The Poisson Process in Quantum Stochastic Calculus

Shayanthan Pathmanathan
Exeter College
University of Oxford

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Given a compensated Poisson process $(X_t)_{t\geq 0}$ based on $(\Omega, \mathcal{F}, \mathbb{P})$, the Wiener-Poisson isomorphism $\mathcal{W} : \mathfrak{F}_+(L^2(\mathbb{R}_+)) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ is constructed. We restrict the isomorphism to $\mathfrak{F}_+(L^2[0,1])$ and prove some novel properties of the Poisson exponentials $\mathcal{E}(f) := \mathcal{W}(e(f))$. A new proof of the result $\Lambda_t + A_t + A_t^\dagger = \mathcal{W}^{-1}\widehat{X_t}\mathcal{W}$ is also given. The analogous results for $\mathfrak{F}_+(L^2(\mathbb{R}_+))$ are briefly mentioned.

The concept of a compensated Poisson process over $\mathbb{R}_+$ is generalised to any measure space $(M, \mathcal{M}, \mu)$ as an isometry $I : L^2(M, \mathcal{M}, \mu) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ satisfying certain properties. For such a generalised Poisson process we recall the construction of the generalised Wiener-Poisson isomorphism, $\mathcal{W}_I : \mathfrak{F}_+(L^2(M)) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$, using Charlier polynomials. Two alternative constructions of $\mathcal{W}_I$ are also provided, the first using exponential vectors and then deducing the connection with Charlier polynomials, and the second using the theory of reproducing kernel Hilbert spaces.

Given any measure space $(M, \mathcal{M}, \mu)$, we construct a canonical generalised Poisson process $I : L^2(M, \mathcal{M}, \mu) \to L^2(\Delta, \mathcal{B}, \mathbb{P})$, where Δ is the maximal ideal space, with $\mathcal{B}$ the completion of its Borel σ-field with respect to $\mathbb{P}$, of a C^*-algebra $\mathcal{A} \subseteq \mathfrak{B}(\mathfrak{F}_+(L^2(M)))$. The Gelfand transform $\mathcal{A} \to \mathfrak{B}(L^2(\Delta))$ is unitarily implemented by the Wiener-Poisson isomorphism $\mathcal{W}_I : \mathfrak{F}_+(L^2(M)) \to L^2(\Delta)$. This construction only uses operator algebra theory and makes no *a priori* use of Poisson measures.

A new Fock space proof of the quantum Ito formula for $(\Lambda_t + A_t + A_t^\dagger)_{0\leq t\leq 1}$ is given. If $(F_t)_{0\leq t\leq 1}$ is a real, bounded, predictable process with respect to a compensated Poisson process $(X_t)_{0\leq t\leq 1}$, we show that if $M_t = \int_0^t F_s dX_s$, then on $\mathsf{E}_{\text{lb}} := \text{linsp}\{e(f) : f \in L^2_{\text{lb}}[0,1]\}$,

$$\mathcal{W}^{-1}\widehat{M_t}\mathcal{W} = \int_0^t \mathcal{W}^{-1}\widehat{F_s}\mathcal{W}(d\Lambda_s + dA_s + dA_s^\dagger),$$

and that $(\mathcal{W}^{-1}\widehat{M_t}\mathcal{W})_{0\leq t\leq 1}$ is an essentially self-adjoint quantum semimartingale. We prove, using the classical Ito formula, that if $(J_t)_{0\leq t\leq 1}$ is a regular self-adjoint quantum semimartingale, then $(\mathcal{W}\widehat{M_t}\mathcal{W}^{-1} + J_t)_{0\leq t\leq 1}$ is an essentially self-adjoint quantum semimartingale satisfying the quantum Duhamel formula, and hence the quantum Ito formula. The equivalent result for the sum of a Brownian and Poisson martingale, provided that the sum is essentially self-adjoint with core E_{lb}, is also proved.

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Contents

Chapter 1

Introduction

Brownian motion $(W_t)_{t\geq 0}$ on a complete probability space $(\Omega_{\mathfrak{w}}, \mathcal{F}_{\mathfrak{w}}, \mathbb{P}_{\mathfrak{w}})$, first constructed by Wiener in [71], satisfies the properties

i) for $(t_1, \ldots, t_n) \in \mathbb{R}_+^n$, the joint law of $(W_{t_1}, \ldots, W_{t_n})$ is Gaussian with mean zero and covariance $\mathbb{E}[W_{t_k} W_{t_l}] = \min(t_k, t_l)$,

ii) $t \mapsto W_t(\omega)$ is continuous a.s..

It is well-known that with its natural filtration $(W_t)_{t\geq 0}$ is a square integrable martingale with $\langle W, W \rangle_t = t$. Using this property we can define multiple stochastic integrals

$$J^{(n)} : L^2(D_n) \longrightarrow L^2(\Omega_{\mathfrak{w}}, \mathcal{F}_{\mathfrak{w}}, \mathbb{P}_{\mathfrak{w}}),$$

which are isometries such that for $m \neq n$, $J^{(m)}(L^2(D_m))$ is orthogonal to $J^{(n)}(L^2(D_n))$ (see (3.1.1) for the definition of D_n). Multiple stochastic integrals were first defined in [72], however the more commonly used definition is due to the work of Ito, first for real integrals in [30] and later for complex integrals in [31]. We can use the multiple stochastic integrals to construct the isometry

$$\mathfrak{w} : \mathfrak{F}_+(L^2(\mathbb{R}_+)) \longrightarrow L^2(\Omega_{\mathfrak{w}}, \mathcal{F}_{\mathfrak{w}}, \mathbb{P}_{\mathfrak{w}}),$$

where $\mathfrak{F}_+(L^2(\mathbb{R}_+))$ is the symmetric Fock space over $L^2(\mathbb{R}_+)$. This decomposition agrees with the one obtained in [13]. If the σ-field $\mathcal{F}_{\mathfrak{w}}$ is generated by $(W_t)_{t\geq 0}$, then $\mathfrak{w}$ is surjective and is called the *Wiener interpretation of Fock space.* The connection with symmetric Fock space was first presented in [61].

Quantum stochastic calculus, originally constructed in [28], is the non-commutative extension of Ito calculus. If $(F_t)_{t\geq 0}$ is a predictable process with respect to the natural filtration of Brownian motion and is bounded on finite intervals, then $(0, (\mathfrak{w}^{-1}\widehat{F_t}\mathfrak{w})_{t\geq 0}, (\mathfrak{w}^{-1}\widehat{F_t}\mathfrak{w})_{t\geq 0}, 0)$, where $\widehat{F_t}$ is the operator of multiplication by F_t, is a suitable quantum stochastic integrand and for all $f \in L^2_{\text{lb}}(\mathbb{R}_+)$,

$$\mathfrak{w}^{-1}\widehat{M_t}\mathfrak{w}e(f) = \left(\int_0^t \mathfrak{w}^{-1}\widehat{F_s}\mathfrak{w}dA_s + \mathfrak{w}^{-1}\widehat{F_s}\mathfrak{w}dA_s^\dagger\right) e(f), \tag{1.0.1}$$

where $M_t = \int_0^t F_s dW_s$. In particular, on the exponential vectors of locally bounded square integrable functions, multiplication by Brownian motion at time t corresponds to the operator $A_t + A_t^\dagger$ (see (2.1.1) for the definition of A_t and $A_t^\dagger$). Consequently many results in quantum stochastic calculus are motivated by the isomorphism $\mathfrak{w}$, in particular the establishing of a quantum Ito formula generalising the classical Ito formula

$$f(W_t) = f(0) + \int_0^t f'(W_s)dW_s + \frac{1}{2}\int_0^t f''(W_s)ds.$$

1.1 Probabilistic interpretations of Fock space

The Wiener interpretation of Fock space over $L^2(\mathbb{R}_+)$ can be generalised. Suppose $(X_t)_{t\geq 0}$ is a real square integrable martingale on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to its natural filtration $(\mathcal{F}_t)_{t\geq 0}$ and $\mathcal{F}$ is generated by $(X_t)_{t\geq 0}$. Then $(X_t)_{t\geq 0}$ is a *normal martingale* if $(X_t^2 - t)_{t\geq 0}$ is a martingale for the filtration $(\mathcal{F}_t)_{t\geq 0}$, that is $\langle X, X\rangle_t = t$. For such a martingale, we say a measurable process $(F_t)_{t\geq 0}$ is *square integrable* if

$$\int_0^t \mathbb{E}[|F_s|^2]ds < \infty \quad \text{for all } t \geq 0.$$

A normal martingale, $(X_t)_{t\geq 0}$, is said to satisfy the *predictable representation property* if for all $Y \in L^2(\Omega, \mathcal{F}, \mathbb{P})$,

$$Y = \mathbb{E}[Y] + \int_0^\infty F_s dX_s$$

for some predictable process $(F_t)_{t\geq 0}$ such that

$$\int_0^\infty \mathbb{E}[|F_s|^2]ds < \infty.$$

Since the multiple stochastic integrals defined for Brownian motion, $(W_t)_{t\geq 0}$, only depend on the property $\langle W, W\rangle_t = t$, for any normal martingale we may define multiple stochastic integrals

$$J^{(n)} : L^2(D_n) \longrightarrow L^2(\Omega, \mathcal{F}, \mathbb{P}),$$

which are isometries such that $J^{(m)}(L^2(D_m))$ is orthogonal to $J^{(n)}(L^2(D_n))$ for $m \neq n$. Hence we have an isometry

$$\mathfrak{w}_X : \mathfrak{F}_+(L^2(\mathbb{R}_+)) \longrightarrow L^2(\Omega, \mathcal{F}, \mathbb{P}). \tag{1.1.1}$$

These isometries give the various probabilistic interpretations of Fock space. The *chaotic space* of $(X_t)_{t\geq 0}$ is defined to be $CS(X) := \mathfrak{w}_X(\mathfrak{F}_+(L^2(\mathbb{R}_+)))$. We say $(X_t)_{t\geq 0}$ has the *chaotic representation property* if $CS(X) = L^2(\Omega, \mathcal{F}, \mathbb{P})$. It can easily be deduced that if $(X_t)_{t\geq 0}$ has the chaotic representation property then it has the predictable representation property. In particular, Brownian motion has the chaotic and predictable representation property. The normal martingales are in some sense generalisations of Brownian motion producing different interpretations of Fock space. Consequently, quantum stochastic calculus should be considered as both an extension and unification of classical stochastic calculus.

If $(X_t)_{t\geq 0}$ is a normal martingale and $X_t \in L^4(\Omega)$, then $([X,X]_t - \langle X,X\rangle_t)_{t\geq 0}$ is a square integrable martingale. Hence if $(X_t)_{t\geq 0}$ has the predictable representation property then

$$[X,X]_t - \langle X,X\rangle_t = \int_0^t \Psi_s dX_s, \tag{1.1.2}$$

for some square integrable predictable process $(\Psi_t)_{t\geq 0}$. The equation (1.1.2) is called a *structure equation* for $(X_t)_{t\geq 0}$. From [21], we have the following uniqueness of solution (in law) results:

- $\Psi_s = 0$, $(X_t)_{t\geq 0}$ is Brownian motion;
- $\Psi_s = c$, $(X_t)_{t\geq 0}$ is the compensated Poisson process with jump size c and intensity $\frac{1}{c^2}$;
- $\Psi_s = \beta X_{s-}$, $(X_t)_{t\geq 0}$ is the Azéma martingale with parameter β.

Our work will primarily be concerned with the case when $\Psi_s = 1$, the Poisson interpretation of Fock space, in which case $(X_t)_{t\geq 0}$ is the compensated Poisson process with intensity 1. It turns out that the compensated Poisson process also has the chaotic representation property. Poisson multiple stochastic integrals are in many ways not as well-known as the Wiener multiple integrals, however they have been around for just as long. The Poisson chaotic expansion was first discovered by Wiener under the name of 'discrete chaos' in [72, §11] and developed more fully in [73], although as in the Wiener case, the modern definition of multiple Poisson integrals is influenced by the work of Ito (see [32]). Extension of multiple Poisson integrals to more general spaces can be found in [64] and [39]. The construction which we give in our work is based on these two articles.

1.2 The Wiener-Ito isomorphism

The Wiener interpretation of Fock space over $L^2(\mathbb{R}_+)$ is an example of the Wiener-Ito isomorphism. Before describing the construction of the Wiener-Ito isomorphism we discuss the

Hermite polynomials, which are closely related to the isomorphism. The *Hermite polynomials* $(h_n)_{n=0}^{\infty}$ are defined by the generating function

$$e^{tz-t^2/2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} h_n(z),$$

for all t and z in $\mathbb{C}$. It can be shown that

$$\int_{-\infty}^{\infty} h_m(x)h_n(x)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx = n!\,\delta_{m,n},$$

and thus $(h_n)_{n=0}^{\infty}$ is an orthogonal sequence in $L^2(\mathbb{R}, \mathcal{B}, \gamma)$, where if $E \in \mathcal{B}$,

$$\gamma(E) = \frac{1}{\sqrt{2\pi}} \int_E e^{-\frac{x^2}{2}}\,dx.$$

We can also show that the Hermite polynomials are total in $L^2(\mathbb{R}, \mathcal{B}, \gamma)$. The Charlier polynomials (Definition 4.2.1) are the analogous polynomials for Poisson variables.

By a *Gaussian Hilbert space* we mean a closed subspace of $L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P})$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, consisting of centred Gaussian random variables. Given a Gaussian Hilbert space $\mathcal{H}$, following the notations of [34, Chapter 2], we define

$$\mathcal{P}_n := \{p(\xi_1, \dots, \xi_m) : p \text{ is a real polynomial of degree} \le n,\ \xi_j \in \mathcal{H}\}$$

and let

$$\mathcal{H}^{:n:} := \overline{\mathcal{P}_n} \cap \overline{\mathcal{P}_{n-1}}^{\perp} \subseteq L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P}),$$

taking $\mathcal{P}_{-1} := \{0\}$. $\mathcal{H}^{:n:}$ is called the *nth Wiener chaos.* From [34, Theorem 2.6] we know that

$$\bigoplus_{n=0}^{\infty} \mathcal{H}^{:n:} = L^2_{\mathbb{R}}(\Omega, \sigma(\mathcal{H}), \mathbb{P}),$$

where $\sigma(\mathcal{H})$ is the σ-field generated by the random variables in $\mathcal{H}$. If $\xi_1, \dots, \xi_n \in \mathcal{H}$, the *Wick product* is defined to be

$$:\xi_1 \dots \xi_n: := \pi_n(\xi_1 \dots \xi_n),$$

where π_n is the orthogonal projection onto $\mathcal{H}^{:n:}$. This is well-defined since all moments of a Gaussian random variable exist. It follows from [34, Theorem 3.9] that if $\xi_1, \dots, \xi_n$ and $\eta_1, \dots, \eta_n$ are in $\mathcal{H}$,

$$\langle :\xi_1 \dots \xi_n:, :\eta_1 \dots \eta_n: \rangle = \sum_{\sigma \in S_n} \prod_{j=1}^{n} \langle \xi_j, \eta_{\sigma(j)} \rangle,$$

and from [34, Corollary 3.27] that the set of Wick products $:\xi_1 \ldots \xi_n:$ with $\xi_1, \ldots, \xi_n \in \mathcal{H}$ form a total subset of $\mathcal{H}^{:n:}$. Therefore $\mathcal{H}^{:n:}$ is a concrete realisation of the nth symmetric tensor product $\mathcal{H}^{\otimes_s^n}$ and thus we have an identification of $\mathfrak{F}_+(\mathcal{H})$, the real symmetric Fock space over $\mathcal{H}$, with $L^2_{\mathbb{R}}(\Omega, \sigma(\mathcal{H}), \mathbb{P})$.

The Hilbert spaces we use will have inner products which are linear on the right and conjugate linear on the left. Let $H_{\mathbb{R}}$ be a real Hilbert space and H its complexification. Then we define a *Gaussian field* over H to be an isometry $I : H \to L^2(\Omega, \mathcal{F}, \mathbb{P})$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, such that for each $h \in H_{\mathbb{R}}$, $I(h)$ is a centred Gaussian variable. As before we shall assume that the probability space is complete and $\mathcal{F} = \sigma(I(H) \cup \mathfrak{N})$ where $\mathfrak{N}$ is the collection of null sets in $\mathcal{F}$. If $(W_t)_{t\geq 0}$ is Brownian motion, then $\mathfrak{w}|_{L^2(\mathbb{R}_+)}$ is a Gaussian field over $L^2(\mathbb{R}_+)$. Note that given a Gaussian field $I : H \to L^2(\Omega, \mathcal{F}, \mathbb{P})$, $I(H_{\mathbb{R}})$ is a Gaussian Hilbert space and $H_{\mathbb{R}} \cong I(H_{\mathbb{R}})$. From [26, Remark 2.0] we know that

$$\mathfrak{F}_+(H) = \mathfrak{F}_+(H_{\mathbb{R}}) \oplus i\mathfrak{F}_+(H_{\mathbb{R}}).$$

Therefore from the comments above we have the following theorem.

Theorem 1.2.1 *Suppose $I : H \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a Gaussian field. Then there exists an isometric isomorphism, called the Wiener-Ito isomorphism,*

$$\mathfrak{w}_I : \mathfrak{F}_+(H) \longrightarrow L^2(\Omega, \mathcal{F}, \mathbb{P}),$$

such that

$$\mathfrak{w}_I : h_1 \otimes_s \ldots \otimes_s h_n \longmapsto \frac{1}{\sqrt{n!}} : I(h_1) \ldots I(h_n) :,$$

for $h_1, \ldots, h_n \in H_{\mathbb{R}}$.

This result can be found in [61, Theorem 3] and [18, Proposition 5.2]. From [34, Theorem 3.21] it follows that if $\{h_1, \ldots, h_k\}$ is an orthonormal set in $H_{\mathbb{R}}$ and $\sum_{j=1}^k p_j = n$,

$$\mathfrak{w}_I(h_1^{\otimes_s^{p_1}} \otimes_s \ldots \otimes_s h_k^{\otimes_s^{p_k}}) = \frac{1}{\sqrt{n!}} h_{p_1}(I(h_1)) \ldots h_{p_k}(I(h_k)). \tag{1.2.1}$$

Thus this definition can be used to provide an alternative construction of $\mathfrak{w}_I$. From (1.2.1) we can deduce that the Wiener interpretation of Fock space is the Wiener-Ito isomorphism induced by $\mathfrak{w}|_{L^2(\mathbb{R}_+)}$.

The exponential vectors in $\mathfrak{F}_+(L^2(\mathbb{R}_+))$ are crucial in the construction of quantum stochastic integrals. Consequently it is important to investigate the behaviour of exponential vectors under $\mathfrak{w}_I$. If $I : H \to L^2(\Omega)$ is a Gaussian field, then for $h \in H$ we define the *Wick exponential* to be

$$\mathcal{E}_{\mathfrak{w}_I}(h) := \mathfrak{w}_I(e(h)) = e^{I(h) - \frac{\mathbb{E}[I(h)^2]}{2}}, \tag{1.2.2}$$

where $e(h)$ is the Fock space exponential vector of h. It can easily be deduced that the Wick exponentials satisfy

$$\mathcal{E}_{\mathfrak{w}_I}(h_1)\mathcal{E}_{\mathfrak{w}_I}(h_2) = e^{\langle \overline{h_1}, h_2 \rangle}\mathcal{E}_{\mathfrak{w}_I}(h_1 + h_2).$$

It is possible to construct the isomorphism $\mathfrak{w}_I$ using just the exponential vectors (see [69, §4]). In Section 4.3 we shall do a similar construction for the Poisson situation.

1.3 Motivation

The aim of this thesis is to establish results for Poisson processes which are known to hold for Gaussian processes. In this section we present the results for Brownian motion and Gaussian random variables which motivated the results in this thesis.

In the various probabilistic interpretations of Fock space a classical Ito formula exists. It is desirable to obtain an Ito formula in quantum stochastic calculus, which generalises and unifies the various probabilistic interpretations. The quantum Ito formula (Definition 2.3.1) for regular self-adjoint quantum semimartingales was established in [66]. It has remained an open problem whether it is possible to establish an Ito formula for unbounded essentially self-adjoint quantum semimartingales. This problem is partially tackled in [67] by considering a matrix formulation of quantum stochastic calculus and proving the Ito formula for various matrix quantum stochastic integrals (see also [9]). However, this method requires the extension of quantum stochastic calculus to domains larger than the original exponential domains of the Hudson-Parthasarathy formulation.

Another possible way to obtain unbounded quantum semimartingales which satisfy the quantum Ito formula is to consider perturbations of classical martingales, which satisfy the classical Ito formula, by regular self-adjoint quantum semimartingales. This approach is motivated by results obtained for perturbations of self-adjoint operators on Hilbert spaces and on the theory of perturbations of unitary groups. The method of proof used to show that an essentially self-adjoint quantum semimartingale, M, satisfies the quantum Ito formula is to deduce it from the quantum Duhamel formula for the unitary group $(e^{ip\overline{M_t}})_{p\in\mathbb{R}}$. Consequently, it would be expected that perturbations of classical martingales will also satisfy the quantum Duhamel formula, from which the quantum Ito formula can be deduced.

In [68], Vincent-Smith shows that if J is a regular self-adjoint quantum semimartingale and if 'Brownian motion', $(A_t + A_t^\dagger)_{0\le t\le 1}$, is perturbed by J, then the perturbed quantum semimartingale, $(J_t + A_t + A_t^\dagger)_{0\le t\le 1}$, is essentially self-adjoint and satisfies the quantum Ito formula [68, Theorem 3.4]. The proof of this result approximates $A_t + A_t^\dagger$ by $f_n(A_t + A_t^\dagger)$ where f_n is bounded and uses the classical Ito formula to approximate $e^{ip\overline{M_t}}$ by $e^{ipM_t^{(n)}}$ where $M^{(n)} = (J_t + f_n(A_t + A_t^\dagger))_{0\le t\le 1}$ is a regular quantum semimartingale. It is shown in [67, §13] that the result also holds for perturbations of Brownian martingales of the form $(\int_0^t F_s dW_s)_{0\le t\le 1}$, where $(F_t)_{0\le t\le 1}$ is a bounded predictable process. Notice that from (1.0.1) that this classical martingale can be represented as a quantum stochastic integral. Classical Poisson martingales, via the Poisson interpretation of Fock space, can also be represented as quantum stochastic integrals (see Theorem 6.2.4). We prove that perturbations of classical Poisson martingales also satisfy the quantum Ito formula.

Given that there exists an isomorphism from symmetric Fock space over $L^2(\mathbb{R}_+)$ generated by a Poisson process which is analogous to the isomorphism $\mathfrak{w}$ induced by Brownian motion, it is natural to ask if there is a map analogous to the Wiener-Ito isomorphism. Such an isomorphism, which we call the Wiener-Poisson isomorphism, does exist. However we need to restrict ourselves from general complexified Hilbert spaces H to $L^2(M, \mathcal{M}, \mu)$, where $(M, \mathcal{M}, \mu)$ is a measure space, and consider generalised Poisson processes, $I : L^2(M, \mathcal{M}, \mu) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$, which are proper generalisations of a Poisson process over $\mathbb{R}_+$. We shall explain the reasons for this in Section 4.1. The construction of these isomorphisms was first done in [64] for non-atomic measure spaces and later in [39] for Polish spaces which may have atoms. We give a construction of this isomorphism for general measure spaces using Charlier polynomials. Our approach is slightly different to the two articles mentioned, in that we use well-known results for the isomorphism on $\mathfrak{F}_+(L^2(\mathbb{R}_+))$ to obtain results for the isomorphism on $\mathfrak{F}_+(L^2(M))$.

In [69], given a complexified Hilbert space, H, Vincent-Smith gives a canonical construction of a Gaussian field. He constructs a unital commutative C^*-algebra, $\mathcal{A} \subseteq \mathfrak{B}(\mathfrak{F}_+(H))$, with cyclic vector $e(0)$ and then considers the Segal spatial isomorphism [62, Scholium 9.1],

$$\mathscr{S} : \mathfrak{F}_+(H) \longrightarrow L^2(\Delta, \mathcal{B}, \mathbb{P}),$$

which implements the Gelfand transform. Here, the measure $\mathbb{P}$, which occurs in the construction of the Segal spatial isomorphism, is a probability measure. It turns out that $\mathscr{S}|_H$ is a Gaussian field and the Wiener-Ito isomorphism it induces is $\mathscr{S}$. Thus, Vincent-Smith is able to produce a Gaussian field using C^*-algebra theory, rather than probability theory.

In this thesis, we shall do a similar argument for the Poisson distribution. Given any measure space $(M, \mathcal{M}, \mu)$, we construct a unital commutative C^*-algebra $\mathcal{A} \subseteq \mathfrak{B}(\mathfrak{F}_+(L^2(M)))$ with cyclic vector $e(0)$ and look at the associated Segal spatial isomorphism $\mathscr{S}$. Analogously to the Gaussian case, $\mathscr{S}|_{L^2(M)}$ is a generalised Poisson process and $\mathscr{S}$ its Wiener-Poisson isomorphism. It should be noted that in both cases the probability measure $\mathbb{P}$ occurs naturally in the theory of C^*-algebras and no *a priori* use of Gaussian or Poisson measures is required.

1.4 Overview of thesis

There are two main threads to this thesis. The first is proving the quantum Ito formula for perturbed classical Poisson martingales, and the second is the use of the Segal spatial isomorphism to construct Poisson processes. Consequently, this thesis can divided into Chapters 2, 3, 6, which prove the perturbation result and Chapters 3, 4, 5, which describe our construction of Poisson processes. The material in Chapter 3 is crucial to both ideas and provides many of the tools we need. It contains a large collection of well-known results on the Poisson interpretation of Fock space.

Chapter 2 gives details of results in quantum stochastic calculus. We begin the chapter with a brief account of the Hudson-Parthasarathy construction of quantum stochastic integrals on exponential domains of $\mathfrak{F}_+(L^2[0,1])$. A definition of the quantum Duhamel formula for essentially self-adjoint quantum semimartingales, including an extra measurability condition which always holds for regular self-adjoint quantum semimartingales, is given. The quantum Ito formula is deduced from the quantum Duhamel formula (Theorem 2.3.3), as in the regular case. This result is useful because in order to show an essentially self-adjoint quantum semimartingale satisfies the quantum Ito formula we only need to show that it satisfies the quantum Duhamel formula. The chapter follows the approach of [66], although we need to take more care when dealing with unbounded quantum semimartingales.

In Chapter 3 we discuss the Poisson interpretation of Fock space over $L^2(\mathbb{R}_+)$. If $(X_t)_{t\geq 0}$ is a compensated Poisson process on $(\Omega, \mathcal{F}, \mathbb{P})$, the construction of the isomorphism

$$\mathcal{W} : \mathfrak{F}_+(L^2(\mathbb{R}_+)) \longrightarrow L^2(\Omega, \mathcal{F}, \mathbb{P}),$$

an example of a map of the type given in (1.1.1), is described. Since we only consider quantum stochastic integrals on $\mathfrak{F}_+(L^2[0,1])$ we restrict ourselves to the compensated Poisson process over $[0,1]$. We show that the operators of multiplication by adapted measurable processes give rise to suitable quantum stochastic integrands via the isomorphism $\mathcal{W}$. The properties, such as the multiplication formula, of Poisson exponentials $\mathcal{E}(f) := \mathcal{W}(e(f))$ are examined.

We prove the novel result that $\mathcal{E}_{\text{lb}}$, the linear span of the Poisson exponentials of bounded functions on $[0, 1]$, is dense in $L^p(\Omega)$ for $1 \leq p < \infty$, with the σ-field generated by $(X_t)_{0 \leq t \leq 1}$. This result is useful, since it can be used to show that certain operators of multiplication on $L^2(\Omega)$ are essentially self-adjoint with core $\mathcal{E}_{\text{lb}}$. The result that $\mathcal{W}$ intertwines $\Lambda_t + A_t + A_t^\dagger$ and $\widehat{X_t}$ is well-known. We prove this result using the Poisson multiplication formula and the classical Ito product formula, which is different from the usual proofs found in the literature.

The main aim of Chapter 4 is to construct the isomorphism for the Poisson distribution which is equivalent to the Wiener-Ito isomorphism for Gaussian variables alluded to in Section 1.2. We begin by giving the definition of a generalised Poisson process over a measure space $(M, \mathcal{M}, \mu)$ as an isometry $I : L^2(M, \mathcal{M}, \mu) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$, which generalises the notion of a compensated Poisson process over $\mathbb{R}_+$. The definition is slightly more general than the usual definition of a Poisson random measure. We introduce the Charlier polynomials and use them to construct the isometry

$$\mathcal{W}_I : \mathfrak{F}_+(L^2(M)) \longrightarrow L^2(\Omega, \mathcal{F}, \mathbb{P})$$

in Theorem 4.2.7. The results in Chapter 3 are then used to establish formulae for certain exponential vectors $\mathcal{W}_I(e(f))$ which are proved in [64] for non-atomic measure spaces. These formulae are then used to prove the surjectivity of $\mathcal{W}_I$ (Theorem 4.2.12), when the σ-field $\mathcal{F}$ is generated by the isometry I. The isomorphism will be called the generalised Wiener-Poisson isomorphism. This approach is different to those of [64] and [39]. We also give a construction of the isomorphism using a restricted exponential domain and the connection with the Charlier polynomials deduced from this. The construction is similar to the one for the Wiener-Ito isomorphism in [69, §4]. Another approach to the Wiener-Poisson isomorphism using the theory of reproducing kernel Hilbert spaces is briefly discussed. We finish the chapter by considering the discrete chaos spaces $\mathcal{W}_I(L^2(M)^{\otimes_s^n})$ and establishing some formulae which are proved in [64]. Although our formulae are not as general, the new proofs we give are much easier to follow.

Chapter 5 is devoted to the canonical construction of generalised Poisson processes, using operator algebra theory, modifying the arguments in [69, §5]. A unital commutative C^*-algebra $\mathcal{A} \subseteq \mathfrak{B}(\mathfrak{F}_+(L^2(M)))$ is constructed and is shown to have cyclic vector $e(0)$. The Segal spatial isomorphism

$$\mathscr{S} : \mathfrak{F}_+(L^2(M)) \longrightarrow L^2(\Delta, \mathcal{B}, \mathbb{P}),$$

associated with the C^*-algebra $\mathcal{A}$ is shown to be the Wiener-Poisson isomorphism of the generalised Poisson process $\mathscr{S}|_{L^2(M)}$ (Theorem 5.2.5 and Theorem 5.3.1). It should be noted that the probability measure $\mathbb{P}$ occurs naturally. The final part of the chapter considers the way in which the generalised Poisson processes we construct are related when two measure spaces are measure isomorphic.

The concluding chapter, Chapter 6, is concerned with showing that perturbations of certain classical martingales satisfy the quantum Duhamel formula and hence the quantum Ito formula. As mentioned before, for convenience we work on $L^2[0,1]$ rather than on $L^2(\mathbb{R}_+)$. We begin with a new Fock space proof of the Ito formula for $(\Lambda_t + A_t + A_t^\dagger)_{0\leq t\leq 1}$. Then we prove the well-known result on the representation of classical Poisson martingales as quantum stochastic integrals (Theorem 6.2.4) and show that the quantum Duhamel and Ito formulae for these representations can be deduced from the classical Ito formula. The main result of this chapter is Theorem 6.3.1, which says that perturbations of certain classical martingales by regular self-adjoint quantum semimartingales are essentially self-adjoint quantum semimartingales which satisfy the quantum Duhamel formula. We then look at mixed Brownian-Poisson martingales, that is the sum of a classical Brownian and Poisson martingale and prove that if the sum is also essentially self-adjoint then perturbations of it also satisfy the quantum Duhamel formula.

Chapter 2

Quantum Semimartingales

In this chapter we present a very brief account of the construction of quantum stochastic integrals, more details of which can be found in [28], [47], [51] and [10]. We also give an extension of the definition of the quantum Duhamel and the quantum Ito formulae for essentially self-adjoint quantum semimartingales and show that with our definitions the quantum Duhamel formula implies the quantum Ito formula.

2.1 Quantum stochastic integrals

We shall define our integrals on the symmetric Fock space over $L^2[0,1]$, $\mathfrak{F}_+(L^2[0,1])$, and shall not use an initial space. Given $f \in L^2[0,1]$ the *exponential vector* of f is

$$e(f) := (1, f, \ldots, \frac{f^{\otimes n}}{\sqrt{n!}}, \ldots).$$

Note that because the exponential vectors are linearly independent, a linear operator can be defined on the linear span of a set of exponential vectors by defining its action on each exponential vector. We define

$$\mathsf{E} := \operatorname{linsp}\{e(f) : f \in L^2[0,1]\};$$
$$\mathsf{E}_{\mathrm{lb}} := \operatorname{linsp}\{e(f) : f \in L^\infty[0,1]\}.$$

The notation E_{lb} is used since if we work on $\mathfrak{F}_+(L^2(\mathbb{R}_+))$ instead, the analogous space will consist of the linear span of exponential vectors of square integrable locally bounded functions and not just bounded functions. Following [2] and [66] we shall define our integrals over E_{lb} although they can be constructed on

$$\mathsf{E}(S) := \operatorname{linsp}\{e(f) : f \in S\},$$

where S is any admissible set in $L^2[0,1]$. That is, S is a dense subspace of $L^2[0,1]$ such that for all $t \in [0,1]$, $f_t := \mathbb{1}_{[0,t]} f \in S$. Such a set $\mathsf{E}(S)$, where S is an admissible set, is dense in $\mathfrak{F}_+(L^2[0,1])$ [51, Corollary 19.5]. In particular, both E and E_{lb} are dense in $\mathfrak{F}_+(L^2[0,1])$.

If $P_{[0,t]}$ is the orthogonal projection of $L^2[0,1]$ onto $L^2[0,t]$, we define

$$\Lambda_t := \lambda(P_{[0,t]}), \quad A_t := a(\mathbb{1}_{[0,t]}), \quad A_t^\dagger := a^\dagger(\mathbb{1}_{[0,t]}), \tag{2.1.1}$$

where λ, a and $a^\dagger$ are defined in [28, §2] and [47, §IV.1.4,6]. $(\Lambda_t)_{0\le t\le 1}$, $(A_t)_{0\le t\le 1}$ and $(A_t^\dagger)_{0\le t\le 1}$, the *gauge*, *annihilation* and *creation processes*, are often called the *basic processes.*

For each $t \in [0,1]$, there exists a unique unitary map

$$\mathfrak{u}_t : \mathfrak{F}_+(L^2[0,t]) \otimes \mathfrak{F}_+(L^2(t,1]) \longrightarrow \mathfrak{F}_+(L^2[0,1])$$

such that

$$\mathfrak{u}_t : e(f) \otimes e(g) \longmapsto e(f+g). \tag{2.1.2}$$

Consequently, $\mathfrak{F}_+(L^2[0,t])$ can be considered a subspace of $\mathfrak{F}_+(L^2[0,1])$ by identifying it with $\mathfrak{u}_t(\mathfrak{F}_+(L^2[0,t]) \otimes 1)$. Similarly, $\mathfrak{F}_+(L^2(t,1])$ can also be considered a subspace of $\mathfrak{F}_+(L^2[0,1])$. A family of linear operators $F = (F_t : t \in [0,1])$ whose domains contain E_{lb} is said to be an *adapted process* if for all $f \in L^\infty[0,1]$, $t \mapsto F_t e(f)$ is strongly measurable and for all $t \in [0,1]$,

$$\begin{cases} F_t e(f_t) \in \mathfrak{F}_+(L^2[0,t]) \\ F_t e(f) = \mathfrak{u}_t(F_t e(f_t) \otimes e(\mathbb{1}_{(t,1]} f)) \end{cases} \tag{2.1.3}$$

with the identification of $\mathfrak{F}_+(L^2[0,t])$ as a subspace of $\mathfrak{F}_+(L^2[0,1])$. If F_t is a bounded operator for each t, the measurability of $t \mapsto F_t\phi$, for any $\phi \in \mathfrak{F}_+(L^2[0,1])$, and $t \mapsto \|F_t\|$ follow from the definition of an adapted process. Our definition of quantum stochastic integrals follow that of [2, §II].

Definition 2.1.1 *Let E, F, G and H be adapted processes such that for all $f \in L^\infty[0,1]$,*

$$\int_0^1 (|f(s)|^2\|E_s e(f)\|^2 + |f(s)|\|F_s e(f)\| + \|G_s e(f)\|^2 + \|H_s e(f)\|)ds < \infty. \tag{2.1.4}$$

Then the quantum stochastic integral of the quadruple (E,F,G,H) is the unique adapted process $(M_t : t \in [0,1])$ defined on E_{lb} such that for all f, $g \in L^\infty[0,1]$,

$$\langle e(f), M_t e(g)\rangle = \int_0^t \langle e(f), (\overline{f(s)}g(s)E_s + g(s)F_s + \overline{f(s)}G_s + H_s)e(g)\rangle ds. \tag{2.1.5}$$

We write

$$M_t = \int_0^t E_s d\Lambda_s + F_s dA_s + G_s dA_s^\dagger + H_s ds$$

and denote the integral of (E,F,G,H) as $M(E,F,G,H)$ or as M if the dependence on the integrands is clear or unimportant.

The uniqueness is immediate since $\mathsf{E}_{\rm lb}$ is dense in $\mathfrak{F}_+(L^2[0,1])$. The existence can be found in [28, Theorem 4.4] and [47, §VI.1.6,7] amongst others. Notice that if the adjoint processes (E^*, G^*, F^*, H^*) are also adapted processes such that for all $f \in L^\infty[0,1]$,

$$\int_0^1 (|f(s)|^2 \|E_s^* e(f)\|^2 + |f(s)| \|G_s^* e(f)\| + \|F_s^* e(f)\|^2 + \|H_s^* e(f)\|) ds < \infty, \tag{2.1.6}$$

then whenever $t \in [0,1]$, on $\mathsf{E}_{\rm lb}$,

$$M_t^* = \int_0^t E_s^* d\Lambda_s + G_s^* dA_s + F_s^* dA_s^\dagger + H_s^* ds.$$

Consequently, in this case M_t has a densely defined adjoint and is therefore closable.

Since the original paper of Hudson and Parthasarathy [28], there have been various different reformulations and extensions of quantum stochastic integrals, primarily in an attempt to extend the integral to domains larger than the span of exponential vectors. One approach uses the classical Ito calculus (see [6]), although the definition is implicit and consists of a series of classical stochastic differential equations. Another approach (see [7] and [42]) uses non-causal stochastic analysis and Malliavin calculus. Recently a new definition which combines the original Hudson-Parthasarathy definition and the above two approaches has been formulated (see [5] and [3]). We shall use the original Hudson-Parthasarathy definition of quantum stochastic integrals on exponential vectors because we shall be investigating essential self-adjointness of quantum stochastic integrals and this is easier when the domains of the operators are known.

Attal's spaces of quantum semimartingales are first introduced in [2]. Regular quantum semimartingales, which are bounded quantum semimartingales, have the property that they satisfy the quantum Ito product formula and form a $*$-algebra under operator multiplication.

Definition 2.1.2 *Let $1 \leq p \leq \infty$ and $\mathfrak{H} := \mathfrak{F}_+(L^2[0,1])$. Then $L^p([0,1]; \mathfrak{B}(\mathfrak{H}))$ consists of bounded processes $(F_t : t \in [0,1])$ such that $t \mapsto F_t\phi$ is strongly measurable for each $\phi \in \mathfrak{F}_+(L^2[0,1])$ and $s \mapsto \|F_s\|$ is in $L^p[0,1]$. As usual, two processes in $L^p([0,1]; \mathfrak{B}(\mathfrak{H}))$ are identified if they are equal almost everywhere. A quadruple (E, F, G, H) is said to be Bochner integrable if*

$$E \in L^\infty([0,1]; \mathfrak{B}(\mathfrak{H})), \quad F, G \in L^2([0,1]; \mathfrak{B}(\mathfrak{H})), \quad H \in L^1([0,1]; \mathfrak{B}(\mathfrak{H})).$$

Note that if we are working over $\mathbb{R}_+$ we consider the Bochner-Lebesgue spaces $L^p_{\rm loc}(\mathbb{R}_+; \mathfrak{B}(\mathfrak{H}))$ with $\mathfrak{H} := \mathfrak{F}_+(L^2(\mathbb{R}_+))$ instead. Using the Pettis measurability theorem (see [17, Chapter 2, Theorem 1.2]), we can easily deduce that it is sufficient in the above definition to only require that $t \mapsto F_t e(f)$ is measurable for each $f \in L^\infty[0,1]$. Given $F \in L^p([0,1];\mathfrak{B}(\mathfrak{H}))$ for some $p \geq 1$, for each $t \in [0,1]$ we can define the integral of F by

$$\left(\int_0^t F_s ds\right)\phi := \int_0^t F_s \phi ds, \tag{2.1.7}$$

whenever $\phi \in \mathfrak{F}_+(L^2[0,1])$. From the definition of Bochner-Lebesgue spaces this operator is well-defined and is bounded with norm $t^{1/q}\|F\|_p$, where $\frac{1}{p} + \frac{1}{q} = 1$ (see [67, §2]). From now on, whenever we write down an integral in $\mathfrak{B}(\mathfrak{F}_+(L^2[0,1]))$ it will be defined by (2.1.7), in the strong operator sense, rather than as a Bochner integral in the non-separable space $\mathfrak{B}(\mathfrak{F}_+(L^2[0,1]))$.

Definition 2.1.3 *A quantum stochastic integral process* $(M_t : t \in [0,1])$ *is said to be bounded if* M_t *is bounded on* E_{lb} *for each* $t \in [0,1]$*. We define*

$$\mathcal{S}' := \{M(E,F,G,H) : (E,F,G,H) \text{ is adapted and Bochner integrable}\},$$
$$\mathcal{S} := \{M \in \mathcal{S}' : M_t \text{ is bounded for each } t \in [0,1]\}.$$

Elements of $\mathcal{S}'$ *are called quantum semimartingales, while elements of* $\mathcal{S}$ *are said to be regular quantum semimartingales.*

Elements of $\mathcal{S}'$ are easy to construct, since any Bochner integrable quadruple (E,F,G,H) gives rise to an element of $\mathcal{S}'$. For example taking $E = H = 0$ and $F = G = I$ gives $M_t = A_t + A_t^\dagger$, which corresponds via the Wiener-Ito isomorphism to multiplication by Brownian motion (see (1.0.1)). It is quite difficult to construct elements in $\mathcal{S}$, because given (E,F,G,H) the construction of M_t does not ensure it is bounded. However, Attal has shown that there exists a bijection between the space of quantum semimartingales and regular quantum semimartingales (see [4, Theorem 4.6]).

One way of constructing elements of $\mathcal{S}$ is as solutions of quantum stochastic differential equations. For example, in [28, §7] unitary solutions $(U_t : t \in [0,1])$ of

$$U_t = I + \int_0^t (\alpha_1 - 1)U_s d\Lambda_s + \alpha_2 U_s dA_s - \alpha_1\overline{\alpha_2}U_s dA_s^\dagger + (i\alpha_3 - \frac{1}{2}|\alpha_2|^2)U_s ds,$$

where α_1, $\alpha_2 \in \mathbb{C}$ with $|\alpha_1| = 1$ and $\alpha_3 \in \mathbb{R}$, are constructed. More results on solutions of quantum stochastic differential equations can be found in [23], [24] and [65]. Another method of constructing elements of $\mathcal{S}$ is to use probabilistic interpretations of Fock space. In [2, Theorem 4], Attal shows that certain bounded classical stochastic integrals can be represented as regular quantum semimartingales. In Section 6.1 we shall give details in the case of the Poisson interpretation of Fock space.

We now give an example of an element of $\mathcal{S}$ using 'upside-down' Brownian motion. Given a Brownian motion $(W_t)_{0\leq t\leq 1}$ we know from the classical Ito formula that

$$e^{iW_t} = 1 + i\int_0^t e^{iW_s} dW_s - \frac{1}{2}\int_0^t e^{iW_s} ds.$$

Thus, if $\mathfrak{w} : \mathfrak{F}_+(L^2[0,1]) \to L^2(\Omega_{\mathfrak{w}}, \mathcal{F}_{\mathfrak{w}}, \mathbb{P}_{\mathfrak{w}})$ is the Wiener interpretation of Fock space (see (1.0.1)),

$$\mathfrak{w}^{-1}\widehat{e^{iW_t}}\mathfrak{w} = I + i\int_0^t \mathfrak{w}^{-1}\widehat{e^{iW_s}}\mathfrak{w}(dA_s + dA_s^\dagger) - \frac{1}{2}\int_0^t \mathfrak{w}^{-1}\widehat{e^{iW_s}}\mathfrak{w} ds,$$

where $\widehat{e^{iW_t}}$ denotes the bounded operator of multiplication by e^{iW_t} on $L^2(\Omega_{\mathfrak{w}})$. Now, the *Fourier-Wiener* transform is the unitary map

$$\begin{aligned} \mathscr{F} : \mathfrak{F}_+(L^2[0,1]) &\longrightarrow \mathfrak{F}_+(L^2[0,1]) \\ e(f) &\longmapsto e(if). \end{aligned}$$

More details of this map can be found in [47, §IV.2.2] and [34, §13.1]. It can easily be shown that on E_{lb},

$$iA_t - iA_t^\dagger = \mathscr{F}^{-1}(A_t + A_t^\dagger)\mathscr{F}.$$

Consequently on E_{lb},

$$iA_t - iA_t^\dagger = \mathscr{F}^{-1}(A_t + A_t^\dagger)\mathscr{F} = \mathscr{F}^{-1}\mathfrak{w}^{-1}\widehat{W_t}\mathfrak{w}\mathscr{F}. \tag{2.1.8}$$

In fact, $\overline{iA_t - iA_t^\dagger}$ is unitarily equivalent via $\mathfrak{w}\mathscr{F}$ to $\widehat{W_t}$. Hence we have that $D(\overline{iA_t - iA_t^\dagger}) = D((\mathfrak{w}\mathscr{F})^{-1}\widehat{W_t}\mathfrak{w}\mathscr{F})$ and on this common domain (2.1.8) holds with $iA_t - iA_t^\dagger$ replaced by its closure [34, Theorem 13.21 iii)]. The operator $iA_t - iA_t^\dagger$ is often called 'upside-down' Brownian motion (see [47, §IV.2.3]). It can then be shown in a similar way to the usual Brownian motion that

$$(\mathfrak{w}\mathscr{F})^{-1}\widehat{e^{iW_t}}\mathfrak{w}\mathscr{F} = I + i\int_0^t (\mathfrak{w}\mathscr{F})^{-1}\widehat{e^{iW_s}}\mathfrak{w}\mathscr{F}(idA_s - idA_s^\dagger) - \frac{1}{2}\int_0^t (\mathfrak{w}\mathscr{F})^{-1}\widehat{e^{iW_s}}\mathfrak{w}\mathscr{F} ds.$$

Therefore $(\mathfrak{w}^{-1}\widehat{e^{iW_t}}\mathfrak{w} + (\mathfrak{w}\mathscr{F})^{-1}\widehat{e^{iW_t}}\mathfrak{w}\mathscr{F} - 2I)_{0\leq t\leq 1}$ is a regular quantum semimartingale.

It is possible to give a characterisation of elements of $\mathcal{S}$ which only depends on $(M_t)_{0\leq t\leq 1}$ without direct reference to the integrands of the quantum semimartingale. We shall not include any details of this, since we shall not use this characterisation in our work, however details can be found in [2, Theorem 7], which extends [52, Theorem 3.10]. The integral representation of a quantum semimartingale is unique (see [41, Corollary 1.3] or [65, Lemma 4.7]). If the quadruple (E, F, G, H) is Bochner integrable, then the adjoint quadruple (E^*, G^*, F^*, H^*) is also Bochner integrable and thus satisfies (2.1.6). Therefore, if $M \in \mathcal{S}'$, M_t is closable for each $t \in [0, 1]$. The space $\mathcal{S}$ is stable under composition and the adjoint operation [2, Theorem 5], and many results have been obtained concerning it. Operator processes in $\mathcal{S}$ have the advantage that the operators can be extended to the whole of $\mathfrak{F}_+(L^2[0, 1])$ by the boundedness property. The extension $\overline{M_t|_{\mathsf{E}_{\text{lb}}}}$ agrees with the quantum stochastic integral if it is defined on the whole of E. This is because if $M_t|_{\mathsf{E}}$ is the quantum stochastic integral constructed on E then $\overline{M_t|_{\mathsf{E}_{\text{lb}}}} \subseteq \overline{M_t|_{\mathsf{E}}}$ ($M_t|_{\mathsf{E}}$ is also closable by exactly the same reasons as for $M_t|_{\mathsf{E}_{\text{lb}}}$) and since $D(\overline{M_t|_{\mathsf{E}_{\text{lb}}}}) = \mathfrak{F}_+(L^2[0, 1])$, we must have that on E, $\overline{M_t|_{\mathsf{E}_{\text{lb}}}} = M_t|_{\mathsf{E}}$. It should also be noted that the bounded extension also agrees with the Ito calculus formulation [2, Corollary 2]. For the non-causal stochastic analysis definition, a similar argument using the closability of the operators, shows that the bounded extension is also compatible with this formulation.

We say that $M \in \mathcal{S}$ is a process of self-adjoint operators if the extension of M_t to the whole of Fock space is self-adjoint for each $t \in [0, 1]$. However, when dealing with $\mathcal{S}'$ self-adjointness needs more care since we need to specify the domain we are working on.

Definition 2.1.4 *Suppose $M \in \mathcal{S}'$. Then we say that M is a process of essentially self-adjoint operators if for each $t \in [0, 1]$, M_t is an essentially self-adjoint operator with core E_{lb}. We define*

$$\mathcal{S}'_{\text{sa}} := \{M \in \mathcal{S}' : M \textit{ is a process of essentially self-adjoint operators}\},$$
$$\mathcal{S}_{\text{sa}} := \{M \in \mathcal{S} : M \textit{ is a process of self-adjoint operators}\}.$$

When working with unbounded quantum semimartingales the domains are important and this is the reason we shall assume all our integrals are defined on E_{lb}. For $M \in \mathcal{S}'_{\text{sa}}$ we define $\overline{M} := (\overline{M_t} : t \in [0, 1])$. If $M = M(E, F, G, H) \in \mathcal{S}'$ is a symmetric operator then by the uniqueness of the integral representations $E = E^*$, $G = F^*$ and $H = H^*$. In particular this holds for all $M \in \mathcal{S}'_{\text{sa}}$. Unlike for $\mathcal{S}'$, it is not easy to find quantum semimartingales in $\mathcal{S}'_{\text{sa}}$, since symmetric operators are not always essentially self-adjoint. One method of constructing such elements is to use the Wiener interpretation of Fock space, $\mathfrak{w} : \mathfrak{F}_+(L^2[0, 1]) \to L^2(\Omega_{\mathfrak{w}})$. If $(F_t)_{0 \le t \le 1}$ is a real predictable process on $(\Omega_{\mathfrak{w}}, \mathcal{F}_{\mathfrak{w}}, \mathbb{P}_{\mathfrak{w}})$, bounded on each finite interval, and

$\widehat{F_t}$ is the operator of multiplication by F_t on $L^2(\Omega_{\mathfrak{w}})$, then for $t \in [0,1]$,

$$M_t = \int_0^t \mathfrak{w}^{-1}\widehat{F_s}\mathfrak{w}(dA_s + dA_s^\dagger)$$

is a well-defined quantum semimartingale and $\mathfrak{w}M_t\mathfrak{w}^{-1}$ is equivalent to multiplication by $\int_0^t F_s dW_s$ on $\mathfrak{w}(\mathsf{E}_{\rm lb})$. Since $\int_0^t F_s dW_s \in L^4(\Omega_{\mathfrak{w}})$ (see [36, Exercise 3.25]) and $\mathfrak{w}(\mathsf{E}_{\rm lb})$ is dense in $L^4(\Omega_{\mathfrak{w}})$, the operator of multiplication by $\int_0^t F_s dW_s$ is self-adjoint with core $\mathfrak{w}(\mathsf{E}_{\rm lb})$. Thus M_t is essentially self-adjoint with core $\mathsf{E}_{\rm lb}$ and hence $(M_t)_{0\le t\le 1} \in \mathcal{S}'_{\rm sa}$ (see [67, §13]). Given a class of processes in $\mathcal{S}'_{\rm sa}$ we can construct more examples of essentially self-adjoint quantum semimartingales by perturbing the class by regular quantum semimartingales. This perturbation technique is used frequently in Functional Analysis and Quantum Field Theory. In Chapter 6 we shall consider perturbations of classical semimartingales.

2.2 The quantum Duhamel formula

If $(W_t)_{0\le t\le 1}$ is Brownian motion then a classical Ito formula exists. Namely, if $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable,

$$f(W_t) = f(0) + \int_0^t f'(W_s)dW_s + \frac{1}{2}\int_0^t f''(W_s)ds, \tag{2.2.1}$$

where the stochastic integral could be in the extended sense (see [16, §2.6] or [49, §1.1.3]). It is natural to ask if this can be extended to quantum stochastic calculus. That is, given an essentially self-adjoint quantum semimartingale M, can we obtain a formula for $f(\overline{M})$? In [8, Theorem 4.4] it is shown that if the classical Duhamel formula holds, that is for all $p \in \mathbb{R}$,

$$e^{ipW_t} = 1 + ip\int_0^t e^{ipW_s}dW_s - \frac{p^2}{2}\int_0^t e^{ipW_s}ds, \tag{2.2.2}$$

then it is possible to deduce the classical Ito formula, (2.2.1), for functions in the set

$$C^{2+}_{\rm loc}(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} \in L^1(\mathbb{R}) : p \mapsto p^2\hat{f}(p) \in L^1(\mathbb{R})\}, \tag{2.2.3}$$

where the Fourier transform $\hat{f}$ of f is defined by

$$\hat{f}(p) := \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ipx}dx.$$

This is deduced by using the Fourier inversion formula

$$f(W_t) = \int_{-\infty}^{\infty} \hat{f}(p)e^{ipW_t}dp,$$

substituting (2.2.2) and then justifying the interchange of integrals via a Fubini type lemma [8, Lemma 4.3]. From the Ito formula for functions $f \in C^{2+}_{\text{loc}}(\mathbb{R})$, it is possible to deduce the general Ito formula by using sequences $f_n \in C^{2+}_{\text{loc}}(\mathbb{R}) \cap C^{\infty}_{\text{c}}(\mathbb{R})$ which approximate a more general function f. In the quantum case we would like to establish a formula for $e^{ip\overline{M_t}}$ like (2.2.2), called the quantum Duhamel formula, from which we can deduce a quantum Ito formula for certain functions.

The existence of a quantum Duhamel formula and a quantum Ito formula has been proved for regular quantum semimartingales in [66]. For any $M \in \mathcal{S}$, the Ito formula for polynomials is established by induction [66, Lemma 4.1] and then extended to analytic functions [66, Theorem 4.1]. The result for analytic functions is used to establish the quantum Duhamel formula for e^{ipM_t} [66, Proposition 5.1]. Then if $M \in \mathcal{S}_{\text{sa}}$, an Ito formula for $f(M_t)$, whenever $f \in C^{2+}_{\text{loc}}(\mathbb{R})$, is obtained in [66, Theorem 6.2], using the quantum Duhamel formula and the Fourier transform. For bounded quantum semimartingales all these results are possible primarily due to the fact that since M_t is bounded for each $t \in [0, 1]$, two such semimartingales can be composed. For general quantum semimartingales such compositions are not possible, however functional calculus can be used to define $f(\overline{M_t})$, for bounded continuous functions f, whenever $M \in \mathcal{S}'_{\text{sa}}$. We would like to extend the formulae obtained for regular self-adjoint quantum semimartingales to certain $M \in \mathcal{S}'_{\text{sa}}$. In this section, we give a meaning to the statement that $M \in \mathcal{S}'_{\text{sa}}$ satisfies the quantum Duhamel formula and deduce that if $M \in \mathcal{S}'_{\text{sa}}$ satisfies the quantum Duhamel formula then this implies that M satisfies the quantum Ito formula. This has been done for regular quantum semimartingales in [66, Theorem 6.2].

Given subspaces $V_1 \subseteq \mathfrak{F}_+(L^2[0, t])$ and $V_2 \subseteq \mathfrak{F}_+(L^2(t, 1])$ we shall denote by $V_1 \underline{\otimes} V_2$ the algebraic tensor product. Let T be a densely defined operator on $\mathfrak{F}_+(L^2[0, t])$ with domain $D(T)$. Suppose that $T \otimes I$ is an essentially self-adjoint operator on $\mathfrak{F}_+(L^2[0, t]) \otimes \mathfrak{F}_+(L^2(t, 1])$ with domain $D(T)\underline{\otimes}X$ where X is a dense subspace of $\mathfrak{F}_+(L^2(t, 1])$. Then we would like to investigate if T is essentially self-adjoint on $\mathfrak{F}_+(L^2[0, t])$ and if this is true whether $e^{ip(\overline{T \otimes I})} = e^{ip\overline{T}} \otimes I$.

Lemma 2.2.1 *Let T be as above. Then $T : D(T) \to \mathfrak{F}_+(L^2[0, t])$ is an essentially self-adjoint operator.*

PROOF: We need to prove that T is closable, $D(\overline{T}) = D(T^*)$ and that for all $\phi \in D(\overline{T})$, $\overline{T}\phi = T^*\phi$. Note that if $\phi \in D(\overline{T})$, then $\phi \otimes 1 \in D(\overline{T \otimes I})$ with $(\overline{T \otimes I})(\phi \otimes 1) = \overline{T}\phi \otimes 1$.

Suppose ϕ, $\psi \in D(T)$. Then by the essential self-adjointness of $T \otimes I$,

$$\begin{aligned}\langle \phi, T\psi \rangle &= \langle \phi \otimes 1, (\overline{T \otimes I})(\psi \otimes 1) \rangle \\ &= \langle (T \otimes I)(\phi \otimes 1), \psi \otimes 1 \rangle \\ &= \langle T\phi, \psi \rangle.\end{aligned}$$

Thus T is a symmetric operator and hence is closable. If $\phi \in D(\overline{T})$ and $\psi \in D(T)$,

$$\begin{aligned}\langle \phi, T\psi \rangle &= \langle \phi \otimes 1, (\overline{T \otimes I})(\psi \otimes 1) \rangle \\ &= \langle (\overline{T \otimes I})(\phi \otimes 1), \psi \otimes 1 \rangle \\ &= \langle \overline{T}\phi, \psi \rangle.\end{aligned}$$

Therefore $D(\overline{T}) \subseteq D(T^*)$ and $\overline{T} \subseteq T^*$. Conversely, if $\phi \in D(T^*)$, for all $\psi \in D(T)$ and $\theta \in X$,

$$\begin{aligned}\langle \phi \otimes 1, (T \otimes I)(\psi \otimes \theta) \rangle &= \langle \phi, T\psi \rangle \langle 1, \theta \rangle \\ &= \langle T^*\phi, \psi \rangle \langle 1, \theta \rangle \\ &= \langle T^*\phi \otimes 1, \psi \otimes \theta \rangle.\end{aligned}$$

Thus $\phi \otimes 1 \in D((T \otimes I)^*) = D(\overline{T \otimes I})$, and hence there exists $(\phi_n) \subseteq D(T \otimes I)$ such that $\phi_n \to \phi \otimes 1$ and $(T \otimes I)\phi_n$ converges. Let P_t be the orthogonal projection of $\mathfrak{F}_+(L^2[0,1])$ onto $\mathfrak{u}_t(\mathfrak{F}_+(L^2[0,t]) \otimes 1)$. If $\psi \in D(T \otimes I)$, then $P_t\psi \in D(T \otimes I)$ and $P_t(T \otimes I)\psi = (T \otimes I)P_t\psi$. Consequently, since $\mathfrak{F}_+(L^2[0,t]) \otimes 1 \cong \mathfrak{F}_+(L^2[0,t])$, $P_t\phi_n \to \phi$ in $\mathfrak{F}_+(L^2[0,t])$ and $T(P_t\phi_n)$ converges. Therefore, $\phi \in D(\overline{T})$ and $D(T^*) \subseteq D(\overline{T})$. $\square$

The above lemma is a type of converse to [70, Theorem 8.33]. The fact that $P_t(T \otimes I) = (T \otimes I)P_t$ on $D(T \otimes I)$ can be used as part of an alternative definition of adaptedness. Recall that the Malliavin gradient on Fock space is defined by

$$\begin{aligned}\nabla : D(\sqrt{N}) &\longrightarrow L^2([0,1]; \mathfrak{F}_+(L^2[0,1])) \\ \phi = (\phi_n) &\longmapsto \nabla_t\phi := (\sqrt{n}\phi_n(\cdot, t)),\end{aligned}$$

where

$$D(\sqrt{N}) = \{\phi \in \mathfrak{F}_+(L^2[0,1]) : \sum_{n=0}^{\infty} n\|\phi_n\|^2 < \infty\}$$

and ϕ_n is interpreted as an element of $L^2_{\text{sym}}([0,1]^n)$. If $P_t : \mathfrak{F}_+(L^2[0,1]) \to \mathfrak{u}_t(\mathfrak{F}_+(L^2[0,t]) \otimes 1)$ is the orthogonal projection of $\mathfrak{F}_+(L^2[0,1])$ onto $\mathfrak{u}_t(\mathfrak{F}_+(L^2[0,t]) \otimes 1)$ as in the above proof, then $P\nabla : D(\sqrt{N}) \to L^2([0,1]; \mathfrak{F}_+(L^2[0,1]))$, defined by $(P\nabla\phi)(t) = P_t(\nabla_t\phi)$, is a bounded operator. We define $D := \overline{P\nabla}$ and $D_t\phi := (D\phi)(t)$. For each $\phi \in \mathfrak{F}_+(L^2[0,1])$, for almost all $t \in [0,1]$ (depending on the element ϕ), $D_t\phi \in \mathfrak{F}_+(L^2[0,1])$. Then in [5, Proposition 3.5] and [3, Proposition 1.4] it is shown that an operator with domain E_{lb} is adapted at time t if and only if on E_{lb},

$$\begin{cases} P_tF &= FP_t \\ D_uF &= FD_u \text{ a.e. } u \geq t. \end{cases}$$

More details can be found in [5, §3] and [3, §I.1.2].

Lemma 2.2.2 *Let T be as above. Then for each $p \in \mathbb{R}$, $e^{ip(\overline{T\otimes I})} = e^{ip\overline{T}} \otimes I$.*

PROOF: It can easily be shown that $\overline{T\otimes I}$ is equal to the closure of the operator $\overline{T} \otimes I$ with domain $D(\overline{T})\underline{\otimes}\mathfrak{F}_+(L^2(t,1])$. Thus, since $\overline{T}$ is self-adjoint we obtain the result by applying [70, Theorem 8.35]. □

Corollary 2.2.3 *If $M \in \mathcal{S}'_{\text{sa}}$ and $f \in L^\infty[0,1]$, then for each $p \in \mathbb{R}$ and $t \in [0,1]$,*

$$\begin{cases} e^{ip\overline{M_t}}e(f_t) &\in \mathfrak{F}_+(L^2[0,t]) \\ e^{ip\overline{M_t}}e(f) &= \mathfrak{u}_t(e^{ip\overline{M_t}}e(f_t) \otimes e(f\mathbb{1}_{(t,1]})) \end{cases}$$

PROOF: This follows immediately from the lemma above since for each $t \in [0,1]$, M_t is in the same form as $T \otimes I$ in the lemma. □

Therefore, if $t \mapsto e^{ip\overline{M_t}}$ is strongly measurable, then $(e^{ipM_t} : t \in [0,1])$ is an adapted process. We do not know if this measurability condition is true for all $M \in \mathcal{S}'_{\text{sa}}$.

Lemma 2.2.4 *Suppose $M = M(E, F, F^*, H) \in \mathcal{S}'_{\text{sa}}$ and that the map $(p,t) \mapsto e^{ip\overline{M_t}}$ is strongly measurable as a function $\mathbb{R} \times [0,1] \to \mathfrak{B}(\mathfrak{F}_+(L^2[0,1]))$. Then $\overline{M_t} + E_t$ is self-adjoint and $(p,t) \mapsto e^{ip(\overline{M_t}+E_t)}$ is strongly measurable.*

PROOF: By the Kato-Rellich theorem (see [55, Theorem X.12] or [70, Theorem 5.28]), $\overline{M_t}+E_t$ is self-adjoint. Since E_t is a bounded self-adjoint operator and $t \mapsto E_t$ is strongly measurable, the map $(p,t) \mapsto e^{ipE_t}$ is strongly measurable. By [70, Theorem 7.40] in the strong sense,

$$e^{ip(\overline{M_t}+E_t)} = \lim_{n\to\infty}(e^{i(p/n)\overline{M_t}}e^{i(p/n)E_t})^n.$$

However by [66, Lemma 3.3], $(p,t) \mapsto (e^{i(p/n)\overline{M_t}}e^{i(p/n)E_t})^n$ is strongly measurable. Consequently, $e^{ip(\overline{M_t}+E_t)}$ is the strong limit of these operators and so the result follows. □

For regular self-adjoint quantum semimartingales the quantum Duhamel formula represents $(e^{ipM_t})_{0\le t\le 1}$ as a regular quantum semimartingale. If M belongs to $\mathcal{S}'_{\mathrm{sa}}$ the situation is more delicate and we include an extra measurability condition as part of the definition of the quantum Duhamel formula, which always holds in the bounded case.

Definition 2.2.5 *Let $M = M(E, F, F^*, H) \in \mathcal{S}'_{\mathrm{sa}}$. Then we say M satisfies the quantum Duhamel formula if the mapping $(p, t) \mapsto e^{ip\overline{M_t}}$ is strongly measurable as a function $\mathbb{R}\times[0, 1] \to \mathfrak{B}(\mathfrak{F}_+(L^2[0, 1]))$, and for each $p \in \mathbb{R}$ and $t \in [0, 1]$,*

$$e^{ip\overline{M_t}} = I + \int_0^t E_{\exp(ipM)}d\Lambda + F_{\exp(ipM)}dA + G_{\exp(ipM)}dA^\dagger + H_{\exp(ipM)}ds, \tag{2.2.4}$$

where

$$\begin{aligned} E_{\exp(ipM)}(s) &:= e^{ip(\overline{M_s}+E_s)} - e^{ip\overline{M_s}},\\ F_{\exp(ipM)}(s) &:= ip\int_0^1 e^{ip(1-u)\overline{M_s}}F_s e^{ipu(\overline{M_s}+E_s)}du,\\ G_{\exp(ipM)}(s) &:= ip\int_0^1 e^{ip(1-u)(\overline{M_s}+E_s)}F_s^* e^{ipu\overline{M_s}}du,\\ H_{\exp(ipM)}(s) &:= ip\int_0^1 e^{ip(1-u)\overline{M_s}}H_s e^{ipu\overline{M_s}}du\\ &\quad + (ip)^2\int_0^1\int_0^1 u e^{ip(1-u)\overline{M_s}}F_s e^{ipu(1-v)(\overline{M_s}+E_s)}F_s^* e^{ipuv\overline{M_s}}dudv. \end{aligned} \tag{2.2.5}$$

The above definition requires some explanation. Firstly, for convenience we write $E_{\exp(ipM)}d\Lambda$ instead of $E_{\exp(ipM)}(s)d\Lambda_s$ etc. in (2.2.4) and this notation will always be used when stating the quantum Duhamel formula. The integrals in (2.2.5) are in the strong sense of (2.1.7), that is, for example, for each $\phi \in \mathfrak{F}_+(L^2[0, 1])$,

$$\left(\int_0^1 e^{ip(1-u)\overline{M_s}}F_s e^{ipu(\overline{M_s}+E_s)}du\right)\phi = \int_0^1 e^{ip(1-u)\overline{M_s}}F_s e^{ipu(\overline{M_s}+E_s)}\phi du,$$

where the integral on the right-hand side is a Bochner integral over $\mathfrak{F}_+(L^2[0, 1])$. The measurability condition imposed ensures that the integrals in (2.2.5) exist. For example, when considering $F_{\exp(ipM)}(s)$, we know by Lemma 2.2.4, that for each $p \in \mathbb{R}$ and $\phi \in \mathfrak{F}_+(L^2[0, 1])$,

$$(u, s) \longmapsto e^{ip(1-u)\overline{M_s}}F_s e^{ipu(\overline{M_s}+E_s)}\phi$$

is strongly measurable and furthermore

$$\|e^{ip(1-u)\overline{M_s}}F_s e^{ipu(\overline{M_s}+E_s)}\phi\| \le \|F_s\|\|\phi\|. \tag{2.2.6}$$

Consequently for all $s \in [0,1]$,

$$u \longmapsto e^{ip(1-u)\overline{M_s}} F_s e^{ipu(\overline{M_s}+E_s)} \phi$$

is integrable on $[0,1]$ and by Fubini's theorem,

$$s \longmapsto F_{\exp(ipM)}(s)\phi = ip \int_0^1 e^{ip(1-u)\overline{M_s}} F_s e^{ipu(\overline{M_s}+E_s)} \phi du$$

is strongly measurable. The same argument works for the other integrands. Given this strong integrability, it is now clear from the properties of the Bochner integral that the integrands in (2.2.4) are adapted processes. It also follows from inequalities like (2.2.6), that the quadruple $(E_{\exp(ipM)}, F_{\exp(ipM)}, G_{\exp(ipM)}, H_{\exp(ipM)})$ is Bochner integrable. Therefore the measurability condition imposed in the definition ensures that both sides of (2.2.4) are well-defined.

If $M \in \mathcal{S}_{\text{sa}}$ then the requirement that $(p,t) \mapsto e^{ipM_t}$ is strongly measurable always holds. This is because when M_t is bounded for each $t \in [0,1]$,

$$e^{ipM_t} = \sum_{n=0}^{\infty} \frac{(ip)^n}{n!} M_t^n,$$

where the sum converges in the norm operator topology. Therefore since for each n and $\phi \in \mathfrak{F}_+(L^2[0,1])$, $t \mapsto M_t^n \phi$ is measurable,

$$(p,t) \longmapsto e^{ipM_t}\phi = \sum_{n=0}^{\infty} \frac{(ip)^n}{n!} M_t^n \phi$$

is strongly measurable. When dealing with unbounded essentially self-adjoint quantum semimartingales we do not know if this measurability condition is always satisfied. However from [70, Theorem 8.30] we do know that for each $t \in [0,1]$ there exists a dense subset of analytic vectors of $\overline{M_t}$, say $\mathcal{D}_t$, such that for each $\phi \in \mathcal{D}_t$,

$$e^{ip\overline{M_t}}\phi = \sum_{n=0}^{\infty} \frac{(ip)^n}{n!} \overline{M_t}^n \phi.$$

Unfortunately, the subspace $\mathcal{D}_t$ may not be the same for each t.

2.3 The quantum Ito formula

In [66, Theorem 6.2] the Ito formula for regular quantum semimartingales was established for functions in the set $C^{2+}_{\text{loc}}(\mathbb{R})$ (see (2.2.3) for the definition of $C^{2+}_{\text{loc}}(\mathbb{R})$). Note that if $f \in C^{2+}_{\text{loc}}(\mathbb{R})$,

f is assumed to be continuous, and from the condition on $\hat{f}$ and the Fourier inversion theorem, f is also bounded. It is known that if $f \in C^{2+}_{\mathrm{loc}}(\mathbb{R})$ and T is a self-adjoint operator on $\mathfrak{F}_+(L^2[0,1])$,

$$f(T) = \int_{-\infty}^{\infty} \hat{f}(p) e^{ipT} dp, \tag{2.3.1}$$

where the integral is again in the strong sense.

Definition 2.3.1 *Let $M = M(E, F, F^*, G) \in \mathcal{S}'_{\mathrm{sa}}$. We say that M satisfies the quantum Ito formula if for all $f \in C^{2+}_{\mathrm{loc}}(\mathbb{R})$ and $t \in [0,1]$,*

$$f(\overline{M_t}) = f(0)I + \int_0^t E_{f(M)} d\Lambda + F_{f(M)} dA + F^*_{f(M)} dA^\dagger + H_{f(M)} ds, \tag{2.3.2}$$

where

$$\begin{aligned}
E_{f(M)}(s) &:= \int_{-\infty}^{\infty} \hat{f}(p)(e^{ip(\overline{M_s}+E_s)} - e^{ip\overline{M_s}}) dp, \\
F_{f(M)}(s) &:= \int_{-\infty}^{\infty} ip\hat{f}(p) \left(\int_0^1 e^{ip(1-u)\overline{M_s}} F_s e^{ipu(\overline{M_s}+E_s)} du \right) dp, \\
F^*_{f(M)}(s) &:= \int_{-\infty}^{\infty} ip\hat{f}(p) \left(\int_0^1 e^{ip(1-u)(\overline{M_s}+E_s)} F^*_s e^{ipu\overline{M_s}} du \right) dp, \\
H_{f(M)}(s) &:= \int_{-\infty}^{\infty} ip\hat{f}(p) \left(\int_0^1 e^{ip(1-u)\overline{M_s}} H_s e^{ipu\overline{M_s}} du \right) dp \\
&\quad + \int_{-\infty}^{\infty} (ip)^2 \hat{f}(p) \left(\int_0^1 \int_0^1 u e^{ip(1-u)\overline{M_s}} F_s e^{ipu(1-v)(\overline{M_s}+E_s)} F^*_s e^{ipuv\overline{M_s}} du dv \right) dp.
\end{aligned} \tag{2.3.3}$$

Again, the notational convention as for the quantum Duhamel formula is used and the integrals in (2.3.3) are in the strong sense. The above definition should be interpreted as the integrals in (2.3.3) and quantum stochastic integral in (2.3.2) all exist and that equality holds in (2.3.2). Note that since $\hat{f}(p)$ is $\overline{\hat{f}(-p)}$, we have $E_{f(M)}(s) = E_{f(M)}(s)^*$, $F^*_{f(M)}(s) = F_{f(M)}(s)^*$ and $H_{f(M)}(s) = H_{f(M)}(s)^*$. Thus if $M \in \mathcal{S}'_{\mathrm{sa}}$ satisfies the quantum Ito formula, then for all $f \in C^{2+}_{\mathrm{loc}}(\mathbb{R})$, $f(M) = (f(M_t))_{0 \le t \le 1}$ is a regular self-adjoint quantum semimartingale. It should also be noted that if $M \in \mathcal{S}'_{\mathrm{sa}}$ is such that $(p,t) \mapsto e^{ip\overline{M_t}}$ is measurable, then the right-hand side of (2.3.2) always exists.

When dealing with regular quantum semimartingales it can be deduced from the fact that M satisfies the quantum Duhamel formula, that it satisfies the quantum Ito formula. This is proved by using a Fubini type theorem [66, Theorem 6.1] to justify the change of integrals in (2.3.1) when the formula for e^{ipM_t} from (2.2.4) is substituted in (see [66, Theorem 6.2]). Given the slightly stronger definition of the quantum Duhamel formula in the unbounded case, we can also deduce that if $M \in \mathcal{S}'_{\rm sa}$ satisfies the quantum Duhamel formula, it satisfies the quantum Ito formula.

Lemma 2.3.2 *Let* $\{\mathtt{E}(p,s), \mathtt{F}(p,s), \mathtt{G}(p,s), \mathtt{H}(p,s) : p \in \mathbb{R},\ s \in [0,1]\}$ *be a set of densely defined operators, whose domains contain* $\mathsf{E}_{\rm lb}$ *for each* p *and* s *such that*

a) for all $f \in L^\infty[0,1]$ *the functions*

$$(p,s) \longmapsto \mathtt{E}(p,s)e(f), \quad (p,s) \longmapsto \mathtt{F}(p,s)e(f),$$
$$(p,s) \longmapsto \mathtt{G}(p,s)e(f), \quad (p,s) \longmapsto \mathtt{H}(p,s)e(f)$$

are Bochner integrable over $\mathbb{R} \times [0,1]$*,*

b) for almost all p*, the processes*

$$s \longmapsto \mathtt{E}(p,s), \quad s \longmapsto \mathtt{F}(p,s), \quad s \longmapsto \mathtt{G}(p,s), \quad s \longmapsto \mathtt{H}(p,s)$$

are adapted,

c) for all $s \in [0,1]$ *the Bochner integral over* $\mathbb{R}$ *of the functions*

$$p \longmapsto \mathtt{E}(p,s), \quad p \longmapsto \mathtt{F}(p,s), \quad p \longmapsto \mathtt{G}(p,s), \quad p \longmapsto \mathtt{H}(p,s)$$

are processes E*,* F*,* G *and* H *which satisfy (2.1.4),*

d) for almost all p*,*

$$\mathtt{M}(p,t) = \int_0^t \mathtt{E}(p,s)d\Lambda_s + \mathtt{F}(p,s)dA_s + \mathtt{G}(p,s)dA_s^\dagger + \mathtt{H}(p,s)ds$$

is a regular quantum semimartingale such that $p \mapsto \mathtt{M}(p,t)$ *is Bochner integrable for all* $t \in [0,1]$ *with integral* $M(t)$*.*

Then

$$M(t) = \int_0^t E(s)d\Lambda_s + F(s)dA_s + G(s)dA_s^\dagger + H(s)ds.$$

PROOF: Note that by Fubini's theorem, the Bochner integrals E, F, G and H exist for almost all s. Otherwise it is assumed that they are taken to be zero. Taking $\mathtt{F}(p,s)$ for example, by Fubini, for $f \in L^\infty[0,1]$,

$$s \longmapsto F(s)e(f) = \int_{-\infty}^{\infty} \mathtt{F}(p,s)e(f)dp$$

is strongly measurable. Hence, from the properties of Bochner integrals E, F, G and H are adapted processes. Condition c) ensures that (E, F, G, H) are suitable integrands. Suppose that $\mathtt{K}$ is one of the functions $\{\mathtt{E}, \mathtt{F}, \mathtt{G}, \mathtt{H}\}$ with corresponding $K \in \{E, F, G, H\}$ and integrator $A^\epsilon \in \{\Lambda, A, A^\dagger, s\}$. Let f, $g \in L^\infty[0,1]$ and choose the appropriate $h \in \{\bar{f}g, g, \bar{f}, 1\}$. Then

$$\begin{aligned}\int_0^t \langle e(f), h(s)K(s)e(g)\rangle ds &= \int_0^t \langle e(f), h(s)\int_{-\infty}^\infty \mathtt{K}(p,s)e(g)dp\rangle ds \\ &= \int_0^t \int_{-\infty}^\infty \langle e(f), h(s)\mathtt{K}(p,s)e(g)\rangle dpds \\ &= \int_{-\infty}^\infty \int_0^t \langle e(f), h(s)\mathtt{K}(p,s)e(g)\rangle dsdp \\ &= \int_{-\infty}^\infty \langle e(f), \int_0^t \mathtt{K}(p,s)dA_s^\epsilon e(g)\rangle dp.\end{aligned}$$

The interchange of the integrals is allowed by Fubini's theorem. Now let $N = N(E, F, G, H)$ be the quantum stochastic integral of (E, F, G, H). Then by (2.1.5),

$$\begin{aligned}\langle e(f), N_t e(g)\rangle &= \int_0^t \langle e(f), (\overline{f(s)}g(s)E(s) + g(s)F(s) + \overline{f(s)}G(s) + H(s))e(g)\rangle ds \\ &= \int_{-\infty}^\infty \langle e(f), \left(\int_0^t \mathtt{E}(p,s)d\Lambda_s + \mathtt{F}(p,s)dA_s + \mathtt{G}(p,s)dA_s^\dagger + \mathtt{H}(p,s)ds\right) e(g)\rangle dp \\ &= \langle e(f), M(t)e(g)\rangle.\end{aligned}$$

The last equality holds because we are assuming that the Bochner integral of $p \mapsto \mathtt{M}(p,t)$ exists. Since $\mathsf{E}_{\rm lb}$ is dense in $\mathfrak{F}_+(L^2[0,1])$ the result follows. □

Theorem 2.3.3 *If $M \in \mathcal{S}'_{\rm sa}$ satisfies the quantum Duhamel formula, then M satisfies the quantum Ito formula.*

Proof: We know from (2.3.1) that

$$f(\overline{M_t}) - f(0)I = \int_{-\infty}^\infty \hat{f}(p)(e^{ip\overline{M_t}} - I)dp.$$

Therefore, substituting in the formula for $e^{ip\overline{M_t}}$ we get

$$f(M_t) - f(0)I = \int_{-\infty}^\infty \left(\int_0^t \mathtt{E}(p,s)d\Lambda_s + \mathtt{F}(p,s)dA_s + \mathtt{G}(p,s)dA_s^\dagger + \mathtt{H}(p,s)ds\right) dp,$$

where

$$\begin{aligned}
\mathtt{E}(p,s) &= \hat{f}(p)(e^{ip(\overline{M_s}+E_s)} - e^{ip\overline{M_s}}),\\
\mathtt{F}(p,s) &= ip\hat{f}(p)\int_0^1 e^{ip(1-u)\overline{M_s}}F_s e^{ipu(\overline{M_s}+E_s)}du,\\
\mathtt{G}(p,s) &= ip\hat{f}(p)\int_0^1 e^{ip(1-u)(\overline{M_s}+E_s)}F_s^* e^{ipu\overline{M_s}}du,\\
\mathtt{H}(p,s) &= ip\hat{f}(p)\int_0^1 e^{ip(1-u)\overline{M_s}}H_s e^{ipu\overline{M_s}}du\\
&\quad + (ip)^2\hat{f}(p)\int_0^1\int_0^1 ue^{ip(1-u)\overline{M_s}}F_s e^{ipu(1-v)(\overline{M_s}+E_s)}F_s^* e^{ipuv\overline{M_s}}dudv.
\end{aligned}$$

If we can show that the family $\{\mathtt{E}(p,s), \mathtt{F}(p,s), \mathtt{G}(p,s), \mathtt{H}(p,s)\}$ satisfy the condition of Lemma 2.3.2 we obtain the result. We shall show this is true for $\mathtt{F}(p,s)$, the other cases following similarly. From the measurability condition in the quantum Duhamel formula, if

$$\tilde{f}(p,u,s) := ip\hat{f}(p)e^{ip(1-u)\overline{M_s}}F_s e^{ipu(\overline{M_s}+E_s)},$$

we know that for $\phi \in \mathfrak{F}_+(L^2[0,1])$,

$$(p,u,s) \longmapsto \tilde{f}(p,u,s)\phi$$

is measurable as a map $\mathbb{R}\times[0,1]\times[0,1] \to \mathfrak{F}_+(L^2[0,1])$. Furthermore

$$\|\tilde{f}(p,s,u)\phi\| \le |p||\hat{f}(p)|\|F_s\|\|\phi\|.$$

Therefore by Fubini's theorem

$$(p,s) \longmapsto \mathtt{F}(p,s)\phi = \int_0^1 \tilde{f}(p,s,u)\phi du$$

is measurable and

$$\|\mathtt{F}(p,s)\phi\| \le |p||\hat{f}(p)|\|F_s\|\|\phi\| \in L^1(\mathbb{R}\times[0,1]).$$

Consequently, $(p,s)\mapsto \mathtt{F}(p,s)\phi$ is Bochner integrable over $\mathbb{R}\times[0,1]$. Furthermore

$$\|F(s)\phi\| = \left\|\int_{-\infty}^{\infty}\left(\int_0^1 \tilde{f}(p,s,u)\phi du\right)dp\right\| \le \int_{-\infty}^{\infty}|p||\hat{f}(p)|dp\|F_s\|\|\phi\|.$$

Therefore condition c) is satisfied. The proofs for $\mathtt{E}$, $\mathtt{G}$ and $\mathtt{H}$ are similar. Also $p\mapsto \mathtt{M}(p,t)$, as defined in Lemma 2.3.2, is Bochner integrable with integral $f(\overline{M_t})$. Therefore, we may apply Lemma 2.3.2 to obtain the result. □

The quantum stochastic integrals considered in this chapter have been defined on $\mathfrak{F}_+(L^2[0,1])$. However, they can easily be constructed on $\mathfrak{F}_+(L^2(\mathbb{R}_+))$ instead. All the results concerning unbounded essentially self-adjoint semimartingales can be seen to hold in this case as well. When we look at perturbations of classical semimartingales in Chapter 6, we shall only consider integrals defined on $\mathfrak{F}_+(L^2[0,1])$ in order to simplify our proofs.

2.4 Convergence of quantum semimartingales

We finish this chapter by stating convergence results which will be useful later. A proof of the first result can be found in [67, Proposition 2.1].

Proposition 2.4.1 *Let $M = M(E, F, G, H)$ be a quantum semimartingale, and let $M^{(n)} = M^{(n)}(E^{(n)}, F^{(n)}, G^{(n)}, H^{(n)})$ be a sequence of regular quantum semimartingales such that*

i) $E_t^{(n)}$, $F_t^{(n)}$, $G_t^{(n)}$ *and* $H_t^{(n)}$ *converge strongly to* E_t, F_t, G_t *and* H_t *respectively, for almost all* $t \in [0,1]$,

ii) $\sup\{\|M_t^{(n)}\| : t \in [0,1], n = 1, 2, \ldots\} = K < \infty$,

iii) $\sup_n \|E_t^{(n)}\| = \alpha(t) \in L^\infty[0,1]$,

iv) $\sup_n(\|F_t^{(n)}\| + \|G_t^{(n)}\|) = \beta(t) \in L^2[0,1]$,

v) $\sup_n \|H_t^{(n)}\| = \gamma(t) \in L^1[0,1]$.

Then

a) M *is a regular quantum semimartingale with* $\|M_t\| \leq K$ *for all* $t \in [0,1]$,

b) $M_t^{(n)}$ *converges strongly to* M_t *for all* $t \in [0,1]$.

Some convergence results for self-adjoint operators on Hilbert spaces are now stated for completeness. The propositions below can be found in [56, Theorem VIII.25, Theorem VIII.21] and are useful since for a self-adjoint operator T they allow e^{ipT} to be approximated by e^{ipT_n} where T_n is a sequence of self-adjoint operators which converge in some way to T.

Proposition 2.4.2 *Let T_n and T be self-adjoint operators on a Hilbert space H and suppose that D is a common core for T and all T_n. If $T_n\phi \to T\phi$ for each $\phi \in D$, then $T_n \to T$ in the strong resolvent sense.*

Proposition 2.4.3 *Let T_n and T be self-adjoint operators on a Hilbert space H. Then $T_n \to T$ in the strong resolvent sense if and only if e^{ipT_n} converges strongly to e^{ipT} for each real p.*

Corollary 2.4.4 *Let T_n and T be self-adjoint operators on a Hilbert space H and suppose that D is a common core for T and all T_n. If $T_n\phi \to T\phi$ for each $\phi \in D$, then e^{ipT_n} converges strongly to e^{ipT} for each real p.*

Our approach to proving that a perturbation of a classical Poisson martingale, M say, satisfies the quantum Duhamel formula is by approximating the martingale by a sequence of self-adjoint regular quantum semimartingales, $M^{(n)}$, and then using Proposition 2.4.1. Corollary 2.4.4 gives the tool we need to show that the quantum Duhamel integrands of $M^{(n)}$ converge, as in the hypothesis of Proposition 2.4.1, to the quantum Duhamel integrands of M.

Chapter 3

The Poisson Interpretation of Fock Space

If $(N_t)_{t\geq 0}$ is a Poisson process, which exists on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we can construct an isomorphism

$$\mathcal{W} : \mathfrak{F}_+(L^2(\mathbb{R}_+)) \longrightarrow L^2(\Omega, \mathcal{F}, \mathbb{P}).$$

We begin this chapter by providing a brief outline of the construction of this map and examine some of its properties. Since this isomorphism is well-known, we omit most of the proofs. In the later sections we investigate the Poisson exponentials which are important in quantum stochastic calculus. We also look at the use of the isomorphism $\mathcal{W}$ in constructing examples of quantum semimartingales.

3.1 The Wiener-Poisson isomorphism

There are various different but equivalent definitions of a Poisson process. We use the definition which is most appropriate for our work.

Definition 3.1.1 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. We say that $(N_t)_{t\geq 0}$ is a Poisson process (with unit jump size and intensity 1) based on $(\Omega, \mathcal{F}, \mathbb{P})$ if*

i) $N_0 = 0$ a.s.,

ii) for $0 \leq s < t < \infty$, $N_t - N_s$ is a Poisson random variable with mean $t - s$, that is $N_t - N_s$ takes values in $\mathbb{N}_0$ such that

$$\mathbb{P}(N_t - N_s = n) = \frac{(t-s)^n e^{-(t-s)}}{n!},$$

iii) for $0 \le t_0 < t_1 < \ldots < t_n < \infty$,

$$\{N_{t_0}, N_{t_j} - N_{t_{j-1}} : j = 1, \ldots, n\}$$

is a set of independent random variables.

Given a Poisson process there exists a version such that a.s. the paths are increasing and are constant except for jumps of size 1, of which there are finitely many in each bounded interval and infinitely many in $[0, \infty)$ [19, §VIII.4]. For such a version, given any $t > 0$, the paths are a.s. continuous at t [22, Theorem 3.2 1)]. By taking the right limit we obtain a version of the Poisson process with the same properties as above, but with paths which are a.s. right continuous with left-hand limits. The version whose paths are a.s. right continuous with left-hand limits is referred to as the càdlàg version, and we shall assume that we are always using this version. This version is unique up to indistinguishability.

If $(N_t)_{t\ge 0}$ is a Poisson process, we define,

$$\begin{aligned}
\mathcal{F}_t &:= \sigma(\{N_u : u \le t\} \cup \mathfrak{N});\\
\mathcal{F}_{[t,s]} &:= \sigma(\{N_u - N_t : t \le u \le s\} \cup \mathfrak{N});\\
\mathcal{F}_{[t} &:= \sigma(\{N_u - N_t : t \le u\} \cup \mathfrak{N});\\
\mathcal{F}_{\infty} &:= \sigma(\{N_u : 0 \le u\} \cup \mathfrak{N}),
\end{aligned}$$

where $\mathfrak{N}$ is the collection of $\mathbb{P}$-null sets in Ω. The filtration $(\mathcal{F}_t)_{t\ge 0}$ satisfies the usual conditions (see [15, §2.3 Theorem 4]).

Definition 3.1.2 *Let* $(N_t)_{t\ge 0}$ *be a Poisson process based on* $(\Omega, \mathcal{F}, \mathbb{P})$*. Then the martingale* $\{(X_t)_{t\ge 0}, (\mathcal{F}_t)_{t\ge 0}\}$*, where* $X_t := N_t - t$*, is said to be a compensated Poisson process (with unit jump size and intensity 1).*

Notice that by the properties of our version of $(N_t)_{t\ge 0}$ mentioned above, $(X_t)_{t\ge 0}$ is a process of finite variation. We also note that $L^2(\Omega, \mathcal{F}_t, \mathbb{P}) = L^2(\Omega, \sigma(X_u : u \le t), \mathbb{P})$ (see [15, §2.3 Lemma]), similarly for $\mathcal{F}_{[t,s]}$, $\mathcal{F}_{[t}$ and $\mathcal{F}_{\infty}$.

Suppose $\{(Y_t)_{t\ge 0}, (\mathcal{G}_t)_{t\ge 0}\}$ is a square integrable martingale. By the *first increasing process* of $(Y_t)_{t\ge 0}$, we mean the process $(\langle Y, Y\rangle_t)_{t\ge 0}$, which is the unique predictable, right continuous, increasing process such that $(Y_t^2 - \langle Y, Y\rangle_t)_{t\ge 0}$ is a martingale [44, Proposition 17.2]. The process $([Y, Y]_t)_{t\ge 0}$ is called the *quadratic variation process* of $(X_t)_{t\ge 0}$. It is the càdlàg increasing process such that for any $t \ge 0$ and any sequence of partitions of $[0, t]$,

$$\pi_n : 0 = t_0^n < t_1^n < \ldots < t_{k(n)}^n = t,$$

with

$$\delta\pi_n = \sup_{1\le j\le k(n)} |t_j^n - t_{j-1}^n| \to 0 \quad \text{as } n\to\infty,$$

we have

$$[Y,Y]_t = \operatorname*{problim}_{n\to\infty} \sum_{j=1}^{k(n)} (Y_{t_j^n} - Y_{t_{j-1}^n})^2$$

(see [44, Theorem 18.6]). From the classical Ito product formula we know that

$$[Y,Y]_t = Y_t^2 + 2\int_0^t Y_{s-}dY_s,$$

which is often used as an alternative definition of the quadratic variation process.

If $(X_t)_{t\geq 0}$ is a compensated Poisson process, then it follows from the independence of the increments that $(X_t^2 - t)_{t\geq 0}$ is a martingale and hence $\langle X,X\rangle_t = t$. Given $t > 0$ consider the sequence of partitions

$$\pi_n : 0 < \frac{t}{n} < \ldots < \frac{(n-1)t}{n} < t.$$

Then $\delta\pi_n \to 0$ as $n\to\infty$. If $\omega\in\Omega$ is such that $N_t(\omega)$ is monotone increasing with only finitely many jumps in $[0,t]$, then for large n, since each subinterval $((j-1)t/n, jt/n]$ of the partition has at most one jump and the number of such subintervals with jumps is $N_t(\omega)$,

$$\begin{aligned}\sum_{j=1}^n (X_{jt/n}(\omega) - X_{(j-1)t/n}(\omega))^2 &= \sum_{j=1}^n (N_{jt/n}(\omega) - N_{(j-1)t/n}(\omega) - \frac{t}{n})^2\\ &= \sum_{j=1}^n (N_{jt/n}(\omega) - N_{(j-1)t/n}(\omega))^2\\ &\qquad - 2\sum_{j=1}^n (N_{jt/n}(\omega) - N_{(j-1)t/n}(\omega))\frac{t}{n} + \frac{t^2}{n}\\ &= N_t(\omega) - 2\frac{t}{n}N_t(\omega) + \frac{t^2}{n}.\end{aligned}$$

Since for the version of the Poisson process we are using the paths are a.s. increasing with only finitely many jumps of size 1 in $[0,t]$, we have that a.s.

$$\sum_{j=1}^n (X_{jt/n} - X_{(j-1)t/n})^2 \to N_t \quad \text{as } n\to\infty.$$

Hence for a compensated Poisson process

$$[X,X]_t = t + X_t.$$

This is equation (1.1.2) with $\Psi_s = 1$.

We now briefly describe the isomorphism between $\bigoplus_{n=0}^{\infty} L^2(D_n)$ and $L^2(\Omega, \mathcal{F}, \mathbb{P})$, where the simplex

$$D_n := \{(t_1, \ldots, t_n) \in \mathbb{R}^n : 0 \leq t_1 < \ldots < t_n\} \tag{3.1.1}$$

is endowed with Lebesgue measure. This isomorphism is often called the *Poisson interpretation of Fock space*. The following theorem can be found in [45].

Theorem 3.1.3 *Let $(X_t)_{t\geq 0}$ be a compensated Poisson process on $(\Omega, \mathcal{F}, \mathbb{P})$. Then there exists a unique isometry*

$$J = \bigoplus_{n=0}^{\infty} J^{(n)} : \bigoplus_{n=0}^{\infty} L^2(D_n) \longrightarrow L^2(\Omega, \mathcal{F}, \mathbb{P}),$$

such that if

$$f(t_1, \ldots, t_n) = \mathbb{1}_{(a_1,b_1]}(t_1) \ldots \mathbb{1}_{(a_n,b_n]}(t_n),$$

with $0 \leq a_1 \leq b_1 \leq \ldots \leq a_n \leq b_n$, then

$$J^{(n)}(f) = (X_{b_1} - X_{a_1}) \ldots (X_{b_n} - X_{a_n}). \tag{3.1.2}$$

Notice that for $m \neq n$, $J^{(m)}(L^2(D_m))$ is orthogonal to $J^{(n)}(L^2(D_n))$. In fact we shall consider a slightly modified map. If $L^2_{\text{sym}}(\mathbb{R}^n_+)$ denotes the set of symmetric square integrable functions on $\mathbb{R}^n_+$, then the map $\phi^{(n)} : f \mapsto \sqrt{n!}f|_{D_n}$ is a surjective isometric isomorphism $\phi^{(n)} : L^2_{\text{sym}}(\mathbb{R}^n_+) \to L^2(D_n)$. If we let $\Phi = \bigoplus_{n=0}^{\infty} \phi^{(n)}$, the composition

$$\mathcal{W} := J \circ \Phi : \bigoplus_{n=0}^{\infty} L^2_{\text{sym}}(\mathbb{R}^n_+) \longrightarrow L^2(\Omega, \mathcal{F}, \mathbb{P})$$

is an isometry. We define $I^{(n)} := J^{(n)} \circ \phi^{(n)}$, and therefore $\mathcal{W} = \bigoplus_{n=0}^{\infty} I^{(n)}$. This map is preferred since it does not use the order structure of $\mathbb{R}_+$ and can be generalised to a more general measure space (see Section 4.2). Since $L^2_{\text{sym}}(\mathbb{R}^n_+)$ is a concrete realisation of the symmetric tensor product $L^2(\mathbb{R}_+)^{\otimes_s^n}$ given by

$$(f_1 \otimes_s \ldots \otimes_s f_n)(t_1, \ldots, t_n) = \frac{1}{n!} \sum_{\sigma \in S_n} f_1(t_{\sigma(1)}) \ldots f_n(t_{\sigma(n)}),$$

where $f_1, \ldots, f_n \in L^2(\mathbb{R}_+)$, we can say $\mathfrak{F}_+(L^2(\mathbb{R}_+)) \cong \bigoplus_{n=0}^{\infty} L^2_{\text{sym}}(\mathbb{R}^n_+)$. Consequently we actually have an isomorphism

$$\mathcal{W} : \mathfrak{F}_+(L^2(\mathbb{R}_+)) \longrightarrow L^2(\Omega, \mathcal{F}, \mathbb{P}),$$

which we shall call the *Wiener-Poisson isomorphism*.

Recall that, given $f \in L^2(\mathbb{R}_+)$, the exponential vector of f, $e(f) \in \mathfrak{F}_+(L^2(\mathbb{R}_+))$, is defined by

$$e(f) := (1, f, \frac{f^{\otimes 2}}{\sqrt{2!}}, \dots, \frac{f^{\otimes n}}{\sqrt{n!}}, \dots).$$

When considered as an element of $L^2_{\text{sym}}(\mathbb{R}^n_+)$, $f^{\otimes n}(t_1, \dots, t_n) = f(t_1) \dots f(t_n)$. As quantum stochastic integrals are defined on exponential vectors, we need to consider the random variables $\mathcal{W}(e(f))$ in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. From [46, §II.2] (see also [44, Theorem 29.2]), we have the result below.

Theorem 3.1.4 *Let $f \in L^2(\mathbb{R}_+)$, and define $f_t := \mathbb{1}_{[0,t]} f$. There is a unique version $(Z_t)_{t \geq 0}$ of $(\mathcal{W}(e(f_t)))_{t \geq 0}$ which satisfies*

$$Z_t = 1 + \int_0^t f(s) Z_{s-} dX_s \quad \textit{for all } t \geq 0. \tag{3.1.3}$$

Moreover if $\mathcal{E}(f) := \mathcal{W}(e(f))$ and $\int_0^\infty f(s) dX_s := \mathcal{W}(f)$, we have

$$\mathcal{E}(f) = \exp\{\int_0^\infty f(s) dX_s\} \prod_{s \geq 0} (1 + f(s) \Delta X_s) e^{-f(s) \Delta X_s} \quad \textit{a.s.}. \tag{3.1.4}$$

The Z_{s-} in (3.1.3) can be replaced by Z_s, because we can integrate adapted measurable, as well as predictable, square integrable processes with respect to the compensated Poisson process (see [16, §3.3]). We prefer to retain the suffix $s-$ since it is consistent with the notation used for other square integrable martingales. Notice that $\overline{\mathcal{E}(f)} = \mathcal{E}(\overline{f})$ and that the version of $(Z_t)_{t \geq 0}$ satisfying the exponential equation (3.1.3) is the unique càdlàg version of the martingale $(\mathcal{E}(f_t))_{t \geq 0}$ (see [57, Theorem II.2.9]). If $f \in L^2(\mathbb{R}_+)$ is locally bounded with compact support, then

$$\int_0^\infty f(s) dX_s = \sum_{s \geq 0} f(s) \Delta X_s - \int_0^\infty f(s) ds,$$

because in this case the stochastic integral agrees a.s. with the Lebesgue-Stieltjes integral [44, Theorem 24.4 3°]. Consequently, for all such f we can write

$$\mathcal{E}(f) = \exp\{-\int_0^\infty f(s) ds\} \prod_{s \geq 0} (1 + f(s) \Delta X_s). \tag{3.1.5}$$

The following result will be useful later on, since it shows that if $t_j \in \mathbb{R}_+$ and $\alpha_j \in \mathbb{R}$ for $j = 1, \dots, n$, $\exp\{i \sum_{j=1}^n \alpha_j X_{t_j}\}$ is in the linear span of the exponential vectors.

Lemma 3.1.5 *Let $\alpha = \sum_{j=1}^{n} \alpha_j \mathbb{1}_{(a_j,b_j]}$ where $\alpha_j \in \mathbb{R}$ and $0 \le a_j \le b_j$. Then*

$$\mathcal{E}(e^{i\alpha} - 1) = \exp\{-\int_0^\infty (e^{i\alpha} - 1 - i\alpha)ds + i\sum_{j=1}^{n} \alpha_j (X_{b_j} - X_{a_j})\}. \tag{3.1.6}$$

PROOF: Notice that $e^{i\alpha} - 1$ equals $e^{i\alpha_j} - 1$ on $(a_j, b_j]$ and is zero outside these intervals. Therefore by (3.1.5), we have

$$\begin{aligned}
\mathcal{E}(e^{i\alpha} - 1) &= \exp\{-\int_0^\infty (e^{i\alpha} - 1)ds\} \prod_{s\ge 0} (1 + (e^{i\alpha(s)} - 1)\Delta X_s) \\
&= \exp\{-\int_0^\infty (e^{i\alpha} - 1)ds\} \prod_{s\ge 0, \Delta X_s \ne 0} e^{i\alpha(s)} \\
&= \exp\{-\int_0^\infty (e^{i\alpha} - 1)ds\} \exp\{i\sum_{s\ge 0} \alpha(s)\Delta X_s\} \\
&= \exp\{-\int_0^\infty (e^{i\alpha} - 1)ds\} \exp\{i\sum_{j=0}^{n} \alpha_j (N_{b_j} - N_{a_j})\}.
\end{aligned}$$

Since $X_t = N_t - t$, (3.1.6) follows. □

Corollary 3.1.6 *If $t_j \in \mathbb{R}_+$ for $j = 1, \ldots, n$,*

$$\exp\{i\sum_{j=1}^{n} \alpha_j X_{t_j}\} \in \text{linsp}\{\mathcal{E}(e^{i\alpha} - 1) : \alpha \in L^{\text{step}}_{\mathbb{R}}(\mathbb{R}_+)\}$$

for all $\alpha_j \in \mathbb{R}$.

PROOF: Taking $\alpha = \sum_{j=1}^{n} \alpha_j \mathbb{1}_{(0,t_j]}$ in Lemma 3.1.5, because $X_0 = 0$,

$$\mathcal{E}(e^{i\alpha} - 1) = \exp\{-\int_0^\infty (e^{i\alpha} - 1 - i\alpha)ds\} \exp\{i\sum_{j=1}^{n} \alpha_j X_{t_j}\}.$$

Therefore the claim holds. □

The isometry $\mathcal{W}$ actually maps $\mathfrak{F}_+(L^2(\mathbb{R}_+))$ onto $L^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$. This can be proved in various ways, but in [21, Proposition 4], Émery shows that the exponential vectors of Lemma 3.1.5 are total in $L^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$ to prove surjectivity. It should be noted that he uses his 'Émery Ito formula' to obtain the expression (3.1.6).

Theorem 3.1.7 *The map $\mathcal{W} : \mathfrak{F}_+(L^2(\mathbb{R}_+)) \to L^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$ is surjective. Under this map we can identify*

$$\mathfrak{F}_+(L^2[0,t]) \cong L^2(\Omega, \mathcal{F}_t, \mathbb{P});$$
$$\mathfrak{F}_+(L^2[t,s]) \cong L^2(\Omega, \mathcal{F}_{[t,s]}, \mathbb{P});$$
$$\mathfrak{F}_+(L^2[t,\infty)) \cong L^2(\Omega, \mathcal{F}_{[t}, \mathbb{P}).$$

The identification of the appropriate subspaces can be obtained by the same argument as in [21, Proposition 4] by restricting the functions $u \in L^2(\mathbb{R}_+)$ in the proof to having supports in the appropriate intervals.

3.2 The Poisson exponentials

From now on we shall restrict ourselves to working over $L^2[0,1]$ since our quantum stochastic integrals will always be defined on $\mathfrak{F}_+(L^2[0,1])$, and consider the isomorphism

$$\mathcal{W} : \mathfrak{F}_+(L^2[0,1]) \longrightarrow L^2(\Omega, \mathcal{F}_1, \mathbb{P}).$$

There will be no confusion in denoting $\mathcal{F}_1$ by $\mathcal{F}$ and $L^p(\Omega, \mathcal{F}, \mathbb{P})$ by $L^p(\Omega)$. Although we are restricting ourselves to $\mathfrak{F}_+(L^2[0,1])$, all the arguments can be transformed to $L^2(\mathbb{R}_+)$ by making appropriate modifications (see Section 3.4). In this section we look at some of the properties of the exponential vectors in $L^2(\Omega)$, which will be useful later on and compare them to the corresponding properties of the exponentials in the Wiener interpretation of Fock space. Recall from (1.2.2) that in the Wiener interpretation of Fock space, that is the isometric isomorphism $\mathfrak{w} : \mathfrak{F}_+(L^2[0,1]) \to L^2(\Omega_\mathfrak{w}, \mathcal{F}_\mathfrak{w}, \mathbb{P}_\mathfrak{w})$, where a version of $(\mathfrak{w}(\mathbb{1}_{[0,t]}))_{0\le t\le 1}$ gives Brownian motion,

$$\mathcal{E}_\mathfrak{w}(f) = \exp\{\mathfrak{w}(f) - \frac{1}{2}\int_0^1 f(s)^2 ds\}, \tag{3.2.1}$$

where $\mathcal{E}_\mathfrak{w}(f) := \mathfrak{w}(e(f))$. More details can be found in [34, §III.2].

We begin with a lemma that is immediate from (3.1.4), which also holds in the Wiener case.

Lemma 3.2.1 *Let f, $g \in L^2[0,1]$ have disjoint supports. Then $\mathcal{E}(f)\mathcal{E}(g) = \mathcal{E}(f+g)$.*

Recall that the Hilbert space tensor product, $\mathfrak{F}_+(L^2[0,t]) \otimes \mathfrak{F}_+(L^2(t,1])$ is the completion of the algebraic tensor product $\mathfrak{F}_+(L^2[0,t]) \underline{\otimes} \mathfrak{F}_+(L^2(t,1])$ with respect to the norm induced by the inner product

$$\langle \phi_1 \otimes \phi_2, \psi_1 \otimes \psi_2 \rangle := \langle \phi_1, \psi_1 \rangle \langle \phi_2, \psi_2 \rangle.$$

Furthermore,

$$\mathcal{T} : \mathfrak{F}_+(L^2[0,t]) \times \mathfrak{F}_+(L^2(t,1]) \longrightarrow \mathfrak{F}_+(L^2[0,t]) \underline{\otimes} \mathfrak{F}_+(L^2(t,1])$$

given by

$$\mathcal{T} : (\phi, \psi) \longmapsto \phi \otimes \psi$$

is a bounded bilinear map. The Hudson-Parthasarathy construction of quantum stochastic integrals relies heavily on the identification $\mathfrak{F}_+(L^2[0,t]) \otimes \mathfrak{F}_+(L^2(t,1]) \cong \mathfrak{F}_+(L^2[0,1])$, via the isomorphism $\mathfrak{u}_t$ (see (2.1.2)). Let

$$\tilde{\mathfrak{u}}_t := \mathfrak{u}_t|_{\mathfrak{F}_+(L^2[0,t])\underline{\otimes}\mathfrak{F}_+(L^2(t,1])} \circ \mathcal{T} : \mathfrak{F}_+(L^2[0,t]) \times \mathfrak{F}_+(L^2(t,1]) \longrightarrow \mathfrak{F}_+(L^2[0,1]). \qquad (3.2.2)$$

Note that $\tilde{\mathfrak{u}}_t$ is a continuous bilinear map. From Lemma 3.2.1 above we can easily deduce the well-known fact that the isomorphism $\mathcal{W}$ transforms $\tilde{\mathfrak{u}}_t$ into ordinary multiplication. The exact same result holds in the Wiener case.

Proposition 3.2.2 *Let $\mathfrak{m}_t : L^2(\Omega, \mathcal{F}_t) \times L^2(\Omega, \mathcal{F}_{[t,1]}) \to L^2(\Omega, \mathcal{F})$ be the multiplication map $(X, Y) \mapsto XY$. Then the diagram*

$$\begin{array}{ccc} \mathfrak{F}_+(L^2[0,t]) \times \mathfrak{F}_+(L^2(t,1]) & \xrightarrow{\tilde{\mathfrak{u}}_t} & \mathfrak{F}_+(L^2[0,1]) \\ {\scriptstyle \mathcal{W}\times\mathcal{W}}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{W}} \\ L^2(\Omega, \mathcal{F}_t) \times L^2(\Omega, \mathcal{F}_{[t,1]}) & \xrightarrow{\mathfrak{m}_t} & L^2(\Omega, \mathcal{F}) \end{array}$$

commutes, that is $\mathfrak{m}_t(\mathcal{W} \times \mathcal{W}) = \mathcal{W}\tilde{\mathfrak{u}}_t$.

PROOF: Note that $\mathfrak{m}_t$ is a continuous map. If $\phi \in \mathfrak{F}_+(L^2[0,t])$ and $\psi \in \mathfrak{F}_+(L^2(t,1])$,

$$\mathfrak{m}_t(\mathcal{W} \times \mathcal{W}) : (\phi, \psi) \longmapsto \mathcal{W}\phi\mathcal{W}\psi.$$

From Lemma 3.2.1, if $f \in L^2[0,t]$ and $g \in L^2(t,1]$,

$$\begin{aligned} \mathcal{W}\tilde{\mathfrak{u}}_t(e(f), e(g)) &= \mathcal{W}(e(f+g)) = \mathcal{E}(f+g) \\ &= \mathcal{E}(f)\mathcal{E}(g) = \mathcal{W}(e(f))\mathcal{W}(e(g)) \\ &= \mathfrak{m}_t(\mathcal{W} \times \mathcal{W})(e(f), e(g)). \end{aligned}$$

Thus, since $\{e(f) : f \in L^2[0,t]\}$ and $\{e(f) : f \in L^2(t,1]\}$ are total in $\mathfrak{F}_+(L^2[0,t])$ and $\mathfrak{F}_+(L^2(t,1])$ respectively, and the maps $\mathfrak{m}_t(\mathcal{W} \times \mathcal{W})$ and $\mathcal{W}\tilde{\mathfrak{u}}_t$ are both continuous, the result follows. $\square$

Given a random variable on any probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, the corresponding multiplication operator on $L^2(\Omega')$ will be important later on. For clarity we make the following definition.

Definition 3.2.3 *Let X be a complex random variable on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$. Then we denote by $\widehat{X}$ the operator of multiplication by X on $L^2(\Omega')$. That is, $\widehat{X} : D(\widehat{X}) \to L^2(\Omega')$, where*

$$D(\widehat{X}) := \{Y \in L^2(\Omega') : XY \in L^2(\Omega')\},$$

such that

$$\widehat{X}(Y)(\omega) = X(\omega)Y(\omega).$$

We shall also use the notation $X\widehat{\ }$ to denote the operator of multiplication by X.

$\widehat{X}$ is a closed operator and from [56, §VIII.3 Proposition 1], if X is real-valued then $\widehat{X}$ is a self-adjoint operator on $L^2(\Omega')$. We shall now return to the Poisson space $(\Omega, \mathcal{F}, \mathbb{P})$.

Lemma 3.2.4 *Suppose $(F_t)_{0\le t\le 1}$ is an adapted, measurable process, and let $\widehat{F_t}$ be the operator of multiplication by F_t on $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Then if*

$$\mathcal{D}(\widehat{F_t}) := D(\widehat{F_t}) \cap L^2(\Omega, \mathcal{F}_t, \mathbb{P}) = \{X \in L^2(\Omega, \mathcal{F}_t) : F_t X \in L^2(\Omega, \mathcal{F}_t, \mathbb{P})\},$$

we have that $\mathfrak{u}_t(\mathcal{W}^{-1}(\mathcal{D}(\widehat{F_t}))\underline{\otimes}\mathfrak{F}_+(L^2(t,1])) \subseteq D(\mathcal{W}^{-1}\widehat{F_t}\mathcal{W})$. Moreover, if $\phi \in \mathcal{W}^{-1}(\mathcal{D}(\widehat{F_t}))$ and $\psi \in \mathfrak{F}_+(L^2(t,1])$,

$$\mathcal{W}^{-1}\widehat{F_t}\mathcal{W}(\mathfrak{u}_t(\phi \otimes \psi)) = \mathfrak{u}_t(\mathcal{W}^{-1}\widehat{F_t}\mathcal{W}\phi \otimes \psi),$$

that is on $\mathcal{W}^{-1}(\mathcal{D}(\widehat{F_t}))\underline{\otimes}\mathfrak{F}_+(L^2(t,1])$,

$$\mathcal{W}^{-1}\widehat{F_t}|_{\mathcal{D}(\widehat{F_t})}\mathcal{W} \otimes I = \mathfrak{u}_t^{-1}(\mathcal{W}^{-1}\widehat{F_t}\mathcal{W})\mathfrak{u}_t.$$

Proof: Suppose $\phi \in \mathcal{W}^{-1}(\mathcal{D}(\widehat{F_t}))$ and $\psi \in \mathfrak{F}_+(L^2(t,1])$. Then by Proposition 3.2.2, $\mathcal{W}\mathfrak{u}_t(\phi\otimes\psi) = \mathcal{W}\tilde{\mathfrak{u}}_t(\phi,\psi) = \mathcal{W}\phi\mathcal{W}\psi$. By the independence of $\mathcal{F}_t$ and $\mathcal{F}_{[t,1]}$, $\mathcal{W}\phi\mathcal{W}\psi \in D(\widehat{F_t})$, therefore $\mathfrak{u}_t(\phi \otimes \psi) \in D(\mathcal{W}^{-1}\widehat{F_t}\mathcal{W})$, and

$$\begin{aligned}
\mathcal{W}^{-1}\widehat{F_t}\mathcal{W}(\mathfrak{u}_t(\phi \otimes \psi)) &= \mathcal{W}^{-1}(F_t\mathcal{W}\phi\mathcal{W}\psi) \\
&= \mathcal{W}^{-1}\mathfrak{m}_t(\mathcal{W} \times \mathcal{W})(\mathcal{W}^{-1}F_t\mathcal{W}\phi, \psi) \\
&= \mathcal{W}^{-1}\mathcal{W}\tilde{\mathfrak{u}}_t(\mathcal{W}^{-1}\widehat{F_t}\mathcal{W}\phi, \psi) \\
&= \mathfrak{u}_t(\mathcal{W}^{-1}\widehat{F_t}\mathcal{W}\phi \otimes \psi).
\end{aligned}$$

Since $\mathcal{W}^{-1}(D_t)\underline{\otimes}\mathfrak{F}_+(L^2(t,1])$ is spanned by elements of the form $\phi \otimes \psi$, we arrive at the result. □

The above result can be interpreted as saying that the operator of multiplication by F_t on $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is the ampliation of the operator of multiplication by F_t on $L^2(\Omega, \mathcal{F}_t, \mathbb{P})$. Using this result we now construct quantum semimartingales from classical stochastic processes. We refer the reader to [16, Chapters 2,3] and [38, Chapter 3] for definitions and results for classical processes.

Theorem 3.2.5 *Let $(F_t)_{0\le t\le 1}$ be a bounded, adapted, measurable process on $(\Omega, \mathcal{F}, \mathbb{P})$. If $\widehat{F_t}$ is the operator of multiplication by F_t on $L^2(\Omega, \mathcal{F}, \mathbb{P})$, then $(\mathcal{W}^{-1}\widehat{F_t}\mathcal{W})_{0\le t\le 1}$ is an adapted process of bounded operators such that $t \mapsto \|\mathcal{W}^{-1}\widehat{F_t}\mathcal{W}\|_{\mathfrak{B}(\mathfrak{F}_+(L^2[0,1]))}$ is in $L^\infty[0,1]$. Hence if $(H_t)_{0\le t\le 1}$ is another bounded, adapted, measurable process and*

$$M_t = \int_0^t \mathcal{W}^{-1}\widehat{F_s}\mathcal{W}(d\Lambda_s + dA_s + dA_s^\dagger) + \int_0^t \mathcal{W}^{-1}\widehat{H_s}\mathcal{W}ds, \tag{3.2.3}$$

then $(M_t)_{0\le t\le 1}$ is a quantum semimartingale.

Proof: Since F_t is bounded for each t, $\widehat{F_t}$ is a bounded operator on $L^2(\Omega)$. Let $f \in L^2[0,1]$ be bounded. If $\mathcal{D}(\widehat{F_t})$ is as in Lemma 3.2.4, $\mathcal{D}(\widehat{F_t}) = L^2(\Omega, \mathcal{F}_t)$, and hence by Lemma 3.2.4, since $\mathfrak{u}_t(e(f_t) \otimes e(f\mathbb{1}_{(t,1]})) = e(f)$, we have $e(f) \in D(\mathcal{W}^{-1}\widehat{F_t}\mathcal{W})$ and

$$\mathcal{W}^{-1}\widehat{F_t}\mathcal{W}(e(f)) = \mathfrak{u}_t(\mathcal{W}^{-1}\widehat{F_t}\mathcal{W}(e(f_t)) \otimes e(f\mathbb{1}_{(t,1]})).$$

Therefore to show that the process $(\mathcal{W}^{-1}\widehat{F_t}\mathcal{W})_{0\le t\le 1}$ is adapted, we only need to show strong measurability of $t \mapsto \widehat{F_t}\mathcal{E}(f)$. Since $L^2(\Omega)$ is separable, by the Pettis measurability theorem [17, Chapter 2, Theorem 1.2], we only need to show that for all $X \in L^2(\Omega)$,

$$t \longmapsto \int_\Omega XF_t\mathcal{E}(f)d\mathbb{P}$$

is measurable. This is true due to Fubini's theorem. The final part follows immediately because $|F_t| \le K$ for some constant K, thus $\|\widehat{F_t}\|_{\mathfrak{B}(L^2(\Omega))} \le K$ for all $t \in [0,1]$ and therefore the map $t \mapsto \|\mathcal{W}^{-1}\widehat{F_t}\mathcal{W}\|_{\mathfrak{B}(\mathfrak{F}_+(L^2[0,1]))} = \|\widehat{F_t}\|_{\mathfrak{B}(L^2(\Omega))}$ is in $L^\infty[0,1]$. □

In the Wiener interpretation of Fock space, from the definition of the Wiener exponential (see (3.2.1)), we obtain the multiplication formula

$$\mathcal{E}_{\mathfrak{w}}(f)\mathcal{E}_{\mathfrak{w}}(g) = e^{\langle \bar{f}, g\rangle}\mathcal{E}_{\mathfrak{w}}(f+g),$$

for all f, $g \in L^2[0,1]$. As mentioned in [47, §IV.3.6], we do not have such an elegant multiplication formula in the Poisson case, and certain conditions on f and g are required. In order that an analogous formula holds in the Poisson case we shall usually work with bounded functions. We let

$$\mathcal{E} := \text{linsp}\{\mathcal{E}(f) : f \in L^2[0,1]\};$$
$$\mathcal{E}_{\text{lb}} := \text{linsp}\{\mathcal{E}(f) : f \in L^\infty[0,1]\}.$$

As previously mentioned the corresponding spaces in $\mathfrak{F}_+(L^2[0,1])$ are represented by E and E_{lb} respectively. We shall work over $\mathcal{E}_{\text{lb}}$, because a multiplication formula for exponentials of bounded functions can be proved. The following formula can also be proved using Yor's multiplication formula for stochastic exponentials, $\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X+Y+[X,Y])$ (see [75, §2.1 Proposition 4]), however we prefer to make use of the explicit formula for Poisson exponentials instead, because by using this approach we do not need to use the Doléans formula for a general exponential semimartingale. It should be noted that the multiplication formula for Poisson exponentials holds for f, $g \in L^2[0,1]$ satisfying weaker conditions than boundedness. However in the work we do, boundedness is the natural condition to impose.

Proposition 3.2.6 *Let f, $g \in L^\infty[0,1]$. Then*

$$\mathcal{E}(f)\mathcal{E}(g) = e^{\langle \bar{f}, g\rangle}\mathcal{E}(f+g+fg). \tag{3.2.4}$$

PROOF: Notice that as f, $g \in L^\infty[0,1]$, $fg \in L^\infty[0,1]$, and hence the right-hand side of (3.2.4) is well-defined. By (3.1.5) we have

$$\begin{aligned}
\mathcal{E}(f)\mathcal{E}(g) &= \exp\{-\int_0^\infty f(s)ds\}\prod_{s\geq 0}(1+f(s)\Delta X_s) \\
&\quad \times \exp\{-\int_0^\infty g(s)ds\}\prod_{s\geq 0}(1+g(s)\Delta X_s) \\
&= \exp\{-\int_0^\infty (f(s)+g(s))ds\} \\
&\quad \times \prod_{s\geq 0}(1+f(s)\Delta X_s + g(s)\Delta X_s + f(s)g(s)(\Delta X_s)^2) \\
&= e^{\langle \bar{f}, g\rangle}\exp\{-\int_0^\infty (f(s)+g(s)+f(s)g(s))ds\} \\
&\quad \times \prod_{s\geq 0} 1 + (f(s)+g(s)+f(s)g(s))\Delta X_s),
\end{aligned}$$

where the last equality comes from the fact that $\Delta X_s = 0$ or 1. As $fg \in L^\infty[0,1]$ we have the required formula. □

Corollary 3.2.7 $\mathcal{E}_{\text{lb}} \subseteq L^p(\Omega)$ *for all* $1 \le p < \infty$.

PROOF: We prove this by induction. Let $f \in L^\infty[0,1]$ and assume $\mathcal{E}(f)^n = k_n \mathcal{E}(g_n)$ for some constant k_n and some $g_n \in L^\infty[0,1]$. This is clearly true when $n = 0$. Then by (3.2.4) we have

$$\begin{aligned}
\mathcal{E}(f)^{n+1} &= \mathcal{E}(f)^n \mathcal{E}(f) \\
&= k_n \mathcal{E}(g_n)\mathcal{E}(f) \\
&= k_n e^{\langle \bar{f}, g_n \rangle} \mathcal{E}(f + g_n + f g_n) \\
&= k_{n+1} \mathcal{E}(g_{n+1})
\end{aligned}$$

where $k_{n+1} = k_n e^{\langle \bar{f}, g_n \rangle}$ and $g_{n+1} = f + g_n + f g_n$. Therefore the induction is valid and thus $\mathcal{E}(f)^n \in L^2(\Omega)$ for all $n \ge 0$. Hence $\mathcal{E}_{\text{lb}} \subseteq L^p(\Omega)$ for all $1 \le p < \infty$. □

Corollary 3.2.8 *If* $f \in L^\infty[0,1]$, *then*

$$\mathcal{E}(f)^2 = e^{\langle \bar{f}, f \rangle} \mathcal{E}(2f + f^2)$$

and therefore

$$\mathbb{E}[|\mathcal{E}(f)|^4] = e^{2\text{Re}\langle \bar{f}, f \rangle + \|2f + f^2\|^2}.$$

PROOF: We have from (3.2.4) that $\mathcal{E}(f)^2 = e^{\langle \bar{f}, f \rangle} \mathcal{E}(2f + f^2)$. Therefore, using the fact that $\langle \mathcal{E}(g), \mathcal{E}(h) \rangle = e^{\langle g, h \rangle}$ for all g, $h \in L^2[0,1]$,

$$\begin{aligned}
\mathbb{E}[|\mathcal{E}(f)|^4] &= e^{2\text{Re}\langle \bar{f}, f \rangle} \langle \mathcal{E}(2f + f^2), \mathcal{E}(2f + f^2) \rangle \\
&= e^{2\text{Re}\langle \bar{f}, f \rangle} e^{\|2f + f^2\|^2},
\end{aligned}$$

which gives the required formula. □

In fact $\mathcal{E}_{\text{lb}}$ can be shown to be dense in $L^p(\Omega)$ for $1 \le p < \infty$. Our proof of this is similar to that in [21, Proposition 4 iii)] for the case $L^2(\Omega)$, however we prove a more general result (see Proposition 3.2.10) using the approach of [34, Lemma 2.7]. When considering the Wiener case, since the exponential multiplication formula holds for all f, $g \in L^2[0,1]$, or by direct calculation, we obtain $\mathcal{E} \subseteq L^p(\Omega)$, and consequently the following results can be stated for $\mathcal{E}$ instead of $\mathcal{E}_{\text{lb}}$.

Lemma 3.2.9 *Suppose* $X \in L^1(\Omega, \mathcal{F}(\{X_{t_1}, \dots, X_{t_n}\} \cup \mathfrak{N}))$, *for some* $t_j \in [0,1]$. *Then if* $\mathbb{E}[X \exp\{i \sum_{j=1}^n \alpha_j X_{t_j}\}] = 0$ *for all* $\alpha_j \in \mathbb{R}$, *then* $X = 0$ *a.s..*

Proof: We may assume that a.s. $X = f(X_{t_1}, \ldots, X_{t_n})$ for some $f \in L^1(\mathbb{R}^n, \mu)$ where μ is the distribution function of $\{X_{t_1}, \ldots, X_{t_n}\}$. Then for $(y_1, \ldots, y_n) \in \mathbb{R}^n$, we obtain

$$\begin{aligned}
0 &= \mathbb{E}[X \exp\{-i \sum_{j=1}^{n} y_j X_{t_j}\}] \\
&= \mathbb{E}[f(X_{t_1}, \ldots, X_{t_n}) \exp\{-i \sum_{j=1}^{n} y_j X_{t_j}\}] \\
&= \int_{\mathbb{R}^n} f(x_1, \ldots, x_1) \exp\{-i \sum_{j=1}^{n} y_j x_j\} d\mu(x) \\
&= \widehat{f d\mu}(y_1, \ldots, y_n).
\end{aligned}$$

Hence the Fourier transform of $f d\mu$ is zero, and by the injectivity of the Fourier transform, we have $f = 0$ μ-a.e., that is $X = 0$ a.s.. □

Proposition 3.2.10 *Suppose $X \in L^1(\Omega)$ and $\mathbb{E}[X \exp\{i \sum_{j=1}^{n} \alpha_j X_{t_j}\}] = 0$ for all $n \in \mathbb{N}$, $\alpha_j \in \mathbb{R}$ and $t_j \in [0,1]$. Then $X = 0$ a.s..*

Proof: X must be measurable with respect to some $\mathcal{F}(\{X_{t_j}\}_{j=1}^{\infty} \cup \mathfrak{N})$. We define $\mathcal{F}_n := \mathcal{F}(\{X_{t_1}, \ldots, X_{t_n}\} \cup \mathfrak{N})$. Then $\mathbb{E}[X|\mathcal{F}_n] \in L^1(\Omega, \mathcal{F}_n)$, and if $\alpha_j \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[X|\mathcal{F}_n] \exp\{i \sum_{j=1}^{n} \alpha_i X_{t_j}\}] &= \mathbb{E}[\mathbb{E}[X \exp\{i \sum_{j=1}^{n} \alpha_j X_{t_j}\} | \mathcal{F}_n]] \\
&= \mathbb{E}[X \exp\{i \sum_{j=1}^{n} \alpha_j X_{t_j}\}] = 0.
\end{aligned}$$

Thus by Lemma 3.2.9, $\mathbb{E}[X|\mathcal{F}_n] = 0$ a.s.. Now by the martingale convergence theorem, $\mathbb{E}[X|\mathcal{F}_n] \to X$ a.s. as $n \to \infty$. Therefore $X = 0$ a.s.. □

Using the above we can obtain the density of $\mathcal{E}_{\mathrm{lb}}$ in $L^p(\Omega)$ for $1 \le p < \infty$. This result does not appear in any of the literature.

Theorem 3.2.11 *$\mathcal{E}_{\mathrm{lb}}$ is dense in $L^p(\Omega)$ for all $1 \le p < \infty$.*

Proof: From Corollary 3.1.6, we have that

$$\mathcal{E}' := \mathrm{linsp}\{\exp\{i \sum_{j=1}^{n} \alpha_j X_{t_j}\} : n \in \mathbb{N},\ (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n,\ (t_1, \ldots, t_n) \in [0,1]^n\}$$

is a subset of $\mathcal{E}_{\mathrm{lb}}$. For $p > 1$, if $\frac{1}{p} + \frac{1}{q} = 1$, we know that $L^p(\Omega)^* = L^q(\Omega)$. Hence to prove the density of $\mathcal{E}_{\mathrm{lb}}$ in $L^p(\Omega)$ we only need to show that if $X \in L^q(\Omega)$, and $\mathbb{E}[X \exp\{i \sum_{j=1}^{n} \alpha_j X_{t_j}\}] =$

0 for all $n \in \mathbb{N}$, $\alpha_j \in \mathbb{R}$ and $t_j \in [0,1]$, then $X = 0$ a.s.. However, as $L^q(\Omega) \subseteq L^1(\Omega)$, this comes immediately from Proposition 3.2.10. The density of $\mathcal{E}_{\rm lb}$ in $L^1(\Omega)$ follows from the density of $\mathcal{E}_{\rm lb}$ in $L^p(\Omega)$ for $p > 1$. □

Corollary 3.2.12 *If $Y \in L^p_{\mathbb{R}}(\Omega)$ for some $p > 2$ and $\widehat{Y}$ denotes the operator of multiplication by Y, as in Definition 3.2.3, then $\widehat{Y}$ is essentially self-adjoint with core $\mathcal{E}_{\rm lb}$.*

PROOF: Find q such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Then $\mathcal{E}_{\rm lb}$ is dense in $L^q(\Omega)$ by Theorem 3.2.11. Hence by [56, §VIII.3, Proposition 2], $\widehat{Y}$ is essentially self-adjoint with core $\mathcal{E}_{\rm lb}$. □

The above result is useful, because if we can show certain multiplication operators by random variables in $L^p(\Omega)$ ($p > 2$) can be represented as quantum stochastic integrals on $\mathsf{E}_{\rm lb}$, then we can say that these quantum stochastic integrals and perturbations of them by regular self-adjoint quantum semimartingales are essentially self-adjoint with core $\mathsf{E}_{\rm lb}$. Once self-adjointness is established, we can investigate if the quantum Ito formula holds for these quantum stochastic integrals.

Notice that Proposition 3.2.10 proves a stronger result than the density of $\mathcal{E}_{\rm lb}$ in $L^p(\Omega)$. It proves that $\mathcal{E}'$ is dense in $L^p(\Omega)$. It can also be used to prove that $\mathcal{E}'$ is weak* dense in $L^\infty(\Omega)$, since $L^1(\Omega)^* = L^\infty(\Omega)$.

3.3 The Poisson process in $\mathfrak{F}_+(L^2[0,1])$

We begin with a definition, which succinctly describes the situation when two closed densely defined operators on two Hilbert spaces are unitarily equivalent.

Definition 3.3.1 *Let H_1 and H_2 be Hilbert spaces and $U : H_1 \to H_2$ be a unitary operator. Let T_1 and T_2 be closed densely defined operators on H_1 and H_2 respectively. We say U intertwines T_1 and T_2 if U maps $D(T_1)$ onto $D(T_2)$ and on $D(T_1)$, $T_1 = U^{-1}T_2U$.*

It is well-known that given a compensated Poisson process $(X_t)_{0 \le t \le 1}$, $\mathcal{W}$ intertwines $\Lambda_t + A_t + A_t^\dagger$ (where Λ_t, A_t and $A_t^\dagger$ are defined in (2.1.1)) and the multiplication operator $\widehat{X_t}$. One proof of this result is by showing that

$$e^{iu(\Lambda_t + A_t + A_t^\dagger)} = \mathcal{W}^{-1} e^{iu\widehat{X_t}} \mathcal{W}$$

and then using the uniqueness of the Stone infinitesimal generator of a unitary group. A rough outline of this argument, containing a couple of infelicities, is given in [47, §IV.2.5]. We now flesh it out. The proof involves direct calculation of the action of the two exponential operators on $\mathcal{E}_{\text{lb}}$. From [47, §IV.2.5] (see also Proposition 5.1.4), given $u \in \mathbb{R}$, $f \in L^\infty[0,1]$,

$$e^{iu(\Lambda_t + A_t + A_t^\dagger)} e(f) = \exp\{(e^{iu} - 1)\int_0^t f(s)ds + t(e^{iu} - 1 - iu)\} \\ \times e(e^{iu\mathbb{1}_{[0,t]}} f + (e^{iu} - 1)\mathbb{1}_{[0,t]}).$$

Notice that since $X_t = N_t - t$ and $\Delta X_s = 0$ or 1,

$$e^{iuX_t} = e^{iu(N_t - t)} = e^{-iut} \prod_{s \le t, \Delta X_s \neq 0} e^{iu}.$$

Therefore using (3.1.5),

$$\begin{aligned} e^{iuX_t}\mathcal{E}(f) = &\ \exp\{-\int_0^1 f(s)ds - iut\} \\ &\times \prod_{s \le t, \Delta X_s \neq 0} e^{iu}(1 + f(s)\Delta X_s) \prod_{s > t}(1 + f(s)\Delta X_s) \\ = &\ \exp\{-\int_0^1 f(s)ds - iut\} \\ &\times \prod_{s \le t}(1 + (e^{iu}f(s) + e^{iu} - 1)\Delta X_s) \prod_{s > t}(1 + f(s)\Delta X_s) \\ = &\ \exp\{(e^{iu} - 1)\int_0^t f(s)ds + t(e^{iu} - 1 - iu)\} \\ &\times \mathcal{E}(e^{iu\mathbb{1}_{[0,t]}} f + (e^{iu} - 1)\mathbb{1}_{[0,t]}). \end{aligned} \tag{3.3.1}$$

Using the action of $e^{iu(\Lambda_t + A_t + A_t^\dagger)}$ on E_{lb} described above, we deduce that for each $f \in L^\infty[0,1]$,

$$e^{iu(\Lambda_t + A_t + A_t^\dagger)} e(f) = \mathcal{W}^{-1} e^{iuX_t} \mathcal{W} e(f).$$

Since E_{lb} is dense in $\mathfrak{F}_+(L^2[0,1])$ and $e^{iu(\Lambda_t + A_t + A_t^\dagger)}$ and $\mathcal{W}^{-1} e^{iu\widehat{X_t}} \mathcal{W}$ are bounded operators, the equality of the operators follows.

Another proof of this result can be obtained by using the Attal-Meyer extension of quantum stochastic integrals, which says that if

$$M_t = \int_0^t E_s d\Lambda_s + F_s dA_s + G_s dA_s^\dagger + H_s ds$$

with suitable integrands, then for each $f \in L^\infty[0,1]$, M_t satisfies the classical stochastic differential equation

$$\begin{aligned} M_t\mathcal{E}(f_t) = &\int_0^t f(s) M_s \mathcal{E}(f_s) dX_s + \int_0^t f(s) E_s \mathcal{E}(f_s) dX_s \\ &+ \int_0^t f(s) F_s \mathcal{E}(f_s) ds + \int_0^t G_s \mathcal{E}(f_s) dX_s + \int_0^t H_s \mathcal{E}(f_s) ds, \end{aligned} \tag{3.3.2}$$

where for the integrals with respect to dX_s, we take the predictable projections of the integrands (see [6] or [2] for more details). In [3, Theorem II.1] this formula is used to obtain the result that $\mathcal{W}$ intertwines $\Lambda_t + A_t + A_t^\dagger$ and $\widehat{X_t}$.

We give a new proof of the relation between $\Lambda_t + A_t + A_t^\dagger$ and $\widehat{X_t}$ using the classical Ito product formula and the multiplication formula for Poisson exponentials. Meyer's proof cannot be extended to more general classical Poisson martingales which will be introduced in Section 6.1. It is possible to generalise Attal's proof, but the argument we give only requires the Hudson-Parthasarathy construction of quantum stochastic integrals and can also be generalised.

Proposition 3.3.2 *Let $f \in L^\infty[0,1]$, and let $(Z_t)_{0\le t\le 1}$ be the càdlàg version of $(\mathcal{E}(f_t))_{0\le t\le 1}$. Then*

$$X_t Z_t = \int_0^t (f(s)X_{s-}Z_{s-} + Z_{s-} + f(s)Z_{s-})dX_s + \int_0^t f(s)Z_s ds, \tag{3.3.3}$$

where the stochastic integrands are square integrable processes. In particular

$$\mathbb{E}[X_t\mathcal{E}(f_t)] = \int_0^t \mathbb{E}[f(s)\mathcal{E}(f_s)]ds. \tag{3.3.4}$$

Proof: Applying the classical Ito product formula and using (3.1.3), in the extended integral sense,

$$\begin{aligned} X_t Z_t &= \int_0^t Z_{s-}dX_s + \int_0^t X_{s-}dZ_s + [X,Z]_t \\ &= \int_0^t Z_{s-}dX_s + \int_0^t X_{s-}dZ_s + \int_0^t f(s)Z_{s-}d[X,X]_s, \end{aligned}$$

Note that since Z_{s-} is the predictable projection of the martingale Z_s, we know that for a.a. s in $[0,1]$, $Z_{s-} = Z_s$ a.s.. Similarly for X_s. From Jensen's inequality, for all $s \in [0,1]$,

$$\mathbb{E}[|X_s|^4] \le \mathbb{E}[|X_1|^4], \quad \mathbb{E}[|Z_s|^4] \le \mathbb{E}[|Z_1|^4].$$

Therefore

$$\begin{aligned} \int_0^t \mathbb{E}[|X_{s-}f(s)Z_{s-}|^2]ds &= \int_0^t \mathbb{E}[|X_s f(s)Z_s|^2]ds \\ &\le \|f\|_\infty^2 \int_0^t \mathbb{E}[|X_s Z_s|^2]ds \\ &\le \|f\|_\infty^2 \int_0^t \mathbb{E}[|X_s|^4]^{\frac{1}{2}}\mathbb{E}[|Z_s|^4]^{\frac{1}{2}}ds \\ &\le \|f\|_\infty^2 \int_0^t \mathbb{E}[|X_1|^4]^{\frac{1}{2}}\mathbb{E}[|Z_1|^4]^{\frac{1}{2}}ds \\ &\le \|f\|_\infty^2 \mathbb{E}[|X_1|^4]^{\frac{1}{2}} e^{\mathrm{Re}\langle \bar{f}, f\rangle + \frac{\|2f+f^2\|^2}{2}}, \end{aligned}$$

where we use Corollary 3.2.8 in the last inequality. Hence

$$\int_0^t X_{s-}dZ_s = \int_0^t f(s)X_{s-}Z_{s-}dX_s,$$

where the integrand on the right-hand side is a square integrable process. Furthermore as $[X,X]_s = s + X_s$, and $(s,\omega) \mapsto f(s)Z_{s-}(\omega)$ is in $L^2([0,1]\times\Omega)$,

$$\int_0^t f(s)Z_{s-}d[X,X]_s = \int_0^t f(s)Z_{s-}ds + \int_0^t f(s)Z_{s-}dX_s,$$

because the stochastic integral agrees with the pathwise Lebesgue-Stieltjes integral [44, Theorem 24.4 3°]. Since for all t in $[0,1]$, $\int_0^t f(s)Z_s ds = \int_0^t f(s)Z_{s-}ds$ a.s., we have shown that formula (3.3.3) holds. As all the integrands with respect to dX_s are square integrable processes, the expectation formula follows immediately, as the expectation of a Poisson stochastic integral of a square integrable predictable process is zero. □

Proposition 3.3.3 *On* E_{lb} *we have* $\Lambda_t + A_t + A_t^\dagger = \mathcal{W}^{-1}\widehat{X_t}\mathcal{W}$.

Proof: From the definition of a Poisson process we know that $\mathcal{F}_t$ and $\mathcal{F}_{[t,1]}$ are independent. Thus from (3.2.4) and (3.3.4) we can deduce that for f, $g \in L^\infty[0,1]$,

$$\begin{aligned}
\langle e(f), \mathcal{W}^{-1}\widehat{X_t}\mathcal{W}e(g)\rangle &= \langle \mathcal{E}(f), \widehat{X_t}\mathcal{E}(g)\rangle = \mathbb{E}[X_t\mathcal{E}(\bar{f})\mathcal{E}(g)] \\
&= \mathbb{E}[X_t\mathcal{E}(\bar{f}+g+\bar{f}g)]e^{\langle f,g\rangle} \\
&= \mathbb{E}[X_t\mathcal{E}((\bar{f}+g+\bar{f}g)\mathbb{1}_{[0,t]})]\mathbb{E}[\mathcal{E}((\bar{f}+g+\bar{f}g)\mathbb{1}_{(t,1]})]e^{\langle f,g\rangle} \\
&= \int_0^t \mathbb{E}[(\overline{f(s)}+g(s)+\overline{f(s)}g(s))\mathcal{E}((\overline{f}+g+\overline{f}g)_t)]ds \\
&\quad \times \mathbb{E}[\mathcal{E}((\overline{f}+g+\overline{f}g)\mathbb{1}_{(t,1]})]e^{\langle f,g\rangle} \\
&= \int_0^t \langle \mathcal{E}(f), (g(s)+\overline{f(s)}+\overline{f(s)}g(s))\mathcal{E}(g)\rangle ds \\
&= \langle e(f), (\Lambda_t + A_t + A_t^\dagger)e(g)\rangle
\end{aligned}$$

As E_{lb} is dense in $\mathfrak{F}_+(L^2[0,1])$ we arrive at the fact that for all $g \in L^\infty[0,1]$, $(\Lambda_t + A_t + A_t^\dagger)e(g) = \mathcal{W}^{-1}\widehat{X_t}\mathcal{W}e(g)$. □

To prove that the domains of the two operators also agree, we show that E_{lb} is a core for $\Lambda_t + A_t + A_t^\dagger$. In order to show this, we prove that if $k \geq 0$, E_{lb} is dense in the space

$$\mathbb{D}^{k,2} := \{\phi = (\phi_n) \in \mathfrak{F}_+(L^2[0,1]) : \sum_{n=0}^{\infty}(n+1)^k\|\phi_n\|^2 < \infty\} \tag{3.3.5}$$

with the Hilbert space norm

$$\|\phi\|_{k,2}^2 := \sum_{n=0}^{\infty}(n+1)^k\|\phi_n\|^2.$$

For $\phi = (\phi_n) \in \mathbb{D}^{k,2}$, we define $(\mathcal{N}+1)^{\frac{k}{2}}\phi_n := ((n+1)^{\frac{k}{2}}\phi_n)$, and hence $\|\phi\|_{k,2} = \|(\mathcal{N}+1)^{\frac{k}{2}}\phi\|$. If $k > 0$ and the spaces $\mathbb{D}^{-k,2}$ are defined as above, then they will not be complete. Thus for $k > 0$ we must define $\mathbb{D}^{-k,2}$ to be the completion of $\mathfrak{F}_+(L^2[0,1])$ with respect to the norm

$$\|\phi\|_{-k,2}^2 := \sum_{n=0}^{\infty}(n+1)^{-k}\|\phi_n\|^2.$$

The spaces $\mathbb{D}^{k,2}$ are examples of the well-known Malliavin or Gaussian Sobolev spaces (see [34, §15.5] and [43, §2.3]), defined on Fock space rather than on L^p-spaces induced by a Gaussian Hilbert space. The following result has been proved in the Gaussian Hilbert space setting in [34, Theorem 15.110]. Janson actually proves density for $\mathbb{D}^{k,p}$, $1 \leq p < \infty$. In our situation, $p = 2$, we have an elementary Fock space proof.

Lemma 3.3.4 *If $k \geq 0$ then E_{lb} is dense in $\mathbb{D}^{k,2}$.*

Proof: As $\mathbb{D}^{k,2}$ is a Hilbert space, we only need to show that $\mathsf{E}_{\text{lb}}^{\perp_{k,2}} = \{0\}$. Suppose $\phi = (\phi_n) \in \mathbb{D}^{k,2}$ and $\phi \in \mathsf{E}_{\text{lb}}^{\perp_{k,2}}$. Then for all $f \in L^\infty[0,1]$ and $t \in \mathbb{R}$,

$$\langle \phi, \mathcal{E}(tf)\rangle_{k,2} = \sum_{n=0}^{\infty} \frac{t^n}{\sqrt{n!}}(n+1)^k\langle \phi_n, f^{\otimes^n}\rangle.$$

The power series converges for all real t. Thus $\langle \phi_n, f^{\otimes^n}\rangle = 0$ for all $f \in L^\infty[0,1]$. By the totality of $\{f^{\otimes^n} : f \in L^\infty[0,1]\}$ in $L^2_{\text{sym}}(\mathbb{R}^n_+)$, we have that $\phi_n = 0$. Hence $\phi = 0$. □

Theorem 3.3.5 *$\mathcal{W}$ intertwines $\Lambda_t + A_t + A_t^\dagger$ and $\widehat{X_t}$.*

Proof: From Proposition 3.3.3,

$$(\Lambda_t + A_t + A_t^\dagger)|_{\mathsf{E}_{\text{lb}}} = \mathcal{W}^{-1}\widehat{X_t}\mathcal{W}|_{\mathsf{E}_{\text{lb}}}. \tag{3.3.6}$$

Now, $\Lambda_t + A_t + A_t^\dagger$ is essentially self-adjoint with core $\mathfrak{F}_+(L^2[0,1])_{00}$, the subspace of $\mathfrak{F}_+(L^2[0,1])$ consisting of vectors with finite expansions. However, we know from Lemma 3.3.4 that if $\phi \in \mathfrak{F}_+(L^2[0,1])_{00}$, there exists $\phi_n \in \mathsf{E}_{\text{lb}}$ such that $\|(\mathcal{N}+1)(\phi - \phi_n)\| \to 0$ and $\|\phi - \phi_n\| \to 0$ as $n \to \infty$. Therefore

$$\begin{aligned}\|(\Lambda_t + A_t + A_t^\dagger)(\phi - \phi_n)\| &\leq \|\Lambda_t(\phi - \phi_n)\| + \|A_t(\phi - \phi_n)\| + \|A_t^\dagger(\phi - \phi_n)\| \\ &\leq 3\|(\mathcal{N}+1)(\phi - \phi_n)\| \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned}$$

This shows that E_{lb} is a core for $\Lambda_t + A_t + A_t^\dagger$. Since $X_t \in L^4(\Omega)$, E_{lb} is also a core for $\mathcal{W}^{-1}\widehat{X_t}\mathcal{W}$. If $\phi \in D(\Lambda_t + A_t + A_t^\dagger)$, then there exist $(\phi_n) \subseteq \mathsf{E}_{\text{lb}}$ such that $\phi_n \to \phi$ and $(\Lambda_t + A_t + A_t^\dagger)\phi_n \to (\Lambda_t + A_t + A_t^\dagger)\phi$ as $n \to \infty$. From (3.3.6), we have that $\mathcal{W}\phi_n \to \mathcal{W}\phi$ and $\widehat{X_t}(\mathcal{W}\phi_n) \to \mathcal{W}(\Lambda_t + A_t + A_t^\dagger)\phi$ as $n \to \infty$. Hence $\mathcal{W}\phi \in D(\widehat{X_t})$ and $\mathcal{W}^{-1}\widehat{X_t}\mathcal{W}\phi = (\Lambda_t + A_t + A_t^\dagger)\phi$. Similarly, if $Y \in D(\widehat{X_t})$, $\mathcal{W}^{-1}Y \in D(\Lambda_t + A_t + A_t^\dagger)$. Therefore $\mathcal{W}$ maps $D(\Lambda_t + A_t + A_t^\dagger)$ onto $D(\widehat{X_t})$ and on $D(\Lambda_t + A_t + A_t^\dagger)$, $\Lambda_t + A_t + A_t^\dagger = \mathcal{W}^{-1}\widehat{X_t}\mathcal{W}$. □

3.4 The Poisson process in $\mathfrak{F}_+(L^2(\mathbb{R}_+))$

As mentioned before, the results obtained for $L^2[0,1]$ can be extended to $L^2(\mathbb{R}_+)$, with some modifications. We have worked over $[0,1]$ to simplify the arguments in order to illuminate the underlying ideas. The main simplification occurs because the subspace $L^\infty[0,1]$ of $L^2[0,1]$ is an algebra under pointwise multiplication, while the corresponding subspace $L^2_{\text{lb}}(\mathbb{R}_+)$ of $L^2(\mathbb{R}_+)$ is not. In this section our Wiener-Poisson isomorphism $\mathcal{W}$ acts on $\mathfrak{F}_+(L^2(\mathbb{R}_+))$ and not just $\mathfrak{F}_+(L^2[0,1])$ and our unitary map $\mathfrak{u}_t$ also changes in the appropriate way. Our complete probability space will be $(\Omega, \mathcal{F}, \mathbb{P})$ and we shall denote $\mathcal{F}_\infty$ by $\mathcal{F}$ and $L^p(\Omega, \mathcal{F}, \mathbb{P})$ by $L^p(\Omega)$.

The first two results showing that $\mathcal{W}$ transforms $\tilde{\mathfrak{u}}_t$, defined in an analogous way to (3.2.2), into ordinary multiplication and that classical stochastic processes can be used to construct quantum semimartingales remain the same, since the exponential multiplication formula in Lemma 3.2.1 for functions with disjoint supports still holds. As before if X is a complex random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, $\widehat{X}$ is used to represent the operator of multiplication by X with domain $D(\widehat{X})$ (see Definition 3.2.3).

Proposition 3.4.1 *Let $\mathfrak{m}_t : L^2(\Omega, \mathcal{F}_t) \times L^2(\Omega, \mathcal{F}_{[t}) \to L^2(\Omega, \mathcal{F})$ be the multiplication map $(X, Y) \mapsto XY$. Then $\mathfrak{m}_t(\mathcal{W} \times \mathcal{W}) = \mathcal{W}\tilde{\mathfrak{u}}_t$.*

Theorem 3.4.2 *Let $(F_t)_{t\geq 0}$ be an adapted, measurable process on $(\Omega, \mathcal{F}, \mathbb{P})$, bounded on each finite interval. If $\widehat{F_t}$ is the operator of multiplication by F_t on $L^2(\Omega, \mathcal{F}, \mathbb{P})$, then $(\mathcal{W}^{-1}\widehat{F_t}\mathcal{W})_{t\geq 0}$ is an adapted process of bounded operators such that $t \mapsto \|\mathcal{W}^{-1}\widehat{F_t}\mathcal{W}\|_{\mathfrak{B}(\mathfrak{F}_+(L^2(\mathbb{R}_+)))}$ is in $L^\infty_{\text{loc}}(\mathbb{R}_+)$. Hence if $(H_t)_{t\geq 0}$ is another adapted, measurable process, bounded on finite intervals, and*

$$M_t = \int_0^t \mathcal{W}^{-1}\widehat{F_s}\mathcal{W}(d\Lambda_s + dA_s + dA_s^\dagger) + \int_0^t \mathcal{W}^{-1}\widehat{H_s}\mathcal{W}ds,$$

then $(M_t)_{t\geq 0}$ is a quantum semimartingale.

Slight modifications are needed when considering the more general multiplication formula. For Proposition 3.2.6 to hold with $[0,1]$ replaced by $\mathbb{R}_+$ we require $fg \in L^2(\mathbb{R}_+)$. This does not hold in general if $f,\, g \in L^2_{\mathrm{lb}}(\mathbb{R}_+)$. Consequently, we introduce the space $L^\infty_{\mathrm{c}}(\mathbb{R}_+)$ of bounded functions on $\mathbb{R}_+$ with compact support, and we let

$$\mathcal{E}_{\mathrm{lb,c}} := \mathrm{linsp}\{\mathcal{E}(f) : f \in L^\infty_{\mathrm{c}}(\mathbb{R}_+)\}.$$

Note that $L^\infty_{\mathrm{c}}(\mathbb{R}_+)$ is an admissible subspace, in the sense of [28].

Proposition 3.4.3 *Let $f,\, g \in L^\infty_{\mathrm{c}}(\mathbb{R}_+)$. Then*

$$\mathcal{E}(f)\mathcal{E}(g) = e^{\langle \bar{f}, g\rangle}\mathcal{E}(f + g + fg).$$

PROOF: The proof is the same as that in Proposition 3.2.6, since if $f,\, g \in L^\infty_{\mathrm{c}}(\mathbb{R}_+)$, then $fg \in L^\infty_{\mathrm{c}}(\mathbb{R}_+)$. □

Corollary 3.4.4 $\mathcal{E}_{\mathrm{lb,c}} \subseteq L^p(\Omega)$ *for all* $1 \le p < \infty$.

PROOF: This comes from applying the above proposition as in Corollary 3.2.7. □

We do not, when working over $L^2(\mathbb{R}_+)$, have that $\mathcal{E}_{\mathrm{lb}} \subseteq L^p(\Omega)$. However it should be noted that if $f \in L^2_{\mathrm{lb}}(\mathbb{R}_+)$, for each $t > 0$, $f_t \in L^\infty_{\mathrm{c}}(\mathbb{R}_+)$ and that if Y is a random variable measurable with respect to $\mathcal{F}_t$ for some $t > 0$, then by the independence of $\mathcal{F}_t$ and $\mathcal{F}_{[t}$, $\mathcal{E}_{\mathrm{lb}} \subseteq D(\widehat{Y})$ if and only if $\mathcal{E}_{\mathrm{lb,c}} \subseteq D(\widehat{Y})$.

Theorem 3.4.5 $\mathcal{E}_{\mathrm{lb,c}}$ *is dense in* $L^p(\Omega)$ *for* $1 \le p < \infty$.

PROOF: This is proved in exactly the same way as Theorem 3.2.11, using the result that if $X \in L^1(\Omega)$ and $\mathbb{E}[X \exp\{i \sum_{j=1}^n \alpha_j X_{t_j}\}] = 0$ for all $n \in \mathbb{N}$, $\alpha_j \in \mathbb{R}$ and $t_j \in \mathbb{R}_+$, then $X = 0$ a.s.. This can be proved as before using the injectivity of the Fourier transform and the martingale convergence theorem. □

Corollary 3.4.6 *If $Y \in L^p_{\mathbb{R}}(\Omega)$ for some $p > 2$ and $\widehat{Y}$ is the operator of multiplication by Y, $\widehat{Y}$ is essentially self-adjoint with core $\mathcal{E}_{\mathrm{lb,c}}$. In particular, if Y is measurable with respect to $\mathcal{F}_t$, then $\widehat{Y}$ is essentially self-adjoint with core $\mathcal{E}_{\mathrm{lb}}$.*

PROOF: The first part follows in the same way as Corollary 3.2.12 follows from Theorem 3.2.11. For the last part, because Y is measurable with respect to $\mathcal{F}_t$, and $\mathcal{E}_{\mathrm{lb,c}} \subseteq D(\widehat{Y})$, it follows that $\mathcal{E}_{\mathrm{lb}} \subseteq D(\widehat{Y})$. Therefore since $\widehat{Y}$ is essentially self-adjoint with core $\mathcal{E}_{\mathrm{lb,c}}$, it is clearly essentially self-adjoint with core $\mathcal{E}_{\mathrm{lb}}$. □

The result about $\mathcal{W}$ intertwining $\Lambda_t + A_t + A_t^\dagger$ and $\widehat{X_t}$ for all $t \in \mathbb{R}_+$ also holds. The proof in the $[0, 1]$ case with appropriate changes can be used.

Theorem 3.4.7 *For all $t \geq 0$, $\Lambda_t + A_t + A_t^\dagger = \mathcal{W}^{-1}\widehat{X_t}\mathcal{W}$.*

PROOF: $X_t \in L^p(\Omega)$ for all $1 \leq p < \infty$ and X_t is measurable with respect to $\mathcal{F}_t$. Therefore by Corollary 3.4.6, $\mathcal{E}_{\mathrm{lb}} \subseteq D(\widehat{X_t})$. Arguing as in Theorem 3.3.5, if we can show

$$(\Lambda_t + A_t + A_t^\dagger)|_{\mathsf{E}_{\mathrm{lb}}} = \mathcal{W}^{-1}\widehat{X_t}\mathcal{W}|_{\mathsf{E}_{\mathrm{lb}}},$$

the result follows since it can be shown that E_{lb} is a core for both the operators in question. To show the operators agree on E_{lb} the same argument as in Proposition 3.3.3 can be used. □

3.5 Poisson processes with different intensities

In this chapter, we have only talked about Poisson processes with intensity 1, and we shall in general only consider such processes. However, it is easy to generalise to Poisson processes with different intensities. We shall give brief details, more results can be found in [47, §IV.2.4-5, §IV.3.6].

Let $(N_t^c)_{t\geq 0}$ be a Poisson process with unit jump size and intensity $\frac{1}{c^2}$ based on a complete probability space $(\Omega^c, \mathcal{F}^c, \mathbb{P}^c)$. Then $X_t^c := cN_t^c - \frac{1}{c}t$, $(X_t)_{t\geq 0}$ is said to be a *compensated Poisson process with jump size c and intensity* $\frac{1}{c^2}$. If $\mathcal{F}_t^c$ is defined in an analogous way to the $c = 1$ case, $\{(X_t^c)_{t\geq 0}, (\mathcal{F}_t^c)_{t\geq 0}\}$ is a martingale such that $\langle X^c, X^c\rangle_t = t$ and

$$[X^c, X^c]_t = t + cX_t^c.$$

The normalisation we use is not standard but is chosen so that $\langle X^c, X^c\rangle_t = t$. In classical stochastic analysis the process $(N_t^c - \frac{1}{c^2}t)_{t\geq 0}$, called the compensated Poisson process with jump size 1 and intensity $\frac{1}{c^2}$ ([16, §1.9 Example 1]), is often used. As for the case when $c = 1$ there exists a unitary map

$$\mathcal{W}^c : \mathfrak{F}_+(L^2[0,1]) \longrightarrow L^2(\Omega^c, \mathcal{F}_1^c, \mathbb{P}^c).$$

The product formula for the exponentials $\mathcal{E}^c(f) := \mathcal{W}^c(e(f))$, becomes for all f, $g \in L^\infty[0,1]$,

$$\mathcal{E}^c(f)\mathcal{E}^c(g) = e^{\langle \bar{f}, g\rangle}\mathcal{E}^c(f + g + \frac{1}{c}fg).$$

The result of Section 3.3 becomes $c\Lambda_t + A_t + A_t^\dagger = (\mathcal{W}^c)^{-1}\widehat{X_t^c}\mathcal{W}^c$ instead.

Chapter 4

Generalised Poisson Processes

Recall from Section 3.1 that if $(N_t)_{t\geq 0}$ is a Poisson process based on $(\Omega, \mathcal{F}, \mathbb{P})$ and we let $I = \mathcal{W}|_{L^2(\mathbb{R}_+)} : L^2(\mathbb{R}_+) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$, then I is an isometry. It can be shown that if $(E_j)_{j=1}^n$ are disjoint Borel sets in $\mathbb{R}_+$ with finite Lebesgue measure and $\alpha = \sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}$ with $\alpha_j \in \mathbb{R}$, by approximating α by step functions and using the dominated convergence theorem,

$$\mathbb{E}[e^{iI(\alpha)}] = \exp\{\int_0^\infty (e^{i\alpha} - 1 - i\alpha) ds\}.$$

Hence from a Poisson process we may construct an isometry $I : L^2(\mathbb{R}_+) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ such that

i) if $E \in \mathcal{B}_{\mathbb{R}_+}$ with $m(E) < \infty$, then $I(\mathbb{1}_E) + m(E)$ has a Poisson distribution with mean $m(E)$,

ii) if $(E_j)_{j=1}^n \subseteq \mathcal{B}_{\mathbb{R}_+}$ are disjoint with $m(E_j) < \infty$, then $\{I(\mathbb{1}_{E_1}), \ldots, I(\mathbb{1}_{E_n})\}$ are independent random variables,

where $\mathcal{B}_{\mathbb{R}_+}$ are the Borel sets in $\mathbb{R}_+$ and m is the Lebesgue measure on $\mathbb{R}_+$. Conversely, for every such isometry if we let $N_t = I(\mathbb{1}_{[0,t]}) + t$, then $(N_t)_{t\geq 0}$ is a Poisson process. This formulation using the isometry is preferred since it does not depend on the order structure of $\mathbb{R}_+$. The aim of this chapter is to generalise this definition of a Poisson process to more general measure spaces $(M, \mathcal{M}, \mu)$ and to construct the analogous isomorphism to the Wiener-Poisson isomorphism. We shall also investigate some of the properties of this isomorphism.

4.1 Definition and properties of generalised Poisson processes

For convenience we make the following definition.

Definition 4.1.1 *A random variable on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is said to be a compensated Poisson variable, or to have a compensated Poisson distribution, with intensity* λ *if* $X + \lambda$ *has a Poisson distribution with mean* λ.

The definition below generalises the concept of a Poisson process on $\mathbb{R}_+$.

Definition 4.1.2 *Let* $(M, \mathcal{M}, \mu)$ *be a measure space. A generalised Poisson process over* $(M, \mathcal{M}, \mu)$ *is a linear isometry* $I : L^2(M, \mathcal{M}, \mu) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$, *where* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a complete probability space, such that*

i) if $E \in \mathcal{M}$ *with* $\mu(E) < \infty$, *then* $I(\mathbb{1}_E)$ *has a compensated Poisson distribution with intensity* $\mu(E)$,

ii) if $(E_j)_{j=1}^n \subseteq \mathcal{M}$ *are pairwise disjoint with* $\mu(E_j) < \infty$, *then* $\{I(\mathbb{1}_{E_1}), \ldots, I(\mathbb{1}_{E_n})\}$ *are independent random variables.*

For all $E \in \mathcal{M}$ *such that* $\mu(E) < \infty$, *we shall denote* $I(\mathbb{1}_E)$ *by* X_E.

Closely related definitions of what we call generalised Poisson processes, can be found in [22, Chapter 3], [64, §1], [60, §2] and [39, Definition 1]. Notice that if $E \in \mathcal{M}$ with $\mu(E) < \infty$, and we let $N_E = I(\mathbb{1}_E) + \mu(E)$, then the collection of random variables $\{N_E : E \in \mathcal{M} \text{ with } \mu(E) < \infty\}$ has the properties that N_E has a Poisson distribution with mean $\mu(E)$ and if $(E_j)_{j=1}^n \subseteq \mathcal{M}$ are disjoint sets with $\mu(E_j) < \infty$, then $\{N_{E_1}, \ldots, N_{E_n}\}$ are independent with $N_{\cup E_j} = \sum_{j=1}^n N_{E_j}$ a.s.. Conversely, suppose we have a family of random variables $\{N_E : E \in \mathcal{M} \text{ with } \mu(E) < \infty\}$ satisfying the properties above. Then by defining $I(\mathbb{1}_E) = N_E - \mu(E)$, and extending by linearity and the isometry property, we have a generalised Poisson process over $(M, \mathcal{M}, \mu)$. We shall from now on assume that $\mathcal{F} = \sigma(\{X_E : E \in \mathcal{M} \text{ with } \mu(E) < \infty\} \cup \mathfrak{N})$.

Given a Poisson process we can construct an isometric isomorphism

$$\mathcal{W} : \mathfrak{F}_+(L^2(\mathbb{R}_+)) \longrightarrow L^2(\Omega, \mathcal{F}, \mathbb{P}).$$

In Section 4.2 we shall construct an analogous map

$$\mathcal{W}_I : \mathfrak{F}_+(L^2(M)) \longrightarrow L^2(\Omega, \mathcal{F}, \mathbb{P}),$$

for a generalised Poisson process, $I : L^2(M, \mathcal{M}, \mu) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$. Our construction is more general than the work of [64] and [39]. Both of them work with σ-finite measure spaces and with Poisson random measures $\{N_E : E \in \mathcal{M} \text{ with } \mu(E) < \infty\}$. These have the properties that for each $\omega \in \Omega$, $E \mapsto N_E(\omega)$ is a measure on $(M, \mathcal{M})$, if $E \in \mathcal{M}$ with $\mu(E) < \infty$ then

N_E has a Poisson distribution with mean $\mu(E)$ and if $(E_j)_{j=1}^n \subseteq \mathcal{M}$ are disjoint sets with $\mu(E_j) < \infty$ then $\{N_{E_1}, \ldots, N_{E_n}\}$ are independent. However as Liebscher mentions in [39, Remark 1], the construction of the Wiener-Poisson isomorphism only requires the hypothesis which we have used to define a generalised Poisson process, that is we only require σ-additivity in the square-mean sense. In the later article [40], in which Liebscher constructs isomorphisms between Fock space and $L^2(\Omega)$ generated by more general distributions, [40, Definition 1.1] which he uses is analogous to our definition. Using a generalised Poisson process to construct the Wiener-Poisson isomorphism is in some sense more natural than using Poisson random measures, since the construction does not require the pointwise measure property. In Chapter 5, given a measure space, using the Gelfand transform of a naturally occurring C^*-algebra we construct a canonical generalised Poisson process, rather than a Poisson random measure, again indicating that in our work the definition we use is the correct one.

For a generalised Poisson process, we do not know if there is a version of the process $\{X_E : E \in \mathcal{M} \text{ with } \mu(E) < \infty\}$ such that for each $\omega \in \Omega$,

$$E \longmapsto X_E(\omega) + \mu(E)$$

is a σ-additive set function on $\{E \in \mathcal{M} : \mu(E) < \infty\}$. However, a Poisson process $I : L^2(\mathbb{R}_+) \to L^2(\Omega)$ does have a version $\{X_E : E \in \mathcal{B}_{\mathbb{R}_+} \text{ with } m(E) < \infty\}$ which produces a Poisson random measure. If $N_t = X_t + t$, we can choose the version of the Poisson process $(N_t)_{t \geq 0}$ whose paths are increasing and right continuous. For each $\omega \in \Omega$ let

$$S(\omega) = \{s \in \mathbb{R}_+ : N_s(\omega) - N_{s-}(\omega) \neq 0\}.$$

Then $S(\omega)$ is a countable set and

$$N_t(\omega) = \sum_{s_n \in S(\omega)} \mathbb{1}_{[0,t]}(s_n).$$

Therefore, by approximating Borel sets by open sets, which is possible because m is a regular measure on $\mathbb{R}_+$, we have that for $E \in \mathcal{B}_{\mathbb{R}_+}$ with $\mu(E) < \infty$,

$$N_E(\omega) := X_E(\omega) + m(E) = \sum_{s_n \in S(\omega)} \mathbb{1}_E(s_n).$$

Therefore if $E = \bigcup_{j=1}^{\infty} E_j$ with $(E_j) \subseteq \mathcal{B}_{\mathbb{R}_+}$ pairwise disjoint and $\mu(E) < \infty$, for $\omega \in \Omega$,

$$\begin{aligned} N_E(\omega) &= \sum_{s_n \in S(\omega)} \mathbb{1}_E(s_n) = \sum_{s_n \in S(\omega)} \mathbb{1}_{\cup_{j=1}^{\infty} E_j}(s_n) \\ &= \sum_{s_n \in S(\omega)} \sum_{j=1}^{\infty} \mathbb{1}_{E_j}(s_n) = \sum_{j=1}^{\infty} \sum_{s_n \in S(\omega)} \mathbb{1}_{E_j}(s_n) \\ &= \sum_{j=1}^{\infty} N_{E_j}(\omega). \end{aligned}$$

Thus we have obtained a Poisson random measure from $I : L^2(\mathbb{R}_+) \to L^2(\Omega)$. Note that this argument relies heavily on the topology and order structure of $\mathbb{R}_+$.

Surgailis deals with non-atomic measure spaces, which allows him to construct the isomorphism without the use of Charlier polynomials. Liebscher considers measure spaces which may contain atoms, but requires M to be a Polish space and $\mathcal{M}$ to be its Borel σ-field. We construct the Wiener-Poisson isomorphism, $\mathcal{W}_I$, without either of these restrictions in Theorem 4.2.12, in fact we do not even require $(M, \mathcal{M}, \mu)$ to be σ-finite. Our proof of the surjectivity of $\mathcal{W}_I$ uses the Poisson exponentials, $\mathcal{W}_I(e(f))$ with $f \in L^2(M)$, rather than the space of polynomials of X_E, unlike [64] and [39]. In Section 4.3 we give a construction of $\mathcal{W}_I$ using the exponential vectors and then deriving the connection with the Charlier polynomials, rather than starting with the Charlier polynomials and then deducing properties of the Poisson exponentials, which we do in Section 4.2.

Definition 4.1.3 *Suppose* $(M, \mathcal{M}, \mu)$ *is a measure space. Then we define*

$$L^{\mathcal{S}}_{\mathbb{R}}(M) := \{\sum_{j=1}^{n} \alpha_j \mathbb{1}_{E_j} : n \in \mathbb{N},\ \alpha_j \in \mathbb{R},\ (E_j)_{j=1}^{n} \subseteq \mathcal{M} \text{ disjoint with } \mu(E_j) < \infty\}.$$

This set will be useful for our work on generalised Poisson processes. Notice that $L^{\mathcal{S}}_{\mathbb{R}}(M)$ is dense in $L^p_{\mathbb{R}}(M)$ for $1 \leq p < \infty$ [58, Theorem 3.13]. The vectors $e(e^{i\alpha}-1)$ with $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$, will play a crucial role in our work on the Poisson process. The following properties of generalised Poisson processes can be easily deduced.

Proposition 4.1.4 *Suppose* $I : L^2(M, \mathcal{M}, \mu) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ *is a generalised Poisson process. Then*

i) if $I_{\mathbb{R}} = I|_{L^2_{\mathbb{R}}(M)}$, $I_{\mathbb{R}} : L^2_{\mathbb{R}}(M) \to L^2_{\mathbb{R}}(\Omega)$ *and* $I = I_{\mathbb{R}} \oplus iI_{\mathbb{R}}$,

ii) if $f \in L^1(M) \cap L^2(M)$ *and* $f \geq 0$, $I(f) + \int_M f d\mu \geq 0$ *a.s.*, *iii) if f, $g \in L^2(M)$ and $fg = 0$, $I(f)$ and $I(g)$ are independent.*

Lemma 4.1.5 *If $I : L^2(M, \mathcal{M}, \mu) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a generalised Poisson process and $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M) \oplus iL^{\mathcal{S}}_{\mathbb{R}}(M)$, then*

$$\mathbb{E}[e^{I(\alpha)}] = \exp\{\int_M (e^{\alpha} - 1 - \alpha)d\mu\}. \tag{4.1.1}$$

Notice that we could equivalently define a generalised Poisson process $I : L^2(M) \to L^2(\Omega)$ as an isometry such that (4.1.1) holds for all $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$. The formula (4.1.1) holds for other functions $f \in L^2(M)$, which can be deduced by approximating f by simple functions.

Proposition 4.1.6 *Let $I : L^2(M, \mathcal{M}, \mu) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ be a generalised Poisson process. If $f \in L^2(M)$ and*

$$\int_{\mathrm{Re}f>1} \exp\{2\mathrm{Re}f\}d\mu < \infty, \tag{4.1.2}$$

then $e^f - 1 - f \in L^1(M)$, $e^{I(f)} \in L^2(\Omega)$ and

$$\mathbb{E}[e^{I(f)}] = \exp\{\int_M (e^f - 1 - f)d\mu\}. \tag{4.1.3}$$

Proof: The proof follows that of [64, Proposition 2.2], but we fill in the details. Note that $\mu(\{m : |f(m)| > 1\}) < \infty$ and thus

$$|(e^f - 1 - f)\mathbb{1}_{\{|f|>1\}}| \le (e + 1 + |f|)\mathbb{1}_{\{\mathrm{Re}f\le 1 \cap |f|>1\}} + (e^{2\mathrm{Re}f} + 1 + |f|)\mathbb{1}_{\{\mathrm{Re}f>1\cap|f|>1\}} \in L^1(M).$$

Also, we have that $(e^z - 1 - z)/z^2 \to 1/2$ as $z \to 0$. Hence, there exists $K > 0$ such that

$$|(e^f - 1 - f)\mathbb{1}_{\{|f|\le 1\}}| \le K|f\mathbb{1}_{\{|f|\le 1\}}|^2 \in L^1(M).$$

Thus $e^f - 1 - f \in L^1(M)$. Choose a sequence $(f_j) \subseteq L^{\mathcal{S}}_{\mathbb{R}}(M) \oplus iL^{\mathcal{S}}_{\mathbb{R}}(M)$ such that $f_j \to f$ a.s. and in $L^2(M)$ with $(\mathrm{Re}f_j)^{\pm} \le (\mathrm{Re}f)^{\pm}$, $(\mathrm{Im}f_j)^{\pm} \le (\mathrm{Im}f)^{\pm}$. We may suppose that $I(f_j) \to I(f)$ a.s. as well. Then

$$\begin{aligned}|e^{I(f_k)} - e^{I(f_j)}|^2 &= (e^{I(f_k)} - e^{I(f_j)})(e^{I(\bar{f}_k)} - e^{I(\bar{f}_j)})\\ &= e^{I(2\mathrm{Re}f_k)} + e^{I(2\mathrm{Re}f_j)} - 2\mathrm{Re}\ e^{I(f_k+\bar{f}_j)}.\end{aligned}$$

Thus from (4.1.1),

$$\mathbb{E}[|e^{I(f_k)} - e^{I(f_j)}|^2] = \exp\{\int_M (e^{2\mathrm{Re}f_k} + e^{2\mathrm{Re}f_j} - 2\mathrm{Re}\ e^{(f_k+\bar{f}_j)})d\mu\}.$$

On $\{m : |f(m)| \le 1\}$, since $|f_j|$, $|f_k| \le |f|$,

$$|e^{2\mathrm{Re}f_k} + e^{2\mathrm{Re}f_j} - 2\mathrm{Re}\ e^{(f_k+\bar{f}_j)}| \le K|f_k - f_j|^2,$$

for some constant K. Hence

$$\int_{|f|\leq 1}(e^{2\mathrm{Re}f_k}+e^{2\mathrm{Re}f_j}-2\mathrm{Re}\ e^{(f_k+\bar{f}_j)})d\mu\to 0\ \text{ as } j,k\to\infty.$$

If $\mathrm{Re}f(m)\leq 1$, then $\mathrm{Re}f_j(m)$, $\mathrm{Re}f_k(m)\leq 1$, therefore on $\{m:\mathrm{Re}f(m)<1\}$,

$$|e^{2\mathrm{Re}f_k}+e^{2\mathrm{Re}f_j}-2\mathrm{Re}\ e^{(f_k+\bar{f}_j)}|\leq 4e^2.$$

Furthermore, if $\mathrm{Re}f(m)>1$ then $\mathrm{Re}f_j(m)$, $\mathrm{Re}f_k(m)\leq \mathrm{Re}f(m)$ and on $\{m:\mathrm{Re}f(m)>1\}$,

$$|e^{2\mathrm{Re}f_k}+e^{2\mathrm{Re}f_j}-2\mathrm{Re}\ e^{(f_k+\bar{f}_j)}|\leq 4e^{2\mathrm{Re}f}.$$

Thus from (4.1.2) and the dominated convergence theorem,

$$\int_{|f|>1}(e^{2\mathrm{Re}f_k}+e^{2\mathrm{Re}f_j}-2\mathrm{Re}\ e^{(f_k+\bar{f}_j)})d\mu\to 0\ \text{ as } j,k\to\infty.$$

Hence $(e^{I(f_j)})$ is a Cauchy sequence in $L^2(\Omega)$ converging a.s. to $e^{I(f)}$. Consequently $e^{I(f)}\in L^2(\Omega)$. Moreover, by the dominated convergence theorem

$$\begin{aligned}\mathbb{E}[e^{I(f)}] &= \lim_{j\to\infty}\mathbb{E}[e^{I(f_j)}]\\ &= \lim_{j\to\infty}\exp\{\int_M(e^{f_j}-1-f_j)d\mu\}\\ &= \exp\{\int_M(e^f-1-f)d\mu\},\end{aligned}$$

which gives the required formula. □

The converse of the above result also holds. We omit the proof, which can be found in the proof of [64, Proposition 2.2].

Proposition 4.1.7 *Suppose $f\in L^2(M)$ and $e^{I(f)}\in L^2(\Omega)$. Then*

$$\int_{|f|>1}\exp\{2\mathrm{Re}f\}d\mu<\infty.$$

We can use similar arguments to Proposition 4.1.6 and the moment generating function formula (4.1.3) to prove another result.

Proposition 4.1.8 *Suppose $f\in\bigcap_{1\leq p<\infty}L^p(M)$. Then $I(f)\in L^p(\Omega)$ for all $p<\infty$.*

PROOF: For $f \in L^1(M) \cap L^2(M)$ put $J(f) = I(f) + \int_M f d\mu$. We may assume $f \geq 0$, otherwise we can write f as a linear combination of such functions. If $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$, then from (4.1.3), for $p \in \mathbb{N}$,

$$\mathbb{E}[J(\alpha)^p] = \int_M m_p(\alpha) d\mu$$

for some polynomial m_p. Now let $(f_j) \subseteq L^{\mathcal{S}}_{\mathbb{R}}(M)$ be an increasing sequence of functions such that $f_j \geq 0$ and $f_j \to f$ a.s. and in $L^p(M)$. Then for $j \leq k$,

$$\mathbb{E}[|J(f_k) - J(f_j)|^p] = \mathbb{E}[(J(f_k) - J(f_j))^p] = \int_M m_p(f_k - f_j) d\mu.$$

Since we know that $f \in \bigcap_{1 \leq p < \infty} L^p(M)$, by the dominated convergence theorem we have shown that $(J(f_j))$ is a Cauchy sequence in $L^p(\Omega)$ which converges in $L^2(\Omega)$ to $J(f)$. Thus $J(f) \in L^p(\Omega)$ and hence $I(f) \in L^p(\Omega)$. The result follows for all $p < \infty$ since $L^q(\Omega) \subseteq L^p(\Omega)$ for $q \geq p$. □

The following density result will be useful when proving that the image of the linear span of the Fock space exponential vectors under the generalised Wiener-Poisson isomorphism is dense.

Proposition 4.1.9 *The set* $\{\exp\{i \sum_{j=1}^n \alpha_j X_{E_j}\} : n \in \mathbb{N},\ \alpha_j \in \mathbb{R},\ E_j \in \mathcal{M} \text{ with } \mu(E_j) < \infty\}$ *is total in* $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

PROOF: The argument follows that of Theorem 3.2.11, and therefore we shall only briefly outline the details. First, suppose that $X \in L^1(\Omega, \sigma(\{X_{E_1}, \ldots, X_{E_n}\} \cup \mathfrak{N}), \mathbb{P})$ is such that $\mathbb{E}[X \exp\{i \sum_{j=1}^n \alpha_j X_{E_j}\}] = 0$ for all $\alpha_j \in \mathbb{R}$. Then by the injectivity of the Fourier transform $X = 0$ a.s. (see the proof of Lemma 3.2.9). Hence, using the martingale convergence theorem as in Proposition 3.2.10, we can deduce that if $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{E}[X \exp\{i \sum_{j=1}^n \alpha_j X_{E_j}\}] = 0$ for all $n \in \mathbb{N}$, $\alpha_j \in \mathbb{R}$ and $E_j \in \mathcal{M}$ with $\mu(E_j) < \infty$, then $X = 0$ a.s.. The proof of the totality of the set in question can be completed in the same way as the proof of Theorem 3.2.11. □

Examples of generalised Poisson processes do exist. In [37], given a σ-finite measure space, Kingman gives a canonical construction of a Poisson random measure on the space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the set of integer or infinite valued measures on $(M, \mathcal{M})$ and $\mathcal{F}$ is the smallest σ-field on Ω such that for each $E \in \mathcal{M}$, $\nu \mapsto \nu(E)$ is measurable as a map from Ω to $\mathbb{N}_0 \cup \{\infty\}$. Kingman's construction uses Kolmogorov's consistency theorem and makes *a priori* use of

Poisson measures. An outline of the construction can be found in [33, §4], [29, §1.8] and [74, §II.7,10].

Generalised Poisson processes are in some sense an analogue of Gaussian fields [34, Definition 1.19]. We do not have an equivalent definition of a Gaussian Hilbert space in the Poisson situation, since non-trivial scalar multiples of (compensated) Poisson random variables never have a (compensated) Poisson distribution. Hence, a possible definition of a 'Poisson Hilbert space' could be that it is the closure, in $L^2(\Omega)$, of the linear span of a set of random variables with compensated Poisson distributions. However, to obtain an isometry between a general (complexified) Hilbert space, H, and a so-called Poisson Hilbert space, we need to specify which elements of H map to compensated Poisson variables. Therefore by working over $L^2(M, \mathcal{M}, \mu)$ and requiring $I(\mathbb{1}_E)$ to have a compensated Poisson distribution whenever $\mu(E) < \infty$, we can overcome this problem, and so have a canonical definition.

Our definition of a generalised Poisson process is actually analogous to that of a Gaussian stochastic integral defined in [34, Definition 7.16]. In [34, Theorem 7.25], given a Gaussian stochastic integral $I : L^2(M, \mathcal{M}, \mu) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$, the Wiener-Ito isometry is constructed from $\mathfrak{F}_+(L^2(M, \mathcal{M}, \mu))$ onto $L^2(\Omega, \sigma(\{I(f) : f \in L^2(M, \mathcal{M}, \mu)\} \cup \mathfrak{N}), \mathbb{P})$. The Wiener-Poisson isomorphism we construct is analogous to this isometry. As in the case of $\mathfrak{F}_+(L^2[0,1])$ the exponential vectors in $\mathfrak{F}_+(L^2(M))$ are defined in the same way, and again we shall denote the linear span of these vectors by E .

4.2 Generalised Wiener-Poisson isomorphisms

In order to construct $\mathcal{W}_I$ we introduce the Charlier polynomials, which are an analogue of the Hermite polynomials in the Gaussian case. We use the definition from [39, §1].

Definition 4.2.1 *For $f : \mathbb{N}_0 \to \mathbb{C}$, we let*

$$\Delta f(n) = f(n) - f(n-1)$$

with $f(-1) = 0$. If for all $t \in \mathbb{C}$ the function $T_t : \mathbb{N}_0 \to \mathbb{C}$ is

$$T_t(n) = \frac{t^n e^{-t}}{n!},$$

we define the Charlier polynomial of degree n to be the unique polynomial in two variables such that for all $x \in \mathbb{N}_0$,

$$C_n(t, x) = (-t)^n \frac{(\Delta)^n T_t(x)}{T_t(x)}.$$

The definition above does produce a polynomial since the denominator cancels out and can be extended for all $(t,x) \in \mathbb{C}^2$ by taking $x! = x(x-1)!$. We should point out that various different definitions of Charlier polynomials are used, mostly differing by a constant factor. From [39, Proposition 1], we have the following properties (see also [22, Chapter 3] and [54, §2]).

Proposition 4.2.2 *The Charlier polynomials satisfy the following properties:*

i) for all $\alpha \geq 0$, the sequence $(C_n(\alpha,\cdot))_{n\in\mathbb{N}_0}$ is a complete orthogonal system in $l^2(\mathbb{N}_0, T_\alpha)$,

ii) for all $\alpha \geq 0$, $\mathbb{E}[C_n(\alpha,\cdot)^2] = n!\alpha^n$ in $l^2(\mathbb{N}_0, T_\alpha)$,

iii) for all $z \in \mathbb{C}$, $x \in \mathbb{N}_0$ and $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} C_n(\alpha, x) = e^{-\alpha z}(1+z)^x,$$

iv) for all α, $\beta \geq 0$ and n, x, $y \in \mathbb{N}_0$,

$$\sum_{j=0}^{n} \binom{n}{j} C_{n-j}(\alpha, x) C_j(\beta, y) = C_n(\alpha+\beta, x+y).$$

These properties of the Charlier polynomials can be used to construct Wiener-Poisson isomorphisms. The construction is the same as in [39, §3] (see also [40]). However, Liebscher works only on Polish spaces with a measure structure on their Borel σ-field. We shall give an extension of these results to general measure spaces.

Given disjoint sets $(E_j)_{j=1}^k \subseteq \mathcal{M}$ with $\mu(E_j) < \infty$, and $(p_j)_{j=1}^k$ natural numbers with $\sum_{j=1}^k p_j = n$, we define for $\phi = \mathbb{1}_{E_1}^{\otimes p_1} \otimes \ldots \otimes \mathbb{1}_{E_k}^{\otimes p_k}$,

$$L^{(n)}(\phi) := \prod_{j=1}^{k} C_{p_j}(\mu(E_j), X_{E_j} + \mu(E_j)). \tag{4.2.1}$$

For $\sigma \in S_n$ and $f_1, \ldots, f_n \in L^2(M)$ we define $(f_1 \otimes \ldots \otimes f_n)^\sigma := f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(n)}$. Then we can extend $L^{(n)}$ by setting $L^{(n)}(\phi^\sigma) := L^{(n)}(\phi)$. If

$$H_0^n := \text{linsp}\{\mathbb{1}_{E_1} \otimes \ldots \otimes \mathbb{1}_{E_n} : E_i = E_j \text{ or } E_i \cap E_j = \emptyset\},$$

then $L^{(n)}$ can be linearly extended to a well-defined map on H_0^n. The proof of the well-definedness of $L^{(n)}$ is outlined in [39, Lemma 3]. We fill in the details using

$$\sum_{r=1}^{n} \sum_{j_r=1}^{N} \equiv \sum_{j_1}^{N} \cdots \sum_{j_n}^{N}$$

to simplify notation.

Lemma 4.2.3 *Suppose $(E_j)_{j=1}^N$ is a collection of disjoint sets in $\mathcal{M}$ with $\mu(E_j) < \infty$. Then if*

$$\sum_{r=1}^{n} \sum_{j_r=1}^{N} \alpha_{j_1,\ldots,j_n} \mathbb{1}_{E_{j_1}} \otimes \ldots \otimes \mathbb{1}_{E_{j_n}} = 0,$$

for all $k = 1, \ldots, n$ and $j_k = 1, \ldots, N$, either $\alpha_{j_1,\ldots,j_n} = 0$ or $\prod_{r=1}^n \mu(E_{j_r}) = 0$.

Proof: This follows immediately from taking the inner product with $\mathbb{1}_{E_{j_1}} \otimes \ldots \otimes \mathbb{1}_{E_{j_n}}$. □

Corollary 4.2.4 *If $(E_j)_{j=1}^N \subseteq \mathcal{M}$ are as above, and*

$$\sum_{r=1}^{n} \sum_{j_r=1}^{N} \alpha_{j_1,\ldots,j_n} \mathbb{1}_{E_{j_1}} \otimes \ldots \otimes \mathbb{1}_{E_{j_n}} = \sum_{r=1}^{n} \sum_{j_r=1}^{N} \beta_{j_1,\ldots,j_n} \mathbb{1}_{E_{j_1}} \otimes \ldots \otimes \mathbb{1}_{E_{j_n}},$$

then

$$\sum_{r=1}^{n} \sum_{j_r=1}^{N} \alpha_{j_1,\ldots,j_n} L^{(n)}(\mathbb{1}_{E_{j_1}} \otimes \ldots \otimes \mathbb{1}_{E_{j_n}}) = \sum_{r=1}^{n} \sum_{j_r=1}^{N} \beta_{j_1,\ldots,j_n} L^{(n)}(\mathbb{1}_{E_{j_1}} \otimes \ldots \otimes \mathbb{1}_{E_{j_n}}).$$

Lemma 4.2.5 *Let $(E_j)_{j=1}^n \subseteq \mathcal{M}$ with $\mu(E_j) < \infty$ be such that $E_i \cap E_j = \emptyset$ or $E_i = E_j$ for all $i, j = 1, \ldots, n$ and suppose $l_1, \ldots, l_k \in \{1, \ldots, n\}$ are distinct such that $E_{l_1} = \ldots = E_{l_k} = E$, but $E_j \cap E = \emptyset$ for $j \neq l_r$. If $E = F_1 \cup \ldots \cup F_m$ where $(F_j)_{j=1}^m \subseteq \mathcal{M}$ are disjoint and $\sigma \in S_n$ maps $\{1, \ldots, k\}$ onto $\{l_1, \ldots, l_k\}$. Then*

$$\mathbb{1}_{E_1} \otimes \ldots \otimes \mathbb{1}_{E_n} = \sum_{r=1}^{k} \sum_{j_r=1}^{m} (\mathbb{1}_{F_{j_1}} \otimes \mathbb{1}_{F_{j_2}} \otimes \ldots \otimes \mathbb{1}_{F_{j_k}} \otimes \mathbb{1}_{E_{\sigma(k+1)}} \otimes \ldots \otimes \mathbb{1}_{E_{\sigma(n)}})^{\sigma^{-1}}.$$

Furthermore

$$L^{(n)}(\mathbb{1}_{E_1} \otimes \ldots \otimes \mathbb{1}_{E_n}) = \sum_{r=1}^{k} \sum_{j_r=1}^{m} L^{(n)}((\mathbb{1}_{F_{j_1}} \otimes \mathbb{1}_{F_{j_2}} \otimes \ldots \otimes \mathbb{1}_{F_{j_k}} \otimes \mathbb{1}_{E_{\sigma(k+1)}} \otimes \ldots \otimes \mathbb{1}_{E_{\sigma(n)}})^{\sigma^{-1}}).$$

Proof: By the linearity of the tensor product,

$$\begin{aligned}(\mathbb{1}_{E_1} \otimes \ldots \otimes \mathbb{1}_{E_n})^{\sigma} &= \mathbb{1}_E^{\otimes k} \otimes \mathbb{1}_{E_{\sigma(k+1)}} \otimes \ldots \otimes \mathbb{1}_{E_{\sigma(n)}} \\ &= \sum_{r=1}^{k} \sum_{j_r=1}^{m} \mathbb{1}_{F_{j_1}} \otimes \mathbb{1}_{F_{j_2}} \otimes \ldots \otimes \mathbb{1}_{F_{j_k}} \otimes \mathbb{1}_{E_{\sigma(k+1)}} \otimes \ldots \otimes \mathbb{1}_{E_{\sigma(n)}},\end{aligned}$$

which establishes the first part. Consequently, from Proposition 4.2.2 iv)

$$\begin{aligned}L^{(n)}(\mathbb{1}_{E_1} \otimes \ldots \otimes \mathbb{1}_{E_n}) &= L^{(k)}(\mathbb{1}_E^{\otimes k}) L^{(n-k)}(\mathbb{1}_{E_{\sigma(k+1)}} \otimes \ldots \otimes \mathbb{1}_{E_{\sigma(n)}}) \\ &= \sum_{l_1+\ldots+l_m=k} \lambda_{l_1,\ldots,l_m} L^{(l_1)}(\mathbb{1}_{F_1}^{\otimes l_1}) \ldots L^{(l_m)}(\mathbb{1}_{F_m}^{\otimes l_m}) \\ &\qquad \times L^{(n-k)}(\mathbb{1}_{E_{\sigma(k+1)}} \otimes \ldots \otimes \mathbb{1}_{E_{\sigma(n)}}),\end{aligned}$$

where $\lambda_{l_1,\ldots,l_m}$ is the coefficient of $x_1^{l_1}\ldots x_m^{l_m}$ in the expansion $(x_1+\ldots+x_m)^k$. However, by the definition of the multinomial coefficient we obtain the second expression. $\square$

Proposition 4.2.6 *$L^{(n)}$ is a well-defined map on H_0^n.*

PROOF: Suppose

$$\phi = \sum_{j=1}^{m} \alpha_j \mathbb{1}_{E_{j,1}} \otimes \ldots \otimes \mathbb{1}_{E_{j,n}},$$

where for each $j = 1, \ldots m$, $E_{j,k} = E_{j,l}$ or $E_{j,k} \cap E_{j,l} = \emptyset$ and $\mu(E_{j,k}) < \infty$ for all $k, l = 1, \ldots, n$. Let $(E_j)_{j=1}^N$ be a finer partition of $(E_{j,k})_{j=1,k=1}^{m,n}$. By applying Lemma 4.2.5 to $\mathbb{1}_{E_{j,1}} \otimes \ldots \otimes \mathbb{1}_{E_{j,n}}$ we may find constants $\beta_{j_1,\ldots,j_n}$ such that

$$\phi = \sum_{r=1}^{n} \sum_{j_r=1}^{N} \beta_{j_1,\ldots,j_n} \mathbb{1}_{E_{j_1}} \otimes \ldots \otimes \mathbb{1}_{E_{j_n}},$$

and

$$L(\phi) := \sum_{j=1}^{m} \alpha_j L^{(n)}(\mathbb{1}_{E_{j,1}} \otimes \ldots \otimes \mathbb{1}_{E_{j,n}}) = \sum_{r=1}^{n} \sum_{j_r=1}^{N} \beta_{j_1,\ldots,j_n} L^{(n)}(\mathbb{1}_{E_{j_1}} \otimes \ldots \otimes \mathbb{1}_{E_{j_n}}).$$

Suppose that

$$\phi = \sum_{r=1}^{n} \sum_{j_r=1}^{N} \gamma_{j_1,\ldots,j_n} \mathbb{1}_{E_{j_1}} \otimes \ldots \otimes \mathbb{1}_{E_{j_n}},$$

for some constants $\gamma_{j_1,\ldots,j_n}$. Then by Corollary 4.2.4, we have that

$$L(\phi) = \sum_{r=1}^{n} \sum_{j_r=1}^{N} \beta_{j_1,\ldots,j_n} L^{(n)}(\mathbb{1}_{E_{j_1}} \otimes \ldots \otimes \mathbb{1}_{E_{j_n}}) = \sum_{r=1}^{n} \sum_{j_r=1}^{N} \gamma_{j_1,\ldots,j_n} L^{(n)}(\mathbb{1}_{E_{j_1}} \otimes \ldots \otimes \mathbb{1}_{E_{j_n}}).$$

Consequently, if we have two different representations of $\phi \in H_0^n$, by choosing a partition finer than both we obtain that $L^{(n)}$ is well-defined. $\square$

If $S : L^2(M)^{\otimes^n} \to L^2(M)^{\otimes_s^n}$ is the symmetrisation operator

$$S(f_1 \otimes \ldots \otimes f_n) := f_1 \otimes_s \ldots \otimes_s f_n = \frac{1}{n!} \sum_{\sigma \in S_n} (f_1 \otimes \ldots \otimes f_n)^{\sigma},$$

we have the following result.

Theorem 4.2.7 *Let $(M, \mathcal{M}, \mu)$ be a measure space and that $I : L^2(M, \mathcal{M}, \mu) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a generalised Poisson process. Then with the above definitions the map $L^{(n)} : S(H_0^n) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ satisfies*

$$\|L^{(n)}(\phi)\|^2 = n!\|\phi\|^2,$$

and thus extends to a continuous linear map $L^{(n)} : L^2(M)^{\otimes_s^n} \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ with the property that for $m \neq n$, $L^{(m)}(L^2(M)^{\otimes_s^m})$ is orthogonal to $L^{(n)}(L^2(M)^{\otimes_s^n})$. If we define $I^{(n)} := \frac{1}{\sqrt{n!}} L^{(n)}$, the map

$$\mathcal{W}_I := \bigoplus_{n=0}^{\infty} I^{(n)} : \mathfrak{F}_+(L^2(M)) \cong \bigoplus_{n=0}^{\infty} L^2(M)^{\otimes_s^n} \longrightarrow L^2(\Omega, \mathcal{F}, \mathbb{P})$$

is an isometry and will be called the Wiener-Poisson isomorphism associated with the generalised Poisson process $I : L^2(M, \mathcal{M}, \mu) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Proof: The proof is the same as in [39, Proposition 4], thus we shall only outline it. If $\phi = \mathbb{1}_{E_1}^{\otimes_s^{p_1}} \otimes_s \ldots \otimes_s \mathbb{1}_{E_k}^{\otimes_s^{p_k}}$, where $(E_j)_{j=1}^k \subseteq \mathcal{M}$ are mutually disjoint with $\mu(E_j) < \infty$, then

$$\|\phi\|^2 = \frac{1}{n!} \sum_{\sigma \in S_n} \langle \mathbb{1}_{E_1}^{\otimes p_1} \otimes \ldots \otimes \mathbb{1}_{E_k}^{\otimes p_k}, (\mathbb{1}_{E_1}^{\otimes p_1} \otimes \ldots \otimes \mathbb{1}_{E_k}^{\otimes p_k})^{\sigma} \rangle = \frac{1}{n!} \prod_{j=1}^{k} p_j! \mu(E_j)^{p_j}.$$

From Proposition 4.2.2 ii), we have

$$\|L^{(n)}(\phi)\|^2 = \prod_{j=1}^{k} p_j! \mu(E_j)^{p_j}.$$

Thus $\|L^{(n)}(\phi)\|^2 = n! \|\phi\|^2$. For $\psi \in S(H_0^n)$, we may write

$$\psi = \sum_{l=1}^{n} \sum_{j_l=1}^{N} \alpha_{j_1,\ldots,j_n} \mathbb{1}_{E_{j_1}} \otimes_s \ldots \otimes_s \mathbb{1}_{E_{j_n}},$$

for some constants $\alpha_{j_1,\ldots,j_n}$ and $(E_j)_{j=1}^N \subseteq \mathcal{M}$ disjoint. Then since $\mathbb{1}_{E_{i_1}} \otimes_s \ldots \otimes_s \mathbb{1}_{E_{i_n}}$ and $\mathbb{1}_{E_{j_1}} \otimes_s \ldots \otimes_s \mathbb{1}_{E_{j_n}}$ are orthogonal in $L^2(M)^{\otimes_s^n}$ and $L^{(n)}(\mathbb{1}_{E_{i_1}} \otimes_s \ldots \otimes_s \mathbb{1}_{E_{i_n}})$ and $L^{(n)}(\mathbb{1}_{E_{j_1}} \otimes_s \ldots \otimes_s \mathbb{1}_{E_{j_n}})$ are orthogonal in $L^2(M, \mathcal{M}, \mu)$ for $i \neq j$, the result follows for all $\psi \in L^2(M)^{\otimes_s^n}$. The fact that $L^{(m)}(S(H_0^m))$ is orthogonal to $L^{(n)}(S(H_0^n))$ for $m \neq n$ follows from the orthogonality of the Charlier polynomials. □

This construction agrees with the normal construction of Poisson multiple stochastic integrals given by a generalised Poisson process over an atomless measure space (see [64, §1]), since $C_1(t, x) = x - t$ and the diagonals in M^n have zero measure. In particular this is true when $M = \mathbb{R}_+$. In this case, if $I : L^2(\mathbb{R}_+) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a generalised Poisson process, then clearly $(N_t)_{t \geq 0}$, where $N_t = I(\mathbb{1}_{[0,t]}) + t$, is a Poisson process, and hence by Section 3.1 we may construct the isomorphism $\mathcal{W}$. Note that if $a_1 \leq b_1 \leq \ldots \leq a_n \leq b_n$ and we let

$$f(t_1, \ldots, t_n) = \mathbb{1}_{(a_1,b_1]}(t_1) \ldots \mathbb{1}_{(a_n,b_n]}(t_n) \in L^2(D_n),$$

then functions of the form $\sum_{\sigma \in S_n} f^\sigma$ are total in $L^2_{\mathrm{sym}}(\mathbb{R}^n_+)$, a concrete realisation of $L^2(\mathbb{R}_+)^{\otimes^n_s}$. We have from (3.1.2) that

$$\begin{aligned}\mathcal{W}(\sum_{\sigma \in S_n} f^\sigma) &= J^{(n)}(\sqrt{n!} f) \\ &= \sqrt{n!}(X_{b_1} - X_{a_1}) \dots (X_{b_n} - X_{a_n}),\end{aligned}$$

and from (4.2.1)

$$\begin{aligned}\mathcal{W}_I(\sum_{\sigma \in S_n} f^\sigma) &= \frac{1}{\sqrt{n!}} L^{(n)}(\sum_{\sigma \in S_n} (\mathbb{1}_{(a_1,b_1]} \otimes \dots \otimes \mathbb{1}_{(a_n,b_n]})^\sigma) \\ &= \sqrt{n!} L^{(n)}(\mathbb{1}_{(a_1,b_1]} \otimes \dots \otimes \mathbb{1}_{(a_n,b_n]}) \\ &= \sqrt{n!} \prod_{j=1}^n C_j((b_j - a_j), X_{b_j} - X_{a_j} + (b_j - a_j)) \\ &= \sqrt{n!}(X_{b_1} - X_{a_1}) \dots (X_{b_n} - X_{a_n}).\end{aligned}$$

Hence since both operators are bounded, $\mathcal{W}_I = \mathcal{W}$. Results for the Wiener-Poisson isomorphism when working over $\mathbb{R}_+$ are well-known and have been presented in Chapter 3. We would like to use these results in order to deduce results about the generalised Wiener-Poisson isomorphisms.

Lemma 4.2.8 *Let $I : L^2(M, \mathcal{M}, \mu) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ be a generalised Poisson process. Suppose $(E_j)_{j=1}^n \subseteq \mathcal{M}$ are disjoint sets with $\mu(E_j) < \infty$, $0 \leq t_0 \leq t_1 \leq \dots \leq t_n$ are such that $\mu(E_j) = t_j - t_{j-1}$ and $(N_t)_{t \geq 0}$ is a Poisson process on $(\Omega', \mathcal{F}', \mathbb{P}')$. For $\alpha_j \in \mathbb{C}$, define*

$$\alpha := \sum_{j=1}^n \alpha_j \mathbb{1}_{E_j} \in L^2(M), \quad \alpha' := \sum_{j=1}^n \alpha_j \mathbb{1}_{(t_{j-1}, t_j]} \in L^2(\mathbb{R}_+).$$

Then if

$$\mathcal{E}(\alpha') = f(X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}) \in L^2(\Omega', \mathcal{F}', \mathbb{P}')$$

for some Borel function f,

$$\mathcal{W}_I(e(\alpha)) = f(X_{E_1}, \dots, X_{E_n}) \in L^2(\Omega, \mathcal{F}, \mathbb{P}).$$

Proof: Note that $\alpha^{\otimes^k} \in S(H_0^k)$, and hence from (4.2.1),

$$\mathcal{W}_I(\alpha^{\otimes^k}) = \frac{1}{\sqrt{k!}} L^{(k)}(\alpha^{\otimes^k}) = P_k(X_{E_1}, \dots, X_{E_n}),$$

for some polynomial P_k. However as the construction of $L^{(k)}$ is the same for each generalised Poisson process I, we also have that

$$\mathcal{W}(\alpha'^{\otimes^k}) = P_k(X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}).$$

By independence, $\{X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}\}$ has the same distribution as $\{X_{E_1}, \ldots, X_{E_n}\}$. Therefore

$$\begin{aligned}
&\mathbb{E}_{(\Omega,\mathcal{F},\mathbb{P})}[|f(X_{E_1}, \ldots, X_{E_n}) - \sum_{k=0}^{m} \mathcal{W}_I(\frac{\alpha^{\otimes^k}}{\sqrt{n!}})|^2] \\
&= \mathbb{E}_{(\Omega',\mathcal{F}',\mathbb{P}')}[|f(X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}) - \sum_{k=0}^{m} \mathcal{W}(\frac{\alpha'^{\otimes^k}}{\sqrt{n!}})|^2] \\
&\to 0 \ \text{ as } m \to \infty.
\end{aligned}$$

Hence, we have the result. □

Lemma 4.2.9 *Suppose $(E_j)_{j=1}^n \subseteq \mathcal{M}$ are as above. Then if $\alpha = \sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}$ with $\alpha_j \in \mathbb{R}$,*

$$\mathcal{W}_I(e(e^{i\alpha} - 1)) = \exp\{-\int_M (e^{i\alpha} - 1 - i\alpha)d\mu + iI(\alpha)\}. \tag{4.2.2}$$

Proof: Let $0 \leq t_0 \leq t_1 \leq \ldots \leq t_n$ be such that $t_j - t_{j-1} = \mu(E_j)$. Then if $\alpha' = \sum_{j=1}^n \alpha_j \mathbb{1}_{(t_{j-1},t_j]}$, and $(N_t)_{t\geq 0}$ is a Poisson process, by Lemma 3.1.5,

$$\begin{aligned}
\mathcal{E}(e^{i\alpha'} - 1) &= \exp\{-\int_0^\infty (e^{i\alpha'} - 1 - i\alpha')ds + i\sum_{j=1}^n \alpha_j(X_{t_j} - X_{t_{j-1}})\} \\
&= \exp\{-\sum_{j=1}^n (e^{i\alpha_j} - 1 - i\alpha_j)(t_j - t_{j-1}) + i\sum_{j=1}^n \alpha_j(X_{t_j} - X_{t_{j-1}})\}.
\end{aligned}$$

Therefore we may apply Lemma 4.2.8 to get

$$\mathcal{E}(e^{i\alpha} - 1) = \exp\{-\sum_{j=1}^n (e^{i\alpha_j} - 1 - i\alpha_j)\mu(E_j) + i\sum_{j=1}^n \alpha_j X_{E_j}\},$$

because $t_j - t_{j-1} = \mu(E_j)$. Since

$$\sum_{j=1}^n (e^{i\alpha_j} - 1 - i\alpha_j)\mu(E_j) = \int_M (e^{i\alpha} - 1 - i\alpha)d\mu, \quad \sum_{j=1}^n \alpha_j X_{E_j} = I(\alpha),$$

we have (4.2.2). □

Corollary 4.2.10 *The set $\{\mathcal{W}_I(e(e^{i\alpha} - 1)) : \alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)\}$ is total in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.*

Proof: If $(E_j)_{j=1}^n \subseteq \mathcal{M}$ with $\mu(E_j) < \infty$, then if $\alpha_j \in \mathbb{R}$,

$$\exp\{i \sum_{j=1}^{n} \alpha_j X_{E_j}\} \in \operatorname{linsp}\{\mathcal{W}_I(e(e^{i\alpha} - 1)) : \alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)\}.$$

Thus the result follows immediately from Proposition 4.1.9. □

The formula (4.2.2) agrees with that given in [64, Corollary 2.1] for the usual construction of multiple Poisson integrals over non-atomic measure spaces. However, in our proof we use the definition of the Poisson exponential for a Poisson process over $\mathbb{R}_+$, and then obtain the formula (4.2.2) for more general Poisson processes. The formula [64, (2.11)] also holds in the non-atomic case. Surgailis has shown that

$$e^{I(f)} = \exp\{\int_M (e^f - 1 - f) d\mu\} \mathcal{W}_I(e(e^f - 1)), \tag{4.2.3}$$

by showing

$$\langle e^{I(f)}, I^{(n)}(\mathbb{1}_{E_1} \otimes_s \ldots \otimes_s \mathbb{1}_{E_n})\rangle = \langle \exp\{\int_M (e^f - 1 - f) d\mu\} e(e^f - 1), \mathbb{1}_{E_1} \otimes_s \ldots \otimes_s \mathbb{1}_{E_n}\rangle,$$

where $(E_j)_{j=1}^n \subseteq \mathcal{M}$ are disjoint with $\mu(E_j) < \infty$, since vectors of the form $\mathbb{1}_{E_1} \otimes_s \ldots \otimes_s \mathbb{1}_{E_n}$ are total in $\mathfrak{F}_+(L^2(M))$ when $(M, \mathcal{M}, \mu)$ is a non-atomic measure space. This argument cannot be used when $(M, \mathcal{M}, \mu)$ has atoms. However, we can use Corollary 4.2.10 and the formulae (4.1.3) and (4.2.2) to obtain (4.2.3).

Proposition 4.2.11 *Let $f \in L^2(M)$ be such that*

$$\int_{\mathrm{Re} f > 1} \exp\{2\mathrm{Re} f\} d\mu < \infty.$$

Then $e^{I(f)} \in L^2(\Omega)$ and

$$e^{I(f)} = \exp\{\int_M (e^f - 1 - f) d\mu\} \mathcal{W}_I(e(e^f - 1)). \tag{4.2.4}$$

Proof: From Proposition 4.1.6 we know that $e^{I(f)} \in L^2(M)$. By similar arguments to the proof of Proposition 4.1.6 we can establish that $e^f - 1 \in L^2(M)$, thus the right-hand side of (4.2.4) exists. If $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$, it follows from (4.1.3) that

$$\begin{aligned}
\langle e^{I(f)}, \mathcal{W}_I(e(e^{i\alpha} - 1))\rangle &= \exp\{-\int_M (e^{i\alpha} - 1 - i\alpha) d\mu\} \mathbb{E}[e^{I(\bar{f}) + iI(\alpha)}] \\
&= \exp\{-\int_M (e^{i\alpha} - 1 - i\alpha) d\mu\} \exp\{\int_M (e^{\bar{f} + i\alpha} - 1 - \bar{f} - i\alpha) d\mu\} \\
&= \exp\{\int_M (e^{\bar{f}} - 1 - \bar{f}) d\mu\} \exp\{\int_M (e^{\bar{f}} - 1)(e^{i\alpha} - 1) d\mu\} \\
&= \langle \exp\{\int_M (e^f - 1 - f) d\mu\} e(e^f - 1), e(e^{i\alpha} - 1)\rangle.
\end{aligned}$$

Since $\{\mathcal{W}_I(e(e^{i\alpha}-1)) : \alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)\}$ is total in $L^2(\Omega)$, (4.2.4) holds. □

From Corollary 4.2.10, it follows that $\mathcal{W}_I(\mathsf{E})$ is dense in $L^2(\Omega)$ and thus we can easily deduce the surjectivity of the map $\mathcal{W}_I$ onto $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Although the density of the linear span of elements $\exp\{i\sum_{j=1}^n \alpha_j X_{E_j}\} \subseteq \mathcal{W}_I(\mathsf{E})$ is probably well-known, this approach to proving the surjectivity of generalised Wiener-Poisson isomorphisms appears to be new. Alternative ways of showing the surjectivity of $\mathcal{W}_I$ are described after Corollary 4.5.2.

Theorem 4.2.12 *$\mathcal{W}_I$ maps $\mathfrak{F}_+(L^2(M))$ onto $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Thus the Wiener-Poisson isomorphism $\mathcal{W}_I : \mathfrak{F}_+(L^2(M)) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ is an isometric isomorphism.*

Proof: From Corollary 4.2.10, $\mathcal{W}_I(\mathsf{E})$ is dense in $L^2(\Omega, \sigma(\{X_E : E \in \mathcal{M} \text{ with } \mu(E) < \infty\} \cup \mathfrak{N}), \mathbb{P})$. Since $\mathcal{W}_I$ is an isometry surjectivity follows. □

Corollary 4.2.13 ([39, Theorem 5]) *Let M be a Polish space and $\mathcal{M}$ its Borel σ-field. Then if $I : L^2(M) \to L^2(\Omega)$ is a generalised Poisson process, $\mathcal{W}_I : \mathfrak{F}_+(L^2(M)) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ is an isometric isomorphism.*

If $(M, \mathcal{M}, \mu)$ is a σ-finite measure space, then $L^2(M^n)^{\otimes_s^n} \cong L^2_{\text{sym}}(M^n)$. Thus we have an isometry

$$I^{(n)} = \mathcal{W}_I|_{L^2(M^n)^{\otimes_s^n}} : L^2_{\text{sym}}(M^n) \longrightarrow L^2(\Omega, \mathcal{F}, \mathbb{P}),$$

and we can define multiple Poisson integrals for $f \in L^2(M^n)$ by

$$\int_{M^n} f dX^n := I^{(n)}(Sf).$$

The surjectivity of $\mathcal{W}_I$ is equivalent to saying that each random variable $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ has a unique expansion of the form

$$X = \sum_{n=0}^{\infty} \int_{M^n} f_n dX^n,$$

where $f_n \in L^2_{\text{sym}}(M^n)$ and

$$\sum_{n=0}^{\infty} \int_{M^n} |f_n|^2 d\mu^n = \mathbb{E}[|X|^2] < \infty.$$

If $E \in \mathcal{M}$ with $\mu(E) < \infty$ and P_E denotes the orthogonal projection on $L^2(M)$ consisting of multiplication by $\mathbb{1}_E$, then as in [28, §2] or [47, §IV.1.4,6], we can define operators $\lambda(P_E)$, $a(\mathbb{1}_E)$ and $a^\dagger(\mathbb{1}_E)$ on $\mathfrak{F}_+(L^2(M))$, which we shall write as Λ_E, A_E and $A_E^\dagger$ respectively. It is true that $\mathcal{W}_I$ intertwines $\Lambda_E + A_E + A_E^\dagger$ and $\widehat{X_E}$, however we shall leave the proof until Section 5.1.

4.3 Construction of $\mathcal{W}_I$ using exponential vectors

It is possible to construct the isomorphism $\mathcal{W}_I$ using the exponential vectors $e(e^{i\alpha} - 1)$ with $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$ and the formula (4.2.2), similar to the approach in [10, Proposition 14], although we shall construct our map initially on a more restricted exponential domain than Biane. In [8, §7] and [67, §4] the Wiener-Ito isomorphism is constructed in an analogous way using the full set of exponential vectors. In the Poisson case we restrict ourselves to a smaller set of exponential vectors since we know explicitly the formula for $\mathcal{W}_I(e(e^{i\alpha} - 1))$ with $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$. The following density result is based on the proof of [28, Proposition 6.2 d)].

Lemma 4.3.1 *Let $(M, \mathcal{M}, \mu)$ be any measure space. Then the set $\{e(e^{i\alpha} - 1) : \alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)\}$ is total in $\mathfrak{F}_+(L^2(M))$.*

PROOF: Suppose that for $j = 1, \ldots, n$, $f_j : \mathbb{R} \to L^2(M)$ is differentiable with derivative f_j'. Then

$$\frac{d}{d\theta}(f_1(\theta) \otimes_s \ldots \otimes_s f_n(\theta)) = \sum_{j=1}^{n} f_1(\theta) \otimes_s \ldots \otimes_s f_j'(\theta) \otimes_s \ldots \otimes_s f_n(\theta).$$

Let $\beta \in L^{\mathcal{S}}_{\mathbb{R}}(M)$ be such that $\beta = \sum_{j=1}^{k} \beta_j \mathbb{1}_{E_j}$, where $(E_j)_{j=1}^{k} \subseteq \mathcal{M}$ are disjoint sets with $\mu(E_j) < \infty$ and $\beta_j \in \mathbb{R}$. If we define $f : \mathbb{R} \to L^2(M)$ by

$$f(\theta) = e^{i\theta\beta} - 1 = \sum_{j=1}^{k} (e^{i\theta\beta_j} - 1)\mathbb{1}_{E_j},$$

it follows that

$$f^{(m)}(\theta) = \sum_{j=1}^{k} (i\beta_j)^m e^{i\theta\beta_j} \mathbb{1}_{E_j}.$$

In particular if $E = \bigcup_{j=1}^{k} E_j$, then for all $\theta \in \mathbb{R}$,

$$\|f(\theta)\| \le 2\mu(E), \quad \|f^{(m)}(\theta)\| \le \|\beta^m\|_\infty \mu(E).$$

We now differentiate $\theta \mapsto e(f(\theta))$ in $\mathfrak{F}_+(L^2(M))$. By using the mean value theorem we have

$$\begin{aligned}\left\| \frac{f(\theta + h)^{\otimes n} - f(\theta)^{\otimes n}}{h} \right\| &\le n \sup_{\phi \in \mathbb{R}} \{\|f(\phi)\|^{n-1} \|f'(\phi)\|\} \\ &\le n\|\beta\|_\infty 2^{n-1} \mu(E)^n.\end{aligned}$$

By the properties of Fock space,

$$\lim_{h\to 0}\left\|\sum_{n=1}^{\infty}\frac{f(\theta+h)^{\otimes n}-f(\theta)^{\otimes n}}{\sqrt{n!}h}-\frac{nf(\theta)^{\otimes_s^{n-1}}\otimes_s f'(\theta)}{\sqrt{n!}}\right\|^2$$
$$=\lim_{h\to 0}\sum_{n=1}^{\infty}\left\|\frac{f(\theta+h)^{\otimes n}-f(\theta)^{\otimes n}}{\sqrt{n!}h}-\frac{nf(\theta)^{\otimes_s^{n-1}}\otimes_s f'(\theta)}{\sqrt{n!}}\right\|^2.$$

By an application of the series version of the dominated convergence theorem the limit and sum can be interchanged and thus the above becomes

$$\sum_{n=1}^{\infty}\lim_{h\to 0}\left\|\frac{f(\theta+h)^{\otimes n}-f(\theta)^{\otimes n}}{\sqrt{n!}h}-\frac{nf(\theta)^{\otimes_s^{n-1}}\otimes_s f'(\theta)}{\sqrt{n!}}\right\|^2,$$

which can easily be seen to be zero. Therefore

$$\frac{d}{d\theta}e(f(\theta))=\sum_{n=1}^{\infty}\frac{nf(\theta)^{\otimes_s^{n-1}}\otimes_s f'(\theta)}{\sqrt{n!}}.$$

Arguing similarly for the second derivative we get

$$\left(\frac{d}{d\theta}\right)^2 e(f(\theta))=\sum_{n=1}^{\infty}\left(\frac{n(n-1)f(\theta)^{\otimes_s^{n-2}}\otimes_s f'(\theta)^{\otimes_s^2}}{\sqrt{n!}}+\frac{nf(\theta)^{\otimes_s^{n-1}}\otimes_s f''(\theta)}{\sqrt{n!}}\right).$$

Continuing in this way we obtain that

$$\left(-i\frac{d}{d\theta}\right)^n e(f(\theta))\Big|_{\theta=0},$$

which lies in the closure of the linear span of $\{e(e^{i\alpha}-1):\alpha\in L^{\mathcal{S}}_{\mathbb{R}}(M)\}$, is of the form

$$(*,*,\ldots,*,\sqrt{n!}\beta^{\otimes n},0,0,\ldots).$$

Thus for all β of this form, we have that

$$\beta^{\otimes n}\in\overline{\operatorname{linsp}\{e(e^{i\alpha}-1):\alpha\in L^{\mathcal{S}}_{\mathbb{R}}(M)\}}.$$

As these vectors are total in $\mathfrak{F}_+(L^2(M))$, the result follows. □

In fact we know this density results holds by Corollary 4.2.10, since $\mathcal{W}_I$ is an isomorphism. However, a different proof is given since we need to use this result to give an alternative construction of $\mathcal{W}_I$. Given the density of

$$\mathsf{E}'':=\operatorname{linsp}\{e(e^{i\alpha}-1):\alpha\in L^{\mathcal{S}}_{\mathbb{R}}(M)\},$$

if we can define a bounded linear operator on E'' then we can extend it to a bounded linear operator on $\mathfrak{F}_+(L^2(M))$. Also, given the linear independence of the exponential vectors, we only need to define the operator on each exponential vector and extend it by linearity to E''.

Theorem 4.3.2 *Define a linear operator* $\mathscr{W}_I : \mathsf{E}'' \to L^2(\Omega)$ *by setting*

$$\mathscr{W}_I(e(e^{i\alpha}-1)) = \exp\{-\int_M (e^{i\alpha}-1-i\alpha)d\mu + iI(\alpha)\}. \tag{4.3.1}$$

Then $\mathscr{W}_I$ *is a well-defined isometry and thus can be uniquely extended to an isometric isomorphism*

$$\mathscr{W}_I : \mathfrak{F}_+(L^2(M)) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$$

such that $\mathscr{W}_I|_{L^2(M)} = I$.

Proof: Note that if $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$, $\mathscr{W}_I(\alpha) \in L^2(\Omega)$. Suppose α, $\beta \in L^{\mathcal{S}}_{\mathbb{R}}(M)$, then $\beta - \alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$ and from Lemma 4.1.5,

$$\begin{aligned}
\langle \mathscr{W}_I(e(e^{i\alpha}-1)), \mathscr{W}_I(e(e^{i\beta}-1))\rangle &= \exp\{-\int_M (e^{-i\alpha}-1+i\alpha)d\mu\} \\
&\quad \times \exp\{-\int_M (e^{i\beta}-1-i\beta)d\mu\}\mathbb{E}[e^{iI(\beta-\alpha)}] \\
&= \exp\{-\int_M (e^{i\beta}+e^{-i\alpha}+i(\alpha-\beta)-2)d\mu\}\mathbb{E}[e^{iI(\beta-\alpha)}] \\
&= \exp\{-\int_M (e^{i\beta}+e^{-i\alpha}+i(\alpha-\beta)-2)d\mu\} \\
&\quad \times \exp\{\int_M (e^{i(\beta-\alpha)}-1-i(\beta-\alpha))d\mu\} \\
&= \exp\{\int_M (e^{i\beta}-1)(e^{-i\alpha}-1)d\mu\} \\
&= \langle e(e^{i\alpha}-1), e(e^{i\beta}-1)\rangle.
\end{aligned}$$

Hence if α, $\beta \in L^{\mathcal{S}}_{\mathbb{R}}(M)$ are such that $e^{i\alpha}-1 = e^{i\beta}-1$ then

$$\mathbb{E}[|\mathscr{W}_I(e(e^{i\alpha}-1)) - \mathscr{W}_I(e(e^{i\beta}-1))|^2] = \|e(e^{i\alpha}-1) - e(e^{i\beta}-1)\|^2 = 0.$$

Thus $\mathscr{W}_I : \mathsf{E}'' \to L^2(\Omega)$ is a well-defined isometry and hence by Lemma 4.3.1 we can extend it to an isometry $\mathscr{W}_I : \mathfrak{F}_+(L^2(M)) \to L^2(\Omega)$. Since $\mathscr{W}_I(\mathsf{E}'')$ is dense in $L^2(\Omega)$, the map $\mathscr{W}_I$ is surjective and we have an isometric isomorphism. For the last part, recall from the proof of Lemma 4.3.1 that for $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$,

$$-i\frac{d}{d\theta}e(e^{i\theta\alpha}-1)\Big|_{\theta=0} = \alpha.$$

Since $\mathscr{W}_I$ is a unitary map, (4.3.1) implies that in $L^2(\Omega)$,

$$\exp\{-\int_M (e^{i\theta\alpha}-1-i\theta\alpha)d\mu + i\theta I(\alpha)\}$$

is differentiable with respect to θ at $\theta = 0$. Thus in $L^2(\Omega)$,

$$-i\frac{d}{d\theta}\exp\{-\int_M (e^{i\theta\alpha} - 1 - i\theta\alpha)d\mu + i\theta I(\alpha)\}\Big|_{\theta=0} = I(\alpha).$$

The above can also be proved directly using the dominated convergence theorem. Therefore differentiating both sides of (4.3.1) and multiplying by $-i$ gives that on $L^{\mathcal{S}}_{\mathbb{R}}(M)$, $\mathscr{W}_I = I$. Hence by the totality of $L^{\mathcal{S}}_{\mathbb{R}}(M)$ in $L^2(M)$ the claim follows. □

From the definition of $\mathscr{W}_I$ we can deduce that $\mathscr{W}_I|_{L^2(M)^{\otimes^n_s}} = \frac{1}{\sqrt{n!}}L^{(n)}$, where $L^{(n)}$ is as described previously.

Lemma 4.3.3 *If $E \in \mathcal{M}$ with $\mu(E) < \infty$ then*

$$\mathscr{W}_I(\mathbb{1}_E^{\otimes m}) = \frac{1}{\sqrt{m!}}C_m(\mu(E), X_E + \mu(E)).$$

In particular for all $\theta \in \mathbb{R}$,

$$\mathscr{W}_I(e(e^{i\theta\mathbb{1}_E} - 1)) = \sum_{n=0}^{\infty}\frac{(e^{i\theta} - 1)^n}{n!}C_n(\mu(E), X_E + \mu(E)).$$

Proof: The proof is by induction. When $m = 0$, $\mathscr{W}_I(1) = \mathscr{W}_I(e(0)) = 1$. Assume

$$\mathscr{W}_I(\mathbb{1}_E^{\otimes k}) = \frac{1}{\sqrt{k!}}C_k(\mu(E), X_E + \mu(E))$$

for all $k < m$. For all $\theta \in \mathbb{R}$ we know from Proposition 4.2.2 iii) that pointwise

$$\sum_{n=0}^{\infty}\frac{(e^{i\theta} - 1)^n}{n!}C_n(\mu(E), X_E + \mu(E)) = \exp\{-(e^{i\theta} - 1 - i\theta)\mu(E) + i\theta X_E\} \tag{4.3.2}$$
$$= \mathscr{W}_I(e(e^{i\theta\mathbb{1}_E} - 1)).$$

However, from Proposition 4.2.2 ii) we know that

$$\sum_{n=0}^{\infty}\frac{|(e^{i\theta} - 1)^n|^2}{n!^2}\|C_n(\mu(E), X_E + \mu(E))\|^2 = \sum_{n=0}^{\infty}\frac{|(e^{i\theta} - 1)^n|^2}{n!^2}n!\mu(E)^n < \infty.$$

Thus the sum in (4.3.2) also converges in $L^2(\Omega)$ to $\mathscr{W}_I(e(e^{i\theta\mathbb{1}_E} - 1))$. Since $\mathscr{W}_I$ is a unitary map,

$$\left(-i\frac{d}{d\theta}\right)^m\sum_{n=0}^{\infty}\frac{(e^{i\theta} - 1)^n}{n!}C_n(\mu(E), X_E + \mu(E)) = \mathscr{W}_I(\left(-i\frac{d}{d\theta}\right)^m e(e^{i\theta\mathbb{1}_E} - 1)).$$

Term by term differentiation is valid. Hence,

$$\sum_{n=0}^{\infty}\left(-i\frac{d}{d\theta}\right)^m \frac{(e^{i\theta}-1)^n}{n!}\Big|_{\theta=0} C_n(\mu(E), X_E+\mu(E)) = \sum_{n=0}^{\infty}\left(-i\frac{d}{d\theta}\right)^m \frac{(e^{i\theta}-1)^n}{\sqrt{n!}}\Big|_{\theta=0}\mathscr{W}_I(\mathbb{1}_E^{\otimes^n}).$$

Due to the fact that

$$\left(\frac{d}{d\theta}\right)^m (e^{i\theta}-1)^n\Big|_{\theta=0} = 0 \ \text{ for all } m<n,$$

the lemma follows by the induction hypothesis. □

Theorem 4.3.4 *If $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$ and $\alpha = \sum_{j=1}^r \alpha_j \mathbb{1}_{E_j}$ where $(E_j)_{j=1}^r \subseteq \mathcal{M}$ are disjoint with $\mu(E_j) < \infty$ and $\alpha_j \in \mathbb{R}$, then $\mathscr{W}_I(\alpha^{\otimes^n})$ is*

$$\frac{1}{\sqrt{n!}}\sum_{l_1+\ldots+l_r=n} \lambda_{l_1,\ldots,l_r}\alpha_1^{l_1}\ldots\alpha_r^{l_r} C_{l_1}(\mu(E_1), X_{E_1}+\mu(E_1))\ldots C_{l_r}(\mu(E_r), X_{E_r}+\mu(E_r)), \tag{4.3.3}$$

where $\lambda_{l_1,\ldots,l_r}$ is the coefficient of $x_1^{l_1}\ldots x_r^{l_r}$ in the expansion $(x_1+\ldots+x_r)^n$.

Proof: When $n=0$ the result holds since $\mathscr{W}_I(1)=1$. Assume the formula holds for all $m<n$. Then by polarisation it follows that for $m<n$,

$$\mathscr{W}_I(\mathbb{1}_{F_1}^{\otimes_s^{p_1}} \otimes_s \ldots \otimes_s \mathbb{1}_{F_k}^{\otimes_s^{p_k}}) = \frac{1}{\sqrt{m!}} L^{(m)}(\mathbb{1}_{F_1}^{\otimes^{p_1}} \otimes \ldots \otimes \mathbb{1}_{F_k}^{\otimes^{p_k}}), \tag{4.3.4}$$

where $(F_j)_{j=1}^k \subseteq \mathcal{M}$ are disjoint with $\mu(F_j)<\infty$, $\sum_{j=1}^k p_j = m$ and $L^{(m)}$ is as previously defined. Let α be as in the statement of the theorem and put $\mathcal{F}_\alpha := \sigma(\{X_{E_1},\ldots,X_{E_r}\}\cup\mathfrak{N})$. From the proof of Lemma 4.3.1 we have

$$\left(-i\frac{d}{d\theta}\right)^n \mathscr{W}_I(e(e^{i\theta\alpha}-1))\Big|_{\theta=0} = \mathscr{W}_I(f_0,\ldots,f_{n-1},\sqrt{n!}\alpha^{\otimes^n},0,\ldots), \tag{4.3.5}$$

where for $i=0,\ldots,n-1$,

$$f_i = \sum_{k=1}^{i}\sum_{j_k=1}^{r} \beta_{j_1,\ldots,j_i}\mathbb{1}_{E_{j_1}} \otimes_s \ldots \otimes_s \mathbb{1}_{E_{j_i}},$$

for some constants $\beta_{j_1,\ldots,j_i}$. From the definition of $\mathscr{W}_I$ and (4.3.4) it follows that $\mathscr{W}_I(\alpha^{\otimes^n}) \in L^2(\Omega,\mathcal{F}_\alpha,\mathbb{P})$. For $l\geq 0$ define

$$H^{(l)} := \operatorname{linsp}\{L^{(l)}(\mathbb{1}_{E_1}^{\otimes^{k_1}} \otimes \ldots \otimes \mathbb{1}_{E_r}^{\otimes^{k_r}}) : \sum_{j=1}^r k_j = l\}. \tag{4.3.6}$$

Then by the completeness and orthogonality of the Charlier polynomials (Proposition 4.2.2 i)), we can deduce

$$L^2(\Omega, \mathcal{F}_\alpha, \mathbb{P}) = \bigoplus_{l=0}^{\infty} H^{(l)}.$$

Since $L^2(M)^{\otimes_s^m}$ is orthogonal to $L^2(M)^{\otimes_s^n}$ in $\mathfrak{F}_+(L^2(M))$ for $m \neq n$, $\mathscr{W}_I(L^2(M)^{\otimes_s^m})$ is orthogonal to $\mathscr{W}_I(L^2(M)^{\otimes_s^n})$ for $m \neq n$. Therefore from (4.3.4), $\mathscr{W}_I(\alpha^{\otimes^n})$ is orthogonal to $H^{(m)}$ for $m < n$. We shall denote the expression (4.3.3) by $K^{(n)}(\alpha)$. If $\sum_{j=1}^r k_j = n$ then

$$\begin{aligned}
&\langle \sqrt{n!} K^{(n)}(\alpha), L^{(n)}(\mathbb{1}_{E_1}^{\otimes k_1} \otimes \ldots \otimes \mathbb{1}_{E_r}^{\otimes k_r})\rangle \\
&\qquad = \sum_{l_1+\ldots+l_r=n} \lambda_{l_1,\ldots,l_r} \alpha_1^{l_1} \ldots \alpha_r^{l_r} \langle L^{(n)}(\mathbb{1}_{E_1}^{\otimes l_1} \otimes \ldots \otimes \mathbb{1}_{E_r}^{\otimes l_r}), L^{(n)}(\mathbb{1}_{E_1}^{\otimes k_1} \otimes \ldots \otimes \mathbb{1}_{E_r}^{\otimes k_r})\rangle.
\end{aligned}$$

By the orthogonality of the Charlier polynomials the inner product in the above sum vanishes unless $l_j = k_j$ for $j = 1, \ldots, r$. Consequently

$$\langle \sqrt{n!} K^{(n)}(\alpha), L^{(n)}(\mathbb{1}_{E_1}^{\otimes k_1} \otimes \ldots \otimes \mathbb{1}_{E_r}^{\otimes k_r})\rangle = \lambda_{k_1,\ldots,k_r} \alpha_1^{k_1} \ldots \alpha_r^{k_r} \| L^{(n)}(\mathbb{1}_{E_1}^{\otimes k_1} \otimes \ldots \otimes \mathbb{1}_{E_r}^{\otimes k_r})\|^2.$$

Now for $m \geq n$ and $\sum_{j=1}^r k_j = m$ from the definition of $\mathscr{W}_I$,

$$\begin{aligned}
&\left(\frac{d}{d\theta}\right)^n \langle \mathscr{W}_I(e(e^{i\theta\alpha}-1)), L^{(m)}(\mathbb{1}_{E_1}^{\otimes k_1} \otimes \ldots \otimes \mathbb{1}_{E_r}^{\otimes k_r})\rangle \Big|_{\theta=0} \\
&\qquad = \left(\frac{d}{d\theta}\right)^n \prod_{j=1}^r \langle \mathscr{W}_I(e(e^{i\theta\alpha_j \mathbb{1}_{E_j}} - 1)), L^{(k_j)}(\mathbb{1}_{E_j}^{\otimes k_j})\rangle \Big|_{\theta=0} \\
&\qquad = \sum_{l_1+\ldots+l_r=n} \lambda_{l_1,\ldots,l_r} \prod_{j=1}^r \left(\frac{d}{d\theta}\right)^{l_j} \langle \mathscr{W}_I(e(e^{i\theta\alpha_j \mathbb{1}_{E_j}} - 1)), L^{(k_j)}(\mathbb{1}_{E_j}^{\otimes k_j})\rangle \Big|_{\theta=0}.
\end{aligned}$$

If $l_j < k_j$ for some j, then by Lemma 4.3.3,

$$\left(\frac{d}{d\theta}\right)^{l_j} \langle \mathscr{W}_I(e(e^{i\theta\alpha_j \mathbb{1}_{E_j}} - 1)), L^{(k_j)}(\mathbb{1}_{E_j}^{\otimes k_j})\rangle \Big|_{\theta=0} = 0,$$

and therefore the product in the previous expression vanishes. In particular this is true when

$m > n$. If $m = n$ by Lemma 4.3.3,

$$\left(-i\frac{d}{d\theta}\right)^n \langle \mathscr{W}_I(e(e^{i\theta\alpha}-1)), L^{(n)}(\mathbb{1}_{E_1}^{\otimes k_1} \otimes \ldots \otimes \mathbb{1}_{E_r}^{\otimes k_r})\rangle \Big|_{\theta=0}$$

$$= \lambda_{k_1,\ldots,k_r} \prod_{j=1}^{r} \left(-i\frac{d}{d\theta}\right)^{k_j} \langle \mathscr{W}_I(e(e^{i\theta\alpha_j \mathbb{1}_{E_j}}-1)), L^{(k_j)}(\mathbb{1}_{E_j}^{\otimes k_j})\rangle \Big|_{\theta=0}$$

$$= \lambda_{k_1,\ldots,k_r} \prod_{j=1}^{r} \langle \sqrt{k_j!}\alpha_j^{k_j} \mathscr{W}_I(\mathbb{1}_{E_j}^{\otimes k_j}), L^{(k_j)}(\mathbb{1}_{E_j}^{\otimes k_j})\rangle$$

$$= \lambda_{k_1,\ldots,k_r} \alpha_1^{k_1} \ldots \alpha_r^{k_r} \prod_{j=1}^{r} \|L^{(k_j)}(\mathbb{1}_{E_j}^{\otimes k_j})\|^2$$

$$= \lambda_{k_1,\ldots,k_r} \alpha_1^{k_1} \ldots \alpha_r^{k_r} \|L^{(n)}(\mathbb{1}_{E_1}^{\otimes k_1} \otimes \ldots \otimes \mathbb{1}_{E_r}^{\otimes k_r})\|^2.$$

Therefore by (4.3.5) for all m, whenever $\sum_{j=1}^r k_j = m$,

$$\langle \mathscr{W}_I(\alpha^{\otimes^n}), L^{(m)}(\mathbb{1}_{E_1}^{\otimes k_1} \otimes \ldots \otimes \mathbb{1}_{E_r}^{\otimes k_r})\rangle = \langle K^{(n)}(\alpha), L^{(m)}(\mathbb{1}_{E_1}^{\otimes k_1} \otimes \ldots \otimes \mathbb{1}_{E_r}^{\otimes k_r})\rangle.$$

Hence by (4.3.6) since $\mathscr{W}_I(\alpha^{\otimes^n}) \in L^2(\Omega, \mathcal{F}_\alpha, \mathbb{P})$, the result follows by induction. □

Corollary 4.3.5 *Suppose $(F_j)_{j=1}^r \subseteq \mathcal{M}$ are disjoint with $\mu(F_j) < \infty$, and $(k_j)_{j=1}^r$ are natural numbers with $\sum_{j=1}^r k_j = m$, then*

$$\mathscr{W}_I(\mathbb{1}_{F_1}^{\otimes_s^{k_1}} \otimes_s \ldots \otimes_s \mathbb{1}_{F_r}^{\otimes_s^{k_r}}) = \frac{1}{\sqrt{m!}} C_{k_1}(\mu(F_1), X_{F_1} + \mu(F_1)) \ldots C_{k_r}(\mu(F_r), X_{F_r} + \mu(F_r)).$$

PROOF: The result follows from the previous theorem by polarisation, similarly to (4.3.4). □

Note that the above corollary gives a new way of showing that the maps $L^{(n)}$ are well-defined. For $\alpha_j \in L^{\mathcal{S}}_{\mathbb{R}}(M)$,

$$\mathscr{W}_I(\alpha_1 \otimes_s \ldots \otimes_s \alpha_n) = \frac{1}{\sqrt{n!}} L^{(n)}(\alpha_1 \otimes \ldots \otimes \alpha_n).$$

and therefore $\mathcal{W}_I = \mathscr{W}_I$. Thus we have shown it is possible to construct the Wiener-Poisson isomorphism from just the exponentials. In the Gaussian case, if $I : L^2(M) \to L^2(\Omega_{\mathfrak{w}}, \mathcal{F}_{\mathfrak{w}}, \mathbb{P}_{\mathfrak{w}})$ is a Gaussian field, we know that if $f \in L^2(M)$ is of norm 1,

$$\mathfrak{w}_I(f^{\otimes^n}) = \frac{1}{\sqrt{n!}} h_n(I(f)),$$

where h_n is the nth Hermite polynomial. Such a general formula does not exist in the Poisson case, since only for very special functions $f \in L^2(M)$ can $\mathcal{W}_I(f^{\otimes^n})$ be written in terms of Charlier polynomials.

4.4 Reproducing kernel Hilbert spaces

Fock space can also be constructed using reproducing kernel Hilbert spaces. The theory of reproducing kernel Hilbert spaces was first developed by Aronszajn in [1]. A brief summary of the main results is presented in [34, Appendix F].

Definition 4.4.1 *Let T be a set. Then a covariance on T is a function $C : T \times T \to \mathbb{C}$ such that*

i) $C(s,t) = \overline{C(t,s)}$,

ii) $\sum_{i,j=1}^{n} \overline{\lambda_i}\lambda_j C(t_i,t_j) \geq 0$ *for all finite subsets* $\{t_j\}_{j=1}^{n} \subseteq T$ *and* $\{\lambda_j\}_{j=1}^{n} \subseteq \mathbb{C}$.

Theorem 4.4.2 ([34, Theorem F.2]) *Suppose that $C : T \times T \to \mathbb{C}$ is a covariance function on T. Then there exists a unique Hilbert space $K(C)$ of functions on T, called the reproducing kernel Hilbert space of C, such that*

i) $C(t,\cdot) \in K(C)$,

ii) $f(t) = \langle C(t,\cdot), f\rangle$.

Lemma 4.4.3 ([34, Theorem F.3]) *Let $K(C)$ be the reproducing kernel Hilbert space of $C : T \times T \to \mathbb{C}$. Then $\{C(t,\cdot)\}_{t\in T}$ is total in $K(C)$ and*

$$C(s,t) = \langle C(s,\cdot), C(t,\cdot)\rangle.$$

If H is any Hilbert space then $\Gamma_H : H \times H \to \mathbb{C}$ defined by

$$\Gamma_H(h,h') := e^{\langle h,h'\rangle}$$

is a covariance function and hence there exists a reproducing kernel Hilbert space $K(\Gamma_H)$. From the above lemma it follows that the mapping

$$\Gamma_H(h,\cdot) \longmapsto e(h)$$

induces an isometric isomorphism from $K(\Gamma_H)$ onto $\mathfrak{F}_+(H)$.

Proposition 4.4.4 ([34, Theorem F.5]) *Let H be a Hilbert space and $\{h_t\}_{t\in T}$ a total set in H. Define a map R from H into the space of all functions on T, by $R(h)(t) = \langle h_t, h\rangle$. Then R is injective and if we define an inner product on $R(H) := \{R(h) : h \in H\}$ by*

$$\langle f,g\rangle := \langle R^{-1}(f), R^{-1}(g)\rangle_H,$$

which makes $R : H \to R(H)$ into an isometry, then $R(H)$ becomes a reproducing kernel Hilbert space with covariance

$$C(s,t) = \langle h_s, h_t \rangle.$$

Using this result Chatterji and Mandrekar gave a construction of the Wiener-Ito isomorphism $\mathfrak{w}_I : K(\Gamma_H) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ induced by a Gaussian field $I : H \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ (see [14, Corollary 6.1]). If $\mathcal{E}_{\mathfrak{w}_I}(h) := e^{I(h) - \frac{1}{2}\|h\|^2}$, then $\{\mathcal{E}_{\mathfrak{w}_I}(h)\}_{h \in H}$ is a total set in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Consequently, from the above theorem we can construct the reproducing kernel Hilbert space of the covariance function

$$C(h, h') = \langle \mathcal{E}_{\mathfrak{w}_I}(h), \mathcal{E}_{\mathfrak{w}_I}(h') \rangle = e^{\langle h, h' \rangle} = \Gamma_H(h, h').$$

Therefore

$$K(\Gamma_H) = \{R(X) : R(X)(h) = \langle \mathcal{E}_{\mathfrak{w}_I}(h), X \rangle,\ X \in L^2(\Omega)\}$$

with the inner product

$$\langle R(X), R(Y) \rangle = \langle X, Y \rangle.$$

Hence, we have an isomorphism $\mathfrak{w} : K(\Gamma_H) \to L^2(\Omega_{\mathfrak{w}}, \mathcal{F}_{\mathfrak{w}}, \mathbb{P}_{\mathfrak{w}})$ given by $e^{\langle h, \cdot \rangle} \mapsto \mathcal{E}_{\mathfrak{w}}(h)$, which by the identification of $e^{\langle h, \cdot \rangle}$ and $e(h)$ is the Wiener-Ito isomorphism. More details of this construction of the Wiener-Ito isomorphism can be found in [35]. We would like to construct the Wiener-Poisson isomorphism in a similar fashion. Since we only work with a restricted exponential domain for the Poisson case, we need the following lemma.

Lemma 4.4.5 ([34, Theorem F.8]) *Suppose $K(C)$ is the reproducing kernel Hilbert space of the covariance $C : T \times T \to \mathbb{C}$ and that T' is a subset of T such that the only function in $K(C)$ which vanishes on all of T' is the zero function. Then the set*

$$K(C') = \{f|_{T'} : f \in K(C)\}$$

with inner product

$$\langle f|_{T'}, g|_{T'} \rangle_{K(C')} = \langle f, g \rangle_{K(C)}$$

is the reproducing kernel Hilbert space of the covariance $C|_{T' \times T'}$.

Lemma 4.4.6 *Let $f \in K(\Gamma_H)$. Then $h \mapsto f(h)$ is a continuous map from H to $\mathbb{C}$.*

Proof: Suppose h, $h' \in H$, then from Lemma 4.4.3,

$$\begin{aligned}\|\Gamma_H(h, \cdot) - \Gamma_H(h', \cdot)\|^2_{K(\Gamma_H)} &= \langle \Gamma_H(h, \cdot), \Gamma_H(h, \cdot) \rangle_{K(\Gamma_H)} + \langle \Gamma_H(h', \cdot), \Gamma_H(h', \cdot) \rangle_{K(\Gamma_H)} \\ &\quad - \langle \Gamma_H(h, \cdot), \Gamma_H(h', \cdot) \rangle_{K(\Gamma_H)} - \langle \Gamma_H(h', \cdot), \Gamma_H(h, \cdot) \rangle_{K(\Gamma_H)} \\ &= \Gamma_H(h, h) + \Gamma_H(h', h') - \Gamma_H(h, h') - \Gamma_H(h', h).\end{aligned}$$

Thus, if $h_n \to h$ then $\Gamma_H(h_n, \cdot) \to \Gamma_H(h, \cdot)$, which implies that $h \mapsto \Gamma_H(h, \cdot)$ is continuous. Since $f(h) = \langle \Gamma(h, \cdot), f\rangle$ it follows that $h \mapsto f(h)$ is continuous. □

Corollary 4.4.7 *If $f \in K(\Gamma_H)$ and $f|_D = 0$ for some total set D in H, then $f \equiv 0$.*

Now suppose we have a generalised Poisson process $I : L^2(M, \mathcal{M}, \mu) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$. Then consider the set $D = \{e^{i\alpha} - 1 : \alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)\}$ which is total in $L^2(M)$. For $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$ we let

$$\mathcal{E}(e^{i\alpha} - 1) := \exp\{-\int_M (e^{i\alpha} - 1 - i\alpha)d\mu + iI(\alpha)\}.$$

By the same argument as in the proof of Theorem 4.3.2, the above expression is well-defined. From Proposition 4.1.9 the set $\{\mathcal{E}(f) : f \in D\}$ is total in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Furthermore for f, $g \in D$,

$$\langle \mathcal{E}(f), \mathcal{E}(g)\rangle = e^{\langle f,g\rangle} = \Gamma'_{L^2(M)}(f, g),$$

where $\Gamma'_{L^2(M)} = \Gamma_{L^2(M)}|_{D\times D}$. Hence by Proposition 4.4.4,

$$K(\Gamma'_{L^2(M)}) = \{R(X) : R(X)(f) = \langle \mathcal{E}(f), X\rangle,\ X \in L^2(\Omega)\},$$

and we have an isometry $\mathcal{W}_I : K(\Gamma'_{L^2(M)}) \to L^2(\Omega)$ given by $e^{\langle e^{i\alpha}-1,\cdot\rangle} \mapsto \mathcal{E}(e^{i\alpha} - 1)$ for $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$. By Corollary 4.4.7, if $f \in K(\Gamma_{L^2(M)})$ and $f|_D = 0$, then $f \equiv 0$. Therefore we may apply Lemma 4.4.5 to obtain that $K(\Gamma_{L^2(M)}) \cong K(\Gamma'_{L^2(M)})$ with the identification $f \mapsto f|_D$.

Theorem 4.4.8 *Let $I : L^2(M, \mathcal{M}, \mu) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ be a generalised Poisson process. Then there exists a unique isometric isomorphism*

$$\mathcal{W}_I : K(\Gamma_{L^2(M)}) \longrightarrow L^2(\Omega, \mathcal{F}, \mathbb{P}),$$

such that if $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$,

$$\mathcal{W}_I : e^{\langle e^{i\alpha}-1,\cdot\rangle} \longmapsto \mathcal{E}(e^{i\alpha} - 1).$$

4.5 Discrete chaos

When dealing with the Gaussian case, that is if $I : L^2(M) \to L^2(\Omega)$ is a Gaussian field, then for $f_1, \ldots, f_n \in L^2(M)$,

$$\mathfrak{w}_I(\sqrt{n!} f_1 \otimes_s \ldots \otimes_s f_n) = \pi_n(I(f_1) \ldots I(f_n)),$$

where π_n is the orthogonal projection onto the nth chaos space $L^2(M)^{:n:} := \mathfrak{w}_I(L^2(M)^{\otimes_s^n})$. In this section, we shall briefly describe a similar formula for the Wiener-Poisson isomorphism. A formula of this type is given in [64, Proposition 1.1] for the non-atomic case. However, the proof is quite complex and requires a multiplication formula to be established [64, Proposition 3.1]. The result we shall present here is not as general, but the proofs we give are much simpler. Following the notations of [22] we let

$$\begin{aligned}\mathbb{P}_n: =& \operatorname{linsp}\{\prod_{j=1}^k C_{p_j}(\mu(E_j), X_{E_j}+\mu(E_j)) : \sum_{j=1}^k p_j \le n,\ (E_j)_{j=1}^k \subseteq \mathcal{M} \text{ disjoint with } \mu(E_j)<\infty\}\\ =& \operatorname{linsp}\{\prod_{j=1}^k X_{E_j}^{p_j} : \sum_{j=1}^k p_j \le n,\ (E_j)_{j=1}^k \subseteq \mathcal{M} \text{ disjoint with } \mu(E_j)<\infty\}.\end{aligned}$$

Note that

$$\overline{\mathbb{P}_n} = \mathcal{W}_I(\bigoplus_{k=0}^n L^2(M)^{\otimes_s^n}).$$

We let

$$\overline{\mathbb{Q}_n} := \overline{\mathbb{P}_n} \cap \mathbb{P}_{n-1}^{\perp} = \mathcal{W}_I(L^2(M)^{\otimes_s^n}),$$

the *nth discrete chaos*, and define ρ_n to be the orthogonal projection onto $\overline{\mathbb{Q}_n}$.

Proposition 4.5.1 *If $\alpha_1, \ldots, \alpha_n \in L^{\mathcal{S}}_{\mathbb{R}}(M)$ then*

$$\mathcal{W}_I(\sqrt{n!}\alpha_1 \otimes_s \ldots \otimes_s \alpha_n) = \rho_n(I(\alpha_1)\ldots I(\alpha_n)). \tag{4.5.1}$$

Proof: We show first that if $(E_j)_{j=1}^k \subseteq \mathcal{M}$ are disjoint with $\mu(E_j)<\infty$ and $\sum_{j=1}^k p_j = n$,

$$\mathcal{W}_I(\sqrt{n!}\mathbb{1}_{E_1}^{\otimes_s^{p_1}} \otimes_s \ldots \otimes_s \mathbb{1}_{E_k}^{\otimes_s^{p_k}}) = \rho_n(X_{E_1}^{p_1}\ldots X_{E_k}^{p_k}). \tag{4.5.2}$$

Note that $C_n(t,x)$ is a monic polynomial of degree n in x. Therefore

$$\begin{aligned}\mathcal{W}_I(\sqrt{n!}\mathbb{1}_{E_1}^{\otimes_s^{p_1}} \otimes_s \ldots \otimes_s \mathbb{1}_{E_k}^{\otimes_s^{p_k}}) &= C_{p_1}(\mu(E_1), X_{E_1}+\mu(E_1))\ldots C_{p_k}(\mu(E_k), X_{E_k}+\mu(E_k))\\ &= X_{E_1}^{p_1}\ldots X_{E_k}^{p_k} + P_{n-1}(X_{E_1},\ldots,X_{E_k}),\end{aligned}$$

where P_{n-1} is a polynomial in k variables with degree less than n. Hence if $(F_s)_{s=1}^l \subseteq \mathcal{M}$ are disjoint with $\mu(F_s)<\infty$ and $\sum_{s=1}^l q_s = n$, then

$$\begin{aligned}&\langle \mathcal{W}_I(\sqrt{n!}\mathbb{1}_{E_1}^{\otimes_s^{p_1}} \otimes_s \ldots \otimes_s \mathbb{1}_{E_k}^{\otimes_s^{p_k}}), L^{(n)}(\mathbb{1}_{F_1}^{\otimes q_1} \otimes \ldots \otimes \mathbb{1}_{F_l}^{\otimes q_l})\rangle\\ &\qquad= \langle X_{E_1}^{p_1}\ldots X_{E_k}^{p_k} + P_{n-1}(X_{E_1},\ldots,X_{E_k}), L^{(n)}(\mathbb{1}_{F_1}^{\otimes q_1} \otimes \ldots \otimes \mathbb{1}_{F_l}^{\otimes q_l})\rangle\\ &\qquad= \langle X_{E_1}^{p_1}\ldots X_{E_k}^{p_k}, L^{(n)}(\mathbb{1}_{F_1}^{\otimes q_1} \otimes \ldots \otimes \mathbb{1}_{F_l}^{\otimes q_l})\rangle,\end{aligned}$$

since $P_{n-1}(X_{E_1}, \ldots, X_{E_k}) \in \mathbb{P}_{n-1}$ and thus (4.5.2) holds. The general result follows because we may write $\alpha_1 \otimes_s \ldots \otimes_s \alpha_n$ as a linear combination of functions of the form $\mathbb{1}_{E_1}^{\otimes_s^{p_1}} \otimes_s \ldots \otimes_s \mathbb{1}_{E_k}^{\otimes_s^{p_k}}$. $\square$

Corollary 4.5.2 *If $f_1, \ldots, f_n \in \bigcap_{1 \le p < \infty} L^p(M)$ then*

$$\mathcal{W}_I(\sqrt{n!} f_1 \otimes_s \ldots \otimes_s f_n) = \rho_n(I(f_1) \ldots I(f_n)). \tag{4.5.3}$$

PROOF: By Proposition 4.1.8, the right-hand side of (4.5.3) is well-defined. From the proof of Proposition 4.1.8, we may find sequences $(f_{j,m}) \subseteq L^{\mathcal{S}}_{\mathbb{R}}(M)$ such that $f_{j,m} \to f_j$ in $L^2(M)$ and $I(f_{j,m}) \to I(f_j)$ in $L^{2n}(\Omega)$ as $m \to \infty$. Then by Holder's inequality,

$$\begin{aligned}
&\|I(f_1) \ldots I(f_n) - I(f_{1,m}) \ldots I(f_{n,m})\|_2 \\
&\qquad \le \|(I(f_1) - I(f_{1,m}))I(f_2) \ldots I(f_n)\|_2 + \ldots \\
&\qquad\qquad \ldots + \|I(f_{1,m}) \ldots I(f_{n-1,m})(I(f_n) - I(f_{n,m}))\|_2 \\
&\qquad \le \|I(f_1) - I(f_{1,m})\|_{2n} \|I(f_2)\|_{2n} \ldots \|I(f_n)\|_{2n} + \ldots \\
&\qquad\qquad \ldots + \|I(f_{1,m})\|_{2n} \ldots \|I(f_{n-1,m})\|_{2n} \|I(f_n) - I(f_{n,m})\|_{2n} \\
&\qquad \to 0 \quad \text{as } m \to \infty.
\end{aligned}$$

Consequently $I(f_{1,m}) \ldots I(f_{n,m}) \to I(f_1) \ldots I(f_n)$ in $L^2(\Omega)$ as $m \to \infty$ and

$$\begin{aligned}
\mathcal{W}_I(\sqrt{n!} f_1 \otimes_s \ldots \otimes_s f_n) &= \lim_{m \to \infty} \mathcal{W}_I(\sqrt{n!} f_{1,m} \otimes_s \ldots \otimes_s f_{n,m}) \\
&= \lim_{m \to \infty} \rho_n(I(f_{1,m}) \ldots I(f_{n,m})) \\
&= \rho_n(I(f_1) \ldots I(f_n)),
\end{aligned}$$

giving the required formula. $\square$

Note that it is in general not possible to extend the formula (4.5.3) for $\alpha_j \in L^2(M)$, since it may not be the case that $I(\alpha_1) \ldots I(\alpha_n) \in L^2(\Omega)$.

We can also use the space of polynomials $\bigcup_{n=0}^{\infty} \mathbb{P}_n$ to show that if $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$ then $e^{iI(\alpha)} \in \mathcal{W}_I(\mathfrak{F}_+(L^2(M)))$. One possible method follows that of [34, Theorem 2.6]. By the definition of $\mathbb{P}_n$, for all $n \in \mathbb{N}_0$, $\sum_{j=0}^{n} \frac{(iI(\alpha))^j}{j!} \in \bigcup_{n=0}^{\infty} \mathbb{P}_n$. Furthermore,

$$\begin{aligned}
\left| e^{iI(\alpha)} - \sum_{j=0}^{n} \frac{(iI(\alpha))^j}{j!} \right| &\le 1 + \sum_{j=0}^{n} \left| \frac{I(\alpha)^j}{j!} \right| \\
&\le 1 + e^{I(\alpha)} + e^{-I(\alpha)}.
\end{aligned}$$

However, by the properties of the Poisson process $1 + e^{I(\alpha)} + e^{-I(\alpha)} \in L^2(\Omega)$. Thus by the dominated convergence theorem, in $L^2(\Omega)$,

$$\sum_{j=0}^{n} \frac{(iI(\alpha))^j}{j!} \to e^{iI(\alpha)} \quad \text{as } n \to \infty,$$

and hence $e^{iI(\alpha)} \in \mathcal{W}_I(\mathfrak{F}_+(L^2(M)))$. Therefore from Proposition 4.1.9 we may deduce that $\mathcal{W}_I$ is surjective.

The density of $\bigcup_{n=0}^{\infty} \mathbb{P}_n$ in $L^2(\Omega)$, and thus the surjectivity of $\mathcal{W}_I$, also follows from [18, Proposition 2.1], since for all $E \in \mathcal{M}$ with $\mu(E) < \infty$, $e^{X_E} \in L^1(\Omega)$. We prefer to show surjectivity by explicitly calculating $\mathcal{W}_I(e(e^{i\alpha} - 1))$ for $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$, because we make frequent use of this formula in our work.

In [34, §7.3] given a Gaussian stochastic integral, $I_{\mathfrak{w}} : L^2(M, \mathcal{M}, \mu) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$, over a σ-finite measure space, the Skorohod integral

$$\delta_{I_{\mathfrak{w}}} : L^2(M, \mathcal{M}, \mu; L^2(\Omega, \mathcal{F}, \mathbb{P})) \longrightarrow L^2(\Omega, \mathcal{F}, \mathbb{P}),$$

a densely defined operator, is constructed. If $I : L^2(M, \mathcal{M}, \mu) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a generalised Poisson process the Skorohod integral can be constructed in an analogous manner as a linear map

$$\delta_I : L^2(M, \mathcal{M}, \mu; L^2(\Omega, \mathcal{F}, \mathbb{P})) \longrightarrow L^2(\Omega, \mathcal{F}, \mathbb{P}).$$

Similar properties with the appropriate modifications to those obtained in the Gaussian case can also be deduced for the Poisson case, although we shall not give details. We can also construct the Malliavin gradient operator

$$\nabla_I : L^2(\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow L^2(M, \mathcal{M}, \mu; L^2(\Omega, \mathcal{F}, \mathbb{P})),$$

which is the adjoint of the Skorohod integral (see [25, Thèoréme 2] and [42, Theorem 1.3]). In Fock space it is known that the Malliavin gradient

$$\nabla : \mathfrak{F}_+(L^2(M)) \longrightarrow L^2(M, \mathcal{M}, \mu; \mathfrak{F}_+(L^2(M)))$$

has domain $D(\nabla) = \mathbb{D}^{1,2}$ (see (3.3.5)), and for all $f \in L^2(M)$,

$$(\nabla e(f))(m) = f(m)e(f). \tag{4.5.4}$$

Note that for $f \in L^2_{\mathbb{R}}(M)$, by (4.2.4),

$$e^{iI(f)} = \exp\{\int_M (e^{if} - 1 - if)d\mu\} \mathcal{W}_I(e^{if} - 1).$$

Therefore, using (4.5.4) we have that for $f \in L^2_{\mathbb{R}}(M)$,

$$(\nabla_I e^{iI(f)})(m) = (e^{if(m)} - 1)e^{iI(f)}. \tag{4.5.5}$$

This should be compared with the Gaussian case where for $f \in L^2_{\mathbb{R}}(M)$,

$$(\nabla_{I_{\mathfrak{w}}} e^{iI(f)})(m) = if(m)e^{iI(f)}.$$

By a similar argument to Lemma 4.3.1 it can be shown that vectors of the form $e(e^{i\alpha} - 1)$ with $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$ are total in $\mathbb{D}^{1,2}$ and therefore $\{e^{iI(f)} : f \in L^2_{\mathbb{R}}(M)\}$ is a core for ∇_I. Hence the formula (4.5.5) uniquely determines the operator ∇_I. In [50] explicit expressions for the Skorohod integral δ_I and the Malliavin gradient ∇_I for certain special Poisson processes are given.

Chapter 5

Construction of Generalised Poisson Processes

In this chapter we shall assume that $(M, \mathcal{M}, \mu)$ is any measure space. We give a canonical construction of a generalised Poisson process over $(M, \mathcal{M}, \mu)$ using the Gelfand transform of a unital commutative C^*-algebra $\mathcal{A} \subseteq \mathfrak{B}(\mathfrak{F}_+(L^2(M)))$. The ideas here are similar to that in [28, §6], although we actually construct a probability space and an isometry, via the Segal spatial isomorphism (see [62, Scholium 9.2]), which is not done in the article mentioned. We do obtain a generalised Poisson process rather than a Poisson random measure, but we do not need to impose any conditions on the measure space $(M, \mathcal{M}, \mu)$. A similar construction of Gaussian fields can be found in [69].

5.1 The C^*-algebra $\mathcal{A}$

The aim of this section is to give a description of the C^*-algebra $\mathcal{A}$ which we shall be working with. The C^*-algebra we construct is a commutative C^*-subalgebra of the C^*-algebra generated by the Weyl operators in $\mathfrak{B}(\mathfrak{F}_+(L^2(M)))$. We give a brief description of Weyl operators on $\mathfrak{F}_+(H)$, where H is any complex space and refer the reader to [47, §IV.1.9] for more details.

Consider the set G of ordered pairs $\mathcal{U} = (u, U)$ where $u \in H$ and U is a unitary operator on $L^2(M)$. For $f \in H$ set $\mathcal{U}f := Uf + u$. Then the *Weyl operator* $W_{\mathcal{U}}$ is defined on E by

$$W_{\mathcal{U}}(e(f)) := e^{-C_{\mathcal{U}}(f)} e(\mathcal{U}f) \quad \text{where} \quad C_{\mathcal{U}}(f) := \langle u, Uf \rangle + \frac{\|u\|^2}{2},$$

and extended by linearity to E. By the linear independence of the exponential vectors this operator is well-defined. Clearly, $W_{\mathcal{U}}$ maps E into E. If $\mathcal{V} = (v, V) \in G$, then it can be easily deduced that

$$W_{\mathcal{U}}W_{\mathcal{V}} = e^{-i\mathrm{Im}\langle u, Uv\rangle}W_{\mathcal{U}.\mathcal{V}}, \tag{5.1.1}$$

where we define

$$\mathcal{U}.\mathcal{V} := (u + Uv, UV).$$

It can be shown that $W_{\mathcal{U}}$ is an isometry on E and so extends to an isometry on $\mathfrak{F}_+(L^2(M))$. Furthermore (5.1.1) still holds for the extension. By taking $\mathcal{V} = (-U^*u, U^*)$, (5.1.1) implies that $W_{\mathcal{U}}$ is invertible and hence the Weyl operators form a unitary group. The operators in the set $\{W_{\mathcal{U}} : \mathcal{U} = (u, I)\}$ are often called the *pure translation Weyl operators*. The C^*-algebra generated by the pure translation Weyl operators, $\mathcal{A}$ say, is the *C^*-algebra of the CCR*. If we define $W(u) := W_{(u/\sqrt{2}, I)}$ for $u \in H$, then

$$W(u)W(v) = e^{-\frac{i}{2}\mathrm{Im}\langle u,v\rangle}W(u+v) = e^{-i\mathrm{Im}\langle u,v\rangle}W(u)W(v),$$

whenever u, $v \in H$. Thus, $\{W(u) : u \in H\}$ satisfies the *Weyl form of the canonical commutation relations*. The C^*-algebra of the CCR has been studied in detail in [12, §5.2.1.2]. It is shown in [12, Theorem 5.2.8] that if $\mathcal{A}'$ is another C^*-algebra generated by a set $\{W'(u) : u \in H\}$ which also satisfies the Weyl form of the canonical commutation relations and $W'(u)^* = W'(-u)$, then there exists a unique $*$-isomorphism $\phi : \mathcal{A} \to \mathcal{A}'$ such that $W(u) \mapsto W'(u)$.

For each $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$ (see Definition 4.1.3), we define a linear operator on $\mathsf{E} \subseteq \mathfrak{F}_+(L^2(M))$ by

$$V(\alpha) : e(f) \mapsto \exp\{\int_M (e^{i\alpha} - 1 + (e^{i\alpha} - 1)f)d\mu\}e(e^{i\alpha}f + e^{i\alpha} - 1). \tag{5.1.2}$$

As α is real-valued and the exponential vectors are linearly independent in Fock space, this map is well-defined. These maps are just multiples of Weyl operators. To see this, fix $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$ and take

$$\mathcal{U}_\alpha := (e^{i\alpha} - 1, e^{i\alpha}I) \in G.$$

Then it follows immediately that

$$V(\alpha) = e^{i\int_M \sin\alpha d\mu}W_{\mathcal{U}_\alpha}.$$

The multiplication by the phase factor ensures that the formula $V(\alpha)V(\beta) = V(\alpha+\beta)$ holds. These Weyl operators on $\mathfrak{F}_+(L^2(M))$ are also mentioned in [51, Exercise 26.10]. In the Gaussian case the Weyl operators used to construct the analogous C^*-algebra have the unitary operator equal to I and hence are a subset of the pure translation Weyl operators. The maps

$\{V(\alpha) : \alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)\}$ will be used to construct our C^*-algebra $\mathcal{A}$. The following properties can be deduced from the properties of Weyl operators, however for completeness we compute them directly.

Proposition 5.1.1 *Let α, $\beta \in L^{\mathcal{S}}_{\mathbb{R}}(M)$. Then*

i) $V(\alpha)$ extends to a unitary operator on $\mathfrak{F}_+(L^2(M))$,

ii) $V(\alpha)V(\beta) = V(\alpha+\beta)$.

Proof: If we can show that $V(\alpha)$ is an isometry and ii) holds, then since $V(0) = I$, we have that $V(\alpha)$ is surjective, and thus we get the fact that $V(\alpha)$ is unitary. If f, $g \in L^2(M)$,

$$\begin{aligned}
\langle V(\alpha)e(f), V(\alpha)e(g)\rangle &= \exp\{\int_M (e^{-i\alpha} - 1 + (e^{-i\alpha} - 1)\bar{f})d\mu\} \\
&\quad \times \exp\{\int_M (e^{i\alpha} - 1 + (e^{i\alpha} - 1)g)d\mu\} \\
&\quad \times \langle e(e^{i\alpha} f + e^{i\alpha} - 1), e(e^{i\alpha} g + e^{i\alpha} - 1)\rangle \\
&= \exp\{\int_M -(e^{-i\alpha} - 1)(e^{i\alpha} - 1)d\mu\} \\
&\quad \times \exp\{\int_M ((e^{-i\alpha} - 1)\bar{f} + (e^{i\alpha} - 1)g)d\mu\} \\
&\quad \times \exp\{\int_M (e^{-i\alpha}\bar{f} + e^{-i\alpha} - 1)(e^{i\alpha} g + e^{i\alpha} - 1)d\mu\} \\
&= \exp\{\int_M \bar{f} g d\mu\} = \langle e(f), e(g)\rangle.
\end{aligned}$$

Thus $V(\alpha)$ extends to an isometry on $\mathfrak{F}_+(L^2(M))$. To prove ii) if $f \in L^2(M)$,

$$\begin{aligned}
V(\alpha)V(\beta)e(f) &= V(\alpha)\exp\{\int_M (e^{i\beta} - 1 + (e^{i\beta} - 1)f)d\mu\} \\
&\quad \times e(e^{i\beta} f + e^{i\beta} - 1) \\
&= \exp\{\int_M (e^{i\beta} - 1 + (e^{i\beta} - 1)f)d\mu\} \\
&\quad \times \exp\{\int_M (e^{i\alpha} - 1 + (e^{i\alpha} - 1)(e^{i\beta} f + e^{i\beta} - 1))d\mu\} \\
&\quad \times e(e^{i\alpha}(e^{i\beta} f + e^{i\beta} - 1) + e^{i\alpha} - 1) \\
&= \exp\{\int_M (e^{i(\alpha+\beta)} - 1 + (e^{i(\alpha+\beta)} - 1)f)d\mu\} \\
&\quad \times e(e^{i(\alpha+\beta)} f + e^{i(\alpha+\beta)} - 1) \\
&= V(\alpha+\beta)e(f).
\end{aligned}$$

Hence we have that $V(\alpha)V(\beta) = V(\alpha+\beta)$ on E, and thus by continuity on $\mathfrak{F}_+(L^2(M))$. □

Proposition 5.1.2 *Let f, $g \in L^2(M)$, then for $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$,*

$$\langle e(f), V(\alpha)e(g)\rangle = \exp\{\int_M (e^{i\alpha} - 1)(\bar{f} + 1)(g + 1)d\mu\}\langle e(f), e(g)\rangle. \tag{5.1.3}$$

Proof: Using the definition of $V(\alpha)$,

$$\begin{aligned}\langle e(f), V(\alpha)e(g)\rangle &= \exp\{\int_M (e^{i\alpha} - 1 + (e^{i\alpha} - 1)g)d\mu\} \\ &\quad \times \exp\{\int_M \bar{f}(e^{i\alpha}g + e^{i\alpha} - 1)d\mu\} \\ &= \exp\{\int_M (e^{i\alpha} - 1)(1 + \bar{f} + g + \bar{f}g)d\mu\}\langle e(f), e(g)\rangle \\ &= \exp\{\int_M (e^{i\alpha} - 1)(\bar{f} + 1)(g + 1)d\mu\}\langle e(f), e(g)\rangle.\end{aligned}$$

Hence we have the required result. □

Proposition 5.1.3 *Let $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$. Then $(V(t\alpha))_{t\in\mathbb{R}}$ is a strongly continuous unitary group.*

Proof: The fact that $(V(t\alpha))_{t\in\mathbb{R}}$ is a unitary group comes from Proposition 5.1.1. From (5.1.3) we know that for all f, $g \in L^2(M)$, the map

$$t \longmapsto \langle e(f), V(t\alpha)e(g)\rangle$$

is continuous at $t = 0$. Thus, since E is dense in $\mathfrak{F}_+(L^2(M))$ and $V(t\alpha)$ is unitary for each t, by a simple approximation argument it follows that for all ϕ, $\psi \in \mathfrak{F}_+(L^2(M))$,

$$t \longmapsto \langle \phi, V(t\alpha)\psi\rangle$$

is continuous at $t = 0$. Since weak continuity implies strong continuity for unitary groups [76, §IX.1 Theorem], the result follows. □

For $E \in \mathcal{M}$ with $\mu(E) < \infty$, if we define

$$U(t\mathbb{1}_E) := e^{-it\mu(E)}V(t\mathbb{1}_E), \tag{5.1.4}$$

then $(U(t\mathbb{1}_E))_{t\in\mathbb{R}}$ is a strongly continuous unitary group with an infinitesimal generator, which we shall call S. We may now prove that $S = \Lambda_E + A_E + A^\dagger_E$, where Λ_E, A_E and $A^\dagger_E$ are defined at the end of Section 4.3. Consequently we have an explicit expression for action of the unitary group $(e^{it(\Lambda_E + A_E + A^\dagger_E)})_{t\in\mathbb{R}}$ on the exponential vectors in $\mathfrak{F}_+(L^2(M))$. Using this we can deduce that if $\mathcal{W}_I : \mathfrak{F}_+(L^2(M)) \to L^2(\Omega)$ is a generalised Wiener-Poisson isomorphism, then $\mathcal{W}_I$ intertwines $\Lambda_E + A_E + A^\dagger_E$ and $\widehat{X_E}$ by considering the unitary groups they generate. This generalises Theorem 3.3.5 for Poisson processes on $\mathbb{R}_+$ to generalised Poisson processes over $(M, \mathcal{M}, \mu)$.

Proposition 5.1.4 *With the notations above* $D(S) = D(\Lambda_E + A_E + A_E^\dagger)$ *and on their common domain* $S = \Lambda_E + A_E + A_E^\dagger$.

PROOF: If $f,\, g \in L^2(M)$ from (5.1.3) we have

$$\frac{d}{dt}\langle e(f), V(t\mathbb{1}_E)e(g)\rangle = i\int_E e^{it}(\bar{f}+1)(g+1)d\mu\ \exp\{\int_E (e^{it}-1)(\bar{f}+1)(g+1)d\mu\}\langle e(f), e(g)\rangle.$$

Using (5.1.2) we also obtain

$$\begin{aligned}
&\langle V(-t\mathbb{1}_E)e(f), (\Lambda_E + A_E + A_E^\dagger + \mu(E)I)e(g)\rangle \\
&\qquad = \exp\{\int_E (e^{it} - 1 + (e^{it}-1)\bar{f})d\mu\} \\
&\qquad\qquad \times \langle e(e^{-it\mathbb{1}_E} f + (e^{-it}-1)\mathbb{1}_E), (\Lambda_E + A_E + A_E^\dagger + \mu(E)I)e(g)\rangle \\
&\qquad = \exp\{\int_E (e^{it} - 1 + (e^{it}-1)\bar{f})d\mu\} \\
&\qquad\qquad \times \int_E (e^{it}\bar{f} + e^{it} - 1 + g + (e^{it}\bar{f} + e^{it} - 1)g + 1)d\mu \\
&\qquad\qquad \times \langle e(e^{-it\mathbb{1}_E} f + (e^{-it}-1)\mathbb{1}_E), e(g)\rangle \\
&\qquad = \int_E e^{it}(\bar{f}+1)(g+1)d\mu\ \exp\{\int_E (e^{it} - 1 + (e^{it}-1)\bar{f})d\mu\} \\
&\qquad\qquad \times \exp\{\int_M (e^{it\mathbb{1}_E}\bar{f}g + (e^{it}-1)\mathbb{1}_E g)d\mu\} \\
&\qquad = -i\frac{d}{dt}\langle e(f), V(t\mathbb{1}_E)e(g)\rangle.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\frac{d}{dt}\langle e(f), U(t\mathbb{1}_E)e(g)\rangle &= e^{-it\mu(E)}\frac{d}{dt}\langle e(f), V(t\mathbb{1}_E)e(g)\rangle - i\mu(E)e^{-it\mu(E)}\langle e(f), V(t\mathbb{1}_E)e(g)\rangle \\
&= i\langle U(-t\mathbb{1}_E)e(f), (\Lambda_E + A_E + A_E^\dagger)e(g)\rangle. \qquad (5.1.5)
\end{aligned}$$

Therefore since (5.1.5) is continuous, for all $\phi \in \mathsf{E}$,

$$\langle \phi, U(t\mathbb{1}_E)e(g)\rangle = \langle \phi, e(g)\rangle + i\int_0^t \langle U(-s\mathbb{1}_E)\phi, (\Lambda_E + A_E + A_E^\dagger)e(g)\rangle ds.$$

If $\psi \in \mathfrak{F}_+(L^2(M))$, approximating ψ by a sequence $(\psi_n) \subseteq \mathsf{E}$ and using the dominated convergence theorem we get

$$\langle \psi, U(t\mathbb{1}_E)e(g)\rangle = \langle \psi, e(g)\rangle + i\int_0^t \langle U(-s\mathbb{1}_E)\psi, (\Lambda_E + A_E + A_E^\dagger)e(g)\rangle ds.$$

Hence $t \mapsto U(t\mathbb{1}_E)e(g)$ is weakly differentiable with weak derivative $i(\Lambda_E + A_E + A_E^\dagger)e(g)$ at $t = 0$ and by [53, Theorem 2.1.3], $e(g) \in D(S)$ with

$$Se(g) = (\Lambda_E + A_E + A_E^\dagger)e(g).$$

By a similar argument to Theorem 3.3.5, E is a core for $\Lambda_E + A_E + A_E^\dagger$ and the result follows by the maximality of self-adjoint operators. $\square$

Corollary 5.1.5 *Let $(M, \mathcal{M}, \mu)$ be a measure space and $E \in \mathcal{M}$ with $\mu(E) < \infty$. If $I : L^2(M, \mathcal{M}, \mu) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a generalised Poisson process then $\mathcal{W}_I$ intertwines $\Lambda_E + A_E + A_E^\dagger$ and $\widehat{X_E}$.*

PROOF: From Lemma 5.1.4 we know that $e^{it(\Lambda_E + A_E + A_E^\dagger)} = U(t\mathbb{1}_E)$. We prove that

$$U(t\mathbb{1}_E) = e^{it(\Lambda_E + A_E + A_E^\dagger)} = \mathcal{W}^{-1} e^{it\widehat{X_E}} \mathcal{W}, \tag{5.1.6}$$

which by the uniqueness of the infinitesimal generator of a unitary group establishes the result. If $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$ from (4.2.2),

$$\begin{aligned} \mathcal{W}_I e^{-it\mu(E)} V(t\mathbb{1}_E)(e(e^{i\alpha} - 1)) &= e^{-it\mu(E)} \exp\{\int_M e^{i\alpha}(e^{it\mathbb{1}_E} - 1) d\mu\} \\ &\quad \times \mathcal{W}_I(e(e^{it\mathbb{1}_E + i\alpha} - 1)) \\ &= e^{itX_E} \exp\{-\int_M (e^{i\alpha} - 1 - i\alpha) d\mu + iI(\alpha)\} \\ &= e^{it\widehat{X_E}}(\mathcal{W}_I(e(e^{i\alpha} - 1))). \end{aligned}$$

Thus by the continuity of the operators and the density of E'', (5.1.6) holds. $\square$

The proof of the above corollary is just a generalisation of Meyer's argument (see (3.3.1)), to show that $e^{iu(\Lambda_t + A_t + A_t^\dagger)} = \mathcal{W}^{-1} e^{it\widehat{X_t}} \mathcal{W}$. When working on $[0, 1]$ we evaluate both operators on the exponential vectors of locally bounded square integrable functions, however in this corollary we only show explicitly that the two operators agree on a more restricted exponential domain which is still total in $\mathfrak{F}_+(L^2(M))$.

We define $\mathcal{A}$ to be the commutative unital C^*-algebra generated by $\{V(\alpha) : \alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)\}$. From Proposition 5.1.4 we can deduce that $\mathcal{A}$ is quite a naturally occurring C^*-algebra.

Proposition 5.1.6 *$\mathcal{A}$ is the C^*-algebra generated by the set $\{e^{it(\Lambda_E + A_E + A_E^\dagger)} : t \in \mathbb{R},\ E \in \mathcal{M}$ with $\mu(E) < \infty\}$.*

PROOF: Let $\mathcal{A}'$ be the C^*-algebra generated by $\{e^{it(\Lambda_E + A_E + A_E^\dagger)} : t \in \mathbb{R},\ E \in \mathcal{M}$ with $\mu(E) < \infty\}$. Then from Proposition 5.1.4 we have that $\mathcal{A}' \subseteq \mathcal{A}$. Conversely, if $\alpha = \sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}$ where $\alpha_j \in \mathbb{R}$ and $(E_j)_{j=1}^n \subseteq \mathcal{M}$ are disjoint with $\mu(E_j) < \infty$ then

$$V(\alpha) = V(\alpha_1 \mathbb{1}_{E_1}) \ldots V(\alpha_n \mathbb{1}_{E_n}) \in \mathcal{A}',$$

since $V(\alpha_j \mathbb{1}_{E_j}) = e^{i\alpha_j \mu(E_j)} e^{i\alpha_j(\Lambda_{E_j} + A_{E_j} + A_{E_j}^\dagger)} \in \mathcal{A}'$. Thus $\mathcal{A} \subseteq \mathcal{A}'$. $\square$

It should be noted that in [67, §5] when constructing a Gaussian field over a complexified Hilbert space H, the C^*-algebra introduced is in fact the C^*-algebra generated by the elements $e^{it(a(h)+a^\dagger(h))}$ with $h \in H_{\mathbb{R}}$ and $t \in \mathbb{R}$ (see [47, §IV.1.4] for definitions of $a(h)$ and $a^\dagger(h)$). Hence, although our algebra $\mathcal{A}$ does not appear to be as natural as in [67, §5], it is the analogous one for the Poisson case.

The next result is crucial to the construction of our probability space. It shows that the vacuum vector $e(0)$ is cyclic for our C^*-algebra $\mathcal{A}$.

Proposition 5.1.7 *$\{V(\alpha)e(0) : \alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)\}$ is a total set in $\mathfrak{F}_+(L^2(M))$.*

PROOF: By the definition of $V(\alpha)$, for all $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$,

$$e(e^{i\alpha} - 1) \in \operatorname{linsp}\{V(\alpha)e(0) : \alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)\}.$$

Therefore the result holds by Lemma 4.3.1. □

Corollary 5.1.8 *If $a \in \mathcal{A}$ and $ae(0) = 0$, then $a = 0$.*

PROOF: Suppose $a \in \mathcal{A}$ and $ae(0) = 0$. Then for all $b \in \mathcal{A}$, $0 = bae(0) = abe(0)$. As $\{be(0) : b \in \mathcal{A}\}$ is dense in $\mathfrak{F}_+(L^2(M))$, we get $a = 0$. □

The density result above can also be deduced from the properties of unitary groups generated by self-adjoint operators. For $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$ let $M_\alpha : L^2(M) \to L^2(M)$ be the operator of multiplication by α, and denote by Λ_α, A_α and $A^\dagger_\alpha$ the operators $\lambda(M_\alpha)$, $a(\alpha)$ and $a^\dagger(\alpha)$ respectively. Then $\Lambda_\alpha + A_\alpha + A^\dagger_\alpha$ is a self-adjoint operator such that

$$\Lambda_\alpha + A_\alpha + A^\dagger_\alpha = \overline{(\Lambda_\alpha + A_\alpha + A^\dagger_\alpha)|_{\mathfrak{F}_+(L^2(M))_{00}}},$$

where $\mathfrak{F}_+(L^2(M))_{00}$ is the set of vectors with finite Fock space expansion. If $\alpha = \sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}$, where $\alpha_j \in \mathbb{R}$ and $(E_j)_{j=1}^n \subseteq \mathcal{M}$ are disjoint with $\mu(E_j) < \infty$, then

$$\overline{(\Lambda_\alpha + A_\alpha + A^\dagger_\alpha)|_{\mathfrak{F}_+(L^2(M))_{00}}} = \overline{\sum_{j=1}^n \alpha_j (\Lambda_{E_j} + A_{E_j} + A^\dagger_{E_j})}.$$

Therefore by [56, Theorem VIII.31], since the operators $(e^{it\alpha_j(\Lambda_{E_j}+A_{E_j}+A^\dagger_{E_j})})_{j=1}^n$ commute,

$$e^{it(\Lambda_\alpha+A_\alpha+A^\dagger_\alpha)} = \prod_{j=1}^n e^{it\alpha_j(\Lambda_{E_j}+A_{E_j}+A^\dagger_{E_j})} = e^{-it\int_M \alpha d\mu} V(t\alpha).$$

We know that $e(0) \in D((\Lambda_\alpha + A_\alpha + A_\alpha^\dagger)^n)$ for all $n \geq 0$, thus by the differentiability properties of unitary groups,

$$\left(-i\frac{d}{dt}\right)^n e^{it(\Lambda_\alpha + A_\alpha + A_\alpha^\dagger)} e(0)\Big|_{t=0} = (\Lambda_\alpha + A_\alpha + A_\alpha^\dagger)^n e(0)$$
$$= (*, \ldots, *, (A_\alpha^\dagger)^n e(0), 0, \ldots).$$

However, since $(A_\alpha^\dagger)^n e(0) = \sqrt{n!}\alpha^{\otimes_s^n}$, and these vectors are total in $\mathfrak{F}_+(L^2(M))$, the density result above follows. This also gives an alternative proof of Lemma 4.3.1.

Proposition 5.1.7 should be compared to Corollary 4.2.10. As mentioned before the vectors $e(e^{i\alpha}-1)$ or $\mathcal{E}(e^{i\alpha}-1)$ with $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$ are important in the Poisson case. This is not entirely surprising since if X is a random variable which has a Poisson distribution with mean λ, then

$$\mathbb{E}[e^{itX}] = \exp\{\lambda(e^{it} - 1)\}.$$

Therefore $e^{it}-1$ appears naturally when dealing with the Poisson distribution. In the Gaussian case the corresponding exponential vectors are $e(i\alpha)$ and $\mathcal{E}(i\alpha)$ with $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$ (see [21, Proposition 4] with $\phi = 0$ and [69, Proposition 5.4]). If X is a zero mean Gaussian random variable with variance σ^2, then its characteristic function is

$$\mathbb{E}[e^{itX}] = \exp\{\frac{\sigma^2(it)^2}{2}\}.$$

Again, the exponential vectors used to show density in the Gaussian case also appear to be natural.

5.2 The Segal spatial isomorphism $\mathscr{S}$

Before applying the Segal spatial isomorphism to our C^*-algebra to construct a generalised Poisson process we give a brief account of the general theory. If $\mathcal{A}$ is a unital commutative C^*-algebra we can construct the maximal ideal space

$$\Delta := \{\phi : \mathcal{A} \to \mathbb{C} : \phi \text{ linear}, \ \phi \neq 0, \ \phi(ab) = \phi(a)\phi(b)\}.$$

Then Δ with the weak* topology is a compact Hausdorff space and if we define $\widehat{a} : \phi \mapsto \phi(a)$, then $\widehat{a} \in \mathbb{C}(\Delta)$ and the Gelfand transform $a \mapsto \widehat{a}$ is an isometric $*$-isomorphism (see [48, Theorem 1.3.6]). We shall sometimes use $a\widehat{\ }$ to denote $\widehat{a}$. Note that since $\mathcal{A}$ is abelian, the *pure state space* of $\mathcal{A}$, which is the set of extreme points of the state space

$$S := \{\rho \in A^* : \rho(1) = 1, \ \rho \geq 0\},$$

coincides with the maximal ideal space [48, Theorem 5.1.6].

Theorem 5.2.1 ([62, Scholium 9.2]) *Suppose $\mathcal{A} \subseteq \mathfrak{B}(H)$ is a unital commutative C^*-algebra with a cyclic vector x. Then there exists a unique finite, regular measure ν on the Borel sets $\mathcal{B}$ of Δ such that*

$$\langle x, ax \rangle = \int_\Delta \widehat{a} d\nu.$$

Furthermore, there exists a unique unitary isomorphism, called the Segal spatial isomorphism,

$$\mathscr{S} : H \longrightarrow L^2(\Delta, \mathcal{B}, \nu),$$

such that for all $a \in \mathcal{A}$ and $f \in L^2(\Delta)$, $\mathscr{S}(ax) = \widehat{a}$ and $\mathscr{S} a \mathscr{S}^{-1} f = \widehat{a} f$.

Proof: Consider the linear functional $\phi : \mathcal{A} \to \mathbb{C}$ defined by

$$\phi(\widehat{a}) = \langle x, ax \rangle.$$

Then ϕ is a positive linear functional, so by Riesz's theorem there exists a unique regular Borel measure ν on Δ such that

$$\langle x, ax \rangle = \phi(\widehat{a}) = \int_\Delta \widehat{a} d\nu.$$

Since $\nu(\Delta) = \phi(\widehat{1}) = \|x\|^2$ we obtain that the measure is finite. If $ax = bx$ for some a, $b \in \mathcal{A}$, for all $c \in \mathcal{A}$, $(a - b)cx = 0$. Thus since $\{cx : c \in \mathcal{A}\}$ is dense in H, $a = b$. Hence we may construct a linear map

$$\mathscr{S} : \{ax : a \in \mathcal{A}\} \longrightarrow L^2(\Delta, \mathcal{B}, \nu),$$

by defining $\mathscr{S}(ax) = \widehat{a}$. Then for a, $b \in \mathcal{A}$,

$$\begin{aligned} \langle \mathscr{S}(ax), \mathscr{S}(bx) \rangle &= \int_\Delta \bar{\widehat{a}} \widehat{b} d\nu \\ &= \langle x, a^* bx \rangle \\ &= \langle ax, bx \rangle. \end{aligned}$$

By the density of $\{ax : a \in \mathcal{A}\}$ and $\mathbb{C}(\Delta)$ in H and $L^2(\Delta, \mathcal{B}, \nu)$ respectively, $\mathscr{S}$ extends to a unitary map

$$\mathscr{S} : H \longrightarrow L^2(\Delta, \mathcal{B}, \nu).$$

For the last part, if $f \in \mathbb{C}(\Delta)$, $f = \widehat{b}$ for some $b \in \mathcal{A}$. Thus

$$\mathscr{S} a \mathscr{S}^{-1} f = \mathscr{S} a (\mathscr{S}^{-1} \widehat{b}) = \mathscr{S} a (be(0)) = \widehat{ab} = \widehat{a} f.$$

Hence on $\mathbb{C}(\Delta)$, $\widehat{a} = \mathscr{S} a \mathscr{S}^{-1}$ and by density equality holds on $L^2(\Delta, \mathcal{B}, \nu)$. $\square$

Suppose Ω is a compact Hausdorff space and λ is a finite, standard, regular Borel measure on Ω (see [62, p.236 Definitions]). Then the unital commutative C^*-algebra $\mathbb{C}(\Omega)$ can be considered to be a subalgebra of $L^2(\Omega, \mathcal{B}, \lambda)$. It is well-known that for this C^*-algebra, Δ is homeomorphic to Ω (see [48, Theorem 2.1.15]). Since λ is a finite measure, $1 \in L^2(\Omega)$ is a cyclic vector for $\mathbb{C}(\Omega)$. Consequently, we can construct the Segal spatial isomorphism associated with this C^*-algebra, which as expected turns out to be the identity map

$$\mathscr{S} = I : L^2(\Omega, \mathcal{B}, \lambda) \longrightarrow L^2(\Omega, \mathcal{B}, \lambda).$$

Another example of the Segal spatial isomorphism can be found in [69, §5], in which the isomorphism is used to construct Gaussian fields. If $H_{\mathbb{R}}$ is a real Hilbert space and H its complexification, the unital C^*-algebra generated by the set $\{e^{it(a(h)+a^\dagger(h))} : t \in \mathbb{R},\ h \in H_{\mathbb{R}}\}$, can be shown to be commutative with cyclic vector $e(0)$. The Segal spatial isomorphism in this case

$$\mathscr{S} : \mathfrak{F}_+(H) \longrightarrow L^2(\Delta, \mathcal{B}, \mathbb{P}),$$

turns out to be the Wiener-Ito isomorphism of the Gaussian field $\mathscr{S}|_H$.

We now consider the C^*-algebra, $\mathcal{A}$, constructed in Section 5.1. As usual we denote by Δ the maximal ideal space of $\mathcal{A}$. From Proposition 5.1.7 we know that $e(0)$ is a cyclic vector for $\mathcal{A}$. Therefore we can apply Theorem 5.2.1 to $\mathcal{A}$.

Proposition 5.2.2 *Let $\mathcal{B}_\Delta$ be the Borel σ-field of the maximal ideal space Δ of the C^*-algebra $\mathcal{A}$. Then there exists a unique regular, Borel probability measure, $\mathbb{P}_\Delta$, on Δ such that for all $a \in \mathcal{A}$,*

$$\langle e(0), ae(0)\rangle = \int_\Delta \widehat{a} d\mathbb{P}_\Delta.$$

Furthermore, there exists a unique unitary isomorphism

$$\mathscr{S} : \mathfrak{F}_+(L^2(M)) \longrightarrow L^2(\Delta, \mathcal{B}_\Delta, \mathbb{P}_\Delta)$$

such that for all $a \in \mathcal{A}$ and $X \in L^2(\Delta, \mathcal{B}_\Delta, \mathbb{P}_\Delta)$, $\mathscr{S}(ae(0)) = \widehat{a}$ and $\mathscr{S} a \mathscr{S}^{-1} X = \widehat{a} X$.

The result that $\mathbb{P}_\Delta$ is a probability measure comes from the fact that $\mathbb{P}_\Delta(\Delta) = \|e(0)\|^2 = 1$. We shall now complete the probability space $(\Delta, \mathcal{B}_\Delta, \mathbb{P}_\Delta)$ and denote the completion $(\Delta, \mathcal{B}, \mathbb{P})$. Since every X measurable with respect to $\mathcal{B}$ is equal $\mathbb{P}$-a.s. to a function X' measurable with respect to $\mathcal{B}_\Delta$ (see [58, p.169, Lemma 1]), we have that $L^2(\Delta, \mathcal{B}_\Delta, \mathbb{P}_\Delta) = L^2(\Delta, \mathcal{B}, \mathbb{P})$. Hence the isomorphism holds for the 'new' probability space, and from now on we shall work with the completed probability space.

Lemma 5.2.3 *Let I be the restriction of $\mathscr{S}$ to $L^2(M)$. Then if $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$, as an element of $L^2(\Delta, \mathcal{B}, \mathbb{P})$,*

$$\widehat{V(\alpha)} = \exp\{i\big(\int_M \alpha d\mu + I(\alpha)\big)\}. \tag{5.2.1}$$

PROOF: By Proposition 5.1.3 $(V(t\alpha))_{t\in\mathbb{R}}$ is a strongly continuous unitary group. Therefore by Stone's theorem $(\mathscr{S}V(t\alpha)\mathscr{S}^{-1})_{t\in\mathbb{R}}$ has a self-adjoint infinitesimal generator, $\widehat{S}$ say, which can be obtained by a differential limit in the Hilbert space. Differentiating in $L^2(\Omega)$,

$$\begin{aligned}
\frac{d}{dt}\widehat{V(t\alpha)}\Big|_{t=0} &= \frac{d}{dt}\mathscr{S}V(t\alpha)e(0)\Big|_{t=0} \\
&= \mathscr{S}\frac{d}{dt}V(t\alpha)e(0)\Big|_{t=0} \\
&= \mathscr{S}\frac{d}{dt}\{\exp\{\int_M (e^{it\alpha} - 1)d\mu\}e(e^{it\alpha} - 1)\}\Big|_{t=0} \\
&= \mathscr{S}(i\alpha + i\int_M \alpha d\mu) \\
&= i\left(\int_M \alpha d\mu + I(\alpha)\right).
\end{aligned}$$

If $X \in L^\infty(\Delta, \mathcal{B}, \mathbb{P})$,

$$\left\|\frac{\widehat{V(t\alpha)}X - X}{t} - i\left(\int_M \alpha d\mu + I(\alpha)\right)X\right\| \le \|X\|_\infty\left\|\frac{\widehat{V(t\alpha)} - 1}{t} - i\left(\int_M \alpha d\mu + I(\alpha)\right)\right\|.$$

Therefore

$$\frac{d}{dt}\widehat{V(t\alpha)}X\Big|_{t=0} = i\left(\int_M \alpha d\mu + I(\alpha)\right)X.$$

Hence for $X \in L^\infty(\Delta, \mathcal{B}, \mathbb{P})$,

$$\widehat{S}X = \left(\int_M \alpha d\mu + I(\alpha)\right)X.$$

This implies that $\int_M \alpha d\mu + I(\alpha)$, and hence $I(\alpha)$, is a.s. real-valued and as $L^\infty(\Delta, \mathcal{B}, \mathbb{P})$ is a core for the self-adjoint operator $\int_M \alpha d\mu + I(\alpha)$, by the maximality of self-adjoint operators, we have that the two operators agree on their common domain. Therefore we can deduce that

$$\widehat{V(t\alpha)} = \exp\{it\big(\int_M \alpha d\mu + I(\alpha)\big)\}.$$

Letting $t = 1$, gives (5.2.1). □

Lemma 5.2.4 $\mathcal{B} = \sigma(\{X_E : E \in \mathcal{M}$ *with* $\mu(E) < \infty\} \cup \mathfrak{N})$, *where* $X_E := I(\mathbb{1}_E)$.

PROOF: The inclusion of the null sets occurs because X_E is only defined a.s., so the statement makes sense irrespective of which representative is chosen for X_E. From Proposition 5.1.7 we have that $\{\mathscr{S}(V(\alpha)e(0)) : \alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)\}$ is dense in $L^2(\Delta, \mathcal{B}, \mathbb{P})$. However as

$$\mathscr{S}(V(\alpha)e(0)) = \exp\{i(\int_M \alpha d\mu + I(\alpha))\}$$

and we have completed $\mathcal{B}$ with respect to $\mathbb{P}$,

$$\mathcal{B} \subseteq \sigma(\{X_E : E \in \mathcal{M} \text{ with } \mu(E) < \infty\} \cup \mathfrak{N}).$$

The converse inclusion is trivial. □

Collecting all the results of this section together we arrive at the following theorem.

Theorem 5.2.5 *The map $I : L^2(M, \mathcal{M}, \mu) \to L^2(\Delta, \mathcal{B}, \mathbb{P})$ is a generalised Poisson process.*

PROOF: If $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$ and $\mu(\alpha) = \int_M \alpha d\mu$, then

$$\begin{aligned}\mathbb{E}[e^{i(I(\alpha)+\mu(\alpha))}] &= \langle 1_\Delta, \widehat{V(\alpha)}\rangle \\ &= \langle \widehat{V(0)}, \widehat{V(\alpha)}\rangle \\ &= \langle e(0), V(\alpha)e(0)\rangle \\ &= \exp\{\int_M (e^{i\alpha} - 1)d\mu\}.\end{aligned}$$

In particular, if $E \in \mathcal{M}$ with $\mu(E) < \infty$, $X_E + \mu(E)$ has a Poisson distribution with mean $\mu(E)$. If $(E_j)_{j=1}^n \subseteq \mathcal{M}$ are disjoint sets with $\mu(E_j) < \infty$, then

$$\mathbb{E}[e^{\sum_{j=1}^n it_j(X_{E_j}+\mu(E_j))}] = \mathbb{E}[e^{i(I(\alpha)+\mu(\alpha))}],$$

where $\alpha = \sum_{j=1}^n t_j \mathbb{1}_{E_j}$. Therefore from above,

$$\begin{aligned}\mathbb{E}[e^{i(I(\alpha)+\mu(\alpha))}] &= \exp\{\int_M (e^{i\alpha} - 1)d\mu\} \\ &= \exp\{\sum_{j=1}^n (e^{it_j} - 1)\mu(E_j)\} \\ &= \prod_{j=1}^n \exp\{\mu(E_j)(e^{it_j} - 1)\} \\ &= \prod_{j=1}^n \mathbb{E}[e^{it_j(X_{E_j}+\mu(E_j))}].\end{aligned}$$

Hence the joint characteristic function splits into a product of the characteristic function of each random variable, and thus $\{X_{E_1}, \ldots, X_{E_n}\}$ are independent. The isometry property of I comes from Proposition 5.2.2. □

5.3 Properties of $\mathscr{S}$

We have shown that $I = \mathscr{S}|_{L^2(M)}$ is a generalised Poisson process. Hence, from Section 4.2 there exists a Wiener-Poisson isomorphism $\mathcal{W}_I$. Not surprisingly, $\mathcal{W}_I$ and $\mathscr{S}$ turn out to be the same map.

Theorem 5.3.1 *With the definitions above, $\mathscr{S} = \mathcal{W}_I$.*

PROOF: From Proposition 4.3.1, $\{e(e^{i\alpha} - 1) : \alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)\}$ is total in $\mathfrak{F}_+(L^2(M))$. In order to show the equality of the two maps we only need to show they agree on $e(e^{i\alpha} - 1)$ for each $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$. Using Lemma 5.2.3, and the definition of $V(\alpha)$,

$$\begin{aligned}\mathscr{S}(e(e^{i\alpha} - 1)) &= \exp\{-\int_M (e^{i\alpha} - 1)d\mu\}\mathscr{S}V(\alpha)e(0)\\ &= \exp\{-\int_M (e^{i\alpha} - 1)d\mu\}\widehat{V(\alpha)}\\ &= \exp\{-\int_M (e^{i\alpha} - 1 - i\alpha)d\mu + iI(\alpha)\}.\end{aligned}$$

However from (4.2.2),

$$\mathcal{W}_I(e(e^{i\alpha} - 1)) = \exp\{-\int_M (e^{i\alpha} - 1 - i\alpha)d\mu + I(\alpha)\}.$$

Thus $\mathscr{S}(e(e^{i\alpha} - 1)) = \mathcal{W}_I(e(e^{i\alpha} - 1))$. □

We finish this section by investigating how our construction behaves for two different measure spaces $(M, \mathcal{M}, \mu)$ and $(M', \mathcal{M}', \mu')$. It turns out that if the spaces $(M, \mathcal{M}, \mu)$ and $(M', \mathcal{M}', \mu')$ are isomorphic as measure spaces, then the two constructions are very closely related. However, we cannot obtain these conclusions from just a Hilbert space isometry between $L^2(M, \mathcal{M}, \mu)$ and $L^2(M', \mathcal{M}', \mu')$. We shall use two different notions of equivalence of measure spaces, one definition requiring the measurable sets to be identified and the other requiring the underlying spaces M and M' to be identified.

If $(M, \mathcal{M}, \mu)$ is a measure space we define an equivalence relation on $\mathcal{M}$ by

$$E \sim F \ \Leftrightarrow \mu(E\Delta F) = 0,$$

where $E\Delta F$ is the symmetric difference. We denote by $\mathcal{M}[\mu]$ the set of equivalence classes of $\sim$. The complement and countable union of elements in $\mathcal{M}[\mu]$ can be defined in a natural way and μ can be considered a map on $\mathcal{M}[\mu]$.

Definition 5.3.2 *Let $(M, \mathcal{M}, \mu)$ and $(M', \mathcal{M}', \mu')$ be two measure spaces. Then a measure isomorphism is a bijective map $T : \mathcal{M}[\mu] \to \mathcal{M}'[\mu']$ such that*

$$T([E]\backslash[F]) = T([E])\backslash T([F]), \quad T(\bigcup_{n=0}^{\infty}[E_n]) = \bigcup_{n=0}^{\infty} T([E_n]),$$

and

$$\mu([E]) = \mu'(T([E])),$$

whenever E, F and $E_n \in \mathcal{M}$. $(M, \mathcal{M}, \mu)$ and $(M', \mathcal{M}', \mu')$ are said to be measure isomorphic if there exists a measure isomorphism between the two spaces.

This definition follows that of [27, p.167]. From this definition we can easily obtain the result below.

Lemma 5.3.3 *Let $(M, \mathcal{M}, \mu)$ and $(M', \mathcal{M}', \mu')$ be isomorphic measure spaces with measure isomorphism $T : \mathcal{M}[\mu] \to \mathcal{M}'[\mu']$. Then there exists an isometric isomorphism $I_T : L^2(M) \to L^2(M')$ defined by the extension of*

$$I_T : \sum_{j=1}^{n} \alpha_j \mathbb{1}_{E_j} \longmapsto \sum_{j=1}^{n} \alpha_j \mathbb{1}_{T(E_j)},$$

where $E_j \in \mathcal{M}$ with $\mu(E_j) < \infty$ and $T(E_j) \in T([E_j])$.

If $(M, \mathcal{M}, \mu)$ is a non-atomic σ-finite measure space, then $(M, \mathcal{M}, \mu)$ is measure isomorphic to $([0, \mu(M)), \mathcal{B}_{[0,\mu(M))}, m)$ [27, §41 Theorem C, Exercise 6]. We can also introduce the notion of a measure preserving map (see [27, p.162]), which is a stronger definition than a measure isomorphism.

Definition 5.3.4 *Let $(M, \mathcal{M}, \mu)$ and $(M', \mathcal{M}', \mu')$ be two measure spaces. Then a bijective map $\Psi : M \to M'$ is said to be measure preserving if both Ψ and Ψ^{-1} are measurable and for all $E \in \mathcal{M}'$, $\mu'(E) = \mu(\Psi^{-1}(E))$.*

Given this definition, from [27, §39 Theorem C] or [63, Lemma 5.0.1] we arrive at the next lemma.

Lemma 5.3.5 *Let $(M, \mathcal{M}, \mu)$ and $(M', \mathcal{M}', \mu')$ be measure spaces and $\Psi : M \to M'$ a measure preserving map. Then there exists an isometric isomorphism $I_\Psi : L^2(M) \to L^2(M')$ such that $I_\Psi(f) = f \circ \Psi^{-1}$.*

Notice that if $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega', \mathcal{F}', \mathbb{P}')$ are probability spaces with measure preserving map $\Psi : \Omega \to \Omega'$, then if $X \in L^0(\Omega)$, X and $X \circ \Psi^{-1}$ have the same distribution. This follows because

$$\begin{aligned}\mathbb{P}'(X \circ \Psi^{-1} \leq x) &= \mathbb{P}'(\{\omega' \in \Omega' : X(\Psi^{-1}(\omega')) \leq x\}) \\ &= \mathbb{P}(\Psi^{-1}\{\omega' \in \Omega' : X(\Psi^{-1}(\omega')) \leq x\}) \\ &= \mathbb{P}(\{\Psi^{-1}(\omega') : X(\Psi^{-1}(\omega')) \leq x\}) \\ &= \mathbb{P}(\{\omega : X(\omega) \leq x\}) = \mathbb{P}(X \leq x).\end{aligned}$$

Therefore the map I_Ψ also preserves distributions as well as the Hilbert space norms.

From now on we shall assume that $(M, \mathcal{M}, \mu)$ and $(M', \mathcal{M}', \mu')$ are measure isomorphic with measure isomorphism T. If I_T is the isometric isomorphism defined in Lemma 5.3.3, we have an induced isometric isomorphism

$$\mathfrak{F}_+(I_T) : \mathfrak{F}_+(L^2(M)) \longrightarrow \mathfrak{F}_+(L^2(M')),$$

defined by

$$f_1 \otimes_s \ldots \otimes_s f_n \longmapsto I_T f_1 \otimes_s \ldots \otimes_s I_T f_n$$

and whose inverse, $\mathfrak{F}_+(I_T^{-1})$, is defined in the same way (see [34, Appendix E] for more details).

Proposition 5.3.6 *Let $\mathcal{A} \subseteq \mathfrak{B}(\mathfrak{F}_+(L^2(M)))$ and $\mathcal{A}' \subseteq \mathfrak{B}(\mathfrak{F}_+(L^2(M')))$ be the C^*-algebras constructed in Section 5.1 for $(M, \mathcal{M}, \mu)$ and $(M', \mathcal{M}', \mu')$ respectively. Then for $\alpha \in L^{\mathcal{S}}_{\mathbb{R}}(M)$, $\mathfrak{F}_+(I_T)V(\alpha)\mathfrak{F}_+(I_T^{-1}) = V(I_T(\alpha))$. In particular, $\mathfrak{F}_+(I_T)\mathcal{A}\mathfrak{F}_+(I_T^{-1}) = \mathcal{A}'$.*

Proof: Suppose $\alpha' \in L^{\mathcal{S}}_{\mathbb{R}}(M')$ and let $\alpha = I_T^{-1}(\alpha')$. If $\beta' \in L^{\mathcal{S}}_{\mathbb{R}}(M')$ and $\beta = I_T^{-1}(\beta')$,

$$\begin{aligned}\mathfrak{F}_+(I_T)V(\alpha)\mathfrak{F}_+(I_T^{-1})e(\beta') &= \mathfrak{F}_+(I_T)V(\alpha)e(I_T^{-1}\beta') \\ &= \mathfrak{F}_+(I_T)V(\alpha)e(\beta) \\ &= \exp\{\int_M (e^{i\alpha} - 1 + (e^{i\alpha} - 1)\beta)d\mu\} \\ &\quad \times \mathfrak{F}_+(I_T)e(e^{i\alpha}\beta + e^{i\alpha} - 1) \\ &= \exp\{\int_M (e^{i\alpha} - 1 + (e^{i\alpha} - 1)\beta)d\mu\} \\ &\quad \times e(I_T(e^{i\alpha}\beta + e^{i\alpha} - 1)) \\ &= \exp\{\int_{M'} (e^{i\alpha'} - 1 + (e^{i\alpha'} - 1)\beta')d\mu'\} \\ &\quad \times e(e^{i\alpha'}\beta' + e^{i\alpha'} - 1) \\ &= V(\alpha')e(\beta').\end{aligned}$$

Since $L^{\mathcal{S}}_{\mathbb{R}}(M')$ is total in $L^2(M')$, by the continuity of the two operators we have

$$\mathfrak{F}_+(I_T)V(\alpha)\mathfrak{F}_+(I_T^{-1}) = V(\alpha').$$

The last part follows immediately. □

Thus the C^*-algebras $\mathcal{A}$ and $\mathcal{A}'$ are unitarily equivalent to each other. Consequently, if Δ and Δ' are the Gelfand spaces of $\mathcal{A}$ and $\mathcal{A}'$, by the definition of the Gelfand spaces being the maximal ideal spaces with the induced weak* topology, there exists a homeomorphism $\psi : \Delta \to \Delta'$ such that $(\mathfrak{F}_+(I_T)a\mathfrak{F}_+(I_T^{-1}))^\frown \circ \psi = \widehat{a}$.

Lemma 5.3.7 *If $(\Delta, \mathcal{B}_\Delta, \mathbb{P}_\Delta)$ and $(\Delta', \mathcal{B}_{\Delta'}, \mathbb{P}_{\Delta'})$ are as constructed in Proposition 5.2.2, then the homeomorphism $\psi : \Delta \to \Delta'$ is a measure preserving map.*

Proof: As ψ is a homeomorphism, ψ and ψ^{-1} are measurable with respect to the Borel σ-fields. Hence we can define the push forward measure of $\mathbb{P}_\Delta$ under ψ, $\psi_*\mathbb{P}_\Delta$, by $\psi_*\mathbb{P}_\Delta(E) = \mathbb{P}_\Delta(\psi^{-1}(E))$ for $E \in \mathcal{B}_{\Delta'}$ (see [63, §5.0]). Since ψ is a homeomorphism, it follows that $\psi_*\mathbb{P}_\Delta$ is a regular Borel measure on $(\Delta', \mathcal{B}_{\Delta'})$. If $a \in \mathcal{A}$ and $a' = \mathfrak{F}_+(I_T)a\mathfrak{F}_+(I_T^{-1})$, then $\widehat{a'} \circ \psi = \widehat{a}$, hence by [63, Lemma 5.0.1],

$$\begin{aligned}
\phi(\widehat{a'}) &= \langle e(0), a'e(0)\rangle \\
&= \langle e(0), ae(0)\rangle \\
&= \int_\Delta \widehat{a} d\mathbb{P}_\Delta \\
&= \int_{\Delta'} \widehat{a'} d\psi_*\mathbb{P}_\Delta.
\end{aligned}$$

Therefore by the uniqueness of $\mathbb{P}_{\Delta'}$ we must have $\mathbb{P}_{\Delta'} = \psi_*\mathbb{P}_\Delta$. □

Corollary 5.3.8 *If $(\Delta, \mathcal{B}, \mathbb{P})$ and $(\Delta', \mathcal{B}', \mathbb{P}')$ are the completions of the probability spaces $(\Delta, \mathcal{B}_\Delta, \mathbb{P}_\Delta)$ and $(\Delta', \mathcal{B}_{\Delta'}, \mathbb{P}_{\Delta'})$ respectively, $\psi : \Delta \to \Delta'$ is a measure preserving map between the two completed spaces.*

We shall denote by $\mathscr{S} : \mathfrak{F}_+(L^2(M)) \to L^2(\Delta, \mathcal{B}, \mathbb{P})$ and $\mathscr{S}' : \mathfrak{F}_+(L^2(M')) \to L^2(\Delta', \mathcal{B}', \mathbb{P}')$ the Segal spatial isomorphisms of Proposition 5.2.2, and by I_ψ the map induced by the measure preserving map ψ as in Lemma 5.3.5.

Theorem 5.3.9 *The diagram*

$$\begin{array}{ccc}
\mathfrak{F}_+(L^2(M)) & \xrightarrow{\mathscr{S}} & L^2(\Delta, \mathcal{B}, \mathbb{P}) \\
{\scriptstyle \mathfrak{F}_+(I_T)}\big\downarrow & & \big\downarrow{\scriptstyle I_\psi} \\
\mathfrak{F}_+(L^2(M')) & \xrightarrow{\mathscr{S}'} & L^2(\Delta', \mathcal{B}', \mathbb{P}')
\end{array}$$

commutes, that is $I_\psi \mathscr{S} = \mathscr{S}'\mathfrak{F}_+(I_T)$*. In particular, the restriction of the map* $\mathscr{S}'\mathfrak{F}_+(I_T)$ *to* $L^2(M)$ *gives a generalised Poisson process over* $(M, \mathcal{M}, \mu)$*.*

PROOF: Since $\{V(\alpha)e(0) : \alpha \in L^{\infty,+}_{\mathbb{R},\mathcal{S}}\}$ is total in $\mathfrak{F}_+(L^2(M))$ (Proposition 5.1.7), we only need to show that the two maps coincide on this set. However,

$$\begin{aligned}\mathscr{S}'\mathfrak{F}_+(I_T)(V(\alpha)e(0)) &= \mathscr{S}'\mathfrak{F}_+(I_T)(V(\alpha)(\mathfrak{F}_+(I_T^{-1})e(0')))\\ &= (\mathfrak{F}_+(I_T)V(\alpha)\mathfrak{F}_+(I_T^{-1}))\widehat{\ }\\ &= \widehat{V(\alpha)} \circ \psi^{-1}\\ &= I_\psi(\widehat{V(\alpha)}) = I_\psi \mathscr{S}(V(\alpha)e(0)),\end{aligned}$$

and thus the diagram commutes. By the comment after Lemma 5.3.5, since the map I_ψ also preserves distributions, the claim that $\mathscr{S}'\mathfrak{F}_+(I_T)|_{L^2(M)}$ is a generalised Poisson process comes immediately from the fact that $\mathscr{S}|_{L^2(M)}$ is. □

Notice that if instead of a measure isomorphism, we have a Hilbert space isometric isomorphism $J : L^2(M, \mathcal{M}, \mu) \to L^2(M', \mathcal{M}', \mu')$, it may not be possible to construct the map I_ψ, because it may not be true in this case that $\mathfrak{F}_+(J)\mathcal{A}\mathfrak{F}_+(J^{-1}) = \mathcal{A}'$. It is possible to construct an isometry $\mathscr{S}'\mathfrak{F}_+(J)\mathscr{S}^{-1} : L^2(\Delta, \mathcal{B}, \mathbb{P}) \to L^2(\Delta', \mathcal{B}', \mathbb{P}')$. However, this map need not be induced by a measure preserving map between the two probability spaces. If it was, then by the same argument as in the proof of Theorem 5.3.9, $\mathscr{S}'\mathfrak{F}_+(J)|_{L^2(M)}$ would be a generalised Poisson process. However, if $E \in \mathcal{M}$ with $\mu(E) < \infty$, then $\mathscr{S}'\mathfrak{F}_+(J)(\mathbb{1}_E) = \mathscr{S}'(J(\mathbb{1}_E))$ need not have a compensated Poisson distribution, if $J(\mathbb{1}_E)$ is not a characteristic function. Hence $\mathscr{S}'\mathfrak{F}_+(J)|_{L^2(M)}$ need not be a generalised Poisson process, and thus $\mathscr{S}'\mathfrak{F}_+(J)\mathscr{S}^{-1}$ may not be induced by a measure preserving map between $(\Delta, \mathcal{B}, \mathbb{P})$ and $(\Delta', \mathcal{B}', \mathbb{P}')$.

We have discovered that if the measure spaces $(M, \mathcal{M}, \mu)$ and $(M', \mathcal{M}', \mu')$ are isomorphic as measure spaces, then the corresponding probability spaces constructed in Section 5.2 are the same, up to a measure preserving isomorphism. Thus, the construction in Section 5.2 depends on the underlying measure structure, rather that on the Hilbert space structure, unlike the construction in [69] for the Gaussian case. This is not surprising, since as mentioned before, in the Poisson case, in order to construct an isomorphism we needed to make a choice as to which elements of a Hilbert space map to compensated Poisson variables. We chose it so that the characteristic functions in $L^2(M, \mathcal{M}, \mu)$ map to compensated Poisson variables. Thus, it is expected that the probability spaces $(\Delta, \mathcal{B}, \mathbb{P})$ constructed by the Gelfand transform rely on the measure structure.

If we take $(M, \mathcal{M}, \mu)$ to be the Lebesgue space $(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+}, m)$ and let $N_t := \mathscr{S}(\mathbb{1}_{[0,t]}) + t$, then we have the usual Poisson process. In this case $\mathscr{S}$ becomes the Poisson interpretation of Fock space. Now suppose $M = \mathbb{R}_+ \times \{1, \ldots, n\}$ and endow M with the product of the Lebesgue and counting measure. Then in this example, if we set $X^j_{t_j} = \mathscr{S}((\mathbb{1}_{[0,1]}, j)) + t_j$, $(X^1_{t_1}, \ldots, X^n_{t_n})$ becomes an n-dimensional Poisson process.

Our construction can also be used to produce any set of independent Poisson variables without the use of Kolmogorov's theorem. Suppose M is a set and we have a function $\lambda : M \to \mathbb{R}_+$. We let

$$M_N := \{m \in M : \lambda(m) = 0\}.$$

For $E \subseteq M$ we define $\mu(E) := \infty$ if $E \cap M_N^c$, where M_N^c is the complement of M_N, is uncountable and

$$\mu(E) := \sum_{m_j \in E \cap M_N^c} \lambda(m_j),$$

otherwise. It can be shown that if $\mathcal{M}$ is the collection of all subsets of M, then $(M, \mathcal{M}, \mu)$ is a measure space. Hence applying our construction we obtain an isometry $I : L^2(M) \to L^2(\Delta)$. If we let

$$N_m := I(\mathbb{1}_{\{m\}}) + \lambda(m) = I(\mathbb{1}_{\{m\}}) + \mu(\{m\}),$$

then $\{N_m\}_{m \in M}$ is a collection of independent Poisson variables such that $\mathbb{E}[N_m] = \lambda(m)$.

The process could be constructed using Kolmogorov's theorem. We would consider the product $\prod_{m \in M} \mathbb{N}_m$ and the measures

$$\mathbb{P}_E((n_1, \ldots, n_k)) = \prod_{j=1}^{k} \frac{\lambda(m_j)^{n_j} e^{-\lambda(m_j)}}{n_j!},$$

on $\prod_{m \in E} \mathbb{N}_m$ where $E = \{m_1, \ldots, m_k\}$. The measures $\mathbb{P}_E$ form a consistent family of probability measures and we can apply Kolmogorov's theorem [74, Theorem 6.3], to construct our process. The Gelfand space can be constructed by interpreting the maximal ideal space as $\Delta \subseteq \prod_{a \in \mathcal{A}} \mathbb{C}_a$ and using Tychonoff's theorem. Consequently, in our construction the use of product spaces is hidden within the Gelfand space and does not appear explicitly.

Chapter 6

Perturbations of Classical Semimartingales

In [68, §3] it is proved that perturbations of Brownian motion, $(A_t + A_t^\dagger)_{0\le t\le 1}$, by regular quantum semimartingales satisfy the quantum Duhamel formula. In this chapter we show that certain classical Poisson martingales can be represented as essentially self-adjoint quantum semimartingales and that perturbations of these quantum semimartingales by regular self-adjoint quantum semimartingales also satisfy the quantum Duhamel formula. We finish by showing that perturbations of essentially self-adjoint mixed Brownian-Poisson quantum semimartingales satisfy the quantum Duhamel formula.

6.1 The Ito formula for $(\Lambda_t + A_t + A_t^\dagger)_{0\le t\le 1}$

Given a compensated Poisson process $(X_t)_{0\le t\le 1}$, by Émery's Ito formula [21, Proposition 2], we have the classical Duhamel formula

$$e^{ipX_t} = 1 + \int_0^t (e^{ip(X_s+1)} - e^{ipX_s})dX_s + \int_0^t (e^{ip(X_s+1)} - e^{ipX_s} - ipe^{ipX_s})ds,$$

for all $p \in \mathbb{R}$. Consequently, since $\mathcal{W}$ intertwines $\Lambda_t + A_t + A_t^\dagger$ and $\widehat{X_t}$ we can use the formula above to deduce a formula for $e^{ip(\Lambda_t+A_t+A_t^\dagger)}$. However, in this section we shall deduce the Duhamel formula for $(\Lambda_t + A_t + A_t^\dagger)_{0\le t\le 1}$, using the Fock space structure, without using the Poisson interpretation of Fock space and classical stochastic calculus. We cannot apply the formula from [66, Proposition 5.1] directly since $(\Lambda_t + A_t + A_t^\dagger)_{0\le t\le 1}$ is not a regular quantum semimartingale.

Recall from (5.1.4) the definition of $U(p\mathbb{1}_{[0,t]})$ on $\mathfrak{F}_+(L^2[0,1])$. We define for $p \in \mathbb{R}$ and

$t \in [0,1]$,

$$U(p,t) := U(p\mathbb{1}_{[0,t]}) = e^{ip(\Lambda_t + A_t + A_t^\dagger)}.$$

If $f \in L^2[0,1]$, it follows from (5.1.3) that $(p,t) \mapsto U(p,t)e(f)$ is weakly measurable and hence strongly measurable by the Pettis measurability theorem.

Lemma 6.1.1 *For all $p \in \mathbb{R}$ the quantum stochastic integral*

$$\begin{aligned} M(p,t) := \int_0^t (e^{ip}U(p,s) - U(p,s))(d\Lambda_s + dA_s + dA_s^\dagger) \\ + \int_0^t (e^{ip}U(p,s) - U(p,s) - ipU(p,s))ds \end{aligned}$$

is well-defined and the process $(M(p,t))_{0\le t\le 1}$ is a quantum semimartingale. Furthermore for f, $g \in L^\infty[0,1]$,

$$\begin{aligned} \langle e(f), M(p,t)e(g)\rangle = \int_0^t \Big[((\overline{f(s)}+1)(g(s)+1)(e^{ip}-1)e^{-ips} - ipe^{-ips}) \\ \times \exp\{\int_0^s (e^{ip}-1)(\overline{f(x)}+1)(g(x)+1)dx\}\Big]ds \ \langle e(f), e(g)\rangle. \end{aligned} \tag{6.1.1}$$

Proof: From the comments above the integrands are strongly measurable and from (5.1.2) or by Corollary 2.2.3 the integrands are adapted. Since $\|U(p,s)\| \le 1$ the quantum stochastic integrals in the lemma exist and the integral processes are in fact quantum semimartingales. By (2.1.5),

$$\begin{aligned} \langle e(f), M(p,t)e(g)\rangle = \int_0^t \{\langle e(f), (\overline{f(s)} + g(s) + \overline{f(s)}g(s))(e^{ip}-1)U(p,s)e(g)\rangle \\ + \langle e(f), (e^{ip} - 1 - ip)U(p,s)e(g)\rangle\}ds. \end{aligned} \tag{6.1.2}$$

It follows from (5.1.3) that

$$\langle e(f), U(p,s)e(g)\rangle = e^{-ips}\exp\{\int_0^s (e^{ip}-1)(\overline{f(x)}+1)(g(x)+1)dx\} \ \langle e(f), e(g)\rangle. \tag{6.1.3}$$

Substituting this in (6.1.2), the last part follows. □

Lemma 6.1.2 *For all $p \in \mathbb{R}$ the map $t \mapsto \langle e(f), U(p,t)e(g)\rangle$ is differentiable a.e. on $[0,1]$ with derivative*

$$\begin{aligned} u(p,t) = ((\overline{f(t)}+1)(g(t)+1)(e^{ip}-1)e^{-ipt} - ipe^{-ipt}) \\ \times \exp\{\int_0^t (e^{ip}-1)(\overline{f(x)}+1)(g(x)+1)dx\} \ \langle e(f), e(g)\rangle. \end{aligned}$$

Furthermore if f and g are continuous $t \mapsto \langle e(f), U(p,t)e(g)\rangle$ is differentiable everywhere on $[0,1]$, and

$$\langle e(f), U(p,t)e(g)\rangle - \langle e(f), e(g)\rangle = \int_0^t u(p,s)ds. \tag{6.1.4}$$

PROOF: If $f,\ g \in L^\infty[0,1]$ then from (6.1.3) and the fundamental theorem of calculus for Lebesgue integrable functions, $t \mapsto \langle e(f), U(p,t)e(g)\rangle$ is differentiable a.e. with derivative $u(p,t)$. If f and g are continuous then the integrand in (6.1.3) is continuous. Thus applying the fundamental theorem for continuous functions (see [58, Theorem 7.21]) gives differentiability everywhere and hence we obtain (6.1.4). □

Lemma 6.1.3 *If $f,\ g$ are continuous functions on $[0,1]$, then for all $p \in \mathbb{R}$ and $t \in [0,1]$,*

$$\langle e(f), U(p,t)e(g)\rangle = \langle e(f), (I + M(p,t))e(g)\rangle.$$

PROOF: This follows immediately from (6.1.1) and (6.1.4). □

Theorem 6.1.4 *On E_{lb}, for all $p \in \mathbb{R}$ and $t \in [0,1]$,*

$$\begin{aligned} e^{ip(\Lambda_t + A_t + A_t^\dagger)} = I &+ \int_0^t (e^{ip(\Lambda_s + A_s + A_s^\dagger + I)} - e^{ip(\Lambda_s + A_s + A_s^\dagger)})(d\Lambda_s + dA_s + dA_s^\dagger) \\ &+ \int_0^t (e^{ip(\Lambda_s + A_s + A_s^\dagger + I)} - e^{ip(\Lambda_s + A_s + A_s^\dagger)} - ipe^{ip(\Lambda_s + A_s + A_s^\dagger)})ds. \end{aligned}$$

PROOF: Let S be the set of continuous functions on $[0,1]$. Then $\mathsf{E}(S)$ is dense in $\mathfrak{F}_+(L^2[0,1])$. Therefore by Lemma 6.1.3, $M(p,t) + I = U(p,t)$ on $\mathsf{E}(S)$. Since $M(p,t)|_{\mathsf{E}_{\text{lb}}}$ is closable and $U(p,t)$ is bounded, we must have $U(p,t) = I + M(p,t)$ on E_{lb}. The theorem follows by substituting in the definition of $U(p,t)$. □

Given the strong measurability of $(p,t) \mapsto U(p,t)$ the theorem above implies that the quantum semimartingale $(\Lambda_t + A_t + A_t^\dagger)_{0 \le t \le 1}$ satisfies the quantum Duhamel formula. If we let $M_t = \Lambda_t + A_t + A_t^\dagger$, then $(M_t)_{0\le t\le 1} = M(I,I,I,0)$. Hence considering $F_{\exp(ipM)}$ for example, given any $\phi \in \mathfrak{F}_+(L^2[0,1])$,

$$\begin{aligned} F_{\exp(ipM)}(s)\phi &= ip\int_0^1 e^{ip(1-u)\overline{M_s}} I e^{ipu(\overline{M_s}+I)}\phi du \\ &= ipe^{ip\overline{M_s}}\phi \int_0^1 e^{ipu} du \\ &= (e^{ip(\overline{M_s}+I)} - e^{ip\overline{M_s}})\phi. \end{aligned}$$

Similarly for the other coefficients $E_{\exp(ipM)}$, $G_{\exp(ipM)}$ and $H_{\exp(ipM)}$.

Theorem 6.1.5 $(\Lambda_t + A_t + A_t^\dagger)_{0\le t\le 1}$ *satisfies the quantum Duhamel formula.*

Corollary 6.1.6 $(\Lambda_t + A_t + A_t^\dagger)_{0\le t\le 1}$ *satisfies the quantum Ito formula.*

Using arguments similar to that above and using the inversion formula for Fourier transforms, if $M_t = \Lambda_t + A_t + A_t^\dagger$ then we can write the quantum Ito formula, when $f \in C^{2+}_{\mathrm{loc}}(\mathbb{R})$, as

$$f(M_t) = f(0)I + \int_0^t (f(M_s + I) - f(M_s))dM_s + \int_0^t (f(M_s + I) - f(M_s) - f'(M_s))ds.$$

By similar methods to [66, §7], it is possible to deduce the classical Ito formula for $f(X_t)$ whenever $f \in C^{2+}_{\mathrm{loc}}(\mathbb{R})$. Therefore this method gives a Fock space proof of Ito's formula for a compensated Poisson process.

This approach can also be used to prove the Ito formula for $(A_t + A_t^\dagger)_{0\le t\le 1}$, without using classical stochastic analysis. From [47, §IV.2.9] it can easily be deduced that for $p \in \mathbb{R}$ and $f \in L^2[0,1]$,

$$e^{ip(A_t + A_t^\dagger)}e(f) = e^{-p^2 t + ip\int_0^t f(s)ds}e(f + ip\mathbb{1}_{[0,t]}).$$

Then by the same argument as for $\Lambda_t + A_t + A_t^\dagger$, we can show that if f and g are continuous on $[0,1]$,

$$\langle e(f), e^{ip(A_t+A_t^\dagger)}e(g)\rangle = \langle e(f), \left(I + \int_0^t ipe^{ip(A_s+A_s^\dagger)}(dA_s + dA_s^\dagger) + \frac{1}{2}\int_0^t (ip)^2 e^{ip(A_s+A_s^\dagger)}ds \right) e(g)\rangle.$$

Consequently on E_{lb},

$$e^{ip(A_t+A_t^\dagger)} = I + \int_0^t ipe^{ip(A_s+A_s^\dagger)}(dA_s + dA_s^\dagger) + \frac{1}{2}\int_0^t (ip)^2 e^{ip(A_s+A_s^\dagger)}ds,$$

which is the quantum Duhamel formula for $(A_t + A_t^\dagger)_{0\le t\le 1}$ and thus $(A_t + A_t^\dagger)_{0\le t\le 1}$ satisfies the quantum Ito formula, which if $M_t = A_t + A_t^\dagger$ and $f \in C^{2+}_{\mathrm{loc}}(\mathbb{R})$ can be written as

$$f(M_t) = f(0)I + \int_0^t f'(M_s)dM_s + \frac{1}{2}\int_0^t f''(M_s)ds.$$

6.2 Classical Poisson martingales

In this section we shall consider the representation of some classical multiplication operators as quantum stochastic integrals. We shall always consider classical stochastic integrals with respect to a compensated Poisson process $(X_t)_{0\le t\le 1}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ (see Chapter 3 for more details).

Definition 6.2.1 *Let $(F_t)_{0\le t\le 1}$ be a bounded, predictable process. Then the càdlàg martingale $(M_t)_{0\le t\le 1}$, where*

$$M_t = \int_0^t F_s dX_s,$$

is said to be the classical Poisson martingale associated with $(F_t)_{0\le t\le 1}$.

Note that unlike in [2, Theorem 4], we do not require M_t to be bounded. This is because we shall work with quantum semimartingales rather than just regular quantum semimartingales.

Lemma 6.2.2 *If $(M_t)_{0\le t\le 1}$ and $(F_t)_{0\le t\le 1}$ are as in Definition 6.2.1, then $M_t \in L^4(\Omega)$ for each $t \in [0,1]$. Hence for $t \in [0,1]$, $\mathbb{E}[|M_t|^4] \le \mathbb{E}[|M_1|^4]$.*

Proof: Note that for all $t \le 1$, $\mathbb{E}[M_1|\mathcal{F}_t] = M_t$. Consequently if we prove that $M_1 \in L^4(\Omega)$, then by Jensen's inequality, the lemma is proved. By the classical Ito product formula,

$$M_1^2 = \int_0^1 2M_{s-}dM_s + [M,M]_1. \tag{6.2.1}$$

For some constant K, $|F_s| \le K$ for all $s \in [0,1]$, thus $(s,\omega) \mapsto 2M_{s-}(\omega)F_s(\omega)$ is in $L^2([0,1]\times\Omega)$, and hence since $dM_s = F_s dX_s$,

$$\int_0^1 2M_{s-}dM_s = \int_0^1 2M_{s-}F_s dX_s \in L^2(\Omega).$$

Also, because $[M,M]_1 = \int_0^1 F_s^2 d[X,X]_s = \int_0^1 F_s^2(ds + dX_s)$,

$$[M,M]_1 = \int_0^1 F_s^2(ds + dX_s) \in L^2(\Omega).$$

Therefore from (6.2.1), $M_1^2 \in L^2(\Omega)$, hence $M_1 \in L^4(\Omega)$. □

Corollary 6.2.3 *Let $(F_t)_{0\le t\le 1}$ be a real, bounded, predictable process and $(M_t)_{0\le t\le 1}$ the associated Poisson martingale. Then for $t \in [0,1]$, $\mathcal{W}^{-1}\widehat{M_t}\mathcal{W}$ is an essentially self-adjoint operator with core E_{lb}.*

Proof: This follows immediately from Corollary 3.2.12, because from Lemma 6.2.2 for all $t \in [0,1]$, $M_t \in L^4(\Omega)$. □

Let $f \in L^\infty[0,1]$ and $(F_t)_{0\le t\le 1}$, $(M_t)_{0\le t\le 1}$ be as above, with $|F_t| \le K$ and suppose that $(Z_t)_{0\le t\le 1}$ is the càdlàg version of $(\mathcal{E}(f_t))_{0\le t\le 1}$. We know that for a.a. $s \in [0,1]$, $M_{s-} = M_s$ and $Z_{s-} = Z_s$ a.s.. Since $\mathbb{E}[|Z_s|^2] = \mathbb{E}[|\mathcal{E}(f_s)|^2] = e^{\|f_s\|^2}$,

$$\int_0^1 \mathbb{E}[|F_s Z_{s-}|^2]ds \le K^2 \int_0^1 \mathbb{E}[|Z_s|^2]ds \le K^2 e^{\|f\|^2}, \tag{6.2.2}$$

and

$$\int_0^1 \mathbb{E}[|F_s f(s) Z_{s-}|^2] ds \le K^2 \|f\|_\infty^2 \int_0^t \mathbb{E}[|Z_s|^2] ds \le K^2 \|f\|_\infty^2 e^{\|f\|^2}. \tag{6.2.3}$$

From Corollary 3.2.8 and Lemma 6.2.2 it also follows that

$$\begin{aligned}\int_0^1 \mathbb{E}[|M_{s-} f(s) Z_{s-}|^2] ds &\le \|f\|_\infty^2 \int_0^1 \mathbb{E}[|M_s Z_s|^2] ds \\ &\le \|f\|_\infty^2 \int_0^1 \mathbb{E}[|M_s|^4]^{\frac{1}{2}} \mathbb{E}[|Z_s|^4]^{\frac{1}{2}} ds \\ &\le \|f\|_\infty^2 \mathbb{E}[|M_1|^4]^{\frac{1}{2}} e^{\|f\|^2 + \frac{\|2f+f^2\|^2}{2}}. \end{aligned} \tag{6.2.4}$$

By the classical Ito formula we have

$$\begin{aligned} M_t \mathcal{E}(f_t) = &\int_0^t F_s Z_{s-} dX_s + \int_0^t M_{s-} f(s) Z_{s-} dX_s \\ &+ \int_0^t F_s f(s) Z_{s-} dX_s + \int_0^t F_s f(s) Z_{s-} ds. \end{aligned} \tag{6.2.5}$$

By the inequalities (6.2.2), (6.2.3) and (6.2.4), the integrands in (6.2.5) with respect to dX_s are in $L^2([0,1] \times \Omega)$. Therefore the stochastic integrands in (6.2.5) are square integrable processes and so

$$\mathbb{E}[M_t \mathcal{E}(f_t)] = \int_0^t \mathbb{E}[f(s) F_s \mathcal{E}(f_s)] ds. \tag{6.2.6}$$

Using (6.2.6), by exactly the same argument as in Proposition 3.3.3, we can conclude that for $f, g \in \mathsf{E}_{\mathrm{lb}}$,

$$\langle \mathcal{E}(g), M_t \mathcal{E}(f) \rangle = \langle e(g), \int_0^t \mathcal{W}^{-1} \widehat{F_s} \mathcal{W} (d\Lambda_s + dA_s + dA_s^\dagger) e(f) \rangle.$$

Notice that by Theorem 3.2.5, since a predictable process is adapted and measurable (see [16, §2.2]), the integral on the right-hand side is well-defined. By the density of E_{lb}, we have a representation of classical Poisson integrals as quantum stochastic integrals.

Theorem 6.2.4 *Let $(F_t)_{0 \le t \le 1}$ and $(M_t)_{0 \le t \le 1}$ be as above. If $t \in [0,1]$, on E_{lb},*

$$\mathcal{W}^{-1} \widehat{M_t} \mathcal{W} = \int_0^t \mathcal{W}^{-1} \widehat{F_s} \mathcal{W} (d\Lambda_s + dA_s + dA_s^\dagger).$$

This result is a slight generalisation of [2, Theorem 4], since we do not require M_t to be bounded for each t. A proof of this result can also be obtained using (3.3.2) (see [3, Theorem II.1]).

Since the time integral is just a Bochner integral, if $(H_t)_{0 \le t \le 1}$ is a bounded, adapted, measurable process, then multiplication by $\int_0^t H_s ds \in L^\infty(\Omega)$ is represented as the quantum stochastic integral $\int_0^t \mathcal{W}^{-1} \widehat{H_s} \mathcal{W} ds$ on E_{lb}.

Corollary 6.2.5 *Let $(F_t)_{0\le t\le 1}$ and $(H_t)_{0\le t\le 1}$ be bounded, adapted, measurable processes, with $(F_t)_{0\le t\le 1}$ predictable. Then if*

$$N_t = \int_0^t F_s dX_s + \int_0^t H_s ds,$$

on E_{lb},

$$\mathcal{W}^{-1}\widehat{N_t}\mathcal{W} = \int_0^t \mathcal{W}^{-1}\widehat{F_s}\mathcal{W}(d\Lambda_s + dA_s + dA_s^\dagger) + \int_0^t \mathcal{W}^{-1}\widehat{H_s}\mathcal{W}ds.$$

Consequently $(\mathcal{W}^{-1}\widehat{N_t}\mathcal{W})_{0\le t\le 1}$ is a quantum semimartingale.

Corollary 6.2.6 *If $(F_t)_{0\le t\le 1}$ is a real, bounded, predictable process, $M_t = \int_0^t F_s dX_s$, and $f : \mathbb{R} \to \mathbb{C}$ is a bounded, twice continuously differentiable function, with bounded derivative, then on E_{lb},*

$$\begin{aligned}\mathcal{W}^{-1}\widehat{f(M_t)}\mathcal{W} = f(0)I &+ \int_0^t \mathcal{W}^{-1}(f(M_s+F_s) - f(M_s))\widehat{\ }\mathcal{W}(d\Lambda_s + dA_s + dA_s^\dagger) \\ &+ \int_0^t \mathcal{W}^{-1}(f(M_s+F_s) - f(M_s) - F_s f'(M_s))\widehat{\ }\mathcal{W}ds.\end{aligned} \tag{6.2.7}$$

PROOF: By Émery's Ito formula [21, Proposition 2],

$$\begin{aligned}f(M_t) = f(0) &+ \int_0^t (f(M_{s-}+F_s) - f(M_{s-}))dX_s \\ &+ \int_0^t (f(M_{s-}+F_s) - f(M_{s-}) - F_s f'(M_{s-}))ds.\end{aligned}$$

Consequently, as the integrands above are all bounded and predictable, by Corollary 6.2.5 we have (6.2.7) with M_{s-} instead of M_s. However, $M_{s-} = M_s$ a.s., for a.a. $s \in [0,1]$, therefore if $g : \mathbb{R} \to \mathbb{R}$ is a Borel function, as operators $\mathcal{W}^{-1}\widehat{g(M_{s-})}\mathcal{W} = \mathcal{W}^{-1}\widehat{g(M_s)}\mathcal{W}$ for a.a. $s \in [0,1]$. Thus the representation (6.2.7) holds. □

If X is a real-valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, then from [56, Theorem VIII.4], if $f : \mathbb{R} \to \mathbb{R}$ is a bounded Borel function, then $f(\mathcal{W}^{-1}\widehat{X}\mathcal{W}) = \mathcal{W}^{-1}\widehat{f(X)}\mathcal{W}$. Consequently, (6.2.7) can be written as

$$\begin{aligned}f(\mathcal{W}^{-1}\widehat{M_t}\mathcal{W}) = f(0)I &+ \int_0^t (f(\mathcal{W}^{-1}(M_s+F_s)\widehat{\ }\mathcal{W}) - f(\mathcal{W}^{-1}\widehat{M_s}\mathcal{W}))(d\Lambda_s + dA_s + dA_s^\dagger) \\ &+ \int_0^t (f(\mathcal{W}^{-1}(M_s+F_s)\widehat{\ }\mathcal{W}) - f(\mathcal{W}^{-1}\widehat{M_s}\mathcal{W}) - \mathcal{W}^{-1}\widehat{F_s}\mathcal{W}f'(\mathcal{W}^{-1}\widehat{M_s}\mathcal{W}))ds.\end{aligned} \tag{6.2.8}$$

Also, we note that any convergence results in spectral theory, concerning the convergence of the sequence $(f_n(\mathcal{W}^{-1}\widehat{X}\mathcal{W}))$, where $f_n : \mathbb{R} \to \mathbb{R}$ are bounded real-valued functions, can be applied to the sequence $(\mathcal{W}^{-1}\widehat{f_n(X)}\mathcal{W})$ instead.

Using (6.2.8) we can now prove that if $(M_t)_{0\le t\le 1}$ is the classical Poisson martingale associated to a real, bounded, predictable process, then $(\mathcal{W}^{-1}\widehat{M_t}\mathcal{W})_{0\le t\le 1}$, which is essentially self-adjoint with core E_{lb} for each t (see Corollary 6.2.3), satisfies the quantum Duhamel formula.

Theorem 6.2.7 *Let $(F_t)_{0\le t\le 1}$ be a real, bounded, predictable process, and $(M_t)_{0\le t\le 1}$ the associated classical Poisson martingale. Then if*

$$\mathcal{W}^{-1}\widehat{M_t}\mathcal{W} = \int_0^t \mathcal{W}^{-1}\widehat{F_s}\mathcal{W}(d\Lambda_s + dA_s + dA_s^\dagger),$$

$(\mathcal{W}^{-1}\widehat{M_t}\mathcal{W})_{0\le t\le 1}$ satisfies the quantum Duhamel formula.

Proof: First, we want to show that $(p,t) \mapsto e^{ipM_t}$ is strongly measurable as a map in $\mathfrak{B}(L^2(\Omega))$. This follow from the Pettis measurability theorem and Fubini's theorem, because for all X, $Y \in L^2(\Omega)$,

$$(p,t,\omega) \mapsto e^{ipM_t(\omega)}X(\omega)Y(\omega)$$

is a measurable function $\mathbb{R}\times[0,1]\times\Omega \to \mathbb{R}$. Since $f(x) = e^{ipx}$ satisfies the hypothesis of Corollary 6.2.6, we have by (6.2.8),

$$\begin{aligned} e^{ip\mathcal{W}^{-1}\widehat{M_t}\mathcal{W}} = I &+ \int_0^t (e^{ip\mathcal{W}^{-1}(M_s+F_s)^\frown\mathcal{W}} - e^{ip\mathcal{W}^{-1}\widehat{M_s}\mathcal{W}})(d\Lambda_s + dA_s + dA_s^\dagger) \\ &+ \int_0^t (e^{ip\mathcal{W}^{-1}(M_s+F_s)^\frown\mathcal{W}} - e^{ip\mathcal{W}^{-1}\widehat{M_s}\mathcal{W}} - ip\mathcal{W}^{-1}\widehat{F_s}\mathcal{W}e^{ip\mathcal{W}^{-1}\widehat{M_s}\mathcal{W}})ds. \end{aligned}$$

We now need to show that the integrands above coincide with the integrands $E_{\exp(ip\mathcal{W}^{-1}\widehat{M}\mathcal{W})}$, $F_{\exp(ip\mathcal{W}^{-1}\widehat{M}\mathcal{W})}$, $G_{\exp(ip\mathcal{W}^{-1}\widehat{M}\mathcal{W})}$ and $H_{\exp(ip\mathcal{W}^{-1}\widehat{M}\mathcal{W})}$ in the quantum Duhamel formula. We shall just show it for $F_{\exp(ip\mathcal{W}^{-1}\widehat{M}\mathcal{W})}$, the others follow similarly. For $X \in L^2(\Omega)$,

$$\begin{aligned} \mathcal{W}F_{\exp(ip\mathcal{W}^{-1}\widehat{M}\mathcal{W})}(s)\mathcal{W}^{-1}X &= \mathcal{W}ip\int_0^1 e^{ip(1-u)\mathcal{W}^{-1}\widehat{M_s}\mathcal{W}}\mathcal{W}^{-1}\widehat{F_s}\mathcal{W}e^{ipu\mathcal{W}^{-1}(M_s+F_s)^\frown\mathcal{W}}du\mathcal{W}^{-1}X \\ &= ip\int_0^1 \mathcal{W}e^{ip(1-u)\mathcal{W}^{-1}\widehat{M_s}\mathcal{W}}\mathcal{W}^{-1}\widehat{F_s}\mathcal{W}e^{ipu\mathcal{W}^{-1}(M_s+F_s)^\frown\mathcal{W}}\mathcal{W}^{-1}Xdu \\ &= ip\int_0^1 e^{ip(1-u)M_s}F_s e^{ipu(M_s+F_s)}Xdu, \end{aligned}$$

where this is a Bochner integral in $L^2(\Omega)$. However by [20, Theorem III.11.17], we can change it (a.s.) to a pointwise integral, that is to integrate with respect to u with ω fixed. All the operators in the integrand are multiplication operators, therefore they commute. Hence

$$\begin{aligned}\mathcal{W}F_{\exp(ip\mathcal{W}^{-1}\widehat{M}\mathcal{W})}(s)\mathcal{W}^{-1}X &= ip\int_0^1 F_s e^{ipM_s+ipuF_s}du X \\ &= ipe^{ipM_s}F_s X\int_0^1 e^{ipuF_s}du \\ &= (e^{ip(M_s+F_s)} - e^{ipM_s})X.\end{aligned}$$

Thus,

$$F_{\exp(ip\mathcal{W}^{-1}\widehat{M}\mathcal{W})}(s) = e^{ip\mathcal{W}^{-1}(M_s+F_s)\hat{}\,\mathcal{W}} - e^{ip\mathcal{W}^{-1}\widehat{M}_s\mathcal{W}}.$$

Similarly for the other terms in the quantum Duhamel formula. □

By the method used in the above proof, and the inversion formula for the Fourier transform, we can also deduce the quantum Ito formula for these martingales from the classical Ito formula.

Proposition 6.2.8 *Let $(F_t)_{0\leq t\leq 1}$ and $(M_t)_{0\leq t\leq 1}$ be as above. If*

$$\mathcal{W}^{-1}\widehat{M_t}\mathcal{W} = \int_0^t \mathcal{W}^{-1}\widehat{F_s}\mathcal{W}(d\Lambda_s + dA_s + dA_s^\dagger),$$

then $(\mathcal{W}^{-1}\widehat{M_t}\mathcal{W})_{0\leq t\leq 1}$ satisfies the quantum Ito formula.

6.3 Perturbations of classical Poisson martingales

We now examine perturbations of the classical Poisson martingales introduced in the previous section by regular self-adjoint quantum semimartingales, and show that these also satisfy the quantum Duhamel formula. The analogous result for Brownian motion has been shown in [68, Theorem 3.4], and for general Brownian martingales in [67, Theorem 13.1]. The proof in the Poisson case differs, because the classical Ito formula is different.

Theorem 6.3.1 *Suppose $(F_t)_{0\leq t\leq 1}$ is a real, bounded, predictable process and $(M_t)_{0\leq t\leq 1}$ the associated classical Poisson martingale, such that on E_{lb},*

$$\mathcal{W}^{-1}\widehat{M_t}\mathcal{W} = \int_0^t \mathcal{W}^{-1}\widehat{F_s}\mathcal{W}(d\Lambda_s + dA_s + dA_s^\dagger).$$

Furthermore, let

$$J_t = \int_0^t R_s d\Lambda_s + S_s dA_s + S_s^* dA_s^\dagger + U_s ds$$

be such that $(J_t)_{0\le t\le 1}$ is a regular self-adjoint quantum semimartingale. Then, if $N_t = \mathcal{W}^{-1}\widehat{M_t}\mathcal{W} + J_t$, the essentially self-adjoint quantum semimartingale $N = (N_t|_{\mathsf{E}_{\mathrm{lb}}} : t \in [0,1])$ satisfies the quantum Duhamel formula.

PROOF: By Corollary 6.2.3 we know that for $t \in [0,1]$, $\mathcal{W}^{-1}\widehat{M_t}\mathcal{W}$ is an essentially self-adjoint operator with core E_{lb}. Therefore by the Kato-Rellich theorem (see [55, Theorem X.12] or [70, Theorem 5.28]), $N_t = \mathcal{W}^{-1}\widehat{M_t}\mathcal{W} + J_t$ is self-adjoint and $N_t = \overline{N_t|_{\mathsf{E}_{\mathrm{lb}}}}$. For $n = 1, 2, \ldots$, choose a $C^\infty(\mathbb{R})$ function, $f_n : \mathbb{R} \to \mathbb{R}$, with $f_n = 0$ on $(-\infty, 4n] \cup [4n, \infty)$, $f(x) = x$ for $x \in [-n, n]$, such that $-1 \le f_n', f_n'' \le 1$. Then f_n satisfies the hypothesis of Corollary 6.2.6 and therefore

$$\begin{aligned} f_n(\mathcal{W}^{-1}\widehat{M_t}\mathcal{W}) = &\int_0^t (f_n(\mathcal{W}^{-1}(M_s + F_s)\widehat{\ }\mathcal{W}) - f_n(\mathcal{W}^{-1}\widehat{M_s}\mathcal{W}))(d\Lambda_s + dA_s + dA_s^\dagger) \\ &+ \int_0^t (f_n(\mathcal{W}^{-1}(M_s + F_s)\widehat{\ }\mathcal{W}) - f_n(\mathcal{W}^{-1}\widehat{M_s}\mathcal{W}))ds \\ &- \int_0^t \mathcal{W}^{-1}\widehat{F_s}\mathcal{W} f_n'(\mathcal{W}^{-1}\widehat{M_s}\mathcal{W}))ds. \end{aligned}$$

Hence $(f_n(\mathcal{W}^{-1}\widehat{M_t}\mathcal{W}))_{0\le t\le 1}$ is a regular self-adjoint quantum semimartingale. Let

$$\begin{aligned} E_s^{(n)} &= R_s + f_n(\mathcal{W}^{-1}(M_s + F_s)\widehat{\ }\mathcal{W}) - f_n(\mathcal{W}^{-1}\widehat{M_s}\mathcal{W}), \\ F_s^{(n)} &= S_s + f_n(\mathcal{W}^{-1}(M_s + F_s)\widehat{\ }\mathcal{W}) - f_n(\mathcal{W}^{-1}\widehat{M_s}\mathcal{W}), \\ G_s^{(n)} &= S_s^* + f_n(\mathcal{W}^{-1}M_s + F_s)\widehat{\ }\mathcal{W}) - f_n(\mathcal{W}^{-1}\widehat{M_s}\mathcal{W}), \\ H_s^{(n)} &= U_s + f_n(\mathcal{W}^{-1}(M_s + F_s)\widehat{\ }\mathcal{W}) - f_n(\mathcal{W}^{-1}\widehat{M_s}\mathcal{W}) - \mathcal{W}^{-1}\widehat{F_s}\mathcal{W} f_n'(\mathcal{W}^{-1}\widehat{M_s}\mathcal{W}). \end{aligned}$$

If $X \in L^2(\Omega)$, then by the mean value theorem, there exists a function $\theta : \Omega \to [0,1]$, such that

$$\begin{aligned} \|(f_n(M_s + F_s) - f_n(M_s))X\|^2 &= \int_\Omega |f_n(M_s + F_s) - f_n(M_s)|^2 |X|^2 d\mathbb{P} \\ &= \int_\Omega |F_s(\omega)|^2 |f_n'(M_s(\omega) + \theta(\omega)F_s(\omega))|^2 |X(\omega)|^2 d\mathbb{P}(\omega) \\ &\le \int_\Omega |F_s(\omega)|^2 |X(\omega)|^2 d\mathbb{P}(\omega) \\ &\le \|F\|_\infty^2 \|X\|^2, \end{aligned}$$

where $\|F\|_\infty = \sup_{t\in[0,1],\ \omega\in\Omega} |F_t(\omega)|$. Therefore

$$\|f_n(\mathcal{W}^{-1}(M_s + F_s)\widehat{\ }\mathcal{W}) - f_n(\mathcal{W}^{-1}\widehat{M_s}\mathcal{W})\| \le \|F\|_\infty. \tag{6.3.1}$$

Also, since $|f_n'| \leq 1$, it follows that $\|f_n'(\mathcal{W}^{-1}\widehat{M_s}\mathcal{W})\| \leq 1$ for all n. Hence

$$\begin{aligned} \|E_s^{(n)}\| &\leq \|R_s\| + \|F\|_\infty; \\ \|F_s^{(n)}\|,\ \|G_s^{(n)}\| &\leq \|S_s\| + \|F\|_\infty; \\ \|H_s^{(n)}\| &\leq \|U_s\| + 2\|F\|_\infty. \end{aligned} \tag{6.3.2}$$

Define

$$N_t^{(n)} = \int_0^t E_s^{(n)} d\Lambda_s + F_s^{(n)} dA_s + G_s^{(n)} dA_s^\dagger + H_s^{(n)} ds.$$

Then $N_t^{(n)} = f_n(\mathcal{W}^{-1}\widehat{M_t}\mathcal{W}) + J_t$. By the spectral theorem [56, Theorem VIII.5 c)], for $\phi \in \mathsf{E}_{\text{lb}}$,

$$\lim_{n\to\infty} f_n(\mathcal{W}^{-1}\widehat{M_t}\mathcal{W})\phi = \mathcal{W}^{-1}\widehat{M_t}\mathcal{W}\phi, \tag{6.3.3}$$

$$\lim_{n\to\infty} f_n(\mathcal{W}^{-1}(M_t + F_t)\widehat{\ }\mathcal{W})\phi = \mathcal{W}^{-1}(M_t + F_t)\widehat{\ }\mathcal{W}\phi. \tag{6.3.4}$$

Thus we have for $\phi \in \mathsf{E}_{\text{lb}}$,

$$\lim_{n\to\infty} N_t^{(n)}\phi = N_t\phi, \quad \lim_{n\to\infty} (N_t^{(n)} + E_t^{(n)})\phi = (N_t + R_t + \mathcal{W}^{-1}\widehat{F_t}\mathcal{W})\phi.$$

Therefore, by Corollary 2.4.4, since E_{lb} is a core for N_t and $N_t + R_t + \mathcal{W}^{-1}\widehat{F_t}\mathcal{W}$, the following is true in the strong operator sense

$$\lim_{n\to\infty} e^{ipN_t^{(n)}} = e^{ipN_t}, \quad \lim_{n\to\infty} e^{ip(N_t^{(n)} + E_t^{(n)})} = e^{ip(N_t + R_t + \mathcal{W}^{-1}\widehat{F_t}\mathcal{W})}. \tag{6.3.5}$$

Notice that this implies that the maps

$$(p,t) \mapsto e^{ipN_t}, \quad (p,t) \mapsto e^{ip(N_t + R_t + \mathcal{W}^{-1}\widehat{F_t}\mathcal{W})},$$

are strongly measurable. Since $\|f_n'\|_\infty \leq 1$ and f_n' converges pointwise to 1, by another application of the spectral theorem [56, Theorem VIII.5 d)], $f_n'(\mathcal{W}^{-1}\widehat{M_t}\mathcal{W})$ converges strongly to I. From (6.3.3) and (6.3.4) we know that if $\phi \in \mathsf{E}_{\text{lb}}$,

$$(f_n(\mathcal{W}^{-1}(M_t + F_t)\widehat{\ }\mathcal{W}) - f_n(\mathcal{W}^{-1}\widehat{M_t}\mathcal{W}))\phi \to \mathcal{W}^{-1}\widehat{F_t}\mathcal{W}\phi \ \text{ as } n \to \infty.$$

If $\psi \in \mathfrak{F}_+(L^2[0,1])$, then choose $\phi \in \mathsf{E}_{\text{lb}}$ such that $\|\phi - \psi\| < \epsilon/4(\|F\|_\infty + 1)$. Since $(f_n(\mathcal{W}^{-1}(M_t + F_t)\widehat{\ }\mathcal{W}) - f_n(\mathcal{W}^{-1}\widehat{M_t}\mathcal{W}))\phi$ converges to $\mathcal{W}^{-1}\widehat{F_t}\mathcal{W}\phi$, there exists N such that

$$\|(f_n(\mathcal{W}^{-1}(M_t + F_t)\widehat{\ }\mathcal{W}) - f_n(\mathcal{W}^{-1}\widehat{M_t}\mathcal{W}) - \mathcal{W}^{-1}\widehat{F_t}\mathcal{W})\phi\| \leq \epsilon/2$$

whenever $n \geq N$. Then for $n \geq N$ by (6.3.1),

$$\begin{aligned}
&\|(f_n(\mathcal{W}^{-1}(M_t + F_t)^\frown \mathcal{W}) - f_n(\mathcal{W}^{-1}\widehat{M_t}\mathcal{W}) - \mathcal{W}^{-1}\widehat{F_t}\mathcal{W})\psi\| \\
&\qquad \leq \|(f_n(\mathcal{W}^{-1}(M_t + F_t)^\frown \mathcal{W}) - f_n(\mathcal{W}^{-1}\widehat{M_t}\mathcal{W}) - \mathcal{W}^{-1}\widehat{F_t}\mathcal{W})\phi\| \\
&\qquad + \|((f_n(\mathcal{W}^{-1}(M_t + F_t)^\frown \mathcal{W}) - f_n(\mathcal{W}^{-1}\widehat{M_t}\mathcal{W}) - \mathcal{W}^{-1}\widehat{F_t}\mathcal{W}))(\psi - \phi)\| \\
&\qquad \leq \frac{\epsilon}{2} + 2\|F\|_\infty \frac{\epsilon}{4(\|F\|_\infty + 1)} \\
&\qquad < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Therefore $f_n(\mathcal{W}^{-1}(M_t + F_t)^\frown \mathcal{W}) - f_n(\mathcal{W}^{-1}\widehat{M_t}\mathcal{W})$ converges strongly to $\mathcal{W}^{-1}\widehat{F_t}\mathcal{W}$. This can also be shown using the dominated convergence theorem on $(\Omega, \mathcal{F}, \mathbb{P})$. Hence in the strong operator sense

$$\lim_{n\to\infty} E_t^{(n)} = R_t + \mathcal{W}^{-1}\widehat{F_t}\mathcal{W}, \quad \lim_{n\to\infty} F_t^{(n)} = S_t + \mathcal{W}^{-1}\widehat{F_t}\mathcal{W},$$
$$\lim_{n\to\infty} G_t^{(n)} = S_t^* + \mathcal{W}^{-1}\widehat{F_t}\mathcal{W}, \quad \lim_{n\to\infty} G_t^{(n)} = U_t.$$

From [66, Proposition 5.1], we know that the regular quantum semimartingale $(N_t^{(n)})_{0\leq t\leq 1}$ satisfies the quantum Duhamel formula. Therefore

$$e^{ipN_t^{(n)}} = I + \int_0^t E_{\exp(ipN^{(n)})}d\Lambda + F_{\exp(ipN^{(n)})}dA + G_{\exp(ipN^{(n)})}dA^\dagger + H_{\exp(ipN^{(n)})}ds,$$

where

$$\begin{aligned}
E_{\exp(ipN^{(n)})}(s) &= e^{ip(N_s^{(n)}+E_s^{(n)})} - e^{ipN_s^{(n)}}, \\
F_{\exp(ipN^{(n)})}(s) &= ip\int_0^1 e^{ip(1-u)N_s^{(n)}} F_s^{(n)} e^{ipu(N_s^{(n)}+E_s^{(n)})}du, \\
G_{\exp(ipN^{(n)})}(s) &= ip\int_0^1 e^{ip(1-u)(N_s^{(n)}+E_s^{(n)})} G_s^{(n)} e^{ipuN_s^{(n)}}du, \\
H_{\exp(ipN^{(n)})}(s) &= ip\int_0^1 e^{ip(1-u)N_s^{(n)}} H_s^{(n)} e^{ipuN_t^{(n)}}du \\
&\quad + (ip)^2\int_0^1\int_0^1 ue^{ip(1-u)N_s^{(n)}} F_s^{(n)} e^{ipu(1-v)(N_s^{(n)}+E_s^{(n)})} G_s^{(n)} e^{ipuvN_s^{(n)}}dudv.
\end{aligned}$$

To finish the proof we would like to apply Proposition 2.4.1 to show that for all $\phi \in \mathsf{E}_{\mathrm{lb}}$,

$$\left(\int_0^t E_{\exp(ipN^{(n)})}d\Lambda + F_{\exp(ipN^{(n)})}dA + G_{\exp(ipN^{(n)})}dA^\dagger + H_{\exp(ipN^{(n)})}ds\right)\phi$$

converges to

$$\left(\int_0^t E_{\exp(ipN)}d\Lambda + F_{\exp(ipN)}dA + G_{\exp(ipN)}dA^\dagger + H_{\exp(ipN)}ds\right)\phi.$$

The strong convergence of the integrands will only be shown for $F_{\exp(ipN^{(n)})}$. However the arguments for the other integrands are very similar. We have

$$\|pe^{i(1-u)pN_t^{(n)}}F_t^{(n)}e^{iup(N_t^{(n)}+E_t^{(n)})}\| \leq \|pF_t^{(n)}\| \leq |p|(\|S_t\| + \|F\|_\infty).$$

Operator multiplication is continuous on norm bounded sets for the strong operator topology [11, Proposition 2.4]. Therefore for all $\phi \in \mathfrak{F}_+(L^2[0,1])$,

$$\lim_{n\to\infty} pe^{ip(1-u)N_t^{(n)}}F_t^{(n)}e^{ipu(N_t^{(n)}+E_t^{(n)})}\phi = pe^{ip(1-u)N_t}(S_t + \mathcal{W}^{-1}\widehat{F_t}\mathcal{W})e^{ipu(N_t+E_t+F_t)}\phi.$$

Applying the dominated convergence theorem for Bochner integrals, we get

$$\lim_{n\to\infty} F_{\exp(ipN^{(n)})}(t)\phi = \lim_{n\to\infty} F_{\exp(ipN)}(t)\phi.$$

Similar arguments show that $E_{\exp(ipN^{(n)})}$, $G_{\exp(ipN^{(n)})}$ and $H_{\exp(ipN^{(n)})}$ converge strongly to $E_{\exp(ipN)}$, $G_{\exp(ipN)}$ and $H_{\exp(ipN)}$ respectively. Hence condition i) of Proposition 2.4.1 is satisfied. Condition ii) is immediate in view of the fact that we are dealing with unitary operators. Finally, since $\|F_{\exp(ipN^{(n)})}(t) + G_{\exp(ipN^{(n)})}(t)\| \leq |p|(\|F_t^{(n)}\| + \|G_t^{(n)}\|)$ and $\|H_{\exp(ipN^{(n)})}(t)\| \leq |p|(\|H_t^{(n)}\| + \|F_t^{(n)}\|\|G_t^{(n)}\|)$ using (6.3.2),

$$\begin{aligned}
\|E_{\exp(ipN^{(n)})}(t)\| &\leq 2 \in L^\infty[0,1],\\
\|F_{\exp(ipN^{(n)})}(t) + G_{\exp(ipN^{(n)})}(t)\| &\leq |p|(2\|S_t\| + 2\|F\|_\infty) \in L^2[0,1],\\
\|H_{\exp(ipN^{(n)})}(t)\| &\leq |p|(\|U_t\| + 2\|F\|_\infty\\
&\qquad + \|S_t\|^2 + 2\|S_t\|\|F\|_\infty + \|F\|_\infty^2) \in L^1[0,1].
\end{aligned}$$

Thus, conditions iii), iv) and v) of Proposition 2.4.1 are also satisfied. Therefore we can conclude that for $\phi \in \mathsf{E}_{\mathrm{lb}}$,

$$\begin{aligned}
\lim_{n\to\infty} e^{ipN_t^{(n)}}\phi &= \lim_{n\to\infty}\bigg(I + \int_0^t E_{\exp(ipN^{(n)})}d\Lambda + F_{\exp(ipN^{(n)})}dA\\
&\qquad\qquad + G_{\exp(ipN^{(n)})}dA^\dagger + H_{\exp(ipN^{(n)})}ds\bigg)\phi\\
&= \bigg(I + \int_0^t E_{\exp(ipN)}d\Lambda + F_{\exp(ipN)}dA\\
&\qquad\qquad + G_{\exp(ipN)}dA^\dagger + H_{\exp(ipN)}ds\bigg)\phi.
\end{aligned}$$

By (6.3.5), we also know that for $\phi \in \mathsf{E}_{\mathrm{lb}}$, $\lim_{n\to\infty} e^{ipN_t^{(n)}}\phi = e^{ipN_t}\phi$, therefore

$$e^{ipN_t} = I + \int_0^t E_{\exp(ipN)}d\Lambda + F_{\exp(ipN)}dA + G_{\exp(ipN)}dA^\dagger + H_{\exp(ipN)}ds,$$

and $N = \mathcal{W}^{-1}\widehat{M}\mathcal{W} + J$ satisfies the quantum Duhamel formula. □

Corollary 6.3.2 *If* $(\mathcal{W}^{-1}\widehat{M_t}\mathcal{W})_{0\le t\le 1}$ *and* $(J_t)_{0\le t\le 1}$ *are as above,* $(\mathcal{W}^{-1}\widehat{M_t}\mathcal{W}+J_t)_{0\le t\le 1}$ *satisfies the quantum Ito formula.*

Corollary 6.3.3 *If* $\mathcal{W}^{-1}\widehat{X_t}\mathcal{W} = \Lambda_t + A_t + A_t^\dagger$ *and* $(J_t)_{0\le t\le 1}$ *is a regular quantum semimartingale, then* $(\mathcal{W}^{-1}\widehat{X_t}\mathcal{W}+J_t)_{0\le t\le 1}$ *satisfies the quantum Duhamel and quantum Ito formulae.*

Proof: This follows immediately from the above theorem, since $X_t = \int_0^t 1dX_s$, and 1 is a real, bounded, predictable process. □

Using the results obtained on perturbations of Poisson martingales, we can deduce an Ito formula for the 'compensated Poisson process with non-commuting drift', an analogue of the result [68, Theorem 1.1].

Theorem 6.3.4 *Let* $(U_t)_{0\le t\le 1}$ *be a process of bounded, adapted, self-adjoint operators on* $\mathfrak{F}_+(L^2[0,1])$*, belonging to* $L^1([0,1],\mathfrak{B}(\mathfrak{F}_+(L^2[0,1])))$*. Define*

$$J_t = \int_0^t U_s ds,$$

and put $N_t = \mathcal{W}^{-1}\widehat{X_t}\mathcal{W} + J_t$*. If* $f \in C^{2+}_{\mathrm{loc}}(\mathbb{R})$*, then*

$$\begin{aligned} f(N_t) = f(0) &+ \int_0^t (f(N_s+I) - f(N_s))dN_s \\ &+ \int_0^t (f(N_s+I) - f(N_s) - f'(N_s))ds \\ &+ \int_0^t \int_{-\infty}^{\infty} ip\widehat{f}(p)\left(\int_0^1 e^{ip(1-u)N_s}[U_s, e^{ipuN_s}]du\right) dpds. \end{aligned} \tag{6.3.6}$$

Proof: This is an application of Corollary 6.3.2, with $R = S = S^* = 0$. $(N_t)_{0\le t\le 1}$ satisfies the quantum Ito formula,

$$f(N_t) = f(0) + \int_0^t E_{f(N)}d\Lambda + F_{f(N)}dA + F^*_{f(N)}dA^\dagger + H_{f(N)}ds,$$

where the integrands are those in the definition of the quantum Ito formula. From the definition, $E_{f(N)}(s) = f(N_s+I) - f(N_s)$. By the spectral theorem [55, Theorem VIII.5 a)] (see also [59, Theorem 13.24 b)]), for all q, $r \in \mathbb{R}$,

$$e^{iqN_s}e^{ir(N_s+I)} = e^{i(q+r)N_s+irI}.$$

Therefore if $\phi \in \mathfrak{F}_+(L^2[0,1])$,

$$\begin{aligned} F_{f(N)}(s)\phi &= \int_{-\infty}^{\infty} ip\widehat{f}(p)\left(\int_0^1 e^{ip(1-u)N_s}e^{ipu(N_s+I)}\phi du\right)dp \\ &= \int_{-\infty}^{\infty} ip\widehat{f}(p)e^{ipN_s}\left(\int_0^1 e^{ipu}\phi du\right)dp \\ &= \int_{-\infty}^{\infty} \widehat{f}(p)(e^{ip(N_s+I)} - e^{ipN_s})\phi dp \\ &= (f(N_s+I) - f(N_s))\phi. \end{aligned}$$

Therefore $F_{f(N)}(s) = f(N_s+I)-f(N_s)$ and since f is real-valued $F^*_{f(N)}(s) = f(N_s+I)-f(N_s)$. By a similar argument to that above, we have

$$(f(N_s+I) - f(N_s))U_s = \int_{-\infty}^{\infty} ip\widehat{f}(p)\left(\int_0^1 e^{ip(1-u)N_s}e^{ipuN_s}U_s du\right)dp.$$

Thus, again arguing as for $F_{f(N)}$,

$$\begin{aligned} H_{f(N)}(s) = &\int_{-\infty}^{\infty} \widehat{f}(p)\left(\int_0^1 e^{ip(1-u)N_s}[U_s, e^{ipuN_s}]du\right)dp \\ &+ (f(N_s+I) - f(N_s))U_s + f(N_s+I) - f(N_s) - f'(N_s). \end{aligned}$$

Since $dN_s = d\Lambda_s + dA_s + dA_s^\dagger + U_s ds$, substituting in $E_{f(N)}$, $F_{f(N)}$, $F^*_{f(N)}$ and $H_{f(N)}$ into the quantum Ito formula gives the result. □

We remark that this argument cannot be generalised to $N_t = J_t + \mathcal{W}^{-1}\widehat{M_t}\mathcal{W}$ where $(M_t)_{0\le t\le 1}$ is the classical Poisson martingale associated with the process $(F_t)_{0\le t\le 1}$, because we would then require e^{iqN_s} and $e^{ir(N_s+\mathcal{W}^{-1}\widehat{F_s}\mathcal{W})}$ to commute with $\mathcal{W}^{-1}\widehat{F_s}\mathcal{W}$, which may not be the case. When U_s commutes with N_s, for example when U_s is a multiplication operator on $L^2(\Omega)$, the formula (6.3.6) reduces to the usual Ito formula, since in this case $[U_s, e^{ipuN_s}] = 0$.

6.4 Mixed quantum semimartingales

As mentioned, the perturbation result proved in the previous section also holds for Brownian martingales. In this section we examine if this theorem holds if we perturb a Poisson martingale added to a Brownian martingale. Let $\mathfrak{w} : \mathfrak{F}_+(L^2[0,1]) \to L^2(\Omega_\mathfrak{w}, \mathcal{F}_\mathfrak{w}, \mathbb{P}_\mathfrak{w})$ be the Wiener interpretation of Fock space. If X is a complex random variable on $(\Omega_\mathfrak{w}, \mathcal{F}_\mathfrak{w}, \mathbb{P}_\mathfrak{w})$, we define $\widehat{X}$ to be the operator of multiplication by X on $L^2(\Omega_\mathfrak{w}, \mathcal{F}_\mathfrak{w}, \mathbb{P}_\mathfrak{w})$ (see Definition 3.2.3). If X is real-valued then $\mathfrak{w}^{-1}\widehat{X}\mathfrak{w}$ is a self-adjoint operator and $f(\mathfrak{w}^{-1}\widehat{X}\mathfrak{w}) = \mathfrak{w}^{-1}\widehat{f(X)}\mathfrak{w}$. Recall that if $(F_t)_{0\le t\le 1}$ is a bounded predictable process on $(\Omega_{\mathfrak{w}}, \mathcal{F}_{\mathfrak{w}}, \mathbb{P}_{\mathfrak{w}})$, then $(\mathfrak{w}^{-1}\widehat{F_t}\mathfrak{w})_{0\le t\le 1}$ is a suitable quantum stochastic integrand and if we let $M_t = \int_0^t F_s dW_s$, then from (1.0.1),

$$\mathfrak{w}^{-1}\widehat{M_t}\mathfrak{w} = \int_0^t \mathfrak{w}^{-1}\widehat{F_s}\mathfrak{w}(dA_s + dA_s^\dagger).$$

Theorem 6.4.1 *Let $(F_{1,t})_{0\le t\le 1}$ and $(F_{2,t})_{0\le t\le 1}$ be real, bounded, predictable processes on $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega_{\mathfrak{w}}, \mathcal{F}_{\mathfrak{w}}, \mathbb{P}_{\mathfrak{w}})$ respectively. Define*

$$M_{1,t} = \int_0^t F_{1,s}dX_s, \quad M_{2,t} = \int_0^t F_{2,s}dW_s.$$

Suppose that the mixed quantum semimartingale $(M_t)_{0\le t\le 1}$ given by

$$\begin{aligned} M_t &= \mathcal{W}^{-1}\widehat{M_{1,t}}\mathcal{W} + \mathfrak{w}^{-1}\widehat{M_{2,t}}\mathfrak{w} \\ &= \int_0^t \mathcal{W}^{-1}\widehat{F_{1,s}}\mathcal{W}(d\Lambda_s + dA_s + dA_s^\dagger) + \int_0^t \mathfrak{w}^{-1}\widehat{F_{2,s}}\mathfrak{w}(dA_s + dA_s^\dagger), \end{aligned}$$

is essentially self-adjoint with core E_{lb}. If J is a regular self-adjoint quantum semimartingale given by

$$J_t = \int_0^t R_s d\Lambda_s + S_s dA_s + S_s^* dA_s^\dagger + U_s ds,$$

then $N_t = M_t + J_t$ is essentially self-adjoint with core E_{lb}, and $N = (N_t|_{\mathsf{E}_{\text{lb}}} : t \in [0,1])$ satisfies the quantum Duhamel formula.

PROOF: The proof follows that of Theorem 6.3.1, thus we shall only give an outline. We treat $\mathcal{W}^{-1}\widehat{M_{1,t}}\mathcal{W}$ and $\mathfrak{w}^{-1}\widehat{M_{2,t}}\mathfrak{w}$ separately. Since we are given that M_t is essentially self-adjoint with core E_{lb}, by the Kato-Rellich theorem $M_t + J_t$ is also essentially self-adjoint with core E_{lb}. Define functions $f_n : \mathbb{R} \to \mathbb{R}$ as in the proof of Theorem 6.3.1. Then as previously we have

$$\begin{aligned} f_n(\mathcal{W}^{-1}\widehat{M_{1,t}}\mathcal{W}) = &\int_0^t (f_n(\mathcal{W}^{-1}(M_{1,s} + F_{1,s})\widehat{\ }\mathcal{W}) - f_n(\mathcal{W}^{-1}\widehat{M_{1,s}}\mathcal{W}))(d\Lambda_s + dA_s + dA_s^\dagger) \\ &+ \int_0^t (f_n(\mathcal{W}^{-1}(M_{1,s} + F_{1,s})\widehat{\ }\mathcal{W}) - f_n(\mathcal{W}^{-1}\widehat{M_{1,s}}\mathcal{W}))ds \\ &- \int_0^t \mathcal{W}^{-1}\widehat{F_{1,s}}\mathcal{W}f_n'(\mathcal{W}^{-1}\widehat{M_{1,s}}\mathcal{W}))ds. \end{aligned}$$

By the Ito formula for Brownian motion, and the representation of classical Brownian martingales as quantum stochastic integrals discussed above,

$$f_n(\mathfrak{w}^{-1}\widehat{M_{2,t}}\mathfrak{w}) = \int_0^t \mathfrak{w}^{-1}\widehat{F_{2,s}}\mathfrak{w}f_n'(\mathfrak{w}^{-1}\widehat{M_{1,s}}\mathfrak{w})(dA_s + dA_s^\dagger) + \frac{1}{2}\int_0^t (\mathfrak{w}^{-1}\widehat{F_{2,s}}\mathfrak{w})^2 f_n''(\mathfrak{w}^{-1}\widehat{M_{2,s}}\mathfrak{w})ds.$$

Thus $(f_n(\mathcal{W}^{-1}\widehat{M_{1,t}}\mathcal{W}) + f_n(\mathfrak{w}^{-1}\widehat{M_{2,t}}\mathfrak{w}))_{0\le t\le 1}$ is a regular quantum semimartingale. Let

$$\begin{aligned}
E_s^{(n)} &= R_s + f_n(\mathcal{W}^{-1}(M_{1,s}+F_{1,s})^{\frown}\mathcal{W}) - f_n(\mathcal{W}^{-1}\widehat{M_{1,s}}\mathcal{W}),\\
F_s^{(n)} &= S_s + f_n(\mathcal{W}^{-1}(M_{1,s}+F_{1,s})^{\frown}\mathcal{W}) - f_n(\mathcal{W}^{-1}\widehat{M_{1,s}}\mathcal{W}) + \mathfrak{w}^{-1}\widehat{F_{2,s}}\mathfrak{w} f_n'(\mathfrak{w}^{-1}\widehat{M_{2,s}}\mathfrak{w}),\\
G_s^{(n)} &= S_s^* + f_n(\mathcal{W}^{-1}(M_{1,s}+F_{1,s})^{\frown}\mathcal{W}) - f_n(\mathcal{W}^{-1}\widehat{M_{1,s}}\mathcal{W}) + \mathfrak{w}^{-1}\widehat{F_{2,s}}\mathfrak{w} f_n'(\mathfrak{w}^{-1}\widehat{M_{2,s}}\mathfrak{w}),\\
H_s^{(n)} &= U_s + f_n(\mathcal{W}^{-1}(M_{1,s}+F_{1,s})^{\frown}\mathcal{W}) - f_n(\mathcal{W}^{-1}\widehat{M_{1,s}}\mathcal{W})\\
&\qquad - \mathcal{W}^{-1}\widehat{F_{1,s}}\mathcal{W} f_n'(\mathcal{W}^{-1}\widehat{M_{1,s}}\mathcal{W}) + \frac{1}{2}(\mathfrak{w}^{-1}\widehat{F_{2,s}}\mathfrak{w})^2 f_n''(\mathfrak{w}^{-1}\widehat{M_{2,s}}\mathfrak{w}).
\end{aligned}$$

Using (6.3.1), and the fact that $|f_n'| \le 1$ and $|f_n''| \le 1$, we have

$$\begin{aligned}
\|E_s^{(n)}\| &\le \|R_s\| + \|F_{1,s}\|_\infty;\\
\|F_s^{(n)}\|,\ \|G_s^{(n)}\| &\le \|S_s\| + \|F_{1,s}\|_\infty + \|F_{2,s}\|_\infty;\\
\|H_s^{(n)}\| &\le \|U_s\| + 2\|F_{1,s}\|_\infty + \frac{1}{2}\|F_{2,s}\|_\infty.
\end{aligned} \tag{6.4.1}$$

As f_n is bounded, if

$$N_t^{(n)} = \int_0^t E_s^{(n)} d\Lambda_s + F_s^{(n)} dA_s + G_s^{(n)} dA_s^\dagger + H_s^{(n)} ds,$$

then $N_t^{(n)} = f_n(\mathcal{W}^{-1}\widehat{M_{1,t}}\mathcal{W}) + f_n(\mathfrak{w}^{-1}\widehat{M_{2,t}}\mathfrak{w}) + J_t$, and $(N_t^{(n)})_{0\le t\le 1}$ is a regular quantum semimartingale. From exactly the same arguments used in Theorem 6.3.1, in the strong operator topology

$$\lim_{n\to\infty} e^{ipN_t^{(n)}} = e^{ipN_t}, \quad \lim_{n\to\infty} e^{ip(N_t^{(n)}+E_t^{(n)})} = e^{ip(N_t+R_t+\mathcal{W}^{-1}\widehat{F_t}\mathcal{W})}, \tag{6.4.2}$$

giving the strong measurability of the maps

$$(p,t) \mapsto e^{ipN_t}, \quad (p,t) \mapsto e^{ip(N_t+R_t+\mathcal{W}^{-1}\widehat{F_t}\mathcal{W})}.$$

We can also deduce that in the strong limit

$$\begin{aligned}
&\lim_{n\to\infty} E_t^{(n)} = R_t + \mathcal{W}^{-1}\widehat{F_{1,t}}\mathcal{W}, &&\lim_{n\to\infty} F_t^{(n)} = S_t + \mathcal{W}^{-1}\widehat{F_{1,t}}\mathcal{W} + \mathfrak{w}^{-1}\widehat{F_{2,t}}\mathfrak{w},\\
&\lim_{n\to\infty} G_t^{(n)} = S_t^* + \mathcal{W}^{-1}\widehat{F_{1,t}}\mathcal{W} + \mathfrak{w}^{-1}\widehat{F_{2,t}}\mathfrak{w}, &&\lim_{n\to\infty} G_t^{(n)} = U_t.
\end{aligned}$$

Using the above and (6.4.1), we can apply Proposition 2.4.1 to obtain that for $\phi \in \mathsf{E}_{\mathrm{lb}}$,

$$\begin{aligned}
\lim_{n\to\infty} e^{ipN_t^{(n)}}\phi = \bigg(I + \int_0^t & E_{\exp(ipN)} d\Lambda + F_{\exp(ipN)} dA\\
&+ G_{\exp(ipN)} dA^\dagger + H_{\exp(ipN)} ds\bigg)\phi.
\end{aligned}$$

Therefore from (6.4.2) we can deduce $(N_t)_{0\le t\le 1}$ satisfies the quantum Duhamel formula. □

Corollary 6.4.2 *If* $(M_t)_{0\le t\le 1}$ *and* $(J_t)_{0\le t\le 1}$ *are as above,* $(M_t+J_t)_{0\le t\le 1}$ *satisfies the quantum Ito formula.*

It should be noted that the assumption $\mathcal{W}^{-1}\widehat{M_{1,t}}\mathcal{W}+\mathfrak{w}^{-1}\widehat{M_{2,t}}\mathfrak{w}$ is essentially self-adjoint with core E_{lb} is required. This property does not automatically follow, even though each of the operators which are added together do satisfy this property.

Examples of essentially self-adjoint mixed semimartingales for which the above result can be applied do exist. If $F_{1,s} = 0$ or $F_{2,s} = 0$, we obtain a classical Poisson or Brownian martingale. Also if either $M_{1,s}$ or $M_{2,s}$ is bounded for each $s \in [0,1]$ as a function on $(\Omega, \mathcal{F}, \mathbb{P})$ or $(\Omega_{\mathfrak{w}}, \mathcal{F}_{\mathfrak{w}}, \mathbb{P}_{\mathfrak{w}})$, then the hypothesis of the theorem holds by Kato-Rellich. Another example is obtained by taking $F_{1,s} = 1$ and $F_{2,s} = 1$. Then on E_{lb},

$$M_t = \Lambda_t + 2(A_t + A_t^\dagger) = 2(\frac{1}{2}\Lambda_t + A_t + A_t^\dagger).$$

However from Section 3.5, we know that $\frac{1}{2}\Lambda_t + A_t + A_t^\dagger$ and $\widehat{X_t^{1/2}}$ are intertwined by $\mathcal{W}^{1/2}$. Since $\widehat{X_t^{1/2}}$ is essentially self-adjoint with core E_{lb}, the same is true of $\Lambda_t + 2(A_t + A_t^\dagger)$.

For a slightly less trivial example, consider the case when $F_{1,s} = \mathbb{1}_{[1/2,1]}(s)$ and $F_{2,s} = \mathbb{1}_{[0,1/2]}(s)$. Then

$$M_t = \begin{cases} A_t + A_t^\dagger & 0 \le t \le \frac{1}{2} \\ \Lambda_t - \Lambda_{1/2} + A_t + A_t^\dagger & \frac{1}{2} < t \le 1. \end{cases}$$

The essential self-adjointness of M_t for $0 \le t \le 1/2$ is immediate, since it corresponds to multiplication by Brownian motion via the Wiener-Ito isomorphism. We can use Nelson's analytic vector theorem (see [55, Theorem X.39] or [70, Theorem 8.31]) to show self-adjointness for $1/2 < t \le 1$. Note that

$$\begin{aligned} \|A_t|_{L^2[0,1]^{\otimes_s^n}}\| &\le \sqrt{n+1}; \\ \|A_t^\dagger|_{L^2[0,1]^{\otimes_s^n}}\| &\le \sqrt{n+1}; \\ \|\Lambda_t|_{L^2[0,1]^{\otimes_s^n}}\| &\le n. \end{aligned}$$

If $\phi \in L^2[0,1]^{\otimes_s^n}$, then

$$\|(\Lambda_t - \Lambda_{1/2} + A_t + A_t^\dagger)^k\phi\| \le 4^k(n+k)!\|\phi\|.$$

Therefore

$$\sum_{k=0}^{\infty} \frac{\|(\Lambda_t - \Lambda_{1/2} + A_t + A_t^\dagger)^k\phi\|}{k!}|t|^k \le \sum_{k=0}^{\infty} \frac{(n+k)!}{k!}(4|t|)^k < \infty \quad \text{if } |t| < \frac{1}{4}.$$

Thus by Nelson's analytic vector theorem $\Lambda_t - \Lambda_{1/2} + A_t + A_t^\dagger$ is essentially self-adjoint. Since $\mathfrak{F}_+(L^2[0,1])_{00}$ is invariant under $\Lambda_t - \Lambda_{1/2} + A_t + A_t^\dagger$, by [55, Theorem 8.31 Corollary 2], $\mathfrak{F}_+(L^2[0,1])_{00}$ is a core for $\Lambda_t - \Lambda_{1/2} + A_t + A_t^\dagger$. Now $\mathsf{E}_{\mathrm{lb}} \subseteq D(\Lambda_t - \Lambda_{1/2} + A_t + A_t^\dagger)$, and from Lemma 3.3.4, E_{lb} is dense in $\mathbb{D}^{2,2}$. Hence given $\phi \in \mathfrak{F}_+(L^2[0,1])_{00}$, there exists $\phi_n \in \mathsf{E}_{\mathrm{lb}}$ such that $\|\phi - \phi_n\| \to 0$ and $\|(\mathcal{N}+1)(\phi - \phi_n)\| \to 0$ as $n \to \infty$. Consequently

$$\begin{aligned} \|(\Lambda_t - \Lambda_{1/2} + A_t + A_t^\dagger)(\phi - \phi_n)\| &\leq 4\|(\mathcal{N}+1)(\phi - \phi_n)\| \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Therefore E_{lb} is a core for M_t, and we can apply Theorem 6.4.1 to it.

We can also consider the case when $F_{1,s} = c_1$ and $F_{2,s} = c_2 - c_1$, where c_1, $c_2 \in \mathbb{R}$. Then

$$M_t = c_1 \Lambda_t + c_2(A_t + A_t^\dagger).$$

By a similar argument to that above, it can be shown that M_t satisfies the hypothesis of Theorem 6.4.1.

Bibliography

[1] N. Aronszajn, Theory of reproducing kernels, *Transactions of the American Mathematical Society* **68** (1950), 337–404.

[2] S. Attal, An algebra of non-commutative bounded semimartingales: square and angle quantum brackets, *Journal of Functional Analysis* **124** (1994), 292–332.

[3] S. Attal, Classical and quantum stochastic calculus: survey article, *Quantum Probability Communications, QP-PQ, Volume X*, World Scientific, Singapore, 1998, pp. 1–52.

[4] S. Attal, The structure of quantum semimartingale algebras, *Journal of Operator Theory* **46** (2001), 391–410.

[5] S. Attal and J.M. Lindsay, Quantum stochastic calculus with maximal operator domains, Preprint, 1997.

[6] S. Attal and P. A. Meyer, Interprétation probabiliste et extension des intégrales stochastiques non commutatives, *Séminaire de Probabilités XXVII*, Lecture Notes in Mathematics 1557, Springer-Verlag, Berlin, 1993, pp. 312–327.

[7] V.P. Belavkin, A quantum nonadapted Itô formula and stochastic analysis in Fock scale, *Journal of Functional Analysis* **102** (1991), 414–447.

[8] A.C.R. Belton, A proof of Itô's formula and the Wiener-Itô isomorphism, Unpublished, Oxford, 1996.

[9] A.C.R. Belton, A matrix formulation of quantum stochastic calculus, DPhil thesis, Oxford, 1998.

[10] P. Biane, Calcul stochastique non-commutatif, *Lectures on probability theory : Ecole d'Eté de probabilités de Saint-Flour XXIII - 1993*, Lecture Notes in Mathematics 1608, Springer-Verlag, Berlin, 1995, pp. 1–96.

[11] O. Bratteli and D.W. Robinson, *Operator algebras and quantum statistical meachanics I*, Springer-Verlag, New York, 1979.

[12] O. Bratteli and D.W. Robinson, *Operator algebras and quantum statistical meachanics II*, Springer-Verlag, New York, 1981.

[13] R.H. Cameron and W.T. Martin, The orthogonal development of non-linear functionals in series of Fourier-Hermite functionals, *Annals of Mathematics* **48** (1947), 385–392.

[14] S.D. Chatterji and V. Mandrekar, Equivalence and singularity of Gaussian measures and applications, *Probabilistic Analysis and Related Topics, Volume 1*, Academic Press, New York, 1978, pp. 169–197.

[15] K.L. Chung, *Lectures from Markov processes to Brownian motion*, Springer-Verlag, New York, 1982.

[16] K.L. Chung and R.J. Williams, *Introduction to stochastic integration*, Birkhäuser, Boston, 1983.

[17] J. Diestel and J.J. Uhl, *Vector measures*, American Mathematical Society, Providence, Rhode Island, 1977.

[18] R.L. Dobrushin and R.A. Minlos, Polynomials in linear random functions, *Russian Mathematical Surveys* **32** (1977), 71–127.

[19] J.L. Doob, *Stochastic processes*, Wiley, New York, 1990.

[20] N. Dunford and J.T. Schwartz, *Linear operators: Part 1*, Interscience, New York, 1958.

[21] M. Émery, On the Azéma martingale, *Séminaire de Probabilités XXIII*, Lecture Notes in Mathematics 1372, Springer-Verlag, Berlin, 1989, pp. 66–87.

[22] D. D. Engel, *The multiple stochastic integral*, Memoirs of the American Mathematical Society 38 No. 265, American Mathematical Society, Providence, 1982.

[23] M.P. Evans, Existence of quantum diffusions, *Probability Theory and Related Fields* **81** (1989), 473–483.

[24] F. Fagnola, On quantum stochastic differential equations with unbounded coefficients, *Probability Theory and Related Fields* **86** (1990), 501–516.

[25] B. Gaveau and P. Trauber, L'intégrale stochastique comme opérateur de divergence dans l'espace fonctionnel, *Journal of Functional Analysis* **46** (1982), 230–238.

[26] A. Guichardet, *Symmetric Hilbert spaces and related topics*, Lecture Notes in Mathematics 261, Springer-Verlag, Berlin, 1972.

[27] P.R. Halmos, *Measure theory*, Springer-Verlag, New York, 1988.

[28] R.L. Hudson and K.R. Parthasarathy, Quantum Ito's formula and stochastic evolutions, *Communications in Mathematical Physics* **93** (1984), 301–323.

[29] N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion processes*, North-Holland, Amsterdam, 1981.

[30] K. Ito, Multiple Wiener integral, *Journal of the Mathematical Society of Japan* **3** (1951), 157–169.

[31] K. Ito, Complex multiple Wiener integral, *Japanese Journal of Mathematics* **22** (1952), 63–86.

[32] K. Ito, Spectral type of shift transformations of differential processes with stationary increments, *Transactions of the American Mathematical Society* **81** (1956), 253–263.

[33] P. Jagers, Aspects of random measures and point processes, *Advances in Probability and Related Topics, Volume 3*, Dekker, New York, 1974, pp. 179–239.

[34] S. Janson, *Gaussian Hilbert spaces*, Cambridge University press, Cambridge, 1997.

[35] G. Kallinpur, The role of reproducing kernel Hilbert spaces in the study of Gaussian processes, *Advances in Probability, Volume 2*, Dekker, New York, 1970, pp. 49–83.

[36] I. Karatzas and S.E. Shreve, *Brownian motion and stochastic calculus*, Springer-Verlag, New York, 1988.

[37] J.F.C. Kingman, Completely random measures, *Pacific Journal of Mathematics* **21** (1967), 59–78.

[38] P.E. Kopp, *Martingales and stochastic integrals*, Cambridge University Press, Cambridge, 1984.

[39] V. Liebscher, On the isomorphism of Poisson space and symmetric Fock space, *Quantum Probability and Related Topics, QP-PQ, Volume IX*, World Scientific, Singapore, 1994, pp. 295–300.

[40] V. Liebscher, Multiple integrals and the isomorphism between a general white noise space and a symmetric Fock space, *Quantum Probability Communications, QP-PQ, Volume X*, World Scientific, Singapore, 1998, pp. 245–259.

[41] J.M. Lindsay, Independence for quantum stochastic integrators, *Quantum Probability and Related Topics, QP-PQ, Volume VI*, World Scientific, Singapore, 1991, pp. 325–332.

[42] J.M. Lindsay, Quantum and non-causal stochastic calculus, *Probability Theory and Related Fields* **97** (1993), 65–80.

[43] P. Malliavin, *Stochastic analysis*, Springer-Verlag, Berlin, 1997.

[44] M. Métivier, *Semimartingales : a course on stochastic processes*, de Gruyter, Berlin, 1982.

[45] P. A. Meyer, Notions sur les intégrales multiples, *Séminaire de Probabilités X*, Lecture Notes in Mathematics 511, Springer-Verlag, Berlin, 1976, pp. 321–331.

[46] P. A. Meyer, Éléments de probabilités quantiques, *Séminaire de Probabilités XX, 1984/85*, Lecture Notes in Mathematics 1204, Springer-Verlag, Berlin, 1986, pp. 186–312.

[47] P. A. Meyer, *Quantum probability for probabilists*, 2nd ed., Lecture Notes in Mathematics 1538, Springer-Verlag, Berlin, 1995.

[48] G.J. Murphy, *C^*-algebras and operator theory*, Academic Press, London, 1990.

[49] D. Nualart, *The Malliavin calculus and related topics*, Springer-Verlag, New York, 1995.

[50] D. Nualart and J. Vives, Anticipative calculus for the Poisson process based on Fock space, *Séminaire de Probabilités XXIV*, Lecture Notes in Mathematics 1426, Springer-Verlag, Berlin, 1990, pp. 154–165.

[51] K.R. Parthasarathy, *An introduction to quantum stochastic calculus*, Birkhäuser, Basel, 1992.

[52] K.R. Parthasarathy and K.B. Sinha, Stochastic integral representations of bounded quantum semimartingales in Fock space, *Journal of Functional Analysis* **67** (1986), 126–151.

[53] A. Pazy, *Semigroups of linear operators and applications to partial differential operators*, Springer-Verlag, New York, 1983.

[54] N. Privault, Multiple stochastic integral expansions of arbitrary Poisson jump time functionals, *Statistics and Probability Letters* **43** (1999), 179–188.

[55] M. Reed and B. Simon, *Methods of modern mathematical physics II: Fourier analysis, self-adjointness*, Academic Press, New York, 1975.

[56] M. Reed and B. Simon, *Methods of modern mathematical physics I: Functional analysis*, Revised ed., Academic Press, New York, 1980.

[57] D. Revuz and M. Yor, *Continuous martingales and Brownian motion*, Springer-Verlag, Berlin, 1991.

[58] W. Rudin, *Real and complex analysis*, 3rd ed., McGraw-Hill, Singapore, 1987.

[59] W. Rudin, *Functional analysis*, 2nd ed., McGraw-Hill, Singapore, 1991.

[60] J. Ruiz de Chávez, Espaces de Fock pour les processus de Wiener et de Poisson, *Séminaire de Probabilités XIX, 1983/84*, Lecture Notes in Mathematics 1123, Springer-Verlag, Berlin, 1985, pp. 230–241.

[61] I. Segal, Tensor algebras over Hilbert spaces I, *Transactions of the American Mathematical Society* **81** (1956), 106–134.

[62] I.E. Segal and R.A. Kunze, *Integrals and operators*, 2nd ed., Springer-Verlag, Berlin, 1978.

[63] D.W. Stroock, *A concise introduction to the theory of integration*, 3rd ed., Birkhäuser, Boston, 1999.

[64] D. Surgailis, On multiple Poisson integrals and associated equilibrium Markov processes, *Probability and Mathematical Statistics* **3** (1984), 217–239.

[65] G.F. Vincent-Smith, Unitary quantum stochastic evolutions, *Proceedings of the London Mathematical Society (3)* **63** (1991), 401–425.

[66] G.F. Vincent-Smith, The Itô formula for quantum semimartingales, *Proceedings of the London Mathematical Society (3)* **75** (1997), 671–720.

[67] G.F. Vincent-Smith, The Itô formula for essentially self-adjoint quantum semimartingales, Unpublished, Oxford, 1998.

[68] G.F. Vincent-Smith, The Itô formula for perturbed Brownian martingales, *Bulletin of the London Mathematical Society* **32** (2000), 736–745.

[69] G.F. Vincent-Smith, The Gelfand transform and the Wiener-Itô-Segal Isomorphism, Preprint, Oxford, 2001.

[70] J. Weidmann, *Linear operators in Hilbert spaces*, Springer-Verlag, New York, 1980.

[71] N. Wiener, Differential space, *Journal of Mathematics and Physics* **2** (1923), 131–174.

[72] N. Wiener, The homogeneous chaos, *American Journal of Mathematics* **55** (1938), 897–936.

[73] N. Wiener and A. Winter, The discrete chaos, *American Journal of Mathematics* **65** (1943), 279–298.

[74] D. Williams, *Diffusions, Markov processes and martingales, Volume 1 : Foundations*, 1st ed., Wiley, Chichester, 1979.

[75] M. Yor, Sur les intégrales stochastique optionelles et une suite remarquable de formules de exponentielles, *Séminaire de Probabilités X*, Lecture Notes in Mathematics 511, Springer-Verlag, Berlin, 1976, pp. 481–500.

[76] K. Yosida, *Functional analysis*, 6th ed., Springer-Verlag, Berlin, 1980.