

# How anisotropy beats fractality in two-dimensional on-lattice DLA growth

Denis S. Grebenkov<sup>1,2,\*</sup> and Dmitry Beliaev<sup>3,†</sup>

<sup>1</sup>*Laboratoire de Physique de la Matière Condensée (UMR 7643),  
CNRS – Ecole Polytechnique, University Paris-Saclay, 91128 Palaiseau, France*

<sup>2</sup>*Interdisciplinary Scientific Center Poncelet (ISCP),<sup>‡</sup>  
Bolshoy Vlasievskiy Pereulok 11, 119002 Moscow, Russia*

<sup>3</sup>*Mathematical Institute, University of Oxford,  
Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford, OX2 6GG, UK*  
(Dated: Received: October 27, 2017/ Revised version:)

We study the fractal structure of Diffusion-Limited Aggregation (DLA) clusters on the square lattice by extensive numerical simulations (with clusters having up to  $10^8$  particles). We observe that DLA clusters undergo strongly anisotropic growth, with the maximal growth rate along the axes. The naive scaling limit of a DLA cluster by its diameter is thus deterministic and one-dimensional. At the same time, on all scales from the particle size to the size of the entire cluster it has non-trivial box-counting fractal dimension which corresponds to the overall growth rate which, in turn, is smaller than the growth rate along the axes. This suggests that the fractal nature of the lattice DLA should be understood in terms of fluctuations around one-dimensional backbone of the cluster.

PACS numbers: 05.40.-a, 05.50.+q, 05.65.+b, 05.10.Ln  
Keywords: DLA, non-equilibrium growth, anisotropy, fractals

## I. INTRODUCTION

Diffusion Limited Aggregation (DLA) was first introduced by T. A. Witten and L. M. Sander [1, 2] as a model of irreversible colloidal aggregation and then rapidly became a basic model of non-equilibrium growth phenomena such as electrodeposition and dendritic growth, viscous fingering in fluids, dielectric breakdown, mineral deposition, bacterial colony growth, pattern formation, to name but a few [3–11]. The growth is driven by a Laplacian field and is modeled by adding particles, one at a time, to a growing cluster via either a random walk on a lattice, or Brownian motion. In spite of these very simple growth rules, only a few rigorous mathematical results about DLA are available [12, 13]. Most properties of both on-lattice and off-lattice DLA clusters are known either from numerical simulations, or from theoretical approximations (see [14–21] and references therein). In particular, numerical simulations have revealed that DLA clusters on the square lattice are inhomogeneous [22–24], anisotropic [17, 25–28] and multifractal [29, 30]; their properties are lattice dependent (i.e., nonuniversal) [27]; their scaling is not determined by a single exponent [24, 31]; and the involved “exponents” change with the number of particles suggesting a transient regime [24, 27]. To some extent, all these properties are caused by the local anisotropy of the lattice growth rules. As a consequence, even the mere existence

of the scaling limit of the on-lattice DLA remains controversial. This situation contrasts with the significant progress made over the last decade in the analysis of other lattice models such as percolation and Ising models. The identification of stochastic Loewner evolution (SLE) processes as the scaling limit of lattice models led to numerous breakthrough discoveries in this field of statistical physics and mathematics [32–34].

In this paper, we provide theoretical arguments and extensive numerical simulations to shed a light onto the scaling limit of on-lattice DLA clusters. Our main conclusion is that the naive but widely used scaling limit, in which the cluster is rescaled by its diameter, is a deterministic one-dimensional cross-like shape. Figuratively speaking, anisotropy of the cluster beats fractality, resulting in a trivial, non-fractal limit. To explain this point, let us consider a graph of a one-dimensional random walk (with unit-size steps) versus the number of steps  $t$ . This is a random curve on the plane. Rescaling of this curve by its diameter (which is equal to  $t$  in this setting) yields a trivial deterministic limit – the unit interval. In order to obtain a nontrivial limit (the Brownian path), anisotropic rescaling has to be performed, by  $t$  and  $\sqrt{t}$ , along the horizontal and vertical axes, respectively. While the choice of rescaling factors is elementary for this toy example, the proper rescaling of an anisotropic on-lattice DLA cluster remains unknown.

The scaling properties of DLA clusters are usually characterized by two observables: the growth rate  $\beta$  and the fractal dimension  $D$ . Most authors compute the latter using the former. Indeed, if one covers a *regular* fractal of diameter 1 by disks of size  $\epsilon$ , then the number of disks scales as  $N \propto \epsilon^{-D}$ , where  $D$  is the Minkowski dimension of the fractal. Rescaling the fractal by  $\epsilon^{-1}$  yields the diameter  $\propto N^D$ . Hence the growth rate is the inverse of the dimension. This relation that was first put

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<sup>‡</sup>International Joint Research Unit – UMI 2615 CNRS/ IUM/ IITP RAS/ Steklov MI RAS/ Skoltech/ HSE, Moscow, Russian Federation

\*Electronic address: [denis.grebenkov@polytechnique.edu](mailto:denis.grebenkov@polytechnique.edu)

†Electronic address: [belyaev@maths.ox.ac.uk](mailto:belyaev@maths.ox.ac.uk)

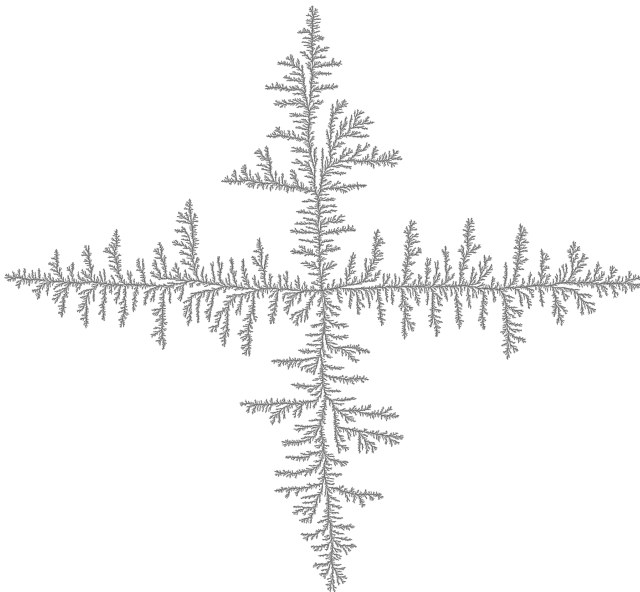


FIG. 1: A large DLA cluster with 145 999 976 particles. In this coarse-grained picture of resolution  $2048 \times 2048$ , each pixel represents a  $64 \times 64$  block of the original cluster image of size  $2^{17} \times 2^{17}$ .

forward by H. E. Stanley for percolation clusters [35] (see also [36]), was often used to get the fractal dimension of both on-lattice and off-lattice DLA clusters by computing the growth rate for the radius of gyration (e.g., [27]). It is important to stress, however, that this relation does not hold in general, it is valid only under some regularity assumptions. The simplest counter-example is an aggregate of  $2t$  particles, half of them forming a disk of radius  $\propto \sqrt{t}$ , and the other half forming an interval of length  $\propto t$ . For this aggregate the fractal dimension is 2 (determined by the disk) but the growth rate is 1 (determined by the interval). The naive rescaling by the diameter  $\propto t$  results in a trivial limit (the unit interval) because the part with the higher dimension but smaller growth rate (the disk) is shrunk and thus fully eliminated in the limit  $t \rightarrow \infty$ .

To our knowledge, the above regularity assumption and the consequent equality between the inverse of the growth rate  $\beta$  and the fractal dimension  $D$  were never properly verified for the on-lattice DLA. The first goal of our work is to check this important equality. Although we obtain slightly different numerical values for  $D$  and  $1/\beta$  (see below), they cannot be distinguished within the numerical accuracy. The second goal consists in emphasizing the role of anisotropy. For this purpose, we introduce the angular growth rate and show that DLA clusters grow faster along the axes of the square lattice. In particular this implies that if the DLA cluster is rescaled by its diameter, then the scaling limit becomes deterministic and one-dimensional. In other words, the parts of the DLA cluster with lower growth rates are eliminated, as

in the above example with a disk and an interval. One can interpret this result as a kind of the law of large numbers for DLA clusters. On the other hand, branches of DLA exhibit a strong pre-fractal behavior that suggests that fluctuations of DLA branches around the axes may have non-trivial scaling limit. This observation can be interpreted as an analogue of the central limit theorem.

## II. NUMERICAL RESULTS

Our strategy to support the above claims consists in two parts: (i) numerical computation of both the growth rate  $\beta$  and the fractal dimension  $D$ , and (ii) profound analysis of the cluster anisotropy. For this purpose, we adapted a bias-free algorithm by Y. E. Loh to generate DLA clusters on the square lattice [37]. The growth of each cluster was stopped when it reaches the edges of the square computational domain,  $2^{\ell_{\max}} \times 2^{\ell_{\max}}$ , with a prescribed scale  $\ell_{\max}$ . As a result, the number of particles in various clusters is not identical. We generated 100 clusters with  $\ell_{\max} = 16$  that have the minimal and the maximal number of particles 41 003 402 and 51 514 999, respectively. We also generated one larger cluster with 145 999 976 particles by setting  $\ell_{\max} = 17$  (Fig. 1). To our knowledge, this is the largest *on-lattice* DLA cluster ever generated (in contrast, off-lattice DLAs of similar sizes have been reported earlier, e.g., [38]).

### A. Fractal dimension versus growth rate

Knowing the history of growth of each generated cluster, we compute two conventional characteristics: the cluster radius,  $\mathcal{R}(t)$ , and the radius of gyration,  $R(t)$ , as functions of the cluster size  $t$  (i.e., the number of particles)

$$\mathcal{R}(t) = \max_{1 \leq k \leq t} \left\{ \sqrt{x_k^2 + y_k^2} \right\}, \quad (1)$$

$$R(t) = \left( \frac{1}{t} \sum_{k=1}^t (x_k^2 + y_k^2) \right)^{\frac{1}{2}}, \quad (2)$$

where  $(x_k, y_k)$  are the coordinates of the  $k$ -th attached particle (with the seed point of the cluster,  $(x_0, y_0)$ , being located at the origin). We checked that  $\mathcal{R}(t)$  and  $R(t)$  behave similarly on the numerically accessible scales, and differ by a factor around 1.5. For this reason, we focus on the radius of gyration which exhibits less fluctuations. Figure 2a illustrates a power law growth of  $R(t)$  with the cluster size  $t$  for one DLA cluster. For this cluster, we get the growth exponent  $\beta \approx 0.594$ . This value is by 2% larger than the earlier reported growth exponent of off-lattice DLA, 0.583 [39]. For the same cluster, we also compute its box-counting dimension by evaluating the number  $\mathcal{N}_\ell$  of non-empty boxes at scale  $\ell$ , ranging from 1 (the size of one particle) to  $2^{\ell_{\max}}$  (the

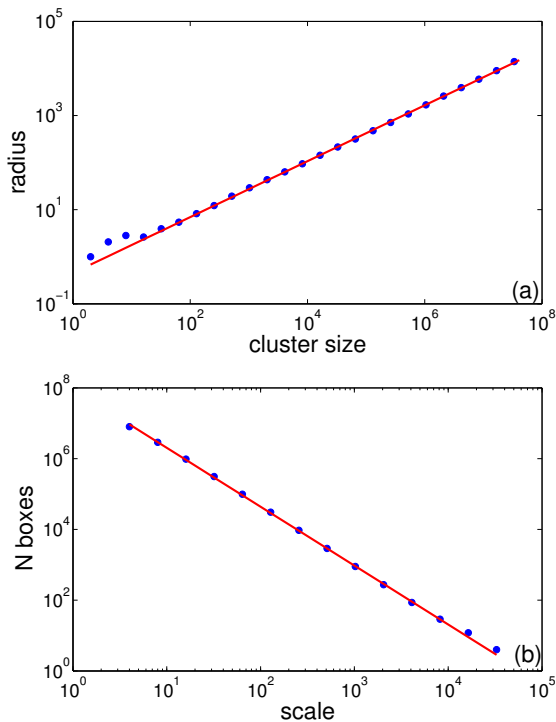


FIG. 2: (Color online) **(a)** Radius of gyration  $R(t)$  as a function of the cluster size  $t$  for one cluster (symbols). A linear fit at loglog scale (line),  $\ln R(t) = 0.594 \ln t - 0.796$ , was obtained over sizes from  $2^{10}$  to  $2^{25}$ . **(b)** Number of non-empty boxes,  $\mathcal{N}_\ell$ , at scale  $\ell$ , for the same cluster. Solid line shows a linear fit at loglog scale,  $\ln \mathcal{N}_\ell = -1.664 \ln \ell + 18.351$ , obtained for scales ranging between  $2^2$  and  $2^{13}$ .

size of the whole cluster). Figure 2b shows a power law scaling  $\mathcal{N}_\ell \propto \ell^{-D}$ , which enables us to determine the Minkowski (box-counting) dimension  $D$ . For this DLA cluster, we get  $D \approx 1.664$ . We conclude that the fractal dimension is smaller than  $1/\beta \approx 1.684$  by 1%. In order to check the relevance of this difference, we repeated the above analysis for 100 independently generated clusters. We obtain the empirical mean and standard deviation for two exponents:

$$D = 1.666 \pm 0.004, \quad \beta = 0.596 \pm 0.004. \quad (3)$$

The average fractal dimension  $D$  is smaller than the average of the inverse of the growth rate,  $1/\beta = 1.678 \pm 0.011$ , by only 0.7%, and this difference is below the statistical uncertainty. For this reason, we cannot exclude the relation  $D = 1/\beta$  for on-lattice DLA clusters. We also found that the box-counting fractal dimension  $D$  is remarkably close to the theoretical value  $5/3$  predicted by mean field theories [14, 15], an analytical diamond-shaped model [17], and a continuous-time random walk theory [16].

## B. Role of anisotropy

Various measures have been introduced in 1980's to characterize the anisotropy of DLA clusters on the square lattice [17, 25–28]. We propose another quantity, the *angular growth rate*, which is particularly adapted to study anisotropic but mostly star-like structures. We cover the plane by  $n_s$  equal sectors  $S_1, \dots, S_{n_s}$  (of angle  $2\pi/n_s$ ) centered at the origin (the center of the cluster), and define the angular radius of gyration up to cluster size  $t$ :

$$R_\theta(t) = \left( \frac{1}{n_\theta(t)} \sum_{k=1}^t (x_k^2 + y_k^2) \mathcal{I}_{(x_k, y_k) \in S_\theta} \right)^{\frac{1}{2}}, \quad (4)$$

where  $\mathcal{I}_{(x_k, y_k) \in S_\theta}$  is equal 1 if the point  $(x_k, y_k)$  belongs to the sector  $S_\theta$  of a discretized polar angle  $\theta$ , and zero otherwise, while  $n_\theta(t)$  is the number of cluster points belonging the sector  $S_\theta$  up to  $t$ . The angular growth rate,  $\beta_\theta$ , is defined from the expected power law scaling:  $R_\theta(t) \propto t^{\beta_\theta}$  as  $t \rightarrow \infty$ . In this way, one can probe whether the growth rate depends on the direction and, in particular, whether the growth rates along the square lattice axes and along the diagonals are different.

The left column of Fig. 3 shows the progressive growth of the largest DLA cluster shown in Fig. 1. One can clearly see how an isotropic structure of a small cluster (with  $10^4$  particles) slowly evolves into the cross-like anisotropic structure of larger clusters (e.g., with  $10^7$  particles). For comparison, the right column of Fig. 3 shows a grayscale representation of the density of points, averaged over 100 DLA clusters, at the same  $t$ . The average density is defined as the sum of indicator functions of 100 independently generated clusters. For small clusters ( $t = 10^4$  and below), the density is almost isotropic, meaning that the typical cluster has almost a round shape. For larger clusters with  $t = 10^5$ , the diamond shape emerges, indicating a directional preferential growth along the four axes. At  $t = 10^6$  and  $t = 10^7$ , the diamond shape progressively transforms into a cross-like shape. These features are particularly well seen by looking at two contours: the outer contour showing the maximally distant points from the center, and the inner contour showing the angular radius of gyration. These two contours were computed by identifying the points of all 100 clusters that lie within a sector between angles  $\theta$  and  $\theta + \delta\theta$  (with the angular resolution  $\delta\theta = 1^\circ$ , i.e.,  $n_s = 360$ ). In each sector, the distance between the center and the most distant point, and the angular radius of gyration  $R_\theta(t)$ , were computed and then plotted versus the polar angle  $\theta$  from 0 to  $360^\circ$ . The outer and inner contours illustrate respectively the positions of extreme and average points of DLA clusters. Remarkably, these two contours evolve with the cluster size in a very similar way. Note that the evolution of a commonly observed diamond-like structure of the square DLA clusters into a cross-like shape with four distinct arms was first conjectured by Meakin [27] and then confirmed by numerical simulations on large clusters in [24]. Moreover, the scal-

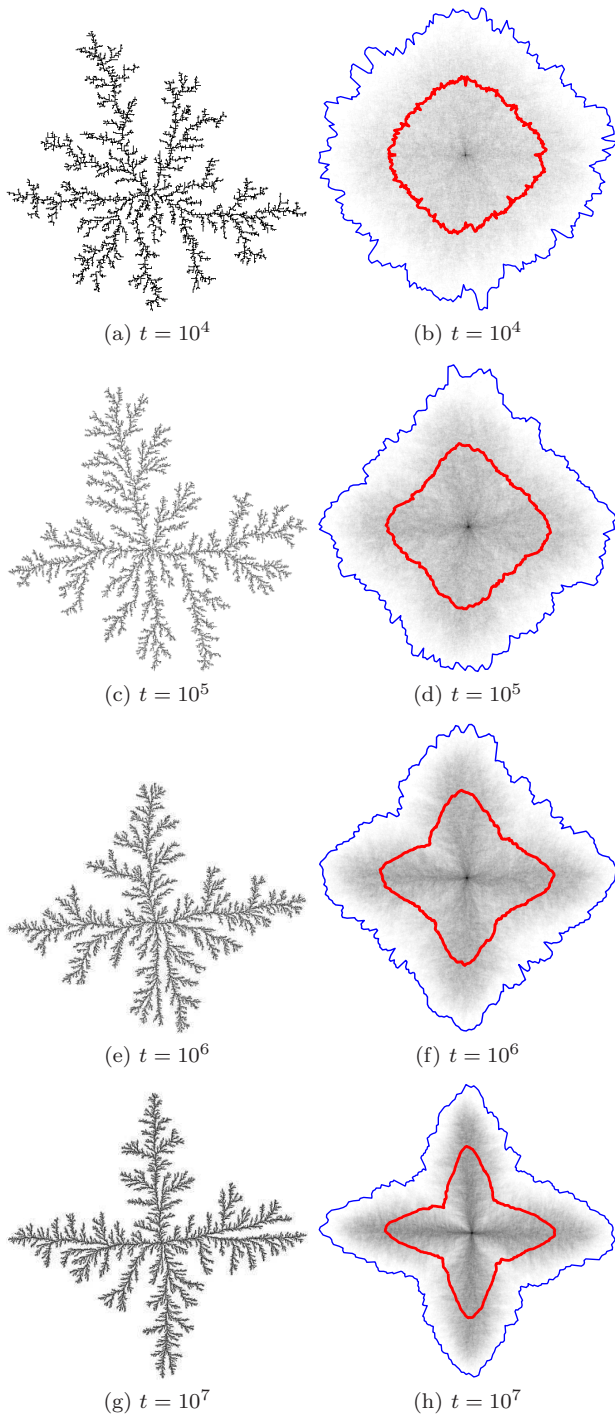


FIG. 3: (Color online). **Left column:** A DLA cluster at various cluster sizes  $t$ :  $10^4$ ,  $10^5$ ,  $10^6$ , and  $10^7$ , from the top to the bottom (coarse-grained  $512 \times 512$  pictures); **Right column:** Grayscale representation of the density of points, averaged over 100 DLA clusters, at the same  $t$ ; thin outer contour shows the maximally distant points from the center (an angular version of the maximal distance  $\mathcal{R}$  defined by Eq. (1)); thick inner contour shows the angular radius of gyration, with the angular resolution of  $1^\circ$ .

ing exponents for the length and width of the four main arms were claimed to be different [16, 19, 24]. We note however that the arguments elaborated in these papers rely on (over)simplified assumptions (e.g., the diamond-like limiting shape of DLA clusters), whereas predictions of the scaling exponents were sometimes different. While there was no doubt about anisotropic character of the on-lattice DLA growth, its explanations remained rather controversial.

After this visual inspection, we proceed to quantify the anisotropic effects. Figure 4a shows how the angular radius of gyration depends on the direction  $\theta$  at different cluster sizes  $t$ . One can see how the anisotropy is progressively established (with four maxima along axes and four minima along diagonal directions). For comparison, Fig. 4b shows the angular radius of gyration  $R_\theta(t)$  for an average cluster obtained by superimposing 100 clusters. As expected, this plot resembles that for one cluster but the average over 100 clusters yields smoother curves. The emergence of anisotropy is particularly clear at semilogarithmic scale (Fig. 4c): flat profiles of  $R_\theta(t)$  versus  $\theta$  at small cluster sizes  $t$  progressively become uneven, with prominent peaks in four axial directions.

In order to reveal the different growth along axes and diagonals, we aggregate the angular radii of gyration  $R_\theta(t)$  for 4 directions of square lattice axes to define  $R_{\text{axis}}(t)$ , and for 4 diagonal directions to define  $R_{\text{diag}}(t)$ . Since the number of points in each sector is significantly smaller than in the whole cluster, fluctuations are much stronger. To reduce fluctuations, we choose relatively large sectors of angle  $11.25^\circ$  (with  $n_s = 32$ ) and we average the aggregated radii over 100 DLA clusters. The resulting axial and diagonal radii  $R_{\text{axis}}(t)$  and  $R_{\text{diag}}(t)$  are shown in Fig. 5a. One can see the faster growth along the axes than along the diagonals, with the growth rates 0.612 and 0.535, respectively. Finally, Fig. 5b presents the angular growth rate  $\beta_\theta$  obtained by linear fits at loglog scale of  $R_\theta(t)$  versus  $t$  (to reduce fluctuations, the angular radius of gyration was averaged over 100 DLA clusters). We observe variations of  $\beta_\theta$  from 0.53 to 0.61, the minimal and maximal growth rates corresponding to the diagonals and to the axes, respectively.

### III. DISCUSSION

With the aid of extensive numerical simulations, we have shown that DLA clusters on the square lattice exhibit strong anisotropic behavior driven by the local aggregation rules. In particular, the growth rate depends on direction, with the maximal growth rate along the axes and the minimal one along the diagonals. This implies that after rescaling by cluster's diameter, the mean size of the cluster in all non axial directions converges to zero hence the scaling limit becomes deterministic and one-dimensional. On the other hand, on all scales from the particle size to the size of the entire cluster it has non-trivial box-counting fractal dimension which corresponds



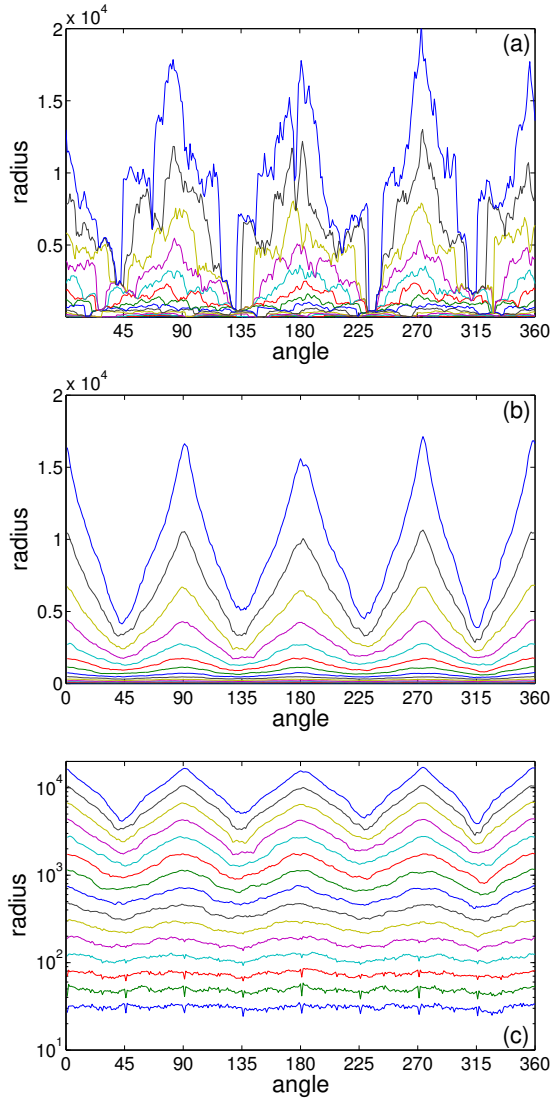


FIG. 4: (Color online) **(a)** Angular radius of gyration  $R_\theta(t)$  as a function of the angle  $\theta$  for one cluster, at cluster sizes  $t = 2^{11}, 2^{12}, \dots, 2^{25}$  (corresponding curves are arranged from bottom to top). **(b,c)** Angular radius of gyration  $R_\theta(t)$  as a function of the angle  $\theta$ , averaged over 100 clusters, at cluster sizes  $t = 2^{11}, 2^{12}, \dots, 2^{25}$ , at linear **(b)** and semilogarithmic **(c)** scales. For all plots, we set  $n_s = 360$ .

to the overall growth rate of the cluster. The latter is a sort of a (non-arithmetic) average of the angular growth rates. This average fully ignores distinctions between growth rates in different directions and thus partly neglects anisotropic effects, in particular, the fastest growth along the axes. This suggests that the fractal nature of the lattice DLA should be understood in terms of fluctuations around one-dimensional backbone of the cluster.

The crucial impact of the lattice anisotropy on the DLA growth naturally disappears for the intrinsically isotropic off-lattice DLA that had also been extensively investigated (see [38–44] and references therein). This

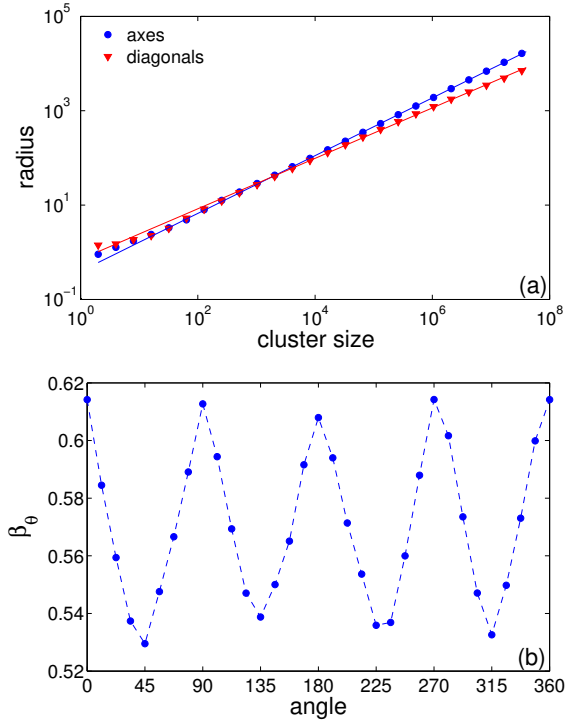


FIG. 5: (Color online) **(a)** Aggregated angular radii of gyration for 4 axes,  $R_{\text{axis}}(t)$ , and for 4 diagonals,  $R_{\text{diag}}(t)$ , averaged over 100 DLA clusters, at cluster sizes  $t = 1, 2, 2^2, \dots, 2^{25}$ . Linear fits at loglog scale (lines) are:  $\ln R_{\text{axis}}(t) = 0.612 \ln t - 0.923$  and  $\ln R_{\text{diag}}(t) = 0.535 \ln t - 0.340$ . **(b)** The angular growth rate  $\beta_\theta$  as a function of the angle  $\theta$ , obtained from linear fits at loglog scale of  $R_\theta(t)$  versus  $t$ . For both plots, we set  $n_s = 32$ .

leads to an important question: how to explain the qualitative difference between on-lattice and off-lattice DLA? There are two evident distinctions. First, the growth of the on-lattice DLA is controlled by the discrete harmonic measure versus the continuous one for the off-lattice model. We believe that this is not really an issue, since the discrete measure converges rapidly to the continuous one as the size of the aggregate increases [45, 46]. The second reason is that the models have different rules for the *local* particle attachment. The effect of the local rules on the anisotropy of the clusters has been reported earlier [47]. Given a cluster, the distribution of places where a new particle will get close to the cluster is almost the same for particles performing random walk and for particles performing Brownian motion. By the concentration of the harmonic measure, the particle which is started near the cluster will attach to the cluster very close to its starting position, with a large probability. The main difference is that for the lattice there are very few places where the particle can attach, especially in the vicinity of a “tip” in the cluster. This has two consequences: (i) the on-lattice particle is more likely to attach to the tip in the direction of the fastest growth than the

Brownian particle would do, and (ii) a growing tip of an on-lattice cluster is less likely to be split into two competing branches. As a result, the branches of off-lattice DLA are more wiggly. In summary, although the growth rules for on-lattice and off-lattice DLA look similar, the lattice anisotropy greatly affects the structure of on-lattice DLA and its fractal properties. In particular, the fractal dimension of the square lattice DLA,  $1.666 \pm 0.004$  (that we computed numerically by box-counting method) is smaller than that of the off-lattice DLA,  $1.715 \pm 0.004$  (computed numerically in [39]).

We conclude that the naive but widely used scaling limit of on-lattice DLA fails due to anisotropy. The future analysis needs to account for anisotropic effects and

to potentially focus on individual branches of large DLA clusters. Our results present thus the first step towards finding a proper rescaling of DLA clusters that is crucial to understand the fractal properties of the on-lattice DLA model and its scaling limit.

### Acknowledgments

DG acknowledges the support under Grant No. ANR-13-JSV5-0006-01 of the French National Research Agency. DB acknowledges the support under EPSRC Fellowship EP/M002896/1.

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