Essays on Two-Sector Matching, Status Rewards and Liability

Paweł Gola

Hertford College

University of Oxford

A thesis submitted for the degree of

Doctor of Philosophy

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This thesis consists of three self-contained chapters.

Chapter 1 develops a two-sector, bivariate matching model, in which each sector uses a different dimension of skill in the production process. I show there exists a unique assignment of agents to sectors and derive comparative statics. The main result is that if jobs are scarce, both an increase in sector one skills' spread and a technological improvement increase the supply of talent in sector one, but decrease it in sector two. In sector two, this raises wages and wage inequality. In sector one, the effects are ambiguous in general, but wages increase for the most and decrease for the least talented agents.

Chapter 2 studies the impact of social status on occupational sorting in a two-sector matching framework. Talent is two-dimensional and thus status is not a zero-sum game; it depends both on occupational prestige and within-sector rank (local status). I show that the weights with which these two components enter – the structure of status – crucially influence the way in which agents self-select into sectors and argue that it is likely that these weights differ across occupations. The more important are the individual components of status in a sector, or the less important the collective component, the better the agents who join that industry, which has important implications for total payoffs, wage levels and inequality, and profits. I also show that the stable assignment is typically inefficient, which is driven by the distortion of relative status rewards, not status concerns *per se*.

Chapter 3 investigates whether directors of companies should have limited liability. I develop a three-player model in which: (a) debtholders and equityholders are defined by their control rights and (b) the project is run by the directors. The main result is that increased liability for directors forces them to internalise more of the downside risk of the project and hence reduces their risk-taking. This is optimal if over-investment was a problem initially. I show that the extent to which over-investment is a problem depends on how well debtholders are protected compared to equityholders. If debtholders are strong, increased liability can cause under-investment.

Wordcount: approximately 63000

I dedicate this thesis to my late grandmother, Joanna.

Meiner Oma.

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Contents

Chapter 1

Supply and Demand in a Two-Sector Matching Model

1.1 Introduction

In a standard Becker (1973) or Sattinger (1979) matching market, labour is heterogeneous and thus workers with different levels of talent are not perfect substitutes. The degree to which talent is differentiated has a major impact on wages: the more *similar* workers are, the more often they compete for the same jobs, which decreases wages. This effect is present with a single market, but if there are more sectors, it has also quite striking general equilibrium implications.

Consider two sectors, engineering and law, each using a different dimension of skill. Imagine that the government wants to encourage talented students to pursue engineering. In order to achieve that, it decides to improve students' engineering skills by increasing the number of obligatory maths classes in high school[1]. Such policy, although very natural, can lead to the opposite outcome than the one desired. Suppose that the returns to maths training diminish in talent and jobs are scarce. If that is the case, then the least talented students benefit more from additional classes, which makes engineering skills less differentiated and decreases wages for the highly talented engineers[2]. As a result, more of the talented students join the other sector, which increases talent supply in law and lawyers' wages fall as well. The only workers who benefit, are the least talented engineers, as they end up working for better firms.

As we can see, a simple and seemingly natural supply side intervention can lead to quite striking and often undesired outcomes. The goal of this paper is to clarify what are the consequences of various supply and demand side changes on (a) the flows of talent (b) wage levels and inequality, (c) profits and (d) total surplus – in the sector in which they have occurred, as well as in other industries.

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[1] For the sake of argument I will assume, quite implausibly, that the effect this has on the skills used in law is negligible.

[2] If jobs are abundant, then some firms are not on the market. An improvement of engineering skills can allow some of those firms to enter the market, and through this, increase demand for talent. The end effect on wages and market flows would be ambiguous.

In particular, I consider two sectors and focus on changes in the interdependence of skills, skill marginals and distributions of productivity (technology).

In Section 1.2, I present the model, characterise the unique stable assignment of agents to sectors and derive the associated wages. The *original formulation* of the model allows for general, continuous distributions of agents' skills and firms' productivity. The distribution of skill is bivariate: one of them – X – is used in sector one whereas the other – Y – in sector two. This captures the idea that a different skillsset is required in banking than in manufacturing (or academia). A match between an agent and a firm produces a surplus, according to a sector-specific surplus function. In line with the matching literature, I require that these surplus functions are (weakly) supermodular[3]. This implies that wages in each sector are identical to those holding under positive and assortative matching (PAM) – given the distribution of skill in that sector – which is then used to find out how agents sort into sectors.

The original formulation is perhaps the most natural, but the marginals of skills and the distributions of productivity are largely meaningless. They are just units of measurement, the choice of which is completely arbitrary and does not affect any of the real variables. For this reason, I apply probability integral transformation to all relevant random variables and work with their ranks. I interpret those ranks as *talents* and *firms*, rather than skills and productivities. This transformation results in the *copula formulation*, which is not only the most natural normalisation of the model, but also makes it more tractable.

Section 1.3 focuses on changes in the interdependence of skills. For the symmetric case, I show that if the joint distribution of skill becomes more *concordant*, then in each sector the distribution of talent deteriorates in first order stochastic dominance sense. I show further that if the marginals of the original skill distri-

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[3] Specifically, I allow for all strictly supermodular functions and certain types of weakly supermodular ones.

bution remain unchanged, wages rise for any talent level in each sector, whereas profits and total surplus fall. As the lowest wage remains constant, this means that the difference between top and bottom earnings increases both in absolute and relative terms.

In Section 1.4, I keep the copula of the skill distribution constant, and study the effects of changes in the marginal distribution of skill. I define the spread of skills' marginal in terms of the spread of surplus functions, which captures well the idea of talent differentiation. In particular, I say that the marginal of skill becomes *more spread out in real terms*, if surplus becomes more spread out for all firms, in the sense of Bickel and Lehmann (1979)[4].

I start the analysis by considering the simpler case of scarce jobs. The differentiation of talent (spread of skills) is just one way in which a change in skill marginals affects wages. If skill levels improve, some additional firms might become competitive, which increases demand for this particular dimension of talent. With scarce jobs, sector sizes are fixed and the second channel is switched off. Thus, it is only skills' spread that matters, not levels. More spread out skills attract talent to sector one and decrease talent supply in sector two; as jobs are scarce, this translates into first order stochastic dominance changes in talent distributions. Wages improve in sector two for any talent level, but more so for more talented agents. In sector one, the most talented agents' wages increase, as the effect of greater dissimilarity of workers dominates the effect of increased talent supply. The least talented agents, however, receive lower wages, as they are pushed down the ladder by newcomers. Wage inequality spreads with people who change occupations and thus increases in both sectors. This suggests that a riise in wage inequality in one industry can be driven by changes that happen exclusively in another sector. Profits decrease in sector two, as firms face tougher

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[4] A distribution becomes more spread out in Bickel-Lehman sense, if the difference of quantiles increases for any two ranks.

competition for talent. Total surplus and sector one profits depend not only on the spread, but also on skill levels: in particular, both increase if skills become more spread out and improve for all agents who previously worked in sector one. I call such a change a *surplus increasing real spread.*

If jobs are abundant, then sector sizes are endogenous and skill levels start playing a role. Because of that, I focus on surplus increasing real spreads, as described above, which increase both the dissimilarity of workers and the demand for talent. Therefore, they increase talent supply in sector one and decrease it in sector two. Profits decrease in sector two and increase in sector one, as does total surplus. The decreased talent supply results in higher wages for all sector two agents and the greater spread and improved levels of skills increase top sector one wages: the impact on lowest sector one wages is ambiguous, as is the effect on wage inequality, both because of the changes in demand for talent. To study this in more detail, I then focus on two special cases of surplus increasing spreads, in an attempt to isolate the effects of increased spread and improved surplus.

The first special case, isolated spread, keeps the skill level of the least talented of previously matched agents constant, whilst increasing skills' spread. The effects of that are very similar to those of a real spread in the scarce jobs case, in that inequality increases in both sectors and the wage of the least talented sector one agents is likely (although no longer certain) to fall. The second special case, isolated improvement, keeps the spread constant, whilst increasing skill levels for all workers. This makes all workers better off, but has ambiguous effect on wage inequality. To see why, consider a standard, multiplicative surplus function: $\Pi^1(x, z) = xz$. Then an isolated improvement is equivalent to a right shift of X by some constant. Suppose originally the levels of X are very low and the top agents work in sector two. As we shift X, sector one firms start competing for top workers, which increases their bargaining power and thus inequality. However, if we shift X far enough to the right, only sector one firms remain in competition for top workers, which decreases their bargaining power (compared to the intermediate case) and wage inequality.

In Section 1.5 I study the effects of a first order stochastic improvement in productivity. As the surplus function is supermodular, this increases the difference in surplus produced by agents of any two talents, for any firm: and thus again results in a more spread out surplus and more dissimilar workers in sector one. Hence, the effects of a technological improvement on talent supply and wages are qualitatively the same as the effects of a surplus increasing real spread: and thus sector two wages, top sector one wages and total surplus increase, whilst the changes in lowest sector one wages and wage inequality are ambiguous. A change in productivity distribution affects also the degree of firms' similarity. Therefore, an improvement in productivity can be bad for sector one firms, if it makes them more similar. For scarce jobs it is still true that a technological improvement results in a fall in lowest sector one wages and an increase in inequality. Thus, we have the surprising result that a technological improvement in sector one will be bad for some sector one workers, but good for all sector two workers; and that it increases wage inequality in sector two. It follows that an increase in wage inequality in one sector can be driven by technological improvements in some entirely different industry.

All proofs that do not appear in the main body can be found in the Appendix.

1.1.1 Related Literature

This paper builds on the work of Becker (1973), Sattinger (1979) and Roy (1951) by combining their approaches towards matching and self-selection, respectively, within one framework[5]. In fact, my model nests Sattinger-like, one sector matching models and Roy-like, two sector comparative advantage models within one

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[5] Sattinger (1979) itself is a continuous version of the famous marriage market model by Becker (1973).

framework. It is also closely related to Costrell and Loury (2004), as it extends their comparative statics analysis to a two-sector setting.

In the broadest of terms, this paper is related to the optimal assignment literature, started by Shapley and Shubik (1971) for the discrete and Gretsky, Ostroy, and Zame (1992) for the continuous case[6]. In particular, my model can be reformulated as a special case of the problem considered in the latter paper: therefore, their existence results hold for my model in principle. In general, this literature relies on optimal transportation theory to find optimal assignments and then uses the fact that, without frictions, the optimal assignment is stable. I characterise the stable assignment directly, using fixed point methods, which allows me to derive comparative statics. An additional advantage, utilised in Chapter 2, is that this approach can handle certain types of externalities.

To the best of my knowledge, the only two papers to consider multisector endogenous matching are McCann, Shi, Siow, and Wolthoff (2012) and Grossman, Helpman, and Kircher (2013)[7]. In both papers matching is one-to-many rather than one-to-one, as here, so these are different models mathematically – but they are very different also in terms of economics. McCann et al. (2012) have a complicated model, with three markets and schooling. This additional elements come at the cost of using specific functional forms and not providing any comparative statics results. Grossman et al. (2013) focus on the impact of trade liberalisation, rather than changes in skill and productivity distributions, and ability in their model is one-dimensional.

My model is in principle a multivariate matching problem. Most of this literature is focused on marriage markets (Anderson, 2003; Chiappori, Oreffice, and

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[6] Another important paper in this literature is Chiappori, McCann, and Nesheim (2010), who extend the results from Gretsky et al. (1992) to a production economy.

[7] There is a number of papers on matching in the trade literature (Costinot and Vogel, 2010; Sampson, 2014) that have multiple sectors – usually a continuum of them. Those sectors, however, differ mostly in what they produce, rather than in the skills and technologies they use and therefore are not really sectors in the sense considered here.

Quintana-Domeque, 2011, 2012); however, a recent paper by Lindenlaub (2014) investigates multivariate matching in labour markets and its implications for wage dispersion. This paper differs from my work in that it has one sector (possibly the whole economy) in which two different skills are used and the demand for these skills differs across firms. Because of that, the questions asked by Lindenlaub are different than those considered here: in particular, as the technology used by each firm is identical *given* their task requirements/productivities, she does not consider changes that are restricted to a subset of firms (a sector), which is the focus of this paper. Another difference is that Lindenlaub solves the important quadratic-Gaussian special case explicitly – which does not seem possible in my setting – but does not provide general comparative statics results.

1.2 Model

There are two populations: agents and firms. Agents have two separate skills – X and Y –, given by a bivariate distribution $F(x, y) : [x_l, x_h] \times [y_l, y_h] \to [0, 1]$, which is twice continuously differentiable and has strictly positive density for all (x, y) in the support of $F(\bullet)$. Firms are divided into two sectors, one and two. Sector $i \in \{1, 2\}$ firms have productivity Z^i, given by a univariate, strictly increasing and continuously differentiable distribution $H_{Z^i}(z^i) : [z^i_l, z^i_h] \to [0, 1]$. A match between an agent and a firm in sector one produces a surplus of $\Pi^1(x, z^1)$ and a match between an agent and a firm in sector two produces a surplus of $\Pi^2(y, z^2)$, where $\Pi^1 : [x_l, x_h] \times [z^1_l, z^1_h] \to \mathbf{R}^+, \Pi^2 : [y_l, y_h] \times [z^2_l, z^2_h] \to \mathbf{R}^+$ are (a) twice continuously differentiable with (b) $\Pi^1_x, \Pi^2_y > 0$, $\Pi^1_z, \Pi^2_z \geq 0$ and (c) Π^1_{xz}, Π^2_{yz} that are weakly positive and such that if $\Pi^i_{xz}(x, z) > 0$ for any (x, z), then $\Pi^i_{xz}(x', z') > 0$ for any $(x', z') > (x, z)$. Note that only the x-coordinate of workers' skill matters in sector 1 and only the y-coordinate matters in sector 2. The mass of agents is normalised to 1; the mass of sector i firms is $R^i > 0$.

Any agent and firm are free not to match anyone, in which case they receive a reservation utility/profit normalised to 0.

Note that property (c) allows for all surplus functions that are strictly supermodular over the entire domain, but also for a certain type of weakly supermodular functions. In particular, surpluses that depend only on worker's skill meet all conditions – this implies that my setting nests Roy-like models, which will be shown formally in Section 1.2.3.

1.2.1 Copula Formulation

The above *original formulation* of the matching problem is perhaps most natural, but not very convenient to work with. Instead, I apply *probability integral transformation* to all random variables and work with the copula of the original skill distribution most of the time. I call this the *copula formulation.*

Denote the marginals of $F(x, y)$ as $F_X(x)$ and $F_Y(y)$ and define the talents $U = F_X(X)$ and $V = F_Y(Y)$. F's unique (by Sklar's Theorem) copula is given by:

$$C(u, v) = F(F_X^{-1}(u), F_Y^{-1}(v)).$$

As $F_{xy}(\bullet) > 0$, it follows that $F_X(\cdot)$ and $F_Y(\cdot)$ are strictly increasing and differentiable, which implies that so are $F_X^{-1}(\cdot)$ and $F_Y^{-1}(\cdot)$. It follows that $C(\bullet)$ is continuously differentiable and $C_{uv}(\bullet)$ exists, is continuous and strictly positive.

Let us now apply probability integral transformation to productivities and define $H = H_{Z^1}^{-1}(Z^1) = H_{Z^2}^{-1}(Z^2)$; clearly, H is standard uniform distributed. Note that any firm type can be uniquely defined by the vector (h, i), where i denotes the sector the firm belongs to. Therefore, whenever an agent with talent vector (u, v) is matched with a firm with productivity (h, i), they produce a surplus of:

$$\pi^1(u, h) = \Pi(F_X^{-1}(u), H_{Z^1}^{-1}(h)),$$

$$\pi^2(v,h) = \Pi(F_Y^{-1}(v), H_{Z^2}^{-1}(h)),$$

for $i = 1$ and $i = 2$, respectively. We can easily see that $\pi^1(\bullet)$ and $\pi^2(\bullet)$ inherit properties (a) to (c).

1.2.2 (Stable) Matchings and Assignments

Before we can proceed, the basic objects of the analysis need to be defined. Note that all the definitions – and generally the remainder of this paper, unless stated otherwise – refer to the copula formulation.

Definition 1.1. A *matching* consists of a subset of matched agents $A_A \subset [0,1] \times [0,1]$, a subset of matched firms $A_F \subset [0,1] \times \{1,2\}$ and a matching function, $\zeta : A_A \to A_F$.

Any matching is required to satisfy a 'measure consistency' property, which means that the mass of any subset of matched firms needs to be equal to the subset of agents they are matched with (see Legros and Newman, 2002, p. 929). Formally, for any measurable set $S \subset A_F$ define its partition into the set of all sector one firms: $B^1 = \{(h,1) \in B\}$ and all sector two fims: $B^2 = \{(h,2) \in B\}$, then:

Definition 1.2. A matching is *measure consistent* if for any measurable $B \subset A_F$ and its preimage $\zeta^{-1}(B) \subset A_A$:

$$\int\int_{\zeta^{-1}(B)} C_{uv}(u,v)\ \mathrm{d}u\ \mathrm{d}v = R^1 \int_{B^1} 1\ \mathrm{d}h + R^2 \int_{B^2} 1\ \mathrm{d}h.$$

Define a *payoff scheme* as a pair of mappings: $w : [0,1]^2 \to \mathbf{R}^+$ and $r : [0,1] \times \{0,1\} \to \mathbf{R}^+$. The first mapping – w – will be interpreted as wages and the second – r – as profits. As any unmatched firm and agent produce zero and the sum of the firm's profit and the worker's wage in any given match cannot exceed the surplus they produce, we can define payoff schemes that are feasible for a given matching:

Definition 1.3. Given a matching, any associated payoff scheme is *feasible* iff:

$$\begin{aligned}&\text{for all } (u,v,h,i),\ \text{such that } \zeta(u,v)=(h,i): \quad w(u,v)+r(h,i)\leq \pi^i(u,v,h)\\&\text{for all } (u,v)\notin A_A \quad w(u,v)=0\\&\text{for all } (h,i)\notin A_F \quad r(h,i)=0.\end{aligned}$$

We can also define stable matchings, so measure consistent matchings in which there exists no agent-firm pair that would prefer to be assigned with each other rather than with their current matches.

Definition 1.4. A matching $(A_A, A_F, \zeta(\bullet))$ is *stable* if and only if it is measure consistent and there exists a payoff scheme that is feasible given $(A_A, A_F, \zeta(\bullet))$, such that for any (u,v,h,i):

$$w(u,v)+r(h,i)\geq \pi^i(u,v,h). \tag{1.1}$$

Most of the time my focus will be on the way in which workers are assigned to sectors. This information can be easily inferred from a matching:

Definition 1.5. Any matching $(A_A, A_F, \zeta(\bullet))$ results in an *assignment*, given by the set A_A and an assignment function $\theta : A_A \to \{1,2\}$:

$$\theta(u,v)=\begin{cases}1 & \text{if } \zeta(u,v)\in A_F^1\\ 2 & \text{if } \zeta(u,v)\in A_F^2.\end{cases}$$

An assignment $(A_A, \theta(\bullet))$ is *stable* if and only if there exists a stable matching that results in $(A_A, \theta(\bullet))$.

1.2.3 Characterisation Strategy

To characterise the stable assignment (and some stable matchings) I deploy a two-step strategy. In the first step, I consider both sectors in partial equilibrium, treating assignment as given. This way I find what are the within-sector stable matchings and associated wage functions. This is very similar to the problem first considered and solved by Sattinger (1979). In the second step, I use those wages to find the stable assignment in a manner somewhat similar to Roy's model.

First Step

In this part, I treat the assignment as given and suppress it from notation. Denote the resulting marginal distribution of U among sector one agents as $G^1(\cdot)$ and the marginal distribution of V among sector two agents as $G^2(\cdot)$, let the *critical abilities* u^c and v^c be the greatest lower bounds of $G^1(\cdot)$ and $G^2(\cdot)$ supports, respectively, and denote the mass of agents in sector i as $M^i \leq R^i$ (by measure consistency). Define the within-sector matching functions as $\zeta^1(u) : [u^c, 1] \to [0, 1]$, $\zeta^2(v) : [v^c, 1] \to [0, 1]$. In particular:

Definition 1.6. The *positive, assortative within-sector matching functions (PAM)* are given by: $P^1(u) = \frac{1}{R^1}(R^1 + M^1(G^1(u) - 1))$ and $P^2(v) = \frac{1}{R^2}(R^2 + M^2(G^2(v) - 1))$.

Then the following result holds.

Proposition 1.1. All feasible wage schemes that can support a stable within sector matching (meet Inequality 1.1) are of the following form:

$$w^1(u) = \int_{u^c}^{u} \pi_u^1(r, P^1(r))\mathrm{d}r + C^1, \tag{1.2}$$

$$w^2(v) = \int_{v^c}^{v} \pi_v^2(r, P^2(r))\mathrm{d}r + C^2, \tag{1.3}$$

where $C^1 \in [0, \pi^1(u^c, \frac{R^1-M^1}{R^1})]$ and $C^2 \in [0, \pi^2(v^c, \frac{R^2-M^2}{R^2})]$.

Proposition 1.1 is a modest generalisation of Sattinger's famous result for strictly supermodular surpluses[8]. With weak supermodularity more matchings might be stable than just PAM – but the fact that $\pi^1_{uh}(\bullet)$ is single-crossing ensures that wages still need to be identical to those holding in PAM. This is the case as for the set of firms and workers for which $\pi^1_{uh}(\bullet) = 0$ the marginal surplus of each agent's talent in any matching is identical to that holding under PAM – and for any pair for which surplus is strictly supermodular PAM is the only stable matching (see the proof of Proposition 1.1, in Appendix 1.A).

Thus, for the purpose of finding the stable assignment, we can proceed as if matching in both sectors was positive and assortative, even if it isn't. Quite clearly, if there are any unmatched agents in a stable matching, the competitive forces will drag wages of the worst agents to zero – and contrary, if there are unmatched firms in any of the sectors, competition will drag the profits of least productive firms to zero.

Lemma 1.1. In any stable matching it has to be the case that: (i) if $R^1 > M^1$ then $w^1(u^c) = \pi^1(u^c, \frac{R^1-M^1}{R^1})$ and if $R^2 > M^2$ then $w^2(v^c) = \pi^2(v^c, \frac{R^2-M^2}{R^2})$; (ii) if $1 - M^1 - M^2 > 0$ then $w^1(u^c) = w^2(v^c) = 0$.

Second Step

Inequality (1.1) in the definition of a stable matching has to hold for any worker-firm pair, so also for within-sector pairs. Therefore, in any stable matching wage functions need to be of the form derived in the previous step. In this part, we will use this fact to find the assignments that result from stable matchings – the

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[8] Legros and Newman (2002) call 'famous' the result that under weakly supermodular surpluses any stable matching can be supported only by the payoff schemes that support PAM. However, they don't provide any references and their Proposition 3 holds only for one-sided matching markets. Sattinger (1979) shows that the cross-derivative of the surplus function needs to be positive, but his argument is true only for strictly positive cross-derivatives.

stable assignments.

Note that Proposition 1.1 and positive surplus functions imply that we can have either unmatched agents or unmatched firms, but never both: otherwise there would be a positive mass of both firms and agents earning zero, which contradicts stability. If there are more firms than agents ($R^1 + R^2 > 1$) some firms will end up unmatched, because of measure consistency – and thus all agents will find a match. Similarly, if firms are scarce ($R^1 + R^2 < 1$) then all will find a match, whereas some agents will not. Finally, if $R^1 + R^2 = 1$, then both all firms and all agents will be matched[9]. These facts combined imply that

$$M^1 + M^2 = \min\{R^1 + R^2, 1\}. \tag{1.4}$$

It is possible that all agents work in a single sector. Trivially, for this to happen it is necessary that $R^i \geq 1$. Lemma 1.1 implies that additionally we need $\pi^i(0, \frac{R^i-1}{R^i}) \geq \pi^j(1,1)$, as otherwise some agents would strictly prefer to work in sector j. As it turns out, these two conditions are not only necessary, but also sufficient.

Proposition 1.2. In any stable assignment we have that $M^i = 1$ if and only if $R^i \geq 1$ and $\left[\pi^i(0, \frac{R^i-1}{R^i}) \geq \pi^j(1,1)\right]$.

Thus, if $R^i \geq 1$ and $\pi^i(0, \frac{R^i-1}{R^i}) \geq \pi^j(1,1)$ the unique stable assignment will have all agents working in one sector. Therefore, we need only to characterise stable assignments for cases where this is not true for any sector.

Definition 1.7. *Condition (d)* is met if both for $(i,j) = (1,2)$ and $(i,j) = (2,1)$ it is the case that $R^i < 1$ or $\pi^i(0, \frac{R^i-1}{R^i}) < \pi^j(1,1)$.

Condition (d) together with measure consistency ensure that $M^i \in (0, R^i]$ – so there is a positive mass of agents in each sector that does not exceed the mass

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[9] Suppose not. Then measure consistency implies that both a positive measure of agents and firms would need to be unmatched, which is impossible.

of firms[10]. The remainder of this subsection will focus on cases where condition (d) is met.

By definitions of u^c and v^c the mass of matched agents with $(u,v) < (u^c, v^c)$ has to be zero. At the same time, any agent with $u > u^c$ or $v > v^c$ receives a strictly positive wage in one of the sectors; therefore all agents with $(u,v) < (u^c, v^c)$ and only such agents remains unmatched[11] and thus:

$$C(u^c, v^c) = \max\{1 - R^1 - R^2, 0\}.$$

Moreover, it has to be the case that agents with (u^c, v^c) need to earn the same wages in each sector, as otherwise some agents would want to relocate.

Lemma 1.2. If condition (d) is met, then $w^1(u^c) = w^2(v^c)$.

Clearly, if one of the sectors offers higher wages in the top end of the distribution, it is able to attract all agents who are highly talented in both dimensions. Define the *star abilities* u^* and v^*:

$$\begin{aligned} u^* = 1 \quad &\text{and} \quad w^1(1) = w^2(v^*) \quad \text{if } w^1(1) \le w^2(1) \\ v^* = 1 \quad &\text{and} \quad w^1(u^*) = w^2(1) \quad \text{if } w^1(1) > w^2(1). \end{aligned}$$

Hence, any agent with $u \in (u^*, 1]$ strictly prefers to join sector one and any agent with $v^* \in (v^*, 1]$ strictly prefers to join sector two. Note also that it is always the case that $\max\{u^*, v^*\} = 1$.

Define the set $\Gamma = [u^c, u^*] \times [v^c, v^*]$. As $R^1, R^2 > 0$, Γ is non-empty[12]. Agents with $(u,v) \in \Gamma$ will join sector one if $w^1(u) \ge w^2(v)$ and sector two if $w^2(v) \ge$

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[10] This follows from Proposition 1.2 and Equation (1.4)

[11] I ignore here agents for whom $u = u^c \vee v = v^c$ as their mass is zero.

[12] As $R^1, R^2 > 0$, it has to be the case that $v^c, u^c < 1$. Wage functions are strictly increasing and therefore $w^2(1) > w^1(u^c)$ and $w^1(1) > w^2(v^c)$, which implies that $v^* > v^c$ and $u^* > u^c$.

$w^1(u)$. Define a function $\psi(v) : [v^c, v^*] \to [u^c, u^*]$ such that:

$$w^1(\psi(v)) = w^2(v). \tag{1.5}$$

and a function $\phi(u) : [u^c, u^*] \to [v^c, v^*]$ such that

$$w^1(u) = w^2(\phi(u)). \tag{1.6}$$

Any agent in Γ will strictly prefer sector one if $u > \psi(v)$ and sector two if $u < \psi(v)$. Moreover, as $w^1(u)$ and $w^2(v)$ are strictly increasing and $w^1(u^c) = w^2(v^c), w^1(u^*) = w^2(v^*)$ it has to be the case that both $\psi(v)$ and $\phi(u)$ exist and are each other inverses: $\phi(\psi(v)) = v$.

Remark 1.1. Note that the quintuple $(u^c, v^c, u^*, v^*, \psi(\bullet))$ fully defines the stable assignment. Agents with $(u, v) < (u^c, v^c)$ are unmatched. Sector 1 is populated by agents with: (i) $v < v^c$ and $u > u^c$; (ii) $(u, v) \in \Gamma$ and $\psi(v) < u$; (iii) $u > u^*$. Sector 2 is populated by agents with: (i) $v > v^c$ and $u < u^c$; (ii) $(u, v) \in \Gamma$ and $\psi(v) > u$; (iii) $v > v^*$. The set of remaining agents is of zero mass.

Thus, given $(u^c, v^c, u^*, v^*, \psi(\bullet))$, we can derive the marginal distributions of talent in each sector. The probability that an agent with talent $V = v$ chooses sector two is $\Pr(\theta(U, v) = 2|v)$. For $v \in [v^c, v^*]$ the probability that a sector two agent has ability lower than v is:

$$G^2(v) = \int_{v^c}^{v} \frac{\Pr(\theta(U, v) = 2|r)}{R^2} \mathrm{d}r,$$

as V's marginal distribution is standard uniform. Consider some arbitrary agent with $v \in (v^c, v^*]$. Such an agent will be in sector 2 as long as $U \leq \psi(v)$, which implies that $\Pr(\theta(U, v) = 2|v) = C_v(\psi(v), v)$[13]. Recalling the definitions of v^c and

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[13] It doesn't matter whether $\psi(v) \geq u$ holds strictly, as the probability of $\psi(v) = u$ is 0 anyway.

v^* we have that:

$$G^2(v) = \begin{cases} 0 & \text{for } v < v^c, \\ \frac{1}{M^2}\int_{v^c}^{v} C_v(\psi(r), r)\mathrm{d}r & \text{for } v \in [v^c, v^*], \\ G^2(v^*) + \frac{1}{M^2}(v - v^*) & \text{for } v > v^*. \end{cases} \tag{1.7}$$

Analogously, for sector one we have:

$$G^1(u) = \begin{cases} 0 & \text{for } u < u^c, \\ \frac{1}{M^1}\int_{u^c}^{u} C_u(r, \psi^{-1}(r))\mathrm{d}r & \text{for } v \in [u^c, u^*], \\ G^1(u^*) + \frac{1}{M^1}(u - u^*) & \text{for } u > u^*. \end{cases} \tag{1.8}$$

Remark 1.2. The relation between sector two rank of any agent with $v \in [v^c, v^*]$ and sector one rank of any agent with $u = \psi(v)$ is negative. Consider all agents with $V \geq v$. By Remark 1.1 all such agents have to be matched; and can work in sector one only if their $U \geq \psi(v)$. Similarly, all agents with $U \geq \psi(v)$ need to be matched; and can work in sector two only if their $V \geq v$. As the mass of all agents with $V \geq v$ or $U \geq \psi(v)$ is $1 - C(\psi(v), v)$ it follows that $M^1\big(1 - G^1(\psi(v))\big) + M^2\big(1 - G^2(v)\big) = 1 - C(\psi(v), v)$. This implies that the fewer agents with $V \geq v$ work in sector two, the more agents with $U \geq \psi(v)$ work in sector one. Thus, there is also a negative relation between the sector two firm with which v is matched under PAM and the sector one firm her counterpart $\psi(v)$ works for, as $P^1(\psi(v)) = \frac{1}{R^1}\Big[R^1 - 1 + C(\psi(v), v) + R^2\big(1 - P^2(v)\big)\Big]$. The intuition is simple: a good match for v implies that there are few people with $(U, V) \geq (v, \psi(v))$ in sector two, which means that many of them work in sector one – which in turn implies that $\psi(v)$'s match will be relatively weak.

If we substitute the marginal distributions back into wage functions and combine all the results so far, we will arrive at the following set of equations and conditions:

$$\text{for } v \in [v^c, v^*]: \quad \int_{u^c}^{\psi(v)} \pi_u^1\Big(t, \int_{u^c}^{t} \frac{R^1 - M^1 + C_u(r,\psi^{-1}(r))}{R^1}\mathrm{d}r\Big)\mathrm{d}t \tag{1.9}$$
$$= \int_{v^c}^{v} \pi_v^2\Big(t, \int_{v^c}^{t} \frac{R^2 - M^2 + C_v(\psi(r),r)}{R^2}\mathrm{d}r\Big)\mathrm{d}t,$$

$$C(u^c, v^c) = \max\{1 - R^1 - R^2, 0\}, \tag{1.10}$$

$$\max\{u^*, v^*\} = 1, \tag{1.11}$$

$$\int_{u^c}^{u^*} C_u(r, \psi^{-1}(r))\mathrm{d}r + 1 - u^* = M^1, \tag{1.12}$$

$$\int_{v^c}^{v^*} C_v(\psi(r), r)\mathrm{d}r + 1 - v^* = M^2, \tag{1.13}$$

$$M^1 + M^2 = \min\{R^1 + R^2, 1\}, \tag{1.14}$$

$$M^1 \in (0, R^1] \text{ and } M^2 \in (0, R^2], \tag{1.15}$$

$$M^1 < R^1 \Rightarrow \pi^1\Big(u^c, \frac{R^1 - M^1}{R^1}\Big) \leq \pi^2\Big(v^c, \frac{R^2 - M^2}{R^2}\Big), \tag{1.16}$$

$$M^2 < R^2 \Rightarrow \pi^2\Big(v^c, \frac{R^2 - M^2}{R^2}\Big) \leq \pi^1\Big(u^c, \frac{R^1 - M^1}{R^1}\Big). \tag{1.17}$$

Equations (1.12) and (1.13) are derived by taking $v = 1$ and $u = 1$ in (1.7) and (1.8), respectively; whereas conditions (1.16) and (1.17) follow from Lemmas 1.1, 1.2 and Proposition 1.1.

Existence and Uniqueness

A solution to the set (1.9)-(1.17) gives us $(u^c, v^c, u^*, v^*, \psi(\bullet))$ and thus fully characterise a stable assignment. This characterisation of the stable assignment will lay at the heart of the subsequent comparative statics analysis (Sections 1.3 to 1.5). Note also that each of these equations and conditions need to hold for any stable assignment and thus any stable assignment has to be represented by a solution to (1.9)-(1.17).

Theorem 1.1. If condition (d) is met, then solution to the set (1.9)-(1.17) exists, is unique and fully characterises the unique stable assignment as specified in

Remark 1.1. If condition (d) is not met, then in the unique stable assignment all agents work in the sector for which $\pi^i(0, \frac{R^i - i}{R^i}) \geq \pi^j(1, 1)$.

The proof relies on constructing a map, the fixed point of which is equivalent to the solution of (1.9) and finding a norm for which this map is a *contraction mapping*[14]. This proves that $\psi(\cdot)$ is unique *given* (u^c, v^c, M^1, M^2) – and also continuous in these variables. Then showing existence and uniqueness is merely a matter of proving that the remaining equations have a unique solution given the function $\psi(u^c, v^c, M^1, M^2)$. The existence of a stable matching – and thus a stable assignment – could be alternatively shown by rewriting the model as a special case of Gretsky et al. (1992). The uniqueness result is, however, new – it does not follow from any of the existing uniqueness results for stable matchings (see e.g. Chiappori et al., 2010; Carlier, 2003), as it is possible that there are multiple stable matchings in this model – but they all result in the same assignment.

Trivially, the existence of a stable assignment implies existence of stable matchings. In particular:

Corollary 1.1. A matching in which agents are assigned to sectors as specified in Theorem 1.1 and are positively and assortatively matched within sectors is always stable. Moreover, if the surplus functions in each sector are strictly supermodular for all possible agent-firms pairs, then this is the only stable matching.

This result follows from Theorem 1.1 and the proof of Proposition 1.1. In the following subsection we will investigate a special case in which the matching specified in Corollary 1.1 is not the only stable one.

Sattinger and Roy

The first step in my characterisation strategy is very similar to Sattinger (1979), the second to Roy (1951). This is not a coincidence: in fact, both one-sector

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[14] The norm I use is Bielecki's norm for a high-enough parameter λ.

matching and Roy-like models are nested within this framework[15].

In the case of Sattinger's model, that's fairly obvious: if condition (d) does not hold, then the model collapses to just one sector. Alternatively, extending the model to cases where $R^i = 0$ would be straightforward and would also trivially imply one sector.

As for Roy-like models, suppose that surplus functions in each sector depend only on the agent's skill, so that $\Pi^1_z(\bullet) = \Pi^2_z(\bullet) = \Pi^1_{xz}(\bullet) = \Pi^2_{yz}(\bullet) = 0$. Suppose further that jobs are abundant in each sector: $R^1, R^2 > 1$. Then by Proposition 1.1 and Lemma 1.1 we have that $w^1(x) = \Pi^1(x)$ and $w^2(y) = \Pi^2(y)$ – the agents receive the entire surplus and their wage does not depend on the assignment, like in Roy's model. Thus, Roy-like models can be seen as two-sector matching models in which all firms from the same sector are identical. In such a case, all within-sector matchings are stable.

1.3 Changes in Interdependence

In this section, I study what happens if the skills used in each sector become more interdependent. To make this problem tractable, I will restrict attention to matching problems with *symmetric copula formulations.* For this special case, an analytical solution can be found trivially.

Definition 1.8. The copula formulation of a matching problem is *symmetric* iff: (i) $C(u, v) = C(v, u)$ for all $(u, v) \in [0, 1]^2$; (ii) $\pi^1(u, h) = \pi^2(u, h)$ for all $(u, h) \in [0, 1]^2$ and (iii) $R^1 = R^2$.

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[15] The actual models by Sattinger and Roy are not, strictly speaking. In the former case, the reason is that Sattinger allows for both firms and agents to be unemployed, which is precluded here by the assumption of positive surpluses – so only some special cases of his model are nested here. In the latter, the reason is that Roy uses bi-variate log-normal distribution of skills, which is not defined over an interval – however, we could get an arbitrarily good approximation of Roy's model, by using bi-variate log-normal distribution, truncated arbitrarily high and arbitrarily close to zero.

Note that a symmetric original formulation implies that the copula formulation is symmetric as well. However, matching problems with asymmetric original formulations can have symmetric copula formulations[16]. Symmetry implies trivially that $u^c = v^c$, $u^* = v^*$ and $\psi(v) = u$ characterises a stable assignement; and as by Theorem 1.1 the stable assignment is unique, Corollary 1.2 follows.

Corollary 1.2. If the copula formulation of a matching problem is symmetric then (i) $v^c = u^c$ is given by the solution to $C(v^c, v^c) = \max\{1 - 2R^1, 0\}$; (ii) $v^* = u^* = 1$ and (iii) $\psi(v) = u$ for $v \in [v^c, v^*]$. This fully characterises the unique stable assignment, as described in Remark 1.1.

In the symmetric case, agents simply choose the sector which uses the skill in which they are more talented. Clearly, this results in identical talent (but not necessarily skill) distributions, wages and sector sizes in both industries.

To formalise the idea of greater interdependence I use the concordance ordering.

Definition 1.9 (Scarsini, 1984)**.** Let F and G be two continuous cdfs of bivariate distributions, with copulas C_F and C_G, respectively. F is *more concordant* then G if and only if $C_F(u, v) \geq C_G(u, y)$ for all $(u, y) \in [0, 1]^2$.

Consider two matching problems with symmetric copula formulations: the *old* and the *new* one. To isolate the impact of changes in interdependence, I assume that all other aspects of the old matching problem – including the marginal distributions of skill – remain unchanged. Thus, the copula formulations of the matching problems differ only in the copula itself and the new copula is more

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[16] For example, consider the following original formulation: the distribution of skill is $F(x, y) = \frac{1}{2}(y - 10)x^3$ with support on $[0, 1] \times [10, 12]$; the firms' productivity is standard uniform distributed in both sectors; the surplus functions are $\Pi^1(x, z^1) = x^3(1 + z^1)$ for $(x, z^1) \in [0, 1]^2$ and $\Pi^2(y, z^2) = \frac{1}{2}(y - 10)(1 + z^2)$ for $(y, z^2) \in [10, 12] \times [0, 1]$ and masses of firms are $R^1 = R^2 > 0$. This is asymmetric. It is easy to verify that $F_X(x) = x^3$ and $F_Y(y) = \frac{y-10}{2}$. Thus, the corresponding copula formulation is described by: $C(u, v) = uv$, $\pi^1(u, h) = uh + u$, $\pi^2(v, h) = vh + v$ and $R^1 = R^2 > 0$, which is symmetric.

concordant than the old one. To formally distinguish between the new and old problem, I introduce a parameter – ρ; the old problem is denoted by ρ_1 and the new one by ρ_2. For example, $u^c(\rho_1)$ is the old critical ability in sector one, whereas $v^*(\rho_2)$ is the new star ability in sector two.

Using this notation, I can provide a natural definition of what constitutes an *increase in the supply of talent.*

Definition 1.10. *Talent supply increases* in sector i if, for any talent level t, the mass of agents with talent greater than t increases. Formally:

$$M^1(\rho_2)(1 - G^i(t, \rho_2)) \geq M^1(\rho_1)(1 - G^i(t, \rho_1)) \qquad \text{for all } t \in [0, 1]. \tag{1.18}$$

Supply of high talent increases in sector i, if there exists such a $t^* < 1$ that Equation (1.18) holds for all $t \in (t^*, 1]$. An increase is strict, if Equation (1.18) holds strictly for some t.

Demand for talent can be defined analogously: so as an increase in the mass of matched firms with productivity greater than h, for all values of h.

Proposition 1.3. Consider two matching problems with symmetric copula formulations. Suppose further that their copula formulations differ only in the distribution of skill, with the new distribution being more concordant than the old one. Then, in each sector: (i) the distribution of talent deteriorates in first order stochastic dominance sense; (ii) wages increase for any talent; (iii) the difference between highest and lowest earnings increases; (iv) profits fall for all firms and (v) the total surplus produced decreases.

Proof. I will prove it for sector one, results for sector two follow from symmetry. (i) As all agents work in one of the sectors or are unmatched, it follows from measure consistency and symmetry that for any $u \geq u^c$ we have:

$$2M^1G^1(u) + \max\{1 - 2R^1, 0\} = C(u, u).$$

In a symmetric problem, sector sizes depend only on R^1, which implies that:

$$2M^1(\rho_1)(G^1(u, \rho_2) - G^1(u, \rho_1)) = C(u, u, \rho_2) - C(u, u, \rho_1).$$

To complete the proof, it suffices to note that the definition of concordance implies that RHS is positive and that $u^c(\rho_2) \leq u^c(\rho_1)$ by Corollary 1.2.

(ii) Recall Equation (1.2). The wage constant, C^1, remains unchanged[17]. Since surplus function, marginals and sector sizes do not change either, by Definition 1.6 wages depend only on within-sector talent distributions and hence, they increase for any $u > u^c(\rho_2)$; for $u \in [0, u^c(\rho^2)]$ they remain constant (and equal to 0).

(iii) Trivial, as $C^1(\rho_2) = C^1(\rho_1)$ (see footnote 17) and $w^1(1, \rho_2) \geq w^1(1, \rho_1)$.

(iv) Denote the worst matched firm in sector i as $P^i = \frac{R^i - M^i}{R^i}$. As the sector size does not change, neither do P^i and A^i_F (the set of matched firms)[18]. All unmatched firms make zero profit and the profit function for all matched firms in sector i is given by (see Sattinger, 1979):

$$r^i(h) = \int_{P^i}^{h} \pi^i_h((P^i)^{-1}(r), r)\mathrm{d}r + C^i_P. \tag{1.19}$$

The profit constant will either remain unchanged (for abundant jobs) or will fall (for scarce jobs, as then $C^i_P = \pi^1(u_c, 0)$ and u^c decreased). Thus, (iv) follows

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[17] If jobs are scarce ($R^1 \leq \frac{1}{2}$), then $C^1 = 0$. If jobs are abundant ($R^1 > \frac{1}{2}$), then $C^1 = \pi^1(u^c, M^1)$ and by Corollary 1.2 we have that $u^c = 0$.

[18] For non-strictly supermodular surplus functions and the abundant jobs case ($R^1 + R^2 > 1$), the set of matched firms is not unique. Hereafter, I will assume that in such cases firms with $h \geq P^i$ become matched. This simplifies notation and is without any loss in generality, as the profits of the relevant firms will be 0.

from (i).

(v) By (i), every firm produces lower surplus. □

The intuition for this result is simple. A more concordant skill distribution means a lower overall supply of talent, as agents can work in at most one sector and there are more of them who are talented in both dimensions. Because of symmetry, this translates into a decrease in talent supply in each sector, which decreases profits and total surplus, but increases wages. The lowest wage, however, remains unchanged and thus the gap between top and bottom wages widens.

1.4 Changes in Marginal Distribution of Skill

So far, I have looked at changes in the copula, whilst keeping the marginals constant. Now, I will study changes in the marginal distribution of one of the skills, whilst keeping the copula constant. As talent differentiation plays an important role in this model, most of the time I will focus on what happens if the marginal distribution becomes more *spread out*.

1.4.1 Partial Orderings of Spread

Throughout this section, I will consider two matching problems that meet all conditions from Section 1.2, including condition (d), and differ only in the marginal distribution of X, with the marginal in the new problem being more spread out than in the old one[19]. A well-known notion of spread, which will be relevant for my analysis is the Bickel-Lehman spread[20].

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[19] If any of the matching problems does not meet condition (d) comparative statics are trivial.

[20] First defined in Bickel and Lehmann (1979) and introduced to economics in Landsberger and Meilijson (1994).

Definition 1.11. A distribution $F_X(x, \rho_2)$ is *(strictly) more spread out in Bickel-Lehman sense* than distribution $F_X(x, \rho_1)$ if:

$$F_X^{-1}(u_2, \rho_2) - F_X^{-1}(u_1, \rho_2)(>) \geq F_X^{-1}(u_2, \rho_1) - F_X^{-1}(u_1, \rho_1) \text{ for all } 0 \leq u_1 < u_2 \leq 1.$$

Bickel-Lehman spread is not invariant under strictly increasing transformation of skill units[21]. This is a problem, as skills are not easy to observe, or define, and hence the choice of a measurement unit is largely arbitrary. Hence, the various measures of skill are not necessarily linear transformations of each other. Take the example of years in education, which is a popular proxy for skill. Under the – extremely strong – assumption that people are homogeneous in the time spent on learning in any given year of education, hours of learning are an increasing transformation of years in education. However, as people tend to spend more time learning in the later stages of their education (think of the difference in workloads between graduate and primary schools!), this transformation is not linear, but convex.

I resolve this issue by directly focusing on the mapping from talents into surplus – which does not depend on skill units. If we do that, then, keeping the production technology constant, a skill distribution becomes more spread out iff the only real variable in the model, surplus, spreads out for all firms. This leads to the following definition of *real spread*.

Definition 1.12. The new distribution of X is *(strictly) more spread out in real terms* than the old distribution of X if the new distribution of surplus $S = \Pi^1(X, z)$ is (strictly) more spread out in Bickel-Lehman sense than the old distribution of S, for all firms $z \in [z_l^1, z_h^1]$.

This is a stronger notion than Bickel-Lehman spread, in the sense that it requires a continuum of increasing transformations of the marginal distribution

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[21] Unless we fix the unit of skill, it is not even clear what linearity of surplus function means.

to become more spread out in the Bickel-Lehman sense. For the special case of a multiplicative surplus function – $\Pi^1(x, z) = xz$ – real spread is equivalent to Bickel-Lehman spread[22].

Given that the surplus function and firm distribution are unchanged, (strict) real spread captures well the idea that talent becomes more differentiated, as it is equivalent to an increase in the marginal surplus of talent for any talent-firm pair ($\pi^1_u(u, h, \rho_2) \geq (>)\pi^1_u(u, h, \rho_1)$ for all $(u, h) \in [0, 1]^2$). Therefore, real spread is invariant under strictly increasing transformations of sector one skill: a change in skill units will in turn change the surplus function and the two effects will cancel each other out. Similarly to the definition of concordance in Section 1.3, the definition of real spread relies on the copula formulation of the matching problem.

1.4.2 Scarce Jobs

I will consider the cases of scarce ($R^1 + R^2 < 1$) and abundant ($R^1 + R^2 \geq 1$) jobs separately. The former is much simpler – and possibly more realistic – and thus serves as a good starting point. In particular, scarcity of jobs implies that lowest wages are zero and that all firms are matched – which in turn means that sector sizes are fixed and the PAM function in each sector is equal to the distribution of skill. The two last facts are implied by Equations (1.9)-(1.17), whereas the first follows trivially from Lemma 1.1.

Corollary 1.3. If $R^1 + R^2 < 1$ then (1) $P^1(u) = G^1(u)$, $P^2(v) = G^2(v)$, (2) $M^i = R^i$ and (3) $w^1(u^c) = w^2(u^c) = 0$.

The following Proposition, as well as all comparative statics results in this paper, follows from Theorem 1.2 in Appendix 1.B, which links changes in the

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[22] More generally, if there exists such a unit of skill that surplus is linear in it for all $z \in [z^1_l, z^1_h]$, then real spread is equivalent to Bickel-Lehman spread of skill (measured in this unit).

surplus function of the copula formulation ($\Delta_\rho \pi_u^1(u,h)(>) \geq 0$ for all $(u,h) \in [0,1]^2$) with changes in the quality of matches in each sector.

Proposition 1.4. If jobs are scarce and the marginal distribution of X becomes more spread out in real terms then (i) the distribution of talent in sector one improves in first order stochastic dominance sense; (ii) the distribution of talent in sector two deteriorates in first order stochastic dominance sense. If, further, the real spread is strict, then supply of talent increases strictly in sector one and falls strictly in sector two.

With scarce jobs, the mass of sector one firms is fixed and hence the overall demand for talent does not change. Thus, changes in the marginal distribution of skill affect the market only through their effect on talent differentiation – that is, through the spread. In other words, with scarce jobs skill levels do not matter[23]. Ignoring reallocation, the more dissimilar sector one workers become, the fewer substitutes there are for agents of any talent, which increases wages. In general equilibrium, this attracts agents from the other sector and results in an increased supply of talent in sector one and a decreased supply of talent in sector two.

To bring the most interesting results into focus, in the discussion on wages I will focus on strict real spreads; all results extend easily to the more general case.

Proposition 1.5. If jobs are scarce and the marginal distribution of sector one skill becomes strictly more spread out in real terms, then (i) in sector two, wages strictly increase for all agents and the higher the talent the greater the increase; (ii) in sector one, wages strictly decrease for a positive mass of the least talented

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[23] To see this clearly, consider the following extreme example (it violates the assumption of continuous skill distribution, but nevertheless it conveys very well the logic of what is happening). Suppose that in the old matching problem, every agent had sector one skill $x(\rho_1) > x_h(\rho_2)$. Thus, workers were identical for sector one firms', which paid them zero. Hence, all talented workers were in sector two. Any spread of sector one marginal results in some sector one workers being paid positive wages and an improvement in talent-pool – despite the strict decrease in skill levels.

agents and strictly increase for a positive mass of the most talented agents; and (iii) in both sectors the gap between highest and lowest wages widens strictly.

Proof. (i) From Theorem 1.2 in Appendix 1.B and Corollary 1.3 follows that a strict spread results in a strict decrease in sector two critical ability v^c. As C^2 remains unchanged and surplus function is supermodular, the increase in wages follows from inspection of Equation (1.1). Note that for any $v'' > v' \geq v^c$ we have:

$$w^2(v'') = \int_{v'}^{v''} \pi_v^2(r, P^2(r))\mathrm{d}r + w^2(v'). \tag{1.20}$$

As $P^2(r)$ increases and surplus is supermodular, it follows that $w^2(v'')$ increases by more than $w^2(v')$.

(ii) I start with top wages. Proposition 1.4 and the proof of Lemma 1.8 (in Appendix 1.B) imply that $u^*(\rho_2) \leq u^*(\rho_1)$ and $v^*(\rho_2) \geq v^*(\rho_1)$. Thus:

$$w^1(u^*(\rho_1), \rho_2) \geq w^1(u^*(\rho_2), \rho_2) = w^2(v^*(\rho_2), \rho_2) \geq w^2(v^*(\rho_1), \rho_2)$$
$$w^2(v^*(\rho_2), \rho_1) \geq w^2(v^*(\rho_1), \rho_1) = w^1(u^*(\rho_1), \rho_1) \geq w^1(u^*(\rho_2), \rho_1)$$

which trivially implies that:

$$w^1(u^*(\rho_1), \rho_2) - w^1(u^*(\rho_1), \rho_1) \geq w^2(v^*(\rho_2), \rho_2) - w^2(v^*(\rho_2), \rho_1). \tag{1.21}$$

Thus, $w^1(u^*(\rho_1)$ strictly increases. For any $u > u^*$ we have that:

$$w^1(u) = \int_{u^*}^{u} \pi_u^1(r, G^1(r))\mathrm{d}r + w(u^*(\rho_1)). \tag{1.22}$$

For $u > u^*(\rho_1)$, $G^1(u)$ does not change; and as real spread implies that $\pi^1(u, h)$ strictly increases, it follows that $w^1(u, \rho_2) > w^1(u, \rho_1)$ for any $u \in [u^*(\rho_1), 1]$.

I will turn now to wages of the least talented agents. By Theorem 1.2 we have that $u^c(\rho_2) > u^c(\rho_1)$. As wages strictly increase in talent, it follows from definition of critical ability that $w^1(u^c(\rho_2), \rho_1) > w^1(u^c(\rho_1), \rho_1) = 0$. Existence of a positive mass of agents for whom wages decrease (increase) follows from continuity of wage functions.

(iii) Follows from (i), (ii) and the fact that with scarce jobs $C^i(\rho_2) = C^i(\rho_1)$. □

In sector two, talent supply falls and therefore wages increase. Sector one wages are affected positively by the increase in real spread and negatively by the higher supply of talent. For agents with highest talent, the effect of the increase in spread dominates and their wages raise[24]. For the least talented agents the increase in talent supply dominates: those agents are pushed down the ladder so much that their wages fall.

As the surplus function is supermodular, the decrease in sector two talent supply benefits more talented agents more[25]. This increases the wage's spread in the Bickel-Lehman sense[26]. In sector one, the raise in wage inequality happens in a weaker sense (the gap between top and bottom wages widens), as there the inequality inducing effect of greater talent differentiation is partially outdone by the increased supply of talent.

Profits depend not only on talent's supply and demand, but also on how much firms can produce and how dissimilar they are. These depend not so much on the spread of skills, as on their levels: the former trivially and the latter as, because of supermodularity, the more skilled the agents are, the greater is the difference in surplus produced by high and low productivity firms and the more dissimilar they become. Therefore, in order to say anything conclusive about sector one

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[24] To see this, note that top wages increase in sector two and yet most talented agents are more likely to join sector one than previously.

[25] In each sector, any agent earns more than her marginally less talented colleague, and the difference is equal to the value of the formers' marginal surplus. Therefore, each agents wage depends on the marginal surplus produced by all workers who are less talented than her. In sector two, the fall in talent supply means that all agents are matched with better firms: as a result, the marginal surpluses improve for all workers, which benefits those at the top more.

[26] Specifically, the new distribution of $w^2(X)$ is more spread out than its old distribution.

profits, we need a stronger condition than real spread.

Definition 1.13. The new distribution of X is a *surplus increasing (strict) real spread* of the old distribution of X if it is a (strict) real spread and $F_X^{-1}(u^c(\rho_1), \rho_2) \geq F_X^{-1}(u^c(\rho_1), \rho_1)$.

I call such spreads surplus increasing as, keeping the firm constant, they increase the skill level – and thus surplus – of any agent who used to work in sector one under the old distribution of skill (so for $u \geq u^c(\rho_1)$)[27].

Proposition 1.6. If jobs are scarce, then a real spread of the marginal distribution of sector one talent decreases profits for all sector two firms. If, further, the real spread is surplus increasing then profits raise for all sector one firms as well and total surplus produced in the economy increases.

Proof. The first claim follows from inspection of Equation (1.19): each firm is matched with a less productive agent, so $(P^2)^{-1}$ decreases for all h and the profit constant falls as well, as it is equal to $\pi^2(v^c, 0)$ and v^c falls by Proposition 1.4.

For surplus increasing real spreads, the improvement in skills for $u \geq u^c(\rho_2)$ implies that $\pi_h^2(u, h)$ increases for all $(u, h) \in [u^c(\rho_1), 1] \times [0, 1]$; as $(P^1)^{-1}(u)$ and C_R^1 increase by Proposition 1.4 and Lemma 1.1, the result wrt sector one profits follows from inspection of Equation (1.19) as well.

As for the last claim, note that, fixing the assignment, a surplus increasing real spread trivially increases total surplus. And as the stable assignment is surplus maximising in this model, the change from the old to new stable assignment has to further improve total surplus[28]. $\square$

In sector two, we have only the general equilibrium effect of decreased talent supply, which depends only on sector one skills spread, not levels. In sector

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[27] This follows trivially from the definitions of surplus increasing, real and Bickel-Lehman spreads.

[28] This is the case, as my model can be rewritten as a special case of the assignment model described in Gretsky et al. (1992) and thus the equivalence of stable and efficient matching showed by them holds for my model as well.

one, firms become more dissimilar, the spread of profits increases and, as the least productive firms produce higher surplus and are better off, profits increase. Relocation increases talent supply and thus enchances this effect.

1.4.3 Abundant Jobs

If jobs are abundant ($R^1 + R^2 \geq 1$), skill levels do play a role, as the set of matched firms is endogenous. If agents become less skilled in the X dimension, sector one firms become less competitive compared to sector two firms and some might be forced to leave the market. This decreases the demand for sector one talent and has a negative impact on wages. As the increase in spread still positively affects wages, the final effect on relocation is ambiguous.

To address this, I focus on surplus increasing real spreads. It follows from Theorem 1.2 in Appendix 1.B that they results in an increase in talent supply in sector one and a fall in sector two.

Proposition 1.7. A surplus increasing real spread of the marginal distribution of X result in an increase in sector one talent supply and a fall in sector two talent supply. If the real spread is strict, then the changes in talent supply are strict as well.

Surplus increasing real spreads have, ignoring relocation, a positive effect on sector one wages, as they increase both talent demand and talent differentiation. Thus, in general equilibrium, they attract more talent to sector one, at the expense of sector two. Note also that it is still the case that an increase if the real spread of X is strong enough, it can increase the supply of high talent in sector one, even if X falls for any talent level[29]. Finally, as both sector one talent demand and supply increase, it expands; sector two shrinks.

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[29] To see this, recall the example from footnote 23 and assume further that the new matching problem is symmetric and that $R^1 > 1$. Then, as long as the old, common skill level is such that $\Pi^1(x(\rho_1), H_{Z^1}^{-1}(\frac{R^1-1}{R^1})) < \Pi^2(y_h, z_h^2)$, we have that there will be a positive mass of agents working in sector two even before the change. As all agents used to be identical for sector one

Lemma 1.3. If jobs are abundant, a surplus increasing real spread of the marginal distribution of X results in an increase of the lowest profit in sector one and a decrease in sector two ($C^1_R(\rho_2) \geq C^1_R(\rho_1)$ and $C^2_R(\rho_2) \leq C^2_R(\rho_1)$).

The shift in the demand for sector one talent happens on the extensive margin, as some new low productivity firms enter the market. Hence, firms' gain from the increase in surplus is always at least as big as the loss from increased demand – and even the lowest profits increase[30].

Proposition 1.8. If jobs are abundant, a surplus increasing real spread of the marginal distribution of X implies that profits increase for all sector one firms and decrease for all sector two firms. Total surplus produced in the economy increases.

Proof. The result wrt sector one (two) profits follows from the definition of an increase (decrease) in supply, the definition of PAM ($P^i(\cdot)$), Lemma 1.3 and inspection of Equation (1.19). The increase in total surplus follows from analogous reasoning as in the proof of Proposition 1.6. □

Hence, the impact of the shift in demand for talent in sector one is dominated by the greater differentiation of firms (implied by an increase in skill levels) and the increase in talent supply. Propostitions 1.6 and 1.8 imply jointly that surplus increasing real spreads always result in an increase in sector one profits and a decrease in sector two profits, irrespective of whether jobs are scarce or abundant.

Proposition 1.9. If jobs are abundant, a surplus increasing real spread of the marginal distribution of X increases all sector two wages, as well as wages of the most talented sector one agents.

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firms, they offered the same wage and all sector two agents were earning more than that. Thus, $v^c(\rho_1) = v^*(\rho_1) < 1 = v^*(\rho_2)$, which means that the most talented agents join sector one.

[30] Differently than the scarce jobs case, even if skill levels are strictly higher it is possible that sector one profits increase weakly – precisely because of the shift in demand.

Proof. Consider $T = A^1_A(\rho_2) \cap A^1_A(\rho_1)$, the set of agents who work in sector two in both matching problems. Denote the least talented of those agents –$\inf_y T$ – as $\max_\rho v^c$. Her wage depends on two factors: positively on the surplus she produces and negatively on its share received by the firm she is matched with. The first factor always increases, as she is matched with a more productive firms. The change in the second factor can be both positive (for $\max_\rho v^c = v^c(\rho_1)$) and negative (for $\max_\rho v^c = v^c(\rho_2)$). If the former is the case, however, then the increase in surplus received by her firm:

$$\Delta_\rho r^2(P^2(v^c(\rho_1)) = \int_{v^c(\rho_2)}^{v^c(\rho_1)} P^2_v(r,\rho_2)\pi^2_h(r, P^2(r,\rho_2)\mathrm{d}r + \Delta_\rho C^2_r,$$

is always less than the increase in the surplus she produces (by Lemma 1.3):

$$\begin{aligned}\pi^2(v^c(\rho_1), P^2(v^c(\rho_1),\rho_2)) - \pi^2(v^c(\rho_1), P^2(\rho_1))& \\ = \int_{v^c(\rho_2)}^{v^c(\rho_1)} P^2_v(r,\rho_2)\pi^2_h(v^c(\rho_1), P^2(r,\rho_2)\mathrm{d}r.&\end{aligned}$$

Thus, $w^2(\max_\rho v^c, \rho_2) - w^2(\max_\rho v^c, \rho_1) \geq 0$ and by inspection of Equation (1.20) we have that $w^2(v'', \rho_2) - w^2(v'', \rho_1) \geq w^2(v', \rho_2) - w^2(v', \rho_1)$, for any $v'' > v' \geq \max_\rho v^c$. It follows that wages increase for all $v \in T$. This and revealed preference imply that all agents who used to work in sector two are better off[31]. As the top wages increase by more in sector one (by the same reasoning as in the proof of Proposition 1.5) it follows that wages increase for most talented sector one workers. □

Thus, again, a change in the marginal distribution of X influences wages in both sectors. In sector two, the effects of an increase in talent supply dominate the fact that some sector two firms leave the market. In sector one, for the

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[31] This is trivial if $v^c(\rho_2) \leq v^c(\rho_1)$. If $v^c(\rho_2) \geq v^c(\rho_1)$ then the agents with $v \in [v^c(\rho_1), v^c(\rho_2))$ will move to sector one; but as the lowest wages are the same in both sectors, they earn more than $w^2(v^c(\rho_2), \rho_2)$, which in turn is greater than their old wage.

most talented workers, the increase in talent demand and greater dissimilarity of workers dominate the increase in talent supply. However, for the least talented workers the end effect is ambiguous. The effects on wage inequality in both sectors are ambiguous, as the changes in talent demand can, under certain conditions, be equality-enhancing.

In order to establish those conditions – and also to show that it is indeed the increase in demand for talent that complicates the analysis – in Sections 1.4.3 and 1.4.3 I will isolate the effects of greater vertical differentiation and of higher demand, by studying isolated spreads and isolated increases in surplus[32].

Isolated Spread

In this section I will focus on a special case of surplus increasing real spread, in an attempt to isolate the effects of the spread from the effects of an increase in surplus. This will allow me to pin down the changes in wage inequality, as well as to demonstrate clearly why the direction of change in wages for least talented agents is ambiguous.

Definition 1.14. The new distribution of X is a *strict isolated spread* of the old distribution of X if it is a strict real spread and $F_X^{-1}(u^c(\rho_1), \rho_2) = F_X^{-1}(u^c(\rho_1), \rho_1)$.

In order to simplify exposition, I assume that the surplus function in sector two is strictly supermodular and that jobs are abundant in each sector ($R^1, R^2 \geq 1$); all results can be easily extended to the more general case.

Proposition 1.10. If jobs are abundant in each sector and the surplus function in sector two is strictly supermodular, then an isolated, strict real spread of the marginal distribution of X implies that (i) in sector two, wages strictly increase for all agents and the higher the talent the greater the increase; (ii) in sector

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[32] For the standard case of multiplicative sector one surplus, any surplus increasing real spread can be partitioned into isolated spread and isolated increase in surplus.

one, wages strictly increase for a positive mass of the most talented agents and, if the critical ability remains constant, strictly decrease for a positive mass of least talented agents; and (iii) in both sectors the gap between highest and lowest wages widens strictly.

Proof. Abundance of jobs in each sector implies that the least talented, matched agents earn the entire surplus (by Lemma 1.1); and as their wages have to be equal across sectors (by Lemma 1.2) we have that:

$$\pi^1(u^c(\rho_k), P^1(\rho_k)) = \pi^2(v^c(\rho_k), P^2(\rho_k)), \qquad \text{for any } k = 1, 2. \tag{1.23}$$

Given that sector one expands and sector two shrinks, this can hold only if $v^c(\rho_2) \leq v^c(\rho_1)$ and $u^c(\rho_2) \geq u^c(\rho_1)$, as with scarce jobs. Moreover, note that if the change in either $u^c(\rho)$ or $v^c(\rho)$ is strict, then this equation can hold only if the change in masses is strict as well. If both $u^c(\rho)$ and $v^c(\rho)$ remain unchanged, then the strict fall in the quality in $u^c(\rho_1)$'s match (by Theorem 1.2 in Appendix 1.A) also implies that the change in mass is strict. Thus, as expected, but contrary to the scarce jobs case, sector one will strictly expand and sector two will strictly shrink.

The analysis for sector two is exactly the same as in the proof of Proposition 1.9. The only difference is that the improvement in wages for $v > v^c(\rho_1)$ is strict, as the surplus function is supermodular and isolated, strict spread implies that $P^2(\rho_2) > P^2(\rho_1)$. This implies that the increase in top sector one wages is strict as well (see Equation (1.21)). The direction of change is still ambiguous for bottom wages in general, but not if $u^c(\rho_1) = u^c(\rho_2)$, in which case the wage of the least talented worker depends only on the surplus she produces: $\pi^1(u^c(\rho_1), P^1(\rho_1))$. As P^1 strictly decreases, so will $u^c(\rho_1)$'s wage.

It follows immediately from Equation (1.20) and Proposition 1.7 that sector two wages raise strictly more for more talented agents. Additionally, the wage of the previously worst agent will increase by more than the lowest wage (as $\Delta_\rho v^c(\rho) \leq 0$): these two facts imply that top sector two wages will increase more than the lowest wage – and as the increase in top sector one wages is even greater (by Equation (1.21)), which completes the proof of (iii). All results that refer to a strictly positive mass of agents follow from the continuity of wage functions. $\square$

Comparing Proposition 1.10 with Proposition 1.5, it becomes clear that as we isolate the effects of the spread from the effects of the improvement in levels, the results become very similar to the scarce jobs case. The only difference is that now the least talented sector one agents can be better off, if the critical ability level increases and the improvement in the skill level for the new critical ability is high enough[33]. The reason is that isolated spread minimises the increase in talent demand, but does not necessarily shut it down completely.

Isolated Improvement

In this section, I will isolate the effect of an increase in surplus from the effect of an increase in real spread. This will allow me to pin down the change in wages of the least talented agents, as well as demonstrate why the change in wage inequality will be ambiguous in each sector.

Definition 1.15. The new distribution of X is an *isolated, strict improvement* of the old distribution of X, if the old and new distributions are real spreads of each other and $x_l(\rho_2) > x_l(\rho_1)$.

An isolated, strict improvement holds real spread constant whilst increasing the skill of all agents: and thus also of agents with talent $u^c(\rho_1)$. Therefore, it is a special case of a strictly increasing real spread. This implies that an isolated, strict improvement deteriorates sector one match quality and improves sector

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[33] There are three factors that influence her wage: the worse match, the higher skill level and the fact that now she earns the entire surplus. The first effect dominates the third one, but not the second one.

two match quality. However, neither of these changes is necessarily strict. An isolated improvement does not change how differentiated talents are, and hence it results in agents' relocation only if the demand for sector one talent changes. This happens only if some additional sector one firms become competitive – so if some of them were not matched before ($M^1(\rho_1) < R^1$) and the increase in skill is high enough to make them more productive than the previously least productive sector two firms ($\pi^1(u^c(\rho_1), P^1(\rho_1), \rho_2) > \pi^2(v^c(\rho_1), P^2(\rho_1))$).

As usual, to simplify exposition I focus on cases where $R^1, R^2 \geq 1$, as well as assume that the surplus function in sector two is strictly supermodular. The former implies that isolated, strict improvements do cause some additional sector one firms to enter the market.

Proposition 1.11. If jobs are abundant in each sector, then an isolated, strict improvement results in a strict increase in sector one talent supply and strict fall in sector two talent supply.

The value added of this result is that with abundant jobs, an improvement in total surplus can cause some agents to relocate even if marginal surplus remains unchanged.

Proposition 1.12. If wages are abundant in each sector, then an isolated, strict improvement in the marginal distribution of X (i) strictly increases wages in both sectors, for all talent levels (ii) decreases the critical ability level in sector one and increases in sector two, with at least one of these changes being strict.

Proof. (ii) As marginal surplus is unchanged, any potential differences in wage inequality are driven entirely by reallocation. Note that Equation 1.10 implies that the critical abilities need to move in opposite directions. Suppose that $u^c(\rho_2) \geq u^c(\rho_1)$ and $v^c(\rho_1) \geq v^c(\rho_2)$. Then, by the same reasoning as in the proof of Proposition 1.10, the difference between $w^2(v^*)$ and C^2 would increase;

and the inverse of this argument implies that the difference between $w^1(u^*)$ and C^1 would decrease. As, $C^1 = C^2$, this implies that $w^1(u^*)$ increases by less than $w^2(v^*)$. This, however, contradicts the fact that more top workers join sector one (specifically Equation (1.21)). Therefore, it has to be the case that $\Delta_\rho u^c(\rho) \leq 0$ and $\Delta_\rho v^c(\rho) \geq 0$, with at least one inequality holding strictly.

(i) The increases in sector two and top sector one wages will be strict, as the increase in v^c implies that $v^c(\rho_2)$ will either produce strictly higher surplus (if $v^c(\rho_2) = v^c(\rho_1)$, by Proposition 1.11) or receive strictly more of it (if $v^c(\rho_2) > v^c(\rho_1)$). Moreover, for isolated, strict improvements in the distribution of X, it has to be the case that all sector one wages increase and thus all agents are better off. To see this, note that the difference between wages of any two agents decreases, as the spread of surplus is constant and all agents are matched with less productive firms (Equation 1.20)). Thus, we have that $\Delta_\rho w^1(u,\rho) > \Delta_\rho w^1(u^*(\rho_1),\rho) > 0$ for all $u \geq u^c(\rho_1)$. □

Unsurprisingly, in sector one the direct increase in demand for talent dominates the general equilibrium effect of increased talent supply. In sector two, there is only the general equilibrium effect of decreased supply present, which results in higher wages. Thus, all agents benefit from an increase in skill levels that does not affect spread.

At first glance, an isolated, strict improvement of X seems to create a contradiction as far as wage inequality is concerned. On one hand, worse (better) matches for sector one (two) agents, imply that wages of highly skilled agents will raise less (more); on the other hand, the difference between top and lowest wage cannot fall in sector one if it increases in sector two. This not-quite-contradiction is caused by the fact that although all agents are better off in absolute terms, the relative position in society falls for some of them. In particular, the agents who used to earn lowest wages in sector one, will not do so any more: and the fact that they will gain more than the top agents, does not mean that the lowest wage will increase more than the top ones.

Population-wise, the direction of change in wage inequality is ambiguous. To simplify exposition, I show why is this the case for multiplicative sector one surplus. Note that for this surplus function any isolated, strict improvement of X is equivalent to a shift of X by some positive constant.

Proposition 1.13. Consider some random variable Q with upper lower bound equal to zero. Suppose that jobs are abundant and, in sector one: (i) firms' productivity is strictly positive, (ii) surplus is multiplicative ($\Pi^1(x, z) = xz$) and (iii) the skill distribution is a righ-shift of Q ($X(\rho) = Q + \rho$, where $\rho \geq 0$). Then there exists a unique inequality-maximising right-shift of Q, denoted as $X(\rho_c)$. Moreover, the gap between the globally highest and lowest wage increases strictly for $\rho \in [0, \rho^c]$ and decreases strictly for $\rho \in [\rho^c, \frac{\Pi^2(y_h, z_h^2)}{z_l^1}]$.

Proof. First, note that for any $\rho \in [0, r_h]$, where $r_h = \frac{\Pi^2(y_h, z_h^2)}{z_l^1}$, the random variable $X(\rho) = Q + \rho$ results in a matching problem that meets conditions (a)-(d) from Section 1.2. Moreover, for $\rho = 0$ we have that $\Pi^1(x_l, z_1) \leq \Pi^2(y_l, z_2)$ for any z_1, z_2 and thus $u^c(0) \geq 0$. It follows trivially from Proposition 1.11 that there has to exist a unique $\rho_c \geq 0$ such that $u^c = v^c = 0$ and that for any $\rho \leq \rho_c$, $v^c = 0$ and $u^c \geq 0$, whereas for any $\rho \geq \rho_c$, $v^c > 0$ and $u^c = 0$.

Take any $0 \leq \rho_1 < \rho_2 \leq \rho^c$: then as the critical ability in sector two remains constant, wage dispersion in that sector has to be greater for ρ^c than ρ_2, which implies that the same is true for sector one. And similarly, for any $\rho_h \geq \rho_2 > \rho_1 \geq \rho^c$ we have that u^c remains constant and thus wage dispersion in sector one is lower for ρ_2 than ρ_1, which implies that the same is true for sector two. This proves the last statement. The first statement follows from that and the fact that for $\rho > \rho_h$ all agents work in sector one and thus wage inequality is constant. □

To understand this result, note that as the vertical structure of the market is constant – as spread is unchanged – the only way top firms can face more competition for top talent is from firms in another industry. As sector one skills improve from very low levels, more sector one firms start competing for high talent. However, as sector one technology keeps improving, eventually it overtakes sector two and some sector two firms cease to compete for top talent, which again increases competition among workers and decreases the top workers' premium.

1.5 Changes in the Distribution of Productivity

In this section, I change the distribution of productivity, whilst keeping the distribution of skills constant. Specifically, I will look at the effects of a first order stochastic improvement in productivity in sector one. This is interpreted as a technological improvement: the most natural reason for an increase in all firms' productivity is the introduction of a more efficient technology.

As in Section 1.4, I consider two matching problems that meet all conditions from Section 1.2, including condition (d). This time the two matching problems differ only in the distribution of sector one productivity, with the new distribution first order stochastically dominating the old one. The notion of first order stochastic dominance is well-known; in order to sharpen some of the following results I additionally define the notion of strict first order stochastic dominance[34].

Definition 1.16. A distribution $H_{Z^1}(z, \rho_2)$ *strictly first order stochastically dominates* distribution $H_{Z^2}(z, \rho_1)$ iff $H_{Z^1}^{-1}(h, \rho_2) > H_{Z^1}^{-1}(h, \rho_1)$ for all $0 \geq h \geq 1$.

Crucially, keeping the surplus function and skill distribution constant, a technological improvement is equivalent to a spread of surplus for all firms – as $\Pi^1(\bullet)$ is supermodular, the difference in surplus produced by agents of any skills increases in the productivity z. Moreover, as surplus increases in productivity,

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[34] Strict first order stochastic dominance requires a strict increase in both the upper lower and the lower upper bounds of the productivity distribution.

technological improvement results in higher surplus.

Corollary 1.4. Suppose the surplus function in sector one is (strictly) supermodular and only productivity distribution changed. Then the following statements are equivalent: (i) $\pi^1_u(u,h,\rho_2) \geq (>) \pi^1_u(u,h,\rho_1)$ and $\pi^1(u,h,\rho_2) \geq (>) \pi^1(u,h,\rho_1)$ for all $(u,h) \in [0,1]^2$ and (ii) the distribution of sector one productivity has (strictly) improved in first order stochastic sense.

Therefore, yet again, a change in the original formulation can be expressed in terms of a change in the copula formulation. Note that both a technological improvement and a surplus increasing real spread of skills are equivalent to an increase in the marginal surplus of talent. This is precisely what Theorem 1.2 in Appendix 1.B requires, and thus the effects of a technological improvement on match talent supply is the same as the effects of more spread out skills.

Proposition 1.14. A first order stochastic improvement in sector one firms' productivity result in an increase in sector one talent supply and a fall in sector two talent supply. If the real spread is strict, then the changes in talent supply are strict as well.

A technological improvement increases talent demand and differentiation, and thus its impact on wages is the same as that of a surplus increasing real spread of the skill distribution. Therefore, both for scarce and abundant jobs, all sector two wages strictly increase – as well as top sector one wages. The effect of technological improvement on lowest sector one wages is ambiguous; however, for scarce jobs, wages of the least talented agents fall for sure. Somewhat surprisingly, this means that an improvement in sector one technology is good for all sector two workers, but can be bad for the weakest sector one agents. Intuitively, with scarce jobs, none of the positive effect of improved technology on the weakest matches is captured by agents; and on top of that, they suffer from being matched with worse firms than before.

Corollary 1.5. If sector one firms' productivity improves in first order stochastic sense, then wages increase for any talent level in sector two and for the most talented sector one workers. If, further, jobs are scarce and the improvement strict, then (iii) wages decrease strictly for a positive mass of least talented sector one workers; (iv) sector two wages increase by strictly more for agents of higher talent and (v) the gap between top and bottom sector one wages widens strictly.

As an improvement in technology increases both the marginal surplus of talent and the total surplus, the effect on wage inequality is ambiguous in general. The intuition is the same as for surplus increasing spreads: as firms in sector one become more productive, they start competing for top workers and thus increase wage inequality; however, once they become much more productive than sector two firms, the latter cannot compete for top workers any more, which decreases their market power and wage inequality.

This logic, however, does not work if jobs are scarce – in which case wage inequality is driven only by talent differentiation and supply. Therefore, a technological improvement raises wage inequality in both sectors. This means that an increase in wage inequality in, say, non-financial sector, could be driven by technological improvement that happens in the financial sector.

A technological improvement has an unambiguous effect only on sector two profits and total surplus. In the former case, the fall in talent supply decreases profits (Equation (1.19)). Total surplus increases, for the usual reasons: fixing the assignment, total surplus increases and moving to the new stable assignment improves it further. The impact of technological improvement on sector one profits is ambiguous, as it depends also on whether firms become more or less similar. If productivity improves but becomes less spread out, firms can produce more, but the bargaining power of the more productive ones deteriorates. In the extreme case of abundant jobs and identical firms, profits are zero: thus, even if the change from the original distribution to the common productivity constitutes an improvement, it still results in a decrease of profits for all sector one firms.

1.6 Conclusions

This paper has developed a tractable, two-sector matching model with two skill dimensions, each used exclusively by one of the sectors. The main insight is that the general equilibrium effects of supply and demand side changes depend crucially on whether they make talent more or less vertically differentiated. The more dissimilar workers are, the fewer substitutes they have, which attracts additional talent from the other sector. Those general equilibrium effects on talent supply can have striking implications for wages in both sectors. For example, if jobs are scarce, an increase in sector one workers dissimilarity increases wages in sector two for everyone, but only for top workers in sector one – with the least talented sector one workers being worse off. This results also in an increase in wage inequality in both sectors. Note that workers can become more dissimilar in response to both supply and demand side changes.

I have also developed a novel, direct characterisation of stable matchings in multi-sector models. This characterisation, which relies on fixed-point rather than optimal transport theory is the basis of my comparative statics analysis. However, this paper does not exploit all of its advantages. In particular, this approach can also handle settings in which the surplus produced by any individual match depends on the assignment of agents to sectors – so settings with inter-sector externalities. I use this in Chapter 2.

Appendix

1.A Stable Matchings and Assignments

Proof of Proposition 1.1. I will prove this Proposition for sector one – the proof for sector two is analogous[35]. First of all, note that any differentiable wage function $w^1(u)$ can be written as $w^1(u) = w^1(u) - w^1(u^c) + w^1(u^c) = \int_{u^c}^{u} w_u(s)\mathrm{d}s + w^1(u^c)$. Take an arbitrary stable within-sector matching function $\zeta_1(u)$. Consider: two agents with ability u and $u + \epsilon$ and two firms with productivities $h = \zeta_1(u)$ and $h = \zeta_1(u + \epsilon)$. Stability implies:

$$w^1(u+\epsilon) + r^1(\zeta_1(u)) \geq \pi^1(u+\epsilon, \zeta_1(u)) \tag{1.24}$$

$$w^1(u) + r^1(\zeta_1(u+\epsilon)) \geq \pi^1(u, \zeta_1(u+\epsilon)) \tag{1.25}$$

$$w^1(u) + r^1(\zeta_1(u)) = \pi^1(u, \zeta_1(u)) \tag{1.26}$$

$$w^1(u+\epsilon) + r^1(\zeta_1(u+\epsilon)) = \pi^1(u+\epsilon, \zeta_1(u+\epsilon)). \tag{1.27}$$

By subtracting (1.26) from (1.24) and (1.27) from (1.25) and then adding (1.24) to (1.25) and rearranging we arrive at:

$$\pi^1(u+\epsilon, \zeta_1(u+\epsilon)) - \pi^1(u, \zeta_1(u+\epsilon)) \geq \pi^1(u+\epsilon, \zeta_1(u)) - \pi^1(u, \zeta_1(u))$$

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[35] For similar proofs of PAM's stability see Becker (1973) or Hopkins (2012). For an alternative approach, see Sattinger (1979).

which can be rewritten as

$$\int_{u}^{u+\epsilon}\int_{\zeta_1(u)}^{\zeta_1(u+\epsilon)} \pi_{uh}(s,r)\mathrm{d}r\mathrm{d}s \geq 0. \tag{1.28}$$

Consider any u^m, such that $\pi_{uh}(u^m, \zeta_1(u^m)) > 0$. As π_{uh} is a derivative of both π_u and π_h, the implications of the mean value theorem apply to π_{uh} wrt to both u and h – and therefore, there have to exist such $u^h > u^l$ and $h^h > h^l$ that $(u^m, \zeta_1(u^m) \in [u^l, u^h] \times [h^l, h^h]$ and $\pi^1_{uh}(u,h) > 0$ for all $(u,h) \in [u^l, u^h] \times [h^l, h^h]$. Note that all firms with $h > \zeta_1(u^m)$ have to be matched[36]. Moreover, by Inequality (1.28) it follows that for any $u > u^m$, $\zeta_1(u) \geq \zeta_1(u^m)$ and for any $u < u^m$, $\zeta_1(u^d) \leq \zeta_1(u^m)$, so all firms with $h > \zeta(u^m)$ have to matched with agents with $u \geq u^m$. Thus, measure consistency implies that $M^1(1 - G^1(u^m)) = R^1(1 - \zeta_1(u^m))$, which after some rearranging results in $\zeta_1(u^m) = P^1(u^m)$, as required.

Define $\Xi := \{u : \pi_{uh}(u, \zeta_1(u)) > 0\}$ and $u^y = \inf(\Xi)$. I will argue that for any $u \leq u^y$, $\zeta_1(u) \leq P^1(u^y)$. Note that it has to be the case that for any $u \in \Xi, \zeta_1(u) = P^1(u)$. It also follows from our previous considerations that for any $u^m \in \Xi$ and $u^s < u^m$ it has to be the case that $\zeta_1(u^s) \leq \zeta_1(u^m)$. Now suppose for some $u^s \notin \Xi, \zeta_1(u^s) > \zeta_1(u^y)$ – then we could always find some $u^m \in \Xi$ arbitrarily close to u^y such that $P^1(u^l) < \zeta_1(u^s)$; contradiction. Let's invoke the additional single-crossing condition and prove the second part of the proposition. Single-crossing of π_{uh} implies that any $u > u^y$ belongs to Ξ; and, trivially, for any $(u,h) \leq (u^y, P^1(u^y))$ it has to be that $\pi_{uh}(u,h) = 0$.

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[36] Suppose not and there exists an unmatched $h' > \zeta(u^m)$. Note that:

$$\pi^1(u^m, h') - \pi^1(u^m, \zeta_1(u^m)) = \int_{\zeta_1(u^m)}^{h'} \pi^1_h(u^m, s)\mathrm{d}s = \int_{\zeta_1(u^m)}^{h'}\int_0^{u^m} \pi^1_{uh}(r,s)\mathrm{d}r + \pi^1_h(0,s)\mathrm{d}s > 0.$$

Then for any feasible payoff scheme we have $w^1(u^m) + r^1(h') = \pi^1(u^m, \zeta(u^m)) - r^1(\zeta_1(u^m)) < \pi^1(u^m, h')$, which contradicts stability.

Now subtract (1.26) from (1.24), (1.27) and (1.25) to get:

$$w^1(u+\epsilon) - w^1(u) \geq \pi^1(u+\epsilon, \zeta_1(u)) - \pi^1(u, \zeta_1(u)) \tag{1.29}$$

$$r^1(\zeta_1(u+\epsilon)) - r^1(\zeta_1(u)) \geq \pi^1(u, \zeta_1(u+\epsilon)) - \pi^1(u, \zeta_1(u)) \tag{1.30}$$

$$w^1(u+\epsilon) + r^1(\zeta_1(u+\epsilon)) - w^1(u) + r^1(\zeta_1(u)) = \pi^1(u+\epsilon, \zeta_1(u+\epsilon)) - \pi^1(u, \zeta_1(u)). \tag{1.31}$$

Note that for any $u > u^y$, $\zeta_1(u) = P^1(u)$. Dividing (1.29), (1.30) and (1.31) by ϵ and taking the limit, we get that the RHS's are, respectively: $\pi_u(u, P^1(u))$, $\pi_h(u, P^1(u))S_u(u))$ and $\pi_h(u, P^1(u))S_u(u)) + \pi_u(u, P^1(u)$. This implies that the two inequalities have to hold with equality and thus we have that for $u > u^y$, $\lim_{\epsilon\to\infty} \frac{w^1(u+\epsilon)-w^1(u)}{\epsilon} = \pi_u(u, P^1(u))$, which in turn implies that $w^1(\cdot)$ is differentiable and $w^1_u(u) = \pi_u(u, P^1(u))$.

Take some arbitrary $u^x < u^y$ and note we can always find some ϵ, such that $u^x + \epsilon < u^y$. It follows that for such ϵ, Inequality (1.28) holds with equality, which we can use to write (1.30) as:

$$r^1(\zeta_1(u+\epsilon)) - r^1(\zeta_1(u)) \geq \pi^1(u^x+\epsilon, \zeta_1(u^x+\epsilon)) - \pi^1(u^x, \zeta_1(u^x+\epsilon)).$$

Let us also rewrite (1.31) as

$$\begin{aligned} &w^1(u+\epsilon) + r^1(\zeta_1(u+\epsilon)) - w^1(u) + r^1(\zeta_1(u)) \\ &= \pi^1(u^x+\epsilon, \zeta_1(u^x+\epsilon)) - \pi^1(u^x, \zeta_1(u^x+\epsilon)) + \pi^1(u+\epsilon, \zeta_1(u)) - \pi^1(u, \zeta_1(u)), \end{aligned}$$

which implies that (1.29) and (1.30) have to hold with equality. By dividing the RHS of (1.29) by ϵ and taking the limit we get $\pi_u(u, \zeta_1(u))$ – which implies that for $u < u^y$, $w^1(u)$ is differentiable and $\pi_u(u, \zeta_1(u)) = w_u(u)$.

Summing up, we have shown so far that for $u < u^y$ and for $u > u^y$, $w_u =$

$\pi_u(u, \zeta_1(u))$, which means – as $u = u^y$ is of measure 0 – that:

$$w^1(u) = \int_{u^c}^{u} \pi_u(s, \zeta_1(s))\mathrm{d}s + w^1(u^c).$$

This means that for our proposition to be correct we require:

$$\int_{u^c}^{u} \pi_u(s, \zeta_1(s))\mathrm{d}s - \int_{u^c}^{u} \pi_u(s, P^1(s))\mathrm{d}s = 0,$$

which can be written as:

$$\int_{u^c}^{u} \int_{\zeta_1(s)}^{P^1(s)} \pi_{uh}(s, r)\mathrm{d}r\mathrm{d}s = 0.$$

And this has to be the case, as for $u > u^m, \zeta_1(u) = P^1(u)$ and for $u \leq u^m$ both $\zeta_1(u) \wedge P^1(u) \leq P^1(u^m)$, which implies that for any h lying between $\zeta_1(u)$ and $P^1(u)$, $\pi_{uh}(u, h) = 0$. Of course, $w^1(u^c)$ is simply the constant C^1; it can't be less than zero, as then agents would be better off unmatched; it can't be greater than $\pi^1(u^c, \frac{R^1-M^1}{R^1})$, because of measure consistency, wage feasibility and the fact that $\pi^1(u, \cdot)$ is non-decreasing. $\square$

Proof of Lemma 1.1. (i) Consider sector one and suppose that the firm with $h = \zeta_1(u^c)$ makes a positive profit and thus $r^1(\zeta_1(u^c)) > 0$. But as $R^1 > M^1$ we can always find an unmatched firm with h_2, such that $\pi^1(u^c, \zeta_1(u^c)) - \pi^1(u^c, h_2) < r^1(\zeta_1(u^c))$. We can then use $\pi^1(u^c, \zeta_1(u^c)) = w^1(u^c) + r^1(\zeta_1(u^c)$ to get $w^1(u^c) - \pi^1(u^c, h_2) = w^1(u^c) + r^1(h_2) - \pi^1(u^c, h_2) < 0$, which contradicts stability. Therefore, $r^1(\zeta_1(u^c)) = 0$, which means that $w^1(u^c) = \pi^1(u^c, \zeta_1(u^c))$. It follows from proof of Proposition 1.1 that $\pi^1(u^c, \zeta_1(u^c)) \leq \pi^1(u^c, \frac{R^1-M^1}{R^1})$. Suppose this holds strictly. Then by property (b) of the surplus function, it follows that $\zeta_1(u^c) < \frac{R^1-M^1}{R^1}$. But measure consistency implies that then there needs to exist some unmatched firm with $h' \geq \frac{R^1-M^1}{R^1}$, which trivially contradicts stability.

Therefore, $\pi^1(u^c, \zeta_1(u^c)) = \pi^1(u^c, \frac{R^1-M^1}{R^1})$. Sector 2 is analogous.

(ii) $1 - M^1 - M^2 > 0$ implies that there are some unmatched agents who earn 0. Suppose that $w^1(u^c) > 0$. But then we can always find some $u \in [0, u^c)$ such that $r^1(\zeta_1(u^c)) < \pi^1(u, \zeta_1(u^c))$ (by continuity of $\pi^1(\bullet)$), which contradicts stability. Therefore $w^1(u^c) = 0$ and the same is true for $w^1(v^c)$.

$\square$

Proof of Proposition 1.2. I start showing "only if" for sector one and then move to "if" – the proofs for sector two are analogous.

R^1 needs to be weakly greater than 1 from measure consistency. Suppose that $\pi^2(1,1) > \pi^1(0, \frac{R^1-1}{R^1})$ and $M^1 = 1$. Therefore, for any arbitrarily small $\epsilon > 0$ there has to exist a positive mass of unmatched sector one firms with $h' \in [1-\epsilon, 1]$. As density is strictly positive over the entire domain and wage and surplus functions are continuous it follows that there has to exist a positive mass of agents such that $w^2(v) + r(h', 1) < \pi^1(u, h')$, which contradicts stability.

Let's move to the "if" part. Suppose we have $\pi^2(1,1) < \pi^1(0, \frac{R^1-1}{R^1}) \wedge R^1 \geq 1$ and $M^1 < 1$. From Equation (1.4) follows that $M^2 > 0$. Hence $v^c < 1$ and there exists a positive measure agents with $v < 1$ who will join sector two. Clearly, for any such agent $w^2(v) < \pi^2(1,1)$. Also, as $M^2 > 0$ it has to be the case that there exists a positive measure of firms with $(h, 1)$ such that $\pi^1(u, h) \geq \pi^1(u, \frac{R^1-1}{R^1})$ and $r(h, 1) = 0$. Therefore for the agents and firms in question we have

$$w^2(v) + r(h,1) < \pi^2(1,1) < \pi^1(0, \frac{R^1-1}{R^1}) \leq \pi^1(u,h),$$

which contradicts stability and concludes the proof. $\square$

Proof of Lemma 1.2. This follows trivially for $R^1 + R^2 < 1$, as then $1 - M^1 - M^2 > 0$ and thus $C_1 = C_2 = 0$, by Lemma 1.1. Therefore I will focus on the cases where $R^1 + R^2 \geq 1$ and thus $C(u^c, v^c) = 0$. This implies that $\min\{u^c, v^c\} = 0$. Suppose that $u^c = 0$ and $w^1(u^c) > w^2(v^c)$; as the payoff in the second sector is continuous and condition (d) ensures $v^c < 1$, there has to exist some $\epsilon > 0$ such that for any $v \in [v^c, v^c + \epsilon]$ we have $w^1(u^c) > w^2(v)$, which means that none of these agents will join sector two – which contradicts v^c's definition.

Now, suppose that $w^1(u^c) < w^2(v^c)$; suppose further that $v^c > 0$. For any $\epsilon > 0$ the mass of agents with $(u, v) \in [0, \epsilon] \times [v^c - \epsilon, v^c]$ is strictly positive – as $C_{uv}(u, v) > 0$ for all (u, v) – and all agents in this set will be working in sector one. However, by continuity of surplus and wage functions it has to be the case that there exists a small enough ϵ that for all agents in this set we have:

$$w^1(u) + \pi^2(v^c, \frac{R^2 - M^2 + v^c}{R^2}) - w^2(v^c) < \pi^2(v, \frac{R^2 - M^2 + v^c}{R^2}),$$

which contradicts stability. Now suppose that $v^c = 0$ as well – there has to exist some $\epsilon > 0$ such that for any $u \in [0, \epsilon]$ we have $w^1(u) < w^2(v^c)$, which contradicts the definition of u^c.

The proof for the case when $v^c = 0$ is analogous. □

Proof of Theorem 1.1. I start by reducing the set of equations and inequalities (1.9)-(1.17), which will be henceforth referred to as *the original set.* Denote the total mass of matched agents, as $M = M^1 + M^2$; this allows us to write M^2 as $M - M^1$. Using this fact, differentiating Equation (1.9), dividing both sides by $\pi^1_u\Big(\psi(v), \frac{1}{R^1}\int_{u^c}^{\psi(v)} C_u(r, \psi^{-1}(r))\mathrm{d}t\Big)$, using the relation from Remark (1.2) and then integrating from v^c to v (and remembering that $\psi(v^c) = u^c$) we get:

$$\psi(v) = u^c + \int_{v^c}^{v} \frac{\pi^2_v\Big(t, \frac{1}{R^2}\big(R^2 - M^2 + \int_{v^c}^{t} C_v(\psi(r), r)\mathrm{d}r\big)\Big)}{\pi^1_u\Big(\psi(t), \frac{1}{R^1}\big(R^1 - M + M^2 + C(\psi(t), t) - \int_{v^c}^{t} C_v(\psi(r), r)\mathrm{d}r\big)\Big)}\mathrm{d}t.$$

This equation still depends on $\psi(\cdot), u^c, v^c$ and indirectly on v^*, so we have not really reduced the system yet. We will get rid of v^* by extending the functions

$C(\bullet)$, $\pi^1(\bullet)$ and $\pi^2(\bullet)$ in a way that allows us to define an extended function $\psi^e(\cdot)$, which uniquely determines $\psi(\cdot)$. The extended functions $C^e(\bullet)$, $\pi^{1e}(\bullet)$ and $\pi^{2e}(\bullet)$ are defined as follows: (1) $C^e : [0, 1 + B] \times [0, 1] \to [0, 1]$

$$C^e(u,v) = \begin{cases} C(u,v) & \text{for } (u,v) \in [0,1] \times [0,1] \\ v & \text{for } (u,v) \in (1, 1+B] \times [0,1], \end{cases}$$

(2): $\pi^{1e}(u,h) : [0, 1 + B] \times [0, \frac{1+R^1}{R^1}] \to \mathbf{R}^+$:

$$\pi^{1e}(u,h) = \begin{cases} \pi^1(u,h) & \text{for } (u,h) \in [0,1]^2 \\ \pi^1(1,h) + (u-1)\pi^1_u(1,h) & \text{for } (u,h) \in (1,B] \times [0,1], \\ \pi^1(u,1) & \text{for } (u,h) \in [0,1] \times (1, \frac{1+R^1}{R^1}], \\ \pi^1(1,1) + (u-1)\pi^1_u(1,1) & \text{for } (u,h) \in (1,B] \times (1, \frac{1+R^1}{R^1}], \end{cases}$$

(3): $\pi^{2e}(v,h) : [0,1] \times [0, 1 + \frac{1}{R^2}] \to \mathbf{R}^+$:

$$\pi^{2e}(v,h) = \begin{cases} \pi^2(v,h) & \text{for } (u,h) \in [0,1]^2 \\ \pi^2(v,1) & \text{for } (u,h) \in [0,1] \times (1, 1+\frac{1}{R^2}], \end{cases}$$

where $B = \frac{\max \pi^2_v}{\min \pi^1_u}$. The idea behind these extensions is to get functions that will be defined also for $\psi^e(v) > 1$ and such that $C^e(\cdot, v)$, $C^e_v(\cdot, v)$, $\pi^{1e}_u(\cdot,\cdot)$ and $\pi^{2e}_v(v,\cdot)$ are Lipschitz continuous[37]; denote their Lipschitz-constants as L^1, L^2 ,L^3, L^4 and L^5 respectively.

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[37] We will do this in detail for $C^e_v(u,v)$ – the reasoning for the other two is analogous. $C^e_v(u,v) : [0, 1+B] \times [0,1] \to [0,1]$:

$$C^e_v(u,v) = \begin{cases} C_v(u,v) & \text{for } (u,v) \in [0,1] \times [0,1] \\ 1 & \text{for } (u,v) \in (1, 1+B] \times [0,1], \end{cases}$$

is clearly continuous in u. It is equally easy to see that the function $C^e_v(\cdot, v)$ is differentiable almost everywhere and its derivative is Lebesque integrable. It is also the case that for any

Now we can define the extended function $\psi^e(v) : [v^c, 1] \in [u^c, 1+B]$:

$$\psi^e(v) = u^c + \int_{v^c}^{v} \frac{\pi_v^{2e}\Big(t, \frac{R^2 - M^2 + \int_{v^c}^{t} C_v^e(\psi(r),r)\mathrm{d}r}{R^2}\Big)}{\pi_u^{1e}\Big(\psi^e(t), \frac{R^1 - M + M^2 + C^e(\psi^e(t),t) - \int_{v^c}^{t} C_v^e(\psi^e(r),r)\mathrm{d}r}{R^1}\Big)}\mathrm{d}t, \tag{1.32}$$

which together with:

$$M = \min\{R^1 + R^2, 1\} \tag{1.33}$$

$$1 - M = C^e(u^c, v^c), \tag{1.34}$$

$$M^2 = \int_{v^c}^{1} C_v^e(\psi(r), r)\mathrm{d}r, \tag{1.35}$$

$$v^* = \sup\{v \in [v^c, 1] : \psi^e(v) \leq 1\}, \tag{1.36}$$

$$u^* = \psi^e(v^*) \tag{1.37}$$

$$M^2 \in [\max\{0, M - R^1\}, \min\{1, R^2\}] \tag{1.38}$$

and (1.16) to (1.17) rewritten in terms of the extended functions, constitute the *modified set* of equations.

Lemma 1.4. The relation between the original and the modified set is as follows: (a) if ψ^e solves the modified set then its restriction to $[v^c, \sup\{v \in [v^c, 1] : \psi^e(v) \leq 1\}]$ solves the original one and (b) if a function $\psi(v) : [v^c, v^*] \to [u^c, u^*]$ solves the original set then we can always find its extension $\psi^e(v) : [v^c, 1] \to [u^c, 1+B]$ that solves the modified one.

Proof. Note that if $v^* = \sup\{v \in [v^c, 1] : \psi^e(v) \leq 1\}$, then $\max\{\psi^e(v^*), v^*\} = 1$,

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$(u, v) \in (1, 1+B] \times [0, 1]$ we have:

$$C_v^e(a, v) + \int_a^1 C_{uv}^e(r, v)\mathrm{d}r + \int_1^u 0\mathrm{d}r = 1,$$

which means that $C_v^e(\cdot, v)$ is absolutely continuous. Moreover, as $C^e(\bullet)$ is twice continuously differentiable and any continuous function defined on a compact set is bounded it follows that $C_v^e(\cdot, v)$ is essentially bounded; and a differentiable almost everywhere, absolutely continuous function with an essentially bounded derivative is Lipschitz-continuous.

as required. For $v > v^*$, we have $\psi^e(v) > 1$ and thus $C_v^e(\psi(v), v) = 1$, which shows that Equation (1.35) is equivalent to Equation (1.12). For $v \leq v^*$ we have that

$$\int_{v^c}^{v} C_v^e(\psi(r), r)\mathrm{d}r \leq M^2,$$

$$C(\psi^e(v), v) - \int_{v^c}^{v} C_v^e(\psi(r), r)\mathrm{d}r \leq M - M^2,$$

which means that the original and extended $C(\bullet)$, $C_v(\bullet)$, $\pi_u^1(\bullet)$ and $\pi_v^2(\bullet)$ are identical; and thus if (1.32) is met, (1.9) has to be met too. The equivalence of Equations (1.37) and (1.12) follows from the relation in Remark 1.2 and its equivalent for $\psi^{-1}(\cdot)$. The equivalence of all other equations is trivial.

Claim (b) is trivial for $\psi(v^*) = 1$, as then ψ and ψ^e coincide. For $\psi(v^*) < 1$ I claim that

$$\psi^e(v) = \begin{cases} \psi(v) & \text{for } v \in [v^c, v^*] \\ 1 + \int_{v^*}^{v} \frac{\pi_v^2(t, \frac{1}{R^2}(R^2+1-v))}{\pi_u^1(1,1)}\mathrm{d}t & \text{for } v > v^* \end{cases}$$

solves (1.32)-(1.38) (and ψ is clearly its restriction for $v \in [v^c, v^*]$). Equation (1.32) is trivially met for $v \leq v^*$ and to see that it is met for $v > v^*$ it suffices to substitute for π^{1e}, π^{2e}, $C^e(u, v)$, $C_v^e(u, v)$ and use the fact that $\psi^e(v^*) = u^* = 1$. All the other equations are met trivially. □

Thus, if the solution to the modified set exists and is unique, then the solution to the original set also exists and is unique. Now we will focus on showing that the former is indeed the case. Define the set:

$$K = \{d \in C[0, 1] : |d(v) - 1| \leq 1 + B\},$$

where $C[0, 1]$ is the set of all continuous functions that map from $[0, 1]$. The constant function $d(v) = 1$ lies in K and hence the set is non-empty. Define a norm, $||\cdot||_\lambda$ on $C[0,1]$:

$$||h||_\lambda = \sup\nolimits_{[0,1]} e^{-\lambda v}|h(v)|,$$

where λ is some weakly positive number. K is a complete metric space for this norm.[38]

Endow the sets $[0,1]^2$ and $[\max\{0, M-R^1\}, \min\{1, R^2\}]$ with the Euclidean norm and define a mapping $T : K \times [0,1]^2 \times [\max\{0, M-R^1\}, \min\{1, R^2\}] \to K$:

$$(Td)(v, v^c, u^c, M^2) = \begin{cases} u^c & \text{for } v < v^c \\ u^c + \int_{v^c}^{v} \frac{\pi_v^{2e}(t, \frac{R^2 - M^2 + \int_{v^c}^{t} C_v^e(d(r),r)}{R^2}\mathrm{d}r)}{\pi_u^{1e}(d(t), \frac{R^1 + M - M^2 + C^e(d(t),t) - \int_{v^c}^{t} C_v^e(d(r),r)\mathrm{d}r}{R^1}\mathrm{d}r)}\mathrm{d}t & \text{for } v \geq v^c. \end{cases}$$

Note that this map is well-defined, as for any $v^c \in [0,1]$ and $d \in K$:

$$\frac{R^2 - M^2 + \int_{v^c}^{t} C_v^e(d(r),r)}{R^2}\mathrm{d}r \leq \int_{v^c}^{t} \frac{R^2+1}{R^2}\mathrm{d}r \leq \frac{1}{R^2} + 1$$

$$\frac{R^1 - M + M^2 + C(d(t),t) - \int_{v^c}^{t} C_v^e(d(r),r)\mathrm{d}r}{R^1} \leq \frac{R^1 + C(d(t),t)}{R^1} \leq \frac{1}{R^1} + 1;$$

and that it is continuous in v, v^c, u^c and M^2. It is also the case that for $v \geq v^c$:

$$|[(Td)(v, v^c, u^c, M^2) - 1| \leq \int_{v^c}^{v} B\mathrm{d}t + |u^c - 1| \leq 1 + B,$$

and for $v < v^c$:

$$|[(Td)(v, v^c, u^c, M^2) - 1| \leq |u^c - 1| \leq 1 + B,$$

so indeed $T(K) \subset K$. Finally, it should be clear that for any (v^c, u^c, M^2) the restriction of any fixed point of $(Td)(\bullet)$ to $[v^c, 1]$ gives us the solution to (1.32)

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[38] If we endowed K with the sub-norm, then K would be a closed subspace of $C[0,1]$; since $C[0,1]$ is complete in the sub-norm, so is K. And it was shown by Bielecki (1956) that the $||\cdot||_\lambda$ norm is equivalent to the sup-norm for any $C[a,b]$ – and thus if K is a complete metric space for the sub-norm it is also a complete metric space for $||\cdot||_\lambda$.

and that any solution to (1.32) can be easily extended into a fixed point of $(Td)(\bullet)$. Therefore, it suffices to show that there exists such a λ that for any $(v^c, u^c, M^2) \in [0,1]^2 \times [\max\{0, M - R^1\}, \min\{1, R^2\}]$, $Td(\bullet)$ is a contraction wrt to the norm $|| \cdot ||_\lambda$ to show that (1.32) has a unique solution for any feasible (u^c, v^c, M^2).

Let us drop (v^c, u^c, M^2) from the definition of the map (remembering that we are keeping them constant) and enhance our notation by new maps: $M^2 : [v^c, 1] \times K \to [0,1], P^2 : [v^c, 1] \times K \to [0, 1 + \frac{1}{R^2}]$ and $P^1 : [0, B] \times K \to [0, 1 + \frac{1}{R^1}]$:

$$\begin{aligned}
(M^2 d)(v) &= \int_{v^c}^{v} C^e_v(d(r), r)\mathrm{d}r, \\
(P^2 d)(v) &= \frac{R^2 - M^2 + (M^1 d)(v)}{R^2}, \\
(P^1 d)(d(v)) &= \frac{R^1 - M + M^2 + C^e(d(v), v) - (M^2 d)(v)}{R^1}.
\end{aligned}$$

Take any any $t \geq v^c$ and any $d_1, d_2 \in S$ and for any map $(fd)(t)$ denote $(fd_1)(t) - (fd_2)(t)$ as $\Delta_d(fd)(t)$. Then we have:

$$\begin{aligned}
|\Delta_d(M^2 d)(t)| &= |\int_{v^c}^{t} C^e_v(d_1(r), r) - C^e_v(d_2(r), r)\mathrm{d}r \qquad (1.39)\\
&\leq \int_{v^c}^{t} |C^e_v(d_1(r), r) - C^e_v(d_2(r), r)|\mathrm{d}r \leq \int_{v^c}^{t} L_2 |d_1(r) - d_2(r)|\mathrm{d}r| \\
&= L_2 \int_{v^c}^{t} e^{\lambda r} e^{-\lambda r} |d_1(r) - d_2(r)|\mathrm{d}r \leq L_2 ||d_1 - d_2||_\lambda \int_{v^c}^{t} e^{\lambda r}\mathrm{d}r \\
&= \frac{L_2}{\lambda} ||d_1 - d_2||_\lambda (e^{\lambda t} - e^{\lambda v^c}) \leq \frac{L_2}{\lambda} ||d_1 - d_2||_\lambda e^{\lambda t},
\end{aligned}$$

which can be used to establish:

$$|\Delta_d(P^2 d)(t)| \leq \frac{L_2}{\lambda R^2} ||d_1 - d_2||_\lambda e^{\lambda t} \qquad (1.40)$$

$$|(P^1 d_1)(d_1(t)) - (P^1 d_2)(d_2(t))| = |\frac{C^e(d_1(v), v) - C^e(d_2(v), v) - \Delta_d(M^2 d)(v)}{R^1}| \qquad (1.41)$$

$$
\begin{aligned}
&\leq \frac{1}{R^1}(|C^e(d_1(v),v) - C^e(d_2(v),v)| + |\Delta_d(M^2d)(v)|) \\
&\leq \frac{L_2}{\lambda R^1}||d_1 - d_2||_\lambda e^{\lambda t} + \frac{L^1}{R^1}|d_1(t) - d_2(t)|.
\end{aligned}
$$

Denote sup$\pi_v^2(v,h) = L_6$, inf$\pi_u^1(u,h) = L_7$ and note that continuity of π_u^1 and π_v^2 and the fact that $\pi_u^1 > 0$ imply that both L^6 and L^7 are finite. Using all this, we can write, for any $v \geq v^c$ and any $d_1, d_2 \in S$:

$$
\begin{aligned}
|\Delta_d(Td)(v)| &= |\int_{v^c}^{v} \frac{\pi_v^{2e}(t,(P^2d_1)(t))}{\pi_u^{1e}(d_1(r),(P^1d_1)(d_1(t)))} - \frac{\pi_v^{2e}(t,(P^2d_2)(t))}{\pi_u^{1e}(d_2(r),(P^1d_2)(d_2(t)))}\mathrm{d}t| \\
&\leq \int_{v^c}^{v} |\frac{\pi_v^{2e}(t,(P^2d_1)(t))}{\pi_u^{1e}(d_1(r),(P^1d_1)(d_1(t)))} - \frac{\pi_v^{2e}(t,(P^2d_2)(v,))}{\pi_u^{1e}(d_1(r),(P^1d_1)(d_1(t)))} \\
&\quad + \frac{\pi_v^{2e}(t,(P^2d_2)(t))}{\pi_u^{1e}(d_1(r),(P^1d_1)(d_1(t)))} - \frac{\pi_v^{2e}(t,(P^2d_2)(t))}{\pi_u^{1e}(d_2(r),(P^1d_2)(d_2(t)))}|\mathrm{d}t \\
&\leq \int_{v^c}^{v} \frac{|\pi_v^{2e}(t,(P^2d_1)(t)) - \pi_v^{2e}(t,(P^2d_2)(t))|}{L_7} \\
&\quad + L_6(|\frac{\pi_u^{1e}(d_1(r),(P^1d_1)(d_1(t))) - \pi_u^{1e}(d_2(r),(P^1d_2)(d_2(t)))}{\pi_u^{1e}(d_1(r),(P^1d_1)(d_2(t)))\pi_u^{1e}(d_2(r),(P^1d_2)(d_2(t)))}|\mathrm{d}t \\
&\leq \int_{v^c}^{v} \frac{L_5}{L_7}|\Delta_d(P^2d)(t)| \\
&\quad + \frac{L_6}{L_7^2}[|\pi_u^{1e}(d_1(r),(P^1d_1)(d_1(t))) - \pi_u^{1e}(d_2(t),(P^1d_1)(d_1(t))|] \\
&\quad + \frac{L_6}{L_7^2}[|\pi_u^{1e}(d_2(t),(P^1d_1)(d_1(t)) - \pi_u^{1e}(d_2(r),(P^1d_2)(d_2(t)))|]\mathrm{d}t \\
&\leq \int_{v^c}^{v} \frac{L_5L_2}{\lambda L_7R^2}||d_1 - d_2||_\lambda e^{\lambda(t-v^c)} + \frac{L_3L_6}{L_7^2}|d_1(t) - d_2(t)| \\
&\quad + \frac{L_4L_6}{L_7^2}|(P^1d_1)(d_1(t)) - P^1d_2)(d_2(t))|\mathrm{d}t \\
&\leq \frac{L_5L_2}{\lambda^2L_7R^2}||d_1 - d_2||_\lambda e^{\lambda v} + \frac{L_3L_6}{\lambda L_7^2}||d_1 - d_2||_\lambda e^{\lambda v} \\
&\quad + \int_{v^c}^{v} \frac{L_4L_6}{L_7^2}(\frac{L_2}{\lambda R^1}||d_1 - d_2||_\lambda e^{\lambda(t-v^c)} + \frac{L_1}{R^1}|d_1(t) - d_2(t)|)\mathrm{d}t \\
&\leq \frac{1}{\lambda}||d_1 - d_2||_\lambda e^{\lambda v}\Big[\frac{L_5L_2}{\lambda L_7R^2} + \frac{L_3L_6}{L_7^2} + \frac{L_4L_6}{L_7^2}\Big(\frac{L_2}{\lambda R^1} + \frac{L_1}{R^1}\Big)\Big]
\end{aligned}
$$

Now, for $v < v^c$ this has to hold as well, as then $|(Td_1)(v) - T(d_2)(v)| = 0$;

therefore, for any $v \in [0,1]$ we have that:

$$\Delta_d(Td)(v)| \leq \frac{1}{\lambda}||d_1 - d_2||_\lambda e^{\lambda v}\Big[\frac{L_5 L_2}{\lambda L_7 R^2} + \frac{L_3 L_6}{L_7^2} + \frac{L_4 L_6}{L_7^2}\Big(\frac{L_2}{\lambda R^1} + \frac{L_1}{R^1}\Big)\Big].$$

Dividing both sides of that by $e^{\lambda v}$ we get:

$$e^{-\lambda v}|\Delta_d(Td)(v)| \leq \frac{1}{\lambda}||d_1 - d_2||_\lambda \Big[\frac{L_5 L_2}{\lambda L_7 R^2} + \frac{L_3 L_6}{L_7^2} + \frac{L_4 L_6}{L_7^2}\Big(\frac{L_2}{\lambda R^1} + \frac{L_1}{R^1}\Big)\Big]$$

which, by taking sup on both sides implies that:

$$||(Td_1)(t) - T(d_2)(t)||_\lambda \leq \frac{1}{\lambda}||d_1 - d_2||_\lambda \Big[\frac{L_5 L_2}{\lambda L_7 R^2} + \frac{L_3 L_6}{L_7^2} + \frac{L_4 L_6}{L_7^2}\Big(\frac{L_2}{\lambda R^1} + \frac{L_1}{R^1}\Big)\Big]. \tag{1.42}$$

Therefore, there has to exist a high enough λ for which our map $(Td)(v)$ is a contraction in the metric space $(S, ||\cdot||_\lambda)$ – which, by Banach's Fixed-Point Theorem means that $(Td)(v)$ has a unique fixed point, which in turn means that Equation (1.32) has a single solution for any given $(v^c, u^c, M^2) \in [0,1]^2 \times [\max\{0, M - R^1\}, \min\{1, R^2\}]$. Note that Equation (1.42) does not depend on (v^c, u^c, M^2) – and thus, by standard results (see e.g. Hasselblatt and Katok, 2003, p. 68) it follows that as $(Td)(v, v^c, u^c, M^2)$ is continuous in v^c, u^c and M^2 the fixed point – and thus the solution of (1.32) – is continuous in them as well.

Denote the fixed point of $(Td)(\cdot, v^c, u^c, M^2)$ as $d^*(\cdot, v^c, u^c, M^2)$ – then the following result holds:

Lemma 1.5. Keeping the other two parameters constant, $d^*(\cdot, v^c, u^c, M^2)$ is weakly decreasing in v^c and M^2 and weakly increasing in u^c for all v's. Moreover, for some v's, $d^*(\cdot, v^c, u^c, M^2)$ is strictly decreasing in v^c and M^2 (strictly increasing in u^c).

Proof. I start with the claims regarding $d(v, \cdot, u^c, M^2)$. To simplify notation, I drop u^c and M^2 from all functions, as they are fixed constant for the proof of the claims regarding v^c. Take any $v_2^c > v_1^c \in [0,1]$ and denote $d^*(v, v_2^c) - d^*(v, v_1^c)$ as $\Delta_{v^c} d^*(v, v^c)$ and:

$$\begin{aligned}
M^2(v, v^c) &= \int_{v^c}^{v} C_v(d^*(r, v^c), r)\mathrm{d}r, \\
P^2(v, v^c) &= \frac{R^2 - M^2 + M^2(v, v^c)}{R^2}, \\
P^1(d^*(v, v^c), v^c) &= \frac{R^1 - 1 + M^2 + C(d^*(v, v^c), r) - M^2(v, v^c)}{R^1}.
\end{aligned}$$

Then for any $v \geq v_2^c$ we have:

$$\begin{aligned}
\Delta_{v^c} d^*(v, v^c) &= \Delta_{v^c} d^*(v_2^c, v^c) \\
&\quad + \int_{v_2^c}^{v} \frac{\pi_v^{2e}(t, P^2(v_2^c, t))}{\pi_u^{1e}(d^*(t, v_2^c), P^1(v_2^c, d^*(t, v_2^c)))} - \frac{\pi_v^{2e}(t, P^2(v_1^c, t))}{\pi_u^{1e}(d^*(t, v_1^c), P^1(v_1^c, d^*(t, v_1^c)))}\mathrm{d}t.
\end{aligned}$$

It is trivial that for any $v \in [v_1^c, v_2^c)$ we have $\Delta_{v^c} d^*(v, v^c) < 0$, which proves the second (strict) part of this claim. Thus, we only need to show now that $\Delta_{v^c} d^*(v, v^c) \leq 0$ for all $v \in [v_2^c, 1]$. Suppose not. Then the set $\Omega^{gen} = \{v \in [v_2^c, 1] : \Delta_{v^c} d^*(v, v^c) > 0\}$ has to be non-empty. Then we have that for $v^g = \inf \Omega^{gen}$, $\Delta_{v^c} d^*(v^g, v^c) = 0$ and $\Delta_{v^c} d_v^*(v^g, v^c) > 0$. The sign of $\Delta_{v^c} d_v^*(v^g, v^c)$ depends only on the signs of[39]:

$$\pi_v^{2e}(v^g, P^2(v_2^c, v^g)) - \pi_v^{2e}(v^g, P^2(v_1^c, v^g))$$

---

[39] To see this, note that:

$$\begin{aligned}
\Delta_{v^c} d_v^*(v^g, v^c) &= \frac{\pi_v^{2e}(v^g, P^2(v_2^c, v^g))}{\pi_u^{1e}(d^*(v^g, v_2^c), P^1(v_2^c, d^*(v^g, v_2^c)))} - \frac{\pi_v^{2e}(v^g, P^2(v_1^c, v^g))}{\pi_u^{1e}(d^*(v^g, v_1^c), P^1(v_1^c, d^*(v^g, v_1^c)))} \\
= \frac{\pi_v^{2e}(v^g, P^2(v_2^c, v^g)) - \pi_v^{2e}(v^g, P^2(v_1^c, v^g))}{\pi_u^{1e}(d^*(v^g, v_2^c), P^1(v_2^c, d^*(v^g, v_2^c)))} \\
+\pi_v^{2e}(v^g, P^2(v_1^c, v^g)) &\frac{\pi_u^{1e}(d^*(v^g, v_1^c), P^1(v_1^c, d^*(v^g, v_1^c))) - \pi_u^{1e}(d^*(v^g, v_2^c), P^1(v_2^c, d^*(v^g, v_2^c)))}{\pi_u^{1e}(d^*(v^g, v_1^c), P^1(v_1^c, d^*(v^g, v_1^c)))\pi_u^{1e}(d^*(v^g, v_2^c), P^1(v_2^c, d^*(v^g, v_2^c)))}.
\end{aligned}$$

and

$$\pi_u^{1e}(d^*(v^g, v_1^c), P^1(v_1^c, d^*(v^g, v_1^c))) - \pi_u^{1e}(d^*(v^g, v_2^c), P^1(v_2^c, d^*(v^g, v_2^c))).$$

However, as $\Delta_{v^c} d^*(v^g, v^c) = 0$ and both surplus functions are weakly supermodular, these in turn depend only on the sign of $M^2(v_2^c, v^g) - M^2(v_1^c, v^g)$. As for any $v \leq v^g$ it was the case that $\Delta_{v^c} d^*(v^g, v^c) \leq 0$ and $v_2^c \geq v_1^c$, it follows that: $M^2(v_2^c, v^g) - M^2(v_1^c, v^g) \leq 0$ and thus:

$$\Delta_{v^c} d_v^*(v, v^c) \leq 0,$$

which means that Ω^{gen} has to be empty and proves our first claim.

The proof of the claim wrt M^2 is analogous[40]. As for the claim regarding u^c, note that for a change in u^c, note that $\Delta_{u^c} d^*(v^c, u^c)$ is positive. The subsequent reasoning is analogous, but with opposite signs the strict decreasingness follows from $\Delta_{u^c} d^*(v^c, u^c) < 0$ and continuity). $\square$

Everything I derived so far applies both for cases with abundant and scarce jobs. From now on, however, I will consider those cases separately.

Scarce jobs If $R^1 + R^2 < 1$, then $M = R^1 + R^2$, which reduces (1.38) to $M^2 = R^2$. Moreover, we have also that $C(u^c, v^c) = 1 - R^1 - R^2 > 0$ – as for $(u, v) > 0$, $C(\bullet)$ is strictly increasing in both parameters, its inverse with respect to both parameters exists, which allows us to define u^c as a strictly decreasing, continuous function of v^c. Define $\underline{v}$ as $u^c(\underline{v}) = 1$ and note also that as $u^c \in [0, 1]$ Equation (1.34) shrinks the range of feasible v^c's to $[\underline{v}, 1]$. All of this implies that $d^*(v, v^c, u^c, M^2)$ depends only on v and v^c and is decreasing and continuous in v^c

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[40] For $v = v^c$ we have $\Delta_M d^*(v, M^2) = 0$ and $\Delta_M d_v^*(v, M^2) < 0$. The sign of $\Delta_M d_v^*(v^g, M^2)$ depends on $M_1^2 - M_2^2 < 0$ and the difference in $M^2(v, M^2)$, which is weakly negative for the same reasons as above. Thus, $\Delta_M d_v^*(v^e, M^2) \leq 0$, which implies that $d^*(v, a, \cdot)$ will never strictly increase.

– hence, I will denote it as $d^*(v, v^c)$ from now on. Thus, the modified system of equations reduces to:

$$R^2 = \int_{v^c}^{1} C_v^e(d^*(r, v^c), r)\mathrm{d}r.$$

Let us start with existence. The RHS is continuous in v^c, as $d^*(v, v^c))$ is continuous in v^c. For $v^c = \underline{v}$, we have $d^*(v, v^c) \geq 1$ regardless of v and therefore $\int_0^1 C_v^e(d^*(r, v^c), r)\mathrm{d}r = 1$, whereas for $v^c = 1$, we have $\int_1^1 C_v^e(d^*(r, v^c), r)\mathrm{d}r = 0$; thus, a solution to (1.35) (given $R^2 \in (0, 1)$) exists. It is unique, as $d^*(v, \cdot)$ is weakly decreasing for all and strictly decreasing for some v and thus the RHS crosses R^2 only once from above. This completes the proof for $R^1 + R^2 < 1$.

Abundant jobs If $R^1 + R^2 \geq 1$, then $M = 1$ and thus $C(u^c, v^c) = 0$. This implies that either u^c or v^c has to be equal to zero: $\min\{u^c, v^c\} = 0$ – more importantly, this implies that we can't define u^c as a function of v^c, as there are many v^c's for which $u^c = 0$ will give us $C(0, v^c) = 0$. We will deal with this problem by defining the set $\Gamma^c = \{(u^c, v^c) : \min\{u^c, v^c\} = 0\}$, a new variable $a \in [-1, 1]$ and writing u^c and v^c as functions of a:

$$u^c(a) = \begin{cases} -a & \text{for } a \leq 0, \\ 0 & \text{for } a > 0, \end{cases} \qquad v^c(a) = \begin{cases} 0 & \text{for } a \leq 0, \\ a & \text{for } a > 0. \end{cases}$$

Note that for any a, $(u^c(a), v^c(a)) \in \Gamma^c$ and for any $(u^c, v^c) \in \Gamma^c$ there exists a unique a, such that $(u^c(a), v^c(a)) = (u^c, v^c)$. Thus, if there exists a unique a that solves Equation (1.35), there also exists a unique (u^c, v^c) that solves it. Moreover, clearly $v^c(a)$ is continuous and increasing, whereas $u^c(a)$ is continuous and decreasing. Therefore the function $d^*(v, a, M^2) = d^*(v, (v^c(a), u^c(a), M^2)$ is continuous and decreasing in a (and strictly decreasing for some v's). This allows us to write Equation (1.35) as:

$$M^2 = \begin{cases} \int_0^1 C_v^e(d^*(r,a,M^2),r)\mathrm{d}r & \text{for } a<0, \\ \int_a^1 C_v^e(d^*(r,a,M^2),r)\mathrm{d}r & \text{for } a \geq 0. \end{cases}$$

We will argue that given any $M^2 \in [\max\{0,1-R^1\}, \min\{1,R^2\}]$ there exists a unique $a \in [-1,1]$ that meets this equation, and that this a (denoted as $a(M^2)$) is continuous and strictly decreasing in M^2.

Let us start with existence. The RHS is continuous in a, as $d^*(v,a,M^2)$ is continuous in a. For $a=-1$, we have $\int_0^1 C_v^e(d^*(r,a,M^2),r)\mathrm{d}r = 1$, whereas for $a=1$, we have $\int_a^1 C_v^e(d^*(r,a,M^2),r)\mathrm{d}r = 0$; thus, a solution to (1.35) (given $M^2 \in [\max\{0,1-R^1\}, \min\{1,R^2\}]$) exists. It is unique, as $d^*(v,\cdot,M^2)$ is weakly decreasing for all and strictly decreasing for some v and thus the RHS crosses M^2 only once from above.

That $a(M^2)$ is continuous, follows from the fact that $d^*(v,\cdot,\cdot)$ is continuous. It is strictly decreasing in M^2, as the LHS is strictly increasing in M^2 and the RHS is weakly decreasing in M^2 and strictly decreasing in a; thus, if M^2 increases, the only way for Equation (1.35) to be met is for a to decrease. Note that as $a(M^2)$ is uniqe and a defines uniquely (u^c, v^c), there also exist some unique $u^c(M^2)$ and $v^c(M^2)$; the former is non-decreasing in M^2 and the latter non-increasing; and for any $M_2^2 > M_1^2$ we have that $u^c(M_2^2) > u^c(M_1^2)$ or $v^c(M_2^2) < v^c(M_1^2)$.

Thus the modified set reduces to:

$$M^2 > 1-R^1 \Rightarrow \pi^1\Big(u^c(M^2), \frac{R^1-1+M^2}{R^1}\Big) \leq \pi^2\Big(v^c(M^2), \frac{R^2-M^2}{R^2}\Big) \tag{1.43}$$

$$M^2 < R^2 \Rightarrow \pi^1\Big(u^c(M^2), \frac{R^1-1+M^2}{R^1}\Big) \geq \pi^2\Big(v^c(M^2), \frac{R^2-M^2}{R^2}\Big) \tag{1.44}$$

$$M^2 \in [\max\{0,1-R^1\}, \min\{1,R^2\}]. \tag{1.45}$$

First of all, note that $u^c(0)=0$, $v^c(0)=1$, $u^c(1)=1$ and $v^c(1)=0$. Condition (1.43) -(1.44) will be trivially met if there exists some $M^2 \in [\max\{0, 1 - R^1\}, \min\{1, R^2\}]$ such that:

$$\pi^1\Big(u^c(M^2), \frac{R^1 - 1 + M^2}{R^1}\Big) = \pi^2\Big(v^c(M^2), \frac{R^2 - M^2}{R^2}\Big).$$

If there doesn't, then it has to be the case that either (a) $LHS > RHS$ for all $M^2 \in [\max\{0, 1 - R^1\}, \min\{1, R^2\}]$ or (b) $RHS > LHS$ for all $M^2 \in [\max\{0, 1 - R^1\}, \min\{1, R^2\}]$. However, (a) can take place only if $\max\{0, 1 - R^1\} = 1 - R^1$ – as $LHS > RHS$ for $M^2 = 0$ violates condition (d). And for $M^2 = 1 - R^1$, $LHS > RHS$ meets (1.43) -(1.44), as the first inequality doesn't have to hold. For similar reasons, (b) can only take place if $\min\{1, R^2\} = R^2$ – in which case $RHS > LHS$ meets (1.43) -(1.44). Thus, existence of a solution to (1.43) -(1.44) follows, which implies that a solution to the modified set exists, which implies that a solution to the original set exists, which completes the existence proof.

As for uniqueness, remember that we have shown so far that $d^*(v, a(M^2), M^2)$ is unique. Thus, it suffices to show that there exists only one M^2 that meets (1.43) -(1.44). Define the set of all $M^2 \in [\max\{0, 1 - R^1\}, \min\{1, R^2\}]$ that meet (1.43) -(1.44) as Ω^M and consider $\min \Omega^M = M_1^2$. Note that M_1^2 exists as Ω^M is non-empty and $\pi^1(\cdot, \cdot)$, $\pi^2(\cdot, \cdot)$, $v^c(\cdot)$ and $u^c(\cdot)$ are continuous. Suppose $M_1^2 = \min\{1, R^2\}$ – then the solution is unique. Now suppose that $M_1^2 < \min\{1, R^2\}$, which implies that for any $M_2^2 \in \Omega^M$ such that $M_2^2 > M_1^2$ we need to have:

$$\pi^1\Big(u^c(M_2^2), \frac{R^1 - 1 + M_2^2}{R^1}\Big) \leq \pi^2\Big(v^c(M_2^2), \frac{R^2 - M_2^2}{R^2}\Big)$$

and for M_1^2 we have:

$$\pi^1\Big(u^c(M_1^2), \frac{R^1 - 1 + M_1^2}{R^1}\Big) \geq \pi^2\Big(v^c(M_1^2), \frac{R^2 - M_1^2}{R^2}\Big).$$

But this is a contradiction, as $\pi_u^1 > 0$, $\pi_h^1 \geq 0$, $\pi_v^2 > 0$, $\pi_h^2 \geq$, $u^c(\cdot)$ is weakly increasing, $v^c(\cdot)$ is weakly decreasing and $u^c(M_2^2) > u^c(M_1^2) \vee v^c(M_2^2) < v^c(M_1^2)$. Thus M_2^2 doesn't exist, which implies that M_1^2 is the only element in Ω^M, which completes the proof. $\square$

1.B Comparative Statics

To simplify what follows, we need to first introduce new notation. The difference between the old and new values of any object O is denoted as $\Delta_\rho O$. The greater of the old and new values of any object O is denoted as $\max O$. Thus, for instance, the change in sector one size is $\Delta_\rho M^1(\rho)$ and the greater critical ability in sector two is $\max v^c(\rho)$.

Definition 1.17. A change in surplus (CiS) from $\pi^1(\bullet, \rho_1)$ to $\pi^1(\bullet, \rho_2)$ in sector one is *(strictly) worker-friendly* if $\pi_u^1(u, h, \rho_2) \geq (>) \pi_u^2(u, h, \rho_1)$ for all (u, h).

The two matching problems have the *(strong) impossibility property* if sector two can (weakly) expand only if its critical ability grows.

Definition 1.18. The matching problems $(Q(\rho_1), Q(\rho_2))$ have *(strong) impossibility property* if it is impossible that $v^c(\rho_2) < (\leq) v^c(\rho_1)$ and $\Delta_\rho M^2(\rho) > (\geq) 0$.

The Theorem below is the driving force of all my comparative statics results.

Theorem 1.2. Suppose $(Q(\rho_1),\ Q(\rho_2))$ exhibit the impossibility property. Then $\Delta_\rho \pi_u^1(\bullet, \rho) \geq 0$ for all (u, h) results in (i) $P^2(v, \rho_2) \geq P^2(v, \rho_1)$ for all v and (ii) $P^1(u, \rho_2) \leq P^1(u, \rho_1)$ for all u. If the property is strong, then (i) holds strictly for a positive measure of v and (ii) for a positive measure of u. If $\Delta_\rho \pi_u^1(\bullet, \rho) > 0$ for all (u, h), then (iii) $P^2(v, \rho_2) > P^2(v, \rho_1)$ for all $v \in [\max v^c(\rho), \max v^*(\rho))$ and (iv) $P^1(u, \rho_2) < P^1(u, \rho_1)$ for all $u \in [\max u^c(\rho), \max u^*(\rho))$.

Proof of Theorem 1.2. The results for sector two will be proved in a series of lemmas and the result for sector one follows from an analogous reasoning (details can be found at the end of the proof)[41]. I will start, however, by defining four sets of sector two skill levels, which will be of crucial importance throughout the entire proof:

$$\Xi^0 = \{v \in [\max v^c(\rho), \min v^*(\rho)] : \psi(v, \rho_2) = \psi(v, \rho_1) \wedge P^2(v, \rho_2) \leq P^2(v, \rho_1)\}$$
$$\Xi^1 = \{v \in [\max v^c(\rho), \min v^*(\rho)] : \psi(v, \rho_2) \leq \psi(v, \rho_1) \wedge P^2(v, \rho_2) < P^2(v, \rho_1)\}$$
$$\Xi^2 (= \{v \in [\max v^c(\rho), \min v^*(\rho)) : \psi(v, \rho_2) \leq \psi(v, \rho_1) \wedge P^2(v, \rho_2) \leq P^2(v, \rho_1)\}.$$

Lemma 1.6. (Strictly) Worker-friendly technological change in sector one implies that Ξ^1 (Ξ^2) is empty.

Proof of Lemma 1.6. Remember that $P_v^2(v) = \frac{\psi_v(v) C_{uv}(\psi(v), v)}{R^2}$. Take any $v_0 \in \Xi^0$ Note that by Remark 1.2 we have $\Delta_\rho P^1(\psi(v_0, \rho_1), \rho) \geq 0$. Then we have:

$$\begin{aligned}\Delta_\rho w_u^1(\psi(v_0, \rho_1), \rho) = {} & \Delta_\rho \pi_u^1(\psi(v_0, \rho_1), P^1(\psi(v_0, \rho_1), \rho_2), \rho) \\ & + \int_{P^2(\psi(v_0, \rho_1), \rho_1)}^{P^2(\psi(v_0, \rho_1), \rho_2)} \pi_{uh}^1(\psi(v_0, \rho_1), r, \rho_1) \mathrm{d}r \geq (>) 0,\end{aligned}$$

as $\Delta_\rho \pi_u^1(u, h, \rho) \geq (>) 0$ for any (u, h), $\pi^1(\bullet)$ is supermodular and $\Delta_\rho P^1(\psi(v_0, \rho_1), \rho) \geq 0$. Whereas for v_0 we have:

$$\Delta_\rho w_v^2(v_0, \rho) = \int_{P^2(v_0, \rho_1)}^{P^2(v_0, \rho_2)} \pi_{vh}^2(v_0, r) \mathrm{d}r \leq 0,$$

as $\pi^2(\bullet)$ is supermodular and $\Delta_\rho P^2(v_0, \rho) \leq 0$. By differentiating Equation (1.5) wrt to v for both ρ_2 and ρ_1, taking differences and rearranging, we arrive at:

$$\Delta_\rho \psi_v(v_0, \rho) = \frac{1}{w_u^1(\psi(v_0, \rho_1), \rho_2)} \left[\Delta_\rho w_v^2(v_0, \rho) - \psi_v(v_0, \rho_1) \Delta_\rho w_u^1(\psi(v_0, \rho_1)\right],$$

from which follows trivially that $\Delta_\rho \psi_v(v_0, \rho) \leq (<) 0$.

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[41] The results for sector one follows also directly from the results for sector two, but the proof of that is more space consuming.

Take any $v_1 \in \Xi^1$. Suppose that $\Delta_\rho \psi(v,\rho) \leq 0$ for all $v \in [v_1, \min v^*(\rho)]$, which implies that $v^*(\rho_2) > v^*(\rho_1)$. Remember that both for ρ_1 and ρ_2 we need to have $P^2(1) = 1$ and thus $\Delta_\rho P^2(1,\rho) = 0$. Therefore:

$$\begin{aligned}0 &= \Delta_\rho P^2(v_1,\rho) + \frac{\int_{v_1}^{v^*(\rho_2)} C_v(\psi(v,\rho_2),v)\mathrm{d}v - \int_{v_1}^{v^*(\rho_1)} C_v(\psi(v,\rho_1),v)\mathrm{d}v - \Delta_\rho v^*(\rho)}{R^2} \\ &= \Delta_\rho P^2(v_1,\rho_1) - \int_{v_1}^{v^*(\rho_1)} \int_{\psi(v,\rho_2)}^{\psi(v,\rho_1)} \frac{C_{uv}(s,v)}{R^2}\mathrm{d}s\mathrm{d}v - \int_{v^*(\rho_2)}^{v^*(\rho_1)} \frac{1 - C_v(\psi(v,\rho_2),v)}{R^2}\mathrm{d}v.\end{aligned}$$

Note that as $\psi(v,\rho_2) \leq 1$ it follows that $C_v(\psi(v,\rho_2),v) \leq 1$; hence we have that the two latter terms on the RHS are weakly and the first is strictly negative – contradiction. Therefore there needs to exist some $v \in (v_1, \min v^*(\rho)]$ such that $\Delta_\rho \psi(v,\rho) > 0$ for Ξ^1 to be non-empty. Denote the set of all such v's as Ξ^3; then it follows that $\inf \Xi^3 \in \Xi^0$[42]. But under $\Delta_\rho \pi_u^1(u,h,\rho) \geq (>)0$ for all (u,h) for any $v \in \Xi^0$, $\Delta_\rho \psi_v(v,\rho) \leq (<)0$, which contradicts $v = \inf \Xi^3$. Thus Ξ^1 has to be empty and the result for worker-friendly CiS stands.

Now consider any $v_2 \in \Xi^2$. Note that under $\Delta_\rho \pi_u^1(u,h,\rho) > 0$ for all (u,h) there has to exist some arbitrarily small $\epsilon > 0$ such that $v_2 + \epsilon < \min v^*(\rho)$ and for any $v \in (v_2; v_2 + \epsilon]$ we have $\Delta_\rho \psi(v,\rho) < 0$. This follows from continuity if $\Delta_\rho \psi(v_2,\rho) < 0$ and from the fact that if $\Delta_\rho \psi_v(v_2,\rho) = 0$ then $v_2 \in \Xi^0$ and $\Delta_\rho \psi_v(v_2,\rho) < 0$. Therefore, trivially, $\Delta_\rho P^2(v_2+\epsilon,\rho) < 0$ and thus $v_2 + \epsilon \in \Xi^1$, which means that Ξ^2 has to be empty and concludes the proof. □

Lemma 1.7. Suppose Ξ^1 is empty. Consider some $v_e \in [\max v^c(\rho), \min v^*(\rho)]$. Then $\Delta_\rho P^2(v_e,\rho) \geq 0$ implies $\Delta_\rho P^2(v_e,\rho) \geq 0$ for all $v \in [v_e, \min v^*(\rho)]$. If Ξ^2 is empty, then additionally $\Delta_\rho P^2(v_e,\rho) > 0$ implies $\Delta_\rho P^2(v_e,\rho) > 0$ for all $v \in [v_e, \min v^*(\rho)]$.

Proof. I will start with the first claim. Suppose it is false. Then the set $\Upsilon^1 = \{v \in [v_e, \min v^*(\rho)] : \Delta_\rho P^2(v,\rho) < 0\}$ has to be non-empty. Take some $v^1 \in \Upsilon^1$

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[42] By continuity of $\Delta_\rho \psi(v,\rho)$, which follows from continuity of $\psi(v)$.

and define $\Upsilon^2 = \{v \in [v_e, v_1] : \Delta_\rho P^2(v,\rho) \geq 0\}$. By continuity of $\Delta_\rho P^2(v,\rho)$ the point $v^2 = \max \Upsilon^2$ exists and is $< v^1$. Therefore, for any $v \in (v^2, v^1]$ we have $\Delta_\rho P^2(v,\rho) < 0$. However, as:

$$\Delta_\rho P^2(v^1) = \Delta_\rho P^2(v^2,\rho) + \frac{1}{R^2}\int_{v^2}^{v^1}\int_{\psi(r,\rho_1)}^{\psi(r,\rho_2)} C_{uv}(s,r)\mathrm{d}s\mathrm{d}r,$$

this implies that there exists some $v_1 \in (v^2, v^1]$ such that $\Delta_\rho \psi(v_1,\rho) < 0$ and thus $v_1 \in \Xi^1$ – contradiction.

Let us move to the second claim. Again, suppose it is false. Then the set $\Upsilon^3 = \{v \in [v_e, \min v^*(\rho)] : \Delta_\rho P^2(v,\rho) \leq 0\}$ has to be non-empty; but as $\Delta_\rho P^2(v,\rho)$ is continuous in v, the non-emptiness implies that $v^3 = \min \Upsilon^3$ exists. Additionally, $v^3 > v_e$, as $\Delta_\rho P^2(v_e,\rho) > 0$. Define a new set $\Upsilon^4 = \{v \in [v_e, v^3] : \Delta_\rho \psi(v,\rho) \leq 0\}$ and $v^4 = \max \Upsilon^4$; by definition of v^3, for any $v < v^3 \wedge \in \Upsilon^4$ we have that $\Delta_\rho P^2(v,\rho) > 0$. As $[v_e, v^3]$ is a compact set and $\Delta_\rho \psi(v,\rho)$ is continuous v^4 won't exist only if Υ^4 is empty; but an empty Υ^4 implies that $\Delta_\rho \psi(v,\rho) > 0$ for any $v \in [v_e, v^3]$, which in turn means that $\Delta_\rho P^2(v^3,\rho) > 0$, which contradicts the definition of v^3. Therefore v^4 needs to exist. Now suppose that $v^4 < v^3$; then we have $\Delta_\rho P^2(v^4,\rho) > 0$ and for any $v \in (v^4, v^3], \Delta_\rho \psi(v,\rho) > 0$, which implies that $\Delta_\rho P^2(v^3) > 0$ and also contradicts the definition of v^3. Therefore it has to be the case that $v^3 = v^4$; but this implies that $\Delta_\rho(\psi(v^3,\rho)) \leq 0$ and $\Delta_\rho P^2(v^3,\rho) \leq 0$, which contradicts emptiness of Ξ^2 □

Lemma 1.8. $\Delta_\rho P^2(\min v^*(\rho),\rho) \geq 0$ implies that (i) for any $v > \min v^*(\rho)$ we have $\Delta_\rho P^2(v,\rho) \geq 0$ and (ii) for all $v \in [\min v^*(\rho), \max v^*(\rho))$ we have $\Delta_\rho P^2(v,\rho) > 0$.

Proof. Note that $\Delta_\rho P^2(\min v^*(\rho),\rho) > (\geq) 0$ implies that $v^*(\rho_2) > (\geq) v^*(\rho_1)$[43].

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[43] To see this, denote the ρ_i for which $v^*(\rho_i) = \max v^*(\rho)$ as ρ_m; then, as $\Delta_\rho P^2(1,\rho) = 0$, we have:

$$0 = \Delta_\rho P^2(\min v^*(\rho),\rho_1) + \frac{1}{R^2}\int_{v^*(\rho_2)}^{v^*(\rho_1)} \frac{1 - C_v(\psi(v,\rho_m),v)}{R^2}\mathrm{d}v.$$

Thus, if $\Delta_\rho P^2(\min v^*(\rho), \rho) = 0$ then $\min v^*(\rho) = \max v^c(\rho)$ and the second claim follows trivially. Whereas if $\Delta_\rho P^2(\min v^*(\rho), \rho) > 0$ then $v^*(\rho_2) > v^*(\rho_1)$ and by the fact that all agents with $v \in (v^*, 1]$ join sector two for sure it follows that for $v \in (v^*(\rho_1), v^*(\rho_2))$ we also have $\Delta_\rho P^2(v, \rho) > 0$. Claim (i) for $v > \max v^*(\rho)$ follows easily from the aforementioned property of v^*. $\square$

Lemma 1.9. The (strong) impossibility property implies that if $v^c(\rho_2) < (\leq)v^c(\rho_1)$ then $\Delta_\rho P^2(v^c(\rho_1), \rho) > 0$.

Proof. This follows trivially from the fact that $\Delta_\rho v^c(\rho) < (\leq)0$ implies that $G^2(v^c(\rho_1), \rho) > (\geq)0$, the fact that:

$$\Delta_\rho P^2(v, \rho) = \frac{1}{R^2}\big((G^2(v, \rho_2) - 1)\Delta_\rho M^2(\rho) + M^2(\rho_1)\Delta_\rho G^2(v, \rho)\big) \tag{1.46}$$

and the fact that $v^c(\rho_1) < 1$ and thus $G^2(v, \rho_2) - 1 < 0$. $\square$

Lemma 1.10. Empty Ξ^1 and impossibility property imply jointly that $\Delta_\rho P^2(\max v^c(\rho), \rho) \geq 0$. If either CiS is strictly worker-friendly or the property is strong then this inequality holds strictly.

Proof. Suppose (strong) impossibility property holds. Define a set $\Xi^5 = \{v \in [\max v^c(\rho), \max v^*(\rho)) : \Delta_\rho \psi(v_5, \rho) < 0 \wedge \Delta_\rho P^2(v_5, \rho) \leq 0\}$. By continuity, there has to exist some some arbitrarily small $\epsilon > 0$ such that $v_5 + \epsilon \in \Xi^1$; thus, by Lemma 1.6, worker-friendly CiS implies that Ξ^5 has to be empty.

If $v^c(\rho_2) < (\leq)v^c(\rho_1)$ – then by Lemma 1.9 we have $\Delta_\rho P^2(\max v^c(\rho), \rho) > 0$. If $v^c(\rho_2) \geq v^c(\rho_1)$ and $\max v^c(\rho) \geq \min v^*(\rho)$, then – as $v^*(\rho) > v^c(\rho)$ – it has to be that $v^*(\rho_2) > v^c(\rho_2) > v^*(\rho_1)$. But as all agents with $v > v^*(\rho)$ join sector two, this implies $\Delta_\rho P^2(v^c(\rho_2), \rho) > 0$.

As $1 - C_v(\psi(v, \rho_m), v) \geq 0$, the fact that $\Delta_\rho P^2(\min v^*(\rho), \rho) > (\geq)0$ implies that for this to hold we need $v^*(\rho_2) > (\geq)v^*(\rho_1)$.

Thus, we only need to show the result for $\max v^c(\rho) < \min v^*(\rho)$ and $v^c(\rho_2) \geq (>) v^c(\rho_1)$. As $\Delta_\rho M^3(\rho) = 0$ we have $C(u^c(\rho_1), v^c(\rho_1)) = C(u^c(\rho_2), v^c(\rho_2))$ and thus $\Delta_\rho v^c(\rho) \geq (>)0$ implies $\Delta_\rho u^c(\rho) \leq 0$. As $\psi(v^c(\rho), \rho) = u^c(\rho)$ and $\psi(v, \rho)$ is strictly increasing for any ρ we have: $\psi(v^c(\rho_2), \rho_1) \geq (>) u^c(\rho_1)$, $u^c(\rho_1) \geq u^c(\rho_2)$ and $u^c(\rho_2) = \psi(v^c(\rho_2), \rho_2)$, which trivially implies that:

$$\Delta_\rho \psi(v^c(\rho_2), \rho) \leq (<)0.$$

If second property holds, then this inequality holds weakly, which together with emptyΞ^1 implies $\Delta_\rho P^2(v^c(\rho_1), \rho) \geq 0$. If the second property is strong, then as Ξ^5 is empty, because of emptyΞ^1, we have that $\Delta_\rho \psi(v^c(\rho_2), \rho) < 0$ implies $\Delta_\rho P^2(\max v^c(\rho), \rho) > 0$. If Ξ^2 is empty, then we have that $\Delta_\rho \psi(v^c(\rho_2), \rho) \leq 0$ implies $\Delta_\rho P^2(\max v^c(\rho), \rho) > 0$, which concludes the proof. □

Lemma 1.11. Empty Ξ^1 and impossibility properties imply jointly that for any $v < \max v^c(\rho)$, $\Delta_\rho P^2(v, \rho) \geq 0$.

Proof. Suppose $\Delta_\rho v^c(\rho) < 0$ – then for all $v < \max v^c(\rho)$ we have trivially that $G^2(v^c(\rho_1), \rho) \geq 0$ and by impossibility property that $\Delta_\rho M^2(\rho) \leq 0$. Thus, the claim follows from Equation (1.46). Now suppose that $\Delta_\rho v^c(\rho) \geq 0$. This implies that for any $v \leq v^c(\rho_2)$ it is the case that $\Delta_\rho G^2(v^c(\rho_2), \rho) = 0 - G^2(v, \rho_2) \leq 0$ and this expression is decreasing in v. As by Lemma 1.10 $\Delta_\rho P^2(v^c(\rho_2), \rho) \geq 0$ it follows from Equation (1.46) that $\Delta_\rho P^2(v, \rho) \geq 0$ for all $v < \max v^c(\rho)$, as required. Note that this implies also that $\Delta_\rho P^2(0, \rho) = -\Delta M^2(\rho) \geq 0$. □

Lemma 1.12. For all $v \in [\max v^c(\rho), \min v^*(\rho)]$, if $P^2(v, \rho_2) \geq (>) P^2(v, \rho_1)$ then $P^1(\psi(v, \rho_2), \rho_2) \leq (<) P^1(\psi(v, \rho_2), \rho_1)$.

Proof. From Remark 1.2, Definition 1.6 and Equation (1.8) follows that:

$$\begin{aligned}\Delta_\rho P^2(\psi(v,\rho_2),\rho) = &-\frac{1}{R^1}R^2\Delta_\rho P^2(v,\rho)\\ &+\frac{1}{R^1}\Big[\int_{\psi(v,\rho_1)}^{\psi(v,\rho_2)} C_u\big(r,v\big)\mathrm{d}r - \int_{\psi(v,\rho_1)}^{\psi(v,\rho_2)} C_u\big(r,\phi(r,\rho_1)\big)\mathrm{d}r\Big].\end{aligned}$$

If $\psi(v,\rho_2) \geq \psi(v,\rho_1)$ then for any $r \in [\psi(v,\rho_1),\psi(v,\rho_2)]$, $\phi(r,\rho_1) \geq v$ and my claim follows. If $\psi(v,\rho_2) < \psi(v,\rho_1)$ then for any $r \in [\psi(v,\rho_2),\psi(v,\rho_1)]$, $\phi(r,\rho_1) < v$ and my claim follows as well. $\square$

All results for sector two follow easily from Lemmas 1.6, 1.7, 1.8, 1.10 and 1.11 as well as continuity of $\Delta_\rho P^2(\cdot,\rho)$. As Lemma 1.8 has an exact sector one analogue, the sector one results for $u \geq \max u^c(\rho)$ follow from sector two results and Lemma 1.12. The results for $u < \max u^c(\rho)$ follow from reasoning analogous to that in proof of Lemma 1.11 once we note that $\Delta_\rho M^2(\rho) \leq 0$ implies $\Delta_\rho M^1(\rho) \geq 0$. $\square$

To prove Propositions 1.4, 1.7 and 1.11, in each case it suffices to show that the impossibility property holds (strongly for Proposition 1.11), as (strict) real spread implies (strictly) worker-friendly CiS and then the results follows from Theorem 1.2. Note also that surplus increasing real spread implies $\Delta_\rho\pi^1(u^c(\rho_1),P^1(\rho_1),\rho) \geq 0$ and an isolated, strict improvement implies $\Delta_\rho\pi^1(u^c(\rho_1),P^1(\rho_1),\rho) > 0$.

Proof of Proposition 1.4. The impossibility property is met, as $\Delta_\rho M^2(\rho) = 0$. $\square$

Proof of Proposition 1.7. Suppose the impossibility property does not hold, then $\Delta_\rho M^2(\rho) > 0$ and $\Delta_\rho v^c(\rho) < 0$, which implies that $\Delta_\rho M^1(\rho) < 0$, $\Delta_\rho u^c(\rho) \leq 0$ and trivially $\Delta_\rho P^2(\rho) < 0$ and $\Delta_\rho P^1(\rho) > 0$. Sector two expansion implies $M^2(\rho_1) < R^2$; sector one shrinkage implies $R^1 > M^1(\rho_2)$, and thus from (1.16)-

(1.17) follows that:

$$\pi^1(u^c(\rho_2), P^2(\rho_2), \rho_2) \leq \pi^2(v^c(\rho_2), P^2(\rho_2)) \tag{1.47}$$

$$\pi^2(v^c(\rho_1), P^2(\rho_1)) \leq \pi^1(u^c(\rho_1), P^2(\rho_1), \rho_1). \tag{1.48}$$

Given that $\pi^2_v > 0$ and $\pi^2_h \geq 0$, we have that RHS of (1.47) is strictly less than the LHS of (1.48) and therefore $\pi^1(u^c(\rho_2), P^2(\rho_2), \rho_2) < \pi^1(u^c(\rho_1), P^2(\rho_1), \rho_1)$. However, as $\Delta_\rho \pi^1(u^c(\rho_1), P^1(\rho_1), \rho) \geq 0$, $\pi^1_u > 0$ and $\pi^1_h \geq 0$ this is impossible and impossibility property holds. □

Proof of Proposition 1.11. Suppose strong impossibility property does not hold, then $\Delta_\rho M^2(\rho) \geq 0$ and $\Delta_\rho v^c(\rho) \leq 0$, which implies that $\Delta_\rho M^1(\rho) \leq 0$, $\Delta_\rho u^c(\rho) \geq 0$ and trivially $\Delta_\rho P^2(\rho) \leq 0$ and $\Delta_\rho P^1(\rho) > 0$. This and Equations (1.16)-(1.17) give us:

$$\pi^1(u^c(\rho_1), P^2(\rho_1), \rho_1) \geq \pi^1(u^c(\rho_2), P^2(\rho_2), \rho_2),$$

which together with the isolated, strict improvement implies:

$$\pi^1(u^c(\rho_1), P^1(\rho_1), \rho_2) > \pi^1(u^c(\rho_2), P^2(\rho_2), \rho_2).$$

This is impossible as $\pi^1_u > 0$ and $\pi^1_h \geq 0$. □

Proof of Lemma 1.3. In sector one, there are two possibilities: $M^1(\rho_1) < R^1$ and $M^1(\rho_1) = R^1$. If the former is the case, then $C^1_R(\rho_1) = 0$ and the result follows trivially. If the latter is true, then $M^1(\rho_2) = R^1$ and by Proposition 1.7 we have $u^c(\rho_2) \geq u^c(\rho_1)$, $v^c(\rho_2) \leq v^c(\rho_1)$, $P^1(\rho_1) = P^2(\rho_2)$ and $P^2(\rho_2) = P^2(\rho_1)$. This implies that $C^1_R(\rho_2) \geq C^1_R(\rho_1)$, as by measure consistency and Lemma 1.1:

$$C^1_R(\rho_i) = \pi^1(u^c(\rho_i), P^1(\rho_i)) - \pi^2(v^c(\rho_i), P^2(\rho_i)),$$

for $i = 1, 2$[44]. The result for sector two follows from analogous reasoning, but the two cases are $M^2(\rho_2) < R^2$ and $M^2(\rho_2) = R^2$. $\square$

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[44] Clearly, for $R^1 + R^2 = 1$ the profit constant does not change.

Chapter 2

Occupational Sorting and the Structure of Status

2.1 Introduction

Economic and sociological literatures on status tend to emphasise different aspects of status concerns. Economists put more accent on its individual, relative components[1]. Sociologists, on the other hand, usually focus on its collective aspect, occupational prestige in particular[2]. There are reasons to believe that individual status depends not only on the rank in the society en large, but also on the rank in smaller reference groups (local status, see Chapter 2 of Frank, 1985b), such as occupation. These two status components both influence and are influenced by occupational sorting. However, they differ strongly in the sign of this influence – a very strong talent-pool in a profession increases its prestige, but also means that impressing one's peers is more difficult, which affects local status negatively.

The goal of this paper is to show that, indeed, occupational sorting – and, hence, wages and profits – depends crucially on the weight agents put on local status and occupational prestige; but also that there are good reasons to think that these weights differ across occupations. I focus on the scarce jobs case of the two-sector matching model from Chapter 1 and extend it by allowing agents' payoffs to depend not only on wages, but also on status rewards[3]. The status reward in each sector depends on three standard components: *local status*, *public status* and *occupational prestige*. The first two depend on agents' individual characteristics (within sector rank and sector-specific talent, respectively), whereas the last depends on the average talent in the sector. Keeping occupational prestige constant, *status structure* – so the vector of status' components weights – determines

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[1] See e.g. Frank (1984b), Bakshi and Chen (1996), Ireland (2001), Hopkins and Kornienko (2004), Hopkins and Kornienko (2009).

[2] Marshall (1964, p. 193) writes that "The mass of evidence suggests that occupation is generally regarded as an index of social status, probably the most important single index . . . ".

[3] Scarce jobs mean that there are at least as many agents as firms. To ensure that all firms are matched, I assume additionally that the lowest output that can be produced by any match is higher than the sum of firm's and agent's reservation payoffs.

the spread of the status reward in a sector, but not its level.

The higher the weight of the individual components of status (local and public status) or the lower the weight of the collective component (occupational prestige), the more spread out is the status reward. As jobs are scarce and the lowest payoff in each sector is fixed at the reservation payoff, in partial equilibrium an increase in status' spread increases the total payoff of each agent. In general equilibrium, this attracts more talent from sector two and, hence, increases talent supply in sector one and decreases it in sector two. The change in talent supply influences occupational prestige, payoff levels and inequality, wage levels and inequality, and profits. In particular, a fall in the weight of occupational prestige in sector one increases its prestige, which can, in some cases, increase the status reward of all agents in that sector. This suggests that the prestige of an occupation is the result of that profession being able to attract top talent, rather than the reason for it.

Of course, as all these results hold for changes in the status structure of just one sector, they are meaningful only if status structure can indeed differ across occupations. This is likely to be the case. In Chapter 2 of Frank (1985b) evidence is provided that the extent to which people care about their rank in a particular group depends on the intensity of interaction with members of this group. Moreover, both Marshall (1964) and Fershtman, Murphy, and Weiss (1996) argue that the importance of occupational prestige is driven by informational constraints: specifically, by our inability to grant status based on individual achievements, which is caused by the lack of both information about them and the expertise to assess them correctly. Thus, membership of a particular group is often used as a proxy. This suggests that the individual components of status should be more important in sectors where information about individual achievements is easily available. To see whether this is indeed the case, I microfound status rewards. In this microfoundation, status is granted in face to face meetings, based on available information. I show that, indeed, local status weight depends positively on insiders' information and the frequency of meeting with one's peers, whereas the weight of occupational prestige depends negatively on the quality of outsiders' information.

I assume that status rewards depend only on agents' talent, not income, and only on the talent dimension that is specific to their sector. The former is motivated by the sociologists findings that although occupational prestige is positively correlated with both average wages and average education, the latter is more important (see e.g. Hauser and Warren, 1997). The latter is equivalent to assuming that a doctor's status depends on how good his medical skills are and an economist's status depends on his talent toward economics. As natural as this sounds, it has an important implication – namely, that status is not a zero-sum game. Thus, the average status reward in the economy depends on the assignment of agents to sectors: assigning agents to sectors in which they are highly talented makes them feel more appreciated and thus creates an additional surplus. The idea that the number of dimensions on which agents can be ranked is positively related with average self-esteem can be tracked back to Robert Nozick[4].

Both occupational prestige and local status concerns create externalities, albeit of opposite signs: the entry of a high talent individual increases prestige of that profession and hence is good for everyone; at the same time, it pushes some incumbents down the ladder and decreases their local status rewards. Thus, typically, the stable assignment of agents to sectors is inefficient. It is worth noting that this inefficiency is driven by the distortion of relative status rewards, not status concerns themselves: in fact, the efficient assignment is equivalent to the stable one of a matching problem with identical output functions, but zero

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[4] Nozick, (1974 [reprinted, 2006, Chapter 8]) postulated that the elimination of some dimensions on which agents can be ranked would reduce the aggregate well-being. A similar effect is true in my model, as in the symmetric case an increase in the interdependence between the talent dimensions results in a fall in average status reward (I do not show this formally, but it follows trivially from a reasoning analogous to that in Section 3 of Chapter 1).

weights on local status and occupational prestige. Hence, an output maximising assignment is inefficient as well, as it does not take into account that, as status is not a zero-sum game, total status surplus depends on the assignment. Finally, recalling my microfoundation, to ensure efficiency we need both perfect information and equal frequency of meetings with peers and outsiders – neither of which seems particularly likely.

As mentioned earlier, changes to status structure that result in a greater spread of sector one status reward increase talent supply in sector one and decrease it in sector two. This increases wages in sector two, but its impact on sector one wages is ambiguous: the increased talent supply has a negative effect, but the change in status structure means that the least talented agents might need a higher compensation for they decreased status reward. The change in the difference between globally highest and lowest wages is ambiguous, as the gap between top and bottom wages widens in sector two and shrinks in sector one. Interestingly, the impact on total payoffs (so wage plus status rewards) is much clearer: they rise in sector two and for top sector one agents, but fall for the least talented sector one workers. The gap between most and least well-off agents widens in both sectors and thus also in the entire economy. Thus, not only do changes in status structure affect wages and wage inequality, they can also affect them in the opposite direction than total payoffs. In other words, the society can become more egalitarian, even though wage inequality raises; and vice versa.

Smith (1776) and Fershtman and Weiss (1993) have postulated that, all other things equal, an agent's wage should be negatively related to the prestige of her occupation. The logic of their argument still holds in my model, but it turns out that status structure belongs to the things that need to be kept constant. In particular, it is possible that an increase in the spread of sector one status increases its prestige, decreases the prestige of the other occupation *and* increases all wages in sector one by more than the highest increase in sector two wages.

The reason is, of course, that the increase in spread can dramatically decrease the reward from the individual status components for the least talented agents, which can outweigh the changes in occupational prestige. This could at least partially account for the mixed empirical evidence of the negative relation between wages and occupational prestige (see Fershtman and Weiss, 1993).

The fall in sector two talent supply affects negatively profits in the second industry. The impact on sector one profits is much less clear cut: they benefit from the increase in talent supply, but the least productive firms might need to compensate their workers for the decreased status rewards, which would then force the more productive firms to increase wages as well[5]. In general, the more productive firms are more likely to gain from an increase in status reward spread, but both a fall and increase in profits for all firms is possible. This suggests that there is some room for profit increasing manipulations of the status structure on firms part, for example by establishing (or abolishing) industry wide rankings and awards, or promoting (or discouraging) inter-profession socialisation.

Finally, I show also that, other than in Roy (1951) or Weiss and Fershtman (1992), the ranking of occupations with respect to prestige depends not only on skill intensity, which I represent by the spread of output, but also on status structures[6]. Although, for equally sized sectors, the ranking in the efficient assignment does depend only on the spread of outputs, in the stable assignment a sufficiently spread out status reward in the less skill-intense sector will invert the ranking. Thus, a high enough weight on local status can attract the most talented agents to a sector where their skills are wasted.

The rest of this paper is organised as follows. In the remainder of this section, I review the literature most relevant for this study. In Section 2.2 I set up the

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[5] Otherwise, as a within-occupation move does not change an agent's status reward, the more talented workers would be willing to leave for less productive firms.

[6] Roy (1951) does not write about occupational prestige, but inferior and superior occupations, which seems to be a fairly similar concept.

model, introduce status reward and characterise the unique stable assignment. In Section 2.3 I characterise the efficient assignment and discuss why the stable assignment is typically inefficient. In Section 2.4 I derive the comparative statics results for spread increasing changes in status structure. Section 2.5 studies the determinants of occupational prestige rankings and Section 2.6 concludes.

2.1.1 Related Literature

This paper is most closely related to the body of work by Chaim Fershtman, Yoram Weiss and co-authors (Weiss and Fershtman, 1992; Fershtman and Weiss, 1993; Fershtman, Murphy, and Weiss, 1996), who show that inclusion of social rewards into a general equilibrium framework can vastly enhance both the economists understanding of the effects of cultural differences on wages and growth, and the sociologists understanding of what determines occupational prestige rankings. My paper is similar in spirit, but significantly enhances their analysis by considering a setting in which talent and status are not one-dimensional; there are concerns for local status as well as occupational prestige, sectors differ in their status structures and firms are not identical.

Another closely related body of work is that by Robert Frank (especially Chapters 2 and 3 of Frank, 1985a and its companion papers Frank, 1984a and Frank, 1984b), who considers the impact of local status concerns on internal wage structures and the sorting of agents to organisations. Frank interprets local status as status within firm, the economics of which are quite different than that of within-occupation local status. A firm takes into account the effect of their hiring decisions on the well-being of other employees, and thus internalises the externalities produced by local status concerns. An occupation consists of workers employed by many independent firms, neither of which considers the effect of their hiring decisions on everyone else in that profession. Thus, within-firm relative concerns influence mostly internal wage spreads, whereas within-occupation relative concerns affect mostly occupational sorting and only indirectly wage structures.

There is a number of papers in the sorting literature concerned with the peer effect. In the model by de Bartolome (1990) families care about schooling, the quality of which depends both on peers and expenditure on schooling. Children's ability is binary in that model. Becker and Murphy (2000) also use a two-type model to discuss the implications of the peer effect on residential sorting. The closest to my work is the paper by Damiano, Li, and Suen (2010), in which ability is continuous, both peer effect and relative concerns are present, and agents sort into two organisations with fixed capacity. As ability is one-dimensional, the stable equilibrium has one of the organisations attracting all most able workers and the other one attracting all the least talented workers, with workers of medium ability joining both organisations.

Finally, this paper contributes methodologically to the multivariate matching literature, by demonstrating that the characterisation of stable matchings developed in Chapter 1 can be applied to settings with externalities. The presence of externalities renders the standard linear programming method (see e.g. Chiappori, Oreffice, and Quintana-Domeque, 2011; McCann, Shi, Siow, and Wolthoff, 2012; Lindenlaub, 2014) useless, as it finds only the efficient matchings and externalities make the stable matching inefficient. For the same reason, in settings with externalities existence of the stable matching is not guaranteed by the results from Gretsky et al. (1992). To the best of my knowledge, my existence and uniqueness results are the first for a multivariate matching model with externalities.

2.2 Model

There are two populations: agents and firms. Agents have two separate skills – X and Y –, given by a bivariate distribution $F(x, y) : [x_l, x_h] \times [y_l, y_h] \to [0, 1]$, which is twice continuously differentiable and has strictly positive density for all (x, y) in the support of $F(\bullet)$. Firms are divided into two sectors, one and two. Sector $i \in \{1, 2\}$ firms have productivity Z^i, given by a univariate, strictly increasing and continuously differentiable distribution $H_{Z^i}(z^i) : [z^i_l, z^i_h] \to [0, 1]$. A match between an agent and a firm in sector one produces an *output* of $\Pi^2(x, z^1)$ and a match between an agent and a firm in sector two produces an output of $\Pi^2(y, z^2)$. Additionally, agents receive a *status reward* $\mathrm{T}^i(x, y, \Theta)$, which depends not only on their skills and sector, but also on the *assignment* of agents to sectors (see Definition 2.3 below). Thus, the total surplus produced in a match is given by $S^1(x, y, z^1, \Theta) = \Pi^1(x, z^1) + \mathrm{T}^1(x, y, \Theta)$ in sector 1 and by $S^2(x, y, z^2, \Theta) = \Pi^2(y, z^2) + \mathrm{T}^1(x, y, \Theta)$ in sector 2. The mass of agents is normalised to 1; the mass of sector i firms is $R^i > 0$ and jobs are scarce: $R^1 + R^2 \leq 1$. Any agent and firm are free not to match anyone, in which case they produce a reservation output normalised to 0; unmatched agents receive also a reservation status reward $\bar{\mathrm{T}}$. For simplicity, I assume that, for any Θ and $i \in \{1, 2\}$, $\inf \Pi^i \geq \bar{\mathrm{T}} - \inf \mathrm{T}^i$. This ensures that the output is always high enough to make any match preferable to remaining unmatched.

The status reward functions are specified in detail in Section 2.2.3. The output functions $\Pi^1 : [x_l, x_h] \times [z^1_l, z^1_h] \to \mathbf{R}^+, \Pi^2 : [y_l, y_h] \times [z^2_l, z^2_h] \to \mathbf{R}^+$ are assumed to be (a) twice continuously differentiable with (b) $\Pi^1_x, \Pi^2_y > 0$, $\Pi^1_z, \Pi^2_z \geq 0$ and (c) Π^1_{xz}, Π^2_{yz} that are weakly positive and such that if $\Pi^i_{xz}(x, z) > 0$ for any (x, z), then $\Pi^i_{xz}(x', z') > 0$ for any $(x', z') > (x, z)$.

2.2.1 Copula Formulation

The above *original formulation* of the matching problem is not convenient to work with. Therefore, as in Chapter 1, I apply the probability integral transformation to all random variables and work with the copula of the original skill distribution most of the time[7]. I call this the *copula formulation.*

Denote the marginals of $F(x, y)$ as $F_X(x)$ and $F_Y(y)$ and define the talents $U = F_X(X)$ and $V = F_Y(Y)$. F's unique (by Sklar's Theorem) copula is given by:

$$C(u, v) = F(F_X^{-1}(u), F_Y^{-1}(y)).$$

As $F_{xy}(\bullet) > 0$, it follows that $F_X(\cdot)$ and $F_Y(\cdot)$ are strictly increasing and differentiable, which implies that so are $F_X^{-1}(\cdot)$ and $F_Y^{-1}(\cdot)$. Given that, it follows that $C(u, v)$ is twice continuously differentiable and has strictly positive density.

Let us now apply probability integral transformation to productivities and define $H = H_{Z^1}^{-1}(Z^1) = H_{Z^2}^{-1}(Z^2)$; clearly, H is standard uniform distributed. Note that any firm type can be uniquely defined by the vector (h, i), where i denotes the sector the firm belongs to. Therefore, whenever an agent with talent vector (u, v) is matched with a firm with productivity (h, i), they produce a surplus of:

$$s^1(u, v, h, \Theta) = \pi^1(u, h) + \tau^1(u, v, \Theta) = \Pi(F_X^{-1}(u), H_{Z^1}^{-1}(h)) + \mathrm{T}^1(F_X^{-1}(u), F_Y^{-1}(v), \Theta),$$

$$s^2(u, v, h, \Theta) = \pi^2(v, h) + \tau^2(u, v, \Theta) = \Pi(F_Y^{-1}(v), H_{Z^2}^{-1}(h)) + \mathrm{T}^2(F_X^{-1}(u), F_Y^{-1}(v), \Theta),$$

for $i = 1$ and $i = 2$, respectively. We can easily see that the output functions $\pi^1(\bullet)$ and $\pi^2(\bullet)$ inherit from $\Pi^1(\bullet)$ and $\Pi^2(\bullet)$ properties (a) to (c). I call the tuple $\{\pi^1(\bullet), \pi^2(\bullet)\}$ the *output structure.*

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[7] Here this is even more natural, as the status rewards are going to depend on ranks only.

2.2.2 (Stable) Matchings and Assignments

The basic objects of my analysis are analogous to those in Chapter 1. Note that all the definitions – and generally the remainder of this section, unless stated otherwise – refer to the copula formulation.

Definition 2.1. A *matching* consists of a subset of matched agents $A_A \subset [0,1] \times [0,1]$, a subset of matched firms $A_F \subset [0,1] \times \{1,2\}$ and a matching function, $\zeta : A_A \to A_F$.

Any matching needs to satisfy a 'measure consistency' property, which requires that the mass of any subset of matched firms is equal to the subset of agents they are matched with (see Legros and Newman, 2002, p. 929). Formally, for any measurable set $B \subset A_F$ define its partition into the set of all sector one firms: $B^1 = \{(h,1) \in B\}$ and all sector two firms: $B^2 = \{(h,2) \in B\}$.

Definition 2.2. A matching is *measure consistent* if for any measurable $B \subset A_F$ and its preimage $\zeta^{-1}(B) \subset A_A$:

$$\int\int_{\zeta^{-1}(B)} C_{uv}(u,v)\ \mathrm{d}u\ \mathrm{d}v = R^1 \int_{B^1} 1\ \mathrm{d}h + R^2 \int_{B^2} 1\ \mathrm{d}h.$$

Most of the time my focus will be on the way in which workers are assigned to sectors. This information can be easily inferred from a matching.

Definition 2.3. Any matching $(A_A, A_F, \zeta(\bullet))$ results in an *assignment* Θ, given by the set A_A and an assignment function $\theta : A_A \to \{1,2\}$:

$$\theta(u,v) = \begin{cases} 1 & \text{if } \zeta(u,v) \in A_F^1 \\ 2 & \text{if } \zeta(u,v) \in A_F^2. \end{cases}$$

Define a *payoff scheme* as a pair of mappings: $w : [0,1]^2 \to \mathbf{R}^+$ and $r : [0,1] \times \{0,1\} \to \mathbf{R}^+$. The first mapping – w – will be interpreted as wages and the second – r – as profits. Note that the total payoff agents receive is equal to the sum of their wage and status reward. As any unmatched firm and agent produce zero output and the sum of the firm's profit and the worker's wage in any given match cannot exceed the output they produce, we can define payoff schemes that are feasible for a given matching.

Definition 2.4. Given a matching, any associated payoff scheme is *feasible* iff:

$$\begin{aligned}
&\text{for all } (u,v,h,i), \text{ such that } \zeta(u,v)=(h,i): \quad w(u,v)+r(h,i)\leq \pi^i(u,v,h)\\
&\text{for all } (u,v)\notin A_A \quad w(u,v)=0\\
&\text{for all } (h,i)\notin A_F \quad r(h,i)=0.
\end{aligned}$$

We can also define stable matchings, so measure consistent matchings in which there exists no agent-firm pair that would prefer to be assigned with each other rather than with their current matches.

Definition 2.5. A matching $(A_A, A_F, \zeta(\bullet))$, which results in an assignment Θ, is *stable* if and only if it is measure consistent and there exists a payoff scheme that is feasible given $(A_A, A_F, \zeta(\bullet))$, such that for any (u,v,h,i):

$$w(u,v)+\tau^i(u,v,\Theta)+r(h,i)\geq s^i(u,v,h,\Theta). \tag{2.1}$$

An assignment $\Theta = (A_A, \Theta(\bullet))$ is *stable* if and only if there exists a stable matching that results in $(A_A, \Theta(\bullet))$.

2.2.3 Status Rewards

The only substantial difference between the model from Chapter 1 and this one is the introduction of a status reward. Following Frank (1985a) and Marshall (1964), my status reward function consists of three components: *local status, public status* and *occupational prestige*[8]. Status depends only on the talent coordinate relevant for production in that sector – so the u-coordinate in sector one and the v-coordinate in sector two.

For a given assignment Θ, denote the marginal distribution of U among sector one workers as $G^1(\cdot, \Theta)$ and its mean as $\bar{u}(\Theta)$; similarly, the marginal distribution of V among sector two workers is $G^1(\cdot, \Theta)$ and its mean $\bar{v}(\Theta)$. Then the status reward functions are given by:

$$\tau^1(u, \Theta) = \Big[l^1\big(2G^1(u, \Theta) - 1\big) + (1 - p^1)(2u - 1) + p^1\big(2\bar{u}(\Theta) - 1\big)\Big]k,$$

$$\tau^2(v, \Theta) = \Big[l^2\big(2G^2(v, \Theta) - 1\big) + (1 - p^2)(2v - 1) + p^2\big(2\bar{v}(\Theta) - 1\big)\Big]k,$$

where $k > 0$, $l^i \geq 0$ and $p^i \in [0, 1]$. The first term, within-sector rank, represents local status. The second term, talent, stands for public status. The last term, the average talent in the sector, is interpreted as occupational prestige. Note that, even though the weights of status components differ across sectors, the average status reward in each sector is always equal to occupational prestige. Thus, conditional on occupational prestige, the weights determine the spread of the status reward, rather than its level (which depends on parameter k and is the same in both sectors). The vector $\{l^i, p^i\}$ denotes the *status structure in sector i*.

Frank (1985a) discusses at length why the relative importance of local status depends on the intensity of contacts within the reference group. Marshall (1964) and Fershtman et al. (1996) note that the better the information about an individuals' achievements and talents, the less important occupational prestige is, relative to local and public status[9]. Both these effects, but the latter espe-

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[8] Frank uses the term 'global status' instead of public status and 'halo effect' instead of occupational prestige. Marshall only uses the term occupational prestige, but he describes both local and public status without naming them.

[9] For example, Marshall (1964, p. 181) writes that "[...] a person may be recognised as a representative of a particular group or social class. It is obvious that it is only in terms such as these that we can speak about the social status of a group, for instance of teachers. But an individual teacher may, by virtue of personality and attributes not characteristic of the group,

cially, strongly suggest that status components' weights vary across occupations, as sectors differ in how much interaction with co-workers is required and how well are achievements publicised. In order to provide some structure for those loose observations, I will next microfound the status rewards.

Microfoundation

My microfoundation is based on the premise that status originates from face-to-face meetings[10]. Whenever a matched agent – a worker – meets someone, she is ranked by that person, based on the available information[11]. If the worker is ranked high by the other person, she receives a positive utility; if she is ranked low, she receives a negative utility. I assume that the ranking is based on the talent dimension relevant for the worker's sector: sector one workers are ranked based on the u-coordinate of talent and sector two workers based on the v-coordinate[12].

Matched agents meet people both during and after work. During work, they meet only agents from the same sector; after work, they are equally likely to meet anyone. Sectors differ both in the quality of information about workers' talents and the frequency of work-meetings; the number of meetings after work is the same for both sectors and normalised to one. The information the worker's peers have about her achievements is at least as good as the information of the outsiders. A detailed description of the information structure and the way in which status is awarded can be found in Appendix 2.A, together with derivations.

In such a framework, in any stable assignment the expected status reward takes the form stated above, with $l^i = f^i + (n^i - o^i)R^i$ and $p^i = 1 - o^i$, where

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acquire a rather different social status within a community in which he is well known."

[10] See Chapter 2 of Frank (1985a) for a review of evidence that status comparisons are intensified by face-to-face meetings.

[11] The exact assumptions about the status of unmatched workers are not crucial, as long as they result in a common status reward for all unemployed agents. The most natural way to achieve that is to assume no information about unmatched agents' talents, which should be approximately true for most talent dimensions.

[12] Note that this implies that if the ranking exercise is mutual, it is possible that both agents receive positive status utility. Thus, status is not a zero-sum game in my model.

$o^i \in [0,1]$ stands for the quality of outsiders' information, $n^i \in [o^i, 1]$ for the quality of insiders' information and $f^i \geq 0$ is the number of work-meetings[13]. Hence, the importance of local status depends positively on the quality of insiders' information, frequency of work-meetings and the size of the sector; and negatively on the quality of outsiders' information. These relations are consistent with what the literature postulates, but it is worth elaborating on the negative impact of outsiders information on local status. Even in after work meetings, insiders are able to rank the worker more precisely than outsiders – hence, it is more important how one ranks against peers than outsiders. However, as outsiders information improves, the difference between their and insiders' information falls, which decreases the relative importance of local status. Nevertheless, the spread of the status reward increases, as more people than before are able to rank the worker precisely.

2.2.4 Characterisation Strategy

To characterise the stable matching in this model I use the same two step strategy as in Chapter 1. First, I fix the assignment and consider each sector in partial equilibrium, which allows me to find the within-sector stable matchings and associated payoffs. Then, I use those payoff functions to characterise the stable assignment.

First Step

Fix and suppress Θ, let the *critical abilities* u^c and v^c be the upper lower bounds of $G^1(\cdot)$ and $G^2(\cdot)$ supports, respectively, and denote the mass of agents in sector i as

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[13] More generally, in non-stable assignments, $l^i = f^i + (n^i - o^i)M^i$, where M^i is the mass of sector i workers. Thus, technically, this microfoundation implies that the weight on local status depends on the assignment; however, as for measure consistent assignments $\tau^1(u^c)$ is lowest for $M^1 = R^1$, in any stable assignment the mass of sector one agents is fixed at R^1 (see Section 2.2.4). Same reasoning applies for sector two.

$M^i \leq R^i$ (by measure consistency). Define the within-sector matching functions as $\zeta^1(u) : [u^c, 1] \to [0,1]$, $\zeta^2(v) : [v^c, 1] \to [0,1]$. In particular:

Definition 2.6. The *positive, assortative within-sector matching functions (PAM)* are: $P^1(u) = \frac{1}{R^1}(R^1 + M^1(G^1(u) - 1))$ and $P^2(v) = \frac{1}{R^2}(R^2 + M^2((G^2(v) - 1))$.

Then the following result holds:

Proposition 2.1. All feasible wage schemes that can support a stable within-sector matching (meet Inequality 2.1) are of the following form:

$$w^1(u) = \int_{u^c}^{u} \pi_u^1(r, P^1(r))\mathrm{d}r + C^1, \tag{2.2}$$

$$w^2(v) = \int_{v^c}^{v} \pi_v^2(r, P^2(r))\mathrm{d}r + C^2, \tag{2.3}$$

where $C^1 \in [\bar{\mathrm{T}} - \tau^1(u^c), \pi^1(u^c, \frac{R^1 - M^1}{R^1})]$ and $C^2 \in [\bar{\mathrm{T}} - \tau^2(v^c), \pi^2(v^c, \frac{R^2 - M^2}{R^2})]$.

Proof. For any sector i agent and any sector i firm, Inequality 2.1 takes the form:

$$w(u, v) + r(h, i) \geq \pi^i(u, v, h)$$

and thus, as only one of the talent dimensions matters for output, the expressions for $w^1(u)$ and $w^2(v)$ follow from standard results[14]. $C^1 + \tau^1(u^c)$ cannot be less than $\bar{\mathrm{T}}$, as u^c would prefer to be unmatched. C^1 cannot be greater than $\pi^1(u^c, \frac{R^1 - M^1}{R^1})$, because of measure consistency, wage feasibility and the fact that $\pi^1(u, \cdot)$ is non-decreasing.$\pi^1(u^c, \frac{R^1 - M^1}{R^1}) \geq \bar{\mathrm{T}} - \tau^1(u^c)$ follows from the fact that $\inf_{u,h} \pi^1 \geq \bar{\mathrm{T}} - \inf_u \tau^1$ for any Θ. □

As agents' status does not depend on the specific firm they are matched with, but only on the sector, firms still have to pay competitive wages, given the distribution of talent in the industry[15].

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[14] For strictly supermodular outputs from Sattinger (1979), for weakly supermodular outputs from Proposition 1 in Chapter 1.

[15] The possibility that agents' status depends also on the firm they are matched with is very

Second Step

Proposition 2.1 holds for all assignments, including the stable ones. In this part, I will use this to derive further conditions that hold in stable assignments and, hence, I suppress Θ from notation again.

As jobs are scarce, and any match is preferable to remaining unmatched, it follows from measure consistency that all firms end up matched[16]. Hence, $M^1 = R^1$ and $M^2 = R^2$ – sector sizes are fixed. Note that this implies that $P^i(\cdot) = G^i(\cdot)$, so the firm an agent is matched with depends only on her sectoral rank. By definitions of u^c and v^c the mass of matched agents with $(u, v) < (u^c, v^c)$ has to be zero. At the same time, any agent with $u > u^c$ or $v > v^c$ will receive a payoff strictly greater than $\bar{\mathrm{T}}$[17]. Therefore, all agents with $(u, v) < (u^c, v^c)$ and only such agents will remain unmatched and thus:

$$C(u^c, v^c) = 1 - R^1 - R^2. \tag{2.4}$$

Moreover, it has to be the case that agents with (u^c, v^c) need to earn identical payoffs in both sector, as otherwise some agents would want to relocate.

Lemma 2.1. In any stable assignment $w^1(u^c) + \tau^1(u^c) = w^2(v^c) + \tau^2(v^c)$. Additionally, if $R^1 + R^2 < 1$, then $w^1(u^c) + \tau^1(u^c) = \bar{\mathrm{T}}$.

If one of the sectors offers higher wages at the top end of the distribution, it attracts all the agents who are highly talented in both dimensions. Define the

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real and interesting, but outside the scope of this paper.

[16] Suppose not. Then, by measure consistency we have positive masses of unmatched firms and agents. This contradicts stability, as for any Θ and $i \in \{1, 2\}$, $\inf \Pi^i \geq \bar{\mathrm{T}} - \inf \mathrm{T}^i$ and the output functions are strictly increasing in talent.

[17] By Proposition 2.1 and the facts that output is strictly increasing in talent and status reward nondecreasing.

star abilities u^* and v^*:

$$u^* = 1 \text{ and } w^1(1) + \tau^1(1) = w^2(v^*) + \tau^2(v^*), \quad \text{if } w^1(1) + \tau^1(1) \leq w^2(1) + \tau^2(1)$$
$$v^* = 1 \text{ and } w^1(u^*) + \tau^1(u^*) = w^2(1) + \tau^2(1), \quad \text{if } w^1(1) + \tau^1(1) > w^2(1) + \tau^2(1).$$

Hence, stability requires that any agent with $u \in (u^*, 1]$ joins sector one and any agent with $v^* \in (v^*, 1]$ joins sector two. Note also that it is always the case that

$$\max\{u^*, v^*\} = 1. \tag{2.5}$$

Define the set $\Gamma = [u^c, u^*] \times [v^c, v^*]$. As $R^1, R^2 > 0$, Γ is non-empty[18]. Agents with $(u, v) \in \Gamma$ will join sector one if $w^1(u) \geq w^2(v)$ and sector two if $w^2(v) \geq w^1(u)$. Define a function $\psi(v) : [v^c, v^*] \to [u^c, u^*]$ such that:

$$w^1(\psi(v)) + \tau^1(\psi(v)) = w^2(v) + \tau^2(v). \tag{2.6}$$

Any agent in Γ strictly prefers sector one if $u > \psi(v)$ and sector two if $u < \psi(v)$.

Remark 2.1. The quintuple $(u^c, v^c, u^*, v^*, \psi(\bullet))$ fully defines the stable assignment, as in Chapter 1. Agents with $(u, v) < (u^c, v^c)$ are unmatched. Sector 1 is populated by agents with: (i) $v < v^c$ and $u > u^c$; (ii) $(u, v) \in \Gamma$ and $\psi(v) < u$; (iii) $u > u^*$. Sector 2 is populated by agents with: (i) $v > v^c$ and $u < u^c$; (ii) $(u, v) \in \Gamma$ and $\psi(v) > u$; (iii) $v > v^*$. The set of remaining agents is of zero mass.

Thus, given $(u^c, v^c, u^*, v^*, \psi(\bullet))$, we can derive the marginal distributions of

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[18] As $R^1, R^2 > 0$, it has to be the case that $v^c, u^c < 1$. Agents' payoff functions are strictly increasing and therefore, by Lemma 2.1, $w^2(1) + \tau^2(1) > w^1(u^c)$ and $w^1(1) + \tau^1(1) > w^2(v^c) + \tau^2(v^c)$, which implies that $v^* > v^c$ and $u^* > u^c$.

talent in each sector (details can be found in Appendix 2.B.1):

$$G^1(u) = \begin{cases} 0 & \text{for } u < u^c, \\ \frac{1}{R^1} \int_{u^c}^{u} C_u(r, \psi^{-1}(r))\mathrm{d}r & \text{for } v \in [u^c, u^*], \\ G^1(u^*) + \frac{1}{R^1}(u - u^*) & \text{for } u > u^*. \end{cases} \tag{2.7}$$

$$G^2(v) = \begin{cases} 0 & \text{for } v < v^c, \\ \frac{1}{R^2} \int_{v^c}^{v} C_v(\psi(r), r)\mathrm{d}r & \text{for } v \in [v^c, v^*], \\ G^2(v^*) + \frac{1}{R^2}(v - v^*) & \text{for } v > v^*. \end{cases} \tag{2.8}$$

Substitute these talent distributions into the wage and status reward functions, then it follows from Equation (2.6) that, for $v \in [v^c, v^*]$:

$$\begin{aligned} &\int_{u^c}^{\psi(v)} \pi_u^1\Big(t, \int_{u^c}^{t} \frac{C_u(r, \psi^{-1}(r))}{R^1}\mathrm{d}r\Big) + 2k\big(l^1 \frac{C_u(t, \psi^{-1}(t))}{R^1} + (1 - p^1)\big)\mathrm{d}t \qquad (2.9) \\ = &\int_{v^c}^{v} \pi_v^2\Big(t, \int_{v^c}^{t} \frac{C_v(\psi(r), r)}{R^2}\mathrm{d}r\Big) + 2k\big(l^1 \frac{C_v(\psi(t), t)}{R^2} + 2(1 - p^2)\big)\mathrm{d}t. \end{aligned}$$

$G^1(1)$ and $G^2(1)$ have to equal 1, which implies:

$$\int_{u^c}^{u^*} C_u(r, \psi^{-1}(r))\mathrm{d}r + 1 - u^* = R^1, \tag{2.10}$$

$$\int_{v^c}^{v^*} C_v(\psi(r), r)\mathrm{d}r + 1 - v^* = R^2. \tag{2.11}$$

A solution to Equations (2.4), (2.5) and (2.9)-(2.11) gives us $(u^c, v^c, u^*, v^*, \psi(\bullet))$ and thus fully characterises a stable assignment. Note that each of these equations needs to hold for any stable assignment and thus any stable assignment has to be represented by a solution to Equations (2.4), (2.5) and (2.9)-(2.11).

Theorem 2.1. A solution to Equations (2.4), (2.5) and (2.9)-(2.11) exists, is unique and fully characterises the unique stable assignment as specified in Remark 2.1.

The main idea behind the proof of this result (see Appendix 2.B.2) is identical to that behind the existence and uniqueness proofs in Chapter 1. I define a map, the fixed point of which is equivalent to the solution of Equation (2.9) and find a norm for which this map is a *contraction mapping*[19]. This proves that $\psi(\cdot)$ is unique *given* (u^c, v^c) – and also continuous in these two variables. Then showing existence and uniqueness is merely a matter of proving that the remaining equations have a unique solution given the function $\psi(u^c, v^c)$. Note that, as there are externalities in this model, existence does not follow from the standard results in Gretsky et al. (1992). In fact, I am not aware of any other existence results for multivariate matching with externalities.

Trivially, the existence of a stable assignment implies existence of stable matchings.

Corollary 2.1. A matching in which agents are assigned to sectors as specified in Theorem 2.1 and are positively and assortatively matched within sectors is always stable. Moreover, if the surplus functions in each sector are strictly supermodular for all possible agent-firms pairs, then this is the only stable matching.

This follows from Theorem 2.1 and the proof of Proposition 1 in Chapter 1.

2.3 Efficiency

The presence of local status concerns and occupational prestige creates externalities, as the decision of any agent to join a given sector affects the surplus produced by all the matches in that industry. Other than in Frank (1985a), these externalities are not internalised, as each occupation consists of many independent and infinitely small firms, and there is no-one to regulate the entry of agents to sector.

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[19] The norm I use is Bielecki's norm for a high-enough parameter λ.

Thus, the stable and efficient assignments are not necessarily equivalent and the question of efficiency needs to be investigated separately.

Definition 2.7. A matching $(A_A, A_F, \zeta(\bullet))$ is *efficient* if and only if it is measure consistent and the total surplus produced under any measure consistent matching $(A'_A, A'_F, \zeta(\bullet)')$ is weakly lower than the total surplus produced under $(A_A, A_F, \zeta(\bullet))$[20].

Thus, a matching is efficient if it maximises the surplus produced in the whole economy (see e.g. Becker, 1973; Gretsky et al., 1992; Chiappori et al., 2010; Lindenlaub, 2014).

Proposition 2.2. The efficient assignment is unique and equivalent to the stable assignment of a matching problem with identical output structure, but zero weights on local status and occupational prestige.

Proof. For any assignement Θ, the most efficient matching that results in Θ is positive and assortative within sectors[21]. The average status reward is always equal to $\bar{u}(\Theta)k$ in sector one and to $\bar{v}(\Theta)k$ in sector two, neither of which depends on the respective status' structures. These two facts combined imply that the total surplus produced in the economy under assignment Θ does not depend on the status components' weight and neither does efficiency. If $l^i = p^i = 0$, then surplus functions do not depend on the assignment and the results from Gretsky et al. (1992) apply. Hence, stable assignments are efficient and efficient assignments are stable. As, by Theorem 2.1, the stable assignment is unique, the uniqueness of the efficient assignment follows. $\square$

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[20] Define a function $s^i(u, v, h, i, \Theta)$, which is equal to $s^1(u, h, \Theta)$ for $i = 1$ and to $s^2(v, h, \Theta)$ for $i = 2$. Then the total surplus produced under $(A_A, A_F, \zeta(\bullet))$ is given by $\int\int_{A_A} s(u, v, \zeta(u, v), \Theta)\ \mathrm{d}u\ \mathrm{d}v$, where Θ is the assignment that results from $(A_A, A_F, \zeta(\bullet))$.

[21] This is the case, as once we fix the assignement, the model becomes equivalent to two separate Becker-Sattinger industries with surplus functions given by $\pi^1(u, h) + \tau^1(u, \Theta)$ and $\pi^2(v, h) + \tau^2(v, \Theta)$; and in a Becker-Sattinger industry PAM is both stable and efficient.

In the absence of local status and occupational prestige, there are no externalities and the stable assignment is efficient. Local status and occupational prestige create an inefficiency as they distort the relative rewards of agents: local status rewards the highly ranked agents too much, whereas occupational prestige does not reward them enough. As the total surplus produced in the economy does not depend on the structures of status, in order to find the efficient assignment it suffices to find the stable assignment for the case of no local status and occupational prestige concerns (see Section 2.2.4).

The inefficiency is caused by the distortion of relative status rewards, rather than status concerns *per se*. If the social planner tried to ignore status concerns and assigned agents to sectors in a way that maximises total output, she would also create an inefficiency. The reason is that status is not a zero-sum game in this model and thus the total status reward in the economy depends on the assignment. In other words, the social planner could make some potentially great artists feel inefficiently unappreciated, if, in her quest for maximal output, she assigned them to sciences or engineering. In fact, even if local status and occupational prestige matter and the stable assignment is inefficient, it is still possible that its total surplus will be higher than in the output maximising assignment.

If we recall the microfoundation of the status reward functions, it becomes clear that the real culprits are imperfect information and the fact that agents care more about the opinion of their peers, as these are the reasons why individual status rewards depend on local status and occupational prestige. In fact, in order to ensure efficiency of the stable matching, we need *both* perfect information about achievements and talents, and equal chances of meeting insiders and outsiders, neither of which is likely at all. This does not mean that the stable assignment is necessarily inefficient: local status and occupational prestige create externalities of opposite signs (and occur in both sectors), so it can, by pure luck, happen that these effects cancel each other out[22]. Nevertheless, other than in Fershtman et al. (1996), perfect information on its own is not enough to ensure efficiency, as it eliminates only occupational prestige, not local status.

2.4 Changes to Status Structure

In this section, I investigate how changes to the status structure affect occupational sorting and through that payoffs, wages and profits. As I have shown in my earlier work, in two sector matching models the spread of surplus is of crucial importance, especially if jobs are scarce. The more spread out surplus is, the weaker the competition between agents, which increases talent supply. This reasoning works equally well with status concerns – and as surplus' spread depends on the spread of status reward, we should expect status structure to play an important role.

Consider two matching problems, the *old* and the *new*, which meet conditions (a)-(c) from Section 2.2, as well as the requirement that every match is worthwhile. To formally distinguish between them, I introduce a parameter – ρ; the old problem is denoted by ρ_1 and the new one by ρ_2. For example, $u^c(\rho_2)$ is the new critical ability in sector one, whereas $\Theta_s(\rho_1)$ is the old stable assignment in sector two. Throughout this section, I assume that the second sector's status and output structures are the same in the old and new problems.

To define a more spread out status reward, I use the notion of spread introduced by Bickel and Lehmann (1979).

Definition 2.8. A distribution $F_X(x, \rho_2)$ is *more spread out in Bickel-Lehman*

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[22] To see this, consider symmetric sectors. Then agents sort into the sector in which they are more talented, which is the efficient outcome.

sense than distribution $F_X(x, \rho_1)$ if:

$$F_X^{-1}(u_2, \rho_2) - F_X^{-1}(u_1, \rho_2) \geq F_X^{-1}(u_2, \rho_1) - F_X^{-1}(u_1, \rho_1) \text{ for all } 0 \leq u_1 < u_2 \leq 1. \tag{2.12}$$

If there exists some x such that Equation (2.12) holds strictly for all $x \leq u_1 < u_2 \leq 1$, then distribution $F_X(x, \rho_2)$ is *more spread out, strictly from the xth quintile.*

As the status reward function $\mathrm{T}(X, \Theta)$ depends on the assignment, our definition of *status reward spread* needs to specify for which assignments does status reward become more spread out. For the results below to hold, it is sufficient that status reward becomes more spread out for the old stable assignment $\Theta(\rho_1)$.

Definition 2.9. Sector one status reward becomes *(strictly) more spread out* if the new distribution of status reward $W = \mathrm{T}(X, \Theta(\rho_1))$ is more spread out in Bickel-Lehman sense (strictly from the $u^c(\rho_1)$th quintile) than the old distribution of W.

Both an increased weight of local status and a lower weight of occupational prestige make status depend more on the individual talent, rather than the sectoral average and thus strictly spread status reward out[23]. These could be caused by improvements in information about talents and achievements – for insiders, outsiders or both – or by an increase of within-occupation interaction and socialisation (recall Section 2.2.3).

Theorem 2.2. If sector one status reward becomes strictly more spread out, then (i) the distribution of talent in sector one improves in first order stochastic dominance sense and (ii) the distribution of talent in sector two deteriorates in first order stochastic dominance sense. The prestige of occupation one strictly increases and the prestige of occupation two falls strictly.

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[23] An increase in the importance of status, k, makes status reward more spread out in both sectors and thus the results below do not hold for it.

Proof. If sector one status reward becomes strictly more spread out, so does sector one surplus, and all results follow immediately from Theorem 2.3 in Appendix 2.C. □

The intuition is very simple. Scarcity of jobs implies that the reservation payoff in sector one is fixed. Hence, ignoring reallocation, a strictly more spread out status reward increases payoffs for all sector one agents. In general equilibrium this attracts additional talent from sector two, increases talent supply in sector one and decreases it in sector two. In other words, the more does status depend on individual achievements in a sector, the more attractive is that sector for highly talented people.

The link between surplus' spread and occupational prestige is also quite interesting, as it implies that the less important occupational prestige is, the more prestigious the occupation. More generally, the prestige of an occupation is not the reason why that profession is able to attract top talent, but its result. This, in turn, is more likely if the information about individual achievements is readily available. Given that this is largely the case in academia, at least for insiders (think of Hirsch-index, citations or ideas.repec rankings), it could be part of the reason why top talent still joins academia, despite lower wages (see Weiss and Lillard, 1978).

The negative effects of surplus' spread on the status reward of lowly ranked agents are at least partially mitigated by the resulting increase in occupational prestige. In fact, it is possible that the general equilibrium effect of higher occupational prestige dominates the direct impact of spread, if the stable assignment is sensitive to small changes in spread and the weight on occupational prestige is high[24]. As the wages paid to the least talented agents depend on their status reward, it follows that the direction of change in sector one wage levels is am-

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[24] I discuss the conditions needed for that in Appendix 2.C.1 in detail, but in general the spread of output in sector one and surplus in sector two need to be low compared to k.

biguous. Much more can be said, however, about sector two wages, as well as total payoffs in both sectors.

To simplify the following discussion on wages and payoffs I focus on the case of strictly scarce jobs ($R^1+R^2<1$); note that all results hold also for $R^1+R^2=1$ if we assume that $C^1+\tau^1(u^c)$ does not change and, additionally, that the surplus function in sector two is strictly supermodular.

Proposition 2.3. If jobs are strictly scarce and sector one status reward becomes strictly more spread out, then (i) both wages and total payoffs strictly increase for all agents in sector two; (ii) in sector one, total payoffs strictly increase for a positive mass of the most talented agents and strictly fall for a positive mass of the least talented agents; (iii) the gap between top and bottom total payoffs increases strictly in both sector and the whole economy, but (iv) the gap between top and bottom wages shrinks strictly in sector one, strictly widens in sector two and its change in the whole economy is ambiguous.

Proof. Firstly, note that Theorem 2.2 implies that v^c decreases and u^c increases. In fact, for strictly scarce jobs both of these changes are strict, which follows from Theorem 2.3 in Appendix 2.C.

(iv) For any $v''>v'\geq v^c$ we have:

$$w^2(v'')=\int_{v'}^{v''}\pi_v^2(r,G^2(r))\ \mathrm{d}r+w^2(v'). \tag{2.13}$$

A deterioration in sector two talent distribution in first order stochastic dominance sense implies that $G^2(r)$ increases for all r and therefore, as surplus is supermodular, $w^2(v'')$ increases more than $w^2(v')$. The strict fall in v^c implies that $w^2(v^c)$ increases by strictly more than C^2 and thus the difference between $w^2(1)$ and C^2 increases strictly. An analogous reasoning holds for sector one, except that there talent distribution improves and, hence, $G^1(r)$ decreases for all r and u^c strictly increases. Clearly, the change in the difference between the overall highest and overall lowest wage can go both ways.

(i) As v^c falls strictly, so does the public status reward of the least talented sector two agent. Since the local status of the lowest ranked agent is always equal to -1 and occupational prestige falls strictly, it follows that $\tau(v^c)$ decreases strictly. Thus, by Lemma 2.1 the lowest wage $w^2(v^c) = C^2$ increases strictly and the strict increase in wages follows from the proof of (iv).

The public status reward remains unchanged for agent of any talent. Thus, the change in total payoff depends on the change in local status reward, occupational prestige and wage. Local status reward increases by Theorem 2.2 and the increase in wage has to be strictly greater than the fall in occupational prestige reward[25]. Therefore, total payoff increases strictly.

(ii) Denote sector i total payoffs as $t^i(\cdot)$. I start at the top. Theorem 2.2 and the proof of Lemma 2.6 (Appendix 2.C) imply that $u^*(\rho_2) \leq u^*(\rho_1)$ and $v^*(\rho_2) \geq v^*(\rho_1)$. Thus:

$$\begin{aligned} t^1(u^*(\rho_1),\rho_2) \geq t^1(u^*(\rho_2),\rho_2) = t^2(v^*(\rho_2),\rho_2) \geq t^2(v^*(\rho_1),\rho_2) \\ t^2(v^*(\rho_2),\rho_1) \geq t^2(v^*(\rho_1),\rho_1) = t^1(u^*(\rho_1),\rho_1) \geq t^1(u^*(\rho_2),\rho_1) \end{aligned}$$

thich trivially implies that:

$$t^1(u^*(\rho_1),\rho_2) - t^1(u^*(\rho_1),\rho_1) \geq t^2(v^*(\rho_2),\rho_2) - t^2(v^*(\rho_2),\rho_1). \tag{2.14}$$

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[25] To see the latter, note that as v^c fell strictly, the increase in $w^2(v^c(\rho_1))$ is strictly greater than the increase in C^2, which in turn is at least as great as the fall in occupational prestige reward. Formally, for any $v \geq v^c(\rho_1)$:

$$\begin{aligned} w^2(v,\rho_2) - w^2(v,\rho_1) \geq w^2(v^c(\rho_1),\rho_2) - w^2(v^c(\rho_1),\rho_1) > C^2(\rho_2) - C^1(\rho_1) \\ = -2k(v^c(\rho_2) - v^c(\rho_1) + \bar{v}(\rho_2) - \bar{v}(\rho_1)) > -2k(\bar{v}(\rho_2) - \bar{v}(\rho_1)). \end{aligned}$$

Thus, $t^1(u^*(\rho_1)$ strictly increases. For any $u > u^*$ we have that:

$$t^1(u) = \int_{u^*}^{u} \pi_u^1(r, G^1(r)) + 2(l^1 - p^1 + 1))\ \mathrm{d}r + w(u^*(\rho_1)). \tag{2.15}$$

For $u > u^*(\rho_1)$, $G^1(u)$ does not change; and as strict spread of status reward implies that l^1 or $1 - p^1$ strictly increase, it follows that $t^1(u, \rho_2) > t^1(u, \rho_1)$ for any $u \in [u^*(\rho_1), 1]$.

I will turn now to payoffs of the least talented agents. Recall that u^c strictly increases. As total payoff strictly increases in talent, it follows from definition of critical ability that $t^1(u^c(\rho_2), \rho_1) > t^1(u^c(\rho_1), \rho_1) = 0 = t^1(u^c(\rho_2), \rho_2)$. Existence of a positive mass of agents for whom wages decrease (increase) follows from continuity of wage functions.

(iii) Follows from (i), (ii) and the fact that with strictly scarce jobs the lowest payoff is fixed at the reservation status. $\square$

The effect on agents' payoffs is similar to that of a spread in surplus on wages in Chapter 1. In sector two, talent supply falls and therefore payoffs increase. Sector one payoffs are affected positively by the increase in status reward spread and negatively by the higher supply of talent. For agents with highest talent, the effect of the increase in spread dominates and their payoffs raise. For the least talented agents the increase in talent supply dominates: those agents are pushed down the ladder so much that their payoffs fall.

As the increase in status reward spread increases top payoffs in both sectors and the lowest payoffs are fixed at the reservation payoff, the gap between best and worst off workers increases. The change in wage inequality between top and bottom earners is, however, ambiguous – and definitely decreases in sector one, because of the increased talent supply. This has two important implications. Firstly, it means that changes in status structure can have an impact not only on wages levels, but also inequality. Secondly, it suggests that with status concerns, wage inequality does not tell the entire story, as it is possible that wage inequality falls (rises), despite a rise (fall) in overall inequality.

The existing literature predicts a negative relationship between the difference in occupational prestige and the difference in wages, so that, all things equal, agents should earn less in more prestigious occupation (see Book I, Chapter X, Part I in (Smith, 1776) or (Fershtman and Weiss, 1993)). This general principle holds here too, but the structures of status belong to the things that need to be equal. As status reward spreads out, the local and public status rewards fall for the least talented agents, which might not be fully compensated by the increase in occupational prestige.

Proposition 2.4. It is possible that, as a result of a sufficiently more spread out status reward in sector one, the lowest increase in sector one wage is greater than the highest increase in sector two wage, despite the fact that sector one becomes more and sector two less prestigious.

Proof. The change in wages in each sector depends on the change in wage constant C^i and the change in wage spread, which for sector one is given by $\int_{u^c}^{u} \pi_u^1(r, G^1(r))\,\mathrm{d}r$. Suppose that both the maximal spread of output, measured by $\pi^i(1,1) - \pi^i(0,0)$, and the maximum status reward $l^i k + p^i k$, are arbitrarily small compared $\pi^i(0,0) - \bar{\mathrm{T}}$, for $i \in \{0,1\}$. Suppose further that the weight of local status increases. Then the increase in C^2 is bounded from above by $2k$ and the increase in C^1 is bounded from below by $2k(l^1(\rho_2) - l^1(\rho_1) - 1)$. Note that wage spread in each sector is bounded from below by 0 and from above by the maximal spread of output. Thus, the increase in wage in sector two is bounded from above by $2k + \pi^2(1,1) - \pi^2(0,0)$ and the increase in wage in sector one is bounded from below by $2k(l^1(\rho_2) - l^1(\rho_1) - 1) + \pi^1(1,1) - \pi^1(0,0)$. It follows that for high enough $l^2(\rho_2)$ the lowest increase in sector one wage (for $w^1(1)$) is greater than the highest increase in sector two wage (for $w^2(1)$).

□

Thus, changes in status structure can result in an increase in both the prestige of an occupation and the wages earned by its members. This can explain to some extent the mixed empirical evidence for the negative link between wages and occupational prestige, as reported in Fershtman and Weiss (1993).

As changes in status structure have a strong impact on occupational sorting, it is natural to ask whether this can be used to cheaply attract more talent to a sector, for example by creating rankings or establishing awards. This question is, in general, out of the scope of this paper – nevertheless, the following result suggests that status structure manipulations could, in certain circumstances, lead to an increase in profits in the industry in which they took place.

Proposition 2.5. If jobs are strictly scarce and sector one status reward becomes strictly more spread out, then (i) profits fall for all sector two firms and (ii) sector one profits can both fall or increase for all firms, but more productive firms always gain more (or lose less).

Proof. (i) The profit function for all matched firms in sector i is given by (see Sattinger, 1979):

$$r^i(h) = \int_0^h \pi^i_h((G^i)^{-1}(r), r)\mathrm{d}r + C^i_P, \tag{2.16}$$

where the profit constant C^i_P is equal to the difference between the output produced by the worst match ($\pi^i(x^c, 0)$) and the wage constant C^i. A v^c falls and C^2 increases, C^2_P has to fall. Thus, it follows from Theorem 2.2 and inspection of Equation (2.16) that profits fall strictly in sector two.

(ii) Equation (2.16) implies that for any $h'' > h'$, we have:

$$r^i(h) = \int_{h'}^{h''} \pi^i_h((G^i)^{-1}(r), r)\mathrm{d}r + r^i(h'),$$

which, by Theorem 2.2 and supermodularity of surplus functions, means that

$r^1(h'', \rho_2) - r^1(h'', \rho_1) \geq r^1(h', \rho_2) - r^1(h', \rho_2)$ and thus the more productive sector one firms gain more than the less productive ones. In general, there are two channels through which sector one profits are affected: the improvement in talent-pool and the ambiguous change in profit constant. The fact that all sector one profits can increase follows immediately from the fact that C^1 can fall (see Appendix 2.C.1). To see that all sector one profits can fall, suppose that both the maximal spread of output, measured by $\pi^i(1,1) - \pi^i(0,0)$, and the maximum status reward $l^1k + p^1k$, are arbitrarily small compared to $\pi^i(0,0) - \bar{\mathrm{T}}$, for $i \in \{0,1\}$. Then a sufficiently large increase in l^i will decrease the profit constant by more than the maximum possible increase in profts spread (by a reasoning analogous to the proof of Proposition 2.4). □

The fall in talent supply in sector two has a negative impact on profits through two channels. Firstly, it increases the competition for talent. Secondly, it decreases the status reward of the least talented agents, which makes the outside option more attractive and forces firms to pay them more. In sector one, on the other hand, the increased talent supply has a positive effect on profits: by decreasing competition and making the outside option less desirable. The increase in status reward spread itself, however, has a negative effect on the lowest status rewards and thus on firm profits. The end effect is ambiguous for all firms, but the more productive ones will always gain more, or at least lose less, than the less productive ones.

2.5 Prestige Rankings

In this section I study how do occupations rank with respect to prestige. There is a large sociological literature on this subject and its perhaps most striking finding is the remarkable stability of those rankings across time and countries (see Treiman, 1977). The economic literature on that subject is much smaller and suggests that the ranking of occupations is driven by the spread of skill distribution (Roy, 1951) or by skill intensity (Weiss and Fershtman, 1992). These explanations are very similar and both capture the fact that in some occupations talent is in higher demand – those profession attract more of it and, hence, are more prestigious. I will show that although this logic is very sound, the ranking of occupations according to prestige depends also on status structures in each sector.

The notion of *more spread out output* captures well the idea that sectors can vary in talent intensity.

Definition 2.10. Output $O(\rho_2)$ is *(strictly) more spread out* than output $O(\rho_1)$ if the distribution of output $O(\rho_2, h)$ is (strictly) more spread out in Bickel-Lehman sense than the distribution of $O(\rho_1, h)$, for firms of all ranks $h \in [0, 1]$.

As in Chapter 1, the spread of output can increase in response to an increase in real spread of the marginal distribution of the relevant skill, an improvement in productivity distribution in first order stochastic sense or changes in the actual output function. In terms of the copula formulation, a (strict) increase in output spread is equivalent to an increase in the marginal output of talent, for any talent-firm pair: $\pi_u^1(u, h, \rho_2) \geq (>) \pi_u^1(u, h, \rho_1)$. Both increases in the spread of output and in the spread of status reward result in a more spread out surplus and thus have similar effect on talent supply and occupational prestige in each sector.

Proposition 2.6. If sector one output becomes strictly more spread out, then (i) the distribution of talent in sector one improves in first order stochastic dominance sense and (ii) the distribution of talent in sector two deteriorates in first order stochastic dominance sense. The prestige of occupation one strictly increases and the prestige of occupation two falls strictly.

Proof. If sector one output becomes strictly more spread out, so does sector one surplus, and all results follow immediately from Theorem 2.3 in Appendix 2.C. □

An increase in the spread of output makes agents more differentiated with respect to talent and decreases competition between them. This increases wages and attracts new talent; thus, talent supply increases in sector one and falls in sector two[26]. Therefore, an increase in the spread of output has the expected impact on occupational prestige: the more spread out output, the more prestigious the profession. However, the analysis in Weiss and Fershtman (1992) suggests something more than that: and namely that the ranking of occupations with respect to prestige depends *only* on the relative skill intensity. In the remainder of this section I study whether this remains true with asymmetric status structures.

The sectors in Weiss and Fershtman (1992) and Roy (1951) are not capacity constrained. To account for the fact that asymmetry in sectors sizes could be an additional factor influencing my results, in what follows I focus on equal sized sectors ($R^1 = R^2$).

Proposition 2.7. If sectors are of equal size, the talent copula is symmetric and output in sector two is strictly more spread out than output in sector one, then, in the efficient assignment, (i) the talent distribution in sector two first order stochastically dominates the talent distribution is sector one and (ii) occupation two is strictly more prestigious than occupation one.

Proof. Consider two matching problems, the old and the new one. The new matching problem, denoted by ρ_2, has identical output functions as the original one, but its status structures are given by $l^i = p^i = 0$. Note that the stable assignment of this problem is identical to the efficient assignment of the original problem. The old problem is symmetric, with $l^i = p^i = 0$ and the output functions

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[26] The effect on wages levels and spread in both sectors is also qualitatively the same as in Chapter 1.

in both sectors equal to $\pi^1(\cdot, h)$, and thus the within-sector distributions of talent are identical. The new problem can be created by (strictly) spreading out old problem's sector two output. Hence, it follows from Proposition 2.6 that (i) $G^2(\cdot, \rho_2)$ first order stochastically dominates $G^1(\cdot, \rho_1)$, which in turn first order stochastically dominates $G^1(\cdot, \rho_2)$ and (ii) $\bar{v}(\rho_2) > \bar{v}(\rho_1) > \bar{u}(\rho_2)$. $\square$

As there is no asymmetry in the way society values the two dimensions of talent, the ranking of occupations with respect to prestige in the efficient assignment depends only on the relative spread of output, as in Weiss and Fershtman (1992). In the stable assignment, however, status structures are going to play an important role.

Proposition 2.8. If sectors are of equal size and status rewards are sufficiently more spread out in sector one than sector two, then, in the stable assignment, (i) the talent distribution in sector one first order stochastically dominates the talent distribution is sector two and (ii) occupation one is strictly more prestigious than occupation two, even if output is more spread out in sector two than one.

Proof. Denote the stable assignment of a symmetric matching problem as Θ_{sym}. As π_v^2 is continuous and defined over a compact set, $\sup \pi_v^2$ is finite and thus, if $t_u^1(u, \Theta_{sym}) - t_v^2(u, \Theta_{sym})$ is sufficiently large for all $u \in [u^c(\Theta_{sym}), 1]]$, then $s_u^1(u, h, \Theta_{sym})$ is greater than $s_v^2(u, h, \Theta_{sym})$ for all $(u, h) \in [u^c(\Theta_{sym}), 1] \times [0, 1]$. Thus, results (i) and (ii) follow by Theorem 2.3 in Appendix 2.C and a reasoning analogous to that in the proof of Proposition 2.7. $\square$

It is possible that the sector with less spread out output attracts inefficiently many talented agents – and thus the ranking of occupations with respect to prestige can get inverted – if local status is sufficiently important in that sector and occupational prestige is sufficiently unimportant[27]. The empirical stability of

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[27] Note that it might be impossible for status to be sufficiently spread out for this ranking reversal to be feasible. This is the case if the output in sector two is sufficiently spread out

prestige rankings implies that either the spread of status rewards is typically less important than the spread of output in determining the spreads of surplus, or that the status structures of occupations are also quite stable. As status structures depend on information about achievements and the degree of inter-occupation interaction, this is quite plausible.

2.6 Conclusions

The individual and collective aspects of status have very different implications for occupational sorting. Therefore, the relative importance of these components – the structure of status – can influence the way in which agents self-select into sectors. Moreover, as the existence of local status concerns and taste for occupational prestige are likely driven by informational constraints, as well as the asymmetry in meetings with peers and outsiders, there are good reasons to believe that status structure differs across sectors. I have show that in my two-sector matching model, an increase in the weight of the individual components or a fall in the weight of the collective component in some sector result in an increase in talent supply in that industry and a fall in talent supply in the other sector. This has, in turn, important implications for total payoffs, wage levels and inequality, and profits in both sectors. I have also demonstrated that the inefficiency of the stable assignment of agents to sectors is caused by local status concerns and taste for occupational prestige, not by status concerns *per se*. Finally, I have shown that if status rewards are sufficiently more spread out in the sector with less spread out output, the efficient ranking of occupations with respect to prestige will get reversed.

The specific implications of changes in status structure for talent supply, wages

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compared to the difference between output level and reservation status ($\pi^1(0,0) - \bar{\mathrm{T}}$), or jobs are only weakly scarce and $(1-p^1)k$ is not high enough.

and profits might depend on my assumptions, the chief of which is that jobs are scarce. Without this assumption the model becomes much less tractable: in particular, the method used in this paper to prove existence and uniqueness would yield only existence results with abundant jobs – the number of equilibria is an open question. Another issue that should be addressed in future research is the relative weighting of status dimension by society; in this paper, they are valued equally highly. An asymmetric, but still exogenous weighting would not complicate the model too much, but neither would add much to our understanding. Allowing for asymmetric and endogenous weightings could, however, help us better understand how output and status structures influence the relative importance of talent dimensions.

Appendix

2.A Microfoundation of Status Rewards

I keep the assignment Θ fixed throughout this section and suppress it from notation. The distribution of talent among sector i agents is $G^i(u,v)$, and its marginals are G^i_V and G^i_U. After agents self-select into sectors, each draws one Judge from all agents, so simply from the copula $C(u,v)$, and f^i Judges from her sector, so from $G^i(u,v)$. Denote the talent vector of the Judge as (u^J, v^J). The Judge observes the sector in which the agent works and some signal about her ability. Then she uses this information to establish how likely it is that the agent ranks higher and, based on that, grants her some positive or negative status utility. The status utility received by an agent (u,v) from a Judge (u^J, v^J) is given by:

$$\tau(u,v,u^J,v^J) = \begin{cases} \big[\Pr(u \geq u^J) - \Pr(u < u^J)\big]k & \text{if } \theta(u,v) = 1, \\ \big[\Pr(v \geq v^J) - \Pr(v < v^J)\big]k & \text{if } \theta(u,v) = 2. \end{cases} \tag{2.17}$$

Information is modelled in the simplest possible way. The true talent of an agent from sector i is observed by a Judge from the same sector with probability n^i, whereas with probability $1-n^i$ only the agent's sector is observed. If the Judge is from the other sector, then the agents talent is observed with probability o^i and with probability $1-o^i$ only her sector is observed. Insiders have better information than outsiders:

$$n^i \geq o^i.$$

Suppose that $M^1, M^2 > 0$[28]. I first derive the expected utility from non-work meetings. Suppose that $\theta(u^J, u^J) = i$ and $\theta(u, v) = 2$. The Judge observes the agents ability with probability o^2 if $i = 1$ and n^2 if $i = 2$ and grants her utility p if $v \geq v^J$ and $-p$ if $v^J > v$. The probability of drawing a Judge who is better than the agent is $G^i_V(v)$. However, with probability $(1 - o^2)$ or$(1 - n^2)$ – depending on her sector – the Judge will not observe the actual ability and only infer it from the fact that the agent works in sector two; in which case the probability that the Judge is better (worse) than the agent is $G^2_V(v^J)$. Finally, the probability of drawing a Judge from sector one is M^1 and the probability of drawing a Judge from sector two is M^2. Thus the expected status utility for a sector two agent from after work meetings is given by:

$$\begin{aligned}\tau^2_a(v) = \Big[&M^1\big[o^2(2G^1_V(v) - 1) + (1 - o^2)\int_0^1 (1 - 2G^1_V(v))g^1_V(v)\ \mathrm{d}v\big] \qquad (2.18)\\ &+M^2\big[n^2(2G^2_V(v) - 1) + (1 - n^2)\int_0^1 (1 - 2G^2_V(v))g^2_V(v)\ \mathrm{d}v\big]\Big]k.\end{aligned}$$

As every agent has to join either sector one or two, we have

$$M^1 G^1_V(v) + M^2 G^2_V(v) = v.$$

Note also that $\int_{y_l}^{y_h}(1 - 2G^2_V(v))g^2_V(v)\ \mathrm{d}v$ is equal to zero. Thus we can write:

$$\begin{aligned}\tau^2_a(v) &= \Big[o^2(2(v - M^2G^2_V(v)) - M^1) + M^2n^2(2G^2_V(v) - 1)\\ &\quad +M^1(1 - o^2)\big[1 - [2G^2_V(v)G^1_V(v)]^1_0 + 2\int_0^1 G^1_V(v)g^2_V(v)\mathrm{d}\ v\big]\Big]k\\ &= \Big[(n^2 - o^2)M^2\big(2G^2_V(v) - 1\big) + o^2\big(2v - 1\big) + (1 - o^2)\big[2\bar{v} - 1\big]\Big]k. \qquad (2.19)\end{aligned}$$

---

[28] This is always the case in any stable matching.

The expected status utility derived from work-meetings is, by analogous reasoning, given by:

$$\tau_w^2(v) = f^2 n^2 \big(2G_V^2(v) - 1\big),$$

and so the total expected status reward is equal to:

$$\tau^2(v) = \Big[\big(n^2 f^2 + (n^2 - o^2)\big) M^2 \big(2G_V^2(v) - 1\big) + o^2\big(2v - 1\big) + (1 - o^2)\big[2\bar{v} - 1\big]\Big] k.$$

And as the problem is symmetric, the status reward in sector one is given by:

$$\tau^1(u) = \Big[\big(n^1 f^1 + (n^1 - o^1)\big) M^1 \big(2G_U^1(u) - 1\big) + o^1\big(2u - 1\big) + (1 - o^1)\big[2\bar{u} - 1\big]\Big] k.$$

2.B Stable Assignments

Proof of Lemma 2.1. This follows trivially for $R^1 + R^2 < 1$, as then $1 - M^1 - M^2 > 0$ and thus $C^1 + \tau^1(u^c) = C^2 + \tau^2(v^c) = \bar{\mathrm{T}}$, by Proposition 2.1. Therefore I will focus on the cases where $R^1 + R^2 = 1$ and thus $C(u^c, v^c) = 0$. This implies that $\min\{u^c, v^c\} = 0$. Suppose that $u^c = 0$ and $C^1 + \tau^1(u^c) > C^2 + \tau^2(v^c)$; as the payoff in the second sector is continuous and $v^c < 1$, there has to exist some $\epsilon > 0$ such that for any $v \in [v^c, v^c + \epsilon]$ we have $C^1 + \tau^1(u^c) > t^2(v) + \tau^2(v)$, which means that none of these agents will join sector two – which contradicts v^c's definition.

Now, suppose that $C^1 + \tau^1(u^c) < C^2 + \tau^2(v^c)$; suppose further that $v^c > 0$. For any $\epsilon > 0$ the mass of agents with $(u, v) \in [0, \epsilon] \times [v^c - \epsilon, v^c]$ is strictly positive – as $C_{uv}(u, v) > 0$ for all (u, v) – and all agents in this set will be working in sector one. However, by continuity of surplus and wage functions it has to be the case that there exists a small enough ϵ that for all agents in this set we have:

$$t^1(u) + \tau^1(u) + r^2(0) = t^1(u) + \tau^1(u) + s^2(v^c, 0) - C^2 - \tau^2(v^c) < s^2(v, 0),$$

which contradicts stability. Now suppose that $v^c = 0$ as well – there has to exist some $\epsilon > 0$ such that for any $u \in [0, \epsilon]$ we have $C^1 + \tau^1(u) < C^2 + \tau^2(v)$, which contradicts the definition of u^c.

The proof for the case when $v^c = 0$ is analogous. □

2.B.1 Talent Distributions

The probability that an agent with talent $V = v$ chooses sector two is $\Pr(\theta(U, v) = 2|v)$. For $v \in [v^c, v^*]$ the probability that a sector two agent has ability lower than v is:

$$G^2(v) = \int_{v^c}^{v} \frac{\Pr(\theta(U, v) = 2|r)}{R^2} \mathrm{d}r,$$

as V's marginal distribution is standard uniform. Consider some arbitrary agent with $v \in (v^c, v^*]$. Such an agent will be in sector 2 as long as $\psi(v) \geq u$, which implies that $\Pr(\theta(U, v) = 2|v) = C_v(\psi(v), v)$[29]. Recalling the definitions of v^c and v^*, expression (2.8) follows. The talent distribution in sector one can be derived by an analogous reasoning.

2.B.2 Existence and Uniqueness

Proof of Theorem 2.1. The proof of this Theorem is very similar to the proof of Theorem 1 in Chapter 1. I start by reducing the set of equations (2.4), (2.5), (2.9)-(2.11), which will be henceforth referred to as *the original set.* Consider any $r \in [v^c, v^*]$; then, by differentiating $C(\psi(r), r)$ rearranging and integrating from v^c to v, we arrive at:

$$R^1 G^1(\psi(v)) + R^2 G^2(v) = C(\psi(v), v). \tag{2.20}$$

---

[29] It doesn't matter whether $\psi(v) \geq u$ holds strictly, as the probability of $\psi(v) = u$ is 0 anyway.

An analogous procedure for $C(r, \psi^{-1}(r)$ gives

$$R^1 G^1(u) + R^2 G^2(\psi^{-1}(u)) = C(u, \psi^{-1}(u)).$$

This, (2.10) and (2.11) imply that $\psi(v^*) = u^*$, which follows also from the definitions of u^* and v^*.

By differentiating Equation (2.9), rearranging, using Equation (2.20) and $o^i = 1 - p^i$ and then integrating from v^c to v (and remembering that $\psi(v^c) = u^c$) we get:

$$\psi(v) = u^c + \int_{v^c}^{v} \frac{\pi_v^2\Big(t, \frac{\int_{v^c}^{t} C_v(\psi(r),r)\mathrm{d}r}{R^2}\Big) + 2k\big(\frac{l^2}{R^2} C_v(\psi(t),t) + o^2\big)}{\pi_u^1\Big(\psi(t), \frac{C(\psi(t),t) - \int_{v^c}^{t} C_v(\psi(r),r)\mathrm{d}r}{R^1}\Big) + 2k\big(\frac{l^1}{R^1} C_u(\psi(t),t) + o^1\big)} \mathrm{d}t.$$

This still depends on $\psi(\cdot), u^c, v^c$ and indirectly on v^*. I will eliminate v^* by extending the functions $C(\bullet)$, $C_v(\bullet)$, $C_u(\bullet)$, $\pi^1(\bullet)$ and $\pi^2(\bullet)$ in a way that allows to define an extended function $\psi^e(\cdot)$, which uniquely determines $\psi(\cdot)$. The extended functions $C^e(\bullet)$, $C_v^e(\bullet)$, $C_u^e(\bullet)$, $\pi^{1e}(\bullet)$ and $\pi^{2e}(\bullet)$ are defined as follows:
(1) $C^e : [0, 1 + B] \times [0, 1] \to [0, 1]$

$$C^e(u, v) = \begin{cases} C(u, v) & \text{for } (u, v) \in [0, 1] \times [0, 1] \\ v & \text{for } (u, v) \in (1, 1 + B] \times [0, 1], \end{cases}$$

(2): $C_v^e(u, v) : [0, 1 + B] \times [0, 1] \to [0, 1]$

$$C_v^e(u, v) = \begin{cases} C_v(u, v) & \text{for } (u, v) \in [0, 1] \times [0, 1] \\ 1 & \text{for } (u, v) \in (1, 1 + B] \times [0, 1], \end{cases}$$

(3) $C_u^e(u,v) : [0, 1 + B] \times [0, 1] \to [0, 1]$

$$C_v^e(u,v) = \begin{cases} C_u(u,v) & \text{for } (u,v) \in [0,1] \times [0,1] \\ C_u(1,v) & \text{for } (u,v) \in (1, 1+B] \times [0,1], \end{cases}$$

(4) $\pi_v^{1e}(u,h) : [0, 1 + B] \times [0, \frac{1}{R^1}] \to \mathbf{R}^+$:

$$\pi_v^{1e}(u,h) = \begin{cases} \pi_u^1(u,h) & \text{for } (u,h) \in [0,1]^2 \\ \pi_u^1(1,h) & \text{for } (u,h) \in (1, B] \times [0,1], \\ \pi_u^1(u,1) & \text{for } (u,h) \in [0,1] \times (1, \frac{1}{R^1}], \\ \pi_u^1(1,1) & \text{for } (u,h) \in (1, B] \times (1, \frac{1}{R^1}], \end{cases}$$

(5): $\pi_u^{2e}(v,h) : [0, 1] \times [0, 1 + \frac{1}{R^2}] \to \mathbf{R}^+$:

$$\pi_u^{2e}(v,h) = \begin{cases} \pi_v^2(v,h) & \text{for } (u,h) \in [0,1]^2 \\ \pi_v^2(v,1) & \text{for } (u,h) \in [0,1] \times (1, \frac{B+1}{R^2}], \end{cases}$$

where $B = \frac{2k(\frac{l^2}{R^2}+o^2)+\max \pi_v^2}{o^1+\min \pi_u^1}$. The idea behind these extensions is to get functions that will be defined also for $\psi^e(v) > 1$ and such that $C^e(\cdot, v)$, $C_v^e(\cdot, v)$, $C_u^e(\cdot, v)$, $\pi_u^{1e}(\cdot, \cdot)$ and $\pi_v^{2e}(v, \cdot)$ are Lipschitz continuous[30]; denote their Lipschitz-constants as L^1, L^2 ,L^3, L^4, L^5 and L^6 respectively. The fact that, for $u > 1$, $C_u^e(\bullet)$ is not a

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[30] I will do this in detail for $C_v^e(u,v)$ – the reasoning for the other two is analogous. $C_v^e(u,v)$ is clearly continuous in u. It is equally easy to see that the function $C_v^e(\cdot, v)$ is differentiable almost everywhere and its derivative is Lebesque integrable. It is also the case that for any $(u,v) \in (1, 1+B] \times [0,1]$ we have:

$$C_v^e(a,v) + \int_a^1 C_{uv}^e(r,v)\mathrm{d}r + \int_1^u 0\mathrm{d}r = 1,$$

which means that $C_v^e(\cdot, v)$ is absolutely continuous. Moreover, as $C^e(\bullet)$ is twice continuously differentiable and any continuous function defined on a compact set is bounded it follows that $C_{vv}^e(\cdot, v)$ is essentially bounded; and a differentiable almost everywhere, absolutely continuous function with an essentially bounded derivative is Lipschitz-continuous.

derivative of $C^e(\bullet)$ does not matter, as $C_u^e(\bullet)$ is clearly an extension of $C_u(\bullet)$ to $[0,1] \times [0, 1+B]$.

Now I can define the extended function $\psi^e(v) : [v^c, 1] \in [u^c, 1+B]$:

$$\psi^e(v) = u^c + \int_{v^c}^{v} \frac{\pi_v^{2e}\Big(t, \frac{\int_{v^c}^{t} C_v^e(\psi^e(r),r)\mathrm{d}r}{R^2}\Big) + 2k\big(\frac{l^2}{R^2} C_v^e(\psi^e(t),t) + o^2\big)}{\pi_u^{1e}\Big(\psi^e(t), \frac{C^e(\psi^e(t),t) - \int_{v^c}^{t} C_v^e(\psi^e(r),r)\mathrm{d}r}{R^1}\Big) + 2k\big(\frac{l^1}{R^1} C_u^e(\psi^e(t),t) + o^1\big)} \mathrm{d}t. \tag{2.21}$$

which together with:

$$1 - R^1 - R^2 = C^e(u^c, v^c), \tag{2.22}$$

$$R^2 = \int_{v^c}^{v^*} C_v^e(\psi(r), r)\mathrm{d}r + 1 - v^*, \tag{2.23}$$

$$v^* = \sup\{v \in [v^c, 1] : \psi^e(v) \leq 1\}, \tag{2.24}$$

$$u^* = \psi^e(v^*) \tag{2.25}$$

constitutes the *modified set* of equations.

Proposition 2.9. The solution to the modified set of equations exists and is unique.

Proof. Define the set:

$$K = \{d \in C[0,1] : |d(v) - 1| \leq 1 + B\},$$

where $C[0,1]$ is the set of all continuous functions that map from $[0,1]$. The constant function $d(v) = 1$ lies in K and hence the set is non-empty. Define a norm, $||\cdot||_\lambda$ on $C[0,1]$:

$$||h||_\lambda = \sup_{[0,1]} e^{-\lambda v} |h(v)|,$$

where λ is some weakly positive number. K is a complete metric space for this norm.[31]

Endow the set $[0,1]^2$ with the Euclidean norm and define mappings $G^2 : K \times [v^c, 1] \times [0,1] \to [0,1], s^2 : K \times [v^c, 1] \times [0,1] \to [0, 2k(\frac{l^2}{R^2} + o^2) + \max \pi_v^2]$, $G^1 : K \times [u^c, 1+B] \times [v^c, 1] \times [0,1]^2 \to [0, \frac{1}{R^1}]$, $s^1 : K \times [u^c, 1+B] \times [v^c, 1] \times [0,1]^2 \to [0, 2k(\frac{l^1}{R^1} + o^1) + \min \pi_u^1]$ and $T : K \times [0,1]^2 \to K$:

$$
\begin{aligned}
(G^2 d)(v, v^c) &= \frac{1}{R^2} \int_{v^c}^{v} C_v^e(d(r), r) \mathrm{d}r, \\
(G^1 d)(u, v, v^c, u^c) &= \frac{C^e(u, v) - R^2 (G^2 d)(v)}{R^1}, \\
(s_v^2 d)(v, v^c) &= \pi_v^{2e}\big(v, (G^2 d)(v, v^c)\big) + 2k\Big(\frac{l^2}{R^2} C_v^e(d(v), v) + o^2\Big) \\
(s_v^1 d)(u, v, v^c) &= \pi_u^{1e}\Big(u, (G^2 d)(u, v, v^c)\Big) + 2k\Big(\frac{l^1}{R^2} C_u^e(d(v), v) + o^1\Big) \\
(Td)(v, v^c, u^c) &= \begin{cases} u^c & \text{for } v < v^c \\ u^c + \int_{v^c}^{v} \frac{(s_v^2 d)(v, v^c)}{(s_v^1 d)(d(v), v, v^c)} \mathrm{d}t & \text{for } v \geq v^c. \end{cases}
\end{aligned}
$$

These maps are well-defined, as for any $v^c \in [0,1]$ and $d \in K$:

$$
\begin{aligned}
(G^2 d)(v, v^c) &\leq \int_{v^c}^{t} \frac{1}{R^2} \mathrm{d}r \leq \frac{1}{R^2} \\
(G^1 d)(u, v, v^c, u^c) &\leq \frac{C(d(t), t)}{R^1} \leq \frac{1}{R^1}.
\end{aligned}
$$

Note that $(Td)(\bullet)$ is continuous in v, v^c and u^c. It is also the case that for $v \geq v^c$:

$$
|[(Td)(v, v^c, u^c, M^2) - 1| \leq \int_{v^c}^{v} B \mathrm{d}t + |u^c - 1| \leq 1 + B,
$$

and for $v < v^c$:

$$
|[(Td)(v, v^c, u^c, M^2) - 1| \leq |u^c - 1| \leq 1 + B,
$$

---

[31] If we endowed K with the sup-norm, then K would be a closed subspace of $C[0,1]$; since $C[0,1]$ is complete in the sup-norm, so is K. And it was shown by Bielecki (1956) that the $|| \cdot ||_\lambda$ norm is equivalent to the sup-norm for any $C[a,b]$ – and thus if K is a complete metric space for the sup-norm it is also a complete metric space for $|| \cdot ||_\lambda$.

so indeed $T(K) \subset K$. Finally, it should be clear that for any v^c, u^c the restriction of any fixed point of $(Td)(\bullet)$ to $[v^c, 1]$ gives us the solution to (2.21) and that any solution to (2.21) can be easily extended into a fixed point of $(Td)(\bullet)$. Therefore, by Banach Fixed-Point Theorem, it suffices to show that there exists such a λ that for any $(v^c, u^c) \in [0,1]^2$, $Td(\bullet)$ is a contraction wrt to the norm $||\cdot||_\lambda$ to show that (2.21) has a unique solution for any feasible (u^c, v^c).

Let us drop (v^c, u^c) from the definition of a map (remembering that we are keeping them constant). Take any any $t \geq v^c$ and any $d_1, d_2 \in S$. For any map $(fd)(t)$ denote $(fd_1)(t) - (fd_2)(t)$ as $\Delta_d(fd)(t)$ and for any map $(fd)(d(t),t)$ denote $(fd_1)(d_1(t),t) - (fd_2)(d_2(t),t)$ as $\Delta_d(fd)(d(t),t)$ Then we have:

$$\begin{aligned}
|\Delta_d(G^2 d)(t)| &= \frac{1}{R^2}\left|\int_{v^c}^{t} C_v^e(d_1(r),r) - C_v^e(d_2(r),r)\mathrm{d}r\right| \qquad (2.26)\\
&\leq \int_{v^c}^{t} \frac{|C_v^e(d_1(r),r) - C_v^e(d_2(r),r)|}{R^2}\mathrm{d}r \leq \int_{v^c}^{t} \frac{L_2}{R^2}|d_1(r) - d_2(r)|\mathrm{d}r\\
&= \frac{L_2}{R^2}\int_{v^c}^{t} e^{\lambda r}e^{-\lambda r}|d_1(r) - d_2(r)|\mathrm{d}r \leq \frac{L_2}{R^2}||d_1 - d_2||_\lambda \int_{v^c}^{t} e^{\lambda r}\mathrm{d}r\\
&= \frac{L_2}{R^2\lambda}||d_1 - d_2||_\lambda (e^{\lambda t} - e^{\lambda v^c}) \leq \frac{L_2}{R^2\lambda}||d_1 - d_2||_\lambda e^{\lambda t},\\
|\Delta_d(s_v^2 d)(t)| &\leq |\pi_v^{2e}(t, (G^2 d_1)(t)) - \pi_v^{2e}(t, (G^2 d_2)(t))| \qquad (2.27)\\
&\quad + \frac{2kl^2}{R^2}|C_v^e(d_1(t),t) - C_v^e(d_2(t),t)|\\
&\leq L_6|\Delta_d(G^2 d)(t)| + \frac{2kl^2 L_2}{R^2}|d_1(t) - d_2(t)|\\
&\leq \frac{L_6 L_2}{R^2\lambda}||d_1 - d_2||_\lambda e^{\lambda t} + \frac{2kl^2 L_2}{R^2}|d_1(t) - d_2(t)|
\end{aligned}$$

which can be used to establish:

$$\begin{aligned}
\Delta_d(G^1 d)(d(t),t) &= \left|\frac{C^e(d_1(v),v) - C^e(d_2(v),v) - R^2\Delta_d(G^2 d)(v)}{R^1}\right| \qquad (2.28)\\
&\leq \frac{1}{R^1}(|C^e(d_1(v),v) - C^e(d_2(v),v)| + |R^2\Delta_d(G^2 d)(v)|)\\
&\leq \frac{L_2}{\lambda R^1}||d_1 - d_2||_\lambda e^{\lambda t} + \frac{L^1}{R^1}|d_1(t) - d_2(t)|,\\
\Delta_d(s_v^1 d(d(t),t) &\leq |\pi_u^{1e}(d_1(t), (G^1 d_1)(d_1(t),t)) - \pi_u^{1e}(d_2(t), (G^1 d_2)(d_2(t),t))| \qquad (2.29)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2kl^1}{R^1}|C_u^e(d_1(t),t)-C_u^e(d_2(t),t)| \\
&\leq |\pi_u^{1e}(d_1(t),(G^1d_1)(d_1(t),t))-\pi_u^{1e}(d_1(t),(G^1d_2)(d_2(t),t))| \\
&\quad +|\pi_u^{1e}(d_1(t),(G^1d_2)(d_2(t),t))-\pi_u^{1e}(d_2(t),(G^1d_2)(d_2(t),t))| \\
&\quad +\frac{2kl^1L_3}{R^1}|d_1(t)-d_2(t)| \\
&\leq \frac{L_2L_5}{\lambda R^1}||d_1-d_2||_\lambda e^{\lambda t}+\Big(\frac{L_1L_5+2kl^1L_3}{R^1}+L_4\Big)|d_1(t)-d_2(t)|
\end{aligned}
$$

Denote $\sup \pi_v^2(v,h)+\frac{2kl^2}{R^2}+o^2=L_7$, $\inf \pi_u^1(u,h)+o^1=L_8$ and note that continuity of π_u^1 and π_v^2 and the fact that $\pi_u^1>0$ imply that both L_7 and L_8 are finite. Using all this, we can write, for any $v\geq v^c$ and any $d_1,d_2\in S$:

$$
\begin{aligned}
|\Delta_d(Td)(v)| &= |\int_{v^c}^{v}\frac{(s_v^2d_1)(t)}{(s_u^1d_1)(d_1(t),t)}-\frac{(s_v^2d_2)(t)}{(s_u^1d_2)(d_2(t),t)}\mathrm{d}t| \\
&= |\int_{v^c}^{v}\frac{(s_v^2d_1)(t)}{(s_u^1d_1)(d_1(t),t)}-\frac{(s_v^2d_2)(t)}{(s_u^1d_1)(d_1(t),t)} \\
&\quad +\frac{(s_v^2d_2)(t)}{(s_u^1d_1)(d_1(t),t)}-\frac{(s_v^2d_2)(t)}{(s_u^1d_2)(d_2(t),t)}\mathrm{d}t|\mathrm{d}t \\
&\leq \int_{v^c}^{v}\Big(\frac{L_1L_5L_7+2kl^1L_3L_7}{R^1L_8^2}+\frac{L_7L_4}{L_8^2}+\frac{2kl^2L_2}{L_8R^2}\Big)|d_1(t)-d_2(t)| \\
&\quad +\Big(\frac{L_6L_2}{L_8R^2\lambda}+\frac{L_2L_5L_7}{\lambda R^1L_8^2}\Big)||d_1-d_2||_\lambda e^{\lambda t}\ \mathrm{d}t \\
&\leq \frac{1}{\lambda}\Big(\frac{L_1L_5L_7+2kl^1L_3L_7}{R^1L_8^2}+\frac{L_7L_4}{L_8^2}+\frac{2kl^2L_2}{L_8R^2}\Big)||d_1-d_2||_\lambda e^{\lambda v} \\
&\quad +\frac{1}{\lambda}\Big(\frac{L_6L_2}{L_8R^2\lambda}+\frac{L_2L_5L_7}{\lambda R^1L_8^2}\Big)||d_1-d_2||_\lambda e^{\lambda t}.
\end{aligned}
$$

For $v<v^c$ this has to hold as well, as then $|(Td_1)(v)-T(d_2)(v)|=0$. Denote $\frac{L_1L_5L_7+2kl^1L_3L_7}{R^1L_8^2}+\frac{L_7L_4}{L_8^2}+\frac{2kl^2L_2}{L_8R^2}=L_9$, then, for any $v\in[0,1]$, we have that:

$$
\Delta_d(Td)(v)|\leq\frac{1}{\lambda}||d_1-d_2||_\lambda e^{\lambda v}\Big(\frac{L_6L_2}{L_8R^2\lambda}+\frac{L_2L_5L_7}{\lambda R^1L_8^2}+L_9\Big).
$$

Dividing both sides of that by $e^{\lambda v}$ we get:

$$
e^{-\lambda v}|\Delta_d(Td)(v)|\leq\frac{1}{\lambda}||d_1-d_2||_\lambda\Big(\frac{L_6L_2}{L_8R^2\lambda}+\frac{L_2L_5L_7}{\lambda R^1L_8^2}+L_9\Big)
$$

which, by taking sup on both sides implies that:

$$||(Td_1)(t) - T(d_2)(t)||_\lambda \leq \frac{1}{\lambda}||d_1 - d_2||_\lambda \Big(\frac{L_6 L_2}{L_8 R^2 \lambda} + \frac{L_2 L_5 L_7}{\lambda R^1 L_8^2} + L_9 \Big). \tag{2.30}$$

Therefore, there has to exist a high enough λ for which our map $(Td)(v)$ is a contraction in the metric space $(S, || \cdot ||_\lambda)$ – which, by Banach's Fixed-Point Theorem means that $(Td)(v)$ has a unique fixed point, which in turn means that Equation (2.21) has a single solution for any given $(v^c, u^c) \in [0, 1]^2$. Note that Equation (2.30) does not depend on (v^c, u^c) – and thus, by standard results (see e.g. Hasselblatt and Katok, 2003, p. 68) it follows that, as $(Td)(v, v^c, u^c)$ is continuous in v^c and u^c, the fixed point – and thus the solution of (2.21) – is continuous in them as well.

Denote the fixed point of $(Td)(\cdot, v^c, u^c)$ as $d^*(\cdot, v^c, u^c)$ – then the following result holds:

Lemma 2.2. Keeping the other parameter constant, $d^*(\cdot, v^c, u^c)$ is weakly decreasing in v^c and weakly increasing in u^c, for all v's, with these relations holding strictly for some v's.

Proof. I start with the claims regarding $d(v, \cdot, u^c)$. To simplify notation, I drop u^c from all functions, as I keep them constant for the proof of the claims regarding v^c. Take any $v_2^c > v_1^c \in [0, 1]$ and denote $d^*(v, v_2^c) - d^*(v, v_1^c)$ as $\Delta_{v^c} d^*(v, v^c)$ and:

$$G^2(v, v^c) = \frac{1}{R^2} \int_{v^c}^{v} C_v(d^*(r, v^c), r) \mathrm{d}r,$$

$$G^1(d^*(v, v^c), v^c) = \frac{C(d^*(v, v^c), r)}{R^1} - G^2(v, v^c).$$

As $d^*(\cdot, v^c)$ is strictly increasing, for any $v \in [v_1^c, v_2^c)$ we have $\Delta_{v^c} d^*(v, v^c) < 0$, which proves the second (strict) part of this claim. Thus, we only need to show now that $\Delta_{v^c} d^*(v, v^c) \leq 0$ for all $v \in [v_2^c, 1]$. Suppose not. Then the set $\Omega^{gen} =$

$\{v \in [v_2^c, 1] : \ \Delta_{v^c} d^*(v, v^c) > 0\}$ has to be non-empty. Denfine $v^g = \inf \Omega^{gen}$, then $\Delta_{v^c} d^*(v^g, v^c) = 0$ and $\Delta_{v^c} d^*_v(v^g, v^c) > 0$. Thus, the sign of $\Delta_{v^c} d^*_v(v^g, v^c)$ depends only on the signs of[32]:

$$\pi_v^{2e}(v^g, G^2(v_2^c, v^g)) - \pi_v^{2e}(v^g, G^2(v_1^c, v^g)) \tag{2.31}$$

and

$$\pi_u^{1e}(d^*(v^g, v_1^c), G^1(v_1^c, d^*(v^g, v_1^c))) - \pi_u^{1e}(d^*(v^g, v_2^c), G^1(v_2^c, d^*(v^g, v_2^c))). \tag{2.32}$$

However, as $\Delta_{v^c} d^*(v^g, v^c) = 0$ and both surplus functions are weakly supermodular, these in turn depend only on the sign of $G^2(v_2^c, v^g) - G^2(v_1^c, v^g)$. As for any $v \leq v^g$ it was the case that $\Delta_{v^c} d^*(v^g, v^c) \leq 0$ and $v_2^c \geq v_1^c$, it follows that: $G^2(v_2^c, v^g) - G^2(v_1^c, v^g) \leq 0$ and thus $\Delta_{v^c} d^*_v(v, v^c) \leq 0$, which means that Ω^{gen} has to be empty and proves our first claim.

As for the claim regarding u^c, note that for a change in u^c, note that $\Delta_{u^c} d^*(v^c, u^c)$ is positive. The subsequent reasoning is analogous, but with opposite signs the strict decreasingness follows from $\Delta_{u^c} d^*(v^c, u^c) < 0$ and continuity). $\square$

Note that this Lemma and Equation (2.24) imply that $v^*(v^c)$ is strictly increasing in v^c and strictly decreasing in u^c. I will finish the proof by considering separately the cases of strictly and weakly scarce jobs.

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[32] To see this, note that:

$$\Delta_{v^c} d^*_v(v^g, v^c) = \frac{\pi_v^{2e}(v^g, G^2(v_2^c, v^g))}{\pi_u^{1e}(d^*(v^g, v_2^c), G^1(v_2^c, d^*(v^g, v_2^c)))} - \frac{\pi_v^{2e}(v^g, G^2(v_1^c, v^g))}{\pi_u^{1e}(d^*(v^g, v_1^c), G^1(v_1^c, d^*(v^g, v_1^c)))}$$
$$= \frac{\pi_v^{2e}(v^g, G^2(v_2^c, v^g)) - \pi_v^{2e}(v^g, G^2(v_1^c, v^g))}{\pi_u^{1e}(d^*(v^g, v_2^c), G^1(v_2^c, d^*(v^g, v_2^c)))}$$
$$+\pi_v^{2e}(v^g, G^2(v_1^c, v^g)) \frac{\pi_u^{1e}(d^*(v^g, v_1^c), G^1(v_1^c, d^*(v^g, v_1^c))) - \pi_u^{1e}(d^*(v^g, v_2^c), G^1(v_2^c, d^*(v^g, v_2^c)))}{\pi_u^{1e}(d^*(v^g, v_1^c), G^1(v_1^c, d^*(v^g, v_1^c)))\pi_u^{1e}(d^*(v^g, v_2^c), G^1(v_2^c, d^*(v^g, v_2^c)))}.$$

Strictly scarce jobs $R^1 + R^2 < 1$ implies $C(u^c, v^c) > 0$. As for $(u, v) > 0$, $C(\bullet)$ is strictly increasing in both parameters, its inverse with respect to both parameters exists, which allows us to define u^c as a strictly decreasing, continuous function of v^c. Define $\underline{v}$ as $u^c(\underline{v}) = 1$ and note also that as $u^c \in [0, 1]$ Equation (2.22) shrinks the range of feasible v^c's to $[\underline{v}, 1]$. All of this implies that $d^*(v, v^c, u^c)$ depends only on v and v^c and is decreasing and continuous in v^c – hence, I will denote it as $d^*(v, v^c)$ from now on. Note that $d^*(v, v^c)$ uniquely determines v^*, which is strictly increasing and continuous in v^c. Thus, the modified system of equations reduces to:

$$R^2 = \int_{v^c}^{v^*(v^c)} C_v^e(d^*(r, v^c), r) + 1 - v^* \mathrm{d}r.$$

Let us start with existence. The RHS is continuous in v^c, as $d^*(v, v^c))$ and $v^*(v^c)$ are continuous in v^c. For $v^c = \underline{v}$, we have $d^*(v, v^c) \geq 1$ regardless of v and therefore $\int_0^1 C_v^e(d^*(r, v^c), r)\mathrm{d}r = 1$, whereas for $v^c = 1$, we have $\int_1^1 C_v^e(d^*(r, v^c), r)\mathrm{d}r = 0$; thus, a solution to (2.23) (given $R^2 \in (0, 1)$) exists. It is unique, as $d^*(v, \cdot)$ is weakly decreasing for all and strictly decreasing for some v and $C_v^e(d^*(r, v^c), r) \leq 1$ for $v \leq v^*$ – thus the RHS crosses R^2 only once from above. This completes the proof for $R^1 + R^2 < 1$.

Abundant jobs If $R^1 + R^2 \geq 1$, then $C(u^c, v^c) = 0$. This implies that either u^c or v^c has to be equal to zero – more importantly, this implies that we can't define u^c as a function of v^c. I deal with this problem by defining the set $\Gamma^c = \{(u^c, v^c) : \min\{u^c, v^c\} = 0\}$, a new variable $a \in [-1, 1]$ and writing u^c and v^c as functions of a:

$$u^c(a) = \begin{cases} -a & \text{for } a \leq 0, \\ 0 & \text{for } a > 0, \end{cases} \qquad v^c(a) = \begin{cases} 0 & \text{for } a \leq 0, \\ a & \text{for } a > 0. \end{cases}$$

Note that for any a, $(u^c(a), v^c(a)) \in \Gamma^c$ and for any $(u^c, v^c) \in \Gamma^c$ there exists a unique a, such that $(u^c(a), v^c(a)) = (u^c, v^c)$. Thus, if there exists a unique a that solves Equation (2.23), there also exists a unique (u^c, v^c) that solves it. Moreover, clearly $v^c(a)$ is continuous and increasing, whereas $u^c(a)$ is continuous and decreasing. Given that, existence and uniqueness follows from an analogous reasoning as above (for details, see the proof of Theorem 1 in Chapter 1). $\square$

Lemma 2.3. The relation between the original and the modified set is as follows: (a) if ψ^e solves the modified set then its restriction to $[v^c, \sup\{v \in [v^c, 1] : \psi^e(v) \leq 1\}]$ solves the original one and (b) if a function $\psi(v) : [v^c, v^*] \to [u^c, u^*]$ solves the original set then we can always find its extension $\psi^e(v) : [v^c, 1] \to [u^c, 1 + B]$ that solves the modified one.

Proof. Note that if $v^* = \sup\{v \in [v^c, 1] : \psi^e(v) \leq 1\}$, then $\max\{\psi^e(v^*), v^*\} = 1$, as required. For $v \leq v^*$ we have that

$$\int_{v^c}^{v} C_v^e(\psi(r), r)\mathrm{d}r \leq R^2,$$

$$C(\psi^e(v), v) - \int_{v^c}^{v} C_v^e(\psi(r), r)\mathrm{d}r \leq R^1,$$

which means that the original and extended $C(\bullet)$, $C_v(\bullet)$, $\pi_u^1(\bullet)$ and $\pi_v^2(\bullet)$ are identical; and thus if (2.21) is met, (2.4) has to be met too. The equivalence of all other equations is trivial.

Claim (b) is trivial for $\psi(v^*) = 1$, as then ψ and ψ^e coincide. For $\psi(v^*) < 1$ the reasoning is slightly more complicated. Consider the following map: T^1 :

$K \times [0,1]^2 \to [u^c, 1+B]$:

$$(T^1 d)(v, v^c, v^*) = \begin{cases} u^c & \text{for } v \in [0, v^c) \\ \psi(v) & \text{for } v \in [v^c, v^*] \\ 1 + \int_{v^*}^{v} \frac{\pi_v^2(t, G^2(v^*) + \int_{v^*}^{t} C^e(d(r), r)\mathrm{d}r) + 2k(\frac{l^2}{R^2} C_v^e(d(t), t) + o_2)}{\pi_u^1(1,1) + 2k(\frac{l^1}{R^1} C_u^e(d(t), t) + o^1)} \mathrm{d}t & \text{for } v \in (v^*, 1]. \end{cases}$$

The restriction of the fixed point of this map to $[v^c, 1]$ solves the modified set of equations and is clearly and extension of $\psi(\cdot)$. By a reasoning analogous to that for $(Td)(v, v^v, u^c)$ in proof of Proposition 2.9, follows that there exists a unique fixed point of the map $(T^1 d)(v, v^c, v^*)$. □

Theorem 2.1 follows from Proposition 2.9 and Lemma 2.3

□

2.C Changes in Status Structure

To simplify what follows, we need to first introduce new notation. The difference between the old and new values of any object O is denoted as $\Delta_\rho O$. The greater of the old and new values of any object O is denoted as $\max O$. Thus, for instance, the change in sector one size is $\Delta_\rho M^1$ and the greater critical ability in sector two is $\max v^c$.

Definition 2.11. Sector one surplus $(s^1(\bullet, \Theta))$ becomes *strictly more spread out* if $s_u^1(u, h, \Theta_s(\rho_1), \rho_2) > s_u^2(u, h, \Theta_s(\rho_1), \rho_1)$ for all $(u, h) \in [u^c(\rho_1), 1] \times [0, 1]$, where $\Theta_s(\rho_1)$ denotes the old stable assignment.

The Theorem below is the driving force of all my comparative statics results.

Theorem 2.3. If sector one surplus becomes strictly more spread out, then (i) $G^2(v, \rho_2) \geq G^2(v, \rho_1)$ for all v and holds strictly for $v \in [\max v^c(\rho), \max v^*(\rho))$; (ii) $G^1(u, \rho_2) \leq G^1(u, \rho_1)$ for all u and holds strictly for $u \in [\max u^c(\rho), \max u^*(\rho))$.

Proof of Theorem 2.3. This Theorem will be proved in a series of lemmas. I start, however, by defining two sets of sector two skill levels, which will be of crucial importance throughout the entire proof:

$$\begin{aligned}
\Xi^0 &= \{v \in [\max v^c(\rho), \min v^*(\rho)] : \psi(v,\rho_2) = \psi(v,\rho_1) \wedge G^2(v,\rho_2) \leq G^2(v,\rho_1)\} \\
\Xi^1 &= \{v \in [\max v^c(\rho), \min v^*(\rho)] : \psi(v,\rho_2) \leq \psi(v,\rho_1) \wedge G^2(v,\rho_2) < G^2(v,\rho_1)\} \\
\Xi^2 &= \{v \in [\max v^c(\rho), \min v^*(\rho)) : \psi(v,\rho_2) \leq \psi(v,\rho_1) \wedge G^2(v,\rho_2) \leq G^2(v,\rho_1)\}.
\end{aligned}$$

I also denote the total payoff sector i agent receives as $t^i(\cdot) = t^i(\cdot) + \tau^i(\cdot)$.

Lemma 2.4. A strict increase in the spread of sector one surplus implies that Ξ^2 is empty.

Proof of Lemma 2.4. Remember that $G^2_v(v) = \frac{\psi(v)C_{uv}(\psi(v),v)}{R^2}$. Take any $v_0 \in \Xi^0$ Note that by Equation (2.20) we have $\Delta_\rho G^1(\psi(v_0,\rho_1)) \geq 0$. Then we have:

$$\begin{aligned}
\Delta_\rho t^1_u(\psi(v_0,\rho_1)) = {} & \Delta_\rho s^1_u(\psi(v_0,\rho_1), G^1(\psi(v_0,\rho_1),\Theta_s,\rho_2)) \\
& + \int_{G^2(\psi(v_0,\rho_1),\rho_1)}^{G^2(\psi(v_0,\rho_1),\rho_2)} \pi^1_{uh}(\psi(v_0,\rho_1), r, \rho_1)\mathrm{d}r > 0,
\end{aligned}$$

as $\Delta_\rho s^1_u(u,h) > 0$ for any (u,h), $\pi^1(\bullet)$ is supermodular and $\Delta_\rho G^1(\psi(v_0,\rho_1)) \geq 0$. Whereas for v_0 we have:

$$\Delta_\rho t^2_v(v_0) = \int_{G^2(v_0,\rho_1)}^{G^2(v_0,\rho_2)} \pi^2_{vh}(v_0,r)\mathrm{d}r \leq 0,$$

as $\pi^2(\bullet)$ is supermodular and $\Delta_\rho G^2(v_0) \leq 0$. By differentiating Equation (2.6) wrt to v for both ρ_2 and ρ_1, taking differences and rearranging, we arrive at:

$$\Delta_\rho \psi_v(v_0) = \frac{1}{t^1_u(\psi(v_0,\rho_1),\rho_2)} \big[\Delta_\rho t^2_v(v_0) - \psi_v(v_0,\rho_1)\Delta_\rho t^1_u(\psi(v_0,\rho_1))\big],$$

from which follows trivially that $\Delta_\rho \psi_v(v_0) < 0$.

Take any $v_1 \in \Xi^1$. Suppose that $\Delta_\rho \psi(v) \leq 0$ for all $v \in [v_1, \min v^*(\rho)]$, which implies that $v^*(\rho_2) > v^*(\rho_1)$. Remember that both for ρ_1 and ρ_2 we need to have $G^2(1) = 1$ and thus $\Delta_\rho G^2(1) = 0$. Therefore:

$$\begin{aligned}
0 &= \Delta_\rho G^2(v_1) + \frac{\int_{v_1}^{v^*(\rho_2)} C_v(\psi(v,\rho_2),v)\mathrm{d}v - \int_{v_1}^{v^*(\rho_1)} C_v(\psi(v,\rho_1),v)\mathrm{d}v - \Delta_\rho v^*(\rho)}{R^2} \\
&= \Delta_\rho G^2(v_1,\rho_1) - \int_{v_1}^{v^*(\rho_1)} \int_{\psi(v,\rho_2)}^{\psi(v,\rho_1)} \frac{C_{uv}(s,v)}{R^2} \mathrm{d}s\mathrm{d}v - \int_{v^*(\rho_2)}^{v^*(\rho_1)} \frac{1 - C_v(\psi(v,\rho_2),v)}{R^2} \mathrm{d}v.
\end{aligned}$$

Note that as $\psi(v,\rho_2) \leq 1$ it follows that $C_v(\psi(v,\rho_2),v) \leq 1$; hence we have that the two latter terms on the RHS are weakly and the first is strictly negative – contradiction. Therefore there needs to exist some $v \in (v_1, \min v^*(\rho)]$ such that $\Delta_\rho \psi(v) > 0$ for Ξ^1 to be non-empty. Denote the set of all such v's as Ξ^3; then it follows that $\inf \Xi^3 \in \Xi^0$[33]. This implies that $\Delta_\rho \psi_v(\inf \Xi^3) < 0$ which contradicts $v = \inf \Xi^3$. Thus, Ξ^1 has to be empty. Now consider any $v_2 \in \Xi^2$. Note that under $\Delta_\rho \pi_u^1(u,h) > 0$ for all (u,h) there has to exist some arbitrarily small $\epsilon > 0$ such that $v_2 + \epsilon < \min v^*(\rho)$ and for any $v \in (v_2; v_2 + \epsilon]$ we have $\Delta_\rho \psi(v) < 0$. This follows from continuity if $\Delta_\rho \psi(v_2) < 0$ and from the fact that if $\Delta_\rho \psi_v(v_2) = 0$ then $v_2 \in \Xi^0$ and $\Delta_\rho \psi_v(v_2) < 0$. Therefore, trivially, $\Delta_\rho G^2(v_2 + \epsilon) < 0$ and thus $v_2 + \epsilon \in \Xi^1$, which means that Ξ^2 has to be empty and concludes the proof. □

Lemma 2.5. Suppose Ξ^2 is empty. Consider some $v_e \in [\max v^c(\rho), \min v^*(\rho)]$. Then $\Delta_\rho G^2(v_e) > 0$ implies (i) $\Delta_\rho G^2(v_e) > 0$ for all $v \in [v_e, \min v^*(\rho))$ and (ii) $G^2(\min v^*) \geq 0$.

Proof. Suppose the (i) is false. Then the set $\Upsilon^3 = \{v \in [v_e, \min v^*(\rho)) : \Delta_\rho G^2(v) \leq 0\}$ has to be non-empty; but as $\Delta_\rho G^2(\cdot)$ is continuous in v[34], the non-emptiness implies that $v^1 = \min \Upsilon^3$ exists. Additionally, $v^1 > v_e$, as $\Delta_\rho G^2(v_e) > 0$. Define a new set $\Upsilon^4 = \{v \in [v_e, v^1] : \Delta_\rho \psi(v) \leq 0\}$ and $v^2 = \max \Upsilon^4$; by definition of v^1,

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[33] By continuity of $\Delta_\rho \psi(v)$, which follows from continuity of $\psi(v)$.

[34] Follows from continuity of $\psi(\cdot)$ and $C_v(\bullet)$.

for any $v < v^1 \wedge \in \Upsilon^4$ we have that $\Delta_\rho G^2(v) > 0$. As $[v_e, v^1]$ is a compact set and $\Delta_\rho\psi(v)$ is continuous v^2 won't exist only if Υ^4 is empty; but an empty Υ^4 implies that $\Delta_\rho\psi(v) > 0$ for any $v \in [v_e, v^1]$, which in turn means that $\Delta_\rho G^2(v^1) > 0$, which contradicts the definition of v^1. Therefore v^2 needs to exist. Now suppose that $v^2 < v^1$; then we have $\Delta_\rho G^2(v^2) > 0$ and for any $v \in (v^2, v^1], \Delta_\rho\psi(v) > 0$, which implies that

$$\Delta_\rho G^2(v^1) = \Delta_\rho G^2(v^2) + \frac{1}{R^2}\int_{v^2}^{v^1}\int_{\psi(r,\rho_1)}^{\psi(r,\rho_2)} C_{uv}(s,r)\mathrm{d}s\mathrm{d}r > 0$$

and also contradicts the definition of v^1. Therefore it has to be the case that $v^1 = v^2$; but this implies that $\Delta_\rho(\psi(v^1)) \leq 0$ and $\Delta_\rho G^2(v^1) \leq 0$, which contradicts the emptiness of Ξ^2. Claim (ii) follows from continuity of $G^2(\cdot)$. $\square$

Lemma 2.6. $\Delta_\rho G^2(\min v^*(\rho)) \geq 0$ implies that (i) for any $v > \min v^*(\rho)$ we have $\Delta_\rho G^2(v) \geq 0$ and (ii) for all $v \in [\min v^*(\rho), \max v^*(\rho))$ we have $\Delta_\rho G^2(v) > 0$.

Proof. Note that $\Delta_\rho G^2(\min v^*(\rho)) > (\geq)0$ implies that $v^*(\rho_2) > (\geq)v^*(\rho_1)$[35]. Thus, if $\Delta_\rho G^2(\min v^*(\rho)) = 0$ then $\min v^*(\rho) = \max v^c(\rho)$ and the second claim follows trivially. Whereas if $\Delta_\rho G^2(\min v^*(\rho)) > 0$ then $v^*(\rho_2) > v^*(\rho_1)$ and by the fact that all agents with $v \in (v^*, 1]$ join sector two for sure it follows that for $v \in (v^*(\rho_1), v^*(\rho_2))$ we also have $\Delta_\rho G^2(v) > 0$. Claim (i) for $v > \max v^*(\rho)$ follows easily from the aforementioned property of v^*. $\square$

Lemma 2.7. Empty Ξ^2 implies that $\Delta_\rho G^2(\max v^c(\rho)) > 0$.

Proof. Suppose not, which implies that $v^c(\rho_2) \geq v^c(\rho_1)$. Firstly, suppose that $\max v^c(\rho) \geq \min v^*(\rho)$. Then – as $v^*(\rho) > v^c(\rho)$ – it has to be that $v^*(\rho_2) >$

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[35] To see this, denote the ρ_i for which $v^*(\rho_i) = \max v^*(\rho)$ as ρ_m; then, as $\Delta_\rho G^2(1) = 0$, we have:

$$0 = \Delta_\rho G^2(\min v^*(\rho), \rho_1) + \frac{1}{R^2}\int_{v^*(\rho_2)}^{v^*(\rho_1)} \frac{1 - C_v(\psi(v,\rho_m), v)}{R^2}\mathrm{d}v.$$

As $1 - C_v(\psi(v,\rho_m), v) \geq 0$, the fact that $\Delta_\rho G^2(\min v^*(\rho)) > (\geq)0$ implies that for this to hold we need $v^*(\rho_2) > (\geq)v^*(\rho_1)$.

$v^c(\rho_2) \geq v^*(\rho_1)$ and thus, by the same reasoning as in the proof of Proposition 2.6, $\Delta_\rho G^2(v^c(\rho_2)) > 0$, which implies $v^c(\rho_1) > v^c(\rho_2)$, contradiction. Thus, it has to be the case that $\max v^c(\rho) < \min v^*(\rho)$. $C(u^c(\rho_1), v^c(\rho_1)) = C(u^c(\rho_2), v^c(\rho_2))$ and thus $\Delta_\rho v^c(\rho) \geq 0$ implies $\Delta_\rho u^c(\rho) \leq 0$. As $\psi(v^c(\rho)) = u^c(\rho)$ and $\psi(v)$ is strictly increasing for any ρ we have: $\psi(v^c(\rho_2), \rho_1) \geq u^c(\rho_1)$, $u^c(\rho_1) \geq u^c(\rho_2)$ and $u^c(\rho_2) = \psi(v^c(\rho_2), \rho_2)$, which trivially implies that:

$$\Delta_\rho \psi(v^c(\rho_2)) \leq 0.$$

But then $v^c(\rho_2) \in \Xi^2$ – contradiction. $\square$

Lemma 2.8. For all $v \in [\max v^c(\rho), \min v^*(\rho)]$, if $G^2(v, \rho_2) \geq (>) G^2(v, \rho_1)$ then $G^1(\psi(v, \rho_2), \rho_2) \leq (<) G^1(\psi(v, \rho_2), \rho_1)$.

Proof. From Equation (2.20), Definition 2.6 and Equation (2.7) follows that:

$$\begin{aligned}\Delta_\rho G^2(\psi(v, \rho_2)) = & -\frac{1}{R^1} R^2 \Delta_\rho G^2(v) \\ & + \frac{1}{R^1} \Big[\int_{\psi(v,\rho_1)}^{\psi(v,\rho_2)} C_u\big(r, v)\big) \mathrm{d}r - \int_{\psi(v,\rho_1)}^{\psi(v,\rho_2)} C_u\big(r, \psi^{-1}(r, \rho_1)\big) \mathrm{d}r \Big].\end{aligned}$$

If $\psi(v, \rho_2) \geq \psi(v, \rho_1)$ then for any $r \in [\psi(v, \rho_1), \psi(v, \rho_2)]$, $\psi^{-1}(r, \rho_1) \geq v$ and my claim follows. If $\psi(v, \rho_2) < \psi(v, \rho_1)$ then for any $r \in [\psi(v, \rho_2), \psi(v, \rho_1)]$, $\psi^{-1}(r, \rho_1) < v$ and my claim follows as well. $\square$

Note that $\Delta_\rho G^2(\max v^c(\rho)) > 0$ trivially implies that $v < \max v^c(\rho)$, $\Delta_\rho G^2(v) \geq 0$. Thus, the results for sector two follow easily from Lemmas 2.4, 2.5, 2.6, 2.7 as well as continuity of $\Delta_\rho G^2(\cdot)$. As Lemma 2.6 has an exact sector one analogue, the sector one results follow from sector two results and Lemma 2.8. $\square$

2.C.1 Decrease in Sector One Wages

I will outline here sufficient conditions for an increase in the spread of sector one status to result in a decrease in sector one wages.

First of all, note that from the proof of Proposition 2.3 we have that any sector one wage increases by at most as much as C^1. Thus, a fall in C^1 implies a fall in all sector one wages. In general, an increase in the spread of surplus has a negative, direct effect and a positive, general equilibrium effect on lowest status rewards. If the maximum spread of output in both sectors is low compared to k, then small changes to the status structure can result in strong general equilibrium effects and the overall effect on wages can be negative. To see this, consider the following example.

The output structure in both sectors is symmetric and output does not depend on firms. In particular, we have $\pi^1(u,h) = bu+k$ and $\pi^2(v,h) = bv+k$, where $b < \frac{k}{99}$. Suppose that local status does not matter in each sector, so that $l^1 = l^2 = 0$. Suppose further that only occupational prestige matters in both sectors initially ($p^1(\rho_1) = p^2 = 1$), but then the its weight falls in sector one ($p^1(\rho_2) = 1 - 99b$). Then from Equation (2.9) we have that $\psi(v,\rho_2) = \frac{1}{100}v + c_1$ and as none of Equations (2.4), (2.5), (2.10), (2.11) depends on output, it follows that the new stable assignment does not depend on b (and neither does the old one, as the old problem is symmetric). The difference in the status reward of the least talented agents – a through that wage constant – does depend on b, however: the lower b, the higher the new status reward. In particular, for arbitrarily low b, the change in $\tau^1(v^c)$ is arbitrarily close to the increase in $\bar{u}k$. Thus, for arbitrarily low b wages fall for all sector one agents.

Chapter 3

Directors Liability and Firm's Risk Taking

3.1 Introduction

In every limited liability company a fundamental conflict of interest between the debtholders and equityholders can arise. The equityholders participate in losses early, but later they are protected by limited liability. Because of that, when the firm incurs losses and part of the equity vanishes, so does shareholders' participation in future losses. At the same time, assuming standard debt contracts, their participation in potential profits is very likely to grow or at least decrease slower[1]. In the extreme, if all equity is gone – so the firm is on border of insolvency – shareholders will always want the company to continue old projects or undertake new ones, no matter how efficient those are; simply because they can lose nothing more. The creditors, on the other hand, who will incur all potential losses if the projects are continued and only a part of the profits, are likely to be overly conservative.

Usually, however, neither the equityholders nor debtholders formally decide whether to continue trading or not: it is the choice of the companies' directors. It is safe to assume that by making it, they will be mostly concerned with their own interest – and their gains from continuing or going bust will be determined by their employment contract. In most applications, shareholders have at least formal control over how the management's contracts are designed. Of course, as Aghion and Tirole (1997) pointed out, formal control is not necessarily equivalent to real control. Indeed, if the debtholders are sufficiently strong/well informed they might have some considerable degree of real control over how the directors' contracts are designed, especially in bad states of the world – that could happen via covenants or insolvency procedures itself. However, creditors are not always banks and they are not always strong. For instance, they might be the firm's suppliers who gave it a trade credit, or employees, who receive their payment at

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[1] By 'standard' I mean contracts, which require the company to pay the debtholders some fixed amount of money, rather than a fraction of profits.

the end of the month; or even depositors, if the firm itself is a bank. Finally, it could be that the owner of the company runs it by him- or herself, in which case there will be no employment contract in the first place. In all these cases directors could have incentives to continue trading even if that is not efficient. With imperfect information about the firm's prospects this could easily result in over-investment, in a manner similar to de Meza and Webb (1987).

This potential inefficiency stems from shareholders' limited liability and could be prevented by its extension. Limited liability, however, has many other advantages – most importantly it significantly decreases transaction costs, as noted by Jensen and Meckling (1976) – which might make it too valuable to give up. The example of such countries as Poland, Germany or the UK shows that there exists an alternative solution. In all these countries there are regulations in place that make the directors' of limited liability companies liable for part of the firm's debts in the case of insolvency[2]. To see why this can help, consider the extreme – the directors being fully liable and having sufficient funds to meet their liability. In such a case, they would choose to continue trading only if the contract gave them a big enough part of the profits to compensate their losses: but if the project was inefficient, there would not be enough profit to do so. Thus, over-investment would not be possible.

This gives us a flavour of the reasons for which many countries introduce some degree of directors' liability. However, as directors' liability is equivalent to debtholders protection – the more liable the directors are, the more of the investment will the debtholders recover if the project fails – it also has certain drawbacks. One can easily imagine that strong debtholders, who have significant real control over the company or superior information, will require less protection

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[2] In Poland, this liability is regulated by article 299 of Commercial Companies Code as well as articles 11 and 21 of Bankruptcy and Reorganisation Law; in the UK, it is specified in Section 214 of Insolvency Act 1986; in Germany the appropriate laws are paragraph 15a of Insolvenzordnung and paragraph 64 of GmbH-Gesetz.

than the weak ones; they will have other means to make sure that inefficient projects are not launched. If this is the case, there is little need for extended directors' liability – in fact, it could even lead to under-investment, as it will generally decrease risk taking. The fact that the degree to which directors are held liable varies greatly between countries hints that some of the costs resulting from directors' liability are significant[3].

The main focus of this chapter is to investigate when the advantages of introducing directors' liability outweighs the disadvantages. To that end, I develop a three-player game in which the Principal (representing shareholders) has a business project. In order to launch it, she needs to first secure financing from the Creditors (representing debtholders) and, second, hire an Agent (representing the firm's directors), to run it. The Agent has some illiquid wealth, which can be used in the future to meet any legal obligations, but cannot be invested in the project directly: that wealth should be thought of as either durable consumption goods, such as a car or house, or expected future income. If the project fails, the legal system transfers part of this wealth to the Creditors. For most of the analysis I assume (a) that private contracts with regard to the Agent's wealth cannot be written and (b) that the Principal cannot borrow more than what is needed to launch the projects – so that she cannot lower the *de facto* level of directors' liability.

Crucially, the Principal's project can be of two types: one has a higher and the other a lower probability of success, with the potential profit and loss being the same for both types. In other words, one is safer and on average more profitable, the other one is riskier and less profitable. Projects are represented by the capital that is needed to launch them and the potential profits and losses[4].

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[3] For instance, in Poland, if the directors filed in insolvency too late, they are automatically held fully liable for any losses uncovered by the firm's assets; whereas in the UK the degree to which they are liable is at court's discretion – in practice it's very rarely 100%.

[4] I also assume that in the case of a failure, the company loses all the capital that was invested into it.

This approach is equivalent to the more standard one, in which a project is defined by the investment that is needed to launch it, and the amount of money it yields in each state. Additionally, I focus only on the extreme case: when the Principal does not have any cash she could invest into the project. This is done mainly for expositional clarity, as relaxing this assumption would not change the basic results, but it can be also interpreted as a company on the border of bankruptcy trying to launch a new project.

In the main variant of the model, the Principal first offers the Creditors a financing contract, then (after they accept) learns the type of the project and finally offers the Agent an employment contract[5]. I interpret this as the debtholders being weak – as the timing does not allow them to condition their financing decision on the employment contract – and the shareholders being strong, as they have the same information as the directors. In particular, as the Principal has all bargaining power in this model and the same information as the Agent, she can be interpreted as an entrepreneur who runs the company herself. At the time the Principal *de facto* decides whether the project should be run, the financing contract has already been drawn and – as it cannot be conditioned on the project's type – the Principal might find the low type project profitable, even if its actual net present value is negative. This is essentially a commitment problem, as the past-self Principal would prefer not to run this project – but the only way she can commit to that is through paying high enough interests, which is very costly. However, by introducing directors' liability the government provides that commitment device for free and, therefore, decreases risk taking and restores efficiency.

The second variant considers the other extreme: a very weak Principal, who

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[5] The fact that the financing contract is signed before the type of the project is drawn is crucial, as it implies that the Creditors don't know the projects type when they make the financing decisions. The Principal does not know it either at that time, but that assumption is needed just for simplicity; relaxing it wouldn't change the basic results, but would require us to change the equilibrium concept.

designs the employment contract at the same time as the financing one. As the Creditors know what the employment contract is going to be, they know also in which case the Agent will accept it: in other words, they know already which types of the project will be launched. Because of this, the Principal can credibly commit to not launching inefficient projects. In equilibrium, she will always choose to do so – hence, in this setting, no over-investment is possible. Underinvestment, on the other hand, is an issue, due to Agent's superior information, which allows him to receive a rent if both projects are launched. For some efficient risky projects, this rent can be greater than the project's expected profit – in which case the Principal chooses not to launch it. Introducing Agent's legal liability only amplifies this problem. Both these findings combined imply that with weak shareholders and strong debtholders no directors' liability is the optimal policy.

The third variant of the model considers a mix of the previous two settings: both the Creditors and the Principal are relatively weak, as the Principal designs the employment contract after financing is secured, but before the project's type is determined. In such a case, if the Agent's reservation utility is not too high, the optimal policy makes the directors' liable for a positive fraction of the debt[6]. That result is trivial in a sense – it is a combination of the previous two cases: the lack of Creditors' control over the employment contract creates the possibility of over-investment, whereas Agent's superior information paired with sufficiently restrictive law causes underinvestment – but it is also highly interesting for interpretational reasons.

Depositors in banks are relatively weak creditors, with no possibilities to impose any covenants on the bank, or in any other way control management's remuneration; a similar argument can be made for investors in mutual or pension funds. At the same time, it is very likely that shareholders in such institutions

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[6] Optimal in the sense, that every efficient project will be launched, and an equilibrium will exist in which every inefficient one will not be launched with certainty.

have limited knowledge about the firm's dealings. In other words, a great deal of financial sector will be best described by the third variant of the model. That suggests that introducing some degree of directors' liability in this sector could eliminate excessive risk-taking. Moreover, as in this case the optimal degree of liability – as a fraction of total debt – will be definitely less than 100%, it is more likely that the optimal policy will be feasible, compared to the main variant[7]. The reason is that even the combined wealth of all directors in a bank will be most likely much less than the amount of deposits held in that bank: as consequence, if the optimal policy would require the directors' to repay 100% of the firm's debts, it would almost certainly not be feasible in this context.

Finally, I extend the model by allowing private contracts to be drawn. If these contracts are costless and unrestricted, then the private contract equilibrium always results in an efficient outcome, irrespective of the legal system. However, if contracts that decrease the level of real liability are more costly, then setting a high level of legal liability can be particularly damaging. Vice versa, if contracts that increase real liability are more costly, there is an additional reason to set a high level of legal liability. As banks have millions of creditors, they are likely to fall into the latter category. This provides another reason why introducing high levels of directors' liability could be justified in the financial sector. Generally, if private contracts are not costless, introducing personal liability for directors of limited liability companies will decrease the risk taken by that company. Whether that is optimal or not, depends on whether there was too much or too little risk in the system, which in my model depends on the timing. Alternatively, we can interpret this in terms of investment: more directors' liability will result in less investment, which – depending on the setting – may or may not be efficient.

This Chapter is organised as follows. In the following section I discuss the

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[7] Especially that in real life management will be most likely risk averse, which is ignored in my models. I will discuss that issue in more detail in Section 3.4.1.

relevant literature. Section 3.2 sets up the model and discusses the crucial assumptions. In Section 3.3 all variants of the model are solved and some basic welfare results are given. Section 3.4 primarily discusses optimal policy design and feasibility; it also explains why the results could be equivalently interpreted as an outcome of a private contract than a government's policy. Section 3.5 concludes.

3.1.1 Related Literature

The model presented here is most related to the ones in de Meza and Webb (1987) and the basic model in Chapter 6 of Tirole (2006). Apart from some minor differences, there is also a crucial one: the model in this thesis introduces a third party. The party which has the formal right to the investment project (the shareholders) is not the same party that will run it (the Agent). At the same time the investors are not treated as a single entity but separately. To some extent that is also done in de Meza and Webb (1987) but only *ex post*, based on the securities revenue streams. Here – as there is no equity – the shareholders are defined *ex ante*, in the terms of their control rights only. This modelling choice is more appropriate when companies other than start-ups are considered; that is companies where the management consists of hired professionals rather than the original entrepreneur. Another crucial feature of my model is the discreteness of project types – because of this, Creditors can earn a rent, despite receiving a 'take-it-or-leave -it' offer. This is driven by the fact that the low type project can be drawn with a non-zero probability and, hence, could not happen in a continuous model, as the one in de Meza and Webb (1987).

The fact that Creditors can earn a rent only if they do not know the employment contract is counter-intuitive: especially if we interpret such timing as the Creditors being weak. That draws some parallels to the result by Meyer and Vickers (1997), who show – in a completely different setting – that better information on Principals part can lead to an *ex ante* loss in efficiency. The mechanism in their paper is very different than here, but in both cases if the decision maker knows more at the time the choice is made, he might be worse off.

Papers by Aghion and Bolton (1992) and Dewatripont and Tirole (1994) explore the problem of optimal securities design with respect to control rights. In both of these papers control rights connected with a given security are endogenous and allocated at the financing stage – whereas in my model the control rights (understood here as the control over the design of employment contract) are given at the beginning of the game. However, even with endogenous control rights over-investment is an issue – as giving control to another party can allow them to earn rents, it is costly and, hence, introducing directors' legal liability will still be optimal for some parameter ranges.

Stiglitz and Weiss (1981) have shown that asymmetric information can lead to underinvestment, if the distributions of various projects' profits have equal means, but different variances, whereas de Meza and Webb (1987) have pointed out out that over-investment will occur if the mean differs across the projects as well. That result was generalised by Bernhardt (2000), who shows that with additive (mean zero) shocks to profits the system will be plagued with under-investment, whereas multiplicative shocks (shocks changing the expected profit) will lead to over-investment. I show that in a three-player model under- and over-investment can also be caused by the order in which contracts are written, even if the distribution of profits does not change.

The impact of the order in which financing contracts are written on the equilibrium behaviour did not draw much attention in the literature. Bernhardt (2000) notes in the very last paragraph of the paper that the timing of the financing game influences the results, but he does not expand on this. Jensen and Meckling (1976), however, in the section on debt, point out that if their manager-owner first issues debt and then decides which investment opportunity to choose (so cannot commit to it in advance) and only later sells the equity claim on the market, the less efficient project might be chosen in equilibrium. Despite the different terminology and rights of players in their model, it is worth noting that such sequence of actions is very close to the one in my main variant – the reason is that selling equity, *de facto* determines the manager's remuneration. The result that changing the sequence in which contracts are written in the financing game can create different types of inefficiencies is – to my knowledge – new, as Jensen and Mecklings alternative timing does not create inefficiency.

As I mentioned before, the main focus of this thesis is to investigate the impact of directors' liability on their firm's risk taking and efficiency: I am not aware of any other paper concerned with this issue. The problem of restoring efficiency in a system suffering from under- or over-investment was, however, addressed in a number of papers. De Meza and Webb (1987) show that a tax on interest rates would result in higher equilibrium interest rate and, in turn, in less investment; similarly, a interest rate subsidy would lower the equilibrium interest rates and increase investment – depending on whether under- or over-investment is the issue of concern, subsidies or taxes should be used. Becker and Fuest (2007) and Miglo (2007) show that similar effect can be achieved by the use of corporate taxation. As noted before, extended liability on the shareholders part would also solve the problem of over-investment: but at the cost of increased transaction costs. My contribution is to show that none of these policies are necessary, over-investment can be also ruled out by introducing directors' liability[8]. This policy may be, in many cases, less controversial and easier to implement than the alternatives. It is also important to understand that such seemingly different tools as taxation and rules on liability interact. For instance, in Poland, directors' liability coexists with capital gains tax, which – as depositors are banks creditors – is *de facto* an

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[8] And underinvestment by introducing some sort of directors' insurance. See Section 3.4.2.

interest rate tax; in that view the fact that the financial crisis didn't really affect Poland is less surprising – but it also raises the question whether there is too little investment in Poland in normal times.

3.2 The model

3.2.1 Players

There are three players – all of them risk neutral – and Nature. All players have perfect information about other players' utility functions and the structure of the game, and all are rational.

1. The Principal has a business project, which is defined by a binomial bet with probability p^i of winning X ($X \geq 0$) and probability $1 - p^i$ of losing Y ($Y \geq 0$), where $i =$ H,L is Principal's type, $p^H > p^L \neq 0$. For the project to be launched, Principal needs to obtain capital Y from the Creditors and hire the Agent to run the project. Apart from that, Principal chooses how the potential gain will be divided, by offering the Agent and Creditors contracts, as specified below. Her reservation payoff is normalised to 0.

2. The Agent doesn't have any cash at hand either, but he has some illiquid wealth $T \geq Y$, which can be used in future to meet legal obligations[9]. The Agent learns Principal's type at the same time as the Principal and has to decide whether to run the project or not. Her reservation payoff is $Z > 0$.

3. The Creditors' role is mostly passive: they need only to decide whether they are willing to finance the project or not. They never observe the project's type. Their reservation payoff is normalised to 0.

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[9] That wealth could be a car, the director's private house, or even future income.

4. Nature moves twice. Firstly, it draws the project's type, which is high with probability $\alpha \in (0,1)$ and low with $1-\alpha$. Later, it draws the binomial bet, which defines the project.

3.2.2 Contracts and legal systems

In general the only future event that is verifiable and thus can be specified in a contract is the project's outcome, so either profit or loss.

Employment contract The contract between Principal and Agent to run project i, is defined by the wage W^i – the share of profit Agent gets in the case of success. If Y is observed, there can't be any payment between Principal and Agent, as none of them has any cash in that state of the world.

Financing contract The contract between Principal and Creditors is defined by interest R – the share of profit Creditors get in the case of success. It is assumed that Principal can borrow only Y and nothing more. Clearly, if X is realised, Creditors receive R and the money they lent Principal is returned to them.

Legal system No contracts are allowed between the Agent and Creditors, but there might be transfers between them: these are determined by the legal system. The legal system is able to use Agent's wealth to pay Creditors back. If -Y is observed, a fraction δ of it is transferred from the Agent to Creditors. In other words the transfer from Agent to Creditors is δY, where $\delta \in [0,1]$. Note, that as we assumed $T \geq Y$, there is always enough wealth for that.

3.2.3 Timing of the game

The timing of the main variant of the model is the following:

- Principal chooses R, which constitutes the “take it or leave it” financing contract offer for Creditors.
- Creditors decide whether to Finance or Not.
    - If not, the game ends. Principal and Creditors earn 0, Agent earns $Z > 0$.
- Nature draws Principal’s type: high with probability α and low with probability $1 - \alpha$.
- Principal and Agents observe Principal’s type. Principal chooses W, which constitutes her “take it or leave it” employment contract offer for A.
- Agent decides whether to Accept or Not.
    - If not, the game ends. Principal and Creditors earn 0, Agent earns $Z > 0$.
- Nature draws whether the project will succeed or fail.
    - If it succeeds, the game ends and X is divided according to the contracts W and R, and the capital Y is returned to C. Agent’s net payoff is W; Creditors’ net payoff is R; Principal’s net payoff is X-W-R.
    - If it fails, the game ends and Creditors’ capital is lost. Additionally, the legal system transfers δY from the Agent to the Principal. Agent’s net payoff is -δ Y; Creditors’ net payoff is -(1-δ) Y; Principal’s net payoff is 0.

Two alternative timings are considered as well: they differ with respect to when Principal chooses W. In the variant with weak Creditors and Principal, the employment contract is chosen after Creditors’ move, but before Nature draws the

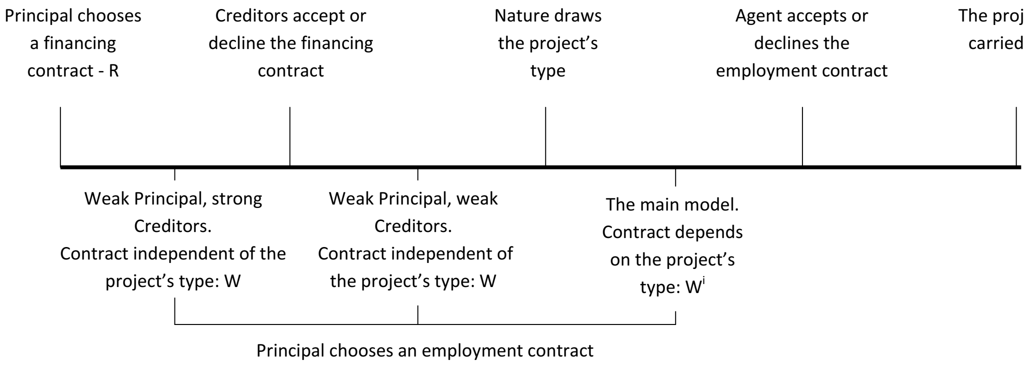

Figure 3.1: Timing

project's type. In the variant with strong Creditors and weak Principal, employment contract is chosen before Creditors decided whether to finance or not. The differences between the various timings are depicted in Figure 3.1.

As a benchmark, I will use a model in which Nature moves first and draws the project's type; and all the players immediately observe the outcome of that move, so that there is no uncertainty with regard to the project's type when any decision is made.

3.2.4 Strategies

I specify here the strategies for the main variant only – the strategies for the other variants follow naturally, but are also described informally in Sections 3.3.3 and 3.3.4.

Principal The Principal moves twice. Her choice of R begins the game and cannot be conditioned on anything; at the time she chooses W, however, she knows both R and the project's type. Because of that, her strategy should specify what R does she play at the beginning; and also what W will be played later, for all possible pairs (R, i) – but as R is chosen by Principal that can be reduced to i only. All in all, Principal's strategy is defined by the triple:

$$S_P = (R, W^L, W^H).$$

Creditors The Creditors have to decide whether to Finance, Not Finance or mix – and at the time they move, they know only R. Creditors' strategy has to specify the probability of Financing for any possible R. Define the probability that Creditors Finance as γ: then any Creditors' strategy can be written as:

$$S_C = \gamma(R),$$

where $\gamma : \mathbb{R} \to [0, 1]$.

Agent The Agent moves only once, and at that time he knows the project's type – i, interest – R and the wage – W. Because of that, Agent's strategy has to assign every triple (i, R, W) a probability that W will be accepted. Define that probability as β, then any Agent's strategy can be written as:

$$S_A = \beta(i, R, W),$$

where $\beta : \{0, 1\} \times \mathbb{R}^2 \to [0, 1]$.

3.2.5 Assumptions

The model presented here is highly stylised and as such relies on many simplifying assumptions. For the sake of clarity I enumerate those that have an impact on the results. A more detailed discussion on the consequences of relaxing some of them can be found in Sections 3.4 and 3.5.

1. The Agent's wealth is higher than the maximal potential loss, so that the

Agent can always meet his legal obligation. (Section 3.4.1)

2. All players are assumed to be risk neutral. (Section 3.4.2)
3. There is no equity in the model.
4. In all variants of the model other than the benchmark I assume that the net present value of the high type project is positive.
5. No private contracts with respect to Agent's wealth can be written. (Section 3.4.2)
6. The relative timing of the financing and employment contract is exogenous.
7. The distribution of project types is discrete.
8. The Principal has the whole bargaining power: both with respect to the financing and employment contract.
9. The Agent's reservation utility is strictly positive.
10. Principal cannot borrow more than Y. (Section 3.4.2)

3.3 Solution

The model is written in such a way, that every continuation game in every variant of the model is a proper subgame: because of that, Subgame Perfect Equilibrium can and is used as the equilibrium concept.

A note on notation Take some hypothetical parameter (or variable, or function) F_k^i. Then the lower subscript denotes the variant the parameter refers to: 1 stands for the main variant, 2 for the version where Creditors are strong, and 3 for the variation with both Principal and Creditors being weak. The upper subscript defines the parameter within the given model. For instance: R_3^* stands for commitment interest in the third model. If there is no lower subscript, it means that this parameter is the same in all three versions of the model – for instance R' or R''. I will also call a situation where the game doesn't end before i type project's outcome is drawn, as "launching project i".

3.3.1 The benchmark model

Consider the benchmark model, that is the one, where Nature draws the project's type before financing contract is designed, and all players observe it. That implies that both Principal and Creditors can condition all their choices on the observed type, and so $S_P = (R^L, R^H, W^L, W^H)$ and $S_C = \gamma(R, i)$, where $\gamma : \{0, 1\} \times \mathbb{R} \to [0, 1]$; Agent's strategy set is the same as described above. The *net present value* of a project's type is the difference between the average payoff of a project's type and the Agent's reservation utility.

Definition 3.1. Project's $i \in \{L, H\}$ net present value is given by $\mu^i - Z = p^i X - (1 - p^i)Y - Z$. The net expected value of the entire project is given by $\mu - Z = qX + (1 - q)Y - Z$, where $q = \alpha p^H + (1 - \alpha)p^L$.

As information is perfect in the benchmark model, the legal system does not affect the equilibrium outcome.

Proposition 3.1. In the benchmark case, in the unique Subgame Perfect Equilibrium the project of type $i \in \{L, H\}$ is launched if its net present value is positive. The Agent and Creditors receive in expectation their reservation payoffs of Z and 0, respectively, and the Principal receives the entire net present value of the project.

A proof can be found in the Appendix. The reasoning is simple: the higher the liability level, the lower the interest rate charged by the Creditors, but also the higher the wage that the Agent needs to be paid to run a project of a given type. As all players are risk neutral, these two effects cancel each other out and the equilibrium is efficient for any level of liability.

3.3.2 The main model

In the main version of the model, the distinction between the Agent and the Principal is somewhat superficial: as the Principal has the same information set as the Agent and gives him a 'take-it-or-leave-it-offer', she *de facto* decides whether the project should be run or not – but, in doing so, is constrained by the Agent's reservation payoff. Thus, an alternative, and perhaps more natural, way to interpret the main model is to think of an entrepreneur who both owns and runs the company. Viewed in this light, the timing assumptions are natural: at the time when financing is secured the present-self entrepreneur does not know exactly how profitable her project is going to be. Then, as the project is already being developed, she receives some additional information and re-evaluates whether the project is worth the time. However, as the interests she pays on debt do not reflect this new information, the expected payoff of the future-self entrepreneur and the project's net expected value are not the same.

Principal's decision whether to launch the project or abandon it depends crucially on the interest rate R. Define the *commitment interest* as such an interest rate that the future Principal is indifferent between running and abandoning the riskier project.

Definition 3.2. The *commitment interest rate* R^* is defined as the rate for which the future Principal is indifferent between running the low type project and abandoning it and, in the main model, $R_1^* = X - \frac{(1-p^L)\delta Y + Z}{p^L}$.

The Creditors will finance the project only if they at least break even in expectation. The interest rate needed for this to happen depends on which projects are run by the Principal in the following subgame. The interests needed to break even if the Principal is anticipated to run both projects are called *the average interests* and the interests needed to break even if the Principal is anticipated to run only the safer project are called the *benchmark interests*.

Definition 3.3. The *average interest rate* R' is defined as the rate for which the Creditors break even in expectation if the Principal runs both projects: $R' = \frac{1-q}{q}(1-\delta)Y$. The *benchmark interest rate* R'' is defined as the rate for which the Creditors break even in expectation if the Principal runs only the high type project: $R'' = \frac{1-p^H}{p^H}(1-\delta)Y$.

Define additionally the *highest mixing probability* as the probability of the low project being launched that makes the Creditors break even if the commitment interest rate is played.

Definition 3.4. The *highest mixing probability* is the highest probability of the low project being launched for which the commitment interest rates results in (weakly) positive payoff to Creditors, and in the main model is given by $\bar{\rho}_1 = \frac{\alpha p^H(R_1^* - R'')}{(\alpha-1)(\mu^L - Z)}$.

The equilibria of this game are fully determined by the three interest rates defined above.

Proposition 3.2. In the main model, three types of Subgame Perfect Equilibria are possible. (1) If the commitment interest rate is greater than the average interest rate, then the average interest rate is played and both projects are run with probability one. (2) If the commitment interests lie between the average and benchmark interests, then the commitment interests are played, the high type project is run with probability one and the low type project is run with probability $\rho \in [0, \bar{\rho}_1]$. (3) If the commitment interests are less than the benchmark interests, then the benchmark interests are played, the high type project is run with probability one and the low type project is not run at all.

A formal proof can be found in the Appendix, but the reasoning behind this result is very simple. The Principal always proposes the lowest interest rate the Creditors are willing to accept, but can credibly commit to not running the low type project only by offering an interest payment at least as high as commitment interests. Thus, if the commitment interests are higher than the average interests, committing towards not launching the project is too costly; hence, the Principal offers the average interests and runs both projects. If the commitment interests are less than the benchmark ones, the Principal has to offer the latter, as the Creditors are never going to accept an interest payment lower than that. In each of these two cases, the Agent and the Creditors receive they reservation payoffs of Z and zero, respectively, whilst the Principal captures the entire surplus, which is equal to $\mu - Z$ in the former and $\alpha(\mu^H - Z)$ in the latter case.

The interim case, where commitment interests lay between the other two, is the most interesting one. The Principal offers the commitment interests rate – anything more would lower her payoff, whereas offering anything less would mean that the low quality project is launched and the Creditors would not finance.

Corollary 3.1. In the main model, if equilibrium of the second type is played, then (i) the Agent receives her reservation payoff in expectation; (ii) the Creditors receive an expected payoff equal to $E(V_C, \rho) = \alpha p^H(R^* - R'') + \rho(1-\alpha)(\mu^L - Z)$ and (iii) the Principal's expected payoff is given by: $E(V_P) = \alpha(\mu^H - Z) - E(V_C, 0)$.

Proof. As the Principal has all bargaining power and the same information set as the Agent, she can always condition the wage on the type of project and prevent the Agent from receiving any rent. Thus, total surplus[10] is split between the Creditors and the Principal and equal to $\alpha(\mu^H - Z) + \rho(1-\alpha)(\mu^L - Z)$. Suppose

---

[10] Note that by *total surplus* I mean the average net present value of the launched projects, weighted by the probability of that type of project being drawn. *Total welfare* is then simply total surplus plus the Agent's reservation payoff.

$\rho = 0$, then Creditors' expected payoff is $\alpha(p^H R_1^* - (1-p^H)(1-\delta)Y)$, which, after some rearranging, gives $\alpha p^H(R_1^* - R'')$. This gives us Principal's payoff for $\rho = 0$ and, as the Principal is indifferent between launching the low type project and not (by definition of commitment interests), also for all $\rho \in [0, \bar{\rho_1}]$. To show (ii), it suffices to subtract the Principal's payoff from total surplus. □

With commitment interests higher than benchmark interest rate, if the Principal decides to never launch the project ($\rho = 0$), the Creditors receive a strictly positive expected payoff. This does not have to happen, as the Principal could decide to mix and run the low type project with some positive probability. However, the fact that the Principal has the entire bargaining power and yet the Creditors can earn a rent is quite interesting. The reason is that the interest rate plays the role of a costly commitment device – the Principal is willing to offer Creditors such high payment in order to credibly commit not to launch the low type project. Note that, as at the commitment interest rate the Principal is indifferent between launching and abandoning the project, her payoff does not depend on ρ.

Corollary 3.2. The commitment interest rate falls faster with the liability level than average and benchmark interests. Thus, the higher the level of liability, the more likely it is that equilibria (2) and (3), respectively, are played. In particular, if we define δ_1^1 as the level of liability for which the commitment interest are equal to average interests and δ_1^2 as the level of liability for which the commitment interest are equal to average interests, then for any $\delta < \delta_1^1$ equilibrium (1) is played, for any $\delta \in [\delta_1^1, \min\{1, \delta_1^2\})$ equilibria (2) are played and for any $\delta \in [\delta_1^2, 1]$ equilibrium (3) is played.

Proof. From Definitions 3.2 and 3.3 we have that the gradients of R^*, R' and R'' are, respectively, $\frac{1-p^L}{p^L}$, $\frac{1-q}{q}$ and $\frac{1-p^H}{p^H}$. Thus, as $p^H > p^L$, the Proposition follows from the definition of q, Proposition 3.2 and simple algebra. □

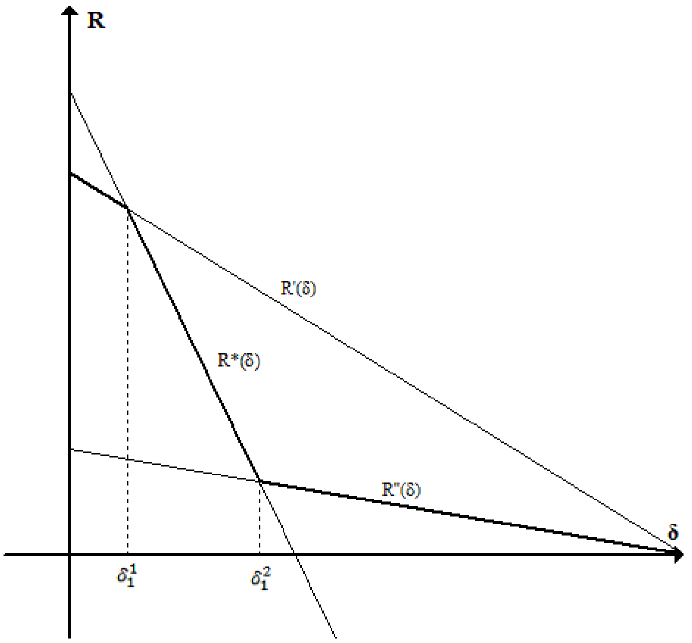

Figure 3.1: $R_1^*(\delta)$, $R'(\delta)$ and $R''(\delta)$ for $\alpha\mu^H + (1-\alpha)\mu^L > \frac{q}{p^L}Z$.

From the Principal's/Agent's point of view, an increase in the liability level has the strongest impact on profitability of the low type project: both for the high type project and the full project the probability of failure – and thus being liable for part of the losses – is strictly lower. Thus, the higher the liability level, the less likely is the launch of the low type project.

Figure 3.1 depicts the three interests rates as a function of δ for a case where the net present value of the low type project is negative, but the probability of the low type project succeeding is not too low. These two conditions together ensure that δ_1^2, δ_1^2 are in $(0, 1)$, which means that the type of equilibrium depends on the legal system. With no legal system ($\delta = 0$) there is over-investment, as the low type project is launched despite its negative npv. Introducing a relatively low level of Agent's liability, $\delta < \delta_1^1$, does not change the equilibrium outcome, as credible commitment is still too expensive.

For $\delta = \delta_1^1$ the commitment and average interest are equal; in other words, for these interest the Creditors would be willing to finance even if both types of projects were launched, whereas the Principal is indifferent between launching the risky project and not. In that very point P can launch in equilibrium the low project with any probability $\rho \in [0,1]$. Note that this means that the efficient equilibrium in which the Creditors earn a rent suddenly becomes possible. Therefore, if we introduce sufficiently high Agent's liability, efficient equilibria become possible even if none were achievable without the legal system.

For $\delta \in (\delta_1^1; \delta_1^2)$ there are multiple equilibria of type (2). For $\delta = \delta_1^2$ the commitment interest are equal to the benchmark ones and in the only equilibrium the Principal never runs the low type project, as otherwise Creditors would not break even and hence would not accept. For $\delta > \delta_1^2$ the third equilibrium is played. Note that for $\delta = 1$, so the full liability case, the average and benchmark interests are both equal to zero as the informed parties (Principal/Agent) bear the entire risk. Hence, if we introduce Agent's liability above a certain threshold, we can reach an unique efficient equilibrium, even if only inefficient equilibria were possible without the legal system.

Corollary 3.3. If the npv of the low type project is positive, then $\delta_1^1 \geq 1$ and the low type project is launched for any $\delta \in [0,1]$. If the npv of the low type project is sufficiently low ($\mu^L - Z \leq \frac{p^H - p^L}{p^H} Y$), then $\delta_1^1 \leq 0$ and the low type project is not launched for any $\delta \in [0,1]$.

Proof. From definition of δ_1^1 and some rearranging we get that:

$$\frac{p^L - q}{q}(1 - \delta_1^1)Y = \mu^L - Z.$$

As $p^L < q$, a positive npv of the low type project implies that $\delta_1^1 \geq 1$. By definition of δ_1^2 we get $\mu^L - Z \leq \frac{p^H - p^L}{p^H}(1-\delta)Y$ and thus both claim follow from Corolary 3.2. □

If the net present value of the less profitable project is either positive or too low, then the commitment device is not necessary. In the first case, that is because the Principal's present-self wants both projects to be launched; in the latter case because launching the project is not tempting enough for the Principal's future-self. In each case the resulting equilibrium is efficient.

Proposition 3.3. In the main model, overall welfare – the sum of all players' payoffs – is weakly increasing in liability, in the sense that both the highest and lowest equilibrium welfare increases weakly. Principal's expected payoff is increasing in δ, but the highest possible payoff Creditors receive is not necessarily monotone in δ, and has a discontinuity at δ_1^1.

Proof. If the npv of the low type project is negative, then only equilibrium (1) is possible and thus total welfare and players' payoffs do not change. Therefore, I will focus on the case of inefficient low type project.

In equilibrium (1) total welfare is equal to the npv of the full project, in equilibrium (2) it is equal to $\alpha(u^H - Z) + \rho(1-\alpha)(u^L - Z) + Z$ and in equilibrium (3) to $\alpha\mu^H + (1-\alpha)Z$. Therefore, the results wrt total welfare follow from Corollary 3.2 and the definition of $\bar{\rho}_1$. Principal's payoffs are equal to total surplus in equilibria (1) and (3) and to $\alpha(\mu^H - Z) - \alpha p^H(R_1^* - R'')$ in equilibrium (2) and thus, by Corollary 3.2 are weakly increasing in liability. Creditors' highest possible payoff is 0 in equilibria (1) and (3) and $\alpha p^H(R^* - R'')$ in equilibrium (2), which means that it increases as we move from any $\delta < \delta_1^1$ to any $\delta \in [\delta_1^1, \delta_1^2)$ and then gradually decreases, and has a discontinuity at δ_1^1.

□

Let me make two observations to clarify this result. Firstly, even though the interests rate cannot be conditioned on the project's type, the high type project always gives a higher expected payoff to the Principal then the low-type project. Secondly, the high type project subsidises the low type project through the common interest rate and thus the low type project always yields a higher expected payoff to the Principal than its net present value. Thus, in the main model there is no reason for under-investment and the only source of inefficiency is over-investment. Therefore, as an increase in liability decreases the likelihood of an investment in the less profitable project, overall welfare increases with liability.

Interestingly, although directors' liability should, in principle, protect the Creditors, the maximal payoff they can achieve is not monotone in liability. In particular, if the npv of the risky project is negative, then, for intermediate values of δ, Creditors can earn a rent by providing the Principal a commitment device, which she can use to not launch the welfare decreasing project. If liability is low, then the commitment device is too costly for the Principal. If liability is high, then the Principal is committed to not launching the risky project anyway and does not need the costly commitment device. Thus, the legal system can provide the Principal with a *free* commitment device.

3.3.3 Weak Principal, strong Creditors

In this variant of the model, the Principal chooses the employment contract at the same time as the wage contract. I interpret this as a proxy for strong Creditors who are able to ensure that the financing conditions depend on the employment contract. This implies that there is no time-inconsistency problem on the Principal's part. However, as the Principal and Agent do not have the same information set any more, this variant of the model cannot be reinterpreted as one with an entrepreneur who both owns and runs the firm. As the Principal does not observe the project type and can only condition the wage on the outcome, over-investment is not an issue any more, but under-investment can be. Increases in liability levels still make the launch of the low-type project less likely, which now leads (weakly) less efficient equilibria.

In comparison to the main model Principal's and Creditors strategy sets change. As Principal chooses both the financing and employment contracts at the very beginning of the game, she can't condition her choice on anything, and all her strategies can be written as: $S_P = (R, W)$. Creditors, on the other hand, at the time they move know the employment contract additionally, and can condition their choice on that too. Because of that any their strategies can be written as $S_C = \gamma(R, W)$, where $\gamma : \mathbb{R}^2 \to [0, 1]$. The Agent's strategy set does not change. The strategy sets are clearly affected by the assumption that P can only borrow Y, which in this variant has an impact on the results. For a discussion on relaxing this assumption, see Section 3.4.2.

Define the *i type wage*, as a wage contract that makes the Agent indifferent between accepting the rejecting, if the project turns out to be of type i.

Definition 3.5. For $i \in \{H, L\}$, the *i type wage* is defined a the wage for which the Agent is indifferent between the employment contract and the reservation payoff:

$$W^i = \frac{1}{p^i}\big(Z + (1 - p^i)\delta Y\big).$$

If the low type wage is offered, but the project turns out to be of high type, the Agent earns an *information rent.*

Definition 3.6. The *information rent* is defined as the rent earned by the agent if the low type wage is offered, but the project is of high type: $I = \frac{p^H - p^L}{p^L}\big(Z + \delta Y\big)$.

The (Subgame Perfect) equilibria of this game are fully determined by the npv of the low type project, the information rent and the probability of the project being of high type.

Proposition 3.4. In the variant with information asymmetry but no commitment problem, there are three types of Subgame Perfect Equilibria. (1) If the low type project is profitable enough compared to the information rent ($(1-\alpha)(\mu^L - Z) > \alpha I$) then low type wage is offered and both projects are run with probability 1. (2) If the entire expected surplus created by running the low type projects is captured by the Agent through information rent ($(1-\alpha)(\mu^L - Z) = \alpha I$), then the Principal mixes between the high and low type wages with any probability $\rho \in [0,1]$, high type project is launched with probability one and low type project with probability ρ. (3) If the low type project is not profitable enough compared to the information rent ($(1-\alpha)(\mu^L - Z) > \alpha I$), then the high type wage is offered and only the high type project is launched.

Proof. By backward induction. By definition of i type wages, it follows that if the Agent observes that the project is of type i, she accepts iff $W \geq W^{*i}$. If A observes high type project he will Accept both W^{*L} and W^{*H}, but if he observes low type project, only W^{*L} will be accepted, as for any $\delta \in [0,1]$, $W^{*L} > W^{*H}$[11].

As the Creditors observe W, the lowest R they are willing to accept is: R' if $W \geq W^{*L}$ and R'' if $W \in [W^{*H}; W^{*L})$. If they observe any other W, they will be willing to accept any R, as the project won't be launched for sure.

If the Principal wants only the high type project to be launched, she offers (W^H, R'') and if she wants both projects to be run, she offers (W^L, R'). The latter contract results in an additional expected surplus of $(1-\alpha)(\mu^L - Z)$, but also in the Agent earning the information rent with probability α. Thus, the Principal prefers the latter option if $(1-\alpha)(\mu^L - Z)$ is strictly greater than αI, the latter if it's strictly smaller and is indifferent otherwise. $\square$

When commitment is not an issue, over-investment cannot occur, as the Principal would never offer the Agent a contract that would incentivise her to run an inefficient project. On the other hand, as the Principal is not able to condition the wage on the project's type, in order to incentivise the Agent to run a low-type, efficient project, she needs to propose a contract that enables him to earn a rent. If this rent outweighs the additional surplus, the low type project is not launched. Thus, information asymmetry can cause under-investment. This

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[11] All of that applies only for NE outcomes of the subgame starting with A's decision.

problem will be only enhanced by the legal system.

Proposition 3.5. The higher the level of director's liability, the less likely it is that the low type project is launched. Define δ_2^1 as the liability level for which equilibrium (2) occurs, then (i) total welfare and Principal's expected payoffs decrease weakly with liability and (ii) Agent's expected payoffs are not monotone in liability, with a discontinuity at δ_2^1.

Proof. The first result follows immediately from Proposition 3.4 and the fact that information rent strictly increases in δ. Result (ii) follows from the fact that the Agent earns αI for $\delta \in [0, \delta_2^1]$ and 0 otherwise; note that αI is strictly increasing in δ. The result with respect to total welfare follows from Proposition 3.4 and the fact that for $\delta \in [0, 1]$ equilibrium (3) can only happen if the npv of the low type project is positive. As for the results with respect to to Principal's expected payoff, note that if the npv of the low type project is negative, then the Principal's payoff is equal to total welfare and does not change. If the npv is positive, then $E(V_P) = \mu - Z - \alpha I$ for $\delta \in [0, \delta_2^1]$, which is strictly decreasing in δ, and $E(V_P) = \mu^H - Z$ for $\delta \in (\delta_2^1, 1]$. □

It is still the case that an increase in directors' liability exposes the Agent to more downside risk and thus decreases risk taking: however, as too much risk taking cannot be an issue, more liability (weakly) decreases efficiency. Surprisingly, some increases in liability can strictly increase the Agent's expected payoff. The reason is that the more liable the Agent is, the more does he need to be paid to accept the employment contract if the project is low type *in comparison* to a high type project – and thus his information rent increases.

3.3.4 Weak Principal and Creditors

Now the employment contract is designed/ can be changed after the financing contract is signed, but not after the project's type is determined. Therefore, both commitment and asymmetric information problems are present and the impact of the legal system on efficiency depends on the net present value of the low type project. This case describes well some financial institutions: the Creditors – e.g. deposit holders – have no control over the employment contract and thus the shareholders are tempted to change directors' bonuses in a way that induces more risk taking; but at the same time, the shareholders do not know the details of the project themselves and need to pay an information rent to the directors.

The Principal can condition her choice of W only on R – but as R is chosen by her as well, P's strategies can be written as $S_P = (R, W)$, which is exactly the same as in Section 3.3.3. At the same time, Creditors do not know the details of the financing contract when they make their move – because of that their strategies' set is the same as in the main variant. The average and benchmark interest rates, as well as low and high type wages and information rent have the same values as previously, but the commitment interest rate changes compared to the main model. In order to launch the low type project, the Principal needs to pay a rent to the Agent, and thus the commitment interest rate in this variant of the model is lower and given by:

$$R_3^* = R_1^* - \frac{\alpha}{p^L(1-\alpha)} I. \tag{3.1}$$

As the highest mixing probability depends on the difference between commitment and benchmark interests, it becomes $\bar{\rho}_2 = \frac{\alpha p^H (R_3^* - R'')}{(\alpha-1)(\mu^L - Z)}$. As the updated commitment interest rate accounts for all the dynamics introduced by the wage contract, the Subgame Perfect Equilibria of this variant depend only on the relation between the average, benchmark and commitment interest rates.

Proposition 3.6. If both asymmetric information and commitment problem are present, then three types of Subgame Perfect Equilibria are possible. (1) If the commitment interest rate is greater than the average interest rate ($R_3^* \geq R'$), then the average interest rate is played and both projects are run with probability one. (2) If the commitment interests lie between the average and benchmark interests ($R_3^* \in (R'', R')$), then the commitment interests are played, the high type project is run with probability one and the low type project is run with probability $\rho \in [0, \bar{\rho}_3]$. (3) If the commitment interests are less than the benchmark interests ($R_3^* \leq R''$), then the benchmark interests are played, the high type project is run with probability one and the low type project is not run at all.

Proof. Analogous to the proof of Proposition 3.2. □

The basic intuition is exactly the same as for Proposition 3.2. Therefore, the impact of an increase in liability on the probability of the low type project being run is also qualitatively the same as in Section 3.3.2.

Proposition 3.7. The commitment interest rate falls faster with the liability level than average and benchmark interests. Thus, the higher the level of liability, the more likely it is that equilibria (2) and (3), respectively, are played. In particular, if we define δ_3^1 as the level of liability for which the commitment interest are equal to average interests and δ_3^2 as the level of liability for which the commitment interest are equal to average interests, then for any $\delta < \delta_3^1$ equilibrium (1) is played, for any $\delta \in [\delta_3^1, \min\{\delta_3^2, 1\}]$ equilibria (2) are played and for any $\delta \in (\delta_3^2, 1]$ equilibrium (3) is played.

Proof. The information rent increases in δ and therefore the commitment interests fall faster than in the model with just commitment issues; thus, by the proof of Corollary 3.2 commitment interests fall faster than average interests, which fall faster than benchmark interests. The remainder of the Proposition follows trivially. □

As liability level goes up, the low type project becomes less profitable for the Principal faster than the high-type project – compared to Section 3.3.2 this effect is even stronger, because of the information rent. Thus, in all three variants an increases in directors' liability decreases the likelihood of the low type project being launched. However, as there are both commitment issues and asymmetric information in this variant, the impact of the legal system on efficiency is not signed in general and depends on the npv of the low type project.

Proposition 3.8. In equilibrium (1), the Agent earns $\alpha I + Z$ in expectation and the Principal earns the remainder of total welfare, so $\mu - Z - \alpha I$. In equilibrium (2), the Creditors earn $E(V_C, \rho) = \alpha p^H(R^* - R'') + \rho(1-\alpha)(\mu^L - Z)$, the Principal earns $\mu^H - Z - E(V_C, 0)$ and the Agent receives $Z + \rho\alpha I$. In equilibrium (3), the Agent and Creditors receive their reservation payoffs and the Principal receives the entire surplus, $\mu^H - Z$. Therefore, the highest and lowest possible equilibrium total welfare levels are (i) increasing in liability if the net present value of the low type project is negative and (ii) decreasing in liability if npv of the low type project is negative. The Agent's and Principal's payoffs, as well as the highest possible Creditors' payoff, are not monotone in liability. In particular, Principal's payoff first decreases and then increases in liability.

Proof can be found in Appendix 3.A. If the low type project is inefficient then increasing liability (weakly) improves welfare; if it is efficient, then the opposite is true. All players can potentially earn rents – this happens if the net present value of the low type project is negative, but not too negative and the liability level is neither too low nor too high. In such a case, Creditors can earn a rent, as they provide the Principal with a costly commitment device and the Agent earn the information rent, with probability $\alpha\rho$, where $\rho \in (0, \bar{\rho})$. The Principal's payoff initially decreases in liability, as she is forced to pay a higher information rent, but then increases in it, as a high enough liability eliminates the need of paying any rents. Whether her payoff is higher with full liability or with no legal system at all depends on the parameters.

3.3.5 All variants

So far all three variants were – with minor exceptions – treated separately. They are, however, called variants for a reason: they share most of the structure – and what they don't share, the timing of employment contract's design, can be treated as yet another exogenous variable in the model. Thus, it is worthwhile to compare the evolution of different variables across all the variants, in order to understand better the influence of the order in which contracts are written/ the relative strength of players on interest and risk taking.

Corollary 3.4. For any level of Agent's liability δ the resulting equilibrium interest – $R^e(\delta)$ – will be weakly lowest in the variant with strong Creditors and weakly highest in the main variant, so that $R^e_1(\delta) \geq R^e_3(\delta) \geq R^e_2(\delta)$.

Proof can be found in the Appendix. With weak Creditors the rent can serve the additional role of a costly commitment device. If, further, the Principal is strong, then the Agent does not earn any information rent from running the low type project, which makes launching the low type project more tempting for the future-self Principal and, because of that, the commitment device becomes even more expensive.

Interest rates are closely connected to the probability that the risky project is launched: the higher the probability of the low type project being launched, the higher interest rate the Creditors charge. Because of that, the ranking of these probabilities is the same as the ranking of equilibrium interest.

Corollary 3.5. For any level of Agent's liability δ the highest possible probability with which the risky project can be launched – $\bar{\rho}(\delta)$ will be weakly lowest in the variant with strong Creditors and weakly highest in the main variant, so that $\bar{\rho}_1(\delta) \geq \bar{\rho}_3(\delta) \geq \bar{\rho}_2(\delta)$.

Proof in the Appendix. As the Principal's position becomes relatively stronger the firm takes more risk; and *vice versa*[12]. This is another reason why the relative strength of the players/ the order in which contracts are written influences strongly the efficiency of the system.

3.4 Discussion

In this section I address the question of the optimal level of directors' liability in a number of setups. Then I discuss in detail different aspects of the feasibility of such a policy. After that I extend the model, first by introducing risk aversion and then by reinterpreting the optimal policy as the equilibrium outcome of a game in which directors' liability is decided as part of the financing contract and flat wages are feasible.

3.4.1 Policy

Consider a policymaker who cares only about the efficiency of launched projects or – equivalently – about overall welfare; he can act before the game starts and choose any δ between 0 and 1.

Corollary 3.6. If under-investment is not possible, setting as high as possible liability level is optimal. If over-investment is not possible, setting as low as possible liability level is optimal.

This follows immediately from Corollary 3.2 and Proposition 3.7. As under-investment is caused by asymmetric information between the Principal and the Agent and under-investment by commitment problems, in settings where only on of these is an issue policy design is trivial. However, if both these problems can occur, so in the variant with weak Creditors and Principal, the optimal policy

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[12] Stronger in the sense that if the Principal knows less than the Agent, but can change the employment contract after the financing one is signed, he has relatively stronger position/ more control than when he knows less and cannot change the employment contract.

depends on the net present value of the low-type project. For negative net present value, high liability levels are optimal; if the low-type project is efficient, however, low liability is preferable. In general, high liability is not always optimal, even with risk neutral players.

Of course, any policymaker is highly unlikely to know the exact net present value of the project. For that reason, I will briefly consider the question of optimal policy design with an imperfectly informed policymaker. In particular, I consider a policymaker who does not know the return of a successful project and therefore does not know the net present value of the low-type project[13]. This lack of knowledge is, clearly, problematic only if both information asymmetry and commitment are an issue: otherwise either full or no liability is always an optimal policy.

Corollary 3.7. If there are both information asymmetry and commitment problems, then there exists a unique liability level for which efficient equilibria exist for any $X \geq \frac{1}{p^H}(Z + (1 - p^H)Y)$. This *optimal policy* is the liability level for which the average interests are equal to the commitment interests if the net present value of the low type project is equal to zero:

$$\delta_3^* = 1 - (1 + \frac{Z}{Y})\frac{q}{q + (1 - \alpha)p^L}.$$

Proof. Define the low type project for $\bar{X} = \frac{1}{p^L}(Z + (1 - p^L)Y)$ as the *marginal project*. By definition of δ_3^* and Proposition 3.6 it follows that for $X = \bar{X}$ both launching and not launching the low type project is an equilibrium outcome. As the commitment interest rate increases in X and the average and banchmark interest rates do not depend on X, it follows that for any $X > \bar{X}$ and δ_3^* the commitment interest rate is above the average interests and thus the low type project is launched, which is efficient. For any $X < \bar{X}$ the commitment interest

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[13] Knows $\{Y, p^H, p^L, \alpha\}$, but not X.

rate is lower than average interest rate and hence there exists an equilibrium in which the low -type project is not launched, which is also efficient. Consider any $\delta < \delta_3^*$; then there exists some $\epsilon > 0$, such that for $X = \bar{X} - \epsilon$ the inefficient low-type project is launched with probability one. Now, consider any $\delta > \delta_3^*$, so that the highest probability with which the marginal project can be launched in equilibrium is less than 1. Hence, there has to be some strictly efficient project that will be launched with probability strictly lower than one in any possible equilibrium. □

Note that the optimal policy only ensures that an efficient equilibrium exists, but not that it is unique. The optimal liability level is strictly less than one, which shows that an imperfectly informed policy maker is not going to introduce full liability as long as there is any possibility of asymmetric information between management and shareholders. However, if underinvestment is an issue then this optimal policy might be infeasible, as δ_3^* can be strictly less than zero. In such a case, the second best is to set the liability level to zero.

The result that over-investment can be always prevented depends crucially on the assumption that the Agent's wealth is sufficient to cover all the losses. This is unlikely to be the case in many – if not most – applications, as even if we include Agent's future income in his wealth, it might not be enough to repay all creditors of a publicly held company – especially that in most countries individuals can also default and release themselves from any obligations[14]. To study this formally, I assume that $\frac{T}{Y} = t \in (0, 1)$. As all parties are aware of that, t becomes the upper-bound of feasible liability levels.

Corollary 3.8. If the net present value of the low-type project is negative, the optimal policy is more likely to be feasible if there is asymmetric information

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[14] That is another way in which the real liability differs across countries. For instance, in the UK the individual insolvency law is much more lenient than in Germany – and that's another reason why the *de facto* δ is likely to be much higher in the latter.

between Agent and Principal. If the net present value of the low-type project is positive, then the optimal policy is more likely to be feasible if the Principal is time-inconsistent.

This follows immediately from the results in Section 3.3.5, Corollary 3.2 and Proposition 3.7. Corollary 3.8 implies that the first best outcome is most likely to be achieved through optimal policy if both information asymmetry and commitment problems are present. This can be seen even better if we again consider a policymaker who knows everything, but the value of the positive outcome.

Corollary 3.9. If only one of (a) asymmetric information or (b) commitment problems is an issue then, for $t < 1$, there exists no feasible liability level that ensures existence of efficient equilibria for all $X \geq \frac{1}{p^H}(Z + (1 - p^H)Y)$. If both of these problems coexist, then the optimal policy is feasible as long as $\delta_3^* \in [0, z]$.

Proof. The first statement wrt information asymmetry follows immediately from previous considerations. For commitment problems, note that for the marginal project (see proof of Corollary 3.7) the commitment interests are weakly lower than the average interests only for $\delta = 1$. As only the commitment interests depend on X and increase in it, it follows that if $t < 1$ then there exist some $X > \bar{X}$ such that for $\delta = t$, the commitment interests are higher than the average ones. The second statement follows trivially from Corollary 3.7. $\square$

In general, Corollary 3.9 reinforces the message that the case where both over- and under-investment is possible is the one where policy is most likely to play a crucial role. Note that a lot of firms in the financial sector could fall into this category – the dispersed depositors/investors cannot really impose anything on the shareholders who control the board, who nevertheless don't have as detailed knowledge about the firm's operations as its directors. On the other hand, if the shareholders know (almost) as much as the directors – for instance in some family-held companies – their wealth could be too limited for the optimal level of liability to be feasible. Similarly, if the debtholders are strong – so are, for instance, banks – and have a significant degree of control over the company, then even introducing the second-best level of liability might not be enough to eliminate underinvestment.

A separate question is the political feasibility of an increase in directors' liability. One particular concern policymakers are likely to have is that an increase in liability will result in higher wages for management in the good state of the world, as they need to be compensated for the increased exposure to downside risk. This is a legitimate concern, but is has to be noted that the increase in liability can also prevent the low-type project from being launched, which reduces the overall downside risk. Moreover, in settings with asymmetric information a high enough level of liability can prevent management from earning the information rent, which will have a further, negative effect on equilibrium wages. Therefore, in some cases, it is possible that an increase in liability can decrease the wage received by the management/directors[15].

3.4.2 Extensions

Risk aversion

Suppose the Agent is risk averse, with a standard utility function U, such that $U' > 0$ and $U'' < 0$. His risk aversion creates an additional source of welfare loss from a high liability level – the agency cost[16]. In particular, *given* the type of the project, the higher the Agent's liability, the less risk is shared with the risk neutral Creditors and the higher is the agency cost. However, this implies that

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[15] In fact, it is even possible that the wage is lower with full than with no liability. To see this, consider the variant with both information and commitment problems and suppose that $p^H = 1$ and the npv of the low-type project is negative. Then the wage with full liability is Z, which is the lowest possible equilibrium wage.

[16] The agency cost could be avoided if the Agent was paid a flat wage equal to Z. However, for this to happen, δ would need to be negative.

high liability makes the low type project even less profitable for the Principal and thus the commitment interest fall faster than with risk neutrality. Therefore, it is still the case that higher levels of liability mean that the riskier project is less likely to be launched – which, in turn, decreases the agency cost. Thus, the impact of increased liability on the agency cost cannot be signed[17].

Hence, in some cases, Agent's risk aversion can make the introduction of directors' liability even more desirable. The set of optimal policies will be smaller, however, and consist of just one element – as increasing liability above the level necessary for preventing the launch of the risky project would inevitably lead to a welfare loss. Apart from that, as the commitment interests are lower if the Agent is risk averse, the level of liability needed to prevent over-investment is more likely to be feasible.

Private contracts

In this extension, I consider the possibility that transfers between the Agent and Creditors result not only from legal regulations, but also from private contracts between the parties. Denote the *private* transfer from the Agent to Creditors as B and the *legal* transfer from Agent to Creditors as L. The Agent's payoff in the case of the project's failure is $-(L+B)$ and Creditors' payoff is $L+B-Y$. Rewriting B as $\frac{B}{Y}Y = bY$ and L as $\frac{L}{Y}Y = lY$, we get $l+b=\delta$.

Hence, in this richer setting δ denotes the level of directors' real liability, determined by the law and private arrangements. Note that the results from Section 3.3 can be treated as the solution of a subgame of a more general game, in which firstly the legal liability is set by the policymaker and then the private transfer is chosen by the Principal. This does not result in any loss of generality,

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[17] It is obvious that it can be positive, if the change in liability levels does not result in a change in launch probabilities. To see that it can be negative, consider $p^H = 1$. Then there is no agency cost from launching the high type project – but if the low type project is run, then there is a strictly positive agency cost. It follows that an increase in liability that prevents the riskier project from being launched decreases the average agency cost.

as it does not matter if private transfer is chosen before interests rates or are they chosen simultaneously, as long as Creditors know both the interest rate and the real liability level at the time they move. Consider the extreme case, in which there is no legal liability and the Principal is free to choose any level of private transfer between zero and one – then the analysis in Section 3.3 tells us what happens once the private transfer is determined.

In this extension, it is the Principal, not the policymaker, who chooses the level of liability and she is not concerned with total welfare, but her own expected payoff. If only commitment problems or information asymmetry are present, then (by Propositions 3.3 and 3.5) total welfare and Principal's payoffs react to changes in liability in the same direction and equilibrium contracts are a subset of optimal policies and result in optimal outcomes. If both these problems are present, then the efficiency of the private contract equilibrium depends on parameter values. In general, the Principal will seek to design a contract which will commit her to launching only projects with positive npv, but at the same time will not allow any other parties to earn rents. As liability cannot be negative, the latter can be achieved only if the low type project is not launched. For risky projects with negative net present value such outcome is efficient and the equilibrium private contract belongs again to the set of optimal policies. However, if the npv of the low type project is positive, then the Principal might prefer to commit towards not launching the risky project, if the information rent she would need to pay is high enough. Therefore, if private transfers are restricted to be positive, private contract equilibrium might result in a suboptimal level of real liability.

In the absence of a legal system, private contracts making the Agent liable for firm's debts should emerge if there is a commitment problem on shareholders part. As hinted a couple of times previously, from the three countries which regulations I have studied with most detail, the UK ones seem to be most lenient – it is most likely that such contracts could be found worthwhile to write there. And indeed, there is some evidence from the UK that directors' individual wealth is sometimes used as collateral for firms' credits. That takes the form of "personal guarantees" – a quick google search shows this to be a common practice in the UK, but also in other countries, like Australia[18]. Note that a private contract obliging the entrepreneurs to partly cover firm's losses in the event of a failure is somewhat similar to the standard requirement that borrowers put in their own wealth in order to borrow, which is usually explained by moral hazard (see Holmstrom and Tirole, 1997).

If the Principal was able to pay the Agent a flat wage, equal to his reservation payoff, then the low type project could be run without the Agent earning an information rent and the inefficiency would be eliminated. Of course, this would require paying the Agent for failure and thus implies a negative level of real liability. Therefore, if negative private transfers are feasible ($b \in (-\infty, 1]$), then private contract equilibrium is always efficient.

Proposition 3.9. Define Ω as the set of real liability levels that maximise Principal's highest possible payoff and Θ as the set of real liability levels that maximise highest possible total welfare. Then, in all variants, if $\delta \in (-\infty, 1]$, $\Omega \subset \Theta$.

Proof. In the main variant, for $\delta = 1$ total welfare is maximised and the Principal receives the entire surplus. In the variant with only information asymmetry present, for $\delta = -\frac{Z}{Y}$ maximises total welfare and the Principal receives the entire surplus. In the intermediate variant there are two possibilities. If the npv of the low type project is negative, then $\delta = 1$ maximises total welfare and eliminates both Creditors' and Agnet's rent. If the npv is positive, then $\delta = -\frac{Z}{Y}$ achieves the same. $\square$

The reason why the private contract equilibrium is always efficient and thus

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[18] For the former see, for instance, http://www.business-lawfirm.co.uk/Articles/Company/DirectorsTips-for-avoiding-difficulties-with-personal-guarantees.aspx and for the latter http://bakerlove.com.au/take-sting-personal-guarantees/.

results in an optimal level of real liability, is that, with unrestricted contracts, the Principal can prevent the other players from earning any rents and, therefore, her goals are aligned with those of a policymaker who cares only about efficiency. Note also that the Principal can always set an efficient level of real liability, even if the legal liability level was not set at the optimal level.

However, Proposition 3.9 does not mean that the level of legal liability is unimportant. Firstly, the Principal's ability to design private contracts that increase directors' liability is likely to be limited. For instance, a fixed cost of designing a single private liability contract together with a large number of creditors could make the entire cost of private contracts prohibitive – again, think of banks and deposit holders[19]. Moreover, contracts with negative transfers could be even more costly. To see this, note that in this model, Creditors reservation payoff does not depend on the amount of money borrowed. For non-negative private transfers this is without loss in generality, as the sum of money borrowed by the Principal is fixed. However, if the Principal wanted to pay the Agent a fixed wage, which is necessary to eliminate the information rent if the low type project is run, then she would need to borrow more than Y and pay a higher fee to Creditors[20].

Therefore, optimal policy design will crucially depend on what types of private contracts are feasible/ cheap. Consider again the financial sector: the firms there have presumably a relatively easy access to cheap capital, so they will be able to outdo the effects of the legal liability being set to high (unless the law would make the directors' fully liable for all the uncovered debts) – but at the same time, it

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[19] A careful reader might wonder, if the cost of execution wouldn't be an issue also if there is legal liability and weak/dispersed creditors. It might be indeed, but it seems also very plausible to presume that someone with enough cash would start purchasing these debt claims and prosecuting the directors for all of them. .

[20] To see that clearly, suppose that Creditors' reservation payment is 0 if they lend Y, but μ^H if they lend any sum bigger than Y. That is enough to ensure that Principal would never borrow more than Y (as by doing so she would always get a negative payment, whilst she receives positive payoffs in any equilibrium with b=0) – and *de facto* her choice of b would be restricted to positive values (assuming these are feasible).

might be very costly for them to write contracts with the depositors/investors that will make the directors' liable for the debts. In such a case, setting a high level of legal liability seems optimal, as its negative effects will be outdone but its positive effects can't be replicated by the use of private contract.

3.5 Conclusions

This chapter offers several insights on when is introducing director's legal liability advisable. First of all, if there is a commitment issue on the part of shareholders – that is, whenever Creditors are weak and cannot condition financing on the details of the employment contract – introducing legal liability might be optimal. Without such policy, shareholders might not be able to credibly commit to offering the directors contracts that will disincentivise them from running inefficient projects. This is particularly likely to happen if the shareholders are well-informed about the firms dealings. Note that if the shareholders are also running the company – so in the case of an entrepreneur – both conditions are fulfilled. However, in such a case it is also likely that the optimal level of liability is not feasible, as the directors might not have enough wealth to cover a high enough part of the losses.

If, however, creditors are strong, then commitment and over-investment are not an issue, but under-investment can be. In such a case, introducing director's liability is likely to lead to a less efficient outcome. The optimal level of liability is most probable to be feasible if the shareholders are not well-informed and have a commitment problem. Finally, the optimal level of legal liability depends also crucially on the type of private contracts that can be written. If contracts that make directors' more liable are cheap and easy to write, then there is little need for a policy that introduces legal liability. On the contrary, if shareholders have access to cheap credit and can decrease the level of real liability, then even setting a too high level of liability is relatively harmless.

Appendix

3.A Proofs

Proof of Proposition 3.1. I will solve the game by backward induction. Consider the subgame that starts with A deciding whether to Accept or Reject. The payoff from accepting is given by:

$$V_A = p^i W - (1 - p^i)\delta Y, \tag{3.2}$$

whereas the payoff from rejecting is Z. Because of that, the only Nash Equilibria of that subgame require A to Accept whenever $V_A > Z$, Reject if $V_A < Z$, and mix with some $\beta \in [0, 1]$ if $V_A = Z$. For now I will consider the NE with $\beta = 1$, and will return to the other cases later.

The next subgame starts with Principal learning her type, and choosing W^i. Depending on the choice of W^i, Principal will get one of the two following expected payoffs:

$$E(V_{P1}) = p^i(X - W^i - R^i) \quad \text{if } V_A \geq 0. \tag{Category 1}$$

$$E(V_{P2}) = 0 \quad \text{if } V_A < 0 \tag{Category 2}$$

Clearly the optimal W^i in the first case is: $W^{i*} = \frac{(1-p^i)\delta Y+Z}{p^i}$; in the second case the choice of contract doesn't matter. In equilibrium Principal will choose the optimal contract from the category that gives her the higher payoff, possibly mixing in the case of indifference.

The next subgame begins with C's decision whether to Finance or Not. Let ρ be equal to the probability that in equilibrium Principal's choice of W will fall in Category 1, given R^i. Then C's payoff is given by:

$$E(V_C) = \rho(p^i R^i - (1 - p^i)(1 - \delta)Y)$$

If C decides not to Finance, they will always receive 0. If $E(V_C) = 0$ all strategies where C Finances with $\gamma \in [0, 1]$ constitute a SPE of that subgame. For now I will consider only $\gamma = 1$.

The last subgame starts with Principal choosing R^i. Let me start with arguing that in SPE, Principal will never offer R which would give C an expected payoff greater than 0. If they did so, by – say – offering $R^{i1} > R''^{i}$, they could always do better by offering $R^{i1} - \epsilon$, so that R^{i1} can be never played in equilibrium. As long as $\rho > 0$, $R^i = \frac{1-p^i}{p^i}(1 - \delta)Y$. For such R^i, $E(VP_1) = \mu^i - Z$ – as long as $\mu^i - Z > 0$, ρ has to be 1 and we have found an SPE indeed. If $\mu^i - Z < 0$, then Principal will either choose W^i from Equation Category 1 or such R^i that C won't finance: in either case, in SPE she will get 0. If $\mu^i - Z$=0, then ρ can be both 0 and greater than 0 – but in either case Principal will get 0.

What remains is to show what happens, if in the case of C and A being indifferent between two options, they decide to make other choice than the ones considered above. I will show that such strategies either don't constitute an SPE, or the SPE they constitute give the same payoffs as those specified above.

Let me firstly consider the case when $\mu^i - Z > 0$. Lets say that C mixes with probability γ – then Principal's payoff from playing $R^i = \frac{1-p^i}{p^i}(1 - \delta)Y$ is equal $\gamma(\mu^i - Z)$. But in such a case, for any $\gamma < 1$ Principal can always do better by playing $R^i = \frac{1-p^i}{p^i}(1 - \delta)Y + \epsilon$, where ϵ is some very small number, and receiving

$\mu^i - Z - p\epsilon$. Moreover, however small ϵ is, Principal can always do even better by choosing some even smaller mark-up over R^i: because of that, there is no equilibrium for $\gamma < 1$. On the other hand, if $\mu < 0$, then if C mixes, Principal's profit from choosing a strategy from Category 1 will be strictly negative: she can always do better by – for instance – choosing $R^i < 0$ and forcing C not to finance, so the SPE will yield the same profit as specified in the Proposition. If $\mu^i - Z = 0$ then there are also equilibria in which C mixes – in fact, if $\mu^i - Z = 0$ then every player can mix at any stage: as everyone will receive they reservation utilities anyway. Similar logic applies for Agent's choice whether to Accept or Reject. □

Proof of Proposition 3.2. (1) $R' < R_1^*$, so for R' both projects yield positive expected payoff to future-self Principal and both will be launched. As Principal's payoffs from each project are decreasing in R, Principal's general payoff is decreasing in R any $R > R'$ (i.e. R_1^*) is dominated by R’[21]. At the same time, from definition of R' any $R < R'$ (i.e. R”, as R”≤R’ for any $\delta \leq 1$) will not be accepted by Creditors, so R' has to be chosen in equilibrium.

(2) As $R^* \leq R'$ it is preferred to R’. Any $R < R^*$ (i.e. R”) will not be accepted by Creditors, hence R^* has to be played in any equilibrium. There are multiple equilibria, as Principal is indifferent between launching the low type project and not. As in equilibrium C's payoff has to be greater than 0, we can find the range of equilibrium ρ by solving the following inequality:

$$(\alpha p^H + (1-\alpha)\rho p^L)R^* - (\alpha(1-p^H) + \rho(1-\alpha)(1-p^L))(1-\delta)Y \geq 0$$

and remembering that $\rho \in [0,1]$. The solution is $\rho \in [0, \bar{\rho}_1]$.

(3) As $R^* < R''$, for R'' the low type project will be launched with probability

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[21] To be precise, it is weakly decreasing, but it doesn't matter, as it is strictly decreasing for any $R < X - \frac{(1-p^H)\delta Y + Z}{p^H}$, and R' and R_1^* belongs to that category.

0, and hence R'' has to be played in equilibrium (it is prefered to any R>R" and any R<R" won't be accepted by C). ☐

Proof of Proposition 3.8. (1) As both projects are launched, Creditors do not earn any rent by definition of R' and the low type wage is played. Hence, the Agent earn $\alpha I+Z$. By definition of total welfare, the Principal's payoff is $\mu-Z-\alpha I$. By Proposition 3.7, for $\delta\in[0,\delta_3^1)$, (a) Principal's payoff is strictly decreasing; (b) Agent's payoff is strictly increasing and (c) total welfare and Creditors payoffs are constant.

(2) Creditors and Principal's expected payoffs follow from a reasoning identical to that in the proof of Corollary 3.1. Agent's receive $I+Z$ if the low type project is launched and Z otherwise, so $\rho\alpha I+Z$ on average. By Proposition 3.7 and simple algebra, for $\delta\in[\delta_3^1,\min\{\delta_3^2,1\})$, Principal's payoff is strictly increasing, whereas Creditors highest possible payoff is strictly decreasing.

(3) Only high type project is launched, Creditors do not earn a rent by definition of R'', Agent's do not earn a rent by definition of W^H and thus Principal earns μ^H-Z; payoffs and total welfare do not change with liability.

Comparative statics: Principal's payoff is first decreasing and then increasing in δ, by (1) and (2). Agent's payoff is firstly strictly increasing and positive and then, in (3), equal to 0, so not monotone in general. Creditors highest possible payoff is zero and then and then becomes positive and decreasing, so is not monotone. Highest possible total welfare is equal to μ in (1) and to $\alpha\mu^H+(1-\alpha)Z$ in (2) and (3) and thus is increasing if the npv on the low type project is negative and decreasing otherwise. The lowest possible eqm welfare is the same in (1) and (3) and $\alpha\mu^H+(1-\alpha)(\bar\rho_3\mu^L+(1-\bar\rho_3)Z)$ and, by definition of $\bar\rho_3$ and proof of Proposition 3.7 is also increasing (decreasing) for low type projects with negative (positive) npv.

☐

Proof of Corollary 3.4. The fact that $R_1^e \geq R_3^e$ follows trivially from the fact that $R_3^* < R_1^*$ and Propositions3.2 and 3.6.

From definitions of δ_3^2 and δ_2^1 and simple algebra follows that:

$$\frac{\alpha}{(1-\alpha)} \frac{p^H - p^L}{p^L}(Z + \delta_2^1 Y) = \frac{\alpha}{(1-\alpha)} \frac{p^H - p^L}{p^L}(Z + \delta_3^2 Y) - \frac{p^H - p^L}{p^H}(1 - \delta_3^2)Y,$$

which implies that $\delta_3^2 > \delta_2^1$ and thus it follows from Propositions 3.5 and 3.7 that $R_3^e \geq R_2^e$.

□

Proof of Corollary 3.5. Let us, for now, restrict the attention to $\delta \in [0, 1)$. Then the maximum probability with which the risky project can be launched for some R is defined by:

$$(\alpha p^H + (1-\alpha)\bar{\rho} p^L)R - (\alpha(1-p^H) + \bar{\rho}(1-\alpha)(1-p^L))(1-\delta)Y = 0. \tag{3.3}$$

By differentiating this equation with respect to R, we get:

$$\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}R} = \frac{-p^L \bar{\rho}}{p^L R - (1-p^L)(1-\delta)Y}.$$

The numerator is negative, as $\bar{\rho} \in [0, 1]$. The denominator is negative as long as $R < \frac{1-p^L}{p^L}(1-\delta)Y$. We know that for any $\delta \in [0, 1)$, the equilibrium interest for any timing belong to $[R''; R']$ – and as R' is less than $\frac{1-p^L}{p^L}(1-\delta)Y$, $\bar{\rho}$ will increase with interest in the interval of interest. That, together with Corollary 3.4, establishes the result for $\delta \in [0, 1)$.

For $\delta = 1$ the equilibrium interest in every variant are 0, so Equation 3.3 is true for any $\bar{\rho}$ – and thus it won't help us in proving the result. We know, however, from previous considerations, that for $\delta = 1$ the possibility of over-investment is eliminated, so:

- In the main variant all efficient projects are launched with prob. 1 and all inefficient ones are not launched.

- In the variant with strong Creditors the maximum probability of all projects for which $\mu^L - Z \geq \frac{\alpha}{1-\alpha}\frac{p^H - p^L}{p^L}(Y + Z)$ being launched is 1, for all other it's 0.

- In the intermediate model the maximum probability of all projects for which $\mu^L - Z \geq \frac{\alpha}{1-\alpha}\frac{p^H - p^L}{p^L}(Y + Z)$ being launched is 1, for all other it's 0.

Because of that, for any given project, the probability that it will be launched, is weakly highest for the main variant and weakly lowest for the other two, as required. $\square$

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