

# **Theoretical analysis of the viscosity correction factor for heat transfer in pipe flow**

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## Abstract

In most heat transfer applications, knowledge of the viscosity variation is important. Thus viscosity correction factors have been researched and proposed for almost a century. One of the most successful relations was reported by Sieder-Tate in 1936, which has been widely used in engineering analysis and design. In this study, we have improved on the Sieder-Tate relation, following a classical theoretical analysis of the thermal boundary layer. An exact solution to the viscosity correction factor was obtained which shows that the Sieder-Tate correction factor over-predicts the heat transfer coefficient in the case of cold wall (cooling) and does not hold properly for hot wall (heating). We have found that a relation of  $\left(\frac{\mu_\infty}{\mu_w}\right)^{0.254}$  (in case of cooling) and  $\left(\frac{\mu_\infty}{\mu_w}\right)^{0.087}$  in the case of heating is better than the Sieder-Tate factor of  $\left(\frac{\mu_\infty}{\mu_w}\right)^{0.14}$  (uniformly applied to both the heating and cooling cases). Here  $\mu_\infty$  and  $\mu_w$  are the bulk and wall viscosity coefficients respectively. The theoretical analysis also shows that the above correction factors are limited to small values of  $\ln\left(\frac{\mu_w}{\mu_\infty}\right)$  (for cold wall) and  $\ln\left(\frac{\mu_\infty}{\mu_w}\right)$  (for hot wall). However a general solution has been obtained and the correlation developed by Petukhov (1970) closely matches the exact solution for the case of cold wall cooling of a fluid.

**Keywords:** *Sieder-Tate, heat transfer, boundary layer, viscosity correction factor, Nusselt number.*

## 1. Introduction

Convective heat transfer (heating or cooling) of fluids in a pipe flow is inevitably present in any processing industry, for example in heat exchangers, coolant flows, power generators, etc. However, design of such systems or any heat transfer equipment requires, amongst other inputs, knowledge of heat transfer coefficients which depend on the properties of the fluid, flow characteristics and geometry. The viscosity of the fluid depends strongly on temperature, and so the thermal gradient across the boundary layer creates a spatial variation in viscosity. In the case of cooling, the dynamic viscosity at the wall ( $\mu_w$ ) is larger than the bulk viscosity ( $\mu_\infty$ ), whereas when the fluid is heated,  $\frac{\mu_w}{\mu_\infty} < 1$ . One of the early correlations reported by Sieder and Tate (1936) includes what is probably the most widely used viscosity correction factor in engineering heat transfer design. They suggested that  $\left(\frac{\mu_\infty}{\mu_w}\right)^{0.14}$  should be the viscosity correction factor to be applied to the Nusselt number ( $Nu$ ) for both heating and

cooling situations. This was obtained by correlating heat transfer data with the mainstream fluid properties and temperature. In the work by Petukhov (1970), the viscosity correction factor obtained empirically by non-linear fitting to experimental data generated two exponents: 0.11 in the case of heating, and 0.25 in the case of cooling.

The later work by Field (1990) presents viscosity corrections obtained theoretically from classical boundary layer analysis. Field found that the correction factors, for small or moderate values of  $\frac{\mu_\infty}{\mu_w}$ , are in the form of  $\left(\frac{\mu_\infty}{\mu_w}\right)^p$ ; in the case of cooling,  $p = 0.1$  whereas for heating,  $p = 0.27$ . These correction factors were considered superior to those obtained by Sieder-Tate (1936) and Petukhov (1970), and the influence of viscosity variations on condensation was explored by Field (1992). However, in deriving the correction factors, two inherent assumptions were made with respect to the thermal boundary layer. In following the classical heat transfer analysis they were: (i) the spatial variation of the temperature is linear in space and (ii) the shear stress is assumed to be constant. Improving on these two assumptions, was the motivation of the present work.

For a Newtonian fluid the theoretical analysis by Yang (1962) generated an index of 0.11. As noted by Joshi (1978) there is a substantial discrepancy between this result and that of Shannon and Depew (1969) who performed a similar analysis for the case of a uniform heat flux boundary condition; their results gave an index of 0.3 at the entrance to the tube decreasing to 0.14 in the fully developed region. Joshi (1978) concluded that further work was needed to clarify the situation but the focus of his own work was on non-Newtonian fluids. That Yang (1962) and Shannon and Depew (1969) had significantly different results may well be due to the former considering only heated tubes. The present work clearly considers the separate cases of a heated wall and a cold wall. In the present work, we have assumed that the thermal boundary layer is smaller than the hydrodynamic boundary layer,  $Pr > 1$ , which is true for most liquids (other than liquid metals) and concentrated upon Newtonian fluid in the laminar regime. We have considered here a fully developed hydrodynamic profile and a developing thermal boundary layer. Furthermore, for many liquids such as oils, the thermal boundary layer will be thin compared to the thickness of the hydrodynamic boundary layer because the ratio  $(\delta_T/\delta)$  depends upon  $Pr^{-1/2}$ . As a result much of the thermal boundary layer will be within the laminar sub-layer; we return to this point later.

## 2. Theoretical analysis

In classical thermal boundary layer analysis for the estimation of the heat transfer coefficient, a quadratic temperature profile is considered, as given by Schlichting and Gersten (2003), Hartnett et al. (1998), Kay & Nedderman (1985)

$$\theta = 2\tilde{y} - \tilde{y}^2, \quad (1)$$

where  $\theta = \frac{T_w - T}{T_w - T_\infty}$  is the scaled temperature,  $T_w$  is the boundary wall temperature and  $T_\infty$  is the bulk temperature. Here  $\tilde{y} = y/\delta_T$  is the scaled y-coordinate from the wall with respect to the thermal boundary layer thickness. In extending the classical heat transfer analysis of Kay and Nedderman(1985) which did not allow for spatial viscosity variations, Field (1990) made two assumptions:

- (i) to account for the linear temperature dependence on the viscosity of the fluid in the thermal layer ( $\tilde{y} < 1$ ), the viscosity is considered dependent on the spatial variable ( $\tilde{y}$ ) as

$$\frac{\mu}{\mu_w} = \exp(-\alpha\tilde{y}), \quad (2)$$

where  $\mu_w$  is the viscosity of the fluid at the wall ( $\tilde{y} = 0$ ) and  $\alpha$  is a function of the ratio of the change of the viscosity between the bulk and the wall such that  $\alpha = \ln(\mu_w/\mu_\infty)$ . When the wall cools the fluid,  $\alpha > 0$ ; and when the fluid is heated  $\alpha < 0$ .

- (ii) The shear stress,  $\tau$  is considered constant within the thermal boundary layer,  $0 < \tilde{y} < 1$ , which is reasonable if the velocity is linear within the thermal boundary layer, and holds satisfactorily for large Prandtl number.

However, for  $Pr > 1$ , which is true for most liquid flows (Lienhard IV & Lienhard V, 2017), we relax both of the above assumptions. We define the viscosity dependence on temperature of the fluid as

$$\frac{\mu}{\mu_w} = \exp(-\alpha\theta) = \exp[-\alpha(2\tilde{y} - \tilde{y}^2)], \quad (3)$$

and considering an isothermal flow of a Newtonian fluid, the shear stress linearly decreases with distance from the wall

$$\tau = \tau_w(1 - \tilde{y}), \quad (4)$$

where  $\tau_w$  is the wall shear stress. Using the stress-strain constitutive relationship of a Newtonian fluid  $\tau = \mu \frac{du}{dy}$ , together with Eqs. (3) and (4), we can write

$$(1 - \tilde{y}) = \exp(-\alpha\theta) \frac{d\tilde{u}}{d\tilde{y}}, \quad (5)$$

where  $\tilde{u} = u \frac{\mu_w}{\tau_w \delta_T}$ . The standard expression for the thermal thickness is given by (Kay and Nedderman, 1985)

$$\theta_T = \frac{\tau_w \delta_T^2}{\mu_w u_1} \int_0^1 \tilde{u}(1 - \theta) d\tilde{y}, \quad (6)$$

where  $u_1$  is the maximum fluid velocity scale. Balancing heat conduction across the wall to the net outflow of enthalpy, the heat transfer analogue of the momentum equation in the boundary layer is, as noted in Lienhard IV & Lienhard V (2017):

$$\frac{k}{\delta_T} \left( \frac{d\theta}{d\tilde{y}} \right)_{\tilde{y}=0} = \rho c_p u_1 \frac{d\theta_T}{dx}, \quad (7)$$

where  $k$  is the thermal conductivity of the wall,  $\rho$  is the fluid density (assumed constant) and  $c_p$  is the specific heat transfer capacity of the fluid. Eq. (7) is analogous to thermal energy conservation in that there is a balance between the rate of thermal energy being carried away by the boundary layer flow and the rate of heat transfer at the wall. Since,  $\left( \frac{d\theta}{d\tilde{y}} \right)_0 = 2$  from Eq. (1), we can integrate Eq. (7), to find, using Eq. (6), an expression for  $\delta_T$ :

$$\delta_T = \left[ \frac{3\mu_w k x}{\rho c_p \tau_w \psi_0} \right]^{\frac{1}{3}}, \quad (8)$$

where  $\psi_0 = \int_0^1 \tilde{u}(1 - \theta) d\tilde{y}$  and  $\tilde{u}$  is obtained from Eq. (5). By definition, the heat transfer coefficient ( $h$ ) is related to the heat flux across the wall ( $q_0$ ) as

$$h(T_w - T_\infty) = q_0 = k \left( \frac{dT}{dy} \right)_0. \quad (9)$$

Thus, the expression for the average heat transfer coefficient across the laminar fluid film is

$$\bar{h} = \frac{1}{L} \int_0^L h dx = \left( \frac{3\rho c_p k^2}{4L} \right)^{\frac{1}{3}} \left( \frac{12\tau_w}{\mu_w} \psi_0 \right)^{\frac{1}{3}}. \quad (10)$$

Introducing the dimensionless groups  $Re = \frac{\rho u_m d_e}{\mu_\infty}$  and  $Nu = \frac{\bar{h} d_e}{k}$ , and taking the wall shear stress

$$\tau_w = \mu_\infty \left( \frac{du}{dy} \right)_0 = \frac{8u_m \mu_\infty}{d_e}, \quad (11)$$

applicable for a laminar flow in a pipe, we can rearrange Eq. (10) in the form

$$Nu = 1.816 \left( RePr \frac{de}{L} \right)^{\frac{1}{3}} F = 1.816 \left( RePr \frac{de}{L} \right)^{\frac{1}{3}} \left( 12\psi_0 \frac{\mu_\infty}{\mu_w} \right)^{\frac{1}{3}}, \quad (12)$$

where  $d_e$  is the hydraulic diameter,  $u_m$  is the mean fluid velocity and the correction factor

$F = \left( 12\psi_0 \frac{\mu_\infty}{\mu_w} \right)^{\frac{1}{3}}$ . Eq. (12) is the theoretical alternative to the famous Sieder-Tate (1936)

correlation for heat transfer in a laminar flow

$$Nu = 1.86 \left( RePr \frac{de}{L} \right)^{\frac{1}{3}} \left( \frac{\mu_\infty}{\mu_w} \right)^{0.14}. \quad (13)$$

## 2.1 Determination of $\psi_0$

The key to obtaining a numerical estimate for  $Nu$  in Eq. (12) is the calculation of  $\psi_0$  which is

$$\psi_0 = \int_0^1 (1 - \theta) \tilde{u} d\tilde{y} = \int_0^1 (1 - \theta) \left[ \int_0^{\tilde{y}} (1 - \tilde{y}) \exp(\alpha\theta) d\tilde{y} \right] d\tilde{y}, \quad (14)$$

where  $\theta$  is defined by Eq. (1) and  $\tilde{u}$  is obtained from Eq. (5). The analytical solution of Eq. (14) can be given in terms of the error function, for wall cooling ( $\alpha > 0$ )

$$\psi_0 = \frac{1}{24\alpha} \left[ \frac{3e^\alpha \sqrt{\pi} \operatorname{erf}(\sqrt{\alpha})}{\alpha^{3/2}} - \frac{6}{\alpha} - 4 \right], \quad (15)$$

where  $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$ , and for wall heating ( $\alpha < 0$ )

$$\psi_0 = \frac{1}{24\alpha} \left[ -\frac{3e^\alpha \sqrt{\pi} \operatorname{erfi}(\sqrt{|\alpha|})}{|\alpha|^{3/2}} - \frac{6}{\alpha} - 4 \right], \quad (16)$$

where  $\operatorname{erfi}(z)$  is the imaginary error function,  $\operatorname{erfi}(z) = -i \operatorname{erf}(iz)$ . However, it must be emphasized that since the error functions cannot be simplified further, one cannot obtain greater physical insight regarding the influence of the viscosity variation upon Eq. (12).

Consequently, we approximate the quadratic profile of  $\theta$  with piecewise linear functions. For example, in two equal intervals in  $\tilde{y}$ ,  $\theta$  is represented by two piecewise linear functions (being continuous at  $\tilde{y} = 1/2$ ) as

$$\theta = \begin{cases} \frac{3}{2}\tilde{y} & \forall 0 \leq \tilde{y} \leq \frac{1}{2} \\ \frac{1}{2}(1 + \tilde{y}) & \forall \frac{1}{2} \leq \tilde{y} \leq 1 \end{cases} \quad (17)$$

leading to the explicit expression,

$$\psi_0 = \frac{1}{216\alpha^4} \left[ 4(3\alpha^2 + 20\alpha + 48) \exp\left(\frac{3}{4}\alpha\right) - 45\alpha^3 - 126\alpha^2 - 224\alpha - 192 \right] + \frac{1}{8\alpha^4} \left[ 192 \exp(\alpha) - \alpha^2(\alpha + 2) \exp\left(\frac{\alpha}{2}\right) - 4(\alpha^2 + 12\alpha + 48) \exp\left(\frac{3}{4}\alpha\right) \right], \quad (18)$$

where  $\alpha = \ln\left(\frac{\mu_w}{\mu_\infty}\right)$  as already defined in Eq. (2). For  $\alpha < 1$ , we can have a series expansion of the exponentials in Eq. (18) up to  $O(\alpha^5)$  in the numerator, and thus represent the correction factor  $F$  in Eq. (12) as

$$F = \left(12 \frac{\mu_\infty}{\mu_w} \psi_0\right)^{\frac{1}{3}} \approx \left[\frac{11}{16} e^{-\alpha} \left(1 + \frac{21}{88}\alpha\right)\right]^{\frac{1}{3}} \approx 0.882 \left[e^{-\alpha} e^{\frac{21}{88}\alpha}\right]^{\frac{1}{3}} \approx 0.882 \left(\frac{\mu_\infty}{\mu_w}\right)^{0.254}. \quad (19)$$

which contrasts with the correction factor in the Sieder-Tate relation (Eq. 13) but is close to the relation reported by Petukhov (1970) for the case of cooling duties (where the exponent reported was 0.25). The above relation gives an insight into heat transfer for all cooling conditions ( $\alpha < 0$ ). In the case of heating, the viscosity close to the wall is less than the bulk viscosity, and it is counterintuitive to consider Eq. (11) for  $\tau_w$ . If the value of  $\tau_w$  was based on the bulk viscosity, the implication is that the velocity gradient would continuously increase as  $y \rightarrow 0$  because  $\mu_w$  is smaller than the mainstream viscosity ( $\mu_\infty$ ). It is probable that under these conditions the wall shear stress ( $\tau_w$ ) would be based upon the mean (geometric) value  $\sqrt{\mu_\infty \mu_w}$ . Thus, in this case of heating, the correction factor  $F$  in Eq. (19) becomes

$$F = 0.882 \left[e^{-\frac{\alpha}{2}} e^{\frac{21}{88}\alpha}\right]^{\frac{1}{3}} = 0.882 \left(\frac{\mu_\infty}{\mu_w}\right)^{0.0871} \quad (20)$$

Calculations for the solution of  $\psi_0$  by approximating it using three piecewise linear functions are presented in Appendix A. Moreover, a generalised solution of  $\psi_0$  considering  $n$  piecewise linear functions, using a recursive technique is also described in Appendix B.

Another aspect to the problem, is what happens when we consider the piecewise approximations as equal divisions in  $\theta$  instead of  $\tilde{y}$  (as presented before). To analyse this, we consider two equal intervals in  $\left(\text{being continuous at } \tilde{y} = 1 - \frac{1}{\sqrt{2}}\right)$ , represented by two piecewise linear functions

$$\theta = \begin{cases} \frac{1}{2-\sqrt{2}} \tilde{y} & \forall 0 \leq \tilde{y} \leq 1 - \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} (\tilde{y} - 1) + 1 & \forall 1 - \frac{1}{\sqrt{2}} \leq \tilde{y} \leq 1 \end{cases} \quad (21)$$

Proceeding as before, but using the above form in the integration in Eq. (14), and subsequently performing a series expansion of the exponentials (for  $\alpha < 1$ ) we have (for a cold wall)

$$F = 0.83 \left( \frac{\mu_\infty}{\mu_w} \right)^{0.215}, \quad (22)$$

and in case of heating (or hot wall), following the analysis above for the wall shear stress,

$$F = 0.83 \left( \frac{\mu_\infty}{\mu_w} \right)^{0.049}. \quad (22)$$

Values of the correction factors ( $F$ ) following series expansions (for  $\alpha < 1$ ) in the case of heating and cooling for various number of piecewise approximations with equal divisions in  $\tilde{y}$  and  $\theta$  are presented in Table 1.

**Table 1:** Correction factors ( $F$ ) obtained using different piecewise linear approximations of the quadratic profile in  $\theta$ , using a series expansion for small  $\alpha$  ( $< 1$ ).

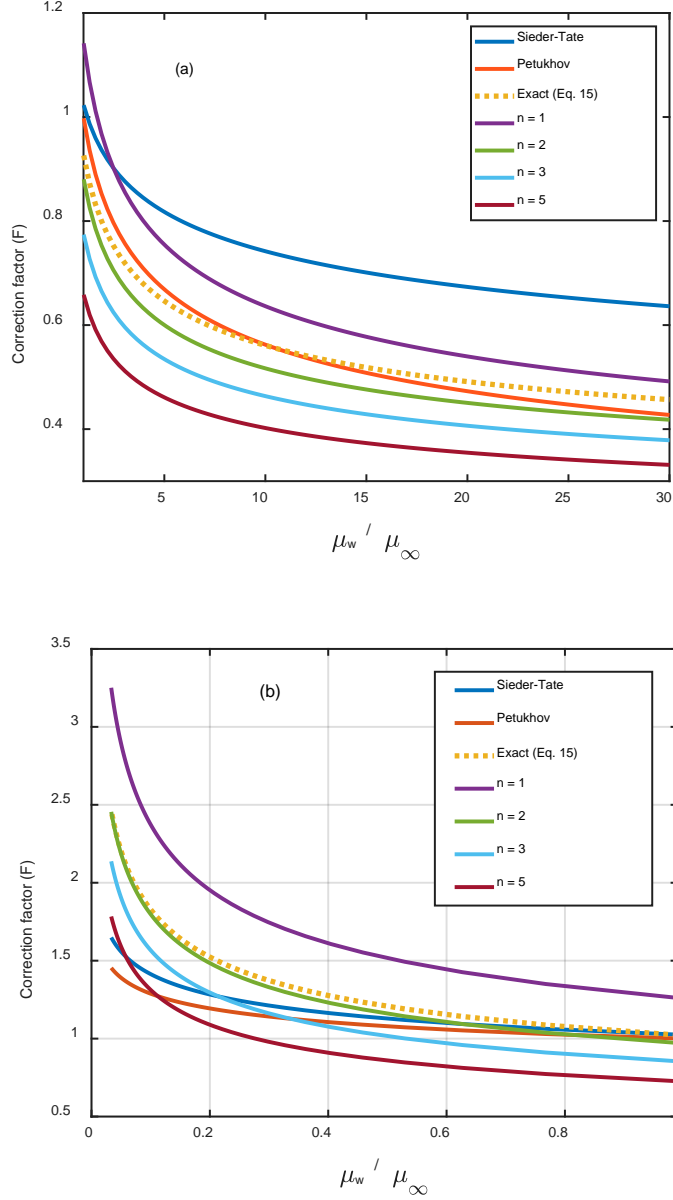
Number of piecewise linear approximations	Cooling (cold wall)		Heating (hot wall)	
	Equal $\tilde{y}$ division	Equal $\theta$ division	Equal $\tilde{y}$ division	Equal $\theta$ division
1	$1.14(\mu_\infty/\mu_w)^{0.267}$	Same	$1.14(\mu_\infty/\mu_w)^{0.1}$	Same
2	$0.88(\mu_\infty/\mu_w)^{0.254}$	$0.83(\mu_\infty/\mu_w)^{0.215}$	$0.88(\mu_\infty/\mu_w)^{0.087}$	$0.83(\mu_\infty/\mu_w)^{0.049}$
3	$0.78(\mu_\infty/\mu_w)^{0.247}$	$0.70(\mu_\infty/\mu_w)^{0.227}$	$0.78(\mu_\infty/\mu_w)^{0.080}$	$0.70(\mu_\infty/\mu_w)^{0.061}$
4	$0.71(\mu_\infty/\mu_w)^{0.242}$	$0.83(\mu_\infty/\mu_w)^{0.206}$	$0.71(\mu_\infty/\mu_w)^{0.076}$	$0.83(\mu_\infty/\mu_w)^{0.039}$

### 3. Discussion

We would like to re-emphasize here that we have considered a fully developed hydrodynamic profile. Fig. 1a compares the results of the different approaches against the exact solution for a wide range of  $\frac{\mu_\infty}{\mu_w}$  values. The curves (based on the piecewise approximations) presented here are for any general value of  $\alpha$  (and are not restricted to a small range, as in the

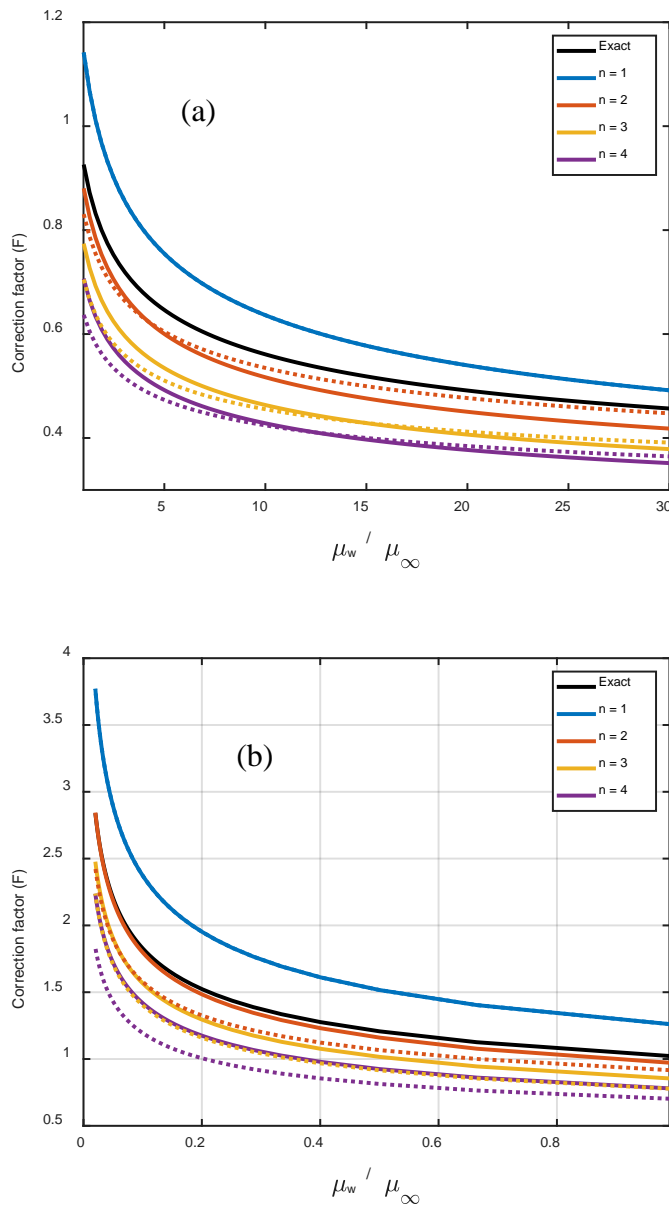


case of series expansions). For small  $\alpha$ , the curves relate closely to the corresponding relations stated in Table 1 (equal  $\tilde{y}$  divisions). Interestingly the correlation developed by Petukhov (1970) closely matches the exact solution, for a cold wall cooling the fluid ( $\alpha > 0$ ); the range of  $\frac{\mu_\infty}{\mu_w}$  values certainly covers the range found in practice. With a quadratic temperature profile three quarters of the temperature change, and thus (given the exponential relationship between viscosity and temperature) more than three quarters of the viscosity variation, occurs across half of the thermal boundary layer. It is well known that for  $Pr > 1$ , the thermal boundary layer is thin because its thickness to that of the hydrodynamic boundary layer ( $\delta_T/\delta$ ) depends upon  $Pr^{-1/2}$  (Nellis and Klein, 2009). Thus for many liquids, but not gases, much of the thermal boundary layer will be within the laminar sub-layer (e.g. if  $Pr > 100$ , three quarters of  $\delta_T$  will be in a region that is  $< 0.05\delta$ ). We therefore do not find it surprising that the correlation developed by Petukhov (1970), principally with data for turbulent flow closely matches the exact solution for a cold wall cooling a fluid ( $\alpha > 0$ ) because Petukhov's data varied in the range  $130 < Pr < 140$ . In the present work, the results of the two-piecewise approximation of the temperature profile hold best for both the heating and cooling scenarios. However, as the number of piecewise approximations is increased, the comparison with the exact solution worsens. This occurs because of increasing the number of piecewise functions of the temperature profile ( $\theta$ ) does not improve the value of the integral function used in the Eq. (14). For heating (Fig. 1b), the result obtained using the two-piecewise relation lies almost over (on top of) the exact numerical solution, and the correction factor is  $\left(\frac{\mu_\infty}{\mu_w}\right)^{0.087}$  (Eq. 20) for small  $\alpha$ . It may be noted that the predictions by Petukhov (1970) for the heated wall do not hold satisfactorily in the case of high wall temperature, when the dynamic viscosity at the wall is much smaller than the bulk viscosity. For a single piecewise function, the quadratic profile is just approximated as linear, and the results are similar to the analysis made by Field (1990). The results from the (single) linear approximation ( $n = 1$ ) always overpredict (in both hot and cold wall cases) the correction factor and hence the heat transfer coefficient. Also, it can be observed from both the figures that the predictions of the correction factor by Sieder-Tate relation are better for a heated rather than a cold wall.



**Fig. 1:** (a) Viscosity correlation factors for a cold wall, cooling the fluid, solving Eq. (14) in case of  $\alpha > 0$  and (b) for a hot wall heating the fluid ( $\alpha < 0$ ). The correction factor by Petukhov (1970) is  $\left(\frac{\mu_\infty}{\mu_w}\right)^{0.25}$  for cooling and  $\left(\frac{\mu_\infty}{\mu_w}\right)^{0.11}$  for heating. The Sieder-Tate relation for viscosity correction is  $\left(\frac{\mu_\infty}{\mu_w}\right)^{0.14}$  for both heating and cooling. Exact solutions are given by Eq. (15) and (16) for the cold and hot wall respectively. Note that the curves plotted are for any general value of  $\alpha$  (and not restricted to only the small range, as in the case of the series expansions).

We next examine whether having equal divisions in  $\theta$  instead of equal divisions in  $\tilde{y}$  for the piecewise approximations, improves the fit to the exact solution. Fig. 2 provides quantitative comparison of the results obtained from the two approaches (equal divisions in  $\tilde{y}$  and  $\theta$ ). Fig. 2a shows that for cooling, the predictions using equal divisions in  $\theta$  are marginally close to the exact profile compared to the corresponding results for equal divisions in  $\tilde{y}$ , when  $\frac{\mu_w}{\mu_\infty}$  is larger than 10 for the two piecewise approximations. However in the case of a hot wall, the results of the equal divisions in  $\theta$  are inferior by comparison to the equal  $\tilde{y}$  divisions.



**Fig. 2:** Viscosity correction factors for the case of (a) cooling and (b) heating using different numbers of piecewise linear approximations with equal divisions in  $\tilde{y}$  (solid lines in the

figure) and equal divisions in  $\theta$  (dotted lines in the figure). Note that the curves plotted are for any general value of  $\alpha$  (and are not restricted to small values, as with the case of series expansions).

As a penultimate discussion point it must be mentioned that the reason why the numerical results match better for two piecewise linear approximations to the quadratic  $\theta$  profile is not understood from a physical view and remains an open question. Finally it is suggested, following an observation of Whitaker (1972) that the main reason that the Sieder-Tate relationship has been widely used in engineering design is that, in the course of the design, construction, and operation of most heat transfer equipment there will be considerable uncertainty and there is therefore little need for sophisticated correlations. Simple correction factors of the same form are still appropriate today. However a distinction between the heating and cooling cases is justifiable theoretically, reflects the data with more accuracy and is straightforward.

#### 4. Conclusions

This study presents the results of the classical analysis of the viscosity correction factor for both cold and hot walls, for  $Pr > 1$ , which applies to most liquids (other than liquid metals). Considering the actual quadratic (spatial) temperature profile, an exact solution of the heat transfer coefficient is obtained, and is compared to previously developed relations by Sieder-Tate (1936) and Petukhov (1970), and to solutions using the piecewise approximations of the quadratic temperature profile. Key conclusions are:

- (i) An exact solution of the heat transfer coefficient is obtained considering the viscosity variation effect as given in Eqs. (15, 16) following a first principles analysis using boundary layer theory (Kay and Nedderman, 1985).
- (ii) The relations by Petukhov (1970), which provide correction factors of  $\left(\frac{\mu_\infty}{\mu_w}\right)^{0.11}$  for cooling and  $\left(\frac{\mu_\infty}{\mu_w}\right)^{0.25}$  for heating are better than the Sieder-Tate factors, because these are closer to the exact solutions.
- (iii) Use of a two-piecewise linear approximation (in equal division of the spatial coordinate) for the quadratic temperature profile gives a better result than either the linear approximation or an increasing number of piecewise linear functions. The physical

basis for **this** cannot currently be explained. Using the two-piecewise approximation, we have found that  $\left(\frac{\mu_\infty}{\mu_w}\right)^{0.254}$  (for cooling) and  $\left(\frac{\mu_\infty}{\mu_w}\right)^{0.087}$  is better than the Sieder-Tate factor of  $\left(\frac{\mu_\infty}{\mu_w}\right)^{0.14}$  (for both heating and cooling).

- (iv) Use of an equal division with respect to the temperature variable rather than the spatial ordinate, did not improve the predictions.
- (v) Viscosity correction factors in the form of  $\left(\frac{\mu_\infty}{\mu_w}\right)^p$  can only be obtained for small values of  $\ln\left(\frac{\mu_w}{\mu_\infty}\right)$  for a cold wall and  $\ln\left(\frac{\mu_\infty}{\mu_w}\right)$  for a hot wall (typically less than 1). The generalised solutions reported in this work for large  $\left(\frac{\mu_w}{\mu_\infty} \text{ or } \frac{\mu_\infty}{\mu_w} \leq 30\right)$  values of the viscosity ratios.

## Nomenclature

$c_p$	Specific heat transfer capacity of the fluid, $J.kg.K$
$d_e$	Hydraulic diameter of the pipe, $m$
$F$	Correction factor, as defined in Eq. (12)
$h$	Heat transfer coefficient, $W/m^2K$
$\bar{h}$	Spatially averaged heat transfer coefficient, $W/m^2K$
$k$	Thermal conductivity of the boundary wall, $W/m.K$
$Nu$	Nusselt number, defined as $\frac{\bar{h}d_e}{k}$
$p$	Exponent for the viscosity relation $\mu_\infty/\mu_w$
$Pr$	Prandtl number, defined as $c_p\mu_\infty/k$
$q_0$	Heat flux across the wall, $W/m^2$
$Re$	Reynolds number, defined as $\frac{\rho u_m d_e}{\mu_\infty}$
$T$	Temperature of fluid, $K$
$T_\infty$	Temperature in the bulk fluid, $K$
$T_w$	Temperature of the boundary wall, $K$
$u$	Fluid velocity, $m/s$
$\tilde{u}$	Non-dimensional fluid velocity, equals $u \frac{\mu_w}{\tau_w \delta_T}$

$u_1$	Maximum fluid velocity, $m/s$
$u_m$	Mean fluid velocity in the channel, $m/s$
$x$	Abscissa of the coordinate, $m$
$y$	Co-ordinate from the wall, $m$
$\tilde{y}$	Scaled $y$ -coordinate with respect to $\delta_T$

#### *Greek symbols*

$\alpha$	Logarithm of the viscosity ratio, equals $\ln(\mu_w/\mu_\infty)$
$\delta$	Hydrodynamic boundary layer thickness, $m$
$\delta_T$	Thermal boundary layer thickness, $m$
$\theta$	Non-dimensional temperature, equals $\frac{T_w - T}{T_w - T_\infty}$
$\mu$	Viscosity of the fluid, $Pa \cdot s$
$\mu_\infty$	Viscosity of the bulk fluid, $Pa \cdot s$
$\mu_w$	Viscosity of the fluid at the wall ( $y = 0$ ), $Pa \cdot s$
$\rho$	Density of the fluid, $kg/m^3$
$\tau$	Shear stress of the fluid, $Pa$
$\tau_w$	Wall shear stress, $Pa$
$\psi_0$	Non-dimensional parameter, equals $\int_0^1 \tilde{u}(1 - \theta)d\tilde{y}$

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## Appendix A: Viscosity correction factor (F) using 3 piecewise linear approximations

The three piecewise linear approximations,

$$\theta = \begin{cases} \frac{5}{3}\tilde{y} & \forall 0 \leq \tilde{y} \leq \frac{1}{3} \\ \tilde{y} + \frac{2}{9} & \forall \frac{1}{3} \leq \tilde{y} \leq \frac{2}{3} \\ \frac{\tilde{y}}{3} + \frac{2}{3} & \forall \frac{2}{3} \leq \tilde{y} \leq 1 \end{cases} \quad (\text{A.1})$$

lead to

$$\psi_0 = -\frac{1}{20250\alpha^4} [182250(4\alpha - 27)e^\alpha - 375(9\alpha^3 + 137\alpha^2 + 516\alpha - 12960)e^{8\alpha/9} + (1250\alpha^3 + 5715\alpha^2 + 22752\alpha + 47628)e^{5\alpha/9} + 9(325\alpha^3 + 1005\alpha^2 + 1728\alpha + 1458)]. \quad (\text{A.2})$$

Considering an asymptotic expansion of  $\psi_0$  for  $\alpha$  small but  $> 0$  (cooling), we can represent  $F$  in Eq. (11) as

$$F = \left(12 \frac{\mu_\infty}{\mu_w} \psi_0\right)^{\frac{1}{3}} \approx 0.78(\mu_\infty/\mu_w)^{0.247}. \quad (\text{A.3})$$



## Appendix B: Recursive relation for $\theta_{kn}$ and generalised solution of $\psi_0$ based on $n$ piecewise linear functions

We know from Eq. (1) that  $\theta = 2\tilde{y} - \tilde{y}^2$  and the boundary conditions are  $\theta(1) = 1$  and  $\theta(0) = 0$ . Considering the notation for  $\theta_{kn}$  where  $k$  represents the sequence in the  $n$  –piecewise linear approximations

$$\theta_{11} = \theta(1)\tilde{y}, \quad (\text{B.1})$$

$$\theta_{12} = \frac{\theta(1/2)\tilde{y}}{1/2}, \quad (\text{B.2})$$

$$\text{and so, } \theta_{1n} = \frac{\theta(1/n)\tilde{y}}{1/n} = n \left[ \frac{2}{n} - \left( \frac{1}{n} \right)^2 \right] \tilde{y} = \left( 2 - \frac{1}{n} \right) \tilde{y}. \quad (\text{B.3})$$

Now, using the correlation of matching slopes for the equation of a line connecting two points, we can say

$$\theta_{22} = \left[ \frac{\theta(1) - \theta(1/2)}{1/2} \right] \left( \tilde{y} - \frac{1}{2} \right) + \theta_{12} \left( \frac{1}{2} \right), \quad (\text{B.4})$$

$$\theta_{23} = \left[ \frac{\theta(2/3) - \theta(1/3)}{1/3} \right] \left( \tilde{y} - \frac{1}{3} \right) + \theta_{13} \left( \frac{1}{3} \right), \quad (\text{B.5})$$

$$\text{and so, } \theta_{2n} = \left[ \frac{\theta(2/n) - \theta(1/n)}{1/n} \right] \left( \tilde{y} - \frac{1}{n} \right) + \theta_{1n} \left( \frac{1}{n} \right) = \left( 2 - \frac{3}{n} \right) \tilde{y} + \frac{2}{n^2}. \quad (\text{B.6})$$

Similarly, for  $k = 3$ ,

$$\theta_{33} = \left[ \frac{\theta(1) - \theta(2/3)}{1 - 2/3} \right] \left( \tilde{y} - \frac{2}{3} \right) + \theta_{23} \left( \frac{2}{3} \right) \quad (\text{B.7})$$

$$\theta_{34} = \left[ \frac{\theta(3/4) - \theta(2/4)}{3/4 - 2/4} \right] \left( \tilde{y} - \frac{2}{4} \right) + \theta_{24} \left( \frac{2}{4} \right) \quad (\text{B.8})$$

$$\text{and so, } \theta_{3n} = \left[ \frac{\theta(3/n) - \theta(2/n)}{1/n} \right] \left( \tilde{y} - \frac{2}{n} \right) + \theta_{2n} \left( \frac{2}{n} \right) = \left( 2 - \frac{5}{n} \right) \tilde{y} + \frac{6}{n^2}. \quad (\text{B.9})$$

Therefore, we can represent, for  $1 \leq k \leq n$

$$\theta_{kn} = \left[ 2 - \frac{2k-1}{n} \right] \tilde{y} + \frac{k(k-1)}{n^2} = [2(n-k) + 1] \frac{\tilde{y}}{n} + \frac{k(k-1)}{n^2} \quad (\text{B.10})$$

where  $n$  is the number of piecewise functions and  $k(\leq n)$  is the sequence or the  $k^{\text{th}}$  piecewise function from  $\tilde{y} = 0$ . Hence, from Eq. (14) we get

$$\psi_0 = \sum_{\substack{k=1 \\ k \leq n}}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} (1 - \theta_{kn}) \left[ \int_{\frac{k-1}{n}}^{\tilde{y}} (1 - \tilde{y}) \exp(\alpha \theta_{kn}) d\tilde{y} \right] d\tilde{y} = \sum_{\substack{k=1 \\ k \leq n}}^n \zeta(k, n, \alpha), \quad (\text{B.12})$$

where  $\zeta(k, n, \alpha) = \frac{1}{2(2(k-n)-1)^3 n^3 \alpha^4} e^{\frac{(k-1)(2n-k+1)\alpha}{n^2}} [A_1 + A_2 + A_3 + A_4 + A_5 + A_6]$ , for

$$A_1 = \alpha^3 \{n(40k^4 - 96k^3 + 90k^2 - 40k + 7) - (2k-1)^2(2k^3 - 4k^2 + 3k - 1)\},$$

$$A_2 = -2n^6 \left( e^{\frac{\alpha(2n-2k+1)}{n^2}} - 1 \right) (2\alpha^2 + 4\alpha + 3),$$

$$A_3 = 2n^5 \alpha \left\{ 4\alpha^2 + (\alpha + 1)(8k - 1) e^{-\frac{\alpha(2k-2n-1)}{n^2}} - 8k\alpha + 9\alpha - 8k + 7 \right\},$$

$$A_4 = 2n^3 \alpha^2 \left\{ 7 + 8 \left( e^{\frac{\alpha(2n-2k+1)}{n^2}} - 1 \right) k^3 + 15\alpha - 24k(1 + 2\alpha) + k^2 \left( 27 - 3e^{\frac{\alpha(2n-2k+1)}{n^2}} + 40\alpha \right) \right\},$$

$$A_5 = -n^2 \alpha^2 \left\{ 4 \left( e^{\frac{\alpha(2n-2k+1)}{n^2}} - 1 \right) k^4 - 2k^3 \left( e^{\frac{\alpha(2n-2k+1)}{n^2}} - 40\alpha - 9 \right) - 20\alpha - 24k^2(6\alpha + 1) + 2k(45\alpha + 7) - 3 \right\},$$

$$A_6 = -2n^4 \alpha \left\{ 4k^2(1 + 3\alpha) \left( e^{\frac{\alpha(2n-2k+1)}{n^2}} - 1 \right) - 3(1 + 2\alpha)^2 + k \left( 7 + 27\alpha + 20\alpha^2 - (1 + 3\alpha) e^{\frac{\alpha(2n-2k+1)}{n^2}} \right) \right\}.$$

The above expression for  $\psi_0$  is a generic solution for any number of (linear) piecewise functions