the zero set of some family of (deterministic) functions but find that this behaviour cannot be tightly controlled in all cases. One way of overcoming such an obstacle, is to consider a probability measure on the functions of interest and derive results for 'typical' behaviour according to this measure.

This approach has recently been taken to analysing Hilbert's 16th problem: understanding the number and arrangement of components of real algebraic hypersurfaces on projective space (see, for example, [96, 36, 64]). To be more specific, one can define a real algebraic hypersurface as the zero set in $\mathbb{R}P^n$ of a homogeneous polynomial of degree d in $n+1$ variables. There are, of course, many ways of defining a measure on such sets. One particular family of such measures which has been well studied is the Kostlan ensemble, which can be constructed as follows. We fix $n \in \mathbb{N}$ and let

$f_d : \mathbb{R}^{n+1} \to \mathbb{R}$ be the Gaussian field given by

$$f_d(x) = \sum_{|J|=d} \sqrt{\binom{d}{J}} a_J x^J$$

where $J = (j_0, \ldots, j_n)$ is a multi-index with $|J| = j_0 + \cdots + j_n$, $\binom{d}{J} = \frac{d!}{j_0! \cdot \cdots \cdot j_n!}$ and $\{a_J\}$ are independent standard normal random variables. Each realisation of f_d is homogeneous of degree d, and so one can equivalently consider the restriction of this field to the unit sphere $\mathbb{S}^n$. If the domain of $\{f_d\}$ is extended to $\mathbb{C}^{n+1}$ (using the same formula with complex Gaussian variables) then it is known as the Fubini-Study ensemble.

This measure on homogeneous polynomials can be motivated very naturally as follows. The set

$$\left\{ \sqrt{\binom{d}{J}} a_J x^J \right\}_{|J|=d}$$

can be taken to define an orthonormal basis for a Hilbert space of degree d homogeneous polynomials in $n+1$ variables, so that f_d is the canonical Gaussian measure on this Hilbert space. The inner product of this space is then equal (up to a constant factor $\sqrt{d!}$) to the restriction of the inner product of the Bargmann-Fock space (see [6]):

$$\langle f, g \rangle = \frac{1}{\pi^{n+1}} \int_{\mathbb{C}^{n+1}} f(z)\overline{g(z)} e^{-\|z\|^2} \, dz.$$

It can be shown that this is the unique inner product on homogeneous polynomials that is invariant under unitary transformations, so that the Kostlan ensemble is the unique unitary invariant Gaussian ensemble of homogeneous polynomials. These two properties (Gaussianity and unitary invariance) justify the interpretation of the Kostlan ensemble as modelling 'typical' homogeneous polynomials in a natural way (although there exist other Gaussian ensembles of homogeneous polynomials which are orthogonally invariant [55, 56], and these have also been studied [35]).

A particularly important case is the behaviour of polynomials with large degree, and interesting properties have been derived in this limit ([11]). The exhibition of nice behaviour in this regime can be intuitively understood by the fact that the Kostlan ensemble f_d has a local scaling limit as $d \to \infty$. Specifically, in a neighbourhood of any point of the sphere $\mathbb{S}^n$, if one maps to the plane using the inverse of the exponential map, then the (appropriately rescaled) covariance function converges to the Gaussian kernel. That is for $x_0 \in \mathbb{S}^n$ and $x, y \in \mathbb{R}^n$, as $d \to \infty$

$$\kappa_{x_0,d}(x,y) := \mathrm{Cov}\left(f_d\left(\exp_{x_0}\left(\frac{x}{\sqrt{d}}\right)\right), f_d\left(\exp_{x_0}\left(\frac{y}{\sqrt{d}}\right)\right)\right)) \to \exp\left(-\frac{\|x-y\|^2}{2}\right).$$

This has motivated interest in the centred Gaussian field $f : \mathbb{R}^n \to \mathbb{R}$ with covariance function $K(x) = \exp(-\|x\|^2/2)$ which is known as the Bargmann-Fock field. This scaling limit can be analysed as a field in its own right, and is useful for understanding the behaviour of the Kostlan ensemble for large degree d. It can also be constructed explicitly as a series; in the planar case this takes the form

$$f(x) = e^{-\frac{\|x\|^2}{2}} \sum_{i,j=0}^{\infty} a_{ij} \frac{1}{\sqrt{i!j!}} x_1^i x_2^j \qquad \text{for } x = (x_1, x_2) \in \mathbb{R}^2$$

where the a_{ij} are independent standard normal random variables. With probability one, the series converges locally uniformly, and so is analytic. In later sections we will see that this field has become an important object of study in the topic of level sets of Gaussian fields, particularly in the planar case.

1.2.3 Percolation and the Bogomolny-Schmit conjecture

Many of the interesting questions that one can ask about excursion/level sets of Gaussian fields involve long range connections; for instance, one may ask about the probability of an excursion set containing an unbounded component. This type of question is generally very difficult to answer, and in particular is not amenable to many of the standard tools of analysis (this will be explained more fully in the next subsection). However long range connection properties have been historically well studied in the simpler setting of percolation for discrete lattice models. As a result of this, many physicists used heuristics based on such models to help understand and make predictions for level sets of Gaussian fields. This connection has greatly influenced the subsequent research on geometry of Gaussian fields; both through the methods used and the type of question asked. More recently, the desire to make this connection rigorous (and so understand the universal properties of these classes of models) has actually served as independent motivation for studying the geometry of Gaussian fields. In this subsection, we give an overview of discrete percolation models and their heuristic connection with Gaussian fields.

Percolation models are used to understand long range connection properties of random or disordered media. The archetypal model is Bernoulli bond percolation on the square lattice, which is defined as follows: we form a graph with vertices at each point of $\mathbb{Z}^2$ and edges between all nearest neighbours, then declare each edge to be open independently with probability p, and closed otherwise. The resulting open sub-graph is a realisation of the model. Similar models have been used to represent the spread of forest fires, where the vertices represent trees in the forest and open edges represent fire spreading between them [66]. For this application, there is natural interest in understanding how far the fire may spread and how this depends on the local probability p of fire spreading from one tree to the next. This is typical of the kind of question asked more generally of percolation models. The most classic result for percolation models is the existence of a phase transition for long range connections: there exists a critical probability p_c such that for $p > p_c$ connections occur on infinite scales and for $p < p_c$ connections occur only on finite scales.

Theorem 1.2.1 (Harris-Kesten Theorem [44, 51])**.** *For Bernoulli bond percolation on $\mathbb{Z}^2$ with parameter p, if $p \leq 1/2$ then with probability one there is no unbounded open connected component. If $p > 1/2$ then with probability one there exists a unique unbounded open connected component.*

Clearly this model is somewhat idealised, compared with what might be used in direct applications. Indeed there are many ways of altering such a model, for example by changing the underlying graph or by adding some dependence between different edges. One could question the practical utility of proving results for a specific model if these proofs cannot be adapted to more general situations. As a result, universality phenomena are well studied within percolation. Intuitively, a result is universal if it holds for a wide range of different models. Many percolation models of the same dimension are believed to have universal properties based on non-rigorous physical arguments and numerics, although much less is known rigorously (see [39, Chapter 9] and references therein).

One way of justifying predictions of universality, in two dimensions, is by considering scaling limits. The Schramm-Loewner evolution with parameter κ (SLE(κ)) is a family of random planar curves defined using a Brownian motion (of diffusivity κ) and Loewner's differential equation (see [60] for more details). It is known that a variety of random models defined on two dimensional lattices converge to SLE(κ) (where κ depends on the model) when the plane is rescaled. In particular, this is true for the interface between open and closed clusters for certain percolation models [101], and is believed to hold for a much wider variety of models [102]. If two percolation models have SLE(κ) as a common scaling limit, then their large scale behaviour should be very similar and so the collection of models converging to SLE(κ), for each value of κ, can be thought of as a universality class.

Bearing this conjectured universality in mind, and the fact that percolation models were historically well studied, it was natural for physicists to try to understand the geometry of excursion sets of Gaussian fields by relating them to these simpler models.

To this end they developed a heuristic connection based on critical points of the field. Specifically, one can think of the local maxima of a Gaussian field above some level ℓ as defining a random lattice. If two local maxima are close to one another, then intuitively there should be a saddle point between them, which can be thought of as an edge of the lattice. If the saddle point is above height ℓ then the corresponding edge can be though of as open (i.e. the two nearby local maxima should be connected in $\{f \geq \ell\}$, see Figure 1.1). The first explicit description of this analogy that we can

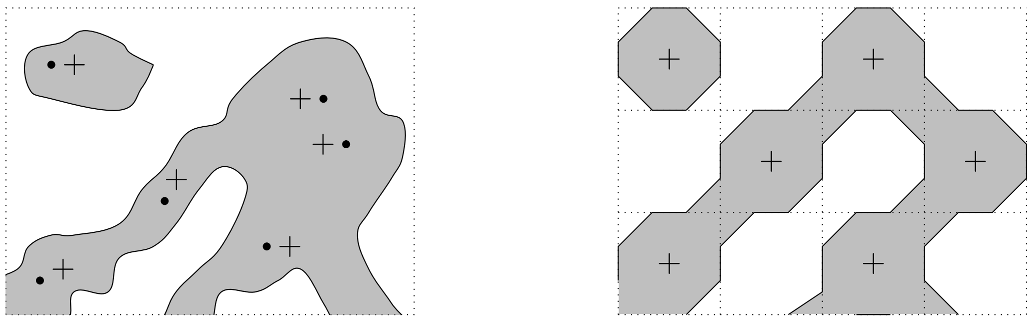


Figure 1.1: The heuristic correspondence between a Gaussian field excursion set and percolation (on a rotated copy of $\mathbb{Z}^2$, in this example). Local maxima (left) and vertices (right) are denoted by '+'.

find was given by Weinrib [106] in 1982 although some elements of this idea, due to Longuet-Higgins [65], date back to at least 1957.

The analogy between Gaussian fields and discrete models is not particularly helpful for rigorous arguments, as the joint distribution of the critical points of a Gaussian field is very complicated, however it has had a large influence on the analysis of Gaussian fields. In [18], Bogomolny and Schmit argued that the nodal sets of the Random Plane Wave can be well modelled by open components of critical Bernoulli bond percolation on the square lattice. Based on this, they then (non-rigorously) derived that the number of nodal set components in a large area should be asymptotically Gaussian with an explicit mean and variance. These claims became known as the Bogomolny-Schmit conjecture, and drew a lot of subsequent attention from physicists and mathematicians. The conjecture was thought of as potentially very interesting, if true, but was regarded with some scepticism because of two objections. The first, is that the correlations of the Random Plane Wave decay slowly with distance (points of the field at a distance R typically have correlations of order $R^{-1/2}$ when R is large) in contrast with the independence of different edges in Bernoulli percolation. Bogomolny and Schmit [19] used a non-rigorous heuristic, known as the Harris criterion, to argue that the oscillation of the covariance function results in cancellations, and so the Random Plane Wave should be in the same universality class as Bernoulli percolation. However this heuristic is considered to be somewhat unconvincing by many mathematicians. The second objection to the Bogomolny-Schmit conjecture is that the choice of the square lattice is not well justified. This is less important for large scale behaviour (as Bernoulli percolation on a general regular lattice should be in the same universality class as the same process on the square lattice) however it is important for the explicit constants given in the conjecture, which depend on the lattice.

Subsequently, the mean prediction of the Bogomolny-Schmit conjecture was tested numerically. Different authors confirmed that this prediction is inaccurate by about 5% (with a very low standard error) [79, 54, 8], therefore the precise claim of the conjecture seems to be incorrect. However it was also shown in [8], that the probability of a nodal set of Random Plane Wave joining opposite sides of a given rectangle (these are known as 'crossing events', and are defined formally later) is a good match for the predicted probability of the corresponding events for Bernoulli percolation. This is significant as these probabilities are expected to be universal, and so offer convincing evidence that the nodal set of the Random Plane Wave may be in the same universality class as critical Bernoulli percolation.

These tantalising ideas have motivated a great deal of work (which we describe in Section 1.4) trying to rigorously understand the percolation properties of Gaussian fields, however many open questions remain.

1.3 Methods and tools

In this section we outline some of the most important techniques used for analysing the geometry of smooth Gaussian fields. These techniques can be broadly split into two classes; analytic methods and percolation-type arguments, based on whether the geometric quantity to be studied is 'local' or not.

In the context of excursion sets of Gaussian fields, a geometric quantity depending on a field f is said to be local if its value on a domain can be determined by partitioning the domain and considering the values of f on each element of the partition separately. (We are solely interested in this property for heuristic purposes, and so we do not attempt to define it formally.) So, for example, the volume of an excursion set is local, as it is simply the sum of the volume of the excursion sets in each element of a partition. However the number of connected components of an excursion set is non-local, because when looking at elements of a partition *separately*, one cannot tell which components in subdomains join together (see Figure 1.2). Equivalently, one can think of local quantities as those which are invariant under rearranging a partition of an excursion set (see Figure 1.3); this captures the notion of evaluating the functional on elements of the partition separately, although we will not attempt to be precise in defining which types of rearrangement are valid.

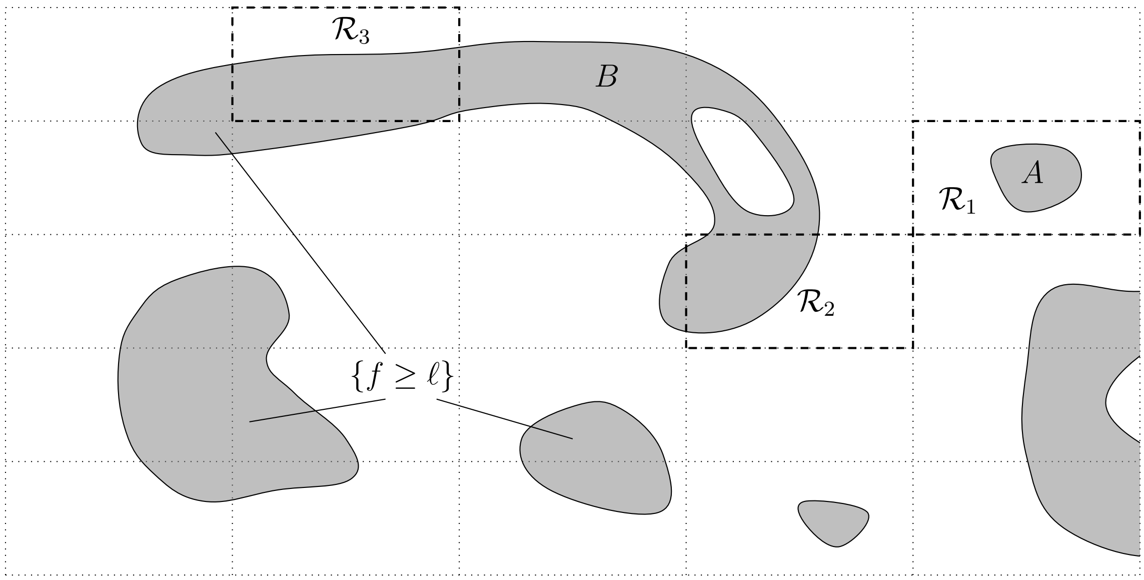


Figure 1.2: Consider counting the number of excursion set components using this partition of rectangles. The component A is contained in $\mathcal{R}_1$, so, it is clear from looking only at $\mathcal{R}_1$ that this increases the total number of components by one. However B intersects multiple elements of the partition, so, for example, when looking at $\mathcal{R}_2$ or $\mathcal{R}_3$ individually, it is unclear whether these sets are connected and hence how they contribute to the overall number of components.

An important class of local geometric functionals are the Lipschitz-Killing curvatures (also known as Minkowski functionals or intrinsic volumes). In general, for a 'nice' d-dimensional set M (such as a Whitney stratified manifold) there are $d+1$ Lipschitz-Killing curvatures $\mathcal{L}_0(M), \dots, \mathcal{L}_d(M)$ with the 0-th, $(d-1)$-th and d-th curvatures corresponding to the Euler characteristic, half the surface area and the volume of M respectively. The remaining Lipschitz-Killing curvatures do not generally have simple interpretations, but $\mathcal{L}_k(M)$ can be thought of as a k-dimensional measure. These objects are fundamental, as they form a basis for all finitely additive, monotone functionals on basic complexes which are invariant under rigid motions (this is known as Hadwiger's characterisation theorem, see [53, Chapter 9]). For the excursion sets of planar Gaussian fields (on some compact domain), there are three Lipschitz-Killing curvatures, which roughly correspond to the area, half the perimeter

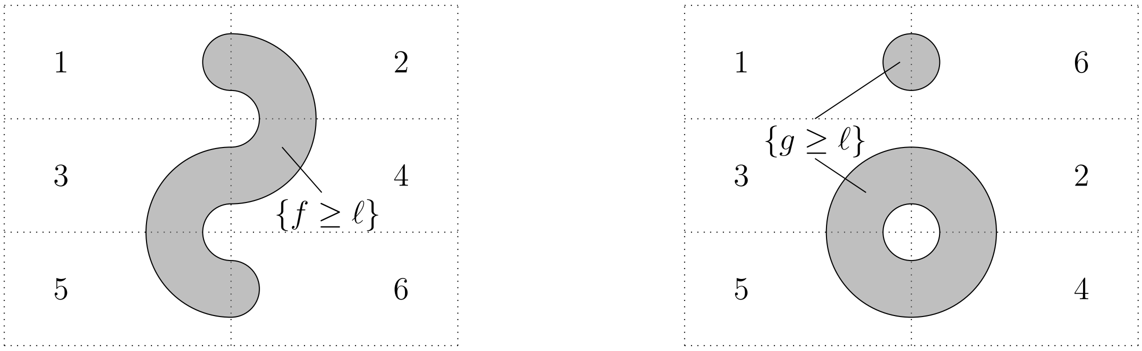


Figure 1.3: The excursion sets of the two functions f and g can be obtained from one another by a rearrangement of elements of this partition (indicated by the numbering of the rectangles), hence any local functional must be identical on these two sets. However the number of connected components of the two excursion sets differs, and so this cannot be a local quantity.

and the Euler characteristic of the set (there is some correction for boundary effects). It may not be obvious from the topological definition of the Euler characteristic that this is a local functional, however this follows from the 'inclusion-exclusion' property of the functional. Alternatively, the Poincaré-Hopf theorem decomposes the Euler characteristic into an alternating sum of critical points, which are local.

Typical examples of non-local quantities include the number of unbounded components of an excursion set or the event that an excursion set component intersects opposite sides of a rectangle. These quantities usually involve macro- or meso-scopic connections between excursion sets. The most important example for our purposes is the number of excursion/level set components in a large, bounded domain.

From the above rough definition, it should be clear, in principle, why locality is such a useful property: it allows one to take arbitrarily fine partitions, and so apply integral/analytic methods. Non-local quantities are usually much more challenging to study, and often the only tools available are those from percolation theory, which are based on controlling short range interactions of fields/lattice models. In Section 1.4, we show that the results for local geometric functionals of Gaussian fields are generally much stronger than those for non-local functionals.

1.3.1 Analytic methods

1.3.1.1 The Kac-Rice formula

The Kac-Rice formula gives an exact expression for moments of level set volumes for random fields in terms of their covariance structure. The simplest version of this formula, which gives the expected number of zeros of a one dimensional random field, can help in developing an intuitive understanding of its more general form. Consider a differentiable function $f : [a,b] \to \mathbb{R}$ and let $x_0 \in [a,b]$ be an isolated zero of f which is non-degenerate (that is, $f'(x_0) \neq 0$). Let U be some small neighbourhood of x_0, then by a Taylor approximation, the graph of f is well approximated on $U \times \mathbb{R}$ by a line through $(x_0, 0)$ with gradient $f'(x_0)$. We would therefore expect that

$$\int_U |f'(x)| \, \frac{1}{2\epsilon} \mathbb{1}_{|f(x)| \leq \epsilon} \, dx \approx 1.$$

Hence, expanding the domain of this integral to $[a,b]$, gives an approximate count for the number of zeroes of f (assuming non-degeneracy of the zeroes). By taking ϵ to zero, this count becomes more accurate and results in Kac's counting formula.

Proposition 1.3.1 (Kac's counting formula [83])**.** *Let $f : [a,b] \to \mathbb{R}$ be continuously differentiable such that all zeroes of f are non-degenerate and $f(a), f(b) \neq 0$, then the number of zeroes of f is given by*

$$\lim_{\epsilon \to 0} \int_a^b |f'(x)| \, \frac{1}{2\epsilon} \mathbb{1}_{|f(x)| \leq \epsilon} \, dx.$$

Applying this formula to a random function $f : [a,b] \to \mathbb{R}$ (assuming some regularity for the following manipulations) we obtain that the expected number of zeroes of the field is equal to

$$\begin{aligned} \mathbb{E}\left(\lim_{\epsilon \to 0} \int_a^b |f'(x)| \, \frac{1}{2\epsilon} \mathbb{1}_{|f(x)| \leq \epsilon} \, dx \right) &= \int_a^b \lim_{\epsilon \to 0} \mathbb{E}\left(|f'(x)| \, \frac{1}{2\epsilon} \mathbb{1}_{|f(x)| \leq \epsilon} \right) dx \\ &= \int_a^b \mathbb{E}\left(|f'(x)| \mid f(x) = 0 \right) p_{f(x)}(0) \, dx \end{aligned}$$

where $p_{f(x)}$ denotes the density of $f(x)$. These arguments can be made rigorous under assumptions on the distribution of the function f (see [83] which also contains a nice introduction to the one-dimensional Kac-Rice formula). In this thesis we will repeatedly use the following version of the Kac-Rice formula which holds in higher dimensions.

Theorem 1.3.2 (Kac-Rice formula [1, Corollary 11.2.2])**.** *Let $T \subset \mathbb{R}^d$ be compact such that ∂T has finite Hausdorff $(d-1)$-measure. Let $A \subset \mathbb{R}^k$ be open such that ∂A has Hausdorff dimension $k-1$. Let $f : T \to \mathbb{R}^d$ and $g : T \to \mathbb{R}^k$ be centred Gaussian fields such that f is almost surely continuously differentiable and for each $t \in T$ the vector $(f(t), \nabla f(t), g(t))$ is a non-degenerate Gaussian vector. Assume also that the covariance functions of each element of ∇f and g are uniformly Hölder continuous.*

Then for $u \in \mathbb{R}^d$, letting N_u denote the number of points $t \in T$ such that $f(t) = u$ and $g(t) \in A$, one has

$$\mathbb{E}\left(N_u\right) = \int_T \mathbb{E}\left(\left|\det \nabla f(t)\right| \mathbb{1}_{g(t)\in A} \,\middle|\, f(t) = u\right) p_{f(t)}(u)\, du$$

where $p_{f(t)}(u)$ denotes the probability density of $f(t)$ at the point u.

This formula can be generalised in many ways: to volumes of higher dimensional sets (i.e. the level sets of a function $f : \mathbb{R}^d \to \mathbb{R}^k$ where $k < d$), to higher order moments, to non-Gaussian fields or to fields defined on manifolds (see [5, Chapter 6] for details).

1.3.1.2 The Wiener Chaos expansion

The Wiener Chaos expansion is an abstract orthogonal decomposition of Gaussian Hilbert spaces which has recently been used to obtain strong characterisations of local geometric variables for Gaussian fields. The basic decomposition holds in great generality: let $\mathcal{H}$ be a closed, linear space of centred, jointly Gaussian random variables defined on the common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(H_n)_{n\in\mathbb{N}}$ be the Hermite polynomials on $\mathbb{R}$ defined by $H_0 \equiv 1$ and $H_k(t) = tH_{k-1}(t) - H'_{k-1}(t)$ for $k \geq 1$, so that in particular $H_0(t) = 1$, $H_1(t) = t$ and $H_2(t) = t^2 - 1$. For $q \in \{0, 1, 2, \dots\}$, the q-th Wiener Chaos associated with $\mathcal{H}$, denoted C_q, is defined as the closed linear span (in $L^2(\mathbb{P})$) of all random variables of the form $H_q(\xi)$ where $\xi \in \mathcal{H}$ with $\mathrm{Var}(\xi) = 1$. It can be shown that the chaoses are orthogonal in $L^2(\mathbb{P})$ and that

$$L^2(\Omega, \sigma(\mathcal{H}), \mathbb{P}) = \bigoplus_{q=0}^{\infty} C_q.$$

That is, any square integrable functional F which is measurable with respect to $\sigma(\mathcal{H})$ has a unique L^2-series expansion

$$F = \sum_{q=0}^{\infty} F[q]$$

where $F[q]$ is the projection of F onto the q-th chaos C_q. Note that $F[0] = \mathbb{E}(F)$. Further details of this decomposition can be found in [85, Chapter 2] or, with a slightly different approach, in [48, Chapter 2].

This decomposition has been successfully applied to the Lipschitz-Killing curvatures of Gaussian excursion sets (see the detailed references in Section 1.4.1). This analysis relies on expressing the curvatures as functionals of finite Gaussian vectors.

By arguments similar to those used to derive the Kac-Rice formula, one has the formal equality

$$L_{\ell,R} := \text{Length}\left(\{f = \ell\} \cap [-R,R]^2\right) = \int_{[-R,R]^2} \|\nabla f(x)\| \; \delta(f(x) - \ell) \; dx$$

where $\delta(\cdot)$ denotes the Dirac delta function. (This equality can be made rigorous by a limiting argument using the co-area formula.) The chaos expansion of this variable can be computed by taking the expansion of the integrand for each value of x and integrating the result:

$$L_{\ell,R} = \sum_{q=0}^{\infty} L_{\ell,R}[q].$$

One can then obtain asymptotic results for the length of the level set by analysing the behaviour of the chaos projections individually as $R \to \infty$. Analogous expansions for other Lipschitz-Killing curvatures may be studied similarly.

The behaviour of the chaos projections depends heavily on the covariance structure of the underlying field. For fields with fast correlation decay, each chaos projection typically has the same order of variance as $R \to \infty$, and central limit theorems can be obtained for the Lipschitz-Killing curvatures [58, 57]. For certain fields (including some models of eigenfunctions of the Laplacian) the chaos expansion is dominated by the first few projections. That is, the variance of one of the first few terms is of higher order than the variance of all remaining terms, which can result in non-central limit theorems [93].

The key to these arguments, is the local expression for the length, which simplifies the problem to considering the chaos expansion of $\|\nabla f(x)\|\delta(f(x))$. Since this is an L^2 function of the Gaussian vector $(\nabla f(x), f(x))$, its expansion is tractable. In contrast, a non-local quantity such as the number of excursion set components also has a Wiener chaos expansion, however there is no feasible way of evaluating it.

By the Poincaré-Hopf theorem, the Euler characteristic of an excursion set has a local expression as the alternating sum of critical points of different indices. This allows its chaos expansion to be analysed in a similar manner to the level set length above (in any dimension). By Hadwiger's formula ([1, Chapter 13]), the other Lipschitz-Killing curvatures of excursion sets can be expressed as an integral over the Euler characteristic of lower dimensional slices of the excursion set. Therefore the chaos expansion of the other curvatures can be analysed using that of the Euler characteristic (see [57]).

1.3.2 Percolation methods

Many methods from percolation have recently been adapted to the setting of continuous Gaussian fields. We outline some of these methods here and give corresponding definitions in the continuous setting. In Section 1.4.2, we discuss the known percolation results for Gaussian fields.

A common strategy in percolation arguments, is to construct infinite scale events (which one is interested in) from finite scale ones (which are easier to analyse). Two quintessential examples of such finite events are 'box-crossings' and 'arm events'. For concreteness, consider a realisation of Bernoulli percolation on $\mathbb{Z}^2$ with parameter p (as defined in Section 1.2.3). A rectangle $[a,b] \times [c,d] \subset \mathbb{R}^2$ is said to have a left-right crossing if there exists a path of open edges inside the rectangle which joins $\{a\} \times [c,d]$ to $\{b\} \times [c,d]$ (see Figure 1.4a) and we denote this event as $\mathcal{C}^{\mathrm{Perc}}_{[a,b]\times[c,d]}$. This definition applies to other two dimensional percolation models and can also be

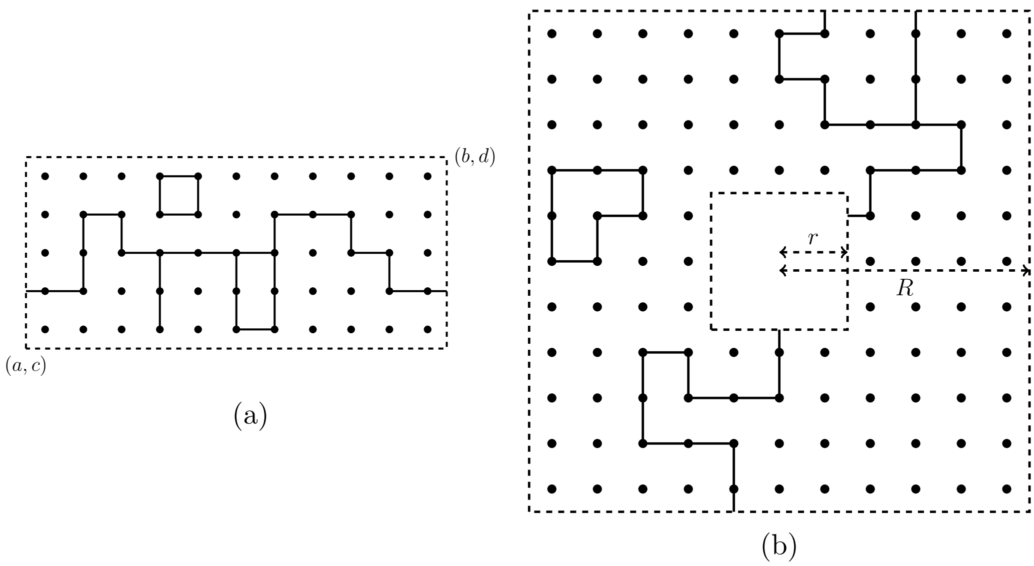


Figure 1.4: A left-right crossing of $[a,b] \times [c,d]$ and a four-arm event on $A_{r,R}$. (Solid lines represent open edges.)

generalised to other shapes. For $k \in \mathbb{N}$, a k-arm event is said to occur for an annulus $A_{r,R} := [-R,R]^2/[-r,r]^2$ if there exist k different paths in $A_{r,R}$ joining $\partial([-r,r]^2)$ to $\partial([-R,R]^2)$ which are alternatively open and closed when considered in a clockwise order (see Figure 1.4b). The most important of these events are one-arm and four-arm.

These events have clear analogues for Gaussian fields (recalling that the excursion set above a level ℓ can be thought of as corresponding to the open subgraph of a percolation realisation with parameter p). A rectangle $[a,b]\times[c,d]$ is said to have a left-right crossing in $\{f\geq\ell\}$ if there exists a component of $\{f\geq\ell\}\cap([a,b]\times[c,d])$ which intersects $\{a\}\times[c,d]$ and $\{b\}\times[c,d]$ (see Figure 1.5). A similar definition can be made for arm events. These will be discussed more rigorously in Chapters 2 and 3.

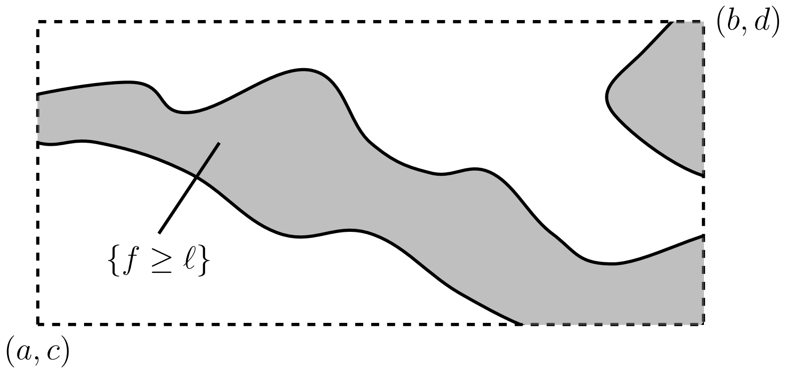


Figure 1.5: A left-right crossing of $[a,b]\times[c,d]$ in $\{f\geq\ell\}$.

In order to analyse infinite events using such definitions, the typical method is to bound the probability of these finite events and then find a way of analysing the probability of joint events. For Bernoulli bond percolation, the following bounds on probabilities are classic.

Theorem 1.3.3 (Sharp phase transition [51, 95, 99])**.** *Consider Bernoulli percolation on $\mathbb{Z}^2$ with parameter p and let $c>0$ be given. If $p<1/2$ then there exists $c_1,c_2>0$ depending on p such that for all $R>0$*

$$\mathbb{P}\left(\mathcal{C}^{\mathrm{Perc}}_{[0,R]\times[0,cR]}\right)\leq c_1e^{-c_2R}.$$

If $p=1/2$ then there exists $c_1\in(0,1)$ such that for all $R>0$

$$c_1<\mathbb{P}\left(\mathcal{C}^{\mathrm{Perc}}_{[0,R]\times[0,cR]}\right)<1-c_1. \qquad \text{(RSW)}$$

If $p>1/2$ then there exists $c_1,c_2>0$ depending on p such that for all $R>0$

$$\mathbb{P}\left(\mathcal{C}^{\mathrm{Perc}}_{[0,R]\times[0,cR]}\right)\geq 1-c_1e^{-c_2R}.$$

The second result is known as an RSW estimate after Russo, Seymour and Welsh [95, 99]. The exponential bounds are due to Kesten [51] and in fact they are equivalent to one another by self-duality of the square lattice (see [39, Chapter 11]). These results are referred to as a 'sharp' phase transition, as they imply the phase transition of the model (Theorem 1.2.1) but also give stronger quantitative bounds. The regimes $p < 1/2$, $p = 1/2$ and $p > 1/2$ are referred to as sub-critical, critical and super-critical respectively. The probabilities of arm events can often be bounded exponentially in sub-critical or super-critical regimes (using a slight generalisation of the results for box crossing). At criticality, the probabilities are more difficult to analyse, but results for some two dimensional models are known [61] (and conjectured to be universal).

The next step in analysing infinite events is to consider the probability of a combination of finite events. There are two common ways of doing this. The first is to simply use a union bound. For example, consider an infinite sequence of rectangles with dimensions 2^{n+1} by 2^n for $n > n_0$ which are alternatively horizontally or vertically oriented and have a corner at the origin (see Figure 1.6). If each of these rectangles has an open crossing in the longer direction, then there exists an infinite open cluster. By the exponential bounds above and the union bound, one can show that this happens with positive probability for all $p > 1/2$ (after choosing n_0 appropriately).

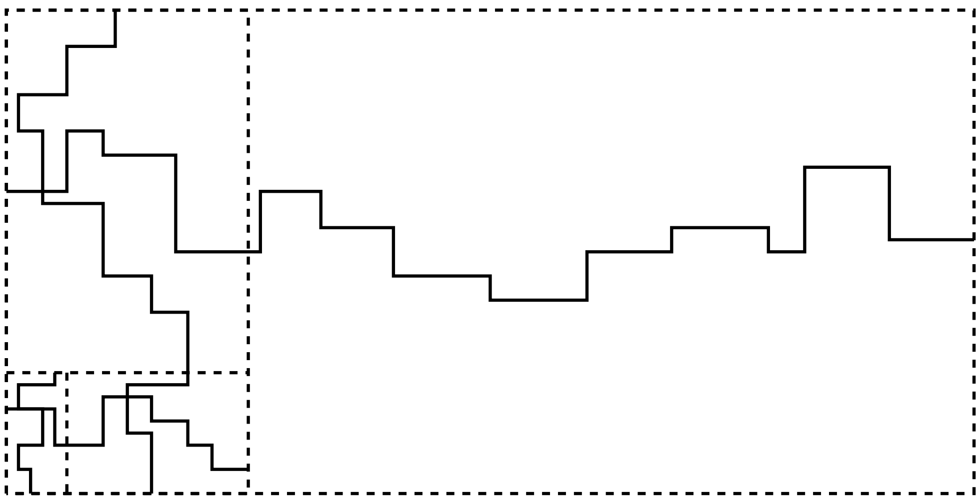


Figure 1.6: If each dashed rectangle in this infinite sequence has an open crossing along its longer side, then there exists an unbounded open cluster.

The second tool for controlling the probability of combined events is known as the Fortuin-Kasteleyn-Ginibre (FKG) inequality. An event A depending on a realisation of Bernoulli percolation is said to be increasing if the following holds: let ω_1 be a realisation of the model such that A occurs and ω_2 be another realisation such that all open edges of ω_1 are also open in ω_2, then ω_2 is also in A. Roughly speaking this says that an event is increasing if opening more edges does not stop the event from occurring. Box crossing events are a typical example of increasing events.

Theorem 1.3.4 (FKG inequality [34])**.** *If A and B are increasing events for Bernoulli bond percolation with parameter p, then*

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B).$$

This property is also known as positive association, and is shared by many other percolation models, although there are important examples for which it fails (such as the random cluster model with certain parameter values, see [40, Chapter 3]). By taking an intersection of multiple crossing events, one can create interesting and complicated path events. Then, using the FKG inequality and RSW estimates, it can be shown that the probability of such events on any scale is bounded away from zero (and one) at criticality.

Many of the arguments used to prove the results in this section rely on the discrete nature of bond percolation and so are not directly applicable to continuous Gaussian fields. One approach to adapting these arguments is to consider discretisations of the Gaussian field. Specifically, one considers a discrete lattice with a fine mesh such as $\delta\mathbb{Z}^2$ for $\delta > 0$ small, and declares each point x of the lattice to be open if $f(x) \geq \ell$ and closed otherwise. This defines a dependent site percolation model ([39, Chapter 1] gives an explanation of site percolation) to which standard percolation arguments can be applied. One must then argue that the topology of the discretisation of $\{f \geq \ell\}$ matches that of its continuous analogue with high probability as $\delta \to 0$. Two properties of the Gaussian field are typically required for this argument: non-negative correlations (that is, $K \geq 0$) which ensures that the FKG inequality holds for the discretised model and fast correlation decay ($K(x) \to 0$ at some rate as $|x| \to \infty$) which takes the place of independence when analysing the discretised model instead of Bernoulli percolation.

1.4 Literature review

In this section we survey some current results on the geometry of smooth Gaussian excursion sets, with a focus on those which are more relevant to later chapters of this thesis.

1.4.1 Local geometric functionals

The most studied local geometric functionals of smooth Gaussian fields are the Lipschitz-Killing curvatures and the number of critical points. The statistics of these quantities are generally analysed for one of two regimes; fields with fast decay of correlations or Gaussian Laplace eigenfunctions.

In the former case, the behaviour of these local functionals can be characterised quite generally: for fields with integrable covariance functions and a (continuous) spectral density which is positive at the origin, the Lipschitz-Killing curvatures of an excursion set in a domain have mean and variance proportional to the volume of the domain, and obey a central limit theorem with Gaussian limit [57, 78]. In particular this holds in any dimension and for excursion sets above any level. The same statement is true for the total number of critical points in a domain [84]. These results have all been proven using the Wiener Chaos expansion with the approach outlined in Section 1.3.1.2.

Local geometric functionals for Gaussian Laplace eigenfunctions have been shown to exhibit somewhat more complicated, and interesting, behaviour; necessitating separate study for different fields, functionals and levels. We focus here on Laplace eigenfunctions on the two-dimensional sphere, which is the most well studied model, to give a flavour of the known results. For this manifold, the spectrum of the Laplacian is discrete and degenerate (has repeated eigenvalues). Specifically, the eigenvalues of the Laplacian on the 2-dimensional sphere are of the form $n(n+1)$ for $n \in \mathbb{N}$ with multiplicity $(2n+1)$. The n-th Random Spherical Harmonic is then defined as

$$f_n(x) = \sqrt{\frac{4\pi}{2n+1}} \sum_{i=-n}^{n} a_{n,i} Y_{n,i}(x) \quad x \in \mathbb{S}^2$$

where $Y_{n,-n}, \dots, Y_{n,n}$ is an $L^2(\mathbb{S}^2)$-orthonormal basis of the eigenspace with eigenvalue $n(n+1)$ and the $a_{n,i}$ are standard complex Gaussian variables which are independent except for the relation $\overline{a}_{n,i} = (-1)^n a_{n,-i}$ (which ensures that the field takes real values). This is a natural way of constructing a random eigenfunction of the Laplacian on the sphere (see [80] for more details). One is typically interested in the behaviour of this model as $n \to \infty$, which is called the high energy limit. It can be shown that, after rescaling, the covariance function of this field converges locally to the covariance function of the Random Plane Wave (with some multiplicative correction factor), so that the Random Plane Wave can be thought of as the scaling limit of Random Spherical Harmonics [107]. In particular, we will heuristically think of f_n for large n as being comparable to the Random Plane Wave restricted to a domain with area of order n^2 (since then the domain size is of order n^2 relative to the wavelength for both models).

Functional	Order of mean	Order of variance	
$\mathcal{L}_0(\{f_n \geq \ell\})$ (Euler characteristic)	n^2	n^3	for $\ell \neq 0, \pm 1$
		$O(n^2 \log^2 n)$	for $\ell = 0, \pm 1$
$\mathcal{L}_1(\{f_n \geq \ell\})$ (Boundary length)	n^2	n	for $\ell \neq 0$
		$\log n$	for $\ell = 0$
$\mathcal{L}_2(\{f_n \geq \ell\})$ (Area)	n^2	$1/n$	for $\ell \neq 0$
		$1/n^2$	for $\ell = 0$
$N_{\text{crit}}(\{f_n \geq \ell\})$ (Number of critical points)	n^2	n^3	for $\ell \neq 0$
		$n^2 \log n$	for $\ell = 0$

Table 1.1: The asymptotic behaviour of local functionals for excursion sets of the n-th Random Spherical Harmonic as $n \to \infty$.

Since the excursion sets of a Random Spherical Harmonic are two-dimensional, three Lipschitz-Klling Curvatures can be defined on them: the area, boundary length and Euler characteristic of $\{f_n \geq \ell\}$ for $\ell \in \mathbb{R}$. Separate central limit theorems have been proven for each of these functionals (at generic fixed levels ℓ) as $n \to \infty$ using the Wiener Chaos expansion (see [69, 70], [92, 68] and [21] respectively). A central limit theorem has also been proven for the number of critical points of Random Spherical Harmonics above a fixed level [22]. In contrast with the case of fields exhibiting fast correlation decay, the statistics of these functionals are quite varied. Specifically, whilst the mean of each functional is of order n^2 as $n \to \infty$ for any level ℓ, the order of the variances for generic levels depends on the functional. Moreover, for each functional, there is a small number of levels at which the order of the variance is lower than for generic levels (see Table 1.1). This phenomenon was first derived non-rigorously by Berry for the length of the nodal set of the Random Plane Wave [14] and is known as 'Berry cancellation'.

These results are proven using the Wiener Chaos expansion. For generic levels, it can be shown that each functional considered above is dominated by the projection onto its second chaos (i.e. the variance of the sum of all higher order projections has lower order variance and so is negligible in the limit). Furthermore, for each functional φ, the projection onto the second chaos takes the form

$$\varphi(\{f_n \geq \ell\})[2] = g_\varphi(\ell, n) \int_{\mathbb{S}^2} f_n^2(x) - 1 \, dx = g_\varphi(\ell, n) \left(\|f_n\|^2_{L^2(\mathbb{S}^2)} - \mathbb{E}\left(\|f_n\|^2_{L^2(\mathbb{S}^2)} \right) \right)$$

for some explicit, deterministic function g_φ which depends on the functional (in the case of the Euler characteristic there is also an error term which is $O_{\mathbb{P}}(n)$). The reduced order of variance at anomalous levels is then a consequence of this form: at $\ell = 0$ (and $\ell = \pm 1$ for the Euler characteristic) g_φ is identically zero and so the functional is dominated by higher order chaoses. Another consequence of this analysis is that at all generic levels, all of these local functionals are perfectly correlated asymptotically, since they are all proportional to the same random quantity $\int_{\mathbb{S}^2} f_n^2(x) - 1\, dx$ which is known as the sample bispectrum. In particular this can be interpreted as saying that for generic levels, the asymptotic fluctuations in geometric quantities are driven by fluctuations in the L^2 norm of the field. All of these points are discussed further in [68, 22, 23].

1.4.2 Percolation results

There have been a number of papers analysing the long range connectivity properties of excursion sets of Gaussian fields. Most results have been obtained for the case of two-dimensional fields with fast decay of correlation.

The first significant analysis was performed in a series of papers from 1983-1986 [73, 74, 75] challenging some of the non-rigorous heuristics used by physicists. These works analysed stationary random fields, with general distributions, defined on $\mathbb{Z}^d$ and $\mathbb{R}^d$ for arbitrary dimensions d. They derived conditions which ensure that the field is 'dominated' at sufficiently high or low levels by Bernoulli percolation with sufficiently large or small p respectively. For a field f satisfying these conditions, there exists $\ell_1 < \ell_2$ such that with probability one, $\{f \geq \ell_1\}$ contains a unique unbounded component and $\{f \geq \ell_2\}$ contains only bounded components. In particular, this holds if f is a C^2 Gaussian field with covariance function satisfying some integrability conditions.

A much stronger characterisation of the percolation phase transition was obtained in [4] for a very specific class of smooth, stationary two-dimensional fields (which differ greatly from the Gaussian fields considered in this thesis). The arguments in this work exploited the natural lattice-like structure of these models along with standard discrete percolation techniques.

A significant breakthrough then occurred in 1996: in [3] it was shown that for two dimensional, ergodic fields f satisfying the FKG inequality, all components of $\{f = \ell\}$ are bounded, simultaneously for all $\ell \in \mathbb{R}$, almost surely. In particular, this holds for Gaussian fields with a non-negative covariance function which decays to zero at infinity (these conditions imply the FKG inequality and ergodicity respectively). A corollary of of this result for Gaussian fields, is that $\{f \geq 0\}$ almost surely has no unbounded component, so this paper actually proves the first half of the phase transition for Gaussian fields with positive, decaying correlations. The results of this work are proven using quite soft topological arguments adapted from the discrete percolation literature.

Interest in percolation properties of Gaussian fields within the mathematics community has resurfaced more recently. In part, this is due to the work of Nazarov and Sodin in [80], making rigorous some of the claims of the Bogomolny-Schmit conjecture (this is discussed in the next subsection). In 2016, Beffara and Gayet [7] extended the results of [3] in a quantitative manner: they proved RSW estimates (as described in Section 1.3.2) for fields with non-negative covariance functions which decay sufficiently quickly (specifically, such that $K(x) \leq C|x|^{-\alpha}$ for some $C > 0$ and $\alpha \geq 325$). As a consequence, they obtained polynomially decaying bounds on both one-arm events and the size of the excursion set component containing the origin. Their result is based on a discretisation argument, showing that, with high probability, the topology of the zero set of the field is determined by the signs of the field on a lattice with fine mesh. Later work [9, 90] improved the discretisation scheme used in this paper, which extended the result to fields satisfying the above conditions for $\alpha > 16$ and finally for $\alpha > 4$.

The natural next step in analysing the percolation properties of Gaussian excursion sets is the second half of the phase transition: proving that for all $\ell < 0$ there exists a unique unbounded component of $\{f \geq \ell\}$ almost surely. This was proven for the Bargmann-Fock field by Rivera and Vanneuville [91]. In particular, they proved exponential bounds on box crossing probabilities analogous to those in Section 1.3.2, thus establishing a sharp phase transition. This analysis is also based on discretisation arguments and methods from discrete percolation adapted to correlated models. It also used some specific properties of the Bargmann-Fock field, so the arguments would be difficult to generalise to other models.

The existence of a sharp phase transition was later proven for fields with positive correlations such that $K(x) \leq C|x|^{-\alpha}$ for some $C > 0$ and $\alpha > 2$ under some technical assumptions on the spectral measure of the field [77]. This proof was based on a white noise representation, which provides a way of approximating a given Gaussian field in a compact domain by continuous fields which depend on only a finite number of independent Gaussian variables. More recently, some of the stronger technical assumptions required for this result have been removed [89].

Evidently, there has been impressive progress in recent years in understanding the percolation properties of smooth Gaussian fields. The most significant implication of this line of enquiry so far, is that, under general conditions, planar fields with positive, quickly decaying correlations exhibit a sharp phase transition closely analogous to that of discrete Bernoulli percolation. However many open questions remain; in particular it is unclear how generally the percolation phase transition should be expected to hold for Gaussian fields. Once can easily construct trivial examples of two-dimensional fields for which the phase transition at level zero fails (for example fields which are constant in one direction or exhibit periodic behaviour), however there are currently no known counterexamples amongst smooth, planar Gaussian fields satisfying 'natural' conditions (i.e. something like ergodicity or correlations decaying to zero). In [4] some smooth non-Gaussian counterexamples were constructed by imposing a lattice structure analogous to certain discrete models. We also note that very little is known about the percolation properties of smooth Gaussian fields in higher dimensions.

1.4.3 Number of excursion set components

The number of connected components of excursion/level sets is an important quantity in the geometry of Gaussian fields, as outlined in Section 1.2, and is the main subject of this thesis. In this subsection, we highlight some of the results which form the foundation of the original work in Chapters 2–4.

The first rigorous analysis of this quantity, dates back to 1962 [104]. In this work, Swerling considered $N_{ES}(f,R,\ell)$, the number of components of $\{f\geq\ell\}$ contained in $B(R)$ where f is a smooth, stationary, two-dimensional Gaussian field. He showed that this number was bounded above and below by certain local functionals, the means of which are explicitly computable using the Kac-Rice formula, and hence obtained the bounds

$$c(\ell)R^2\leq\mathbb{E}(N_{ES}(f,R,\ell))\leq C(\ell)R^2\qquad\text{for }R>R_0$$

for two explicit, non-negative functions $c(\cdot),C(\cdot)$. We discuss these bounds more fully in Chapter 2 (see Proposition 2.1.10). One weakness of this result, is that $c(0)=0$ (although $c(\ell)>0$ for $\ell\neq0$). This shortcoming was addressed by Malevich [67] who showed that, for fields with a non-negative covariance function which decays polynomially with distance, the same bound holds for some (non-explicit) $c(0)>0$.

Arguably the most significant breakthrough in this topic occurred in 2009, when Nazarov and Sodin [80] proved that the rescaled number of nodal sets of Random Spherical Harmonics converges exponentially around its limiting mean. Specifically, they showed that for any $\epsilon > 0$ there exist $C, c > 0$ such that

$$\mathbb{P}\left(\left|\frac{N_{LS}(f_n, 0)}{n^2} - a\right| \geq \epsilon\right) \leq Ce^{-Cn}$$

where $N_{LS}(f_n, 0)$ denotes the number of nodal components of the n–th Random Spherical Harmonic f_n and $a > 0$ is an absolute constant. Their proof is based on the Gaussian isoperimetric inequality and uses some deterministic properties of nodal sets of Laplace eigenfunctions, so it is unclear whether this result can be extended to other models or levels. We note that weaker concentration bounds have also been established for general fields using different techniques [90, 10].

This work was partially motivated by the Bogomolny-Schmit conjecture, and inspired a great deal of future work on non-local/percolation properties of Gaussian fields, by demonstrating that precise results in this direction were feasible. In particular several works have obtained bounds on the expected Betti numbers (which includes the number of connected components) of nodal sets for various general Gaussian ensembles on manifolds (see [64, 35, 37] and references therein).

The most relevant result in the literature, for the later chapters of this thesis, is also due to Nazarov and Sodin. In 2015 they proved a law of large numbers for the number of nodal set components of Gaussian fields in arbitrary dimensions [81, 103]. Specifically, they proved the following: if f is a smooth, ergodic, stationary Gaussian field defined on $\mathbb{R}^d$, $\Omega \subset \mathbb{R}^d$ is a bounded, open, convex set and $N_{LS}(R \cdot \Omega, 0)$ is the number of components of $\{f = 0\}$ contained in $R \cdot \Omega := \{x \in \mathbb{R}^d \mid (1/R)x \in \Omega\}$, then

$$\frac{N_{LS}(R \cdot \Omega, 0)}{R^d \cdot \mathrm{Vol}(\Omega)} \to c \quad \text{as } R \to \infty$$

for some constant $c \geq 0$ which depends on the distribution of the field, where convergence occurs in mean and almost surely. This result holds equally well for the number of excursion or level set components at an arbitrary level, with different limiting constants, but was stated only for the nodal set. We note that when f is the Random Plane Wave, the limiting constant c corresponds to the explicit mean prediction of the Bogomolny-Schmit conjecture (Section 1.2.3). Nazarov and Sodin also prove an analogous statement for families of Gaussian fields, defined on manifolds, whose covariance functions have local scaling limits. The principle behind their result, is that when partitioning a very large domain into large subdomains, most nodal components will be contained in a single subdomain (i.e. they will not intersect multiple subdomains). Therefore the total number of components is equal to the sum of the number in each subdomain plus an error term of lower order. This property is sometimes described as semi-locality, as it is akin to a large-scale version of the local property, described in Section 1.3. Nazarov and Sodin's proof takes advantage of this property through some integral geometric estimates and an ergodic argument.

The results of Nazarov and Sodin have been extended in a number of directions. For non-ergodic, planar fields, it has been shown that the mean number of nodal set components in a large region scales like the area of the region:

$$\mathbb{E}\left(N_{LS}\left(\left[0,R^2\right],0\right)\right)=cR^2+O(R)\quad \text{as } R\to\infty$$

(see [59]). In this paper, it was also argued that the limiting constant c (which depends on the distribution of f) is better viewed as a functional over ν, the spectral measure of the field. The authors then prove that this functional is weak-$*$ continuous with respect to ν.

Another generalisation was proven by Beliaev and Wigman in [12]. Let Ω be a compact, Riemannian n-manifold and $\{\phi_i\}_{i=0}^{\infty}$ an orthonormal basis of eigenfunctions of its Laplacian such that

$$\Delta\phi_i+t_i^2\phi_i=0 \tag{1.1}$$

where $0=t_0\leq t_1\leq t_2\leq\dots$. For $\alpha\in[0,1)$, the α-random band limited functions are then defined as

$$f_{\alpha,T}=\sum_{\alpha T\leq t_i\leq T}c_i\phi_i\qquad \text{for } T>0$$

where the c_i are independent standard Gaussians. For $\alpha=1$

$$f_{1,T}=\sum_{T-\eta(T)\leq t_i\leq T}c_i\phi_i$$

where $\eta(T)\to\infty$ as $T\to\infty$ but $\eta(T)=o(T)$. Random Spherical Harmonics are a particular example of random band limited functions with $\alpha=1$. Beliaev and Wigman proved the existence of the asymptotic density of nodal domains with given size for these models. (A nodal domain is a connected component of the complement of the nodal set.) Specifically, if $N_{LS}(f,0,t)$ denotes the number of nodal domains of f with volume at most t, then as $T\to\infty$

$$\frac{N_{LS}(f_{\alpha,T},0,\frac{t}{T^{\frac{n}{2}}})}{N_{LS}(f_{\alpha,T},0)}\xrightarrow{L^1}\Psi_\alpha(t)$$

for t outside of some countable (possibly empty) set $\mathcal{T}$. The function Ψ_α is shown to be non-decreasing, continuous on $\mathbb{R}\backslash\mathcal{T}$ and universal across manifolds (of the same dimension).

Another result somewhat related to the number of excursion/level set components was proven by Sarnak and Wigman. They considered the topologies (or nestings) of nodal sets for random band limited functions and showed that these have an asymptotic law which is universal across manifolds [97]. The asymptotic law has full support, implying that in the limit 'any topology' will occur with positive probability.

A natural next step in understanding the behaviour of the number of excursion/level set components is to consider its variance. Currently, very little is rigorously known for this statistic. Since the number of excursion or level set components in a domain is bounded above by the number of critical points of the field in that domain (which is a local quantity), it is possible to deduce somewhat trivial upper bounds on the variance. For a generic stationary, planar Gaussian field, a Kac-Rice computation, performed in [29, 31], shows that for each compact domain Ω, there exists $c = c(\ell) > 0$ such that, for all sufficiently large R,

$$\mathrm{Var}(N_{LS}(R \cdot \Omega, \ell)) \leq cR^4$$

where $N_{LS}(R \cdot \Omega, \ell)$ denotes the number of components of $\{f = \ell\}$ contained in $R \cdot \Omega$. (An analogous bound holds for excursion sets.) This bound can be improved slightly in one important case: the exponential concentration result for Random Spherical Harmonics proven in [80] implies that for some $c > 0$

$$\mathrm{Var}(N_{LS}(f_n, 0)) \leq cn^{4-2/15}$$

where $N_{LS}(f_n, 0)$ is the number of nodal components of the n-th Random Spherical Harmonic.

The only non-trivial lower bound on the variance of either the number of excursion or level sets is the recently announced result of Nazarov and Sodin [82] that $\mathrm{Var}(N_{LS}(f_n, 0))$ grows as a positive power of n for certain families of Gaussian fields on the sphere (including Random Spherical Harmonics). The exponent in their bound is not explictly calculated and expected to be suboptimal. It is unclear whether their methods extend to studying non-zero level sets or excursion sets. The sparsity of results on the variance of the number of excursion set components, along with its potential use for statistical testing (see Section 1.2), makes it a good candidate for further research.

1.5 Statement of purpose

The aim of this thesis, is to contribute towards an improved understanding of the geometry of Gaussian fields, with an emphasis on properties related to percolation and statistical testing. To do this, we study the statistics of the number of connected components of planar Gaussian excursion/level sets in large domains. We focus on planar fields mainly for simplicity: there are many open questions about the behaviour of this quantity, even in this case. For reasons outlined in Section 1.2, we are particularly interested in the specific cases of the Random Plane Wave and the Bargmann-Fock field.

Chapter 2

The Mean Functional for the number of Excursion Sets

2.1 Main results

In this chapter, we begin our analysis of the number of excursion/level set components of planar Gaussian fields in large domains. It has been shown by Nazarov and Sodin that these quantities satisfy a law of large numbers, which implies the existence of functionals describing the asymptotic density of excursion and level set components at a given level. We derive an integral representation of these functionals which provides an alternative method of analysing them. This will be the foundation of our work in later chapters. The representation is based on a deterministic decomposition of excursion sets in terms of critical points which is analogous to the Poincaré-Hopf theorem for the Euler characteristic. We also derive some basic properties of the mean functionals, including symmetries, upper/lower bounds and continuity. Finally, we explicitly compute the functionals for a simple class of fields.

2.1.1 Integral representation

Throughout this thesis we consider a planar Gaussian field satisfying the following assumption:

Assumption 2.1.1. *The Gaussian field $f : \mathbb{R}^2 \to \mathbb{R}$ is stationary, normalised (i.e. has mean zero and variance one) and satisfies:*

1. *For some $\eta > 0$, $f \in C^{2+\eta}_{loc}(\mathbb{R}^2)$ almost surely;*
2. *$\nabla^2 f(0)$ is a non-degenerate Gaussian vector (here, and later on, we treat $\nabla^2 f$ as a vector of distinct partial derivatives (f_{xx}, f_{xy}, f_{yy}));*

3. For each $t \in \mathbb{R}^2$, if $f(t) - f(0)$ is a non-degenerate Gaussian variable then $(f(t) - f(0), \nabla f(t), \nabla f(0))$ is a non-degenerate Gaussian vector.

We define the spectral measure ν of f by

$$\mathbb{E}(f(x)f(0)) = K(x) = \int_{\mathbb{R}^2} e^{it\cdot x} d\nu(t).$$

These assumptions are quite minimal, and natural for our analysis; they impose enough topological regularity for our later arguments. They can alternatively be stated in terms of the spectral measure ν. By Kolmogorov's theorem (Theorem A.1.2), the first condition holds provided that $K \in C^{4+\eta'}_{\mathrm{loc}}(\mathbb{R}^2)$ for some $\eta' > 2\eta$ which is equivalent to requiring $\int_{\mathbb{R}^2} |t|^{4+\eta'} d\nu(t) < \infty$. By Lemma A.2.3, the second condition is equivalent to the measure ν not being supported on the union of two lines through the origin. The third condition holds provided that the support of ν is not too degenerate; in particular it holds if the support of ν contains an open set.

The first contribution of this chapter is to extend the results of Nazarov-Sodin [81] and Kurlberg-Wigman [59], which describe the mean behaviour of the number of nodal set components of a Gaussian field, to excursion/level sets at arbitrary levels. For $u \in \mathbb{R}^2$ and $R > 0$ let $B(u, R)$ be the ball of radius R centred at u and let $B(R) = B(0, R)$. Let $N_{ES}(f, R, \ell) = N_{ES}(R, \ell)$ be the number of connected components of $\{f \geq \ell\}$ contained in $B(R)$ (i.e. those which intersect $B(R)$ but not $\partial B(R)$). Let $N_{LS}(f, R, \ell) = N_{LS}(R, \ell)$ be the corresponding quantity for the level set $\{f = \ell\}$.

Theorem 2.1.2. *Let f be a Gaussian field satisfying Assumption 2.1.1. For each $\ell \in \mathbb{R}$, there exist $c_{ES}(\nu, \ell), c_{LS}(\nu, \ell) \geq 0$ such that*

$$\mathbb{E}[N_{ES}(R, \ell)] = c_{ES}(\nu, \ell) \cdot \pi R^2 + O(R) \quad \text{and} \quad \mathbb{E}[N_{LS}(R, \ell)] = c_{LS}(\nu, \ell) \cdot \pi R^2 + O(R)$$

as $R \to \infty$. The constants implied by the $O(\cdot)$ notation may depend on ν but are independent of ℓ. If f is also ergodic, then

$$\frac{N_{ES}(R, \ell)}{\pi R^2} \to c_{ES}(\nu, \ell) \quad \text{and} \quad \frac{N_{LS}(R, \ell)}{\pi R^2} \to c_{LS}(\nu, \ell)$$

as $R \to \infty$, almost surely and in L^1.

The limiting values $c_{ES}(\nu, \ell)$ and $c_{LS}(\nu, \ell)$ can be viewed as functionals over ν and ℓ which specify the asymptotic mean density of excursion and level set components respectively. We address the positivity of these functionals in the next subsection. When the field f is fixed, we simplify notation to $c_{ES}(\ell)$ and $c_{LS}(\ell)$.

Remark 2.1.3. The use of the domain $B(R)$ in the definitions of $N_{ES}(R, \ell)$ and $N_{LS}(R, \ell)$ is mainly for simplicity, and after minor modifications the proof of Theorem 2.1.2 would go through equally well for rescaled copies of any bounded reference domain $\Omega \subset \mathbb{R}^2$, provided that Ω contains a neighbourhood of the origin and $\partial\Omega$ is piecewise smooth (and probably more generally as well). Moreover the limiting constants c_{LS} and c_{ES} would not depend on Ω (after replacing the scaling factor πR^2 by $\mathrm{Area}(\Omega) R^2$).

Remark 2.1.4. Theorem 2.1.2 can also be applied to lower excursion sets (the components of $\{x \in \mathbb{R}^2 : f(x) \leq \ell\}$) since the distribution of f is symmetric and $\{x \in \mathbb{R}^2 : f(x) \leq \ell\} = \{x \in \mathbb{R}^2 : -f(x) \geq -\ell\}$.

Theorem 2.1.2 is, in isolation, only a modest improvement on previous results; it could be proven by slightly adapting the analysis in [81] and [59]. The most substantive contribution of this chapter, is to relate the functionals c_{ES} and c_{LS} to the density of critical points of f of different types and at different levels. This relationship depends on a topological classification of the saddle points of the field into two types:

Definition 2.1.5. Let x_0 be a saddle point of a function $g : \mathbb{R}^2 \to \mathbb{R}$ such that there are no other critical points at the same level as x_0 (that is, if x_1 is another critical point of g, then $g(x_1) \neq g(x_0)$). We say that x_0 is a *lower connected saddle* if it is in the closure of only one component of $\{g < g(x_0)\}$. Similarly, x_0 is said to be *upper connected* if it is in the closure of only one component of $\{g > g(x_0)\}$. (See Figure 2.1.)

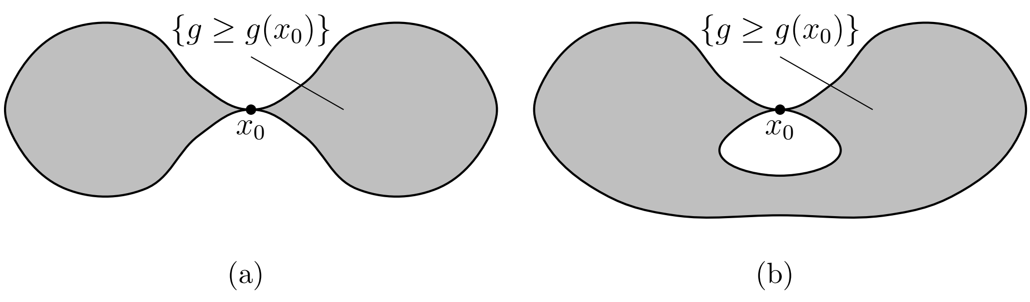


Figure 2.1: The excursion set of a function with a lower connected (2.1a) and upper connected (2.1b) saddle point respectively.

Interestingly, this definition was used as far back as 1870, by Maxwell [71], to understand topographical properties of landscapes.

We say that a Gaussian field f satisfying Assumption 2.1.1 is *periodic* if there exists $x \neq 0$ with $K(x) = 1$ and is *aperiodic* otherwise (this name comes from the fact that realisations of such a field are invariant under translation by nx for $n \in \mathbb{Z}$). A Gaussian field which is aperiodic, almost surely has no two critical points at the same level (see Lemma 2.2.4), and we will show that for such fields, all saddle points of f satisfy exactly one of the two conditions in Definition 2.1.5. In the case that f is periodic, a more general definition for classifying saddle points as upper or lower connected is required, which is given in Section 2.3.

Previous work has shown that, for sufficiently regular isotropic Gaussian fields, the expected number of local maxima, local minima or saddle points with value in a certain interval can be expressed as the integral of an explicit level density function [27, 26]. In Section 2.3, we prove the following version of this result, which applies to more general Gaussian fields and also isolates the upper and lower connected saddles, but does not explicitly identify the densities. This proposition uses Definition 2.3.1 for upper and lower connected saddle points (which coincides with Definition 2.1.5 for aperiodic fields).

Proposition 2.1.6. *Let f be a Gaussian field satisfying Assumption 2.1.1. Then there exist non-negative functions $p_{m^+}, p_{m^-}, p_{s^+}, p_{s^-}, p_s \in L^1(\mathbb{R})$ such that the following holds. Let $\Omega \subset \mathbb{R}^2$ be compact and $\partial\Omega$ have finite Hausdorff-1 measure. Let $\ell \in \mathbb{R}$ and let $N_{m^+}(\ell)$, $N_{m^-}(\ell)$, $N_{s^+}(\ell)$, $N_{s^-}(\ell)$ and $N_s(\ell)$ denote the number of local maxima, local minima, upper connected saddles, lower connected saddles and saddles of f in Ω with level above ℓ respectively. Then*

$$\mathbb{E}[N_h(\ell)] = \mathrm{Area}(\Omega) \int_\ell^\infty p_h(x)\, dx$$

for $h = m^+, m^-, s^+, s^-, s$. Furthermore, these functions can be chosen to satisfy the relations $p_{m^+}(x) = p_{m^-}(-x)$, $p_{s^+}(x) = p_{s^-}(-x)$ and $p_{s^-} + p_{s^+} = p_s$, and such that p_{m^+}, p_{m^-} and p_s are continuous.

The main theorem of this chapter gives an explicit expression for the functionals c_{LS} and c_{ES} in terms of the densities introduced in Proposition 2.1.6. This representation implies the absolute continuity of these functionals with respect to the level. In the case of spectral measures with compact support, we also show joint continuity of these functionals with respect to both the level and the spectral measure.

Let $\mathcal{P}_c$ denote the set of spectral measures ν associated with a Gaussian field satisfying Assumption 2.1.1 such that the support of ν is contained in the closure of

$B(1)$. By rescaling the axes, the results we state for $\mathcal{P}_c$ can be shown to hold for any collection of spectral measures with support in a compact set.

Theorem 2.1.7. *Let f be a Gaussian field satisfying Assumption 2.1.1 and let p_{m^+}, p_{m^-}, p_{s^+}, p_{s^-} denote the densities specified in Proposition 2.1.6. Then*

$$c_{ES}(\nu,\ell) = \int_\ell^\infty p_{m^+}(x) - p_{s^-}(x)\,dx \tag{2.1}$$

and

$$c_{LS}(\nu,\ell) = \int_\ell^\infty p_{m^+}(x) - p_{s^-}(x) + p_{s^+}(x) - p_{m^-}(x)\,dx \tag{2.2}$$

and hence c_{LS} and c_{ES} are absolutely continuous in ℓ. In addition, c_{LS} and c_{ES} are jointly continuous in $(\nu,\ell) \in \mathcal{P}_c \times \mathbb{R}$ where $\mathcal{P}_c$ is given the weak-$*$ topology (i.e. the topology of probabilistic weak convergence).

Remark 2.1.8. Theorems 2.1.2, and 2.1.7 can be generalised to many examples of non-Gaussian, stationary random fields, since the proof requires only that the field satisfies certain topological properties almost surely and is sufficiently regular to apply the Kac-Rice formula (see the more general version of this result stated in Proposition 2.2.6 below).

It is also likely that Theorems 2.1.2 and 2.1.7 could be generalised to higher dimensions d, with the analogues of (2.1) and (2.2) still valid using a more general definition of lower and upper connected saddle points as subsets of critical points with index $d-1$ and 1 respectively. Since some of the topological arguments in the proof increase in complexity in higher dimensions, in the interest of simplicity we do not pursue this generalisation here.

The primary motivation for Theorem 2.1.7 is to provide a new tool with which to analyse the asymptotic mean number of level/excursion sets. Since the densities p_{m^+}, p_{m^-} and p_s are in principle known by the Kac-Rice formula, this result demonstrates that the study of the functionals c_{ES} and c_{LS} can be reduced to an analysis of the density p_{s^-} (or, equivalently, $p_{s^+} = p_s - p_{s^-}$). In this chapter, we use some simple properties of p_{s^-} to derive basic identities for the functionals c_{ES} and c_{LS}. In Chapter 3 we perform a more detailed analysis of p_{s^-} in order to address more subtle questions regarding the smoothness and overall shape of c_{ES} and c_{LS}.

Theorem 2.1.7 has an intuitive geometric interpretation. This theorem is based on a deterministic relationship between the excursion sets and critical points of sufficiently regular planar functions, which is closely related to Morse theory. The excursion set of such a function above a level, deforms continuously as the level changes, provided it does not pass through a critical point. In particular, there is no change in the number of components of the excursion set. When passing through a critical point, the topology of the excursion set changes in a way that is predicted by Morse theory and depends on the index of the critical point. For local maxima and local minima, the number of components of the excursion set changes in a consistent way: decreasing by one and remaining unchanged respectively, as the level increases. For saddle points, the change in the number of components is determined by whether it is upper connected or lower connected (see Figure 2.2).

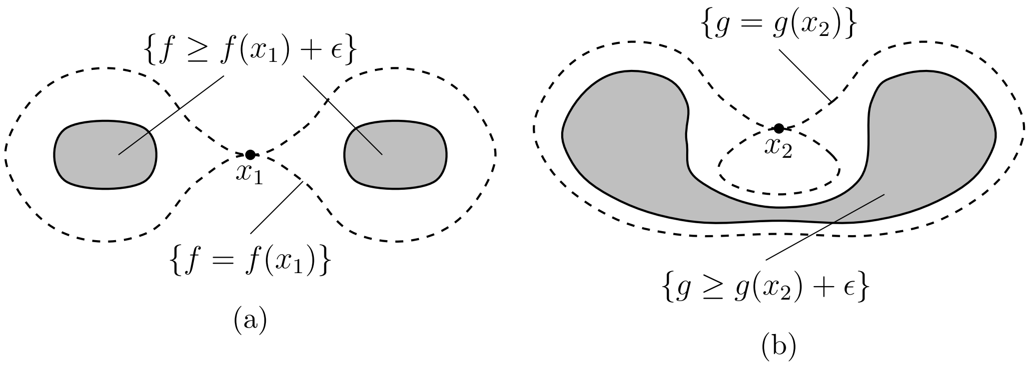


Figure 2.2: The number of excursion set components increases by one on passing through the lower connected saddle x_1 (2.2a) and is constant on passing through the upper connected saddle x_2 (2.2b).

If we consider some regular function g defined on $B(R)$ at level ℓ and then raise the level above the maximum of g, this reduces the number of excursion set components to zero. Therefore summing over the levels above ℓ at which the number of components changes, which must correspond to particular types of critical points by the arguments above, we see that

$$N_{ES}(g, R, \ell) \approx N_{m^+}(R, \ell) - N_{s^-}(R, \ell)$$

where $N_{m^+}(R, \ell)$ and $N_{s^-}(R, \ell)$ denote the number of local maxima and lower connected saddles respectively of g in $B(R)$ above level ℓ. A corresponding expression holds for level sets. This equality is approximate because of boundary effects, however for Gaussian fields, these become negligible as the domain size increases. Ultimately, Theorem 2.1.7 is a probabilisitic expression, in the setting of Gaussian fields, of this deterministic relationship between excursion set components and critical points of various types.

This relationship is strongly analogous to the Poincaré-Hopf theorem for the Euler characteristic which states that for a regular function g

$$\varphi(\{g \geq \ell\} \cap B(R)) \approx N_{m^+}(R, \ell) - N_s(R, \ell) + N_{m^-}(R, \ell)$$

where $\varphi(A)$ denotes the Euler characteristic of the set A and $N_s(R,\ell)$ and $N_{m^-}(R,\ell)$ denote the number of saddle points and local minima respectively of g in $B(R)$ above level ℓ. Once again this equality is only approximate due to boundary effects. The Poincare-Hopf theorem is useful in the analysis of Gaussian fields, because it shows that the Euler characteristic of excursion sets is a local quantity. This permits precise calculation of the statistics of the Euler characteristic as described in Section 1.4. The number of excursion set components is non-local, so no equivalent representation is possible. However our method provides an intermediate result: it separates out the local and non-local aspects of the number of excursion set components, by defining this quantity in terms of critical points (which are local) classified in a way that is non-local.

2.1.2 Symmetries and bounds

Theorem 2.1.7 has some immediate consequences which also have geometric interpretations.

Corollary 2.1.9. *Let f be a Gaussian field satisfying Assumption 2.1.1, for all $\ell \in \mathbb{R}$*

(1) $c_{LS}(\ell) = c_{LS}(-\ell)$,

(2) $c_{ES}(\ell) + c_{ES}(-\ell) = c_{LS}(\ell)$,

(3) $c_{ES}(\ell) - c_{ES}(-\ell) = c_{EC}(\ell) := (2\pi)^{-3/2}\sqrt{\det \nabla^2 K(0)}\, \ell\, e^{-\ell^2/2}$.

The first point says that the density of level set components is symmetric about zero, which naturally follows from the symmetry of the Gaussian distribution. Noting that $\{f \geq -\ell\}$ has the same distribution as $\{f \leq \ell\}$, (again by symmetry of the Gaussian distribution) the second point can be interpreted as saying that, on large scales, the number of level set components of f is approximately equal to the number of upper excursion set components (i.e. components of $\{f \geq \ell\}$) plus the number of lower excursion set components (i.e. components of $\{f \leq \ell\}$). In fact, it can be shown that for any sufficiently regular deterministic function, this equality holds up to boundary effects (this argument is given explicitly later, in the proof of Lemma 2.2.5). The first two points of Corollary 2.1.9 follow immediately from Theorem 2.1.7 and the identities in Proposition 2.1.6.

The notation c_{EC} in the third point of Corollary 2.1.9 is intended to denote the mean Euler characteristic functional for excursion sets, which takes the explicit form given above. Intuitively, this point reflects the definition of the Euler characteristic of a two dimensional set, as the number of connected components (i.e. components of $\{f \geq \ell\}$) minus the number of 'holes' (i.e. components of $\{f \leq \ell\}$).

The equality in this point was proven for isotropic fields in [104] using a winding number calculation. We present a different proof using Theorem 2.1.7, which simplifies the calculation in the non-isotropic case and also highlights the relation to the Euler characteristic of the excursion set. Specifically, the proof is as follows: using the integral representation (2.1) and the symmetries $p_{m^+}(x) = p_{m^-}(-x)$, $p_{s^+}(x) = p_{s^-}(-x)$ and $p_{s^-} + p_{s^+} = p_s$, one sees that

$$\begin{aligned} c_{ES}(\ell) - c_{ES}(-\ell) &= \int_{\ell}^{\infty} p_{m^+}(x) - p_s(x) + p_{m^-}(x)\, dx \\ &= \mathbb{E}(N_{m^+}(\ell) - N_s(\ell) + N_{m^-}(\ell)) \end{aligned}$$

where $N_{m^+}(\ell)$, $N_s(\ell)$ and $N_{m^-}(\ell)$ are the number of critical points above level ℓ as defined in Proposition 2.1.6 for $\Omega = [0,1]^2$. Lemma 11.7.1 of [1] states that this alternating sum is equal to the expression given in Corollary 2.1.9, thus completing the proof.

By the Poincaré-Hopf theorem, this alternating sum is essentially the expected Euler characteristic of the excursion set $\{x \in [0,1]^2 \mid f(x) \geq \ell\}$. When working on finite subsets of the plane, boundary effects must be considered, but these become negligible as the area of the subset increases. Formally, if we let $\varphi(A)$ denote the Euler characteristic of a set A, then Theorem 11.7.2 of [1] gives an expression for the expected Euler characteristic of an excursion set on a cube (including boundary effects) which implies that

$$\mathbb{E}(N_{m^+}(\ell) - N_s(\ell) + N_{m^-}(\ell)) = \lim_{R\to\infty} \frac{1}{R^2}\mathbb{E}\left(\varphi\left(\{x \in [0,R]^2 : f(x) \geq \ell\}\right)\right).$$

Hence the expression $c_{EC}(\ell)$ above, can be interpreted as the asymptotic mean Euler characteristic of excursion sets in an analogous way to c_{ES} or c_{LS}.

The third point of Corollary 2.1.9 gives a non-trivial lower bound for $c_{ES}(\ell)$ at positive values of ℓ (since $c_{ES}(-\ell) \geq 0$). This can be combined with some local estimates to give upper and lower bounds on c_{ES} and c_{LS} for most values of ℓ. This is most easily formulated when f is an isotropic Gaussian field. Recall that, in this case its covariance function may be expressed as $K(x) = k(|x|^2)$. It is shown in [26] that

$$\chi := \frac{-k'(0)}{\sqrt{k''(0)}} \in \left(0, \sqrt{2}\right] \quad \text{and} \quad \xi^2 := \frac{-k'(0)}{k''(0)} \in [0,\infty) \tag{2.3}$$

parameterise the critical point densities p_{m^+}, p_{m^-} and p_s of isotropic fields in any dimension. These parameters can equivalently be calculated using the covariance function K:

$$\chi = \frac{-\sqrt{3}K_{11}(0)}{\sqrt{K_{1111}(0)}} \quad \text{and} \quad \xi^2 = \frac{-6K_{11}(0)}{K_{1111}(0)}.$$

It can also be shown that, for planar isotropic fields, if $\chi = \sqrt{2}$ then $K(x) = J_0(\sqrt{8/\xi^2}\,|x|)$, where J_0 is the 0-th Bessel function. Therefore in this case, f is simply the Random Plane Wave (possibly with a rescaling of space).

Proposition 2.1.10 (Swerling [104])**.** *Let f be an isotropic Gaussian field satisfying Assumption 2.1.1, then for $\ell \geq 0$*

$$c_{LS}(\ell) \leq \frac{\chi^2}{\pi\xi^2}\,\phi(\ell)\left(\frac{2\sqrt{3-\chi^2}}{\chi}\,\phi\left(\frac{\chi\ell}{\sqrt{3-\chi^2}}\right) + \ell\left(2\Phi\left(\frac{\chi\ell}{\sqrt{3-\chi^2}}\right) - 1\right)\right). \tag{2.4}$$

where ϕ and Φ denote the standard normal probability density and cumulative density functions respectively.

In particular, if $\chi^2 > \frac{6e}{2e+\pi} \approx 1.9$ then by Corollary 2.1.9, $c_{LS}(0) < c_{LS}(1)$ so that c_{LS} is at least bimodal (that is, it has at least two local maxima).

This result is proven using the method of 'flip points'. Specifically, for any fixed direction u, the number of level set components at level ℓ in a finite region is bounded above by half the number of points t in the region such that $f(t) = \ell$ and $\partial_u f(t) = 0$ where ∂_u denotes the partial derivative in the direction u. The expected number of such points can be computed using the Kac-Rice formula. For isotropic fields, this expectation is invariant over u and is given by the right hand side of (2.4). This method could also be applied to non-isotropic fields, and the bound could be optimised over the direction u, but we omit this for simplicity. The final comment of this result follows from the fact that for an isotropic field

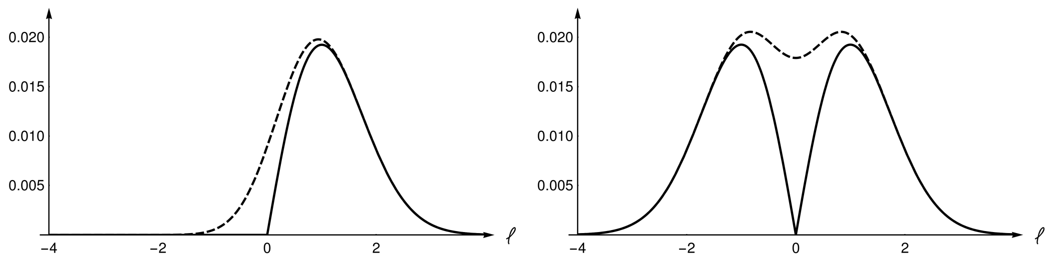
$$\nabla^2 K(0) = \begin{pmatrix} 2k'(0) & 0 \\ 0 & 2k'(0) \end{pmatrix}.$$

It is also possible to construct an upper bound on c_{LS} using the inequality $0 \leq p_{s^-} \leq p_s$ and the densities p_{m^+}, p_s and p_{m^-}, which are explicitly known for isotropic fields. However this bound is larger than the bound in (2.4) at all levels ℓ.

By combining Corollary 2.1.9 and Proposition 2.1.6, one can derive upper and lower bounds on c_{ES} and c_{LS} for most values of ℓ. This is illustrated in Figure 2.3, which shows such bounds for the Random Plane Wave (for which $\chi = \sqrt{2}$ and $\xi^2 = 8$).

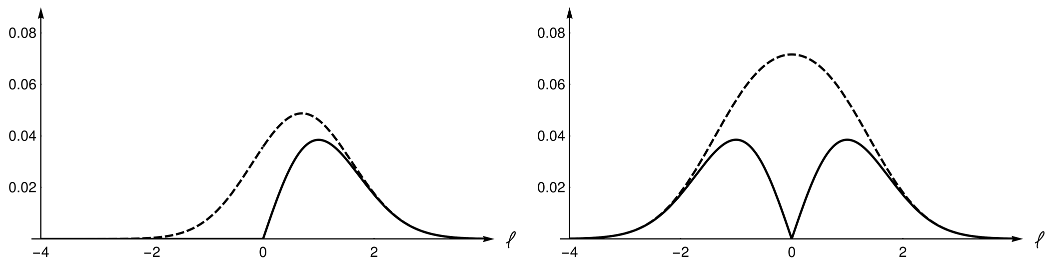
Although these bounds are not particularly tight for ℓ close to 0, they quickly become accurate as $|\ell|$ increases. For $\ell \geq 1$ both the upper and lower bounds on $c_{LS}(\ell)$ are within 5.1% of the true value (since the upper bound is within 5.1% of the lower bound) while for $\ell \geq 1.5$, both bounds are within 0.6% of the true value.

The bounds for other isotropic fields are similar, although for fields with lower values of χ, the percentage difference between the upper and lower bounds on $c_{LS}(\ell)$ is a bit larger. Figure 2.4 shows the corresponding bounds on c_{ES} and c_{LS} for the Bargmann-Fock field, (recall that this is the field with covariance kernel $K(x) = \exp(-|x|^2/2)$) for which $\chi = 1$ and $\xi^2 = 2$. In this case, for $\ell \geq 1$ the upper and lower bounds on $c_{LS}(\ell)$ are within 40% of the true value, and the corresponding accuracies for $\ell \geq 1.5$, $\ell \geq 2$ and $\ell \geq 2.5$ are 14%, 5% and 2% respectively.



(a) (b)

Figure 2.3: Subfigures 2.3a and 2.3b show lower bounds (solid) and upper bounds (dashed) for $c_{ES}(\nu, \ell)$ and $c_{LS}(\nu, \ell)$ respectively in the Random Plane Wave case, for which $\chi = \sqrt{2}$ and $\xi^2 = 8$.



(a) (b)

Figure 2.4: Subfigures 2.4a and 2.4b show lower bounds (solid) and upper bounds (dashed) for $c_{ES}(\nu, \ell)$ and $c_{LS}(\nu, \ell)$ respectively, where ν is the spectral measure of the Bargmann-Fock field, for which $\chi = 1$ and $\xi^2 = 2$.

The positivity of c_{ES} and c_{LS} is extremely important for understanding the large scale behaviour of excursion/level sets of a Gaussian field. When these functionals are positive, Theorem 2.1.2 implies that the number of excursion or level set components in a domain of area R^2 is, on average, of order R^2. If $c_{ES}(\ell) = 0$ then, by stationarity,

$\{f \geq \ell\}$ almost surely has no compact components (and the analogous statement holds for level sets). Corollary 2.1.9 states that $c_{ES}(\ell) > 0$ for all $\ell > 0$, and so $c_{LS}(\ell) > 0$ for all $\ell \neq 0$, however the positivity of $c_{LS}(0)$ and $c_{ES}(\ell)$ for $\ell \leq 0$ is not addressed by this method. Nazarov and Sodin use a technique known as the barrier method to prove sufficient conditions for positivity of $c_{LS}(0) = 2c_{ES}(0)$ which hold quite generally [81]. In Chapter 3 we give an alternative proof of the positivity of $c_{LS}(0)$ (under stronger assumptions) which follows easily from other results derived therein. One weakness of all of these results, is that the bound on $c_{LS}(0)$ is not quantitative. There is only one case in which an explicit non-zero lower bound on $c_{LS}(0)$ is known: for the Random Plane Wave, it has been shown that $c_{LS}(0) \geq 1.1 \times 10^{-5}$ (see [46]) although numerical simulations suggest that $c_{LS}(0) \approx 0.0589/(4\pi) \approx 0.00469$ (see [8] and references therein).

2.1.3 An explicit example

A natural question at this stage, is whether one can derive explicit values for the mean functionals c_{ES} and c_{LS} for any particular fields and levels. For certain degenerate cases, this question has a trivial answer: fields with spectral measure supported on a line through the origin are known to be almost surely constant in one direction, and so have no compact level domains. Therefore both mean functionals are identically zero for such fields. There is only one non-degenerate case for which this question has been answered: it has been shown that fields with spectral measure supported on precisely four points almost surely have no compact nodal domains, so that $c_{ES}(0) = c_{LS}(0) = 0$ [59].

In this subsection, we extend the previous example slightly. Specifically, the methods developed in this chapter allow us to fully characterise the large scale behaviour of the number of excursion/level set components (at all levels) of fields with spectral measure supported on four or five points. In particular, this allows us to derive explicit expressions for the two mean functionals. We note that such fields do not satisfy Assumption 2.1.1, and so we cannot apply Theorems 2.1.2 or 2.1.7. Nonetheless, we are able to directly verify that the conclusions of these theorems also hold in this degenerate case.

Recall that a random variable Y is said to be Rayleigh distributed with parameter $\sigma^2 > 0$, denoted $Y \sim \mathrm{Ray}(\sigma)$, if $\mathbb{P}(Y \leq x) = 1 - e^{-x^2/(2\sigma^2)}$ for all $x \geq 0$.

Proposition 2.1.11. *Let f be the Gaussian field with spectral measure*

$$\nu = a\delta_0 + \frac{b}{2}(\delta_u + \delta_{-u}) + \frac{c}{2}(\delta_v + \delta_{-v})$$

where $b, c > 0$, $a = 1 - b - c \geq 0$ and $u, v \in \mathbb{R}^2$ are linearly independent. There exists a version of this field such that

$$N_{ES}(R,\ell) = |u \times v| \cdot \pi R^2 \cdot \mathbb{1}\{|Y_1 - Y_2| < \ell + X_0 \leq Y_1 + Y_2]\} + O_{L^1(\mathbb{P})}(R)$$
$$N_{LS}(R,\ell) = |u \times v| \cdot \pi R^2 \cdot \mathbb{1}\{|Y_1 - Y_2| < |\ell + X_0| \leq Y_1 + Y_2]\} + O_{L^1(\mathbb{P})}(R)$$

as $R \to \infty$, where $\times$ denotes the cross product, $X_0 \sim \mathcal{N}(0,a)$, $Y_1 \sim \mathrm{Ray}\left(\sqrt{b}\right)$, $Y_2 \sim \mathrm{Ray}\left(\sqrt{c}\right)$ and X_0, Y_1, Y_2 are independent. For a sequence of random variables Z_R, the notation $Z_R = O_{L^1(\mathbb{P})}(R)$ means that $\sup_R \mathbb{E}(|Z_R/R|) < \infty$.

As a consequence of this;

$$\mathbb{E}[N_{ES}(R,\ell)] = c_{ES}(\ell) \cdot \pi R^2 + O(R) \quad \text{and} \quad \mathbb{E}[N_{LS}(R,\ell)] = c_{LS}(\ell) \cdot \pi R^2 + O(R)$$

where

$$c_{ES}(\ell) = |u \times v| \cdot \mathbb{P}\left(|Y_1 - Y_2| < \ell + X_0 \leq Y_1 + Y_2\right),$$
$$c_{LS}(\ell) = |u \times v| \cdot \mathbb{P}\left(|Y_1 - Y_2| < |\ell + X_0| \leq Y_1 + Y_2\right).$$

Moreover the constants implied by the $O(\cdot)$ notation in these expressions are independent of ℓ. In particular, $c_{LS}(\ell) = 0$ if and only if $\ell = a = 0$.

Furthermore, there exist functions p_{m^+}, p_{s^-}, p_{m^-}, p_{s^+} and p_s satisfying the conclusions of Proposition 2.1.6, and these are defined by

$$p_{m^+}(x) = p_{m^-}(-x) = |u \times v| \cdot p_{X_0+Y_1+Y_2}(x)$$
$$p_{s^-}(x) = p_{s^+}(-x) = |u \times v| \cdot p_{X_0+|Y_1-Y_2|}(x)$$

where p_Z denotes the probability density of a random variable Z. Therefore the equalities in the conclusion of Theorem 2.1.7 hold for f.

Figures 2.5 and 2.6 show c_{ES} and c_{LS} for fields described in this proposition with different values of a, b, c when $u = (1,0)$ and $v = (0,1)$. These examples pose some interesting questions about the general properties of the mean excursion/level functionals.

Figures 2.5 and 2.6 suggest that the derivatives of c_{ES} and c_{LS} need not be continuous at zero when $a = 0$. This can be verified analytically since

$$c'_{ES}(\ell) = p_{s^-}(\ell) - p_{m^+}(\ell)$$

and these densities are explicitly known. In particular, it can be shown that $p_{Y_1+Y_2}$ is everywhere continuous, whereas $p_{|Y_1-Y_2|}$ has a jump discontinuity at zero (but is continuous elsewhere). So c'_{ES} is continuous except at zero. An analogous calculation

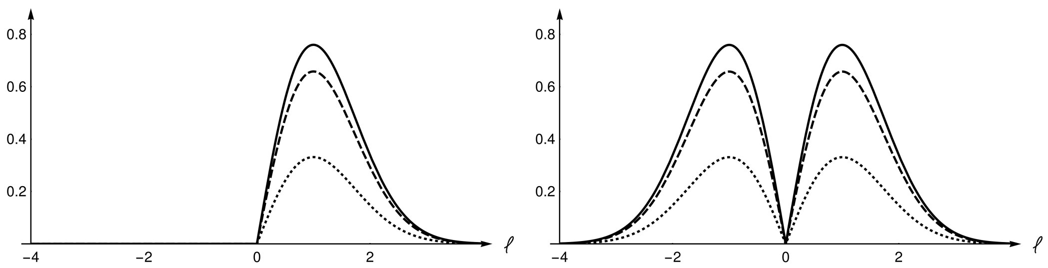


Figure 2.5: The functions $c_{ES}(\ell)$ (left) and $c_{LS}(\ell)$ (right) with $a = 0$ for $b - c = 0$ (solid), $b - c = 0.5$ (dashed) and $b - c = 0.9$ (dotted) respectively.

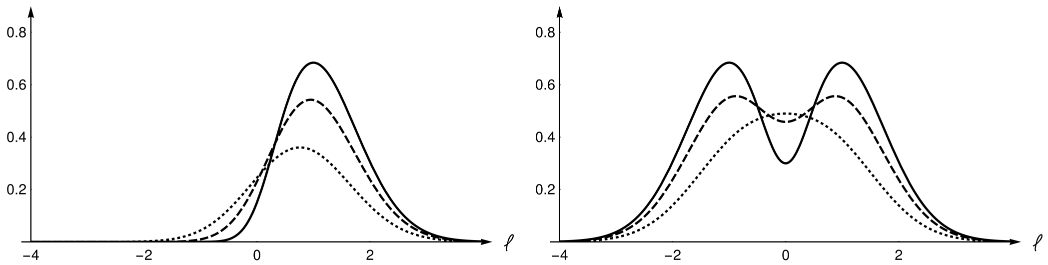


Figure 2.6: The functions $c_{ES}(\ell)$ (left) and $c_{LS}(\ell)$ (right) with $b = c$ for $a = 0.1$ (solid), $a = 0.3$ (dashed) and $a = 0.6$ (dotted) respectively.

permits the same conclusion for c'_{LS}. On the other hand, when $a \neq 0$, c_{ES} and c_{LS} are continuously differentiable everywhere. A major question arising from the analysis in this chapter is whether c_{LS} and c_{ES} are continuously differentiable more generally (i.e. for other fields). The analogous question for discrete percolation models has important consequences. We address this further in Chapter 3.

Figure 2.5 also suggests that c_{LS} is bimodal whenever $a = 0$ (we know that it is at least bimodal, since $c_{LS}(0) = 0$ and $c_{LS}(\ell) > 0$ for $\ell \neq 0$ in this case), however for $a > 0$, c_{LS} can be bimodal or unimodal. Combining this observation with Proposition 2.1.10, raises the question of determining the overall shape of c_{ES} and c_{LS} for general Gaussian fields, and, in particular, the number of critical points of this functional. We will begin to answer these questions in Chapter 3. In Chapter 4, we will see that these answers have important consequences for the behaviour of the variance of the number of excursion/level sets.

2.1.4 Summary of the rest of the chapter

The remainder of this chapter is organised as follows. In Section 2.2, we state the main (completely deterministic) topological lemma underlying our results (Lemma 2.2.5) and combine this with Proposition 2.1.6 to prove Theorems 2.1.2 and 2.1.7 and their corollaries. In Section 2.3, we give the full definition of upper/lower connected saddle points and prove Proposition 2.1.6. In Section 2.4, we prove Lemma 2.2.5 via a series of steps, and also prove Proposition 2.1.11, thereby completing the proof of all results in the chapter.

2.2 Proof of the main results

The first step in proving Theorems 2.1.2 and 2.1.7, is to translate our assumptions on the Gaussian field f into topological properties; these properties differ slightly depending on whether or not f is periodic.

Definition 2.2.1. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a function which is not constant in any direction; that is, there is no $v \in \mathbb{R}^2 \backslash \{0\}$ such that for all $x \in \mathbb{R}^2$ and $a \in \mathbb{R}$, $g(x) = g(x + av)$. We say that g is *doubly periodic* if there exist two linearly independent vectors $y, z \in \mathbb{R}^2$ such that for all $x \in \mathbb{R}^2$, $g(x + y) = g(x) = g(x + z)$. We say that g is *singly periodic* if it is not doubly periodic and there exists $y \in \mathbb{R}^2 \backslash \{0\}$ such that for all $x \in \mathbb{R}^2$, $g(x) = g(x + y)$. If neither of these conditions holds then we say that g is *aperiodic*.

We note that this definition applies to deterministic functions. We say that a random field f is *doubly periodic* if there exist linearly independent vectors $y, z \in \mathbb{R}^2$ such that with probability one, $f(x + y) = f(x) = f(x + z)$ for all $x \in \mathbb{R}^2$. We make an analogous definition for singly periodic fields, and we say that a random field is *aperiodic* if it is neither doubly periodic nor singly periodic.

For a stationary, C^2 random field f which is almost surely not constant in any direction, we claim that f is doubly periodic if and only if there exists $A \in GL(2, \mathbb{R})$ such that

$$\{Ax : f(x) = f(0) \text{ a.s.}\} = \mathbb{Z}^2.$$

If the latter condition holds, then clearly f is doubly periodic by definition. To verify the converse statement, we note that $S := \{x : f(x) = f(0) \text{ a.s.}\}$ is an additive subgroup of $\mathbb{R}^2$: if $f(y) = f(0) = f(x)$ almost surely then by stationarity $f(x) = f(x + y)$ almost surely. If this subgroup is discrete (i.e. $S \cap B(0, \epsilon) = \{0\}$ for some $\epsilon > 0$) then it is a free Abelian group of rank at most 2. By definition of being doubly periodic it must have rank at least two, therefore showing that S is discrete completes the proof of the claim.

Suppose S is not discrete, so there exists a sequence $x_n \in \mathbb{R}^2 \cap S$ such that $x_n \to 0$. By restricting to a subsequence, we may assume that the argument of x_n converges to some $\theta \in [0, 2\pi)$. Since f is almost surely constant on the union of this sequence and the origin, we see that the derivative of f at 0 in the direction θ is almost surely constant. However by stationarity, this implies that f is almost surely constant in the direction θ which gives a contradiction.

A similar argument shows that a stationary, C^2 random field f which is almost surely not constant in any direction is singly periodic if and only if it is not doubly periodic and there exists $A \in GL(2, \mathbb{R})$ such that $\{Ax : f(x) = f(0) \text{ a.s.}\} = \mathbb{Z} \times \{0\}$.

For our purposes, it is more natural to consider periodic functions as being defined on the torus or cylinder, and so we will specify the topological properties of Gaussian fields in terms of these domains.

For any doubly periodic C^2 function $g : \mathbb{R}^2 \to \mathbb{R}$ we choose two linearly independent vectors $y, z \in \mathbb{R}^2$ such that each vector in $\{x : g(x + x_0) = g(x_0) \ \forall x_0 \in \mathbb{R}^2\}$ can be expressed as a linear combination of y and z with constant coefficients. Such vectors exist because this set forms a discrete lattice (which can be seen by repeating the arguments above in the deterministic case). The function g is then specified entirely by its values on the parallelogram

$$P := \left\{x \in \mathbb{R}^2 : x = ty + sz, \ t, s \in [0, 1)\right\}.$$

We call this the *associated parallelogram* for g and call y, z *periodic vectors* for g (they are not unique, for instance we could also choose $-y, -z$). By identifying P with the 2-dimensional torus $\mathbb{T}$, we let $g_{\mathbb{T}}$ be g defined on the torus. By the choice of y, z above, when g is a stationary random field there are no two distinct points in P, on which g is almost surely equal.

If g is singly periodic, then we can choose $y \in \mathbb{R}^2 \backslash \{0\}$ satisfying the conditions in Definition 2.2.1 with minimum distance to the origin; we call this a *periodic vector* for g. By rotating the axes we may assume that g is periodic in the direction of the x-axis (i.e. $y = (y_1, 0)$) so that g is specified entirely by its values on $[0, y_1] \times \mathbb{R}$. This strip can be identified with the infinite cylinder $\mathcal{C} := S^1 \times \mathbb{R}$ under the quotient relationship that $x \sim z$ if and only if $x - z = (my_1, 0)$ for some $m \in \mathbb{Z}$. We will often work with the compact subset $\mathcal{C}(n) := [0, y_1] \times [-n, n]$ under the same quotient relationship. We let $g_{\mathcal{C}}$ and $g_{\mathcal{C}(n)}$ denote g restricted to these surfaces under the previous identification.

We next introduce some elements of Morse theory, in particular the concept of a Morse function on a manifold; here we follow [43]. We emphasise that, unless explicitly stated, a manifold M may or may not have a boundary ∂M. We also emphasise that we will only ever work with M being the plane $\mathbb{R}^2$, the closure of the ball $B(R)$, the torus $\mathbb{T}$, or the cylinders $\mathcal{C}$ and $\mathcal{C}(n)$, and so it is sufficient to bear these examples in mind.

Definition 2.2.2. Let M be an n-dimensional Riemannian manifold and $f \in C^2(M)$. (If $\partial M \neq \emptyset$, we take this to mean that for any coordinate chart ψ, the function $f \circ \psi^{-1}$ can be extended to a C^2 function on an open subset of $\mathbb{R}^n$. In particular this means that $f|_{\partial M}$ is twice continuously differentiable.) We say that f is *Morse* if the following hold (with any condition depending on ∂M holding implicitly if M has no boundary):

1. The critical points of f and $f|_{\partial M}$ are non-degenerate;
2. None of the critical points of f are contained in ∂M;
3. If ∇f is tangent to ∂M at p and u is a unit vector normal to ∂M at p, then $\partial_{\nabla f(p)} \partial_u f(p) \neq 0$ where $\partial_{\nabla f(p)}$ and ∂_u denote the directional derivatives in the directions $\nabla f(p)$ and u respectively;
4. The critical points of f and $f|_{\partial M}$ all occur at distinct levels.

If M is a manifold without boundary this simplifies to the requirement that all critical points of f are non-degenerate and occur at distinct levels.

We note that in the above definition $f|_{\partial M}$ is viewed as a function on the $(n-1)$-dimensional manifold ∂M, so that a critical point of $f|_{\partial M}$ is not necessarily a critical point of f.

Using the definition of a Morse function, we next introduce a set of conditions that a stationary random field must satisfy in order for our main results to hold. In the lemma that immediately follows, we will claim, in particular, that these conditions are satisfied for Gaussian fields satisfying Assumption 2.1.1, but we have chosen to isolate these conditions to illustrate the limited role of Gaussianity in the proof of the main results.

We say that a point $t \in \partial M$ is a *tangent point* of f, if t is a critical point of the restricted function $f|_{\partial M}$ (the name comes from the fact that the level set $\{f = f(t)\}$ will be tangent to ∂M at t). We will also use the term 'tangent point' to describe a local extrema of the restriction of f to a finite union of line segments. Let $N_{\mathrm{tang}}(A)$ denote the number of tangent points of f in A (where A is the boundary of a manifold or a finite union of line segments) and $N_{\mathrm{crit}}(M)$ the number of critical points of f in M. We will only use these terms when A and M are compact, so that by the first point in the definition of a Morse function, the number of critical and tangent points will be finite.

Assumption 2.2.3. *The random field f is stationary and satisfies the following:*

1. *If f is aperiodic, then for each $R > 0$, $f|_{\overline{B(R)}}$ is almost surely Morse (taking a union over $R \in \mathbb{N}$, this implies also that f is almost surely Morse on $\mathbb{R}^2$);*
2. *If f is doubly periodic, then $f_{\mathbb{T}}$ is almost surely Morse;*
3. *If f is singly periodic, then for each $n \in \mathbb{N}$, $f_{\mathcal{C}(n)}$ is almost surely Morse (taking a union over $\mathbb{N}$, this implies that $f_{\mathcal{C}}$ is almost surely Morse);*
4. *If f is periodic and $L \subset \mathbb{R}^2$ is a line segment of length R, then $\mathbb{E}(N_{\mathrm{tang}}(L)) = c_\theta R$ for a constant $c_\theta > 0$ that depends only on the direction of L;*
5. *$\mathbb{E}(N_{\mathrm{crit}}(B(R))) = cR^2$ for a constant $c > 0$;*
6. *$\mathbb{E}(N_{\mathrm{tang}}(\partial B(R))) = O(R)$ as $R \to \infty$.*

Lemma 2.2.4. *Let f be a Gaussian field satisfying Assumption 2.1.1, then it also satisfies Assumption 2.2.3.*

Proof. All six conditions are established using variations of the Kac-Rice formula (see Section 1.3.1.1 for background on this result).

(1)-(3). We first note that, since $f \in C^2_{\mathrm{loc}}(\mathbb{R}^2)$ almost surely by assumption, $f|_M \in C^2(M)$ almost surely whenever M is $\overline{B(R)}$, $\mathbb{T}$ or $\mathcal{C}(n)$. Corollary 11.3.5 of [1] states that $f|_M$ has the first two properties of a Morse function if there exists a countable atlas $\{\psi_i\}_{i\in I}$ such that for each $i \in I$, the covariance kernel of $f^{(i)} := f|_M \circ \psi_i^{-1}$ is C^{4+} (on its domain of definition) and the Gaussian vectors $(\partial_1 f^{(i)}, \partial_{11} f^{(i)}, \partial_{12} f^{(i)})$ and $(\partial_2 f^{(i)}, \partial_{22} f^{(i)}, \partial_{12} f^{(i)})$ are non-degenerate at each point (where ∂_1 denotes taking the derivative with respect to the first variable, ∂_{11} denotes taking the second derivative with respect to the first variable and we make analogous definitions for ∂_2, ∂_{22} and ∂_{12}). For each choice of M that we consider (i.e. $\overline{B(R)}$, $\mathbb{T}$ or $\mathcal{C}(n)$) it is clear that we can choose a finite atlas of charts ψ_i and that $f|_M \circ \psi_i^{-1}$ will be a translation of f (restricted to some open set) for each such chart. Therefore the covariance kernel of $f^{(i)}$ is a restriction of K, which is C^{4+} by assumption, and all that remains is to show that $(\partial_1 f(t), \partial_{11} f(t), \partial_{12} f(t))$ and $(\partial_2 f(t), \partial_{22} f(t), \partial_{12} f(t))$ are non-degenerate Gaussian vectors for any $t \in \mathbb{R}^2$. By assumption $\nabla f(t)$ and $\nabla^2 f(t)$ are non-degenerate Gaussian vectors for any $t \in \mathbb{R}^2$. It is a standard fact, for Gaussian fields with constant variance, that $\nabla f(t)$ and $\nabla^2 f(t)$ are independent for fixed t (see Lemma A.2.2) and this proves the necessary non-degeneracy. (This part of the result does not depend on whether f is periodic.)

We note that $f|_{\overline{B(R)}}$ has the third property of a Morse function provided that there is no $\theta \in [0, 2\pi]$ such that

$$0 = g_1(\theta) := \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}^T \nabla f(p)$$
$$0 = g_2(\theta) := \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}^T \nabla^2 f(p) \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix}$$

where $p = (R\cos(\theta), R\sin(\theta))$ (see Figure 2.7).

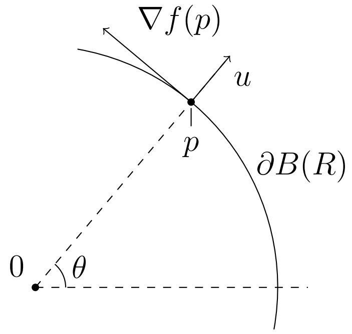


Figure 2.7: If $p = (R\cos(\theta), R\sin(\theta))$ and $\nabla f(p)$ is orthogonal to $u := (\cos(\theta), \sin(\theta))$, we require that $\partial_{\nabla f(p)} \partial_u f(p) \neq 0$.

If we assumed that f is C^3, then Bulinskaya's lemma would imply that no such θ exists almost surely. Since we assume only that f is $C^{2+\eta}$, we use a different argument which is adapted from the proof of [1, Lemma 11.2.11]. Since $f \in C^{2+\eta}_{\text{loc}}(\mathbb{R}^2)$ almost surely, g_2 is almost surely η-Hölder continuous. Therefore for each $\epsilon > 0$ we can find $C_\epsilon, D_\epsilon > 0$ such that, with probability at least $1 - \epsilon$, on $\partial B(R)$ the η-Hölder norm of g_2 is at most C_ϵ and each element of $\nabla^2 f$ is bounded in absolute value by D_ϵ; we denote this event by A_ϵ.

Now let I_{mj} be a collection of open intervals in $\mathbb{R}$ such that $[0, 2\pi] \subset \cup_j I_{mj}$ for each m, $\sum_j \text{Length}(I_{mj}) \to 2\pi$ as $m \to \infty$ and $\text{Length}(I_{mj}) \to 0$ uniformly in j as $m \to \infty$. Furthermore let θ_{mj} be the midpoint of each I_{mj} and define

$$E_{mj} = \{\exists \theta \in I_{mj} : g_1(\theta) = g_2(\theta) = 0\}.$$

On the event $A_\epsilon \cap E_{mj}$ we note that

$$|g_1(\theta_{mj})| \leq c_1 D_\epsilon \text{Length}(I_{mj}) \quad \text{and} \quad |g_2(\theta_{mj})| \leq c_1 C_\epsilon \text{Length}(I_{mj})^\eta$$

for some universal constant $c_1 > 0$. Therefore

$$\begin{aligned}
&\mathbb{P}(A_\epsilon \cap E_{mj}) \\
&\quad \leq \int_{|x|\leq c_1 D_\epsilon \mathrm{Length}(I_{mj})} \mathbb{P}\left(|g_2(\theta_{mj})| \leq c_1 C_\epsilon \mathrm{Length}(I_{mj})^\eta | g_1(\theta_{mj}) = x\right) p_{g_1(\theta_{mj})}(x)\, dx \\
&\quad = \int_{|x|\leq c_1 D_\epsilon \mathrm{Length}(I_{mj})} \mathbb{P}\left(|g_2(\theta_{mj})| \leq c_1 C_\epsilon \mathrm{Length}(I_{mj})^\eta\right) p_{g_1(\theta_{mj})}(x)\, dx
\end{aligned} \tag{2.5}$$

where $p_{g_1(\theta_{mj})}$ is the probability density function of $g_1(\theta_{mj})$ and we have used the fact that $g_1(\theta_{mj})$ is independent of $g_2(\theta_{mj})$. Each $g_i(\theta)$ is a linear combination of elements of a non-degenerate Gaussian vector and so has non-zero variance, therefore on the compact region $[0, 2\pi]$ these variances are bounded away from zero. Since these are Gaussian fields, their univariate densities are bounded above by a constant times the inverse of their standard deviations, so we see that the density of each $g_i(\theta_{mj})$ is bounded above uniformly in m and j. We can therefore sum the expression on either side of (2.5) over j. Since $\mathrm{Length}(I_{mj})$ converges to zero uniformly in j, so does the integrand in each term of the sum, and so we conclude that

$$\sum_j \mathbb{P}(A_\epsilon \cap E_{mj}) \to 0$$

as $m \to \infty$. Since this is true for any $\epsilon > 0$ and $\mathbb{P}(A_\epsilon) \geq 1 - \epsilon$, we conclude that there is almost surely no $\theta \in [0, 2\pi]$ such that $g_1(\theta) = g_2(\theta) = 0$, and hence f almost surely has the third property of a Morse function.

The proof that $f_{\mathcal{C}(n)}$ satisfies the third property of a Morse function is near identical. The argument above can be repeated for the boundary $S^1 \times \{-n, n\}$. Finally, $f_{\mathbb{T}}$ trivially satisfies the third property of a Morse function.

Next we consider the fourth property of a Morse function. We first suppose f is aperiodic and define $h_1 : T_k \to \mathbb{R}^5$ where $T_k = \left\{(s,t) \in \overline{B(R)}^2 \,\middle|\, |s-t| \geq 1/k\right\}$ by

$$h_1(s,t) = \begin{pmatrix} \nabla f(s) \\ \nabla f(t) \\ f(s) - f(t) \end{pmatrix}.$$

Bulinskaya's lemma (Lemma A.2.1) states that $\mathbb{P}(h_1^{-1}(0) = \emptyset) = 1$ provided that $h_1 \in C^1(T_k)$ almost surely and the univariate densities of h_1 are bounded uniformly in T_k. By assumption, $h_1 \in C^1_{\mathrm{loc}}(\mathbb{R}^2)$ and so, since h_1 is Gaussian, we need only show that the determinant of the covariance matrix of $(\nabla f(s), \nabla f(t), f(s) - f(t))$ is bounded away from zero on T_k. Since f is aperiodic, $f(s) - f(t)$ is a non-degenerate

Gaussian variable for all $s \neq t$, and so by Assumption 2.1.1 (in particular using stationarity of f), the determinant of this covariance matrix is non-zero for all $s \neq t$. Since this determinant is continuous in (s,t) it is bounded away from zero on the compact set T_k. Taking the countable union of T_k for $k \in \mathbb{N}$ shows that almost surely there are no points $s,t \in B(R)$ with $s \neq t$ such that $\nabla f(s) = \nabla f(t) = 0$ and $f(s) = f(t)$.

Let $f_\partial(\theta)$ be a parametrisation of $f|_{\partial B(R)}$. To show that f has no two tangent points at the same level and no tangent points at the level of any critical point, we apply identical arguments to the following two functions:

$$h_2(\theta,\omega) = \begin{pmatrix} f_\partial'(\theta) \\ f_\partial'(\omega) \\ f_\partial(\theta) - f_\partial(\omega) \end{pmatrix} \quad \text{and} \quad h_3(t,\theta) = \begin{pmatrix} \nabla f(t) \\ f_\partial'(\theta) \\ f(t) - f_\partial(\theta) \end{pmatrix}.$$

This completes the proof that $f|_{\overline{B(R)}}$ is a Morse function when f is aperiodic. When f is doubly periodic, we can repeat these arguments with

$$T_k = \left\{(s,t) \in P_k^2 : |s-t| \geq 1/k\right\}$$

where $P_k = \{ay + bz : a, b \in [0, 1-1/k]\}$ and y, z are the periodic vectors of f, to show that $f_{\mathbb{T}}$ is Morse. Finally in the singly periodic case we consider $C_k = [0, y_1 - 1/k] \times [-n, n]$ and $T_k = \{(s,t) \in C_k^2 : |s-t| \geq 1/k\}$ where $y = (y_1, 0)$ is the periodic vector of f. In both of these cases, the choice of domain ensures that $f(t) - f(s)$ is non-degenerate, so that Assumption 2.1.1 can be used. This completes the proof of the first three points of the lemma.

(4). Let L be a line segment of length R and u a unit length vector parallel to L. Applying the Kac-Rice formula (Theorem 1.3.2) to $f|_L$ and using the stationarity of f along with the independence of $\nabla f(t)$ and $\nabla^2 f(t)$, we see that

$$\begin{aligned}\mathbb{E}(N_{\mathrm{tang}}(L)) &= \int_L \mathbb{E}\left(|\partial_{uu} f(t)| \mid \partial_u f(t) = 0\right) p_{\partial_u f(t)}(0)\, dt \\ &= R\, p_{\partial_u f(0)}(0) \mathbb{E}\left(|\partial_{uu} f(0)|\right) =: c_\theta R\end{aligned}$$

where $p_{\partial_u f(0)}$ is the probability density of $\partial_u f(0)$, and $c_\theta < \infty$ since $\partial_{uu} f(0)$ can be expressed as a linear combination of components of the non-degenerate Gaussian vector $\nabla^2 f(0)$ and so has finite moments of all orders.

(5). Applying the same Kac-Rice formula to f, and using the same arguments as above, shows that

$$\begin{aligned}\mathbb{E}(N_{\mathrm{crit}}(B(R))) &= \int_{B(R)} \mathbb{E}\left(\left|\det \nabla^2 f(t)\right| \mid \nabla f(t) = 0\right) p_{\nabla f(t)}(0)\, dt \\ &= \pi R^2 p_{\nabla f(0)}(0) \mathbb{E}\left(\left|\det \nabla^2 f(0)\right|\right) =: cR^2.\end{aligned}$$

(6). We now explicitly parametrise $f|_{\partial B(R)}$ as $g(\theta) := f(R\cos(\theta), R\sin(\theta))$ and note that

$$\begin{aligned} g'(\theta) =& - R\sin(\theta)\partial_1 f(p) + R\cos(\theta)\partial_2 f(p) \\ g''(\theta) =& - R\cos(\theta)\partial_1 f(p) - R\sin(\theta)\partial_2 f(p) \\ &+ R^2\sin^2(\theta)\partial_{11} f(p) - 2R^2\sin(\theta)\cos(\theta)\partial_{12} f(p) + R^2\cos^2(\theta)\partial_{22} f(p) \end{aligned} \tag{2.6}$$

where $p = (R\cos(\theta), R\sin(\theta))$. Since $\nabla f(p)$ and $\nabla^2 f(p)$ are independent, a simple calculation shows that $(g'(\theta), g''(\theta))$ has non-degenerate distribution, so we can apply the Kac-Rice formula (Theorem 1.3.2) to show that

$$\begin{aligned} \mathbb{E}(N_{\text{tang}}(\partial B(R))) &= \mathbb{E}\left(\#\{\theta \in [0, 2\pi] \mid g'(\theta) = 0\}\right) \\ &= \int_0^{2\pi} \mathbb{E}\left(|g''(\theta)| \mid g'(\theta) = 0\right) p_{g'(\theta)}(0)\, d\theta. \end{aligned} \tag{2.7}$$

Since f is stationary

$$\mathrm{Var}\left(g'(\theta)\right) = R^2 \mathbb{E}\left(\left(-\sin(\theta)\,\partial_1 f(0) + \cos(\theta)\,\partial_2 f(0)\right)^2\right).$$

The expectation on the right side of this equation is non-zero for each θ, since $\nabla f(0)$ is non-degenerate, and by compactness is bounded away from zero uniformly in θ. Therefore, by considering the density of a Gaussian random variable, there exists $c_0 > 0$ such that $p_{g'(\theta)}(0) \leq c_0/R$ for all $\theta \in [0, 2\pi]$ and $R > 0$.

Using this observation, along with the fact that $\nabla f(0)$ is independent of $\nabla^2 f(0)$, and substituting (2.6) into (2.7), we see that there exists a constant $c_1 > 0$ independent of θ and R such that

$$\begin{aligned} &\mathbb{E}(N_{\text{tang}}(\partial B(R))) \\ &\qquad \leq c_1 R \int_0^{2\pi} \mathbb{E}\left(|\partial_{11} f(0)| + |\partial_{22} f(0)| + |\partial_{12} f(0)|\right) d\theta \\ &\qquad\quad + c_1 \int_0^{2\pi} \mathbb{E}\left(|\partial_1 f(0)| + |\partial_2 f(0)| \,\Big|\, \sin(\theta)\partial_1 f(p) = \cos(\theta)\partial_2 f(p)\right) d\theta \\ &\qquad = O(R). \end{aligned}$$

$\square$

We now introduce the deterministic relationship between level sets and critical points that is the foundation of our results. We denote the number of critical points of f in $B(R)$ of type h with level in $[\ell_1, \ell_2]$ by $N_{h,R}[\ell_1, \ell_2]$, where $h = m^+, m^-, s^+, s^-$ denotes local maxima, local minima, upper connected saddles and lower connected saddles respectively (for an aperiodic function that is Morse on $\mathbb{R}^2$, Definition 2.1.5

defines the upper/lower connected saddle points; the definitions in the general case will be given in Section 2.3). Let us recall our earlier notation: for a planar function f, we denote the number of components of $\{f \geq \ell\}$ and $\{f = \ell\}$ in $B(R)$ by $N_{ES}(R, \ell)$ and $N_{LS}(R, \ell)$ respectively.

In the case of a singly periodic function $f : \mathbb{R}^2 \to \mathbb{R}$ we also need to define internal and external rectangular tilings of $B(x, R)$ which have a finite number of horizontal and vertical line segments as boundary. Let $(y_1, 0)$ be the periodic vector of f, and let $S(n_1, n_2) = [n_1 y_1, (n_1 + 1)y_1] \times [n_2, n_2 + 1]$ where $n_1, n_2 \in \mathbb{Z}$. We define $B_{\text{int}}(x, R)$ to be the union over $n_1, n_2 \in \mathbb{Z}$ of all $S(n_1, n_2)$ contained in $B(x, R)$. Similarly we define $B_{\text{ext}}(x, R)$ to be the union over $n_1, n_2 \in \mathbb{Z}$ of all $S(n_1, n_2)$ which intersect $B(x, R)$.

Lemma 2.2.5. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a deterministic function. Suppose first that f is aperiodic and assume that f and $f|_{\overline{B(R)}}$ are Morse. Then there exists an absolute constant $c > 0$ such that for each $\ell \in \mathbb{R}$,*

$$N_{ES}(R, \ell) = N_{m^+,R}[\ell, \infty) - N_{s^-,R}[\ell, \infty) + \gamma_{\ell,R} \tag{2.8}$$

and

$$N_{LS}(R, \ell) = N_{m^+,R}[\ell, \infty) - N_{s^-,R}[\ell, \infty) + N_{s^+,R}(\ell, \infty) - N_{m^-,R}(\ell, \infty) + \delta_{\ell,R} \tag{2.9}$$

where

$$\max\{|\gamma_{\ell,R}|, |\delta_{\ell,R}|\} \leq c N_{\text{tang}}(\partial B(R)). \tag{2.10}$$

Suppose instead that $f : \mathbb{R}^2 \to \mathbb{R}$ is doubly periodic with associated parallelogram P and assume that $f_{\mathbb{T}}$ is Morse and $f|_{\partial P}$ has a finite number of local extrema. Then there exists a constant $c_f > 0$ depending only on P, such that, for each $R > 0$ and $\ell \in \mathbb{R}$, (2.8) *and* (2.9) *hold with* (2.10) *replaced by*

$$\max\{|\gamma_{\ell,R}|, |\delta_{\ell,R}|\} \leq c_f \cdot \left(N_{\text{crit}}(P)\, R + N_{\text{tang}}(\partial B(R))\right).$$

Finally, suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is singly periodic with periodic vector $(y, 0)$ and assume that $f_{\mathcal{C}(n)}$ is Morse for each $n \in \mathbb{Z}$ and that $N_{\text{tang}}(\{0\} \times [-2R, 2R]) < \infty$. Then (2.8) *and* (2.9) *still hold with* (2.10) *replaced by*

$$\max\{|\gamma_{\ell,R}|, |\delta_{\ell,R}|\} \leq c_f \Big(N_{\text{tang}}(\partial B_{int}(R)) + N_{\text{tang}}(\partial B_{ext}(R)) + N_{\text{crit}}(B(R + r_f) \backslash B(R - r_f)) \Big)$$

where c_f and r_f are constants that depend only on the periodic vector $y = (y_1, 0)$ of f (so in particular, they are independent of ℓ and R).

The proof of Lemma 2.2.5 is contained in Section 2.4. The error term in the above estimate can be intuitively understood as the result of boundary effects from working with the domain $B(R)$ in the definition of N_{LS} and N_{ES} (i.e. from accounting for excursion/level set components which intersect $\partial B(R)$). Although it appears quite complicated (especially in the case of singly periodic functions), when applied to the Gaussian field f, it is bounded in expectation by $O(R)$ as a result of Lemma 2.2.4.

We are now ready to state the core technical results of this chapter; versions of Theorems 2.1.2 and 2.1.7 that hold for arbitrary (i.e. non-Gaussian) random fields using only the properties contained in Assumption 2.2.3.

Proposition 2.2.6. *Let f be a random field satisfying Assumption 2.2.3. For each $\ell \in \mathbb{R}$*

$$\mathbb{E}\left(N_{ES}(R,\ell)\right) = c_{ES}(\ell) \cdot \pi R^2 + O(R) \quad \textit{and} \quad \mathbb{E}\left(N_{LS}(R,\ell)\right) = c_{LS}(\ell) \cdot \pi R^2 + O(R)$$

as $R \to \infty$, where

$$c_{ES}(\ell) := \frac{1}{\pi}\mathbb{E}(N_{m^+,1}[\ell,\infty) - N_{s^-,1}[\ell,\infty))$$

and

$$c_{LS}(\ell) := \frac{1}{\pi}\mathbb{E}\left(N_{m^+,1}[\ell,\infty) - N_{s^-,1}[\ell,\infty) + N_{s^+,1}(\ell,\infty) - N_{m^-,1}(\ell,\infty)\right).$$

The constants implied by the $O(\cdot)$ notation may depend on the distribution of f but are independent of ℓ. If, in addition, f is ergodic, then

$$\frac{1}{\pi R^2}N_{ES}(R,\ell) \xrightarrow{L^1,a.s.} c_{ES}(\ell) \quad \textit{and} \quad \frac{1}{\pi R^2}N_{LS}(R,\ell) \xrightarrow{L^1,a.s.} c_{LS}(\ell).$$

Remark 2.2.7. Since f is stationary, we could equivalently define $c_{ES}(\ell)$ in Proposition 2.2.6 as

$$c_{ES}(\ell) = \frac{1}{\pi R^2}\mathbb{E}\left(N_{m^+,R}[\ell,\infty) - N_{s^-,R}[\ell,\infty)\right)$$

for any $R > 0$ or make an analogous definition for $c_{LS}(\ell)$.

Together with Lemma 2.2.4 and Proposition 2.1.6, this proposition implies Theorem 2.1.2 and also the first two parts of Theorem 2.1.7.

Proof of Proposition 2.2.6. We prove the statements given for excursion sets. The analogous statements for level sets follow from a near identical argument. As f is stationary, the expected number of critical points of f of a particular type in a domain is proportional to the area of the domain. Since the field satisfies Assumption 2.2.3, we can apply Lemma 2.2.5, and taking expectations yields

$$\mathbb{E}\left(N_{ES}(R,\ell)\right) = c_{ES}(\ell)\cdot \pi R^2 + O(R).$$

Sending $R\to\infty$ proves the first two statements of the proposition.

The remainder of the proof follows the general roadmap of Nazarov and Sodin's arguments in [81]. Suppose that f is ergodic, and let $N^{(u)}_{h,R}$ denote the number of critical points of type h in $B(u,R)$ with level in $[\ell,\infty)$ for $h=m^+,s^+$ and level in (ℓ,∞) for $h=m^-,s^-$ respectively. The 'sandwich estimate' of [81] (Lemma 1) can be slightly altered to show that for any $r\in(0,R)$

$$\frac{1}{\pi R^2}\int_{B(R-r)}\frac{N^{(u)}_{h,r}}{\pi r^2}\,du \le \frac{N^{(0)}_{h,R}}{\pi R^2} \le \frac{1}{\pi R^2}\int_{B(R+r)}\frac{N^{(u)}_{h,r}}{\pi r^2}\,du.$$

Applying Wiener's ergodic theorem, (see [81, Section 6.1]) both of these integrals converge almost surely and in L^1 to the limit $\mathbb{E}\left(N^{(0)}_{h,1}\right)/\pi$. So in particular, $N^{(0)}_{h,R}/(\pi R^2)$ has the same limit, and

$$\frac{1}{\pi R^2}\left(N^{(0)}_{m^+,R} - N^{(0)}_{s^-,R}\right) \xrightarrow{L^1} c_{ES}(\ell).$$

Applying Lemma 2.2.5 with the bound on $\mathbb{E}(|\gamma_{\ell,R}|)$ implied by Assumption 2.2.3 shows that

$$\frac{1}{\pi R^2}\mathbb{E}\left(\left|N_{ES}(R,\ell) - N^{(0)}_{m^+,R} + N^{(0)}_{s^-,R}\right|\right) = o(1)$$

as $R\to\infty$. Combining these results completes the proof of L^1 convergence.

We now extend our notation, by defining $N^{(u)}_{ES,r}$ to be the number of components of $\{f=\ell\}$ in $B(u,r)$. The original 'sandwich estimate' of [81] states that

$$\frac{1}{\pi R^2}\int_{B(R-r)}\frac{N^{(u)}_{ES,r}}{\pi r^2}\,du \le \frac{N^{(0)}_{ES,R}}{\pi R^2} \le \frac{1}{\pi R^2}\int_{B(R+r)}\frac{\tilde{N}^{(u)}_{ES,r}}{\pi r^2}\,du$$

where $\tilde{N}^{(u)}_{ES,r}$ is the number of components of $\{f\ge\ell\}$ which intersect $\overline{B(u,r)}$. Since

$$N^{(u)}_{ES,r} \le \tilde{N}^{(u)}_{ES,r} \le N^{(u)}_{ES,r} + \#\{x\in\partial B(u,r)\mid f(x)=\ell\}$$

we can rearrange this estimate as

$$\begin{aligned}\left|\frac{N^{(0)}_{ES,R}}{\pi R^2} - \frac{1}{\pi R^2}\int_{B(R)}\frac{N^{(u)}_{ES,r}}{\pi r^2}\,du\right| \le& \frac{1}{\pi R^2}\int_{B(R+r)\backslash B(R-r)}\frac{N^{(u)}_{ES,r}}{\pi r^2}\,du \\ &+ \frac{1}{\pi R^2}\int_{B(R+r)}\frac{\#\{x\in\partial B(u,r)\mid f(x)=\ell\}}{\pi r^2}\,du.\end{aligned} \tag{2.11}$$

Applying Lemma 2.2.5 inside the integral term on the left hand side of (2.11) shows that

$$\begin{aligned}
&\left|\frac{N^{(0)}_{ES,R}}{\pi R^2}-\frac{1}{\pi R^2}\int_{B(R)}\frac{N^{(u)}_{m^+,r}}{\pi r^2}-\frac{N^{(u)}_{s^-,r}}{\pi r^2}du\right|\\
&\qquad\leq\frac{1}{\pi R^2}\int_{B(R+r)\backslash B(R-r)}\frac{N^{(u)}_{ES,r}}{\pi r^2}\,du\\
&\qquad\quad+\frac{1}{\pi R^2}\int_{B(R+r)}\frac{\#\{x\in\partial B(u,r)\mid f(x)=\ell\}}{\pi r^2}\,du\\
&\qquad\quad+\frac{c_f}{\pi R^2}\int_{B(R)}\frac{N_{\mathrm{crit}}(P)\cdot r+N_{\mathrm{tang}}(B(u,r))}{\pi r^2}\,du\\
&\qquad\quad+\frac{c_f}{\pi R^2}\int_{B(R)}\frac{N_{\mathrm{tang}}(B_{\mathrm{int}}(u,r))+N_{\mathrm{tang}}(B_{\mathrm{ext}}(u,r))}{\pi r^2}\,du\\
&\qquad\quad+\frac{c_f}{\pi R^2}\int_{B(R)}\frac{N_{\mathrm{crit}}(B(u,r+r_f)\backslash B(u,r-r_f))}{\pi r^2}\,du.
\end{aligned}\tag{2.12}$$

(The upper bound here is a result of adding the upper bounds on the error terms in Lemma 2.2.5 in the cases that f is aperiodic, doubly periodic and singly periodic respectively, so that we can deal with all three cases at once.) From Wiener's ergodic theorem, the integral term within the absolute value signs will converge almost surely to $c_{ES}(\ell)$ and the first integral term on the right hand side will converge almost surely to zero. Applying the same argument to the remaining integral terms shows that they will each converge to a constant, $\Psi_1(r)$, $\Psi_2(r)$, $\Psi_3(r)$ and $\Psi_4(r)$ respectively, where

$$\begin{aligned}
\Psi_1(r)&=\frac{1}{\pi r^2}\mathbb{E}\left(\#\{x\in\partial B(r)\mid f(x)=\ell\}\right),\\
\Psi_2(r)&=\frac{c_f\mathbb{E}(N_{\mathrm{crit}}(P))}{\pi r}+\frac{c_f}{\pi r^2}\mathbb{E}\left(N_{\mathrm{tang}}(B(r))\right),\\
\Psi_3(r)&=\frac{c_f}{\pi r^2}\left(\mathbb{E}\left(N_{\mathrm{tang}}(B_{\mathrm{int}}(r))\right)+\mathbb{E}\left(N_{\mathrm{tang}}(B_{\mathrm{ext}}(r))\right)\right),\\
\Psi_4(r)&=\frac{c_f}{\pi r^2}\mathbb{E}\left(N_{\mathrm{crit}}(B(u,r+r_f)\backslash B(u,r-r_f))\right).
\end{aligned}$$

We now fix $r>0$ and take the limit superior of (2.12) as $R\to\infty$ to show that

$$\limsup_{R\to\infty}\left|\frac{N^{(0)}_{ES,R}}{\pi R^2}-c_{ES}(\ell)\right|\leq\Psi_1(r)+\Psi_2(r)+\Psi_3(r)+\Psi_4(r)\tag{2.13}$$

almost surely. Since the number of boundary points of f at level ℓ is deterministically bounded above by a constant times the number of tangent points, we see that $\Psi_1(r)\leq c_1\Psi_2(r)$ for some $c_1>0$. Assumption 2.2.3 implies that $\Psi_2(r)+\Psi_3(r)+\Psi_4(r)=o(1)$ as $r\to\infty$. Therefore we may take a countable sequence $r_n\to\infty$, such that (2.13) holds almost surely for each r_n and the right hand side of (2.13) becomes arbitrarily small, to show that $N^{(0)}_{ES,R}/(\pi R^2)$ converges almost surely to $c_{ES}(\ell)$. $\square$

To complete the proof of Theorem 2.1.7, it remains only to show the joint continuity of c_{ES} and c_{LS} with respect to the level and spectral measure.

Proof of Theorem 2.1.7 (Joint continuity). By Prokhorov's theorem, we know that the weak-$*$ topology on $\mathcal{P}_c$ is metrisable, and so we can use the sequential definition of continuity for c_{LS} and c_{ES}. Let (ν_n, ℓ_n) be a sequence in $\mathcal{P}_c \times \mathbb{R}$ converging to (ν, ℓ). By the triangle inequality

$$|c_{LS}(\nu_n, \ell_n) - c_{LS}(\nu, \ell)| \leq |c_{LS}(\nu_n, \ell_n) - c_{LS}(\nu_n, \ell)| + |c_{LS}(\nu_n, \ell) - c_{LS}(\nu, \ell)|. \quad (2.14)$$

Theorem 1.3 of [59] states that the second term on the right hand side converges to 0 with n in the special case $\ell = 0$. However the proof of this theorem can be repeated verbatim, using non-centred Gaussian fields, to show that $|c_{LS}(\nu_n, \ell) - c_{LS}(\nu, \ell)| \to 0$.

Now assume that $\ell_n \leq \ell$. By the first part of Theorem 2.1.7, we see that

$$|c_{LS}(\nu_n, \ell_n) - c_{LS}(\nu_n, \ell)| \leq (4/\pi)\mathbb{E}_{\nu_n}\left(N^{(n)}_{\mathrm{crit}}[\ell_n, \ell]\right) \quad (2.15)$$

where $N^{(n)}_{\mathrm{crit}}[\ell_n, \ell]$ denotes the number of critical points of f_n (the Gaussian field with spectral measure ν_n) in the circle of radius 1 with level in $[\ell_n, \ell]$. By the Kac-Rice theorem (Theorem 1.3.2)

$$\begin{aligned}
\mathbb{E}_{\nu_n}\left(N^{(n)}_{\mathrm{crit}}[\ell_n, \ell]\right) &= \int_{B(1)} \mathbb{E}\left(\left|\det \nabla^2 f_n(t)\right| \mathbb{1}_{f_n(t)\in[\ell_n,\ell]} \;\middle|\; \nabla f_n(t) = 0\right) p_{\nabla f_n(t)}(0)\, dt \\
&= \pi\, \mathbb{E}\left(\left|\det \nabla^2 f_n(0)\right| \mathbb{1}_{f_n(0)\in[\ell_n,\ell]}\right) p_{\nabla f_n(0)}(0) \\
&\leq \mathbb{E}\left(\left|\det \nabla^2 f_n(0)\right|^2\right)^{\frac{1}{2}} \mathbb{P}\left(f_n(0) \in [\ell_n, \ell]\right)^{\frac{1}{2}} \frac{1}{2} \frac{1}{\sqrt{\det \mathrm{Cov}(\nabla f_n(0))}}
\end{aligned} \quad (2.16)$$

where we have used the independence of a field and its gradient at a point, along with the Cauchy-Schwarz inequality. It is well known that the covariance structure of a stationary Gaussian field and its derivatives at a point can be expressed in terms of the spectral measure (see [1, Chapter 5]). For example,

$$\mathrm{Var}(\partial_1 f_n(0)) = \int_{\mathbb{R}^2} \lambda_1^2 \, d\nu_n(\lambda)$$

with similar expressions for other derivatives and covariances. Since all spectral measures we are considering are supported on $\overline{B(1)}$, the definition of weak-$*$ convergence implies that the covariance structure associated with each ν_n at the origin converges to that of ν. Specifically, if f denotes the field with spectral measure ν, then

$$\mathrm{Cov}(f_n(0), \nabla f_n(0), \nabla^2 f_n(0)) \to \mathrm{Cov}(f(0), \nabla f(0), \nabla^2 f(0)).$$

Applying this to (2.16), we see that $\det \mathrm{Cov}(\nabla f_n(0))$ is bounded away from 0 in n. Similarly $\mathbb{E}\left(|\det \nabla^2 f_n(0)|^2\right)$ is uniformly bounded above in n. Finally we note that $\mathrm{Cov}(f_n(0))$ is uniformly bounded away from 0, so that choosing ℓ_n sufficiently close to ℓ ensures that $\mathbb{P}\left(f_n(0) \in [\ell_n, \ell]\right)$ is arbitrarily small. An identical argument works for $\ell_n \geq \ell$, and so combining these observations with (2.14)–(2.16) shows that

$$|c_{LS}(\nu_n, \ell_n) - c_{LS}(\nu, \ell)| \xrightarrow{n\to\infty} 0$$

as required. The proof of continuity of $c_{ES}(\nu, \ell)$ is almost identical. Although Theorem 1.3 of [59] is stated only for level sets, the proof of this result can be adapted to apply to excursion sets with no changes of any substance. □

2.3 Critical point densities

In this section we prove Proposition 2.1.6. We begin by giving the general definition for lower and upper connected saddle points (of two-dimensional functions). Let M be a manifold and let $A \subset M$. We say that A is *simple* if it is compact, connected and every loop in A (i.e. every continuous map $h : S^1 \to A$) is M-contractible. We will be solely interested in the case that A is a component of an excursion set of $f : M \to \mathbb{R}$. For such A, and in the case that M is simply connected (e.g. for $M = \mathbb{R}^2$ or $B(R)$), the condition of being simple is just the same as being bounded.

Definition 2.3.1. Let M be a 2-dimensional Riemannian manifold without boundary and let $f : M \to \mathbb{R}$ be a Morse function with a saddle point $x \in M$ such that $f(x) = \ell$. For $c \leq \ell$, let A_c denote the component of $\{f \geq c\}$ containing x and for $c > \ell$, let $A_c = A_\ell \cap \{f \geq c\}$. We say that x is *lower connected* if either of the following conditions hold for all $\epsilon > 0$ sufficiently small:

1. $A_{\ell-\epsilon}$ is simple and $A_{\ell+\epsilon}$ consists of two simple components; or

2. $A_{\ell-\epsilon}$ is not simple but $A_{\ell+\epsilon}$ has a simple component.

We say that x is *upper connected* if it is a lower connected saddle for $-f$.

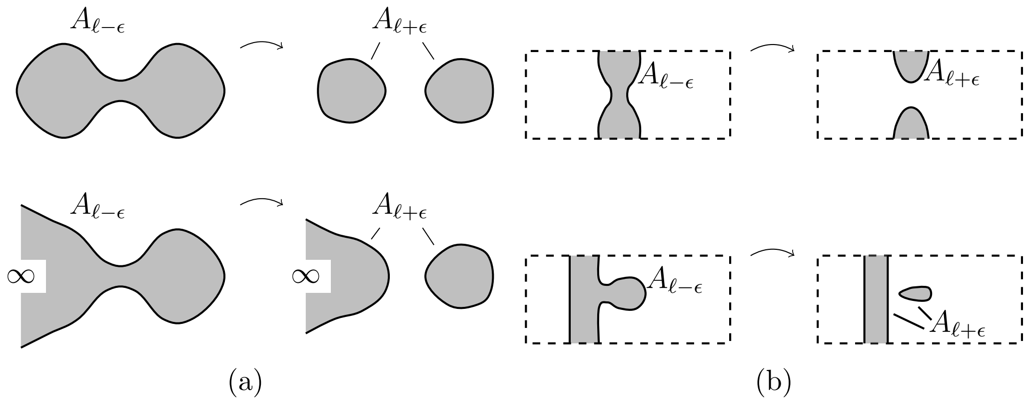


Figure 2.8: Subfigures 2.8a and 2.8b give two examples of the excursion sets (shaded) at a level below and above a lower connected saddle point in $\mathbb{R}^2$ and $\mathbb{T}$ respectively (the dashed rectangles in 2.8b are identified with the torus by the standard quotient relation).

When f is an aperiodic Gaussian field, by Lemma 2.2.4 it will be Morse on $\mathbb{R}^2$ so we may take this as our Riemannian manifold and use the definition of lower/upper connected saddle points directly (we show in Section 2.4 that this definition coincides with that given in Section 2.1). If f is doubly periodic, recall that it is completely specified by its values on the associated parallelogram P, which we identify with the torus. We then say that a saddle point of f is lower connected if it corresponds to a lower connected saddle of $f_{\mathbb{T}}$ by the definition above (see Figure 2.8). Similarly if f is singly periodic, we say that a saddle point of f is lower connected if it corresponds to a lower connected saddle point of $f_{\mathcal{C}}$. Upper connected saddle points are defined analogously. Figure 2.9 shows how lower connected saddle points of a doubly periodic field affect the number of excursion sets of the field on the plane.

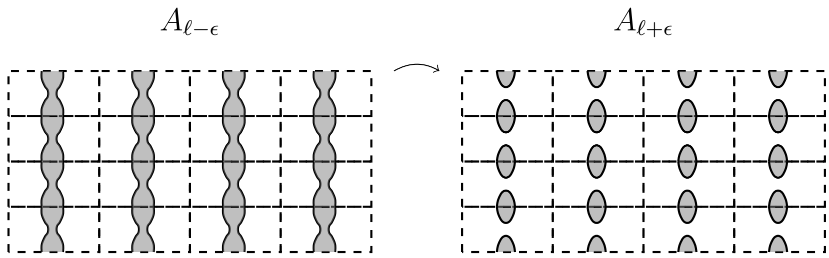


Figure 2.9: Passing through the lower connected saddle points of a doubly periodic function increases the number of compact excursion sets.

The following lemma shows that our above definitions partition the set of saddle points in all cases of interest.

Lemma 2.3.2. *Let f be a random field satisfying Assumption 2.2.3. With probability one, all saddle points of f are either upper connected or lower connected but not both.*

This result is proven in Section 2.4. We are now ready to prove Proposition 2.1.6.

Proof of Proposition 2.1.6. Let f be a Gaussian field satisfying Assumption 2.1.1. We fix a compact $\Omega \subset \mathbb{R}^2$ such that $\partial\Omega$ has finite Hausdorff-1 measure and consider $N_h(\ell)$, the number of critical points of $f|_\Omega$ of type h with value greater than ℓ, where $h = m^-, s, m^+$ corresponds to local minima, saddle points and local maxima respectively. If $(f(0), \nabla^2 f(0))$ has a non-degenerate distribution, then by the Kac-Rice formula (Theorem 1.3.2)

$$\begin{aligned}\mathbb{E}(N_h(\ell)) &= \int_\Omega \mathbb{E}\left(\left|\det \nabla^2 f(t)\right| \mathbb{1}_{\mathrm{Index}\nabla^2 f(t)=i, f(t)>\ell} \;\middle|\; \nabla f(t) = 0\right) p_{\nabla f(t)}(0)\, dt \\ &= \mathrm{Area}(\Omega) p_{\nabla f(0)}(0) \mathbb{E}\left(\left|\det \nabla^2 f(0)\right| \mathbb{1}_{\mathrm{Index}\nabla^2 f(0)=i, f(0)>\ell}\right) \\ &= \mathrm{Area}(\Omega) p_{\nabla f(0)}(0) \int_\ell^\infty \mathbb{E}\left(\left|\det \nabla^2 f(0)\right| \mathbb{1}_{\mathrm{Index}\nabla^2 f(0)=i} \;\middle|\; f(0) = u\right) \phi(u)\, du\end{aligned} \tag{2.17}$$

where i is the index corresponding to h and ϕ denotes the standard Gaussian probability density. The third equality follows from the definition of conditioning on a Gaussian variable. If $(f(0), \nabla^2 f(0))$ has a degenerate distribution then $f(0)$ can be expressed as a linear combination of the elements of $\nabla^2 f(0)$ almost surely. Substituting in this expression for $f(0)$ allows us to apply Kac-Rice and the arguments above to derive (2.17) in this case too. Then, by Gaussian regression,

$$\mathbb{E}\left(\left|\det \nabla^2 f(0)\right| \mathbb{1}_{\mathrm{Index}\nabla^2 f(0)=i} \;\middle|\; f(0) = u\right)$$

is a continuous function of u (see, for example, Proposition 1.2 of [5]). This proves the existence and continuity of the densities p_{m^+}, p_{m^-} and p_s. The fact that $p_{m^+}(x) = p_{m^-}(-x)$ follows from the symmetry of f.

Since $\mathbb{E}(N_s[0, \ell,]) = \int_0^\ell p_s(x) dx$, we know that $\mathbb{E}(N_s[0, \ell])$ is absolutely continuous in ℓ. It is also clear that for any $\ell_1 < \ell_2$, $\mathbb{E}(N_{s^+}[\ell_1, \ell_2]) \leq \mathbb{E}(N_s[\ell_1, \ell_2])$ so $\mathbb{E}(N_{s^+}[0, \ell])$ is absolutely continuous in ℓ. Therefore there exists a function $p_{s^+} : \mathbb{R} \to [0, \infty)$ such that

$$\mathbb{E}(N_{s^+}[0, \ell]) = \int_0^\ell p_{s^+}(x)\, dx.$$

Since $\mathbb{E}(N_s[0, \infty))$ is finite, the monotone convergence theorem shows that $p_{s^+} \in L^1(\mathbb{R})$. By symmetry of the Gaussian distribution, and the definition of lower connected saddles, this also shows the existence of $p_{s^-}(x) = p_{s^+}(-x)$. The fact that $p_{s^-} + p_{s^+} = p_s$ follows from Lemma 2.3.2. $\square$

2.4 Topological lemmas

In this section we prove Lemma 2.2.5 using deterministic topological arguments. We also establish Lemma 2.3.2 and Proposition 2.1.11 using similar methods, thereby completing the proof of all results in the chapter.

To prove Lemmas 2.2.5 and 2.3.2 we require several results from Morse theory which we now introduce. We note that several aspects of this theory require us to work with compact manifolds, and so in the aperiodic and singly periodic cases we work with $B(R)$ and $\mathcal{C}(n)$ rather than directly with $\mathbb{R}^2$ and $\mathcal{C}$; in the doubly periodic case, by contrast, we work directly with the torus $\mathbb{T}$. We also emphasise that we only ever work with the five examples of manifolds just mentioned, so it is sufficient to have them in mind.

Recall that, for a Morse function f defined on a manifold M, the tangent points of f are defined as the critical points of $f|_{\partial M}$.

Theorem 2.4.1 ([43, Theorem 7])**.** *Let M be a compact n-dimensional Riemannian manifold and let $f : M \to \mathbb{R}$ be a Morse function. If f has no critical or tangent points with value in $[a, b]$ then $\{f \geq b\}$ is a deformation retract of $\{f \geq a\}$.*

We define a k-cell to be a copy of the closed unit ball in $\mathbb{R}^k$ and temporarily denote this by B_k. If Y is a topological space, then we define the following operation to be 'attaching a k-cell to Y'. First we find a continuous function $g : \partial B_k \to Y$, then we take the disjoint union $Y \sqcup B_k$ and identify each point in ∂B_k with its image under g. By attaching a 0-cell, we simply mean taking the disjoint union of Y and a single point.

Theorem 2.4.2 ([43, Theorem 8])**.** *Let M be a compact 2-dimensional Riemannian manifold and let $f : M \to \mathbb{R}$ be a Morse function. If t is a critical point of f of index k with $f(t) = \ell$, then for $\epsilon > 0$ sufficiently small, $\{f \geq \ell - \epsilon\}$ is homotopy equivalent to $\{f \geq \ell + \epsilon\}$ with a $(2 - k)$-cell attached. If t is a tangent point and ℓ, ϵ are defined in the same way, then $\{f \geq \ell - \epsilon\}$ is homotopy equivalent to either $\{f \geq \ell + \epsilon\}$ or $\{f \geq \ell + \epsilon\}$ with a k-cell attached for some $k \in \{0, 1, 2\}$.*

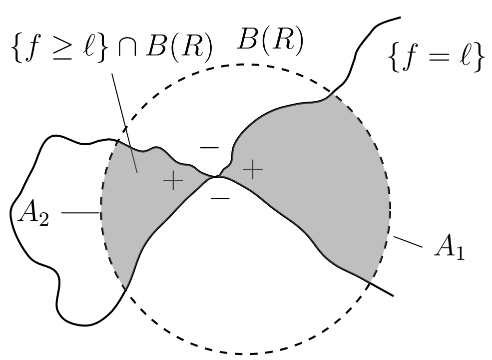
These theorems are proven using methods very similar to those of the standard proofs for manifolds without boundary, which can be found in almost any text on Morse theory.

We now work towards a proof of Lemma 2.2.5. First we will need another definition which, in the case of aperiodic or singly periodic fields, identifies a subset of the upper/lower connected saddle points which have unfavourable topological properties (for our purposes). Recall that if M is a manifold, then $A \subset M$ is said to be simple if A is compact, connected and every loop in A is M-contractible. In the case that $M = \mathbb{R}^2$ or $B(R)$ and A is an excursion set component, this is just the condition that A is bounded.

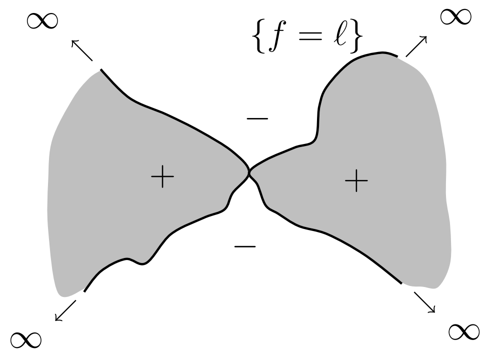
Definition 2.4.3. Let M_∞ be a Riemannian 2-manifold without boundary, such that $f_\infty \in C^2_{\mathrm{loc}}(M_\infty)$ is a Morse function on M_∞, and let M be a compact submanifold of M_∞ with boundary, such that $f := f_\infty|_M$ is a Morse function on M. Let $x_0 \in M$ be a saddle point of f at level ℓ and let A_c be defined as in Definition 2.3.1 for f_∞. If x_0 is lower connected, then we say that it is *four-arm in M* if, for all $\epsilon > 0$ sufficiently small, all of the simple components of $A_{\ell+\epsilon}$ intersect ∂M. Similarly, we say that an upper connected saddle is *four-arm in M* if it satisfies the previous condition for $-f$ (and $-\ell$).

We say that x_0 is an *infinite-four-arm saddle point* if $A_{\ell+\epsilon}$ has two unbounded components for all $\epsilon > 0$ sufficiently small. (We also say that infinite-four-arm saddle points are four-arm in M.)

Remark 2.4.4. In the proof of Lemma 2.3.2, we will show that when $M_\infty = \mathbb{R}^2$ (the most important case for our analysis), the level set at the level of an infinite-four-arm saddle takes the form shown in Figure 2.10b. This corresponds to the way an 'infinite-four-arm event' is typically defined in the percolation literature: as two disjoint paths joining a point to infinity which are separated by two 'dual' paths joining the same point to infinity. For other choices of M_∞ (such as $M_\infty = \mathcal{C}$) the level set at the level of an infinite-four-arm saddle may look quite different, so that this terminology is less intuitive.



(a) A lower connected saddle point that is four-arm in $\overline{B(R)}$.



(b) An infinite-four-arm saddle point in $\mathbb{R}^2$.

Figure 2.10: Different types of four-arm saddle points.

Let us explain the importance of four-arm saddles. If f is aperiodic, then its lower connected saddle points are defined so as to correspond to an increase in the number of excursion set components in $\mathbb{R}^2$. However if such a saddle point is four-arm in $\overline{B(R)}$, then this increase cannot be observed from inside $\overline{B(R)}$ since the excursion sets that are created when passing through the saddle intersect $\partial B(R)$ (see Figure 2.10a). The case when f is singly periodic is similar (in the case that f is doubly periodic we do not have to worry about such saddles). Infinite-four-arm saddle points will be relevant when we prove Lemma 2.3.2. Fortunately we can control the number of four-arm saddles which occur in terms of the boundary behaviour of f.

Lemma 2.4.5. *Let M_∞ be a Riemannian 2-manifold without boundary such that $f_\infty \in C^2_{loc}(M_\infty)$. Let M be a compact submanifold of M_∞ with boundary such that $f := f_\infty|_M$ is Morse. Let $N_{\text{4-arm}}(M)$ be the number of saddle points of f_∞ contained in M which are four-arm in M (including infinite-four-arm). Then*

$$N_{\text{4-arm}}(M) \leq 3N_{\text{tang}}(\partial M).$$

Intuitively this bound holds because as we raise the level past a saddle which is four-arm in $\overline{B(R)}$, we separate (at least) two components of $\{f \geq \ell\} \cap \partial B(R)$ from inside $\{f \geq \ell\} \cap \overline{B(R)}$ (in Figure 2.10a the two boundary components A_1 and A_2 will be in different excursion set components above level ℓ). The total number of such separations which may occur is bounded above by the number of boundary excursion components at different levels, which in turn is bounded by the number of tangent points. This is formalised in the proof below.

Proof. We describe an algorithm which will create an injective mapping from the lower connected saddle points which are four-arm in M to the tangent points of f in ∂M. Let $x_1, \dots, x_m$ denote all such saddle points arranged in order of increasing level, i.e. $f(x_i) < f(x_{i+1})$. We fix $\epsilon > 0$ such that there are no critical or tangent points (other than x_i) with value in $[f(x_i) - \epsilon, f(x_i) + \epsilon]$ for any i and these intervals are disjoint for different i. For each $i = 1, \dots, m$, we define $S_1^{(i)}, \dots, S_{m_i}^{(i)}$ to be the simple components of $\{f \geq f(x_i) + \epsilon\}$ which intersect ∂M.

We proceed by induction. Let A_- denote the component of $\{f \geq f(x_1) - \epsilon\}$ containing x_1. Since x is lower connected and four-arm in M, A_- will contain some $S_j^{(1)}$ (it may contain two such elements: see Figure 2.11), and so we make a preliminary association between x_1 and $S_j^{(1)}$.

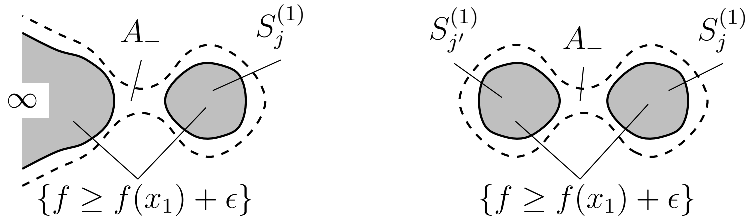


Figure 2.11: Since x_1 is a lower connected saddle point, A_- may contain either one (left) or two (right) simple component of $\{f \geq f(x_1) + \epsilon\}$.

We now assume inductively that for each $x_1, \dots, x_n$ we have either a preliminary association with some $S_j^{(n)}$ (which differs across points) or a final association to some tangent point with value less than $f(x_n)$.

We consider x_{n+1} and let A_- denote the component of $\{f \geq f(x_{n+1}) - \epsilon\}$ containing x_{n+1}. First suppose that A_- is contained in some $S_j^{(n)}$ which has a preliminary association with some x_i (where $i \leq n$). Then A_- must be simple, as it is a subset of $S_j^{(n)}$, and so, by the definition of a lower connected four-arm saddle point, this means that A_- must contain two different components $S_p^{(n+1)}$ and $S_q^{(n+1)}$. We then make a preliminary association between x_{n+1} and $S_p^{(n+1)}$ and make a new preliminary association between x_i and $S_q^{(n+1)}$ (so we remove the old association between x_i and $S_j^{(n)}$). If A_- is contained in some $S_j^{(n)}$ which does not have a preliminary association or if A_- is not contained in any $S_j^{(n)}$ then we choose some $S_p^{(n+1)}$ contained in A_- (which must exist since x_{n+1} is four-arm in M) and make a preliminary association between this and x_{n+1}.

Now suppose that x_i, where $i \leq n$, has a preliminary association with $S_j^{(n)}$. If the maximum level of any point in $S_j^{(n)}$ is less than $f(x_{n+1})$ then we make a final association between x_i and the highest such point in $\partial M \cap S_j^{(n)}$, which is by definition a tangent point. We also remove the preliminary association between x_i and $S_j^{(n)}$. Otherwise, $S_j^{(n)}$ must be a superset of $S_k^{(n+1)}$ for some k, and we then make a new preliminary association between x_i and $S_k^{(n+1)}$. We perform this process for all preliminary associations which remain after the step described in the previous paragraph, which completes the inductive step. This algorithm will cease after the m-th step, at which stage if a point x_i has a preliminary association with some $S_j^{(n+1)}$, we make a final association between x_i and the highest tangent point of $S_j^{(n+1)}$.

The end result of this algorithm is an injective mapping, defined by the final associations, from the set of lower connected saddles which are four-arm in M to the tangent points of f. An identical argument applied to $-f$ creates such a map for the upper connected saddles which are four-arm in M. A very similar algorithm can be applied to the infinite-four-arm saddles of f, except that the inductive step is slightly simpler. Combining these three results gives the statement of the lemma (with the explicit factor three). $\square$

Our next result is a preliminary version of Lemma 2.2.5 for Morse functions on manifolds without boundary. This setting allows us to prove the desired relationship between the number of excursion set components and critical points in the simpler case that all critical points have different levels.

Lemma 2.4.6. *Let M_∞ be a Riemannian 2-manifold without boundary such that $f \in C^2_{loc}(M_\infty)$ is Morse. Let M be a compact submanifold of M_∞ and $f := f_\infty|_M$ be Morse. For $\ell \in \mathbb{R}$, let $N_{\mathrm{simple}}(\ell)$ be the number of simple components of $\{f \geq \ell\}$ which do not intersect the boundary of M and for $\ell_1 < \ell_2$, let $N_{m^+}[\ell_1, \ell_2)$ and $N_{s^-}[\ell_1, \ell_2)$, be the number of local maxima and lower connected saddle points respectively of f with level in $[\ell_1, \ell_2)$. Then*

$$N_{\mathrm{simple}}(\ell_1) - N_{\mathrm{simple}}(\ell_2) = N_{m^+}[\ell_1, \ell_2) - N_{s^-}[\ell_1, \ell_2) + \zeta \tag{2.18}$$

where

$$|\zeta| \leq N_{\text{4-arm}}(M, [\ell_1, \ell_2)) + 2N_{\mathrm{tang}}(\partial M, [\ell_1, \ell_2)) \leq 5N_{\mathrm{tang}}(\partial M) \tag{2.19}$$

and $N_{\text{4-arm}}(M, [\ell_1, \ell_2))$, $N_{\mathrm{tang}}(\partial M, [\ell_1, \ell_2))$ denote the number of four-arm saddles (i.e. those which are four-arm in M, including infinite-four-arm) and tangent points of f respectively with level in $[\ell_1, \ell_2)$. So in particular, if M has no boundary then $\zeta = 0$.

The proof involves applying the Morse theorems to each critical/tangent point to derive the change in the number of simple components at the corresponding levels and then summing these changes.

Proof. First suppose that f has no critical or tangent points with level in $[a, b]$. By Theorem 2.4.1, $\{f \geq b\}$ is a deformation retract of $\{f \geq a\}$ under some map $h : \{f \geq a\} \times [0, 1] \to \{f \geq a\}$. In particular, this is also true for each component of $\{f \geq b\}$ and the component of $\{f \geq a\}$ in which it is contained. Since h is a homotopy, the number of simple components is the same in each set. If A is a component of $\{f \geq a\}$ which intersects ∂M, then we claim that $A \cap \{f \geq b\}$ also intersects ∂M. Suppose this is not true, then by considering the infimum ℓ^* of $\ell \in [a, b]$ such that $A \cap \{f \geq \ell\}$ does not intersect ∂M and taking a sequence of points in $A \cap \{f \geq \ell^* - 1/n\} \cap \partial M$ we see that $A \cap \{f \geq \ell^*\}$ contains a tangent point at level $\ell^* \in [a, b]$, which is a contradiction. Combining all these observations, we see that N_{simple} is constant on intervals which contain no critical or tangent points.

Now let x be a critical or tangent point of f at level c and take $\epsilon > 0$ small enough to apply Lemma 2.4.2. We consider in turn the different types of critical or tangent point and calculate $N_{\text{simple}}(c-\epsilon) - N_{\text{simple}}(c+\epsilon)$. Let $A_{c-\epsilon}$ denote the component of $\{f \geq c-\epsilon\}$ containing x and $A_{c+\epsilon} = A_{c-\epsilon} \cap \{f \geq c+\epsilon\}$. We note that to determine $N_{\text{simple}}(c-\epsilon) - N_{\text{simple}}(c+\epsilon)$ it is enough to consider $A_{c-\epsilon}$ and $A_{c+\epsilon}$, since by the arguments in the previous paragraph, the number of simple components of $\{f \geq \ell\}$ in $M \backslash A_{c-\epsilon}$ which do not intersect ∂M will be constant as ℓ varies in $[c-\epsilon, c+\epsilon]$.

If x is a local maximum, then by Lemma 2.4.2, $A_{c-\epsilon}$ is homotopy equivalent to $A_{c+\epsilon}$ with a 0-cell attached. So in particular $\{f \geq c-\epsilon\}$ has one more component than $\{f \geq c+\epsilon\}$, and this extra component is M-contractible. Since the extra component contains a local maximum, (which cannot be in ∂M) for $\epsilon > 0$ sufficiently small, this component is disjoint from ∂M and so $N_{\text{simple}}(c-\epsilon) - N_{\text{simple}}(c+\epsilon) = 1$.

If x is a local minimum, then $A_{c-\epsilon}$ is homotopy equivalent to $A_{c+\epsilon}$ with a 2-cell attached. Attaching a 2-cell does not change the number of components of $\{f \geq c+\epsilon\}$, and does not affect whether the component it is attached to is simple or not. Once again since the local minimum is not in ∂M, restricting ϵ sufficiently small ensures that the attached 2-cell does not intersect ∂M and so $N_{\text{simple}}(c-\epsilon) - N_{\text{simple}}(c+\epsilon) = 0$.

Next we suppose that x is a saddle point, so that by Lemma 2.4.2, $A_{c-\epsilon}$ is homotopy equivalent to $A_{c+\epsilon}$ with a 1-cell attached. So in particular, $A_{c+\epsilon}$ consists of either one or two components. First we note that if $A_{c-\epsilon}$ is simple and does not intersect ∂M, then any components of $A_{c+\epsilon}$ have both these properties, therefore $N_{\text{simple}}(c-\epsilon) - N_{\text{simple}}(c+\epsilon) \leq 0$. If $A_{c-\epsilon}$ is simple but intersects ∂M, then by repeating the sequential argument in the first paragraph of this proof we see that $A_{c+\epsilon}$ must intersect ∂M and so cannot consist of two simple components disjoint from the boundary. Furthermore, if $A_{c-\epsilon}$ is not simple, then by the above homotopy, $A_{c+\epsilon}$ cannot consist of two simple components. Combining these two observations show that $N_{\text{simple}}(c-\epsilon) - N_{\text{simple}}(c+\epsilon) \geq -1$. So the number of simple components disjoint from ∂M is either constant or increases by one on passing through the saddle point. If x is not a lower connected saddle, then by definition $N_{\text{simple}}(c-\epsilon) - N_{\text{simple}}(c+\epsilon) = 0$, if x is a lower connected saddle which is not four-arm in M, then $N_{\text{simple}}(c-\epsilon) - N_{\text{simple}}(c+\epsilon) = -1$ and if x is a lower connected saddle which is four-arm in M, then $N_{\text{simple}}(c-\epsilon) - N_{\text{simple}}(c+\epsilon) = 0$.

Finally let x be a tangent point, so that $A_{c-\epsilon}$ is homotopy equivalent to $A_{c+\epsilon}$ with a k-cell attached for some $k \in \{0, 1, 2\}$. In each case, $A_{c+\epsilon}$ has at most two simple components disjoint from ∂M, so $|N_{\text{simple}}(c-\epsilon) - N_{\text{simple}}(c+\epsilon)| \leq 2$.

Now suppose that there are no critical or tangent points at levels ℓ_1 and ℓ_2. Then $N_{\text{simple}}(\ell_1)-N_{\text{simple}}(\ell_2)$ is equal to the sum of the finite number of jumps $N_{\text{simple}}(c-\epsilon)-N_{\text{simple}}(c+\epsilon)$ at each level $c \in (\ell_1, \ell_2)$ with a critical/tangent point. By the analysis of different types of critical/tangent point above, we deduce (2.18) along with the first inequality of (2.19). The second inequality of (2.19) follows immediately from Lemma 2.4.5.

In the case that there is a critical/tangent point at level ℓ_1 or ℓ_2, we note that

$$\{f \geq \ell\} = \bigcap_{\epsilon>0} \{f \geq \ell - \epsilon\}$$

and that the intersection of a decreasing family of compact, connected sets is connected, so that $\{f \geq \ell\}$ and $\{f \geq \ell - \epsilon\}$ have the same number of components for ϵ sufficiently small. Repeating the earlier argument for tangent points shows that these sets have the same number of components which do not intersect ∂M. Then, if f has a critical point x at level ℓ, applying the Morse lemma ([72, Lemma 2.2]) on a neighbourhood of x, and Theorem 2.4.1 outside this neighbourhood, shows that $N_{\text{simple}}(\ell - \epsilon) = N_{\text{simple}}(\ell)$. (If no such point exists, we simply apply Theorem 2.4.1 to M.) Therefore we can apply the above arguments to deduce (2.18) and (2.19) for the levels $\ell_1 - \epsilon$ and $\ell_2 - \epsilon$ such that N_{simple} is constant on $[\ell_1 - \epsilon, \ell_1]$ and $[\ell_2 - \epsilon, \ell_2]$. Since there are no critical/tangent points with level in $[\ell_1 - \epsilon, \ell_1)$ or $[\ell_2 - \epsilon, \ell_2)$, this proves the lemma. □

With the above preliminary lemma, the proof of Lemma 2.2.5 in the case of aperiodic functions is straightforward.

Proof of Lemma 2.2.5 in the case of aperiodic functions. Let f be an aperiodic function satisfying the assumptions of Lemma 2.2.5. Applying Lemma 2.4.6 with $M = \overline{B(R)}$ and $M_\infty = \mathbb{R}^2$ and taking $\ell_2 \to \infty$ gives exactly the stated relationship for excursion sets. To complete the proof, we prove a corresponding relationship for level sets. We fix a level ℓ and let $f_R = f|_{\overline{B(R)}}$, we construct a graph on the vertex set

$$V := \{\text{Components of } \{f_R \geq \ell\}\} \cup \{\text{Components of } \{f_R \leq \ell\}\}$$

by declaring two vertices to be joined by an edge if they have non-empty intersection. Clearly each edge corresponds to a component of $\{f_R = \ell\}$. This graph is acyclic and connected, so is by definition a tree. Hence the number of edges is one less than the number of vertices:

$$\begin{aligned}\#\{\text{Components of } \{f_R = \ell\}\} =& \#\{\text{Components of } \{f_R \geq \ell\}\} \\ &+ \#\{\text{Components of } \{f_R \leq \ell\}\} - 1.\end{aligned}$$

The number of components of $\{f_R = \ell\}$ which intersect $\partial B(R)$ is bounded above by the number of tangent points of f in $\partial B(R)$, and the same bound holds for the components of $\{f_R \geq \ell\}$ and $\{f_R \leq \ell\}$. Therefore we can express the equation above as

$$N_{LS}(f, R, \ell) = N_{ES}(f, R, \ell) + N_{ES}(-f, R, -\ell) + \delta_{\ell,R}$$

where $|\delta_{\ell,R}| \leq 4N_{\mathrm{tang}}(\partial B(R))$. Applying the first part of this lemma to each of the N_{ES} terms here then completes the proof. □

For the periodic cases, the argument is a little more technical since we cannot apply Lemma 2.4.6 to $B(R)$ directly. Instead, we tile $B(R)$ with translated parallelograms or rectangles, apply Lemma 2.4.6 to $f_{\mathbb{T}}$ or $f_{\mathcal{C}(n)}$ on each translated domain and aggregate the results.

Proof of Lemma 2.2.5 in the case of periodic functions. Let f be a doubly or singly periodic function satisfying the assumptions of Lemma 2.2.5. It suffices to prove the excursion set relationship, since the level set relationship will follow by the same argument as in the aperiodic case.

Suppose f is doubly periodic with periodic vectors $y, z \in \mathbb{R}^2$ and associated parallelogram P. By assumption, $f_{\mathbb{T}}$ is a Morse function almost surely. Suppose that $\{f_{\mathbb{T}} \geq \ell\}$ has N_{simple} simple components.

Let A be a component of $\{f \geq \ell\}$ and A' be the corresponding component of $\{f_{\mathbb{T}} \geq \ell\}$, we wish to show that A is bounded if and only if A' is simple. If A' contains a non-$\mathbb{T}$-contractible loop, then there exists a path in A joining some point x to $x + n_1 y + n_2 z$ for some integers n_1, n_2 not both equal to zero. Since f is invariant under translations by y, z, the translation of this path joining $x + n_1 y + n_2 z$ to $x + 2n_1 y + 2n_2 z$ will be contained in $\{f \geq \ell\}$. By iterating this argument and concatenating the paths we find an unbounded path in $\{f \geq \ell\}$ started at $x \in A$. Therefore if A is compact, we see that A' cannot contain a non-$\mathbb{T}$-contractible loop, and so it is simple.

Since $f|_{\partial P}$ has a finite number of local extrema, $\{f \geq \ell\} \cap \partial P$ has a finite number of components. Therefore, if A intersects the boundary of a translated parallelogram $P + n_1 y + n_2 z$ where $n_1, n_2 \in \mathbb{Z}$, then it must contain the translation of one of these boundary components. So if A is unbounded, it must contain a path joining two translated versions of the same boundary component. We can choose this path so that it joins two points which differ by $m_1 y + m_2 z$ for some $m_1, m_2 \in \mathbb{Z}$ which are not both zero. We note that the quotient of this path is a loop on the torus. Considering the quotient map as a covering map, it is a standard result (see, for example, [20, Corollary 3.6]) that the lift of any contractible loop to a covering space must also be a contractible loop. We therefore conclude that the quotient of the path we have constructed is a non-$\mathbb{T}$-contractible loop and so A' is not simple.

We can tile $\mathbb{R}^2$ with the translated parallelograms $P+n_1y+n_2z$ where $n_1, n_2 \in \mathbb{Z}$. We associate each bounded component A of $\{f \geq \ell\}$ to a particular translation $P+n_1y+n_2z$ of P such that $(P+n_1y+n_2z)\cap A \neq \emptyset$, N_{simple} components are mapped to each translated parallelogram and if A is associated with P, then $A+n_1y+n_2z$ is associated with $P+n_1y+n_2z$. If A is a component of $\{f \geq \ell\}$ contained in $B(R)$ then it must be associated with a parallelogram which intersects $B(R)$, therefore

$$N_{ES}(R,\ell) \leq N_{\text{simple}} \cdot \#\{\text{Translations of } P \text{ in } B(R+d)\} \tag{2.20}$$

where $d = \operatorname{diam}(P)$. Similarly, each compact component of $\{f \geq \ell\}$ which is associated with a parallelogram inside $B(R)$ must be in $B(R)$ unless it intersects $\partial B(R)$, but the number of such intersections is bounded above by the number of tangent points of f to $B(R)$. Therefore

$$N_{\text{simple}} \cdot \#\{\text{Translations of } P \text{ in } B(R)\} - N_{\text{tang}}(\partial B(R)) \leq N_{ES}(R,\ell) \tag{2.21}$$

The number of translated parallelograms contained in the ball of a given radius can be approximated by a generalisation of Gauss' circle problem. It is shown in [62] that

$$\#\{\text{Translations of } P \text{ in } B(R)\} = \frac{\pi R^2}{\operatorname{Area}(P)} + o(R)$$

as $R \to \infty$. We also note that $N_{\text{simple}} \leq N_{\text{crit}}(P)$ since each component must contain a critical point. Combining these two facts with (2.20) and (2.21),

$$N_{ES}(R,\ell) = N_{\text{simple}} \cdot \frac{\pi R^2}{\operatorname{Area}(P)} + O(N_{\text{crit}}(P) \cdot R + N_{\text{tang}}(\partial B(R)))$$

as $R \to \infty$, where the constant implied by the $O(\cdot)$ notation depends only on P.

Now suppose that $\{f_{\mathbb{T}} \geq \ell\}$ contains m local maxima. Reasoning as above, it is clear that

$$\begin{aligned} m \cdot \#\{\text{Translations of } P \text{ in } B(R)\} &\leq N_{m^+,R}[\ell,\infty) \\ &\leq m \cdot \#\{\text{Translations of } P \text{ in } B(R+d)\} \end{aligned}$$

and so

$$N_{m^+,R}[\ell,\infty) = m \cdot \frac{\pi R^2}{\operatorname{Area}(p)} + O(N_{\text{crit}}(P) \cdot R)$$

with an analogous result holding for lower connected saddles. Applying Lemma 2.4.6 to $f_{\mathbb{T}}$ (i.e. with the setting $M_\infty = M = \mathbb{T}$ and $\ell_2 \to \infty$) then shows that

$$N_{ES}(R,\ell) = N_{m^+,R}[\ell,\infty) - N_{s^-,R}[\ell,\infty)) + O(N_{\mathrm{crit}}(P)\cdot R + N_{\mathrm{tang}}(\partial B(R)))$$

as $R \to \infty$ where the constant in the $O(\cdot)$ notation is independent of ℓ, as required (although it may depend on P).

Now suppose that f is singly periodic with periodic vector $y = (y_1, 0)$ and define $\mathcal{R}(n) = [ny_1, (n+1)y_1) \times \mathbb{R}$. By assumption, f almost surely has a finite number of tangent points in $[0, y_1] \times \{n_1\}$ or $\{n_2 y_1\} \times [-R, R]$ for any $n_1, n_2 \in \mathbb{Z}$ or $R > 0$. Repeating the argument from the doubly periodic case then shows that the compact components of $\{f \geq \ell\}$ correspond precisely to the simple components of $\{f_{\mathcal{C}} \geq \ell\}$.

As before, we can associate each compact component of $\{f \geq \ell\}$ with a translated rectangle $\mathcal{R}(n)$ for $n \in \mathbb{Z}$, in such a way that if A is a compact component of $\{f \geq \ell\}$ which is mapped to $\mathcal{R}(n)$ then A intersects $\mathcal{R}(n)$, and if $A' = A + y$ then A' maps to $\mathcal{R}(n+1)$.

We recall our earlier definitions: if $S(n_1, n_2) = [n_1 y_1, (n_1+1)y_1] \times [n_2, n_2+1]$ where $n_1, n_2 \in \mathbb{Z}$, then we define $B_{\mathrm{int}}(x, R)$ to be the union over $n_1, n_2 \in \mathbb{Z}$ of all $S(n_1, n_2)$ contained in $B(x, R)$. Similarly we define $B_{\mathrm{ext}}(x, R)$ to be the union over $n_1, n_2 \in \mathbb{Z}$ of $S(n_1, n_2)$ which intersect $B(x, R)$. We now fix $R > 0$ and define m_n for $n \in \mathbb{Z}$ to be the largest integer such that

$$[ny_1, (n+1)y_1] \times [-m_n, m_n] \subset B(R)$$

and note that under the standard quotient relationship, this rectangle can be identified with $\mathcal{C}(m_n)$.

Let $N_{\mathrm{simple}}(f_{\mathcal{C}(m_n)}, \ell)$ denote the number of simple components of $\{f|_{\mathcal{C}(m_n)} \geq \ell\}$ which do not intersect $\partial\mathcal{C}(m_n)$. By the mapping described above, there will be at least $N_{\mathrm{simple}}(f_{\mathcal{C}(m_n)}, \ell)$ compact components of $\{f \geq \ell\}$ which are associated with $\mathcal{R}(n)$ and intersect $[ny_1, (n+1)y_1] \times [-m_n, m_n]$. If any of these components intersect $\partial B_{\mathrm{int}}(R)$, then they will do so at distinct components of $\{f|_{\partial B_{\mathrm{int}}(R)} \geq \ell\}$ and so in particular the number of such components intersecting the boundary is at most the number of tangent points in $\partial B_{\mathrm{int}}(R)$. Therefore

$$N_{ES}(R,\ell) \geq \sum_n N_{\mathrm{simple}}(f_{\mathcal{C}(m_n)}, \ell) - N_{\mathrm{tang}}(\partial B_{\mathrm{int}}(R))$$

We can then apply Lemma 2.4.6 to $f_{\mathcal{C}(m_n)}$ (i.e. with the setting $M_\infty = \mathcal{C}$ and $M = \mathcal{C}(m_n)$) for each n to get this inequality in terms of local maxima and lower connected saddles points. Let $r = \max\{\sqrt{2}, y_1\}$. Then, since $B_{\text{int}}(R)$ covers $B(R-r)$, we see that

$$N_{ES}(R,\ell) \geq N_{m^+}(R,\ell) - N_{s^-}(R,\ell) - 6N_{\text{tang}}(\partial B_{\text{int}}(R)) - N_{\text{crit}}(B(R)\backslash B(R-r))$$

This gives the required lower bound for $N_{ES}(R,\ell)$. The upper bound follows from a very similar argument. First we define M_n to be the largest integer such that

$$[ny_1,(n+1)y_1]\times[M_n-1,M_n]\subset B_{\text{ext}}(R)$$

so that $B(R)$ is covered by the finite union of rectangles $[ny_1,(n+1)y_1]\times[-M_n,M_n]$. Then

$$N_{ES}(R,\ell)\leq\sum_n N_{\text{simple}}(f_{\mathcal{C}(M_n)},\ell)$$

and by applying Lemma 2.4.6 to each cylinder we see that

$$N_{ES}(R,\ell) \leq N_{m^+}(R,\ell) - N_{s^-}(R,\ell) + 5N_{\text{tang}}(\partial B_{\text{ext}}(R)) + N_{\text{crit}}(B(R+r)\backslash B(R))$$

as required. □

The following refinement of Lemma 2.2.5 is not needed in this chapter, but will be useful in Chapter 4.

Corollary 2.4.7. *Let $f:\mathbb{R}^2\to\mathbb{R}$ be an aperiodic deterministic function such that f and $f|_{\overline{B(R)}}$ are Morse. Let $\ell_1<\ell_2$ and assume that $f|_{\overline{B(R)}}$ has no critical or tangent points at level ℓ_1 or ℓ_2, then*

$$N_{ES}(R,\ell_1) - N_{ES}(R,\ell_2) = N_{m^+,R}[\ell_1,\ell_2) - N_{s^-,R}[\ell_1,\ell_2) + \gamma_{\ell_1,\ell_2,R} \tag{2.22}$$

and

$$\begin{aligned} N_{LS}(R,\ell_1) - N_{LS}(R,\ell_2) =& N_{m^+,R}[\ell_1,\ell_2) - N_{s^-,R}[\ell_1,\ell_2) \\ &+ N_{s^+,R}(\ell_1,\ell_2] - N_{m^-,R}(\ell_1,\ell_2] + \delta_{\ell_1,\ell_2,R} \end{aligned} \tag{2.23}$$

where

$$\begin{aligned} \max\{|\gamma_{\ell_1,\ell_2,R}|,|\delta_{\ell_1,\ell_2,R}|\} &\leq 2N_{\text{4-arm}}(B(R),[\ell_1,\ell_2]) + 4N_{\text{tang}}(\partial B(R),[\ell_1,\ell_2]) \\ &\leq 10N_{\text{tang}}(\partial B(R)). \end{aligned} \tag{2.24}$$

If f is instead an aperiodic Gaussian field satisfying Assumption 2.1.1, then for each $R>0$ and $\ell_1<\ell_2$, with probability one (2.22), (2.23) *and* (2.24) *hold.*

The key difference between this result and Lemma 2.2.5 in the aperiodic case, is that the error term is bounded by the number of critical/tangent points in a finite range of levels. The arguments needed to prove these two results (Lemma 2.2.5 for aperiodic functions and Corollary 2.4.7) are very similar, however because of the difference in the error terms and the assumption that ℓ_1 and ℓ_2 are not critical levels, it turns out to be easier to give separate proofs rather than trying to combine them.

Proof. If f is an aperiodic random field satisfying Assumption 2.1.1, then applying Bulinskaya's lemma (Lemma A.2.1) to

$$(f-\ell,\nabla f):\overline{B(R)}\to\mathbb{R}^3 \qquad \text{and} \qquad (f-\ell,\nabla_\partial f):\partial B(R)\to\mathbb{R}^2$$

(where ∇_∂ denotes the derivative in the direction tangent to the boundary) for $\ell=\ell_1,\ell_2$ shows that $f|_{\overline{B(R)}}$ almost surely has no critical/tangent points at level ℓ_1 or ℓ_2. Therefore the statement of the present corollary for random fields follows from the deterministic result and Lemma 2.2.4. The deterministic statement of the corollary for excursion sets is simply an application of Lemma 2.4.6 (with $M_\infty=\mathbb{R}^2$ and $M=\overline{B(R)}$). It remains to prove the result for level sets.

We fix a deterministic f and $\ell_1<\ell_2$ satisfying the conditions of the corollary. Since $f|_{\overline{B(R)}}$ has no critical/tangent points at level ℓ_1 or ℓ_2, each component of the level set $\{f=\ell\}$ (for $\ell=\ell_1,\ell_2$) contained in $B(R)$ is a closed C^2-curve which does not intersect $\partial B(R)$. Each such curve must 'surround' a component of either $\{f\geq\ell\}$ or $\{f\leq\ell\}$ contained in $B(R)$ (i.e. the curve will be the part of the boundary of the excursion set component which separates its interior from infinity). Moreover, every component of $\{f\geq\ell\}$ or $\{f\leq\ell\}$ contained in $B(R)$ will have a component of $\{f=\ell\}$ contained in $B(R)$ as its outer boundary. We therefore have

$$\begin{aligned}N_{LS}(R,\ell_1)-N_{LS}(R,\ell_2)=&N_{ES}(R,\ell_1)-N_{ES}(R,\ell_2)\\&+N_{ES}(-f,R,-\ell_1)-N_{ES}(-f,R,-\ell_2)\end{aligned}$$

where $N_{ES}(-f,R,-\ell)$ denotes the number of components of $\{-f\geq-\ell\}=\{f\leq\ell\}$ contained in $B(R)$ (and the other terms are defined, as earlier, for f). We note that by symmetry, local maxima and lower connected saddle points of $-f$ correspond to local minima and upper connected saddle points respectively of f. Then applying Lemma 2.4.6 to the right hand side of the above expression verifies (2.23) with (2.24). □

We next complete the proof of Lemma 2.3.2. Again we shall separate the argument into the aperiodic case and the periodic cases, so that the reader interested only in the aperiodic case can access the simplest version of the argument. Since in this case we work only with the manifolds $\mathbb{R}^2$ and $B(R)$, we replace the condition that an excursion set component A is simple with the equivalent condition that it is bounded.

Proof of Lemma 2.3.2 in the case of aperiodic fields. Let f be an aperiodic stationary field satisfying Assumption 2.2.3. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a Morse realisation of f and assume that $g|_{\overline{B(n)}}$ is Morse for all $n \in \mathbb{N}$. Let x be a lower connected saddle of g at level ℓ.

The first step is to show that, with probability one, x is not also an upper connected saddle of g. For $\epsilon > 0$ sufficiently small, let $A_{\ell-\epsilon}$ be the component of $\{g \geq \ell - \epsilon\}$ containing x and $A_{\ell+\epsilon} = A_{\ell-\epsilon} \cap \{g \geq \ell + \epsilon\}$. From the definition of a lower connected saddle point, we have three cases to consider. In the first two cases we use the fact that g is Morse to (deterministically) rule out the possibility of x being upper connected; we show that the third case occurs with zero probability and can therefore be neglected.

1)$A_{\ell-\epsilon}$ is unbounded and $A_{\ell+\epsilon}$ has one bounded and one unbounded component.

Let S denote the bounded component of $A_{\ell+\epsilon}$. We start by choosing n sufficiently large that $\overline{B(n)}$ contains a neighbourhood of S and a neighbourhood of x. We let $A'_{\ell-\epsilon}$ be the component of $\{g|_{\overline{B(n)}} \geq \ell - \epsilon\}$ containing x and $A'_{\ell+\epsilon}$ denote $A'_{\ell-\epsilon} \cap \{g \geq \ell + \epsilon\}$. We can apply Theorem 2.4.2 to $-g$ to deduce that $\overline{B(n)} \cap \{g \leq \ell + \epsilon\}$ is homotopy equivalent to $\overline{B(n)} \cap \{g \leq \ell - \epsilon\}$ with a 1-cell, denoted γ, attached. By reducing ϵ we can ensure that $A'_{\ell+\epsilon}$ has two components, one of which is bounded, and so both end-points of γ must be contained in the same component of $\overline{B(n)} \cap \{g \leq \ell - \epsilon\}$. This will in turn be contained in a component of $\{g \leq \ell - \epsilon\}$ which we denote by B. Clearly $B \cup \gamma$ is bounded if and only if B is, therefore $\overline{B(n)} \cap \{g \leq \ell - \epsilon\}$ and $\overline{B(n)} \cap \{g \leq \ell + \epsilon\}$ have the same number of bounded components. This holds for any n sufficiently large (possibly after reducing ϵ) but if x were upper connected we could find m large enough that $\overline{B(m)} \cap \{g \leq \ell - \epsilon\}$ has one more bounded component than $\overline{B(m)} \cap \{g \leq \ell + \epsilon\}$ (this argument is stated more formally in the proof of Lemma 2.4.6). Therefore x is not an upper connected saddle point.

2) $A_{\ell-\epsilon}$ is bounded and $A_{\ell+\epsilon}$ has two bounded components.

The arguments in the previous case are also valid in this case where S is chosen to be either of the two components of $A_{\ell+\epsilon}$.

3)$A_{\ell-\epsilon}$ is unbounded and $A_{\ell+\epsilon}$ is bounded and connected.
We will show that when f is aperiodic, it almost surely has no saddle points of this type. We fix $R > 0$ and let $g_R = g|_{B(R)}$ which we assume is Morse. For $y \in \mathbb{R}^2$ and $\epsilon > 0$, we define $A_{y,-\epsilon}$ to be the component of $\{g \geq g(y) - \epsilon\}$ containing y and $A_{y,\epsilon} = A_{y,-\epsilon} \cap \{g \geq g(y) + \epsilon\}$. Let $x_1, \ldots, x_n$ be the saddle points of g_R for which $A_{x_i,-\epsilon}$ is unbounded but $A_{x_i,\epsilon}$ is bounded for all $\epsilon > 0$ sufficiently small. We then choose a fixed ϵ sufficiently small that this condition holds for each i and such that g_R has no other critical points and no tangent points in $\{g_R(x_i) - 3\epsilon \leq g_R \leq g_R(x_i) + 3\epsilon\}$ for any i. This ensures the intervals $[g(x_i) - \epsilon, g(x_i) + \epsilon]$ are non-overlapping for $i = 1, \ldots, n$. Finally we assume that the x_i are ordered so that $g(x_i) < g(x_j)$ for $i < j$.

Since $A_{x_i,-\epsilon}$ is unbounded for each i, we see that

$$B_i^- := A_{x_i,-\epsilon} \cap \partial B(R)$$

is non-empty for each i. Since there are no tangent points with level in $[g_R(x_i) - \epsilon, g_R(x_i) + \epsilon]$, we can repeat the arguments in the proof of Lemma 2.4.6 to show that

$$B_i^+ := A_{x_i,\epsilon} \cap \partial B(R)$$

is non-empty and has the same number of components as B_i^- for each $i = 1, \ldots, n$ (see Figure 2.12). Suppose there exists a point $y \in B_i^- \cap B_j^-$ for some $i < j$. Since $A_{x_j,-\epsilon}$ is unbounded and path connected, there exists an unbounded path started at y and contained in $\{g \geq g(x_j) - \epsilon)\}$. Since $y \in A_{x_i,\epsilon}$ which is path connected, this path must also be contained in $A_{x_i,\epsilon}$ which is bounded. This contradiction implies that $B_1^-, \ldots, B_n^-$ are disjoint. Since each of these sets must have a local maximum, we see that $n \leq N_{\mathrm{tang}}(\partial B(R))$. Since f is stationary, we know that the expected number of lower connected saddle points x contained in $B(R)$ for which $A_{x,-\epsilon}$ is unbounded and $A_{x,\epsilon}$ is bounded for $\epsilon > 0$ sufficiently small, is $c_0 R^2$ for some $c_0 \geq 0$. However by the argument just given, this number is bounded above by the expected number of tangent points of f in $\partial B(R)$ which is $O(R)$ by Assumption 2.2.3. Hence $c_0 = 0$ so f almost surely has no saddle points of this type.

We have shown that the set of lower connected saddles of f is almost surely disjoint from the set of upper connected saddles. Using the same notation as above, we will now show that any saddle point of g must be lower connected or upper connected, thus completing the proof of the lemma.

For a small neighbourhood N of x, $\{g > \ell\} \cap N$ and $\{g < \ell\} \cap N$ each have two components. Since g is Morse, it has no other critical points at level ℓ and so the level set $\{g = \ell\}$ consists of non-intersecting curves except for the component

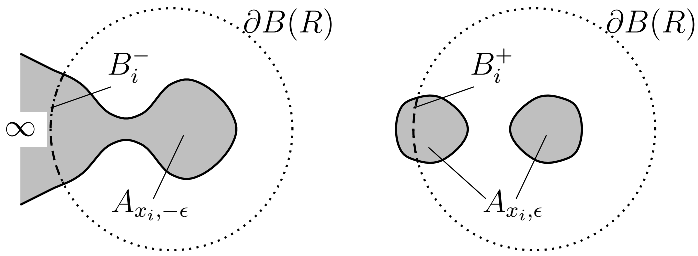


Figure 2.12: We show that almost surely an unbounded excursion set component of f cannot split into two bounded excursions set components by passing through a saddle point.

which contains x. Since g has no infinite-four-arm saddles, there are two mutually exclusive possibilities to consider; either there exists a path in $\{g < \ell\}$ joining the two components of $\{g < \ell\} \cap N$ or there exists a path in $\{g > \ell\}$ joining the two components of $\{g > \ell\} \cap N$.

We now show that in the first case x will be lower connected. By symmetry, this will imply that in the second case, x is upper connected and so complete the proof for aperiodic fields. We fix a path γ in $\{g < \ell\}$ connecting the components of $\{g < \ell\} \cap N$. From the existence of γ it is clear that $A_\ell \backslash \{x\}$ has two components B_1, B_2 and that without loss of generality, B_1 is bounded (see Figure 2.13).

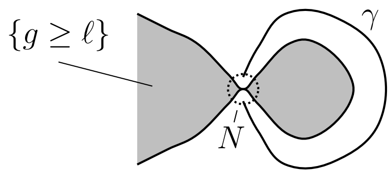


Figure 2.13: The existence of the path γ implies that x is a lower connected saddle point.

For $\epsilon > 0$ sufficiently small, $A_{\ell+\epsilon}$ therefore has at least one bounded component $B_1 \cap \{g \geq \ell+\epsilon\}$. If $A_{\ell-\epsilon}$ is compact for some $\epsilon > 0$, then clearly B_2 is also bounded and so $A_{\ell+\epsilon}$ has two bounded components so x is lower connected. If $A_{\ell-\epsilon}$ is unbounded for arbitrarily small ϵ we can assume that $A_{\ell+\epsilon}$ has an unbounded component, by the argument given above, and so x is again lower connected. We note that this also proves the equivalence of Definition 2.3.1 in the aperiodic case and Definition 2.1.5. $\square$

Proof of Lemma 2.3.2 in the case of periodic fields. The arguments here are similar to those in the aperiodic case (i.e. replacing the condition of boundedness with the more general condition of simplicity). Let f be a stationary field satisfying Assumption 2.2.3. Let $g : M \to \mathbb{R}$ be a Morse realisation of f restricted to M, where M is one of $\mathbb{T}$ or $\mathcal{C}$ depending on whether f is doubly periodic or singly periodic. We also assume that $g|_{\mathcal{C}(n)}$ is Morse for all $n \in \mathbb{N}$ in the singly periodic case. Let x be a lower connected saddle of g at level ℓ.

Again the first step is to show that, with probability one, x is not an upper connected saddle of g. For $\epsilon > 0$ sufficiently small, let $A_{\ell-\epsilon}$ be the component of $\{g \geq \ell - \epsilon\}$ containing x and $A_{\ell+\epsilon} = A_{\ell-\epsilon} \cap \{g \geq \ell + \epsilon\}$. Again we have three cases to consider, the first two of which are proven deterministically:

1)$A_{\ell-\epsilon}$ is not simple and $A_{\ell+\epsilon}$ has one simple and one non-simple component.

Let S denote the simple component of $A_{\ell+\epsilon}$. If f is singly periodic, we set $M_c = \mathcal{C}(n)$ where n is sufficiently large that M_c contains a neighbourhood of S and a neighbourhood of x. If f is doubly periodic we take $M_c = M = \mathbb{T}$. We define $A'_{\ell-\epsilon}$ to be the component of $\{g|_{M_c} \geq \ell - \epsilon\}$ containing x and $A'_{\ell+\epsilon} := A'_{\ell-\epsilon} \cap \{g|_{M_c} \geq \ell + \epsilon\}$. By Theorem 2.4.2 applied to $-g$, we deduce that $M_c \cap \{g \leq \ell + \epsilon\}$ is homotopy equivalent to $M_c \cap \{g \leq \ell - \epsilon\}$ with a 1-cell, denoted γ, attached. By reducing ϵ we can ensure that $A'_{\ell+\epsilon}$ has two components, one of which is simple, and so both end-points of γ must be contained in the same component of $M_c \cap \{g \leq \ell - \epsilon\}$. This will then be contained in a component of $\{g \leq \ell - \epsilon\}$ which we denote by B. If B is not simple, then clearly $B \cup \gamma$ is not simple. Suppose that B is simple but $B \cup \gamma$ is not simple, then $B \cup \gamma$ must contain a loop, denoted ζ, which is not M-contractible and in particular ζ must intersect γ. However B is path connected and $B \cup \gamma$ 'surrounds' a region which is homotopy equivalent to S and so must be simple (See Figure 2.14 for an example of these sets in the doubly periodic case).

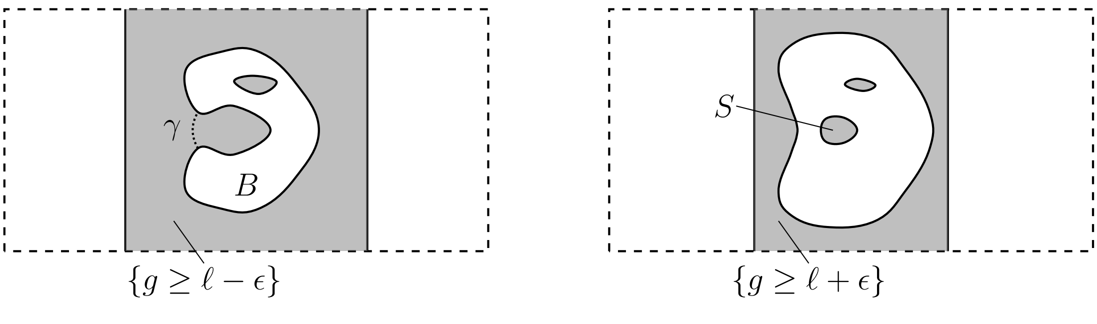


Figure 2.14: An example of the excursion sets at a level below (left) and above (right) a saddle point at level ℓ for which $A_{\ell-\epsilon}$ is not simple and $A_{\ell+\epsilon}$ has one simple and one non-simple component when $M = \mathbb{T}$.

Therefore ζ is homotopy equivalent to a loop which is contained in B and so B is not simple, which is a contradiction. We have shown that $B \cup \gamma$ is simple if and only if B is, therefore $M_c \cap \{g \leq \ell - \epsilon\}$ and $M_c \cap \{g \leq \ell + \epsilon\}$ have the same number of simple components. This holds for any M_c sufficiently large (possibly after reducing ϵ) but if x were upper connected we could find M_c such that $M_c \cap \{g \leq \ell - \epsilon\}$ has one more simple component than $M_c \cap \{g \leq \ell + \epsilon\}$ (again, we note that this argument is stated more formally in the proof of Lemma 2.4.6). We conclude that x is not an upper connected saddle point.

2) $A_{\ell-\epsilon}$ is simple and $A_{\ell+\epsilon}$ has two simple components.
Once again, the arguments in the previous case are also valid in this case where S is chosen to be either of the two components of $A_{\ell+\epsilon}$.

3)$A_{\ell-\epsilon}$ is not simple and $A_{\ell+\epsilon}$ is simple (in particular, connected).
We can repeat the argument given in this section of the proof for aperiodic fields to show that if f is singly periodic then $A_{\ell-\epsilon}$ must be bounded. Clearly in the doubly periodic case, $A_{\ell-\epsilon} \subset \mathbb{T}$ must also be bounded. We therefore choose a compact domain $M_c \subset M$ of the form $\mathbb{T}$ or $\mathcal{C}(n)$ for some n which contains a neighbourhood of $A_{\ell-\epsilon}$. Let $B_1, \ldots, B_n$ denote the components of $\{g \leq \ell+\epsilon\} \cap M_c$ which intersect $A_{\ell+\epsilon}$. Since $A_{\ell+\epsilon}$ is simple, we know that at most one of these sets is not simple and the remainder are. (One of the sets will 'surround' $A_{\ell+\epsilon}$ which in turn will 'surround' the remaining sets.) Without loss of generality, we assume B_1 is the 'surrounding' set, which may not be simple (and we will show that it is not simple). By Theorem 2.4.2, $A_{\ell-\epsilon}$ is homotopy equivalent to $A_{\ell+\epsilon}$ with a 1-cell, denoted γ, attached. Clearly γ is contained in B_1 and B_1 is not simple, otherwise $A_{\ell+\epsilon} \cup \gamma$ would be simple. If $M = \mathcal{C}$ then $B_1 \backslash \gamma$ has two components and each of these components contains a loop which is not $\mathcal{C}$-contractible (since $A_{\ell+\epsilon} \cup \gamma$ is compact and so separated from ∞ and $-\infty$ by $B_1 \backslash \gamma$). If $M = \mathbb{T}$ then $B_1 \backslash \gamma$ may have one or two components, in either case each component will contain a non-$\mathbb{T}$-contractible loop (to see this, consider a path on either side of γ which then traverses the boundary of $A_{\ell+\epsilon}$). This means that passing through the saddle point x can only create non-simple components of $\{g \leq \ell - \epsilon\}$, so x is not upper connected.

This completes the proof that the sets of upper and lower connected saddle points of f are almost surely disjoint. We continue to use the notation defined above, and we now show that any saddle point of g must be lower connected or upper connected. As in the aperiodic case, Lemma 2.4.5 and Assumption 2.2.3 allow us to conclude that f almost surely has no infinite four-arm saddles.

Suppose that f is doubly periodic, so that g is defined on the torus $\mathbb{T}$. We fix a small neighbourhood N of x such that $\{g > \ell\} \cap N$ has precisely two components denoted N_1, N_2. If $\gamma : [0,1] \to \mathbb{T}$ is a path contained in $\{g > \ell\} \cup \{x\}$, we say that

γ *cuts through* x if $x \in \gamma[0,1]$ and for every $t \in (0,1)$ such that $\gamma(t) = x$, for $\epsilon > 0$ sufficiently small $\gamma((t-\epsilon, t)) \subset N_1$ and $\gamma((t, t+\epsilon)) \subset N_2$ (intuitively, this means that the image of γ just before hitting x is always on the same side of the saddle point; see Figure 2.15). We make an analogous definition for paths contained in $\{g < \ell\} \cup \{x\}$.

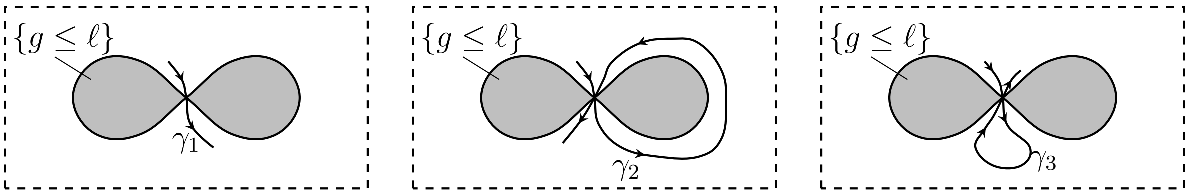


Figure 2.15: The paths γ_1 and γ_2 cut through x but γ_3 does not.

First we suppose that there exists a $\mathbb{T}$-contractible loop γ in $\{g < \ell\} \cup \{x\}$ which cuts through x. Since x is a saddle point, it is clear that $A_{\ell+\epsilon}$ must have two components, one of which is surrounded by γ and so is simple. If $A_{\ell-\epsilon}$ is simple, then both components of $A_{\ell+\epsilon}$ must be simple. If $A_{\ell-\epsilon}$ is not simple, we know from Lemma 2.4.6 that $A_{\ell+\epsilon}$ can contain at most one simple component. In either case, we see that x is lower connected. By symmetry, x is upper connected if there exists a $\mathbb{T}$-contractible loop in $\{g > \ell\} \cup \{x\}$ cutting through x.

Now we suppose that all loops cutting through x are non-$\mathbb{T}$-contractible. We define C_+ to be the union of $\{x\}$ and the components of $\{g > \ell\}$ which x is in the closure of. We define C_- to be the analogous set for $\{g < \ell\}$. At least one of C_+ or C_- must contain a non-contractible loop cutting through x (in order to stop any paths in C_+ cutting through x from joining to form a loop, there must be a loop in $\mathbb{T} \backslash C_+$ cutting through x which blocks them, and since x is the only critical point at level ℓ, such a 'blocking' loop can be found in C_-). We also note that both C_+ and C_- must contain non-$\mathbb{T}$-contractible loops (if C_+ was simple, then we could find a $\mathbb{T}$-contractible loop in C_- cutting through x, and this argument is symmetric).

Suppose that γ_1 is a loop contained in C_+ which cuts through x. If every non-$\mathbb{T}$-contractible loop in C_+ intersects x, then $C_+ \backslash \{x\}$ is simple, so $A_{\ell+\epsilon}$ is simple and hence x is lower connected. Therefore we may assume that C_+ contains a non-$\mathbb{T}$-contractible loop γ_2 which does not intersect x. Let ζ_1 and ζ_2 be two loops which generate the fundamental group of $\mathbb{T}$, we denote the concatenation of n copies of ζ_1 and m copies of ζ_2 by $n\zeta_1 + m\zeta_2$. We suppose $\gamma_1 \simeq n_1\zeta_1 + m_1\zeta_2$ and $\gamma_2 \simeq n_2\zeta_1 + m_2\zeta_2$ where $\simeq$ denotes being $\mathbb{T}$-homotopic. It is known (see [33, Section 1.2.3]) that any loop homotopic to $n_1\zeta_1 + m_1\zeta_2$ must intersect any loop homotopic to $n_2\zeta_1 + m_2\zeta_2$ unless $n_1 m_2 = n_2 m_1$.

If $n_1m_2 \neq n_2m_1$ then we know that any non-contractible loop in C_- must intersect either γ_1 or γ_2. Since $C_- \cap C_+ = \{x\}$ clearly such a loop must intersect γ_1 at x. By the previous paragraph this means that $C_- \backslash \{x\}$ is simple, and so x is upper connected.

If $n_1m_2 = n_2m_1$ then by choosing a path ξ in $C_+ \backslash \{x\}$ which joins γ_1 to γ_2 we consider the concatenated loop

$$n_2\gamma_1 + \xi + n_1(-\gamma_2) + (-\xi)$$

where $-$ denotes inverting the direction of the path. This path is contained in C_+ and cuts through x (since γ_1 does but γ_2 and ξ do not hit x) and since $n_1m_2 = n_2m_1$ this loop is $\mathbb{T}$-contractible. However this contradicts the above supposition, so this case is not possible. This completes the proof in the doubly periodic case. The proof in the singly periodic case is simply a repetition of parts of the proof for aperiodic and doubly periodic fields, so we omit it. □

All that remains is to complete the calculations for the special class of degenerate fields.

Proof of Proposition 2.1.11. We recall that a random variable Y is Rayleigh distributed with parameter $\sigma > 0$ if

$$\mathbb{P}(Y \leq x) = 1 - e^{-x^2/(2\sigma^2)}$$

for all $x \geq 0$, and we denote this as $Y \sim \mathrm{Ray}(\sigma)$. If $Y \sim \mathrm{Ray}(\sigma)$ and θ is an independent uniform-$[0, 2\pi]$ random variable then it is well known that $Y\cos(\theta) \sim \mathcal{N}(0, \sigma^2)$.

Let f be the Gaussian field with spectral measure

$$\nu = a\delta_0 + \frac{b}{2}(\delta_u + \delta_{-u}) + \frac{c}{2}(\delta_v + \delta_{-v})$$

where $a + b + c = 1$ so that the covariance function of f is

$$K(x) = a + b\cos(2\pi u \cdot x) + c\cos(2\pi v \cdot x).$$

Then f has the representation

$$f(x) = X_0 + Y_1\cos(2\pi u \cdot x + \theta_1) + Y_2\cos(2\pi v \cdot x + \theta_2) \tag{2.25}$$

where $X_0 \sim \mathcal{N}(0, a)$, $Y_1 \sim \mathrm{Ray}(\sqrt{b})$, $Y_2 \sim \mathrm{Ray}(\sqrt{c})$, the θ_i are uniformly distributed on $[0, 2\pi]$ and all of these random variables are independent. (By well known properties of the Rayleigh distribution, the field defined by the right hand side of (2.25) will have Gaussian finite dimensional distributions and simple calculations show that the covariance function of this field is K.)

Let

$$p = \frac{1}{u_1v_2 - u_2v_1}\begin{pmatrix} v_2 \\ -v_1 \end{pmatrix} \qquad q = \frac{1}{u_1v_2 - u_2v_1}\begin{pmatrix} u_2 \\ -u_1 \end{pmatrix}$$

and $P = \{tp + sq : t, s \in [0,1]\}$. Then P is the parallelogram associated with f and we consider $f_{\mathbb{T}}$ to be the restriction of f to P when P is identified with the two dimensional torus. By rotating the axes, we may assume that $u_1 > 0$ (since this does not affect the definition of c_{LS}). Some basic calculations show that, on the event

$$\{Y_1 \neq 0\} \cap \{Y_2 \neq 0\} \cap \{Y_1 \pm Y_2 \neq 0\},$$

$f_{\mathbb{T}}$ has four critical points which occur at different levels and are all non-degenerate. Therefore $f_{\mathbb{T}}$ is almost surely a Morse function. Moreover, the critical points of $f_{\mathbb{T}} - X_0$ occur at levels $-Y_1 - Y_2, -|Y_1 - Y_2|, |Y_1 - Y_2|, Y_1 + Y_2$. Clearly the critical points at levels $Y_1 + Y_2$ and $-Y_1 - Y_2$ are a local maximum and local minimum respectively. We can use Lemma 2.4.6 to characterise the other two critical points and the number of simple components of $\{f_{\mathbb{T}} - X_0 \geq \ell\}$, denoted $N_{\mathrm{simple}}(\ell)$, at different levels. Specifically, since $\{f_{\mathbb{T}} - X_0 > Y_1 + Y_2\} = \emptyset$ and $\{f_{\mathbb{T}} - X_0 \geq -Y_1 - Y_2\} = \mathbb{T}$, we see that $N_{\mathrm{simple}}(\ell) = 0$ whenever $|\ell| > Y_1 + Y_2$. Then, since the critical points at level $Y_1 + Y_2$ and $-Y_1 - Y_2$ must be a local maximum and local minimum respectively, applying Lemma 2.4.6 shows that $N_{\mathrm{simple}}(\ell)$ does not change as ℓ passes through $-Y_1 - Y_2$ and decreases by one as ℓ passes through $Y_1 + Y_2$. Therefore

$$N_{\mathrm{simple}}(\ell) = \begin{cases} 0, & \text{if } \ell > Y_1 + Y_2, \\ 1, & \text{if } \ell \in (|Y_1 - Y_2|, Y_1 + Y_2], \\ 0, & \text{if } \ell \in [-Y_1 - Y_2, -|Y_1 - Y_2|), \\ 0, & \text{if } \ell < -Y_1 - Y_2. \end{cases}$$

Now assume that $\theta_1 = \theta_2 = 0$. Then by traversing the parallelogram P across one of its edges (depending on which of Y_1, Y_2 is bigger) we can find a closed path which is not $\mathbb{T}$-contractible, on which $f_{\mathbb{T}} - X_0$ is bounded below by $|Y_1 - Y_2|$, so in particular is positive. This shows that $\{f_{\mathbb{T}} - X_0 \geq 0\}$ has a non-simple component (for general values of θ_1, θ_2, such a path will also exist, but it will be translated). Since $\{f_{\mathbb{T}} - X_0 \geq |Y_1 - Y_2|\}$ consists of a single simple component, this implies that the critical point at level $|Y_1 - Y_2|$ must be a lower connected saddle point. Then by Lemma 2.4.6 we see that

$$N_{\mathrm{simple}}(\ell) = \begin{cases} 1, & \text{if } \ell \in (|Y_1 - Y_2|, Y_1 + Y_2], \\ 0, & \text{otherwise}. \end{cases}$$

It is then clear that

$$p_{m^+}(x) = \frac{1}{\text{Area}(P)} p_{X_0+Y_1+Y_2}(x) \quad \text{and} \quad p_{s^-}(x) = \frac{1}{\text{Area}(P)} p_{X_0+|Y_1-Y_2|}(x).$$

Arguments given in the proof of Lemma 2.2.5 for periodic functions show that

$$N_{ES}(R,\ell) = \mathbb{1}_{\ell - X_0 \in (|Y_1-Y_2|, Y_1+Y_2]} \cdot \frac{\pi R^2}{\text{Area}(P)} + O(R + N_{\text{tang}}(\partial B(R))).$$

Arguing as in the proof of Lemma 2.2.4 it can be shown that $\mathbb{E}(N_{\text{tang}}(\partial B(R)) = O(R)$ as $R \to \infty$. Then for $\ell \in \mathbb{R}$

$$\mathbb{E}\left(\left|\frac{N_{ES}(R,\ell)}{\pi R^2} - \frac{1}{\text{Area}(P)} \mathbb{1}_{\ell - X_0 \in (|Y_1-Y_2|, Y_1+Y_2]}\right|\right) = O\left(\frac{1}{R}\right),$$

as required. In particular, this convergence shows that

$$c_{ES}(\ell) := \lim_{R\to\infty} \frac{1}{\pi R^2} \mathbb{E}(N_{ES}(R,\ell)) = \frac{1}{\text{Area}(P)} \mathbb{P}(\ell - X_0 \in (|Y_1 - Y_2|, Y_1 + Y_2]).$$

We can repeat these arguments for the sum of upper and lower excursion sets to derive the corresponding expression for $c_{LS}(\ell)$. $\square$

Chapter 3

Smoothness and Monotonicity of the Mean Functional

3.1 Introduction

In this chapter we analyse the smoothness and monotonicity (with respect to the level) of the mean functionals c_{ES} and c_{LS} introduced in the previous chapter, which encode the asymptotic density of excursion and level set components respectively for planar Gaussian fields. Our main results (Theorems 3.2.4, 3.2.5 and Corollary 3.2.12) show that, for a wide class of fields, the continuous differentiability of c_{ES} and c_{LS} at ℓ is equivalent to the statement that, if the field is conditioned to have a saddle point at the origin at level ℓ, then almost surely the 'arms' of the saddle (i.e. the four level lines that emanate from the saddle point) do not connect the origin to infinity. Since we can prove that the latter property holds for many fields, we deduce the continuous differentiability of the density functionals.

As discussed in Chapter 1, recent work has established that, in many circumstances, the geometry of Gaussian excursion sets exhibits similar behaviour to discrete percolation models. The results in this chapter can therefore be compared to what is known, and conjectured, about the analogous density functionals for discrete percolation models. Consider Bernoulli bond percolation on $\mathbb{Z}^d$, defined by declaring the edges of the d-dimensional integer lattice to be open independently with probability p and closed otherwise (this is the natural generalisation of Bernoulli percolation on $\mathbb{Z}^2$ as defined in Section 1.2.3). Let C_n denote the number of open clusters that are contained in $[-n,n]^d$. Then it is known ([39, Chapter 4]) that there exists a function $\psi : [0,1] \to [0,1]$ such that

$$\frac{C_n}{(2n)^d} \to \psi(p)$$

as $n \to \infty$, almost surely and in L^1. This is a direct analogue of the definition of $c_{ES}(\ell)$ given in Theorem 2.1.2, and is also proven using an ergodic argument. The smoothness of ψ is of interest because it is related to the percolation phase transition. Specifically, it is conjectured that ψ is analytic on $[0,1]\backslash\{p_c\}$ and twice but not three times differentiable at p_c, where $p_c \in (0,1)$ is the critical probability for the model; this reflects the values of certain 'critical exponents' which are believed to be universal for percolation models (see [39, Chapter 9]). What has been shown rigorously, is that, for all $d \geq 2$, ψ is analytic on $[0, p_c)$ and smooth on $(p_c, 1]$. In the case $d = 2$, it is further known that ψ is analytic on $(p_c, 1]$ and at least twice differentiable at p_c (see [39, Chapter 4]). Somewhat weaker results have been derived for other percolation models, including the Poisson-Boolean model and 'spread-out' percolation models [15].

Since the connectivity of the excursion sets of a wide class of planar Gaussian fields is conjectured, and in some cases known, to undergo a phase transition at $\ell = 0$ that is analogous to the phase transition at p_c for Bernoulli percolation (Section 1.4.2), it is natural to conjecture that, for such fields, c_{ES} and c_{LS} are also analytic on $\mathbb{R}\backslash\{0\}$ and twice but not three times differentiable at 0. Our proof of the continuous differentiability of c_{ES} and c_{LS} can be seen as a first step in this direction.

Despite the connections to classical percolation theory, the method we use to prove differentiability of the density functionals is quite different. In Bernoulli percolation, the starting point is the equality

$$\psi(p) = \mathbb{E}_p\left(|C^*|^{-1}\right),$$

where $|C^*|$ is the number of vertices in the open cluster at the origin and $\mathbb{E}_p$ denotes expectation for the percolation model with parameter p. By enumerating clusters, this can be expressed as a power series in p, and the smoothness of ψ can be deduced from bounds on the coefficients in terms of connection probabilities for the cluster at the origin.

This approach does not readily generalise to the setting of Gaussian fields: whilst it can be shown that

$$c_{ES}(\ell) = \mathbb{E}\left(\mathrm{Vol}(C^*)^{-1}\mathbb{1}_{f(0)>\ell}\right),$$

where $\mathrm{Vol}(C^*)$ is the volume of the component of $\{f \geq \ell\}$ containing the origin, it is not known whether the density of $(\mathrm{Vol}(C^*), f(0))$ is jointly continuous ([12] studies a kind of 'ergodic' density for $\mathrm{Vol}(C)$ at the zero level). Instead, our proof of differentiability uses the integral representation for c_{ES} and c_{LS} from Chapter 2. We recall that one of the motivations for Theorem 2.1.7, was to provide a tool with which to study the excursion/level set functionals: since p_{m^+}, p_{m^-}, and $p_s = p_{s^+} + p_{s^-}$ are explicitly known for a wide class of fields, by establishing simple properties of p_{s^-} we can deduce results for c_{ES} and c_{LS}. In this chapter, we consider the function

$$p^*_{s^-}(\ell) := p_s(\ell)\,\mathbb{P}\left(\tilde{f}_\ell \text{ has a lower connected saddle point at the origin}\right), \qquad (3.1)$$

where $\tilde{f}_\ell$ is the field f conditioned to have a saddle point at the origin at level ℓ (in the sense of Palm distributions; see Lemma 3.3.2 for a formal definition) and lower connected saddle points are as defined in Chapter 2. Under mild conditions, we show that $p^*_{s^-}$ defines a version of p_{s^-} (as the latter is defined only up to null sets). By studying $\tilde{f}_\ell$ we are able to deduce properties of $p^*_{s^-}$, and hence of c_{ES} and c_{LS}.

Our study of $\tilde{f}_\ell$ also allows us to derive certain montonicity properties of c_{ES} and c_{LS} (see Propositions 3.2.14–3.2.16), which are informative about the general shape of these functionals and also a key input to our most significant results in Chapter 4.

3.2 Main results

In this chapter we continue to work in the basic setting of fields satisfying Assumption 2.1.1 (and later introduce some stronger assumptions). We are especially interested in two fields satisfying this assumption: the Random Plane Wave and the Bargmann-Fock field. These fields were introduced and motivated in Chapter 1. Recall that the Random Plane Wave has covariance $K(x) = J_0(|x|)$, where J_0 is the 0-th Bessel function, and spectral measure equal to the normalised Lebesgue measure on the unit circle. The Bargmann-Fock field has covariance $K(x) = \exp(-|x|^2/2)$, and Gaussian spectral measure.

3.2.1 Differentiability

Our first set of results concerns the differentiability of c_{ES} and c_{LS}. These require some further assumptions on f.

Assumption 3.2.1. *For all $t \in \mathbb{R}^2\backslash\{0\}$,*

$$\mathrm{Cov}\left((f(t), \nabla f(t)) \,\middle|\, f(0), \nabla f(0), \nabla^2 f(0)\right)$$

is non-degenerate (i.e. this 3×3 matrix has non-zero determinant).

Assumption 3.2.2. *There exist $c, \zeta > 0$ such that, for all $|t| \geq 1$,*

$$\max_{|k|\leq 3} \left|\partial^k K(t)\right| \leq c|t|^{-(1+\zeta)}.$$

Moreover, there exists a neighbourhood V of the origin on which the spectral measure ν has density ρ with respect to Lebesgue measure and $\inf_V \rho > 0$.

Assumption 3.2.3. *For $0 < r < R$, let $\mathrm{Arm}_\ell(r,R)$ denote the 'one-arm event' that there exists a component of $\{f \geq \ell\}$ which intersects both $\partial B(r)$ and $\partial B(R)$. Then there exist $c_1, c_2 > 0$ such that for any $1 < r < R$*

$$\mathbb{P}\left(f \in \mathrm{Arm}_0(r,R)\right) \leq c_1 (r/R)^{c_2}. \tag{3.2}$$

Assumption 3.2.1 is extremely mild; it is satisfied whenever the support of the spectral measure ν contains an open set or a centred circle/ellipse (Lemma A.2.4). In particular, it holds for the Random Plane Wave and the Bargmann-Fock field.

Assumptions 3.2.2 and 3.2.3 are somewhat more restrictive. Assumption 3.2.2 holds for any smooth field with sufficiently nice correlation decay, and in particular holds for the Bargmann-Fock field, but it does not hold for the Random Plane Wave (whose correlations decay only as $|t|^{-1/2}$). It also implies Assumption 3.2.1, by the previous remark.

Assumption 3.2.3 relates to the conjectured properties of the 'percolation universality class', and has been shown to hold for a wide class of fields that includes the Bargmann-Fock field [77]. Moreover it is strongly believed to hold for the Random Plane Wave. We state our results directly in terms of one-arm decay, as it is likely that these bounds will be extended to more fields over time.

The first main result of this chapter is that c_{ES} and c_{LS} are continuously differentiable under the above assumptions:

Theorem 3.2.4. *Let f be a Gaussian field satisfying Assumptions 2.1.1 and 3.2.2–3.2.3. Then c_{ES} and c_{LS} are continuously differentiable on $\mathbb{R}$. In other words, the functions p_{s^-} and p_{s^+} defined (up to null sets) in Proposition 2.1.6 can be chosen to be continuous, and*

$$c'_{ES}(\ell) = -p_{m^+}(\ell) + p_{s^-}(\ell)$$

and

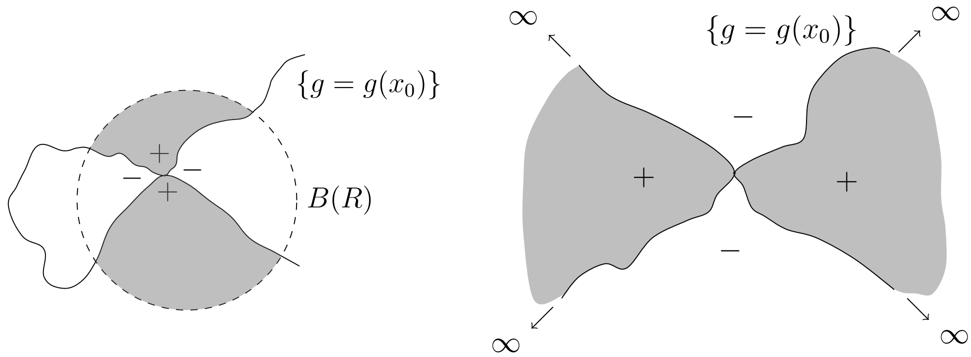
$$c'_{LS}(\ell) = -p_{m^+}(\ell) + p_{s^-}(\ell) - p_{s^+}(\ell) + p_{m^-}(\ell).$$

We emphasise that Theorem 3.2.4 applies to a wide class of fields, including the important case of the Bargmann-Fock field, but does not apply to the Random Plane Wave as stated (although we believe the conclusion to be true).

3.2.1.1 Four-arm saddle points

Theorem 3.2.4 follows from a more general result establishing that, under very mild conditions, the continuous differentiability of c_{ES} and c_{LS} is implied by the decay of certain connection probabilities involving four-arm saddles. We now repeat the definition of four-arm saddle points given (in a more general setting) in Chapter 2.

Let $D \subset \mathbb{R}^2$ be a simply connected domain with piecewise C^1 boundary and let $x_0 \in D$ be a saddle point of $g \in C^2_{\mathrm{loc}}(\mathbb{R}^2)$ such that g has no other critical points at the same level as x_0. We say that x_0 is *four-arm in D* if it is in the closure of two components of $\{x \in D : g(x) > g(x_0)\}$ and two components of $\{x \in D : g(x) < g(x_0)\}$ (see Figure 3.1a); intuitively, a saddle point is four-arm in D if we cannot tell whether it is upper or lower connected by looking at the values of g in D. A saddle point x_0 is said to be *infinite four-arm* if it is in the closure of two components of $\{x \in \mathbb{R}^2 : g(x) > g(x_0)\}$ and two components of $\{x \in \mathbb{R}^2 : g(x) < g(x_0)\}$ (see Figure 3.1b). Four-arm saddle points are somewhat analogous to four-arm events for percolation models (see Section 1.3.2).



(a) An upper connected saddle point that is four-arm in $B(R)$.

(b) An infinite-four-arm saddle point.

Figure 3.1: Four arm saddle points

Recall the conditional field $\tilde{f}_\ell$ (to be formally defined in Lemma 3.3.2) and the functions $p^*_{s^-}$ and p_{s^-} defined in (3.1) and Proposition 2.1.6 respectively.

Theorem 3.2.5. *Let f be a Gaussian field satisfying Assumptions 2.1.1 and 3.2.1. Then $p^*_{s^-} = p_{s^-}$ almost everywhere. Moreover, let $a < b$ and suppose that for all $\ell \in (a, b)$*

$$\mathbb{P}\left(\tilde{f}_\ell \text{ has an infinite four arm saddle at the origin}\right) = 0. \tag{3.3}$$

Then $p^*_{s^-}|_{(a,b)}$ is continuous, and hence c_{ES} and c_{LS} are continuously differentiable on (a, b).

Theorem 3.2.4 follows from Theorem 3.2.5, once we verify condition (3.3) under Assumptions 3.2.2 and 3.2.3. To do so, we first use Assumption 3.2.2 and a Cameron-Martin argument to treat the conditional field $\tilde{f}_\ell$ away from the origin as a perturbation of the unconditioned field f. We then use Assumption 3.2.3 to bound the relevant connection probabilities for the unconditioned field.

As a corollary of Theorem 3.2.5 (actually of its proof), we deduce a bound on the number of saddle points of a Gaussian field that are four-arm inside a ball and whose level lies in a narrow range. This improves a bound that was used in Chapter 2, and is also a key ingredient in the lower bounds on the variance of the number of excursion/level set components given in Chapter 4.

Corollary 3.2.6. *Let f be a Gaussian field satisfying all the assumptions of Theorem 3.2.5. Then there exists a function $\delta_R \to 0$ as $R \to \infty$ and a constant $c > 0$ such that, for each $R > 1$ and $a \leq a_R \leq b_R \leq b$,*

$$\mathbb{E}\left(N_{\text{4-arm}}(R, [a_R, b_R])\right) \leq c \min\left\{\delta_R R^2 (b_R - a_R), R\right\}$$

where $N_{\text{4-arm}}(R, [a_R, b_R])$ is the number of saddle points of f which are four-arm in $B(R)$ and have level in $[a_R, b_R]$.

Remark 3.2.7. In Chapter 2 it was shown that $\mathbb{E}\left(N_{\text{4-arm}}(R)\right) = O(R)$; Corollary 3.2.6 supersedes this bound whenever $b_R - a_R = O(R^{-1})$. It is possible to improve the conclusion of Corollary 3.2.6 further by imposing stronger assumptions on the field. For example, suppose we assume exponential decay of arm probabilities at some non-zero level: for some $\ell^* > 0$ and $\delta \in (0, 1)$, there exist $c_1, c_2 > 0$ such that

$$\mathbb{P}\left(f \in \mathrm{Arm}_{\ell^*}(\delta R, R)\right) \leq c_1 e^{-c_2 R}. \tag{3.4}$$

Then for any $a > \ell^*$ (or $b < -\ell^*$), it is possible to prove that there exists $c > 0$ such that

$$\mathbb{E}\left(N_{\text{4-arm}}(R, [a_R, b_R])\right) \leq c \min\left\{R \log(R)(b_R - a_R), R\right\}.$$

In [77, 89], it is shown that a wide class of fields satisfy (3.4), so this assumption is reasonable. We do not prove this result formally here because Corollary 3.2.6 is simpler to prove, holds for a wider class of fields, and suffices for its intended purpose (see Chapter 4).

In light of Theorem 3.2.5, the fact that (3.4) is expected to hold for a wide class of fields also suggests it should be much easier to prove the differentiability of c_{ES} away from zero, since the probability of four-arm saddles in $B(R)$ should decay exponentially at non-zero levels.

3.2.1.2 The positivity of the level set density

As discussed in Section 3.2, the positivity of the mean excursion/level set functionals is important for determining the asymptotic geometric behaviour of Gaussian excursion/level sets. One nice consequence of the differentiability of c_{ES} and c_{LS}, is that it gives a new, short proof of their positivity in the delicate case $\ell = 0$:

Proposition 3.2.8. *Let f be a Gaussian field satisfying Assumption 2.1.1. Suppose either c_{ES} or c_{LS} is differentiable at 0. Then $c_{ES}(0) > 0$ and $c_{LS}(0) > 0$.*

The positivity of $c_{ES}(0)$ and $c_{LS}(0)$ are already known under quite general assumptions ([81, 45] give a variety of sufficient conditions, whose union can be checked to exhaust fields satisfying Assumptions 2.1.1 and 3.2.1). We restate this result because it uses a very different method of proof; in particular, it does not rely on the 'barrier method'.

3.2.2 Monotonicity

We next consider monotonicity properties of c_{ES} and c_{LS}. We begin by analysing the ratio

$$\frac{p^*_{s-}(\ell)}{p_s(\ell)} = \mathbb{P}\left(\tilde{f}_\ell \text{ has a lower connected saddle point at the origin}\right),$$

which we intuitively expect to be non-decreasing: if we condition on the origin being a saddle point at increasing heights, it seems more likely that it should be lower connected. This can be made rigorous under some additional assumptions, and allows us to deduce regions on which $c_{ES}(\ell)$ and $c_{LS}(\ell)$ are monotone.

Assumption 3.2.9. *The field f is isotropic (i.e. its law is invariant under rotations) and hence its covariance function can be expressed as $K(x) = k(|x|^2)$. Then*

$$\chi := \frac{-k'(0)}{\sqrt{k''(0)}} \geq 1. \tag{3.5}$$

Furthermore, $(f(0), \nabla^2 f(0))$ is a non-degenerate Gaussian vector, and for all $x \in \mathbb{R}^2$,

$$\mathbb{E}\left(f(x) \,\middle|\, f(0) = 0, \nabla^2 f(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \geq 0, \tag{3.6}$$

$$\mathbb{E}\left(f(x) \,\middle|\, f(0) = 1, \nabla^2 f(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) \leq 1. \tag{3.7}$$

As discussed in Section 2.1.2, the parameter χ is used in [26] to parameterise the density of eigenvalues of $\nabla^2 f$ at critical points and is shown to take values in $(0, \sqrt{2}]$. We note that (3.5) can be replaced by a weaker condition (see Remark 3.5.8) however we do not state the general condition here, as (3.5) is much simpler to verify.

In Section 3.5.1.2, we explain how (3.6) and (3.7) can be translated into explicit properties of the conditional field $\tilde{f}_\ell$. We can also give equivalent versions of (3.6) and (3.7) that are easier to check in practice. If we rescale the domain of f so that $k'(0) = -1$ (note that $k'(0) = 2K_{11}(0)$ and that this rescaling does not affect the value of χ), then it can be shown by Gaussian regression that (3.6) is equivalent to

$$\left(k\left(|x|^2\right) + k'\left(|x|^2\right)\right) k''(0) + \left(x_1^2\left(3k''(0) - 1\right) + x_2^2\left(1 - k''(0)\right)\right) k''\left(|x|^2\right) \geq 0$$

for all $x \in \mathbb{R}^2$ and (3.7) is equivalent to

$$\forall y \geq 0, \qquad \frac{2k''(0)k\left(y\right) + yk''\left(y\right) + k'\left(y\right)}{2k''(0) - 1} \leq 1.$$

From this, it can be verified that specific fields satisfy Assumption 3.2.9, including the Bargmann-Fock field. The Random Plane Wave does not satisfy Assumption 3.2.9; in this case $\chi = \sqrt{2}$ but $(f(0), \nabla^2 f(0))$ is non-degenerate so the conditional expectations in (3.6) and (3.7) are not defined. However we are able to prove the monotonicity of p^*_{s-}/p_s in this case too:

Theorem 3.2.10. *Let f be the Random Plane Wave or a field satisfying Assumptions 2.1.1, 3.2.1 and 3.2.9 (e.g. the Bargmann-Fock field). Then $p^*_{s-}(\ell)/p_s(\ell)$ is non-decreasing in ℓ.*

Given the definition of p^*_{s-}, we will show that Theorem 3.2.10 is a consequence of $\tilde{f}_\ell - \ell$ being stochastically decreasing in ℓ. Our proof of the latter fact differs for the Random Plane Wave and for fields satisfying Assumption 3.2.9 (in the former case it is somewhat simpler, because $(f(0), \nabla^2 f(0))$ is degenerate for the Random Plane Wave).

The monotonicity of p^*_{s-}/p_s has implications for the smoothness of c_{ES} and c_{LS}:

Corollary 3.2.11. *Let f satisfy the conditions of Theorem 3.2.10, then p^*_{s-} has at most a countable set of discontinuities, all of which are jump discontinuities. In particular, c_{ES} and c_{LS} are twice differentiable almost everywhere.*

As discussed in Section 3.1, it seems likely that the stronger result that c_{ES} and c_{LS} are twice differentiable everywhere should hold. There are no known counterexamples to this statement, aside from fields which are periodic or constant in one direction.

Monotonicity of p^*_{s-}/p_s also implies a converse of Theorem 3.2.5:

Corollary 3.2.12. *Let f satisfy the conditions of Theorem 3.2.10, then for every $a < b$ the following are equivalent:*

1. *For all $\ell \in (a, b)$*
$$\mathbb{P}\left(\tilde{f}_\ell \text{ has an infinite four-arm saddle at the origin}\right) = 0;$$
2. *There exists a version of p_{s^-} which is continuous on (a, b);*
3. *$c_{ES}(\cdot)$ is continuously differentiable on (a, b);*
4. *$c_{LS}(\cdot)$ is continuously differentiable on (a, b).*

Remark 3.2.13. Clearly, if any of (1)–(4) hold in Corollary 3.2.12, then by Theorem 3.2.5, the version of $p_{s^-}|_{(a,b)}$ which is continuous is equal to $p^*_{s^-}|_{(a,b)}$.

Finally we use Theorem 3.2.10 to deduce intervals on which c_{ES} and c_{LS} are monotone. We shall state the strongest form of our results only in the case of the Random Plane Wave and Bargmann-Fock field. Let D_+ and D^+ respectively denote the lower and upper right Dini derivatives, that is, for $g : \mathbb{R} \to \mathbb{R}$,

$$D_+ g(x) := \liminf_{\epsilon \downarrow 0} \frac{g(x+\epsilon) - g(x)}{\epsilon} \quad \text{and} \quad D^+ g(x) := \limsup_{\epsilon \downarrow 0} \frac{g(x+\epsilon) - g(x)}{\epsilon}.$$

Proposition 3.2.14. *Let f be the Random Plane Wave. Then*

$$\begin{aligned} D_+ c_{ES}(\ell) &> 0 \quad \text{for } \ell \in (-\infty, 0.87] \\ D^+ c_{ES}(\ell) &< 0 \quad \text{for } \ell \in [1, \infty) \end{aligned}$$

and

$$D^+ c_{LS}(\ell) < 0 \quad \text{for } \ell \in [1, \infty).$$

Proposition 3.2.15. *Let f be the Bargmann-Fock field. Then there exists $\epsilon > 0$ such that*

$$c'_{ES}(\ell) \begin{cases} > 0 & \text{for } \ell \in (-\epsilon, 0.64] \\ < 0 & \text{for } \ell \in [1.03, \infty) \end{cases}$$

and

$$c'_{LS}(\ell) < 0 \quad \text{for } \ell \in [1.03, \infty).$$

These results are proven using the integral representation for c_{ES} and c_{LS} along with the explicitly known densities p_{m^+}, p_s and p_{m^-}. These densities are known for general isotropic fields, and so we obtain similar results in this case. Recall that the covariance function of an isotropic f may be expressed as $K(x) = k(|x|^2)$ for some $k : [0, \infty) \to \mathbb{R}$, and that $\chi := -k'(0)/\sqrt{k''(0)} \in (0\sqrt{2}]$.

Proposition 3.2.16. *Let f be an isotropic field satisfying Assumptions 2.1.1, 3.2.2, 3.2.3 and 3.2.9. Then there exists $\epsilon > 0$ and an explicit constant $C > 0$ such that*

$$c'_{ES}(\ell) \begin{cases} > 0 & \textit{for } \ell \in (-\epsilon, C) \\ < 0 & \textit{for } \ell \in (\sqrt{2}/\chi, \infty) \end{cases}$$

and

$$c'_{LS}(\ell) < 0 \ \ \textit{for } \ell \in (\sqrt{2}/\chi, \infty).$$

The explicit formula for the constant C is quite complicated and is given in the proof of this proposition. However it is straightforward to evaluate this formula for any particular field (as we have done for the Random Plane Wave and Bargmann-Fock field in Propositions 3.2.14 and 3.2.15).

As an intermediate result to Proposition 3.2.14 we require that, for the Random Plane Wave, $c_{ES}(\ell) > 0$ for $\ell \leq 0$. Since this result is not stated elsewhere in the literature, we do so here. The proof uses the 'barrier method' and is near-identical to that of Nazarov-Sodin in [81] in the case $\ell = 0$.

Proposition 3.2.17. *Let f be the Random Plane Wave. Then $c_{ES}(\ell) > 0$ for all $\ell \in \mathbb{R}$.*

3.2.3 Summary of the rest of the chapter

In Section 3.3 we give a formal definition of $\tilde{f}_\ell$, the field f conditioned to have a saddle point at the origin at level ℓ, and derive explicit distributions for $\tilde{f}_\ell$ in special cases. In Section 3.4 we study topological properties of $\tilde{f}_\ell$, and use this to deduce the results outlined in Section 3.2.1. In Section 3.5 we consider stochastic monotonicity properties of $\tilde{f}_\ell$, and complete the proofs of the results in Section 3.2.2.

3.3 The field conditioned to have a saddle at the origin

In this section we consider $\tilde{f}_\ell$, the field f conditioned to have a saddle point at the origin at level ℓ. Using the theory of Palm distributions, we give an explicit representation for $\tilde{f}_\ell$, and in the isotropic case we derive simple expressions for its distribution.

We begin with a general statement expressing $\tilde{f}_\ell$ as (a limit of) a Palm distribution relative to a point process defined by the saddle points of f. Let us first recall the relevant theory of Palm distributions (see [50, Chapter 11] for background). We define a point process ζ to be a random measure on $\mathbb{R}^d$ such that $\zeta(B)$ is integer-valued for every bounded Borel set B. We say that ζ is simple if, with probability one, $\zeta(\{s\}) \leq 1$ for every $s \in \mathbb{R}^d$. We say that it is non-degenerate if $\mathbb{E}(\zeta(B)) > 0$ for every Borel set B with positive Lebesgue measure. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a planar random field and $\mathcal{S}$ a non-degenerate, simple point process on $\mathbb{R}^2$, and suppose that $(g, \mathcal{S})$ are jointly stationary (i.e. this joint distribution is invariant under translations). Fix a bounded Borel set $B \subset \mathbb{R}^2$ such that $0 < \mathbb{E}\left(\#\{s \in B : s \in \mathcal{S}\}\right) < \infty$. Then the *Palm distribution of g relative to $\mathcal{S}$* is defined as the random field $\tilde{g}$ satisfying, for any Borel cylinder set A,

$$\mathbb{P}\left(\tilde{g}(x) \in A\right) = \frac{\mathbb{E}\left(\#\{s \in B : s \in \mathcal{S}, g(x-s) \in A\}\right)}{\mathbb{E}\left(\#\{s \in B : s \in \mathcal{S}\}\right)}. \tag{3.8}$$

This definition is independent of the reference set B, hence we write $\tilde{g} = (g \,|\, \{0\} \in \mathcal{S})$.

Lemma 3.3.1. *Let f be a Gaussian field satisfying Assumption 2.1.1. For $\ell \in \mathbb{R}$ and $\epsilon > 0$, let*

$$\tilde{f}_{[\ell,\ell+\epsilon]} = (f \;| \{0\} \in S[\ell, \ell+\epsilon])$$

be the Palm distribution of f relative to $S[\ell, \ell+\epsilon]$, the point process of saddle points with level in $[\ell, \ell+\epsilon]$. (This point process is non-degenerate by Lemma A.2.5.) Then there exists a random field $\tilde{f}_\ell$ such that, as $\epsilon \to 0$, $\tilde{f}_{[\ell,\ell+\epsilon]}$ converges in distribution to $\tilde{f}_\ell$ in the topology of uniform $C^{2+\eta}$ convergence on compacts.

It is important to distinguish $\tilde{f}_{[\ell,\ell+\epsilon]}$ from the conditioned field

$$(f(t) | \nabla f(0) = 0, \det \nabla^2 f(0) < 0, f(0) \in [\ell, \ell+\epsilon]),$$

which is defined via the distributional limit

$$\lim_{\delta \to 0} (f(t) | f_1(0), f_2(0) \in [0, \delta), \det \nabla^2 f(0) < 0, f(0) \in [\ell, \ell+\epsilon]).$$

The latter is sometimes known as 'vertical window conditioning', and is the standard way of conditioning on part of a random vector (for a Gaussian vector, this conditioning is given explicitly by Gaussian regression, see [5, Proposition 1.2]). By constrast, the former can be thought of as 'horizontal window conditioning', and corresponds to sampling a 'typical' saddle point (i.e. via the counting measure). The difference between these forms of conditioning is elegantly explained in [49].

Using basic properties of Gaussian fields, we can derive explicit representations for $\tilde{f}_{[\ell,\ell+\epsilon]}$ and $\tilde{f}_\ell$:

Lemma 3.3.2. *Let f be a Gaussian field satisfying Assumption 2.1.1 such that $(f(0), \nabla^2 f(0))$ is a non-degenerate Gaussian vector. Define $\alpha : \mathbb{R}^2 \to \mathbb{R}$ and $\beta = (\beta_{11}, \beta_{22}, \beta_{12}) : \mathbb{R}^2 \to \mathbb{R}^3$ to be the unique functions satisfying*

$$\mathbb{E}\left(f(t) \middle| f(0) = u, \nabla^2 f(0) = w\right) = \alpha(t)u + \beta(t) \cdot w$$

for all $u \in \mathbb{R}$, $w \in \mathbb{R}^3$, and define

$$\gamma(s,t) := \mathbb{E}\left(f(s)f(t) \middle| f(0) = 0, \nabla f(0) = 0, \nabla^2 f(0) = 0\right).$$

Then

$$\tilde{f}_{[\ell,\ell+\epsilon]} \overset{d}{=} g + z_{[\ell,\ell+\epsilon]}\alpha + Z_{[\ell,\ell+\epsilon]} \cdot \beta,$$

where g is a centred Gaussian field with covariance function γ, and $\left(z_{[\ell,\ell+\epsilon]}, Z_{[\ell,\ell+\epsilon]}\right)$ is an independent random vector with density[1]

$$p_{\left(z_{[\ell,\ell+\epsilon]}, Z_{[\ell,\ell+\epsilon]}\right)}(x, X) \propto |\det X| \, p_{f(0),\nabla^2 f(0)}(x, X) \, \mathbb{1}_{x \in [\ell,\ell+\epsilon]} \mathbb{1}_{\det X < 0}.$$

Moreover,

$$\tilde{f}_\ell \overset{d}{=} g + \ell\alpha + Z_\ell \cdot \beta$$

where Z_ℓ is a random vector, independent of g, with density

$$p_{Z_\ell}(X) \propto |\det X| \, p_{f(0),\nabla^2 f(0)}(\ell, X) \mathbb{1}_{\det X < 0}.$$

The functions α, β and γ in Lemma 3.3.2 can be computed explicitly via Gaussian regression (see [5, Proposition 1.2]). Specifically, we define

$$v_0 = (f(0), \partial_{11} f(0), \partial_{22} f(0), \partial_{12} f(0))$$

and

$$v = (f(0), \nabla f(0), \partial_{11} f(0), \partial_{22} f(0), \partial_{12} f(0)),$$

and let Σ_0 and Σ be the respective covariance matrices of these vectors. Then

$$(\alpha(t), \beta_{11}(t), \beta_{22}(t), \beta_{12}(t)) = \mathrm{Cov}\,(f(t), v_0)\, \Sigma_0^{-1}$$

and

$$\gamma(s,t) = \mathrm{Cov}\,(f(s), f(t)) - \mathrm{Cov}\,(f(s), v)\, \Sigma^{-1} \mathrm{Cov}\,(f(t), v)'.$$

[1] Here and in the proof of this lemma we treat $Z_{[\ell,\ell+\epsilon]}$ interchangeably as the three-dimensional column vector $(Z_{[\ell,\ell+\epsilon],11}, Z_{[\ell,\ell+\epsilon],22}, Z_{[\ell,\ell+\epsilon],12})$ and the symmetric 2×2 matrix $\begin{pmatrix} Z_{[\ell,\ell+\epsilon],11} & Z_{[\ell,\ell+\epsilon],12} \\ Z_{[\ell,\ell+\epsilon],12} & Z_{[\ell,\ell+\epsilon],22} \end{pmatrix}$; which form we are using will always be clear from context. We also use this convention for Z_ℓ and X (introduced below).

In the case that $(f(0), \nabla^2 f(0))$ is degenerate (which includes the Random Plane Wave), the representations of $\tilde{f}_{[\ell,\ell+\epsilon]}$ and $\tilde{f}_\ell$ in Lemma 3.3.2 must be modified to accommodate this degeneracy; in particular, $\nabla^2 f(0)$ should be considered as a vector consisting of two of its coordinates, chosen so that they are non-degenerate with $f(0)$, and α, β and γ should be defined accordingly. For simplicity we will not state this representation formally; for the Random Plane Wave we state a more precise description below (in Proposition 3.3.4).

Lemmas 3.3.1 and 3.3.2 are essentially derived in [2, Chapter 6]; we repeat this here for completeness, and so that we can extend the arguments slightly.

Proof of Lemmas 3.3.1 and 3.3.2. We consider the case in which $(f(0), \nabla^2 f(0))$ is non-degenerate, since the degenerate case is similar. Let $T = (t_1, \ldots, t_m) \in \mathbb{R}^{2m}$ and $y_1, \ldots, y_m \in \mathbb{R}$. Then by the definition of $\tilde{f}_{[\ell,\ell+\epsilon]}$, and the Kac-Rice theorem (Theorem 1.3.2),

$$
\begin{aligned}
&\mathbb{P}\left(\tilde{f}_{[\ell,\ell+\epsilon]}(t_1) \le y_1, \ldots, \tilde{f}_{[\ell,\ell+\epsilon]}(t_m) \le y_m\right) \\
&\quad = \frac{\mathbb{E}\left(\#\{s \in B(1) : \nabla f(s) = 0, \det \nabla^2 f(s) < 0, f(s) \in [\ell, \ell+\epsilon], f(s+t_i) \le y_i \ \forall i\right)}{\mathbb{E}\left(\#\{s \in B(1) : \nabla f(s) = 0, \det \nabla^2 f(s) < 0, f(s) \in [\ell, \ell+\epsilon]\right)} \\
&\quad = \frac{\mathbb{E}\left(|\det \nabla^2 f(0)| \mathbb{1}_{f(0)\in[\ell,\ell+\epsilon]} \mathbb{1}_{\det \nabla^2 f(0)<0} \prod_{i=1}^m \mathbb{1}_{f(t_i)\le y_i} \middle| \nabla f(0) = 0\right)}{\mathbb{E}\left(|\det \nabla^2 f(0)| \mathbb{1}_{f(0)\in[\ell,\ell+\epsilon]} \mathbb{1}_{\det \nabla^2 f(0)<0} \middle| \nabla f(0) = 0\right)} \\
&\quad = \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_m} \frac{\int_{\mathbb{R}^3} \int_\ell^{\ell+\epsilon} |\det X| p_T(x,0,X,u) \mathbb{1}_{\det X<0} \, dx \, dX}{\int_{\mathbb{R}^3} \int_\ell^{\ell+\epsilon} |\det X| p(x,0,X) \mathbb{1}_{\det X<0} \, dx \, dX} \, du_m \ \ldots \ du_1,
\end{aligned}
$$

where p_T and p denote the densities of $(f(0), \nabla f(0), \nabla^2 f(0), f(t_1), \ldots, f(t_m))$ and $(f(0), \nabla f(0), \nabla^2 f(0))$ respectively. We note that p is non-degenerate since $\nabla f(0)$ is independent of $(f(0), \nabla^2 f(0))$ (Lemma A.2.2) and these vectors are non-degenerate by assumption. The density p_T may be degenerate, in which case we think of it as having atomic mass. Rearranging these terms, we can express the joint density of $(\tilde{f}_{[\ell,\ell+\epsilon]}(t_i) \ : i = 1, \ldots, m)$ as

$$
\varphi_T^{[\ell,\ell+\epsilon]}(u) := \int_{\mathbb{R}^3} \int_\ell^{\ell+\epsilon} \psi_x(X) p_T(x,0,X,u)/p(x,0,X) \, dx \, dX
$$

where

$$
\psi_x(X) = \frac{|\det X| p(x,0,X) \mathbb{1}_{\det X<0}}{\int_{\mathbb{R}^3} \int_\ell^{\ell+\epsilon} |\det X| p(x,0,X) \mathbb{1}_{\det X<0} \, dx \, dX}.
$$

Then the characteristic function of $(\tilde{f}_{[\ell,\ell+\epsilon]}(t_1), \ldots, \tilde{f}_{[\ell,\ell+\epsilon]}(t_m))$ is given by

$$
\hat{\varphi}_T^{[\ell,\ell+\epsilon]}(\theta) = \int_{\mathbb{R}^3} \int_\ell^{\ell+\epsilon} \psi_x(X) \int_{\mathbb{R}^m} e^{i\theta\cdot u} p_T(x,0,X,u)/p(x,0,X) \, du \, dx \, dX. \tag{3.9}
$$

The inner integral of equation (3.9) can be calculated by Gaussian regression (see [5, Proposition 1.2]). Specifically, let $A = (\alpha(t_1), \dots, \alpha(t_m))'$, $B = (\beta(t_1), \dots, \beta(t_m))'$ and $\Gamma = (\gamma(t_i, t_j))_{i,j=1,\dots,m}$. Then

$$\big(f(t_1), \dots, f(t_m) \,\big|\, f(0) = x, \nabla f(0) = 0, \nabla^2 f(0) = X\big) \sim \mathcal{N}(Ax + BX, \Gamma).$$

Since $p_T(x, 0, X, u)/p(x, 0, X)$ is the probability density of this random variable, we can substitute the characteristic function of a Gaussian vector into (3.9) to give

$$\begin{aligned}\hat{\varphi}_T^{[\ell,\ell+\epsilon]}(\theta) &= \int_{\mathbb{R}^3} \int_{\ell}^{\ell+\epsilon} \psi_x(X) e^{i\theta\cdot(Ax+BX)-\frac{1}{2}\theta'\Gamma\theta}\, dx\, dX \\ &= e^{-\frac{1}{2}\theta'\Gamma\theta} \frac{\int_{\mathbb{R}^4} e^{i\theta\cdot(Ax+BX)} |\det X| p(x,0,X) \mathbb{1}_{x\in[\ell,\ell+\epsilon]} \mathbb{1}_{\det X<0}\, dx\, dX}{\int_{\mathbb{R}^4} |\det X| p(x,0,X) \mathbb{1}_{x\in[\ell,\ell+\epsilon]} \mathbb{1}_{\det X<0}\, dx\, dX}.\end{aligned}$$

Since the characteristic function of a random vector uniquely specifies its distribution, we identify the distribution of $\tilde{f}_{[\ell,\ell+\epsilon]}$ as that given in the statement of Lemma 3.3.2 (using the fact that $p(x, 0, X) = p_{f(0),\nabla^2 f(0)}(x, X)$ since $\nabla f(0)$ is independent of $(f(0), \nabla^2 f(0))$).

By inspecting their joint distribution, it is clear that $z_{[\ell,\ell+\epsilon]} \xrightarrow{d} \ell$ and $Z_{[\ell,\ell+\epsilon]} \xrightarrow{d} Z_\ell$ as $\epsilon \to 0$. We now fix a sequence $\epsilon_i \downarrow 0$, and create a coupling of $\tilde{f}_{[\ell,\ell+\epsilon_i]}$ for each i such that each field consists of the same realisation of g and the sequences $\{z_{[\ell,\ell+\epsilon_i]}\}_{i\in\mathbb{N}}$ and $\{Z_{[\ell,\ell+\epsilon_i]}\}_{i\in\mathbb{N}}$ converge almost surely. Since $K \in C^{4+\eta'}_{\mathrm{loc}}(\mathbb{R}^2)$, the same is true of α, β and γ and hence $g \in C^{2+\eta}_{\mathrm{loc}}(\mathbb{R}^2)$ almost surely for the choice of $\eta \in (0, \eta'/2)$ made at the beginning of Section 2.1 (Kolmogorov's theorem: Theorem A.1.2). It is therefore clear that the coupled fields $\tilde{f}_{[\ell,\ell+\epsilon_i]}$ converge almost surely in the $C^{2+\eta}$ topology uniformly on compact sets as $i \to \infty$ to $\tilde{f}_\ell$. This completes the proof of the lemmas. $\square$

We now present simpler descriptions for $\tilde{f}_\ell$ in the case of isotropic fields. In this case it is quite natural to express the Hessian component Z_ℓ in terms of its eigenvalues $\lambda_1 < \lambda_2$ and the argument of the first eigenvector θ. Recall the parameter $\chi = -k'(0)/\sqrt{k''(0)} \in (0, \sqrt{2}]$, where $K(t) = k(|t|^2)$. Again we must distinguish the case in which $(f(0), \nabla^2 f(0))$ is degenerate, which corresponds to $\chi = \sqrt{2}$ and implies that f is (a rescaled version of) the Random Plane Wave.

Proposition 3.3.3. *Let f be an isotropic field satisfying Assumption 2.1.1 such that $\chi < \sqrt{2}$. Then*

$$\tilde{f}_\ell(\cdot) \stackrel{d}{=} g(\cdot) + \ell\alpha(\cdot) + \lambda_1 b_1(\cdot, \theta) + \lambda_2 b_2(\cdot, \theta),$$

where g, α and β are as in Lemma 3.3.2,

$$b_1(t,\theta) = \cos^2(\theta)\beta_{11}(t) + \sin^2(\theta)\beta_{22}(t) + \sin(\theta)\cos(\theta)\beta_{12}(t),$$
$$b_2(t,\theta) = \sin^2(\theta)\beta_{11}(t) + \cos^2(\theta)\beta_{22}(t) + \sin(\theta)\cos(\theta)\beta_{12}(t),$$

θ is an independent random variable, uniform on $[0, 2\pi)$, and (λ_1, λ_2) is an independent random vector with density proportional to

$$q_\ell(x,y) := |x|y(y-x)\mathbb{1}_{y>0>x} \exp\left(-\frac{(x-\mu\ell)^2 + (y-\mu\ell)^2 + 2\tau(x-\mu\ell)(y-\mu\ell)}{2\sigma^2}\right),$$

where

$$\mu = 2k'(0)\ , \quad \sigma^2 = \frac{16k''(0)(2-\chi^2)}{3-\chi^2} \quad \text{and} \quad \tau = \frac{\chi^2 - 1}{3-\chi^2}. \tag{3.10}$$

Proof. Recall the random vector Z_ℓ from Lemma 3.3.2, which we view as a 2×2 symmetric matrix. Let $\lambda_1 < \lambda_2$ be the eigenvalues of Z_ℓ, and let θ be the argument of the eigenvector associated to λ_1. If h denotes the bijection which maps Z_ℓ to $\Lambda := (\lambda_1, \lambda_2, \theta)$, then for any Borel set A

$$\mathbb{P}(\Lambda \in h(A)) = \frac{\mathbb{E}\left(|\det \nabla^2 f(0)|\ \mathbb{1}_{\det \nabla^2 f(0)<0, \nabla^2 f(0)\in A} | f(0) = \ell\right)}{\mathbb{E}\left(|\det \nabla^2 f(0)|\ \mathbb{1}_{\det \nabla^2 f(0)<0} | f(0) = \ell\right)}.$$

Since f is isotropic and $(f(0), \nabla^2 f(0))$ is non-degenerate, [26] derives the density of the ordered eigenvalues of $(\nabla^2 f(0)|f(0) = \ell)$ and the arguments of the corresponding eigenvectors as that given above. □

Proposition 3.3.4. *Let f be the Random Plane Wave. Then*

$$\tilde{f}_\ell \overset{d}{=} g + \ell\alpha + Z^\ell \cdot \beta,$$

where g is a centred Gaussian field with covariance function γ (which is defined as in Lemma 3.3.2), α, β are defined as

$$\alpha(t) = J_0(|t|) + 2\frac{t_1^2 - t_2^2}{|t|^2} J_2(|t|),$$
$$\beta_{11}(t) = 4\frac{t_1^2 - t_2^2}{|t|^2} J_2(|t|), \qquad \beta_{12}(t) = 8\frac{t_1 t_2}{|t|^2} J_2(|t|),$$

and $Z^\ell = (Z_{11}^\ell, Z_{12}^\ell)^t$ is an independent random vector with density

$$\psi_\ell(x,y) \propto (x(x+\ell) + y^2)\mathbb{1}_{(x(x+\ell)+y^2)>0}\ p_{f(0), f_{11}(0), f_{12}(0)}(\ell, x, y).$$

Alternatively, $\tilde{f}_\ell$ has the representation

$$\tilde{f}_\ell(t) = g(t) + \ell \cdot [J_0(|t|) + 2\cos(2(\theta - \arg t))J_2(|t|)] + \lambda \cdot 4\cos(2(\theta - \arg t))J_2(|t|),$$

where $\arg t$ *denotes the argument of* t *and* $(\theta, \lambda) = (\theta, \lambda_\ell)$ *is a random vector, independent of* g, *with density*

$$p_{\lambda_\ell,\theta}(x, y) \propto x(x + \ell)(2x + \ell)e^{-4x(x+\ell)}\mathbb{1}_{x>\max\{0,-\ell\}}\mathbb{1}_{y\in[0,2\pi)}. \tag{3.11}$$

Proof. The representation

$$\tilde{f}_\ell = g + \ell\alpha + Z^\ell \cdot \beta$$

follows from an argument similar to that used to prove Lemma 3.3.2. The functions α and β can be explicitly calculated using Gaussian regression (see [5, Proposition 1.2] for example). Next we note that $(Z^\ell_{11}, Z^\ell_{12})$ is supported on the region for which

$$\det\begin{pmatrix} Z^\ell_{11} & Z^\ell_{12} \\ Z^\ell_{12} & -Z^\ell_{11} - \ell. \end{pmatrix} < 0.$$

Therefore this matrix almost surely has a unique, positive eigenvalue λ and corresponding eigenvector with argument θ. By explicitly diagonalising this matrix, we obtain a formula for λ and θ:

$$\begin{aligned} Z^\ell_{11} + \ell/2 &= (\lambda + \ell/2)\cos(2\theta) \\ Z^\ell_{12} &= (\lambda + \ell/2)\sin(2\theta). \end{aligned}$$

By the standard change of variable formula (and explicitly evaluating ψ_ℓ in terms of the covariance of the Random Plane Wave) we can calculate the joint density of (λ, θ) to be equal to the expression in (3.11). □

3.4 Differentiability of the mean functional

In this section we prove the results stated in Section 3.2.1. We begin by studying the space $C^{2+\eta}_{\mathrm{Reg}}$ of functions $h \in C^{2+\eta}_{\mathrm{loc}}(\mathbb{R}^2)$ which have a non-degenerate critical point at the origin and no other critical points at level $h(0)$. We will also use the space $C^{2+\eta}_{\mathrm{Reg}}(R)$ which is the set of all $h \in C^{2+\eta}_{\mathrm{Reg}}$ such that $h(0)$ is not a critical level of $h|_{\partial B(R)}$. We endow these spaces with the $C^{2+\eta}_{\mathrm{loc}}$ topology.

By showing that $\tilde{f}_\ell \in C^{2+\eta}_{\mathrm{Reg}}(R)$ almost surely, we prove that $\tilde{f}_\ell$ having an upper (or lower) connected saddle point in a compact region is a continuity event, from this we deduce Theorem 3.2.5 (with the other results following as consequences).

Lemma 3.4.1. *If* $h \in C^{2+\eta}_{Reg}$ *has a saddle point at the origin, then this saddle point is either upper connected, lower connected or an infinite four-arm saddle.*

Proof. For a small enough neighbourhood B of the origin, the level set $\{h = h(0)\}$ in $B \setminus \{0\}$ consists of four curves that connect 0 to ∂B (the Morse lemma [72, Lemma 2.2]). If the connected components of these curves in $\mathbb{R}^2 \setminus B$ are all unbounded, the saddle point must be infinite four-arm. If one of them is finite, then by the implicit function theorem it is a simple C^1 curve joining two points on ∂B. Hence the saddle point is either upper connected (if the field takes values larger than $h(0)$ on the outer boundary of the loop) or lower connected (if the field takes values smaller the $h(0)$ on the outer boundary of the loop). □

We now consider saddle points which are upper or lower connected in a compact domain. Specifically, for a C^2 function h with a saddle point x_0 we say that x_0 is R-*lower connected* if it is in the closure of only one component of $\{h < h(x_0)\} \cap B(x_0, R)$. We make an analogous definition for R-upper connected saddles.

Lemma 3.4.2. *Let $s^-(R)$ be the subset of functions $h \in C^{2+\eta}_{Reg}(R)$ such that the origin is an R-lower connected saddle point of h, then $s^-(R)$ is open and closed in $C^{2+\eta}_{Reg}(R)$. The same is true for the set $s^+(R)$ of functions with R-upper connected saddle points.*

Proof. Let $h \in C^{2+\eta}_{\mathrm{Reg}}(R)$ have a saddle point at the origin which is R-lower connected; we will find a neighbourhood around h which contains only functions with such saddle points at the origin. First we choose $r \in (0, 1)$ sufficiently small that h has a four-arm saddle in $B(r)$. Since the origin is a non-degenerate saddle point for h, $\nabla^2 h(0)$ has eigenvalues $\lambda_1 < 0 < \lambda_2$ and corresponding eigenvectors v_1, v_2. We now choose a neighbourhood $N_1 \subset C^{2+\eta}_{\mathrm{Reg}}(R)$ of h (in the topology of uniform $C^{2+\eta}$ convergence) such that for all $g \in N_1$,

$$\partial_{v_1,v_1} g(0) < \lambda_1/2 \quad \text{and} \quad \partial_{v_2,v_2} g(0) > \lambda_2/2.$$

This ensures that each function in N_1 also has a saddle point at the origin.

Next we choose $N_2 \subset C^{2+\eta}_{\mathrm{Reg}}(R)$ such that for each $g \in N_2$, $\|g\|_{2+\eta} \leq 2\|h\|_{2+\eta}$. We consider the four line segments joining 0 to $\partial B(r)$ parallel to v_1 and v_2 and we reduce r relative to $\|h\|_{2+\eta}$ so that for each $g \in N_1 \cap N_2$, the directional derivative of g on this line segment (parallel to the line segment) has constant sign. This ensures that for each such g the saddle point at the origin is four-arm in $B(r)$.

There exist two connected subsets A_1, A_2 of $\partial B(r)$ such that $h < h(0) - 3\epsilon$ on $A_1 \cup A_2$ for some $\epsilon > 0$ and A_1 and A_2 are in different components of $\overline{B(r)} \cap \{h < h(0)\}$ (see Figure 3.2). We next choose a neighbourhood $N_3 \subset C^{2+\eta}_{\mathrm{Reg}}(R)$ of h such that A_1 and A_2 have the same properties for any function $g \in N_3$ with 3ϵ replaced by 2ϵ and $|g(0) - h(0)| < \epsilon$.

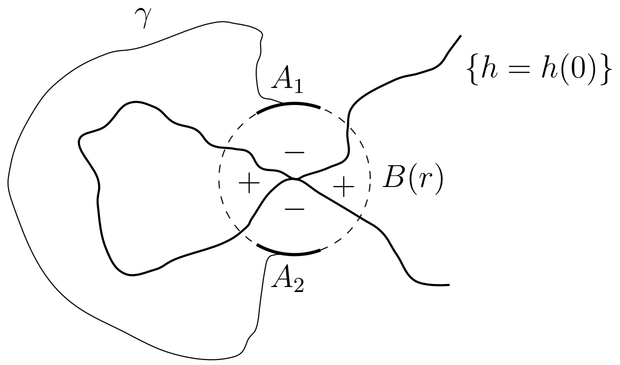


Figure 3.2: Approximating an R-lower connected saddle point in the $C^{2+\eta}\left(\overline{B(R)}\right)$ topology.

By definition of a saddle being R-lower connected, there is a curve γ in $B(R)$ joining A_1 to A_2 in $\{h < h(0)\}$ and h is bounded above by $h(0) - 3\delta$ on γ for some $\delta > 0$. Since γ is compact we can find a neighbourhood N_4 such that $g < h(0) - 2\delta$ on γ for all $g \in N_4$ and $|h(0) - g(0)| < \delta$. Combining these observation, we see that $N := N_1 \cap N_2 \cap N_3 \cap N_4$ is a neighbourhood of h (in $C^{2+\eta}_{\text{Reg}}(R)$) and any $g \in N$ has a saddle point at the origin which is lower connected in $B(R)$ and so the set of functions with such saddle points is open, as required.

The set $C^{2+\eta}_{\text{Reg}}$ can be partitioned into sets of functions which have either a local maximum, a local minimum, a saddle point which is four-arm in $B(R)$ or a saddle point which is R-upper/lower connected at the origin. Arguments which are very similar to those above show that each of these subsets is open, hence proving the statement of the lemma. (For saddle points which are four-arm in $B(R)$, we use the fact that $h(0)$ is not a critical level of $h|_{\partial B(R)}$, which implies that the four level lines emanating from the origin intersect $\partial B(R)$ at different points.) $\square$

We next confirm that $\tilde{f}_\ell \in C^{2+\eta}_{\text{Reg}}(R)$ almost surely:

Lemma 3.4.3. *If f is a Gaussian field satisfying Assumptions 2.1.1 and 3.2.1, then for any $\ell \in \mathbb{R}$ and $R > 0$, $\tilde{f}_\ell \in C^{2+\eta}_{\text{Reg}}(R)$ almost surely.*

Proof. To simplify the presentation we assume that $(f(0), \nabla^2 f(0))$ is non-degenerate; the degenerate case is similar. Recall the representation of $\tilde{f}_\ell$ in Lemma 3.3.2. By the definitions of α, β and γ, $\tilde{f}_\ell$ is almost surely in $C^{2+\eta}_{\text{loc}}(\mathbb{R}^2)$, and has a critical point at the origin at level ℓ. By evaluating the second order derivatives of α, β and γ, it follows that $\nabla^2 \tilde{f}_\ell(0) = Z_\ell$. Since the density of Z_ℓ is identically zero on the region where its determinant is zero, $\det \nabla^2 \tilde{f}_\ell(0) \neq 0$ almost surely, and so the critical point at the origin is non-degenerate.

Next we show that $\tilde{f}_\ell$ almost surely has no other critical points at level ℓ. Let $T_n = \overline{B(n)} \backslash B(\frac{1}{n})$ and consider $(\nabla \tilde{f}_\ell, \tilde{f}_\ell - \ell) : T_n \to \mathbb{R}^3$. Bulinskaya's lemma (Lemma A.2.1) states that $\tilde{f}_\ell$ almost surely has no critical points at level ℓ in T_n, provided that the univariate densities of $(\nabla \tilde{f}_\ell(t), \tilde{f}_\ell(t))$ are bounded in a neighbourhood of $(0, \ell)$ uniformly over $t \in T_n$. Since g and Z_ℓ are independent, the density of $(\nabla \tilde{f}_\ell(t), \tilde{f}_\ell(t))$ is given by the convolution

$$\begin{aligned} p_{\nabla \tilde{f}_\ell(t), \tilde{f}_\ell(t)}(x) &= \int_{\mathbb{R}^3} p_{\nabla g(t), g(t)}(x-u) p_{\nabla (Z_\ell \cdot \beta(t) + \ell \alpha(t)), Z_\ell \cdot \beta(t) + \ell \alpha(t)}(u)\, du \\ &\leq \sup_{x \in \mathbb{R}^3} p_{\nabla g(t), g(t)}(x) \int_{\mathbb{R}^3} p_{\nabla (Z_\ell \cdot \beta(t) + \ell \alpha(t)), Z_\ell \cdot \beta(t) + \ell \alpha(t)}(u)\, du \\ &= \sup_{x \in \mathbb{R}^3} p_{\nabla g(t), g(t)}(x). \end{aligned}$$

Therefore, to show that $p_{\nabla \tilde{f}_\ell(t), \tilde{f}_\ell(t)}$ is bounded, it is sufficient to show that the density of $(\nabla g(t), g(t))$ is bounded uniformly in t. Since these densities are Gaussian, this is equivalent to showing that the determinant of the covariance matrix of $(\nabla g(t), g(t))$ is bounded away from 0 on T_n. However this is the determinant of

$$\mathrm{Cov}\left((\nabla f(t), f(t)) \,\middle|\, f(0), \nabla f(0), \nabla^2 f(0) \right)$$

which is non-degenerate for each $t \in T_n$ by Assumption 3.2.1. Since this determinant is continuous in t, it is bounded away from 0 on the compact set T_n. Taking the countable union of T_n for $n \in \mathbb{N}$ then shows that $\tilde{f}_\ell$ almost surely has no critical points at level ℓ in $\mathbb{R}^2 \backslash \{0\}$.

To verify that $\tilde{f}_\ell|_{\partial B(R)}$ almost surely has no critical points at level ℓ, we apply an identical argument to

$$\left(\begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \cdot \nabla \tilde{f}_\ell(y), \tilde{f}_\ell(y) \right)$$

where $y = (R\cos(\theta), R\sin(\theta))$. This completes the proof that $\tilde{f}_\ell \in C^{2+\eta}_{\mathrm{Reg}}(R)$ almost surely. $\square$

We are now ready to prove Theorem 3.2.5. Let $N^{(R)}_{s^-}[\ell_1, \ell_2]$ denote the number of R-lower connected saddle points of f in $B(1)$ with height in $[\ell_1, \ell_2]$. If $N^{(R)}_{s^-}$ is replaced with N_{s^-} or N_s, we make a corresponding definition for lower connected saddle points or saddle points respectively. Recall that $s^-(R)$ is the subset of functions in $C^{2+\eta}_{\mathrm{Reg}}(R)$ with an R-lower connected saddle point at the origin. We also define s^- and s^+ to be the subsets of $C^{2+\eta}_{\mathrm{Reg}}$ with lower and upper connected saddle points at the origin respectively.

Proof of Theorem 3.2.5. Let f be a field satisfying Assumptions 2.1.1 and 3.2.1. The first step is to show that $p^*_{s^-}$ is lower semi-continuous, by expressing it as the pointwise supremum of a sequence of continuous functions. Let $\ell \in \mathbb{R}$ and $\epsilon > 0$ and we fix $R > 0$. We now claim that

$$\frac{\mathbb{E}\left(N^{(R)}_{s^-}[\ell,\ell+\epsilon]\right)}{\mathbb{E}\left(N_s[\ell,\ell+\epsilon]\right)} = \mathbb{P}\left(\tilde{f}_{[\ell,\ell+\epsilon]} \in s^-(R)\right).$$

We first note that, by Lemma 3.4.2, the event $s^-(R)$ is contained in the Borel σ-algebra generated by the $C^{2+\eta}_{\mathrm{Reg}}(R)$ topology and that $\tilde{f}_{[\ell,\ell+\epsilon]}$ is measurable with respect to this σ-algebra. Furthermore, by an elementary argument (see, for example, [81, Lemma A.1]) this σ-algebra is generated by cylinder sets; those which depend on the value of the function at only finitely many points. Since the distribution of $\tilde{f}_{[\ell,\ell+\epsilon]}$ on cylinder sets is defined in (3.8) as an empirical measure, these two measures must coincide on the σ algebra generated by this π-system. This verifies the claim.

By Lemma 3.3.1, $\tilde{f}_{[\ell,\ell+\epsilon]}$ converges in distribution to $\tilde{f}_\ell$ (in the $C^{2+\eta}_{loc}$ topology) as $\epsilon \to 0$, and since having a saddle point at the origin which is R-lower connected is a continuity event for $\tilde{f}_\ell$ (Lemmas 3.4.2 and 3.4.3), the portmanteau lemma implies that

$$\mathbb{P}\left(\tilde{f}_{[\ell,\ell+\epsilon]} \in s^-(R)\right) \to \mathbb{P}\left(\tilde{f}_\ell \in s^-(R)\right)$$

as $\epsilon \to 0$. By inspecting the form of p_{Z_ℓ}, it is clear that $\tilde{f}_\ell \xrightarrow{d} \tilde{f}_{\ell_0}$ as $\ell \to \ell_0$ in the $C^{2+\eta}_{loc}$ topology. So by applying the portmanteau lemma again, we see that $\mathbb{P}(\tilde{f}_\ell \in s^-(R))$ is continuous in ℓ. Hence the function

$$p^{(R)}_{s^-}(\ell) := p_s(\ell)\mathbb{P}\left(\tilde{f}_\ell \in s^-(R)\right), \tag{3.12}$$

is continuous in ℓ. Now note that

$$\frac{1}{\epsilon}\mathbb{E}\left(N^{(R)}_{s^-}[\ell,\ell+\epsilon]\right) = \frac{\mathbb{E}\left(N_s[\ell,\ell+\epsilon]\right)}{\epsilon}\frac{\mathbb{E}\left(N^{(R)}_{s^-}[\ell,\ell+\epsilon]\right)}{\mathbb{E}(N_s[\ell,\ell+\epsilon])} \to p^{(R)}_{s^-}(\ell)$$

as $\epsilon \to 0$ (by Proposition 2.1.6). Hence $\mathbb{E}\left(N^{(R)}_{s^-}[-\infty,\ell]\right)$ is differentiable in ℓ with derivative $p^{(R)}_{s^-}(\ell)$. We now allow R to vary; since $s^-(R)$ is non-decreasing in R and $\cup_{R>0}s^-(R) = s^-$, taking the limit of (3.12) shows that

$$p^*_{s^-}(\ell) = \lim_{R\to\infty} p^{(R)}_{s^-}(\ell)$$

for each $\ell \in \mathbb{R}$. Hence $p^*_{s^-}$ is indeed a pointwise supremum of continuous functions, and so is lower semi-continuous.

We next prove that $p^*_{s^-} = p_{s^-}$ almost everywhere. Let $a < b$, then since $|p^*_{s^-} - p_{s^-}|$ is bounded, by dominated convergence

$$\begin{aligned}\int_a^b p_{s^-}(x) - p^*_{s^-}(x)\,dx &= \lim_{R\to\infty}\int_a^b p_{s^-}(x) - p^{(R)}_{s^-}(x)\,dx\\ &= \lim_{R\to\infty}\mathbb{E}\left(N_{s^-}[a,b] - N^{(R)}_{s^-}[a,b]\right) = 0,\end{aligned}$$

where in the last line we have used the definition of p_{s^-}, the fundamental theorem of calculus applied to $\mathbb{E}(N^{(R)}_{s^-}[-\infty,\ell])$ (along with the differentiability proven above), and then dominated convergence once again. Since a and b are arbitrary, we conclude that $p^*_{s^-} = p_{s^-}$ almost everywhere.

To finish the proof, we show that (3.3), the condition that $\tilde{f}_\ell$ does not have an infinite four-arm saddle, implies the continuity of $p^*_{s^-}$. Observe that, by repeating the arguments above, we may define the lower semi-continuous function

$$p^*_{s^+}(\ell) := p_s(\ell)\mathbb{P}\left(\tilde{f}_\ell \in s^+\right)$$

which is a version of p_{s^+}. By Lemmas 3.4.1 and 3.4.3, the saddle point of $\tilde{f}_\ell$ at the origin must be either upper connected, lower connected or an infinite four-arm saddle. Therefore

$$\begin{aligned}\mathbb{P}\left(\tilde{f}_\ell \text{ has an infinite four-arm saddle}\right) &= 1 - \mathbb{P}\left(\tilde{f}_\ell \in s^+ \cup s^-\right)\\ &= 1 - \frac{p^*_{s^+}(\ell)}{p_s(\ell)} - \frac{p^*_{s^-}(\ell)}{p_s(\ell)}.\end{aligned} \tag{3.13}$$

(Note that $p_s(\ell) > 0$ by Lemma A.2.5.) Now suppose that (3.3) holds, that is, for all $\ell \in (a,b)$, $\tilde{f}_\ell$ almost surely does not have an infinite four-arm saddle point at the origin. By (3.13) we see that $p^*_{s^+}(\ell) = p_s(\ell) - p^*_{s^-}(\ell)$ for all $\ell \in (a,b)$. Since $p^*_{s^+}$ is lower semi-continuous (and p_s is continuous), we deduce that $p^*_{s^-}$ is upper semi-continuous on (a,b). Hence we have shown that $p^*_{s^-}$ is both upper and lower semi-continuous on (a,b), which completes the result. □

As mentioned previously, Theorem 3.2.4 follows from Theorem 3.2.5 once we verify condition (3.3). This is done in the next lemma:

Lemma 3.4.4. *Let f be a Gaussian field satisfying Assumptions 2.1.1 and 3.2.2–3.2.3. Then for every $\ell \geq 0$ and $r > 0$,*

$$\mathbb{P}\left(\tilde{f}_\ell \in \mathrm{Arm}_\ell(r,R)\right) \to 0 \tag{3.14}$$

as $R \to \infty$. In particular, for all $\ell \in \mathbb{R}$, $\tilde{f}_\ell$ almost surely does not have an infinite four-arm saddle point at the origin.

Proof. We first note that if $\tilde{f}_\ell$ has an infinite four-arm saddle at the origin, then both $\{\tilde{f}_\ell \geq \ell\}$ and $\{\tilde{f}_\ell \leq \ell\} = \{-\tilde{f}_\ell \geq -\ell\}$ have unbounded components containing the origin. Then, since $\tilde{f}_\ell$ and $-\tilde{f}_{-\ell}$ have the same distribution by Lemma 3.3.2, the second claim of this lemma follows from the first.

Since the event $\mathrm{Arm}_\ell(r,R)$ is weakly increasing in r, it is sufficient to prove (3.14) for a sequence $r_R \to \infty$ as $R \to \infty$. This allows us to make use of the fact that, far from the origin, the distribution of $\tilde{f}_\ell$ is close to that of f.

By Assumption 3.2.2 and Lemma A.2.4, we know that $(f(0), \nabla^2 f(0))$ is non-degenerate. Recall the representation for $\tilde{f}_\ell$ in Lemma 3.3.2

$$\tilde{f}_\ell = g + \ell\alpha + Z_\ell \cdot \beta,$$

and recall also the explicit expressions for α, β and the covariance of g derived after this lemma. Since this covariance is expressed as the difference of two positive definite functions, if we let f_1 be a centred Gaussian field with covariance

$$K_1(s,t) = \mathrm{Cov}\left(f(s), v\right) \Sigma^{-1} \mathrm{Cov}\left(f(t), v\right)',$$

then we can decompose $f = g + f_1$, where f_1 and g are independent. Since K_1 can be expressed as a linear combination of $\partial^{k_1} K(s) \partial^{k_2} K(t)$, for $|k_1|, |k_2| \leq 2$, by Assumption 3.2.2 there exists $c_1, \zeta > 0$ such that, for all $r > 1$,

$$\sup_{s,t \notin B(r)} \sup_{|k| \leq 2} |\partial^k K_1(s,t)| \leq c_1 r^{-2(1+\zeta)}. \tag{3.15}$$

Moreover, since α, β can be expressed as a linear combination of $\partial^k K(t)$, for $|k| \leq 2$, by Assumption 3.2.2 there exists $c_2, \zeta > 0$ such that

$$\sup_{|t|>r} |\alpha(t)| \leq c_2 r^{-(1+\zeta)} \quad \text{and} \quad \sup_{|t|>r} \|\beta(t)\|_\infty \leq c_2 r^{-(1+\zeta)}. \tag{3.16}$$

Next, we fix $\ell \geq 0$ and apply a Cameron-Martin argument to the unconditional field f. In particular, by [77, Corollary 3.7] (valid by the condition on the spectral density in Assumption 3.2.2, and since $\mathrm{Arm}_\ell(r,R)$ is an increasing event with respect to the field) there exists $c_3, r_0 > 0$ such that for all $r > r_0$ the following holds: if $F : \mathbb{R}^2 \to \mathbb{R}$ is a continuous random field coupled with f such that

$$\mathbb{P}\left(\|f - F\|_{\infty, A(r,R)} \geq \epsilon\right) \leq \delta$$

where $\|\cdot\|_{\infty, A(r,R)}$ denotes the supremum norm on $A(r,R)$ the centred annulus of inner radius r and outer radius R, then

$$\mathbb{P}\left(F \in \mathrm{Arm}_\ell(r,R)\right) \leq \mathbb{P}\left(f \in \mathrm{Arm}_\ell(r,R)\right) + \delta + c_3 R \epsilon. \tag{3.17}$$

We will apply this bound to

$$F := \tilde{f}_\ell = f - f_1 + \ell\alpha + Z_\ell \cdot \beta$$

Note that, by the union bound,

$$\begin{aligned}
\mathbb{P}\left(\left\|f - \tilde{f}_\ell\right\|_{\infty,A(r,R)} \geq \epsilon\right) \\
&\leq \mathbb{P}\left(\|\ell\alpha + Z_\ell \cdot \beta\|_{\infty,A(r,R)} \geq \epsilon/2\right) + \mathbb{P}\left(\|f_1\|_{\infty,A(r,R)} \geq \epsilon/2\right) \\
&\leq \mathbb{1}\left\{\ell\|\alpha\|_{\infty,A(r,R)} \geq \epsilon/8\right\} + \sum_{i\in\{11,12,22\}} \mathbb{P}\left(|Z_{\ell,i}|\,\|\beta_i\|_{\infty,A(r,R)} \geq \epsilon/8\right) \\
&\quad + \mathbb{P}\left(\|f_1\|_{\infty,A(r,R)} \geq \epsilon/2\right).
\end{aligned} \tag{3.18}$$

We now show that, for a suitable choice of $r = r_R \to \infty$ and $\epsilon = \epsilon_R \to 0$, the three terms in (3.18) all decay to zero as $R \to \infty$.

By (3.16), and since $Z_{\ell,i}$ is almost surely finite, the first two terms in (3.18) converge to zero as long as $\epsilon r^{1+\zeta} \to \infty$. If we assume this convergence is sufficiently fast (to be specified below) then it is a standard estimate for the norm of a Gaussian field that the third term of (3.18) also converges to zero. This argument is essentially the same as [77, Lemma 3.12], but our setting is slightly different so we give a complete proof.

Let $B_x(1)$ denote the ball of radius 1 centred at x. Covering $A(r,R)$ with $O(R^2)$ unit balls, and using the union bound,

$$\mathbb{P}\left(\|f_1\|_{\infty,A(r,R)} \geq \epsilon/2\right) \leq c_3 R^2 \sup_{x\in A(r,R)} \mathbb{P}\left(\|f_1\|_{\infty,B_x(1))} \geq \epsilon/2\right).$$

By the Borell–TIS inequality ([1, Theorem 2.1.1]), for all $u > 0$,

$$\mathbb{P}\left(\|f_1\|_{\infty,B_x(1))} \geq m_x + u\right) \leq 2e^{-u^2/(2\sigma_x^2)},$$

where

$$m_x = \mathbb{E}[\|f_1\|_{\infty,B_x(1)}] \quad \text{and} \quad \sigma_x^2 = \sup_{y\in B_x(1)} K_1(y,y).$$

By Kolmogorov's theorem [81, Appendix A.9], there is a $c_4 > 0$ such that

$$m_x < c_4 \sup_{s,t\in B_x(1)} \sup_{|a_1|,|a_2|\leq 1} \left(\partial^{a_1,a_2} K_1(s,t)\right)^{1/2}.$$

Therefore, by (3.15),

$$\sup_{x\in A(r,R)} m_x < c_5 r^{-1-\zeta} \quad \text{and} \quad \sup_{x\in A(r,R)} \sigma_x^2 < c_5 r^{-2-2\zeta}.$$

Taking $u = \epsilon/4$ and assuming that $\epsilon/4 > c_5 r^{-1-\zeta}$ we have

$$\mathbb{P}\left(\|f_1\|_{\infty,A(r,R)} \geq \epsilon/2\right) \leq 2c_6 R^2 \exp(-c_7\epsilon^2 r^{2+2\zeta}).$$

To finish, we take

$$r = \frac{R}{\log(R)} \quad \text{and} \quad \epsilon = \frac{1}{R\log(R)}$$

and observe that for this choice the right hand side of the estimate above converges to 0 as $R \to \infty$. Combining all of these estimates together we have that the right hand side of (3.18) tends to zero as $R \to \infty$.

Substituting into (3.17), and noting that $r/R \to 0$ and $R\epsilon \to 0$ as $R \to \infty$, proves that $\mathbb{P}(\tilde{f}_\ell \in \mathrm{Arm}_\ell(r,R))$ can be made arbitrarily small, which completes the proof of the lemma. □

Proof of Theorem 3.2.4. This is immediate from Theorem 3.2.5 and Lemma 3.4.4. □

To end the section we prove the remaining results stated in Section 3.2.1, namely Corollary 3.2.6 and Proposition 3.2.8.

Proof of Corollary 3.2.6. By Lemmas 2.2.4 and 2.4.5, $\mathbb{E}(N_{4\text{-arm}}(R)) = O(R)$, so it suffices to prove the other bound here. Recall that $A(R-r,R)$ denotes the annulus of inner radius $R-r$ and outer radius R. We first note that for any $1 < r < R$

$$N_{4-arm}(R,[a_R,b_R]) \leq N_{\mathrm{crit}}\left(A\left(R-r,R\right),[a_R,b_R]\right) + N_{4-arm,r}\left(B\left(R-r\right),[a_R,b_R]\right)$$

where, by a slight abuse of notation, $N_{\mathrm{crit}}\left(A\left(R-r,R\right),[a_R,b_R]\right)$ denotes the number of critical points in $A\left(R-r,R\right)$ which have level in $[a_R,b_R]$, and we denote by $N_{4-arm,r}(B(R-r),[a_R,b_R])$ the number of saddle points $t \in B(R-r)$ which are four-arm in $B(t,r)$ and have level in $[a_R,b_R]$. Using the Kac-Rice theorem (Theorem 1.3.2) and the independence of $(f(0),\nabla^2 f(0))$ and $\nabla f(0)$

$$\begin{aligned}
&\mathbb{E}(N_{\mathrm{crit}}(A(R-r,R),[a_R,b_R])) \\
&\qquad = \int_{A(R-r,R)} \mathbb{E}\left(\left|\det\left(\nabla^2 f(0)\right)\right| \mathbb{1}_{f(0)\in[a_R,b_R]} \middle| \nabla f(0) = 0\right) p_{\nabla f(0)}(0)\, dt \\
&\qquad = c_1\left(R^2 - (R-r)^2\right) \int_{a_R}^{b_R} \mathbb{E}\left(\left|\det\left(\nabla^2 f(0)\right)\right| \middle| f(0) = x\right) p_{f(0)}(x)\, dx \\
&\qquad \leq c_2 Rr \cdot (b_R - a_R)
\end{aligned} \tag{3.19}$$

for some $c_1, c_2 > 0$ independent of R. By stationarity of f

$$\begin{aligned}
\mathbb{E}(N_{4-arm,r}(B(R-r),[a_R,b_R])) &\leq R^2\, \mathbb{E}\left(N_{4\text{-arm},r}(B(1),[a_R,b_R])\right) \\
&= \pi R^2 \int_{a_R}^{b_R} p_s(x) - p^{(r)}_{s^-}(x) - p^{(r)}_{s^+}(x)\, dx
\end{aligned} \tag{3.20}$$

where $p_{s-}^{(r)}$ and $p_{s+}^{(r)}$ are the continuous functions defined as in the proof of Theorem 3.2.5. In this proof it is shown that as $r \to \infty$, $p_{s-}^{(r)} + p_{s+}^{(r)}$ converges pointwise monotonically to $p_{s-}^* + p_{s+}^* = p_s$ which is continuous. Therefore, by Dini's theorem, this convergence is uniform on $[a, b]$, and so for any $\epsilon > 0$, taking r sufficiently large relative to ϵ ensures that the right hand side of (3.20) is bounded above by $\epsilon R^2(b_R - a_R)$. If we choose r depending on R such that $r \to \infty$ but $r/R \to 0$ as $R \to \infty$, then combining (3.19) and (3.20) proves the corollary. □

Proof of Proposition 3.2.8. By Corollary 2.1.9,

$$c_{LS}(\ell) = c_{ES}(\ell) + c_{ES}(-\ell), \tag{3.21}$$

$$c_{EC}(\ell) := c_{ES}(\ell) - c_{ES}(-\ell) = \sqrt{\det \nabla^2 K(0)} \frac{\ell}{(2\pi)^{3/2}} e^{-\ell^2/2}. \tag{3.22}$$

If $c_{LS}(0) = 0$, then it follows from (3.21) that $c_{ES}(0) = 0$. Similarly, if c_{LS} is differentiable at 0, then by (3.21) and the differentiability of c_{EC}, c_{ES} is also differentiable at 0.

It remains to show that if c_{ES} is differentiable at 0 then $c_{ES}(0) \neq 0$. Suppose for the sake of contradiction that $c_{ES}(0) = 0$. Then by the non-negativity of c_{ES}, we have $c'_{ES}(0) = 0$. Hence, by (3.22), $c'_{EC}(0) = 0$. Since c_{EC} has critical points only at $\ell = \pm 1$, we have derived the necessary contradiction. □

Remark 3.4.5. Assuming differentiability of c_{ES} or c_{LS} at ℓ, the above argument actually shows that $c_{LS}(\ell) > 0$ for all $\ell \neq \pm 1$ (although it apparently says nothing about the positivity of $c_{ES}(\ell)$ for $\ell \neq 0$).

3.5 Monotonicity results

In this section, we prove the monotonicity results stated in Section 3.2.2. The main intermediate step is to show that the finite-dimensional projections of $\tilde{f}_\ell - \ell$ are stochastically decreasing in ℓ, which we do in the next subsection.

3.5.1 Stochastic monotonicity

Our analysis differs depending on whether we deal with the Random Plane Wave or a general isotropic field satisfying Assumption 3.2.9, the Random Plane Wave case being somewhat simpler.

3.5.1.1 Stochastic monotonicity for the Random Plane Wave

Let f be the Random Plane Wave. The first step is to show, via explicit calculation, that $\tilde{f}_\ell - \ell$ is stochastically decreasing in ℓ at every point.

By Proposition 3.3.4, $\tilde{f}_\ell$ has the distribution

$$\tilde{f}_\ell(t) = g(t)+\ell\cdot[J_0(|t|) + 2\cos(2(\theta - \arg t))J_2(|t|)]+\lambda\cdot 4\cos(2(\theta-\arg t))J_2(|t|) \quad (3.23)$$

for the random vector (θ, λ) defined in that proposition. To simplify notation, we define

$$\begin{aligned} a &:= a(t,\theta) = 1 - J_0(|t|) - 2\cos(2(\theta - \arg t))J_2(|t|) \\ b &:= b(t,\theta) = 4\cos(2(\theta - \arg t))J_2(|t|). \end{aligned}$$

The key fact leading to stochastic monotonicity is that, by Lemma 3.5.1 below, $a = a(t,\theta) \geq 0$ for all t and θ. This is equivalent to the statement that for all $t \in \mathbb{R}^2$

$$\alpha(t) = \mathbb{E}\left(f(t) \,|\, f(0) = 1, f_{11}(0) = f_{12}(0) = 0\right) \leq 1.$$

For general isotropic fields, we show in Lemma 3.5.6 that Assumption 3.2.9 implies $\alpha(t) \leq 1$ (recall that α has a slightly different definition in the general case, see Lemma 3.3.2).

Lemma 3.5.1. *For all $t \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$,*

$$a = a(t,\theta) = 1 - J_0(|t|) - 2\cos(2(\theta - \arg t))J_2(|t|) \geq 0.$$

Proof. It is sufficient to prove that, for $s \geq 0$,

$$1 - J_0(s) - 2J_2(s) \geq 0 \quad \text{and} \quad 1 - J_0(s) + 2J_2(s) \geq 0.$$

By the identity $2J_1(s)/s = J_0(s) + J_2(s)$ and an explicit uniform bound on $\sqrt{s}|J_n(s)|$ given in [86, Theorem 2.1], the first inequality holds for all $s > 4$. Hence, since $1 - J_0(0) - 2J_2(0) = 0$ and $\frac{d}{ds}(1 - J_0(s) - 2J_2(s)) = J_3(s)$ (which is non-negative for $s \in [0,4]$), the first inequality holds for all $s \geq 0$.

The same bound from [86] shows that the second inequality holds for $s \geq 11$. Since $|J_0| \leq 1$ everywhere and $J_2(s) \geq 0$ for $s \in [0,5] \cup [9,11]$, the inequality also holds on these intervals. We verify the second inequality on the remaining compact set $[5,9]$ by inspection. More precisely, since

$$\left|\frac{d}{ds}(1 - J_0(s) + 2J_2(s))\right| = |2J_1(s) - J_3(s)| \leq 2|J_1(s)| + |J_3(s)| < 2,$$

it suffices to check that $1 - J_0(s) + 2J_2(s) > 0.08$ for all $s \in \{5 + 4i/100 : i = 0, 1, \ldots, 100\}$. $\square$

Remark 3.5.2. We prove the above lemma by somewhat explicit computations. We believe that there might be a more conceptual proof of this statement.

We shall actually show the slightly stronger statement that $\tilde{f}_\ell - \ell$ is pointwise stochastically decreasing conditional on all values of (g, θ):

Lemma 3.5.3. *Let f be the Random Plane Wave. For $t \in \mathbb{R}^2$ and $c \in \mathbb{R}$*

$$\mathbb{P}\left(\tilde{f}_\ell(t) - \ell \leq c \middle| g, \theta\right)$$

is non-decreasing in $\ell \in \mathbb{R}$.

Proof. Given the representation in (3.23), we have

$$\mathbb{P}\left(\tilde{f}_\ell(t) - \ell \leq c \middle| g, \theta\right) = \begin{cases} \mathbb{P}\left(\lambda \leq (c - g(t) + a\ell)/b\right) & \text{if } b(t,\theta) > 0, \\ \mathbb{P}\left(\lambda \geq (c - g(t) + a\ell)/b\right) & \text{if } b(t,\theta) < 0, \\ \mathbb{1}_{a\ell + c - g(t) \geq 0} & \text{if } b(t,\theta) = 0. \end{cases} \tag{3.24}$$

It remains to show that each of the expressions on the right-hand side of (3.24) are non-decreasing in ℓ for all values of g, θ and c. Recall that $a = a(t,\theta) \geq 0$ by Lemma 3.5.1. Hence $\mathbb{1}_{a\ell + c - g(t) \geq 0}$ is clearly non-decreasing in ℓ. Moreover, after integrating (3.11), we see that for a differentiable function $h : \mathbb{R} \to \mathbb{R}$

$$\frac{d}{d\ell}\mathbb{P}\left(\lambda \leq h(\ell)\right) = p_\lambda(h)\left(h'(\ell) + \frac{h(\ell)}{2h(\ell) + \ell}\right) \leq p_\lambda(h)\left(h'(\ell) + 1\right) \tag{3.25}$$

where the last inequality follows from the fact that $p_{\lambda_\ell}(h)$ is zero unless $h > 0 \vee (-\ell)$. Now let $h(\ell) = (c - g(t) + a\ell)/b$, and first suppose $b > 0$. Then $h'(\ell) = a/b > 0$, $h/(2h + \ell) \geq 0$ on the region $h > 0 \vee (-\ell)$ and $p_\lambda(h) \geq 0$ (as a probability density) so (3.25) shows that the left hand side of (3.24) is non-decreasing whenever $b > 0$. Finally we suppose $b < 0$ and note that

$$\begin{aligned} h'(\ell) + 1 = \frac{a(t,\theta) + b(t,\theta)}{b(t,\theta)} &= \frac{1 - J_0(|t|) + 2\cos(2(\theta - \arg t))J_2(|t|)}{b(t,\theta)} \\ &= \frac{a(t, \theta + \pi/2)}{b(t,\theta)} \leq 0. \end{aligned}$$

So once again, (3.25) shows the left hand side of (3.24) is non-decreasing whenever $b < 0$, completing the proof of the lemma. ∎

We now extend this result to finite-dimensional projections of $\tilde{f}_\ell - \ell$. Recall that a random vector $X = (X_1, \ldots, X_n)$ is said to *stochastically dominate* a random vector $Y = (Y_1, \ldots, Y_n)$, written $X \succ Y$, if $\mathbb{E}(g(X)) \geq \mathbb{E}(g(Y))$ for any coordinate-wise increasing $g : \mathbb{R}^n \to \mathbb{R}$. Clearly, if $X \succ Y$ then $X_i \succ Y_i$ for each $i = 1, \dots, n$. The converse is not true in general, but a useful sufficient condition can be formulated using the notion of copulas.

Let $X = (X_1, \dots, X_n)$, where X_i has cumulative density function F_i and induced probability measure $\mathbb{P}_i$. Then Sklar's theorem states that there exists a (unique on $\Pi_{i=1}^n \text{Range}(\mathbb{P}_i)$) function $\text{Cop}_X : [0,1]^n \to [0,1]$, known as the *copula* of X, such that

$$\mathbb{P}(X \in A_1 \times \cdots \times A_n) = \text{Cop}_X(\mathbb{P}_1(A_1), \dots, \mathbb{P}_n(A_n))$$

for all $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})$. The copula is equivalently specified by

$$\text{Cop}_X(u_1, \dots, u_n) = \mathbb{P}(F_1(X_1) \le u_1, \dots, F_n(X_n) \le u_n), \tag{3.26}$$

i.e. Cop_X is the joint cumulative density function of the collection of uniform-[0,1] random variables $F_1(X_1), \dots, F_n(X_n)$.

Theorem 3.5.4 ([98, Theorem 2])**.** *Let $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_n)$ be random vectors with induced marginal probability measures $\mathbb{P}_1, \dots, \mathbb{P}_n$ and $\mathbb{Q}_1, \dots, \mathbb{Q}_n$ respectively. If $\text{Cop}_X = \text{Cop}_Y$, $\Pi_{i=1}^n \text{Range}(\mathbb{P}_i) = \Pi_{i=1}^n \text{Range}(\mathbb{Q}_i)$, and $X_i \succ Y_i$ for each i, then $X \succ Y$.*

Using this theorem, we extend Lemma 3.5.3 to show the stochastic monotonicity of the finite-dimensional projections $\tilde{f}_\ell - \ell$, conditional on any g, θ.

Lemma 3.5.5. *Let f be the Random Plane Wave. For $\ell_1 < \ell_2$ and $t_1, \dots, t_n \in \mathbb{R}^2$,*

$$\left(\tilde{f}_{\ell_1}(t_1) - \ell_1, \dots, \tilde{f}_{\ell_1}(t_n) - \ell_1 \;\middle|\; g, \theta\right) \succ \left(\tilde{f}_{\ell_2}(t_1) - \ell_2, \dots, \tilde{f}_{\ell_2}(t_n) - \ell_2 \;\middle|\; g, \theta\right). \tag{3.27}$$

Proof. By Theorem 3.5.4, it is sufficient to show that the random vectors in (3.27) have the same copula (these copulas are uniquely defined on the same domain, and the stochastic domination of the marginals follows from Lemma 3.5.3).

Fix $\ell \in \mathbb{R}$ and $t_1, \dots, t_n \in \mathbb{R}^2$, and consider the copula

$$\text{Cop}_Z := \text{Cop}_{\tilde{f}_\ell(t_1) - \ell, \dots, \tilde{f}_\ell(t_n) - \ell \big| g, \theta}.$$

By the definition of a and b, we can express

$$\tilde{f}_\ell(t) - \ell = g(t) - \ell a(t, \theta) + \lambda_\ell b(t, \theta)$$

for deterministic functions a, b. Since $g(t_i)$, $\ell a(t_i, \theta)$ and $b(t_i, \theta)$ are constants under the conditioning on (g, θ), and since copulas are invariant under strictly increasing transformations,

$$\text{Cop}_Z = \text{Cop}_{\lambda_\ell \cdot \text{sign}(b(t_1,\theta)), \dots, \lambda_\ell \cdot \text{sign}(b(t_n,\theta)) \big| g, \theta} = \text{Cop}_{\lambda_\ell \cdot \text{sign}(b(t_1,\theta)), \dots, \lambda_\ell \cdot \text{sign}(b(t_n,\theta)) \big| \theta},$$

where the last equality holds since g is independent of $\lambda = \lambda_\ell$ and θ. Notice that the random vector $(\lambda_\ell \cdot \mathrm{sign}(b(t_1,\theta)), \dots, \lambda_\ell \cdot \mathrm{sign}(b(t_n,\theta)) \big| \theta)$ consists of λ_ℓ multiplied by a constant vector (with elements taking values 1, -1 or 0). Hence by considering the alternative characterisation of a copula in (3.26), it is clear that Cop_Z does not depend on the distribution of λ_ℓ, and so is independent of ℓ. $\square$

3.5.1.2 Stochastic monotonicity in the general isotropic case

The copula argument in the Random Plane Wave case relies crucially on the degeneracy of Random Plane Wave, which implies that after conditioning on θ and g, the field depends on the single random variable λ. For general isotropic fields, $\tilde{f}_\ell$ is defined in terms of two eigenvalues, so this argument fails. Instead we use a different method that works with the finite-dimensional projections directly.

Let f satisfy Assumptions 2.1.1, 3.2.1 and 3.2.9. Recall from Proposition 3.3.3 that

$$\tilde{f}_\ell(t) = g(t) + \ell\alpha(t) + \lambda_1 b_1(t,\theta) + \lambda_2 b_2(t,\theta)$$

for α, b_1, b_2 as stated in the proposition (where b_1 and b_2 are defined in terms of β). The role of Assumption 3.2.9 is to ensure the following inequalities hold for α, b_1, b_2:

Lemma 3.5.6. *Let f satisfy Assumptions 2.1.1, 3.2.1 and 3.2.9. For all $t \in \mathbb{R}^2$ and $\theta \in [0, 2\pi)$,*

$$b_1(t,\theta) \geq 0 \ , \quad b_2(t,\theta) \geq 0 \quad \textit{and} \quad \alpha(t) \leq 1.$$

Proof. From the definition of $b_1(t,\theta)$ and β, it is immediate that $b_1(t,0)$ is the quantity given in Assumption 3.2.9 to be non-negative for all values of t. Since f is isotropic, b_1 is non-negative for all values of θ. (By the identity $\cos(\theta) = \sin(\theta + \pi/2)$, this also means that b_2 is non-negative.) Similarly, α is the function given in Assumption 3.2.9 to be bounded above by 1. $\square$

Lemma 3.5.7. *Let f satisfy Assumptions 2.1.1, 3.2.1 and 3.2.9. For any $t_1, \dots, t_n \in \mathbb{R}^2$ and $u_1, \dots, u_n \in \mathbb{R}$,*

$$\mathbb{P}\left(\tilde{f}_\ell(t_i) - \ell \leq u_i, \ \forall i = 1, \dots, n \,\middle|\, g, \theta \right)$$

is non-decreasing in $\ell \in \mathbb{R}$.

Proof. Since the u_i are arbitrary, we may assume $g(t_i) = 0$ for all i. We define the region

$$A_\ell = \left\{ (x,y) \in \mathbb{R}^2 : x b_1(t_i,\theta) + y b_2(t_i,\theta) \leq u_i + (1 - \alpha(t_i))\ell, \ \ \forall i = 1, \dots, n \right\}$$

so that $\tilde{f}_\ell(t_i)-\ell \le u_i$ for all i if and only if $(\lambda_1, \lambda_2) \in A_\ell$. It is enough to prove that the probability of the latter event is non-decreasing in ℓ, because (λ_1, λ_2) is independent of (g, θ). Given the density of (λ_1, λ_2) in Proposition 3.3.3,

$$\begin{aligned}\frac{d}{d\ell}\mathbb{P}\left((\lambda_1, \lambda_2) \in A_\ell\right) &= \frac{d}{d\ell}\frac{\int_{A_\ell} q_\ell(x,y)\ dxdy}{\int_{\mathbb{R}^2} q_\ell(x,y)\ dxdy} \\ &= \frac{\int_{\mathbb{R}^2} q_\ell\ dxdy \cdot \frac{d}{d\ell}\int_{A_\ell} q_\ell\ dxdy - \int_{A_\ell} q_\ell\ dxdy \cdot \frac{d}{d\ell}\int_{\mathbb{R}^2} q_\ell\ dxdy}{\left(\int_{\mathbb{R}^2} q_\ell\ dxdy\right)^2}.\end{aligned}$$

Since $b_1, b_2 \ge 0$ and $\alpha \le 1$ by Lemma 3.5.6, A_ℓ is non-decreasing in ℓ, and so for this derivative to be non-negative it is sufficient that

$$\frac{\int_{A_\ell} \frac{d}{d\ell} q_\ell(x,y)\ dxdy}{\int_{A_\ell} q_\ell(x,y)\ dxdy} \ge \frac{\int_{\mathbb{R}^2} \frac{d}{d\ell} q_\ell(x,y)\ dxdy}{\int_{\mathbb{R}^2} q_\ell(x,y)\ dxdy}. \tag{3.28}$$

By direct evaluation (using Proposition 3.3.3)

$$\frac{d}{d\ell}q_\ell(x,y) = \frac{\mu(1+\tau)}{\sigma^2}(x+y-2\mu\ell)q_\ell(x,y),$$

where μ, τ and σ^2 are defined in (3.10). Since $\mu < 0$, (3.28) is equivalent to

$$\mathbb{E}\left(\lambda_1+\lambda_2 | (\lambda_1, \lambda_2) \in A_\ell\right) \le \mathbb{E}\left(\lambda_1+\lambda_2\right). \tag{3.29}$$

To complete the proof of the lemma, we show that this inequality holds for any possible region A_ℓ. Since the shape of A_ℓ might be quite complicated (see Figure 3.3 for a typical example), we divide the analysis into three cases and in each case show that conditioning on (λ_1, λ_2) being contained in some simple region can only increase the expectation of $\lambda_1+\lambda_2$ relative to conditioning on $(\lambda_1, \lambda_2) \in A_\ell$.

(Case 1). Suppose $b_1(t_i, \theta)/b_2(t_i, \theta) = 1$ for all i, so that the boundary of A_ℓ, denoted ∂A_ℓ, is a line of the form $x+y=c$ for some $c \in \mathbb{R}$. (Note that A_ℓ is a subset of the entire plane not just the upper-left quadrant which is the support of (λ_1, λ_2).) Then trivially

$$\mathbb{E}\left(\lambda_1+\lambda_2 | (\lambda_1, \lambda_2) \in A_\ell\right) = \mathbb{E}\left(\lambda_1+\lambda_2 | \lambda_1+\lambda_2 \le c\right) \le \mathbb{E}\left(\lambda_1+\lambda_2\right)$$

and so (3.29) is verified in this case.

(Case 2). Now suppose that $b_1(t_i, \theta)/b_2(t_i, \theta) \ge 1$ for all i and that this inequality is strict for some i (allowing for the degenerate case that $b_2(t_i, \theta) = 0$). Let $c^* = \mathbb{E}\left(\lambda_1+\lambda_2 | (\lambda_1, \lambda_2) \in A_\ell\right)$ and then we note that the line $x+y=c^*$ must intersect ∂A_ℓ at precisely one point (since the distribution of (λ_1, λ_2) is continuous) which we

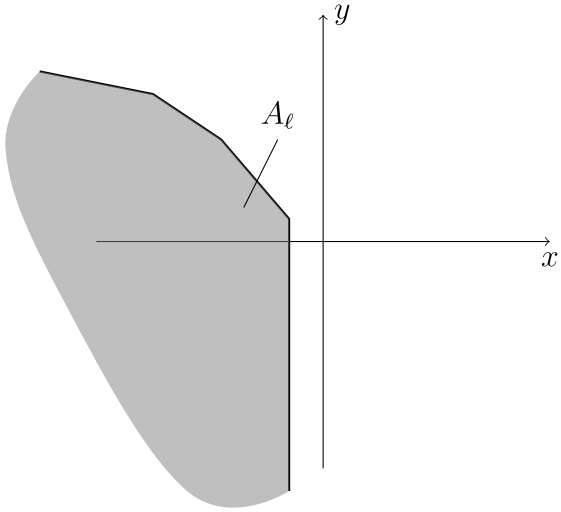


Figure 3.3: A typical example of the region A_ℓ.

denote by (d_1, d_2). We now consider the region $\{x \leq d_1\}$, conditioning on (λ_1, λ_2) lying in this region weakly increases the probability that $\lambda_1 + \lambda_2 = c$ for $c \geq c^*$ and weakly decreases this probability for $c < c^*$ (see Figure 3.4). Therefore

$$\mathbb{E}\left(\lambda_1 + \lambda_2 | (\lambda_1, \lambda_2) \in A_\ell\right) \leq \mathbb{E}(\lambda_1 + \lambda_2 | \lambda_1 \leq d_1). \tag{3.30}$$

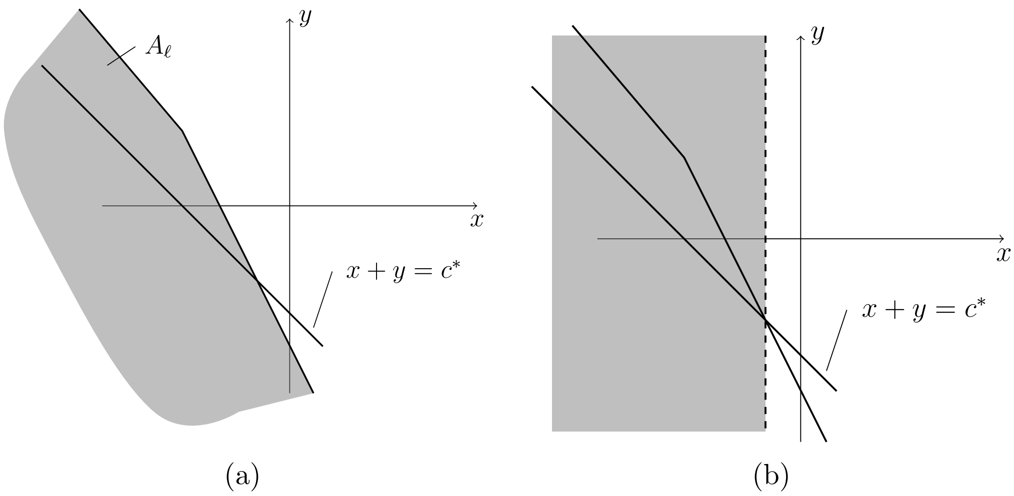


Figure 3.4: When A_ℓ takes the form shown on the left (Case 2), we condition on the region shown on the right, which weakly increases the mean of $\lambda_1 + \lambda_2$.

If $b_1(t_i, \theta)/b_2(t_i, \theta) \leq 1$ for all i and this inequality is strict for some i, then an entirely analogous argument shows that for some d_2

$$\mathbb{E}\left(\lambda_1 + \lambda_2 | (\lambda_1, \lambda_2) \in A_\ell\right) \leq \mathbb{E}(\lambda_1 + \lambda_2 | \lambda_2 \leq d_2). \tag{3.31}$$

(Case 3). Suppose that for some i and j, $b_1(t_i,\theta)/b_2(t_i,\theta) < 1 < b_1(t_j,\theta)/b_2(t_j,\theta)$. Defining c^* as before, we note that the line $x+y=c^*$ must intersect A_ℓ (by definition of c^*) and so must intersect ∂A_ℓ at two points (since the distribution of (λ_1,λ_2) has no atoms). We denote these points by (d_1,d_2) and (e_1,e_2) and without loss of generality take $d_1 < e_1$. We now consider the region $\{x \leq e_1\} \cap \{y \leq d_2\}$. Reasoning as before, conditioning on (λ_1,λ_2) lying in this region weakly increases the probability that $\lambda_1+\lambda_2 = c$ for $c \geq c^*$ and weakly decreases this probability for $c < c^*$ (see Figure 3.5). So in this case

$$\mathbb{E}\left(\lambda_1+\lambda_2|(\lambda_1,\lambda_2)\in A_\ell\right) \leq \mathbb{E}(\lambda_1+\lambda_2|\lambda_1 \leq e_1, \lambda_2 \leq d_2). \tag{3.32}$$

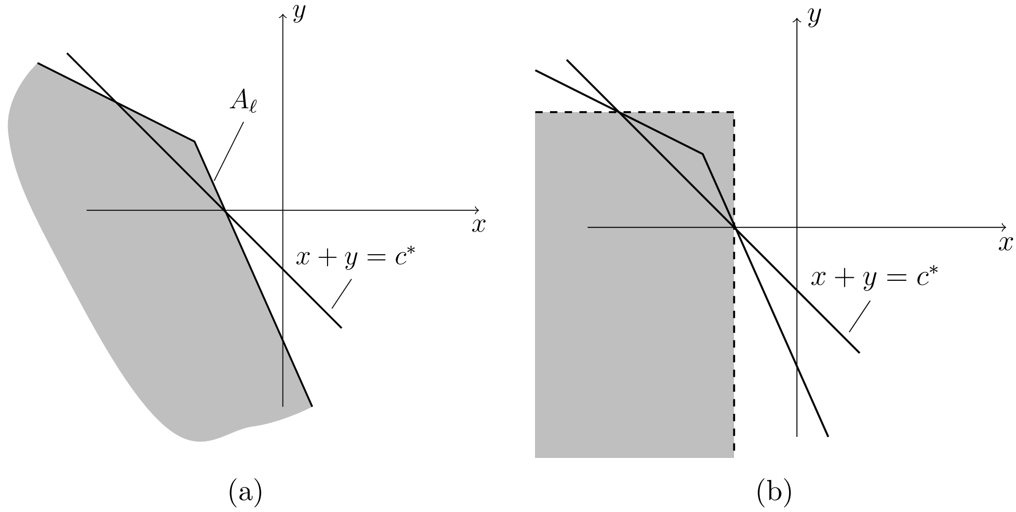


Figure 3.5: When A_ℓ takes the form shown on the left (Case 3), we condition on the region shown on the right, which weakly increases the mean of $\lambda_1+\lambda_2$.

From (3.30), (3.31) and (3.32) we see that, in order to complete the proof of the lemma, we need only verify (3.29) (or, equivalently, (3.28)) when A_ℓ is of the form $\{\lambda_1 \leq c_1\}$, $\{\lambda_2 \leq c_2\}$ or $\{\lambda_1 \leq c_1, \lambda_2 \leq c_2\}$. Furthermore, since $\lambda_2 \geq 0 \geq \lambda_1$, we may assume $c_1 \leq 0$ and $c_2 \geq 0$.

Gromov's theorem ([30, Theorem 1.3]) states that if h_1, h_2 are integrable on $[a,b]$ such that $h_2 > 0$ and $h_1(x)/h_2(x)$ is non-increasing in x then $\int_a^c h_1(x)\,dx/\int_a^c h_2(x)\,dx$ is non-increasing in c. Applying this to

$$h_1(x) := \int_{-\infty}^{c_2} \frac{d}{d\ell} q_\ell(x,y)\,dy \quad \text{and} \quad h_2(x) := \int_{-\infty}^{c_2} q_\ell(x,y)\,dy$$

we see that, provided $h_1(x)/h_2(x)$ is non-increasing, we have that

$$\frac{\int_{-\infty}^{c_1}\int_{-\infty}^{c_2} \frac{d}{d\ell} q_\ell(x,y)\,dydx}{\int_{-\infty}^{c_1}\int_{-\infty}^{c_2} q_\ell(x,y)\,dydx}$$

is non-increasing in c_1. Then taking $c_1 \to \infty$, we see that we need only verify (3.28) when A_ℓ is of the form $\{\lambda_1 \le c_1\}$ or $\{\lambda_2 \le c_2\}$.

It remains to show that $h_1(x)/h_2(x)$ is non-increasing in x, or equivalently, that $\mathbb{E}(\lambda_1 + \lambda_2 | \lambda_1 = x, \lambda_2 \le c_2)$ is non-decreasing in x for $x < 0$. Using the joint density of (λ_1, λ_2) given in Proposition 3.3.3

$$\mathbb{E}(\lambda_1 + \lambda_2 | \lambda_1 = x, \lambda_2 \le c_2) = \frac{\int_0^{c_2} y(y^2 - x^2)\exp(-\frac{1}{2\sigma^2}(y-m)^2)\,dy}{\int_0^{c_2} y(y-x)\exp(-\frac{1}{2\sigma^2}(y-m)^2)\,dy}$$

where $m = (1+\tau)\mu\ell - \tau x$. We next differentiate this expression with respect to x. To simplify the resulting expression, we let Z be a random variable with density proportional to $\exp(-\frac{1}{2\sigma^2}(y-m)^2)\mathbb{1}_{y \in [0,c_2]}$ (i.e. Z is a truncated normal variable). Then using the Leibniz rule for differentiating integrals:

$$\begin{aligned}
&\left(\frac{\int_0^{c_2} y(y-x)\exp(-\frac{1}{2\sigma^2}(y-m)^2)\,dy}{\int_0^{c_2}\exp(-\frac{1}{2\sigma^2}(y-m)^2)\,dy}\right)^2 \left(\frac{d}{dx}\mathbb{E}(\lambda_1 + \lambda_2 | \lambda_1 = x, \lambda_2 \le c_2)\right) \\
&= (\mathbb{E}(Z^2) - x\mathbb{E}(Z)) \left(-3\tau\mathbb{E}\left(Z^2\right) - 2x\mathbb{E}\left(Z\right) + \tau x^2 + \frac{\tau c_2(c_2^2 - x^2)e^{-\frac{(c_2-m)^2}{2\sigma^2}}}{\int_0^{c_2}\exp(-(y-m)^2/(2\sigma^2))\,dy}\right) \\
&\quad - \left(\mathbb{E}(Z^3) - x^2\mathbb{E}(Z)\right)\left(-(2\tau+1)\mathbb{E}(Z) + \tau x + \frac{\tau c_2(c_2 - x)e^{-\frac{(c_2-m)^2}{2\sigma^2}}}{\int_0^{c_2}\exp(-(y-m)^2/(2\sigma^2))\,dy}\right).
\end{aligned}$$

This expression can be rearranged to take the form

$$\begin{aligned}
&\underbrace{(2\tau+1)\mathbb{E}\left(Z^3\right)\mathbb{E}\left(Z\right) - 3\tau\mathbb{E}\left(Z^2\right)^2}_{A} + \underbrace{\tau\mathbb{E}\left(c_2^2 Z^2 (c_2 - Z)\right)\frac{e^{-\frac{(c_2-m)^2}{2\sigma^2}}}{\int_0^{c_2} e^{-\frac{(y-m)^2}{2\sigma^2}}\,dy}}_{B} \\
&- x\left(\underbrace{(2-3\tau)\mathbb{E}\left(Z^2\right)\mathbb{E}\left(Z\right) + \tau\mathbb{E}\left(Z^3\right)}_{C} + \underbrace{\tau\mathbb{E}\left(c_2 Z\left(c_2^2 - Z^2\right)\right)\frac{e^{-\frac{(c_2-m)^2}{2\sigma^2}}}{\int_0^{c_2} e^{-\frac{(y-m)^2}{2\sigma^2}}\,dy}}_{D}\right) \\
&+ x^2\left(\underbrace{\tau\mathbb{E}\left(Z^2\right) + (1-2\tau)\mathbb{E}\left(Z\right)^2}_{E} + \underbrace{\tau c_2\mathbb{E}\left(Z\left(c_2 - Z\right)\right)\frac{e^{-\frac{(c_2-m)^2}{2\sigma^2}}}{\int_0^{c_2} e^{-\frac{(y-m)^2}{2\sigma^2}}\,dy}}_{F}\right).
\end{aligned}$$

Then, since $x < 0$, we need only verify that each of the terms A-F are non-negative. We show this by using two facts: first, that $0 \le Z \le c_2$ (which is true by definition) and second, that $0 \le \tau \le 1$ (which holds because $\tau = (\chi^2 - 1)/(3 - \chi^2)$ and Assumption 3.2.9 implies that $1 \le \chi^2 \le 2$).

These two facts immediately show that B, D and F are non-negative. Furthermore

$$\begin{aligned}
A &\geq (2\tau+1)\left(\mathbb{E}\left(Z^3\right)\mathbb{E}(Z) - \mathbb{E}\left(Z^2\right)^2\right) \\
C &= 2(1-\tau)\mathbb{E}\left(Z^2\right)\mathbb{E}\left(Z\right) + \tau\mathrm{Cov}\left(Z^2, Z\right) \\
E &= (1-\tau)\mathbb{E}\left(Z\right)^2 + \tau\mathrm{Var}(Z)
\end{aligned}$$

(where the first inequality uses $\tau \leq 1$). Applying the Cauchy-Schwarz inequality to $Z^2 = Z^{3/2}Z^{1/2}$ shows that A is non-negative. Using the fact that $Z \geq 0$ (so that Z^2 is an increasing function of Z) implies that $C \geq 0$. Since $\tau \in [0,1]$ we see that $E \geq 0$.

Using Gromov's theorem in the same way as above, shows that in order to verify (3.29) for A_ℓ of the form $\{\lambda_2 \leq c_2\}$ or $\{\lambda_1 \leq c_1\}$ it is enough to show that $\mathbb{E}(\lambda_1+\lambda_2|\lambda_2 = c_2)$ and $\mathbb{E}(\lambda_1+\lambda_2|\lambda_1 = c_1)$ are non-decreasing in $c_2 > 0$ and $c_1 < 0$ respectively. This can be proven using a near identical calculation to that for $\frac{d}{dx}\mathbb{E}(\lambda_1+\lambda_2|\lambda_1 = x, \lambda_2 \leq c_2)$ (the only change is the region on which Z is truncated, which means there will be no terms analogous to B, D and F above). This completes the proof of the lemma. $\square$

Remark 3.5.8. In Assumption 3.2.9, we impose the condition that $\chi \geq 1$. The only point in this chapter at which we use this condition is in the proof of Lemma 3.5.7, in order to show that

$$\mathbb{E}\left(\lambda_1+\lambda_2|\lambda_1 = x, \lambda_2 \leq c_2\right), \quad \mathbb{E}\left(\lambda_1+\lambda_2|\lambda_1 = x\right), \quad \text{and} \quad \mathbb{E}\left(\lambda_1+\lambda_2|\lambda_2 = x\right)$$

are non-decreasing in x (for all $c_2 \geq 0$). Therefore, if an alternative method was found to verify this property (or to verify that (3.29) holds for A_ℓ of the form $\{\lambda_1 \leq c_1\}$, $\{\lambda_2 \leq c_2\}$ and $\{\lambda_1 \leq c_1, \lambda_2 \leq c_2\}$) for fields with $\chi < 1$, then our results (including Theorem 3.2.10) would also hold for such fields.

We expect that it should be possible to extend our results in this way. In the proof of Lemma 3.5.7 we use $\chi \geq 1$ (or equivalently, $\tau \geq 0$) to show that A-F are non-negative. If we explicitly evaluate these terms using the higher order moments of a truncated normal distribution, then numerical calculations suggest that $A+B$, $C+D$ and $E+F$ are non-negative for all relevant values of τ, (i.e. including negative values) which would be sufficient to prove this lemma in such cases. We do not attempt to prove this analytically, because the algebraic expressions involved in these calculation are quite long and we are primarily interested in the case of the Bargmann-Fock field, for which $\tau = 0$.

3.5.2 Proof of Theorem 3.2.10

We now use Lemmas 3.5.5 and 3.5.7 to complete the proof of Theorem 3.2.10, treating the Random Plane Wave case and the general case simultaneously.

Proof of Theorem 3.2.10. We begin by fixing a realisation of g and θ. Let $A(\epsilon, R)$ denote the annulus on the plane centred at the origin with inner radius ϵ and outer radius R. We discretise this region by considering the points with polar coordinates

$$(r_i^{(n)}, \omega_j^{(n)}) := (r_i, \omega_j) := (\epsilon + i2^{-n}(R-\epsilon), \theta + j2^{-n}2\pi)$$

for $i, j = 0, 1, \ldots, 2^n$. We consider these points as a graph by placing an edge between (r_{i_1}, ω_{j_1}) and (r_{i_2}, ω_{j_2}) if and only if $|i_1 - i_2| + |j_1 - j_2| = 1$. We define a site percolation model by declaring the vertex (r_i, ω_j) open if $\tilde{f}_\ell(r_i, \omega_j) - \ell < 0$ (so an edge is open precisely when both of its vertices are open). Let $S_{\epsilon,R,n,\ell}$ denote the event that there is an open path between (ϵ, θ) and $(\epsilon, \theta + \pi)$ in this percolation model.

Let $S_{\epsilon,R,\ell}$ denote the event that $\{\tilde{f}_\ell < \ell\} \cap A(\epsilon, R)$ contains a path joining (ϵ, θ) to $(\epsilon, \theta + \pi)$. We claim that with probability one,

$$\mathbb{1}_{S_{\epsilon,R,\ell}} = \lim_{n\to\infty} \mathbb{1}_{S_{\epsilon,R,n,\ell}}. \tag{3.33}$$

Since $\tilde{f}_\ell$ has no critical points at level ℓ away from the origin (Lemma 3.4.3), the level set $\{\tilde{f}_\ell = \ell\} \cap A(\epsilon, R)$ consists of $C^{2+\eta}$ curves. So in particular, if there exists a path in $\{\tilde{f}_\ell < \ell\} \cap A(\epsilon, R)$ joining (ϵ, θ) to $(\epsilon, \theta + \pi)$, then for n sufficiently large we may assume this path lies on the graph with vertices $(r_i^{(n)}, \omega_j^{(n)})$ as defined above. Hence $\mathbb{1}_{S_{\epsilon,R,\ell}} \leq \liminf_n \mathbb{1}_{S_{\epsilon,R,n,\ell}}$. If there is no path in $\{\tilde{f}_\ell < \ell\} \cap A(\epsilon, R)$ joining (ϵ, θ) to $(\epsilon, \theta + \pi)$ then there are three possibilities: (1) $\tilde{f}_\ell - \ell$ is non-negative at (ϵ, θ) or $(\epsilon, \theta + \pi)$; (2) there exists a path in $\{\tilde{f}_\ell \geq \ell\} \cap A(\epsilon, R)$ joining (ϵ, ω_i) to (ϵ, ω_j) for some $\omega_i \in (\theta, \theta + \pi)$ and $\omega_j \in (\theta - \pi, \theta)$, (here we note that by Lemma 3.4.3, $\tilde{f}_\ell|_{\partial B(\epsilon)}$ has no local extrema at level ℓ and we assume that n is sufficiently large to find such ω_i, ω_j); or (3) there exist two paths in $\{\tilde{f}_\ell \geq \ell\} \cap A(\epsilon, R)$ which join (ϵ, ω_i) and (ϵ, ω_j) respectively to $\partial B(R)$ for ω_i, ω_j as before (See Figure 3.6). In this case, by Lemma 3.4.3 we may assume that the paths intersect $\partial B(R)$ at different points.

In each of these cases, for all n large enough we can construct corresponding paths on the discrete lattice as above which block a discrete path from joining (ϵ, θ) to $(\epsilon, \theta + \pi)$ in $\{\tilde{f}_\ell < \ell\}$ and so $S_{\epsilon,R,n,\ell}$ cannot occur for sufficiently large n. Therefore $\mathbb{1}_{S_{\epsilon,R,\ell}} \geq \limsup_n \mathbb{1}_{S_{\epsilon,R,n,\ell}}$, completing the proof of the claim.

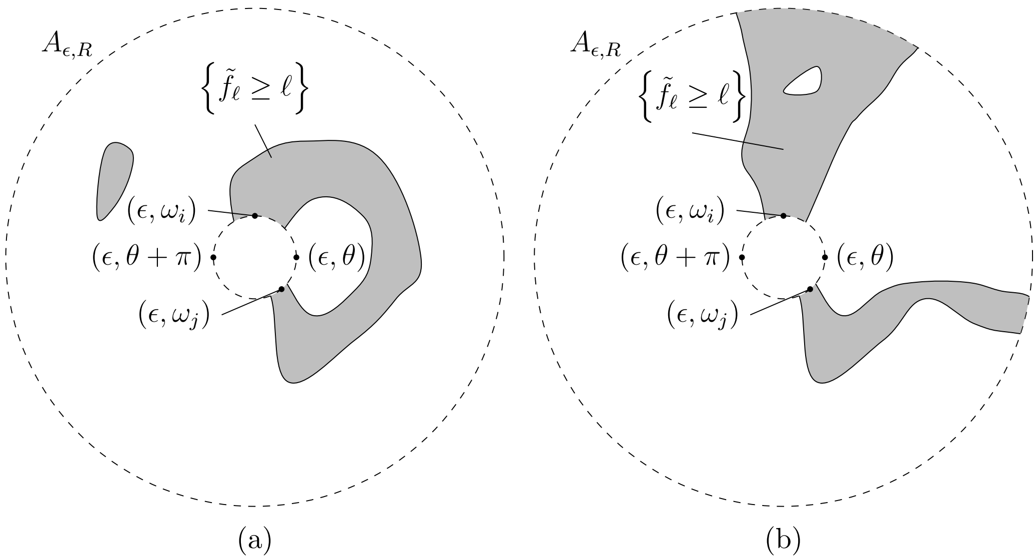


Figure 3.6: Two of the three ways in which $S_{\epsilon,R,\ell}$ can fail, corresponding to cases (2) and (3) above respectively.

Since $S_{\epsilon,R,n,\ell}$ depends on only finitely many points of $\tilde{f}_\ell$ and is a decreasing event, by Lemma 3.5.5 for the Random Plane Wave and Lemma 3.5.7 for general fields

$$\mathbb{P}\left(S_{\epsilon,R,n,\ell_1}|g,\theta\right) \leq \mathbb{P}\left(S_{\epsilon,R,n,\ell_2}|g,\theta\right)$$

for any $\ell_1 < \ell_2$. Then by (3.33) and the bounded convergence theorem

$$\mathbb{P}\left(S_{\epsilon,R,\ell_1}|g,\theta\right) \leq \mathbb{P}\left(S_{\epsilon,R,\ell_2}|g,\theta\right). \tag{3.34}$$

Now let $S_{R,\ell}$ be the event that $\tilde{f}_\ell$ has an R-lower connected saddle point at the origin. Conditional on θ, if this event occurs then so must $S_{\epsilon,R,\ell}$ for ϵ sufficiently small. Conversely, if the saddle point at the origin is not R-lower connected, then it must be four arm in $B(R)$ or R-upper connected. In both of these cases $S_{\epsilon,R,\ell}$ cannot occur for ϵ sufficiently small. We conclude that $\mathbb{1}_{S_{R,\ell}} = \lim_{\epsilon\to 0}\mathbb{1}_{S_{\epsilon,R,\ell}}$ and by applying the bounded convergence theorem to (3.34) we see that

$$\mathbb{P}\left(S_{R,\ell_1}|g,\theta\right) \leq \mathbb{P}\left(S_{R,\ell_2}|g,\theta\right). \tag{3.35}$$

Finally we let S_ℓ be the event that $\tilde{f}_\ell$ has a lower connected saddle point at the origin and note that trivially $S_\ell = \cup_R S_{R,\ell}$. Applying this to (3.35) shows that

$$\mathbb{P}\left(S_{\ell_1}|g,\theta\right) \leq \mathbb{P}\left(S_{\ell_2}|g,\theta\right).$$

Integrating over realisations of g and θ implies that $\mathbb{P}(S_{\ell_1}) \leq \mathbb{P}(S_{\ell_2})$ and so by definition (see the proof of Theorem 3.2.5)

$$\frac{p^*_{s^-}(\ell_1)}{p_s(\ell_1)} \leq \frac{p^*_{s^-}(\ell_2)}{p_s(\ell_2)}.$$

A near identical argument shows that $p^*_{s+}(\ell)/p_s(\ell)$ is non-increasing in ℓ. $\square$

3.5.3 Proof of remaining results

We now prove the remaining results stated in Section 3.2.2., namely Corollaries 3.2.11 and 3.2.12 and Propositions 3.2.14–3.2.17.

Proof of Corollary 3.2.11. Since p^*_{s-}/p_s is monotone, it has at most a countable number of discontinuities, all of which are jump discontinuities. By the continuity of p_s, the same is true of p^*_{s-}. Since c_{ES} is absolutely continuous (Theorem 2.1.7) it is differentiable almost everywhere (see [94, Theorem 7.18]) with derivative $p^*_{s-} - p_{m^+}$. The density p_{m^+} is derived explicitly in [26] and is continuously differentiable. It also follows from monotonicity that p^*_{s-}/p_s is differentiable almost everywhere, and since p_s is smooth (again, from [26]) the same is true of p^*_{s-}, thus showing that c_{ES} is twice differentiable almost everywhere. A similar proof applies to c_{LS}. $\square$

Proof of Corollary 3.2.12. The equivalence of (2)–(4) follows from Theorem 2.1.7, and (1) implies (2) by Theorem 3.2.5, therefore it remains to show that (2) implies (1). Now suppose there exists a version of p_{s^-} which is continuous on (a, b), denoted $\tilde{p}_{s^-}$. Then $\tilde{p}_{s^-}/p_s = p^*_{s-}/p_s$ almost everywhere, and since the former is continuous and the latter is monotone, this equality must hold pointwise on (a, b), so p^*_{s-} is continuous on (a, b). We note that $\tilde{p}_{s^+} := p_s - \tilde{p}_{s^-}$ defines a continuous version of p_{s^+} and arguing as above then shows that p^*_{s+} is continuous on (a, b). Therefore the almost everywhere equality $p^*_{s-} + p^*_{s+} = p_s$ is in fact true for all points in (a, b), and by (3.13) $\tilde{f}_\ell$ almost surely has no infinite four arm saddle at the origin for all $\ell \in (a, b)$. $\square$

Proof of Proposition 3.2.17. We use the 'barrier method', that is, we show that the probability of having at least one component of $\{f \geq \ell\}$ contained in $B(r)$ is strictly positive for some fixed $r > 0$. By linearity of expectation and stationarity of f, this shows that $\liminf_{R\to\infty} \mathbb{E}(N_{ES}(R, \ell))/R^2 > 0$, so in particular $c_{ES}(\ell) > 0$.

It is known that the Random Plane Wave has the orthogonal expansion

$$f(x) = \sum_{m\in\mathbb{Z}} a_m J_{|m|}(r)e^{im\theta}$$

where (r, θ) represents x in polar coordinates, J_m is the m-th Bessel function and $a_m = b_m + ic_m = \overline{a_{-m}}$ with b_0, $(\sqrt{2}b_m)_{m\in\mathbb{N}}$ and $(\sqrt{2}c_m)_{m\in\mathbb{N}}$ independent standard (real) Gaussians and $c_0 = 0$. (This function is clearly Gaussian and can be shown to have the correct covariance structure using Graf's addition theorem for Bessel functions.) Let r be the minimiser of J_0, so $r \approx 3.83$ and $J_0(r) < -0.4$. We note that by considering the power series for the Bessel functions, it can be shown that for $x \in [0,4]$, $|J_m(x)| \leq e^4(2^m/m!)$. Finally we note that J_m is bounded in absolute value by 1 for any m. Now consider the event that

$$a_0 > \min\{|\ell|,1\}, \quad |a_{-1}|,|a_{-2}|,|a_1|,|a_2| \leq C_1 \quad \text{and} \quad \forall |m|>2,\ |a_m| \leq C_2(m!)/4^m.$$

It is easily seen that this event has positive probability, and for appropriately chosen constants $C_1, C_2 > 0$, we see that on this event $f(0) > \ell$ and $f(x) < \ell$ for any x such that $|x| = r$. Therefore f has a component of $\{f \geq \ell\}$ contained in $B(r)$ with positive probability, completing the proof of the result. □

Proof of Proposition 3.2.14. By Corollary 3.2.12 we may assume that $p_{s^-}(\ell)/p_s(\ell)$ is non-decreasing. In [26] it is shown that for the Random Plane Wave

$$\begin{aligned} p_{m^+}(x) &= \frac{1}{4\sqrt{2}\pi^{3/2}}\left((x^2-1)e^{-\frac{x^2}{2}} + e^{-\frac{3x^2}{2}}\right)\mathbb{1}_{x\geq 0} \\ p_s(x) &= \frac{1}{4\sqrt{2}\pi^{3/2}}e^{-\frac{3x^2}{2}}. \end{aligned}$$

In particular, $p_{m^+}(x) = 0$ for $x < 0$, so by Theorem 2.1.7 for $\ell' < \ell \leq 0$

$$c_{ES}(\ell') - c_{ES}(\ell) = \int_{\ell'}^{\ell} -p_{s^-}(x)\,dx.$$

Taking $\ell' \to -\infty$ shows that for $\ell < 0$

$$c_{ES}(\ell) = \int_{-\infty}^{\ell} p_{s^-}(x)\,dx.$$

By Proposition 3.2.17, this is positive for every $\ell < 0$, so in particular there must exist arbitrarily negative x such that $p_{s^-}(x) > 0$. Since $p_{s^-}(\ell)/p_s(\ell)$ is non-decreasing, we conclude that p_{s^-} is strictly positive for all $\ell \in \mathbb{R}$. Since $p_{s^-}(x) = p_{s^+}(-x)$ we also see that $p_{s^+}(x) > 0$ for all x and since $p_{s^-} + p_{s^+} = p_s$ we see that $0 < p_{s^-}(x)/p_s(x) < 1$ for all $x \in \mathbb{R}$. Finally, we note that there must exist a sequence $\ell_n > 0$ with $\ell_n \to 0$ such that $p_{s^-}(\ell_n)/p_s(\ell_n) \geq 1/2$ for all n. Indeed, if this were not true, by monotonicity, there would exist a neighbourhood of 0 on which $p_{s^-}/p_s < 1/2$ and by symmetry $p_{s^+}/p_s < 1/2$ on a possibly smaller neighbourhood, but then there would exist a set of positive measure on which $p_{s^-} + p_{s^+} < p_s$ giving a contradiction.

For $\ell' \leq \ell$ and $\epsilon > 0$

$$\begin{aligned} \frac{1}{\epsilon}\int_{\ell}^{\ell+\epsilon} \frac{p_{s^-}(\ell')}{p_s(\ell')}p_s(x) - p_{m^+}(x)\,dx &\leq \frac{1}{\epsilon}\int_{\ell}^{\ell+\epsilon} p_{s^-}(x) - p_{m^+}(x)\,dx \\ &\leq \frac{1}{\epsilon}\int_{\ell}^{\ell+\epsilon} \frac{p_{s^-}(\ell+\epsilon)}{p_s(\ell+\epsilon)}p_s(x) - p_{m^+}(x)\,dx. \end{aligned}$$

By Theorem 2.1.7 and continuity of p_s, we therefore see that

$$\frac{p_{s^-}(\ell')}{p_s(\ell')}p_s(\ell) - p_{m^+}(\ell) \leq D_+c_{ES}(\ell) \leq D^+c_{ES}(\ell) \leq \frac{p_{s^-}(\ell+\epsilon)}{p_s(\ell+\epsilon)}p_s(\ell) - p_{m^+}(\ell).$$

Since $p_{s^-}/p_s < 1$, evaluating the final inequality using the explicit forms of p_s and p_{m^+} shows that $D^+c_{ES}(\ell) < 0$ whenever $\ell \geq 1$. Since $p_{s^-} > 0$ and $p_{m^+}(\ell) = 0$ for $\ell \leq 0$, taking $\ell' = \ell$ in the first inequality shows that $D_+c_{ES}(\ell) > 0$ for $\ell \leq 0$. If $\ell > 0$ then we may take $\ell' = \ell_n$ as defined above for sufficiently large n. Then by evaluating the densities explicitly we see that $1/2p_s(\ell) - p_{m^+}(\ell) > 0$ for $\ell \in (0, 0.876]$ thus completing the proof of the statements for c_{ES}.

Since $p_{m^-}(x) = 0$ for $x > 0$, we see from Theorem 2.1.7 that

$$\frac{c_{LS}(\ell+\epsilon) - c_{LS}(\ell)}{\epsilon} = \frac{c_{ES}(\ell+\epsilon) - c_{ES}(\ell)}{\epsilon} - \frac{1}{\epsilon}\int_\ell^{\ell+\epsilon} p_{s^+}(x)\,dx$$

for $\ell > 0$. As $D^+c_{ES}(\ell) < 0$ for $\ell \geq 1$ and $p_{s^+} \geq 0$, taking the limit superior here shows that $D^+c_{LS}(\ell) < 0$ (for $\ell \geq 1$). □

Proof of Propositions 3.2.15 and 3.2.16. By Theorem 3.2.4, both c_{ES} and c_{LS} are differentiable, and so by Theorem 2.1.7

$$\begin{aligned} c'_{ES}(\ell) &= p_{s^-}(\ell) - p_{m^+}(\ell) \leq p_s(\ell) - p_{m^+}(\ell) \\ c'_{LS}(\ell) &= p_{m^-}(\ell) + p_{s^-}(\ell) - p_{m^+}(\ell) - p_{s^+}(\ell) \leq p_{m^-}(\ell) + p_s(\ell) - p_{m^+}(\ell). \end{aligned} \tag{3.36}$$

The densities p_{m^-}, p_s and p_{m^+} were derived for isotropic fields satisfying (a weaker version of) Assumption 2.1.1 in [26]. Specifically, these densities are given in terms of $\chi \in (0, \sqrt{2}]$ and $\xi^2 > 0$ (defined in (2.3)) by

$$\begin{aligned} p_{m^+}(x) = p_{m^-}(-x) = \frac{1}{\pi\xi^2}\Bigg(&\chi^2(x^2-1)\phi(x)\Phi\left(\frac{\chi x}{\sqrt{2-\chi^2}}\right) + \frac{\chi x\sqrt{2-\chi^2}}{2\pi}e^{-\frac{x^2}{2-\chi^2}} \\ &+ \frac{\sqrt{2}}{\sqrt{\pi(3-\chi^2)}}e^{-\frac{3x^2}{2(3-\chi^2)}}\Phi\left(\frac{\chi x}{\sqrt{(3-\chi^2)(2-\chi^2)}}\right)\Bigg) \\ p_s(x) = \frac{1}{\pi\xi^2}&\frac{\sqrt{2}}{\sqrt{\pi(3-\chi^2)}}e^{-\frac{3x^2}{2(3-\chi^2)}} \end{aligned}$$

where ϕ and Φ denote the standard normal probability density and cumulative density respectively. (We ignore the case $\chi = \sqrt{2}$ since that corresponds to the Random Plane Wave which is covered by the previous proposition.) By Theorem 2.1.7,

$$c_{LS}(\ell+\epsilon) - c_{LS}(\ell) \leq \int_\ell^{\ell+\epsilon} p_{m^-}(x) + p_s(x) - p_{m^+}(x)\,dx.$$

We denote the integrand above by $I(x)$. Evaluating this expression using the standard Gaussian inequality $1-\Phi(x)\leq(1/x)\phi(x)$ shows that

$$I(x)\leq\frac{1}{\pi\xi^2}\left(\frac{\sqrt{2-\chi^2}}{\pi}\frac{1}{x}\frac{2-\chi^2}{\chi}e^{-\frac{x^2}{2-\chi^2}}-\frac{\chi^2}{\sqrt{2\pi}}(x^2-1)e^{-\frac{x^2}{2}}\right)$$

which is negative for $x>\sqrt{2}/\chi$. Similarly we have

$$c_{ES}(\ell+\epsilon)-c_{ES}(\ell)\leq\int_{\ell}^{\ell+\epsilon}p_s(x)-p_{m^+}(x)\,dx,$$

which is less than our upper bound for $c_{LS}(\ell+\epsilon)-c_{LS}(\ell)$. Hence $c'_{ES}(\ell),c'_{LS}(\ell)<0$ for $\ell>\sqrt{2}/\chi$.

We note that this is a sufficient condition for the derivatives to be negative, chosen for its simplicity. For many fields, the derivatives will be negative on a larger region, and this can be found by using the densities specified above with the appropriate value of χ. For the Bargmann-Fock field, (for which $\chi=1$) substituting these densities into (3.36) shows that $c'_{ES}(\ell),c'_{LS}(\ell)<0$ for $\ell\geq 1.03$, improving on the general bound $\ell>\sqrt{2}/\chi=\sqrt{2}$.

Next we note that $c'_{ES}(0)=p_{s^-}(0)-p_{m^+}(0)$, and by the identities $p_{s^-}(x)=p_{s^+}(-x)$, $p_{s^-}+p_{s^+}=p_s$ almost everywhere and the fact these densities are all continuous, we see that $p_{s^-}(0)=p_s(0)/2$. Evaluating the densities given in [26] at zero shows that $p_s(0)/2>p_{m^+}(0)$ so we conclude that $c'_{ES}(0)>0$. Since c_{ES} is continuously differentiable, we can extend this to a neighbourhood of the origin.

By Theorem 3.2.10, $p_{s^-}(\ell)/p_s(\ell)$ is non-decreasing and so for $\ell>0$

$$\frac{p_{s^-}(\ell)}{p_s(\ell)}\geq\frac{p_{s-}(0)}{p_s(0)}=\frac{1}{2}.$$

Therefore $c'_{ES}(\ell)\geq p_s(\ell)/2-p_{m^+}(\ell)$ for $\ell\geq 0$. Evaluating the densities above then gives an explicit constant C such that this expression is strictly positive for $\ell\leq C$. In the case of the Bargmann-Fock field, $C=0.64$. □

Chapter 4

Variance of the Number of Excursion Sets

4.1 Introduction

In the final chapter of this thesis, we continue analysing the number of Gaussian excursion/level sets in a large domain and progress from studying the mean of this quantity to its variance. We recall from Section 1.2, that this variable has important statistical applications, in fields such as cosmology, and is significant to the development of mathematical theory in helping to unify the percolation theory of Gaussian fields and discrete lattice models. In particular, the Bogomolny-Schmit conjecture makes explicit claims about the variance of the number of nodal and level set components of the Random Plane Wave.

As discussed in Section 1.4.3, results for the variance of the number of excursion/level set components are comparatively lacking. For general fields it is known that there exists $c = c(\ell) > 0$ such that, for all sufficiently large R,

$$\mathrm{Var}(N_{LS}(R,\ell)) < cR^4 \quad \text{and} \quad \mathrm{Var}(N_{ES}(R,\ell)) < cR^4. \tag{4.1}$$

This upper bound has been slightly improved for Random Spherical Harmonics in the case of zero level sets [80]. For a general class of Gaussian ensembles (f_n) defined on the sphere, the variance of the number of nodal set components has been shown to grow at least as a positive power of n, with some unspecified exponent [82].

Given the sparsity of such results, one could attempt to predict the true behaviour of the variance in (at least) two ways: based on the corresponding statistics of other geometric variables or using the similarities between Gaussian fields and discrete percolation models.

For Gaussian fields with fast correlation decay, these methods offer a clear prediction: $\mathrm{Var}(N_{ES}(R,\ell))$ and $\mathrm{Var}(N_{LS}(R,\ell))$ should be of order R^2 as $R \to \infty$. This matches the behaviour of local geometric functionals such as Lipschitz-Killing curvatures or the number of critical points (see Section 1.4). It is also consistent with Bernoulli bond percolation on the square lattice (and other similar models); in this case the number of open connected components in $[-R,R]^2$ has variance of order R^2 for any value of the parameter $p \in (0,1)$. Recall from Section 1.4.2 that planar Gaussian fields with fast correlation decay behave analogously to Bernoulli percolation in other respects.

For Gaussian fields with strong long range dependence, the percolation analogy is not particularly helpful. For example, consider a doubly periodic field which is completely specified by its values on $[0,1]^2$ (i.e. the value of the field is the same at any two points which differ by an integer coordinate translation). It is clear that such a model bears little resemblance to Bernoulli percolation. In Proposition 2.1.11, we gave an example of this type of field (with spectral measure supported on four or five points) and showed that the number of excursion set components in $B(R)$ is R^2 times a single random variable, plus some lower order boundary term, and so has variance of order R^4. This argument could be extended to other doubly periodic fields (without explicitly identifying the random variable) assuming some non-degeneracy. We consider another type of long range dependence, in which this order of variance is attained, later in this chapter (see Theorem 4.2.11).

It is less clear what behaviour to expect in intermediate cases: Gaussian fields with correlations which decay slowly or oscillate. An important instance of this case is the Random Plane Wave. The Bogomolny-Schmit conjecture predicts that the variance of the number of nodal sets of the Random Plane Wave should be of order R^2. In a follow up paper [19], Bogomolny and Schmit make the stronger claim that level sets of the Random Plane Wave at levels close to zero can be modelled by non-critical Bernoulli bond percolation, and give some numerical evidence in favour of this statement. If this is true, it would imply that the variance of the number of excursion/level sets should be of order R^2 for such levels. This is in contrast with the behaviour of local geometric functionals for a closely related model: Random Spherical Harmonics. If f_n is the n-th Random Spherical Harmonic, then the Euler characteristic of $\{f_n \geq \ell\}$ has variance of order n^3 at generic levels (Section 1.4.1). This functional should scale roughly like the number of excursion sets (bearing in mind the definition of the Euler characteristic of a two-dimensional set as the number of components minus the number of 'holes'). Based on this, we might expect the variance of the number of excursion/level set components of Random Plane Wave to be of order R^3 at generic levels.

These heuristic arguments highlight the implications of studying the variance of the number of excursion sets on universality. One of the central goals in the recent literature on percolation results for Gaussian fields, is to understand which fields share a 'universality class' with discrete lattice models (i.e. which of these models have the same large scale behaviour). Depending on how exactly the notion of universality class is made rigorous, it is likely that the number of excursion sets of models in the same class should have the same order of variance. Therefore if this variance can be shown to have different order for two models, then this provides strong evidence that the models are not in the same universality class.

In this chapter, we prove lower bounds on $\mathrm{Var}(N_{LS}(R,\ell))$ and $\mathrm{Var}(N_{ES}(R,\ell))$ which are, conjecturally at least, of the correct order. To summarise our main results, we show that for a wide class of Gaussian fields there exists an exponent $\alpha \in [2,4]$ (depending on the field and explicitly known) such that, for many levels ℓ,

$$\mathrm{Var}(N_{LS}(R,\ell)) > cR^{\alpha} \quad \text{and} \quad \mathrm{Var}(N_{ES}(R,\ell)) > cR^{\alpha}. \tag{4.2}$$

for some $c = c(\ell) > 0$ and all R sufficiently large. The value of $\alpha \in [2,4]$ depends on the behaviour at the origin of the spectral measure of the field. For fields with fast correlation decay and positive spectral density at the origin, the bound (4.2) holds for $\alpha = 2$, whereas for fields whose spectral measure has a singularity at the origin, (4.2) holds for an $\alpha \in (2,4)$ that depends on the rate of blow-up of the singularity. We also study the important special case of the Random Plane Wave, for which we show that (4.2) holds with $\alpha = 3$. We now present these results formally, before discussing them in Section 4.2.3.

4.2 Main results

4.2.1 Fluctuations of the number of excursion/level set components

Throughout this chapter we refine the basic assumptions on our field to the following:

Assumption 4.2.1. *The field f satisfies Assumption 2.1.1 and also the following:*

1. $f \in C^3_{loc}(\mathbb{R}^2)$ *almost surely;*
2. $\mathrm{Cov}(\nabla f(0)) = cI_2$ *where c is a positive constant and I_2 is the 2×2 identity matrix.*

This is only a very slight strengthening of Assumption 2.1.1: the first point simplifies some of the arguments we will make involving Bulinskaya's lemma, but could likely be removed at the cost of lengthening our proofs. The second point is simply a convenient normalisation; for any field satisfying Assumption 2.1.1 we can rotate and rescale the domain $\mathbb{R}^2$ so that this normalisation holds. It will be clear from the statement of our later results that they are not affected by this rescaling (apart from the value of some implicit constants).

Our main results concern the order of fluctuations of N_{ES} and N_{LS}. To formalise this concept we make use of the following definition, taken from [25].

Definition 4.2.2. Let X_n be a sequence of random variables and u_n a sequence of positive real numbers. We say that X_n has *fluctuations of order at least* u_n if there exist $c_1, c_2 > 0$ such that, for all sufficiently large n and all real numbers $a \leq b$ with $b - a \leq c_1 u_n$,

$$\mathbb{P}(a \leq X_n \leq b) \leq 1 - c_2.$$

Similarly, we say that a collection of random variables $(X_R)_{R\geq 0}$ has *fluctuations of order at least* $(u_R)_{R\geq 0}$ if, for any increasing sequence $R_n \to \infty$, X_{R_n} has fluctuations of order at least u_{R_n}.

It is easy to see that if a collection of random variables $(X_n)_{n\geq 0}$ has fluctuations of order at least $(u_n)_{n\geq 0}$ then it has variance of order at least u_n^2, i.e. there exists $c > 0$ such that

$$\mathrm{Var}(X_n) > cu_n^2 \tag{4.3}$$

for all n sufficiently large. On the other hand, having fluctuations of order at least u_n is a strictly stronger statement than (4.3), since the latter is consistent with the bulk of the probability mass concentrating on arbitrarily small scales.

We now present our main results on the fluctuations of N_{ES} and N_{LS}, which are divided into three statements. The first applies to general fields, and in particular fields that either (i) have fast correlation decay and positive spectral density at the origin, or (ii) have a spectral measure with a singularity at the origin. The second concerns the special case of the Random Plane Wave. The third treats a certain class of degenerate fields.

4.2.1.1 General fields

To state our first result we introduce stronger assumptions on fields.

Assumption 4.2.3.

1. *There exists a neighbourhood* $V \subset \mathbb{R}^2$ *of the origin such that the spectral measure* ν *has density* ρ *on* V *and* $\inf_V \rho > 0$.

2. $\max_{|k|\leq 2} \left|\partial^k K(x)\right| \to 0$ *as* $|x| \to \infty$.

We discuss this assumption in more detail after stating our results. For now, we simply note that under this assumption, the support of ν contains an open set, and so the Gaussian vector formed from f and ∇f at any finite number of distinct points is non-degenerate (Lemma A.2.4).

In the case in which there is no singularity in the spectral measure, we need to assume some extra control over the saddle points of the field. We recall that four-arm saddle points are defined in Section 3.2.1.1 and that $N_{\text{4-arm}}(R, [a, b])$ denotes the number of saddle points of f which are four-arm in $B(R)$ and have level in $[a, b]$.

Assumption 4.2.4. *For every* $a \leq b$*, there exists a function* $\delta_R \to 0$ *as* $R \to \infty$ *and a constant* $c > 0$ *such that, for each* $R > 1$ *and* $a \leq a_R \leq b_R \leq b$*,*

$$\mathbb{E}\left(N_{\text{4-arm}}(R, [a_R, b_R])\right) \leq c \min\{\delta_R R^2 (b_R - a_R), R\}.$$

Corollary 3.2.6 gives sufficient conditions for a field to satisfy Assumption 4.2.4. We state this assumption separately, as future work may prove it under weaker conditions.

We can now state our fluctuation lower bound for general fields. We recall that the definition of Dini derivatives was given in Section 3.2.2.

Theorem 4.2.5. *Let* f *be a Gaussian field satisfying Assumptions 4.2.1 and 4.2.3, and define* $g(r) = \inf_{x \in B(2r)} \rho(x)$*. Suppose further that at least one of the following holds:*

1. *The field* f *satisfies Assumption 4.2.4, or*

2. *The spectral measure* ν *has a singularity at the origin, i.e.* $g(r) \to \infty$ *as* $r \to 0$.

If c_{ES} *has a positive lower Dini derivative at* ℓ *(or a negative upper Dini derivative), then* $(N_{ES}(R, \ell))_{R \geq 0}$ *has fluctuations of order at least* $R\sqrt{g(1/R)}$*, and hence variance of order at least* $R^2 g(1/R)$*. The same conclusion holds if we replace* N_{ES} *and* c_{ES} *with* N_{LS} *and* c_{LS} *respectively.*

The variance lower bound $R^2 g(1/R)$ interpolates between R^2 (if the spectral density is bounded at the origin) and $o(R^4)$ (note that $g(1/R) = o(R^2)$ since the spectral density around the origin is integrable). This is consistent with the trivial upper bound in (4.1).

We note that fields with integrable covariance functions have a spectral density which is continuous and bounded (this follows from standard results for the Fourier transform). In this case, Theorem 4.2.5 gives a variance lower bound of order R^2. This matches the order of variance of all Lipschitz-Killing curvatures of excursion sets and also of the number of critical points for such fields (see Section 1.4.1). We therefore strongly expect that this lower bound is of the correct order for such fields.

In Chapter 3 we gave conditions on the field which ensured that c_{ES} and c_{LS} are continuously differentiable; in this case the conditions on Dini derivatives in Theorem 4.2.5 are equivalent to the conditions $c'_{ES}(\ell) \neq 0$ and $c'_{LS}(\ell) \neq 0$.

At this point, it is natural to ask how generally the derivative condition of Theorem 4.2.5 holds. This further justifies our investigation of the shape of the functionals c_{ES} and c_{LS} in Chapter 3. For general fields, we expect that c_{ES} and c_{LS} have non-zero derivative for all but a small, finite number of levels ℓ. We do not have any particularly strong justification for this, however we note that the mean functionals for local geometric quantities of planar Gaussian fields generally have this property and simulations have shown that c_{ES} should be unimodal for some non-degenerate fields (see [87]). We therefore hope that Theorem 4.2.5 can eventually be applied to all but a finite number of levels. However we note that, Theorem 4.2.5 cannot be applied directly to $N_{LS}(R, 0)$, since by symmetry $c'_{LS}(0) = 0$ whenever the derivative is defined. In Section 4.2.3 we give some motivation for why $c'_{ES}(\ell) \neq 0$ and $c'_{LS}(\ell) \neq 0$ are, in a sense, natural conditions for a lower bound on fluctuations.

Using the monotonicity results of Chapter 3, we are able to apply Theorem 4.2.5 for a large range of levels. We illustrate this with the Bargmann-Fock field.

Corollary 4.2.6. *Let f be the Bargmann-Fock field. Then there exists $\epsilon > 0$ such that the following holds. If $\ell \in [-\epsilon, 0.64] \cup [1.03, \infty)$ then $(N_{ES}(R, \ell))_{R \geq 0}$ has fluctuations of order at least R and hence variance of order at least R^2. If $|\ell| \geq 1.03$ then $(N_{LS}(R, \ell))_{R \geq 0}$ has fluctuations of order at least R and hence variance of order at least R^2.*

Proof. Assumptions 4.2.1 and 4.2.3 are trivially satisfied for the Bargmann-Fock field, and Corollary 3.2.6 states that Assumption 4.2.4 is also satisfied. The corollary then follows from Proposition 3.2.15, which states that $c'_{ES}(\ell) \neq 0$ and $c'_{LS}(\ell) \neq 0$ for the respective ranges of levels given above. □

For general isotropic fields we can apply Corollary 3.2.6 and Proposition 3.2.16 to deduce the same fluctuation bounds for a similar range of levels. The exact values of the end-points of the intervals depend on the covariance function of the field.

4.2.1.2 The Random Plane Wave

We now turn to the special case of the Random Plane Wave. This field does not fall within the scope of Theorem 4.2.5, since it does not satisfy Assumption 4.2.3 (its spectral measure is supported on the unit circle). Nevertheless we can prove the following fluctuation bound.

Theorem 4.2.7. *Let f be the Random Plane Wave. If c_{ES} has a positive lower Dini derivative at $\ell \neq 0$ (or a negative upper Dini derivative), then $(N_{ES}(R,\ell))_{R\geq 0}$ has fluctuations of order at least $R^{3/2}$, and hence variance of order at least R^3. The same conclusion holds if we replace N_{ES} and c_{ES} with N_{LS} and c_{LS} respectively.*

The order of variance in this lower bound matches that of two related geometric functionals of Random Spherical Harmonic excursion sets at generic levels: the Euler characteristic and the number of critical points. Since these two quantities are comparable to the number of excursion set components (in a sense these are all zero-dimensional variables, in contrast to, say, the boundary length of excursion sets), we expect that this bound is of the correct order for general levels.

As for the Bargmann-Fock field, our monotonicity bounds in Chapter 3 allow us to apply this result for a specific range of levels, which leads to the following corollary.

Corollary 4.2.8. *Let f be the Random Plane Wave. If $\ell \in (-\infty,0)\cup(0,0.87]\cup[1,\infty)$ then $(N_{ES}(R,\ell))_{R\geq 0}$ has fluctuations of order at least $R^{3/2}$ and hence variance of order at least R^3. If $|\ell| \geq 1$ then $(N_{LS}(R,\ell))_{R\geq 0}$ has fluctuations of order at least $R^{3/2}$ and hence variance of order at least R^3.*

4.2.1.3 Fields with a random level shift

Finally we consider the class of fields whose spectral measure has a delta mass at the origin. In this case, we prove that the variance attains the order of the trivial upper bound in (4.1) for all levels.

Assumption 4.2.9.

1. *For each* $x \in \mathbb{R}^2$*,* $(f(x), f(0), \nabla f(x), \nabla f(0))$ *is a non-degenerate Gaussian vector;*

2. $\max_{|k|\leq 2} \left|\partial^k K(x)\right| \to 0$ *as* $|x| \to \infty$*.*

Assumption 4.2.10. *The stationary Gaussian field* f *has spectral measure* $\nu = \alpha\delta_0 + \nu^*$ *where* $\alpha > 0$ *and* ν^* *is a (positive) measure. If* g *is the Gaussian field with spectral measure* ν^* *then* g *satisfies Assumptions 2.1.1 and 4.2.9.*

Under this assumption, the field f is no longer normalised to have variance one at a point: instead $\mathrm{Var}(f(x)) = 1 + \alpha$ and $\mathrm{Var}(g(x)) = 1$ for all $x \in \mathbb{R}^2$. This is motivated by the fact that we can represent f as

$$f = g + \sqrt{\alpha} Z$$

where Z is a standard Gaussian variable independent of g. This representation follows immediately from considering its covariance function. In order to analyse the level sets of f, we apply results from previous chapters to g and consider the additional effect of shifting the overall level due to Z. It is therefore convenient to normalise g as in our earlier work. Of course, our results apply to any stationary Gaussian field with spectral mass at the origin, one simply has to rescale the level to match the normalisation above.

Theorem 4.2.11. *Let* f *be a Gaussian field satisfying Assumption 4.2.10. For each* $\ell \in \mathbb{R}$*,* $(N_{ES}(f, R, \ell))_{R\geq 0}$ *has fluctuations of order at least* R^2*. Moreover, there exist positive constants* $c_1(\ell), c_2(\ell)$ *such that*

$$c_1 R^4 < \mathrm{Var}(N_{ES}(f, R, \ell)) < c_2 R^4$$

for all $R > 0$ *sufficiently large. The same conclusions hold if we replace* N_{ES} *and* c_{ES} *with* N_{LS} *and* c_{LS} *respectively.*

In this case there is no requirement on the derivative of the mean functional at a given level. Roughly speaking this holds because the level shift Z can always shift the field g to levels at which its mean functional has non-zero derivative (and the collection of such levels has positive measure). Therefore, this result intuitively says that adding a random independent level shift to any non-degenerate Gaussian field (which is equivalent to adding a delta mass to the spectral measure at the origin - see above) ensures that the number of excursion/level set components of the resulting field have variance of maximal order at all levels.

4.2.2 Outline of the method

Before discussing the consequences of our results in detail, it will be helpful to outline our method of proof. The foundation of our argument is a versatile lemma due to Chatterjee.

Lemma 4.2.12 ([25, Lemma 1.2])**.** *Let X and Y be random variables defined on the same probability space. Then, for real numbers $a \leq b$,*

$$\mathbb{P}(a \leq X \leq b) \leq \frac{1}{2}\left(1 + \mathbb{P}\left(|X - Y| \leq b - a\right) + d_{TV}(X, Y)\right),$$

where d_{TV} denotes the total variation distance between the distributions of X and Y.

Definition 4.2.13. Let X_n and Y_n be sequences of random variables defined on the same probability space and let u_n be a sequence of positive real numbers. We say that X_n and Y_n differ by order at least u_n if there exist constants $c_1, c_2 > 0$ such that

$$\mathbb{P}\left(|X_n - Y_n| \geq c_1 u_n\right) \geq c_2$$

for all n sufficiently large.

Corollary 4.2.14. *Let X_n and Y_n be sequences of random variables defined on the same probability space and let u_n be a sequence of positive numbers. If X_n and Y_n differ by order at least u_n and $d_{TV}(X_n, Y_n) \to 0$ as $n \to \infty$, then X_n has fluctuations of order at least u_n.*

We apply Corollary 4.2.14 with $X_R = N_{ES}(R, \ell)$ and $Y_R = N_{ES}(R, \ell + a_R)$ for a certain sequence $a_R \to 0$ as $R \to \infty$. There are two competing requirements on a_R: (i) a_R should decay slowly enough that $N_{ES}(R, \ell)$ and $N_{ES}(R, \ell + a_R)$ differ by a large order; and (ii) a_R must decay quickly enough that $d_{TV}(N_{ES}(R, \ell), N_{ES}(R, \ell + a_R))$ tends to zero.

Let us consider first the order by which $N_{ES}(R, \ell)$ and $N_{ES}(R, \ell + a_R)$ differ. Using the assumption that c_{ES} has non-zero (Dini-)derivative at ℓ, and the convergence to c_{ES} proven in Chapter 2, we show in Lemma 4.3.2 that

$$|\mathbb{E}(N_{ES}(R, \ell) - N_{ES}(R, \ell + a_R))| \gtrsim R^2 a_R.$$

Using a bound on the second moment of the number of critical points in a shrinking height window from [76], (proven using the Kac-Rice theorem) we then show in Lemma 4.3.3 that

$$\left(\mathbb{E}\left((N_{ES}(R, \ell) - N_{ES}(R, \ell + a_R))^2\right)\right)^{1/2} \lesssim R^2 a_R.$$

Since these bounds are of the same order, the second moment method implies that $N_{ES}(R,\ell)$ and $N_{ES}(R,\ell+a_R)$ differ by order at least $R^2 a_R$ (see Proposition 4.3.5).

The next step is to bound the total variation distance between $N_{ES}(R,\ell)$ and $N_{ES}(R,\ell+a_R)$. Our arguments in this step are different for general fields and for the Random Plane Wave.

For general fields (Theorems 4.2.5 and 4.2.11), we view $N_{ES}(R,\ell+a_R)$ as the number of excursion sets of the field $f-a_R$ at level ℓ, and so

$$d_{TV}(N_{ES}(R,\ell),N_{ES}(R,\ell+a_R)) \leq d_{TV}(f,f-a_R).$$

A Cameron-Martin argument then gives an upper bound on this distance in terms of the norm of an (approximately) constant function in the reproducing kernel Hilbert space induced by the field (see (4.12) for the definition of this Hilbert space). By bounding this norm in terms of the behaviour of the spectral measure at the origin, we can prove that $d_{TV}(N_{ES}(R,\ell),N_{ES}(R,\ell+a_R)) \to 0$ as long as

$$a_R \ll \sqrt{g(1/R)}/R,$$

or $a_R \ll 1$ in the case of spectral mass at the origin. Combining this with the previous step, we deduce a fluctuation bound of order

$$R^2 a_R \approx R\sqrt{g(1/R)},$$

or $R^2 a_R \approx R^2$ in the case of spectral mass at the origin.

In the case of the Random Plane Wave (Theorem 4.2.7), the previous approach fails since the constant function cannot be approximated in the reproducing kernel Hilbert space of the Random Plane Wave (which consists of solutions to the Helmholtz equation $\Delta f = -f$). Instead, our approach is to view $N_{ES}(R,\ell+a_R)$ as the number of excursion sets of the field $\ell/(\ell+a_R)f$ at level ℓ (note that this only holds for $\ell \neq 0$, which is why the nodal level is excluded from our results on the Random Plane Wave), and so

$$d_{TV}(N_{ES}(R,\ell),N_{ES}(R,\ell+a_R)) \leq d_{TV}\left(f,\frac{\ell}{\ell+a_R}f\right). \tag{4.4}$$

Using an orthogonal expansion for the Random Plane Wave in terms of Bessel functions (4.20), we show that the topological behaviour of the Random Plane Wave on $B(R)$ is essentially determined by $4R$ i.i.d. standard Gaussian variables. Pinsker's inequality therefore allows us to bound (4.4) in terms of the Kullback-Leibler divergence from one Gaussian vector to another. We recall that, for two probability measures

P, Q such that P is absolutely continuous with respect to Q, the Kullback-Leibler divergence from Q to P is defined as

$$d_{\mathrm{KL}}(P \,||\, Q) = \int_\Omega \log\left(\frac{dP}{dQ}\right)\, dP \tag{4.5}$$

where Ω is the sample space of P and $\frac{dP}{dQ}$ is the Radon-Nikodym derivative of P with respect to Q. This quantity can be computed explicitly for Gaussian vectors, and as a result we show $d_{TV}(N_{ES}(R,\ell), N_{ES}(R,\ell + a_R)) \to 0$ as long as

$$a_R \ll 1/\sqrt{R}.$$

Combining this with the previous step, we deduce a fluctuation bound of order

$$R^2 a_R \approx R^{3/2}.$$

In both cases, the main technical step is to ensure that the approximations (in the general case, approximating the constant function inside the reproducing kernel Hilbert space, and for the Random Plane Wave, truncating the orthogonal expansion) do not radically change the number of excursion set components. To achieve this we apply Morse theory arguments to bound the change by the number of 'quasi-critical points', which we can control with local computations (see the proofs of Lemmas 4.3.8 and 4.3.9 in Section 4.4).

4.2.3 Further discussion and open questions

4.2.3.1 Role of assumptions

Roughly speaking, we have shown that, if a Gaussian field at a given level is statistically similar to the same field at a different level and the asymptotic behaviour of excursion sets at these levels is different, then the number of components fluctuates. One way of interpreting this result, is to say that the fluctuations we have proven are caused by 'level shifts' of the fields. This is most clearly illustrated by Theorem 4.2.11: fields with a true global level shift, as the result of adding an independent Gaussian variable, have the maximal order of fluctuations in the number of excursion set components. For cases like the Bargmann-Fock field, which have fast correlation decay, there is no possibility of having a global level shift. However by assuming the spectral density is bounded away from zero at the origin, we ensure that approximate level shifts on finite domains are possible with some probability. Making the stronger assumption of a spectral singularity at the origin, it becomes more likely for the field to have a level shift and consequently our fluctuation bounds increase.

These ideas can also be related to long-range dependence of the field. Intuitively, one would expect that fields with slower decay of correlation are more likely to exhibit level shifts. This is evident for fields with spectral mass at the origin: such fields have global level shifts and their covariance functions cannot decay to zero. Similarly, Gaussian fields with spectral measure supported on precisely five points (see Proposition 2.1.11) have a (doubly periodic) covariance function which does not decay to zero and attain the maximal order of variance for the number of excursion sets. This result could easily be extended to doubly-periodic fields under some minimal non-degeneracy assumptions.

In the case of spectral singularity at the origin (i.e. $g(r) \to \infty$), the covariance function typically decays too slowly to be integrable. In particular, standard Abel/Tauberian theorems [63, Chapter 1.4] imply that, up to some regularity assumptions, the asymptotics $\rho(x) \sim |x|^{-\alpha}$ as $|x| \to 0$ and $K(x) \sim |x|^{\alpha-2}$ as $|x| \to \infty$ are equivalent for $\alpha \in (0,2)$. Hence, broadly speaking, our results show that if correlations decay polynomially with exponent $\beta \in (0,2)$, then the variance of N_{ES} and N_{LS} grow with order at least $R^{4-\beta}$. This is analogous to known results on fluctuations of certain local functionals of long-range dependent Gaussian processes and fields [63, 100].

The Random Plane Wave also displays long-range dependence: its covariance function decays like $|x|^{-1/2}$ as $|x| \to \infty$. In this case we again observe larger fluctuations of the number of excursion sets (order $R^{3/2}$ compared to the generic R). Assuming this bound is tight, it does not match the behaviour of fields with spectral singularity at the origin, for which covariance decay of order $|x|^{-1/2}$ would be associated with variance of order $R^{7/2}$. This is perhaps unsurprising, given that we approximate level shifts in a different way in this case. Specifically, we approximately parameterise the field in a bounded domain by a finite number of random variables and rescale these, rather than adding a constant to the field. Moreover, the argument for the Random Plane Wave must be different: raising a realisation of this field by a constant causes the Helmholtz equation to break down, and so is extremely degenerate for this field. It is possible, therefore, that the 'reason' for fluctuations in the Random Plane Wave case is different, in some sense.

Finally in the case of fields with integrable covariance functions, we obtain variance lower bounds of minimal order R^2. We note that this bound is not reduced by assuming faster decay of correlations. Hence, any correlations which decay fast enough to be integrable have no large scale effects on the fluctuations of excursion sets (this assumes our bounds are of the correct order). Overall we can conclude that, assuming our bounds are sharp at generic levels, stronger long range dependence is associated with larger fluctuations.

4.2.3.2 Comparison with other geometric variables

At this point, it is interesting to compare our results to those for other geometric properties of Gaussian fields, as described in Section 1.4.1.

As discussed above, we believe our bounds to be of the correct order for generic levels, based on this comparison. However it is known that some geometric functionals exhibit lower variance at certain anomalous levels for particular fields. Specifically, for Random Spherical Harmonics, (which are closely related to the Random Plane Wave) lower order variance occurs for each of the Lipschitz-Killing curvatures at certain levels (including the zero level). This is due to the coefficient of the dominant Wiener Chaos projection vanishing at certain levels.

The same is also known to be true in the case of spectral singularity at the origin [63, Chapter 3]. By contrast, for fields with rapid correlation decay, many terms in the Wiener chaos expansion have fluctuations of leading order (see for instance [32]), and so one should not expect anomalous levels, since that would require many coefficients to vanish simultaneously. This is indeed what occurs for the Lipschitz-Killing curvatures of such fields.

Based on these observations, we conjecture the following:

Conjecture 4.2.15. *Suppose that f satisfies Assumptions 4.2.1, 4.2.3, 4.2.4 and that K is integrable (e.g., the Bargmann-Fock field). Then for all $\ell \in \mathbb{R}$ there exists $c_{var}(\ell) > 0$ such that*

$$\mathrm{Var}(N_{ES}(R,\ell)) \sim c_{var}(\ell) R^2,$$

where $a_R \sim b_R$ denotes that $\lim_{R\to\infty} a_R/b_R = 1$, and the same conclusion is true for $N_{LS}(R,\ell)$.

Conjecture 4.2.16. *Suppose that f satisfies Assumptions 4.2.1 and 4.2.3, and assume that there exist $\alpha \in (0,2)$, $r_0 > 0$ and $0 < c < C$ such that*

$$c|x|^{-\alpha} < \rho(x) < C|x|^{-\alpha}$$

for all $|x| < r_0$. Then there exists a (possibly empty) finite set $\mathcal{L} \subset \mathbb{R}$ and $c_{var}(\ell) > 0$ such that, for all $\ell \notin \mathcal{L}$,

$$\mathrm{Var}(N_{ES}(R,\ell)) \sim c_{var}(\ell) R^{2+\alpha}$$

whereas for all $\ell \in \mathcal{L}$,

$$\mathrm{Var}(N_{ES}(R,\ell)) \ll R^{2+\alpha},$$

and the same conclusion is true for $N_{LS}(R,\ell)$ (with different $\mathcal{L}$ and $c_{var}(\cdot)$). We expect the same conclusion if $\rho(x)$ is of order $|x|^{-\alpha}$ as $|x| \to 0$ in some appropriate sense. If f is the Random Plane Wave, then the same conclusion is true with $R^{2+\alpha}$ replaced with R^3.

Assuming that Conjectures 4.2.15 and 4.2.16 are correct, they give rise to a number of further questions. For simplicity we discuss only the case of excursion sets, but the analogous questions can be asked of level sets.

A first set of questions concerns the anomalous levels $\mathcal{L}$ in the case of the Random Plane Wave or fields with spectral singularity.

Question 4.2.17.

1. Is the set of anomalous levels $\mathcal{L}$ non-empty? What is its cardinality?

2. Let $\mathcal{C}$ denote the set of critical points of the density functional c_{ES}. By Theorems 4.2.5 and 4.2.7, we know that $\mathcal{L} \subseteq \mathcal{C} \cup \{0\}$ for the Random Plane Wave, whereas $\mathcal{L} \subseteq \mathcal{C}$ in the case of spectral singularity case. Are these containments strict?

3. What is the order of $\mathrm{Var}(N_{ES}(R,\ell))$ for $\ell \in \mathcal{L}$? Does it depend on the field and on the level? Is it always of order at least R^2?

A second question concerns the constants $c_{var}(\ell)$ for generic levels $\ell \notin \mathcal{L}$. For the Lipschitz-Killing curvatures of Random Spherical Harmonics, it is known [24, 21] that $c_{var}(\ell)$ is related to the derivative of the first moment (i.e. density) functional $c(\ell)$ via

$$c_{var}(\ell) \propto (\ell\, c'(\ell))^2. \tag{4.6}$$

In particular, levels are anomalous precisely when either $\ell = 0$ or $c'(\ell) = 0$, which are exactly the conditions for which our bound in Theorem 4.2.7 hold. This is evidence that our conditions in Theorem 4.2.7 are quite natural.

We are not aware of any similar results to (4.6) for the Lipschitz-Killing curvatures of general fields, and indeed in general it is difficult to compute the value of c_{var} exactly (even if the density $c(\ell)$ is well-understood for Lipschitz-Killing curvatures [1]). It would be interesting to know if (4.6), or a similar relationship, holds in more generality.

Question 4.2.18. What is the relationship between $c_{var}(\ell)$ and the derivative of the density functional $c'_{ES}(\ell)$? Is $c_{var}(\ell) \propto (\ell\, c'_{ES}(\ell))^2$ for the Random Plane Wave?

The third question involves the asymptotic (normalised) distribution of $N_{ES}(R, \ell)$. For the Lipschitz-Killing curvatures, these are known to be Gaussian in many cases (see Section 1.4.1). Non-Gaussian limit theorems have also been observed in the case of spectral singularity at the origin [63, 100].

Question 4.2.19. Does $N_{ES}(R, \ell)$ have asymptotically Gaussian fluctuations? Does it depend on the field and on the level?

Our final question is somewhat more vague. For Random Spherical Harmonics, it is known that the Lipschitz-Killing curvatures at generic levels are asymptotically proportional to the sample bispectrum;

$$\int_{\mathbb{S}^2} f_n^2(x) - 1\, dx.$$

Fluctuations in these variables can therefore be interpreted as a consequence of fluctuations in the overall L^2 norm of the field (see Section 1.4.1). This appears to be very similar to the interpretation of our lower bounds as arising from 'level shifts'. Unfortunately there seems to be no hope of proving that these phenomena are the same using the Wiener Chaos expansion.

Question 4.2.20. Can we prove a rigorous statement that fluctuations in the number of excursion set components at generic levels are 'caused by level shifts'? Is N_{ES} asymptotically proportional to the sample bispectrum for the Random Plane Wave?

4.2.3.3 Comparison with percolation models

As mentioned in the introduction to this chapter, one of the motivations for our variance results is to compare the behaviour of level sets of Gaussian fields with that of discrete percolation models. We recall that the variance of the number of open clusters of Bernoulli percolation in $[-R, R]^2$ is of order R^2 for each $p \in (0, 1)$. This is consistent with our results for fields with fast decay of correlations, and contributes to the growing literature (see Section 1.4.2) suggesting that such fields are in the same universality class as Bernoulli percolation.

Interestingly, our results contradict some of the auxiliary claims of the Bogomolny-Schmit conjecture. Specifically, we show that for ℓ arbitrarily close to zero, $N_{ES}(R, \ell)$ has variance of order at least R^3 as $R \to \infty$. This is inconsistent with the behaviour of open clusters of Bernoulli percolation (at any level), and so inconsistent with some of the claims made by Bogomolny and Schmit in [19]. However we emphasise that the most important case of the Bogolmony-Schmit conjecture is the critical case, which posits that the nodal set $\{f = 0\}$ of the Random Plane Wave has statistics that match critical Bernoulli percolation ($p = 1/2$). Unfortunately our results do not address this case.

We also note that this contradiction could have been predicted from previously known results. If we let $\varphi(A)$ denote the Euler characteristic of a set $A \subset \mathbb{R}^2$ then

$$\varphi(\{f \geq \ell\} \cap B(R)) \approx N_{ES}(f, R, \ell) - N_{ES}(-f, R, -\ell)$$

where f is the Random Plane Wave and equality holds up to some boundary effects. The stronger claims of the Bogomolny-Schmit conjecture would imply that both terms on the right hand side have variance of order R^2 for $|\ell|$ sufficiently small, so that the variance of the Euler characteristic would be at most order R^2. However the variance of the corresponding Euler characteristic for Random Spherical Harmonics is of cubic order for $\ell \neq 0, \pm 1$. If this matches the behaviour of the Random Plane Wave, it would lead to the described contradiction. As far as we can tell, this observation has not been made previously and regardless, our methods offer a proof of this inconsistency.

Question 4.2.21. What is the order of $\mathrm{Var}(N_{LS}(R, 0))$ and $\mathrm{Var}(N_{ES}(R, 0))$ for the Random Plane Wave? Does it agree with the Bogolmony-Schmit prediction of order R^2?

4.2.3.4 Possible extensions

We expect that there are a number of productive ways in which our techniques and results could be extended.

1. An obvious first step would be to better characterise the shape of c_{ES} and c_{LS}. Proving that these functionals have a non-zero derivative for a larger range of levels than in Corollaries 4.2.6 and 4.2.8 would immediately extend our variance bounds. This is particularly useful as it reduces a question about variance to one about expectation, which should be easier to analyse.

2. One could extend our results to more general fields by finding alternative ways to control the total variation distance between the number of excursion sets at different levels. The method used to prove Theorems 4.2.5 and 4.2.11 would be difficult to generalise greatly: it seems crucial to this argument that constant functions can be approximated in the reproducing kernel Hilbert space of the field, which requires the spectral measure to have mass around the origin. On

the other hand, the proof of Theorem 4.2.7 seems likely to hold for many other fields: the only property of the Random Plane Wave that we use, is that it can be effectively paramaterised on a domain of area R^2 by $4R$ random variables. Generally, if one could show that a field on $B(R)$ is determined by R^α random variables for $\alpha \in (0,2)$ then this method should prove variance bounds of order $R^{4-\alpha}$. As in the case of the Random Plane Wave, this would only apply to non-zero levels.

3. One could find different variables to compare in Chatterjee's lemma. Our choice of $Y_R = N_{ES}(R, \ell + a_R)$ was motivated by our previous results which made an analysis of $X_R - Y_R$ tractable, but perhaps other choices of Y_R might work.

4. Our method could be useful to prove fluctuations bounds on other non-local (or even local) geometric functionals of Gaussian excursion sets. In principle, all we require is that the geometric variable is homotopy-invariant. For example one could consider the number of bounded, doubly-connected excursion set components (i.e. those which look like annuli). This would require an analysis of the corresponding mean functional analogous to that in Chapters 2 and 3. Of course for local functionals, superior results can generally be deduced using the Wiener Chaos expansion.

5. Unfortunately it seems very difficult to extend our method to levels for which $c'_{ES}(\ell) = 0$, even to prove lower order fluctuation bounds. Even if we assume $c''_{ES}(\ell) \neq 0$, the second moment method fails completely (the orders of the first and second moment bounds do not match). It would therefore be necessary to find an alternative proof that $N_{ES}(R, \ell)$ and $N_{ES}(R, \ell + a_R)$ differ by a certain order for such levels ℓ, which seems roughly as difficult as proving the fluctuation bounds on N_{ES} in the first place.

4.3 Main arguments

In this Section we prove our main results (Theorems 4.2.5, 4.2.7 and 4.2.11) following the outline given in Section 4.2.2, subject to two auxiliary results (Lemmas 4.3.8 and 4.3.9) whose proof is deferred to Section 4.4.

Recall that the lower and upper right Dini-derivatives of a function $g : \mathbb{R} \to \mathbb{R}$ at a point x are defined respectively as

$$D_+ g(x) := \liminf_{\epsilon \downarrow 0} \frac{g(x+\epsilon) - g(x)}{\epsilon} \quad \text{and} \quad D^+ g(x) := \limsup_{\epsilon \downarrow 0} \frac{g(x+\epsilon) - g(x)}{\epsilon}. \tag{4.7}$$

The lower and upper left Dini-derivatives are defined respectively as

$$D_-g(x) := \liminf_{\epsilon\downarrow 0} \frac{g(x)-g(x-\epsilon)}{\epsilon} \quad \text{and} \quad D^-g(x) := \limsup_{\epsilon\downarrow 0} \frac{g(x)-g(x-\epsilon)}{\epsilon}. \tag{4.8}$$

For the sake of simplicity, in this section we focus on N_{ES} rather than N_{LS}, and we also assume the level ℓ is such that either $D^+c_{ES}(\ell) < 0$ or $D_+c_{ES}(\ell) > 0$ rather than one of the corresponding conditions for left Dini-derivatives. The arguments are near identical in all of these cases, and we will mention any points of difference.

4.3.1 Varying the level

We first show that $N_{ES}(R_n, \ell)$ and $N_{ES}(R_n, \ell + a_n)$ differ by at least a certain order, for carefully chosen sequences $R_n \to \infty$ and $a_n \to 0$. There are two main inputs into this result.

The first is a deterministic topological link between $N_{ES}(R_n, \ell)$ and $N_{ES}(R_n, \ell + a_n)$ given by Corollary 2.4.7. We recall the statement of this result: if f is an aperiodic Gaussian field satisfying Assumption 2.1.1 (or Assumption 4.2.1) then for any $R > 0$ and $\ell_1 < \ell_2$, with probability one

$$N_{ES}(R, \ell_1) - N_{ES}(R, \ell_2) = N_{m^+,R}[\ell_1, \ell_2) - N_{s^-,R}[\ell_1, \ell_2) + \gamma_{\ell_1,\ell_2,R}$$

and

$$\begin{aligned} N_{LS}(R, \ell_1) - N_{LS}(R, \ell_2) = &N_{m^+,R}[\ell_1, \ell_2) - N_{s^-,R}[\ell_1, \ell_2) \\ &+ N_{s^+,R}(\ell_1, \ell_2] - N_{m^-,R}(\ell_1, \ell_2] + \delta_{\ell_1,\ell_2,R} \end{aligned}$$

where

$$\begin{aligned} \max\{|\gamma_{\ell_1,\ell_2,R}|, |\delta_{\ell_1,\ell_2,R}|\} &\leq 2N_{\text{4-arm}}(B(R), [\ell_1, \ell_2]) + 4N_{\text{tang}}(\partial B(R), [\ell_1, \ell_2]) \\ &\leq 10N_{\text{tang}}(\partial B(R)). \end{aligned}$$

The second input, is a moment bound on the number of critical/tangent points of f in $B(R)$ inside shrinking height windows, taken from [76].

Proposition 4.3.1. *Let f be a Gaussian field satisfying Assumptions 4.2.1 and 4.2.9. Then there exists $c > 0$ such that for all $R > 0$ and $a < b$*

$$\mathbb{E}\left(N_{\text{crit}}(B(R), [a, b])^2\right) < c \min\{R^4(b-a)^2 + R^2(b-a), R^4\}$$

and

$$\mathbb{E}\left(N_{\text{tang}}(\partial B(R), [a, b])^2\right) < c \min\{R^2(b-a)^2 + R(b-a), R^2\}.$$

Proof. The first statement is precisely [76, Theorem 1.3]. We deduce the second statement from [76, Theorem A.4]. This theorem is stated for fields defined on subsets of $\mathbb{R}$, so we divide the boundary of $B(R)$ into two semicircles, and let f_R be the Gaussian process defined on $[0, c_1 R]$ that is the arc-length parameterisation of f restricted to one semicircle, with the domain scaled so that $\mathrm{Var}[f_R'(x)] = 1$ (this scaling is independent of R by the normalisation in Assumption 4.2.1). It is enough to prove that

$$\mathbb{E}\left(N_R([a,b])^2\right) < c_2 \min\left\{R^2(b-a)^2 + R(b-a), R^2\right\},$$

where $N_R([a,b])$ is the number of critical points of f_R with level in $[a,b]$. By [76, Theorem A.4] (which is stated for possibly non-stationary fields) this is true whenever f_R satisfies the following conditions:

1. Each f_R is normalised so that

   $$\mathbb{E}(f_R(x)) = 0\,,\ \mathrm{Var}[f_R(x)] = 1\,,\ \mathrm{Cov}[f_R(x), f_R'(x)] = 0 \quad \text{and} \quad \mathrm{Var}[f_i'(x)] = 1,$$

   and the Gaussian vector $(f_R(x), f_R(y), f_R'(x), f_R'(y))$ is non-degenerate for all distinct $x, y \in [0, c_1 R]$;

2. There exists a constant $c_3 > 0$ such that

   $$\limsup_{R\to\infty} \sup_{x\in[0,c_1R]} \max_{|\alpha|\le 3} \mathrm{Var}[\partial^\alpha f_R(x)] < c_3;$$

3. There exists a constant $c_4 > 0$ such that

   $$\liminf_{R\to\infty} \inf_{x\in[0,c_1R]} \mathrm{Var}[f_R''(x)^2] > 1 + c_4;$$

4. For each $\delta > 0$, there exists a constant $c_5 > 0$ such that

   $$\liminf_{R\to\infty} \inf_{|x-y|\ge\delta} \det(\Sigma_1^R(x,y)) > c_5 \quad \text{and} \quad \liminf_{R\to\infty} \inf_{|x-y|\ge\delta} \det(\Sigma_2^R(x,y)) > c_5,$$

   where $\Sigma_1^R(x,y)$ and $\Sigma_2^R(x)$ denote respectively the covariance matrices of the Gaussian vectors

   $$(f_R(x), f_R(y)) \,|\, (f_R'(x), f_R'(y)) \quad \text{and} \quad (f_R'(x), f_R'(y)).$$

Since $f'_R(x)$ is a linear combination of the elements of $\nabla f(x)$, the first condition is immediate from Assumptions 4.2.1 and 4.2.9.

The second and third conditions hold for the planar field f by Assumption 4.2.1 (see the proof of Theorem 1.3 in [76]), and since the covariance kernel $K_R(x,y) = \mathbb{E}(f_R(x)f_R(y))$ converges uniformly, as $R \to \infty$, in the $C^{3,3}$ norm on compact sets (more precisely, on sets of the form $\{x = R\theta, |y-x| < c\}$), they also hold eventually for f_R as $R \to \infty$.

By Assumption 4.2.9 and stationarity, for any $\delta > 0$, there exists a constant $c_5 > 0$ such that

$$\inf_{|x-y|\geq\delta>0} \det \mathrm{Cov}\left(f(x), f(y) \mid \nabla f(x), \nabla f(y)\right) > c_5,$$
$$\inf_{|x-y|\geq\delta>0} \det \mathrm{Cov}\left(\nabla f(x), \nabla f(y)\right) > c_5.$$

Since $f'_R(x)$ is a linear combination of $\nabla f(x)$ (with uniformly bounded coefficients), this implies the fourth condition above. $\square$

We now work with a Gaussian field f that satisfies Assumptions 4.2.1 and 4.2.9 (we note that the Random Plane Wave satisfies both of these assumptions). To apply the second moment method, we require a lower bound on the mean of the difference $N_{ES}(R_n, \ell) - N_{ES}(R_n, \ell + a_n)$.

Lemma 4.3.2. *Let f be a Gaussian field satisfying Assumptions 4.2.1 and 4.2.9. If $D^+c_{ES}(\ell) < 0$ or $D_+c_{ES}(\ell) > 0$ then there exists $c > 0$ such that, for any positive sequences $a_n \to 0$ and $R_n \to \infty$,*

$$\left|\mathbb{E}\left(N_{ES}(R_n,\ell) - N_{ES}(R_n, \ell + a_n)\right)\right| > cR_n^2 a_n + O(R_n)$$

for all n sufficiently large. If, in addition, f satisfies Assumption 4.2.4 then

$$\left|\mathbb{E}\left(N_{ES}(R_n,\ell) - N_{ES}(R_n, \ell + a_n)\right)\right| > cR_n^2 a_n + O(R_n a_n) + O\left(\sqrt{R_n a_n}\right)$$

for all n sufficiently large. If right Dini derivatives are replaced with left Dini derivatives, then the same conclusion holds on replacing $N_{ES}(R, \ell + a_n)$ with $N_{ES}(R, \ell - a_n)$. These statements also hold if excursion sets are replaced by level sets.

Proof. By Corollary 2.4.7

$$\begin{aligned} N_{ES}(R_n,\ell) - N_{ES}(R_n, \ell + a_n) &= N_{m^+}(B(R_n), [\ell, \ell + a_n)) - N_{s^-}(B(R_n), [\ell, \ell + a_n)) \\ &\quad + O\left(N_{\mathrm{tang}}(\partial B(R_n))\right) \end{aligned}$$

almost surely, and so by Theorem 2.1.7 and Lemma 2.2.4,

$$|\mathbb{E}(N_{ES}(R_n,\ell) - N_{ES}(R_n, \ell + a_n))| \geq |c_{ES}(\ell) - c_{ES}(\ell + a_n)| \cdot \pi R_n^2 + O\left(R_n\right).$$

Applying our assumption on the Dini-derivative of c_{ES} here proves the first part of the lemma.

The tighter inequality in Corollary 2.4.7 states that

$$\begin{aligned} N_{ES}(R_n,\ell)-N_{ES}(R_n,\ell+a_n)=&N_{m^+}(B(R_n),[\ell,\ell+a_n))-N_{s^-}(B(R_n),[\ell,\ell+a_n))\\ &+O\left(N_{\text{tang}}(\partial B(R_n),[\ell,\ell+a_n])\right)\\ &+O\left(N_{\text{4-arm}}(B(R_n),[\ell,\ell+a_n])\right)\end{aligned}$$

almost surely, and so by Theorem 2.1.7 and Assumption 4.2.4,

$$\begin{aligned}|\mathbb{E}(N_{ES}(R_n,\ell)-N_{ES}(R_n,\ell+a_n))|\geq&|c_{ES}(\ell)-c_{ES}(\ell+a_n)|\cdot\pi R_n^2\\ &+O\left(\mathbb{E}(N_{\text{tang}}(\partial B(R_n),[\ell,\ell+a_n]))\right)+o(R_n^2a_n).\end{aligned}$$

Since, by Proposition 4.3.1 and the Cauchy-Schwarz inequality,

$$\mathbb{E}(N_{\text{tang}}(\partial B(R_n),[\ell,\ell+a_n]))=O(R_na_n)+O\left(\sqrt{R_na_n}\right),$$

the lemma is proved. $\square$

We next prove a matching second moment bound.

Lemma 4.3.3. *Let f be a Gaussian field satisfying Assumptions 4.2.1 and 4.2.9. There exists $c>0$ such that, for any positive sequences $a_n\to 0$ and $R_n\to\infty$,*

$$\mathbb{E}\left((N_{ES}(R_n,\ell)-N_{ES}(R_n,\ell+a_n))^2\right)<c\left(R_n^4a_n^2+R_n^2a_n\right), \tag{4.9}$$

and the same conclusion holds for level sets.

Proof. Combine Corollary 2.4.7 and the bounds

$$\mathbb{E}\left(N_{\text{crit}}(B(R),[a,b])^2\right)<c\left(R^4(b-a)^2+R^2(b-a)\right)$$

and

$$\mathbb{E}\left(N_{\text{tang}}(\partial B(R),[a,b])^2\right)<c\left(R^2(b-a)^2+R(b-a)\right).$$

in Proposition 4.3.1. $\square$

Remark 4.3.4. By taking instead the bounds

$$\mathbb{E}\left(N_{\text{crit}}(B(R),[a,b])^2\right)<cR^4\quad\text{and}\quad\mathbb{E}\left(N_{\text{tang}}(\partial B(R),[a,b])^2\right)<cR^2$$

in Proposition 4.3.1, and setting $\ell^2\equiv\infty$ in Corollary 2.4.7, the same argument also establishes the trivial upper bound (4.1) under Assumptions 4.2.1 and 4.2.9. In fact, by applying this argument to a compact domain $B(1)$, covering $B(R)$ with $\approx R^2$ copies of $B(1)$ and controlling boundary components, we can actually derive (4.1) without the second condition $\max_{|\alpha|\leq 2}|\partial_\alpha K(x)|\to 0$ in Assumption 4.2.9.

Armed with matching first and second moment bounds, an application of the second moment method yields a lower bound on the order by which $N_{ES}(\ell, R_n)$ and $N_{ES}(\ell + a_n, R_n)$ differ.

Proposition 4.3.5. *Let f be a Gaussian field satisfying Assumptions 4.2.1 and 4.2.9. Assume that $D^+c_{ES}(\ell) < 0$ or $D_+c_{ES}(\ell) > 0$ and let $a_n \to 0$ and $R_n \to \infty$ be positive sequences. If either of the following conditions hold:*

1. *$R_n a_n \to \infty$ as $n \to \infty$;*
2. *f satisfies Assumption 4.2.4, and $R_n^2 a_n$ is bounded away from zero as $n \to \infty$;*

then $N_{ES}(R_n, \ell)$ and $N_{ES}(R_n, \ell + a_n)$ differ by order at least $R_n^2 a_n$. This statement also holds if excursion sets are replaced by level sets.

Proof. Let

$$X_n = N_{ES}(R_n, \ell) \qquad \text{and} \qquad Y_n = N_{ES}(R_n, \ell + a_n).$$

Under either condition (1) or (2) in the statement of the proposition, Lemma 4.3.2 shows that there is a constant $c_1 > 0$ such that

$$\mathbb{E}|X_n - Y_n| \geq |\mathbb{E}(X_n - Y_n)| > c_1 R_n^2 a_n$$

for all n sufficiently large. Combining this with the Paley-Zygmund inequality

$$\begin{aligned} \mathbb{P}\left(|X_n - Y_n| > (c_1/2) R_n^2 a_n\right) &\geq \mathbb{P}\left(|X_n - Y_n| > \mathbb{E}(|X_n - Y_n|)/2\right) \\ &\geq \frac{(\mathbb{E}|X_n - Y_n|)^2}{4\mathbb{E}\left((X_n - Y_n)^2\right)}. \end{aligned} \tag{4.10}$$

Combining this with Lemma 4.3.3 gives

$$\frac{(\mathbb{E}|X_n - Y_n|)^2}{\mathbb{E}\left((X_n - Y_n)^2\right)} \geq c_2 \frac{|\mathbb{E}(X_n - Y_n)|^2}{R_n^4 a_n^2 + R_n^2 a_n} \geq c_3 \frac{R_n^4 a_n^2}{R_n^4 a_n^2 + R_n^2 a_n} \geq c_4 > 0 \tag{4.11}$$

for constants $c_2, c_3, c_4 > 0$ and all n sufficiently large. Combining (4.10) and (4.11) completes the proof. We make the observation here (for later use) that the constants c_1 and c_4 do not depend on ℓ except through the value of $D_+c_{ES}(\ell)$ (or $D^+c_{ES}(\ell)$ if this quantity is negative). This observation follows immediately from the proof here and the proof of Lemma 4.3.3. □

4.3.2 Bounding the total variation distance

We next bound the total variation distance between the number of excursion sets at different levels, using different arguments for the case of general fields (i.e. fields satisfying the conditions of Theorems 4.2.5 and 4.2.11) and for the Random Plane Wave. This completes the proof of the main results (subject to two auxiliary lemmas, the proofs of which are deferred until Section 4.4).

4.3.2.1 General fields

We begin by recalling some general theory of Gaussian fields (for which we refer to [48]). Recall that to a continuous Gaussian field f, we can associate a Hilbert space of functions $H \subset C(\mathbb{R}^2)$ known as the *reproducing kernel Hilbert space* (RKHS), or Cameron-Martin space, defined as the completion of the space of finite linear combinations of the covariance function K

$$\sum_{1\leq i\leq n} a_i K(s_i,\cdot)\ , \quad a_i \in \mathbb{R}, s_i \in \mathbb{R}^2, n \in \mathbb{N} \tag{4.12}$$

equipped with the inner product

$$\left\langle \sum_{1\leq i\leq n} a_i K(s_i,\cdot), \sum_{1\leq j\leq m} a'_j K(s'_j,\cdot) \right\rangle_H = \sum_{1\leq i\leq n, 1\leq j\leq m} a_i a'_j K(s_i, s'_j).$$

The importance of the RKHS for our purposes, is the following corollary of the Cameron-Martin theorem.

Proposition 4.3.6 ([77, Corollary 3.10])**.** *Let f be a continuous Gaussian field defined on some Euclidean space. For every $h \in H$,*

$$d_{\mathrm{TV}}(f, f-h) \leq \frac{\|h\|_H}{\sqrt{\log 2}}.$$

If the Gaussian field f is stationary, the norm $\|h\|_H$ can be written explicitly in terms of the spectral measure ν. Indeed, we can represent H as the Fourier transform of $L^2_{\mathrm{sym}}(d\nu)$, the space of complex, Hermitian functions which are square integrable with respect to ν. Namely, each $h \in H$ is of the form $\mathcal{F}\left(\hat{h}\, d\nu\right)$ with a unique $\hat{h} \in L^2_{\mathrm{sym}}(d\nu)$, and

$$\langle h_1, h_2\rangle_H = \left\langle \hat{h}_1, \hat{h}_2 \right\rangle_{L^2_{\mathrm{sym}}(d\nu)}.$$

In particular,

$$\|h\|_H = \left\| \hat{h} \right\|_{L^2_{\mathrm{sym}}(d\nu)}.$$

Let us draw two immediate consequences. First, in the case that $\nu(\{0\}) > 0$, the constant function $1 \in H$. Indeed

$$1 = \int e^{2\pi i t \cdot x} \delta_0(x) = \int e^{2\pi i t \cdot x} \phi(x) d\nu(x) = \mathcal{F}(\phi\, d\nu),$$

where $\phi(x) = \mathbb{1}_{x=0}/\nu(\{0\})$. This implies that

$$\|1\|_H^2 = \|\phi\|_{L^2_{\mathrm{sym}}(d\nu)}^2 = \frac{\nu(\{0\})}{\nu(\{0\})^2} = \frac{1}{\nu(\{0\})}. \tag{4.13}$$

Second, in the case that ν has density ρ, we have that $\mathcal{F}\left(\hat{h}\, d\nu\right) = \mathcal{F}\left(\hat{h}\rho\, dx\right)$, i.e. $\hat{h}$ differs from the standard (inverse) Fourier transform $\mathcal{F}^{-1}(h)$ by division by ρ. If $\Omega = \operatorname{supp}\left(\hat{h}\right)$ has finite area, this implies the bound

$$\begin{aligned} \|h\|_H^2 = \left\|\hat{h}\right\|_{L^2_{\mathrm{sym}}(d\nu)}^2 &\le \sup_{\Omega} \left\{|\mathcal{F}^{-1}(h)|^2/\rho\right\} \operatorname{Area}(\Omega) \\ &\le \frac{\sup_{\Omega} \left\{|\mathcal{F}^{-1}(h)|^2\right\} \operatorname{Area}(\Omega)}{\inf_{\Omega} \rho}. \end{aligned} \tag{4.14}$$

We also make use of the following elementary lemma:

Lemma 4.3.7. *Let X_n and Y_n be two sequences of random variables defined on the same probability space and u_n a sequence of positive real numbers. Let A be a fixed event (i.e. independent of n) such that $\mathbb{P}(A) > 0$.*

1. *If X_n has fluctuations of order at least $\delta_n u_n$ for all positive sequences δ_n converging to zero arbitrarily slowly, then X_n has fluctuation of order at least u_n.*
2. *If there exist constants $c_1, c_2 > 0$ such that*

$$\mathbb{P}\left(|X_n - Y_n| \ge c_1 u_n \mid A\right) \ge c_2$$

for all n sufficiently large, then X_n and Y_n differ by order at least u_n.

Proof. We note that by Definition 4.2.2, X_n has fluctuations of order u_n if and only if X_n/u_n has fluctuations of order 1. Therefore when proving the first statement, we may assume $u_n = 1$ for all n.

We fix some positive sequence $\delta_k \to 0$ as $k \to \infty$, and suppose that X_n does not have fluctuations of order at least 1. Then by Definition 4.2.2, for each k we can find $n_k > n_{k-1}$ and $a_{n_k} < b_{n_k}$ such that $b_{n_k} - a_{n_k} \le \delta_k^2$ and

$$\mathbb{P}\left(a_{n_k} \le X_{n_k} \le b_{n_k}\right) > 1 - \delta_k^2. \tag{4.15}$$

By assumption, X_{n_k} has fluctuations of order δ_k. So there exist absolute constants $c_1, c_2 > 0$ such that for all k sufficiently large (so that $\delta_k < c_1$) we have $b_{n_k} - a_{n_k} \leq c_1\delta_k$ and hence

$$\mathbb{P}\big(a_{n_k} \leq X_{n_k} \leq b_{n_k}\big) \leq \delta_k.$$

Provided k is large enough, this contradicts (4.15), so we deduce that X_n has fluctuations of order 1, as required.

Turning to the second statement, for all n sufficiently large

$$c_2\mathbb{P}(A) \leq \mathbb{P}\left(\{|X_n - Y_n| \geq c_1 u_n\} \cap A\right) \leq \mathbb{P}\left(|X_n - Y_n| \geq c_1 u_n\right).$$

Therefore X_n and Y_n satisfy Definition 4.2.13 with constants $c_1, c_2\mathbb{P}(A) > 0$. □

We can now complete the proof of Theorems 4.2.5 and Theorem 4.2.11, beginning with the simpler case of Theorem 4.2.11. For this it will be helpful to extend our previous notation $N_{ES}(R,\ell)$ slightly, defining $N_{ES}(g; R, \ell)$ for $g \in C^2_{\mathrm{loc}}(\mathbb{R}^2)$ to be the number of components of $\{g \geq \ell\}$ contained in $B(R)$ (so that $N_{ES}(R,\ell) = N_{ES}(f; R, \ell)$).

Proof of Theorem 4.2.11. Let f be a Gaussian field satisfying Assumption 4.2.10, which means it can be represented as $f = g + \sqrt{\alpha}Z$ where g has spectral measure ν^* and Z is an independent standard Gaussian variable. We first note that, since $c_{ES}(\nu^*, \ell)$ is absolutely continuous in ℓ (Theorem 2.1.7), it is differentiable for almost every $\ell \in \mathbb{R}$. We also know that $c_{ES}(\nu^*, \ell)$ is not constant in ℓ, since it is positive for some ℓ (Corollary 2.1.9) and tends to zero as $\ell \to -\infty$ (Theorem 2.1.7). Therefore the function $c'_{ES}(\nu^*, \cdot)$, which is defined almost everywhere, is not identically zero. In particular, there exists a set $A \subset \mathbb{R}$ of positive Lebesgue measure and a constant $c_0 > 0$ such that $c_{ES}(\nu^*, \cdot)$ is differentiable for all $\ell \in A$ and

$$\inf_{\ell \in A} c'_{ES}(\nu^*, \ell) \geq c_0 > 0.$$

We now fix sequences $R_n \to \infty$ and $a_n \downarrow 0$, such that $a_n R_n \to \infty$. (We will later choose the convergence of a_n to be arbitrarily slow.) Applying Proposition 4.3.5 to the field g at a level for which $c'_{ES}(\ell) > 0$ (and also assuming this derivative is defined) we can conclude that $N_{ES}(g, R_n, \ell)$ and $N_{ES}(g, R_n, \ell + a_n)$ differ by order at least $R_n^2 a_n$. More specifically we can say that there exists $c_1, c_2 > 0$ such that for all n sufficiently large

$$\mathbb{P}\left(|N_{ES}(g, R_n, \ell) - N_{ES}(g, R_n, \ell + a_n)| \geq c_1 R_n^2 a_n\right) \geq c_2. \tag{4.16}$$

By the observation at the end of the proof of Proposition 4.3.5, if we consider a set of levels ℓ for which $c'_{ES}(\ell)$ is uniformly bounded away from zero, then we can choose $c_1, c_2 > 0$ such that (4.16) holds for all ℓ in this set when n is sufficiently large (how large n is required to be may depend on ℓ).

We now fix an arbitrary $\ell^* \in \mathbb{R}$ and $c_1, c_2 > 0$ such that (4.16) holds for all $\ell \in A$ (when n is sufficiently large). We define the events

$$E_n(u) = \left\{ |N_{ES}(g, R_n, u) - N_{ES}(g, R_n, u + a_n)| \geq c_1 R_n^2 a_n \right\}.$$

Then, recalling that $f = g + \sqrt{\alpha} Z$, we have

$$\begin{aligned}\mathbb{P}\left(|N_{ES}(f, R_n, \ell^*) - N_{ES}(f, R_n, \ell^* + a_n)| \geq c_1 R_n^2 a_n \mid \ell^* - \sqrt{\alpha}Z \in A\right)& \\ = \mathbb{P}\left(E_n(\ell^* - \sqrt{\alpha}Z) \mid \ell^* - \sqrt{\alpha}Z \in A\right)& \\ = \frac{\int_A \mathbb{P}\left(E_n(u)\right) p_{\ell^* - \sqrt{\alpha}Z}(u)\, du}{\mathbb{P}\left(\ell^* - \sqrt{\alpha}Z \in A\right)}&\end{aligned}$$

where $p_{\ell^* - \sqrt{\alpha}Z}$ denotes the (Gaussian) density of $\ell^* - \sqrt{\alpha}Z$. By Fatou's lemma and (4.16), this expression is bounded below by $c_2/2$ for sufficiently large n. Since the event we condition on above ($\ell^* - \sqrt{\alpha}Z \in A$) is independent of n and has positive probability, by Lemma 4.3.7 we conclude that $N_{ES}(f, R_n, \ell^*)$ and $N_{ES}(f, R_n, \ell^* + a_n)$ differ by order at least $R_n^2 a_n$.

By Proposition 4.3.6 and (4.13) there is a $c_0 > 0$ such that,

$$d_{\mathrm{TV}}(N_{ES}(f; R_n, \ell), N_{ES}(f - a_n; R_n, \ell)) \leq d_{\mathrm{TV}}(f, f - a_n) \leq c_0 a_n.$$

Since $a_n \to 0$, the total variation distance converges to zero as $n \to \infty$. Therefore, by Corollary 4.2.14, we conclude that $N_{ES}(R_n, \ell)$ has fluctuations of order at least $R_n^2 a_n$. Since this statement holds for any sequence a_n which converges to zero arbitrarily slowly, it then follows from Lemma 4.3.7 that $N_{ES}(R_n, \ell)$ has fluctuations of order at least R_n^2. By definition, this means that $(N_{ES}(R, \ell))_{R \geq 0}$ has fluctuations of order at least R^2. Combining this with the trivial upper bound on the variance (4.1) (see also Remark 4.3.4) we have the result. Identical arguments apply to level sets. □

We now turn to the general setting of Theorem 4.2.5. Let f be a Gaussian field satisfying Assumptions 4.2.1 and 4.2.3, and recall that Assumption 4.2.3 implies Assumption 4.2.9. We recall also that $g(r) := \inf_{x \in B(2r)} \rho(x)$ where ρ is the spectral density of the field near the origin. Let $R_n \to \infty$ and $\delta_n \to 0$ be positive sequences such that $\delta_n R_n \to \infty$. For each $r > 0$, we define $h_r : \mathbb{R}^2 \to \mathbb{R}$ by

$$h_r(t) = \frac{1}{4r^2} \mathcal{F}\left[\mathbb{1}_{[-r,r]^2}\right](t) = \frac{\sin(2\pi r t_1)}{2\pi r t_1} \frac{\sin(2\pi r t_2)}{2\pi r t_2}. \tag{4.17}$$

We will use h_r, $r \to 0$, to approximate the constant function 1; if we choose positive sequences $R_n \to \infty$ and $r_n \to 0$ such that $r_n R_n \to 0$, then by a Taylor expansion,

$$\|1 - h_{r_n}\|_{C^2(B(R_n))} = O\left(r_n^2 R_n^2\right) \qquad \text{as } n \to \infty. \tag{4.18}$$

In the next lemma we show that this approximation has a negligible effect on the number of excursion sets, that is, the number of excursion sets of $f - a_n$ is well approximated by the number of excursion sets of $f - a_n h_{r_n}$ for an appropriate choice of r_n.

Lemma 4.3.8. *Let f be a Gaussian field satisfying the conditions of Theorem 4.2.5 and fix $\ell \in \mathbb{R}$. Let R_n, r_n and a_n be sequences of positive numbers such that $R_n \to \infty$, $r_n \to 0$, $a_n \to 0$, $r_n R_n \to 0$ and $a_n r_n^2 R_n^2 \to 0$ as $n \to \infty$. Then there exist $c, n_0 > 0$ such that, for all $n > n_0$,*

$$\mathbb{E}\big[|N_{ES}(f - a_n; R_n, \ell) - N_{ES}(f - a_n h_{r_n}; R_n, \ell)|\big] < c a_n r_n^2 R_n^4.$$

The same conclusion holds for level sets.

We defer the proof of Lemma 4.3.8 until Section 4.4. The upshot is that we can apply Proposition 4.3.6 to the field $f - a_n h_{r_n}$, since h_{r_n} is an element of the RKHS H. Indeed, by (4.14) for r sufficiently small

$$\|h_r\|_H \leq \frac{1}{2r\sqrt{g(r)}}. \tag{4.19}$$

We now have all the ingredients needed to prove Theorem 4.2.5.

Proof of Theorem 4.2.5. Let $R_n \to \infty$ and $\delta_n \to 0$ be positive, monotone sequences such that $\delta_n^2 R_n \to \infty$. If $g(r) \to \infty$ as $r \to 0$ we also choose δ_n converging to zero sufficiently slowly that $\delta_n^2 \sqrt{g(\delta_n/R_n)} \to \infty$ as $n \to \infty$. We apply Proposition 4.3.5 with $a_n = \delta_n^2 \sqrt{g(\delta_n/R_n)}/R_n$ and deduce that $N_{ES}(f; R_n, \ell)$ and $N_{ES}(f; R_n, \ell + a_n)$ differ by order at least $R_n^2 a_n = \delta_n^2 R_n \sqrt{g(\delta_n/R_n)}$. Applying Lemma 4.3.8 with $r_n = \delta_n/R_n$ and Markov's inequality we have that, for every $\varepsilon > 0$ and n sufficiently large,

$$\mathbb{P}\left(|N_{ES}(f - a_n; R_n, \ell) - N_{ES}(f - a_n h_{r_n}; R_n, \ell)| > \varepsilon R_n^2 a_n\right) < \frac{c a_n r_n^2 R_n^4}{\varepsilon R_n^2 a_n} = c\delta_n^2/\varepsilon \to 0.$$

Hence $N_{ES}(f; R_n, \ell)$ and $N_{ES}(f - a_n h_{r_n}; R_n, \ell)$ also differ by order at least $R_n^2 a_n$. Moreover, by Proposition 4.3.6 and (4.19), there is a $c_0 > 0$ such that

$$\begin{aligned} d_{\mathrm{TV}}(N_{ES}(f; R_n, \ell), N_{ES}(f - a_n h_{r_n}; R_n, \ell)) &\leq d_{\mathrm{TV}}(f, f - a_n h_{r_n}) \\ &\leq c_0 \|a_n h_{r_n}\|_H \leq \frac{c_0 a_n}{2 r_n \sqrt{g(r_n)}} = \frac{c_0 \delta_n}{2} \to 0. \end{aligned}$$

Therefore, by Corollary 4.2.14, we conclude that $N_{ES}(R_n, \ell)$ has fluctuations of order at least $R_n^2 a_n$. Since δ_n can be chosen to converge to zero arbitrarily slowly, Lemma 4.3.7 implies that $N_{ES}(R_n, \ell)$ has fluctuations of order at least $R_n\sqrt{g(1/R_n)}$, completing the proof of the theorem. $\square$

4.3.2.2 The Random Plane Wave

We now move onto the proof of Theorem 4.2.7. We recall the orthogonal expansion of the Random Plane Wave given in the proof of Proposition 3.2.17:

$$f(x) = \sum_{m\in\mathbb{Z}} a_m J_{|m|}(r) e^{im\theta}, \tag{4.20}$$

where (r, θ) represents x in polar coordinates, J_m is the m-th Bessel function and $a_m = b_m + ic_m = \overline{a_{-m}}$ with b_0, $(\sqrt{2}b_m)_{m\in\mathbb{N}}$ and $(\sqrt{2}c_m)_{m\in\mathbb{N}}$ independent standard (real) Gaussians and $c_0 = 0$. We let f_N denote the truncation of this expansion at N, that is

$$f_N(x) = \sum_{|m|\leq N} a_m J_{|m|}(r) e^{im\theta}. \tag{4.21}$$

Known inequalities for Bessel functions [105, Section 8.5, (9)] state that, for all $\alpha \in (0, 1)$, $m \geq 0$, and $r < \alpha m$,

$$|J_m(r)| \leq c_1 e^{-c_2 m}, \tag{4.22}$$

which means that the terms beyond $m \approx 2R$ are exponentially small inside $B(R)$. In the next lemma we show that these terms have a bounded effect on the number of excursion sets.

Lemma 4.3.9. *Let f be the Random Plane Wave, fix $\ell^* \in \mathbb{R}$, and let $\alpha \in (0, 1)$. Then there exist $c, n_0 > 0$ such that, for all $\ell \in [\ell^* - 1, \ell^* + 1]$, $N \geq n_0$ and $R \leq \alpha N$,*

$$\mathbb{E}\left(|N_{ES}(f; R, \ell) - N_{ES}(f_N; R, \ell)|\right) < c.$$

The proof of Lemma 4.3.9 is deferred to Section 4.4. Armed with this lemma we can complete the proof of Theorem 4.2.7.

Proof of Theorem 4.2.7. Let $R_n \to \infty$ be a positive sequence, and $a_n = \delta_n R_n^{-1/2}$ for some sequence $\delta_n > 0$ which converges to zero slowly enough that $R_n a_n \to \infty$ (we will eventually allow δ_n to converge to zero arbitrarily slowly). Applying Proposition 4.3.5 shows that $N_{ES}(f; R_n, \ell)$ and $N_{ES}(f; R_n, \ell{+}a_n)$ differ by order at least $R_n^2 a_n = \delta_n R_n^{3/2}$.

Now choose $m_n = \lceil 2R_n \rceil$. Applying Lemma 4.3.9 and Markov's inequality shows that for any $\epsilon > 0$

$$\mathbb{P}\left(|N_{ES}(f; R_n, \ell) - N_{ES}(f_{m_n}; R_n, \ell)| > \epsilon R_n^2 a_n\right) \to 0$$

and

$$\mathbb{P}\left(|N_{ES}(f; R_n, \ell + a_n) - N_{ES}(f_{m_n}; R_n, \ell + a_n)| > \epsilon R_n^2 a_n\right) \to 0$$

as $n \to \infty$, so we conclude that $N_{ES}(f_{m_n}; R_n, \ell)$ and $N_{ES}(f_{m_n}; R_n, \ell + a_n)$ also differ by order at least $\delta_n R_n^{3/2}$.

We note that, since $f_{m_n}(t) \geq \ell + a_n$ if and only if $\frac{\ell}{\ell+a_n} f_{m_n}(t) \geq \ell$ (assuming n is sufficiently large that $\ell + a_n$ is bounded away from zero),

$$N_{ES}(f_{m_n}; R_n, \ell + a_n) = N_{ES}\left(\frac{\ell}{\ell + a_n} f_{m_n}; R_n, \ell\right).$$

Since f_{m_n} is parametrised by $2m_n + 1$ independent standard Gaussian variables we see that

$$\begin{aligned}
d_{TV}&\left(N_{ES}(f_{m_n}; R_n, \ell), N_{ES}(f_{m_n}; R_n, \ell + a_n)\right) \\
&= d_{TV}\left(N_{ES}(f_{m_n}; R_n, \ell), N_{ES}\left(\frac{\ell}{\ell + a_n} f_{m_n}; R_n, \ell\right)\right) \\
&\leq d_{TV}\left(\mathcal{N}(0, I_{2m_n+1}), \mathcal{N}\left(0, \left(\frac{\ell}{\ell + a_n}\right)^2 I_{2m_n+1}\right)\right).
\end{aligned}$$

By Pinsker's inequality, the square of the above quantity is at most

$$\frac{1}{2} d_{\mathrm{KL}}\left(\mathcal{N}\left(0, \left(\frac{\ell}{\ell + a_n}\right)^2 I_{2m_n+1}\right) \middle\| \mathcal{N}\left(0, I_{2m_n+1}\right)\right) \tag{4.23}$$

where d_{KL} denotes the Kullback-Leibler divergence defined by (4.5). If $P = \mathcal{N}(0, \Sigma_1)$ and $Q = \mathcal{N}(0, \Sigma_2)$ are two centred k-dimensional Gaussian measures, then it is a standard result that

$$d_{\mathrm{KL}}\left(P \,||\, Q\right) = \frac{1}{2}\left(\log\left(\frac{\det \Sigma_2}{\det \Sigma_1}\right) - k + \mathrm{Tr}(\Sigma_2^{-1}\Sigma_1)\right).$$

Therefore the quantity in (4.23) is equal to

$$\frac{2m_n + 1}{4}\left(\left(\frac{\ell}{\ell + a_n}\right)^2 - 1 - \ln\left(\left(\frac{\ell}{\ell + a_n}\right)^2\right)\right) \leq \frac{c_1}{4}(4R_n + 3)a_n^2$$

for some constant $c_1 > 0$ depending only on ℓ (the inequality follows from a Taylor expansion of the logarithm). This bound converges to zero as $n \to \infty$ and so we can apply Corollary 4.2.14 and conclude that $N_{ES}(f_{m_n}; R_n, \ell)$ has fluctuations of order at least $\delta_n R_n^{3/2}$. Applying Lemma 4.3.9, the same conclusion is true for $N_{ES}(f; R_n, \ell)$. Since δ_n can be chosen to converge to zero arbitrarily slowly, Lemma 4.3.7 implies that $N_{ES}(f; R_n, \ell)$ has fluctuations of order at least $R_n^{3/2}$ as required. □

4.4 Perturbation arguments

In this section we prove Lemmas 4.3.8 and 4.3.9, thus completing the proof of all results in the chapter.

We begin with some heuristics. Let F be a random field on a compact domain D. Our aim is to control the expected difference between the number of components of $\{F \geq 0\}$ and $\{F - p \geq 0\}$, where p is a small (possibly random) perturbation. If p is a constant function taking the value $c > 0$, then the standard methods of Morse theory show that the difference between these two quantities is at most the number of critical points of F with level between 0 and c, since the excursion set $\{F \geq \ell\}$ varies continuously with ℓ unless passing through a critical point of F, in which case the number of components changes by at most one. Since the number of critical points is a local quantity, we can use the Kac-Rice formula to bound its mean.

This same reasoning can be applied to more general perturbations p. Assuming some regularity of F and p (which will be specified in the next subsection) the number of components of $\{F - \alpha p \geq 0\}$ changes continuously with α unless passing through a point at which $\nabla(F - \alpha p) = 0$, and it can be shown that at such points the number of components changes by at most one. Therefore, the difference in the number of excursion sets is bounded above by the number of points at which $\nabla(F - \alpha p) = 0$ for $\alpha \in [0, 1]$ (plus an analogous term which controls boundary effects). Since the number of such points is still a local quantity, an application of the Kac-Rice formula will yield Lemmas 4.3.8 and 4.3.9.

Let us formalise the concepts just described. Let $D \subset \mathbb{R}^2$ be a compact domain with C^1 boundary and let $F, p \in C^2(D)$. We say that $x \in D$ is a quasi-critical point of (F, p) at level $\alpha \in [0, 1]$ if $\nabla(F - \alpha p)(x) = 0$ and $(F - \alpha p)(x) = 0$. For $x \in \partial D$, let $v_\partial(x)$ and $v_{\bar{\partial}}(x)$ denote respectively the unit vectors in the tangent and normal directions to ∂D, and let ∇_∂ and $\nabla_{\bar{\partial}}$ denote the derivatives in these respective directions. We say that $x \in \partial D$ is a quasi-tangent point of (F, p) at level $\alpha \in [0, 1]$ if $\nabla_\partial(F - \alpha p)(x) = 0$ and $(F - \alpha p)(x) = 0$. We denote the number of quasi-critical and quasi-tangent points by $N_{QC}(F,p)$ and $N_{QT}(F,p)$ respectively, i.e.

$$N_{QC}(F,p) = \#\{x \in D : \exists\alpha \in [0,1],\ \nabla(F-\alpha p)(x) = 0, (F-\alpha p)(x) = 0\}$$
$$N_{QT}(F,p) = \#\{x \in \partial D : \exists\alpha \in [0,1],\ \nabla_\partial(F-\alpha p)(x) = 0, (F-\alpha p)(x) = 0\}.$$

We can now state our main perturbation result. We generalise our previous notation $N_{ES}(g;R,\ell)$ slightly, defining $N_{ES}(g;D,\ell)$ to be the number of components of $\{g \geq \ell\}$ contained in a compact domain D (with analogous notation for level sets).

Proposition 4.4.1. *Let $D \subset \mathbb{R}^2$ be a compact domain with C^1 boundary. Suppose that F and p are independent C^3-smooth planar Gaussian fields defined on a neighbourhood of D satisfying the following conditions:*

1. *For each $x, y \in D$, $x \neq y$, the Gaussian vector $(F(x), F(y), \nabla F(x), \nabla F(y))$ is non-degenerate;*
2. *For each $x \in D$, the Gaussian vector $(\nabla F(x), \nabla^2 F(x))$ is non-degenerate;*
3. *For each $x \in \partial D$, both of the Gaussian vectors $(F(x), \nabla_\partial F(x), \nabla_\partial\nabla_{\bar\partial} F(x))$ and $(F(x), \nabla_\partial F(x), \nabla_\partial^2 F(x))$ are non-degenerate;*
4. *Either:*
   - *(a) p is deterministic and the set $\{p = 0\} \cap D$ consists of a union of simple curves and isolated points, which intersect ∂D only at isolated points, or*
   - *(b) For each $x \in D$, the Gaussian vector $(p(x), \nabla p(x))$ is non-degenerate.*

Then there exists $c > 0$, independent of F, p and D, such that with probability one

$$|N_{ES}(F;D,0) - N_{ES}(F-p;D,0)| \leq c\,(N_{QC}(F,p) + N_{QT}(F,p)).$$

The same conclusion holds on replacing N_{ES} with N_{LS}.

We prove Proposition 4.4.1 in the next subsection. Before that, we use it to complete the proofs of Lemmas 4.3.8 and 4.3.9. We state these proofs exclusively for excursion sets; the arguments for level sets are identical.

The simpler case is Lemma 4.3.8, since the perturbation is deterministic.

Proof of Lemma 4.3.8 (given Proposition 4.4.1). For fixed n sufficiently large, we define

$$F = f - \ell - a_n \qquad \text{and} \qquad p = a_n(h_{r_n} - 1).$$

Then by the definition of h_{r_n} in (4.17), the zero set of $p|_{B(R_n)}$ consists of a single point at the origin. Hence we may apply Proposition 4.4.1 to the functions F and p on the domain $D = B(R_n)$ (f satisfies the conditions of the proposition by Assumptions 4.2.1 and 4.2.3 and Lemma A.2.4), and so it is sufficient to prove that there exists a $c > 0$ such that, for n sufficiently large,

$$\mathbb{E}(N_{QC}(F,p)) < c a_n r_n^2 R_n^4 \quad \text{and} \quad \mathbb{E}(N_{QT}(F,p)) < c a_n r_n^2 R_n^3. \tag{4.24}$$

We begin with the first bound in (4.24). Define $G : B(R_n) \times [0,1] \to \mathbb{R}^3$ by

$$G(x,\gamma) = \begin{pmatrix} \nabla(F - \gamma p)(x) \\ (F - \gamma p)(x) \end{pmatrix} \tag{4.25}$$

and let $p_{G(x,\gamma)}$ denote the density of the (non-degenerate) Gaussian vector $G(x,\gamma)$.

We now apply the Kac-Rice formula to G. Note that Theorem 1.3.2 is not applicable here since the mean of G is not constant. Instead we apply [5, Theorem 6.2 and Proposition 6.5] (which require G to by C^2 almost surely) to conclude that

$$\begin{aligned}
&\mathbb{E}(N_{QC}(F,p)) \\
&= \iint\limits_{B(R_n)\times[0,1]} \mathbb{E}\left(\left| \det \begin{pmatrix} \nabla^2(F-\gamma p)(x) & \nabla(F-\gamma p)(x) \\ -\nabla p(x)^t & -p(x) \end{pmatrix} \right| \middle| G(x,\gamma) = 0 \right) p_{G(x,\gamma)}(0) d\gamma dx \\
&\le \pi R_n^2 \sup_{\substack{x \in B(R_n) \\ \gamma \in [0,1]}} \mathbb{E}\left(\left| \det \begin{pmatrix} \nabla^2(F-\gamma p)(x) & \nabla(F-\gamma p)(x) \\ -\nabla p(x)^t & -p(x) \end{pmatrix} \right| \middle| G(x,\gamma) = 0 \right) p_{G(x,\gamma)}(0).
\end{aligned}$$

The density $p_{G(x,\gamma)}(0)$ is bounded above by $c_1/\sqrt{\det \Sigma(x,\gamma)}$, where $\Sigma(x,\gamma)$ is the covariance matrix of $G(x,\gamma)$ and $c_1 > 0$ is an absolute constant. Using the condition $G(x,\gamma) = 0$ (which implies $\nabla(F - \gamma p)(x) = 0$), we have

$$\begin{aligned}
&\mathbb{E}\left(\left| \det \begin{pmatrix} \nabla^2(F-\gamma p)(x) & \nabla(F-\gamma p)(x) \\ -\nabla p(x)^t & -p(x) \end{pmatrix} \right| \middle| G(x,\gamma) = 0 \right) \\
&\qquad = |p(x)| \cdot \mathbb{E}\left(\left| \det\left(\nabla^2(F-\gamma p)(x)\right) \right| \middle| G(x,\gamma) = 0 \right) \\
&\qquad \le c_2 |p(x)| \max_{|k|=2} \max \left\{ \mathbb{E}\left(\left|\partial^k F(x)\right|^2 \middle| G(x,\gamma) = 0 \right), \left|\partial^k p(x)\right|^2 \right\},
\end{aligned}$$

where $c_2 > 0$ is an absolute constant and we have expanded the determinant and applied Hölder's inequality in the final line. By (4.18),

$$\|p\|_{C^2(B(R_n))} = O(a_n (r_n R_n)^2) \to 0 \qquad \text{as } n \to \infty.$$

Hence it suffices to show that

$$\frac{1}{\det(\Sigma(x,\gamma))} \quad \text{and} \quad \max_{|k|=2} \mathbb{E}\left(\left|\partial^k F(x)\right|^2 \middle| G(x,\gamma)=0\right) \tag{4.26}$$

are bounded above, uniformly over n sufficiently large and $(x,\gamma) \in B(R_n) \times [0,1]$.

Considering the first term; since p is deterministic and f is stationary,

$$\Sigma(x,\gamma) = \mathrm{Cov}\begin{pmatrix} \nabla f(x) \\ f(x) \end{pmatrix} = \mathrm{Cov}\begin{pmatrix} \nabla f(0) \\ f(0) \end{pmatrix}$$

which has a non-zero determinant, by Assumption 4.2.1, that clearly does not depend on x or γ.

Now turning to the second term of (4.26), since f is stationary we know that $(f(x), \nabla^2 f(x))$ is independent of $\nabla f(x)$ (Lemma A.2.2) and so for $|k| = 2$

$$\begin{aligned} \mathbb{E}\left(\left|\partial^k F(x)\right|^2 \middle| G(x,\gamma)=0\right) &= \mathbb{E}\left(\left|\partial^k f(x)\right|^2 \middle| f(x) = \ell + a_n + \gamma p(x)\right) \\ &= \mathbb{E}\left(\left|\partial^k f(0)\right|^2 \middle| f(0) = \ell + a_n + \gamma p(x)\right) \end{aligned} \tag{4.27}$$

where the second equality also follows from stationarity of f (recall that p is deterministic). By Gaussian regression ([5, Proposition 1.2]), this final expression is continuous in the variable $\ell + a_n + \gamma p(x)$. Since a_n and $\|p\|_{C^2(B(R_n))}$ converge to zero as $n \to \infty$, we see that this variable has a uniformly bounded range for all n and all $(x,\gamma) \in B(R_n) \times [0,1]$. Therefore (4.27) is uniformly bounded as required. This establishes the first bound in (4.24).

For the second bound in (4.24), a similar application of the Kac-Rice formula on the boundary $\partial D = \partial B(R_n)$ gives the result once we establish that

$$\mathbb{E}\left(\left|\nabla_\partial^2 F(x)\right| \,\middle|\, G_\partial(x,\gamma)=0\right), \quad \left|\nabla_\partial^2 p(x)\right| \quad \text{and} \quad \frac{1}{\det(\Sigma_\partial(x,\gamma))}$$

are bounded above uniformly over n sufficiently large and $(x,\gamma) \in \partial B(R_n) \times [0,1]$, where ∇_∂ denotes the derivative in the direction tangent to the boundary $\partial B(R_n)$,

$$G_\partial(x,\gamma) = \begin{pmatrix} \nabla_\partial(F - \gamma p)(x) \\ (F-\gamma p)(x) \end{pmatrix}$$

and $\Sigma_\partial(x,\gamma)$ denotes the covariance matrix of $G_\partial(x,\gamma)$. These facts are proven using arguments near identical to those given above. □

The proof of Lemma 4.3.9 is slightly more complex because the perturbation is random, and we require some additional auxiliary lemmas (which we state here and prove at the end of the subsection):

Lemma 4.4.2. *Fix $d, n \in \mathbb{N}$. Let X and Y be random variables, let $Z \in \mathbb{R}^d$ be a random vector, and suppose that (X, Y, Z) is jointly Gaussian and centred with (Y, Z) non-degenerate. Then*

$$\sup_{y \in \mathbb{R}} \mathbb{E}\left(|X|^n | Y = y, Z = 0\right) p_{(Y,Z)}(y, 0) \leq \frac{c}{\sqrt{\det(\Sigma)}} \mathbb{E}(X^2)^{n/2} \max\left\{1, \frac{\mathbb{E}(Y^2)^{n/2}}{(\sigma^2_{Y|Z})^{n/2}}\right\},$$

where $c > 0$ is a constant depending only on d and n, Σ and p denote respectively the covariance and density of (Y, Z), and $\sigma^2_{Y|Z}$ denotes the variance of Y conditionally on Z.

Lemma 4.4.3. *For each $\alpha \in (0, 1)$ and $k \in \mathbb{N}$ there exists $c_1, c_2 > 0$ such that, for all $N \geq 1$,*

$$\mathbb{E}\left(\|f - f_N\|^2_{C^k(B(\alpha N))}\right) \leq c_1 e^{-c_2 N}.$$

Proof of Lemma 4.3.9 (given Proposition 4.4.1 and Lemmas 4.4.2 and 4.4.3).
We recall the orthogonal expansion of the Random Plane Wave in (4.20) and its truncation f_N in (4.21). Since the covariance kernels of f_N converge to that of f in the C^∞ on compacts topology, it can be shown that $f_N \to f$ in law in the C^∞ topology on compact sets (see [81, Appendix A]). Let us extend our notation slightly by letting $\bar{N}_{ES}(g; R, \ell)$ denote number of components of $\{g \geq \ell\}$ contained in $B(R) \setminus B(1)$. We first claim that there is a $c > 0$ such that

$$\begin{aligned} \mathbb{E}\left(\left|N_{ES}(f; R, \ell) - \bar{N}_{ES}(f; R, \ell)\right|\right) &< c \\ \mathbb{E}\left(\left|N_{ES}(f_N; R, \ell) - \bar{N}_{ES}(f_N; R, \ell)\right|\right) &< c \end{aligned} \tag{4.28}$$

for all sufficiently large N and $R > 1$. To see this, note that, for $g \in \{f, f_N\}$,

$$\left|N_{ES}(g; R, \ell) - \bar{N}_{ES}(g; R, \ell)\right| \leq N_{\text{crit}}(g, B(1)) + 2N_{\text{tang}}(g, \partial B(1)),$$

which holds because each excursion set component contained in $B(R)$ which intersects $B(1)$ is either contained in $B(1)$ or intersects $\partial B(1)$. In the former case, such a component must contain a local maximum in $B(1)$ and in the latter, it must contain a tangent point in $\partial B(1)$. Then the Kac-Rice formula shows that the quantities $N_{\text{crit}}(g, B(1))$ and $N_{\text{tang}}(g, \partial B(1))$ have a finite mean that is continuous with respect to the $C^2(B(1))$ topology on the covariance kernel of g on $B(1)$ (at least, if g is a C^2 Gaussian field such that $\nabla g(x)$ is non-degenerate on $B(1)$). Since $f_N \to f$ in law in the C^∞ topology on compacts and ∇f_N is eventually non-degenerate on $B(1)$, the result follows.

It remains to show that

$$\mathbb{E}\big[|\bar{N}_{ES}(f;R,\ell) - \bar{N}_{ES}(f_N;R,\ell)|\big] < c$$

uniformly over N sufficiently large and $R \leq \alpha N$. Define $F := f_N - \ell$ and $p := f_N - f$, which are independent C^∞ Gaussian fields. Note that p is centred, and we may check that $(p(x), \nabla p(x))$ is non-degenerate for each $x \neq 0$ and sufficiently large N. Hence we can apply Proposition 4.4.1 to F and p on $D = B(R) \setminus B(1)$. Applying this proposition, it suffices to prove that

$$\mathbb{E}(N_{QC}(F,p)) < c \quad \text{and} \quad \mathbb{E}(N_{QT}(F,p)) < c \tag{4.29}$$

uniformly over N sufficiently large and $R \leq \alpha N$, where $N_{QC}(F,p)$ and $N_{QT}(F,p)$ denote respectively quasi-critical points and quasi-tangent points in $B(R) \setminus B(1)$.

Arguing as in the proof of Lemma 4.3.8 (with one additional application of Hölder's inequality because p is now random), the first bound in (4.29) follows from the Kac-Rice formula once we check the following two conditions. First, for all N sufficiently large

$$\sup_{x \in B(\alpha N) \setminus B(1)} \mathbb{E}\left(|p(x)|^2\right) \leq \frac{1}{N^2}.$$

This condition is immediately implied by Lemma 4.4.3.

Second, we require that

$$\begin{gathered} \max_{|k|=2} \mathbb{E}\left(\left|\partial^k F(x)\right|^4 \middle| G(x,\gamma) = 0\right) p_{G(x,\gamma)}(0) \\ \text{and} \\ \max_{|k|=2} \mathbb{E}\left(\left|\partial^k p(x)\right|^4 \middle| G(x,\gamma) = 0\right) p_{G(x,\gamma)}(0) \end{gathered} \tag{4.30}$$

are bounded above uniformly over $(x,\gamma) \in B(\alpha N) \times [0,1]$ for sufficiently large N, where $G(x,\gamma)$ is defined as in (4.25) with $p_{G(x,\gamma)}$ its density. Since f_N and p are centred Gaussian fields, an application of Lemma 4.4.2 (with the setting $X = \partial^k F(x) = \partial^k f_N(x)$, $Y = (f_N - \gamma p)(x), Z = \nabla(f_N - \gamma p)(x)$) shows that the first quantity in (4.30) is

$$\begin{aligned} &\mathbb{E}\left(\left|\partial^k F(x)\right|^4 \middle| G(x,\gamma) = 0\right) p_{G(x,\gamma)}(0) \\ &\qquad = \mathbb{E}\left(|X|^4 \middle| Y = \ell, Z = 0\right) p_{Y,Z}(0,0) \\ &\qquad \leq \frac{c}{\sqrt{\det(\Sigma(x,\gamma))}} \mathbb{E}\left(\left|\partial^k f_N(x)\right|^2\right)^2 \max\left\{1, \frac{\mathbb{E}\left((f_N(x) - \gamma p(x))^2\right)^2}{(\sigma^2(x,\gamma))^2}\right\}, \end{aligned}$$

where $p_{(Y,Z)}$ is defined as in Lemma 4.4.2, $\Sigma(x,\gamma)$ is the covariance matrix of $G(x,\gamma)$, and $\sigma^2(x,\gamma)$ is the variance of $f_N(x)-\gamma p(x)$ conditional on $\nabla(f_N-\gamma p)(x)=0$. Applying the same argument to the second quantity in (4.30), it is enough to show that each of

$$\begin{gathered}\max_{|k|=2}\max\left\{\mathbb{E}\left(\left|\partial^k f_N(x)\right|^2\right),\mathbb{E}\left(\left|\partial^k p(x)\right|^2\right)\right\},\ \mathbb{E}\left((f_N(x)-\gamma p(x))^2\right),\\ \frac{1}{\det(\Sigma(x,\gamma))},\ \frac{1}{\sigma^2(x,\gamma)}\end{gathered}\tag{4.31}$$

are bounded above uniformly for sufficiently large N and over $(x,\gamma)\in B(\alpha N)\times[0,1]$. By the independence of f_N and p,

$$\begin{gathered}\mathbb{E}((f_N(x)-\gamma p(x))^2)\leq\mathbb{E}\left(f_N(x)^2\right)+\mathbb{E}(p(x)^2)\\ \det(\Sigma(x,\gamma))\geq\det\mathrm{Cov}\begin{pmatrix}\nabla f_N(x)\\ f_N(x)\end{pmatrix},\end{gathered}$$

and so the bound on the first three terms of (4.31) follows from stationarity of f, the convergence of the covariance functions of f_N and the bound on p in Lemma 4.4.3. This convergence also implies that

$$\mathrm{Cov}\begin{pmatrix}\nabla f_N(x)-\gamma p(x)\\ f_N(x)-\gamma p(x)\end{pmatrix}\to\mathrm{Cov}\begin{pmatrix}\nabla f(x)\\ f(x)\end{pmatrix}$$

uniformly on $B(\alpha N)\times[0,1]$ as $N\to\infty$. Therefore

$$\sigma^2(x,\gamma)\to\mathrm{Var}(f(x)\mid\nabla f(x)=0)=\mathrm{Var}(f(0))=1$$

uniformly, proving the required bound on the final term of (4.31).

For the second bound in (4.29), we count the quasi-tangent points on $\partial B(R)$ and $\partial B(1)$ separately. The first quantity is bounded above uniformly by the same argument as for the quasi-critical points applied on the boundary $\partial B(R)$. The second count is bounded above uniformly by the same argument used to establish (4.28). $\square$

Proof of Lemma 4.4.2. The proof is similar to [77, Lemma 3.1]. We denote by $c>0$ a constant that depends only on d and n but may change from line to line. Since a normal random variable $\mathcal{N}(\mu,\sigma^2)$ satisfies $\mathbb{E}(|\mathcal{N}(\mu,\sigma^2)|^n)\leq c\max\{(\sigma^2)^{n/2},|\mu|^n\}$, and since conditioning on part of a Gaussian vector reduces the variance of all coordinates, it suffices to prove that

$$\sup_{y\in\mathbb{R}}|\mathbb{E}(X|Y=y,Z=0)|^n p_{(Y,Z)}(y,0)\leq\frac{c}{\sqrt{\det(\Sigma)}}\frac{\mathbb{E}(X^2|Z=0)^{n/2}\mathbb{E}(Y^2|Z=0)^{n/2}}{(\sigma^2_{Y|Z})^{n/2}}.$$

Recall that, by Gaussian regression,

$$|\mathbb{E}(X|Y=y,Z=0)| = |y|\frac{|\mathbb{E}(XY|Z=0)|}{\sigma^2_{Y|Z}} \quad \text{and} \quad p_{(Y,Z)}(y,0) \le \frac{ce^{-y^2/(2\sigma^2_{Y|Z})}}{\sqrt{\det(\Sigma)}}.$$

Since, for any $\sigma^2 > 0$,

$$\sup_{y\in\mathbb{R}} |y|^n e^{-y^2/(2\sigma^2)} = (\sigma^2 n/e)^{n/2},$$

we deduce that

$$\sup_{y\in\mathbb{R}} |\mathbb{E}(X|Y=y,Z=0)|^n e^{-y^2/(2\sigma^2_{Y|Z})} \le c\frac{|\mathbb{E}(XY|Z=0)|^n}{(\sigma^2_{Y|Z})^{n/2}},$$

and the result follows by the Cauchy-Schwarz inequality. $\square$

Proof of Lemma 4.4.3. Recall from (4.22) that there exist $c_1, c_2 > 0$ such that, for all $m \ge 0$ and $r \le \alpha m$,

$$|J_m(r)| \le c_1 e^{-c_2 m}. \tag{4.32}$$

Applying this to the orthogonal expansion (4.20), we have

$$\mathbb{E}\left(\sup_{x\in B(\alpha N)} |f(x) - f_N(x)|^2\right) \le \mathbb{E}\left(\left(\sum_{|m|>N} c_1|a_m|e^{-c_2|m|}\right)^2\right) \le c_3 e^{-c_4 N}$$

for some $c_3, c_4 > 0$, which gives the result in the case $k = 0$.

For the general case $k \ge 1$, we differentiate in polar coordinates and use Bessel identities to replace the resulting terms by linear combinations of Bessel functions. For instance, since

$$\frac{2mJ_m(r)}{r} = J_{m-1}(r) + J_{m+1}(r) \quad \text{and} \quad 2J'_m(r) = J_{m-1}(r) - J_{m+1}(r)$$

we have that

$$\begin{aligned}\partial_{x_1} f(x) - \partial_{x_1} f_N(x) &= \sum_{|m|>N} a_m e^{im\theta}(J'_{|m|}(r)\cos(\theta) - i\sin(\theta)(m/r)J_{|m|}(r)) \\ &= \sum_{|m|>N} \frac{1}{2}a_m e^{im\theta}((J_{|m|-1}(r) - J_{|m|+1}(r))\cos(\theta) \\ &\qquad\qquad - i\sin(\theta)(-1)^m(J_{|m|-1}(r) + J_{|m|+1}(r))).\end{aligned}$$

Hence by the triangle inequality and (4.32), we have

$$\mathbb{E}\left(\sup_{x\in B(R)} |\partial_{x_1} f(x) - \partial_{x_1} f_N(x)|^2\right) \le c_5 e^{-c_6 N}.$$

The proof for $k > 1$ is similar, and we omit the details. $\square$

4.4.1 Morse theory arguments

In this section we prove Proposition 4.4.1, which is based on the following deterministic result. We say that a quasi-critical point x at level α is non-degenerate if $\det \nabla^2(F-\alpha p)(x) \neq 0$. Similarly, we say that a quasi-tangent point x at level α is non-degenerate if $\nabla^2_\partial(F-\alpha p)(x) \neq 0$.

Lemma 4.4.4. *Let $F, p \in C^2(D)$ be deterministic functions that satisfy the following:*

1. *The quasi-critical and quasi-tangent points of (F,p) are non-degenerate, all occur at distinct levels, are not contained in $\{p=0\}$, and do not occur at level 0 or 1;*
2. *The quasi-critical points of (F,p) are not contained in ∂D;*
3. *The set $\{p=0\}$ consists of a union of simple curves and isolated points, and there are no accumulation points in $\{F=0\}\cap\{p=0\}$;*
4. *Defining $g = F/p$, there is no $x \in \partial D$ such that*

$$v_\partial(x)\cdot(\nabla F - g\nabla p)(x) = 0 \quad \text{and} \quad v_\partial^t(x)(\nabla^2 F - g\nabla^2 p)(x) v_{\bar\partial}(x) = 0.$$

Then there exists $c>0$, independent of F, p and D, such that

$$|N_{ES}(F;D,0) - N_{ES}(F-p;D,0)| \leq c(N_{QC}(F,p) + N_{QT}(F,p)).$$

The same conclusion holds on replacing N_{ES} with N_{LS}.

Given this result, the proof of Proposition 4.4.1 is a straightforward application of Bulinskaya's lemma to various combinations of F, p and their first two derivatives.

Proof of Proposition 4.4.1 (given Lemma 4.4.4). It is sufficient to verify that F and p almost surely satisfy the four conditions in Lemma 4.4.4.
(1). We first verify that, almost surely, the quasi-critical points of (F,p) are at distinct levels. Define $D_n = \{(x,y) \in D^2 : |x-y| \geq 1/n\} \times [0,1]$ and $g_1 : D_n \to \mathbb{R}^6$ by

$$g_1(x,y,\alpha) = \begin{pmatrix} \nabla(F-\alpha p)(x) \\ \nabla(F-\alpha p)(y) \\ (F-\alpha p)(x) \\ (F-\alpha p)(y) \end{pmatrix}.$$

Bulinskaya's lemma (Lemma A.2.1) states that g_1 almost surely does not hit $0 \in \mathbb{R}^6$ at any point in D_n, provided it is almost surely C^1 and that the density of $g_1(x,y,\alpha)$

is bounded on a neighbourhood of 0 uniformly in D_n. We know that $g_1 \in C^1(D_n)$ by assumption and so turn to the second condition. Since F and p are independent, the density of $g_1(x, y, \alpha)$ can be given by a convolution over the densities of

$$(\nabla F(x), \nabla F(y), F(x), F(y)) \qquad \text{and} \qquad \alpha(\nabla p(x), \nabla p(y), p(x), p(y)), \tag{4.33}$$

and therefore it is sufficient to show that the density of either of these vectors is bounded. By assumption, the covariance matrix of the first vector in (4.33) is non-degenerate for every $x \neq y$, and by continuity the determinant of this covariance matrix is bounded away from zero on the compact set D_n. Since this vector is Gaussian, this implies that its density is uniformly bounded above and allows us to apply Bulinskaya's lemma. Taking a countable union over n completes the proof.

Applying the same argument to $g_2 : \partial D^2 \times [0,1] \to \mathbb{R}^4$ and $g_3 : \partial D \times D \times [0,1] \to \mathbb{R}^5$ defined by

$$g_2 := \begin{pmatrix} \nabla_\partial (F - \alpha p)(x) \\ \nabla_\partial (F - \alpha p)(y) \\ (F - \alpha p)(x) \\ (F - \alpha p)(y) \end{pmatrix} \qquad \text{and} \qquad g_3 := \begin{pmatrix} \nabla_\partial (F - \alpha p)(x) \\ \nabla (F - \alpha p)(y) \\ (F - \alpha p)(x) \\ (F - \alpha p)(y) \end{pmatrix},$$

completes the proof that, almost surely, all quasi-critical and quasi-tangent points of (F, p) are at distinct levels.

Similarly, applying this argument to $g_4, g_5 : D \to \mathbb{R}^3$ and $g_6, g_7 : \partial D \to \mathbb{R}^2$ given by

$$g_4 := \begin{pmatrix} \nabla (F - p)(x) \\ (F - p)(x) \end{pmatrix}, g_5 := \begin{pmatrix} \nabla F(x) \\ F(x) \end{pmatrix}, g_6 := \begin{pmatrix} \nabla_\partial (F - p)(x) \\ (F - p)(x) \end{pmatrix}, g_7 := \begin{pmatrix} \nabla_\partial F(x) \\ F(x) \end{pmatrix},$$

shows that (F, p) has no quasi-critical or quasi-tangent points at level 0 or 1 (using the assumption that $(F(x), \nabla F(x))$ is a non-degenerate Gaussian vector for each $x \in D$).

Applying the arguments above to $g_8 : \partial D \times [0,1] \to \mathbb{R}^3$ given by

$$g_8 := \begin{pmatrix} \nabla_\partial^2 (F - \alpha p)(x) \\ \nabla_\partial (F - \alpha p)(x) \\ (F - \alpha p)(x) \end{pmatrix}$$

shows that the quasi-tangent points of (F, p) are non-degenerate almost surely. A slightly different version of Bulinskaya's lemma ([5, Proposition 6.5]) states that since $g_9 : D \times [0,1] \to \mathbb{R}^3$ defined by

$$g_9 := \begin{pmatrix} \nabla (F - \alpha p)(x) \\ (F - \alpha p)(x) \end{pmatrix}$$

is almost surely C^2 and has a univariate probability density which is uniformly bounded, there is almost surely no $(x, \alpha) \in D \times [0, 1]$ such that $g_9(x, \alpha) = 0$ and $\det \nabla^2(F - \alpha p)(x) = 0$. Hence the quasi-critical points of (F, p) are non-degenerate almost surely.

It remains to show that (F, p) has no quasi-critical or quasi-tangent points in $\{p = 0\}$. In the case that $\{p = 0\}$ consists of simple curves and isolated points, which intersects ∂D only at isolated points, we apply Bulinskaya's lemma to

$$g_9 := \begin{pmatrix} \nabla(F - \alpha p)(x) \\ (F - \alpha p)(x) \end{pmatrix} \qquad \text{and} \qquad g_{10} := \begin{pmatrix} \nabla_\partial(F - \alpha p)(x) \\ (F - \alpha p)(x) \end{pmatrix},$$

restricted to $\{p = 0\} \times [0, 1]$ and $\partial D \cap \{p = 0\} \times [0, 1]$ respectively, which gives the results. In the case that the variance of $p(x)$ is non-zero for each x (and so bounded away from zero by compactness), we define $g_{11} : D \times [0, 1] \to \mathbb{R}^4$ and $g_{12} : \partial D \times [0, 1] \to \mathbb{R}^3$ by

$$g_{11} := \begin{pmatrix} \nabla(F - \alpha p)(x) \\ F(x) \\ p(x) \end{pmatrix} \qquad \text{and} \qquad g_{12} := \begin{pmatrix} \nabla_\partial(F - \alpha p)(x) \\ F(x) \\ p(x) \end{pmatrix}.$$

Once again, Bulinskaya's lemma (along with the convolution argument) gives the result.

(2). Applying the same arguments to g_9 restricted to $\partial D \times [0, 1]$ shows that (F, p) almost surely has no quasi-critical points in ∂D.

(3). If p is deterministic, then $\{p = 0\}$ consists of a union of simple curves and isolated points by assumption. Otherwise, applying Bulinskaya's lemma to $(p, \nabla p)$ shows that p almost surely has no critical points in D at level zero. Since p is C^3 by assumption, the implicit function theorem then states that $\{p = 0\} \cap D$ consists of simple C^3 curves. It remains to show that the number of points in $\{F = 0\} \cap \{p = 0\}$ has finite expectation (so in particular, has no accumulation points almost surely). If p is deterministic, we apply the Kac-Rice formula (Theorem 1.3.2) to F restricted to the one-dimensional set $\{p = 0\}$. Since F is C^3 and $(F(x), \nabla F(x))$ is a non-degenerate Gaussian vector for each x, this gives the necessary result. In the case that p is random, we apply the Kac-Rice formula to $(F, p) : D \to \mathbb{R}^2$. This is valid since F and p are C^3 (so the same is true of their covariance functions) and $(F, \nabla F, p, \nabla p)$ is a non-degenerate Gaussian vector at each point of D. The integrand in the Kac-Rice formula is bounded (by compactness of D and non-degeneracy of $(F, \nabla F, p, \nabla p)$), so we conclude that the cardinality of $\{F = 0\} \cap \{p = 0\}$ has finite expectation.

(4). Let $g_{13} : \partial D \times [0,1] \to \mathbb{R}^3$ be given by

$$g_{13}(x,\alpha) = \begin{pmatrix} v_\partial^t(x)(\nabla^2 F - \alpha\nabla^2 p)(x)v_{\bar\partial}(x) \\ v_\partial(x) \cdot (\nabla F - \alpha \nabla p)(x) \\ (F-\alpha p)(x) \end{pmatrix}.$$

Once again, applying Bulinskaya's lemma with the convolution argument from before, and the assumption that $(F(x), \nabla_\partial F(x), \nabla_\partial \nabla_{\bar\partial} F(x))$ is non-degenerate for each x, we conclude that g_{13} almost surely has no zeroes in $\partial D \times [0,1]$ as required. $\square$

To complete the section, we prove Lemma 4.4.4. We will make use of the Morse theorems for manifolds with boundary stated in Section 2.4 (i.e. Theorems 2.4.1 and 2.4.2). We will not require the full strength of these results; we only use the fact that on passing through a critical or tangent point of a Morse function, the number of excursion set components changes by at most one.

The proof of Lemma 4.4.4 is extremely simple whenever $p > 0$ (or $p < 0$) on D. In this case, we can apply the Morse theorems directly to $g := F/p$ to show that

$$|N_{ES}(g; D, 0) - N_{ES}(g, D, 1)| \leq N_{\text{crit}} + N_{\text{tang}}.$$

where N_{crit} and N_{tang} denote respectively the number of critical and tangent points of g in D with level in $[0,1]$. We then observe that $g \geq \alpha$ if and only if $F - \alpha p \geq 0$, and that the critical/tangent points of g with level in $[0,1]$ are exactly the quasi-critical/tangent points of (F,p), which completes the proof.

When p takes positive and negative values, we will instead apply this argument to g restricted to $A := \{p \neq 0\}$ and then argue that the same conclusion holds when we replace $\{F - \alpha p \geq 0\} \cap A$ with $\{F - \alpha p \geq 0\}$. Intuitively this makes sense because the set $\{F - \alpha p \geq 0\} \cap \{p = 0\}$ is invariant under α, and so any changes in the topology of $\{F - \alpha p \geq 0\}$ as α varies must occur in A. This assumes that F and p have some regularity properties: specifically the conditions given in the statement of Lemma 4.4.4.

Proof of Lemma 4.4.4. We begin with the statement for excursion sets. Define $A = \{p \neq 0\}$ and $g : A \to \mathbb{R}$, by $g(x) = F(x)/p(x)$, which by assumption is a Morse function on A (i.e. when viewed as a manifold with boundary in ∂D). The first step is to show that, if g does not contain any critical or tangent points in $\{p \neq 0\}$ with level in $[\alpha_1, \alpha_2]$, then

$$N_{ES}(F - \alpha_1 p; D, 0) = N_{ES}(F - \alpha_2 p; D, 0). \tag{4.34}$$

In other words, $\{F - \alpha_1 p \geq 0\}$ and $\{F - \alpha_2 p \geq 0\}$ have the same number of components that do not intersect ∂D.

So suppose that g has no critical or tangent points with level in $[\alpha_1, \alpha_2]$. Note that, on $\{p > 0\}$, $F - \alpha p \geq 0$ if and only if $g \geq \alpha$. Therefore by Theorem 2.4.1, $\{F - \alpha_2 p \geq 0\} \cap \{p > 0\}$ is a deformation retract of $\{F - \alpha_1 p \geq 0\} \cap \{p > 0\}$. The same reasoning (applied to lower excursion sets) holds on $\{p < 0\}$, and combining these arguments we see that the components of $\{F - \alpha_1 p \geq 0\} \cap A$ are in bijection with the components of $\{F - \alpha_2 p \geq 0\} \cap A$ where one component maps to another if and only if they have non-empty intersection.

We now extend this to the components of $\{F - \alpha_1 p \geq 0\}$ and $\{F - \alpha_2 p \geq 0\}$. Let B be a component of $\{F - \alpha_1 p \geq 0\}$, and let $A_1, \dots, A_m$ be the distinct components of $\{F - \alpha_1 p \geq 0\} \cap A$ in B (since D is compact and g is Morse on A, there are a finite number of such components), with $C_1, \dots, C_m$ the corresponding components of $\{F - \alpha_2 p \geq 0\} \cap A$. Suppose that $\partial A_i \cap \partial A_j$ contains a simple curve $\gamma_{i,j}$ (in $\{p = 0\}$) of positive length. Then we can find a path $\delta_{i,j}$ contained in $A_i \cup A_j \cup \gamma_{i,j}$ joining an arbitrary point in A_i to an arbitrary point in A_j. Moreover, since $\{F = 0\} \cap \{p = 0\}$ has no accumulation points, we may assume that the chosen $\delta_{i,j}$ does not intersect $\{F = 0\}$. Since $F - \alpha_1 p \geq 0$ on $\delta_{i,j}$, and $\delta_{i,j}$ passes through $\{p = 0\}$ and is disjoint from $\{F = 0\}$, by continuity of F and p we can find a restriction $\delta'_{i,j}$ of $\delta_{i,j}$ such that $F - \alpha_2 p \geq 0$ on $\delta'_{i,j}$ and $\delta'_{i,j}$ still has one end-point in each of A_i and A_j. Notice that ∂B consists of simple curves, and the A_i are separated in B by $\{p = 0\}$, which also consists of simple curves. Therefore by applying the above argument to all such i and j, we see that $C_1, \dots, C_m$ are connected in $\{F - \alpha_2 p \geq 0\}$. Reversing the roles of α_1 and α_2, we conclude that the components of $\{F - \alpha_1 p \geq 0\} \cap A$ are connected in $\{F - \alpha_1 p \geq 0\}$ if and only if the corresponding components of $\{F - \alpha_2 p \geq 0\} \cap A$ are connected in $\{F - \alpha_2 p \geq 0\}$. Since the boundaries of the sets $\{F - \alpha_1 p \geq 0\}$ and $\{F - \alpha_2 p \geq 0\}$ consist of simple curves, each component of $\{F - \alpha_1 p \geq 0\}$ or $\{F - \alpha_2 p \geq 0\}$ must intersect A. Hence the components of $\{F - \alpha_1 p \geq 0\}$ and $\{F - \alpha_2 p \geq 0\}$ are also in bijection (where one component maps to another if and only if they have non-empty intersection), and in particular the number of these components is the same.

To finish the proof of (4.34), it remains to show that a component of $\{F - \alpha_1 p \geq 0\}$ hits the boundary ∂D if and only if the same is true of the corresponding component of $\{F - \alpha_2 p \geq 0\}$. Let C be a component of $\{F - \alpha_1 p \geq 0\}$ that does not intersect ∂D. For $\alpha \in [\alpha_1, \alpha_2]$ let C_α be the component of $\{F - \alpha p \geq 0\}$ which intersects C (by the previous paragraph this component exists and is unique). Suppose, for a contradiction, that $C_{\alpha_2} \cap \partial D \neq \emptyset$. We assume part of this intersection occurs in $\{p > 0\}$ (the arguments are similar if part of the intersection occurs in $\{p < 0\}$ and we know that the intersection must occur in $\{p \neq 0\}$). Let α' be the infimum of $\alpha \in [\alpha_1, \alpha_2]$ such that

$$C_\alpha \cap \partial D \cap \{p \geq 0\} \neq \emptyset.$$

We can then choose a sequence $\alpha_i \downarrow \alpha'$ and corresponding points $x_i \in \partial D \cap \{p \geq 0\}$ such that $(F - \alpha_i p)(x_i) \geq 0$. By compactness we can then find $x' \in \partial D \cap \{p \geq 0\}$ such that $\{F - \alpha' p \geq 0\}$ and so we note that $\alpha' > \alpha_1$. Since x' is not a quasi-critical/quasi-tangent point, the implicit function theorem shows that for some $\alpha \in (\alpha^1, \alpha')$, $\{F - \alpha p \geq 0\} \cap \partial D \neq \emptyset$. However this contradicts the definition of α'. We conclude that for all $\alpha \in [\alpha_1, \alpha_2]$, $C_\alpha \cap \partial D = \emptyset$, and the result follows (after reversing the roles of α_1 and α_2 in this argument). Therefore (4.34) is proved.

The second step is to bound the effect of the critical/tangent points of g. So suppose that there is a unique critical or tangent point x of g with level in (α_1, α_2). We assume that $p(x) > 0$ (the following arguments are similar when $p(x) < 0$ and by the conditions of the lemma, $p(x) \neq 0$). Theorem 2.4.2 implies that the number of components of $\{F - \alpha_2 p \geq 0\} \cap A$ differs from the corresponding number for $\{F - \alpha_1 p \geq 0\} \cap A$ by at most one. We now let $A_1, \ldots, A_m$ be the components of $\{F - \alpha_1 p \geq 0\} \cap A$ and let $C_1, \ldots, C_{m+1}$ be the components of $\{F - \alpha_2 p \geq 0\} \cap A$ where C_m and C_{m+1} may be empty. By relabelling, we assume that $x \in A_m$. Applying Theorem 2.4.1 to $A \backslash A_m$ shows that the A_i and C_j contained in $A \backslash A_m$ are in bijection. Therefore by relabelling we may assume that $A_i \cap C_i \neq \emptyset$ for $i = 1, \ldots, m-1$ and that $C_m, C_{m+1} \subset A_m$ (if $p(x) < 0$ this inclusion would be replaced by $A_m, A_{m+1} \subset C_m$ where one or both of A_m and A_{m+1} may be empty). The earlier argument can be repeated to show that if $i, j \neq m$ and $\partial A_i \cap \partial A_j$ contains a curve then C_i and C_j are connected in $\{F - \alpha_2 p \geq 0\}$. If $\partial A_i \cap \partial A_m$ contains a curve, then the same argument shows that C_i must be connected to one of C_m or C_{m+1} in $\{F - \alpha_2 p \geq 0\}$. Therefore, we see that each component of $\{F - \alpha_1 p \geq 0\}$ not containing x intersects a unique component of $\{F - \alpha_2 p \geq 0\}$, the component of $\{F - \alpha_1 p \geq 0\}$ containing x intersects at most two components of $\{F - \alpha_2 p \geq 0\}$ and all the components of $\{F - \alpha_2 p \geq 0\}$ are accounted for by these intersections. We conclude that the number of components of $\{F - \alpha_1 p \geq 0\}$ differs from the corresponding number for $\{F - \alpha_2 p \geq 0\}$ by at most one. Repeating the argument by contradiction above, we see that the components of $\{F - \alpha_1 p \geq 0\}$ which do not contain A_m intersect ∂D if and only if the corresponding components $\{F - \alpha_2 p \geq 0\}$ intersect ∂D. Therefore, regardless of whether the remaining components intersect ∂D or not, the number of components contained in D of $\{F - \alpha_1 p \geq 0\}$ and $\{F - \alpha_2 p \geq 0\}$ differ by at most two. Summing this difference over all critical/tangent points of g (of which there are finitely many, all at different levels since g is Morse) proves the statement of the lemma for excursion sets.

We now show the corresponding result for level sets, making use of the arguments just given. For any $\alpha \in [0, 1]$, we can construct a graph on the vertex set

$$V := \{\text{Components of } \{F - \alpha p \geq 0\}\} \cup \{\text{Components of } \{F - \alpha p \leq 0\}\}$$

by declaring two vertices to be joined by an edge if they have non-empty intersection. Each edge of the graph corresponds to a component of $\{F - \alpha p = 0\}$. This graph is acyclic and connected, and hence is a tree, therefore the number of vertices is one more than the number of edges:

$$\begin{aligned}\#\{\text{Components of } \{F - \alpha p = 0\}\} =& \#\{\text{Components of } \{F - \alpha p \geq 0\}\} \\ &+ \#\{\text{Components of } \{F - \alpha p \leq 0\}\} \\ &- 1.\end{aligned} \tag{4.35}$$

Now suppose that g has no critical/tangent points with level in $[\alpha_1, \alpha_2]$. Again the first step is to show that $\{F - \alpha_1 p = 0\}$ and $\{F - \alpha_2 p = 0\}$ have the same number of components contained in D. By (4.35) and the earlier arguments (which can be applied to $\{F - \alpha p \leq 0\}$), $\{F - \alpha_1 p = 0\}$ and $\{F - \alpha_2 p = 0\}$ have the same number of components, so it remains to check that the same number of these components intersect ∂D. Each component of $\{F - \alpha_1 p \geq 0\}$ or $\{F - \alpha_1 p \leq 0\}$ has a unique outer boundary (that is, the component of its boundary which it shares with the unbounded component of its complement in $\mathbb{R}^2$), which contains at least one component of $\{F - \alpha_1 p = 0\}$. A component of $\{F - \alpha_1 p \geq 0\}$ or $\{F - \alpha_1 p \leq 0\}$ intersects ∂D if and only if its outer boundary does. The argument by contradiction given above shows that this occurs if and only if the corresponding component of $\{F - \alpha_2 p \geq 0\}$ or $\{F - \alpha_2 p \leq 0\}$ intersects ∂D, which occurs if and only if its outer boundary does. Since each component of $\{F - \alpha_i p = 0\}$ is contained in the outer boundary of at least one component of $\{F - \alpha_i p \geq 0\}$ or $\{F - \alpha_i p \leq 0\}$, we have the result.

We turn to the second step, bounding the effect of the critical/tangent points of g. So suppose that g has precisely one critical/tangent point x with level in (α_1, α_2) and assume $p(x) > 0$. Let $A_1, \dots, A_m$ be the components of $\{F - \alpha_1 p \geq 0\}$ and $C_1, \dots, C_{m+1}$ the components of $\{F - \alpha_2 p \geq 0\}$, where $A_i \cap C_i \neq \emptyset$ for $i = 1, \dots, m-1$, either/both of C_m and C_{m+1} may be empty and $A_m \cap C_i = \emptyset$ for $i = 1, \dots, m-2$. The argument in the previous paragraph shows that for $i = 1, \ldots, m-2$, each component of $\{F - \alpha_1 p = 0\}$ in the outer boundary of A_i intersects ∂D if and only if each component of $\{F - \alpha_2 p = 0\}$ in the outer boundary of C_i does. Since A_m (or A_{m-1}) can have at most one component of $\{F - \alpha_1 p = 0\}$ which does not intersect ∂D in its outer boundary (and the same is true for C_m and C_{m+1} with respect to $\{F - \alpha_2 p = 0\}$) we see that the difference between the number of components of $\{F - \alpha_1 p = 0\}$ which are in the outer boundary of some component of $\{F - \alpha_1 p \geq 0\}$ and are contained in D and the corresponding number for $\{F - \alpha_2 p = 0\}$ is at most two. This also holds for $\{F - \alpha_1 p \leq 0\}$, and each component of $\{F - \alpha_i p = 0\}$ is contained in the outer boundary of at least one component of $\{F - \alpha_i p \geq 0\}$ or $\{F - \alpha_i p \leq 0\}$, therefore $|N_{LS}(F - \alpha_1 p; D, 0) - N_{LS}(F - \alpha_2 p; D, 0)| \leq 4$. Summing this result over the critical/tangent points of g proves the result for level sets. $\square$

Appendix A

Smooth Gaussian fields

A.1 Foundations

For the reader's convenience, we state some fundamental properties of Gaussian fields in this appendix. These results are well known and taken from a number of sources, including [1, 81, 5], which may be consulted for further details.

We begin by defining Gaussian random functions: we say that a collection of random variables $(f(x))_{x\in T}$ indexed by some non-empty set T and defined on a common probability space is a Gaussian random function, provided that any finite subcollection of these variables is a Gaussian random vector. We denote this collection of variables by f. A Gaussian random function is said to be centred if $\mathbb{E}(f(x)) = 0$ for all $x \in T$. The covariance function of f, is the function $\kappa : T^2 \to \mathbb{R}$ defined by

$$\kappa(x, y) = \mathrm{Cov}(f(x), f(y)).$$

Since the distribution of a centred Gaussian vector is determined by its covariance matrix, the finite dimensional distributions of a centred Gaussian random function are determined by its covariance function. The covariance function is therefore a succinct way of specifying this distribution, and so many results for Gaussian functions are stated in terms of required properties of the covariance function.

A natural first question is, when do such random functions exist? Or equivalently, for which functions $\kappa : T^2 \to \mathbb{R}$ does there exist a Gaussian random function with κ as its covariance function? It is enough to consider centred random functions, since a deterministic mean can always be added at each point. If κ is a covariance function, then it is clearly symmetric in x and y, and for $x_1, \dots, x_n \in T$ and $a_1, \dots, a_n \in \mathbb{R}$

$$\sum_{i,j=1}^{n} a_i a_j \kappa(x_i, x_j) = \mathbb{E}\left(\sum_{i,j=1}^{n} a_i a_j f(x_i) f(x_j)\right) = \mathbb{E}\left(\left(\sum_{i=1}^{n} a_i f(x_i)\right)^2\right).$$

We therefore see that κ must be positive definite: for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in T$ and $a_1, \ldots, a_n \in \mathbb{R}$

$$\sum_{i,j=1}^{n} a_i a_j \kappa(x_i, x_j) \geq 0.$$

This condition is, in fact, also sufficient for the existence of a Gaussian random function.

Theorem A.1.1 (Kolmogorov's existence theorem [28, Theorem 12.1.2])**.** *Let T be any set and $\kappa : T^2 \to \mathbb{R}$. There exists a centred Gaussian random function $(f(x))_{x \in T}$ with κ as covariance function if and only if κ is symmetric and positive definite.*

Whilst this apparently settles the question of existence, it is actually a somewhat limited result in this context: the interesting questions that can be asked of a Gaussian random function all require some form of measurability in T, which is not guaranteed by the previous theorem. To obtain measurability, it is necessary to impose more structure on T, for example by giving it a topology. In this thesis we only consider (restrictions of) functions defined on Euclidean space and so we now restrict to this case: $T = \mathbb{R}^n$ for some $n \in \mathbb{N}$ (we mention results for fields defined on some simple manifolds in Chapters 1 and 2). When a Gaussian random function is defined on Euclidean space (or more generally, on a manifold) we refer to it as a Gaussian field (this distinction is not uniform across the literature).

To give a specific example in which measurability is a pertinent concern, we consider the supremum of f on some compact set. This is a natural object of interest when studying the geometry of f, but as the supremum of potentially uncountably many random variables, is not necessarily measurable. One obvious solution to this issue, is to work with continuous fields; under this assumption the supremum may be taken over a countable, dense subset, thus ensuring measurability. More generally, a continuous function on Euclidean space is completely determined by its values on a countable dense subset, and so any reasonable topological/geometric events will be measurable. It is possible to ensure measurability without assuming continuity of the field (see [5, Chapter 1] for the notion of a separable process) however continuity is necessary for the events we consider in this thesis anyway.

The classical way of obtaining continuity of a random field is Kolmogorov's theorem [5, Proposition 1.16], which implies the existence of a continuous version of a general random field under a moment condition for points of the field which are close. The following version of this theorem shows that for Gaussian fields, smoothness of the covariance function is essentially equivalent to smoothness of the field.

For $V \subset \mathbb{R}^n$ and $\alpha > 1$, let $C^{\alpha,\alpha}_{\mathrm{loc}}(V)$ denote the set of functions $F : V^2 \to \mathbb{R}$ for which partial derivatives of order at most $\lfloor \alpha \rfloor$ in both variables exist and are Hölder continuous with exponent $\alpha - \lfloor \alpha \rfloor$. We let $C^{\alpha}_{\mathrm{loc}}(V)$ denote the set of functions from V to $\mathbb{R}$ which are α-smooth in the analogous sense.

Theorem A.1.2 (Kolmogorov [81, Sections A.9, A.3], [42, Theorem 3.17])**.** *Let $V \subset \mathbb{R}^n$ be an open set and $K : V^2 \to \mathbb{R}$ be positive definite and symmetric. If $\kappa \in C^{m+\eta',m+\eta'}_{loc}(V)$ for some $m \in \mathbb{N}$ and $\eta \in (0,1)$, then there exists a centred Gaussian field f defined on V with covariance function κ such that $f \in C^{m+\eta}_{loc}(V)$, with probability one, for any $\eta \in (0, \eta'/2)$.*

Conversely, if f is a Gaussian field defined on $V \subset \mathbb{R}^n$ and there exists $m \in \mathbb{N}$ such that with probability one, $x \mapsto f(x)$ is an element of $C^m_{loc}(V)$, then the covariance function of f is an element of $C^{m,m}_{loc}(V)$.

To prove this result, first note that differentiability of the covariance function of some order is equivalent to the existence of L^2 derivatives of the field of the same order (this follows from linearity of expectation and basic properties of Gaussian variables). One can then extend this smoothness to a pathwise result using the standard version of Kolmogorov's theorem. (The converse statement holds since pathwise convergence implies L^2 convergence for jointly Gaussian variables.) We note that tighter smoothness results (using entropy arguments and majorizing measures) are given in [1, Chapter 1], however these are not needed in this thesis.

This theorem then solves our measurability concerns. Let $\mathcal{B}\left(C^{\beta}_{\mathrm{loc}}(V)\right)$ denote the Borel σ-algebra on functions in $C^{\beta}_{\mathrm{loc}}(V)$ (this is the σ-algebra generated by all finite-dimensional cylinder sets). For a Gaussian field f on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the conditions of the first part of Kolmogorov's theorem, it is straightforward to show that (after removing a null set) $f : (\Omega, \mathcal{F}) \to \left(C^{\beta}_{\mathrm{loc}}(V), \mathcal{B}\left(C^{\beta}_{\mathrm{loc}}(V)\right)\right)$ is measurable. Any events of interest in this thesis are measurable by this statement.

In proving measurability, we have also analysed the first important assumption we make on fields in this thesis: smoothness. This is necessary to ensure regularity of excursion/level sets, which many of our arguments depend on, and it is unlikely that our results could be extended beyond this setting. We now introduce the second major assumption on our fields: stationarity. We say that a random field f defined on $\mathbb{R}^n$ is stationary if the finite dimensional distributions of the field are invariant under translations. Specifically, for any $m \in \mathbb{N}$ and $x, x_1, \dots, x_m \in \mathbb{R}^n$

$$(f(x_1), \dots, f(x_m)) \overset{d}{=} (f(x_1 + x), \dots, f(x_m + x)).$$

For a centred Gaussian field, this is equivalent to saying that the covariance function κ takes the form $\kappa(x,y) = K(x-y)$ for some function $K : \mathbb{R}^n \to \mathbb{R}$ and all $x, y \in \mathbb{R}^n$. This assumption is extremely important for our results: many of our arguments involve averaging the behaviour of a field f over some large region, for which stationarity (or something close to it) would seem to be essential.

For stationary continuous Gaussian fields, there is an alternative to the covariance function for specifying the distribution. Bochner's theorem [17] states that a continuous function $K : \mathbb{R}^n \to \mathbb{R}$ is positive definite, if and only if it is the Fourier transform of some finite Borel measure ν defined on $\mathbb{R}^n$; that is, for all $x \in \mathbb{R}^n$

$$K(x) = \int_{\mathbb{R}^n} e^{it\cdot x}\, d\nu(t).$$

When K is the covariance function of a random field, the measure ν is known as the spectral measure of the field. Whenever the field is centred and Gaussian, its distribution is clearly specified by its spectral measure. For a real-valued field, the spectral measure is Hermitian, in the sense that $\nu(A) = \nu(-A)$ for Borel sets A. Whenever the field is normalised so that $K(0) = \mathrm{Var}(f(0)) = 1$ the spectral measure is a probability measure.

There are several reasons for considering the spectral measure of a random field. One motivation is that some results are much easier to state in terms of properties of the spectral measure than in terms of those of the covariance function. A more significant motivation is the following 'spectral decomposition' of the field. For a Borel measure μ on $\mathbb{R}^n$, we define a *complex Gaussian μ-noise* to be a centred Gaussian process W indexed by the Borel sets of $\mathbb{R}^n$ such that for all A, B

$$\mathbb{E}\left(W(A)\overline{W(B)}\right) = \nu(A\cap B) \quad \text{and} \quad W(A\cup B) \overset{\text{a.s.}}{=} W(A) + W(B) - W(A\cap B).$$

Theorem A.1.3 ([1, Theorem 5.4.2])**.** *Let ν be a finite Borel measure on $\mathbb{R}^n$ and W a complex Gaussian ν-noise. Then the random field*

$$f(x) := \int_{\mathbb{R}^n} e^{it\cdot x}\, W(dx)$$

is centred, stationary and Gaussian with covariance function

$$K(x) := \int_{\mathbb{R}^n} e^{it\cdot x}\, d\nu(t).$$

This result can be extended to non-Gaussian fields and to include a partial converse (see [1, Chapter 5]). This representation gives a heuristic way of understanding any stationary continuous Gaussian field as a superposition of sine waves with random amplitudes and directions: if we approximate the ν-noise integral by Lebesgue sums we obtain

$$f(x) \approx \sum_j e^{it_j \cdot x} W(T_j)$$

where the T_j are small n-dimensional cubes partitioning $\mathbb{R}^n$ and $t_j \in T_j$. Writing U and V for the real and imaginary parts of W and ignoring complex terms (which must cancel in the limit since f is real) this approximation becomes

$$f(x) \approx \sum_j \cos(t_j \cdot x) U(T_j) - \sin(t_j \cdot x) V(T_j).$$

We therefore see that f can be decomposed as a sum of sine waves with different directions and wavelengths where the amplitudes are determined (randomly) by the spectral measure.

A.2 Bulinskaya's lemma

In this thesis, we often require our fields to be 'topologically non-degenerate' in some sense, with probability one. For example, we may wish to know that given some fixed value ℓ, a field $f : \mathbb{R}^n \to \mathbb{R}$ does not have any critical points at this level: that is, there is no point x such that $\nabla f(x) = 0$ and $f(x) = \ell$. This situation can be thought of as 'topologically degenerate' in the sense that it imposes $n + 1$ constraints ($\nabla f(x) = 0$ and $f(x) = \ell$) on a set with n degrees of freedom. A result due to Bulinskaya gives conditions on the distribution of a continuous field which ensure that this type of degeneracy is a null event. In the remainder of this appendix, we state Bulinskaya's lemma and give some simple results which facilitate its application.

Lemma A.2.1 (Bulinskaya [1, Lemma 11.2.10]). *Let $T \subset \mathbb{R}^n$ be a compact set and $g : T \to \mathbb{R}^{n+1}$ be a random field such that with probability one, $g \in C^1(T)$. Let $u_0 \in \mathbb{R}^{n+1}$ and suppose that for each $t \in T$, $g(t)$ has a probability density $p_{g(t)}$ and that for some $\epsilon > 0$*

$$\sup_{t \in T} \sup_{u \in B(u_0,\epsilon)} p_{g(t)}(u) < \infty,$$

then with probability one, $g^{-1}(u_0) = \emptyset$.

To apply this to the previous example of a critical point at level ℓ, one can define $g = (f, \nabla f)$ and then needs conditions to ensure that the univariate densities of this field are appropriately bounded. From this and similar examples, we are naturally lead to consider the joint distribution of a Gaussian field and its derivatives at a number of points.

Using the fact that Gaussian spaces are closed, and the definition of the derivative, one immediately obtains the useful fact that derivatives of a Gaussian field (when they exist) are also Gaussian. Rigorously, we can make the following statement: if $T \subset \mathbb{R}^n$ is open and $f : T \to \mathbb{R}$ is a centred Gaussian field with covariance function κ such that $f \in C^m_{\mathrm{loc}}(T)$ with probability one, then for any multi-index α in n variables such that $|\alpha| \leq m$, the field $\partial^\alpha f : T \to \mathbb{R}$ is a centred Gaussian and furthermore

$$\mathbb{E}\left(\partial^\alpha f(x) \partial^\beta f(y)\right) = \partial_x^\alpha \partial_y^\beta \kappa(x,y).$$

for all $x, y \in T$ and $|\beta| \leq m$.

Therefore studying the distribution of f and its derivatives reduces to studying their covariance. In particular, when applying Bulinskaya's lemma to some g which consists of f and its derivatives, it is sufficient to show that the covariance matrix of g is non-degenerate at each point of T (that is, the determinant of the covariance matrix is non-zero at each point). The univariate densities are then bounded above pointwise in T by Gaussianity, and this bound is continuous on T, so that uniform boundedness follows from compactness. (This arguments is spelt out when it is used in the thesis.)

We now verify several claims about non-degeneracy of covariance matrices for Gaussian fields which are used in the thesis. The first result is well known and extremely useful.

Lemma A.2.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 Gaussian field with constant variance, then for any $x \in \mathbb{R}^n$, $f(x)$ is independent of $\nabla f(x)$. In particular, this holds for stationary fields.*

Proof. Without loss of generality we assume that f is centred. Since f and ∇f are jointly Gaussian it is enough to show that they are uncorrelated at each point. For $i \in \{1, \dots, n\}$

$$\mathbb{E}\left(f(x)\frac{\partial}{\partial x_i} f(x)\right) = \frac{\partial}{\partial y_i}\kappa(y,x)\big|_{y=x} = \frac{\partial}{\partial y_i}\kappa(x,y)\big|_{y=x}. \tag{A.1}$$

Since f has constant variance

$$0 = \frac{\partial}{\partial x_i}\kappa(x,x) = \frac{\partial}{\partial y_i}\kappa(y,x)\big|_{y=x} + \frac{\partial}{\partial y_i}\kappa(x,y)\big|_{y=x}$$

and so the terms in (A.1) must equal zero. □

We emphasise that f and ∇f are only independent when evaluated at a single point: they are not independent processes. For the following three claims, we restrict to the planar case. These results are more specific to the setting of this thesis, so could be omitted on a first reading and referred to when needed.

Lemma A.2.3. *Let f be a C^2 stationary planar Gaussian field. Then the spectral measure ν being supported on the union of two lines through the origin is equivalent to the Gaussian vector $\nabla^2 f(0)$ being degenerate.*

Proof. By [1, Chapter 5], for any $s,t\in\mathbb{R}^2$ and $\alpha,\beta,\gamma,\delta\in\mathbb{N}\cup\{0\}$ (such that the following derivatives are defined)

$$\mathbb{E}\left(\frac{\partial^{\alpha+\beta}}{\partial t_1^\alpha \partial t_2^\beta}f(t)\overline{\frac{\partial^{\gamma+\delta}}{\partial s_1^\gamma \partial s_2^\delta}f(s)}\right)=\int_{\mathbb{R}^2}(-ix_1)^\alpha(-ix_2)^\beta e^{-it\cdot x}\overline{(-ix_1)^\gamma(-ix_2)^\delta e^{-is\cdot x}}\,d\nu(x)$$

where ν is the spectral measure of f. Then for $a\in\mathbb{R}^3$,

$$\mathbb{E}\left((a\cdot\nabla^2 f(0))^2\right)=\int_{\mathbb{R}^2}|a_1x_1^2+a_2x_2^2+a_3x_1x_2|^2\,d\nu(x).$$

If $\nabla^2 f(0)$ is degenerate, then we may choose $a\neq 0$ such that this expression is zero, and hence the integrand is identically zero on the support of ν. Hence the support of ν is contained in the zero set of this binary quadratic form, which is contained in the union of two lines through the origin.

Conversely if the support of ν is contained in the union of two lines through the origin, then we may choose $a\neq 0$ such that the zero set of $a_1x_1^2+a_2x_2^2+a_3x_1x_2$ is equal to this union. Hence the integral above will be zero and $\nabla^2 f(0)$ will be degenerate. $\square$

Lemma A.2.4. *Let $f:\mathbb{R}^2\to\mathbb{R}$ be a Gaussian field which is stationary and centred with* $\mathrm{Var}(f(0))=1$ *and covariance function $K\in C^{4+\eta'}_{loc}(\mathbb{R}^2)$ for some $\eta'>0$. If the support of the spectral measure ν contains a centred ellipse (or circle), then Assumptions 2.1.1 and 3.2.1 hold. Moreover, if the support of ν contains an open set then, for any distinct $t_1,\dots,t_n\subset\mathbb{R}^2$, the vector*

$$(f(t_1),\dots,f(t_n),\nabla f(t_1),\dots,\nabla f(t_n),\nabla^2 f(t_1),\dots,\nabla^2 f(t_n))$$

is non-degenerate (so in particular, Assumptions 2.1.1 and 3.2.1 hold).

Proof. Let $a\in\mathbb{R}^9$ and

$$w:=(f(t),\nabla f(t),f(0),\nabla f(0),\nabla^2 f(0)).$$

By the same arguments as in the proof of Lemma A.2.3

$$\mathbb{E}((a\cdot w)^2) = \\ \int_{\mathbb{R}^2} \left|a\cdot(e^{-it\cdot x}, -ix_1e^{-it\cdot x}, -ix_2e^{-it\cdot x}, 1, -ix_1, -ix_2, -x_1^2, -x_1x_2, -x_2^2)\right|^2\, d\nu(x).$$

If f does not satisfy Assumption 3.2.1, then there exists a choice of a such that this expectation is zero and one of the first three elements of a is non-zero. Hence the integrand above must be identically zero on the support of ν. By considering the real and imaginary parts explicitly, the zero set of this integrand cannot contain an ellipse/circle centred at the origin and so neither can the support of μ. By a near-identical argument, and Lemma A.2.3, f also satisfies the non-degeneracy conditions of Assumption 2.1.1. (The other conditions are satisfied by the premise of this lemma.)

By a completely analogous argument we see that if

$$(f(t_1),\dots,f(t_n),\nabla f(t_1),\dots,\nabla f(t_n),\nabla^2 f(t_1),\dots,\nabla^2 f(t_n))$$

is degenerate then some non-trivial linear combination of

$$\begin{aligned}&e^{-it_1\cdot x},\dots,e^{-it_n\cdot x},\\ &ix_1e^{-it_1\cdot x},\dots,ix_1e^{-it_n\cdot x},ix_2e^{-it_1\cdot x},\dots,ix_2e^{-it_n\cdot x}\\ &x_1^2e^{-it_1\cdot x},\dots,x_1^2e^{-it_n\cdot x},x_2^2e^{-it_1\cdot x},\dots,x_2^2e^{-it_n\cdot x},x_1x_2e^{-it_1\cdot x},\dots,x_1x_2e^{-it_n\cdot x}\end{aligned}$$

is identically zero on the support of ν. Since the t_i are distinct, we see that the support of ν cannot contain an open set. $\square$

Lemma A.2.5. *Let f be a Gaussian field satisfying Assumption 2.1.1. Then the density of saddle points p_s defined in Proposition 2.1.6 is non-zero for all $\ell\in\mathbb{R}$.*

Proof. By the Kac-Rice theorem (Theorem 1.3.2), and the independence of $\nabla f(0)$ and $(f(0),\nabla^2 f(0))$,

$$p_s(\ell) = \mathbb{E}\left[\left|\det\nabla^2 f(0)\right|\mathbb{1}_{\det\nabla^2 f(0)<0}\,\middle|\, f(0)=\ell\right]p_{f(0)}(\ell). \tag{A.2}$$

We note that by Gaussian regression

$$\mathrm{Cov}\left(\nabla^2 f(0)\,\middle|\, f(0)=\ell\right) = \mathrm{Cov}\left(\nabla^2 f(0)\right) - \mathrm{Cov}\left(\nabla^2 f(0), f(0)\right)\mathrm{Cov}\left(\nabla^2 f(0), f(0)\right)^t$$

where $\mathrm{Cov}\left(\nabla^2 f(0)\right)$ is a three by three matrix and $\mathrm{Cov}\left(\nabla^2 f(0), f(0)\right)$ is a three-dimensional row vector. Since we assume that $\nabla^2 f(0)$ is non-degenerate, the conditional covariance matrix above is the difference between a rank three and rank one matrix, so must have rank at least two. Therefore $(\nabla^2 f(0)|f(0) = \ell)$ must be supported on either a two or three dimensional (affine) subspace of $\mathbb{R}^3$. This implies that the support of

$$\big(\det \nabla^2 f(0) \,\big|\, f(0) = \ell\big) \tag{A.3}$$

is $\mathbb{R}$, and hence by (A.2) $p_s(\ell) > 0$ for all $\ell \in \mathbb{R}$. $\square$

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