

On counting special Lagrangian homology 3-spheres

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1 Introduction

It is well known [14] that one can define *Gromov–Witten invariants* of symplectic manifolds (M, ω) by choosing a generic compatible metric g and almost complex structure J on M , and counting the *J-holomorphic curves* Σ in M in a given homology class, with signs. The invariants are then essentially independent of the choice of g, J . As *J-holomorphic curves* are *calibrated 2-submanifolds* with respect to the calibration ω on the Riemannian manifold (M, g) , Gromov–Witten invariants arise by counting the simplest nontrivial kind of calibrated submanifold.

In this paper we shall attempt to define a similar invariant I of generic (almost) Calabi–Yau 3-folds (M, J, ω, Ω) by counting another kind of calibrated submanifold, *special Lagrangian 3-submanifolds*, or *SL 3-folds* for short. In fact we shall consider only special Lagrangian *rational homology 3-spheres*, as they occur in 0-dimensional moduli spaces, and can be counted. Let $S(\delta)$ be the set of SL homology 3-spheres in M with homology class δ . Suppose $S(\delta)$ is finite. Then we shall define

$$I(\delta) = \sum_{N \in S(\delta)} w(N), \quad (1)$$

where $w(N)$ is a rational ‘weight function’ depending on the topology of N .

For I to be interesting it should either be unchanged by smooth deformations of the underlying almost Calabi–Yau structure (J, ω, Ω) , or else should transform according to some rigid set of rules as $[\omega]$ and $[\Omega]$ move about in $H^2(M, \mathbb{R})$ and $H^3(M, \mathbb{C})$. Now whether such an invariant can be made

to work, and how it should be defined, depends very much on the singular behaviour of special Lagrangian 3-folds, which is not well understood. Therefore much of this paper is conjectural, though I hope to be able to publish proofs of many of the conjectures in the next few years, unless someone else does first.

Here is why singularities of SL 3-folds are important in this problem. In the moduli space of almost Calabi-Yau structures on M there are certain special real hypersurfaces, determined using the homology of M . At such a hypersurface, some of the SL 3-folds in M will become singular. An SL 3-fold may exist in M only on one side of the hypersurface, and become singular at the hypersurface.

More generally, as we approach the hypersurface from one side, one or more SL 3-folds may collapse down to a singular SL 3-fold, and then on the other side this singular SL 3-fold is replaced by a different collection of one or more SL 3-folds. The key question addressed in this paper is to find a way to define the invariant I so that it is unchanged, or transforms in a controlled way, as we cross these hypersurfaces and the set of SL 3-folds that we are ‘counting’ changes.

In §3–§4 and §5–§6 we study two kinds of singularity that SL 3-folds can develop, modelled respectively on a T^2 -cone in \mathbb{C}^3 , and the union of two SL 3-planes \mathbb{R}^3 in \mathbb{C}^3 . We use these transitions to calculate identities which the weight function $w(N)$ in (1) must satisfy for I to be invariant, or transform nicely, as we pass through the hypersurface. It turns out that the simple weight function $w(N) = |H_1(N, \mathbb{Z})|$ satisfies these identities.

Motivated by this, we formulate a conjecture, Conjecture 7.3, giving a partial definition of I , and a partial statement of the transformation law it should satisfy under deformation. A full definition and transformation law (if the invariant works at all) will have to await a better understanding of singular SL 3-folds. We conclude in §7.6 with a discussion of the relationship of the invariants to String Theory. The author believes that they count objects of significance in String Theory, namely *isolated 3-branes*, and that they should play a rôle in Mirror Symmetry of Calabi–Yau 3-folds.

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2 Special Lagrangian geometry

We now introduce the idea of special Lagrangian submanifolds (SL m -folds), in two different geometric contexts. First, in §2.1, we define SL m -folds in \mathbb{C}^m . Then §2.2 discusses SL m -folds in *almost Calabi–Yau m -folds*, compact Kähler manifolds equipped with a holomorphic volume form which generalize the idea of Calabi–Yau manifolds. Finally, section 2.3 considers the *singularities* of SL m -folds. The principal references for this section are Harvey and Lawson [5] and the author [12].

2.1 Special Lagrangian submanifolds in \mathbb{C}^m

We begin by defining *calibrations* and *calibrated submanifolds*, following Harvey and Lawson [5].

Definition 2.1 Let (M, g) be a Riemannian manifold. An *oriented tangent k -plane* V on M is a vector subspace V of some tangent space $T_x M$ to M with $\dim V = k$, equipped with an orientation. If V is an oriented tangent k -plane on M then $g|_V$ is a Euclidean metric on V , so combining $g|_V$ with the orientation on V gives a natural *volume form* vol_V on V , which is a k -form on V .

Now let φ be a closed k -form on M . We say that φ is a *calibration* on M if for every oriented k -plane V on M we have $\varphi|_V \leq \text{vol}_V$. Here $\varphi|_V = \alpha \cdot \text{vol}_V$ for some $\alpha \in \mathbb{R}$, and $\varphi|_V \leq \text{vol}_V$ if $\alpha \leq 1$. Let N be an oriented submanifold of M with dimension k . Then each tangent space $T_x N$ for $x \in N$ is an oriented tangent k -plane. We say that N is a *calibrated submanifold* if $\varphi|_{T_x N} = \text{vol}_{T_x N}$ for all $x \in N$.

It is easy to show that calibrated submanifolds are automatically *minimal submanifolds* [5, Th. II.4.2]. Here is the definition of special Lagrangian submanifolds in \mathbb{C}^m , taken from [5, §III].

Definition 2.2 Let \mathbb{C}^m have complex coordinates (z_1, \dots, z_m) , and define a metric g , a real 2-form ω and a complex m -form Ω on \mathbb{C}^m by

$$\begin{aligned} g &= |dz_1|^2 + \dots + |dz_m|^2, & \omega &= \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + \dots + dz_m \wedge d\bar{z}_m), \\ & & \text{and } \Omega &= dz_1 \wedge \dots \wedge dz_m. \end{aligned} \tag{2}$$

Then $\text{Re } \Omega$ and $\text{Im } \Omega$ are real m -forms on \mathbb{C}^m . Let L be an oriented real submanifold of \mathbb{C}^m of real dimension m . We say that L is a *special Lagrangian*

submanifold of \mathbb{C}^m , or *SL m -fold* for short, if L is calibrated with respect to $\operatorname{Re} \Omega$, in the sense of Definition 2.1.

Harvey and Lawson [5, Cor. III.1.11] give the following alternative characterization of special Lagrangian submanifolds:

Proposition 2.3 *Let L be a real m -dimensional submanifold of \mathbb{C}^m . Then L admits an orientation making it into an SL submanifold of \mathbb{C}^m if and only if $\omega|_L \equiv 0$ and $\operatorname{Im} \Omega|_L \equiv 0$.*

An m -dimensional submanifold L in \mathbb{C}^m is called *Lagrangian* if $\omega|_L \equiv 0$. Thus special Lagrangian submanifolds are Lagrangian submanifolds satisfying the extra condition that $\operatorname{Im} \Omega|_L \equiv 0$, which is how they get their name.

2.2 Almost Calabi–Yau m -folds and SL m -folds

We shall define special Lagrangian submanifolds not just in Calabi–Yau manifolds, as usual, but in the much larger class of *almost Calabi–Yau manifolds*.

Definition 2.4 Let $m \geq 2$. An *almost Calabi–Yau m -fold*, or *ACY m -fold* for short, is a quadruple (M, J, ω, Ω) such that (M, J) is a compact m -dimensional complex manifold, ω is the Kähler form of a Kähler metric g on M , and Ω is a non-vanishing holomorphic $(m, 0)$ -form on M .

We call (M, J, ω, Ω) a *Calabi–Yau m -fold*, or *CY m -fold* for short, if in addition ω and Ω satisfy

$$\omega^m/m! = (-1)^{m(m-1)/2} (i/2)^m \Omega \wedge \bar{\Omega}. \quad (3)$$

Then for each $x \in M$ there exists an isomorphism $T_x M \cong \mathbb{C}^m$ that identifies g_x, ω_x and Ω_x with the flat versions g, ω, Ω on \mathbb{C}^m in (2). Furthermore, g is Ricci-flat and its holonomy group is a subgroup of $\operatorname{SU}(m)$.

This is not the usual definition of a Calabi–Yau manifold, but is essentially equivalent to it.

Definition 2.5 Let (M, J, ω, Ω) be an almost Calabi–Yau m -fold, and N an oriented real m -dimensional submanifold of M . Fix $\theta \in \mathbb{R}$. We call N a *special Lagrangian submanifold*, or *SL m -fold* for short, with *phase* $e^{i\theta}$ if

$$\omega|_N \equiv 0 \quad \text{and} \quad (\sin \theta \operatorname{Re} \Omega - \cos \theta \operatorname{Im} \Omega)|_N \equiv 0, \quad (4)$$

and $\cos \theta \operatorname{Re} \Omega + \sin \theta \operatorname{Im} \Omega$ is a positive m -form on the oriented m -fold N .

Again, this is not the usual definition of special Lagrangian submanifold, but is essentially equivalent to it. Compared to Definition 2.1 we have introduced two changes:

- (a) the definition now involves a *phase* $e^{i\theta}$, and
- (b) SL m -folds are defined by the *vanishing of forms* ω and $\sin \theta \operatorname{Re} \Omega - \cos \theta \operatorname{Im} \Omega$, rather than as calibrated submanifolds.

The following easy lemma relates the phase of a compact SL m -fold to its homology class.

Lemma 2.6 *Let (M, J, ω, Ω) be an almost Calabi–Yau m -fold, and N a compact SL m -fold in M with phase $e^{i\theta}$. Then*

$$[\Omega] \cdot [N] = R e^{i\theta}, \quad \text{where} \quad R = \int_N \cos \theta \operatorname{Re} \Omega + \sin \theta \operatorname{Im} \Omega > 0, \quad (5)$$

$[\Omega] \in H^m(M, \mathbb{C})$ and $[N] \in H_m(M, \mathbb{Z})$. Thus the homology class $[N]$ determines the phase $e^{i\theta}$ of N .

If we study only SL m -folds N in a fixed homology class in M , then by replacing Ω by $e^{-i\theta} \Omega$ we can suppose that N has phase 1, and so avoid introducing phases. However, we will later need to consider several SL 3-folds N_1, N_2, N_3 in different homology classes in a Calabi–Yau 3-fold M , with different phases, so we can discuss what happens to SL 3-folds as we vary $[\Omega] \in H^3(M, \mathbb{C})$. This is why we introduced change (a).

Using the analogue of Proposition 2.3 one can show that if (M, J, ω, Ω) is a Calabi–Yau manifold, then N is special Lagrangian with phase $e^{i\theta}$ if and only if it is calibrated w.r.t. $\cos \theta \operatorname{Re} \Omega + \sin \theta \operatorname{Im} \Omega$. More generally [12, §9.4], SL m -folds in an almost Calabi–Yau m -fold are calibrated w.r.t. $\cos \theta \operatorname{Re} \Omega + \sin \theta \operatorname{Im} \Omega$, but for a suitably *conformally rescaled* metric g .

Thus, we can give an equivalent definition of SL m -folds in terms of calibrated geometry, and change (b) above is only cosmetic. Nonetheless, in the author’s view the definition of SL m -folds in terms of the vanishing of closed forms is more fundamental than the definition in terms of calibrated geometry, and so should be taken as the primary definition.

The *deformation theory* of special Lagrangian submanifolds was studied by McLean [15, §3], who proved the following result in the Calabi–Yau case with phase 1. The extension to the ACY case is described in [12, §9.5].

Theorem 2.7 *Let (M, J, ω, Ω) be an almost Calabi–Yau m -fold, and N a compact SL m -fold in M . Then the moduli space \mathcal{M}_N of special Lagrangian deformations of N is a smooth manifold of dimension $b^1(N)$, the first Betti number of N . All elements of \mathcal{M}_N have the same phase $e^{i\theta}$, given by $[\Omega] \cdot [N] = Re^{i\theta}$ for $R > 0$.*

Using similar methods one can prove [12, §9.3, §9.5]:

Theorem 2.8 *Let $\{(M, J_t, \omega_t, \Omega_t) : t \in (-\epsilon, \epsilon)\}$ be a smooth 1-parameter family of almost Calabi–Yau m -folds. Let N_0 be a compact SL m -fold in $(M, J_0, \omega_0, \Omega_0)$, and suppose $[\omega_t|_{N_0}] = 0$ in $H^2(N_0, \mathbb{R})$ for all $t \in (-\epsilon, \epsilon)$. Then N_0 extends to a smooth 1-parameter family $\{N_t : t \in (-\delta, \delta)\}$, where $0 < \delta \leq \epsilon$ and N_t is a compact SL m -fold in $(M, J_t, \omega_t, \Omega_t)$, with phase $e^{i\theta_t}$ determined by $[\Omega_t] \cdot [N_0] = R_t e^{i\theta_t}$ for $R_t > 0$.*

Now suppose that (M, J, ω, Ω) is an almost Calabi–Yau 3-fold, and N is a special Lagrangian (rational) homology 3-sphere in M . Then $H^1(N, \mathbb{R}) = H^2(N, \mathbb{R}) = 0$. Thus by Theorem 2.7 the moduli space \mathcal{M}_N has dimension 0, so that N is *rigid* (that is, it admits no nontrivial deformations as an SL 3-fold in M). Also, as $H^2(N, \mathbb{R}) = 0$ the condition $[\omega_t|_N] = 0$ in Theorem 2.8 holds automatically for any family of deformations $(J_t, \omega_t, \Omega_t)$ of the almost Calabi–Yau structure (J, ω, Ω) on M . Thus, Theorem 2.8 shows that N is *stable* under small deformations of (J, ω, Ω) , giving:

Corollary 2.9 *Let (M, J, ω, Ω) be an almost Calabi–Yau 3-fold, and N a special Lagrangian homology 3-sphere in M . Then N is rigid, and stable under small deformations of the almost Calabi–Yau structure (J, ω, Ω) on M .*

One moral of this is that special Lagrangian homology 3-spheres in an almost Calabi–Yau 3-fold may be a good thing to count, as they are isolated and persistent under small deformations. We will discuss this in §7.

2.3 Singularities of SL m -folds

The author’s series of papers [6, 7, 8, 9, 10, 11, 12] is mainly concerned with the study of singularities of SL m -folds in \mathbb{C}^m and in almost Calabi–Yau manifolds. This is done both by the construction of many examples of singular SL m -folds in \mathbb{C}^m , and also through the development of a (partly

conjectural) picture of how families of nonsingular SL m -folds can become singular, particularly when the underlying almost Calabi–Yau manifold is assumed to be *generic*.

We shall now briefly summarize some definitions and conjectures taken from [12, §10]. The author has sketch proofs for the conjectures (at least in complex dimension $m < 6$) and hopes to publish full proofs fairly soon. For more details and motivation, see [12]. Here are some definitions to do with *special Lagrangian cones* in \mathbb{C}^m .

Definition 2.10 A (singular) SL m -fold C in \mathbb{C}^m is called a *cone* if $C = tC$ for all $t > 0$, where $tC = \{tx : x \in C\}$. Let C be an SL cone in \mathbb{C}^m . Then either C is an m -plane \mathbb{R}^m in \mathbb{C}^m , or C is singular at 0. We are interested primarily in SL cones C in which 0 is the only singular point, that is, in which 0 is an *isolated singularity*. Then $\Sigma = C \cap \mathcal{S}^{2m-1}$ is a compact, nonsingular $(m-1)$ -submanifold of \mathcal{S}^{2m-1} . We define the *number of ends at infinity* of C to be the number k of connected components of Σ .

Let C be an SL cone in \mathbb{C}^m with an isolated singularity at 0, and let $\Sigma = C \cap \mathcal{S}^{2m-1}$. Regard Σ as a compact Riemannian manifold, with metric induced from the round metric in \mathcal{S}^{2m-1} . Let $\Delta = d^*d$ be the Laplacian on functions on Σ . Define the *Legendrian index* $\text{l-ind}(C)$ to be the number of eigenvalues of Δ in $(0, 2m)$, counted with multiplicity.

Let the connected components of Σ be $\Sigma_1, \dots, \Sigma_k$. Define the cone C to be *rigid* if for each $j = 1, \dots, k$, the eigenspace of Δ on Σ_j with eigenvalue $2m$ has dimension $\dim SU(m) - \dim G_j$, where G_j is the Lie subgroup of $SU(m)$ preserving Σ_j .

The point of these definitions is that the Legendrian index of C is that Σ is a *minimal Legendrian submanifold* in \mathcal{S}^{2m-1} , and is thus a stationary point of the area functional amongst all Legendrian submanifolds in \mathcal{S}^{2m-1} . The *Legendrian index* is the index of this stationary point. The cone C is the union of one-ended cones C_1, \dots, C_k intersecting at 0, and C is *rigid* if all infinitesimal deformations of C_j as an SL cone come from infinitesimal rotations of C_j by $SU(m)$ matrices, for each j .

Now SL cones are important because they are local models for the simplest kind of singularities of SL m -folds in almost Calabi–Yau m -folds. To understand how singular SL m -folds modelled upon an SL cone C in \mathbb{C}^m can arise as limits of nonsingular SL m -folds, we need to consider SL m -folds L in \mathbb{C}^m asymptotic to C at infinity.

Definition 2.11 Let C be an SL cone in \mathbb{C}^m with isolated singularity at 0, and let $\Sigma = C \cap \mathcal{S}^{2m-1}$, so that Σ is a compact, nonsingular $(m-1)$ -manifold. Let h be the metric on Σ induced by the metric g on \mathbb{C}^m , and r the radius function on \mathbb{C}^m . Define $\iota : \Sigma \times (0, \infty) \rightarrow \mathbb{C}^m$ by $\iota(\sigma, r) = r\sigma$. Then the image of ι is $C \setminus \{0\}$, and $\iota^*(g) = r^2h + dr^2$ is the cone metric on $C \setminus \{0\}$.

Let L be a closed, nonsingular SL m -fold in \mathbb{C}^m . We call L *Asymptotically Conical (AC)* with cone C if there exists a compact subset $K \subset L$ and a diffeomorphism $\phi : \Sigma \times (R, \infty) \rightarrow L \setminus K$ for some $R > 0$, such that $|\phi - \iota| = o(r)$ and $|\nabla^k(\phi - \iota)| = o(r^{1-k})$ as $r \rightarrow \infty$ for $k = 1, 2, \dots$, where ∇ is the Levi-Civita connection of the cone metric $\iota^*(g)$, and $|\cdot|$ is computed using $\iota^*(g)$.

In [12, §10] this notion of Asymptotically Conical is referred to as *weakly Asymptotically Conical*, to distinguish it from a second class of *strongly Asymptotically Conical* SL m -folds which converge to C to order $O(r^{-1})$ rather than $o(r)$. However, we will not need the idea of strongly AC SL m -folds in this paper. The following conjecture [12, Conj. 10.3] is the analogue of Theorem 2.7 for AC SL m -folds.

Conjecture 2.12 *Let L be an AC SL m -fold in \mathbb{C}^m , with cone C , and let k be the number of ends of C at infinity. Then the moduli space \mathcal{M}_L of AC SL m -folds in \mathbb{C}^m with cone C is near L a smooth manifold of dimension $b^1(L) + k - 1 + \text{l-ind}(C)$.*

Our next conjecture [12, Conj. 10.7] is a first approximation to the kinds of deformation results the author expects to hold for singular SL m -folds in almost Calabi–Yau m -folds.

Conjecture 2.13 *Let C be a rigid SL cone in \mathbb{C}^m with an isolated singularity at 0 and k ends at infinity, and L be an AC SL m -fold in \mathbb{C}^m with cone C . Let (M, J, ω, Ω) be a generic almost Calabi–Yau m -fold, and \mathcal{M} a connected moduli space of compact nonsingular SL m -folds N in M .*

Suppose that at the boundary of \mathcal{M} there is a moduli space \mathcal{M}_C of compact, singular SL m -folds with one isolated singular point modelled on the cone C , which arise as limits of SL m -folds in \mathcal{M} by collapsing AC SL m -folds with the topology of L . Then

$$\dim \mathcal{M} = \dim \mathcal{M}_C + b^1(L) + k - 1 + \text{l-ind}(C) - 2m. \quad (6)$$

Suppose we have a suitably generic almost Calabi–Yau m -fold M and a compact, singular SL m -fold N_0 in M , which is the limit of a family of compact nonsingular SL m -folds N in M . We (loosely) define the *index* of the singularities of N_0 to be the codimension of the family of singular SL m -folds with singularities like those of N_0 in the family of nonsingular SL m -folds N . Thus, in the situation of Conjecture 2.13, the index of the singularities is $b^1(L) + k - 1 + \text{l-ind}(C) - 2m$.

More generally, one can work not just with a fixed generic almost Calabi–Yau m -fold, but with a *generic family* of almost Calabi–Yau m -folds. So, for instance, if we have a generic k -dimensional family of almost Calabi–Yau m -folds M , and in each M we have an l -dimensional family of SL m -folds, then in the total $(k+l)$ -dimensional family of SL m -folds we are guaranteed to meet singularities of index at most $k+l$.

Now later in the paper we shall study the behaviour of 0-dimensional families of SL 3-folds in generic 1-dimensional families of almost Calabi–Yau 3-folds. In such families we will only meet singularities of index 1. This is a very useful fact, as it means there will be only a few kinds of singular behaviour to worry about in determining how the invariants we define behave under deformations of the underlying almost Calabi–Yau 3-fold.

One important reason we have chosen to work in almost Calabi–Yau manifolds, rather than just in Calabi–Yau manifolds, is that almost Calabi–Yau manifolds occur in *infinite-dimensional* families. Thus, taking the underlying almost Calabi–Yau manifold to be generic is a very powerful assumption, and should simplify the singular behaviour of SL m -folds considerably. (For instance, one can argue that a compact SL 3-fold in a generic almost Calabi–Yau 3-fold has at most finitely many singular points.) However, Calabi–Yau manifolds only occur in finite-dimensional families, and so working in a generic Calabi–Yau manifold is not that strong an assumption, and probably will not help very much.

3 A model degeneration of SL 3-folds

We now define an explicit SL cone L_0 in \mathbb{C}^3 and three families of AC SL 3-folds L_t^a in \mathbb{C}^3 with cone L_0 , and analyze them in the framework of §2.3.

3.1 Three families of SL 3-folds in \mathbb{C}^3

Let G be the group $U(1)^2$, acting on \mathbb{C}^3 by

$$(e^{i\theta_1}, e^{i\theta_2}) : (z_1, z_2, z_3) \mapsto (e^{i\theta_1} z_1, e^{i\theta_2} z_2, e^{-i\theta_1 - i\theta_2} z_3) \quad \text{for } \theta_1, \theta_2 \in \mathbb{R}. \quad (7)$$

All the G -invariant special Lagrangian 3-folds in \mathbb{C}^3 were written down explicitly by Harvey and Lawson [5, §III.3.A], and studied in more detail in [6, Ex. 5.1] and [9, §4]. Here are some examples of G -invariant SL 3-folds which will be important in what follows.

Definition 3.1 Define a subset L_0 in \mathbb{C}^3 by

$$L_0 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 = |z_2|^2 = |z_3|^2, \\ \text{Im}(z_1 z_2 z_3) = 0, \quad \text{Re}(z_1 z_2 z_3) \geq 0\}. \quad (8)$$

Then L_0 is a *special Lagrangian cone* on T^2 , invariant under the Lie subgroup G of $SU(3)$ given in (7). Let $t > 0$, and define

$$L_t^1 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 - t = |z_2|^2 = |z_3|^2, \\ \text{Im}(z_1 z_2 z_3) = 0, \quad \text{Re}(z_1 z_2 z_3) \geq 0\}. \quad (9)$$

$$L_t^2 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 = |z_2|^2 - t = |z_3|^2, \\ \text{Im}(z_1 z_2 z_3) = 0, \quad \text{Re}(z_1 z_2 z_3) \geq 0\}, \quad (10)$$

$$L_t^3 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 = |z_2|^2 = |z_3|^2 - t, \\ \text{Im}(z_1 z_2 z_3) = 0, \quad \text{Re}(z_1 z_2 z_3) \geq 0\}. \quad (11)$$

Then it can be shown that each L_t^a is a G -invariant, nonsingular, embedded special Lagrangian 3-submanifold in \mathbb{C}^3 diffeomorphic to $\mathcal{S}^1 \times \mathbb{R}^2$, which is *Asymptotically Conical* in the sense of Definition 2.10, with cone L_0 .

Thus the L_t^a for $a = 1, 2, 3$ are *three different* families of AC SL 3-folds in \mathbb{C}^3 asymptotic to the same SL cone L_0 , each family depending on a real parameter $t > 0$. Define subsets D_t^1 and γ_t^1 in \mathbb{C}^3 for $a > 0$ by

$$\gamma_t^1 = \{(t^{1/2} e^{i\theta}, 0, 0) : \theta \in \mathbb{R}\}, \quad D_t^1 = \{(z_1, 0, 0) : z_1 \in \mathbb{C}, \quad |z_1|^2 \leq t\}. \quad (12)$$

Then γ_t^1 is a smooth, oriented \mathcal{S}^1 in L_t^1 , and D_t^1 is a closed, oriented holomorphic disc in \mathbb{C}^3 with area πt and boundary γ_t^1 . The homology class of γ_t^1 generates $H_1(L_t^1, \mathbb{Z}) \cong \mathbb{Z}$. There are similar holomorphic discs D_t^2 with boundary γ_t^2 in L_t^2 , and D_t^3 with boundary γ_t^3 in L_t^3 .

3.2 Using L_0 and the L_t^a as local models

Next we apply the ideas of §2.3 to L_0 and the L_t^a .

Lemma 3.2 *In the notation of §2.3, the SL cone L_0 and AC SL 3-folds L_t^a satisfy $k = 1$, $b^1(L_t^a) = 1$ and $\text{l-ind}(L_0) = 6$, and L_0 is rigid.*

Proof. As L_0 is a T^2 -cone we have $k = 1$, and as L_t^a is diffeomorphic to $\mathcal{S}^1 \times \mathbb{R}^2$ we have $b^1(L_t^a) = 1$. It is not difficult to show that the metric on $\Sigma \cong T^2$ is isometric to the quotient of \mathbb{R}^2 with its flat Euclidean metric by the lattice \mathbb{Z}^2 with basis $2\pi(\frac{\sqrt{2}}{3}, 0)$, $2\pi(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}})$. The eigenvectors of Δ on Σ lift to functions of the form $\sin(\alpha x + \beta y)$, $\cos(\alpha x + \beta y)$ on \mathbb{R}^2 which are invariant under lattice translations.

Calculation shows that the only eigenvalue of Δ in $(0, 6)$ is 2, with multiplicity 6, and so $\text{l-ind}(L_0) = 6$. Similarly, Δ has eigenvalue 6 with multiplicity 6, and as $\dim \text{SU}(3) = 8$ and the subgroup G of $\text{SU}(3)$ preserving L_0 is $\text{U}(1)^2$ with dimension 2, we see that L_0 is rigid as $6 = 8 - 2$. \square

Assuming Conjecture 2.12, we see that the moduli space of AC SL 3-folds L with cone L_0 and $b^1(L) = 1$ is 1-dimensional. It follows that any such L must be G -invariant, as otherwise the G -orbit of L and its rescalings would fill out a moduli space of larger dimension. Thus any such L is one of the L_t^a , and we have:

Corollary 3.3 *Suppose Conjecture 2.12 holds when $m = 3$. Then the moduli space of AC SL 3-folds L in \mathbb{C}^3 with cone L_0 and $b^1(L) = 1$ is $\{L_t^a : a = 1, 2, 3, t > 0\}$.*

Assuming Conjecture 2.13, we deduce:

Corollary 3.4 *Suppose Conjecture 2.13 holds when $m = 3$. Let (M, J, ω, Ω) be a generic almost Calabi–Yau 3-fold, and \mathcal{M} a connected moduli space of compact nonsingular SL 3-folds N in M . Suppose that at the boundary of \mathcal{M} there is a moduli space \mathcal{M}_0 of compact, singular SL 3-folds with one isolated singular point modelled on the cone L_0 , which arise as limits of SL 3-folds in \mathcal{M} by collapsing AC SL 3-folds of the form L_t^a . Then $\dim \mathcal{M} = \dim \mathcal{M}_0 + 1$.*

Thus, in the sense discussed in §2.3, singularities of SL 3-folds modelled on the SL cone L_0 should have *index one*. That is, they should occur in codimension 1 in families of SL 3-folds in a generic almost Calabi–Yau 3-fold, or more generally in generic families of almost Calabi–Yau 3-folds.

3.3 Interpretation in terms of holomorphic discs

Here is an informal way to understand why singularities of SL 3-folds modelled on L_0 should have index one, in terms of *holomorphic discs*. Given a symplectic manifold (M, ω) with compatible almost complex structure J , and a Lagrangian submanifold N in M , one can consider *J -holomorphic discs* D in M with boundary ∂D in N .

The behaviour of such J -holomorphic discs is well understood, because of their application in the Floer homology of Lagrangian submanifolds. It is known (see for instance [2, Th. 3.11]) that if J and N are sufficiently generic, then moduli spaces of J -holomorphic discs are smooth manifolds, with dimension given by a topological formula.

In our case, as N is special Lagrangian the ‘Maslov index’ of D is automatically 0, and as $m = 3$ the formula gives dimension 0 for the moduli space of holomorphic discs. The conclusion is that if (M, J, ω, Ω) is a generic almost Calabi–Yau 3-fold and N an SL 3-fold in M , then holomorphic discs D with boundary ∂D in N are expected to be both *isolated* and *stable*, so that they persist under small deformations of (J, ω, Ω) and N .

In the situation of Corollary 3.4, suppose that N_0 is a singular SL 3-fold in \mathcal{M}_0 with singular point x , and N is an SL 3-fold in \mathcal{M} close to N_0 . Then we expect that N should be modelled near x on some L_t^a for small $t > 0$. (This will be made more precise in §4.) Since there is a holomorphic disc D_t^a in \mathbb{C}^3 with boundary in L_t^a , and such discs are isolated and stable under small perturbations, we expect that there should exist a holomorphic disc D in M with boundary in N modelled on D_t^a .

As D is calibrated with respect to ω , its area is $\int_D \omega$. Thus, the area of D is given *topologically* in terms of relative homology and cohomology by the formula $\text{Area}(D) = [\omega]_{M;N} \cdot [D]_{M;N}$, where $[\omega]_{M;N} \in H^2(M; N, \mathbb{R})$ and $[D]_{M;N} \in H_2(M; N, \mathbb{Z})$. But also, the area of D is necessarily *positive*. When the area shrinks to zero, the boundary \mathcal{S}^1 of D in N is collapsed to a point, and N develops a singularity, a T^2 -cone.

Thus, we have the following picture. Regard $[D]_{M;N}$ as a fixed class $\beta \in H_2(M; N, \mathbb{Z})$. As we continuously vary (M, J, ω, Ω) and N , the class $[\omega]_{M;N}$ varies in $H^2(M; N, \mathbb{R})$. On \mathcal{M} we have $[\omega]_{M;N} \cdot \beta > 0$, and \mathcal{M}_0 is the boundary where $[\omega]_{M;N} \cdot \beta = 0$. So, singularities modelled on the cone L_0 should occur at the zeroes of one topologically determined real function $[\omega]_{M;N} \cdot \beta$ on the moduli space of SL 3-folds, which explains why they happen in codimension one.

The author expects that the SL cone L_0 in \mathbb{C}^3 is the generic local model for singularities of SL 3-folds which occur when the area of a holomorphic disc D with boundary in an SL 3-fold N shrinks to zero.

4 Topological behaviour of these singularities

In §3 we argued that singularities modelled on the SL cone L_0 are of *index one*, in the sense of §2.3. Therefore we expect singularities modelled on L_0 to occur in codimension one in families of SL 3-folds in generic families of almost Calabi–Yau 3-folds. We shall consider such a family, and investigate the topological consequences.

4.1 SL 3-folds with T^2 -cone singularities

In the following condition we set out the situation we want to study, an SL 3-fold in an almost Calabi–Yau 3-fold with one singular point modelled on L_0 , and establish some notation.

Condition 4.1 Let $(M, J_0, \omega_0, \Omega_0)$ be an almost Calabi–Yau 3-fold, and N_0 a compact, embedded, singular SL 3-fold in M with phase $e^{i\theta}$ and one singular point at $x \in M$, locally modelled on L_0 . To be more precise, let $\rho > 0$ be small, B_ρ and \overline{B}_ρ be the open and closed balls of radius ρ about 0 in \mathbb{C}^3 , and (J', ω', Ω') the Euclidean Calabi–Yau structure on $\overline{B}_\rho \subset \mathbb{C}^3$. Suppose there exists a C^1 embedding $\Phi_0 : \overline{B}_\rho \rightarrow M$ which is smooth except perhaps at 0, such that

- (a) $\Phi_0(0) = x$ and $\Phi_0^*(N_0) = L_0 \cap \overline{B}_\rho$;
- (b) $\Phi_0^*(\omega_0) = \omega'$; and
- (c) $\Phi_0^*(J_0) \approx J'$ and $\Phi_0^*(\Omega_0) \approx F e^{i\theta} \Omega'$ near $0 \in \overline{B}_\rho$, for some $F > 0$.

Suppose also that $H_1(N_0, \mathbb{R}) = \{0\}$.

Here are some remarks on this condition:

- As our conclusions will be conjectural, we will not worry very much about the details here – for instance, exactly what sort of approximation is required in part (c), or how many derivatives of Φ_0 exist at x .

- We have chosen the coordinate system Φ_0 to equate the symplectic structures ω_0 on M and ω' on \mathbb{C}^3 . This is possible by the Darboux Lemma. As there are so many symplectomorphisms we can also arrange that $(\Phi_0)^*(N_0) = L_0 \cap \overline{B}_\rho$, possibly at the cost of Φ_0 not being smooth at 0. However, Φ_0 will in general not be holomorphic, so we can only assume that $\Phi_0^*(J_0) \approx J'$ and $\Phi_0^*(\Omega_0) \approx F e^{i\theta} \Omega'$.

Here the factor $e^{i\theta}$ is to compensate for the fact that N_0 has phase $e^{i\theta}$ and L_0 has phase 1, and F is to allow for the fact that as $(M, J_0, \omega_0, \Omega_0)$ is only an *almost* Calabi–Yau 3-fold, so $(J_0, \omega_0, \Omega_0)$ is not necessarily isomorphic to (J', ω', Ω') at x , but only to $(J', \omega', F\Omega')$ for some $F > 0$.

One could instead choose the Φ_0 to be holomorphic coordinates, with $\Phi_0^*(J_0) = J'$ and $\Phi_0^*(\Omega_0) = \Omega'$. But then the best we could hope for would be $\Phi_0^*(\omega_0) \approx F\omega'$ and $\Phi_0^*(N_0) \approx L_0 \cap \overline{B}_\rho$.

- The assumptions that N_0 is embedded and $H_1(N_0, \mathbb{R}) = \{0\}$ are to simplify the calculations involving (relative) homology and cohomology below, and because we will eventually only be interested in special Lagrangian *homology* 3-spheres N , which have $H_1(N, \mathbb{R}) = \{0\}$. The assumptions can be removed without great difficulty.

Suppose Condition 4.1 holds. Define $P = N_0 \setminus \Phi_0(B_\rho)$. Then P is a compact, nonsingular 3-manifold whose boundary is the image under Φ_0 of the intersection of L_0 with the sphere \mathcal{S}_ρ^5 in \mathbb{C}^3 of radius ρ . Now $L_0 \cap \mathcal{S}_\rho^5$ is the orbit of $3^{-1/2}\rho(1, 1, 1)$ under the action of $G = \mathrm{U}(1)^2$ defined in (7). So define $\iota : G \rightarrow P$ by

$$\iota : (e^{i\theta_1}, e^{i\theta_2}) \mapsto \Phi_0\left(\frac{1}{\sqrt{3}}\rho e^{i\theta_1}, \frac{1}{\sqrt{3}}\rho e^{i\theta_2}, \frac{1}{\sqrt{3}}\rho e^{-i\theta_1 - i\theta_2}\right). \quad (13)$$

Then ι is a diffeomorphism $G \rightarrow \partial P$.

Lemma 4.2 *The inclusion $\iota : G \rightarrow P$ induces a map $\iota_* : H_1(G, \mathbb{Z}) \rightarrow H_1(P, \mathbb{Z})$, which has $\mathrm{Ker}(\iota_*) \cong \mathbb{Z}$.*

Proof. As $H_1(G, \mathbb{Z}) \cong \mathbb{Z}^2$ and $\mathrm{Ker}(\iota_*)$ is a subgroup of $H_1(G, \mathbb{Z})$, we see that $\mathrm{Ker}(\iota_*)$ is isomorphic to 0, \mathbb{Z} or \mathbb{Z}^2 . But using Poincaré duality ideas for manifolds with boundary, we can show that the map $H_1(G, \mathbb{R}) \rightarrow H_1(P, \mathbb{R})$ must have image and kernel \mathbb{R} , and this forces $\mathrm{Ker}(\iota_*) \cong \mathbb{Z}$. \square

Using Poincaré duality for P and $H_1(N_0, \mathbb{R}) = \{0\}$ one can also show that $H_1(P, \mathbb{R}) \cong \mathbb{R}$ and $H_2(P, \mathbb{R}) = \{0\}$. Write $Q_0 = N_0 \cap \Phi_0(\overline{B}_\rho)$. Then Q_0 is a singular 3-manifold with boundary $\partial Q_0 = \partial P = \iota(G)$. Topologically Q_0 is the cone on ∂P , and is contractible. We shall now define homology classes ζ, χ and integers k^1, k^2, k^3 which will be important in what follows.

Definition 4.3 Let $\zeta \in H_1(G, \mathbb{Z})$ be a generator for $\text{Ker}(\iota_*) \cong \mathbb{Z}$. Then ζ is unique up to sign. Now $G = \text{U}(1)^2$, so we may identify $H_1(G, \mathbb{Z}) \cong \mathbb{Z}^2$ in the obvious way, such that the maps $\mathcal{S}^1 \rightarrow G$ given by $e^{i\theta} \mapsto (e^{i\theta}, 1)$ and $e^{i\theta} \mapsto (1, e^{i\theta})$ represent the classes in $H_1(G, \mathbb{Z})$ identified with $(1, 0)$ and $(0, 1)$ in \mathbb{Z}^2 respectively. Define $k^1, k^2, k^3 \in \mathbb{Z}$ so that $\zeta \in H_1(G, \mathbb{Z})$ is identified with $(k^1, k^2) \in \mathbb{Z}^2$, and $k^3 = -k^1 - k^2$. Then $k^1 + k^2 + k^3 = 0$.

Let τ be a closed integral 1-chain in ∂P with $[\tau] = \iota_*(\zeta)$ in $H_1(\partial P, \mathbb{Z})$. Then $[\tau] = \iota_*(\zeta) = 0$ in $H_1(P, \mathbb{Z})$, so there exists an integral 2-chain Λ in P with $\partial \Lambda = \tau$. Also τ is a closed integral 1-chain in ∂Q_0 and Q_0 is contractible, so there exists an integral 2-chain Σ in Q_0 with $\partial \Sigma = \tau$. Thus $\Sigma - \Lambda$ is an integral 2-chain without boundary in $N_0 \subset M$. Define $\chi = [\Sigma - \Lambda] \in H_2(M, \mathbb{R})$. Then χ is a homology class in the image of $H_2(M, \mathbb{Z})$ in $H_2(M, \mathbb{R})$. Since $H_2(P, \mathbb{R}) = H_2(Q_0, \mathbb{R}) = \{0\}$ this χ is independent of the choice of Λ, Σ , and so is unique up to the choice of sign of ζ .

Note that as the chain $\Sigma - \Lambda$ representing χ lies in N_0 , which is Lagrangian with respect to ω_0 , we have $[\omega_0] \cdot \chi = 0$, where $[\omega_0] \in H^2(M, \mathbb{R})$ is the *Kähler class* of ω_0 . In the rest of the section we shall consider separately the cases $\chi \neq 0$ and $\chi = 0$.

4.2 The case $\chi \neq 0$: desingularizing N_0

Let $(M, J_0, \omega_0, \Omega_0)$, N_0 and x be as above. We shall now consider the question of when N_0 is the limit as $t \rightarrow 0_+$ of a family of compact, nonsingular SL 3-folds modelled on L_t^a near x . In general this cannot happen in the fixed almost Calabi–Yau 3-fold $(M, J_0, \omega_0, \Omega_0)$. Instead we need to suppose $\chi \neq 0$, and extend $(M, J_0, \omega_0, \Omega_0)$ to a smooth family of almost Calabi–Yau 3-folds $(M, J_t, \omega_t, \Omega_t)$ with $[\omega_t] \cdot \chi = t$. Here is our conjecture.

Conjecture 4.4 *Let Condition 4.1 hold, and k^1, k^2, k^3 and χ be as in Definition 4.3. Suppose $\chi \neq 0$, and $(M, J_0, \omega_0, \Omega_0)$ extends to a smooth family $\{(M, J_t, \omega_t, \Omega_t) : t \in (-\epsilon, \epsilon)\}$ of almost Calabi–Yau 3-folds for some $\epsilon > 0$, where $[\omega_t] \cdot \chi = t$.*

Then there do not exist singular SL 3-folds N_t in $(M, J_t, \omega_t, \Omega_t)$ close to N_0 and of the same topological type for $t \neq 0$. However, for some small $\delta \in (0, \epsilon]$, whenever $a = 1, 2, 3$, $|t| < \delta$ and $k^a t > 0$ there exists a compact, nonsingular, embedded SL 3-fold N_t^a in $(M, J_t, \omega_t, \Omega_t)$ with phase $e^{i\theta_t}$, which depends smoothly on t , and converges to N_0 as $t \rightarrow 0$ in a suitable sense, and near x is locally modelled on $L_{t/k^a\pi}^a$.

To be more precise, whenever $a = 1, 2, 3$, $0 < |t| < \delta$ and $k^a t > 0$, there exists an embedding $\Phi_t^a : \overline{B}_\rho \rightarrow M$ which depends smoothly on t and converges to Φ_0 as $t \rightarrow 0$, and satisfies

- (a) $(\Phi_t^a)^*(N_t^a) = L_{t/k^a\pi}^a \cap \overline{B}_\rho$;
- (b) $(\Phi_t^a)^*(\omega_t) = \omega'$; and
- (c) $(\Phi_t^a)^*(J_t) \approx J'$ and $(\Phi_t^a)^*(\Omega_t) \approx F_t e^{i\theta_t} \Omega'$ for small t , near $0 \in \overline{B}_\rho$, for some $F_t > 0$.

In the rest of this subsection we shall give a partial proof of this conjecture, which justifies the assertion that N_t does not exist for $t \neq 0$ but that N_t^a should exist when $k^a t > 0$, and be locally modelled on $L_{t/k^a\pi}^a$, rather than L_s^a for some other $s > 0$. To complete the proof will require essentially the same analysis needed to solve Conjecture 2.13, and so once Conjecture 2.13 has been proved, completing the conjecture above should be straightforward.

First, the nonexistence of singular SL 3-folds N_t in $(M, J_t, \omega_t, \Omega_t)$ for $t \neq 0$ close to N_0 , and of the same topological type. Suppose such an N_t did exist. Then N_t is homeomorphic to N_0 , and as in §4.1 we can construct a closed integral 2-chain C in N_t close to $\Sigma - \Lambda$ in N_0 . Then $[C] = \chi$ in the image of $H_2(M, \mathbb{Z})$ in $H_2(M, \mathbb{R})$, as χ is a discrete object and is unchanged under small variations. But $\omega_t|_C = 0$, as C lies in N_t which is Lagrangian with respect to ω_t . Hence $[\omega_t] \cdot \chi = 0$, which contradicts the assumptions $[\omega_t] \cdot \chi = t$ and $t \neq 0$.

Suppose that for some small $t \in (-\epsilon, \epsilon)$ there exists a compact nonsingular SL 3-fold N_t^a in $(M, J_t, \omega_t, \Omega_t)$, which is close to N_0 in a suitable sense, and near x is modelled on L_s^a . That is, there should exist an embedding $\Phi_t^a : \overline{B}_\rho \rightarrow M$ satisfying $(\Phi_t^a)^*(N_t^a) = L_s^a \cap \overline{B}_\rho$ and parts (b), (c) above.

Define $P_t^a = N_t^a \setminus \Phi_t^a(B_\rho)$ and $Q_t^a = N_t^a \cup \Phi_t^a(\overline{B}_\rho)$. Then Q_t^a is the image under Φ_t^a of $L_s^a \cap \overline{B}_\rho$. Provided $s < \rho^2$, which we may assume as s is small, Q_t^a is diffeomorphic to $\mathcal{S}^1 \times D^2$, and P_t^a is a compact, nonsingular 3-manifold

with boundary $\partial P_t^a = \partial Q_t^a \cong T^2$. Furthermore, P_t^a is diffeomorphic to P in §4.1. (In our present situation this follows from the assumption that N_t^a is close to N_0 , and in the situation of the conjecture from the fact that P_t^a depends smoothly on t and converges to P as $t \rightarrow 0$.)

Now the boundary of $L_s^a \cap \overline{B}_\rho$ is a G -orbit, so by picking a point in the boundary and using Φ_t^a we may define a diffeomorphism $\iota_t^a : G \rightarrow \partial P_t^a = \partial Q_t^a$, in the same way that we defined ι in §4.1. Furthermore, the diffeomorphism $P \cong P_t^a$ may be chosen to identify ι and ι_t^a .

Let ζ be as in Definition 4.3, and let τ_t^a be a closed integral 1-chain in ∂P_t^a with $[\tau_t^a] = (\iota_t^a)_*(\zeta)$ in $H_1(\partial P_t^a, \mathbb{Z})$. Then $[\tau_t^a] = (\iota_t^a)_*(\zeta) = 0$ in $H_1(P_t^a, \mathbb{Z})$, so there exists an integral 2-chain Λ_t^a in P_t^a with $\partial \Lambda_t^a = \tau_t^a$. In §3.1 we defined a holomorphic disc D_s^a in \mathbb{C}^3 with boundary $\partial D_s^a = \gamma_s^a$ in L_s^a . Thus $\Phi_s^a(\gamma_s^a)$ is an oriented \mathcal{S}^1 in Q_t^a . As $Q_t^a \cong \mathcal{S}^1 \times D^2$ we have $H_1(Q_t^a, \mathbb{Z}) \cong \mathbb{Z}$, and it is easy to see that $[\Phi_s^a(\gamma_s^a)]$ generates $H_1(Q_t^a, \mathbb{Z})$.

Now τ_t^a is a closed 1-chain in ∂Q_t^a , and the integers k^1, k^2, k^3 of Definition 4.3 were chosen to ensure that $[\tau_t^a] = (\iota_t^a)_*(\zeta) = k^a[\Phi_s^a(\gamma_s^a)]$ in $H_1(Q_s^a, \mathbb{Z})$. Therefore τ_t^a is homologous to $k^a \Phi_t^a(\gamma_s^a)$ in Q_t^a , so there exists an integral 2-chain Σ_t^a in Q_t^a with $\partial \Sigma_t^a = \tau_t^a - k^a \Phi_t^a(\gamma_s^a)$.

Consider the integral 2-chain $k^a \Phi_t^a(D_s^a) + \Sigma_t^a - \Lambda_t^a$ in M . As $\partial D_s^a = \gamma_s^a$, $\partial \Sigma_t^a = \tau_t^a - k^a \Phi_t^a(\gamma_s^a)$ and $\partial \Lambda_t^a = \tau_t^a$ it is closed, and so defines a homology class in the image of $H_2(M, \mathbb{Z})$ in $H_2(M, \mathbb{R})$. Now this class is exactly χ , as we constructed it by a small deformation of the construction of χ in Definition 4.3, but χ is a discrete object and so is unchanged by continuous deformations. Therefore we have

$$[\omega_t] \cdot \chi = k^a \int_{\Phi_t^a(D_s^a)} \omega_t + \int_{\Sigma_t^a} \omega_t - \int_{\Lambda_t^a} \omega_t = k^a \int_{D_s^a} \omega' = k^a \pi s,$$

using part (b) of the conjecture and the fact that Σ_t^a and Λ_t^a lie in N_t^a , which is Lagrangian with respect to ω_t .

Now $[\omega_t] \cdot \chi = t$ by definition, so we have shown that $t = k^a \pi s$. But $s > 0$, as L_s^a is only defined for positive s . Thus, if t and k^a are not both zero then N_t^a can only exist if $k^a t > 0$, and then $s = t/k^a \pi$, as we have to prove. The case $t = k^a = 0$ is excluded by the conjecture, and will be discussed in §4.3. This concludes our partial proof of Conjecture 4.4.

4.3 The case $\chi \neq 0$: homology 3-spheres

Next we prove that the 3-manifolds N_t^a above are homology 3-spheres, and calculate the size of $H_1(N_t^a, \mathbb{Z})$.

Proposition 4.5 *In the situation of Conjecture 4.4, both $H_1(N_0, \mathbb{Z})$ and $H_1(N_t^a, \mathbb{Z})$ are finite and satisfy $|H_1(N_t^a, \mathbb{Z})| = |k^a| \cdot |H_1(N_0, \mathbb{Z})|$. Thus, each N_t^a is a (rational) homology 3-sphere.*

Proof. Let $a = 1, 2$ or 3 and t satisfy $|t| < \delta$ and $k^a t > 0$, and let N_t^a be as in Conjecture 4.4. Suppose also that t is small enough that $|t| < |k^a| \pi \rho^2$, to ensure that $Q_t^a \cong \mathcal{S}^1 \times D^2$, and define P_t^a , Q_t^a and ι_t^a as in §4.2. As the N_t^a depend smoothly on t they are all diffeomorphic for fixed a , and so it is enough to prove the proposition for sufficiently small t .

Using the exact sequence

$$H_1(\{x\}, \mathbb{Z}) \rightarrow H_1(N_0, \mathbb{Z}) \rightarrow H_1(N_0; \{x\}, \mathbb{Z}) \rightarrow H_0(\{x\}, \mathbb{Z}) \xrightarrow{\cong} H_0(N_0, \mathbb{Z})$$

and $H_1(\{x\}, \mathbb{Z}) = 0$, we find that $H_1(N_0, \mathbb{Z}) \cong H_1(N_0; \{x\}, \mathbb{Z})$, and by excision and the diffeomorphism $P \cong P_t^a$ we see that $H_1(N_0; \{x\}, \mathbb{Z}) \cong H_1(P_t^a; \partial P_t^a, \mathbb{Z}) \cong H_1(N_t^a; Q_t^a, \mathbb{Z})$. Thus $H_1(N_t^a; Q_t^a, \mathbb{Z}) \cong H_1(N_0, \mathbb{Z})$. Recall that $H_1(N_0, \mathbb{R}) = \{0\}$ by Condition 4.1. It is easy to show using this and facts about manifold topology that $H_1(N_0, \mathbb{Z})$ is finite.

As Q_t^a is diffeomorphic to $\mathcal{S}^1 \times \mathbb{R}^2$ we see that $H_1(Q_t^a, \mathbb{Z})$ is isomorphic to \mathbb{Z} , and is generated by $[\Phi_t^a(\gamma_{t/k^a\pi}^a)]$. But the argument in §4.2 shows that $k^a \Phi_t^a(\gamma_{t/k^a\pi}^a)$ is homologous in Q_t^a to τ_t^a in ∂Q_t^a , and τ_t^a is homologous to 0 in P_t^a . Thus, the image of $|k^a| [\Phi_t^a(\gamma_{t/k^a\pi}^a)]$ in $H_1(N_t^a, \mathbb{Z})$ is zero. Furthermore, no smaller positive multiple of $[\Phi_t^a(\gamma_{t/k^a\pi}^a)]$ can have zero image in $H_1(N_t^a, \mathbb{Z})$, because τ_t^a represents ζ , which by definition generates the kernel of $(\iota_t^a)_*$.

Now consider the exact sequence

$$H_1(Q_t^a, \mathbb{Z}) \rightarrow H_1(N_t^a, \mathbb{Z}) \rightarrow H_1(N_t^a; Q_t^a, \mathbb{Z}) \rightarrow H_0(Q_t^a, \mathbb{Z}) \xrightarrow{\cong} H_0(N_t^a, \mathbb{Z}).$$

We have shown above that $H_1(N_t^a; Q_t^a, \mathbb{Z}) \cong H_1(N_0, \mathbb{Z})$, which is finite, and that the image of $H_1(Q_t^a, \mathbb{Z})$ in $H_1(N_t^a, \mathbb{Z})$ is isomorphic to the cyclic group $\mathbb{Z}_{|k^a|}$. Thus by exactness $H_1(N_t^a, \mathbb{Z})/\mathbb{Z}_{|k^a|} \cong H_1(N_0, \mathbb{Z})$, and so $H_1(N_t^a, \mathbb{Z})$ is finite with $|H_1(N_t^a, \mathbb{Z})| = |k^a| \cdot |H_1(N_0, \mathbb{Z})|$, as we have to prove.

Since $H_1(N_t^a, \mathbb{Z})$ is finite we have $b_1(N_t^a) = 0$, so $b_2(N_t^a) = 0$ by Poincaré duality. But N_t^a is connected (this is part of our definition of manifold) and oriented, as it is special Lagrangian. So N_t^a is by definition a rational homology 3-sphere. \square

Our goal in this paper is to define an invariant of almost Calabi–Yau 3-folds by counting special Lagrangian homology 3-spheres in some appropriate way. We can use the ideas above to draw some conclusions about how to do this. We assume for the moment that Conjecture 4.4 is true.

Because $k^1 + k^2 + k^3 = 0$ and the k^a are not all zero as $\zeta \neq 0$, at least one k^a is positive, and one negative. But the SL 3-fold N_t^a exists for small $t > 0$ if and only if $k^a > 0$, and for small $t < 0$ if and only if $k^a < 0$. So, consider the following three cases:

- (a) Suppose $k_1, k_2 > 0$ and $k_3 < 0$. Then N_t^1 and N_t^2 exist as SL 3-folds for small $t > 0$ but N_t^3 does not, and N_t^3 exists for small $t < 0$ but N_t^1, N_t^2 do not.
- (b) Suppose $k_1 > 0$, $k_2 = 0$ and $k_3 < 0$. Then N_t^1 exists for small $t > 0$, N_t^3 exists for small $t < 0$, and N_t^2 does not exist for any $t \neq 0$.
- (c) Suppose $k_1 > 0$ and $k_2, k_3 < 0$. Then N_t^1 exists for small $t > 0$, and N_t^2, N_t^3 exist for small $t < 0$.

These show that as we deform $(M, J_t, \omega_t, \Omega_t)$, changing the Kähler class $[\omega_t]$, it can happen that two SL homology 3-spheres disappear, and one reappears; or that one disappears, and another reappears; or that one disappears, and two reappear.

In particular, this shows that the number of special Lagrangian homology 3-spheres in M is not invariant under deformations of (M, J, ω, Ω) changing the Kähler class $[\omega]$, even if counted with signs. So, simple counting of SL homology 3-spheres, even with signs, is probably not the right thing to do. However, it is easy to see using Proposition 4.5 that the sum over SL homology 3-spheres N of the weight $|H_1(N, \mathbb{Z})|$ is *unchanged* under transitions of the kind described in Conjecture 4.4.

For instance, in case (a) above, for small $t > 0$ there exist two SL homology 3-spheres N_t^1, N_t^2 , with $|H_1(N_t^1, \mathbb{Z})| = k^1 |H_1(N_0, \mathbb{Z})|$ and $|H_1(N_t^2, \mathbb{Z})| = k^2 |H_1(N_0, \mathbb{Z})|$, and when $t < 0$ there is one SL homology 3-spheres N_t^3 with $|H_1(N_t^3, \mathbb{Z})| = -k^3 |H_1(N_0, \mathbb{Z})|$. But as $-k^3 = k^1 + k^2$ we see that $|H_1(N_t^1, \mathbb{Z})| + |H_1(N_t^2, \mathbb{Z})| = |H_1(N_t^3, \mathbb{Z})|$, so the sum of weights is the same

for small $t > 0$ and $t < 0$. This suggests that the appropriate thing to do is to count SL homology 3-spheres N with weight $|H_1(N, \mathbb{Z})|$, and we will argue this in §7.

We can now discuss the existence of SL 3-folds N_t^a when $k^a = t = 0$, which was passed over in §4.2. The argument of §4.2 shows that for an SL 3-fold N_t^a to exist in $(M, J_t, \omega_t, \Omega_t)$ that is modelled on N_0 away from x and L_s^a near x , we need $t = k^a \pi s$. Thus, if $k^a = 0$ then such SL 3-folds can only exist when $t = 0$. The argument above gives an exact sequence

$$\mathbb{Z} = H_1(Q_t^a, \mathbb{Z}) \rightarrow H_1(N_t^a, \mathbb{Z}) \rightarrow H_1(N_0, \mathbb{Z}) \rightarrow 0,$$

and shows that the generator of $H_1(Q_t^a, \mathbb{Z})$ has order $|k^a|$ in $H_1(N_t^a, \mathbb{Z})$. However, in this case $k^a = 0$, so the map $\mathbb{Z} \rightarrow H_1(N_t^a, \mathbb{Z})$ is *injective*. It follows that $H_1(N_t^a, \mathbb{Z})$ is infinite, the product of \mathbb{Z} with a finite group, and hence that $b^1(N_t^a) = 1$. So N_t^a is not a homology 3-sphere, and by Theorem 2.7 it is not isolated as an SL 3-fold, but occurs in a moduli space of dimension 1.

Thus we are led to the following picture. If $k^a > 0$ then there exists a unique SL 3-fold N_t^a in $(M, J_t, \omega_t, \Omega_t)$ for small $t > 0$. If $k^a < 0$ then there exists a unique SL 3-fold N_t^a in $(M, J_t, \omega_t, \Omega_t)$ for small $-t > 0$. And if $k^a = 0$ then for each small $s > 0$ there exists a unique SL 3-fold N_0^a in $(M, J_0, \omega_0, \Omega_0)$ modelled near x on L_s^a . So in each case the SL 3-folds are indexed by a small positive parameter.

4.4 The case $\chi = 0$: stable SL singularities

In §3.2 we argued that SL singularities modelled on the SL cone L_0 in \mathbb{C}^3 are of *index one*, and should occur in codimension one in families of SL 3-folds in generic families of almost Calabi–Yau 3-folds. However, such local calculations of the ‘index’ of singularities can sometimes give the wrong answer in cases when global topological restrictions come into play. This happens when $\chi = 0$ in the situation of §4.1, and effectively the index of the singularities should be zero rather than one in this case.

Let $\{(M, J_t, \omega_t, \Omega_t) : t \in (-\epsilon, \epsilon)\}$ be a smooth family of almost Calabi–Yau 3-folds deforming $(M, J_0, \omega_0, \Omega_0)$. As $\chi = 0$, we cannot assume that $[\omega_t] \cdot \chi = t$, so the family is essentially arbitrary. The argument in §4.2 that there do not exist singular SL 3-folds N_t in $(M, J_t, \omega_t, \Omega_t)$ for $t \neq 0$ close to N_0 , and of the same topological type, is now no longer valid, and the author conjectures that such N_t do exist for small t .

What about the existence of nonsingular SL 3-folds resolving N_0 ? The argument in §4.2 shows that if for small t there exists a nonsingular SL 3-fold $N_{s,t}^a$ in $(M, J_t, \omega_t, \Omega_t)$ close to N_0 away from x and modelled on L_s^a near x , then $k^a \pi s = [\omega_t] \cdot \chi = 0$. As $s > 0$, this shows that such SL 3-folds $N_{s,t}^a$ cannot exist unless $k^a = 0$. When $k^a = 0$ such $N_{s,t}^a$ can exist, and have $b^1(N_{s,t}^a) = 1$ as in §4.3, so they should occur in 1-parameter families in $(M, J_t, \omega_t, \Omega_t)$ by Theorem 2.7, parametrized by s .

We summarize our conclusions in the following conjecture, which should be provable by the same methods as Conjecture 4.4.

Conjecture 4.6 *Let Condition 4.1 hold, and k^1, k^2, k^3 and χ be as in Definition 4.3. Suppose $\chi = 0$, and $(M, J_0, \omega_0, \Omega_0)$ extends to a smooth family $\{(M, J_t, \omega_t, \Omega_t) : t \in (-\epsilon, \epsilon)\}$ of almost Calabi–Yau 3-folds for some $\epsilon > 0$. Then for some $\delta \in (0, \epsilon]$, N_0 extends to a smooth family of compact, embedded, singular SL 3-folds N_t in $(M, J_t, \omega_t, \Omega_t)$ for $t \in (-\delta, \delta)$, each of which has one singular point locally modelled on L_0 .*

If $k^a = 0$ then for small $t \in (-\epsilon, \epsilon)$ and $s > 0$ there exists a unique compact, nonsingular SL 3-fold $N_{s,t}^a$ in $(M, J_t, \omega_t, \Omega_t)$ close to N_0 away from x , and modelled on L_s^a near x . If $k^a \neq 0$ there do not exist any such $N_{s,t}^a$.

5 Another model degeneration of SL 3-folds

We now describe a family of explicit SL 3-folds $K_{\phi,A}$ in \mathbb{C}^3 . This family was first found by Lawlor [13], was made more explicit by Harvey [4, p. 139–140], and was discussed from a different point of view by the author in [7, §5.4(b)]. Our treatment is based on that of Harvey.

Let $a_1, a_2, a_3 > 0$, and define polynomials $p(x)$, $P(x)$ by

$$p(x) = (1 + a_1 x^2)(1 + a_2 x^2)(1 + a_3 x^2) - 1 \quad \text{and} \quad P(x) = \frac{p(x)}{x^2}.$$

Define real numbers ϕ_1, ϕ_2, ϕ_3 and A by

$$\phi_k = a_k \int_{-\infty}^{\infty} \frac{dx}{(1 + a_k x^2) \sqrt{P(x)}} \quad \text{and} \quad A = \frac{4\pi}{3} (a_1 a_2 a_3)^{-1/2}.$$

Clearly $\phi_k > 0$ and $A > 0$. But writing $\phi_1 + \phi_2 + \phi_3$ as one integral and rearranging gives

$$\phi_1 + \phi_2 + \phi_3 = \int_0^{\infty} \frac{p'(x) dx}{(p(x) + 1) \sqrt{p(x)}} = 2 \int_0^{\infty} \frac{dw}{w^2 + 1} = \pi,$$

making the substitution $w = \sqrt{p(x)}$. So $\phi_k \in (0, \pi)$ and $\phi_1 + \phi_2 + \phi_3 = \pi$. It can be shown that this yields a 1-1 correspondence between triples (a_1, a_2, a_3) with $a_k > 0$, and quadruples $(\phi_1, \phi_2, \phi_3, A)$ with $\phi_k \in (0, \pi)$, $\phi_1 + \phi_2 + \phi_3 = \pi$ and $A > 0$.

For $k = 1, 2, 3$ and $y \in \mathbb{R}$, define $z_k(y)$ by $z_k(y) = e^{i\psi_k(y)} \sqrt{a_k^{-1} + y^2}$, where

$$\psi_k(y) = a_k \int_{-\infty}^y \frac{dx}{(1 + a_k x^2) \sqrt{P(x)}}.$$

Now write $\phi = (\phi_1, \phi_2, \phi_3)$, and define a submanifold $K_{\phi, A}$ in \mathbb{C}^3 by

$$K_{\phi, A} = \{(z_1(y)x_1, z_2(y)x_2, z_3(y)x_3) : y \in \mathbb{R}, x_k \in \mathbb{R}, x_1^2 + x_2^2 + x_3^2 = 1\}. \quad (14)$$

Our next result comes from Harvey [4, Th. 7.78].

Proposition 5.1 *The set $K_{\phi, A}$ defined in (14) is an embedded SL 3-fold in \mathbb{C}^3 diffeomorphic to $\mathcal{S}^2 \times \mathbb{R}$. It is asymptotically conical, with cone the union $\Pi_0 \cup \Pi_\phi$ of two special Lagrangian 3-planes Π_0, Π_ϕ given by*

$$\Pi_0 = \{(x_1, x_2, x_3) : x_j \in \mathbb{R}\}, \quad \Pi_\phi = \{(e^{i\phi_1}x_1, e^{i\phi_2}x_2, e^{i\phi_3}x_3) : x_j \in \mathbb{R}\}. \quad (15)$$

Here is how to interpret the constant A . Using the above notation, define

$$D_{\phi, A} = \{(x_1 e^{i\phi_1/2}, x_2 e^{i\phi_2/2}, x_3 e^{i\phi_3/2}) : x_k \in \mathbb{R}, a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 \leq 1\}. \quad (16)$$

Then $D_{\phi, A}$ is a solid ellipsoid in \mathbb{C}^3 , with boundary in $K_{\phi, A}$. The axes of $D_{\phi, A}$ have lengths $a_k^{-1/2}$ for $k = 1, 2, 3$, and so the volume of $D_{\phi, A}$ is A . Furthermore, $D_{\phi, A}$ is calibrated with respect to $\text{Im}(\Omega_0)$. That is, we can regard $D_{\phi, A}$ as an SL 3-fold of phase i , whereas $K_{\phi, A}$ has phase 1; so that $D_{\phi, A}$ and $K_{\phi, A}$ are both special Lagrangian, but of *perpendicular phase*.

We met a similar situation in §3. There we defined an AC SL 3-fold L_t^a depending on a real parameter $t > 0$, and a holomorphic 2-disc D_t^a with boundary on L_t^a , and area πt . Here we define an AC SL 3-fold $K_{\phi, A}$ depending on a real parameter $A > 0$, and a special Lagrangian 3-disc $D_{\phi, A}$ of phase i , with boundary on $K_{\phi, A}$ and area A .

Next we apply the ideas of §2.3 to $\Pi_0 \cup \Pi_\phi$ and the $K_{\phi, A}$.

Lemma 5.2 *In the notation of §2.3, $\Pi_0 \cup \Pi_\phi$ and $K_{\phi,A}$ satisfy $k = 2$, $b^1(K_{\phi,A}) = 0$ and $\text{l-ind}(K_{\phi,A}) = 6$, and $\Pi_0 \cup \Pi_\phi$ is rigid.*

Proof. As $\Pi_0 \cup \Pi_\phi$ is a cone on two disjoint copies of \mathcal{S}^2 we have $k = 1$, and as $K_{\phi,A}$ is diffeomorphic to $\mathcal{S}^2 \times \mathbb{R}$ we have $b^1(K_{\phi,A}) = 0$. There are two connected components Σ_1, Σ_2 of Σ , each of which is isometric to the unit sphere \mathcal{S}^2 in \mathbb{R}^3 with the round metric.

For both Σ_1, Σ_2 the only eigenvalue of Δ in $(0, 6)$ is 2, with multiplicity 3, and eigenvectors the restriction to \mathcal{S}^2 of linear functions on \mathbb{R}^3 . So $\text{l-ind}(K_{\phi,A}) = 3 + 3 = 6$. Similarly, Δ has eigenvalue 6 with multiplicity 5 on each of Σ_1, Σ_2 , and eigenfunctions the restrictions to \mathcal{S}^2 of harmonic homogeneous quadratic polynomials on \mathbb{R}^3 . As $\dim \text{SU}(3) = 8$ and the subgroup G_j of $\text{SU}(3)$ preserving Σ_j is isomorphic to $\text{SO}(3)$ with dimension 3, we see that $\Pi_0 \cup \Pi_\phi$ is rigid as $5 = 8 - 3$. \square

Assuming Conjecture 2.13, we see that in the sense discussed in §2.3, singularities of SL 3-folds modelled on the SL cones $\Pi_0 \cup \Pi_\phi$ should have *index one*. That is, they should occur in codimension 1 in families of SL 3-folds in a generic almost Calabi–Yau 3-fold, or more generally in generic families of almost Calabi–Yau 3-folds.

Now $\Pi_0 \cup \Pi_\phi$ may be regarded as a singular SL cone in \mathbb{C}^3 with isolated singular point at 0, but it is also the union of two *nonsingular* special Lagrangian 3-planes. In the same way, an embedded singular SL 3-fold N in (M, J, ω, Ω) with one singular point x modelled on $\Pi_0 \cup \Pi_\phi$ is either a nonsingular *immersed* SL 3-fold with one self-intersection point at x , or the union of two nonsingular SL 3-folds N^+, N^- which intersect at x , and have *the same phase*. We shall study this second possibility.

6 Topological behaviour of these singularities

We shall study the following situation.

Condition 6.1 Let $\epsilon > 0$, and $\{(M, J_t, \omega_t, \Omega_t) : t \in (-\epsilon, \epsilon)\}$ be a smooth family of almost Calabi–Yau 3-folds with the same underlying 6-manifold M . Suppose that N_0^+ and N_0^- are compact, embedded, nonsingular special Lagrangian homology 3-spheres in $(M, J_0, \omega_0, \Omega_0)$ with the same phase $e^{i\theta}$, which intersect transversely at one point x , such that the intersection $N_0^+ \cap N_0^-$ is positive in the sense of homology, using the natural orientations on N_0^\pm and M .

We can show using the results of §2.2 that N_0^\pm extend to families of SL 3-folds N_t^\pm in $(M, J_t, \omega_t, \Omega_t)$.

Proposition 6.2 *Suppose Condition 6.1 holds. Then there exists $\delta \in (0, \epsilon]$ such that N_0^+ and N_0^- extend to unique smooth families $\{N_t^\pm : t \in (-\delta, \delta)\}$ of compact, nonsingular, embedded SL 3-folds in $(M, J_t, \omega_t, \Omega_t)$. Each N_t^\pm is diffeomorphic to N_0^\pm and homologous to it in $H_3(M, \mathbb{Z})$, and N_t^+ and N_t^- intersect transversely in one point, where the intersection $N_t^+ \cap N_t^-$ is positive in the sense of homology.*

Proof. As N_0^\pm are homology 3-spheres we have $H^2(N_0^\pm, \mathbb{R}) = 0$, so the condition $[\omega_t|_{N_0^\pm}] = 0$ in $H^2(N_0^\pm, \mathbb{R})$ in Theorem 2.8 holds automatically. Thus, it follows from Theorem 2.8 that for some $\delta \in (0, \epsilon]$ we can extend N_0^\pm to smooth families N_t^\pm of compact, nonsingular, embedded SL 3-folds in $(M, J_t, \omega_t, \Omega_t)$ for $t \in (-\delta, \delta)$. As these are smooth, connected families the N_t^\pm are clearly diffeomorphic and homologous to N_0^\pm .

Now $b^1(N_t^\pm) = b^1(N_0^\pm) = 0$ as N_0^\pm is a homology 3-sphere, so N_t^\pm is rigid by Theorem 2.7, that is, it has no special Lagrangian deformations. Thus the families N_t^\pm are unique. To intersect transversely in one point is an open condition on pairs of submanifolds, so by making δ smaller if necessary we can suppose that N_t^+ and N_t^- intersect transversely in one point, which will be a positive intersection as $[N_t^+] \cap [N_t^-] = [N_0^+] \cap [N_0^-] = 1$. \square

We shall use the following notation.

Definition 6.3 Let Condition 6.1 hold, and use the notation of Proposition 6.2. Define $\chi^\pm = [N_0^\pm]$ in $H_3(M, \mathbb{Z})$. Then $[N_t^\pm] = \chi^\pm$ for $t \in (-\delta, \delta)$. Let the phase of N_t^\pm be $e^{i\theta_t^\pm}$. Then θ_t^\pm is only defined modulo $2\pi\mathbb{Z}$, but we fix it uniquely by requiring that $\theta_0^\pm = \theta$ and θ_t^\pm should depend smoothly on t . Applying Lemma 2.6 to N_t^\pm we see that

$$[\Omega_t] \cdot \chi^\pm = R_t^\pm e^{i\theta_t^\pm} \quad \text{for } t \in (-\delta, \delta), \quad (17)$$

for some $R_t^\pm > 0$ depending smoothly on t . As $\theta_0^+ = \theta_0^- = \theta$, by making $\delta > 0$ smaller if necessary we may assume that

$$|\theta_t^+ - \theta_t^-| < \pi \quad \text{for all } t \in (-\delta, \delta). \quad (18)$$

Then $R_t^+ e^{i\theta_t^+} + R_t^- e^{i\theta_t^-} \neq 0$ for any $t \in (-\delta, \delta)$. Define $R_t > 0$ and θ_t by

$$[\Omega_t] \cdot (\chi^+ + \chi^-) = R_t^+ e^{i\theta_t^+} + R_t^- e^{i\theta_t^-} = R_t e^{i\theta_t} \quad (19)$$

for all $t \in (-\delta, \delta)$. Here θ_t is only defined modulo $2\pi\mathbb{Z}$, but we specify θ_t uniquely by requiring that it depend smoothly on t , and $\theta_0 = \theta$. It is then easy to show that θ_t lies between θ_t^+ and θ_t^- for all $t \in (-\delta, \delta)$.

One can show using a ‘simultaneous diagonalization’ argument that if Π^+, Π^- are SL 3-planes with phase 1 in \mathbb{C}^3 intersecting only at 0 then there exists $B \in \text{SU}(3)$ such that $B\Pi^+ = \Pi_0$ and $B\Pi^- = \Pi_\phi$, where

$$\Pi_0 = \{(x_1, x_2, x_3) : x_j \in \mathbb{R}\} \quad \text{and} \quad \Pi_\phi = \{(e^{i\phi_1}x_1, e^{i\phi_2}x_2, e^{i\phi_3}x_3) : x_j \in \mathbb{R}\}$$

for some $\phi_1, \phi_2, \phi_3 \in (0, \pi)$, where $\phi_1 + \phi_2 + \phi_3 = \pi$ if $\Pi^+ \cap \Pi^-$ is a positive intersection in the sense of homology, and $\phi_1 + \phi_2 + \phi_3 = 2\pi$ if $\Pi^+ \cap \Pi^-$ is a negative intersection. Using this it is easy to prove:

Proposition 6.4 *Suppose Condition 6.1 holds. Then there exists a complex linear isometry $\iota : \mathbb{C}^3 \rightarrow T_x M$ satisfying*

$$\iota^*(\Omega_0) = e^{i\theta} dz_1 \wedge dz_2 \wedge dz_3, \quad \iota^*(T_x N_0^+) = \Pi_0 \quad \text{and} \quad \iota^*(T_x N_0^-) = \Pi_\phi, \quad (20)$$

where $\phi_1, \phi_2, \phi_3 \in (0, \pi)$ with $\phi_1 + \phi_2 + \phi_3 = \pi$, and Π_0, Π_ϕ are given in (15).

This shows that after adjusting the phase by $e^{i\theta}$, the singular SL 3-fold $N_0^+ \cup N_0^-$ is modelled on the SL cone $\Pi_0 \cup \Pi_\phi$ in \mathbb{C}^3 considered in §5. Thus, it is natural to consider whether there exist compact, nonsingular SL 3-folds N_t in $(M, J_t, \omega_t, \Omega_t)$ modelled on $N_0^+ \cup N_0^-$ away from x , and on $K_{\phi, A}$ near x for small $t \in (-\delta, \delta)$ and $A > 0$. Here is our conjectural answer to this question.

Conjecture 6.5 *In the situation of Condition 6.1, with the notation defined above, for each small $t \in (-\delta, \delta)$ with $\theta_t^+ > \theta_t^-$ there exists a unique compact, nonsingular special Lagrangian 3-fold N_t in $(M, J_t, \omega_t, \Omega_t)$ modelled on $N_0^+ \cup N_0^-$ away from x , and on $K_{\phi, A}$ near x , where $A > 0$ is small, depends on t , and is given approximately by*

$$A \approx R_t^+ \sin(\theta_t^+ - \theta_t) = R_t^- \sin(\theta_t - \theta_t^-). \quad (21)$$

Furthermore, N_t is diffeomorphic to the connected sum $N_0^+ \# N_0^-$ and is also a homology 3-sphere, with $|H_1(N_t, \mathbb{Z})| = |H_1(N_t^+, \mathbb{Z})| \cdot |H_1(N_t^-, \mathbb{Z})|$, and $[N_t] = \chi^+ + \chi^- \in H_3(M, \mathbb{Z})$. If t is small and $\theta_t^+ \leq \theta_t^-$ then there do not exist any such SL 3-folds N_t .

In the rest of the section we will give a partial proof of this conjecture, justifying the assertion that N_t should exist only if $\theta_t^+ > \theta_t^-$, the approximate value of A in (21), and the topological claims about N_t . We begin with the topology. If the SL 3-fold N_t exists then it is modelled on $N_0^+ \cup N_0^-$ away from x , and on $K_{\phi,A}$ near x . But $K_{\phi,A}$ is a narrow ‘neck’ diffeomorphic to $\mathcal{S}^2 \times \mathbb{R}$.

Thus, topologically N_t is made by removing the point x from N_0^+ and N_0^- , and joining them together with a small $\mathcal{S}^2 \times \mathbb{R}$ ‘neck’. That is, N_t is the *connected sum* $N_0^+ \# N_0^-$ of N_0^+ and N_0^- . It follows that N_t is a homology 3-sphere, and $H_1(N_t, \mathbb{Z}) \cong H_1(N_t^+, \mathbb{Z}) \times H_1(N_t^-, \mathbb{Z})$, so that $|H_1(N_t, \mathbb{Z})| = |H_1(N_t^+, \mathbb{Z})| \cdot |H_1(N_t^-, \mathbb{Z})|$.

Therefore $b^1(N_t) = 0$, and hence by Theorem 2.7 N_t is *rigid*, so that if it exists it should be unique. (This is not a rigorous argument). It is also obvious that N_t is homologous to $N_0^+ \cup N_0^-$, and so $[N_t] = [N_0^+] + [N_0^-] = \chi^+ + \chi^-$ in $H_3(M, \mathbb{Z})$. Equation (19) then gives $[\omega_t] \cdot [N_t] = R_t e^{i\theta_t}$. So by Lemma 2.6 the phase of N_t is $e^{i\theta_t}$.

Now if N_t has phase $e^{i\theta_t}$ and is modelled on $K_{\phi,A}$ near x , in a similar way to Conjecture 4.4 we expect that for some small $\rho > 0$ there should exist local coordinates $\Phi_t : \overline{B}_\rho \rightarrow M$ near x satisfying the conditions

- (a) $\Phi_t^*(N_t) \approx K_{\phi,A} \cap \overline{B}_\rho$;
- (b) $\Phi_t^*(\omega_t) \approx F_t \omega'$ near $0 \in \overline{B}_\rho$ for some $F_t > 0$; and
- (c) $\Phi_t^*(J_t) = J'$ and $\Phi_t^*(\Omega_t) = e^{i\theta_t} \Omega'$.

Here we have chosen our coordinate system to be holomorphic and to identify Ω_t with $e^{i\theta_t} \Omega'$, which means we can only assume that $\Phi_t^*(N_t) \approx K_{\phi,A}$ and $\Phi_t^*(\omega_t) \approx F_t \omega'$.

Equation (16) defined a special Lagrangian ellipsoid $D_{\phi,A}$ in \mathbb{C}^3 with phase i and boundary in $K_{\phi,A}$. As $\Phi_t^*(N_t) \approx K_{\phi,A} \cap \overline{B}_\rho$ there exists a disc D_t in \mathbb{C}^3 close to $D_{\phi,A}$ with $\partial D_t \subset \Phi_t^*(N_t)$. As D_t is close to $D_{\phi,A}$ we have

$$\int_{D_t} \text{Im } \Omega' \approx \int_{D_{\phi,A}} \text{Im } \Omega' = A, \quad (22)$$

as $D_{\phi,A}$ is calibrated with respect to $\text{Im } \Omega'$ and has area A .

Now $N_t \setminus \Phi_t(\partial D_t)$ has two connected components, say P_t^\pm , where P_t^+ is close to N_0^+ and P_t^- to N_0^- . The closures \overline{P}_t^\pm are compact, oriented 3-manifolds with boundary. Careful consideration of the orientations of N_t

and D_t shows that as 3-chains in M we have

$$\partial(\overline{P}_t^+) = \Phi_t(\partial D_t) \quad \text{and} \quad \partial(\overline{P}_t^-) = -\Phi_t(\partial D_t).$$

Therefore $\overline{P}_t^\pm \pm \Phi_t(D_t)$ is an integral 3-chain in M without boundary, and so defines a homology class in $H_3(M, \mathbb{Z})$. But clearly $\overline{P}_t^\pm \pm \Phi_t(D_t)$ is homologous to N_0^\pm , and so $[\overline{P}_t^\pm \pm \Phi_t(D_t)] = \chi^\pm$.

Consider the integral of the 3-form $\text{Im}(e^{-i\theta_t}\Omega_t)$ over $\overline{P}_t^\pm \pm \Phi_t(D_t)$. Since N_t is special Lagrangian of phase $e^{i\theta_t}$, this form vanishes on N_t and P_t^\pm . Thus

$$\text{Im}(e^{i\theta_t}[\Omega_t] \cdot \chi^\pm) = \pm \int_{\Phi_t(D_t)} \text{Im}(e^{-i\theta_t}\Omega_t) = \pm \int_{D_t} \text{Im}(\Omega') \approx \pm A,$$

using part (c) above and equation (22). Therefore

$$A \approx \text{Im}(e^{-i\theta_t}[\Omega_t] \cdot \chi^+) = -\text{Im}(e^{-i\theta_t}[\Omega_t] \cdot \chi^-).$$

Substituting in equation (17) yields (21), as we have to prove.

Now $|\theta_t^+ - \theta_t^-| < \pi$ by (18), and θ_t lies in between θ_t^+ and θ_t^- . Using these one can show that if $\theta_t^+ > \theta_t^-$ then $\theta_t^+ - \theta_t$ and $\theta_t - \theta_t^-$ lie in $(0, \pi)$ and the (approximate) value for A in (21) is positive, but if $\theta_t^+ \leq \theta_t^-$ then $\theta_t^+ - \theta_t$ and $\theta_t - \theta_t^-$ lie in $(-\pi, 0]$ and the (approximate) value for A is nonpositive. Thus, $\theta_t^+ > \theta_t^-$ corresponds to the condition that $A > 0$, which is part of the definition of $K_{\phi, A}$, and justifies our claim that $\theta_t^+ > \theta_t^-$ should be the necessary and sufficient condition for the existence of N_t for small t . This completes our partial proof of Conjecture 6.5.

Note that Adrian Butscher [1] has proved an analytic result closely related to Conjecture 6.5, but for SL m -folds in \mathbb{C}^m satisfying certain boundary conditions rather than for compact SL m -folds in almost Calabi–Yau m -folds. It seems likely that his analysis can be extended to prove our conjecture.

7 Counting SL homology 3-spheres

The *Gromov–Witten invariants* of a symplectic manifold (M, ω) are defined by counting, with signs, the J -holomorphic curves in M satisfying certain homological conditions. A good introduction to the subject is given by McDuff and Salamon [14]. One important feature of these invariants is that they are very stable under deformations of the choice of almost complex structure J used to define them.

Now it seems a natural (but perhaps optimistic) question to ask whether we can define similar invariants of almost Calabi–Yau 3-folds (M, J, ω, Ω) by counting SL 3-folds N in M satisfying suitable homological conditions. Probably the simplest such condition is to count 3-folds N in some fixed homology class in $H_3(N, \mathbb{Z})$, and we will focus on this. We shall also restrict our attention to (rational) homology 3-spheres, to get zero-dimensional moduli spaces.

Thus we aim to define an invariant as follows. Let (M, J, ω, Ω) be a almost Calabi–Yau 3-fold, let $\delta \in H_3(M, \mathbb{Z})$, and let $S(\delta)$ be the set of special Lagrangian homology 3-spheres N in M with $[N] = \delta$. Suppose $S(\delta)$ is finite, and define

$$I(\delta) = \sum_{N \in S(\delta)} w(N), \quad (23)$$

where w is a *weight function* taking values in a commutative ring R , and $w(N)$ depends only on the topology of N . In this way we define a map $I : H_3(M, \mathbb{Z}) \rightarrow R$, which we consider to be an analogue of the Gromov–Witten invariants. For this invariant to be interesting, we would like it to be stable under deformations of the underlying almost Calabi–Yau 3-fold (M, J, ω, Ω) , or at least to change in a predictable way as we make these deformations.

Therefore we need to know what can happen to special Lagrangian homology 3-spheres as we deform (M, J, ω, Ω) , and especially how they can become singular, appear or disappear. Each such transition may change the set of special Lagrangian homology 3-spheres, and thus the invariant $I(\delta)$. For $I(\delta)$ to be invariant or to transform nicely under these transitions, the weight function w must satisfy some topological identities.

We have already described models for two such transitions in §4 and §6. We will calculate the conditions on w for $I(\delta)$ to be invariant under the change described in §4, and to transform in a certain simple way under the change described in §6. It turns out that the weight function $w(N) = |H_1(N, \mathbb{Z})|$ satisfies both of these conditions.

We also propose corrections to (23) to include multiple covers of special Lagrangian homology 3-spheres and special Lagrangian 3-folds with stable singularities. We summarize our conclusions in Conjecture 7.3, and then discuss the connections between our invariants and String Theory.

7.1 Invariance of $I(\delta)$ under the transitions of §4

In §4.2–§4.3 we explained how, as we deform an almost Calabi–Yau 3-fold $(M, J_t, \omega_t, \Omega_t)$ in a 1-parameter family, three SL 3-folds N_t^1, N_t^2, N_t^3 can converge to the same singular SL 3-fold N_0 with a T^2 -cone singularity. This happens on a hyperplane $[\omega_t] \cdot \chi = 0$ in the Kähler cone, and N_t^a exists as a nonsingular SL 3-fold in (M, J, ω, Ω) if either $k^a > 0$ and $[\omega_t] \cdot \chi > 0$, or $k^a < 0$ and $[\omega_t] \cdot \chi < 0$. It is easy to see that the condition for the invariant $I(\delta)$ given by (23) to be unchanged by this transition is

$$\sum_{a \in \{1,2,3\}: k^a > 0} w(N_t^a) = \sum_{a \in \{1,2,3\}: k^a < 0} w(N_t^a). \quad (24)$$

Now if N is a homology 3-sphere then $H_1(N, \mathbb{Z})$ is a finite group, so $|H_1(N, \mathbb{Z})|$ is a positive integer. Take the commutative ring R to be \mathbb{Z} , and define $w(N) = |H_1(N, \mathbb{Z})|$. Remarkably, it turns out that this weight function, perhaps the simplest nontrivial invariant of N there is, satisfies (24).

Proposition 7.1 *Define an integer-valued invariant w of compact, non-singular 3-manifolds N by $w(N) = |H_1(N, \mathbb{Z})|$ if $H_1(N, \mathbb{Z})$ is finite, and $w(N) = 0$ if $H_1(N, \mathbb{Z})$ is infinite. Then (24) holds for all sets of 3-manifolds N_t^1, N_t^2, N_t^3 constructed as in §4.2.*

The proof follows quickly from the material of §4.

7.2 Transformation of $I(\delta)$ under the transitions of §6

We shall use the following notation.

Definition 7.2 Let M be a compact 6-manifold, and suppose $\chi^+, \chi^- \in H_3(M, \mathbb{Z})$ are linearly independent over \mathbb{R} . Define a subset $W(\chi^+, \chi^-)$ in $H^3(M, \mathbb{C})$ by

$$W(\chi^+, \chi^-) = \{ \Phi \in H^3(M, \mathbb{C}) : (\Phi \cdot \chi^+)(\bar{\Phi} \cdot \chi^-) \in (0, \infty) \}. \quad (25)$$

Then $W(\chi^+, \chi^-)$ is a *real hypersurface* in $H^3(M, \mathbb{C})$, but not a hyperplane. Let $\Phi \in H^3(M, \mathbb{C})$, and write $(\Phi \cdot \chi^+)(\bar{\Phi} \cdot \chi^-) = R e^{i\theta}$, where $R \geq 0$ and $\theta \in (-\pi, \pi]$. Then $\Phi \in W(\chi^+, \chi^-)$ if $R > 0$ and $\theta = 0$. Let $\epsilon > 0$ be small. We say that Φ lies on the *positive side* of $W(\chi^+, \chi^-)$ if $R > 0$ and $\theta \in (0, \epsilon)$, and on the *negative side* of $W(\chi^+, \chi^-)$ if $R > 0$ and $\theta \in (-\epsilon, 0)$.

Section 6 studied a family of almost Calabi–Yau 3-folds $(M, J_t, \omega_t, \Omega_t)$ containing SL homology 3-spheres N_t^\pm , with $[N_t^\pm] = \chi^\pm$ in $H_3(M, \mathbb{Z})$, such that $N_t^+ \cup N_t^-$ is a transverse intersection at a single point, and positive in homology. In the notation above, the phases $e^{i\theta_t^\pm}$ of N_t^\pm are equal exactly when $[\Omega_t] \in W(\chi^+, \chi^-)$, and Conjecture 6.1 says that as $[\Omega_t]$ passes through $W(\chi^+, \chi^-)$ from the negative to the positive side, a new SL 3-fold N_t diffeomorphic to $N_t^+ \# N_t^-$ is created, with $[N_t] = \chi^+ + \chi^-$.

Clearly, N_t contributes $w(N_t) \neq 0$ to the invariant $I(\chi^+ + \chi^-)$ defined in (23). Therefore, $I(\chi^+ + \chi^-)$ will take *different values* on the positive and negative sides of $W(\chi^+, \chi^-)$. So, $I(\delta)$ will *not* be unchanged under deformations of the almost Calabi–Yau structure (J, ω, Ω) on M which change $[\Omega]$ in $H^3(M, \mathbb{C})$. Instead, we hope to arrange for $I(\delta)$ to transform according to some rules involving $[\Omega]$ and the values $I(\alpha)$ for other α in $H_3(M, \mathbb{Z})$.

To see what the simplest of these rules should be, let us generalize the situation of §6, and suppose that $[\Omega_t]$ lies on the negative side of $W(\chi^+, \chi^-)$ for $t \in (-\delta, 0)$, on $W(\chi^+, \chi^-)$ for $t = 0$, and on the positive side of $W(\chi^+, \chi^-)$ for $t \in (0, \delta)$, but that N_0^+ and N_0^- intersect transversely at $k + l$ points x_1, \dots, x_k and y_1, \dots, y_l , where $N_0^+ \cap N_0^-$ is positive at x_j and negative at y_j , in the sense of homology.

Then arguing as in §6, we find that for small $t \in (0, \delta)$ we expect k distinct, immersed SL homology 3-spheres N_t diffeomorphic to $N_0^+ \# N_0^-$, the connected sums of N_0^+ and N_0^- at x_1, \dots, x_k , and for small $t \in (-\delta, 0)$ we expect l distinct, immersed SL homology 3-spheres N_t diffeomorphic to $N_0^+ \# N_0^-$, the connected sums of N_0^+ and N_0^- at y_1, \dots, y_l .

Hence, as $[\Omega_t]$ passes through $W(\chi^+, \chi^-)$ going from the negative to the positive side, we simultaneously create k and destroy l immersed special Lagrangian copies of $N_0^+ \# N_0^-$. Suppose for simplicity that N_0^\pm are the only SL homology 3-spheres in their homology classes χ^\pm . Then $I(\chi^+) = w(N_0^+)$ and $I(\chi^-) = w(N_0^-)$. Write $I(\chi^+ + \chi^-)^+$ for the value of $I(\chi^+ + \chi^-)$ at some point on the positive side of $W(\chi^+, \chi^-)$, and $I(\chi^+ + \chi^-)^-$ for its value at a nearby point on the negative side. Then we have

$$I(\chi^+ + \chi^-)^+ - I(\chi^+ + \chi^-)^- = (k - l) \cdot w(N_0^+ \# N_0^-). \quad (26)$$

If the weight function w satisfies the identity

$$w(N_0^+ \# N_0^-) = w(N_0^+) \cdot w(N_0^-) \quad \text{for all homology 3-spheres } N_0^\pm, \quad (27)$$

where multiplication is in the commutative ring R , then (26) can be written

$$I(\chi^+ + \chi^-)^+ - I(\chi^+ + \chi^-)^- = (\chi^+ \cap \chi^-) \cdot I(\chi^+) \cdot I(\chi^-), \quad (28)$$

as $\chi^+ \cap \chi^- = k - l$ and $w(N_0^+ \# N_0^-) = w(N_0^+) \cdot w(N_0^-) = I(\chi^+) \cdot I(\chi^-)$. Because of the bilinearity of the r.h.s. of (28) in $I(\chi^+)$ and $I(\chi^-)$, it is easy to see that (28) should also hold when there are finitely many SL homology 3-spheres in χ^\pm , and not just one in each.

Thus, (28) gives a formula for how we expect $I(\chi^+ + \chi^-)$ to change as we pass through the hypersurface $W(\chi^+, \chi^-)$. The important thing about this formula is that the values of I on the positive side of $W(\chi^+, \chi^-)$ determine the values of I on the negative side, and vice versa. So although I is not invariant under deformations of (M, J, ω, Ω) which alter the cohomology class $[\Omega]$, it transforms in a completely determined way, and that is more or less as useful.

In §7.1 we proposed the weight function $w(N) = |H_1(N, \mathbb{Z})|$, for N a homology 3-sphere. Now $H_1(N_0^+ \# N_0^-, \mathbb{Z}) \cong H_1(N_0^+, \mathbb{Z}) \times H_1(N_0^-, \mathbb{Z})$ as finite groups, and so

$$|H_1(N_0^+ \# N_0^-, \mathbb{Z})| = |H_1(N_0^+, \mathbb{Z})| \cdot |H_1(N_0^-, \mathbb{Z})|$$

when N_0^\pm are homology 3-spheres. Thus this weight function w satisfies (27), as we wish.

7.3 Including multiple covers

Let N' be an embedded SL homology 3-sphere in an almost Calabi–Yau 3-fold (M, J, ω, Ω) with $[N'] = \chi' \in H_3(M, \mathbb{Z})$, and $\iota : N \rightarrow N'$ be a *finite cover* of N' of degree $d > 1$, and covering group G , so that $|G| = d$. Suppose N is also a homology 3-sphere. Then we can regard N as an *immersed* SL homology 3-sphere in M , with $[N] = d[N'] = d\chi'$ in $H_3(M, \mathbb{Z})$. What contribution should N make to $I(d\chi')$?

We claim that N should contribute $w(N)/d$ to $I(d\chi')$, where w is the weight function for embedded homology 3-spheres. Note that if w takes values in \mathbb{Z} , then $w(N)/d$ takes values in \mathbb{Q} , so that our invariant $I(d\chi')$ will be actually be a rational number. Here is why. Let N'' be another SL homology 3-sphere in M intersecting N' transversely at one point, with $N' \cap N''$ positive. Let $[N''] = \chi''$ in $H_3(M, \mathbb{Z})$. Deform (M, J, ω, Ω) so that $[\Omega]$ passes through $W(\chi', \chi'')$ going from the negative to the positive side.

Since $[N] \cap [N''] = d$ in homology, we expect from §7.2 to create d distinct new SL homology 3-spheres as we pass through $W(\chi', \chi'')$, all diffeomorphic to $N \# N''$.

However, a little thought shows that we actually create only *one* SL copy of $N \# N''$. Effectively this is because the internal symmetry group G of N with $|G| = d$ identifies the d copies of $N \# N''$, so that they all give the same SL 3-fold. In order for (28) to give the correct transformation law for $I(d\chi' + \chi'')$ across $W(\chi', \chi'')$, we must give N the weight $w(N)/d$ rather than $w(N)$.

7.4 Including SL 3-folds with stable singularities

In §4.4 we described a class of SL 3-folds with *stable singularities*. That is, in an almost Calabi–Yau 3-fold (M, J, ω, Ω) one can have SL 3-folds N with one or more T^2 -cone singularities, that persist under small deformations of (J, ω, Ω) . Should such singular SL 3-folds also be counted in the invariant $I(\delta)$, with some appropriate topological weight?

I believe that the answer to this is yes. Furthermore, if the weight for nonsingular SL 3-folds N is $w(N) = |H_1(N, \mathbb{Z})|$, then the weight for singular SL 3-folds may in some circumstances be *negative*. I have not yet sorted all the details of this out, and hope to do so in a future paper. However, here are the ideas which lead me to this conclusion.

In §3–§4 and §5–§6 we described two kinds of ‘index one’ singularity of SL 3-folds, and analysed their topological effects. Now I know of at least two other kinds of ‘index one’ singularity, the first locally modelled on the SL 3-folds of [7, Ex. 7.4], and the second on the singularities described in [8, §6]. The topological effects are complicated to describe.

My calculations indicate that one thing that can happen with the first kind of index one singularity, is that one can simultaneously create, out of nothing, a nonsingular, immersed SL homology 3-sphere N_0 with one point of self-intersection x , and an embedded, singular SL 3-fold N_1 with one T^2 -cone singularity y , of the kind considered in §4.4, and with $k^a = 0$ for some $a = 1, 2, 3$.

In fact, we also create a family of nonsingular, embedded, diffeomorphic SL 3-folds N_t for $t \in (0, 1)$ with $b^1(N_t) = 1$, which converge to N_0 as $t \rightarrow 0_+$ with local model $K_{\phi, A}$ near x , where $A \rightarrow 0_+$ as $t \rightarrow 0_+$, and converge to N_1 as $t \rightarrow 1_-$ with local model L_s^a near y , where $s \rightarrow 0_+$ as $t \rightarrow 1_-$. Then N_t for $t \in [0, 1]$ all have the same homology class $[N_t] = \chi \in H_3(M, \mathbb{Z})$.

Now when it exists, N_0 contributes $w(N_0)$ to $I(\chi)$. In order for the invariant I to be unchanged under this kind of transition we need N_1 to contribute $-w(N_0)$ to $I(\chi)$. It can be shown that $H_1(N_0, \mathbb{Z}) \cong H_1(N_1, \mathbb{Z})$. Thus, if the weight for N_0 is $w(N_0) = |H_1(N_0, \mathbb{Z})|$, as above, then the correct weight for N_1 is $w(N_1) = -|H_1(N_1, \mathbb{Z})|$.

So I conjecture that for SL 3-folds N with one stable T^2 -cone singularity of the kind considered in §4.4 and $k^a = 0$ for some $a = 1, 2, 3$, the appropriate weight is $w(N) = -|H_1(N, \mathbb{Z})|$. I am not yet sure of the answer when the k^a are all nonzero, or there is more than one T^2 -cone singularity.

7.5 A preliminary conjecture

I am now ready to formulate a first guess as to how to define an invariant counting special Lagrangian homology 3-spheres, and what its properties should be under deformations of the underlying almost Calabi–Yau 3-fold.

Conjecture 7.3 *Let (M, J, ω, Ω) be a generic almost Calabi–Yau 3-fold. Then there exists $I : H_3(M, \mathbb{Z}) \rightarrow \mathbb{Q}$ with the following properties:*

- (a) *For each $\delta \in H_3(M, \mathbb{Z})$, let $S(\delta)$ be the set of compact, immersed, possibly singular SL 3-folds N in M , with $[N] = \delta$ and $b^1(N) = 0$. Then $S(\delta)$ is finite, and*

$$I(\delta) = \sum_{N \in S(\delta)} w(N), \quad (29)$$

where $w(N) \in \mathbb{Q}$ is a weight function depending only on the topology of N and its immersion in M .

- (b) *Let N be a nonsingular immersed SL 3-fold in M . Regard N as a nonsingular, compact 3-manifold with immersion $\iota : N \rightarrow M$. Define $G(N)$ to be the group of diffeomorphisms $\phi : N \rightarrow N$ with $\iota \circ \phi = \iota$. Then $G(N)$ is finite, and the weight $w(N)$ is*

$$w(N) = \frac{|H_1(N, \mathbb{Z})|}{|G(N)|}. \quad (30)$$

- (c) *I is unchanged by continuous deformations of (M, J, ω, Ω) that change $[\omega]$ but leave $[\Omega]$ fixed, or multiply $[\Omega]$ by a nonzero complex number. Thus I depends only on the complex structure J on M , and not on the metric g .*

- (d) When we deform (M, J, ω, Ω) so that $[\Omega]$ passes through one of the hypersurfaces $W(\chi^+, \chi^-)$ in $H^3(M, \mathbb{C})$ given in Definition 7.2, the invariant I transforms according to a set of rules that we are not yet able to write down.

One of these rules should be that if χ^+ and χ^- are primitive elements of $H_3(M, \mathbb{Z})$ and $I(\chi^+ + \chi^-)^\pm$ are the values of $I(\chi^+ + \chi^-)$ at two nearby points on the positive and negative sides of $W(\chi^+, \chi^-)$, then

$$I(\chi^+ + \chi^-)^+ - I(\chi^+ + \chi^-)^- = (\chi^+ \cap \chi^-) \cdot I(\chi^+) \cdot I(\chi^-). \quad (31)$$

Here are some remarks on this conjecture. Firstly, in defining Gromov–Witten invariants, J -holomorphic curves are counted *with signs*. But I believe that there is no corresponding way to define the sign of an SL 3-fold, and that SL 3-folds should be counted without signs.

Secondly, one effect of assuming that (M, J, ω, Ω) is *generic* is that the only SL 3-folds that can exist in M are those with *stable singularities*. So there is no need to restrict to SL 3-folds with stable singularities in defining $S(\delta)$.

Thirdly, to understand how $I(\delta)$ transforms under deformations, it is enough to consider smooth 1-parameter families $\{(M, J_t, \omega_t, \Omega_t) : t \in [0, 1]\}$ of almost Calabi–Yau 3-folds, where $(M, J_0, \omega_0, \Omega_0)$ and $(M, J_1, \omega_1, \Omega_1)$ are generic (so that the invariant I is defined when $t = 0$ or 1) and the family itself is generic as a 1-parameter family.

Now by the ideas of §2.3, the only singularities of SL 3-folds N with $b^1(N) = 0$ that we will encounter in such families of almost Calabi–Yau 3-folds are singularities with *index one*. Thus, provided the general picture of §2.3 is correct, to prove Conjecture 7.3 we only need to know about special Lagrangian singularities with index one, and higher codimension singularities can be ignored.

Fourthly, in part (d) we have not given a complete set of rules for the transformation of I on the hypersurfaces $W(\chi^+, \chi^-)$. Here is why. When we pass through the hypersurface $W(\chi^+, \chi^-)$ we expect to create or destroy new SL 3-folds with homology class $\chi^+ + \chi^-$, which are connected sums of 3-folds with homology classes χ^+ and χ^- . But this is only the simplest kind of transition which happens on $W(\chi^+, \chi^-)$.

For instance, if $N_1^+, N_2^+ \in S(\chi^+)$ and $N^- \in S(\chi^-)$, then on $W(\chi^+, \chi^-)$ we may create a new SL homology 3-sphere with homology class $2\chi^+ + \chi^-$, diffeomorphic to the triple connected sum $N_1^+ \# N^- \# N_2^+$. So there should be some change to $I(2\chi^+ + \chi^-)$ on $W(\chi^+, \chi^-)$. Similarly, if a, b are positive integers then we can try and take a multiple connected sum of a elements of $S(\chi^+)$ and b elements of $S(\chi^-)$ to get a new SL homology 3-sphere with homology class $a\chi^+ + b\chi^-$.

However, there is a problem: can we include an SL 3-fold in such a connected sum more than once, and if so, how? The answer to this appears to be rather complex, and I do not yet understand it, which is why I'm not ready to write down a full set of transformation rules for I .

7.6 Relationships with String Theory

I now want to argue that the invariant I postulated above should have an interpretation in String Theory, and may fit into a piece of the Mirror Symmetry story for Calabi–Yau 3-folds which is not understood at present. There are two main reasons for this:

- (a) The invariant I counts objects which are significant in String Theory, namely *isolated 3-branes*; and
- (b) Mirror Symmetry strongly suggests that for a Calabi–Yau 3-fold X there should be discrete, \mathbb{Q} -valued invariants defined on $H_3(X, \mathbb{Z})$ or something like it, which are related under the mirror transform to the Gromov–Witten invariants of the mirror manifold Y . But no such invariants are known at present.

We first discuss reason (a). In String Theory, SL 3-folds correspond roughly to physical objects called *3-branes*. But a 3-brane is not just a 3-dimensional submanifold N ; it also carries with it a complex line bundle over N with a flat $U(1)$ -connection. (For a discussion of this, see for instance Strominger, Yau and Zaslow [16].) We will call a 3-brane *isolated* if it admits no deformations, which happens when N is a rational homology 3-sphere.

If N is a compact 3-manifold, then flat $U(1)$ -connections on N are equivalent to group homomorphisms $H_1(N, \mathbb{Z}) \rightarrow U(1)$. But when $H_1(N, \mathbb{Z})$ is finite, it is easy to show using the theory of finite abelian groups that the number of group homomorphisms $H_1(N, \mathbb{Z}) \rightarrow U(1)$ is exactly $|H_1(N, \mathbb{Z})|$. Hence, if N is a special Lagrangian homology 3-sphere, then there are exactly

$|H_1(N, \mathbb{Z})|$ flat $U(1)$ -connections over N , and so N gives rise to $|H_1(N, \mathbb{Z})|$ isolated 3-branes.

Therefore, for the case of nonsingular, embedded SL 3-folds, *the invariant $I(\delta)$ discussed above counts the number of isolated 3-branes in the homology class δ* . So it is a natural thing to count from the String Theory point of view, although the author formulated Conjecture 7.3 without knowing this.

Next we discuss reason (b). The following basic outline of Mirror Symmetry in String Theory is by now well known, and is described for instance in Greene and Plesser [3]. To a pair (X, S) , where X is a Calabi–Yau 3-fold and S some ‘extra structure’ which we will not worry about, a physicist associates a Super Conformal Field Theory (SCFT). There are believed to exist ‘mirror pairs’ (X, S) and (Y, T) whose SCFT’s are isomorphic under a simple involution of the SCFT structure.

If (X, S) and (Y, T) are such a mirror pair, then $H^{2,1}(X) \cong H^{1,1}(Y)$, and certain cubic forms $I_X^{2,1}$ on $H^{2,1}(X)$ and $I_Y^{1,1}$ on $H^{1,1}(Y)$ agree. Here $I_X^{2,1}$ has a simple definition, depending on the variation of complex structures on X . However, $I_Y^{1,1}$ has a complicated definition, involving an infinite sum over homology classes α in $H_2(Y, \mathbb{Z})$ of the ‘number of rational curves’ in Y with homology class α , multiplied by complex functions of a standard form.

Now, here is the important point. If we accept the mirror conjecture, then $I_X^{2,1}$ can be written as essentially the sum of a power series with integer (or rational) coefficients. *Is there a way of understanding, purely in terms of X and without introducing the mirror Y , why these numbers should be integers, and what they mean?*

The author hopes that this question can be answered as follows. There should be a way to write $I_X^{2,1}$ as a sum over finite collections of elements of $H_3(X, \mathbb{Z})$ of integer or rational invariants similar to the invariant I described in §7.5, multiplied by complex (holomorphic, in a suitable sense) functions of $[\Omega]$ of a standard form. As $[\Omega]$ passes through a hypersurface $W(\chi^+, \chi^-)$, these complex functions may change discontinuously to compensate for changes in I , so that $I_X^{2,1}$ remains continuous. The reason the numbers in the series are integers (or rationals) is then that they count collections of SL 3-folds satisfying certain conditions.

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