

# Products of normal, beta and gamma random variables: Stein characterisations and distributional theory

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## Abstract

In this paper, we extend Stein's method to products of independent beta, gamma, generalised gamma and mean zero normal random variables. In particular, we obtain Stein characterisations for mixed products of these distributions, which include the classical beta, gamma and normal characterisations as special cases. These characterisations, lead us to closed form formulas, involving the Meijer  $G$ -function, for the probability density function and characteristic function of the mixed product of independent beta, gamma and central normal random variables.

**Keywords:** Stein's method, normal distribution, beta distribution, gamma distribution, generalised gamma distribution, products of random variables, Meijer  $G$ -function

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## 1 Introduction

In 1972, Stein [28] introduced a powerful method for deriving bounds for normal approximation. The method rests on the following characterisation of the normal distribution:  $Z \sim N(0, \sigma^2)$  if and only if

$$\mathbb{E}[\sigma^2 f'(Z) - Zf(Z)] = 0 \quad (1.1)$$

for all real-valued absolutely continuous functions  $f$  such that  $\mathbb{E}|f'(Z)|$  exists. This gives rise to the following inhomogeneous differential equation, known as the Stein equation:

$$\sigma^2 f'(x) - xf(x) = h(x) - \mathbb{E}h(Z), \quad (1.2)$$

where  $Z \sim N(0, \sigma^2)$ , and the test function  $h$  is real-valued. The left-hand side of (1.2) is known as the Stein operator. For any bounded test function, a solution  $f$  to (1.2)

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exists (see Stein [29]). Now, evaluating both sides at any random variable  $W$  and taking expectations gives

$$\mathbb{E}[\sigma^2 f'(W) - W f(W)] = \mathbb{E}h(W) - \mathbb{E}h(Z). \quad (1.3)$$

Thus, the problem of bounding the quantity  $\mathbb{E}h(W) - \mathbb{E}h(Z)$  reduces to solving (1.2) and bounding the left-hand side of (1.3).

Over the years, Stein's method has been adapted to many other distributions, such as the Poisson [2], exponential [1], [22], gamma [12] [17], [20] and beta [4], [15]. The first step in extending Stein's method to a new probability distribution is to obtain a Stein equation. For the Beta( $a, b$ ) distribution with density  $\frac{1}{B(a, b)} x^{a-1} (1-x)^{b-1}$ ,  $0 < x < 1$ , where  $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$  is the beta function, a Stein operator commonly used in the literature is

$$\mathcal{A}f(x) = x(1-x)f'(x) + (a - (a+b)x)f(x). \quad (1.4)$$

For the  $\Gamma(r, \lambda)$  distribution with density  $\frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}$ ,  $x > 0$ , the Stein operator

$$\mathcal{A}f(x) = x f'(x) + (r - \lambda x) f(x) \quad (1.5)$$

is often used in the literature. In this paper, we extend Stein's method to products of independent beta, gamma, generalised gamma and central normal random variables. In particular, we obtain natural generalisations of the operators (1.2), (1.4) and (1.5) to products of such random variables.

## 1.1 Products of independent normal, beta and gamma random variables

Fundamental methods for the derivation of the probability density function of products of independent random variables were developed by Springer and Thompson [26]. Using the Mellin integral transform (as suggested by Epstein [7]), the authors obtained explicit formulas for products of independent Cauchy and mean-zero normal variables, and some special cases of beta variables. Building on this work, Springer and Thompson [27] showed that the p.d.f.s of the mixed product of mutually independent beta and gamma variables, and the products of independent central normal variables are Meijer  $G$ -functions (defined in Appendix B).

The p.d.f. of the product  $Z = Z_1 Z_2 \cdots Z_N$  of independent normal random variables  $Z_i \sim N(0, \sigma_i^2)$ ,  $i = 1, 2, \dots, N$ , is given by

$$p(x) = \frac{1}{(2\pi)^{N/2} \sigma} G_{0, N}^{N, 0} \left( \frac{x^2}{2^N \sigma^2} \middle| 0 \right), \quad x \in \mathbb{R}, \quad (1.6)$$

where  $\sigma = \sigma_1 \sigma_2 \cdots \sigma_N$ . If (1.6) holds, we say that  $Z$  has a *product normal* distribution, and write  $Z \sim \text{PN}(N, \sigma^2)$ . The density of the product  $X_1 \cdots X_m Y_1 \cdots Y_n$ , where  $X_i \sim \text{Beta}(a_i, b_i)$  and  $Y_j \sim \Gamma(r_j, \lambda)$  and the  $X_i$  and  $Y_j$  are mutually independent, is given by

$$p(x) = K G_{m, m+n}^{m+n, 0} \left( \lambda^n x \middle| \begin{matrix} a_1 + b_1 - 1, a_2 + b_2 - 1, \dots, a_m + b_m - 1 \\ a_1 - 1, a_2 - 1, \dots, a_m - 1, r_1 - 1, \dots, r_n - 1 \end{matrix} \right), \quad x > 0, \quad (1.7)$$

where

$$K = \lambda^n \prod_{i=1}^m \frac{\Gamma(a_i + b_i)}{\Gamma(a_i)} \prod_{j=1}^n \frac{1}{\Gamma(r_j)},$$

and we adopt the usual convention that the empty product is 1. A random variable with density (1.7) is said to have a *product beta-gamma* distribution. If (1.7) holds with  $n = 0$ , the random variable is said to have a *product beta* distribution, denoted by  $\text{PB}(a_1, b_1, \dots, a_m, b_m)$ ; if (1.7) holds with  $m = 0$ , then we refer to this a *product gamma* distribution, which we denote by  $\text{PG}(r_1, \dots, r_m, \lambda)$ . In this paper, we shall also say that a product of mutually independent beta, gamma and central normal random variables has a *product beta-gamma-normal* distribution.

For the case of two products, (1.6) simplifies to

$$p(x) = \frac{1}{\pi \sigma_1 \sigma_2} K_0 \left( \frac{|x|}{\sigma_1 \sigma_2} \right), \quad x \in \mathbb{R},$$

where  $K_0(x)$  is a modified Bessel function of the second kind (defined in Appendix B). For the product of two gammas, (1.7) can also be written in terms of the modified Bessel function of the second kind (see Malik [19]):

$$p(x) = \frac{2\lambda^{r_1+r_2}}{\Gamma(r_1)\Gamma(r_2)} x^{(r_1+r_2)/2-1} K_{r_1-r_2}(2\lambda\sqrt{x}), \quad x > 0.$$

However, for general  $a_i$  and  $b_i$ , there is no such simplification for the product of two betas.

Peköz et al. [24] extended Stein's method to generalised gamma random variables, denoted by  $\text{GG}(r, \lambda, q)$ , having density

$$p(x) = \frac{q\lambda^r}{\Gamma(\frac{r}{q})} x^{r-1} e^{-(\lambda x)^q}, \quad x > 0. \quad (1.8)$$

For  $G \sim \text{GG}(r, \lambda, q)$ , we have that  $\mathbb{E}G^k = \lambda^{-q} \Gamma((r+k)/q) / \Gamma(r/q)$  and in particular  $\mathbb{E}G^q = \frac{r}{q\lambda^q}$ . Special cases include  $\text{GG}(r, \lambda, 1) = \Gamma(r, \lambda)$  and  $\text{GG}(1, (\sqrt{2}\sigma)^{-1}, 2) = \text{HN}(\sigma^2)$ , where  $\text{HN}(\sigma^2)$  denotes a half-normal random variable:  $|Z|$  where  $Z \sim N(0, \sigma^2)$  (see Döbler [5] for Stein's method for the half normal distribution). Since  $G \stackrel{\mathcal{D}}{=} (\lambda^{q-1}Y)^{1/q}$  for  $Y \sim \Gamma(r/q, \lambda)$ , we can use the product gamma density (1.7) and a change of variables to write down the density of a product of generalised gamma random variables  $\text{GG}(r_i, \lambda, q)$ , denoted by  $\text{PGG}(r_1, \dots, r_n, \lambda, q)$ , in terms of a Meijer  $G$ -function, although we omit this formula. In this paper, we shall also extend Stein's method to products of generalised gamma random variables.

## 1.2 Product distribution Stein characterisations

Recently, Gaunt [10] extended Stein's method to the product normal distribution, obtaining the following Stein operator for the  $\text{PN}(N, \sigma^2)$  distribution:

$$\mathcal{A}f(x) = \sigma^2 A_N f(x) - x f(x), \quad (1.9)$$

where the operator  $A_N$  is given by  $A_N f(x) = x^{-1} T^N f(x)$  and  $Tf(x) = xf'(x)$ . The Stein operator (1.9) generalises the normal Stein operator (1.2) in a natural manner to products. It can be readily seen that (1.9) is a  $N$ -th order differential operator. Such Stein operators are uncommon in the literature with the only other example being the  $N$ -th order operators of Goldstein and Reinert [14], involving orthogonal polynomials, for the normal distribution. However, in recent years, second order operators involving  $f$ ,  $f'$  and  $f''$  have appeared in the literature for the Laplace [25] and Variance-Gamma distributions [6], [11] and the PRR family of [23].

One of the main contributions of this paper is an extension of the product normal Stein characterisation (1.9) to products of beta and gamma random variables, as well as mixed products of beta, gamma and normal random variables (see Propositions 2.7, 2.8 and 2.9). For ease of notation, we define the operators  $T_r f(x) = xf'(x) + rf(x)$  and  $B_{r_1, \dots, r_n} f(x) = T_{r_n} \dots T_{r_1} f(x)$  (note that  $T_0 \equiv T$ ). Then Stein operators for mixed products of the mutually independent random variables  $X \sim \text{PB}(a_1, b_1, \dots, a_m, b_m)$ ,  $Y \sim \text{PG}(r_1, \dots, r_n, \lambda)$  and  $Z \sim \text{PN}(N, \sigma^2)$  can be expressed concisely in terms of the differential operators  $A_N$  and  $B_{r_1, \dots, r_n}$ . We present these Stein operators in Table 1.

Table 1: Stein operators for product distributions		
Product $P$	Stein operator $\mathcal{A}_P f(x)$	Order
$X$	$B_{a_1, \dots, a_m} f(x) - x B_{a_1+b_1, \dots, a_m+b_m} f(x)$	$m$
$Y$	$B_{r_1, \dots, r_n} f(x) - \lambda^n x f(x)$	$n$
$Z$	$\sigma^2 A_N f(x) - x f(x)$	$N$
$XY$	$B_{a_1, \dots, a_m} B_{r_1, \dots, r_n} f(x) - \lambda^n x B_{a_1+b_1, \dots, a_m+b_m} f(x)$	$m + n$
$XZ$	$\sigma^2 B_{a_1, \dots, a_m} A_N B_{a_1, \dots, a_m} f(x) - x B_{a_1+b_1, \dots, a_m+b_m} B_{a_1+b_1-1, \dots, a_m+b_m-1} f(x)$	$2m + N$
$YZ$	$\sigma^2 B_{r_1, \dots, r_n} A_N B_{r_1, \dots, r_n} f(x) - \lambda^{2n} x f(x)$	$2n + N$
$XYZ$	$\sigma^2 B_{a_1, \dots, a_m} B_{r_1, \dots, r_n} A_N B_{r_1, \dots, r_n} B_{a_1, \dots, a_m} f(x) - \lambda^{2n} x B_{a_1+b_1, \dots, a_m+b_m} B_{a_1+b_1-1, \dots, a_m+b_m-1} f(x)$	$2m + 2n + N$

It can be seen that the product beta and product gamma Stein operators reduce to the classical beta and gamma Stein operators when  $m = 1$  and  $n = 1$ , respectively, as was so in the normal case. In Section 2.2.2, we see that for certain parameter values the Stein operators for the products  $XZ$  and  $XYZ$  can be simplified to differential operators of lower order. We give a precise criteria under which this occurs. As an example, if  $b_1 = \dots = b_m$  or  $a_1 + b_1 = \dots = a_m + b_m = 1$  (which include the uniform and arcsine distributions as special cases) the order of the Stein operator decreases by  $m$ .

In Proposition 2.7, we also obtain a characterisation of the generalised gamma distribution which leads to the following PGG( $r_1, \dots, r_n, \lambda, q$ ) Stein operator:

$$\mathcal{A}f(x) = B_{r_1, \dots, r_n} f(x) - (q\lambda^q)^n x^q f(x). \quad (1.10)$$

Taking  $q = 1$  in (1.10) yields the product gamma Stein operator  $\mathcal{A}_Y f(x)$ . Taking  $r_1 = \dots = r_N = 1$ ,  $\lambda = (\sqrt{2}\sigma)^{-1}$  and  $q = 2$  in (1.10) gives the following Stein operator for the product of  $N$  independent half-normal random variables ( $|Z|$  where  $Z \sim \text{PN}(N, \sigma^2)$ ):

$$\mathcal{A}f(x) = \sigma^2 T_1^N f(x) - x^2 f(x),$$

where  $x$  takes values in the interval  $[0, \infty)$ . By allowing  $x$  to take values in  $\mathbb{R}$ , we obtain the following  $\text{PN}(N, \sigma^2)$  Stein operator

$$\mathcal{A}f(x) = \sigma^2 T_1^N f(x) - x^2 f(x),$$

which differs from the  $\text{PN}(N, \sigma^2)$  operator (1.9). Although, making the changes of variables  $g(x) = xf(x)$  we have that  $g'(x) = xf'(x) + f(x)$ , and so

$$A_N g(x) = x^{-1} T_0^N g(x) = T_1^N f(x),$$

from which we recover the Stein operator (1.9).

The product distribution Stein operators that are obtained in this paper are natural generalisations of the classical normal, beta and gamma Stein operators (1.2), (1.4) and (1.5). The operators have a number of interesting properties which are discussed in Remark 2.10. However, despite their elegance, it is in general difficult to solve the corresponding Stein equation and bound the appropriate derivatives of the solution; a further discussion of this is given in Remark 2.14.

The classical normal, beta and gamma Stein equations are first order linear differential equations, and one can obtain uniform bounds for their solutions via elementary calculations. Uniform bounds are available for the first four derivatives of the solution of the  $\text{PN}(2, \sigma^2)$  Stein equation (Gaunt [10]), and in Proposition 2.12 we show that the  $k$ -th derivative of the solution of the  $\text{PG}(r_1, r_2, \lambda)$  Stein equation is uniformly bounded if the first  $k$  derivatives of the test function  $h$  are bounded. Although, for all other cases of product distribution Stein equations we do not have bounds for derivatives of the solution.

However, in Section 3, we consider a novel application of the product beta-gamma-normal Stein characterisation. In Section 3.2, we use the characterisation to obtain a differential equation that the product beta-gamma-normal p.d.f. must satisfy. This allows us to ‘guess’ a formula for the density function, which is then easily verified to indeed be the correct formula through the use of Mellin transforms. This result is new, and obtaining this formula directly using the inverse Mellin transform would have required some quite involved calculations. From our formula we are then able to obtain an expression for the characteristic function of the product normal-beta-gamma distribution, as well as estimates for the tail behaviour of the distribution.

### 1.3 Outline of the paper

We begin Section 2 by establishing some useful properties for the operators  $A_N$  and  $B_{r_1, \dots, r_n}$ . We then obtain Stein characterisations for mixed products of beta, gamma and central normal random variables (Propositions 2.7, 2.8 and 2.9), which lead to the operators of Table 1. In Section 2.2.2, we see that for certain parameter values simpler characterisations can be obtained. In Section 2.3, we consider a Stein equation for the

product of two independent gammas. We solve the equation and show that the  $k$ -th derivative of the solution are uniformly bounded, provided that the first  $k$  derivatives of the test function  $h$  are bounded.

In Section 3, we obtain formulas for the p.d.f. and characteristic function of the the product beta-gamma-normal distribution, as well as an asymptotic formula for the tail behaviour of the distribution. We use the product beta-gamma-normal Stein characterisation to propose a candidate formula for the p.d.f. and then verify it using Mellin transforms.

In Appendix A, we prove some results that were stated in the main text without proof. Finally, Appendix B lists some basic properties of the Meijer  $G$ -function and modified Bessel functions that are used in this paper.

*Notation.* Throughout this paper,  $T$  will denote the operator  $Tf(x) = xf'(x)$  and  $A_N$  will denote the operator  $A_N f(x) = x^{-1}T^N f(x) = \frac{d}{dx}(T^{N-1}f(x))$ . We also let  $T_r$  denote the operator  $T_r f(x) = xf'(x) + rf(x)$  and let  $B_{r_1, \dots, r_n}$  denote the operator  $B_{r_1, \dots, r_n} f(x) = T_{r_n} \cdots T_{r_1} f(x)$ . We shall let  $C^n(I)$  be the space of functions on the interval  $I$  with  $n$  continuous derivatives, and  $C_b^n(I)$  will denote the space of bounded functions on  $I$  with bounded  $k$ -th order derivatives for  $k \leq n$ .

## 2 Stein characterisations for products of normal, beta and gamma random variables

### 2.1 Preliminary results

We begin this section by presenting some useful properties of the operators  $A_N f(x) = x^{-1}T^N f(x)$  and  $B_{r_1, \dots, r_n} f(x) = T_{r_n} \cdots T_{r_1} f(x)$  and establish the existence of some distributional transformations. In proving the existence of these distributional transformations, we establish some facts that will be used in the proof of sufficiency of Propositions 2.7, 2.8 and 2.9.

**Lemma 2.1.** *The operators  $A_N$  and  $B_{r_1, \dots, r_n}$  have the following properties.*

(i) *The operators  $T_r$  and  $T_s$  are commutative, that is,  $T_r T_s f(x) = T_s T_r f(x)$  for all  $f \in C^2(\mathbb{R})$ . Thus, for all  $f \in C^n(\mathbb{R})$ ,  $B_{r_1, \dots, r_n} f(x) = B_{r_{\sigma(1)}, \dots, r_{\sigma(n)}} f(x)$ , where  $\sigma$  is a permutation of the set  $\{1, 2, \dots, n\}$ .*

(ii) *For all  $f \in C^{n+N}(\mathbb{R})$ , the operators  $A_N$  and  $B_{r_1, \dots, r_n}$  satisfy*

$$A_N B_{r_1, \dots, r_n} f(x) = B_{r_1+1, \dots, r_n+1} A_N f(x). \quad (2.1)$$

*Proof.* (i) The first assertion follows since  $T_r T_s f(x) = x^2 f''(x) + (1+r+s)xf'(x) + rsf(x) = T_s T_r f(x)$ , and the second assertion now follows immediately.

(ii) As  $A_1 \equiv \frac{d}{dx}$ , we have  $A_1 T_r f(x) = xf''(x) + (r+1)f'(x) = T_{r+1} A_1 f(x)$ . Thus, on recalling that  $A_N f(x) = \frac{d}{dx}(T_0^{N-1} f(x))$  and using the fact that the operators  $T_r$  and  $T_s$

are commutative, we have

$$\begin{aligned} A_N B_{r_1, \dots, r_n} f(x) &= A_1 T_0^{N-1} T_{r_1} \cdots T_{r_n} f(x) = A_1 T_{r_1} \cdots T_{r_n} T_0^{N-1} f(x) \\ &= T_{r_1+1} A_1 T_{r_2} \cdots T_{r_n} T_0^{N-1} f(x) = T_{r_1+1} \cdots T_{r_n+1} A_1 T_0^{N-1} f(x) \\ &= B_{r_1+1, \dots, r_n+1} A_N f(x), \end{aligned}$$

where an iteration was applied to obtain the penultimate equality.  $\square$

The following fundamental formulas (Luke [18], pp. 24–26) disentangle the iterated operators  $A_N$  and  $B_{r_1, \dots, r_n}$ . For  $f \in C^n(\mathbb{R})$ ,

$$A_N f(x) = \sum_{k=1}^N \left\{ \begin{matrix} N \\ k \end{matrix} \right\} x^{k-1} f^{(k)}(x), \quad (2.2)$$

$$B_{r_1, \dots, r_n} f(x) = \sum_{k=0}^n c_{k,n} x^k f^{(k)}(x), \quad (2.3)$$

where  $\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^n$  are Stirling numbers of the second kind (Olver et al. [21], Chapter 26) and

$$c_{k,n} = \frac{(-1)^n}{n!} \sum_{j=0}^n \frac{(-n)_j}{j!} \prod_{i=1}^n (j + r_i), \quad (2.4)$$

for  $(a)_j = a(a+1) \cdots (a+j-1)$ ,  $(a)_0 = 1$ .

Applying (2.1) and (2.3) gives that, for  $f \in C^{m+n+N}(\mathbb{R})$ ,

$$\begin{aligned} B_{a_1, \dots, a_m} A_N B_{b_1, \dots, b_n} f(x) &= A_N B_{a_1-1, \dots, a_m-1} B_{b_1, \dots, b_n} f(x) \\ &= x^{-1} T_0^N B_{a_1-1, \dots, a_m-1} B_{b_1, \dots, b_n} f(x) \\ &= \sum_{k=1}^{m+n+N} \tilde{c}_{k, m+n+N} x^{k-1} f^{(k)}(x), \end{aligned} \quad (2.5)$$

where the  $\tilde{c}_{k, m+n+N}$  can be computed using (2.4).

We now present formulas for the inverses of the operators  $A_N$  and  $B_{r_1, \dots, r_n}$ . These inverse operators will be used in establishing the existence of some distributional transformations in Lemma 2.5. The inverse of  $A_N$  was found by Gaunt [10] and the result is stated in the following lemma.

**Lemma 2.2.** *Let  $V_N$  be the product of  $N$  independent  $U(0, 1)$  random variables, and define the operator  $G_N$  by  $G_N f(x) = x \mathbb{E} f(x V_N)$ . Then,  $G_N$  is the right-inverse of the operator  $A_N$  in the sense that*

$$A_N G_N f(x) = f(x).$$

*Suppose now that  $f \in C^N(\mathbb{R})$ . Then, for any  $N \geq 1$ ,*

$$G_N A_N f(x) = G_1 A_1 f(x) = f(x) - f(0). \quad (2.6)$$

*Therefore,  $G_N$  is the inverse of  $A_N$  when the domain of  $A_N$  is the space of all  $N$  times differentiable functions  $f$  on  $\mathbb{R}$  with  $f(0) = 0$ .*

**Lemma 2.3.** Let  $\hat{U}_1, \dots, \hat{U}_n$  be independent random variables with distribution function  $u^{r_j}$  on  $(0, 1)$  for  $r_j > 0$ , and define  $\hat{V}_n = \prod_{j=1}^n \hat{U}_j$ . Define the operator  $H_{r_1, \dots, r_n}$  by  $H_{r_1, \dots, r_n} f(x) = (\prod_{k=1}^n r_k)^{-1} \mathbb{E} f(x \hat{V}_n)$ . Then

$$(i) \quad H_{r_1, \dots, r_n} f(x) = H_{r_1} \cdots H_{r_n} f(x).$$

$$(ii) \quad T_r H_s f(x) = f(x) + (r - s) H_s f(x).$$

(iii)  $H_{r_1, \dots, r_n}$  is the right-inverse of the operator  $B_{r_1, \dots, r_n}$  in the sense that

$$B_{r_1, \dots, r_n} H_{r_1, \dots, r_n} f(x) = f(x).$$

(iv) Suppose now that  $f \in C^n(\mathbb{R})$ . Then, for any  $n \geq 1$ ,

$$H_{r_1, \dots, r_n} B_{r_1, \dots, r_n} f(x) = f(x). \quad (2.7)$$

Therefore,  $H_{r_1, \dots, r_n}$  is the inverse of  $B_{r_1, \dots, r_n}$  when the domain of  $B_{r_1, \dots, r_n}$  is  $C^n(\mathbb{R})$ .

*Proof.* (i) We begin by obtaining a useful formula for  $H_{r_1, \dots, r_n} f(x) = (\prod_{k=1}^n r_k)^{-1} \mathbb{E} f(x \hat{V}_n)$ . We have that

$$H_{r_1, \dots, r_n} f(x) = \int_{(0,1)^n} f(x u_1 \cdots u_n) u_1^{r_1-1} \cdots u_n^{r_n-1} du_1 \cdots du_n.$$

By a change of variables  $u_n = \frac{t_n}{x}$  and  $u_j = \frac{t_j}{t_{j+1}}$  for  $1 \leq j \leq n-1$ , this can be written as

$$H_{r_1} f(x) = x^{-r_1} \int_0^x t_1^{r_1-1} f(t_1) dt_1, \quad (2.8)$$

and, for  $n \geq 2$ ,

$$H_{r_1, \dots, r_n} f(x) = x^{-r_n} \int_0^x \int_0^{t_n} \cdots \int_0^{t_2} f(t_1) t_1^{r_1-1} t_2^{r_2-r_1-1} \cdots t_n^{r_n-r_{n-1}-1} dt_1 dt_2 \cdots dt_n.$$

From these representations of  $H_{r_1, \dots, r_n} f(x)$ , it is clear that  $H_{r_1, \dots, r_n} f(x) = H_{r_1} \cdots H_{r_n} f(x)$ .

(ii) We now use the integral representation (2.8) of  $H_s f(x)$  to obtain

$$\begin{aligned} T_r H_s f(x) &= x \frac{d}{dx} \left( x^{-s} \int_0^x t^{s-1} f(t) dt \right) + r x^{-s} \int_0^x t^{s-1} f(t) dt \\ &= -s x^{-s} \int_0^x t^{s-1} f(t) dt + x^{1-s} \cdot x^{s-1} f(x) + r x^{-s} \int_0^x t^{s-1} f(t) dt \\ &= f(x) + (r - s) H_s f(x). \end{aligned}$$

(iii) From part (ii),  $T_r H_r f(x) = f(x)$ . But since  $B_{r_1, \dots, r_n} f(x) = T_{r_n, \dots, r_1} f(x)$  and  $H_{r_1, \dots, r_n} f(x) = H_{r_1} \cdots H_{r_n} f(x)$ , it follows that  $B_{r_1, \dots, r_n} H_{r_1, \dots, r_n} f(x) = f(x)$ .

(iv) We have

$$H_r T_r f(x) = x^{-r} \int_0^x t^{r-1} (t f'(t) + r f(t)) dt = x^{-r} \int_0^x (t^r f(t))' dt = x^{-r} \left[ t^r f(t) \right]_0^x = f(x),$$

and on using a similar argument to part (iii) it follows that  $H_{r_1, \dots, r_n} B_{r_1, \dots, r_n} f(x) = f(x)$ .  $\square$



**Corollary 2.4.** For  $f \in C^{m+n+N}(\mathbb{R})$ ,

$$H_{a_1, \dots, a_m} G_N H_{b_1, \dots, b_n} B_{b_1, \dots, b_n} A_N B_{a_1, \dots, a_m} f(x) = f(x) - f(0).$$

*Proof.* This follows immediately from Lemmas 2.2 and 2.3.  $\square$

We now use the properties of the operators  $A_N$  and  $B_{r_1, \dots, r_n}$  that were obtained above to establish the existence and uniqueness of some distributional transformation that arise naturally in the context of Stein characterisations for products of beta, gamma and central normal random variables. The proof of the following lemma uses a similar argument to the one used by Goldstein and Reinert [13] to prove the existence of the zero bias transformation.

**Lemma 2.5.** (i) Let  $W$  be a random variable with  $0 < \mathbb{E}W^q = \alpha < \infty$ . Then there exists a unique random variable  $W_*$  such that, for all  $f \in C^n(\mathbb{R})$  for which the relevant expectations exist,

$$(q\lambda)^n \mathbb{E}W^q f(W) = \mathbb{E}B_{r_1, \dots, r_n} f(W_*),$$

where  $q, \lambda$  and  $r_1, \dots, r_n$  are positive constants such that  $\alpha = (q\lambda)^{-1} \prod_{k=1}^n r_k$ .

(ii) Let  $W$  be a random variable with  $0 < \mathbb{E}W = \beta < \infty$ . Then there exists a unique random variable  $W_*$  such that, for all  $f \in C^m(\mathbb{R})$  for which the relevant expectations exist,

$$\mathbb{E}W B_{a_1, \dots, a_m} f(W) = \mathbb{E}B_{a_1, \dots, a_m} f(W_*),$$

where  $a_1, b_1, \dots, a_m, b_m$  are positive constants such that  $\beta = \prod_{k=1}^m a_k / (a_k + b_k)$ .

(iii) Let  $W$  be a mean zero random variable with finite, non-zero variance  $\gamma$ . Then there exists a unique random variable  $W_*$  such that, for all  $f \in C^{2m+2n+N}(\mathbb{R})$  for which the relevant expectations exist,

$$\begin{aligned} & \mathbb{E}[\sigma^2 B_{a_1, \dots, a_m} B_{r_1, \dots, r_n} A_N B_{r_1, \dots, r_n} B_{a_1, \dots, a_m} f(W_*) \\ & - \lambda^{2n} W B_{a_1+b_1, \dots, a_m+b_m} B_{a_1+b_1-1, \dots, a_m+b_m-1} f(W)] = 0, \end{aligned}$$

where  $a_1, b_1, \dots, a_m, b_m, r_1, \dots, r_n, \lambda$  and  $\sigma$  are positive constants such that

$$\gamma = \sigma^2 \prod_{j=1}^m \frac{a_j(a_j+1)}{(a_j+b_j)(a_j+b_j+1)} \prod_{k=1}^n \frac{r_k}{\lambda^2}.$$

*Proof.* (i) We define a linear operator  $Q$  by

$$Qf = (q\lambda)^n \mathbb{E}W^q H_{r_1, \dots, r_n} f(W),$$

where  $H_{r_1, \dots, r_n}$  is defined as in Lemma 2.3. As  $\mathbb{E}W < \infty$ , it follows that  $Qf$  exists. To see that  $Q$  is positive, take  $f \geq 0$ . Then  $H_{r_1, \dots, r_n} f(x) \geq 0$ . Hence  $\mathbb{E}W^q H_{r_1, \dots, r_n} f(W) \geq 0$ , and  $Q$  is positive. By the Riesz representation theorem we have  $Qf = \int f d\nu$ , for some unique Radon measure  $\nu$ , which is a probability measure as  $Q1 = 1$ .

We now take  $f(x) = B_{r_1, \dots, r_n} g(x)$ , where  $g \in C^n(\mathbb{R})$ , with derivatives up to  $n$ -th order being continuous with compact support. Then, from (2.7),

$$(q\lambda)^n \mathbb{E}W^q H_{r_1, \dots, r_n} B_{r_1, \dots, r_n} g(W) = (q\lambda)^n \mathbb{E}W^q g(W),$$

which completes the proof.

(ii) The proof is similar to part (i). We define the operator  $R$  by

$$Rf = \mathbb{E}W B_{a_1+b_1, \dots, a_m+b_m} H_{a_1, \dots, a_m} f(W),$$

which exists since  $\mathbb{E}W < \infty$ . To see that  $R$  is positive, take  $f \geq 0$ . By Lemma 2.3,  $T_{a_i+b_i} H_{a_i} f(x) = f(x) + a_i H_{a_i} f(x) \geq 0$ . Hence, by carrying out an iteration we see that  $B_{a_1+b_1, \dots, a_m+b_m} H_{a_1, \dots, a_m} f(x) \geq 0$ . Therefore  $\mathbb{E}W B_{a_1+b_1, \dots, a_m+b_m} H_{a_1, \dots, a_m} f(W) \geq 0$ , and so  $R$  is positive. By the Riesz representation theorem we have  $Rf = \int f d\nu$ , for some unique Radon measure  $\nu$ , which is a probability measure as  $R1 = 1$ .

We now take  $f(x) = B_{a_1, \dots, a_m} g(x)$ , where  $g \in C^m(\mathbb{R})$ , with derivatives up to  $m$ -th order being continuous with compact support. Then, from (2.7),

$$\mathbb{E}W B_{a_1+b_1, \dots, a_m+b_m} H_{a_1, \dots, a_m} B_{a_1, \dots, a_m} g(W) = \mathbb{E}W B_{a_1+b_1, \dots, a_m+b_m} g(W),$$

as required.

(iii) Consider the operator  $S$  defined by

$$Sf = \sigma^{-2} \lambda^{2n} \mathbb{E}W B_{a_1+b_1, \dots, a_m+b_m} B_{a_1+b_1-1, \dots, a_m+b_m-1} \cdots \\ \cdots H_{a_1, \dots, a_m} H_{r_1, \dots, r_n} G_N H_{r_1, \dots, r_n} H_{a_1, \dots, a_m} f(W),$$

which exists because  $\mathbb{E}W^2 < \infty$ . For  $f \geq 0$  we can argue as before to show that  $Sf \geq 0$ . By the Riesz representation theorem we have  $Sf = \int f d\nu$ , for some unique Radon measure  $\nu$ , which is a probability measure as  $S1 = 1$ .

We now take  $f(x) = B_{a_1, \dots, a_m} B_{r_1, \dots, r_n} A_N B_{r_1, \dots, r_n} B_{a_1, \dots, a_m} g(x)$ , where  $g \in C^{2m+2n+N}(\mathbb{R})$ , with derivatives up to  $2m+2n+N$ -th order being continuous with compact support. Then, from Corollary 2.4,

$$\begin{aligned} & \mathbb{E}W B_{a_1+b_1, \dots, a_m+b_m} B_{a_1+b_1-1, \dots, a_m+b_m-1} H_{a_1, \dots, a_m} H_{r_1, \dots, r_n} G_N H_{r_1, \dots, r_n} H_{a_1, \dots, a_m} \cdots \\ & \cdots B_{a_1, \dots, a_m} B_{r_1, \dots, r_n} A_N B_{r_1, \dots, r_n} B_{a_1, \dots, a_m} g(W) \\ & = \mathbb{E}W B_{a_1+b_1, \dots, a_m+b_m} B_{a_1+b_1-1, \dots, a_m+b_m-1} (g(W) - g(0)) \\ & = \mathbb{E}W B_{a_1+b_1, \dots, a_m+b_m} B_{a_1+b_1-1, \dots, a_m+b_m-1} g(W), \end{aligned}$$

since  $\mathbb{E}W = 0$ . The proof is now complete.  $\square$

## 2.2 Stein characterisations

With the preliminary results stated, we are now in a position to obtain Stein characterisations for mixed products of beta, gamma and central normal random variables, which give rise to the product distribution Stein operators of Table 1. From here on we shall suppose that the random variables  $X \sim \text{PB}(a_1, b_1, \dots, a_m, b_m)$ ,  $Y \sim \text{PG}(r_1, \dots, r_n, \lambda)$  and  $Z \sim \text{PN}(N, \sigma^2)$  are mutually independent. We shall also let  $\mathcal{A}_P f(x)$  be the operator for the product distribution  $P$ , as given in Table 1.

### 2.2.1 General parameters

We firstly consider the case of mixed products of beta, gamma and central normal random variables with general parameter values. In Section 2.2.2, we look at particular parameter values under which we can obtain some slightly simpler formulas for product distribution Stein operators. We begin by recalling the product normal Stein characterisation that was obtained by Gaunt [10].

**Proposition 2.6.** *Let  $W$  be a real-valued random variable with mean zero and finite, non-zero variance. Then  $\mathcal{L}(W) = \text{PN}(n, \sigma^2)$  if and only if*

$$\mathbb{E}[\mathcal{A}f_Z(W)] = 0 \quad (2.9)$$

for all  $f \in C^n(\mathbb{R})$  such that the expectation  $\mathbb{E}[\mathcal{A}f_Z(Z)]$  exists.

We now state characterisations for the product beta and product generalised gamma distributions; taking  $q = 1$  gives a product gamma distribution characterisation.

**Proposition 2.7.** *Let  $W$  be a real-valued random variable with  $0 < \mathbb{E}W^q < \infty$ . Then  $\mathcal{L}(W) = \text{PGG}(r_1, \dots, r_n, \lambda, q)$  if and only if*

$$\mathbb{E}[B_{r_1, \dots, r_n} f(W) - (q\lambda^q)^n W^q f(W)] = 0 \quad (2.10)$$

for all  $f \in C^n(\mathbb{R}_+)$  such that the expectation  $\mathbb{E}[B_{r_1, \dots, r_n} f(G) - (q\lambda^q)^n G^q f(G)]$  exists, for  $G \sim \text{PGG}(r_1, \dots, r_n, \lambda, q)$ .

*Proof. Necessity.* We prove necessity by induction on  $n$  and begin by proving the base case  $n = 1$ . The well-known characterisation of the gamma distribution, given in Luk [17], states that if  $U \sim \Gamma(r/q, \lambda)$ , then

$$\mathbb{E}[U f'(U) - (r/q - \lambda U) f(U)] = 0 \quad (2.11)$$

for all differentiable functions  $f$  such that the expectation exists. Now, if  $V \sim \text{GG}(r, \lambda, q)$ , then  $V \stackrel{\mathcal{D}}{=} (\lambda^{q-1} U)^{1/q}$ . Making the change of variables  $V = (\lambda^{q-1} U)^{1/q}$  in (2.11) leads to the following characterising equation for the  $\text{GG}(r, \lambda, q)$  distribution:

$$\mathbb{E}[V f'(V) - (r - q\lambda^q V^q) f(V)] = 0$$

for all differentiable functions  $f$  such that the expectation exists. This can be written as  $\mathbb{E}[T_r f(V) - q\lambda^q V^q f(V)] = 0$ , and so the result is true for  $n = 1$ .

Let us now prove the inductive step. We begin by defining  $W_n = \prod_{i=1}^n V_i$  where  $V_i \sim \text{GG}(r_i, \lambda, q)$  and the  $V_i$  are mutually independent. We observe that  $(T_p f)(ax) = T_p f_a(x)$  where  $f_a(x) = f(ax)$ , and so  $(B_{p_1, \dots, p_l} f)(ax) = B_{p_1, \dots, p_l} f_a(x)$ . By induction assume that  $(q\lambda^q)^n \mathbb{E}W_n g(W_n) = \mathbb{E}B_{r_1, \dots, r_n} g(W_n)$  for all  $g \in C^n(\mathbb{R})$  for some  $n \geq 1$ . Then

$$\begin{aligned} (q\lambda^q)^{n+1} \mathbb{E}W_{n+1}^q f(W_{n+1}) &= (q\lambda^q)^{n+1} \mathbb{E}[V_{n+1}^q \mathbb{E}[W_n^q f_{V_{n+1}}(W_n) \mid V_{n+1}]] \\ &= q\lambda^q \mathbb{E}[V_{n+1}^q \mathbb{E}[B_{r_1, \dots, r_n} f_{V_{n+1}}(W_n) \mid V_{n+1}]] \\ &= q\lambda^q \mathbb{E}[V_{n+1}^q (B_{r_1, \dots, r_n} f)(W_n V_{n+1})] \\ &= q\lambda^q \mathbb{E}[\mathbb{E}[V_{n+1}^q (B_{r_1, \dots, r_n} f_{W_n})(V_{n+1}) \mid W_n]] \\ &= \mathbb{E}[\mathbb{E}[W_n V_{n+1} (B_{r_1, \dots, r_n} f_{W_n})'(V_{n+1}) + r_{n+1} f_{W_n}(V_{n+1}) \mid W_n]] \\ &= \mathbb{E}B_{r_1, \dots, r_{n+1}} f(W_{n+1}). \end{aligned}$$

Thus, necessity has been proved by induction on  $n$ .

*Sufficiency.* In part (i) of Lemma 2.5, we established that there is a unique probability distribution with positive mean such that equation (2.10) holds, and, since the  $\text{PGG}(r_1, \dots, r_n, \lambda, q)$  distribution satisfies (2.10), sufficiency follows.  $\square$

**Proposition 2.8.** *Let  $W$  be a real-valued random variable with  $0 < \mathbb{E}W < \infty$ . Then  $\mathcal{L}(W) = \text{PB}(a_1, b_1, \dots, a_m, b_m)$  if and only if*

$$\mathbb{E}[\mathcal{A}_X f(W)] = 0 \quad (2.12)$$

for all  $f \in C^m((0, 1))$  such that the expectation  $\mathbb{E}[\mathcal{A}_X f(X)]$  exists.

*Proof.* The proof of sufficiency is analogous to the proof of sufficiency of Proposition 2.7, with the only difference being that here we invoke part (ii) of Lemma 2.5. The proof of necessity is also similar and involves an induction on  $m$ . Let  $W_m = \prod_{i=1}^m X_i$  where  $X_i \sim \text{Beta}(a_i, b_i)$  and the  $X_i$  are mutually independent. The base case of the induction  $m = 1$  is the well-known characterisation (1.4) of the beta distribution. By induction assume that  $\mathbb{E}W_m B_{a_1+b_1, \dots, a_m+b_m} g(W_m) = \mathbb{E}B_{a_1, \dots, a_{m+n}} g(W_m)$  for all  $g \in C^m(\mathbb{R})$  for some  $m \geq 1$ . Then

$$\begin{aligned} & \mathbb{E}W_{m+1} B_{a_1+b_1, \dots, a_{m+1}+b_{m+1}} f(W_{m+1}) \\ &= \mathbb{E}[X_{m+1} \mathbb{E}[W_m B_{a_1+b_1, \dots, a_m+b_m} T_{a_{m+1}+b_{m+1}} f_{X_{m+1}}(W_m) \mid X_{m+1}]] \\ &= \mathbb{E}[X_{m+1} \mathbb{E}[B_{a_1, \dots, a_m} T_{a_{m+1}+b_{m+1}} f_{X_{m+1}}(W_m) \mid X_{m+1}]] \\ &= \mathbb{E}[X_{m+1} (T_{a_{m+1}+b_{m+1}} B_{a_1, \dots, a_m} f)(W_m X_{m+1})] \\ &= \mathbb{E}[\mathbb{E}[X_{m+1} (T_{a_{m+1}+b_{m+1}} B_{a_1, \dots, a_m} f_{W_m})(X_{m+1}) \mid W_m]] \\ &= \mathbb{E}[\mathbb{E}[X_{m+1} W_m (B_{a_1, \dots, c_m} f_{W_m})'(X_{m+1}) + a_{m+1} f_{W_m}(X_{m+1}) \mid W_m]] \\ &= \mathbb{E}B_{a_1, \dots, a_{m+1}} f(W_{m+1}), \end{aligned}$$

and so necessity has been proved by induction on  $m$ .  $\square$

We now use the above product beta, gamma and normal characterisations to obtain Stein characterisations for mixed products of such random variables.

**Proposition 2.9.** *We have the following characterisations for mixed products of mutually independent beta, gamma and central normal random variables.*

(i) *Let  $W$  be a real-valued random variable with  $0 < \mathbb{E}W < \infty$ . Then  $\mathcal{L}(W) = \mathcal{L}(XY)$  if and only if*

$$\mathbb{E}[\mathcal{A}_{XY} f(W)] = 0$$

for all  $f \in C^{m+n}(\mathbb{R}_+)$  such that the expectation  $\mathbb{E}[\mathcal{A}_{XY} f(XY)]$  exists.

(ii) *Let  $W$  be a real-valued random variable with mean zero and finite, non-zero variance. Then  $\mathcal{L}(W) = \mathcal{L}(XZ)$  if and only if*

$$\mathbb{E}[\mathcal{A}_{XZ} f(W)] = 0 \quad (2.13)$$

for all  $f \in C^{2m+N}(\mathbb{R})$  such that the expectation  $\mathbb{E}[\mathcal{A}_{XZ} f(XZ)]$  exists.

(iii) Let  $W$  be a real-valued random variable with mean zero and finite, non-zero variance. Then  $\mathcal{L}(W) = \mathcal{L}(YZ)$  if and only if

$$\mathbb{E}[\mathcal{A}_{YZ}f(W)] = 0 \quad (2.14)$$

for all  $f \in C^{2n+N}(\mathbb{R})$  such that the expectation  $\mathbb{E}[\mathcal{A}_{YZ}f(YZ)]$  exists.

(iv) Let  $W$  be a real-valued random variable with mean zero and finite, non-zero variance. Then  $\mathcal{L}(W) = \mathcal{L}(XYZ)$  if and only if

$$\mathbb{E}[\mathcal{A}_{XYZ}f(W)] = 0 \quad (2.15)$$

for all  $f \in C^{2m+2n+N}(\mathbb{R})$  such that the expectation  $\mathbb{E}[\mathcal{A}_{XYZ}f(XYZ)]$  exists.

*Proof.* We begin by considering the proof of sufficiency for these assertions. For part (iv) the proof is analogous to the proofs of sufficiency given in Propositions 2.7 and 2.8, with the only difference being that here we invoke part (iii) of Lemma 2.5. The proof of sufficiency for parts (i), (ii) and (iii) are similar and we omit the details.

To prove necessity we use the characterisations of the product normal, product gamma and product beta distributions that were given in Propositions 2.6, 2.7 and 2.8, respectively. We consider the four assertions separately.

(i) Recall that  $(T_p f)(ax) = T_p f_a(x)$  where  $f_a(x) = f(ax)$ , and so  $(B_{p_1, \dots, p_l} f)(ax) = B_{p_1, \dots, p_l} f_a(x)$ . From the product beta and gamma characterisations we now have

$$\begin{aligned} \lambda^n \mathbb{E}[XY B_{a_1+b_1, \dots, a_m+b_m} f(XY)] &= \lambda^n \mathbb{E}[Y \mathbb{E}[X B_{a_1+b_1, \dots, a_m+b_m} f_Y(X) \mid Y]] \\ &= \lambda^n \mathbb{E}[Y \mathbb{E}[B_{a_1, \dots, a_m} f_Y(X) \mid Y]] \\ &= \lambda^n \mathbb{E}[Y B_{a_1, \dots, a_m} f(XY)] \\ &= \lambda^n \mathbb{E}[\mathbb{E}[Y B_{a_1, \dots, a_m} f_X(Y) \mid X]] \\ &= \mathbb{E}[\mathbb{E}[B_{r_1, \dots, r_n} B_{a_1, \dots, a_m} f_X(Y) \mid X]] \\ &= \mathbb{E}[B_{r_1, \dots, r_n} B_{a_1, \dots, a_m} f(XY)], \end{aligned}$$

as required.

(ii) We begin by noting that, since  $A_N f(x) = \frac{d}{dx}(T_0^{N-1} f(x))$ , we have  $(A_N) f(ax) = a A_N f_a(x)$ . So from our product beta and normal characterisations,

$$\begin{aligned} &\mathbb{E}[X Z B_{a_1+b_1, \dots, a_m+b_m} B_{a_1+b_1-1, \dots, a_m+b_m-1} f(XZ)] \\ &= \mathbb{E}[Z \mathbb{E}[X B_{a_1+b_1, \dots, a_m+b_m} B_{a_1+b_1-1, \dots, a_m+b_m-1} f_Z(X) \mid Z]] \\ &= \mathbb{E}[Z \mathbb{E}[B_{a_1, \dots, a_m} B_{a_1+b_1-1, \dots, a_m+b_m-1} f_Z(X) \mid Z]] \\ &= \mathbb{E}[\mathbb{E}[Z B_{a_1, \dots, a_m} B_{a_1+b_1-1, \dots, a_m+b_m-1} f_X(Z) \mid X]] \\ &= \sigma^2 \mathbb{E}[\mathbb{E}[X A_N B_{a_1, \dots, a_m} B_{a_1+b_1-1, \dots, a_m+b_m-1} f_X(Z) \mid X]] \\ &= \sigma^2 \mathbb{E}[X A_N B_{a_1, \dots, a_m} B_{a_1+b_1-1, \dots, a_m+b_m-1} f(XZ)]. \end{aligned}$$

From Lemma 2.1 we have  $A_N B_{a_1, \dots, a_m} B_{a_1+b_1-1, \dots, a_m+b_m-1} = B_{a_1+b_1, \dots, a_m+b_m} A_N B_{a_1, \dots, a_m}$ .

Applying this formula and the product beta characterisation (2.12) yields

$$\begin{aligned}
& \mathbb{E}[XZB_{a_1+b_1, \dots, a_m+b_m}B_{a_1+b_1-1, \dots, a_m+b_m-1}f(XZ)] \\
&= \sigma^2 \mathbb{E}[XB_{a_1+b_1, \dots, a_m+b_m}A_NB_{a_1, \dots, a_m}f(XZ)] \\
&= \sigma^2 \mathbb{E}[\mathbb{E}[XB_{a_1+b_1, \dots, a_m+b_m}A_NB_{a_1, \dots, a_m}f_Z(X) \mid Z]] \\
&= \sigma^2 \mathbb{E}[\mathbb{E}[B_{a_1, \dots, a_m}A_NB_{a_1, \dots, a_m}f_Z(X) \mid Z]] \\
&= \sigma^2 \mathbb{E}[B_{a_1, \dots, a_m}A_NB_{a_1, \dots, a_m}f(XZ)],
\end{aligned}$$

as required.

(iii) By a similar argument,

$$\begin{aligned}
\lambda^{2n} \mathbb{E}[YZf(YZ)] &= \lambda^{2n} \mathbb{E}[Z\mathbb{E}[Yf_Z(Y) \mid Z]] \\
&= \lambda^n \mathbb{E}[Z\mathbb{E}[B_{r_1, \dots, r_n}f_Z(Y) \mid Z]] \\
&= \lambda^n \mathbb{E}[\mathbb{E}[ZB_{r_1, \dots, r_n}f_Y(Z) \mid Y]] \\
&= \sigma^2 \lambda^n \mathbb{E}[\mathbb{E}[YA_NB_{r_1, \dots, r_n}f_Y(Z) \mid Y]] \\
&= \sigma^2 \lambda^n \mathbb{E}[\mathbb{E}[YA_NB_{r_1, \dots, r_n}f_Z(Y) \mid Z]] \\
&= \sigma^2 \mathbb{E}[\mathbb{E}[B_{r_1, \dots, r_n}A_NB_{r_1, \dots, r_n}f_Z(Y) \mid Z]] \\
&= \sigma^2 \mathbb{E}[B_{r_1, \dots, r_n}A_NB_{r_1, \dots, r_n}f(YZ)].
\end{aligned}$$

(iv) Applying the product beta characterisation (2.12) and the product gamma-normal characterisation (2.15) gives

$$\begin{aligned}
& \lambda^{2n} \mathbb{E}[XYZB_{a_1+b_1, \dots, a_m+b_m}B_{a_1+b_1-1, \dots, a_m+b_m-1}f(XYZ)] \\
&= \lambda^{2n} \mathbb{E}[YZ\mathbb{E}[XB_{a_1+b_1, \dots, a_m+b_m}B_{a_1+b_1-1, \dots, a_m+b_m-1}f_{YZ}(X) \mid YZ]] \\
&= \lambda^{2n} \mathbb{E}[YZ\mathbb{E}[B_{a_1, \dots, a_m}B_{a_1+b_1-1, \dots, a_m+b_m-1}f_{YZ}(X) \mid YZ]] \\
&= \lambda^{2n} \mathbb{E}[\mathbb{E}[YZB_{a_1, \dots, a_m}B_{a_1+b_1-1, \dots, a_m+b_m-1}f_X(YZ) \mid X]] \\
&= \sigma^2 \mathbb{E}[\mathbb{E}[XB_{r_1, \dots, r_n}A_NB_{r_1, \dots, r_n}B_{a_1, \dots, a_m}B_{a_1+b_1-1, \dots, a_m+b_m-1}f_X(YZ) \mid X]] \\
&= \sigma^2 \mathbb{E}[XB_{r_1, \dots, r_n}A_NB_{r_1, \dots, r_n}B_{a_1, \dots, a_m}B_{a_1+b_1-1, \dots, a_m+b_m-1}f(XYZ)].
\end{aligned}$$

We now interchange the order of the operators using part (ii) of Lemma 2.1 and then use our characterisation of the product beta distribution to obtain

$$\begin{aligned}
& \lambda^{2n} \mathbb{E}[XYZB_{a_1+b_1, \dots, a_m+b_m}B_{a_1+b_1-1, \dots, a_m+b_m-1}f(XYZ)] \\
&= \sigma^2 \mathbb{E}[XB_{a_1+b_1, \dots, a_m+b_m}B_{r_1, \dots, r_n}A_NB_{r_1, \dots, r_n}B_{a_1, \dots, a_m}f(XYZ)] \\
&= \sigma^2 \mathbb{E}[\mathbb{E}[XB_{a_1+b_1, \dots, a_m+b_m}B_{r_1, \dots, r_n}A_NB_{r_1, \dots, r_n}B_{a_1, \dots, a_m}f_{YZ}(X) \mid YZ]] \\
&= \sigma^2 \mathbb{E}[\mathbb{E}[B_{a_1, \dots, a_m}B_{r_1, \dots, r_n}A_NB_{r_1, \dots, r_n}B_{a_1, \dots, a_m}f_{YZ}(X) \mid YZ]] \\
&= \sigma^2 \mathbb{E}[B_{a_1, \dots, a_m}B_{r_1, \dots, r_n}A_NB_{r_1, \dots, r_n}B_{a_1, \dots, a_m}f(XYZ)].
\end{aligned}$$

This completes the proof.  $\square$

**Remark 2.10.** We could have obtained first order Stein operators for the product normal, beta and gamma distributions using the density approach of Stein et al. [30] (see also Ley et al. [16] for an extension of the scope of the density method). However, this approach

would lead to complicated operators involving Meijer  $G$ -functions, which, in contrast to our Stein equations, may not be amenable to the use of couplings.

From the formulas (2.2) and (2.3) for the operators  $A_N$  and  $B_{r_1, \dots, r_n}$ , it follows that the product Stein operators of Table 1 are linear ordinary differential operators with simple coefficients. As an example, the Stein operator for the product  $XYZ$  can be written as

$$\mathcal{A}_{XYZ}f(x) = \sigma^2 \sum_{k=1}^{2m+2n+N} \alpha_{k, 2m+2n+N} x^{k-1} f^{(k)}(x) - \lambda^{2n} \sum_{k=0}^{2m} \beta_{k, 2m} x^{k+1} f^{(k)}(x),$$

where the  $\alpha_{k, 2m+2n+N}$  and  $\beta_{k, 2m}$  can be computed using (2.4).

As discussed in the Introduction, Stein operators of order greater than two are not common in the literature; however, our higher order product Stein operators seem to be natural generalisations of the classical normal, beta and gamma Stein operators to products. It is interesting to note that whilst the product beta, gamma and normal Stein operators are order  $m$ ,  $n$  and  $N$ , respectively, the operator for their product is order  $2m + 2n + N$ , whilst one might intuitively expect the order to be  $m + n + N$ . The formula (3.1) of Theorem 3.1 below for the p.d.f. for the product  $XYZ$  sheds light on this, and is discussed further in Remark 3.2. In Section 2.2.2, we shall see that for certain parameter values one can obtain lower Stein operators for the product  $XYZ$ . For example, the operator decreases by  $m$  when  $b_1 = \dots = b_m = 1$ , and this can also be understood from (3.1) and properties of the Meijer  $G$ -function; again, this is discussed in Remark 3.2.

### 2.2.2 Reduced order Stein operators

By Lemma 2.1, we can write the Stein operators for the products  $XZ$  and  $XYZ$  as

$$\mathcal{A}_{XZ}f(x) = \sigma^2 x^{-1} B_{a_1, \dots, a_m} B_{a_1-1, \dots, a_m-1} T_0^N f(x) - x B_{a_1+b_1, \dots, a_m+b_m} B_{a_1+b_1-1, \dots, a_m+b_m-1} f(x)$$

and

$$\begin{aligned} \mathcal{A}_{XYZ}f(x) &= \sigma^2 x^{-1} B_{a_1, \dots, a_m} B_{a_1-1, \dots, a_m-1} B_{r_1, \dots, r_n} B_{r_1-1, \dots, r_n-1} T_0^N f(x) \\ &\quad - \lambda^{2n} x B_{a_1+b_1, \dots, a_m+b_m} B_{a_1+b_1-1, \dots, a_m+b_m-1} f(x). \end{aligned}$$

With this representation, we can write down a simple criterion under which we can obtain Stein operators for the products  $XZ$  and  $XYZ$  with orders less than  $2m + N$  and  $2m + 2n + N$  respectively. For simplicity, we only consider the case of the product  $XYZ$ ; we can treat the operator for product  $XZ$  similarly.

Define sets  $R$  and  $S$  by

$$\begin{aligned} R &= \{a_1 + b_1, \dots, a_m + b_m, a_1 + b_1 - 1, \dots, a_m + b_m - 1\}; \\ S &= \{a_1, \dots, a_m, a_1 - 1, \dots, a_m - 1, r_1, \dots, r_n, r_1 - 1, \dots, r_n - 1, 0, \dots, 0\}, \end{aligned}$$

where it is understood that there are  $N$  zeros in  $S$ . Then if  $|R \cap S| = t$ , the Stein operator  $\mathcal{A}_{XYZ}f(x)$  can be reduced to one of order  $2m + 2n + N - t$ .

To illustrate this criterion, we consider some particular parameter values.

(i)  $b_1 = \dots = b_m = 1$ :  $X$  is product of  $m$  independent  $U(0, 1)$  random variables when also  $a_1 = \dots = a_m = 1$ . Here the Stein operator is

$$\begin{aligned}\mathcal{A}_{XYZ}f(x) &= \sigma^2 x^{-1} B_{a_1-1, \dots, a_m-1} B_{r_1, \dots, r_n} B_{r_1-1, \dots, r_n-1} T_0^N B_{a_1, \dots, a_m} f(x) \\ &\quad - \lambda^{2n} x B_{a_1+1, \dots, a_m+1} B_{a_1, \dots, a_m} f(x),\end{aligned}$$

where we used the fact that the operators  $T_r$  and  $T_s$  are commutative. Taking  $g(x) = B_{a_1, \dots, a_m} f(x)$  then gives the  $(m + 2n + N)$ -th order Stein operator

$$\begin{aligned}\mathcal{A}g(x) &= \sigma^2 x^{-1} B_{r_1, \dots, r_n} B_{r_1-1, \dots, r_n-1} T_0^N B_{a_1, \dots, a_m} g(x) - \lambda^{2n} x B_{a_1+1, \dots, a_m+1} g(x) \\ &= \sigma^2 B_{a_1, \dots, a_m} B_{r_1, \dots, r_n} A_N B_{r_1, \dots, r_n} g(x) - \lambda^{2n} x B_{a_1+1, \dots, a_m+1} g(x).\end{aligned}$$

In the subsequent examples, we shall not write down the resulting lower order Stein operators, although they can be obtained easily by similar calculations.

(ii)  $a_1 + b_1 = \dots = a_m + b_m = 1$ :  $X$  is a product of  $m$  independent arcsine random variables when also  $a_1 = \dots = a_m = 1/2$ . A Stein operator of order  $m + 2n + N$  can again be obtained.

(iii)  $m = n = N$ ,  $a_1 + b_1 = \dots = a_m + b_m = 1$  and  $r_1 = \dots = r_n = 1$ , so that  $X$  and  $Y$  are products of  $m$  arcsine and Exponential(1) random variables respectively. A Stein operator of order  $3m$  can again be obtained.

(iv)  $m = n = N$ ,  $a_1 + b_1 = \dots = a_m + b_m = 1$  and  $r_1 = \dots = r_n = 2$ . A Stein operator of order  $3m$  can be obtained.

## 2.3 A Stein equation for the product of two gammas

In general, for the product distribution Stein equations that are obtained in this paper, it is difficult to solve the equation and bound the appropriate derivatives of the solution. However, for the product normal Stein equation, Gaunt [10] obtained uniform bounds for the first four derivatives of the solution in the case  $N = 2$ . Here we show that, for the  $\text{PG}(r_1, r_2, \lambda)$  Stein equation, under certain conditions on the test function  $h$ , all derivatives of the solution are uniformly bounded. With a more detailed analysis than the one carried out in this paper we could obtain explicit constants; this is discussed in Remark 2.13 below. In Remark 2.14 below, we discuss the difficulties of obtaining such estimates for more general product distribution Stein equations.

Taking  $q = 1$  in the characterisation of the product generalised gamma distribution given in Proposition 2.7 leads to the following Stein equation for the  $\text{PG}(r_1, r_2, \lambda)$  distribution:

$$x^2 f''(x) + (1 + r_1 + r_2) x f'(x) + (r_1 r_2 - \lambda^2 x) f(x) = h(x) - \text{PG}_{r_1, r_2}^\lambda h, \quad (2.16)$$

where  $\text{PG}_{r_1, r_2}^\lambda h$  denotes  $\mathbb{E}h(Y)$ , for  $Y \sim \text{PG}(r_1, r_2, \lambda)$ . The functions  $x^{-(r_1+r_2)/2} K_{r_1-r_2}(2\lambda\sqrt{x})$  and  $x^{-(r_1+r_2)/2} I_{|r_1-r_2|}(2\lambda\sqrt{x})$  (the modified Bessel functions  $I_\nu(x)$  and  $K_\nu(x)$  are defined in Appendix B) form a fundamental system of solutions to the homogeneous equation (this can readily be seen from (B.10)). Therefore, we can use the method of variation of parameters (see Collins [3] for an account of the method) to solve (2.16). The resulting solution is given in the following lemma and its derivatives are bounded in the next proposition. The proofs are given in Appendix A.



**Lemma 2.11.** Suppose  $h : \mathbb{R}_+ \rightarrow \mathbb{R}$  is bounded and let  $\tilde{h}(x) = h(x) - \text{PG}_{r_1, r_2}^\lambda h$ . Then the unique bounded solution  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$  to the Stein equation (2.16) is given by

$$\begin{aligned} f(x) &= -\frac{2K_{r_1-r_2}(2\lambda\sqrt{x})}{x^{(r_1+r_2)/2}} \int_0^x t^{(r_1+r_2)/2-1} I_{|r_1-r_2|}(2\lambda\sqrt{t}) \tilde{h}(t) dt \\ &\quad + \frac{2I_{|r_1-r_2|}(2\lambda\sqrt{x})}{x^{(r_1+r_2)/2}} \int_0^x t^{(r_1+r_2)/2-1} K_{r_1-r_2}(2\lambda\sqrt{t}) \tilde{h}(t) dt \end{aligned} \quad (2.17)$$

$$\begin{aligned} &= -\frac{2K_{r_1-r_2}(2\lambda\sqrt{x})}{x^{(r_1+r_2)/2}} \int_0^x t^{(r_1+r_2)/2-1} I_{|r_1-r_2|}(2\lambda\sqrt{t}) \tilde{h}(t) dt \\ &\quad - \frac{2I_{|r_1-r_2|}(2\lambda\sqrt{x})}{x^{(r_1+r_2)/2}} \int_x^\infty t^{(r_1+r_2)/2-1} K_{r_1-r_2}(2\lambda\sqrt{t}) \tilde{h}(t) dt. \end{aligned} \quad (2.18)$$

**Proposition 2.12.** Suppose  $h \in C_b^k(\mathbb{R}_+)$  and let  $f$  denote the solution (2.17). Then there exist non-negative constants  $C_{0,k}, C_{1,k}, \dots, C_{k,k}$  such that

$$\|f\| \leq C_{0,0} \|\tilde{h}\| \quad \text{and} \quad \|f^{(k)}\| \leq C_{0,k} \|\tilde{h}\| + \sum_{j=1}^k C_{j,k} \|h^{(j)}\|, \quad k \geq 1. \quad (2.19)$$

**Remark 2.13.** The solution  $f$  can be bounded by

$$\begin{aligned} |f(x)| &\leq 2\|\tilde{h}\| \frac{1}{x^{(r_1+r_2)/2}} \int_0^x t^{(r_1+r_2)/2-1} |K_{r_1-r_2}(2\lambda\sqrt{x}) I_{|r_1-r_2|}(2\lambda\sqrt{t}) \\ &\quad - I_{|r_1-r_2|}(2\lambda\sqrt{x}) K_{r_1-r_2}(2\lambda\sqrt{t})| dt, \end{aligned}$$

useful for ‘small’  $x$ , and

$$\begin{aligned} |f(x)| &\leq 2\|\tilde{h}\| \frac{K_{r_1-r_2}(2\lambda\sqrt{x})}{x^{(r_1+r_2)/2}} \int_0^x t^{(r_1+r_2)/2-1} I_{|r_1-r_2|}(2\lambda\sqrt{t}) dt \\ &\quad + 2\|\tilde{h}\| \frac{I_{|r_1-r_2|}(2\lambda\sqrt{x})}{x^{(r_1+r_2)/2}} \int_x^\infty t^{(r_1+r_2)/2-1} K_{r_1-r_2}(2\lambda\sqrt{t}) dt, \end{aligned}$$

useful for ‘large’  $x$ . In the proof of Lemma 2.11, we use asymptotic formulas for modified Bessel functions to show that the above expressions involving modified Bessel functions are bounded for all  $x > 0$ . A more detailed analysis (see Gaunt [9] for an analysis that yields bounds for similar expressions involving integrals of modified Bessel functions) would allow one to obtain an explicit bound, uniform in  $x$ , for these quantities, which would yield an explicit value for the constant  $C_{0,0}$ . By examining the proof of Proposition 2.12, we would then be able to determine explicit values for all  $C_{j,k}$  by a straightforward induction. However, since we do not use the product gamma Stein equation to prove any approximation results in this paper, we omit this analysis.

**Remark 2.14.** For the  $\text{PN}(2, \sigma^2)$  and  $\text{PG}(r_1, r_2, \lambda)$  Stein equations, one can obtain a fundamental system of solutions to the homogeneous equation in terms of modified Bessel functions. These functions are well-understood, meaning that the problem of bounding the derivatives of the solution is reasonably tractable. However, for product distribution Stein equations in general, it is more challenging to bound the derivatives, because the

Stein equation is of higher order and a fundamental system for the homogeneous equation is given in terms of less well-understood Meijer  $G$ -functions (this can be seen from (B.7)), which do not in general reduce to simpler functions. See Gaunt [10], Section 2.3.2 for a detailed discussion of this problem for the product normal case. Obtaining bounds for other product distribution Stein equations is left as an interesting open problem, which if solved would mean that the Stein equations of this paper could be utilised to prove product, beta, gamma and normal approximation results.

### 3 Distributional properties of products of beta, gamma and normal random variables

#### 3.1 Distributional theory

Much of this section is devoted to proving Theorem 3.1 below which gives a formula for the p.d.f. of the product beta-gamma-normal distribution. Throughout this section we shall suppose that the random variables  $X \sim \text{PB}(a_1, b_1, \dots, a_m, b_m)$ ,  $Y \sim \text{PG}(r_1, \dots, r_n, \lambda)$  and  $Z \sim \text{PN}(N, \sigma^2)$  are mutually independent, and denote their product by  $W = XYZ$ .

**Theorem 3.1.** *The p.d.f. of  $W$  is given by*

$$p(x) = K G_{2m, 2m+2n+N}^{2m+2n+N, 0} \left( \frac{\lambda^{2n} x^2}{2^{2n+N} \sigma^2} \left| \begin{array}{c} \frac{a_1+b_1}{2}, \dots, \frac{a_m+b_m}{2}, \\ \frac{a_1}{2}, \dots, \frac{a_m}{2}, \frac{a_1-1}{2}, \dots, \frac{a_m-1}{2}, \\ \dots, \frac{a_1+b_1-1}{2}, \dots, \frac{a_m+b_m-1}{2}, \\ \dots, \frac{r_1}{2}, \dots, \frac{r_n}{2}, \frac{r_1-1}{2}, \dots, \frac{r_n-1}{2}, 0, \dots, 0 \end{array} \right. \right), \quad (3.1)$$

where

$$K = \frac{\lambda^n}{2^{2n+N/2} \pi^{(n+N)/2} \sigma} \prod_{i=1}^m \frac{\Gamma(a_i + b_i)}{2^{b_i} \Gamma(a_i)} \prod_{j=1}^n \frac{2^{r_j}}{\Gamma(r_j)}.$$

We prove this theorem in Section 3.3 by verifying that the Mellin transform of the product  $XYZ$  is the same as the Mellin transform of the density (3.1). However, a constructive proof using the Mellin inversion formula would require more involved calculations. In Section 3.2, we use the product beta-gamma-normal characterisation (Proposition 2.9, part (iv)) to motivate the formula (3.1) as a candidate for the density of the product  $W$ . As far as the author is aware, this is the first time a Stein characterisation has been applied to arrive at a new formula for the p.d.f. of a distribution.

Before proving Theorem 3.1, we note some simple consequences. The product normal p.d.f. (1.6) is an obvious special case of the master formula (3.1), and by using properties of the Meijer  $G$ -function one can also obtain the product beta-gamma density (1.7).

**Remark 3.2.** Let us now recall the sets  $R$  and  $S$  of Section 2.2.2:

$$\begin{aligned} R &= \{a_1 + b_1, \dots, a_m + b_m, a_1 + b_1 - 1, \dots, a_m + b_m - 1\}; \\ S &= \{a_1, \dots, a_m, a_1 - 1, \dots, a_m - 1, r_1, \dots, r_n, r_1 - 1, \dots, r_n - 1, 0, \dots, 0\}, \end{aligned}$$

where there are  $N$  zeros in set  $S$ . By property (B.1) of the Meijer  $G$ -function, it follows that the order of the  $G$ -function in the density (3.1) decreases by  $t$  if  $|R \cap S| = t$ . This

is precisely the same condition under which the order of the Stein operator  $\mathcal{A}_{XYZ}f(x)$  decreases by  $t$ . The reason for this becomes apparent in Section 3.2 when we note that the density (3.1) satisfies the differential equation  $\mathcal{A}_{XYZ}^*p(x) = 0$ , where  $\mathcal{A}_{XYZ}^*$  is an adjoint operator of  $\mathcal{A}_{XYZ}$  with the same order. Hence, the order of the Stein operator decreases precisely when the degree of the  $G$ -function in the density (3.1) decreases.

As an example of this simplification, taking  $b_1 = \dots = b_m = 1$  in (3.1) and simplifying using (B.2), gives the following expression for the density:

$$p(x) = \tilde{K} G_{m,m+2n+N}^{m+2n+N,0} \left( \frac{\lambda^{2n} x^2}{2^{2n+N} \sigma^2} \left| \frac{a_1-1}{2}, \dots, \frac{a_m-1}{2}, \frac{r_1}{2}, \dots, \frac{r_n}{2}, \frac{r_1-1}{2}, \dots, \frac{r_n-1}{2}, 0, \dots, 0 \right. \right),$$

where  $\tilde{K}$  is the normalizing constant. It is instructive to compare this with Example (i) of Section 2.2.2.

Finally, we record two simple corollaries of Theorem 3.1: a formula for the characteristic function of  $W$  and tail estimates for its density.

**Corollary 3.3.** *The characteristic function of  $W$  is given by*

$$\phi(t) = M G_{2m+1,2m+2n+N-1}^{2m+2n+N-1,1} \left( \frac{\lambda^{2n}}{2^{2n+N-2} \sigma^2 t^2} \left| \begin{array}{l} 1, \frac{a_1+b_1+1}{2}, \dots, \frac{a_m+b_m+1}{2}, \dots \\ \frac{a_1+1}{2}, \dots, \frac{a_m+1}{2}, \frac{a_1}{2}, \dots, \frac{a_m}{2}, \dots \\ \dots \frac{a_1+b_1}{2}, \dots, \frac{a_m+b_m}{2} \\ \dots \frac{r_1+1}{2}, \dots, \frac{r_n+1}{2}, \frac{r_1}{2}, \dots, \frac{r_n}{2}, \frac{1}{2}, \dots, \frac{1}{2} \end{array} \right. \right),$$

where

$$M = \frac{1}{\pi^{(n+N-1)/2}} \prod_{i=1}^m \frac{\Gamma(a_i + b_i)}{2^{b_i} \Gamma(a_i)} \prod_{j=1}^n \frac{2^{r_j-1}}{\Gamma(r_j)}.$$

*Proof.* Since the distribution of  $W$  is symmetric about the origin, it follows that the characteristic function  $\phi(t)$  is given by

$$\phi(t) = \mathbb{E}[e^{itW}] = \mathbb{E}[\cos(tW)] = 2 \int_0^\infty \cos(tx) p(x) dx.$$

Evaluating the integral using (B.5) gives

$$\phi(t) = M G_{2m+2,2m+2n+N}^{2m+2n+N,1} \left( \frac{\lambda^{2n}}{2^{2n+N-2} \sigma^2 t^2} \left| \begin{array}{l} \frac{1}{2}, \frac{a_1+b_1}{2}, \dots, \frac{a_m+b_m}{2}, \dots \\ \frac{a_1}{2}, \dots, \frac{a_m}{2}, \frac{a_1-1}{2}, \dots, \frac{a_m-1}{2}, \dots \\ \dots \frac{a_1+b_1-1}{2}, \dots, \frac{a_m+b_m-1}{2}, 0 \\ \dots \frac{r_1}{2}, \dots, \frac{r_n}{2}, \frac{r_1-1}{2}, \dots, \frac{r_n-1}{2}, 0, \dots, 0 \end{array} \right. \right),$$

where

$$M = \frac{2K\sqrt{\pi}}{|t|} = \frac{1}{\pi^{(n+N-1)/2}} \frac{\Gamma(a_i + b_i)}{2^{b_i} \Gamma(a_i)} \prod_{j=1}^n \frac{2^{r_j-1}}{\Gamma(r_j)} \cdot \frac{\lambda^n}{2^{n+N/2-1} \sigma |t|},$$

and simplifying the above expression using (B.2) and then (B.1) completes the proof.  $\square$

**Corollary 3.4.** *The density (3.1) of the random variable  $W$  satisfies the asymptotic formula*

$$p(x) \sim N|x|^\alpha \exp \left\{ - (2n + N) \left( \frac{\lambda^{2n} x^2}{2^{2n+N} \sigma^2} \right)^{1/(2n+N)} \right\}, \quad \text{as } |x| \rightarrow \infty,$$

where

$$N = \frac{(2\pi)^{(2n+N-1)/2}}{(2n+N)^{1/2}} \left( \frac{\lambda^{2n}}{2^{2n+N} \sigma^2} \right)^{\alpha/2} K,$$

with  $K$  defined as in Theorem 3.1, and

$$\alpha = \frac{2}{2n+N} \left\{ \frac{1-3n+N}{2} + \sum_{j=1}^n r_j - \sum_{j=1}^m b_j \right\}.$$

*Proof.* Apply the asymptotic formula (B.3) to the density (3.1).  $\square$

### 3.2 Discovery of Theorem 3.1 via the Stein characterisation

Here we motivate the formula (3.1) for the density  $p$  of the product random variable  $W$ . We do so by using the product beta-gamma-normal Stein characterisation to find a differential equation satisfied by  $p$ .

By part (iv) of Proposition 2.9 we have that

$$\begin{aligned} & \mathbb{E}[\sigma^2 B_{a_1, \dots, a_m} B_{r_1, \dots, r_n} A_N B_{r_1, \dots, r_n} B_{a_1, \dots, a_n} f(W) \\ & - \lambda^{2n} W B_{a_1+b_1, \dots, a_m+b_m} B_{a_1+b_1-1, \dots, a_m+b_m-1} f(W)] = 0 \end{aligned} \quad (3.2)$$

for all  $f \in C^{2m+2n+N}(\mathbb{R})$  such that  $\mathbb{E}|W^{k-1} f^{(k)}(W)| < \infty$  for  $1 \leq k \leq 2m+2n+N$ , and  $\mathbb{E}|W^{k+1} f^{(k)}(W)| < \infty$  for  $0 \leq k \leq 2m$ . By using part (ii) of Lemma 2.1 and that  $A_N f(x) = \frac{d}{dx}(T_0^{N-1} f(x))$ , we can write

$$A_N B_{r_1, \dots, r_n} B_{a_1, \dots, a_m} f(x) = B_{r_1+1, \dots, r_n+1} B_{a_1+1, \dots, a_m+1} T_1^{N-1} f'(x).$$

On substituting into (3.2), we see that the density  $p(x)$  of  $W$  satisfies the equation

$$\begin{aligned} & \int_{-\infty}^{\infty} \left\{ \sigma^2 B_{a_1, \dots, a_m} B_{r_1, \dots, r_n} B_{r_1+1, \dots, r_n+1} B_{a_1+1, \dots, a_m+1} T_1^{N-1} f'(x) \right. \\ & \left. - \lambda^{2n} x B_{a_1+b_1, \dots, a_m+b_m} B_{a_1+b_1-1, \dots, a_m+b_m-1} f(x) \right\} p(x) dx = 0 \end{aligned} \quad (3.3)$$

for all  $f \in C^{2m+2n+N}(\mathbb{R})$  such that  $\mathbb{E}|W^{k-1} f^{(k)}(W)| < \infty$  for  $1 \leq k \leq 2m+2n+N$  and  $\mathbb{E}|W^{k+1} f^{(k)}(W)| < \infty$  for  $0 \leq k \leq 2m$ . In particular, (3.3) holds for all functions  $f$  such that

- (i)  $f \in C^{2m+2n+N}(\mathbb{R})$ ;
- (ii)  $\mathbb{E}|W^{k-1} f^{(k)}(W)| < \infty$  for  $1 \leq k \leq 2m+2n+N$  and  $\mathbb{E}|W^{k+1} f^{(k)}(W)| < \infty$  for  $0 \leq k \leq 2m$ ;
- (iii)  $x^{i+j+2} p^{(i)}(x) f^{(j)}(x) \rightarrow 0$  as  $x \rightarrow \pm\infty$  for all  $i, j$  such that  $0 \leq i+j \leq 2m-1$ ;
- (iv)  $x^{i+j} p^{(i)}(x) f^{(j)}(x) \rightarrow 0$  as  $x \rightarrow \pm\infty$  for all  $i, j$  such that  $0 \leq i+j \leq 2m+2n+N-1$ .

We shall denote the class of functions satisfying (i)–(iv) by  $\mathcal{C}_p$ . It will later become apparent as to why it is helpful to have the additional conditions (iii) and (iv).

We now note that, for differentiable functions  $\phi$  and  $\psi$ ,

$$\begin{aligned} \int_{-\infty}^{\infty} \phi(x) T_r \psi(x) dx &= \int_{-\infty}^{\infty} \phi(x) \{x\psi'(x) + r\psi(x)\} dx = \int_{-\infty}^{\infty} x^{1-r} \phi(x) \frac{d}{dx} (x^r \psi(x)) dx \\ &= \left[ x\phi(x)\psi(x) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x^r \psi(x) \frac{d}{dx} (x^{1-r} \phi(x)) dx \\ &= \left[ x\phi(x)\psi(x) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \psi(x) T_{1-r} \phi(x) dx, \end{aligned} \quad (3.4)$$

provided the integrals exist. A simple calculation shows that

$$T_s(x\phi(x)) = x^2\phi'(x) + (s+1)x\phi(x) = xT_{s+1}\phi(x),$$

and therefore from (3.4) we deduce that

$$\begin{aligned} \int_{-\infty}^{\infty} x\phi(x) T_r \psi(x) dx &= \left[ x^2\phi(x)\psi(x) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \psi(x) T_{1-r}(x\phi(x)) dx \\ &= \left[ x^2\phi(x)\psi(x) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x\psi(x) T_{2-r}\phi(x) dx, \end{aligned} \quad (3.5)$$

if the integrals exist.

We now return to equation (3.3) and use integration by parts and formulas (3.4) and (3.5) to obtain a differential equation that is satisfied by  $p(x)$ . Using (3.5) we obtain

$$\begin{aligned} &\int_{-\infty}^{\infty} xp(x) B_{a_1+b_1, \dots, a_m+b_m} B_{a_1+b_1-1, \dots, a_m+b_m-1} f(x) dx \\ &= \left[ x^2 p(x) B_{a_1+b_1, \dots, a_m+b_m-1} B_{a_1+b_1-1, \dots, a_m+b_m-1} f(x) \right]_{-\infty}^{\infty} \\ &\quad - \int_{-\infty}^{\infty} x T_{2-a_m-b_m} p(x) B_{a_1+b_1-1, \dots, a_m+b_m-1} f(x) dx \\ &= - \int_{-\infty}^{\infty} x T_{2-a_m-b_m} p(x) B_{a_1+b_1-1, \dots, a_m+b_m-1} f(x) dx, \end{aligned}$$

where we used condition (iii) to obtain the last equality. By a repeated application of integration by parts, using formula (3.5) and condition (iii), we arrive at

$$\begin{aligned} &\int_{-\infty}^{\infty} xp(x) B_{a_1+b_1, \dots, a_m+b_m} B_{a_1+b_1-1, \dots, a_m+b_m-1} f(x) dx \\ &= \int_{-\infty}^{\infty} xf(x) B_{3-a_1-b_1, \dots, 3-a_m-b_m} B_{2-a_1-b_1, \dots, 2-a_1-b_1} p(x) dx. \end{aligned}$$

By a similar argument, this time using formula (3.4) and condition (iv), we obtain

$$\begin{aligned} &\int_{-\infty}^{\infty} p(x) B_{a_1, \dots, a_m} B_{r_1, \dots, r_n} B_{r_1+1, \dots, r_n+1} B_{a_1+1, \dots, a_m+1} T_1^{N-1} f'(x) dx \\ &= (-1)^N \int_{-\infty}^{\infty} f(x) \frac{d}{dx} (T_0^{N-1} B_{-a_1, \dots, -a_m} B_{-r_1, \dots, -r_n} B_{1-r_1, \dots, 1-r_n} B_{1-a_1, \dots, 1-a_m} p(x)) dx. \end{aligned}$$

Putting this together we have that

$$\int_{-\infty}^{\infty} \{(-1)^N \sigma^2 x^{-1} T_0^N B_{-a_1, \dots, -a_m} B_{-r_1, \dots, -r_n} B_{1-r_1, \dots, 1-r_n} B_{1-a_1, \dots, 1-a_m} p(x) - \lambda^{2n} x B_{3-a_1-b_1, \dots, 3-a_m-b_m} B_{2-a_1-b_1, \dots, 2-a_m-b_m} p(x)\} f(x) dx = 0$$

for all  $f \in \mathcal{C}_p$ . Since the integral in the above display is equal to zero for all  $f \in \mathcal{C}_p$ , it follows by a slight variation of the fundamental lemma of the calculus of variations (here we have restrictions on the growth of  $f(x)$  in the limits  $x \rightarrow \pm\infty$ ) that  $p(x)$  satisfies the differential equation

$$T_0^N B_{-a_1, \dots, -a_m} B_{-r_1, \dots, -r_n} B_{1-r_1, \dots, 1-r_n} B_{1-a_1, \dots, 1-a_m} p(x) - \sigma^{-2} \lambda^{2n} x^2 B_{3-a_1-b_1, \dots, 3-a_m-b_m} B_{2-a_1-b_1, \dots, 2-a_m-b_m} p(x) = 0. \quad (3.6)$$

We now make a change of variables to transform this differential equation to a Meijer  $G$ -function differential equation (see (B.7)). To this end, let  $y = \frac{\lambda^{2n} x^2}{2^{2n+N} \sigma^2}$ . Then,  $x \frac{d}{dx} = 2y \frac{d}{dy}$  and  $p(y)$  satisfies the differential equation

$$T_0^N B_{-\frac{a_1}{2}, \dots, -\frac{a_m}{2}} B_{-\frac{r_1}{2}, \dots, -\frac{r_n}{2}} B_{\frac{1-r_1}{2}, \dots, \frac{1-r_n}{2}} B_{\frac{1-a_1}{2}, \dots, \frac{1-a_m}{2}} p(y) - y B_{\frac{3-a_1-b_1}{2}, \dots, \frac{3-a_m-b_m}{2}} B_{\frac{2-a_1-b_1}{2}, \dots, \frac{2-a_m-b_m}{2}} p(y) = 0. \quad (3.7)$$

From (B.7) it follows that a solution to (3.7) is

$$p(y) = C G_{2m, 2m+2n+N}^{2m+2n+N, 0} \left( y \left| \frac{a_1+b_1}{2}, \dots, \frac{a_m+b_m}{2}, \frac{a_1+b_1-1}{2}, \dots, \frac{a_m+b_m-1}{2}, \frac{a_1}{2}, \dots, \frac{a_m}{2}, \frac{a_1-1}{2}, \dots, \frac{a_m-1}{2}, \frac{r_1}{2}, \dots, \frac{r_n}{2}, \frac{r_1-1}{2}, \dots, \frac{r_n-1}{2}, 0, \dots, 0 \right. \right),$$

where  $C$  is an arbitrary constant. Therefore, on changing variables, a solution to (3.6) is given by

$$p(x) = \tilde{C} G_{2m, 2m+2n+N}^{2m+2n+N, 0} \left( \frac{\lambda^{2n} x^2}{2^{2n+N} \sigma^2} \left| \frac{a_1+b_1}{2}, \dots, \frac{a_m+b_m}{2}, \frac{a_1+b_1-1}{2}, \dots, \frac{a_m+b_m-1}{2}, \frac{a_1}{2}, \dots, \frac{a_m}{2}, \frac{a_1-1}{2}, \dots, \frac{a_m-1}{2}, \frac{r_1}{2}, \dots, \frac{r_n}{2}, \frac{r_1-1}{2}, \dots, \frac{r_n-1}{2}, 0, \dots, 0 \right. \right),$$

where  $\tilde{C}$  is an arbitrary constant. We can use the integration formula (B.6) to determine a value of  $\tilde{C}$  such that  $\int_{\mathbb{R}} p(x) dx = 1$ . With this choice of  $\tilde{C}$ ,  $p(x) \geq 0$  and so  $p$  is a density function. However, there are  $2m + 2n + N$  linearly independent solutions to (3.6) and whilst our solution  $p$  is indeed a density function, a more detailed analysis would be required to rigorously prove that it is indeed the density function of the product beta-gamma-normal distribution. Since a simple proof that  $p$  is indeed the density function is now available to us via Mellin transforms, we decide to omit such an analysis.

### 3.3 Proof of Theorem 3.1

Firstly, we define the Mellin transform and state some properties that will be useful to us. The Mellin transform of a non-negative random variable  $U$  with density  $p$  is given by

$$M_U(s) = \mathbb{E} U^{s-1} = \int_0^{\infty} x^{s-1} p(x) dx.$$

If the random variable  $U$  has density  $p$  that is symmetric about the origin then we can define the Mellin transform of  $U$  by

$$M_U(s) = 2 \int_0^\infty x^{s-1} p(x) dx.$$

The Mellin transform is useful in determining the distribution of products of independent random variables due to the property that if the random variables  $U$  and  $V$  are independent then

$$M_{UV}(s) = M_U(s)M_V(s). \quad (3.8)$$

*Proof of Theorem 3.1.* It was shown by Springer and Thompson [27] that the Mellin transforms of  $X$ ,  $Y$  and  $Z$  are

$$\begin{aligned} M_X(s) &= \prod_{j=1}^m \frac{\Gamma(a_j + b_j)}{\Gamma(a_j)} \frac{\Gamma(a_j - 1 + s)}{\Gamma(a_j + b_j - 1 + s)}, \\ M_Y(s) &= \frac{1}{\lambda^{n(s-1)}} \prod_{j=1}^n \frac{\Gamma(r_j - 1 + s)}{\Gamma(r_j)}, \\ M_Z(s) &= \frac{1}{\pi^{N/2}} 2^{N(s-1)/2} \sigma^{s-1} \left[ \Gamma\left(\frac{s}{2}\right) \right]^N. \end{aligned}$$

Then, as the random variables are independent, it follows from (3.8) that

$$\begin{aligned} M_{XYZ}(s) &= \prod_{j=1}^m \frac{\Gamma(a_j + b_j)}{\Gamma(a_j)} \frac{\Gamma(a_j - 1 + s)}{\Gamma(a_j + b_j - 1 + s)} \times \frac{1}{\lambda^{n(s-1)}} \prod_{j=1}^n \frac{\Gamma(r_j - 1 + s)}{\Gamma(r_j)} \\ &\times \frac{1}{\pi^{N/2}} 2^{N(s-1)/2} \sigma^{s-1} \left[ \Gamma\left(\frac{s}{2}\right) \right]^N. \end{aligned} \quad (3.9)$$

Now, let  $W$  be a random variable with density (3.1). Since the density of  $W$  is symmetric about the origin, we have

$$\begin{aligned} M_W(s) &= 2 \int_0^\infty x^{s-1} p(x) dx \\ &= \frac{\lambda^n}{2^{2n+N/2} \pi^{(n+N)/2} \sigma} \prod_{j=1}^m \frac{\Gamma(a_j + b_j)}{2^{b_j} \Gamma(a_j)} \prod_{j=1}^n \frac{2^{r_j}}{\Gamma(r_j)} \times \left( \frac{2^{n+N/2} \sigma}{\lambda^n} \right)^s \times \left[ \Gamma\left(\frac{s}{2}\right) \right]^N \\ &\times \prod_{j=1}^m \frac{\Gamma\left(\frac{a_j+s}{2}\right) \Gamma\left(\frac{a_j-1+s}{2}\right)}{\Gamma\left(\frac{a_j+b_j+s}{2}\right) \Gamma\left(\frac{a_j+b_j-1+s}{2}\right)} \prod_{j=1}^n \Gamma\left(\frac{r_j+s}{2}\right) \Gamma\left(\frac{r_j-1+s}{2}\right), \end{aligned} \quad (3.10)$$

where we used (B.6) to compute the integral. On applying the duplication formula  $\Gamma\left(\frac{x}{2}\right) \Gamma\left(\frac{x}{2} + \frac{1}{2}\right) = 2^{1-x} \sqrt{\pi} \Gamma(x)$  to (3.10) we can deduce that the expressions (3.9) and (3.10) are equal. Hence, the Mellin transforms of  $W$  and  $XYZ$  are equal and therefore  $W$  and  $XYZ$  are equal in distribution.  $\square$

## A Further proofs

*Proof of Lemma 2.11.* We begin by proving that there is at most one bounded solution to the  $\text{PG}(r_1, r_2, \lambda)$  Stein equation (2.16). Suppose  $u$  and  $v$  are bounded solutions to (2.16). Define  $w = u - v$ . Then  $w$  is bounded and is a solution to the homogeneous equation

$$x^2 w''(x) + (1 + r_1 + r_2)xw'(x) + (r_1 r_2 - \lambda^2 x)w(x) = 0.$$

From (B.10) it can be readily seen that the general solution is

$$w(x) = Aw_1(x) + Bw_2(x),$$

where

$$w_1(x) = x^{-(r_1+r_2)/2} K_{r_1-r_2}(2\lambda\sqrt{x}) \quad \text{and} \quad w_2(x) = x^{-(r_1+r_2)/2} I_{|r_1-r_2|}(2\lambda\sqrt{x}).$$

From the asymptotic formulas for modified Bessel functions (B.8) and (B.9), it follows that in order to have a bounded solution we must take  $A = B = 0$ , and thus  $w = 0$  and so there is at most one bounded solution to (2.16).

Since (2.16) is an inhomogeneous linear ordinary differential equation, we can use the method of variation of parameters (see Collins [3] for an account of the method) to write down the general solution of (2.16):

$$f(x) = -w_1(x) \int_a^x \frac{w_2(t)\tilde{h}(t)}{t^2 W(t)} dt + w_2(x) \int_b^x \frac{w_1(t)\tilde{h}(t)}{t^2 W(t)} dt, \quad (\text{A.1})$$

where  $a$  and  $b$  are arbitrary constants and  $W(t) = W(w_1, w_2) = w_1 w_2' - w_2 w_1'$  is the Wronskian. From the formula  $W(K_\nu(x), I_\nu(x)) = x^{-1}$  (Olver et al. [21], formula 10.28.2) and a simple computation we have that  $W(w_1(x), w_2(x)) = \frac{1}{2}x^{-1-r_1-r_2}$ . Substituting the relevant quantities into (A.1) and taking  $a = b = 0$  yields the solution (2.17). That the solutions (2.17) and (2.18) are equal follows because  $t^{(r_1-r_2)/2-1} K_{r_1-r_2}(2\lambda\sqrt{t})$  is proportional to the  $\text{PG}(r_1, r_2, \lambda)$  density function.

Finally, we show that the solution (2.17) is bounded if  $h$  is bounded. If  $r_1 \neq r_2$ , then it follows from the asymptotic formulas for modified Bessel functions (see Appendix B.2.3) that the solution is bounded (here we check that the solution is bounded as  $x \downarrow 0$  using (2.17), and to verify that it is bounded as  $x \rightarrow \infty$  we use (2.18)). If  $r_1 = r_2$ , the same argument confirms that the solution is bounded as  $x \rightarrow \infty$ . To deal with the limit  $x \downarrow 0$ , we use the asymptotic formulas  $I_0(x) \sim 1$  and  $K_0(x) \sim -\log(x)$ , as  $x \downarrow 0$ , to obtain

$$\begin{aligned} \lim_{x \downarrow 0} |f(x)| &= \lim_{x \downarrow 0} \frac{2}{x^{(r_1+r_2)/2}} \left| \int_0^x t^{(r_1+r_2)/2-1} [K_0(2\lambda\sqrt{x})I_0(2\lambda\sqrt{t}) \right. \\ &\quad \left. - I_0(2\lambda\sqrt{x})K_0(2\lambda\sqrt{t})] \tilde{h}(t) dt \right| \\ &= \lim_{x \downarrow 0} \frac{1}{x^{(r_1+r_2)/2}} \int_0^x t^{(r_1+r_2)/2-1} [\log(x) - \log(t)] \tilde{h}(t) dt \\ &\leq \|\tilde{h}\| \lim_{x \downarrow 0} \frac{1}{x^{(r_1+r_2)/2}} \int_0^x t^{(r_1+r_2)/2-1} [\log(x) - \log(t)] dt \\ &= \|\tilde{h}\| \lim_{x \downarrow 0} \frac{1}{((r_1+r_2)/2)^2} = \frac{4\|\tilde{h}\|}{(r_1+r_2)^2}, \end{aligned}$$



and so the solution is bounded when  $h$  is bounded. This completes the proof.  $\square$

*Proof of Proposition 2.12.* In this proof, we use a similar approach to the one used in the proof of Proposition 4.2 of Döbler [4]. Denote the Stein operator for the  $\text{PG}(r_1, r_2, \lambda)$  distribution by  $\mathcal{A}_{r_1, r_2, \lambda} f(x)$ , so that the  $\text{PG}(r_1, r_2, \lambda)$  Stein equation is given by

$$\mathcal{A}_{r_1, r_2, \lambda} f(x) = \tilde{h}(x).$$

Now, from the Stein equation (2.16) and a straightforward induction on  $k$ , we have that

$$\begin{aligned} x^2 f^{(k+2)}(x) + (r_1 + r_2 + 2k + 1)x f^{(k+1)}(x) + ((r_1 + k)(r_2 + k) - \lambda^2 x) f^{(k)}(x) \\ = h^{(k)}(x) + k\lambda^2 f^{(k-1)}(x), \end{aligned}$$

which can be written as

$$\mathcal{A}_{r_1+k, r_2+k, \lambda} f^{(k)}(x) = h^{(k)}(x) + k\lambda^2 f^{(k-1)}(x).$$

Now, by Lemma 2.11, there exists a constant  $C_{r_1, r_2, \lambda}$  such that

$$\|f\| \leq C_{r_1, r_2, \lambda} \|\tilde{h}\|.$$

We also note that the test function  $h'(x) + \lambda^2 f(x)$  has mean zero with respect to the random variable  $Y \sim \text{PG}(r_1 + 1, r_2 + 1, \lambda)$ , since by the product gamma characterisation of Proposition 2.7,

$$\mathbb{E}[h'(Y) + \lambda^2 f(Y)] = \mathbb{E}[\mathcal{A}_{r_1+k, r_2+k, \lambda} f'(Y)] = 0.$$

With these facts we therefore have that

$$\begin{aligned} \|f'\| &\leq C_{r_1+1, r_2+1, \lambda} \|h'(x) + \lambda^2 f(x)\| \leq C_{r_1+1, r_2+1, \lambda} (\|h'\| + \lambda^2 \|f\|) \\ &\leq C_{r_1+1, r_2+1, \lambda} (\|h'\| + \lambda^2 C_{r_1, r_2, \lambda} \|\tilde{h}\|). \end{aligned}$$

Repeating this procedure then yields the bound (2.19), as required.  $\square$

## B Properties of the Meijer $G$ -function and modified Bessel functions

Here we define the Meijer  $G$ -function and modified Bessel functions and state some of their properties that are relevant to this paper. For further properties of these functions see Luke [18] and Olver et al. [21].

### B.1 The Meijer $G$ -function

#### B.1.1 Definition

The Meijer  $G$ -function is defined, for  $z \in \mathbb{C} \setminus \{0\}$ , by the contour integral:

$$G_{p,q}^{m,n} \left( z \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} z^{-s} \frac{\prod_{j=1}^m \Gamma(s + b_j) \prod_{j=1}^n \Gamma(1 - a_j - s)}{\prod_{j=n+1}^p \Gamma(s + a_j) \prod_{j=m+1}^q \Gamma(1 - b_j - s)} ds,$$

where  $c$  is a real constant defining a Bromwich path separating the poles of  $F(s + b_j)$  from those of  $F(1 - a_j - s)$  and where we use the convention that the empty product is 1.

### B.1.2 Basic properties

The  $G$ -function is symmetric in the parameters  $a_1, \dots, a_n; a_{n+1}, \dots, a_p; b_1, \dots, b_m; b_{m+1}, \dots, b_q$ . Thus, if one the  $a_j$ 's,  $j = n+1, \dots, p$ , is equal to one of the  $b_k$ 's,  $k = 1, \dots, m$ , the  $G$ -function reduces to one of lower order. For example,

$$G_{p,q}^{m,n} \left( z \left| \begin{matrix} a_1, \dots, a_{p-1}, b_1 \\ b_1, \dots, b_q \end{matrix} \right. \right) = G_{p-1,q-1}^{m-1,n} \left( z \left| \begin{matrix} a_1, \dots, a_{p-1} \\ b_2, \dots, b_q \end{matrix} \right. \right), \quad m, p, q \geq 1. \quad (\text{B.1})$$

The  $G$ -function satisfies the identity

$$z^c G_{p,q}^{m,n} \left( z \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) = G_{p,q}^{m,n} \left( z \left| \begin{matrix} a_1 + c, \dots, a_p + c \\ b_1 + c, \dots, b_q + c \end{matrix} \right. \right). \quad (\text{B.2})$$

### B.1.3 Asymptotic expansion

For  $x > 0$ ,

$$G_{p,q}^{q,0} \left( x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) \sim \frac{(2\pi)^{(\sigma-1)/2}}{\sigma^{1/2}} x^\theta \exp(-\sigma x^{1/\sigma}), \quad \text{as } x \rightarrow \infty, \quad (\text{B.3})$$

where  $\sigma = q - p$  and

$$\theta = \frac{1}{\sigma} \left\{ \frac{1-\sigma}{2} + \sum_{i=1}^q b_i - \sum_{i=1}^p a_i \right\}.$$

### B.1.4 Integration

$$\int_0^\infty e^{\omega x} G_{p,q}^{m,n} \left( \alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx = \omega^{-1} G_{p+1,q}^{m,n+1} \left( \frac{\alpha}{\omega} \left| \begin{matrix} 0, a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right). \quad (\text{B.4})$$

For the conditions under which this formula holds see Luke [18], pp. 166–167.

For  $\alpha > 0$ ,  $\gamma > 0$ ,  $a_j < 1$  for  $j = 1, \dots, n$ , and  $b_j > -\frac{1}{2}$  for  $j = 1, \dots, m$ , we have

$$\int_0^\infty \cos(\gamma x) G_{p,q}^{m,n} \left( \alpha x^2 \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx = \sqrt{\pi} \gamma^{-1} G_{p+2,q}^{m,n+1} \left( \frac{4\alpha}{\gamma^2} \left| \begin{matrix} \frac{1}{2}, a_1, \dots, a_p, 0 \\ b_1, \dots, b_q \end{matrix} \right. \right). \quad (\text{B.5})$$

The following formula follows from Luke [18], formula (1) of section 5.6 and a change of variables:

$$\int_0^\infty x^{s-1} G_{p,q}^{m,n} \left( \alpha x^2 \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx = \frac{\alpha^{-s/2}}{2} \frac{\prod_{j=1}^m \Gamma(b_j + \frac{s}{2}) \prod_{j=1}^n \Gamma(1 - a_j - \frac{s}{2})}{\prod_{j=m+1}^q \Gamma(1 - b_j - \frac{s}{2}) \prod_{j=n+1}^p \Gamma(a_j + \frac{s}{2})}. \quad (\text{B.6})$$

For the conditions under which this formula is valid see Luke, pp. 158–159. In particular, the formula is valid when  $n = 0$ ,  $1 \leq p+1 \leq m \leq q$  and  $\alpha > 0$ .

### B.1.5 Differential equation

The  $G$ -function  $f(z) = G_{p,q}^{m,n} \left( z \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right)$  satisfies the differential equation

$$(-1)^{p-m-n} z B_{1-a_1, \dots, 1-a_p} f(z) - B_{-b_1, \dots, -b_q} f(z) = 0, \quad (\text{B.7})$$

where  $B_{r_1, \dots, r_n} f(z) = T_{r_n} \cdots T_{r_1} f(z)$  for  $T_r f(z) = z f'(z) + r f(z)$ .

## B.2 Modified Bessel functions

### B.2.1 Definitions

The *modified Bessel function of the first kind* of order  $\nu \in \mathbb{R}$  is defined, for all  $x \in \mathbb{R}$ , by

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\nu + k + 1)k!} \left(\frac{x}{2}\right)^{\nu+2k}.$$

The *modified Bessel function of the second kind* of order  $\nu \in \mathbb{R}$  is defined, for  $x > 0$ , by

$$K_\nu(x) = \int_0^\infty e^{-x \cosh(t)} \cosh(\nu t) dt.$$

### B.2.2 Representation in terms of the Meijer $G$ -function

$$\begin{aligned} I_\nu(x) &= i^{-\nu} G_{0,2}^{2,0} \left( -\frac{x^2}{4} \left| \frac{\nu}{2}, -\frac{\nu}{2} \right. \right), \quad x \in \mathbb{R}, \\ K_\nu(x) &= \frac{1}{2} G_{0,2}^{2,0} \left( \frac{x^2}{4} \left| \frac{\nu}{2}, -\frac{\nu}{2} \right. \right), \quad x > 0. \end{aligned}$$

### B.2.3 Asymptotic expansions

$$\begin{aligned} I_\nu(x) &\sim \frac{1}{\Gamma(\nu + 1)} \left(\frac{x}{2}\right)^\nu, \quad x \downarrow 0, \\ K_\nu(x) &\sim \begin{cases} 2^{|\nu|-1} \Gamma(|\nu|) x^{-|\nu|}, & x \downarrow 0, \nu \neq 0, \\ -\log x, & x \downarrow 0, \nu = 0, \end{cases} \end{aligned} \tag{B.8}$$

$$\begin{aligned} I_\nu(x) &\sim \frac{e^x}{\sqrt{2\pi x}}, \quad x \rightarrow \infty, \\ K_\nu(x) &\sim \sqrt{\frac{\pi}{2x}} e^{-x}, \quad x \rightarrow \infty. \end{aligned} \tag{B.9}$$

### B.2.4 Differential equation

The modified Bessel differential equation is

$$x^2 f''(x) + x f'(x) - (x^2 + \nu^2) f(x) = 0. \tag{B.10}$$

The general solution is  $f(x) = A I_\nu(x) + B K_\nu(x)$ .

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