CONVERGENCE AND STABILITY RESULTS FOR THE PARTICLE SYSTEM IN THE STEIN GRADIENT DESCENT METHOD

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Abstract. There has been recently a lot of interest in the analysis of the Stein gradient descent method, a deterministic sampling algorithm. It is based on a particle system moving along the gradient flow of the Kullback-Leibler divergence towards the asymptotic state corresponding to the desired distribution. Mathematically, the method can be formulated as a joint limit of time t and number of particles N going to infinity. We first observe that the recent work of Lu, Lu and Nolen (2019) implies that if $t = O(\log(\log N))$, then the joint limit can be rigorously justified in the Wasserstein distance. Not satisfied with this time scale, we explore what happens for larger times by investigating the stability of the method: if the particles are initially close to the asymptotic state, with distance $O(1/N)$, how long will they remain close? We prove that this happens in algebraic time scales $t = O(\sqrt{N})$ which is significantly better. The exploited method, developed by Caglioti and Rousset for the Vlasov equation, is based on finding a functional invariant for the linearized equation. This allows to eliminate linear terms and arrive at an improved Grönwall-type estimate.

1. Introduction

The Stein gradient descent method is a recently extensively studied algorithm [6, 8, 11, 15, 20, 22, 24–28, 30, 33, 36, 38, 40–42] to sample the probability distribution $\rho_\infty := e^{-V(x)}/Z$ when the normalization constant $Z = \int_{\mathbb{R}^d} e^{-V(x)}\,\mathrm{d}x$ is unknown or difficult to compute. A prominent example is the Bayesian inference [37] used to fit the parameters $\theta \in \Theta$ based on the data D and the a priori distribution of parameters $\pi(\theta)$: the a posteriori distribution

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is given by

$$\mathbb{P}(\theta|D) = \frac{\mathbb{P}(D|\theta)\,\pi(\theta)}{\int_\Theta \mathbb{P}(D|\theta')\,\pi(\theta')\,\mathrm{d}\theta'} \tag{1.1}$$

Compared to the well-known stochastic Metropolis-Hastings algorithm and its variants [1, 17, 29, 31, 32] which requires a huge number of iterations, the Stein algorithm is completely deterministic. In this method, one starts with a measure μ and modifies it via the map

$$T_{\varepsilon,\phi}(x) = x + \varepsilon\,\phi,$$

where ε is a small parameter and ϕ is chosen to minimize the Kullback-Leibler divergence $\mathrm{KL}(T^{\#}_{\varepsilon,\phi}\,\mu||\rho_\infty)$ where for two nonnegative measures μ, ν the Kullback-Leibler divergence is defined as

$$\mathrm{KL}(\mu||\nu) = \begin{cases} \displaystyle\int_{\mathbb{R}^d} \log\left(\frac{\mathrm{d}\mu}{\mathrm{d}\nu}(x)\right)\frac{\mathrm{d}\mu}{\mathrm{d}\nu}(x)\;\mathrm{d}\nu(x) & \text{if } \frac{\mathrm{d}\mu}{\mathrm{d}\nu} \text{ exists,} \\ +\infty & \text{otherwise,} \end{cases} \tag{1.2}$$

and $T^{\#}_{\varepsilon,\phi}\mu$ is the push-forward measure of μ along the map $T_{\varepsilon,\phi}$: $T^{\#}_{\varepsilon,\phi}\mu(A) = \mu(T^{-1}_{\varepsilon,\phi}(A))$ for any Borel set A. The unique minimizer of the Kullback-Leibler divergence corresponds to the desired distribution ρ_∞. The reason for choosing this functional is that its first variation does not depend on the normalization constant Z. Furthermore, this is the only functional with such a property, see [7, Proposition 2.1]. More precisely, ϕ is chosen as a maximizer of the following optimization problem

$$\max_{\phi\in\mathcal{H}}\left\{-\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathrm{KL}(T^{\#}_{\varepsilon,\phi}\,\mu||\rho_\infty)|_{\varepsilon=0} : \|\phi\|_{\mathcal{H}} \le 1\right\}$$

where $\mathcal{H}$ is a sufficiently big Hilbert space. A simple computation (see [24, 25]) shows that

$$-\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathrm{KL}(T^{\#}_{\varepsilon,\phi}\,\mu||\rho_\infty)|_{\varepsilon=0} = \int_{\mathbb{R}^d} (\nabla\log(\rho_\infty)\cdot\phi + \operatorname{div}\phi)\,\mathrm{d}\mu(x)$$

so that we see that the variation does not depend on the normalization constant Z. In the particular case that $\mathcal{H}$ is a reproducing Hilbert space with kernel $K(x-y)$, one can obtain an explicit expression for the optimal ϕ (up to a normalization constant):

$$\phi \propto (\nabla\log(\rho_\infty)\mu) * K - \nabla K * \mu,$$

where $*$ denotes convolution operator $f * g(x) = \int_{\mathbb{R}^d} f(x-y)\,g(y)\,\mathrm{d}y$. In particular, if μ has a particle representation, this (formally) motivates an iterative algorithm: we set

$\mu_0 = \frac{1}{N}\sum_{i=1}^N \delta_{x_0^i}$ and given $\mu_l = \frac{1}{N}\sum_{i=1}^N \delta_{x_l^i}$ from the l-th step, in the $(l+1)$-th step we compute $\mu_{l+1} = \frac{1}{N}\sum_{i=1}^N \delta_{x_{l+1}^i}$ by

$$x_{l+1}^i = x_l^i + \frac{\varepsilon}{N}\sum_{j=1}^N \Big[\nabla \log \rho_\infty(x_l^j)\, K(x_l^i - x_l^j) - \nabla K(x_l^i - x_l^j)\Big] \tag{1.3}$$

(see [24, 25] for more details). This shows that the Stein gradient descent method is simple and attractive for practitioners.

From the analytical point of view, moving from discrete distributions to continuous ones (i.e. sending $N \to \infty$) is a delicate matter. Indeed, the Kullback-Leibler divergence (1.2) is not well-defined for discrete distributions. However, its first variation makes sense for discrete distributions, and thus the algorithm (1.3) is well-defined. In [28], the Stein's method was connected to the ODE system

$$\partial_t x^i(t) = -\frac{1}{N}\sum_{j=1}^N \nabla K(x^i(t) - x^j(t)) - \frac{1}{N}\sum_{j=1}^N K(x^i(t) - x^j(t))\, \nabla V(x^j(t)). \tag{1.4}$$

We notice that the algorithm (1.3) is the time discretization of the ODE (1.4) with the time step ε. Considering the empirical measure $\rho_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i(t)}$, it was proved in [28], that on finite intervals of time, $\rho_t^N \to \rho_t$ in the Wasserstein distance $\mathcal{W}_p$ where ρ_t solves the nonlocal PDE

$$\partial_t \rho_t = \operatorname{div}(\rho_t\, K * (\nabla \rho_t + \nabla V\ \rho_t)). \tag{1.5}$$

More rigorously, by using a Dobrushin-type argument, the authors in [28] established the following stability inequality

$$\mathcal{W}_p(\mu_t, \nu_t) \le C \exp(C \exp(CT)) \mathcal{W}_p(\mu_0, \nu_0) \tag{1.6}$$

for all times $t \in [0, T]$ and measure solutions μ_t, ν_t to (1.5), assuming that $V(x) \approx |x|^p$ for large x (see [28] for more general setting). The estimate (1.6) has also been established for the time discretization of (1.4) [36, Theorem 1]. Having sent $N \to \infty$, one can obtain ρ_∞ as the unique stationary solution of (1.5) by sending $t \to \infty$.

1.1. **Main results.** The paper [28] recasts the Stein method as a limit $N \to \infty$ and then $t \to \infty$. Yet, practical computations involve discretization in space and so, they correspond in fact to the joint limit $N \to \infty$, $t \to \infty$. We first state a result showing that in a certain scaling between N and t, one can rigorously justify the joint limit in the Wasserstein distance $\mathcal{W}_q$. This applies to the potentials having growth $|x|^p$ for large x.

Theorem 1.1 (convergence). *Suppose that K, V satisfy Assumptions 3.1 and 3.2. Let ρ_0 be an initial condition such that $KL(\rho_0|\rho_\infty) < \infty$. Let $\rho_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i(t)}$ where $x_i(t)$ solve (1.4) with $x_i(0)$ such that $\mathcal{W}_p(\rho_0, \rho_0^N) \leq \frac{1}{N}$. Let $N(t) \simeq \exp(2\,C\exp(Ct))$ where C is the constant as in (1.6). Then, for all $q \in [1,p)$*

$$\mathcal{W}_q(\rho_t^{N(t)}, \rho_\infty) \to 0 \text{ as } t \to \infty.$$

We observe that the number of particles is unpractically large compared to the time. To understand what happens for longer time scales, we address the question of stability of the particle system (1.4). Assuming that the initial configuration of particles ρ_0^N is close to the asymptotic state with error $O(\frac{1}{N})$, we ask for how long it remains close. For example, the estimate (1.6) suggests that after time $t = O(\log(\log N))$ or equivalently $N = O(\exp(\exp t))$, the distance $\mathcal{W}_p(\rho_t^N, \rho_\infty)$ is of order 1. Our main result improves this estimate and states that this time is of an algebraic order with respect to N rather than just logarithmic.

Theorem 1.2 (stability estimate). *Suppose that K, V satisfy Assumptions 3.1 and 3.3. Let ρ_t be a measure solution to (1.5). Then, there exists a constant C depending only on K and V such that for all times t that are sufficiently small in the sense that $1 - C\,t\,(t+1)\,\|\rho_0 - \rho_\infty\|_{BL_V^*} > 0$ holds, we have*

$$\|\rho_t - \rho_\infty\|_{BL_V^*} \leq \frac{C\,(t+1)\,\|\rho_0 - \rho_\infty\|_{BL_V^*}}{1 - C\,(t+1)\,t\,\|\rho_0 - \rho_\infty\|_{BL_V^*}} \tag{1.7}$$

where the norm $\|\cdot\|_{BL_V^*}$ is defined in (1.10).

Several comments are in order. First, the conditions on K and V in Assumption 3.3 are quite technical but they allow to consider all smooth, positive definite, sufficiently fast decaying kernels K and potentials V which grow at most like $|x|^2$ for large x. Second, the exploited distance $\|\cdot\|_{\mathrm{BL}_V^*}$ is a weighted modification of the bounded Lipschitz distance (also the flat norm, the Fortet-Mourier distance), commonly used in the analysis of transport-type PDEs (see, for instance, [10] and Section 2 for rigorous definition and related background). Third, we observe that when $\|\rho_0^N - \rho_\infty\|_{\mathrm{BL}_V^*} \leq \frac{1}{N}$, then even for algebraic (with respect to N) time $t \leq \left(\frac{N}{2\,C}\right)^{1/2} - 1$ we have

$$1 - C\,t\,(t+1)\|\rho_0^N - \rho_\infty\|_{\mathrm{BL}_V^*} \geq \frac{1}{2} \tag{1.8}$$

so that with $\widetilde{C} := \left(\frac{2}{C}\right)^{1/2}$ we have

$$\|\rho_t^N - \rho_\infty\|_{\mathrm{BL}_V^*} \leq \frac{\widetilde{C}}{\sqrt{N}}, \qquad 0 \leq t \leq \left(\frac{N}{2\,C}\right)^{1/2} - 1,$$

and so, possible instabilities in the particle system may occur much later compared to the time determined by the estimate (1.6). We remark that we can apply Theorem 1.2 to the solution ρ_t^N because this result applies to any measure solution to (1.5) (see Definition 2.1 for the definition of the measure solution), including both continuous and discrete solutions.

The inspiration for Theorem 1.2 comes from the insightful work of Caglioti and Rousset [3,4] who obtained similar estimates for the Vlasov equation and the vortex method for the 2D Euler equation. The starting point is to consider the dual equation, which is common in the theory of solutions in the space of measures to transport-type PDEs, see for instance the monograph [10]. In our case, we let $\mu_t := \rho_t - \rho_\infty$. Since $\nabla\rho_\infty + \nabla V\,\rho_\infty = 0$, we have

$$\partial_t \mu_t = \operatorname{div}(\mu_t\, K * (\nabla\mu_t + \nabla V\ \mu_t)) + \operatorname{div}(\rho_\infty\, K * (\nabla\mu_t + \nabla V\ \mu_t)). \tag{1.9}$$

To estimate μ_t, we introduce the weighted bounded Lipschitz norm defined for all measures $\nu \in \mathcal{M}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d}(1+V(x))\,\mathrm{d}|\nu|(x) < \infty$ via the formula

$$\|\nu\|_{\mathrm{BL}_V^*} := \sup_{\|g\|_{\mathrm{BL}} \leq 1} \int_{\mathbb{R}^d} g(x)\,(1+V(x))\,\mathrm{d}\nu(x). \tag{1.10}$$

We postpone the relevant technicalities from measure theory and further details to Section 2. Here, the supremum is taken over bounded Lipschitz functions g, see (2.2).

The expression (1.10) allows to estimate $\|\mu_t\|_{\mathrm{BL}_V^*}$ via the duality method. In this approach, for each $g = g(x)$ and $t \geq 0$, we find a function $\varphi^{g,t}(s,x)$ such that

$$\int_{\mathbb{R}^d} g(x)\,(1+V(x))\,\mathrm{d}\mu_t(x) = \int_{\mathbb{R}^d} \frac{\varphi^{g,t}(0,x)}{1+V(x)}\,(1+V(x))\,\mathrm{d}\mu_0(x) \tag{1.11}$$

Taking a supremum over all g such that $\|g\|_{\mathrm{BL}(\mathbb{R}^d))} \leq 1$, we obtain

$$\|\mu_t\|_{\mathrm{BL}_V^*} \leq \sup_{\|g\|_{\mathrm{BL}(\mathbb{R}^d))}\leq 1} \left\| \frac{\varphi^{g,t}(0,x)}{1+V(\cdot)} \right\|_{\mathrm{BL}(\mathbb{R}^d)} \|\mu_0\|_{\mathrm{BL}_V^*} \tag{1.12}$$

so to get the estimate on $\|\mu_t\|_{\mathrm{BL}_V^*}$ we have to control the (RHS) of (1.12). It turns out that if μ_t satisfies (1.9), the right choice for the function $\varphi^{g,t}(s,x)$ with $s \in [0,t]$ is to be the solution of the following PDE

$$\begin{aligned} \partial_s\varphi &= \nabla\varphi \cdot K * (\nabla\mu_s + \mu_s\,\nabla V) + (\nabla\rho_\infty\,\varphi) * \nabla K - (\nabla\rho_\infty\,\varphi) * K \cdot \nabla V \\ &\quad - (\rho_\infty\,\varphi) * \Delta K + (\rho_\infty\,\varphi) * \nabla K \cdot \nabla V, \qquad (1.13) \\ \varphi(t,x) &= g(x)\,(1+V(x)), \end{aligned}$$

see Appendix C. For the sake of simplicity, we will write φ for the solution of (1.13), keeping in mind that it depends on g and t.

At this point, one can see that the main motivation to work in the weighted bounded Lipschitz norm (1.10) is because we have to estimate terms where μ_s is multiplied by an unbounded function which is not admissible for the usual bounded Lipschitz norm (2.3). An example of such term is $\mu_s\,\nabla V$ in the dual problem (1.13).

The crucial part of the argument in [4] is to find a functional of the form

$$\mathcal{Q}(\varphi) \approx \int_{\mathbb{R}^d} w(x)\,|\varphi(x)|^2\,\mathrm{d}x\,,$$

so that it is equivalent to a weighted L^2 norm of φ and it is invariant under the linearized flow associated to (1.13). As the time derivative of $\mathcal{Q}(\varphi)$ vanishes on the linear terms of the dual equation, the estimate on $\mathcal{Q}(\varphi)$ will not yield exponential factors as obtained in (1.6). For the Vlasov and Euler equations, the right choice was $w(x) = |\rho_\infty'(|x|)|$. In our case, we choose

$$\mathcal{Q}(\varphi) = \int_{\mathbb{R}^d} \rho_\infty(x)\,|\varphi(x)|^2\,\mathrm{d}x.$$

While this functional is not necessarily invariant, we prove that there is no positive contribution to its value under the linearized flow associated to (1.13) (see Lemma 5.1). This yields the technical result:

Proposition 1.3 (weighted estimate on φ)**.** *Suppose that K, V satisfy Assumptions 3.1 and 3.3. Let φ be a solution to* (1.13) *with g and $t > 0$ fixed. Then, there exists a constant C depending only on V and K such that*

$$\mathcal{Q}(\varphi(s,\cdot)) \le C\,\|g\|_{L^\infty(\mathbb{R}^d)}\, e^{C\int_s^t \|\mu_u\|_{BL_V^*}\,\mathrm{d}u}.$$

With Proposition 1.3, the proof of Theorem 1.2 is a simple analysis of the explicit formula for solutions to (1.13) together with a Grönwall-type inequality, see Lemma A.2.

We remark that a non-rigorous reason why the functional $\mathcal{Q}$ is important in the analysis of the linearized version of (1.13) is that its dual can be interpreted as a Taylor expansion of the Kullback-Leibler divergence (1.2) around ρ_∞. Indeed, writing $\rho = \rho_\infty + h$ where $\int_{\mathbb{R}^d} h = 0$ (to preserve the mass), we have

$$\int_{\mathbb{R}^d} \rho \log\left(\frac{\rho}{\rho_\infty}\right)\mathrm{d}x \approx \int_{\mathbb{R}^d}\left(h + \frac{h^2}{2\,\rho_\infty}\right)\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^d}\frac{h^2}{\rho_\infty}\,\mathrm{d}x. \tag{1.14}$$

One can wonder how to choose the initial approximation ρ_0^N so that the term $\|\rho_0^N - \rho_\infty\|_{\mathrm{BL}_V^*}$ in (1.8) is sufficiently small. According to [4, Theorem 4], almost every initial configuration satisfies a condition of this type. To illustrate this, for the sake of simplicity, let us restrict the discussion to dimension $d = 2$ (in higher dimensions, one has to modify the $\|\|_{\mathrm{BL}_V^*}$ norm in (1.12) by computing the supremum over test functions whose m-th derivatives are Lipschitz continuous with $m \ge \frac{d}{2}$, see [4, Theorem 4]). Let λ_∞ be the product Lebesgue measure on $(\mathbb{R}^d)^\infty := \mathbb{R}^d \times \mathbb{R}^d \times ...$ (countably many times). Then, from [4, Theorem 4] we know that for all $\alpha \in (0, 1/2)$, there exists a constant $C > 0$ such that for λ_∞-a.e. $\mathbf{x} = (x_1, x_2, ...)$ the empirical measure $\rho^N[\mathbf{x}] = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ satisfy the estimate

$$\|\rho^N[\mathbf{x}] - \rho_\infty\|_{\mathrm{BL}_V^*} \le \frac{C}{N^\alpha}.$$

Notice that this statement excludes a lot of initial configurations. For instance, if $A \subset \mathbb{R}^d$ is a set of measure zero, then $\mathbb{R}^d \times \mathbb{R}^d \times A \times \mathbb{R}^d \times \mathbb{R}^d \times ...$ is of measure zero in $(\mathbb{R}^d)^\infty$. Of course, one would like to obtain estimates that show that the gradient flow improves the initial estimate rather than just does not worsen it too much. This is a difficult problem, far beyond the scope of the current manuscript.

Let us comment on the novelties of the manuscript and put them in the context of other works. From the analytical point of view, estimates of the form (1.7) have only been obtained before by Caglioti and Rousset for the Vlasov equation and for the vorticity formulation of the 2D Euler equation [3, 4]. The method uses the functional $\mathcal{Q}$ which is conservative for the flow of the linearized dual problem and is constructed by the methods of Hamiltonian mechanics. In our case, the functional $\mathcal{Q}$ has a different form and can be rather interpreted via a Taylor expansion of the Kullback-Leibler divergence, see (1.14). More generally, our work shows that the idea of [3, 4] can be possibly applied to a much broader class of PDEs without a Hamiltonian structure. Concerning the particular case of the Vlasov equation, we also mention the work of Han-Kwan and Nguyen [16] who proved a negative result: if f_∞ is an unstable equilibrium of the Vlasov equation (in the sense of so-called Penrose instability condition) and initially $\mathcal{W}_1(\mu_0^N, f_\infty) \approx \frac{1}{N^\alpha}$ for $\alpha > 0$ sufficiently small then $\limsup_{N\to\infty} \mathcal{W}_1(\mu_{T_N}^N, f_\infty) > 0$ for $T_N = O(\log N)$.

From the point of view of numerical analysis and statistics, the only available estimate addressing the convergence of the particle system in the Stein method is (1.6) obtained by Lu, Lu and Nolen in [28] which belongs to the large class of convergence results of mean-field limits for the Vlasov equation and the aggregation equation [2, 9, 18, 19, 21, 35]. First, from their result, we deduce the convergence of the method assuming that $t = O(\log(\log N))$, see Theorem 1.1. Moreover, we provided stability estimates for the longer, more practical timescale $t = O(\sqrt{N})$ which, up to our knowledge, are entirely new. Other interesting results for the Stein method focus on the convergence of (1.3), assuming that the time step ε is sufficiently small and the initial distribution is an absolutely continuous measure [22, 24], or the analysis of the asymptotics $t \to \infty$ via log-Sobolev-type inequalities [11], which is still a program far from being complete. These results crucially assume continuity of the initial distribution ρ_0 to define the Kullback-Leibler divergence (1.2) $\mathrm{KL}(\rho_0|\rho_\infty)$ which is not well-defined for discrete measures ρ_0^N.

The paper is structured as follows. In Section 2, we review the theory of spaces of measures and we define the norm $\|\cdot\|_{\mathrm{BL}_V^*}$. We also define measure solutions to (1.5). In Section 3 we introduce the assumptions on the potential V and the kernel K. In Section 4 we prove Theorem 1.1 while in Section 5 we prove Proposition 1.3 which allows to demonstrate Theorem 1.2 in Section 6.

2. Measure solutions and the functional analytic setting

2.1. **Spaces of measures, the Wasserstein distance, and the weighted bounded Lipschitz distance.** We first introduce the functional analytic framework. The most common notion of distance in the space of measures is probably the Wasserstein distance

$$\mathcal{W}_p(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p \,\mathrm{d}\pi(x,y) \right)^{1/p} \tag{2.1}$$

where $\Pi(\mu,\nu)$ is a set of couplings between μ and ν, i.e. $\pi \in \Pi(\mu,\nu)$ if π is a probability measure on $\mathbb{R}^d\times\mathbb{R}^d$ such that $\pi(A\times\mathbb{R}^d) = \mu(A)$ and $\pi(\mathbb{R}^d\times B) = \nu(B)$. Definition (2.1) requires that $\int_{\mathbb{R}^d}(1+|x|^p)\,\mathrm{d}\mu, \int_{\mathbb{R}^d}(1+|x|^p)\,\mathrm{d}\nu < \infty$. Theorem 1.1 is formulated using the Wasserstein distance.

For Theorem 1.2, we need a notion of distance compatible with the duality method. This will be the bounded Lipschitz distance. To define it, we first introduce the space of bounded Lipschitz functions

$$\mathrm{BL}(\mathbb{R}^d) = \{\psi : \mathbb{R}^d \to \mathbb{R} : \|\psi\|_{L^\infty(\mathbb{R}^d)} < \infty,\, |\psi|_{\mathrm{Lip}} < \infty\}, \tag{2.2}$$

$$\|\psi\|_{\mathrm{BL}} := \|\psi\|_{L^\infty(\mathbb{R}^d)} + \|\psi\|_{\mathrm{Lip}},$$

where

$$|\psi|_{\mathrm{Lip}} = \sup_{x\neq y} \frac{|\psi(x)-\psi(y)|}{|x-y|}.$$

A useful fact is that $|\psi|_{\mathrm{Lip}} \leq \|\nabla \psi\|_{L^\infty(\mathbb{R}^d)}$. In particular, to estimate $\|\psi\|_{\mathrm{BL}(\mathbb{R}^d)}$, it is sufficient to compute $\|\psi\|_{L^\infty(\mathbb{R}^d)}$ and $\|\nabla \psi\|_{L^\infty(\mathbb{R}^d)}$.

Given an arbitrary signed measure $\mu \in \mathcal{M}(\mathbb{R}^d)$, we recall its unique Hahn-Jordan decomposition $\mu = \mu^+ - \mu^-$ where both measures μ^+, μ^- are nonnegative. We define the total variation of μ as a nonnegative measure $|\mu| := \mu^+ + \mu^-$. Then, for any signed measure μ such that $\int_{\mathbb{R}^d} (1 + V(x)) \,\mathrm{d}|\mu|(x) < \infty$, we can define its weighted bounded Lipschitz norm as it was done in (1.10) which is a variant of the bounded Lipschitz norm

$$\|\mu\|_{\mathrm{BL}^*} := \sup_{\|\varphi\|_{\mathrm{BL}} \leq 1} \int_{\mathbb{R}^d} \varphi(x) \,\mathrm{d}\mu(x), \tag{2.3}$$

widely used in the analysis of PDEs, when the total mass is not conserved [5, 12, 14] (otherwise, one can use the Wasserstein distance). In our case, we introduce an additional weight $V(x)$ to address the growth $V(x) \to \infty$ as $|x| \to \infty$. Similar weighted norms were introduced before to remove singularities in the studied problems [13, 39].

We remark that on a bounded domain $\Omega \subset \mathbb{R}^d$, the first Wasserstein distance $\mathcal{W}_1$ is comparable with the bounded Lipschitz norm: asccording to [10, Lemma 1.101], there exists a constant C_Ω such that

$$C_\Omega \, \mathcal{W}_1(\mu, \nu) \leq \|\mu - \nu\|_{\mathrm{BL}^*} \leq \mathcal{W}_1(\mu, \nu).$$

If Ω is unbounded, $C_\Omega = 0$. Concerning the weighted bounded Lipschitz norm, we note that if $\|\varphi\|_{\mathrm{BL}} \leq 1$ and $\left|\frac{\nabla V(x)}{(1+V(x))^2}\right| \leq C_V$ (which is always satisfied in our case due to the growth conditions (3.1)), then $\left\|\frac{\varphi}{1+V}\right\|_{\mathrm{BL}} \leq C_V + 1$ so that

$$\int_{\mathbb{R}^d} \varphi(x) \,\mathrm{d}\mu(x) = \int_{\mathbb{R}^d} \frac{\varphi(x)}{1 + V(x)} \,(1 + V(x)) \,\mathrm{d}\mu(x) \leq \|\mu\|_{\mathrm{BL}_V^*}.$$

Taking the supremum over all φ such that $\|\varphi\|_{\mathrm{BL}} \leq 1$ we deduce

$$\|\mu\|_{\mathrm{BL}^*} \leq (C_V + 1) \, \|\mu\|_{\mathrm{BL}_V^*}.$$

Hence, Theorem 1.2 provides an estimate for the difference $\|\rho_t - \rho_\infty\|_{\mathrm{BL}^*}$ also in the usual bounded Lipschitz norm.

2.2. **Measure solutions to** (1.5)**.** The measure solution is defined as follows.

Definition 2.1 (measure solution)**.** *We say that a family of probability measures* $\{\rho_t\}_{t\in[0,T]}$ *is a measure solution to* (1.5) *if* $t \mapsto \rho_t$ *is continuous (with respect to the narrow topology), for all* $t \geq 0$ *the growth estimate*

$$\sup_{t\in[0,T]} \|\rho_t\|_{BL_V^*} = \sup_{t\in[0,T]} \int_{\mathbb{R}^d} (1 + V(x))\,\mathrm{d}\rho_t(x) \; < +\infty \tag{2.4}$$

is satisfied and for all $\phi \in C_c^\infty([0,\infty)\times\mathbb{R}^d)$

$$\int_0^\infty \int_{\mathbb{R}^d} \partial_t\phi(t,x) - \nabla\phi(t,x)\cdot K*(\nabla\rho_t + \rho_t\,\nabla V)\,\mathrm{d}\rho_t(x)\,\mathrm{d}t + \int_{\mathbb{R}^d} \phi(0,x)\,\mathrm{d}\rho_0(x) = 0. \tag{2.5}$$

Several comments are in order. First, the continuity with respect to the narrow topology means that the map $t \mapsto \int_{\mathbb{R}^d} \psi(x)\,\mathrm{d}\rho_t(x)$ is continuous for all $\psi : \mathbb{R}^d \to \mathbb{R}$ bounded and continuous. Second, the first equality in (2.4) follows by nonnegativity of ρ_t. Third, it is a simple computation to see that $\rho_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i(t)}$, where $x_i(t)$ solves (1.4) with initial condition $x_i(0)$, is a measure solution to (1.5) with initial condition $\rho_0^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i(0)}$. Fourth, due to the narrow continuity of the map $t \mapsto \rho_t$, the weak formulation (2.5) is equivalent with the following: for all times $t > 0$ and all test functions $\phi \in C_c^\infty([0,\infty)\times\mathbb{R}^d)$

$$\begin{aligned}&\int_{\mathbb{R}^d} \phi(t,x)\,\mathrm{d}\rho_t(x) - \int_{\mathbb{R}^d} \phi(0,x)\,\mathrm{d}\rho_0(x)\\ &= \int_0^t \int_{\mathbb{R}^d} \partial_s\phi(s,x)\,\mathrm{d}\rho_s(x)\,\mathrm{d}s - \int_0^t \int_{\mathbb{R}^d} \nabla\phi(s,x)\cdot K*(\nabla\rho_s + \nabla V\,\rho_s)\,\mathrm{d}\rho_s\,\mathrm{d}s.\end{aligned} \tag{2.6}$$

Indeed, fix $t > 0$. For $\delta \in (0, \frac{t}{2})$ let $\eta_\delta \in C_c^\infty([0,\infty))$ be such that $\eta_\delta(s) = 1$ for $s \in [0, t-\delta]$, $\eta_\delta(s) = 0$ for $s \in [t+\delta, \infty)$ and $|\partial_s\eta_\delta(s)| \leq \frac{2}{\delta}$. Consider (2.5) with the test function $\phi(s,x)\,\eta_\delta(s)$. We want to send $\delta \to 0$ and obtain (2.6). It is clear that $\eta_\delta(s) \to \mathbb{1}_{[0,t]}(s)$ (a characteristic function of the interval $[0,t]$) as $\delta \to 0$. Hence, by the dominated convergence theorem we can pass to the limit in all terms except $\int_0^\infty \int_{\mathbb{R}^d} \partial_s\eta_\delta(s)\,\phi(s,x)\,\mathrm{d}\rho_s(x)\,\mathrm{d}s$ which involves the time derivative of η_δ. We claim that

$$\lim_{\delta\to 0} \int_0^\infty \int_{\mathbb{R}^d} \partial_s\eta_\delta(s)\,\phi(s,x)\,\mathrm{d}\rho_s(x)\,\mathrm{d}s = -\int_{\mathbb{R}^d} \phi(t,x)\,\mathrm{d}\rho_t(x).$$

Indeed, using that $\partial_s \eta_\delta(s) = 0$ for $s \notin [t-\delta, t+\delta]$ and $\int_{t-\delta}^{t+\delta} \partial_s \eta_\delta(s)\, \mathrm{d}s = -1$ we can estimate

$$\left| \int_0^\infty \int_{\mathbb{R}^d} \partial_s \eta_\delta(s)\, \phi(s,x)\, \mathrm{d}\rho_s(x)\, \mathrm{d}s + \int_{\mathbb{R}^d} \phi(t,x)\, \mathrm{d}\rho_t(x) \right| =$$
$$= \left| \int_{t-\delta}^{t+\delta} \partial_s \eta_\delta(s) \left(\int_{\mathbb{R}^d} \phi(s,x)\, \mathrm{d}\rho_s(x) - \int_{\mathbb{R}^d} \phi(t,x)\, \mathrm{d}\rho_t(x) \right) \mathrm{d}s \right|$$
$$\leq 4 \sup_{s \in [t-\delta, t+\delta]} \left| \int_{\mathbb{R}^d} \phi(s,x)\, \mathrm{d}\rho_s(x) - \int_{\mathbb{R}^d} \phi(t,x)\, \mathrm{d}\rho_t(x) \right| =: \omega(\delta),$$

where in the last line we used that $|\partial_s \eta_\delta(s)| \leq \frac{2}{\delta}$. By the narrow continuity, the function $s \mapsto \int_{\mathbb{R}^d} \phi(s,x)\, \mathrm{d}\rho_s(x)$ is continuous so $\omega(\delta) \to 0$ as $\delta \to 0$ and the proof is concluded.

Finally, let us notice that due to the growth condition (2.4), the formulation (2.6) is valid for test functions such that

$$|\phi(t,x)|, |\partial_t \phi(t,x)|, |\nabla \phi(t,x)| \leq C\,(1 + V(x))$$

for some constant C.

3. Assumptions on the kernel and the potential

For the sake of clarity, we specify the assumptions for Theorems 1.1 and 1.2 separately.

Assumption 3.1. For both Theorems 1.1 and 1.2 we assume that:

- K is nonnegative, symmetric $K(x) = K(-x)$ and positive-definite, i.e. for all test functions $\xi : \mathbb{R}^d \to \mathbb{R}^d$ we have $\int_{\mathbb{R}^d} K * \xi \cdot \xi \, \mathrm{d}x \geq 0$,
- V is a smooth and nonnegative function,
- there exists $p > 0$, $C > 0$ and $R > 0$ such that for all x with $|x| > R$ we have

$$\begin{aligned} \frac{1}{C}\,(|x|^p - 1) \leq V(x) \leq C\,(|x|^p + 1), \\ \frac{1}{C}\,(|x|^{p-1} - 1) \leq |\nabla V(x)| \leq C\,(|x|^{p-1} + 1), \\ \frac{1}{C}\,(|x|^{p-2} - 1) \leq |\nabla^2 V(x)| \leq C\,(|x|^{p-2} + 1). \end{aligned} \tag{3.1}$$

Assumption 3.2. For Theorem 1.1 we assume additionally (as in [28]) that:

- the condition (3.1) holds with $p > 1$,

- $K \in C^4(\mathbb{R}^d) \cap W^{4,\infty}(\mathbb{R}^d)$,
- there exists a smooth function $K_{1/2}$ such that $K = K_{1/2} * K_{1/2}$ and its Fourier transform $\widehat{K_{1/2}}$ is positive.

Assumption 3.3. For Theorem 1.2 we assume additionally that:

- $K \in W^{3,\infty}(\mathbb{R}^d)$,
- V and K satisfy the following conditions:

$$\sup_{x\in\mathbb{R}^d} \left\| \frac{\nabla V(x)}{1+V(\cdot)} \cdot \nabla K(x-\cdot) \right\|_{\mathrm{BL}}, \sup_{x\in\mathbb{R}^d} \left\| \frac{\nabla V(x)\cdot\nabla V(\cdot)}{1+V(\cdot)} K(x-\cdot) \right\|_{\mathrm{BL}} < \infty. \tag{3.2}$$

The only condition which is difficult to understand is (3.2). Unfortunately, it restricts our reasoning to the case of V which can be at most quadratic at infinity (i.e. $p \leq 2$ in (3.1)).

Lemma 3.4. *Suppose that $V, K \geq 0$, $V \in W^{2,\infty}_{loc}(\mathbb{R}^d)$, $K \in W^{2,\infty}(\mathbb{R}^d)$ and assume that V satisfies* (3.1) *with exponent p. Furthermore, suppose that $|\nabla V|\, K, |\nabla V|\, |\nabla K|, |\nabla V|\, |\nabla^2 K| \in L^{\infty}(\mathbb{R}^d)$. Then, V and K satisfy* (3.2) *if and only if $p \in (0, 2]$.*

The proof is presented in Appendix B.

We conclude with a crucial consequence of (3.2) which provides bounds on the vector field for the transport equation (1.13).

Lemma 3.5. *Suppose that K and V satisfy Assumptions 3.1 and 3.3. Let μ be a measure such that $\|\mu\|_{BL^*_V} < \infty$. Then, there exists a constant C depending only on V and K such that*

$$\|K * (\nabla\mu + \mu\,\nabla V)\|_{L^{\infty}(\mathbb{R}^d)},\ \|\nabla K * (\nabla\mu + \mu\,\nabla V)\|_{L^{\infty}(\mathbb{R}^d)} \leq C\, \|\mu\|_{BL^*_V}. \tag{3.3}$$

Moreover,

$$\|\nabla V \cdot K * (\nabla\mu + \mu\,\nabla V)\|_{L^{\infty}(\mathbb{R}^d)} \leq C\, \|\mu\|_{BL^*_V}. \tag{3.4}$$

The proof is presented in Appendix B.

4. Proof of Theorem 1.1

Here, we prove that in the particular scaling $t \approx \log\log N$, we can pass to the joint limit $t, N \to \infty$. The result is a simple consequence of results in [28].

We recall that in [28, Theorem 2.7] the authors prove that if $\mathcal{W}_p(\rho_0^N, \rho_0) \le \frac{1}{N}$ then

$$\mathcal{W}_p(\rho_t^N, \rho_t) \le C \exp(C\exp(Ct))\, \mathcal{W}_p(\rho_0^N, \rho_0) \le \frac{C\exp(C\exp(Ct))}{N}, \tag{4.1}$$

where C depends only on V and K. Furthermore, using Assumption 3.2, the authors prove that if ρ_0 is an absolutely continuous measure, there exists a unique solution ρ_t to (1.5) which is an absolutely continuous measure for all $t \in [0, \infty)$ and

$$\rho_t \to \rho_\infty \text{ narrowly (as measures)}, \tag{4.2}$$

see [28, Theorem 2.8] (in fact, in [28] the Authors prove the convergence (4.2) only up to a subsequence; the result for the whole sequence was established in [22, Proposition 2, Remark 3]). The target of this Section is to combine (4.1) and (4.2) to prove the Theorem 1.1. We will first upgrade the convergence (4.2).

Lemma 4.1. *Suppose that K and V satisfy Assumptions 3.1 and 3.2. Let $\{\rho_t\}$ be an (absolutely continuous) measure solution to* (1.5) *with initial condition ρ_0 such that $KL(\rho_0|\rho_\infty) < \infty$. Then, for all $1 \le q < p$ we have $\mathcal{W}_q(\rho_t, \rho_\infty) \to 0$ when $t \to \infty$.*

Proof. Since we deal with an absolutely continuous solution ρ_t, we have the inequality $\partial_t \mathrm{KL}(\rho_t|\rho_\infty) \le 0$, see [28, p.3]. Hence, $\mathrm{KL}(\rho_t|\rho_\infty) \le \mathrm{KL}(\rho_0|\rho_\infty) < \infty$ which means that

$$\int_{\mathbb{R}^d} \rho_t \log(\rho_t) + V(x)\rho_t \,\mathrm{d}x \le C.$$

By standard arguments (for instance, splitting the set $\{x \in \mathbb{R}^d : \rho_t \le 1\}$ for two sets: $\{x \in \mathbb{R}^d : \rho_t \le e^{-|x|^p/\sigma}\}$ and $\{x \in \mathbb{R}^d : e^{-|x|^p/\sigma} \le \rho_t \le 1\}$) we have

$$-\int_{\mathbb{R}^d} \rho_t \log^-(\rho_t)\,\mathrm{d}x \le \int_{\mathbb{R}^d} e^{-|x|^p/(2\sigma)}\,\mathrm{d}x + \sigma \int_{\mathbb{R}^d} \rho_t\,|x|^p\,\mathrm{d}x$$

where $\log^-$ is the negative part of log. Hence, choosing σ small enough, using that ρ_t is a probability measure and growth conditions on V in (3.1), we get

$$\sup_{t\in[0,\infty)} \int_{\mathbb{R}^d} (\rho_t|\log(\rho_t)| + |x|^p\rho_t)\,\mathrm{d}x \le C. \tag{4.3}$$

This inequality gives the tightness of the sequence $\{\rho_t\}$ in [28] to prove (4.2) but in our case, it gives us the uniform moment estimate which is relevant in the sequel.

We know that $\rho_t \to \rho_\infty$ narrowly by (4.2). Hence, to prove the lemma, by [34, Theorem 5.11], it is sufficient to prove $\int_{\mathbb{R}^d} |x|^q\rho_t(x)\,\mathrm{d}x \to \int_{\mathbb{R}^d} |x|^q\rho_\infty(x)\,\mathrm{d}x$. Let T_R be the truncation operator defined as

$$T_R(y) = \begin{cases} y & \text{if } |y| \le R, \\ \frac{y}{|y|}\,R & \text{if } |y| > R, \end{cases}$$

so that $|T_R(y)| \le |y|$ and $|T_R(y)| \le R$. Hence,

$$\begin{aligned} &\left|\int_{\mathbb{R}^d} |x|^q(\rho_t - \rho_\infty)\,\mathrm{d}x\right| \le \\ &\qquad \le \left|\int_{\mathbb{R}^d} T_R(|x|^q)(\rho_t - \rho_\infty)\,\mathrm{d}x\right| + \left|\int_{\mathbb{R}^d} (|x|^q - T_R(|x|^q))(\rho_t - \rho_\infty)\,\mathrm{d}x\right|. \end{aligned} \tag{4.4}$$

In the second integral, we can restrict to $|x|^q > R$ so that $|x|^{q-p} < R^{\frac{q-p}{q}}$

$$||x|^q - T_R(|x|^q)| \le 2\,|x|^p\,R^{\frac{q-p}{q}}.$$

By (4.3) and $\int_{\mathbb{R}^d} |x|^p\rho_\infty\,\mathrm{d}x < \infty$, we conclude

$$\left|\int_{\mathbb{R}^d} (|x|^q - T_R(|x|^q))(\rho_t - \rho_\infty)\,\mathrm{d}x\right| \le C\,R^{\frac{q-p}{q}}. \tag{4.5}$$

As $T_R(|x|^q)$ is an admissible test function for the narrow convergence, we deduce from (4.4) and (4.5)

$$\limsup_{t\to\infty} \left|\int_{\mathbb{R}^d} |x|^q(\rho_t - \rho_\infty)\,\mathrm{d}x\right| \le C\,R^{\frac{q-p}{q}}.$$

As R is arbitrary and $q < p$, we send $R \to \infty$ and the proof is concluded. □

Proof of Theorem 1.1. By the triangle inequality, we deduce

$$\mathcal{W}_q(\rho_t^N, \rho_\infty) \le \mathcal{W}_q(\rho_t^N, \rho_t) + \mathcal{W}_q(\rho_t, \rho_\infty).$$

In view of (4.1) and a simple inequality $\mathcal{W}_q(\rho_t^N, \rho_t) \leq \mathcal{W}_p(\rho_t^N, \rho_t)$ (as $q \leq p$), we obtain

$$\mathcal{W}_q(\rho_t^N, \rho_t) \leq \frac{C \exp(C \exp(Ct))}{N}.$$

Hence, if $N = \exp(2\,C \exp(Ct))$ then $\mathcal{W}_q(\rho_t^N, \rho_t) \to 0$ and so, the conclusion follows by Lemma 4.1. □

5. The weighted L^2 estimate (proof of Proposition 1.3)

We first present the crucial cancellation lemma which allows the cancellation of the terms that are linear with respect to φ.

Lemma 5.1. *Let* $\mathcal{Q}(\varphi) := \left(\int_{\mathbb{R}^d} \rho_\infty\, |\varphi|^2 \,\mathrm{d}x\right)^{1/2}$ *and let*

$$f(\varphi) := (\nabla \rho_\infty\, \varphi) * \nabla K - (\nabla \rho_\infty\, \varphi) * K \cdot \nabla V - (\rho_\infty\, \varphi) * \Delta K + (\rho_\infty\, \varphi) * \nabla K \cdot \nabla V.$$

Then,

$$\int_{\mathbb{R}^d} f(\varphi)\, \varphi\, \rho_\infty \,\mathrm{d}x = \int_{\mathbb{R}^d} (\nabla \varphi\, \rho_\infty) * K \cdot (\nabla \varphi\, \rho_\infty) \,\mathrm{d}x \geq 0. \tag{5.1}$$

In particular, if φ solves (1.13)*, then*

$$\partial_t \mathcal{Q}(\varphi) \geq \frac{1}{\mathcal{Q}(\varphi)} \int_{\mathbb{R}^d} \rho_\infty\, \varphi\, \nabla \varphi \cdot K * (\nabla \mu_t + \mu_t\, \nabla V) \,\mathrm{d}x. \tag{5.2}$$

Proof of Lemma 5.1. The most important observation is that $-\rho_\infty \nabla V = \nabla \rho_\infty$. Hence,

$$\begin{aligned}\int_{\mathbb{R}^d} f(\varphi)\, \varphi\, \rho_\infty \,\mathrm{d}x &= \int_{\mathbb{R}^d} (\nabla \rho_\infty\, \varphi) * \nabla K\, \varphi\, \rho_\infty + (\nabla \rho_\infty\, \varphi) * K \cdot \nabla \rho_\infty\, \varphi \,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^d} (\rho_\infty\, \varphi) * \Delta K\, \varphi\, \rho_\infty + (\rho_\infty\, \varphi) * \nabla K \cdot \nabla \rho_\infty\, \varphi \,\mathrm{d}x =: I_1 + I_2.\end{aligned}$$

We observe that $I_1 = -\int_{\mathbb{R}^d} (\nabla \rho_\infty\, \varphi) * K \cdot \nabla \varphi\, \rho_\infty \,\mathrm{d}x$ by integrating by parts in the first term in I_1. Similarly $I_2 = \int_{\mathbb{R}^d} (\rho_\infty\, \varphi) * \nabla K \cdot \nabla \varphi\, \rho_\infty \,\mathrm{d}x$. Now, by standard properties of convolutions, $(\rho_\infty\, \varphi) * \nabla K = (\nabla \rho_\infty\, \varphi) * K + (\rho_\infty\, \nabla \varphi) * K$ so that summing $I_1 + I_2$ we conclude the proof of (5.1) (the nonnegativity follows by the positive definiteness of K). Concerning (5.2), we observe that differentiating in time and using the PDE (1.13)

$$\partial_t \mathcal{Q}(\varphi)\, \mathcal{Q}(\varphi) = \int_{\mathbb{R}^d} \rho_\infty\, \varphi\, f(\varphi) \,\mathrm{d}x + \int_{\mathbb{R}^d} \rho_\infty\, \varphi\, \nabla \varphi \cdot K * (\nabla \mu_t + \mu_t\, \nabla V) \,\mathrm{d}x$$

so that (5.2) follows directly from (5.1). □

Proof of Proposition 1.3. Integrating (5.2) in time from s to t and using the terminal value $\varphi(t,x) = g(x)\,(1+V(x))$ we deduce

$$\mathcal{Q}(\varphi(s,\cdot)) \le \mathcal{Q}(g\,(1+V)) + \\ + \int_s^t \frac{1}{\mathcal{Q}(\varphi(u,\cdot))} \left| \int_{\mathbb{R}^d} \rho_\infty(x)\,\varphi(u,x)\,\nabla\varphi(u,x)\cdot K*(\nabla\mu_u + \mu_u\,\nabla V)\,\mathrm{d}x \right| \mathrm{d}u.$$

The first term can be estimated as follows

$$\mathcal{Q}(g\,(1+V)) = \|g\|_{L^\infty(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} (1+V(x))^2\,\rho_\infty(x)\,\mathrm{d}x \right)^{1/2} \le C\,\|g\|_{L^\infty(\mathbb{R}^d)}.$$

For the second we use $\varphi\,\nabla\varphi = \frac{1}{2}\nabla\varphi^2$ and integrate by parts to get two terms:

$$\int_s^t \frac{1}{\mathcal{Q}(\varphi(u,\cdot))} \left| \int_{\mathbb{R}^d} \nabla\rho_\infty(x)\,\varphi^2(u,x)\cdot K*(\nabla\mu_u + \mu_u\,\nabla V)\,\mathrm{d}x \right| \mathrm{d}u + \\ + \int_s^t \frac{1}{\mathcal{Q}(\varphi(u,\cdot))} \left| \int_{\mathbb{R}^d} \rho_\infty(x)\,\varphi^2(u,x)\,\nabla K*(\nabla\mu_u + \mu_u\,\nabla V)\,\mathrm{d}x \right| \mathrm{d}u.$$

Using $\nabla\rho_\infty = -\rho_\infty\,\nabla V$, we can estimate it by

$$\int_s^t \mathcal{Q}(\varphi(u,\cdot)) \left(\|\nabla V\cdot K*(\nabla\mu_u + \mu_u\,\nabla V)\|_{L^\infty(\mathbb{R}^d)} + \|\nabla K*(\nabla\mu_u + \mu_u\,\nabla V)\|_{L^\infty(\mathbb{R}^d)} \right) \mathrm{d}u.$$

The L^∞ norms above can be bounded by $C\,\|\mu_s\|_{\mathrm{BL}_V^*}$ using Lemma 3.5 so that we obtain

$$\mathcal{Q}(\varphi(s,\cdot)) \le C\,\|g\|_{L^\infty(\mathbb{R}^d)} + C \int_s^t \mathcal{Q}(\varphi(u,\cdot))\,\|\mu_u\|_{\mathrm{BL}_V^*}\,\mathrm{d}u.$$

Using Lemma A.1, we conclude the proof. □

6. BL estimates on φ and proof of Theorem 1.2

The plan is to write the solution to (1.13) explicitly and estimate each term separately. Note that for a general transport equation $\partial_s\varphi = \nabla\varphi\cdot b(s,x) + c(s,x)$, $\varphi(t,x) = g(x)\,(1+V(x))$, the method of characteristics yields the following representation formula

$$\varphi(s,x) = -\int_s^t c(u, X_{s,u}(x))\,\mathrm{d}u + g(X_{s,t}(x))\,(1+V(X_{s,t}(x))),$$

where $X_{s,u}(x)$ is the flow of the vector field b:

$$\partial_u X_{s,u}(x) = -b(u, X_{s,u}(x)), \qquad X_{s,s}(x) = x.$$

Therefore, the solution to (1.13) can be written as

$$\begin{aligned}\varphi(s,x) = &-\int_s^t (\nabla\rho_\infty\,\varphi) * \nabla K(u, X_{s,u}(x))\,\mathrm{d}u + \int_s^t (\nabla\rho_\infty\,\varphi) * K(u, X_{s,u}(x)) \cdot \nabla V(X_{s,u}(x))\,\mathrm{d}u \\ &+ \int_s^t (\rho_\infty\,\varphi) * \Delta K(u, X_{s,u}(x))\,\mathrm{d}u - \int_s^t (\rho_\infty\,\varphi) * \nabla K(u, X_{s,u}(x)) \cdot \nabla V(X_{s,u}(x))\,\mathrm{d}u \\ &+ g(X_{s,t}(x))\,(1 + V(X_{s,t}(x))), \end{aligned} \tag{6.1}$$

where $X_{s,u}(x)$ is the flow of the vector field $-K * (\nabla\mu_u + \mu_u\,\nabla V)$:

$$\partial_u X_{s,u}(x) = -K * (\nabla\mu_u + \mu_u\,\nabla V)(X_{s,u}(x)), \qquad X_{s,s}(x) = x. \tag{6.2}$$

According to (1.11), we need to estimate $\frac{\varphi(0,x)}{1+V(x)}$ uniformly with respect to g. From (6.1) we have

$$\begin{aligned}\frac{\varphi(0,x)}{1+V(x)} = &-\int_0^t \frac{(\nabla\rho_\infty\,\varphi) * \nabla K(u, X_{0,u}(x))}{1+V(x)}\,\mathrm{d}u + \int_0^t (\nabla\rho_\infty\,\varphi) * K(u, X_{0,u}(x)) \cdot \frac{\nabla V(X_{0,u}(x))}{1+V(x)}\,\mathrm{d}u \\ &+ \int_0^t \frac{(\rho_\infty\,\varphi) * \Delta K(u, X_{0,u}(x))}{1+V(x)}\,\mathrm{d}u - \int_0^t (\rho_\infty\,\varphi) * \nabla K(u, X_{0,u}(x)) \cdot \frac{\nabla V(X_{0,u}(x))}{1+V(x)}\,\mathrm{d}u \\ &+ g(X_{0,t}(x))\,\frac{(1 + V(X_{0,t}(x)))}{1+V(x)}. \end{aligned} \tag{6.3}$$

First, we will need a lemma on quantities appearing in (6.3).

Lemma 6.1. *Suppose that K and V satisfy Assumptions 3.1 and 3.3. Let $\{\mu_s\}_{s\in[0,t]}$ be the family of measures and let $X_{s,u}$ be defined by* (6.2)*. Then, there exists a constant C depending only on K and V such that*

$$|\nabla X_{0,s}(x)|\,, \left|\frac{\nabla V(X_{0,s}(x))}{1+V(x)}\right|, \left|\frac{V(X_{0,s}(x))}{1+V(x)}\right|, \left|\frac{\nabla^2 V(X_{0,s}(x))}{1+V(x)}\right| \leq C\, e^{C\,\int_0^s \|\mu_u\|_{BL_V^*}\,\mathrm{d}u}.$$

Proof. Note that

$$X_{0,s}(x) = x - \int_0^s K * (\nabla\mu_u + \mu_u\,\nabla V)(X_{0,u}(x))\,\mathrm{d}u \tag{6.4}$$

so that in particular

$$\nabla X_{0,s}(x) = \mathbb{I}_d - \int_0^s \nabla K * (\nabla\mu_u + \mu_u\,\nabla V)(X_{0,u}(x)) \cdot \nabla X_{0,u}(x)\,\mathrm{d}u,$$

where $\mathbb{I}_d$ is the identity matrix. As $\|\nabla K * (\nabla\mu_u + \mu_u\,\nabla V)\|_{L^\infty(\mathbb{R}^d)} \leq C\,\|\mu_u\|_{\mathrm{BL}_V^*}$ (Lemma 3.5), the estimate on $\nabla X_{0,s}$ follows by Grönwall lemma.

We now proceed to the proof of the estimates involving potential V. We notice that the second term in (6.4) can be estimated by $\int_0^s \|\mu_u\|_{\mathrm{BL}_V^*}\,\mathrm{d}u$ (Lemma 3.5). Now let f be one of the functions V, $|\nabla V|$, $|\nabla^2 V|$ so that the target is to estimate $\frac{f(X_{0,s}(x))}{1+V(x)}$. We want to use the growth conditions (3.1). If f happens to be bounded, the proof is immediately concluded. Otherwise, there exists $q \in \{p, p-1, p-2\}$, $q \geq 0$ such that

$$|f(X_{0,s}(x))| \leq C\,(1+|X_{0,s}(x)|^q) \leq C\,\left(1+|x|^q+\left|\int_0^s K*(\nabla\mu_u+\mu_u\,\nabla V)(X_{0,u}(x))\,\mathrm{d}u\right|^q\right).$$

Using Lemma 3.5 and simple inequality $|x|^q \leq C\,e^{C\,|x|}$ we get

$$\left|\int_0^s K*(\nabla\mu_u+\mu_u\,\nabla V)(X_{0,u}(x))\,\mathrm{d}u\right|^q \leq \left|\int_0^s \|K*(\nabla\mu_u+\mu_u\,\nabla V)\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}u\right|^q \leq C\,e^{C\,\int_0^s \|\mu_u\|_{\mathrm{BL}_V^*}\,\mathrm{d}u},$$

It follows that

$$\left|\frac{f(X_{0,s}(x))}{1+V(x)}\right| \leq C + C\,\frac{|x|^q}{1+V(x)} + C\,\frac{e^{C\,\int_0^s \|\mu_u\|_{\mathrm{BL}_V^*}\,\mathrm{d}u}}{1+V(x)} \leq C\frac{|x|^q}{1+V(x)} + C\,e^{C\,\int_0^s \|\mu_u\|_{\mathrm{BL}_V^*}\,\mathrm{d}u}.$$

To conclude the proof, it remains to observe that because $0 \leq q \leq p$, the term $\frac{|x|^q}{1+V(x)}$ is bounded due to the growth conditions (3.1). $\square$

We proceed to the estimates on φ.

Lemma 6.2. *Suppose that K and V satisfy Assumptions 3.1 and 3.3. Let φ be a solution to* (1.13) *with g and $t>0$ fixed. Then, there exists a constant C depending only on V and K such that*

$$\left\|\frac{\varphi(0,\cdot)}{1+V(\cdot)}\right\|_{L^\infty(\mathbb{R}^d)} \leq C\,(t+1)\,\|g\|_{L^\infty(\mathbb{R}^d)}\,e^{C\,\int_0^t \|\mu_u\|_{BL_V^*}\,\mathrm{d}u}.$$

Proof. Using formula (6.3) and estimating $1 \leq 1 + V(x)$, we have

$$\begin{aligned}\left\|\frac{\varphi(0,\cdot)}{1+V(\cdot)}\right\|_{L^\infty(\mathbb{R}^d)} \leq \int_0^t &\|(\nabla\rho_\infty\,\varphi) * \nabla K(u,\cdot)\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}u \\ &+ \int_0^t \|(\nabla\rho_\infty\,\varphi) * K(u,\cdot)\|_{L^\infty(\mathbb{R}^d)} \left\|\frac{\nabla V(X_{0,u}(\cdot))}{1+V(\cdot)}\right\|_{L^\infty(\mathbb{R}^d)} \mathrm{d}u \\ &+ \int_0^t \|(\rho_\infty\,\varphi) * \Delta K(u,\cdot)\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}u \\ &+ \int_0^t \|(\rho_\infty\,\varphi) * \nabla K(u,\cdot)\|_{L^\infty(\mathbb{R}^d)} \left\|\frac{\nabla V(X_{0,u}(\cdot))}{1+V(\cdot)}\right\|_{L^\infty(\mathbb{R}^d)} \mathrm{d}u \\ &+ \|g\|_{L^\infty(\mathbb{R}^d)} \left\|\frac{1+V(X_{0,t}(\cdot))}{1+V(\cdot)}\right\|_{L^\infty(\mathbb{R}^d)}.\end{aligned}$$

Note that

$$\|\rho_\infty(\cdot)\,\varphi(u,\cdot)\|_{L^1(\mathbb{R}^d)} \leq \|\rho_\infty\|_{L^1(\mathbb{R}^d)}^{1/2}\,\mathcal{Q}(\varphi(u,\cdot)) \leq C\,\mathcal{Q}(\varphi(u,\cdot)), \tag{6.5}$$

$$\|\nabla\rho_\infty(\cdot)\,\varphi(u,\cdot)\|_{L^1(\mathbb{R}^d)} \leq \|\rho_\infty\,|\nabla V|^2\|_{L^1(\mathbb{R}^d)}^{1/2}\,\mathcal{Q}(\varphi(u,\cdot)) \leq C\,\mathcal{Q}(\varphi(u,\cdot)), \tag{6.6}$$

so that due to Proposition 1.3 we have

$$\begin{aligned}\max(\|\rho_\infty(\cdot)\,\varphi(u,\cdot)\|_{L^1(\mathbb{R}^d)}, \|\nabla\rho_\infty(\cdot)\,\varphi(u,\cdot)\|_{L^1(\mathbb{R}^d)}) &\leq C\,\mathcal{Q}(\varphi(u,\cdot)) \\ &\leq C\,\|g\|_{L^\infty(\mathbb{R}^d)}\, e^{C\int_0^t \|\mu_u\|_{\mathrm{BL}_V^*}\,\mathrm{d}u}.\end{aligned}$$

By Young's convolutional inequality, for any function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\begin{aligned}\|(\rho_\infty\,\varphi) * f(u,\cdot)\|_{L^\infty(\mathbb{R}^d)}, \|(\nabla\rho_\infty\,\varphi) * f(u,\cdot)\|_{L^\infty(\mathbb{R}^d)} &\leq \\ \leq C\,\|f\|_{L^\infty(\mathbb{R}^d)}\,\|g\|_{L^\infty(\mathbb{R}^d)}\, &e^{C\int_0^t \|\mu_u\|_{\mathrm{BL}_V^*}\,\mathrm{d}u}.\end{aligned} \tag{6.7}$$

Hence, applying it with $f = K, \Delta K, \partial_{x_i} K$ (for all $i = 1, ..., d$) and using Lemma 6.1 for the potential term, we conclude the proof. □

Lemma 6.3. *Suppose that K and V satisfy Assumptions 3.1 and 3.3. Let φ be a solution to (1.13) with g and $t > 0$ fixed. Then, there exists a constant C depending only on V and K such that*

$$\left\|\nabla\frac{\varphi(0,\cdot)}{1+V(\cdot)}\right\|_{L^\infty(\mathbb{R}^d)} \leq C\,(t+1)\,\|g\|_{BL(\mathbb{R}^d)}\, e^{C\int_0^t \|\mu_u\|_{BL_V^*}\,\mathrm{d}u} \tag{6.8}$$

Proof. As g is a Lipschitz function, its composition with the flow $X_{0,t}(x)$ is differentiable a.e. (see, for instance, [23]). Hence, we can differentiate the formula (6.1). Note that

$$\nabla \frac{\varphi(0,x)}{1+V(x)} = \frac{\nabla\varphi(0,x)}{1+V(x)} - \frac{\varphi(0,x)}{1+V(x)}\,\frac{\nabla V(x)}{1+V(x)}.$$

The second term is bounded by Lemma 6.2 and the assumption on the potential so it is sufficient to estimate $\frac{\nabla\varphi(0,x)}{1+V(x)}$. Differentiating each term in (6.1) with respect to x, dividing by $(1+V(x))$ and estimating $1 \le 1+V(x)$ when there are no terms with the potential V in the numerator, we get

$$\begin{aligned}
\left\|\frac{\nabla\varphi(0,\cdot)}{1+V(\cdot)}\right\|_{L^\infty(\mathbb{R}^d)} &\le \int_0^t \left\|(\nabla\rho_\infty\,\varphi) * \nabla^2 K(u,\cdot)\right\|_{L^\infty(\mathbb{R}^d)} \|\nabla X_{0,u}\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}u \\
&+ \int_0^t \|(\nabla\rho_\infty\,\varphi) * \nabla K(u,\cdot)\|_{L^\infty(\mathbb{R}^d)} \left\|\frac{\nabla V(X_{0,u}(\cdot))}{1+V(\cdot)}\right\|_{L^\infty(\mathbb{R}^d)} \|\nabla X_{0,u}\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}u \\
&+ \int_0^t \|(\nabla\rho_\infty\,\varphi) * K(u,\cdot)\|_{L^\infty(\mathbb{R}^d)} \left\|\frac{\nabla^2 V(X_{0,u}(\cdot))}{1+V(\cdot)}\right\|_{L^\infty(\mathbb{R}^d)} \|\nabla X_{0,u}\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}u \\
&+ \int_0^t \|(\rho_\infty\,\varphi) * \nabla\Delta K(u,\cdot)\|_{L^\infty(\mathbb{R}^d)} \|\nabla X_{0,u}\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}u \\
&+ \int_0^t \left\|(\rho_\infty\,\varphi) * \nabla^2 K(u,\cdot)\right\|_{L^\infty(\mathbb{R}^d)} \left\|\frac{\nabla V(X_{0,u}(\cdot))}{1+V(\cdot)}\right\|_{L^\infty(\mathbb{R}^d)} \|\nabla X_{0,u}\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}u \\
&+ \int_0^t \|(\rho_\infty\,\varphi) * \nabla K(u,\cdot)\|_{L^\infty(\mathbb{R}^d)} \left\|\frac{\nabla^2 V(X_{0,u}(\cdot))}{1+V(\cdot)}\right\|_{L^\infty(\mathbb{R}^d)} \|\nabla X_{0,u}\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}u \\
&+ \|\nabla g\|_{L^\infty(\mathbb{R}^d)} \left\|\frac{1+V(t,X_{0,t}(\cdot))}{1+V(\cdot)}\right\|_{L^\infty(\mathbb{R}^d)} \|\nabla X_{0,t}\|_{L^\infty(\mathbb{R}^d)} \\
&+ \|g\|_{L^\infty(\mathbb{R}^d)} \left\|\frac{\nabla V(X_{0,t}(\cdot))}{1+V(\cdot)}\right\|_{L^\infty(\mathbb{R}^d)} \|\nabla X_{0,t}\|_{L^\infty(\mathbb{R}^d)}\,.
\end{aligned}$$

Now, we obtain (6.8) directly from Lemma 6.1, (6.7) and the fact that $\|\nabla g(\cdot)\|_{L^\infty(\mathbb{R}^d)} \le |g(\cdot)|_{\mathrm{Lip}}$. ☐

Proof of Theorem 1.2. Thanks to Lemmas 6.2 and 6.3 we know that

$$\left\|\frac{\varphi(0,\cdot)}{1+V(\cdot)}\right\|_{\mathrm{BL}(\mathbb{R}^d)} \le C\,(t+1)\,\|g\|_{\mathrm{BL}(\mathbb{R}^d)}\, e^{C\int_0^t \|\mu_u\|_{\mathrm{BL}_V^*}\,\mathrm{d}u}.$$

where the constant C does not depend on g and t. Using (1.12) we obtain

$$\|\mu_t\|_{\mathrm{BL}_V^*} \le C\,(t+1)\,e^{C\int_0^t \|\mu_u\|_{\mathrm{BL}_V^*}\,\mathrm{d}u}\,\|\mu_0\|_{\mathrm{BL}_V^*}.$$

Using Lemma A.2, we arrive at (1.7). □

Appendix A. Grönwall-type inequalities

Lemma A.1 (backward Grönwall's inequality)**.** *Suppose that* $f, g, h : [0,T] \to \mathbb{R}^+$ *such that* h *is nonincreasing,* C *is a nonnegative constant and*

$$f(t) \leq h(t) + C \int_t^T g(s)\, f(s)\, \mathrm{d}s.$$

Then, $f(t) \leq h(t)\, e^{C \int_t^T g(u)\, \mathrm{d}u}$.

Proof. We change variables $u = T - s$ so that

$$f(T-(T-t)) \leq h(T-(T-t)) + C \int_0^{T-t} g(T-u)\, f(T-u)\, \mathrm{d}u.$$

Applying usual Grönwall's inequality to the function $s \mapsto f(T-s)$ (note that the function $s \mapsto h(T-s)$ is nondecreasing) we deduce

$$f(t) = f(T-(T-t)) \leq h(T-(T-t))\, e^{C \int_0^{T-t} g(T-u)\, \mathrm{d}u} = h(t)\, e^{C \int_t^T g(u)\, \mathrm{d}u}.$$

□

Lemma A.2. *Let* $y(t) : [0,\infty) \to \mathbb{R}^+$ *be a continuous function such that*

$$y(t) \leq \alpha(t)\, e^{C \int_0^t y(s)\, \mathrm{d}s}$$

for some $C > 0$ *and nondecreasing, nonnegative function* $\alpha(t)$*. Then,*

$$y(t) \leq \frac{\alpha(t)}{1 - C\, t\, \alpha(t)}$$

whenever $1 - C\, t\, \alpha(t) > 0$*.*

Proof. We slightly adapt the proof from [4, Lemma 3]. We fix $T > 0$ and consider $t \in [0,T]$. Then,

$$y(t) \leq \alpha(T)\, e^{C \int_0^t y(s)\, \mathrm{d}s}.$$

We let $z(t) = \alpha(T)\, e^{C \int_0^t y(s)\, \mathrm{d}s}$ and we note that $z'(t) = C\, z(t)\, y(t) \leq C\, z(t)^2$. Integrating this differential inequality, we get

$$z(t) \leq \frac{z(0)}{1 - t\, z(0)} = \frac{\alpha(T)}{1 - C\, t\, \alpha(T)}.$$

Taking $t = T$, we conclude the proof. □

Appendix B. Technical proofs from Section 3

Proof of Lemma 3.4. Let $g(x,y) = \frac{\nabla V(x)\cdot\nabla V(y)}{1+V(y)}\, K(x-y)$ and $h(x,y) = \frac{\nabla V(x)}{1+V(y)}\,\nabla K(x-y)$. If $p > 2$, we see, taking $x = y$, that g is not bounded which proves that $p \in (0,2]$ is a necessary condition.

Let $p \in (0,2]$. We need to prove that $g, \nabla_y g, h, \nabla_y h \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$. We will use the following inequality

$$|\nabla V(x)| \le C\,|\nabla V(y)| + C\,|\nabla V(x-y)| + C \tag{B.1}$$

which is a consequence of (3.1).

Boundedness of g. Using (B.1) we have

$$\frac{1}{C}\,|g(x,y)| \quad \le \quad \left(\frac{|\nabla V(y)|\,|\nabla V(y)|}{1+V(y)} + \frac{|\nabla V(x-y)|\,|\nabla V(y)|}{1+V(y)} + \frac{|\nabla V(y)|}{1+V(y)}\right) K(x \;-\; y).$$

The first and third terms are controlled since $p \le 2$ while the second uses additionally the control of $|\nabla V|\,K$.

Boundedness of h. We use (B.1) to get

$$\frac{1}{C}|h(x,y)| \le \frac{|\nabla V(y)|}{1+V(y)}\,\nabla K(x-y) + \frac{|\nabla V(x-y)|}{1+V(y)}\,\nabla K(x-y) + \frac{1}{1+V(y)}\,\nabla K(x-y).$$

To conclude, we use boundedness of $\frac{\nabla V}{1+V}$ (which holds for any $p > 0$) and $|\nabla V|\,|\nabla K|$.

Boundedness of $\nabla_y h$. By a direct computation,

$$\nabla_y h(x,y) = -\frac{\nabla V(x)\otimes\nabla V(y)}{(1+V(y))^2}\,\nabla K(x-y) - \frac{\nabla V(x)\otimes\nabla^2 K(x-y)}{1+V(y)} =: R_1 + R_2.$$

Using (B.1) we get

$$\frac{|R_1|}{C} \leq \left| \frac{\nabla V(y) \otimes \nabla V(y)}{(1+V(y))^2} \nabla K(x-y) \right| + \\ + \left| \frac{\nabla V(x-y) \otimes \nabla V(y)}{(1+V(y))^2} \nabla K(x-y) \right| + \frac{|\nabla V(y)|}{(1+V(y))^2} |\nabla K(x-y)|,$$

$$\frac{|R_2|}{C} \leq \left| \frac{\nabla V(y) \otimes \nabla^2 K(x-y)}{1+V(y)} \right| + \left| \frac{\nabla V(x-y) \otimes \nabla^2 K(x-y)}{1+V(y)} \right| + \frac{|\nabla^2 K(x-y)|}{1+V(y)}.$$

All the terms above are bounded because $\frac{\nabla V}{1+V}$, $|\nabla V|\,|\nabla K|$, $|\nabla V|\,|\nabla^2 K|$, $|\nabla^2 K|$ are bounded.

<u>Boundedness of $\nabla_y g$.</u> By a direct computation

$$\nabla_y g(x,y) = \frac{\nabla V(x) \cdot \nabla^2 V(y)}{1+V(y)} K(x-y) - \frac{\nabla V(x) \cdot \nabla V(y)\, \nabla V(y)}{(1+V(y))^2} K(x-y) \\ - \frac{\nabla V(x) \cdot \nabla V(y)}{1+V(y)} \nabla K(x-y) =: P_1 + P_2 + P_3.$$

Concerning the term P_1, we notice that since $p \leq 2$, $|\nabla^2 V| \leq C$ so that P_1 can be estimated by $\frac{|\nabla V(x)|}{1+V(y)} K(x-y) = |h(x,y)|$ which was proved to be bounded above.

Concerning the term P_2, we use (B.1) to get

$$\frac{|P_2|}{C} \leq \|K\|_{L^\infty(\mathbb{R}^d)} \left\| \frac{|\nabla V|^3}{(1+V)^2} \right\|_{L^\infty(\mathbb{R}^d)} + \left\| \frac{|\nabla V|^2}{(1+V)^2} \right\|_{L^\infty(\mathbb{R}^d)} \left(\|\nabla V\, K\|_{L^\infty(\mathbb{R}^d)} + \|K\|_{L^\infty(\mathbb{R}^d)} \right).$$

By the growth conditions (3.1) and $p \leq 2$, $\left\| \frac{|\nabla V|^3}{(1+V)^2} \right\|_{L^\infty(\mathbb{R}^d)}$ is finite and so, P_2 is bounded.

Concerning the term P_3, we argue as in P_2 to get

$$\frac{|P_3|}{C} \leq \|\nabla K\|_{L^\infty(\mathbb{R}^d)} \left\| \frac{|\nabla V|^2}{1+V} \right\|_{L^\infty(\mathbb{R}^d)} + \left\| \frac{|\nabla V|}{1+V} \right\|_{L^\infty(\mathbb{R}^d)} \left(\|\nabla V\, \nabla K\|_{L^\infty(\mathbb{R}^d)} + \|\nabla K\|_{L^\infty(\mathbb{R}^d)} \right).$$

The term $\left\| \frac{|\nabla V|^2}{1+V} \right\|_{L^\infty(\mathbb{R}^d)}$ is bounded because $p \leq 2$ and all the other terms are bounded by assumption. The proof is concluded. □

Proof of Lemma 3.5. Concerning (3.3), we only prove the first estimate. The second can be proved in the same way, replacing K with ∇K. We need to study two terms $\nabla K * \mu$ and

$K*(\mu\,\nabla V)$. For the first one,

$$\|\nabla K*\mu\|_{L^\infty(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\frac{\nabla K(x-y)}{1+V(y)}\,(1+V(y))\,\mathrm{d}\mu(y)\right| \leq$$

$$\leq \sup_{x\in\mathbb{R}^d}\left\|\frac{\nabla K(x-\cdot)}{1+V(\cdot)}\right\|_{\mathrm{BL}}\|\mu\|_{\mathrm{BL}_V^*} \leq \sup_{x\in\mathbb{R}^d}\|\nabla K(x-\cdot)\|_{\mathrm{BL}}\left\|\frac{1}{1+V}\right\|_{\mathrm{BL}}\|\mu\|_{\mathrm{BL}_V^*},$$

where $\frac{1}{1+V}\in \mathrm{BL}(\mathbb{R}^d)$ thanks to the growth condition (3.1). For the second one, we write

$$\|K*(\mu\,\nabla V)\|_{L^\infty(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d}K(x-y)\,\frac{\nabla V(y)}{1+V(y)}\,(1+V(y))\,\mathrm{d}\mu(y)\right| \leq$$

$$\leq \sup_{x\in\mathbb{R}^d}\left\|K(x-\cdot)\,\frac{\nabla V(\cdot)}{1+V(\cdot)}\right\|_{\mathrm{BL}} \leq \sup_{x\in\mathbb{R}^d}\|K(x-\cdot)\|_{\mathrm{BL}}\left\|\frac{\nabla V}{1+V}\right\|_{\mathrm{BL}}\|\mu\|_{\mathrm{BL}_V^*}$$

$$\leq \sup_{x\in\mathbb{R}^d}\|K\|_{\mathrm{BL}}\left\|\frac{\nabla V}{1+V}\right\|_{\mathrm{BL}}\|\mu\|_{\mathrm{BL}_V^*},$$

where $\frac{\nabla V}{1+V}\in \mathrm{BL}(\mathbb{R}^d)$ due to the growth condition (3.1). We proceed to the proof of (3.4) which requires condition (3.2). As before, we write

$$|\nabla V(x)\cdot K*\nabla\mu(x)| = \left|\int_{\mathbb{R}^d}\nabla V(x)\cdot\nabla K(x-y)\,\mathrm{d}\mu(y)\right| \leq$$

$$\leq \sup_{x\in\mathbb{R}^d}\left\|\frac{\nabla V(x)}{1+V(\cdot)}\cdot\nabla K(x-\cdot)\right\|_{\mathrm{BL}}\|\mu\|_{\mathrm{BL}_V^*}.$$

Finally, we conclude

$$|\nabla V(x)\cdot K*(\mu\,\nabla V)(x)| = \left|\int_{\mathbb{R}^d}\nabla V(x)\cdot\nabla V(y)\,K(x-y)\,\mathrm{d}\mu(y)\right| \leq$$

$$\leq \sup_{x\in\mathbb{R}^d}\left\|\frac{\nabla V(x)\cdot\nabla V(\cdot)}{1+V(\cdot)}\,K(x-\cdot)\right\|_{\mathrm{BL}}\|\mu\|_{\mathrm{BL}_V^*}.$$

□

Appendix C. Motivation for the dual equation (1.13)

In this section, we explain how to obtain the dual equation (1.13). Let μ_t satisfy (1.9). Using the weak formulation (2.6), for all $t\geq 0$ and all test functions $\varphi:[0,t]\times\mathbb{R}^d\to\mathbb{R}$ we have

$$\begin{aligned}&\int_{\mathbb{R}^d} \varphi(t,x)\,\mathrm{d}\mu_t(x) - \int_{\mathbb{R}^d} \varphi(0,x)\,\mathrm{d}\mu_0(x)\\ &= \int_0^t \int_{\mathbb{R}^d} \partial_s \varphi(s,x)\,\mathrm{d}\mu_s(x)\,\mathrm{d}s - \int_0^t \int_{\mathbb{R}^d} \nabla\varphi(s,x)\,K * (\nabla \mu_s + \nabla V\ \mu_s)\,\mathrm{d}\mu_s\,\mathrm{d}s\\ &\quad + \int_0^t \int_{\mathbb{R}^d} \varphi(s,x)\ \mathrm{div}\,(\rho_\infty\, K * (\nabla\mu_s + \nabla V\ \mu_s))\,\mathrm{d}x\,\mathrm{d}s\end{aligned}$$

We want to write the last term as an integral with respect to the measure μ_s. Denoting it by L, we have

$$\begin{aligned}L = \int_0^t \int_{\mathbb{R}^d} \varphi(s,x)\,\nabla\rho_\infty\, K * (\nabla\mu_s + \nabla V\ \mu_s)\,\mathrm{d}x\,\mathrm{d}s +\\ + \int_0^t \int_{\mathbb{R}^d} \varphi(s,x)\,\rho_\infty\,\nabla K * (\nabla\mu_s + \nabla V\ \mu_s)\,\mathrm{d}x\,\mathrm{d}s =: L_1 + L_2.\end{aligned}$$

For the term L_1, we observe that $K * \nabla\mu_s = \nabla K * \mu_s$. Using that K is symmetric and ∇K is antisymmetric (i.e. $\nabla K(-x) = -\nabla K(x)$)

$$L_1 = -\int_0^t \int_{\mathbb{R}^d} (\varphi(s,\cdot)\,\nabla\rho_\infty) * \nabla K\ \mathrm{d}\mu_s(x)\,\mathrm{d}s + \int_0^t \int_{\mathbb{R}^d} (\varphi(s,\cdot)\,\nabla\rho_\infty) * K\,\nabla V(x)\,\mathrm{d}\mu_s(x)\,\mathrm{d}s.$$

For the term L_2 we use that $\Delta K(-x) = \Delta K(x)$ and $\nabla K * \nabla\mu_s = \Delta K * \mu_s$ so we obtain

$$L_2 = \int_0^t \int_{\mathbb{R}^d} (\varphi(s,\cdot)\,\rho_\infty) * \Delta K\,\mathrm{d}\mu_s\,\mathrm{d}s - \int_0^t \int_{\mathbb{R}^d} (\varphi(s,\cdot)\,\rho_\infty) * \nabla K \cdot \nabla V\,\mathrm{d}\mu_s\,\mathrm{d}s.$$

Hence, we see that if φ satisfies (1.13), we obtain the desired identity (1.11).

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