

Strange images of profinite groups

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Abstract

We investigate whether a finitely generated profinite group G could have a finitely generated infinite image. A result of Dan Segal shows that this is impossible if G is prosoluble. We prove that such an image does not exist if G is semisimple or *nonuniversal*. We also investigate the existence of dense normal subgroups in G .

1 Introduction

What can the images of a profinite group G be? There are obviously all the continuous images of G but there may well be others. In the first place G may not be finitely generated and then some of its images may be very different from the continuous ones. For example there are examples of profinite groups which are inverse limits of finite perfect groups while $G \neq G^2$, see [10]. However this cannot happen when G is finitely generated: [8] proves that some verbal subgroups of G are automatically closed: for example any term of the lower central series of G . However not all verbal subgroups of G are closed - the second derived subgroup G'' may not be closed, even when G is a finitely generated pro- p group. See [5] for a description of the words w such that $w(G)$ is closed in all finitely generated pro- p groups G .

Let us call a quotient G/N of a profinite group G *strange* if N is not closed in G . The strange images of G are defined to be groups I such that there exists an abstract epimorphism $f : G \rightarrow I$ but there is no continuous such choice for f . Clearly every strange image of G is also a strange quotient but the converse need not hold (for example in $G = C_p^N$).

We are motivated by the following

Theorem 1 (Nikolov & Segal, [8]). *A finitely generated profinite group has no strange finite quotients.*

Corollary 1. *A residually finite image of a finitely generated profinite group G is either finite or uncountable.*

Indeed, if $f : G \rightarrow I$ is a homomorphism with residually finite image I then $\ker f$ is an intersection of subgroups of finite index in G and therefore by Theorem 1 is a closed subgroup of G . Hence I is a profinite group and must be uncountable if it is infinite (say by the Baire category theorem).

It is natural to try to extend this result to other types of images. As a first step we could ask whether a finitely generated profinite group can have a countable infinite image. This may happen, in fact already \mathbb{Z}_p has such an

image: compose the inclusion $\mathbb{Z}_p < \mathbb{Q}_p$ with a surjective linear map $\mathbb{Q}_p \rightarrow \mathbb{Q}$ as a vector space over the rationals.

The following conjecture arose in Blaubeuren in 2007 in discussions with Y. Barnea, E. Breullard, P.E. Caprace, T. Gelander and J. Wilson.

Conjecture 1 (Blaubeuren). *A profinite group G cannot have an infinite finitely generated image.*

Suppose that the homomorphisms $f : G \rightarrow I$ has finitely generated infinite image. Without loss of generality we may assume that G is finitely generated as a profinite group. Let J be the intersection of all subgroups of finite index in I . Since I/J is a residually finite image of G by Corollary 1 we deduce that I/J is finite. Then $H := f^{-1}(J)$ is a subgroup of finite index in G hence by Theorem 1 it is a finitely generated profinite group and J is a finitely generated infinite image of H . Now an easy application of Zorn's lemma shows that J , hence H has an infinite simple image.

This shows that it is enough to prove the Conjecture 1 for a finitely generated *simple* image.

First we consider prosoluble groups. There a theorem of Dan Segal [9] gives the answer.

Theorem 2 (D. Segal). *Suppose G is a prosoluble group generated by d elements. Let a_1, a_2, \dots, a_r generate G modulo G' , i.e. $G = \langle a_1, a_2, \dots, a_r \rangle G'$ (note that G' is closed in G by Corollary 1 of [9]). Then*

$$G' = \left(\prod_{i=1}^r [G, a_i] \right)^{*(72d+46)},$$

where $[G, y] = \{g^{-1}y^{-1}gy \mid g \in G\}$ and for a subset $A \subset G$ and integer t $A^{*t} := A \cdot A \cdots A$ (t times).

So if G/N is a simple infinite quotient of G we have that $G'N = G$ and hence we can choose elements n_1, \dots, n_d from N which generate G modulo G' . Now Theorem 2 gives that every element of G' is a product of $2d(72d + 46)$ conjugates of the elements n_i , hence $G' \leq N$, i.e. $N = G$, contradiction.

Therefore Conjecture 1 holds for prosoluble groups G .

Another class of groups where we can test the validity of Conjecture 1 is the class of semisimple profinite groups, i.e. the Cartesian products of nonabelian finite simple groups. For these groups we have the following result.

Proposition 1. *Every simple image of a Cartesian product G of nonabelian finite simple groups is either finite or uncountable. In particular Conjecture 1 holds for G .*

In fact in Proposition 3 below we shall describe all the simple quotients of G .

A natural generalization of prosoluble groups is the class of *nonuniversal* profinite groups.

For a finite group Γ let $\alpha(\Gamma)$ be the largest integer k such that Γ involves the alternating group A_k , i.e. A_k is an image of a subgroup of Γ . We say that

a profinite group G is *non-universal* if there is an integer k such that $\alpha(\bar{G}) \leq k$ for all finite continuous images \bar{G} of G . In that case we denote by $\alpha(G)$ the maximum of the numbers $\alpha(\bar{G})$. It is easy to see that a profinite group G is non-universal if and only if its continuous finite images don't involve some finite group.

Theorem 3. *Conjecture 1 holds for nonuniversal profinite groups G .*

Proposition 1 together with Theorems 2 and 3 gives the following

Corollary 2. *Let G be a profinite group which has a subnormal chain $G = G_1 \triangleright G_2 \triangleright \cdots \triangleright G_k = 1$ of closed subgroups G_i such that each quotient G_i/G_{i+1} is either semisimple or non-universal. Then G cannot have a finitely generated infinite image.*

1.1 Dense normal subgroups

Notice that when G/N is a strange simple quotient of a profinite group G then N must be a dense subgroup of G .

We now turn to the more general task of investigating dense normal subgroups of profinite groups. This is not such a strange topic as it might appear at first and is of some interest to the study of topologically simple compactly generated locally compact groups, see section 1.2 below.

Of course hoping for a complete classification of these is too much: Any infinite abelian profinite group contains many dense normal subgroups. As another example consider a Cartesian product $\prod_{i=1}^{\infty} S_i$ of profinite or finite groups S_i which contains the direct product $\bigoplus_{i=1}^{\infty} S_i$. So if a profinite group G has an infinite image which is abelian or a Cartesian product of finite groups then G will possess a proper dense normal subgroup. It is surprising that the converse of this holds, at least in the case when G is nonuniversal.

Theorem 4. *Let G be a finitely generated nonuniversal profinite group. Then G has a dense proper normal subgroup if and only if one of the following holds:*

- (a) G/G' is infinite or,
- (b) G maps onto infinitely many finite simple groups.

One might wonder if the condition that G is nonuniversal above can be removed. The following example shows that it is necessary at least if one wants to keep the same conclusion of the theorem.

Consider the semidirect product $G = \prod_{n=5}^{\infty} A_n \rtimes C_2$ where the generator τ of C_2 acts on each of the factors A_n as conjugation by a transposition. Then $G/G' \simeq C_2$ and the only finite simple image of G is C_2 . However it is easy to see that the normal closure of τ in G is a proper dense subgroup.

This shows that when G is universal some extra possibilities should be added to the list in Theorem 4. As a first step one might wish to study this in the case when G is virtually semisimple.

Definition 1. *Let G_* be the intersection of all open normal subgroups K of G such that G/K is abelian or simple.*

Thus G/G_* is the largest quotient of G which has the form $A \times \prod_i S_i$ where A is an abelian profinite group and $\{S_i\}_i$ are nonabelian simple groups. Theorem

4 seems to suggest that any normal dense normal subgroup of G contains G_* . However this is false even for nilpotent groups:

Let $G = U_3(\widehat{\mathbb{Z}})$ be the Heisenberg group of upper 3-by-3 unitriangular matrices with entries in $\widehat{\mathbb{Z}}$. Then $G \simeq \prod_{p \text{ prime}} U_3(\mathbb{Z}_p)$ and just as before this contains the dense normal subgroup $N = \bigoplus_{p \text{ prime}} U_3(\mathbb{Z}_p)$. Since N does not contain G' the quotient G/N is not abelian. Similar examples show that N does not need to contain G_* in some perfect profinite groups G as well.

However in the special case when N contains a finitely generated normal subgroup then indeed $G_* \leq N$, in fact $G' \leq N$.

Theorem 5. *Let G be a finitely generated nonuniversal profinite group and let N be a normal subgroup of G . Suppose that N contains a finitely generated subgroup D whose image is dense in G/G_* . Then G/N is abelian.*

Corollary 3. *Let G be a finitely generated nonuniversal profinite group which maps onto only finitely many finite simple groups (abelian or nonabelian). Then any dense normal subgroup of G contains G' .*

Proof of Corollary 3: The assumption implies that the Frattini subgroup F/G_* of G/G_* is open in G/G_* , i.e. F is open in G . Hence if N is a dense normal subgroup of G then $FN = G$. Therefore we can find finitely many elements of N which generate G/F and hence their images generate a dense subgroup of G/G_* . Therefore Theorem 5 applies. \square

1.2 Topologically simple locally compact groups

As promised we give an application of Corollary 3. A locally compact group G is *locally finitely generated* if its compact open subgroup U is topologically finitely generated, i.e. contains a dense finitely generated subgroup.

Proposition 2. *Let G be a compactly generated totally disconnected locally compact topological group which is locally finitely generated. Assume that G is topologically simple. Then the only proper abstract quotients of G must be abelian.*

Proposition 2 can be viewed as an analogue of the following result in the connected case (cf. Theorem XVI 2.1 in [4]): If N is a dense analytic subgroup of Lie group G there is an abelian subgroup A of G such that $G = NA$.

Proof of Proposition 2. Let U be a compact open subgroup of G . Then U is a profinite group which is finitely generated. Let K be a compact generating set for G . Without loss of generality we may assume that $K = K^{-1} = UKU$. As explained on page 150 of [7] G acts on the regular Schreier graph Γ with vertices G/U and edges (gU, gkU) for $g \in G, k \in K$. Moreover, since G is topologically simple the action is faithful.

Now the valency of every vertex of Γ is $r = |KU/U|$ which is finite. The profinite group U is the stabilizer of a vertex of Γ and is therefore a quotient of the group of rooted automorphisms of the infinite r -regular tree (the universal cover of Γ). It follows that every upper composition factor of U is a quotient of a subgroup of $\text{Sym}(r)$, in particular U is nonuniversal and maps onto finitely many finite simple groups. Now suppose that N is a nontrivial normal subgroup of G . Since G is topologically simple N must be dense in G and hence $N_1 = N \cap U$

is a dense normal subgroup in U . By Corollary 3 it follows that G/N_1 must be abelian. Finally note that $NU = G$ since N is dense in G and therefore $G/N = NU/N \simeq U/N_1$. \square

I wish to thank Pierre-Emmanuel Caprace [1] for pointing out this application to me. In light of this result it will be interesting to know if it is possible that the groups G above can ever have nontrivial abelianization.

The structure of the rest of the article is as follows. Proposition 1 is proved in Section 1 and the proof of Theorem 3 is in Section 3. Theorems 4 and 5 are proved in Section 4. First we need a few definitions.

Notation:

For an element g in a group G the conjugacy class of g is denoted g^G . When $A, B \subset G$ by AB we denote the set $\{ab \mid a \in A, b \in B\}$ and for $t \in \mathbb{N}$ let

$$A^{*t} = A \cdot A \cdots A \quad (t \text{ times}).$$

If $f \in \text{Aut}(G)$ then $[G, f] := \{g^{-1}g^f \mid g \in G\}$. In particular if $N \triangleleft G$ and $a \in G$ then $[N, a] := \{n^{-1}n^a \mid n \in N\}$.

Further let $[N, {}_n G]$ be defined by

$$[N, {}_1 G] = [N, G] = \langle \{[N, g] \mid g \in G\} \rangle, \quad [N, {}_{n+1} G] = [[N, {}_n G], G], \quad (n \geq 1)$$

and finally let

$$[N, {}_\omega G] = \bigcap_{i=1}^{\infty} [N, {}_i G].$$

We call a (finite or profinite) group *semisimple* if it is a Cartesian product of finite simple groups. A finite group G is *quasisimple* if $G = G'$ and $G/Z(G)$ is simple.

A group G is a *central product* of its subgroups $\{S_i\}_{i \in I}$ if $G = \langle \{S_i\}_i \rangle$ and $[S_i, S_j] = 1$ for all distinct $i, j \in I$. Then $G/Z(G)$ is isomorphic to the cartesian product of $S_j/Z(S_j)$. A central product of quasisimple groups is called *quasi-semisimple*. In that case $\text{Aut}(G)$ acts faithfully on $G/Z(G)$.

2 Proof of Proposition 1

Let $\{S_i\}_{i=1}^{\infty}$ be a collection of nonabelian finite simple groups and let $G = \prod_i S_i$. Take an ultrafilter \mathcal{U} on the integers and recall the definition of the *ultralimit*, $\lim_{\mathcal{U}} a_i$ of a bounded sequence $(a_i)_{i=1}^{\infty}$ of real numbers: $\lim_{\mathcal{U}} a_i$ is the unique number α such that for any $\epsilon > 0$ the set of indices $\{i \in \mathbb{N} \mid |a_i - \alpha| < \epsilon\}$ is a member of \mathcal{U} .

Define a function $h = h_{\mathcal{U}} : G \rightarrow [0, 1]$ by

$$h(\mathbf{g}) = \lim_{\mathcal{U}} \frac{\log |g_i^{S_i}|}{\log |S_i|} \quad \text{for any } \mathbf{g} = (g_i)_i \in G = \prod_{i=1}^{\infty} S_i.$$

It is easy to see that $h(ab) \leq h(a) + h(b)$ for any $a, b \in G$ and therefore the set

$$K = K_{\mathcal{U}} := h^{-1}(0)$$

is a normal subgroup of G .

Proposition 3. *The simple images of $G = \prod_{i=1}^{\infty} S_i$ are exactly the quotients $G/K_{\mathcal{U}}$ for some ultrafilter \mathcal{U} on the integers.*

Proof of Proposition 3 As a first step we show that each quotient $G/K_{\mathcal{U}}$ is indeed simple. Suppose that $\mathbf{g} = (g_i)_i \in G$ is outside $K = K_{\mathcal{U}}$. This means that $h(\mathbf{g}) > 0$, i.e. the set

$$A = \{i \in \mathbb{N} \mid \log |g_i^{S_i}| / \log |S_i| > e\}$$

is in \mathcal{U} for some $e > 0$.

Now we use the following result

Theorem 6 (Liebeck & Shalev, [6]). *There is a constant $c > 0$ such that if S is a nonabelian finite simple group and $\mathcal{C} \subset S$ is a conjugacy class in S then*

$$S = \mathcal{C}^{*n},$$

where $n = \lfloor c \frac{\log |S|}{\log |\mathcal{C}|} \rfloor$.

With this result by setting $n = \lfloor c/e \rfloor$ we obtain that $S_i = g_i^{S_i} \cdot g_i^{S_i} \cdots g_i^{S_i}$ (n times) for each $i \in A$, and therefore

$$G = K \cdot (\mathbf{g}^G)^{*n}$$

because $A \in \mathcal{U}$.

Since \mathbf{g} was an arbitrary element from G outside K we see that G/K is simple.

Conversely, suppose that G/H is simple for a proper normal subgroup H of G . For every $\mathbf{t} = (t_i)_i \in H$ and every $\epsilon > 0$ let

$$A(\mathbf{t}, \epsilon) = \left\{ i \in \mathbb{N} \mid \frac{\log |t_i^{S_i}|}{\log |S_i|} < \epsilon \right\} \subseteq \mathbb{N}$$

and let U be the collection of all subsets $A(\mathbf{t}, \epsilon)$ for all $\mathbf{t} \in H$ and all $\epsilon > 0$.

I **claim** that every finite subset of U has nontrivial intersection. Indeed, suppose that

$$A(\mathbf{t}_1, \epsilon_1) \cap A(\mathbf{t}_2, \epsilon_2) \cap \cdots \cap A(\mathbf{t}_k, \epsilon_k) = \emptyset$$

for some $A(\mathbf{t}_i, \epsilon_i) \in U$. Let $\epsilon = \min_i \{\epsilon_i\}$ and write $\mathbf{t}_i = (t_{i,j})_j$ with $t_{i,j} \in S_j$.

It follows that for each integer $j \in \mathbb{N}$ there is some $i \leq k$ such that $j \notin A(\mathbf{t}_i, \epsilon)$, i.e., $\log |t_{i,j}^{S_j}| / \log |S_j| \geq \epsilon$. Put $N = c/\epsilon$, then Theorem 6 gives

$$S_j = \left(t_{i,j}^{S_j} \right)^{*N}.$$

By considering independently each coordinate $j \in \mathbb{N}$ we see that

$$G = \prod_{i=1}^k (\mathbf{t}_i^G)^{*N}.$$

However the elements \mathbf{t}_i are from $H \triangleleft G$ and hence $H = G$. This contradiction proves the claim.

Since the finite collections of elements of U have nontrivial intersection a standard application of Zorn's lemma provides the existence of an ultrafilter \mathcal{U} on \mathbb{N} containing U . Now from the definition of U it follows that $h_{\mathcal{U}}(\mathbf{t}) = 0$ for all $\mathbf{t} \in H$, i.e., $H \leq K_{\mathcal{U}}$. But G/H is simple and therefore $H = K_{\mathcal{U}}$. Proposition 3 is proved.

Proof of Proposition 1 Armed with the description of the simple images of G it is easy to complete the proof.

In the rest of this section the rank of a finite simple group S is its (untwisted) Lie rank if S has Lie type, it is n if $S = A_n$ and 0 otherwise.

There are two cases to consider.

Case 1. For some $m \in \mathbb{N}$ the collection of indices $D(m) := \{i \mid \text{rank}(S_i) \leq m\}$ is in \mathcal{U} .

Since ultrafilter \mathcal{U} is supported on the set $D(m)$, we can assume that $D(m) = \mathbb{N}$. Now a nontrivial conjugacy class \mathcal{C} of a finite simple group S_i has size at least $|S_i|^\epsilon$ for some ϵ which depends only on the rank of S_i . Therefore the quotient $G/K_{\mathcal{U}}$ in this case coincides with the ultraproduct $\prod_i S_i/\mathcal{U}$ and by [2] an ultraproduct of finite sets is either finite or uncountable. So it is not possible for the image to be countably infinite.

Case 2. $D(m) \notin \mathcal{U}$ for all $m \in \mathbb{N}$.

Note that the map $h : G \rightarrow [0, 1]$ induces a well defined map on $G/K_{\mathcal{U}}$. I claim that the image of h is the whole interval $[0, 1]$ and so $G/K_{\mathcal{U}}$ must be uncountable. In turn $h(G) = [0, 1]$ follows from

Proposition 4. *Let S_i be a collection of finite simple groups such that $\text{rank}(S_i) \rightarrow \infty$ as $i \rightarrow \infty$. For any real number $\beta \in [0, 1]$ there is a sequence of elements $g_i \in S_i$ such that $\frac{\log |g_i^{S_i}|}{\log |S_i|} \rightarrow \beta$ as $i \rightarrow \infty$.*

Assuming this for the moment let the elements g_i be as in Proposition 4 and let $\mathbf{g} = (g_i)_i \in G$. Then $h_{\mathcal{U}}(\mathbf{g}) = \beta$ for any nonprincipal ultrafilter \mathcal{U} . Case 2 is completed and so is the proof of Proposition 3. \square

Proof of Proposition 4.

Put $\alpha = 1 - \beta$.

If S_i is an alternating group A_n take g_i to be an even cycle of length about αn in A_n .

If $S = S_i$ is a simple classical group, consider the corresponding quasisimple classical group \tilde{S} acting on its natural module V over a finite field of size q equipped with a bilinear form f (symmetric, sesquilinear, alternating or just equal to 0 in case S_i has type PSL_n). We have $S = \tilde{S}/Z$ where Z is the centre of \tilde{S} and if $g = \tilde{g}Z \in S$ with $\tilde{g} \in \tilde{S}$ we have that

$$|g^S| \leq |\tilde{g}^{\tilde{S}}| \leq |Z||g^S|$$

Since the centre Z of S has asymptotically negligible size compared to the size of S it is enough to find an element $\tilde{g} \in \tilde{S}$ with $\log |\tilde{g}^{\tilde{S}}| \sim \beta \log |\tilde{S}|$.

We can decompose V as $V_0 \oplus V_1 \oplus V_2$ such that:

- $\dim V_0$ is about $\sqrt{\alpha} \dim V$, and $\dim V_1 = \dim V_2$,
- $V_1 \oplus V_2$ is orthogonal to V_0 , and
- The form f is nondegenerate on both V_0 and $V_1 \oplus V_2$ and is isotropic on V_1 and on V_2

Let $\tilde{g} \in \tilde{S}$ be equal to the identity on V_0 and act on each of V_1 and V_2 as a cyclic transformation without fixed vectors. In other words there is a vector $v_i \in V_i$, ($i = 1, 2$) such that $v_i, \tilde{g}v_i, \tilde{g}^2v_i, \dots$ is a basis for V_i .

Now $C_{\tilde{S}}(\tilde{g})$ contains the classical group L on V_0 preserving f and by the choice of $\dim V_0$ we have $\log |L| / \log |\tilde{S}| \sim (\dim V_0 / \dim V)^2$ which tends to α as $\dim V \rightarrow \infty$.

On the other hand if $s \in \tilde{S}$ commutes with \tilde{g} then s must stabilize V_0 , the fixed space of \tilde{g} . Since V_1 and V_2 are cyclic modules for \tilde{g} we have that the action of s on V_1 and V_2 is determined from $s \cdot v_1$ and $s \cdot v_2$. Hence s is completely known from its restriction to V_0 and from the two vectors $sv_1, sv_2 \in V$. Denote by $\text{Gf}(V_0)$ the subgroup of $\text{GL}(V_0)$ which preserves f . We have that $|\text{Gf}(V_0)| \leq q|L|$.

Therefore

$$|L| \leq C_{\tilde{S}}(\tilde{g}) \leq |\text{Gf}(V_0)| |V|^2 \leq q^{1+2\dim V} |L|$$

which gives $\log |G_{\tilde{S}}(\tilde{g})| / \log |\tilde{S}| \sim \log |L| / \log |\tilde{S}| \rightarrow \alpha$ as $\dim V$ tend to infinity.

This easily gives $\log |g^S| / \log |S| \rightarrow 1 - \alpha = \beta$ as the rank of S tends to infinity and proves Proposition 4.

3 Proof of Theorem 3

Let G be a non-universal profinite group with an infinite finitely generated simple quotient G/N . Without loss of generality we may assume that G is generated by say d elements. Let G_1 be the prosoluble residual of G , i.e. the intersection of all open normal subgroups N of G such that G/N is soluble. Let G_2 be the the smallest closed normal subgroup of G_1 such that G_1/G_2 is semisimple. (Alternatively G_2 is the intersection of all open normal subgroups M of G_1 such that G_1/M is simple). Let $G_3 = [G_{2,\omega} G]$. This is a closed subgroup of G by Theorem 1.4 of [8] but strictly speaking we don't need this result here as we could take G_3 as the closure of $[G_{2,\omega} G]$ with the same effect.

The quotient G/G_3 is a pronilpotent-by-semisimple-by-prosoluble group and therefore from Proposition 1 and Theorem 2 we already know that Conjecture 1 holds for it.

We have assumed that N is a normal subgroup of G such that G/N is infinite simple and finitely generated. Since G/G_3 doesn't have infinite finitely generated images it is not possible that $G_3 \leq N$ and therefore $NG_3 = G$. Choose elements n_1, n_2, \dots, n_d from N such that

$$\overline{\langle n_1, \dots, n_d \rangle} G_3 = G$$

Below we shall prove that there is a number t depending on d and $\alpha(G)$ such that

$$G_3 = ([G, n_1] \cdot [G, n_2] \cdots [G, n_d])^{*t} \quad (1)$$

Since $[G, n_i] \subset N$ for each i we see that $G_3 \leq N$ and hence $G = NG_3 = N$, contradiction.

It remains to prove equation (1). Since both sides are closed subsets of G it is enough to show that (1) is true in every finite image of G . So let $\Gamma \geq \Gamma_2 \geq \Gamma_3$ be finite images of G, G_2 and G_3 respectively. It follows that Γ_2 is the semisimple by soluble residual of Γ and that $\Gamma_3 = [G_2, {}_\omega G]$.

Recall the definition of acceptable subgroups in [8], page 178:

A normal subgroup N of Γ is acceptable if

1. $[N, G] = N$, and
2. If $A \leq B \leq N$ are two normal subgroups of G inside N then B/A is not isomorphic to S or $S \times S$ for a nonabelian finite simple group S .

It follows that Γ_3 is an acceptable subgroup of Γ . The first condition is clear. If the second condition fails then $\Gamma/C_\Gamma(A/B)$ is isomorphic to a subgroup of $\text{Aut}(A/B)$ and for a simple group S we have that $\text{Aut}(S)$ and $\text{Aut}(S \times S) = \text{Aut}(S) \wr C_2$ are both semisimple-by-soluble. Therefore $C_\Gamma(A/B)$ contains $\Gamma_2 \geq \Gamma_3$ which implies that A/B is abelian, contradiction.

The following result is proved as Key Theorem (C) in [8].

KEY THEOREM (C) *Let Γ be a d -generated finite group and H an acceptable normal subgroup of G . Suppose that $G = H\langle g_1, \dots, g_r \rangle$. There is a number $t = t(d, \alpha(G))$ which depends only on d and $\alpha(G)$ such that*

$$H = ([H, g_1][H, g_2] \cdots [H, g_r])^{*t}.$$

Now Key Theorem (C) with n_i in place of g_i for $i = 1, 2, \dots, d$ and Γ_3 in place of H gives

$$\Gamma_3 = ([\Gamma_3, n_1][\Gamma_3, n_2] \cdots [\Gamma_3, n_d])^{*t}.$$

Since this holds for any finite image Γ of G equation (1) is proved and Theorem 3 follows. \square .

4 Proofs of Theorems 4 and 5

We shall prove Theorem 5 first and then deduce Theorem 4. We need the following result by Gaschütz [3].

Theorem 7. *Let G be a profinite group generated topologically by d elements. Suppose that M is a closed normal subgroup of G and g_1, \dots, g_k are some $k \geq d$ elements of G such that $\overline{\langle g_1, \dots, g_k \rangle}M = G$.*

Then there exist elements $g'_i \in g_i M$ such that $\overline{\langle g'_1, \dots, g'_k \rangle} = G$.

We shall use this result repeatedly in the following situation: suppose N is a subgroup of G containing elements g_1, \dots, g_d whose images are dense in a quotient G/M of G . Assume that for some normal closed subgroup M_1 of G contained in M we have $M \leq NM_1$. Then we can pick elements $g'_i \in N \cap g_i M$ such that $\overline{\langle g'_1, \dots, g'_d \rangle}M_1 = G$.

Proof of Theorem 5. Let G_i for $i = 1, 2, 3$ be the subgroups defined in Section 3, namely G_1 is the intersection of all open normal subgroups V of G such that G/V is soluble, G_2 is the intersection of the normal open subgroups W of G_1 such that G_1/W is simple, and $G_3 = [G_2, {}^\omega G]$. Let K be the intersection of all open normal subgroups of G with G/K simple and nonabelian. Thus $G_* = G' \cap K$. Put $G_s = G_1 \cap K = G_1 \cap G_*$ and let $G_v = [G_2, G]$. Then $G/G_s = G/G_1 \times G/K$ and the group G_1/G_v is a quasisimple: a perfect central extension of the semisimple group G_1/G_2 .

Step 1: $G'G_1 \leq NG_1$.

By assumption we can find elements g_1, \dots, g_d of N such that $\overline{\langle g_1, \dots, g_d \rangle} G_* = G$. In addition we assume that d is more than the minimal number of topological generators for G . By Theorem 2 applied to G/G_1 and elements $g_i G_1$ it follows that

$$G' = \left(\prod_{i=1}^d [G, g_i] \right)^{*(72d+46)} G_1 \subseteq NG_1,$$

Theorem 7 together with $G'/G_1 \leq NG_1/G_1$ implies that we can replace the elements g_i by some elements $g_{1,i} \in g_i G' \cap N$ such that $\overline{\langle g_{1,1}, \dots, g_{1,d} \rangle} G_s = G$.

Step 2: $G_1 \leq NG_v$.

We can assume that $G_v = 1$ in this step and will aim to show that $G_1 \leq N$. Now $G_1 = G_1/G_v$ is a cover of the semisimple group G_1/G_2 and therefore a central product of finite quasisimple groups. We can write $G_1 = P \circ Q$ as a central product of two quasisimple groups P and Q where $Q = G'_s$ and $G_1 \cap K = G_s = Z(G_1)Q$ and so $G/K \simeq G_1/G_s = P/Z(P)$.

First we shall show that $P \leq N$. The elements $\{g_{1,i}\}_{i=1}^d$ generate a dense subgroup of $G/G_s = G/G_1 \times G_1/G_s$. It follows that in every finite image of \overline{G} of G the images $\overline{g}_{1,i}$ generate a group which acts on the image \overline{P} of P as its full group of inner automorphisms.

We claim that in this case there is a number c depending only on $\alpha(G)$ such that $\overline{P} = (\prod_{i=1}^d [\overline{P}, \overline{g}_{1,i}])^{*c}$ giving that

$$P = \left(\prod_{i=1}^d [P, g_{1,i}] \right)^{*c} \subseteq N. \quad (2)$$

To see this first note that Theorem 6 of Liebeck and Shalev [6] has the following form in the special case when a quasisimple group has bounded Lie rank.

Proposition 5. *Let S be a quasisimple group with $\alpha(S) = \alpha$. Then there is a number $c = c(\alpha) \in \mathbb{N}$ depending only on α such that $S = \mathcal{C}^{*c}$ for any noncentral conjugacy class \mathcal{C} of S .*

The claim (2) now follows from

Proposition 6. *Let T be a finite central product of quasisimple groups and suppose $a_1, \dots, a_d \in \text{Inn}(S)$ be d generators of $\text{Inn}(T)$. Then for $c = c(\alpha(T))$ as above we have*

$$T = \left(\prod_{i=1}^d [T, a_i] \right)^{*c}.$$

Indeed for any quasisimple factor S of T there is at least one a_j which acts as some nontrivial inner automorphism $s \mapsto s^g$ of S . Identifying a_j with the noncentral element g of S we have $[S, a_j] = [S, g] = \mathcal{C}g$ where \mathcal{C} is the conjugacy class of g^{-1} in S . Note that if \mathcal{C}_1 and \mathcal{C}_2 are conjugacy classes of a group S and $g \in S$ then

$$g\mathcal{C}_1 = \mathcal{C}_1g; \quad \mathcal{C}_1\mathcal{C}_2 = \mathcal{C}_2\mathcal{C}_1.$$

The result follows from Proposition 5.

Therefore $P \leq N$.

The proof that $Q \leq N$ relies on the following result whose proof we defer to the next section.

Proposition 7. *Let $l \in \mathbb{N}$ and let $T = S_1S_2 \cdots S_m$ be a central product of quasisimple finite group with $\alpha(T) = \alpha$. Let $c = c(\alpha)$ be the number from Proposition 5. Then if a_1, \dots, a_d are automorphisms of T such that $\cap_{i=1}^d C_T(a_i)$ does not contain any S_j then*

$$T = \left(\prod_{i=1}^d ([T, a_i][T, a_i^{-1}]) \right)^{*c}.$$

This is easily seen to imply that

$$Q \leq \left(\prod_{i=1}^d ([Q, g_{1,i}][Q, g_{1,i}^{-1}]) \right)^{*c} \subseteq N \quad (3)$$

Let S be a quasisimple factor of Q . Suppose that all the elements $g_{1,i}$ centralized S . Since centralizers in profinite groups are closed subgroups and $\langle g_{1,1}, \dots, g_{1,d} \rangle G_s = G$ this gives that $C_G(S)S = G$. Therefore $S/Z(S) \simeq G/C_G(S)$, hence $C_G(S) \geq G_* \geq S$, contradiction. Therefore at least one of the elements $g_{1,i}$ does not centralize S and (3) follows from Proposition 7 as before by passing to the finite images of G .

We showed $G_1/G_v = PQ \leq NG_v/G_v$ i.e. $G_1 \leq NG_v$. Now find elements $g_{2,i} \in N \cap g_{1,i}G_1$ such that $\langle \{g_{2,i}\}_{i=1}^d \rangle G_v = G$.

Step 3: $G_v \leq NG_3$.

By definition $[G_2, {}^\omega G] = G_3$ and by our choices of $g_{2,i}$ we have $\overline{\langle \{g_{2,i}\}_{i=1}^d \rangle G'} = G$. Therefore by the profinite version of Lemma 2.4 of [8] with G/G_3 in place of G and G_2/G_3 in place of H we have

$$G_v = [G_2, G] \leq \left(\prod_{i=1}^d [G_2, g_{2,i}] \right) G_3 \subseteq NG_3.$$

Therefore we can replace $g_{2,i}$ by elements $g_{3,i} \in N \cap g_{2,i}G_v$ such that $\overline{\langle \{g_{3,i}\}_{i=1}^d \rangle G_3} = G$.

Step 4: $G_3 \leq N$.

Now we are in position to apply Key Theorem (C): in every finite image \overline{G} of G the group G_3 maps onto an acceptable normal subgroup $\overline{G_3}$ of \overline{G} and the images of the elements $g_{3,i}$ generate $\overline{G}/\overline{G_3}$. Conclusion:

$$G_3 = [G_3, G] = \left(\prod_{i=1}^d [G_3, g_{3,i}] \right)^{*t} \subseteq N,$$

where $t = t(d, \alpha(G))$ is the number from Key Theorem (C).

Altogether by combining the results of results of Steps 1 to 4 we deduce that $G' \leq N$. Theorem 5 follows. \square

Proof of Theorem 4 The argument at the start of Section 1.1 shows that if G has infinite abelianization or maps onto infinitely many simple groups then it has a proper dense normal subgroup. Conversely suppose that G/G' is finite and G maps onto finitely many simple groups. Then G_* is an open subgroup of G .

Suppose that N is a dense normal subgroup of G . We will show that $N = G$. First since G/G_* is finite it follows that N has a finitely generated subgroup D such that $DG_* = G$. Therefore we can apply Theorem 4 and so $N \geq G'$. But G' is open hence so is N . Therefore $N = G$. \square

4.1 Proof of Proposition 7

Lemma 1. *Let $f \in \text{Aut}(G)$. Then*

$$a^G = \{g^{-1}ag \mid g \in G\} \subseteq [G, f][G, f^{-1}]$$

for all $a \in [G, f^{-1}]$.

Proof: For $x, y \in G$ put $z = xy^f$ and consider

$$[y, f][z, f^{-1}] = y^{-1}x^{-1}x^{f^{-1}}y = [x, f^{-1}]^y.$$

\square

Proof of Proposition 7 Every automorphism of T permutes the subgroups S_i among themselves. In addition if S is a quasisimple group the action of $\text{Aut}(S)$ on its simple factor $S/Z(S)$ is faithful. Let π_j be the projection $T \rightarrow T/C_T(S_j) = S_j/Z(S_j)$.

From the condition on the automorphisms a_i it is easy to show that there exist elements $t_1, \dots, t_d \in T$ with the following property: for any quasisimple subgroup S_j of T , ($j \in \{1, 2, \dots, m\}$) there exists $i \in \{1, \dots, d\}$ such that the projection $\pi_j(v_i)$ of the element $v_i := [t_i, a_i^{-1}]$ on $S_j/Z(S_j)$ is nontrivial. This is obvious if some a_i leaves S_j invariant while acting nontrivially on it. Otherwise some a_i will move S_j to a different subgroup S_l and then we can in fact achieve that the projections π_l of $[t_i, a_i^{-1}]$ on **each** of the groups $S_l/Z(S_l)$ from the orbit of S_j to be nontrivial.

Now by Lemma 1

$$\left(\prod_{i=1}^d v_i^T \right)^{*c} \subseteq \left(\prod_{i=1}^d ([T, a_i][T, a_i^{-1}]) \right)^{*c}.$$

and on the other hand from Proposition 6 and the choice of $v_i = [t_i, a_i^{-1}]$ above we obtain

$$S_j \leq \left(\prod_{i=1}^d v_i^T \right)^{*c}$$

for all $j = 1, \dots, m$. Proposition 7 follows. \square

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