

Causal random geometry from stochastic quantization

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Abstract. In this short note we review a recently found formulation of two-dimensional causal quantum gravity defined through Causal Dynamical Triangulations and stochastic quantization. This procedure enables one to extract the nonperturbative quantum Hamiltonian of the random surface model including the sum over topologies. Interestingly, the generally fictitious stochastic time corresponds to proper time on the geometries.

1. Introduction

Two-dimensional random geometry can either be viewed as a simple toy model for a string theory of fundamental particles or as a playground for four-dimensional quantum gravity. The simplification from the string theory point of view is that the two-dimensional random surfaces we discuss are not embedded in an ambient space, only the intrinsic two-dimensional geometry of the string is considered. Although this removes many features of a full blown string theory, it allows us to focus on the signature of the metric on the surface, also called worldsheet, of the propagating string. In traditional approaches to two-dimensional random geometry, matrix models and Liouville theory, the signature of the metric is considered to be Euclidean whereas genuine string theory is presumed to possess Lorentzian signature worldsheets. Causal Dynamical Triangulations (CDT) describes the random geometry of two-dimensional surfaces with Lorentzian signature without using coordinates, hence avoiding complications due to gauge invariance [1]. The essential difference between causal surfaces and Euclidean surfaces is that the time dependent process where a string splits into two strings is nonsingular in the Euclidean case and singular in the Lorentzian case. This leads to a different assignment of coupling constants and a markedly different continuum theory [2]. In this short note we review a recent observation [3] that the time on the Lorentzian surfaces can be identified with the “fifth time” variable of the stochastic quantization method introduced by Parisi and Wu [4]. Remarkably the splitting and merging dynamics of the string with respect to this stochastic time can be described by an *exactly solvable* Schrödinger like equation [3]! In other words the method of stochastic quantization provides a natural nonperturbative definition of the quantum dynamics of causal random surfaces of arbitrary topology.

2. Stochastic quantization

Stochastic quantization was introduced by Parisi and Wu [4] in 1981 as a quantization scheme for Euclidean field theories (see [5] for a standard review and [6] for an interesting more recent account). The basic idea is to start off with a statistical physics system which evolves according to a, in general, fictitious time t and to observe that there exists a correspondence between certain correlation functions in the equilibrium statistical physics model at $t \rightarrow \infty$ and propagators in the Euclidean field theory. We will outline this correspondence in the simplest case of zero-dimensional Euclidean field theory which is sufficient for our purposes.

Consider a time dependent random variable X_t with initial value $X_0 = x_0$ whose time evolution is given by the Langevin equation

$$\frac{dX_t}{dt} = -f(X_t) + \sqrt{2\Omega} \nu(t), \quad f(x) = \frac{\partial S(x)}{\partial x}, \quad (1)$$

where $\nu(t)$ is Gaussian white noise of width one and $f(x)$ is a conservative drift. Let us now consider the probability density for $X_t = x$ given that $X_0 = x_0$,

$$P(x, x_0, t) = E_\nu \{ \delta(x - X_t) | X_0 = x_0 \}, \quad (2)$$

where E_ν refers to the expectation value with respect to the Gaussian white noise $\nu(t)$. From the Langevin equation (1) one can then derive the so-called Fokker-Planck equation for the probability density

$$\frac{\partial}{\partial t} P(x, x_0, t) = \frac{\partial}{\partial x} \left(\frac{\partial S}{\partial x} + \Omega \frac{\partial}{\partial x} \right) P(x, x_0, t). \quad (3)$$

From this one can directly verify that for $t \rightarrow \infty$,

$$\lim_{t \rightarrow \infty} P(x, x_0, t) = P^{eq}(x) = \frac{1}{Z} e^{-\frac{1}{\Omega} S(x)}, \quad Z = \int dx e^{-\frac{1}{\Omega} S(x)} \quad (4)$$

which is the correct measure arising from the Euclidean path integral with classical action $S(x)$ and thus points towards the correspondence with Euclidean field theory. Here Ω can be seen as the analog of \hbar in the quantum mechanical setting.

In principle one can now use (3) to extract the effective quantum Hamiltonian of the model with classical action $S(x)$. In the following we use this procedure to deduce the effective quantum Hamiltonian of a model of two-dimensional causal quantum gravity which we introduce in the forthcoming section.

3. Two-dimensional causal quantum gravity

The Causal Dynamical Triangulation (CDT) approach to quantum gravity aims to define two-dimensional Lorentzian quantum gravity through a nonperturbative path integral over geometries $[g_{\mu\nu}]$

$$Z(\kappa, \lambda) = \int \mathcal{D}[g_{\mu\nu}] e^{-S[g_{\mu\nu}]}, \quad (5)$$

where the (Euclidean) Einstein-Hilbert action is given by

$$S[g_{\mu\nu}] = -\frac{1}{2\pi G_N} \int \sqrt{\det g_{\mu\nu}} R + \lambda \int \sqrt{\det g_{\mu\nu}}. \quad (6)$$

Here λ denotes the cosmological constant, G_N the Newton's constant and $\kappa = e^{-1/G_N}$ is the dimensionless string coupling.

In the lattice regularization of CDT the path integral is defined as a continuum limit of a sum over triangulated surfaces (see [1] or the recent review [7]). This is done still with Lorentzian

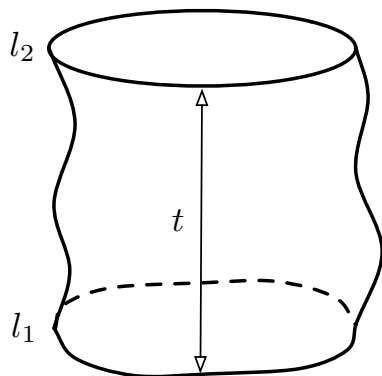


Figure 1. Illustration of the propagator $G_0(l_1, l_2; t)$, describing the amplitude from a fixed spatial boundary of length l_1 to a boundary of length l_2 in time t .

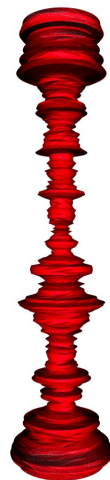


Figure 2. Snapshot from a Monte Carlo simulation of a two-dimensional CDT universe.

signature and a Wick rotation to Euclidean signature is performed at the level of the individual triangulations. In contrast to the Euclidean model defined through Dynamical Triangulations (DT), the Euclidean path integral of the Lorentzian model defined through CDT includes only geometries with a fixed time-sliced structure disallowing for spatial topology changes.

Let us denote the (genus zero) propagator from an initial marked spatial slice of length l_1 to a final unmarked spatial slice of length l_2 in proper time t by $G_0(l_1, l_2; t)$ (see Fig. 1). Let us for convenience also introduce the Laplace transformed propagator

$$G_0(x_1, x_2; t) = \int_0^\infty dl_1 \int_0^\infty dl_2 e^{-x_1 l_1 - x_2 l_2} G_0(l_1, l_2; t). \quad (7)$$

Using the CDT program one can derive the following differential equation for the Laplace transformed propagator

$$\frac{\partial}{\partial t} G_0(x_1, x_2; t) = \frac{\partial}{\partial x_1} \left((\lambda - x_1^2) G_0(x_1, x_2; t) \right). \quad (8)$$

In terms of the length of the spatial boundary the propagator can then also be written as

$$G_0(l_1, l_2; t) = \langle l_2 | e^{-t\mathcal{H}_0} | l_1 \rangle, \quad \mathcal{H}_0 = -l \frac{\partial^2}{\partial l^2} + \lambda l. \quad (9)$$

Here the effective quantum Hamiltonian describes the time evolution of the length of the spatial slice. In particular, one obtains that the average length of a spatial slice as well as its fluctuations are both of the same order $1/\sqrt{\lambda}$ as is also illustrated in Fig. 2.

4. Causal string field theory and stochastic quantization

Viewing each spatial slice as a string one can interpret the two-dimensional fluctuating surfaces considered in the previous section as so-called worldsheets of propagating strings. In principle one can then also allow strings to split and join, thus creating surfaces of arbitrary topology.

Such a so-called string field theory for CDT was introduced in [2, 8] where each splitting and joining of a string is weighted with a factor κ according to the action (6). Using this framework it was observed in [9, 10] that the partition function (5) can then be written as the following simple integral

$$Z(\kappa, \lambda) = \int dx \exp \left[-\frac{1}{g_s} \left(\lambda x - \frac{1}{3} x^3 \right) \right], \quad \kappa = \frac{g_s}{\lambda^{3/2}}, \quad (10)$$

where the integral is formal in the sense that it should be understood as a suitable expansion in powers of the dimensionless string coupling $\kappa = g_s/\lambda^{3/2}$.

In view of (3) and (4) this suggests that the effective quantum Hamiltonian can be derived from a stochastic quantization procedure defined by the following Langevin equation [3]

$$\frac{dX_t}{dt} = - \left. \frac{\partial S(x)}{\partial x} \right|_{x=X_t} + \sqrt{2g_s} \nu(t), \quad S(x) = \lambda x - \frac{1}{3} x^3. \quad (11)$$

The corresponding Fokker-Planck equation reads

$$\frac{\partial}{\partial t} P(x, x_0, t) = \frac{\partial}{\partial x} \left(\lambda - x^2 + g_s \frac{\partial}{\partial x} \right) P(x, x_0, t). \quad (12)$$

One can now directly identify $P(x, x_0, t)$ with the propagator $G(x, x_0, t)$ of the string field theory. By performing an inverse Laplace transformation of (12) one obtains the propagator and Hamiltonian in the “length representation”. These describe the transition amplitude of a string to propagate from a specific initial length l_1 to a certain final length l_2 ,

$$G(l_1, l_2; t) = \langle l_2 | e^{-t\mathcal{H}} | l_1 \rangle, \quad \mathcal{H} = -l \frac{\partial^2}{\partial l^2} + \lambda l - g_s l^2. \quad (13)$$

One observes that Eqs. (12) and (13) exactly yield (8) and (9) for $g_s = 0$. Hence we see that for $g_s = 0$ we exactly obtain the CDT propagator without any spatial or space-time topology changes. Furthermore, the stochastic time t which is in general a fictitious time variable exactly corresponds to the proper time on the worldsheets.

For $g_s > 0$ the effective quantum Hamiltonian (13) describes fluctuating surfaces where spatial slices are allowed to split and joint each weighted by a factor g_s . The corresponding amplitudes thus include a summation over all genera! It is interesting that (13) can be used to explicitly give solutions for some amplitudes. The simplest amplitude to look at is the disc function with is a propagator with one boundary shrunk to zero and integrated over time, i.e.

$$W(l) = \int_0^\infty dt G(l, l_2 = 0; t) \quad (14)$$

From (12) one can derive that $W(l)$ is determined by

$$\mathcal{H} W(l) = 0, \quad (15)$$

where \mathcal{H} is given in (13). This equation can be interpreted as a Wheeler-deWitt equation for the spatial slices. The explicit solution to this equation can be given in terms of Airy functions $\text{Bi}(\cdot)$ and reads

$$W(l) = \frac{\text{Bi} \left(\frac{\lambda}{g_s^{2/3}} - g_s^{1/3} l \right)}{\text{Bi} \left(\frac{\lambda}{g_s^{2/3}} \right)}. \quad (16)$$

This expression corresponds to the following genus expansion in the dimensionless string coupling $\kappa = g_s/\lambda^{3/2}$:

$$W(l) = \text{disc} + \kappa \text{cusp} + \kappa^2 \left(\text{split} + \text{join} \right) + \mathcal{O}(\kappa^3). \quad (17)$$

5. Conclusion

We presented a recently found stochastic quantization formulation of a “causal” random surface model defined through CDT. This formulation is reminiscent of an analogous formulation for Euclidean quantum gravity defined through DT [11, 12]. The stochastic quantization procedure enabled us to determine the effective quantum Hamiltonian (13) of the nonperturbative dynamics including the sum over topologies. A detailed physical interpretation of this Hamiltonian including a discussion on the problem of unboundedness can be found in [3].

In our model the external stochastic time is identified with proper time on the string worldsheets. This special role of an external time variable and unitarity which is inherent to the CDT formulation are very similar to the setup of Hořava-Lifshitz gravity [13] for which the concept of stochastic quantization might also be relevant. This points towards a possible relation between these two models. In fact, it was observed very recently that CDT might not only serve as a lattice regularization of the renormalization group approach to quantum gravity, but also for Hořava-Lifshitz gravity [14].

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