



Volume gradients and homology in towers of residually-free groups

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Abstract We study the asymptotic growth of homology groups and the cellular volume of classifying spaces as one passes to normal subgroups $G_n < G$ of increasing finite index in a fixed finitely generated group G , assuming $\bigcap_n G_n = 1$. We focus in particular on finitely presented residually free groups, calculating their ℓ_2 betti numbers, rank gradient and asymptotic deficiency. If G is a limit group and K is any field, then for all $j \geq 1$ the limit of $\dim H_j(G_n, K)/[G, G_n]$ as $n \rightarrow \infty$ exists and is zero except for $j = 1$, where it equals $-\chi(G)$. We prove a homotopical version of this theorem in which the dimension of $\dim H_j(G_n, K)$ is replaced by the minimal number of j -cells in a $K(G_n, 1)$; this includes a calculation of the rank gradient and the asymptotic deficiency of G . Both the homological and homotopical versions are special cases of general results about the fundamental groups of graphs of *slow* groups. We prove that if a residually free group G is of type FP_m but not of type FP_∞ , then there exists an exhausting filtration by normal subgroups of finite index G_n so that $\lim_n \dim H_j(G_n, K)/[G : G_n] = 0$ for $j \leq m$. If G is of type FP_∞ , then the limit exists in all dimensions and we calculate it.

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1 Introduction

In this article we study the growth of homology groups and the cellular volume of classifying spaces as one passes to subgroups G_n of increasing finite index in a fixed finitely generated group G ; we are particularly interested in finitely presented residually free groups. For the most part we shall restrict our attention to *exhausting normal chains* (also called residual chains), i.e. we shall assume that the finite-index subgroups G_n are normal in G , are nested $G_{n+1} \subset G_n$, and that $\bigcap_{n \geq 0} G_n = 1$. It is easy to see that if the ambient group G is of type FP_m over a field K , then $\dim H_i(G_n, K)/[G : G_n]$ is bounded by a constant; but does this ratio always tend to a limit as $[G : G_n] \rightarrow \infty$, and if so, will the limit be a (significant) invariant of G or merely an artefact of the exhausting normal chain (G_n) that we chose?

The *approximation theorem* of Lück [25] provides an emphatic answer for fields of characteristic zero: if G is finitely presented and of type FP_m over \mathbb{Z} , then $\lim_n \dim H_i(G_n, K)/[G : G_n]$ exists for all $i < m$, the limit is independent of (G_n) , and is equal to the ℓ_2 betti number $\beta_i(G)$. (We refer the reader to [26] for a comprehensive introduction to ℓ_2 invariants, which originate in the work of Atiyah [4].) Much less is known in positive characteristic, although there are some positive results, e.g. [14, 24, 27].

In this article we shall consider homology gradients over arbitrary fields. We shall also study the following homotopical analogues. Given a group B of type \mathbb{F}_m , we define $\text{vol}_m(B)$ to be the least number of m -cells among all classifying spaces $K(B, 1)$ with a finite m -skeleton. For any group B of type \mathbb{F}_m and any field K one has $\dim_K H_m(B, K) \leq \text{vol}_m(B)$. Note that $\text{vol}_k(B)$ is equal to the rank (minimal number of generators) $d(B)$, and $\text{vol}_2(B)$ bounds the *deficiency* $\text{def}(B)$, which is¹ the infimum of $|R| - |X|$ over all finite presentations $\langle X | R \rangle$ of B . If (B_n) is a chain of finite-index subgroups in G , then the limit $\lim_n d(B_n)/[G : B_n]$ is known as the *rank gradient* of G with respect to the chain (B_n) and is often denoted $\text{RG}(G, (B_n))$. Lackenby [23] initiated the study of rank gradient in connection with the study of largeness for 3-manifolds [22] and a more combinatorial point of view was developed in [3]. We define the *m-dimensional volume gradient* of G with respect to the chain (B_n) to be $\lim_n \text{vol}_m(B_n)/[G : B_n]$.

We shall calculate these invariants for *residually free groups*. Let \mathbb{F}_r denote the free group of rank r . A group G is *residually free* if for every $g \in G \setminus \{1\}$ there exists a homomorphism $\phi_g : G \rightarrow \mathbb{F}_2$ such that $\phi_g(g) \neq 1$. The class of finitely generated residually free groups is rather wild, harbouring all manner of pathologies. On the other hand, groups that are *fully residually free* are closely akin to free groups in many ways. By definition, a group is *fully residually free* if for every finite $S \subset G$ there is a homomorphism $\phi_S : G \rightarrow \mathbb{F}_2$ that is injective on S . Such groups are now more commonly called *limit groups*, following Sela [30]. This remarkable class of groups is the class one constructs when one attempts to formulate the notion of an “approximately free group” in various natural ways. For example, they are the finitely generated groups that have the same universal theory as a free group (in the sense

¹ There are different conventions, with many authors taking the opposite sign for the deficiency of an individual presentation, and defining the deficiency of the group to be the supremum of $|X| - |R|$.

of first order logic); they are the groups that arise as Gromov-Hausdorff limits of sequences of marked free groups [13]; and they are the groups one obtains by taking limits of *stable* sequences of homomorphisms from a fixed group to a free group [30]. Basic examples of limit groups are the fundamental groups of closed surfaces of positive genus, free abelian groups, doubles of free groups along maximal cyclic subgroups, and free products of any finite collection of the foregoing groups. Limit groups have finite classifying spaces.

Theorem A (Volume Gradients for Limit Groups) *Let (B_n) be an exhausting chain of finite-index normal subgroups in a limit group G . Then,*

- (1) *Rank Gradient $\frac{d(B_n)}{[G : B_n]} \rightarrow -\chi(G)$ as $n \rightarrow \infty$;*
- (2) *Deficiency Gradient $\frac{\text{def}(B_n)}{[G : B_n]} \rightarrow \chi(G)$ as $n \rightarrow \infty$;*
- (3) *$\frac{\text{vol}_k(B_n)}{[G : B_n]} \rightarrow 0$ as $n \rightarrow \infty$, for all $k \geq 2$.*

Corollary B (Asymptotic Homology of Limit Groups) *Let K be a field. If G is a limit group and (B_n) is an exhausting sequence of normal subgroups of finite index in G , then*

$$\lim_{n \rightarrow \infty} \frac{\dim H_j(B_n, K)}{[G : B_n]} = \begin{cases} -\chi(G) & \text{if } j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that when $\text{char}(K) = p$ we do not assume that $[G : B_n]$ is a power of p (cf. [22]).

Corollary C *If G is a limit group, then $\beta_j^{(2)}(G) = 0$ for $j \neq 1$ and $\beta_1^{(2)}(G) = -\chi(G)$.*

The structure theory of limit groups lends itself well to inductive arguments: there is a hierarchical structure on the class of such groups; free groups, free abelian groups and surface groups lie at the bottom level of the hierarchy, and if one can show that these groups enjoy a certain property then, proceeding by induction, one can deduce that all limit groups satisfy that property provided that the property is preserved under the formation of free products and HNN extensions along cyclic subgroups; see Sect. 2. Given the nature of the induction step, one is drawn naturally into Bass-Serre theory and, in the case of the results stated above, counting arguments involving double coset decompositions of finite-index subgroups; see Sects. 5 and 3.

Correspondingly, our proof of Theorem A proceeds via more technical results that relate the growth of cellular volume in classifying spaces for fundamental groups of graphs of groups to the growth in the vertex and edge groups of the decomposition. The key idea that gives us control in this context is *slowness* for groups. A group G of type F is *slow above dimension 1* if it is residually finite and for every exhausting normal chain of finite-index normal subgroups (B_n) there exists a finite $K(B_n, 1)$ with $r_k(B_n)$ k -cells such that

$$\lim_{n \rightarrow \infty} \frac{r_k(B_n)}{[G : B_n]} = 0$$

for all $k \geq 2$. We say that G is *slow* if it satisfies the additional requirement that the limit exists and is zero for $k = 1$ as well. A key result in Sect. 4 is Proposition 4.5: If

a residually-finite group G is the fundamental group of a finite graph of groups where all of the edge-groups are slow and all of the vertex-groups are slow above dimension 1, then G is slow above dimension 1.

Theorem D *If a residually finite group G of type \mathbb{F} is slow above dimension 1, then with respect to every exhausting normal chain (B_n) ,*

(1) *Rank gradient*

$$\mathrm{RG}(G, (B_n)) = \lim_{n \rightarrow \infty} \frac{d(B_n)}{[G : B_n]} = -\chi(G),$$

(2) *Deficiency gradient*

$$\mathrm{DG}(G, (B_n)) = \lim_{n \rightarrow \infty} \frac{\mathrm{def}(B_n)}{[G : B_n]} = \chi(G).$$

In Sect. 5 we consider a homological condition called K -slowness (Definition 5.1) and prove a homological analogue of Theorem D.

In the second part of this paper we focus on the class of finitely presented residually free groups. This class is much wilder than that of limit groups and the structure theory is correspondingly more awkward. Thus the structure of the arguments in the second half of the paper is more subtle and demanding than those in the first half: there are many layers of arguments using spectral sequences and they draw on finer structural information about the groups involved. Our starting point is the theorem of Bridson, Howie, Miller and Short [8] which states that a finitely presented group is residually free if and only if it can be realised as a subgroup of a direct product of finitely many limit groups so that its projection to each pair of factors is of finite index. Our main result concerning residually free groups is the following.

Theorem E *Let $m \geq 2$ be an integer, let G be a residually free group of type FP_m , and let ρ be the largest integer such that G contains a direct product of ρ non-abelian free groups. Then, there exists an exhausting sequence (B_n) so that for all fields K ,*

- (1) *if G is not of type FP_∞ , then $\lim_n \frac{\dim H_i(B_n, K)}{[G : B_n]} = 0$ for all $0 \leq i \leq m$;*
- (2) *if G is of type FP_∞ then for all $j \geq 1$,*

$$\lim_{n \rightarrow \infty} \frac{\dim H_j(B_n, K)}{[G : B_n]} = \begin{cases} (-1)^\rho \chi(G) & \text{if } j = \rho \\ 0 & \text{otherwise.} \end{cases}$$

Lück's Approximation theorem tells us that when K is a field of characteristic 0, the limits calculated in Theorem E are the ℓ_2 betti numbers of G . In this case, one knows that the limit is independent of the sequence (B_n) , but for fields of positive characteristic we do not know this, nor do we know if the limit exists for an arbitrary exhausting normal chain in G .

We prove Theorem E by using the structure theory of residually free groups to reduce it to a special case of the following result that we hope will have further applications. The proof of this theorem is presented in Sect. 6; it accounts for almost half the length of this paper.

Theorem F Let $G \subseteq G_1 \times \cdots \times G_k$ be a subdirect product of residually-finite groups of type \mathbb{F} , each of which contains a normal free subgroup $F_i < G_i$ such that G_i/F_i is torsion-free and nilpotent. Assume $F_i \subseteq G \cap G_i$. Let $m < k$ be an integer, let K be a field, and suppose that each G_i is K -slow above dimension 1.

If the projection of G to each m -tuple of factors $G_{j_1} \times \cdots \times G_{j_m} < G$ is of finite index, then there exists an exhausting normal chain (B_n) in G so that for $0 \leq j \leq m$,

$$\lim_{n \rightarrow \infty} \frac{\dim H_j(B_n, K)}{[G : B_n]} = 0.$$

Our proof of Theorem F shows that the homology groups $H_j(B_n, K)$ are finite dimensional for $j \leq m$. The Weak Virtual Surjections Theorem [21, Corollary 5.5] implies that this finiteness holds more generally.

One would like to promote Theorem E to a theorem about volume gradients, in the spirit of Theorem 4.6, but for the moment this is obstructed by unresolved conjectures concerning the relationship between finiteness properties of residually free groups and the projections to m -tuples of factors in their existential envelopes (in the sense of [9]). However, in low dimensions these conjectures have been resolved, and that enables us to prove the following theorem, which is the subject of the final section of this paper.

Theorem G Every finitely presented residually free group G that is not a limit group admits an exhausting normal chain (B_n) with respect to which the rank gradient

$$\text{RG}(G, (B_n)) = \lim_{n \rightarrow \infty} \frac{d(B_n)}{[G : B_n]} = 0.$$

Furthermore, if G is of type FP_3 but is not commensurable with a product of two limit groups, (B_n) can be chosen so that the deficiency gradient $\text{DG}(G, (B_n)) = 0$.

The recent work of Abert and Gaboriau [1] on higher-cost for group actions recovers parts (1) and (2) of our Theorem A and establishes similar results for larger classes of groups, including mapping class groups. Earlier work of Abert and Nikolov [2] links rank gradient to cost in the sense of measurable group theory. Part of the original motivation for this work also comes from measurable group theory. Groups G_1 and G_2 are *measure equivalent* if they admit commuting, measure-preserving, free actions on the same probability space. Gaboriau [17] proved that surface groups are measure equivalent to free groups and asked if this was true of all limit groups. This has been answered in the affirmative for elementarily free groups by Bridson, Tweeddale and Wilton [11] but remains open for limit groups in general. Gaboriau [15] proved that if G_1 is measure equivalent to G_2 , then the ℓ_2 betti numbers of the groups are proportional, i.e. there exists a constant c such that $\beta_i^{(2)}(G_1) = c \beta_i^{(2)}(G_2)$ for all $i \geq 1$. The ℓ_2 betti numbers of a finitely generated free group are zero except in dimension 1, where $\beta_1^{(2)}(F_r) = -\chi(F)$, so Corollary C is compatible with a positive answer to Gaboriau's question.

2 Limit groups and residually free groups

In this section we isolate the basic properties of residually free groups and limit groups that we need in later sections, providing references where the reader unfamiliar with these fascinating groups can find more details.

2.1 ω -residually free towers and splittings of limit groups

Limit groups have several equivalent definitions, each highlighting a different aspect of their nature. In the introduction we defined them to be the finitely generated fully residually free groups, but for the practical purposes of proving our theorems it is most useful to work with one of the less intuitively-appealing definitions: a *limit group* is a finitely generated subgroup of an ω -residually free tower groups (abbreviated ω -rft group); see [31] Theorem 1.1 and [19].

ω -rft spaces of height $h \in \mathbb{N}$ are defined by an induction on h and, by definition, an ω -rft group is the fundamental group of an ω -rft space. A height 0 tower is the 1-point union of a finite collection of circles, closed hyperbolic surfaces and tori (of arbitrary dimension), except that the closed surface of Euler characteristic -1 is excluded. An ω -rft space Y of height h is obtained from an ω -rft space Y_0 of height $h - 1$ by adding either (1) a torus T of some dimension, attached to Y_0 by identifying a coordinate circle in T with any loop c in Y_0 such that $[c]$ generates a maximal cyclic subgroup of $\pi_1 Y_0$, or (2) a connected, compact surface S that is either a punctured torus or has Euler characteristic at most -2 , where the attachment identifies each boundary component of S with a homotopically non-trivial loop in Y_0 , chosen so that there exists a retraction $r : Y \rightarrow Y_0$ sending $\pi_1 S$ to a non-abelian subgroup of $\pi_1 Y_0$.

By definition, the *height of a limit group* G is the minimal height of an ω -rft group that has a subgroup isomorphic to G . Limit groups of height 0 are free products of finitely many free abelian groups and of surface groups of Euler characteristic at most -2 . The Seifert-van Kampen Theorem decomposes an ω -rft group as a 2-vertex graph of groups with cyclic edge group, where one of the vertices is an ω -rft group of lesser height, the other is free or free-abelian of finite rank at least 2, and the edge groups are cyclic. Thus one can apply Bass-Serre theory to decompose an arbitrary limit group as follows—see [7, Lemma 1.3].

Lemma 2.1 *If G is a limit group of height $h \geq 1$, then G is the fundamental group of a finite bipartite graph of groups Δ in which the edge groups are cyclic; the vertex groups fall into two types corresponding to the bipartite partition of vertices: type (i) vertex groups are isomorphic to subgroups of a limit group of height $h - 1$; type (ii) vertex groups are all free or all free-abelian.*

In the first part of this paper, the only properties of a limit group that we shall use are residual finiteness and the following decomposition property:

Corollary 2.2 *If a class of groups \mathcal{C} contains all finitely generated free abelian and surface groups and is closed under the formation of amalgamated free products and HNN extensions with cyclic amalgamated groups, then \mathcal{C} contains all limit groups.*

In the second part of the paper we shall need the following additional property of limit groups, which was established by Kochloukova [20].

Theorem 2.3 *Every limit group G has a normal subgroup F that is free with G/F torsion-free and nilpotent.*

2.2 Residually free groups and subdirect products

By definition, a group G is residually free if it is isomorphic to a subgroup of an unrestricted direct product of free groups. In general, one requires infinitely many factors in this direct product, even if G is finitely generated. For example, the fundamental group of a closed orientable surface Σ is residually free but it cannot be embedded in a finite direct product of free groups if $\chi(\Sigma) < 0$, since $\pi_1 \Sigma$ does not contain \mathbb{Z}^2 and is not a subgroup of a free group. However, Baumslag, Myasnikov and Remeslennikov [6, Corollary 19] proved that one can force the enveloping product to be finite at the cost of replacing free groups by *limit groups* (see also [19, Corollary 2] and [30, Claim 7.5]).

Bridson, Howie, Miller and Short [8, 9] characterized the *finitely presented residually free groups* as follows:

Theorem 2.4 *A finitely presented group G is residually free if and only if it can be embedded in a direct product of finitely many limit groups $G \hookrightarrow D := \Lambda_1 \times \cdots \times \Lambda_n$ so that the intersection with each factor is non-trivial and the projection $p_{ij}(G) < \Lambda_i \times \Lambda_j$ to each pair of factors is a subgroup of finite index.*

*Moreover**, in these circumstances, there is a subgroup of finite index $D_0 < D$ such that G contains the $(n - 1)$ -st term of the lower central series of D_0 .*

The “only if” implication in the first sentence of the above theorem was generalized by Kochloukova [20] as follows.

Theorem 2.5 *Let G be a subdirect product of non-abelian limit groups and let $s \geq 2$ be an integer. If G is of type FP_s , then the projection of G to the direct product of each s -tuple of these limit groups has finite index.*

We shall also need Theorem A of [8].

Theorem 2.6 *Every residually free group of type FP_∞ is a subgroup of finite index in a direct product of limit groups.*

3 Bass-Serre theory and cellular volume

We assume that the reader is familiar with Bass-Serre theory as laid out in [32] and with the more topological interpretation described in [29]. We recall some of the basic features of this theory and fix our notation.

For us, a graph X consists of two sets V (the vertices) and E (the unoriented edges, or 1-cells). There are maps $\iota : E \rightarrow V$ and $\tau : E \rightarrow V$, and we allow $\iota(e) = \tau(e)$. We require that the graph be connected in the sense that the equivalence relation generated by $[\iota(e) \sim \tau(e) : e \in E]$ has only one equivalence class in V . There are two sets

of groups: the *vertex groups* G_v , indexed by V , and the *edge groups* G_e , indexed by E , together with monomorphisms $\iota_e : G_e \rightarrow G_{\iota(e)}$ and $\tau_e : G_e \rightarrow G_{\tau(e)}$. A graph of groups \mathcal{G} consists of the above data. It is termed finite if $V \cup E$ is finite. Serre associates to these data a *fundamental group* $G = \pi_1 \mathcal{G}$ and a left action of G on a tree $\tilde{\mathcal{G}}$ so that (modulo some natural identifications) the topological quotient² is X and the pattern of isotropy groups and inclusions corresponds to the original edge and vertex groups $G_v, G_e < G$.

If $B < G$ is a subgroup, then the graph of groups $B \backslash \tilde{\mathcal{G}}$, which has fundamental group B , has vertex groups $\{G_v^g \cap B \mid v \in V, BgG_v \in B \backslash G/G_v\}$ and edge groups $\{G_e^g \cap B \mid e \in E, BgG_e \in B \backslash G/G_e\}$. In particular, if B is normal and of finite index, then for each $v \in V$ there are $|G/G_vB|$ vertices in $B \backslash \tilde{\mathcal{G}}$ where the vertex group is a conjugate of $G_v \cap B$, and for each $e \in E$ there are $|G/G_eB|$ vertices where the edge group is a conjugate of $G_e \cap B$.

3.1 A model for $K(G, 1)$ when $G = \pi_1 \mathcal{G}$

Given explicit CW models for the classifying spaces $K(G_v, 1)$ and $K(G_e, 1)$, one can realise the monomorphisms ι_e and τ_e by cellular maps, $I_e : K(G_e, 1) \rightarrow K(G_v, 1)$ and $T_e : K(G_e, 1) \rightarrow K(G_v, 1)$. Attaching the ends of $K(G_e, 1) \times [0, 1]$ to $K(G_{\iota(e)}, 1) \amalg K(G_{\tau(e)}, 1)$ by means of these maps, for each $e \in E$, we obtain an explicit CW model for $K(G, 1)$.

Note that for each $k \geq 1$, the set of k -cells in $K(G, 1)$ is in bijection with the union of the sets of k -cells in $K(G_v, 1)$ [$v \in V$] together with the $(k-1)$ -cells in $K(G_e, 1)$ [$e \in E$], where an open $(k-1)$ -cell c in $K(G_e, 1)$ contributes the open k -cell $c \times (0, 1)$ to $K(G, 1)$. We single this simple observation out as a lemma because it is central to what follows. In order to state this lemma we need the following terminology.

Definition 3.1 Let G be a group. A sequence of non-negative integers $(r_k)_{k \geq 1}$ is a *volume vector* for G if there is a classifying space $K(G, 1)$ that, for all $k \in \mathbb{N}$, has exactly r_k open k -cells.

When we are discussing several groups, we shall abuse notation by writing the entries of such a vector as $r_k(G)$ but this is *not* meant to imply that $r_k(G)$ is an invariant of G .

Note that if G is type F_∞ then $\text{vol}_k(G)$ is the infimum of $r_k(G)$ over all volume vectors for G .

Lemma 3.2 Let \mathcal{G} be a finite graph of groups and let $G = \pi_1 \mathcal{G}$. With the notation established above, suppose that $(r_k(G_v))$ is a volume vector for G_v ($v \in V$) and $(r_k(G_e))$ is a volume vector for G_e ($e \in E$). For $k \geq 1$, let

$$r_k(G) := \sum_{v \in V} r_k(G_v) + \sum_{e \in E} r_{k-1}(G_e).$$

Then $(r_k(G))$ is a volume vector for G .

² For $H < G$, we write $H \backslash \tilde{\mathcal{G}}$ to denote the quotient graph of groups, which records the isotropy groups and their inclusions as well as the topological quotient.

We are interested in what happens to $\text{vol}_k(G)$ as we pass to subgroups of increasing index, and we shall do this by constructing suitably-controlled volume vectors. The most obvious way of getting models $K(B_n, 1)$ for subgroups $B_n < G$ is to simply take the corresponding covering spaces of a fixed model for G . However, this model is not efficient enough for our purposes: in general it has too many cells. A simple example that illustrates this is the case $G = \mathbb{Z}^r$: the number of k -cells in the cover corresponding to $B_n < G$ goes to infinity as $[G : B_n] \rightarrow \infty$, but $\text{vol}_k(B_n)$ remains constant since $B_n \cong \mathbb{Z}^r$ for all n .

To avoid the phenomenon illustrated by this example, given a finite-index subgroup $B < G$ we first pass to the covering graph-of-groups $B \backslash \tilde{\mathcal{G}}$, where $\tilde{\mathcal{G}}$ is the universal covering (tree) for \mathcal{G} . In the graph of groups $B \backslash \tilde{\mathcal{G}}$, for each $v \in V$ the vertices lying above v are indexed by the double cosets $B \backslash G / G_v$, and the vertex group at the vertex indexed by BgG_v is $B^g \cap G_v$. Likewise, the edges of $B \backslash \tilde{\mathcal{G}}$ are indexed by $\coprod_{e \in E} B \backslash G / G_e$ and the edge groups have the form $B^g \cap G_e$. We now assemble $K(B, 1)$ from classifying spaces for the edge and vertex groups, as described in paragraph (3.1). If B is normal, then we take the same classifying space above each of the vertices indexed by a fixed vertex or edge of \mathcal{G} .

We are interested only in the case where B is normal. In that case, the above discussion shows that the vertices of the finite graph of groups $B \backslash \tilde{\mathcal{G}}$ are index by cosets G / BG_v , and the edges by G / BG_e , and from Lemma 3.2 we obtain the following count:

Proposition 3.3 *Let G be the fundamental group of a finite graph of groups with vertex groups G_v ($v \in V$) and edge groups G_e ($e \in E$). Let $B < G$ be a normal subgroup of finite index. Suppose that volume vectors $(r_k(B \cap G_v))$ and $(r_k(B \cap G_e))$ are given and define*

$$r_k(B) := \sum_{v \in V} [G : BG_v] r_k(B \cap G_v) + \sum_{e \in E} [G : BG_e] r_{k-1}(B \cap G_e).$$

Then $(r_k(B))$ is a volume vector for B .

It is clear from the above formula that we will need to count cosets carefully. The following trivial observation is useful in this regard.

Remark 3.4 Let G be a group, let $H < G$ be a subgroup and let $B \triangleleft G$ be a normal subgroup of finite index. Then $\frac{[G:BH]}{[G:B]} = \frac{1}{[H:B \cap H]}$.

4 Volume gradient, slow groups and hierarchies

Definition 4.1 A group G of type F is *slow above dimension 1* if it is residually finite and for every chain of finite-index normal subgroups (B_n) with $\bigcap_n B_n = \{1\}$, there exist volume vectors $(r_k(B_n))_k$ with only finitely many non-zero entries, so that

$$\lim_{n \rightarrow \infty} \frac{r_k(B_n)}{[G : B_n]} = 0$$

for all $k \geq 2$.

G is *slow* if it satisfies the additional requirement that the limit exists and is zero for $k = 1$ as well.

The following theorem was stated in the introduction as Theorem D.

Theorem 4.2 *If a residually finite group G of type \mathbb{F} is slow above dimension 1, then with respect to every exhausting normal chain (B_n) ,*

(1) *Rank gradient*

$$\mathrm{RG}(G, (B_n)) = \lim_{n \rightarrow \infty} \frac{d(B_n)}{[G : B_n]} = -\chi(G),$$

(2) *Deficiency gradient*

$$\mathrm{DG}(G, (B_n)) = \lim_{n \rightarrow \infty} \frac{\mathrm{def}(B_n)}{[G : B_n]} = \chi(G).$$

Lemma 4.3 *Let G be a residually-finite group of type \mathbb{F} with an exhausting normal chain (B_n) . Suppose that G is slow above dimension 1 and choose volume vectors $(r_k(B_n))_k$ as in the definition. Then*

$$\lim_{n \rightarrow \infty} \frac{r_0(B_n) - r_1(B_n)}{[G : B_n]} = \chi(G),$$

and for every field K

$$\lim_{n \rightarrow \infty} \frac{H_1(B_n, K)}{[G : B_n]} = -\chi(G).$$

Proof As B_n is of type \mathbb{F} , it has an Euler characteristic, which may be calculated from a finite $K(B_n, 1)$ with $r_k(B_n)$ cells of dimension k ,

$$\chi(B_n) = r_0(B_n) - r_1(B_n) + r_2(B_n) - r_3(B_n) + \cdots + (-1)^{k_0} r_{k_0}(B_n).$$

Euler characteristic is multiplicative in the sense that $\chi(B_n) = [G : B_n] \chi(G)$. To obtain the first equality, we divide by $[G : B_n]$ and let $n \rightarrow \infty$.

Towards the second equality, observe that since the homology of B_n can be computed from the cellular chain complex of $K(B_n, 1)$, we have $r_k(B_n) \geq \dim H_k(B_n, K)$, and therefore $\lim_n \dim H_k(B_n, K)/[G : B_n] = 0$ for $k \geq 2$ (by slowness). Thus the second equality can be obtained by calculating $\chi(B_n)$ as the alternating sum of betti numbers (omitting the coefficients K)

$$[G : B_n] \chi(G) = \chi(B_n) = 1 - \dim H_1(B_n) + \cdots + (-1)^{k_0} \dim H_{k_0}(B_n),$$

then dividing through by $[G : B_n]$ and taking the limit. \square

Proof of Theorem 4.2 First we prove (1). Let $r_k(B_n)$ be as in Lemma 4.3. Now, B_n is a quotient of the fundamental group of the 1-skeleton of $K(B_n, 1)$, which is a free group of rank $r_1(B_n) - r_0(B_n) + 1$, so

$$\dim H_1(B_n, \mathbb{Q}) \leq d(B_n) \leq r_1(B_n) - r_0(B_n) + 1.$$

If we divide by $[G : B_n]$ and take the limit, both sides will converge to $-\chi(G)$, by Lemma 4.3.

Turning to the proof of (2), we remind the reader that the *deficiency* $\text{def}(\Gamma)$ is the infimum of $|R| - |X|$ over all possible finite presentations $\langle X \mid R \rangle$ of Γ .

From the 2-skeleton of any $K(B_n, 1)$ we get a group presentation with $r_1(B_n) - r_0(B_n) + 1$ generators and $r_2(B_n)$ relators, so

$$r_2(B_n) - r_1(B_n) + r_0(B_n) - 1 \geq \text{def}(B_n).$$

And from [10, Lemma 2] we have

$$\begin{aligned} \text{def}(B_n) &\geq d(H_2(B_n, \mathbb{Z})) - \text{rk}_{\mathbb{Q}} H_1(B_n, \mathbb{Z}) \\ &= d(H_2(B_n, \mathbb{Z})) - \dim H_1(B_n, \mathbb{Q}) \\ &\geq \dim H_2(B_n, \mathbb{Q}) - \dim H_1(B_n, \mathbb{Q}). \end{aligned}$$

Thus

$$r_2(B_n) - r_1(B_n) + r_0(B_n) - 1 \geq \text{def}(B_n) \geq \dim H_2(B_n, \mathbb{Q}) - \dim H_1(B_n, \mathbb{Q}).$$

We divide by $[G : B_n]$ and let n go to infinity. Since G is slow above dimension 1, both $r_2(B_n)/[G : B_n]$ and $\dim H_2(B_n)/[G : B_n]$ tend to zero, and by Lemma 4.3 the limit of what remains on each side tends to $\chi(G)$. \square

Examples 4.4 Easy examples of slow groups include finitely generated torsion-free nilpotent groups. The trivial group is slow. Free groups are slow above dimension 1, trivially. Surface groups are slow above dimension 1 because a finite-index subgroup of a surface group is again a surface group, so for any finite-index subgroup B we have the volume vector $1, d(B), 1, 0, \dots$

A far greater range of examples is provided by the following construction.

4.1 Hierarchies

Given a class \mathcal{C}_0 of residually-finite groups, we define $\mathcal{S}(\mathcal{C}_0) = \bigcup_n \mathcal{C}_n$ inductively by decreeing that \mathcal{C}_{n+1} consist of all residually-finite groups that can be expressed as the fundamental group of a finite graph of groups with edge-groups that are slow and vertex-groups that lie in \mathcal{C}_n . The following proposition tells us that if the groups in \mathcal{C}_0 are slow above dimension 1, then the groups in $\mathcal{S}(\mathcal{C}_0)$ are as well. Note that this includes the statement that free products of groups that are slow above dimension 1 are again slow above dimension 1.

Proposition 4.5 *If a residually-finite group G is the fundamental group of a finite graph of groups where all of the edge-groups are slow and all of the vertex-groups are slow above dimension 1, then G is slow above dimension 1.*

Proof Given a sequence of finite index normal subgroups (B_n) exhausting G , we build classifying spaces $K(B_n, 1)$ as in Proposition 3.3 so that they have volume vectors $(r_k(B_n))$ satisfying

$$r_k(B_n) = \sum_{v \in V} [G : B_n G_v] r_k(B_n \cap G_v) + \sum_{e \in E} [G : B_n G_e] r_{k-1}(B_n \cap G_e), \quad (4.1)$$

where G_v and G_e are the vertex and edge groups in the given decomposition of G . By hypothesis, the G_v are slow above dimension 1, so for all $k \geq 2$ we have

$$\lim_{n \rightarrow \infty} \frac{r_k(B_n \cap G_v)}{[G_v : B_n \cap G_v]} = 0.$$

But, as we noted in Remark 3.4, $[G_v : B_n \cap G_v] = [G : B_n] / [G : B_n G_v]$, so dividing equality (4.1) by $[G : B_n]$, the first sum becomes

$$\sum_{v \in V} \frac{r_k(B_n \cap G_v)}{[G_v : B_n \cap G_v]},$$

which converges to 0 as $n \rightarrow \infty$. Likewise, since G_e is assumed to be slow (in all dimensions including 1), we have

$$\frac{1}{[G : B_n]} \sum_{e \in E} [G : B_n G_e] r_{k-1}(B_n \cap G_e) = \sum_{e \in E} \frac{r_{k-1}(B_n \cap G_e)}{[G_e : B_n \cap G_e]} \rightarrow 0$$

as $n \rightarrow \infty$. Thus $\frac{r_k(B_n)}{[G : B_n]} \rightarrow 0$ as $n \rightarrow \infty$ for all $k \geq 2$, as required. \square

We saw in Example 4.4 that all finitely generated free groups, surface groups, and free-abelian groups of finite rank are slow above dimension 1, and Corollary 2.2 tells us that if \mathcal{C}_0 contains these groups then $\mathcal{S}(\mathcal{C}_0)$ contains all *limit groups*. Thus Proposition 4.5 implies:

Theorem 4.6 *All limit groups are slow above dimension 1.*

4.2 Proof of Theorem A

Immediate from Theorems 4.2 and 4.6.

5 Homological slowness

In this section we present homological analogues of the results in the previous section. The results for residually free groups that are the main focus of this article can be

deduced using these homological results rather than the homotopical ones, but this is not the point of presenting this variation on our earlier theme. The real justification is that these homological results apply to more groups. For example, one does not require the groups to have classifying spaces with finite skeleta. Also, if an amenable group G of type \mathbb{F} is residually finite and the group algebra KG does not have zero divisors, then one knows that G is K -slow in the following sense for any field K [24, Theorem 0.2(ii)] but we do not know that it must be slow. Thus one can use amenable groups among the building blocks for groups built in a hierarchical manner by repeated application of Proposition 5.3.

Definition 5.1 Let K be a field and let G be a residually finite group. G is K -slow above dimension 1 if for every chain of finite-index normal subgroups (B_n) with $\bigcap_n B_n = \{1\}$,

$$\lim_{n \rightarrow \infty} \frac{\dim_K H_j(B_n, K)}{[G : B_n]} = 0$$

for all $j \geq 2$.

G is K -slow if it satisfies the additional requirement that the limit exists and is zero for $j = 1$ as well.

The argument given in Lemma 4.3 establishes the following result.

Lemma 5.2 Let K be a field and let G be a residually finite group of type \mathbb{F} with an exhausting normal chain (B_n) . If G is K -slow above dimension 1 then

$$\lim_{n \rightarrow \infty} \frac{\dim H_1(B_n, K)}{[G : B_n]} = -\chi(G).$$

Proposition 5.3 Let K be a field. If a residually-finite groups G is the fundamental group of a finite graph of groups where all of the edge-groups are K -slow and all of the vertex-groups are K -slow above dimension 1, then G is K -slow above dimension 1.

Proof Given a sequence of finite index normal subgroups (B_n) exhausting G , we build classifying spaces $K(B_n, 1)$ as in Proposition 3.3. Our aim is to decompose these spaces and use the Mayer-Vietoris sequence of this decomposition to establish the following inequality for all $j \geq 2$ (expressed in the notation of Sect. 3.1, with the coefficients K omitted),

$$\begin{aligned} \dim_K H_j(B_n) &\leq \sum_{v \in V} [G : B_n G_v] \dim_K H_j(B_n \cap G_v) \\ &\quad + \sum_{e \in E} [G : B_n G_e] \dim_K H_j(B_n \cap G_e) \\ &\quad + 2 \sum_{e \in E} [G : B_n G_e] \dim_K H_{j-1}(B_n \cap G_e). \end{aligned} \quad (5.1)$$

The proof can then be completed as in Proposition 4.5: we divide by $[G : B_n]$, let $n \rightarrow \infty$ and use K -slowness (and a simple coset counting identity) to conclude that $\lim_n \dim_K H_j(B_n, K)/[G : B_n] = 0$, as claimed.

The desired decomposition of $K(B_n, 1) = X_n \cup Y_n$ is obtained as follows. In paragraph 3.1 we described how to assemble $K(B_n, 1)$ from vertex spaces $K(B_n \cap G_v, 1)$ and edge spaces $K(B_n \cap G_e, 1) \times [0, 1]$ according to the template of the graph of groups $B_n \backslash \tilde{\mathcal{G}}$, where \mathcal{G} is the given graph of groups with $G = \pi_1 \mathcal{G}$, and $\tilde{\mathcal{G}}$ is its universal cover. We define X_n to be the subset consisting of the images of the open edge spaces $K(B_n \cap G_e, 1) \times (0, 1)$, together with the underlying graph $B_n \backslash \tilde{\mathcal{G}}$ (the addition of which makes X_n connected). We define Y'_n to be the union of the vertex spaces together with the underlying graph $B_n \backslash \tilde{\mathcal{G}}$. We then expand Y'_n into each edge space $K(B_n \cap G_e, 1) \times (0, 1)$, adding a small cylinder $K(B_n \cap G_e, 1) \times (0, \epsilon)$ at each end of the edge and obtain Y_n this way. Note that Y'_n deformation retracts onto Y_n , and $X_n \cap Y_n$ is easy to describe: it consists of the underlying graph $B_n \backslash \tilde{\mathcal{G}}$ plus two disjoint copies of $K(B_n \cap G_e, 1) \times (0, \epsilon)$ for each edge where the edge group is $B_n \cap G_e$.

In this construction, X_n is homotopic to the 1-point union of the graph $B_n \backslash \tilde{\mathcal{G}}$ and, for each edge $e \in E$, disjoint copies of $K(B_n \cap G_e, 1)$ indexed by $B_n \backslash G/G_e$. And Y_n is homotopic to the 1-point union of $B_n \backslash \tilde{\mathcal{G}}$ and, for each vertex $v \in V$, disjoint copies of $K(B_n \cap G_v, 1)$ indexed by $B_n \backslash G/G_v$. Thus for each $j \geq 2$, omitting the coefficients K , we have

$$H_j(Y_n) = \bigoplus_{v \in V} H_j(B_n \cap G_v)^{[G : B_n G_v]}, \quad H_j(X_n) = \bigoplus_{e \in E} H_j(B_n \cap G_e)^{[G : B_n G_e]},$$

and $H_j(X_n \cap Y_n) = H_j(X_n) \oplus H_j(Y_n)$. For $j \geq 3$, the required estimate (5.1) is now immediate from the exactness of the Mayer-Vietoris sequence

$$\cdots \rightarrow H_j(X_n, K) \oplus H_j(Y_n, K) \rightarrow H_j(B_n, K) \rightarrow H_{j-1}(X_n \cap Y_n, K) \rightarrow \cdots$$

In the calculation of H_1 , the graph $B_n \backslash \tilde{\mathcal{G}}$ contributes equally to X_n , Y_n and $X_n \cap Y_n$, augmenting each of the above formulae with a summand $H_1(B_n \backslash \tilde{\mathcal{G}}, K)$. The map on homology induced by the inclusions of $X_n \cap Y_n$ maps the summand $H_1(B_n \backslash \tilde{\mathcal{G}})$ of $H_1(X_n \cap Y_n)$ isomorphically to the diagonal of the summand $H_1(B_n \backslash \tilde{\mathcal{G}}) \oplus H_1(B_n \backslash \tilde{\mathcal{G}})$ in $H_1(X_n) \oplus H_1(Y_n)$. In particular, the $H_1(B_n \backslash \tilde{\mathcal{G}})$ summand contributes nothing to the kernel of $H_1(X_n \cap Y_n) \rightarrow H_1(X_n) \oplus H_1(Y_n)$, so the exactness of the Mayer-Vietoris sequence gives us the desired estimate in the case $j = 2$ as well. \square

6 The Proof of Theorem F

We now turn our attention towards residually free groups that are not limit groups. These form a much larger and wilder class of groups. In the next section we shall use the structure theory of residually free groups, as summarised in Sect. 2, to deduce Theorem E from the following more general result, which was stated as Theorem F in the introduction.

Theorem 6.1 *Let $G \subseteq G_1 \times \cdots \times G_k$ be a subdirect product of residually-finite groups of type \mathbb{F} , each of which contains a normal free subgroup $F_i < G_i$ such that*

G_i/F_i is torsion-free and nilpotent. Assume $F_i \subseteq G \cap G_i$. Let $m < k$ be an integer, let K be a field, and suppose that each G_i is K -slow above dimension 1.

If the projection of G to each m -tuple of factors $G_{j_1} \times \cdots \times G_{j_m} < G$ is of finite index, then there exists an exhausting normal chain (B_n) in G so that for $0 \leq j \leq m$,

$$\lim_{n \rightarrow \infty} \frac{\dim H_j(B_n, K)}{[G : B_n]} = 0.$$

The proof of this theorem occupies the whole of this section. We break it into four steps. First, we construct exhausting chains (B_n) of finite-index subgroups in G that are carefully adapted to our purposes. We then state two technical propositions—Propositions Ψ and Ω —that provide key estimates in the basic spectral sequence argument that we use to prove Theorem 6.1. This basic spectral sequence argument is carried out in Sect. 6.4, after which we return to Propositions Ψ and Ω and prove them. These proofs require extensive spectral sequence calculations that are much more involved than the ‘basic’ one referred to above.

We present an outline of the basic spectral sequence argument immediately so that the reader can see where we are going and what we are doing. We assume that the reader is familiar with the Lyndon-Hochschild-Serre (LHS) spectral sequence in homology associated to a short exact sequence of groups (see [12], p. 171). In the case where the coefficient module is the field K with trivial action, the E^2 -page of the LHS spectral sequence associated to $1 \rightarrow N \rightarrow \Gamma \rightarrow Q \rightarrow 1$ has (p, q) -term $E_{p,q}^2 = H_p(Q, H_q(N, K))$ and the terms $E_{p,q}^\infty$ with $p + q = s$ are the composition factors of a series for $H_s(\Gamma, K)$.

6.1 The spectral sequence argument we want to use

Given G as in Theorem 6.1, we will construct carefully an exhausting normal chain (B_n) so that (after lengthy argument) we obtain enough control on the dimension of the homology groups $H_p(B_n/(B_n \cap N), K)$ and $H_q(B_n \cap N, K)$ with p and q in a suitable range, where N is the direct product of the free groups F_i in the statement of Theorem 6.1. We then argue as follows:

Lemma 6.2 *Let (B_n) be an exhausting normal chain for G , let $N \triangleleft G$ be a normal subgroup and let K be a field. Fix an integer s . Suppose for all non-negative integers α, q with $\alpha + q = s$ and every n we have $\dim H_\alpha(B_n/B_n \cap N, H_q(B_n \cap N, K)) < \infty$. Suppose further that, for all such (α, q) ,*

$$\lim_{n \rightarrow \infty} \frac{\dim H_\alpha(B_n/B_n \cap N, H_q(B_n \cap N, K))}{[G : B_n]} = 0.$$

Then

$$\lim_{n \rightarrow \infty} \frac{\dim H_s(B_n, K)}{[G : B_n]} = 0.$$

Proof The LHS spectral sequence

$$E_{\alpha,q}^2 = H_{\alpha}(B_n/B_n \cap N, H_q(B_n \cap N, K))$$

converges via α to $H_{\alpha+q}(B_n, K)$. Hence

$$\dim H_s(B_n, K) \leq \sum_{\alpha+q=s} \dim H_{\alpha}(B_n/B_n \cap N, H_q(B_n \cap N, K)).$$

□

6.2 Special filtrations of subdirect products

We remind the reader that $G < G_1 \times \cdots \times G_k$ is called a *subdirect product* of the groups G_i if it projects onto each factor, i.e. $\pi_i(G) = G_i$ for all $1 \leq i \leq k$. It is called a *full subdirect product* if, in addition, $G \cap G_i \neq 1$ for all $1 \leq i \leq k$.

Notation Given a direct product $G_1 \times \cdots \times G_k$ and indices $\mathcal{I} = \{i_1, \dots, i_m\}$ with $1 \leq i_1 < \cdots < i_m \leq k$ we denote the canonical projection by

$$\pi_{\mathcal{I}} : G_1 \times \cdots \times G_k \rightarrow G_{i_1} \times \cdots \times G_{i_m},$$

or π_{i_1, \dots, i_m} , if it is appropriate to be more expansive.

For a group H and an integer d , we write $H^{[d]}$ to denote the subgroup generated by $\{h^d \mid h \in H\}$.

Lemma 6.3 *Let $\Gamma = G_1 \times \cdots \times G_k$ be a direct product of finitely-generated residually-finite groups. Let $G < \Gamma$ be a subdirect product. Assume that for all $1 \leq j \leq k$ there is a free group $F_j < G \cap G_j$ that is normal in G_j with G_j/F_j torsion-free and nilpotent. Let $N = F_1 \times \cdots \times F_k \subseteq G$. Let $m \leq k$ be an integer and assume that $\pi_{j_1, \dots, j_m}(G)$ has finite index in $G_{j_1} \times \cdots \times G_{j_m}$ for all $1 \leq j_1 < \cdots < j_m \leq k$.*

Then, one can exhaust G by a chain of finite-index normal subgroups (B_n) such that :

- (1) $B_n \cap N = (B_n \cap F_1) \times \cdots \times (B_n \cap F_k)$ for every n ;
- (2) $\bigcap_n B_n N = N$;
- (3) $\bigcap_n \pi_j(B_n) = 1$ for all $1 \leq j \leq k$;
- (4) $B_n \cap F_j = \pi_j(B_n) \cap F_j$ for all n and all $1 \leq j \leq k$;
- (5) $F_j = \bigcap_n \pi_j(B_n) F_j$ for all $1 \leq j \leq k$;
- (6) *there exists a positive integer δ such that for all n and $1 \leq j_1 < \cdots < j_m \leq k$*

$$\left(\prod_{i=1}^m \pi_{j_i}(B_n)^{[\delta]} \right) \left(B_n \cap \prod_{i=1}^m F_{j_i} \right) \subseteq \pi_{j_1, \dots, j_m}(B_n) \subseteq \prod_{i=1}^m \pi_{j_i}(B_n)$$

and $\left(\prod_{i=1}^m \pi_{j_i}(B_n)^{[\delta]} \right) (B_n \cap \prod_{i=1}^m F_{j_i})$ is a normal subgroup of finite index in $\prod_{i=1}^m \pi_{j_i}(B_n)$ such that the quotient group is a subquotient of Γ/N .

Furthermore, if every G_j is residually p -finite for a fixed prime p then every B_n can be chosen to have p -power index in G .

Proof Every torsion-free finitely generated nilpotent group is residually p (see [5]). Thus we may exhaust each G_j/F_j by a sequence of normal subgroups of p -power index, and pulling these back to G_j we obtain normal subgroups of p -power index $(S_{j,i})_i$ in G_j such that $\cap_i S_{j,i} F_j = F_j$. Next, since G_j is a residually finite there is a filtration $(K_{j,i})_i$ of G_j by normal subgroups of finite index so that $\cap_i K_{j,i} = 1$ and $K_{j,i} \subseteq S_{j,i}$. If G_j is residually finite p -group we can assume that $K_{j,i}$ has p -power index in G_j . In any case, $F_j \subseteq \cap_i K_{j,i} F_j \subseteq \cap_i S_{j,i} F_j = F_j$, hence

$$\bigcap_i K_{j,i} F_j = F_j. \quad (6.1)$$

Define

$$A_i = K_{1,i} \times \cdots \times K_{k,i} \quad \text{and} \quad \tilde{B}_i = A_i \cap G.$$

Then by (6.1)

$$\begin{aligned} \bigcap_i A_i N &= \bigcap_i (K_{1,i} F_1 \times \cdots \times K_{k,i} F_k) = \left(\bigcap_i K_{1,i} F_1 \right) \times \cdots \times \left(\bigcap_i K_{k,i} F_k \right) \\ &= F_1 \times \cdots \times F_k = N \end{aligned} \quad (6.2)$$

and since $N \subseteq G$ and

$$K_{j,i} \cap F_j \subseteq A_i \cap F_j = A_i \cap G \cap F_j = \tilde{B}_i \cap F_j \quad (6.3)$$

we have

$$\begin{aligned} \tilde{B}_i \cap N &= A_i \cap G \cap N = A_i \cap N = (K_{1,i} \cap F_1) \times \cdots \times (K_{k,i} \cap F_k) \subseteq \\ &(\tilde{B}_i \cap F_1) \times \cdots \times (\tilde{B}_i \cap F_k) \subseteq \tilde{B}_i \cap N. \end{aligned} \quad (6.4)$$

Writing n_i for the exponent of the finite group G/\tilde{B}_i we have

$$G^{[n_i]} \subseteq \tilde{B}_i. \quad (6.5)$$

(Note that if $K_{j,i}$ has p -power index in G_j for all j then n_i is a power of p .)

Inductively, for each i we choose $s_i \in \mathbb{Z}$ so that s_i divides s_{i+1} and s_i is divisible by p^{a_i} where a_i goes to infinity as i goes to infinity. (If all n_i are powers of p then we choose s_i to be powers of p .) We will impose some extra conditions on s_i later.

Finally we are able to define

$$B_i := G^{[s_i]}(\tilde{B}_i \cap N). \quad (6.6)$$

From (6.5) and (6.6) we have

$$B_i \subseteq \tilde{B}_i \quad \text{and} \quad B_i \cap N = \tilde{B}_i \cap N. \quad (6.7)$$

And since $(G/N)^{[s_i]}$ has finite index in the nilpotent G/N while $\tilde{B}_i \cap N$ has finite index in N , we see that $B_i < G$ is a normal subgroup of finite index.

(1) From (6.4) and (6.7) we have

$$B_i \cap N = \tilde{B}_i \cap N = (\tilde{B}_i \cap F_1) \times \cdots \times (\tilde{B}_i \cap F_k),$$

and

$$B_i \cap F_j = (B_i \cap N) \cap F_j = (\tilde{B}_i \cap N) \cap F_j = \tilde{B}_i \cap F_j. \quad (6.8)$$

Then

$$B_i \cap N = \tilde{B}_i \cap N = (B_i \cap F_1) \times \cdots \times (B_i \cap F_k). \quad (6.9)$$

(2) We have $B_i \subseteq \tilde{B}_i \subseteq A_i$, so from (6.2) we deduce $N \subseteq \bigcap_i B_i N \subseteq \bigcap_i A_i N = N$ hence

$$\bigcap_i B_i N = N. \quad (6.10)$$

Also,

$$\bigcap_i B_i \subseteq \bigcap_i A_i = \left(\bigcap_i K_{1,i} \right) \times \cdots \times \left(\bigcap_i K_{k,i} \right) = 1. \quad (6.11)$$

Thus (B_i) is an exhausting filtration of G .

(3) Consider $\bigcap_i \pi_j(B_i)$. Note that by (6.4) and (6.6)

$$\begin{aligned} \pi_j(B_i) &= \pi_j(G^{[s_i]})\pi_j(\tilde{B}_i \cap N) = G_j^{[s_i]}\pi_j((K_{1,i} \cap F_1) \times \cdots \times (K_{k,i} \cap F_k)) = \\ &= G_j^{[s_i]}(K_{j,i} \cap F_j) \subseteq G_j^{[s_i]}N. \end{aligned} \quad (6.12)$$

Recall that G/N is a finitely generated torsion-free nilpotent group, so is residually p -finite and $\bigcap_t (G/N)^{[t]} = 1$, whenever t runs through an increasing sequence of p -powers. Then by (6.12) $\bigcap_i \pi_j(B_i) \subseteq \bigcap_i G_j^{[s_i]}N \subseteq \bigcap_i G_j^{[p^{a_i}]}N \subseteq N$, so

$$\bigcap_i \pi_j(B_i) = \bigcap_i (\pi_j(B_i) \cap N). \quad (6.13)$$

Note that by (6.12)

$$\begin{aligned} \pi_j(B_i) \cap N &= (G_j^{[s_i]}(K_{j,i} \cap F_j)) \cap N \\ &= (G_j^{[s_i]} \cap N)(K_{j,i} \cap F_j) = (G_j^{[s_i]} \cap F_j)(K_{j,i} \cap F_j). \end{aligned} \quad (6.14)$$

Now we specify the choice of s_i more tightly, multiplying our original choice by the exponents of the finite group $G_j/K_{j,i}$ if necessary to ensure that for all i and all $1 \leq j \leq k$ we have

$$G_j^{[s_i]} \subseteq K_{j,i}. \quad (6.15)$$

Then by (6.14)

$$\pi_j(B_i) \cap N = (G_j^{s_i} \cap F_j)(K_{j,i} \cap F_j) = K_{j,i} \cap F_j \subseteq K_{j,i}, \quad (6.16)$$

hence by (6.13), (6.16) and the definition of $K_{j,i}$

$$\bigcap_i \pi_j(B_i) = \bigcap_i (\pi_j(B_i) \cap N) \subseteq \bigcap_i K_{j,i} = 1.$$

(4) By (6.3), (6.8) and (6.16)

$$K_{j,i} \cap F_j \subseteq \tilde{B}_i \cap F_j = B_i \cap F_j \subseteq \pi_j(B_i) \cap N = K_{j,i} \cap F_j$$

and so

$$B_i \cap F_j = \pi_j(B_i) \cap F_j. \quad (6.17)$$

(5) By (6.12)

$$\bigcap_i \pi_j(B_i) F_j = \bigcap_i G_j^{[s_i]} (K_{j,i} \cap F_j) F_j = \bigcap_i G_j^{[s_i]} F_j = F_j.$$

(6) Define

$$Y_i := \pi_{j_1}(B_i) \times \cdots \times \pi_{j_m}(B_i) \text{ and } X_i = Y_i \cap N.$$

Note that Y_i is the term on the right in the statement of item (6) We claim that X_i is equal to the second bracketed term on the left. Indeed, from (6.9) and (6.17) we have

$$B_i \cap \prod_{t=1}^m F_{j_t} = \prod_{t=1}^m (B_i \cap F_{j_t}) = \prod_{t=1}^m (\pi_{j_t}(B_i) \cap F_{j_t}) = X_i \triangleleft Y_i. \quad (6.18)$$

To see that the middle term of item (6) is contained in Y_i , we use (6.9) and (6.6) to calculate:

$$\begin{aligned} \pi_{j_1, \dots, j_m}(B_i) &= \pi_{j_1, \dots, j_m}(G^{[s_i]}) \pi_{j_1, \dots, j_m}(\tilde{B}_i \cap N) \\ &= (\pi_{j_1, \dots, j_m}(G))^{[s_i]} \pi_{j_1, \dots, j_m} \left(\prod_{l=1}^k (B_l \cap F_l) \right) \end{aligned}$$

$$\begin{aligned}
&= (\pi_{j_1, \dots, j_m}(G))^{[s_i]} \pi_{j_1, \dots, j_m} \left(\prod_{t=1}^m (B_i \cap F_{j_t}) \right) \\
&= (\pi_{j_1, \dots, j_m}(G))^{[s_i]} X_i \\
&\subseteq \left(\prod_{t=1}^m G_{j_t}^{[s_i]} \right) \left(\prod_{t=1}^m (B_i \cap F_{j_t}) \right) \\
&= \prod_{t=1}^m G_{j_t}^{[s_i]} (B_i \cap F_{j_t}) = \prod_{t=1}^m \pi_{j_t}(B_i) = Y_i. \quad (6.19)
\end{aligned}$$

At this point we have proved that

$$B_i \cap \prod_{t=1}^m F_{j_t} = X_i \subseteq \pi_{j_1, \dots, j_m}(B_i) \subseteq Y_i$$

and that

$$\pi_{j_1, \dots, j_m}(B_i) = (\pi_{j_1, \dots, j_m}(G))^{[s_i]} X_i.$$

So to complete the proof of the inclusions displayed in (6) it only remains to establish the existence of δ such that

$$\prod_{t=1}^m \pi_{j_t}(B_i)^{[\delta]} \subseteq (\pi_{j_1, \dots, j_m}(G))^{[s_i]} X_i. \quad (6.20)$$

By (6.12) and since $K_{j_t, i} \cap F_{j_t} \subseteq B_i \cap F_{j_t}$ (see (6.3))

$$\pi_{j_t}(B_i)^{[\delta]} (B_i \cap F_{j_t}) = (G_{j_t}^{[s_i]} (K_{j_t, i} \cap F_{j_t}))^{[\delta]} (B_i \cap F_{j_t}) = (G_{j_t}^{[s_i]})^{[\delta]} (B_i \cap F_{j_t}).$$

Hence, since $\prod_{t=1}^m (B_i \cap F_{j_t}) \subseteq X_i$, (6.20) is equivalent to

$$\prod_{t=1}^m (G_{j_t}^{[s_i]})^{[\delta]} \subseteq (\pi_{j_1, \dots, j_m}(G))^{[s_i]} X_i, \quad (6.21)$$

so we will be done if we can find δ to do this.

To this end, assume for a moment that we have found δ such that

$$\prod_{t=1}^m (G_{j_t}^{[s_i]})^{[\delta]} \subseteq (\pi_{j_1, \dots, j_m}(G))^{[s_i]} N. \quad (6.22)$$

Then, recalling that we chose s_i so that $G_j^{[s_i]} \subseteq K_{j, i} = \pi_j(A_i)$, for all j , we would have

$$\prod_{t=1}^m (G_{j_t}^{[s_i]})^{[\delta]} \subseteq ((\pi_{j_1, \dots, j_m}(G))^{[s_i]} N) \cap \prod_{t=1}^m K_{j_t, i},$$

and since N is normal in $\prod_{j=1}^k G_j$, the right hand side equals

$$(\pi_{j_1, \dots, j_m}(G))^{[s_i]} \left(N \cap \prod_{t=1}^m K_{j_t, i} \right).$$

Finally (6.4), (6.9) assure us that the second term equals $B_i \cap \prod_{t=1}^m F_{j_t, i}$, which is X_i , so we have completed the required proof of (6.21), modulo assumption (6.22), which we shall prove using the following simple fact.

Claim Let Q be a finitely generated torsion-free nilpotent group and let $Q_0 < Q$ be a subgroup of finite index. Then there exists δ so that for every natural number s ,

$$(Q^{[s]})^{[\delta]} \subseteq Q_0^{[s]}.$$

Proof of Claim Since every subgroup in a nilpotent group is subnormal, an obvious induction on $[Q : Q_0]$ reduces us to the case where Q_0 is normal with index a prime number q . We must show that the exponent of the group $D_s = Q^{[s]}/Q_0^{[s]}$ can be bounded independently of s . As D_s is of bounded nilpotency class this is equivalent to the exponent of the abelianization of D_s having an upper bound independent of s . This abelianization is finitely generated with at most $d(Q)$ generators and each generator has exponent q since $Q^{[q]} \subseteq Q_0$. This completes the proof of the claim.

Returning to the proof of (6.22), recall that by hypothesis, for our fixed integer $m \leq k$, each projection of the form $\pi_{j_1, \dots, j_m}(G)$ has finite index in $G_{j_1} \times \dots \times G_{j_m}$. Thus we can apply the Claim with $Q = \prod_{t=1}^m G_{j_t} N/N$ and $Q_0 = \pi_{j_1, \dots, j_m}(G) N/N$. This completes the proof that the promised inclusions in the statement of part (6) of the lemma hold.

Note that (6.18) establishes the normality and subquotient properties we were required to prove. The finite index property follows from the fact that every finitely generated nilpotent group of finite exponent is finite. The additional p -power property claimed in the last sentence of the statement is easily verified by following the stages of the construction.

This completes the proof of the lemma. \square

6.3 Proposition Ψ and Proposition Ω

From now until the end of Sect. 6 we fix G to be a group satisfying the assumptions of Theorem F. We will work exclusively with a special filtration, i.e. a particular exhausting sequence (B_i) of finite-index normal subgroups, constructed to satisfy the conditions of Lemma 6.3. (The fact that the index $[G : B_i]$ can be a power of a fixed prime p if each G_i is residually p will, however, play no role.) We shall maintain the notation of Lemma 6.3 (in particular δ is the integer whose existence was established there). We also fix, for the duration of the section, an arbitrary

$$\mathcal{I} = \{j_1, \dots, j_m\}$$

with $1 \leq j_1 < \cdots < j_m \leq k$, and define

$$\Gamma_{\mathcal{I}} := G_{j_1} \times \cdots \times G_{j_m}$$

$$\Lambda_i := (F_{j_1} \cap B_i) \times \cdots \times (F_{j_m} \cap B_i),$$

$$S_i := \pi_{j_1}(B_i)^{[\delta]}(B_i \cap F_{j_1}) \times \cdots \times \pi_{j_m}(B_i)^{[\delta]}(B_i \cap F_{j_m}),$$

$$D_i := S_i / \Lambda_i$$

and

$$C_i := \pi_{\mathcal{I}}(B_i) / \Lambda_i.$$

By Lemma 6.3 (6), $D_i \subseteq C_i$.

Lemma 6.4 *There is a positive integer b , independent of \mathcal{I} , so that for all i ,*

$$|C_i / D_i| \leq b.$$

Proof Each C_i / D_i is a finite nilpotent group, so to bound its cardinality it is enough to bound the size of a minimal generating set, the nilpotency class and the exponent. Lemma 6.3(6) tells us that C_i / D_i is a subquotient of the finitely generated nilpotent group $(G_1 \times \cdots \times G_k) / N$. Since $(G_1 \times \cdots \times G_k) / N$ has finite rank there is an upper bound, independent of i , on the number of elements required to generate C_i / D_i . The nilpotency class of $(G_1 \times \cdots \times G_k) / N$ is an upper bound on the nilpotency class of C_i / D_i , and by Lemma 6.3(6) the exponent of C_i / D_i divides δ . \square

The proof of Theorem F depends on the following two technical results about asymptotics of homology groups. Recall that $m = |\mathcal{I}|$.

Proposition Ψ *For all positive integers α and q with $\alpha + q \leq m - 1$ we have*

$$\lim_{i \rightarrow \infty} \frac{1}{[\Gamma_{\mathcal{I}} : S_i]} \dim H_{\alpha}(D_i, H_q(\Lambda_i, K)) = 0$$

and for $\alpha + q = m$ we have

$$\limsup_{i \rightarrow \infty} \frac{1}{[\Gamma_{\mathcal{I}} : S_i]} \dim H_{\alpha}(D_i, H_q(\Lambda_i, K)) < \infty.$$

Proposition Ω *Suppose that for $\alpha + q \leq m$ we have*

$$\limsup_{i \rightarrow \infty} \frac{1}{[\Gamma_{\mathcal{I}} : \pi_{\mathcal{I}}(B_i)]} \dim H_{\alpha}(C_i, H_q(\Lambda_i, K)) < \infty.$$

Then for $\alpha + q \leq m$

$$\lim_{i \rightarrow \infty} \frac{1}{[G : B_i]} \dim H_{\alpha}(B_i / (N \cap B_i), H_q(\Lambda_i, K)) = 0.$$

Remark 6.5 The hypothesis of Theorem F that the summands G_j are K -slow above dimension 1 enters the proof of Proposition Ψ in an essential way but is not required in the proof of Proposition Ω .

6.4 Proposition Ψ and Proposition Ω imply Theorem F

Let C_i and D_i be as in Sect. 6.3 and let b be the constant of Lemma 6.4. The output of Proposition Ψ controls the homology of D_i while Proposition Ω requires as input control on the homology of C_i . The following lemma bridges this gap.

Lemma 6.6 *For every $s \in \mathbb{N}$, there exists a constant $c = c(s, b)$ so that if M is a $K C_i$ -module with $\dim H_\alpha(D_i, M) < \infty$ for all $\alpha \leq s$, then*

$$\dim H_\alpha(C_i, M) \leq c \max_{t \leq \alpha} \{\dim H_t(D_i, M)\}.$$

Proof Consider the LHS spectral sequence $H_\gamma(C_i/D_i, H_t(D_i, M))$ converging to $H_{\gamma+t}(C_i, M)$. Since C_i/D_i is a finite group of order at most b there is a free resolution of the trivial $\mathbb{Z}[C_i/D_i]$ -module \mathbb{Z} with finitely generated modules in every dimension. We fix one such resolution for every possible finite group of order at most b and define c_1 to be the least upper bound on the number of generators of the free modules up to dimension s in these resolutions. Then, for all $\gamma \leq s$ we have

$$\dim H_\gamma(C_i/D_i, H_t(D_i, M)) \leq c_1 \dim H_t(D_i, M).$$

As one passes from the E_2 page of the spectral sequence to the E_∞ page, the dimension of the K -module in each coordinate does not increase, so the filtration of $H_\alpha(C_i, M)$ corresponding to the antidiagonal $\gamma + t = \alpha$ on the E_∞ page allows us to estimate

$$\dim H_\alpha(C_i, M) \leq \sum_{\gamma+t=\alpha} \dim H_\gamma(C_i/D_i, H_t(D_i, M)) \leq c_1 \sum_{\gamma+t=\alpha} \dim H_t(D_i, M).$$

Thus

$$\dim H_\alpha(C_i, M) \leq (c_1 s) \max_{t \leq \alpha} \{\dim H_t(D_i, M)\} \text{ for } \alpha \leq s,$$

and setting $c = c_1 s$ we are done. \square

We now turn to the main argument. We are trying to get into a situation where we can apply Lemma 6.2. From Lemma 6.4 we have

$$\frac{[\Gamma_{\mathcal{I}} : S_i]}{[\Gamma_{\mathcal{I}} : \pi_{\mathcal{I}}(B_i)]} = [C_i : D_i] \leq b. \quad (6.23)$$

In the light of this, we can combine Proposition Ψ , Proposition Ω , using Lemma 6.6 with $M = H_q(\Lambda_i, K)$, to deduce that for $\alpha + q \leq m$,

$$\lim_{i \rightarrow \infty} \frac{\dim H_\alpha(B_i/(N \cap B_i), H_q(\Lambda_i, K))}{[G : B_i]} = 0. \quad (6.24)$$

Now, by Lemma 6.3(1),

$$B_i \cap N = (F_1 \cap B_i) \times \cdots \times (F_k \cap B_i). \quad (6.25)$$

Since $F_j \cap B_i$ is a subgroup of a free group F_j , it is free itself and $H_s(F_j \cap B_i, K) = 0$ for $s \geq 2$. Then by the Künneth formula [28, Theorem 11.31]

$$H_q(B_i \cap N, K) = \oplus_{\mathcal{J}} H_1(F_{l_1} \cap B_i, K) \otimes_K \cdots \otimes_K H_1(F_{l_q} \cap B_i, K) \quad (6.26)$$

where the sum is over all $\mathcal{J} = \{l_1, \dots, l_q\}$ with $1 \leq l_1 < \cdots < l_q \leq k$, while

$$H_q(\Lambda_i, K) = \oplus_{\mathcal{J} \subseteq \mathcal{I}} H_1(F_{l_1} \cap B_i, K) \otimes_K \cdots \otimes_K H_1(F_{l_q} \cap B_i, K). \quad (6.27)$$

By (6.24) and (6.27), for $\mathcal{J} = \{l_1, \dots, l_q\} \subseteq \mathcal{I}$ we have

$$\lim_{i \rightarrow \infty} \frac{1}{[G : B_i]} \dim H_\alpha(B_i/N \cap B_i, H_1(F_{l_1} \cap B_i, K) \otimes_K \cdots \otimes_K H_1(F_{l_q} \cap B_i, K)) = 0.$$

As the above holds for all choices of \mathcal{I} , we deduce from (6.26) that

$$\lim_{i \rightarrow \infty} \frac{1}{[G : B_i]} \dim H_\alpha(B_i/N \cap B_i, H_q(N \cap B_i, K)) = 0 \text{ for all } \alpha + q \leq m.$$

This is the estimate that we need as input for the simple spectral sequence argument Lemma 6.2, so the proof of Theorem F is complete. \square

6.5 Proof of Proposition Ω

In our proof of Proposition Ω we shall need the estimates presented in the following two lemmas. These estimates relate the dimension of homology groups of nilpotent groups to Hirsch length.

Lemma 6.7 *For every $q \in \mathbb{N}$ there is a polynomial $f_q \in \mathbb{Z}[x]$ so that for every finitely-generated torsion-free nilpotent group M ,*

$$\dim H_q(M, K) \leq f_q(h(M)),$$

where $h(M)$ is the Hirsch length of M .

Proof Let N be a central normal subgroup of M . As in our previous arguments, from the LHS spectral sequence for the short exact sequence $1 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 1$

we have

$$\dim H_q(M, K) \leq \sum_{\alpha+\beta=q} \dim H_\alpha(M/N, H_\beta(N, K)).$$

As N is central in M , the action of M/N (via conjugation) on N (hence $H_*(N, K)$) is trivial, so

$$H_\alpha(M/N, H_\beta(N, K)) \cong H_\alpha(M/N, K) \otimes_K H_\beta(N, K), \quad (6.28)$$

and

$$\dim H_q(M, K) \leq \sum_{\alpha+\beta=q} \dim(H_\alpha(M/N, K) \otimes_K H_\beta(N, K)).$$

By [5, Corollary 2.11] there is a central series $(M_i)_i$ for M with all quotients M_i/M_{i-1} infinite cyclic. Arguing by induction on the length $s+1 = h(M)$ of this series and making repeated applications of (6.28), for $M = M_s$ we have

$$\begin{aligned} \dim H_q(M, K) &\leq \sum_{\alpha_0+\dots+\alpha_s=q} \dim(H_{\alpha_s}(M_s/M_{s-1}, K) \otimes_K \dots \otimes_K \\ &\quad \times H_{\alpha_i}(M_i/M_{i-1}, K) \otimes_K \dots \otimes_K H_{\alpha_0}(M_0, K)) \end{aligned}$$

But $M_i/M_{i-1} \cong \mathbb{Z}$, so $H_{\alpha_i}(M_i/M_{i-1}, K) = 0$ if $\alpha_i > 1$ and is K in dimensions 0 and 1. Thus, defining $M_{-1} = 0$ we have

$$\begin{aligned} \dim H_q(M, K) &\leq \sum_{\alpha_0+\dots+\alpha_s=q} \prod_{0 \leq i \leq s} \dim H_{\alpha_i}(M_i/M_{i-1}, K) \\ &\leq \sum_{\alpha_i \leq 1, \alpha_0+\dots+\alpha_s=q} 1 \\ &= \binom{s+1}{q} = f_q(s+1), \end{aligned}$$

where $f_q(x) = x(x-1)\dots(x-q+1)/q!$. □

Lemma 6.8 *Let Q be a torsion-free finitely generated nilpotent group, H a subgroup of Q , V a KH -module and M a normal subgroup of H such that M acts trivially on V . Then, for every integer s there is a constant β depending only on s and the Hirsch length $h(Q)$ such that*

$$\dim H_s(H, V) \leq \beta \sum_{0 \leq p \leq s} \dim H_p(H/M, V).$$

Proof Consider the LHS spectral sequence

$$E_{p,q}^2 = H_p(H/M, H_q(M, V))$$

converging to $H_{p+q}(H, V)$. Since M acts trivially on V we have

$$H_q(M, V) = H_q(M, K) \otimes_K V,$$

where H/M acts diagonally on the tensor product. Since H is nilpotent, it acts nilpotently on M (by conjugation) and hence H acts nilpotently on $H_q(M, K)$, so there is a filtration of $H_q(M, K)$ by KH -submodules such that H acts trivially on the quotients of this filtration; we denote these sections W_1, \dots, W_j .

$$\dim H_p(H/M, H_q(M, K) \otimes_K V) \leq \sum_{1 \leq i \leq j} \dim H_p(H/M, W_i \otimes_K V) \quad (6.29)$$

and since H acts trivially on W_i we have that, as a $K[H/M]$ -module, $W_i \otimes_K V$ is a direct sum of $\dim(W_i)$ copies of V . Thus

$$\dim H_p(H/M, W_i \otimes_K V) \leq \dim W_i \dim H_p(H/M, V). \quad (6.30)$$

Note that

$$\sum_i \dim W_i = \dim H_q(M, K),$$

so

$$\sum_i \dim W_i \dim H_p(H/M, V) = \dim H_q(M, K) \dim H_p(H/M, V). \quad (6.31)$$

Combining (6.29), (6.30) and (6.31) we have

$$\begin{aligned} \dim E_{p,q}^2 &= \dim H_p(H/M, H_q(M, K) \otimes_K V) \\ &\leq \dim H_q(M, K) \dim H_p(H/M, V). \end{aligned} \quad (6.32)$$

By Lemma 6.7, there is a polynomial $f_q(x)$ depending only on q , so that the dimension of $H_q(M, K)$ is bounded above by $f_q(h(M))$. Let β be the maximum of $f_q(z)$, where $0 \leq q \leq s$ and $0 \leq z \leq h(Q)$. Then by (6.32)

$$\dim E_{p,q}^2 \leq \dim H_q(M, K) \dim H_p(H/M, V) \leq \beta \dim H_p(H/M, V).$$

Finally,

$$\dim H_s(H, V) \leq \sum_{p+q=s} \dim E_{p,q}^2 \leq \sum_{0 \leq p \leq s} \beta \dim H_p(H/M, V).$$

□

Proof of Proposition Ω

We saw in Lemma 6.3 that for $\mathcal{I} = \{j_1, \dots, j_m\}$ we have

$$\pi_{\mathcal{I}}(B_i \cap N) = \Lambda_i := (F_{j_1} \cap B_i) \times \cdots \times (F_{j_m} \cap B_i). \quad (6.33)$$

$\pi_{\mathcal{I}}$ induces a map

$$\rho_{\mathcal{I}} : B_i / (B_i \cap N) \rightarrow C_i = \pi_{\mathcal{I}}(B_i) / \pi_{\mathcal{I}}(B_i \cap N).$$

This map is surjective and we denote its kernel by M_i . Define

$$W_i = H_q(\Lambda_i, K).$$

Note that W_i is a C_i -module via the conjugation action of $\pi_{\mathcal{I}}(B_i)$ and W_i is a $B_i / (B_i \cap N)$ -module via the map $\rho_{\mathcal{I}}$.

These two incarnations of W_i are what we must understand, since the passage from the input of Proposition Ω to the output involves changing $H_{\alpha}(C_i, W_i)$ to $H_{\alpha}(B_i / (B_i \cap N), W_i)$, changing denominators, and sharpening the limit from finite to zero.

Concerning the denominators, observe that for $P = \prod_{j \notin \mathcal{I}} F_j$ we have $\pi_{\mathcal{I}}(P) = 1$, so $\pi_{\mathcal{I}}(P B_i) = \pi_{\mathcal{I}}(B_i)$. Then, recalling that $\Gamma_{\mathcal{I}} := G_{j_1} \times \cdots \times G_{j_m}$, we have

$$[G : P B_i] \geq [\pi_{\mathcal{I}}(G) : \pi_{\mathcal{I}}(B_i)] = \frac{[\Gamma_{\mathcal{I}} : \pi_{\mathcal{I}}(B_i)]}{[\Gamma_{\mathcal{I}} : \pi_{\mathcal{I}}(G)]}$$

and so

$$\begin{aligned} \frac{[\Gamma_{\mathcal{I}} : \pi_{\mathcal{I}}(B_i)]}{[G : B_i]} &\leq \frac{[\Gamma_{\mathcal{I}} : \pi_{\mathcal{I}}(G)][G : P B_i]}{[G : B_i]} \\ &= \frac{[\Gamma_{\mathcal{I}} : \pi_{\mathcal{I}}(G)]}{[P : B_i \cap P]}. \end{aligned}$$

We have assumed that $|\mathcal{I}| = m < k$, so $P \neq 1$ and

$$0 \leq \lim_{i \rightarrow \infty} \frac{[\Gamma_{\mathcal{I}} : \pi_{\mathcal{I}}(B_i)]}{[G : B_i]} \leq \lim_{i \rightarrow \infty} \frac{[\Gamma_{\mathcal{I}} : \pi_{\mathcal{I}}(G)]}{[P : B_i \cap P]} = 0. \quad (6.34)$$

By definition, M_i is the kernel of $\rho_{\mathcal{I}}$ and hence acts trivially on W_i . By hypothesis, G/N is torison-free nilpotent and M_i is a subgroup of $B_i / (B_i \cap N) \cong B_i N / N \subseteq G/N$. We apply Lemma 6.8 with $H = B_i / (B_i \cap N)$, $Q = G/N$ and $M = M_i$ to find a constant β depending on $h(Q)$ and m so that for $C_i = H/M_i$ and all $\alpha \leq m$

$$\dim H_{\alpha}(B_i / (B_i \cap N), W_i) \leq \beta \sum_{0 \leq p \leq \alpha} \dim H_p(C_i, W_i).$$

Then

$$\begin{aligned} \frac{\dim H_\alpha(B_i/(B_i \cap N), W_i)}{[G : B_i]} &\leq \frac{\beta}{[G : B_i]} \sum_{0 \leq j \leq \alpha} \dim H_j(C_i, W_i) \\ &= \beta \frac{\sum_{0 \leq j \leq \alpha} \dim H_j(C_i, W_i)}{[\Gamma_{\mathcal{I}} : \pi_{\mathcal{I}}(B_i)]} \frac{[\Gamma_{\mathcal{I}} : \pi_{\mathcal{I}}(B_i)]}{[G : B_i]}. \end{aligned} \quad (6.35)$$

Since for $\alpha + q \leq m$ we have

$$\limsup_{i \rightarrow \infty} \frac{\dim H_\alpha(C_i, W_i)}{[\Gamma_{\mathcal{I}} : \pi_{\mathcal{I}}(B_i)]} < \infty$$

we obtain by (6.34) and (6.35) that

$$\lim_{i \rightarrow \infty} \frac{\dim H_\alpha(B_i/N \cap B_i, W_i)}{[G : B_i]} = 0 \text{ for } \alpha + q \leq m$$

as required. \square

6.6 Proof of Proposition Ψ

We need the following weak form of slowness for nilpotent groups. This is a special case of [24, Theorem 0.2(ii)] and can be proved by a straightforward induction on Hirsch length (cf. Lemma 7.2).

Lemma 6.9 *If N is a finitely generated nilpotent group and K is a field then N is K -slow, i.e. for every exhausting normal chain (B_i) and every $m \geq 0$*

$$\lim_{i \rightarrow \infty} \frac{\dim H_m(B_i, K)}{[N : B_i]} = 0.$$

The direct summands G_j in the statement of Theorem F are assumed to satisfy the hypotheses of the following lemma.

Lemma 6.10 *Let K be a field, let G be a finitely generated residually finite group that is K -slow above dimension 1 and let F be a normal subgroup of G such that G/F is nilpotent and F is free. Let $(L_i)_{i \geq 1}$ be an exhausting sequence of normal subgroups of finite index in G such that $\bigcap_i L_i F = F$. Then, for $q \geq 1$*

$$\lim_{i \rightarrow \infty} \frac{1}{[G : L_i]} \dim H_q(L_i/F \cap L_i, (L_i \cap F)^{ab} \otimes_{\mathbb{Z}} K) = 0,$$

and

$$\limsup_{i \rightarrow \infty} \frac{1}{[G : L_i]} \dim H_0(L_i/F \cap L_i, (L_i \cap F)^{ab} \otimes_{\mathbb{Z}} K) < \infty.$$

Proof K -slowness means that for $q \geq 2$

$$\lim_{i \rightarrow \infty} \frac{\dim H_q(L_i, K)}{[G : L_i]} = 0. \quad (6.36)$$

And since $\dim H_1(L_i, K)$ is bounded above by $d(L_i)$, the number of generators required to generate L_i ,

$$\limsup_{i \rightarrow \infty} \frac{\dim H_1(L_i, K)}{[G : L_i]} < \infty. \quad (6.37)$$

The group that we must understand is

$$H_q(L_i/(F \cap L_i), (L_i \cap F)^{ab} \otimes_{\mathbb{Z}} K) = H_q(L_i/(F \cap L_i), H_1(L_i \cap F, K))$$

which is the $E_{q,1}^2$ term of the LHS spectral sequence

$$E_{\alpha,\beta}^2 = H_{\alpha}(L_i/(L_i \cap F), H_{\beta}(L_i \cap F, K))$$

converging via α to $H_{\alpha+\beta}(L_i, K)$. We denote the differentials

$$d_{\alpha,\beta}^k : E_{\alpha,\beta}^k \rightarrow E_{\alpha-k,\beta+k-1}^k.$$

Since F is free $E_{\alpha,\beta}^k = 0$ for $\beta \notin \{0, 1\}$, so the sequence stabilizes on the E^3 page and

$$E_{q,1}^{\infty} = E_{q,1}^3 = \text{coker } d_{q+2,0}^2$$

is a direct summand of $H_{q+1}(L_i, K)$. Thus

$$\begin{aligned} \dim E_{q,1}^2 &\leq \dim E_{q,1}^3 + \dim E_{q+2,0}^2 \\ &\leq \dim H_{q+1}(L_i, K) + \dim H_{q+2}(\bar{L}_i, K), \end{aligned} \quad (6.38)$$

where $\bar{L}_i := L_i F / F \cong L_i / (L_i \cap F)$. By hypothesis, (\bar{L}_i) is an exhausting sequence of normal subgroups of finite index in G/F , so by Lemma 6.9 for $s \geq 0$ we have

$$\lim_{i \rightarrow \infty} \frac{\dim H_s(\bar{L}_i, K)}{[G/F : \bar{L}_i]} = 0. \quad (6.39)$$

Now, $[G/F : \bar{L}_i] \leq [G : L_i]$, so dividing through (6.38) by $[G : L_i]$ and letting $i \rightarrow \infty$, from (6.36) and (6.39) for $q \geq 1$ we get

$$\lim_{i \rightarrow \infty} \frac{\dim E_{q,1}^2}{[G : L_i]} = 0,$$

while for $q = 0$ using (6.37) we get

$$\limsup_{i \rightarrow \infty} \frac{\dim E_{0,1}^2}{[G : L_i]} \leq \limsup_{i \rightarrow \infty} \frac{\dim H_1(L_i, K)}{[G : L_i]} < \infty,$$

as required. \square

Proof of Proposition Ψ

To simplify the notation, we relabel so that $\mathcal{I} = \{1, \dots, m\}$, and for each $j \in \mathcal{I}$ we define

$$A_{j,i} := \pi_j(B_i)^{[\delta]}(B_i \cap F_j).$$

Recall from (6.19) that $\pi_j(B_i) = G_j^{[s_i]}(B_i \cap F_j)$ and hence

$$A_{j,i} = (G_j^{[s_i]}(B_i \cap F_j))^{[\delta]}(B_i \cap F_j) = (G_j^{[s_i]})^{[\delta]}(B_i \cap F_j)$$

is a normal subgroup of finite index in G_j . From Lemma 6.3 (5) and (6) we have

$$\bigcap_{i \geq 1} A_{j,i} F_j = F_j \quad \text{for all } 1 \leq j \leq k \quad (6.40)$$

and

$$A_{j,i} \cap F_j = B_i \cap F_j. \quad (6.41)$$

And the notation that we used to state Proposition Ψ was (dropping the subscript \mathcal{I})

$$\Gamma = G_1 \times \dots \times G_m, \quad \Lambda_i = (A_{1,i} \cap F_1) \times \dots \times (A_{m,i} \cap F_m),$$

$$S_i = A_{1,i} \times \dots \times A_{m,i}, \quad \text{and } D_i = S_i / \Lambda_i.$$

We shall prove Proposition Ψ by examining the LHS spectral sequences for the short exact sequences $1 \rightarrow \Lambda_i \rightarrow S_i \rightarrow D_i \rightarrow 1$. The $E_{\alpha,\beta}^2$ term of this spectral sequence is

$$\mathcal{E}_i(\alpha, \beta) := H_\alpha(D_i, H_\beta(\Lambda_i, K))$$

and what we must prove is that

$$\lim_{i \rightarrow \infty} \frac{\mathcal{E}_i(\alpha, \beta)}{[\Gamma : S_i]} = 0 \quad \text{for } \alpha + \beta \leq m - 1$$

and for $\alpha + \beta = m$

$$\limsup_{i \rightarrow \infty} \frac{\mathcal{E}_i(\alpha, \beta)}{[\Gamma : S_i]} < \infty.$$

We shall always assume that $\beta \leq m$. The F_j , being free, have homological dimension 1, so by Künneth formula

$$H_\beta(\Lambda_i, K) \cong \bigoplus_{1 \leq j_1 < j_2 < \dots < j_\beta \leq m} (A_{j_1, i} \cap F_{j_1})^{ab} \otimes \dots \otimes (A_{j_\beta, i} \cap F_{j_\beta})^{ab} \otimes K.$$

(Here, and throughout, tensor products are over \mathbb{Z} unless indicated otherwise.) Thus

$$\mathcal{E}_i(\alpha, \beta) \cong \bigoplus_{1 \leq j_1 < j_2 < \dots < j_\beta \leq m} W_{\alpha, j_1, \dots, j_\beta, i},$$

where

$$W_{\alpha, j_1, \dots, j_\beta, i} = H_\alpha(D_i, (A_{j_1, i} \cap F_{j_1})^{ab} \otimes \dots \otimes (A_{j_\beta, i} \cap F_{j_\beta})^{ab} \otimes K).$$

We will be done if we can show, for fixed $\alpha \geq 0$, that

$$\lim_{i \rightarrow \infty} \frac{\dim W_{\alpha, j_1, \dots, j_\beta, i}}{[\Gamma : S_i]} = 0 \quad \text{for } \beta < m$$

and

$$\limsup_{i \rightarrow \infty} \frac{\dim W_{\alpha, j_1, \dots, j_\beta, i}}{[\Gamma : S_i]} < \infty \quad \text{for } \beta = m.$$

To prove this let us fix one sequence $1 \leq j_1 < j_2 < \dots < j_\beta \leq m$. Without loss of generality we can assume that $j_i = i$ for all $1 \leq i \leq \beta$. Define

$$R_{1, i} = A_{1, i} / (A_{1, i} \cap F_1) \times \dots \times A_{\beta, i} / (A_{\beta, i} \cap F_\beta)$$

and

$$R_{2, i} = A_{\beta+1, i} / (A_{\beta+1, i} \cap F_{\beta+1}) \times \dots \times A_{m, i} / (A_{m, i} \cap F_m).$$

Set

$$V_i = (A_{1, i} \cap F_1)^{ab} \otimes \dots \otimes (A_{\beta, i} \cap F_\beta)^{ab} \otimes K.$$

We will need the following generalised version of the Künneth formula: for $k = 1, 2$, let T_k be a group and let M_k be a $K[T_k]$ -module, then

$$H_\alpha(T_1 \times T_2, M_1 \otimes_K M_2) = \bigoplus_{\alpha_1 + \alpha_2 = \alpha} H_{\alpha_1}(T_1, M_1) \otimes_K H_{\alpha_2}(T_2, M_2).$$

To prove this formula, one takes a deleted projective resolution \mathcal{P}_k of M_k as a $K[T_k]$ -module and observes that the complex $\mathcal{P}_1 \otimes_K \mathcal{P}_2$ is a deleted projective resolution of $M_1 \otimes_K M_2$ as a $K[T_1 \times T_2]$ -module (see [28, Theorem 10.81] for details).

With this formula in hand, we have

$$\begin{aligned} W_{\alpha,1,\dots,\beta,i} &= H_{\alpha}(R_{1,i} \times R_{2,i}, V_i) \\ &= \oplus_{\alpha_1+\alpha_2=\alpha} H_{\alpha_1}(R_{1,i}, V_i) \otimes_K H_{\alpha_2}(R_{2,i}, K) \end{aligned} \quad (6.42)$$

and if we define $\Sigma_{k_1,\dots,k_{\beta},i}$ to be

$$\begin{aligned} &H_{k_1}(A_{1,i}/(A_{1,i} \cap F_1), (A_{1,i} \cap F_1)^{ab} \otimes K) \otimes_K \dots \\ &\otimes_K H_{k_{\beta}}(A_{\beta,i}/(A_{\beta,i} \cap F_{\beta}), (A_{\beta,i} \cap F_{\beta})^{ab} \otimes K) \end{aligned}$$

then

$$H_{\alpha_1}(R_{1,i}, V_i) \cong \oplus_{k_1+\dots+k_{\beta}=\alpha_1} \Sigma_{k_1,\dots,k_{\beta},i}. \quad (6.43)$$

Thus

$$\begin{aligned} &\dim H_{\alpha_1}(R_{1,i}, V_i) \\ &= \sum_{k_1+\dots+k_{\beta}=\alpha_1} \prod_{1 \leq t \leq \beta} \dim H_{k_t}(A_{t,i}/(A_{t,i} \cap F_t), (A_{t,i} \cap F_t)^{ab} \otimes K). \end{aligned} \quad (6.44)$$

Lemma 6.10 assures us that

$$\limsup_{i \rightarrow \infty} \frac{\prod_{1 \leq t \leq \beta} \dim H_{k_t}(A_{t,i}/(A_{t,i} \cap F_t), (A_{t,i} \cap F_t)^{ab} \otimes K)}{[G_1 \times \dots \times G_{\beta} : A_{1,i} \times \dots \times A_{\beta,i}]} < \infty. \quad (6.45)$$

And by combining (6.44) and (6.45) we deduce that

$$\limsup_{i \rightarrow \infty} \frac{\dim H_{\alpha_1}(R_{1,i}, V_i)}{[G_1 \times \dots \times G_{\beta} : A_{1,i} \times \dots \times A_{\beta,i}]} < \infty. \quad (6.46)$$

If $\beta = m$ then $R_{2,i} = 1$, so (6.46) completes the proof in this case.

Henceforth we assume that $\beta < m$. In this case, $R_{2,i}$ is non-trivial. Indeed, by (6.41)

$$R_{2,i} \cong (A_{\beta+1,i}F_{\beta+1}/F_{\beta+1}) \times \dots \times (A_{m,i}F_m/F_m)$$

is a normal subgroup of finite index in the infinite nilpotent group $\tilde{N} = (G_{\beta+1}/F_{\beta+1}) \times \dots \times (G_m/F_m)$. Moreover (6.40) tells us that these subgroups of finite index exhaust \tilde{N} , so we can apply Lemma 6.9 to deduce that

$$\lim_{i \rightarrow \infty} \dim H_{\alpha_2}(R_{2,i}, K) / [G_{\beta+1} \times \dots \times G_m : (A_{\beta+1,i}F_{\beta+1}) \times \dots \times (A_{m,i}F_m)] = 0$$

for every $\alpha_2 \geq 0$, hence

$$\lim_{i \rightarrow \infty} \dim H_{\alpha_2}(R_{2,i}, K) / [G_{\beta+1} \times \dots \times G_m : A_{\beta+1,i} \times \dots \times A_{m,i}] = 0. \quad (6.47)$$

We now have all that we need to complete the proof. From (6.42) we have

$$\dim W_{\alpha,1,\dots,\beta,i} = \sum_{\alpha_1+\alpha_2=\alpha} \dim H_{\alpha_1}(R_{1,i}, V_i) \cdot \dim H_{\alpha_2}(R_{2,i}, K).$$

We divide both sides by

$$\begin{aligned} [G_1 \times \cdots \times G_m : A_{1,i} \times \cdots \times A_{m,i}] &= \prod_{j=1}^m [G_j : A_{j,i}] \\ &= \left(\prod_{j=1}^{\beta} [G_j : A_{j,i}] \right) \left(\prod_{j=\beta+1}^m [G_j : A_{j,i}] \right) \end{aligned}$$

and let $i \rightarrow \infty$. We proved that in (6.46) that, when normalised by $\prod_{j=1}^{\beta} [G_j : A_{j,i}]$ the terms involving $R_{1,i}$ remain bounded, while (6.47) assures us that, when normalised by $\prod_{j=\beta+1}^m [G_j : A_{j,i}]$, the terms involving $R_{2,i}$ tend to zero. Thus

$$\lim_{i \rightarrow \infty} \frac{\dim W_{\alpha,1,\dots,\beta,i}}{[G_1 \times \cdots \times G_m : A_{1,i} \times \cdots \times A_{m,i}]} = 0$$

as required, and the proofs of Proposition Ψ and Theorem **F** are complete. \square

7 Proof of Theorem **E**

We recall the statement of Theorem **E**.

Theorem 7.1 *Let $m \geq 2$ be an integer, let G be a residually free group of type FP_m , and let ρ be the largest integer such that G contains a direct product of ρ non-abelian free groups. Then, there exists an exhausting sequence (B_n) so that for all fields K ,*

- (1) *if G is not of type FP_{∞} , then $\lim_n \frac{\dim H_i(B_n, K)}{[G : B_n]} = 0$ for all $0 \leq i \leq m$;*
- (2) *if G is of type FP_{∞} then for all $j \geq 1$,*

$$\lim_{n \rightarrow \infty} \frac{\dim H_j(B_n, K)}{[G : B_n]} = \begin{cases} (-1)^{\rho} \chi(G) & \text{if } j = \rho \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.4 allows us to regard an arbitrary finitely presented residually free group G as a full subdirect product of limit groups $G < G_1 \times \cdots \times G_k$ and Theorem 2.3 tells us that the G_i are free-by-(torsion free nilpotent) as required in Theorem **F**. If the G_i are all non-abelian, then item (1) of the above theorem is immediate consequence of Theorem **F** and Theorem 2.5. If one of the factors G_i is abelian, then the intersection of G with this factor is central and free abelian, so the Euler characteristic of G is zero and the limits in the statement of the theorem are also zero, by virtue of the following simple lemma. (Note that if G is of type F_m then so is the quotient of G by any finitely generated normal abelian subgroup.)

Lemma 7.2 Let $1 \rightarrow Z \rightarrow G \xrightarrow{p} Q \rightarrow 1$ be a central extension, with $Z \cong \mathbb{Z}$ and both G and Q residually finite of type F_m . Then, for all fields K there is an exhausting normal chain (B_n) in G by finite-index normal subgroups so that for all $s \leq m$

$$\lim_{n \rightarrow \infty} \frac{\dim H_s(B_n, K)}{[G : B_n]} = 0.$$

Proof The first thing to note is that for any sequence of finite index subgroups (C_n) in Q and any $s \leq m$ we have

$$\limsup_{n \rightarrow \infty} \dim H_s(C_n, K)/[Q : C_n] < \infty. \quad (7.1)$$

Indeed Q has a classifying space BQ with finite m -skeleton and $\dim H_s(C_n, K)$ is bounded by the number of s -cells in the $[Q : C_n]$ -sheeted covering space of BQ corresponding to C_n , which has $r_s[Q : C_n]$ cells of dimension s , where r_s is the number of s -cells in BQ .

Let (A_n) and (D_n) be exhausting chains of finite-index normal subgroups in G and Q respectively. Let $B_n = A_n \cap p^{-1}D_n$ and let $\bar{B}_n = p(B_n)$. Then we have a central extension $1 \rightarrow Z_n \rightarrow B_n \rightarrow \bar{B}_n \rightarrow 1$ with $Z_n = Z \cap B_n$, and from the LHS spectral sequence we have $\dim H_s(B_n, K) \leq \dim H_s(\bar{B}_n, K) + \dim H_{s-1}(\bar{B}_n, H_1(Z_n, K))$. But $H_1(Z_n, K)$ is the trivial $K\bar{B}_n$ -module K , because the action of B_n on Z by conjugation is trivial. Thus

$$\dim H_s(B_n, K) \leq \dim H_s(\bar{B}_n, K) + \dim H_{s-1}(\bar{B}_n, K).$$

The proof is completed by dividing this equality through by $[G : B_n]$ and letting n go to infinity, using (7.1) twice and noting that $[Z : Z_n] = [G : B_n]/[Q : \bar{B}_n]$ tends to infinity. \square

It remains to consider the case where G is of type FP_∞ . Theorem 2.6 says that G has a subgroup of finite index $H = H_1 \times \cdots \times H_r$ where the H_i are limit groups. Let (B_i) be an exhausting normal chain in G such that each B_i is contained in H and decompose as $B_i = (B_i \cap H_1) \times \cdots \times (B_i \cap H_r)$. Then by the Künneth formula and by Corollary B applied for each H_i we have

$$\begin{aligned} \lim_{i \rightarrow \infty} \frac{\dim H_j(B_i, K)}{[G : B_i]} &= \frac{1}{[G : H]} \sum_{j_1 + \cdots + j_r = j} \prod_{1 \leq s \leq r} \lim_{i \rightarrow \infty} \frac{\dim H_{j_s}(B_i \cap H_s, K)}{[H_s : B_i \cap H_s]} \\ &= \frac{1}{[G : H]} \sum_{j_1 + \cdots + j_r = j} \prod_{1 \leq s \leq r} (-\delta_{1, j_s} \chi(H_s)) \\ &= \frac{1}{[G : H]} (-1)^r \delta_{j, r} \chi(H) = (-1)^r \delta_{j, r} \chi(G). \end{aligned}$$

A limit group does not contain a direct product of two or more non-abelian free groups, and every non-abelian limit group contains a non-abelian free group, so $r = \rho$ unless

one or more of the H_i is abelian. If some H_i is abelian, then $\chi(H_i) = \chi(G) = 0$. This completes the proof of Theorem E. \square

8 Rank gradient and deficiency gradient for residually free groups

Let G be a group and let (B_i) be an exhausting normal chain for G . We are interested in the rank gradient

$$\text{RG}(G, (B_i)) = \lim_{i \rightarrow \infty} \frac{d(B_i)}{[G : B_i]}$$

and the deficiency gradient

$$\text{DG}(G, (B_i)) = \lim_{i \rightarrow \infty} \frac{\text{def}(B_i)}{[G : B_i]}.$$

The first limit exists because, for any nested sequence of subgroups of finite index, the sequence $(d(B_i) - 1)/[G : B_i]$ is non-increasing and bounded below by 0. The second limit exists because the sequence $(\text{def}(B_i) + 1)/[G : B_i]$ is non-increasing and bounded below by $-\text{RG}(G, (B_i))$; cf. proof of Lemma 8.2. Deficiency gradients have recently been studied by Kar and Nikolov [18].

The following lemma is known but we include a simple proof for the reader's convenience.

Lemma 8.1 *Let G be a finitely generated residually finite group with a finitely generated infinite normal subgroup N such that G/N is infinite and residually finite.*

- (1) *There exist exhausting normal chains (B_i) in G such that $(B_i N / N)$ is an exhausting normal chain in G/N .*
- (2) *For any such chain, $\text{RG}(G, (B_i)) = 0$.*

Proof To see that filtrations (B_i) of the desired form exist, note that if (H_i) and (D_i) are exhausting normal chains for G and $Q := G/N$, and if $p : G \rightarrow Q$ is the canonical projection, then $B_i = H_i \cap p^{-1}(D_i)$ has the required properties.

The proof of (2) is similar to Lemma 7.2. It relies on the standard fact that if Λ is a subgroup of index k in a group Γ , then $d(\Lambda) - 1 \leq k(d(\Gamma) - 1)$.

Let $N_i = N \cap B_i$ and $Q_i = B_i / N_i$. Then $[G : B_i] = [N : N_i][Q : Q_i]$ and $d(B_i) \leq d(N_i) + d(Q_i) \leq (d(N) - 1)[N : N_i] + 1 + (d(Q) - 1)[Q : Q_i] + 1$. Hence

$$\begin{aligned} \frac{d(B_i)}{[G : B_i]} &\leq \frac{1}{[G : B_i]} ((d(N) - 1)[N : N_i] + 1 + ((d(Q) - 1)[Q : Q_i] + 1)) \\ &\leq \frac{d(N)}{[Q : Q_i]} + \frac{d(Q)}{[N : N_i]} + \frac{2}{[G : B_i]}. \end{aligned}$$

Letting i go to infinity we conclude that $\text{RG}(G, (B_i)) = 0$. \square

Moving up one dimension we have:

Lemma 8.2 *Let G be a finitely presented residually finite group with a finitely presented infinite normal subgroup N such that $Q = G/N$ is infinite and residually finite. Let (B_i) be an exhausting normal chain for G , let $N_i = N \cap B_i$ and $Q_i = B_i N/N$ and assume that $\bigcap_i Q_i = 1$.*

If $\text{RG}(N, (N_i)) = 0$ then $\text{DG}(G, (B_i)) = 0$.

Proof There is a standard procedure that, given finite presentations $N_i = \langle X_i \mid R_i \rangle$ and $Q_i = \langle Y_i \mid S_i \rangle$ will construct a finite presentation $B_i = \langle X_i \cup Y_i \mid R_i, \tilde{S}_i, T_i \rangle$ where $|\tilde{S}_i| = |S_i|$ and $|T_i| = |d(N_i)| \cdot |Y_i|$.

In more detail, one lifts the canonical projection from the free group $F(Y_i) \rightarrow Q_i = B_i/N_i$ to obtain $\mu : F(Y_i) \rightarrow B_i$, then proceeds as follows. For each $\sigma \in S_i$ one chooses a word $u_\sigma \in F(X_i)$ such that $\mu(\sigma)u_\sigma$ equals $1 \in B_i$; then $\tilde{S}_i \subset F(X_i \cup Y_i)$ is defined to consist of the words σu_σ . To define T_i , one first fixes a generating set \underline{X}_i for N_i with $|\underline{X}_i| = d(N_i)$. Then, for each $x \in \underline{X}_i$ one chooses a word $\eta_x \in F(X_i)$ such that $x = \eta_x$ in N_i and for each $y \in Y_i$ one chooses a word $v_{xy} \in F(X_i)$ so that $v_{xy} = \mu(y)x\mu(y)^{-1}$ in B_i . The set T_i consists of the words $y\eta_x y^{-1}v_{xy}^{-1}$. (This process, although well-defined, is not algorithmic because there is no algorithm that, given a finite presentation, can identify a generating set of minimal cardinality for the group presented.)

Given a finitely presented group $\Gamma = \langle \Upsilon \mid \Sigma \rangle$ and a subgroup $\Gamma_i < \Gamma$ of index k , one obtains a presentation $\langle \Upsilon_i \mid \Sigma_i \rangle$ with $|\Upsilon_i| - 1 = k(|\Upsilon| - 1)$ and $|\Sigma_i| = k|\Sigma|$ by the Reidemeister-Schreier rewriting process. (Topologically, this amounts to taking a k -sheeted covering of the standard 2-complex for $\langle \Upsilon \mid \Sigma \rangle$ and collapsing a maximal tree in the 1-skeleton.) In particular,

$$\lim_{i \rightarrow \infty} \frac{|\Upsilon_i|}{[\Gamma : \Gamma_i]} = |\Upsilon| - 1 \quad \text{and} \quad \lim_{i \rightarrow \infty} \frac{|\Sigma_i|}{[\Gamma : \Gamma_i]} = |\Sigma|. \quad (8.1)$$

We fix finite presentations $N = \langle X \mid R \rangle$ and $Q = \langle Y \mid S \rangle$ and apply the construction of the previous paragraph to construct presentations $N_i = \langle X_i \mid R_i \rangle$ and $Q_i = \langle Y_i \mid S_i \rangle$, and from these we construct a presentation $\langle X_i \cup Y_i \mid R_i, \tilde{S}_i, T_i \rangle$ for B_i . By definition, the deficiency of this presentation is an upper bound on the deficiency of B_i , so

$$\begin{aligned} \text{def}(B_i) &\leq |T_i| + (|R_i| - |X_i|) + (|S_i| - |Y_i|) \\ &= d(N_i) |Y_i| + (|R_i| - |X_i|) + (|S_i| - |Y_i|). \end{aligned}$$

Dividing by $[G : B_i] = [N : N_i] \cdot [Q : Q_i]$ we get

$$\begin{aligned} \frac{1}{[G : B_i]} \text{def}(B_i) &\leq \frac{d(N_i)}{[N : N_i]} \frac{|Y_i|}{[Q : Q_i]} + \frac{1}{[Q : Q_i]} \frac{|R_i| - |X_i|}{[N : N_i]} \\ &\quad + \frac{1}{[N : N_i]} \frac{|S_i| - |Y_i|}{[Q : Q_i]}. \end{aligned}$$

Taking the limit $i \rightarrow \infty$, the second and third summands on the right tend to zero, by (8.1), while the first tends to $\text{RG}(N, (N_i)) \cdot (|Y| - 1)$, which is zero by hypothesis. Thus $\text{DG}(G, (B_i)) \leq 0$.

On the other hand, for any finitely presented group, $\text{def}(\Gamma) \geq -d(\Gamma)$, because Γ has a finite presentation on a set of $d(\Gamma)$ generators. Thus $-d(B_i) \leq \text{def}(B_i)$ and

$$-\text{RG}(G, (B_i)) \leq \text{DG}(G, (B_i)) \leq 0.$$

The first term is zero, by Lemma 8.1, so the lemma is proved. \square

The following theorem can be viewed as a homotopical version of Theorem E in low dimensions. We believe that the condition that G is of type FP_3 is too strong and that type FP_2 is enough (equivalently, finite presentability) as in the homological case, i.e. Theorem E, part (1). It also seems likely that higher dimensional analogues of this result hold, but we cannot resolve this problem because we do not have a complete characterisation of the residually free groups (i.e. subdirect products of limit groups) that are of type FP_m for $m \geq 3$. More specifically, the following conjecture remains open: for a full subdirect product $H \leq G_1 \times \cdots \times G_n$ with each G_i a non-abelian limit group, H is of type FP_m for some $m \leq n$ if and only if for every $1 \leq j_1 < \cdots < j_m \leq n$ the index of $\pi_{j_1, \dots, j_m}(H)$ in $G_{j_1} \times \cdots \times G_{j_m}$ is finite.

Theorem 8.3 *Every finitely presented residually free group G that is not a limit group admits an exhausting normal chain (B_n) with respect to which the rank gradient*

$$\text{RG}(G, (B_n)) = \lim_{n \rightarrow \infty} \frac{d(B_n)}{[G : B_n]} = 0.$$

Furthermore, if G is of type FP_3 but is not commensurable with a product of two limit groups, (B_n) can be chosen so that the deficiency gradient $\text{DG}(G, (B_n)) = 0$.

Proof We use Theorem 2.4 to embed G as a full subdirect product of limit groups $G \leq G_1 \times \cdots \times G_m$ such that each of the projections $\pi_{j_1, j_2}(G) < G_{j_1} \times G_{j_2}$ is of finite index; as G is not a limit group, $m \geq 2$. There will be an abelian factor G_i if and only if G has a non-trivial (free-abelian) centre. If G has such a centre, the theorem follows immediately from Lemmas 8.1 and 8.2, so henceforth we assume that the G_i are all non-abelian.

Proposition 3.2(3) of [9] states that $M := G \cap (G_1 \times \cdots \times G_{m-1})$ is finitely generated. The quotient $G/M = G_m$ is residually finite. Hence, by Lemma 8.1, there is an exhausting normal chain (B_i) with $\text{RG}(G, (B_i)) = 0$, as required.

If G is of type FP_3 but not virtually a product of two limit groups then $m \geq 3$ and Theorem 2.5 tells us for every $1 \leq j_1 < j_2 < j_3 \leq m$ the projection $\pi_{j_1, j_2, j_3}(G)$ has finite index in $G_{j_1} \times G_{j_2} \times G_{j_3}$. In particular this holds for $1 \leq j_1 < j_2 < j_3 = m$, so for every $1 \leq j_1 < j_2 \leq m-1$ we see that $p_{j_1, j_2}(M)$ has finite index in $G_{j_1} \times G_{j_2}$. It follows from Theorem 2.4 (or the Virtual Surjections Theorem of [9]) that M is finitely presented. To complete the proof, we want to appeal to Lemma 8.2 with $M = N$ and $Q = G_m$, but first we must construct a chain (B_i) as described in that lemma. To this end, we fix exhausting normal chains (D_i) , (A_i) and (\bar{Q}_i) for G_1 , G and G_m , respectively.

In the first step of the construction we follow the proof of Lemma 8.1 with $M \rightarrow \pi_1(M)$ playing the role of the map $G \rightarrow G/N$ that was considered there, and with $H_i := A_i \cap M$. As in that proof, $\text{RG}(M, (M_i)) = 0$, where $M_i := H_i \cap \pi_1^{-1}(D_i)$.

Finally, we define $B_i = A_i \cap \pi_1^{-1}(D_i) \cap \pi_m^{-1}(\overline{Q}_i)$. This is an exhausting normal chain for G with $M_i = B_i \cap M$. Moreover, $Q_i := B_i/M_i \subset \overline{Q}_i$, so $\bigcap_i Q_i = 1$. Thus Lemma 8.2 applies and the theorem is proved. \square

Remark 8.4 The exceptions made in the statement of Theorem 8.3 are necessary: if G_i , $i = 1, 2$, is a limit group then Theorem A(1) tells us that the rank gradient of G_i is $-\chi(G_i)$, from which it is easy to deduce that if $G = G_1 \times G_2$ is a product of two limit groups then its deficiency gradient is $\chi(G_1)\chi(G_2)$.

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