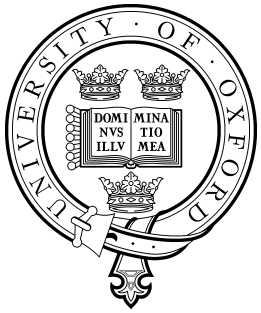
Utility indifference pricing and hedging - Temperature modeling and its significance in the construction of weather derivatives

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1 Introduction

This dissertation is divided in two parts. The first part presents some theory and examples of utility indifference pricing and hedging in incomplete markets. The second part provides some analysis of temperature modeling which is directly linked with the so called weather derivatives, which give an example of an incomplete market described in the first part.

In complete financial markets any contingent claim C can be replicated by trading. We can produce a replicating portfolio of the traded assets , which has wealth process X_t and by assuming a planing horizon $0 \leq t \leq T$, it satisfies $X_T = C$. Hence no arbitrage arguments imply that the option price at time $t = 0$ is the initial value of the portfolio, found by taking the expectation of the claim C under the unique risk-neutral measure of the complete market. In an incomplete financial market not all of the contingent claims can be replicable. Pricing and hedging a claim may be done by embedding the problem in a utility maximization framework, which the agent's preferences towards genuine risk are specified.

This dissertation will use an optimal hedging method which aims to reduce the risk associated with the trading of a derivative on an underlying asset that is not traded, in other words we are aiming to reduce the basis risk. Therefore the financial market which pricing and hedging will take place in, is incomplete. We will use the utility indifference valuation method, which was initially introduced my Hodges and Neuberger [5], in order to price and hedge a derivative on a non-traded asset.

The utility indifference pricing technique was widely used in academic literature during the last years. A marginal version of utility-based pricing was introduced by Davis [2], which gives the utility indifference price for an infinitesimal position in claims, the marginal utility-based price. This was extended by himself after a few years, in pricing weather derivatives [3]. Musiela and Zaphiropoulou [10] used the utility indifference pricing method and gave examples under exponential utility preferences. Monoyios [7], [8], [9] used this technique to study the performance of utility-based strategies for hedging basis risk. He also found analytical prices and hedging strategies under market incompleteness and extended his research by introducing uncertainty in the model parameters.

In the first part of this dissertation we will introduce the main theory associated with utility maximization framework and optimal pricing and hedging in an incomplete market. We will follow papers by Monoyios [8], [9], by introducing two correlated assets, a traded and a non-traded, and try to find pricing and hedging formulas for a

contingent claim written on the non-traded asset, using the utility indifference pricing technique under exponential utility preferences. Monoyios in his papers assumed that both assets follow lognormal distribution. We will take his thoughts a step further by setting the non-traded asset to follow a mean reverting Ornstein-Uhlenbeck (OU) process with constant, known parameters.

We will find a perturbation expansion for the utility indifference price in the case that the claim is just a put option. Equations of the hedging errors under the naive and optimal hedging strategies will be found, and comparison between the two terminal hedging errors distributions will be made.

The second part of the dissertation will deal with the basic ideas and models behind weather derivatives. Nowadays many companies are exposed to the so called 'weather risk'. For example gas companies have reduced profits if a winter month is unusually warm since they sell less gas than expected. Ski resorts also lose money if the temperature is particularly high and the snow melts. Therefore weather derivatives provide protection against the weather risk.

Weather derivatives may be defined as financial instruments with a value that is a contingent claim on an underlying weather index. They can take the form of future contracts, swaps or options. Typically they use as an index either the Heating Degree Days (HDD), or the Cooling Degree Days (CDD). We are planning to use HHD as our index, which is defined by $\text{HDD}_i = \max(18 - T_i)$ on day i , where T_i is the mean (average of maximum and minimum) temperature on day i measured in degrees of Celsius. Define (see Davis [3]) the accumulated Heating Degree Days over a month (31-day) period ending at time t by $X_t = \sum_{i=0}^{30} \text{HDD}_{t-i}$. Contracts are usually written with X_t as the underlying asset. Since the market is illiquid and incomplete, an appropriate style of pricing these contracts is the one described in the fist part of the dissertation

Recent literature provides many techniques which have been introduced mainly from time series analysis, in order to model weather temperature and give pricing formulas for various types of weather derivatives. As stated above, Davis [3], wrote a paper by extending his own previous work, on pricing weather derivatives by marginal value. He also provided some serious evidence that the Accumulated HDDs at a terminal value can be (empirically) modeled by a lognormal distribution. F.Benth and J.Benth [1] proposed an OU process with seasonal volatility to model the daily average temperatures, and calculated some explicit pricing formulas for future contracts. Zapranis and Alexandridis [13] used neural networks in order to examine the temperature time-dependent speed of mean reversion and provide pricing equations for

temperature futures. Xu, Odening and Musshoff [12] wrote an empirical paper on indifference pricing of weather derivatives and Manfredo and Richards [6] analyzed the role of weather derivatives in reducing basis risk.

Deeply motivated by the result that Davis [3] found about the lognormality of the accumulated HDD, we will try to replicate what he found using the following approach. We will model the deviation of mean daily temperature from long-run mean using the simplest version of OU process with constant coefficients and check by simulation weather the continuous version of X_t indeed follows lognormal distribution. We will try to model our de-seasonalised and de-trended simulated data by a first order autocorrelation model and then compare our results with the analysis of real daily temperature data (following [1]). Finally will find the X_t for a certain month using 226 years of real data and argue whether or not a lognormal distribution is the more appropriate or reasonable to use.

The remainder of this dissertation is presented as follows. The first part of the dissertation is contained in Sections 2 and 3. In Section 1 the basis risk model will be introduced and utility indifference pricing and hedging formulas for a general claim will be found. Also we will deal separately with the perfect correlation case, and we will find the residual risk processes for the naive and optimal hedging strategies. Here, by the term 'naive' strategy we mean the Black-Scholes-style hedge which takes the traded asset as a perfect proxy for the non-traded asset. In Section 3 we will derive power series expansions for the indifference price in the case that the contingent claim is a put. Also we will derive some analytical formulas for the claim price in the perfect correlation case and compare the hedging errors at maturity obtained by the naive and optimal hedging. The second part of the dissertation is in Section 4. We will model the daily temperature and analyze some real weather data. Finally the accumulated HDD will be modeled and results about its distribution will be expressed. Section 5 will conclude the paper giving some basic extension ideas for future research.

2 The model

In this Section we will follow [7], [8], [9] in order to find utility indifference pricing and hedging formula for a claim on a non-traded asset. Firstly we consider a basis risk model involving two assets, one traded and one non-traded asset. Take for simplicity the interest rate r to be equal to zero and introduce a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$. On that space the stock price $S := (S_t)_{0\leq t\leq T}$, which is our traded asset, follows a lognormal process given by

$$\mathrm{d}S_t = \sigma S_t(\lambda\,\mathrm{d}t + \mathrm{d}B_t) \tag{1}$$

where $\sigma > 0$ and λ are assumed to be known constants and $B = (B_t)_{0\le t\le T}$ is a multidimensional Brownian motion. Introduce the non-traded asset price $Y := (Y_t)_{0\le t\le T}$ which is assumed to follow a mean reverting OU process given by

$$\mathrm{d}Y_t = a(b - Y_t)\,\mathrm{d}t + c\,\mathrm{d}W_t \tag{2}$$

where $a, b, c > 0$ are known constants, and $W = (W_t)_{0\le t\le T}$ is a correlated Brownian motion such that

$$\mathrm{d}\langle B, W\rangle_t = \rho\,\mathrm{d}t, \qquad W = \rho B + \sqrt{1-\rho^2}B^{\perp}$$

B, $B^{\perp}$ are two independent Brownian motions and ρ is the correlation coefficient, $\rho \in (-1, 1)$.

Now introduce the minimal martingale measure $\mathbb{Q}^{\mathcal{M}}$ such that its density satisfies

$$Z_t := \left.\frac{\mathrm{d}\mathbb{Q}^{\mathcal{M}}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \mathcal{E}(-\lambda B)_t \qquad 0 \le t \le T$$

So under $\mathbb{Q}^{\mathcal{M}}$ the traded asset price process is given by

$$\mathrm{d}S_t = \sigma S_t\,\mathrm{d}B_t^{\mathbb{Q}^{\mathcal{M}}} \tag{3}$$

We also have that

$$\begin{aligned} W_t^{\mathbb{Q}^{\mathcal{M}}} &= W_t + \int_0^t \rho\lambda\,\mathrm{d}u \\ \Rightarrow\ \mathrm{d}W_t^{\mathbb{Q}^{\mathcal{M}}} &= \mathrm{d}W_t + \rho\lambda\,\mathrm{d}t \\ \Rightarrow\ \mathrm{d}W_t &= \mathrm{d}W_t^{\mathbb{Q}^{\mathcal{M}}} - \rho\lambda\,\mathrm{d}t \end{aligned}$$

therefore, under $\mathbb{Q}^{\mathcal{M}}$ the non-traded asset price process is given by

$$\mathrm{d}Y_t = \big[a(b - Y_t) - c\rho\lambda\big]\,\mathrm{d}t + c\,\mathrm{d}W_t^{\mathbb{Q}^{\mathcal{M}}} \tag{4}$$

2.1 Utility indifference pricing and hedging

As mentioned above we will consider a European claim on Y which has payoff $h(Y_T)$ at time T. If $\rho = 1$ then the both assets are traded and the market is complete. We

will deal with this case later. If $|\rho| \neq 1$ then the claim cannot be perfectly replicated and hence the market is incomplete. We will introduce an agent, in the incomplete market case, who takes a position (long or short) in n claims at some time $t \leq T$. Using the utility indifference valuation approach analyzed by [7], [8], [9], [10], we will firstly try to find a price and an optimal hedging strategy for the above claim.

Let the agent be able to trade the stock using a self-financing portfolio, with initial endowment x and $H_t := (H_t)_{0 \leq t \leq T}$ shares of S at time $t \leq T$. So the portfolio wealth process $X = (X_t)_{0 \leq t \leq T}$ is given by

$$X_t = x + (H \cdot S)_t = x + \int_0^t H_u \, \mathrm{d}S_u, \qquad 0 \leq t \leq T \tag{5}$$

Define $\pi := (\pi_t)_{0 \leq t \leq T}$ as the wealth process in the stock S, so $\pi_t = H_t S_t$. Then the wealth process $X^\pi \equiv X$ also satisfies the SDE

$$\mathrm{d}X_t = \sigma \pi_t (\lambda \, \mathrm{d}t + \, \mathrm{d}B_t) \tag{6}$$

Suppose that the agent has risk preferences expressed by an exponential utility function $U(x)$ with constant risk aversion parameter α

$$U(x) = -\exp(-\alpha x), \qquad \alpha > 0 \tag{7}$$

The utility function U is assumed to be continuously differentiable, strictly increasing, strictly concave, and satisfies the Inada conditions:

$$U'(-\infty) = \infty, \qquad U'(\infty) = 0$$

The agent wishes to maximize his expected utility of terminal wealth at maturity T with a random wealth of n units of the contingent claim. So at time $t \in [0, T]$ he wishes to maximize the function $J^{(n)}$ defined by

$$J^{(n)}(t, x, y; \pi) = \mathbb{E}\left[U(X_T + nh(Y_T) | X_t = x, Y_t = y\right]$$

Thus the value function $u^{(n)}(t, x, y)$ is defined as the supremum over all admissible strategies[1] π of $J^{(n)}(t, x, y; \pi)$:

$$u^{(n)}(t, x, y) := \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[U(X_T + nh(Y_T) | X_t = x, Y_t = y\right] \tag{8}$$

with terminal condition:

[1]The set of admissible strategies π is a $\mathbb{Q}$-martingale for all $\mathbb{Q}$ equivalent to $\mathbb{P}$ with finite entropy.

$$u^{(n)}(T,x,y) = U\big(x + nh(y)\big) \tag{9}$$

Denote by $\pi^{*,n}$ the optimal trading strategy that achieves a supremum in (16), and let $X^{*,n}$ be the corresponding optimal wealth process. Also assume that the random endowment $nh(Y_T)$ is bounded below in order for the supremum in (16) to be well-defined.

Next step is to give the definitions of the utility indifference price and optimal hedging strategy

Definition 1. (Utility indifference price)
The utility indifference price per claim $p^{(n)}(t,x,y)$ at time t is the price such that the investor is indifferent between holding and not holding the claims. So $p^{(n)}(t,x,y)$ is defined by

$$u^{(n)}(t, x - np^{(n)}, y) = u^{(0)}(t,x,y)$$

Definition 2. (Optimal Hedging Strategy)
The optimal hedging strategy for n units of the contingent claim is given by

$$\pi_t^{(H)} := \pi_t^{*,n} - \pi_t^{*,0} \qquad 0 \le t \le T$$

where $\pi_t^{*,n}$ is the optimal trading strategy defined above.

Now we will show and explain a detailed solution of the above optimization problem as presented in Monoyios [7], [8], [9] using a technique developed by Zariphopoulou [14].

Zariphopoulou [14] showed that if the SDEs followed by a traded and a non-traded asset S, Y respectively are

$$\begin{aligned} &\mathrm{d}S_t = \mu(Y_t,t)S_t\,\mathrm{d}t + \sigma(Y_t,t)S_t\,\mathrm{d}W_t^1, \quad \mathrm{d}Y_t = \hat{b}(Y_t,t)\,\mathrm{d}t + \hat{a}(Y_t,t)\,\mathrm{d}W_t^2, \\ &\mathrm{d}\langle W^1, W^2\rangle_t = \rho\,\mathrm{d}t \end{aligned}$$

with value function $u(t,x,y) = \sup_\pi \mathbb{E}\left[U(X_T,Y_T)|X_t = x, Y_t = y\right]$ then the HJB equation for the value function is

$$\begin{aligned} &u_t + \frac{1}{2}\hat{a}^2 u_{yy} + \hat{b}u_y + rxu_x + \max_\pi\left[\frac{1}{2}\sigma^2\pi^2 u_{xx} + \rho\pi\sigma\hat{a}u_{xy} + (\mu - r)\pi u_x\right] = 0 \\ &u(T,x,y) = U(x,y) \end{aligned}$$

where the subscripts under u denote differentiation with respect to each parameter. In our case we have that:

$$\mu(Y_t,t)=\sigma\lambda,\quad \sigma(Y_t,t)=\sigma,\quad \hat{b}(Y_t,t)=a(b-Y_t),\quad \hat{a}(Y_t,t)=c$$
$$\mathrm{d}W_t^1=\mathrm{d}B_t,\quad \mathrm{d}W_t^2=\mathrm{d}W_t,\quad r=0,\quad U(x,y)=-\exp(-\alpha(x+nh(y))$$

So by substituting them above, the HJB equation for the value function $u\equiv u_t^{(n)}(t,x,y)$ becomes

$$u_t+\mathcal{L}_Y u+\max_{\pi}\left[\frac{1}{2}\sigma^2\pi^2u_{xx}+\rho\pi\sigma c u_{xy}+\sigma\lambda\pi u_x\right]=0$$
$$u(T,x,y)=U\big(x+nh(y)\big)=-e^{-\alpha(x+nh(y))}$$

where $\mathcal{L}_Y u=\frac{1}{2}c^2u_{yy}+a(b-y)u_y$. To find the optimal trading strategy $\pi^{*,n}$ we have to take the first derivative with respect to π of whatever it is inside the square brackets and set it equal to zero:

$$\sigma^2\pi^{*,n}u_{xx}+\rho\sigma c u_{xy}+\sigma\lambda u_x=0$$

Therefore by solving this equation we find that the optimal trading strategy is given by

$$\pi^{*,n}=-\left(\frac{\lambda u_x+\rho\sigma c u_{xy}}{\sigma u_{xx}}\right) \tag{10}$$

So HJB equation becomes

$$\begin{aligned}
&u_t+\mathcal{L}_Y u+\frac{1}{2}\sigma^2\left(\frac{\lambda u_x+\rho\sigma c u_{xy}}{\sigma u_{xx}}\right)^2u_{xx}-\rho\sigma c\left(\frac{\lambda u_x+\rho\sigma c u_{xy}}{\sigma u_{xx}}\right)u_{xy}\\
&-\sigma\lambda\left(\frac{\lambda u_x+\rho\sigma c u_{xy}}{\sigma u_{xx}}\right)u_x=0\\
\Rightarrow\ &u_t+\mathcal{L}_Y u+\frac{\lambda^2(u_x)^2+2\lambda\rho c u_x u_{xy}+\rho^2c^2(u_{xy})^2}{2u_{xx}}-\frac{\rho c(\lambda u_x u_{xy}+\rho c(u_{xy})^2)}{u_{xx}}\\
&-\frac{\lambda^2(u_x)^2+\lambda\rho c u_x u_{xy}}{u_{xx}}=0\\
\Rightarrow\ &u_t+\mathcal{L}_Y u-\frac{\lambda^2(u_x)^2+2\lambda\rho c u_x u_{xy}+\rho^2c^2(u_{xy})^2}{2u_{xx}}=0
\end{aligned}$$

and takes its final form :

$$u_t+\mathcal{L}_Y u-\frac{(\lambda u_x+\rho c u_{xy})^2}{2u_{xx}}=0 \tag{11}$$

with terminal condition:

$$u(T,x,y) = U\big(x + nh(y)\big) = -e^{-\alpha(x+nh(y))} \tag{12}$$

where $u \equiv u_t^{(n)}(t,x,y)$, $\mathcal{L}_Y u = \frac{1}{2}c^2 u_{yy} + a(b-y)u_y$ and the subscripts under u represent the derivatives with respect to the corresponding variables.

The solution $u \equiv u^{(n)}$ to the HJB equation (11) is presented by the following Proposition (Henderson [4], Monoyios [8],[9], Musiela and Zariphopoulou [10])

Proposition 1. *The value function $u \equiv u^{(n)}$ and the indifference price $p \equiv p^{(n)}$ given $X_t = x, \quad Y_t = y$ for $0 \le t \le T$ are given by*

$$u^{(n)}(t,x,y) = -e^{-\alpha x - \frac{1}{2}\lambda^2(T-t)}[f(t,y)]^{1/(1-\rho^2)} \tag{13}$$

$$f(t,y) = \mathbb{E}^{\,\mathbb{Q}^{\mathcal{M}}}[e^{-\alpha(1-\rho^2)nh(Y_T)}|Y_t = y] \tag{14}$$

$$p^{(n)}(t,y) = -\frac{1}{\alpha n(1-\rho^2)}\log\big[f(t,y)\big] \tag{15}$$

Proof. We use the distortion technique (Zariphopoulou [14]) to find the solution of the HJB equation. Let

$$u^{(n)}(t,x,y) \equiv u = -e^{-\alpha x}v(t,y)^{\delta} \tag{16}$$

where δ is a constant that we are going to choose. Then we have the corresponding derivatives

$$\begin{aligned}
&u_t = -e^{-\alpha x}\delta v^{\delta-1}v_t, \quad u_x = \alpha e^{-\alpha x}v^{\delta}, \quad u_{xx} = -\alpha^2 e^{-\alpha x}v^{\delta}\\
&u_y = -e^{-\alpha x}\delta v^{\delta-1}v_y, \quad u_{yy} = -e^{-\alpha x}\delta v^{\delta-1}v_{yy} - e^{-\alpha x}\delta(\delta-1)v^{\delta-2}(v_y)^2\\
&u_{xy} = \alpha e^{-\alpha x}\delta v^{\delta-1}v_y
\end{aligned}$$

Plug them in equation (11) and expand the terms

$$\begin{aligned}
&-e^{-\alpha x}\delta v^{\delta-1}v_t - \frac{1}{2}\big(-e^{-\alpha x}\delta v^{\delta-1}v_{yy} - e^{-\alpha x}\delta(\delta-1)v^{\delta-2}(v_y)^2\big)\\
&-a(b-y)e^{-\alpha x}\delta v^{\delta-1}v_y + \frac{(\lambda\alpha e^{-\alpha x}v^{\delta} + \rho c\alpha e^{-\alpha x}\delta v^{\delta-1}v_y)^2}{2\alpha^2 e^{-\alpha x}v^{\delta}} = 0\\
\Rightarrow\ &-v_t - \frac{1}{2}c^2 v_{yy} - \frac{1}{2}c^2(\delta-1)v^{-1}(v_y)^2 - a(b-y)v_y\\
&+\frac{\lambda^2 v^{2\delta} + 2\lambda\rho c\delta v^{2\delta-1}v_y + \rho^2 c^2\delta^2 v^{2\delta-2}(v_y)^2}{2\delta v^{2\delta-1}} = 0
\end{aligned}$$

Gathering the terms of v_y, and $v^{-1}v_y$ together, we have that

$$v_t + \frac{1}{2}v_{yy} + \big[a(b-y) - \lambda\rho c\big]v_y - \frac{\lambda^2}{2\delta}v + \frac{c^2}{2}\big[\delta - 1 - \rho^2\delta\big]v^{-1}v_y = 0$$

In order for the above PDE to become linear, we set the $v^{-1}v_y$ term equal to zero:

$$\delta - 1 - \rho^2\delta = 0 \Rightarrow \quad \delta = \frac{1}{1-\rho^2} \tag{17}$$

So by this choice of the parameter δ we are left with the linear parabolic PDE equation

$$v_t + \frac{1}{2}v_{yy} + \big[a(b-y) - \lambda\rho c\big]v_y - \frac{\lambda^2}{2\delta}v = 0 \tag{18}$$

The terminal condition for this equation is found from equation (12):

$$\begin{aligned} u(T,x,y) &= -e^{-\alpha x - \alpha n h(y)} \\ \Rightarrow -e^{-\alpha x}v(T,y)^\delta &= -e^{-\alpha x - \alpha n h(y)} \\ \Rightarrow v(T,y) &= e^{-\frac{\alpha}{\delta}nh(y)} \end{aligned} \tag{19}$$

Now, using Feyman-Kac Theorem, the solution of equation (18) is:

$$\begin{aligned} v(t,y) &= \mathbb{E}^{\,\mathbb{Q}^{\mathcal{M}}}[e^{-\int_t^T \frac{\lambda^2}{2\delta}\,\mathrm{d}u}e^{-\frac{\alpha}{\delta}nh(Y_T)}|Y_t = y] \\ &= \mathbb{E}^{\,\mathbb{Q}^{\mathcal{M}}}[e^{-\frac{\lambda^2}{2\delta}(T-t)}e^{-\frac{\alpha}{\delta}nh(Y_T)}|Y_t = y] \end{aligned}$$

where Y_t follows the SDE equation (4) under $Q^{\mathcal{M}}$, and δ is given by (17). Thus by substituting this into (16) we get

$$\begin{aligned} u^{(n)}(t,x,y) &= -e^{-\alpha x}v(t,y)^\delta \\ &= -e^{-\alpha x}\mathbb{E}^{\,\mathbb{Q}^{\mathcal{M}}}[e^{-\frac{\lambda^2}{2\delta}(T-t)}e^{-\frac{\alpha}{\delta}nh(Y_T)}|Y_T = y]^\delta \\ &= -e^{-\alpha x - \frac{1}{2}\lambda^2(T-t)}\big[\mathbb{E}^{\,\mathbb{Q}^{\mathcal{M}}}[e^{-\alpha(1-\rho^2)nh(Y_T)}|Y_t = y]\big]^{1/(1-\rho^2)} \end{aligned}$$

and therefore (13), (14) are proved. Now to prove (15) we will use the utility indifference price Definition 1:

$$\begin{aligned} & u^{(n)}(t, x - np^{(n)}, y) = u^{(0)}(t,x,y) \\ \Rightarrow & -e^{-\alpha(x-np^{(n)}) - \frac{\lambda^2}{2}(T-t)}f(t,y)^{1/(1-\rho^2)} = -e^{-\alpha x - \frac{\lambda^2}{2}(T-t)} \\ \Rightarrow & e^{\alpha n p^{(n)}}f(t,y)^{1/(1-\rho^2)} = 1 \\ \Rightarrow & p^{(n)}(t,y) = -\frac{1}{\alpha n(1-\rho^2)}\log\big[f(t,y)\big] \end{aligned}$$

where $f(t,y)$ is given by (14) □

Our next goal is to find the PDE that the price per claim $p^{(n)}(t,y)$ follows. By rearranging (15) we get that:

$$f(t,y)^{1/(1-\rho^2)} = e^{-\alpha n p^{(n)}(t,y)}$$

Plug this into (13) to get:

$$\begin{aligned} u^{(n)}(t,x,y) &= -e^{-\alpha x - \frac{1}{2}\lambda^2(T-t)}[f(t,y)]^{1/(1-\rho^2)} \\ &= -e^{-\alpha x - \frac{1}{2}\lambda^2(T-t) - \alpha n p^{(n)}(t,y)} \\ &= -e^{-\omega(t,x,y)} \end{aligned}$$

where $\omega(t,x,y) = \alpha x + \frac{1}{2}\lambda^2(T-t) + \alpha n p^{(n)}(t,y)$. Now find the partial derivatives of $u \equiv u^{(n)}$ as above:

$$\begin{aligned} &u_t = -\frac{1}{2}\lambda^2 e^{\omega} + \alpha n e^{\omega}(p_t^{(n)}), \quad u_x = \alpha e^{\omega}, \quad u_{xx} = -\alpha^2 e^{\omega} \\ &u_y = \alpha n e^{\omega} p_y^{(n)}, \quad u_{yy} = \alpha n e^{\omega} p_{yy}^{(n)} - \alpha^2 n^2 e^{\omega}(p_y^{(n)})^2, \quad u_{xy} = -\alpha^2 n e^{\omega} p_y^{(n)} \end{aligned}$$

Substitute these back in HJB equation (11):

$$\begin{aligned} & u_t + \frac{1}{2}c^2 u_{yy} + a(b-y)u_y - \frac{(\lambda u_x + \rho c u_{xy})^2}{2u_{xx}} = 0 \\ \Rightarrow & -\frac{1}{2}\lambda^2 e^{\omega} + \alpha n e^{\omega}(p_t^{(n)}) + \frac{1}{2}c^2(\alpha n e^{\omega} p_{yy}^{(n)} - \alpha^2 n^2 e^{\omega}(p_y^{(n)})^2) \\ & + a(b-y)\alpha n e^{\omega} p_y^{(n)} + \frac{\left(\lambda\alpha e^{\omega} - \rho c \alpha^2 n e^{\omega} p_y^{(n)}\right)^2}{2\alpha^2 e^{\omega}} = 0 \end{aligned}$$

and after some rearrangements we get the PDE for $p^{(n)}(t,y)$:

$$p_t^{(n)} + \big[a(b-y) - \lambda\rho c\big]p_y^{(n)} + \frac{1}{2}c^2 p_{yy}^{(n)} - \frac{1}{2}c^2 n\alpha(1-\rho^2)(p_y^{(n)})^2 = 0 \tag{20}$$

with terminal condition [2]:

$$p^{(n)}(T,y) = h(y) \tag{21}$$

Note that for $n = 0$ we have a linear PDE and therefore using the Feyman-Kac Theorem we can find the *marginal price* for the claim:

$$\hat{p}(t,y) = \lim_{n\to 0} p^{(n)}(t,y) = \mathbb{E}^{\mathbb{Q}^{\mathcal{M}}}[h(Y_T)|Y_t = y]$$

where Y_t satisfies the SDE (4) under $\mathbb{Q}^{\mathcal{M}}$

[2] trivial since the price at time T is the same as the payoff

Now we wish to find the optimal hedging strategy $\pi^{(H)}$ in terms of the claim price, but first we have to find the optimal trading strategy. We find the required derivatives of $u \equiv u^{(n)}(t,x,y)$ using (13)

$$u_x = \alpha e^{-\alpha x - \frac{\lambda^2}{2}(T-t)} f^{\frac{1}{1-\rho^2}}, \quad u_{xx} = -\alpha^2 e^{-\alpha x - \frac{\lambda^2}{2}(T-t)} f^{\frac{1}{1-\rho^2}}$$
$$u_{xy} = \frac{\alpha}{1-\rho^2} e^{-\alpha x - \frac{\lambda^2}{2}(T-t)} f^{\frac{\rho^2}{1-\rho^2}} f_y$$

and substitute the back in (10) to get

$$\pi^{*,n} = -\left(\frac{\lambda\alpha e^{-\alpha x - \frac{\lambda^2}{2}(T-t)} f^{\frac{1}{1-\rho^2}} + \rho c \frac{\alpha}{1-\rho^2} e^{-\alpha x - \frac{\lambda^2}{2}(T-t)} f^{\frac{\rho^2}{1-\rho^2}} f_y}{-\sigma\alpha^2 e^{-\alpha x - \frac{\lambda^2}{2}(T-t)} f^{\frac{1}{1-\rho^2}}} \right)$$
$$= -\left(\frac{\lambda\alpha f^{\frac{1}{1-\rho^2}} + \rho c \frac{\alpha}{1-\rho^2} f^{\frac{\rho^2}{1-\rho^2}} f_y}{-\sigma\alpha^2 f^{\frac{1}{1-\rho^2}}} \right)$$

Now differentiate equation (15) with respect to y

$$p_y^{(n)} = -\frac{1}{\alpha n(1-\rho^2)} \frac{f_y}{f}$$
$$\Rightarrow f_y = -\alpha n(1-\rho^2) f p_y^{(n)}$$

and plug this into the above equation for $\pi^{*,n}$ to get

$$\pi^{*,n} = -\left(\frac{\lambda\alpha f^{\frac{1}{1-\rho^2}} + \rho c \frac{\alpha}{1-\rho^2} f^{\frac{\rho^2}{1-\rho^2}} \alpha n(1-\rho^2) f p_y^{(n)}}{-\sigma\alpha^2 f^{\frac{1}{1-\rho^2}}} \right)$$
$$= -\left(\frac{\lambda\alpha - \rho c \alpha^2 n p_y^{(n)}}{-\sigma\alpha^2} \right)$$
$$= \frac{\lambda}{\sigma\alpha} - \frac{n\rho c}{\sigma} p_y^{(n)}$$

By using Definition 2 we can find the optimal hedging strategy:

$$\pi_t^{(H)} := \pi_t^{*,n} - \pi_t^{*,0}$$
$$= \frac{\lambda}{\sigma\alpha} - \frac{n\rho c}{\sigma} \frac{\partial p^{(n)}}{\partial y} - \frac{\lambda}{\sigma\alpha}$$
$$= -\frac{n\rho c}{\sigma} \frac{\partial p^{(n)}}{\partial y}$$

and by using the fact that $\pi_t^{(H)} := S_t H_t^{(H)}$ we find that the optimal hedging strategy for a position in n claims is to hold $H_t^{(H)}$ shares at time $t \in [0,T]$ given by

$$H_t^{(H)} = -n\rho \frac{c}{\sigma} \frac{1}{S_t} \frac{\partial p^{(n)}}{\partial y}(t, Y_t) \tag{22}$$

2.2 Residual risk process

In this subsection we suppose the agent trades n claims at $t = 0$ for price $p^{(n)}(0, Y_0)$ and hedges this position using the optimal hedging strategy (22). The agent's overall position has value process $R := (R_t)_{0 \le t \le T}$ given by

$$R_t = X_t^{(H)} + np^{(n)}(t, Y_t)$$

Therefore

$$\mathrm{d}R_t = \mathrm{d}X_t^{(H)} + n\,\mathrm{d}p^{(n)}(t, Y_t) \tag{23}$$

where $X^{(H)} := (X_t^{(H)})_{0 \le t \le T}$ is the value of the hedging portfolio in S and satisfies

$$X_t^{(H)} = x + \int_0^t H_s^{(H)}\,\mathrm{d}S_s$$

where x is the initial endowment. This implies that

$$\mathrm{d}X_t^{(H)} = H_t^{(H)}\,\mathrm{d}S_t \tag{24}$$

with initially

$$X_0^{(H)} = -np^{(n)}(0, Y_0) \quad (= x)$$

Plug (24) into (23) and use Ito formula on the $\mathrm{d}p^{(n)}$ term to get:

$$\begin{aligned}
\mathrm{d}R &= H^{(H)}\,\mathrm{d}S + n\,\mathrm{d}P^{(n)} \\
&= H^{(H)}\big(\sigma S(\lambda\,\mathrm{d}t + \mathrm{d}B)\big) + n\big[p_t^{(n)}\,\mathrm{d}t + p_y^{(n)}\,\mathrm{d}Y + \frac{1}{2}p_{yy}^{(n)}\,\mathrm{d}\langle Y\rangle\big] \\
&= -n\rho\frac{c}{\sigma}\frac{1}{S}p_y^{(n)}\big[\sigma S(\lambda\,\mathrm{d}t + \mathrm{d}B)\big) + n\big(p_t^{(n)}\,\mathrm{d}t + p_y^{(n)}(a(b-Y)\,\mathrm{d}t + c\,\mathrm{d}W) \\
&\quad + \frac{1}{2}p_{yy}^{(n)}c^2\,\mathrm{d}t\big] \\
&= -n\rho c p_y^{(n)}(\lambda\,\mathrm{d}t + \mathrm{d}B) + n\big[p_t^{(n)}\,\mathrm{d}t + p_y^{(n)}a(b-Y)\,\mathrm{d}t \\
&\quad + cp_y^{(n)}(\rho\,\mathrm{d}B + \sqrt{1-\rho^2}\,\mathrm{d}B^{\perp}) + \frac{1}{2}p_{yy}^{(n)}c^2\,\mathrm{d}t\big]
\end{aligned}$$

but using (20):

$$p_t^{(n)} = -\big[a(b-y) - \lambda\rho c\big]p_y^{(n)} - \frac{1}{2}c^2p_{yy}^{(n)} + \frac{1}{2}c^2n\alpha(1-\rho^2)(p_y^{(n)})^2$$

So by substituting $p_t^{(n)}$ in the equation above we get:

$$\begin{aligned}
\mathrm{d}R = -n\rho c p_y^{(n)}(\lambda\,\mathrm{d}t + \mathrm{d}B) + n\big[&-\big[a(b-y) - \lambda\rho c\big]p_y^{(n)}\,\mathrm{d}t - \frac{1}{2}c^2p_{yy}^{(n)}\,\mathrm{d}t \\
&+ \frac{1}{2}c^2n\alpha(1-\rho^2)(p_y^{(n)})^2\,\mathrm{d}t + p_y^{(n)}a(b-Y)\,\mathrm{d}t + cp_y^{(n)}\rho\,\mathrm{d}B \\
&+ cp_y^{(n)}\sqrt{1-\rho^2}\,\mathrm{d}B^{\perp} + \frac{1}{2}p_{yy}^{(n)}c^2\,\mathrm{d}t\big]
\end{aligned}$$

and finally the residual risk (or hedging error) process $R_t \equiv R(t, Y_t)$ satisfies the equation

$$\mathrm{d}R_t = \frac{1}{2}c^2 n^2 (1-\rho^2)\left(\frac{\partial p^{(n)}}{\partial y}\right)^2 \mathrm{d}t + cn\sqrt{1-\rho^2}\frac{\partial p^{(n)}}{\partial y}\,\mathrm{d}B_t^{\perp} \tag{25}$$

with $R_0 = 0$. The last term of this equation is interpreted as the unhedgeable component of risk.

2.3 Perfect correlation case

Recall the equations, under $\mathbb{Q}^{\mathcal{M}}$, satisfied by the traded and non-traded assets S, Y respectively:

$$\begin{aligned}\mathrm{d}S_t &= \sigma S_t\,\mathrm{d}B_t^{\mathbb{Q}^{\mathcal{M}}}\\ \mathrm{d}Y_t &= \big[a(b-Y_t) - c\rho\lambda\big]\,\mathrm{d}t + c\,\mathrm{d}W_t^{\mathbb{Q}^{\mathcal{M}}}\end{aligned}$$

If $\rho = 1$, the market becomes complete and Y is effectively a traded asset. So by no arbitrage arguments the $\mathbb{Q}^{\mathcal{M}}$-drift of Y must be equal to zero. Therefore

$$a(b - Y_t) - c\rho\lambda = 0 \tag{26}$$

and in fact the relationship $W = \rho B + \sqrt{1-\rho^2}B^{\perp}$ implies that $W = B$.

Our aim now is to derive a formula for the claim price process $P(t, Y_t)$ for $0 \le t \le T$. The function $P \equiv P(t, Y_t)$ is smooth enough to apply the Ito formula, so that

$$\begin{aligned}\mathrm{d}P &= P_t\,\mathrm{d}t + P_y\,\mathrm{d}Y + \frac{1}{2}P_{yy}\,\mathrm{d}\langle Y\rangle\\ &= P_t\,\mathrm{d}t + P_y\big[a(b-Y)\,\mathrm{d}t + c\,\mathrm{d}W\big] + \frac{1}{2}P_{yy}c^2\,\mathrm{d}t\end{aligned}$$

and therefore

$$\mathrm{d}P(t, Y_t) = \big[P_t + a(b-Y)P_y + \frac{1}{2}c^2 P_{yy}\big]\,\mathrm{d}t + cP_y\,\mathrm{d}B \tag{27}$$

where we used the fact that if $\rho = 1$ then $W = B$. Now recall the wealth process

$$X_t = x + \int_0^t H_u\,\mathrm{d}S_u$$

Let $X_T = -nh(Y_T)$ and in particular $X_t = -nP(t, Y_t)$. So we can get two equations for the wealth process X_t:

$$\begin{aligned}\mathrm{d}X_t &= H_t\,\mathrm{d}S_t\\ \mathrm{d}X_t &= -n\,\mathrm{d}P(t, Y_t)\end{aligned}$$

These imply that

$$-n\left[\left[P_t + a(b-Y)P_y + \frac{1}{2}c^2 P_{yy}\right]\mathrm{d}t + cP_y\,\mathrm{d}B\right] = H_t^{(P)}\sigma S_t(\lambda\,\mathrm{d}t + \mathrm{d}B_t)$$

where we substitute the expression for $\mathrm{d}P$ found above using the Ito formula, and the SDE equation for the asset S_t. Taking the $\mathrm{d}B_t$ terms equal to zero we get that

$$-ncP_y = \sigma S_t H_t^{(P)} \Rightarrow H_t^{(P)} = -n\frac{c}{\sigma}\frac{1}{S_t}\frac{\partial P}{\partial y} \tag{28}$$

where $H_t^{(P)}$ denotes the perfect hedge. By taking the $\mathrm{d}t$ terms of above equation equal to zero we get that

$$\begin{aligned} &- nP_t - na(b-Y)P_y - \frac{1}{2}nc^2 P_{yy} = -\sigma S\lambda n\frac{c}{\sigma}\frac{1}{S}P_y \\ \Rightarrow\ &P_t + \frac{1}{2}nc^2 P_{yy} + \big[a(b-Y) - \lambda c\big]P_y = 0 \end{aligned}$$

and finally using (26), we get the PDE equation satisfied by the claim price $P(t, Y_t)$:

$$P_t + \frac{1}{2}P_{yy} = 0 \tag{29}$$

with the terminal condition

$$P(T, y) = h(y)$$

By using the Feyman-Kac Theorem we can write the solution to the above PDE as an expectation:

$$P(t, y) = \mathbb{E}^{\mathbb{Q}}[h(Y_T)|Y_t = y] \tag{30}$$

where Y_t satisfies $\mathrm{d}Y_t = c\,\mathrm{d}W_t^{\mathbb{Q}}$ under $\mathbb{Q}$. Note that $\mathbb{Q}$ is the unique measure when the market is complete.

By using the naive (BS-style) strategy under imperfect correlation, we would like to find the SDE satisfied by the hedging error (residual risk) process $R^N \equiv R^N(t, Y_t)$ (for brevity we often omit the arguments of functions). The agent's overall position has value process $R^N := (R_t^N)_{0\le t\le T}$ given by

$$R_t^N = X_t^{(P)} + nP(t, Y_t)$$

where

$$X_t^{(P)} = x + \int_0^t H_u^{(P)}\,\mathrm{d}S_u$$

So

$$\begin{aligned}
\mathrm{d}R^N &= \mathrm{d}X^{(P)} + n\,\mathrm{d}P(t,Y_t) \\
&= H^{(P)}\,\mathrm{d}S + n\,\mathrm{d}P(t,Y_t) \\
&= H^{(P)}\big(\sigma S(\lambda\,\mathrm{d}t + \mathrm{d}B)\big) \\
&\quad + n\big[\big(P_t + a(b-Y)P_y + \frac{1}{2}c^2P_{yy}\big)\,\mathrm{d}t + cP_y\,\mathrm{d}B\big] \quad by \quad (27) \\
&= -n\frac{c}{\sigma}\frac{1}{S}P_y\sigma S(\lambda\,\mathrm{d}t + \mathrm{d}B) \\
&\quad + n\big[\big(P_t + a(b-Y)P_y + \frac{1}{2}c^2P_{yy}\big)\,\mathrm{d}t + cP_y\,\mathrm{d}B\big] \quad by \quad (28) \\
&= -ncP_y(\lambda\,\mathrm{d}t + \mathrm{d}B) + n\big[\big(P_t + a(b-Y)P_y + \frac{1}{2}c^2P_{yy}\big)\,\mathrm{d}t + cP_y\,\mathrm{d}B\big]
\end{aligned}$$

But (29) implies that $P_t = -\frac{1}{2}P_{yy}$. Therefore

$$\begin{aligned}
\mathrm{d}R^N &= -ncP_y(\lambda\,\mathrm{d}t + \mathrm{d}B) + n\big[\big(-\frac{1}{2}P_{yy} + a(b-Y)P_y + \frac{1}{2}c^2P_{yy}\big)\,\mathrm{d}t + cP_y\,\mathrm{d}B\big] \\
&= n[a(b-Y) - \lambda c]P_y\,\mathrm{d}t + ncP_y[\,\mathrm{d}W - \mathrm{d}B] \\
&= n[a(b-Y) - \lambda c]P_y\,\mathrm{d}t + ncP_y[\rho\,\mathrm{d}B + \sqrt{1-\rho^2}\,\mathrm{d}B^{\perp} - \mathrm{d}B]
\end{aligned}$$

where we assume that $\rho \to 1$. So the naive hedging error $R_t^N \equiv R^N(t,Y_t)$ follows the SDE:

$$\mathrm{d}R_t^N = n[a(b-Y_t) - \lambda c]\frac{\partial P}{\partial y}\,\mathrm{d}t + nc\frac{\partial P}{\partial y}\big[(\rho-1)\,\mathrm{d}B + \sqrt{1-\rho^2}\,\mathrm{d}B^{\perp}\big] \qquad (31)$$

where $R_0^N = 0$ and $P \equiv P(t,Y_t)$.

3 Optimal versus naive hedging

In this Section we would like to compare the hedging of a claim using the optimal hedging strategy (22) against the use of the naive hedging strategy (28). In order to do that we will compare the distributions of the terminal hedging errors R_T and R_T^N, by simulating the corresponding SDEs (25) and (31), in the case that the agent sells a put option on the non-traded asset. Before doing that though, we will try and find some analytic formulas for the claim prices and their derivatives with respect to Y, in the case of perfect and non-perfect correlation. This procedure has been done in earlier papers [7], [8], [9] for the lognormal case.

3.1 Power series expansions for the indifference price and hedge

In order to simulate R via the SDE (25) we will find and use some analytic approximations for $p^{(n)}(t,y)$ and $p_y^{(n)}(t,y)$ in the form of power series expansions in powers of $d = -\alpha n(1-\rho^2)$.

Recall the result form Proposition 1:

$$p^{(n)}(t,y) = \frac{1}{d}\log\left[\mathbb{E}^{\mathbb{Q}^{\mathcal{M}}}[e^{dh(Y_T)}|Y_t = y]\right] \tag{32}$$

For a random variable X, its cumulant generating function (CGF) $C_X(d)$ is defined as

$$C_X(d) := \log \mathbb{E}\,\exp(dX)$$

and therefore by using Taylor series expansion on the exponent, the CGF takes the form:

$$C_X(d) = \sum_{j=1}^{\infty} \frac{1}{j!} k_j(X) d^j$$

where $k_j(X)$ is the j^{th} cumulant of X. Define the raw and central moments of X :

$$m_j(X) := \mathbb{E}\,(X^j) \qquad \mu_j(X) := \mathbb{E}\,[(X - m_1)^j], \quad j \in \mathbb{N}$$

Then we can write the cumulants of X as functions of the raw and central moments of X for $j \in \{1,2,3,4,5\}$ (see Monoyios [8]):

$$\begin{aligned}
k_1(X) &= m_1(X) \\
k_2(X) &= \mu_2(X) \\
k_3(X) &= \mu_3(X) \\
k_4(X) &= \mu_4(X) - 3\mu_2^2(X) \\
k_5(X) &= \mu_5(X) - 10\mu_2(X)\mu_3(X)
\end{aligned}$$

These can also expressed as functions of the raw moments only:

$$\begin{aligned}
k_1 &= m_1 \\
k_2 &= m_2 - m_1^2 \\
k_3 &= m_3 - 3m_1m_2 + 2m_1^3 \\
k_4 &= m_4 - 4m_1m_3 - 3m_2^2 + 12m_1^2m_2 - 6m_1^4 \\
k_5 &= m_5 - 5m_1m_4 - 10m_2m_3 + 20m_1^2m_3 + 30m_1m_2^2 - 60m_1^3m_2 + 24m_1^5
\end{aligned} \tag{33}$$

where we suppress the dependence on X for simplicity. Next step is to derive an analytic approximation to the pricing function $p^{(n)}(t,y)$ by noticing its relation with the CGF of the payoff under the minimal measure $\mathbb{Q}^{\mathcal{M}}$.

Proposition 2. *(Monoyios [8], [9])*
The indifference pricing function $p^{(n)}(t,y)$ has power series expansion

$$p^{(n)}(t,y) = \sum_{j=1}^{5} \frac{1}{j!} k_j(h(Y_T)) d^{j-1} + O(d^5) \tag{34}$$

where $d = -\alpha n(1-\rho^2)$ and k_j is the j^{th} cumulant of the payoff $h(Y_T)$ under the minimal measure $\mathbb{Q}^{\mathcal{M}}$, conditional on $Y_t = y$. The expansion is valid for the model parameter satisfying

$$\mathbb{E}^{\mathbb{Q}^{\mathcal{M}}}[e^{dh(Y_T)}|Y_t = y] \leq 2 \tag{35}$$

Proof. Expand the exponential inside (32) and use linearity of the expectation to get:

$$\begin{aligned} p^{(n)}(t,y) &= \frac{1}{d} \log \left[\mathbb{E}^{\mathbb{Q}^{\mathcal{M}}}[e^{dh(Y_T)}|Y_t = y]\right] \\ &= \frac{1}{d} \log \left[\mathbb{E}^{\mathbb{Q}^{\mathcal{M}}}\big(1 + \sum_{j=1}^{\infty} \frac{d^j}{j!} h(Y_T)^j | Y_t = y\big)\right] \\ &= \frac{1}{d} \log \big(1 + \sum_{j=1}^{\infty} \frac{d^j}{j!} m_j(h(Y_T))\big) \end{aligned}$$

where $m_j(h(Y_T)) = \mathbb{E}^{\mathbb{Q}^{\mathcal{M}}}[h(Y_T)^j|Y_t = y]$. Now apply Taylor expansions of $\log(1+x)$ provided $|x| \leq 1$. Therefore

$$p^{(n)}(t,y) = \sum_{j=1}^{5} \frac{1}{j!} k_j(h(Y_T)) d^{j-1} + O(d^5)$$

provided

$$\sum_{j=1}^{\infty} \frac{d^j}{j!} m_j(h(Y_T)) \leq 1$$

which is implied by (35). □

We have managed to produce very accurate series for the claim price $p^{(n)}(t,y)$, which can also be differentiated term by term in order to produce an analytic approximation for $p_y^{(n)}(t,y)$. Note that the leading-order term of the above series gives the marginal price. It is very easy to observe the fact that the claim price depends on the

cumulants of the payoff and hence on it's moments m_j. As a consequence $p_y^{(n)}$ will depend on the moment's partial derivative with respect to y, $\partial m_j := \partial m_j/\partial y$. Therefore our plan is to find m_j, ∂m_j analytically and hence find $p^{(n)}(t,y)$ and $p_y^{(n)}(t,y)$.

Let $\partial g_j := \frac{\partial g_j}{\partial y}$, where g_j denotes any of the m_j, k_j, μ_j. By differentiating (34) with respect to y we get that

$$\frac{\partial p^{(n)}}{\partial y}(t,y) = \sum_{j=1}^{5} \frac{1}{j!} \partial k_j(h(Y_T)) d^{j-1} + O(d^5) \tag{36}$$

By differentiating (33) with respect to y, we can get expressions of ∂k_j in terms of ∂m_j for $j \in \{1,2,3,4,5\}$:

$$\begin{aligned}
\partial k_1 &= \partial m_1 \\
\partial k_2 &= \partial m_2 - 2m_1\partial m_1 \\
\partial k_3 &= \partial m_3 - 3(m_1\partial m_2 + m_2\partial m_1 - 2m_1^2\partial m_1) \\
\partial k_4 &= \partial m_4 - 4(m_1\partial m_3 + m_3\partial m_1) - 6m_2\partial m_2 \\
&\quad + 12m_1(m_1\partial m_2 + 2m_2\partial m_1 - 2m_1^2\partial m_1) \\
\partial k_5 &= \partial m_5 - 5(m_1\partial m_4 + m_4\partial m_1) - 10(m_2\partial m_3 + m_3\partial m_2) \\
&\quad + 20(m_1\partial m_3 + 2m_3\partial m_1) + 30m_2(2m_1\partial m_2 + m_2\partial m_1) \\
&\quad - 60m_1^2(m_1\partial m_2 + 3m_2\partial m_1 - 2m_1^2\partial m_1)
\end{aligned} \tag{37}$$

All these appeared in Monoyios [8], [9]. Bellow we show some new results when Y follows an OU-process.

Consider the claim to be a put option on the non-traded asset , such that

$$h(y) = (K-y)^+ = \max(K-y, 0) \tag{38}$$

where $K > 0$ is the strike price. Our aim now is to find a closed form approximation for the moment m_j and its derivative with respect to y, ∂m_j. Firstly we must find the distribution of the terminal value of the non-traded asset price Y_T.

Recall the non-traded asset price dynamics under the minimal measure:

$$\begin{aligned}
\mathrm{d}Y_t &= \left[a(b-Y_t) - c\rho\lambda\right]\mathrm{d}t + c\,\mathrm{d}W_t^{\mathbb{Q}^{\mathcal{M}}} \\
&= (ab - \rho\lambda c - aY_t)\,\mathrm{d}t + c\,\mathrm{d}W_t^{\mathbb{Q}^{\mathcal{M}}}
\end{aligned}$$

From now on we will work under the minimal measure $\mathbb{Q}^{\mathcal{M}}$. Apply the Ito formula to $X_t = e^{at}Y_t$:

$$\begin{aligned}
\mathrm{d}X_t &= ae^{at}Y_t\,\mathrm{d}t + e^{at}\,\mathrm{d}Y_t \\
&= ae^{at}Y_t\,\mathrm{d}t + e^{at}(ab - \rho\lambda c)\,\mathrm{d}t - ae^{at}Y_t\,\mathrm{d}t + ce^{at}\,\mathrm{d}W_t
\end{aligned}$$

Therefore

$$\mathrm{d}(e^{at}Y_t) = (ab - \rho\lambda c)e^{at}\,\mathrm{d}t + ce^{at}\,\mathrm{d}W_t$$

Integrate from t to T and consider that at time t, $Y_t = y$

$$\begin{aligned}
e^{aT}Y_T - e^{at}Y_t &= \int_t^T (ab - \rho\lambda c)e^{au}\,\mathrm{d}u + c\int_t^T e^{au}\,\mathrm{d}W_u \\
\Rightarrow Y_T &= e^{-a(T-t)}y + (ab - \rho\lambda c)e^{-aT}\left[\frac{e^{au}}{a}\right]_t^T + ce^{-aT}\int_t^T e^{au}\,\mathrm{d}W_u \\
&= e^{-a(T-t)}y + \frac{(ab - \rho\lambda c)}{a}e^{-aT}(e^{aT} - e^{at}) + c\int_t^T e^{-a(T-u)}\,\mathrm{d}W_u \\
\Rightarrow Y_T &= e^{-a(T-t)}y + \frac{(ab - \rho\lambda c)}{a}(1 - e^{-a(T-t)}) + c\int_t^T e^{-a(T-u)}\,\mathrm{d}W_u
\end{aligned}$$

The last term of Y_T is a stochastic (Ito) integral and therefore it is normally distributed with mean zero and variance given by:

$$\begin{aligned}
&\mathbb{E}\left(\int_t^T e^{-a(T-u)}\,\mathrm{d}W_u\right)^2 \\
&= \mathbb{E}\left(\int_t^T e^{-2a(T-u)}\,\mathrm{d}u\right) \\
&= \left[\frac{e^{-2a(T-u)}}{2a}\right]_t^T \\
&= \frac{1 - e^{-2a(T-t)}}{2a}
\end{aligned}$$

by using the Ito isometry. This implies that Y_T follows normal distribution with mean given by:

$$\mathbb{E}\,Y_T = e^{-a(T-t)}y + \frac{(ab - \rho\lambda c)}{a}(1 - e^{-a(T-t)}) := \gamma(t, y)$$

and variance given by

$$\begin{aligned}
Var(Y_T) &= \mathbb{E}\left[Y_T - \mathbb{E}\left(Y_T\right)\right]^2 \\
&= \mathbb{E}\left(c\int_t^T e^{-a(T-u)}\,\mathrm{d}W_u\right)^2 \\
&= \frac{c^2}{2a}\big(1 - e^{-2a(T-t)}\big) \\
&:= \Sigma^2
\end{aligned}$$

So $Y_T \sim N(\gamma, \Sigma^2)$ given that at time t, $Y_t = y$. Now we have the tools we need in order to find a closed form approximation of m_j initially, and then ∂m_j under the assumption that the claim is a put option on the non-traded asset.

Recall that

$$m_j = \mathbb{E}^{\mathbb{Q}^{\mathcal{M}}}[h^j(Y_T)|Y_t = y] \tag{39}$$

where $h(y) = (K - y)^+$. Consider:

$$\begin{aligned}\big(h(Y_T)\big)^j &= \big((K - Y_T)^+\big)^j \\ &= (K - Y_T)^j \mathbb{I}_{Y_T \leq K} \\ &= \sum_{l=0}^{i} \binom{j}{l} (-1)^l K^{j-l} Y_T^l \mathbb{I}_{Y_T \leq K}\end{aligned}$$

Hence by plugging this into (39) we get

$$\begin{aligned}m_j &= \mathbb{E}^{\mathbb{Q}^{\mathcal{M}}}[h^j(Y_T)|Y_t = y] \\ &= \sum_{l=0}^{i} \binom{j}{l} (-1)^l K^{j-l} \mathbb{E}^{\mathbb{Q}^{\mathcal{M}}}[Y_T^l \mathbb{I}_{Y_T \leq K}|Y_t = y]\end{aligned}$$

So now we need to evaluate

$$\mathcal{G}_l = \mathbb{E}^{\mathbb{Q}^{\mathcal{M}}}[Y_T^l \mathbb{I}_{Y_T \leq K}|Y_t = y] \tag{40}$$

$$= \frac{1}{\sqrt{2\pi\Sigma^2}} \int_{-\infty}^{K} x^l e^{-\frac{(x-\gamma)^2}{2\Sigma^2}} \, \mathrm{d}x \tag{41}$$

where $Y_T \sim N(\gamma, \Sigma^2)$.

In order to compute $\mathcal{G}_l$ it will be very helpful to define a new function Ψ_l where

$$\Psi_l = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\beta} w^l e^{-\frac{w^2}{2}} \, \mathrm{d}w$$

We proceed by integration by parts in order to examine the behavior of this function:

$$\begin{aligned}\Psi_l &= -\frac{1}{\sqrt{2\pi}} \left[w^{l-1} e^{-\frac{w^2}{2}} \right]_{-\infty}^{\beta} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\beta} w^{l-2}(l-1) e^{-\frac{w^2}{2}} \, \mathrm{d}w \\ &= -\frac{1}{\sqrt{2\pi}} \beta^{l-1} e^{-\frac{\beta^2}{2}} + \frac{l-1}{\sqrt{2\pi}} \int_{-\infty}^{\beta} (-1) w^{l-3} \, \mathrm{d}(e^{-\frac{w^2}{2}}) \\ &= -\frac{1}{\sqrt{2\pi}} \beta^{l-1} e^{-\frac{\beta^2}{2}} - \frac{l-1}{\sqrt{2\pi}} \left[w^{l-3} e^{-\frac{w^2}{2}} \right]_{-\infty}^{\beta} \\ &\quad + \frac{(l-1)(l-3)}{\sqrt{2\pi}} \int_{-\infty}^{\beta} w^{l-4} e^{-\frac{w^2}{2}} \, \mathrm{d}w \\ &= -\frac{1}{\sqrt{2\pi}} \beta^{l-1} e^{-\frac{\beta^2}{2}} - \frac{(l-1)}{\sqrt{2\pi}} \beta^{l-3} e^{-\frac{\beta^2}{2}} \\ &\quad + \frac{(l-1)(l-3)}{\sqrt{2\pi}} \int_{-\infty}^{\beta} w^{l-4} e^{-\frac{w^2}{2}} \, \mathrm{d}w\end{aligned}$$

Hence it is easy for one to prove, by induction, the following Proposition:

Proposition 3. *Consider the function* $\Psi_l = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\beta} w^l e^{-\frac{w^2}{2}} \, \mathrm{d}w$.
Then for l *odd:*

$$\Psi_l = -\frac{1}{\sqrt{2\pi}} e^{-\frac{\beta^2}{2}} \big[\beta^{l-1} + (l-1)\beta^{l-3} + (l-1)(l-3)\beta^{l-5} + \ldots + (l-1)(l-3)\cdots 4\beta^2 + (l-1)(l-3)\cdots 4 \cdot 2\big],$$

for l *even:*

$$\Psi_l = -\frac{1}{\sqrt{2\pi}} e^{-\frac{\beta^2}{2}} \big[\beta^{l-1} + (l-1)\beta^{l-3} + (l-1)(l-3)\beta^{l-5} + \ldots + (l-1)(l-3)\cdots 5\beta^3 + (l-1)(l-3)\cdots 5 \cdot 3 \cdot \beta\big] + (l-1)(l-3) \cdot 5 \cdot 3 \cdot \Phi(\beta),$$

and for $l = 0$*:*

$$\Psi_0 = \Phi(\beta)$$

where $\Phi(\cdot)$ *denotes the standard cumulative normal distribution function.*

Now lets go back to the calculation of $\mathcal{G}_l$. Consider (41) and make the following change of variable: Let

$$\begin{aligned} &\frac{x-\gamma}{\Sigma} = w \Rightarrow \mathrm{d}w = \frac{\mathrm{d}x}{\Sigma} \\ &x = K \Rightarrow w = \frac{K-\gamma}{\Sigma} \\ &x = -\infty \Rightarrow w = -\infty \quad since \quad \Sigma > 0 \end{aligned}$$

Hence by making the above substitutions:

$$\begin{aligned} \mathcal{G}_l &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{K-\gamma}{\Sigma}} (w\Sigma + \gamma)^l e^{-\frac{w^2}{2}} \, \mathrm{d}w \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{q} \sum_{k=0}^{l} \binom{l}{k} (\Sigma w)^{l-k} \gamma^k e^{-\frac{w^2}{2}} \, \mathrm{d}w \\ &= \sum_{k=0}^{l} \binom{l}{k} \Sigma^{l-k} \gamma^k \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{q} w^{l-k} e^{-\frac{w^2}{2}} \, \mathrm{d}w \end{aligned}$$

where $q = \frac{K-\gamma}{\Sigma}$. Therefore we get the final form of $\mathcal{G}_l$ as:

$$\mathcal{G}_l = \sum_{k=0}^{l} \binom{l}{k} \Sigma^{l-k} \gamma^k \Psi_{l-k}(q) \tag{42}$$

where Ψ_{l-k} is given by Proposition 3. By differentiating this with respect to y we get $\partial\mathcal{G}_l \equiv \frac{\partial\mathcal{G}_l}{\partial y}$ as:

$$\partial\mathcal{G}_l = \sum_{k=0}^{l} \binom{l}{k} \Sigma^{l-k}\gamma^{k-1}\big[k\partial\gamma\Psi_{l-k}(q) + \gamma\partial\Psi_{l-k}(q)\big] \tag{43}$$

where $\partial\gamma \equiv \frac{\partial\gamma}{\partial y} = e^{-a(T-t)}$ and $\partial\Psi_{l-k}(q) \equiv \frac{\partial\Psi_{l-k}}{\partial y}(q)$ will be found explicitly a bit later. Therefore we can write the final forms of m_j and ∂m_j as

$$m_j = \sum_{l=0}^{i} \binom{j}{l} (-1)^l K^{j-l}\mathcal{G}_l \tag{44}$$

$$\partial m_j = \sum_{l=0}^{i} \binom{j}{l} (-1)^l K^{j-l}\partial\mathcal{G}_l \tag{45}$$

.

What is left now is to find explicit forms for $\Psi_i(q) \equiv \Psi_i$ for $i = 0, 1, 2, 3, 4, 5$ using the Proposition 3:

$$\begin{aligned}
\Psi_0 &= \Phi(q) \\
\Psi_1 &= -\frac{1}{\sqrt{2\pi}} e^{-\frac{q^2}{2}} \\
\Psi_2 &= -\frac{1}{\sqrt{2\pi}} e^{-\frac{q^2}{2}} q + \Phi(q) \\
&= q\Psi_1 + \Psi_0 \\
\Psi_3 &= -\frac{1}{\sqrt{2\pi}} e^{-\frac{q^2}{2}} \big[q^2 + 2\big] \\
&= \big[2 + q^2\big]\Psi_1 \\
\Psi_4 &= -\frac{1}{\sqrt{2\pi}} e^{-\frac{q^2}{2}} \big[q^3 + 3q\big] + 3\Phi(q) \\
&= q^3\Psi_1 + 3\Psi_2 \\
\Psi_5 &= -\frac{1}{\sqrt{2\pi}} e^{-\frac{q^2}{2}} \big[q^4 + 4q^2 + 4 \cdot 2\big] \\
&= q^4\Psi_1 + 4\Psi_3
\end{aligned} \tag{46}$$

and explicit forms for $\partial\Psi_i(q) \equiv \partial\Psi_i$ for $i = 0, 1, 2, 3, 4, 5$ by differentiating Ψ_i:

$$\begin{aligned}
\partial\Psi_0 &= \Phi'(q)\frac{\partial q}{\partial y} \\
&= -\frac{1}{\sqrt{2\pi}}e^{-\frac{q^2}{2}}\frac{1}{\Sigma}\partial\gamma \\
&= \frac{\Psi_1}{\Sigma}\partial\gamma \\
\partial\Psi_1 &= -q\frac{1}{\sqrt{2\pi}}e^{-\frac{q^2}{2}}\frac{1}{\Sigma}\partial\gamma \\
&= q\partial\Psi_0 \\
\partial\Psi_2 &= -\frac{1}{\Sigma}\partial\gamma\Psi_1 + q\partial\Psi_1 + \partial\Psi_0 \\
&= q\partial\Psi_1 \\
\partial\Psi_3 &= [2+q^2]\partial\Psi_1 - \frac{2}{\Sigma}q\Psi_1\partial\gamma \\
&= [2+q^2]\partial\Psi_1 - 2q\partial\Psi_0 \\
\partial\Psi_4 &= -3q^2\frac{1}{\Sigma}\partial\gamma\Psi_1 + q^3\partial\Psi_1 + 3\partial\Psi_2 \\
&= -3q^2\partial\Psi_0 + q^3\partial\Psi_1 + 3\partial\Psi_2 \\
\partial\Psi_5 &= -4q^3\frac{1}{\Sigma}\partial\gamma\Psi_1 + q^4\partial\Psi_1 + 4\partial\Psi_3 \\
&= -4q^3\partial\Psi_0 + q^4\partial\Psi_1 + 4\partial\Psi_3
\end{aligned} \tag{47}$$

Summing up, our goal is to find

$$\frac{\partial p^{(n)}}{\partial y}(t,y) = \sum_{j=1}^{5}\frac{1}{j!}\partial k_j(h(Y_T))d^{j-1}$$

So we can write the derivatives of ∂k_j in terms of m_j and ∂m_j using (37). Then, for the case that the claim is a put option, m_j and ∂m_j can be found using (44), (45) with the help of (42), (43) and (46), (47).

3.2 Finding analytical formula for claim price in the perfect correlation case

In order to simulate R^N via (31) we need to find an analytic representation of the claim price $P(t,y)$ and its derivative with respect to y $P_y(t,y)$ when the traded and non-traded assets are perfectly correlated. So we aim to find the expectation in (30) when $h(y) = (K-y)^+$. Recall that

$$P(t,y) = \mathbb{E}^{\mathbb{Q}^{\mathcal{M}}}[h(Y_T)|Y_t = y]$$

where Y_t satisfies

$$\mathrm{d}Y_t = c\,\mathrm{d}W_t^{\mathbb{Q}^{\mathcal{M}}}$$

under $\mathbb{Q}^{\mathcal{M}}$. We can solve this SDE to find Y_T given at time t, $Y_t = y$:

$$Y_T = y + cW_{T-t}^{\mathbb{Q}^{\mathcal{M}}}$$

Therefore we can see that Y_T is normally distributed with mean y and variance $c^2(T-t)$: $Y_T \sim N(y, c^2(T-t))$. Now we can proceed to find $P(t,y)$:

$$\begin{aligned} P(t,y) &= \mathbb{E}^{\mathbb{Q}^{\mathcal{M}}}[(k - Y_T)\mathbb{I}_{Y_t \le K} | Y_t = y] \\ &= \frac{1}{\sqrt{2\pi c^2(T-t)}} \int_{-\infty}^{K} (K-x) e^{-\frac{(x-y)^2}{2c^2(t-t)}}\,\mathrm{d}x \end{aligned}$$

Make the following change of variable:

$$\begin{aligned} &\frac{x-y}{c\sqrt{T-t}} = w \\ &\mathrm{d}x = c\sqrt{T-t}\,\mathrm{d}w \\ &x = K \Rightarrow w = \frac{K-y}{c\sqrt{T-t}} := s \\ &x = -\infty \Rightarrow w = -\infty \end{aligned}$$

By plugging these into the previous equation we get:

$$\begin{aligned} P(t,y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} (K - c\sqrt{T-t}w - y) e^{-\frac{w^2}{2}}\,\mathrm{d}w \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} (K-y) e^{-\frac{w^2}{2}}\,\mathrm{d}w - \frac{c\sqrt{T-t}}{\sqrt{2\pi}} \int_{-\infty}^{s} w e^{-\frac{w^2}{2}}\,\mathrm{d}w \\ &= (K-y)\Phi(s) + c\sqrt{\frac{T-t}{2\pi}} \left[e^{-\frac{w^2}{2}} \right]_{-\infty}^{s} \end{aligned}$$

Therefore the claim price is given by:

$$P(t,y) = (K-y)\Phi(s) + c\sqrt{\frac{T-t}{2\pi}} e^{-\frac{s^2}{2}} \tag{48}$$

where $\Phi(\cdot)$ denotes the standard cumulative normal distribution function and $s = \frac{K-y}{c\sqrt{T-t}}$.

Now we can get the derivative of P with respect to y by just differentiating the expression we found above:

$$\begin{aligned} P_y(t,y) &= -\Phi(s) + (K-y)\Phi'(s)\frac{\partial s}{\partial y} + c\sqrt{\frac{T-t}{2\pi}}\frac{s}{c\sqrt{T-t}}e^{-\frac{s^2}{2}} \\ &= -\Phi(s) + (K-\gamma)\frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{2}}\left(-\frac{1}{c\sqrt{T-t}}\right) + (K-\gamma)\frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{2}}\left(\frac{1}{c\sqrt{T-t}}\right) \end{aligned}$$

and so P_y is given by:

$$\frac{\partial P}{\partial y} = -\Phi(s) \tag{49}$$

3.3 Comparing the hedging errors at maturity

At this point, having analytical expressions of the naive and optimal claim prices (and their derivatives), we will simulate the residual risk SDEs that the hedging errors R and R^N follow:

$$\begin{aligned} \mathrm{d}R_t &= \frac{1}{2}c^2n^2(1-\rho^2)\left(\frac{\partial p^{(n)}}{\partial y}\right)^2 \mathrm{d}t + cn\sqrt{1-\rho^2}\frac{\partial p^{(n)}}{\partial y}\,\mathrm{d}B_t^{\perp} \\ \mathrm{d}R_t^N &= n[a(b-Y_t)-\lambda c]\frac{\partial P}{\partial y}\,\mathrm{d}t + nc\frac{\partial P}{\partial y}\left[(\rho-1)\,\mathrm{d}B + \sqrt{1-\rho^2}\,\mathrm{d}B^{\perp}\right] \end{aligned}$$

and compare the terminal hedging errors R_T and R_T^N. We suppose that the agent sells a put option ($n=-1$) on the non-traded asset Y at time $t=0$. Figure 1 shows the optimal and naive hedging error distribution which is generated from 10,000 asset price histories. We take $\rho = 0.75$ and the relative risk aversion coefficient $\alpha = 0.01$. The rest of the parameters are as follow:

$$\begin{gathered} S_0 = 100, \quad Y_0 = 100, \quad K = 110, \quad \lambda = 0.4, \quad \sigma = 0.3, \\ a = 2, \quad b = 100, \quad c = 0.25, \quad T = 1year \end{gathered}$$

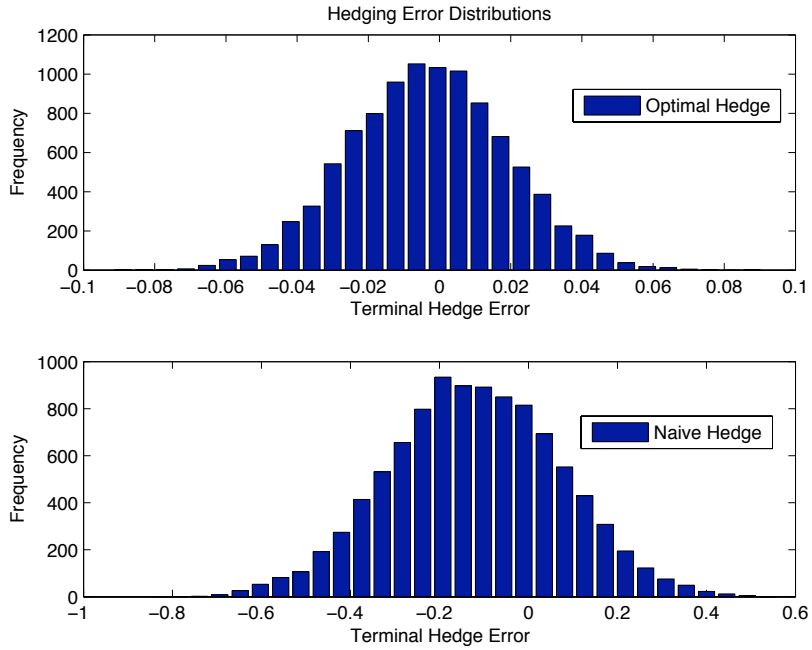


Figure 1: Hedging errors distributions over 10,000 simulated asset price histories. Hedging of a short put ($n = -1$)

The summary statistics for the hedge error distributions are as follows

$$\max(R_T) = 0.0878$$
$$\min(R_T) = -0.0889$$
$$mean(R_T) = -2.0995e - 04$$
$$sd(R_T) = 0.0222$$
$$median(R_T) = -9.0624e - 05$$

$$\max(R_T^N) = 0.5369$$
$$\min(R_T^N) = -0.8303$$
$$mean(R_T^N) = -0.0988$$
$$sd(R_T^N) = 0.1921$$
$$median(R_T^N) = -0.1002$$

From these we can see that the optimal hedge error distribution has a higher mean, higher median and a lower standard deviation than the naive hedge error distribution. Therefore we can conclude that the optimal hedge is much better than

the naive hedge. Nevertheless, safe conclusions are difficult to be drawn since the computations and calculations we shown rely on a rather very strong assumption, that the drifts and variances of the asset processes are constants and known. These coefficients are very difficult to be estimated in real life.

4 Weather Derivatives

In this second part of the dissertation we will model the temperature using a mean-reverting OU process. We will follow Davis paper [3] and try to compare the de-trended and de-seasonalised simulated temperature data with de-trended and de-seasonalised real temperature data. We will try to show, as Davis did in his paper, that the accumulated heating degree days (HDD) empirically follow a lognormal distribution by simulation and by using time series analysis of real temperature data.

4.1 A model for the temperature

Let $W = (W_t)_{0\leq t\leq T}$ be a standard Brownian motion defined on the complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$. We suppose that the temperature at time t, $T(t) \equiv T_t$, can be modeled by an OU-process:

$$\mathrm{d}T_t = a(b - T_t)\,\mathrm{d}t + c\,\mathrm{d}W_t \tag{50}$$

where $c > 0, a, b$ are constants.This is clearly not the best model. The reason we chose the coefficients to be constant is because we would like to find the simplest model we can for the temperature, which will give some reasonable numerical results, comparable to those derived from real temperature data. Solve the SDE (50) by applying the Ito formula to $L_t = e^{at}T_t$:

$$\begin{aligned}
\mathrm{d}L_t &= ae^{at}T_t\,\mathrm{d}t + e^{at}\,\mathrm{d}T_t \\
&= ae^{at}T_t\,\mathrm{d}t + e^{at}(a(b - T_t)\,\mathrm{d}t + c\,\mathrm{d}W_t) \\
&= abe^{at}\,\mathrm{d}t + ce^{at}\,\mathrm{d}W_t
\end{aligned}$$

$$\begin{aligned}
\Rightarrow e^{at}T_t - T_0 &= \int_0^t abe^{au}\,\mathrm{d}u + \int_0^t ce^{au}\,\mathrm{d}W_u \\
\Rightarrow e^{at}T_t &= T_0 + b(e^{at} - 1) + c\int_0^t e^{au}\,\mathrm{d}W_u
\end{aligned}$$

and therefore the solution to (50) is

$$T_t = e^{-at}T_0 + b(1 - e^{-at}) + c\int_0^t e^{a(u-t)}\,\mathrm{d}W_u \tag{51}$$

Now we would like to reformulate the time-continuous dynamics into a time series. If we let again $L_t = e^{at}T_t$ then we find as before that:

$$\mathrm{d}L_t = abe^{at}\,\mathrm{d}t + ce^{at}\,\mathrm{d}W_t$$

Integrate this from time t up to time $t+1$:

$$\begin{aligned}
L_{t+1} - L_t &= \int_t^{t+1} abe^{au}\,\mathrm{d}u + \int_t^{t+1} ce^{au}\,\mathrm{d}W_u \\
\Rightarrow e^{a(t+1)}T_{t+1} - e^{at}T_t &= ab\left[\frac{e^{au}}{a}\right]_t^{t+1} + c\int_t^{t+1} e^{au}\,\mathrm{d}W_u \\
\Rightarrow T_{t+1} &= e^{-a}T_t + b(e^{a(t+1)}e^{-a(t+1)} - e^{at}e^{-a(t+1)}) + ce^{-a(t+1)}\int_t^{t+1} e^{au}\,\mathrm{d}W_u
\end{aligned}$$

and so

$$T_{t+1} = e^{-a}T_t + b(1 - e^{-a}) + ce^{-a}\int_t^{t+1} e^{-a(t-u)}\,\mathrm{d}W_u \tag{52}$$

Let $Z_t = \int_t^{t+1} e^{-a(t-u)}\,\mathrm{d}W_u$. Z_t is a stochastic integral and therefore normally distributed with mean zero and variance given by:

$$\begin{aligned}
Var(Z_t) &= \mathbb{E}\,Z_t^2 - (\mathbb{E}\,Z_t)^2 \\
&= \mathbb{E}\left(\int_t^{t+1} e^{-a(t-u)}\,\mathrm{d}W_u\right)^2 \\
&= \int_t^{t+1} e^{-2a(t-u)}\,\mathrm{d}u \qquad (Ito \quad isometry) \\
&= e^{-2at}\left[\frac{e^{2au}}{2a}\right]_t^{t+1} \\
&= \frac{e^{2a} - 1}{2a}
\end{aligned}$$

Therefore $\epsilon_t := \frac{Z_t}{\sqrt{\frac{e^{2a}-1}{2a}}} \sim N(0,1)$ and hence (52) becomes:

$$\begin{aligned}
T_{t+1} &= e^{-a}T_t + b(1 - e^{-a}) + ce^{-a}\sqrt{\frac{e^{2a} - 1}{2a}}\epsilon_t \\
&= e^{-a}T_t + b(1 - e^{-a}) + c\sqrt{\frac{1 - e^{-2a}}{2a}}\epsilon_t
\end{aligned}$$

$$\Rightarrow T_{t+1} = \alpha T_t + \beta + \gamma\epsilon_t \tag{53}$$

where

$$\alpha = e^{-a}, \qquad \beta = b(1 - e^{-a}), \qquad \gamma = c\sqrt{\frac{1 - e^{-2a}}{2a}}$$

Now we will de-trend the model. Subtract and add x from the left hand side of (53) and subtract and add αx from the right hand side of (53):

$$T_{t+1} - x + x = \alpha(T_t - x) + \alpha x + \beta + \gamma\epsilon_t$$
$$\Rightarrow D_{t+1} = \alpha D_t + (\alpha - 1)x + \beta + \gamma\epsilon_t$$

where $D_t = T_t - x$. Set

$$(\alpha - 1)x + \beta = 0 \Rightarrow x = \frac{\beta}{1 - \alpha} \Rightarrow x = b$$

and substitute it back to the previous equation to get a first-order autoregression model for the de-trended simulated temperature D_t :

$$D_{t+1} = \alpha D_t + \gamma\epsilon_t \tag{54}$$

where $D_t = T_t - b$.

Next step is to simulate 200 years of daily temperature data using equation (50) and perform a standard linear regression of today's de-trended simulated temperature against previous day's simulated temperature. We use the following values for the coefficients:

$$a = 3, \quad b = 10, \quad c = 0.25$$

As the above calculations suggest, Figure 2 shows that the detrended simulated temperature process is normally distributed. Using Matlab (see Figure 3) we perform the linear regression of the simulated detrended temperatures and find the following expected values for the coefficients of (54):

$$\alpha = 0.9880 \qquad \gamma = 1.7657 \cdot 10^{-4}$$

Figure 4 shows a normal probability plot of the residuals. As we can see all the points almost lie on the straight line which implies that the residuals are normally distributed. Independence of the residuals is shown in Figure 5 by plotting the autocorrelation function of the residuals. The value one on the plot determines perfect correlation and the value zero independence. As we can see autocorrelation function lies in a regime near zero.

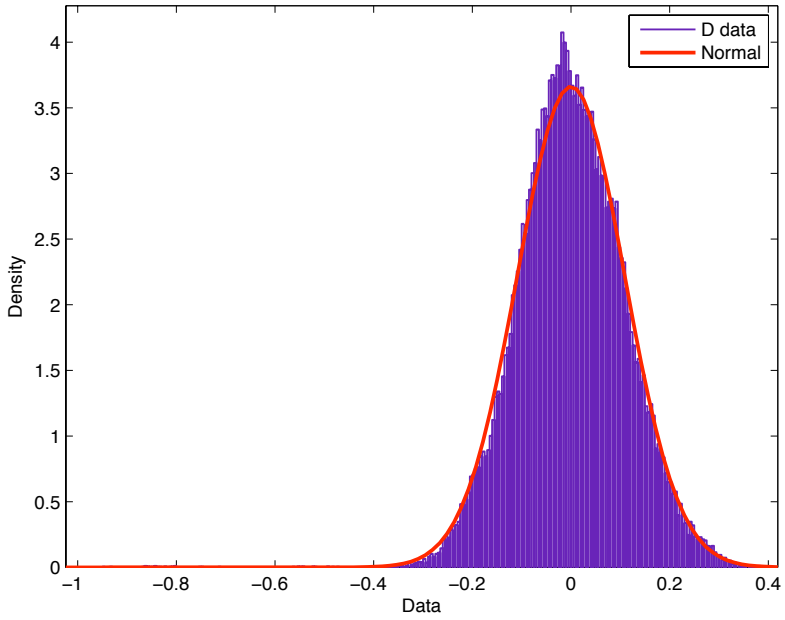


Figure 2: Histogram of simulated detrended temperatures

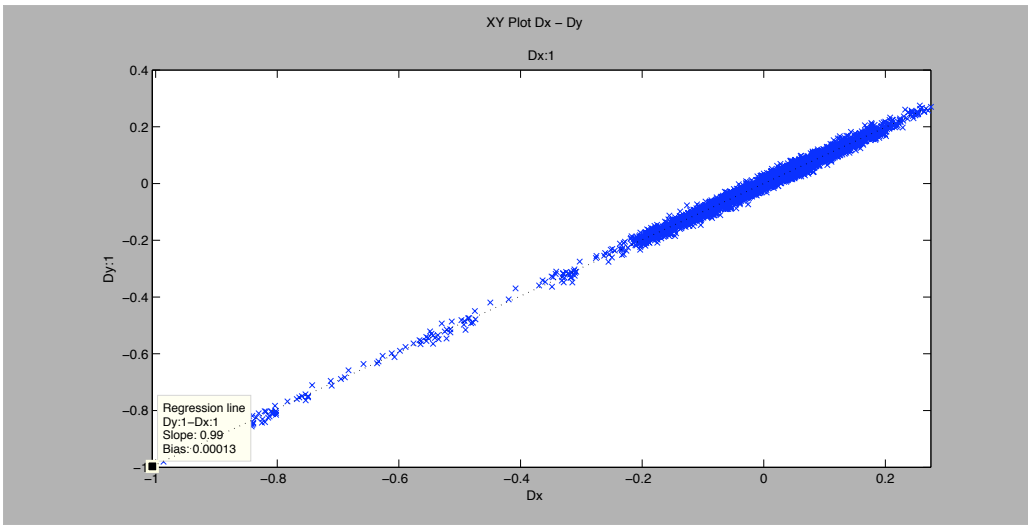


Figure 3: Linear regression of D_{t+1} against D_t (simulated temperature data)

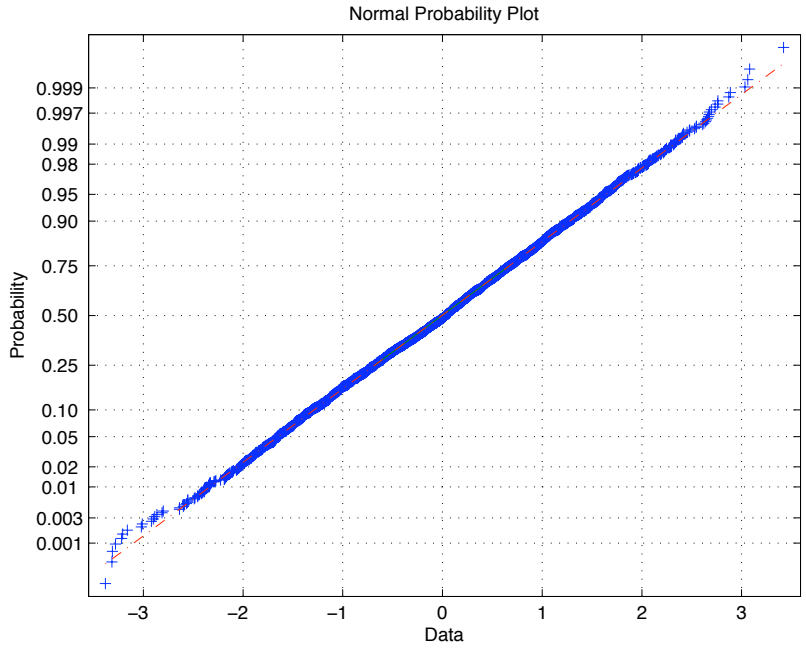


Figure 4: Normal probability plot of residuals ϵ_t

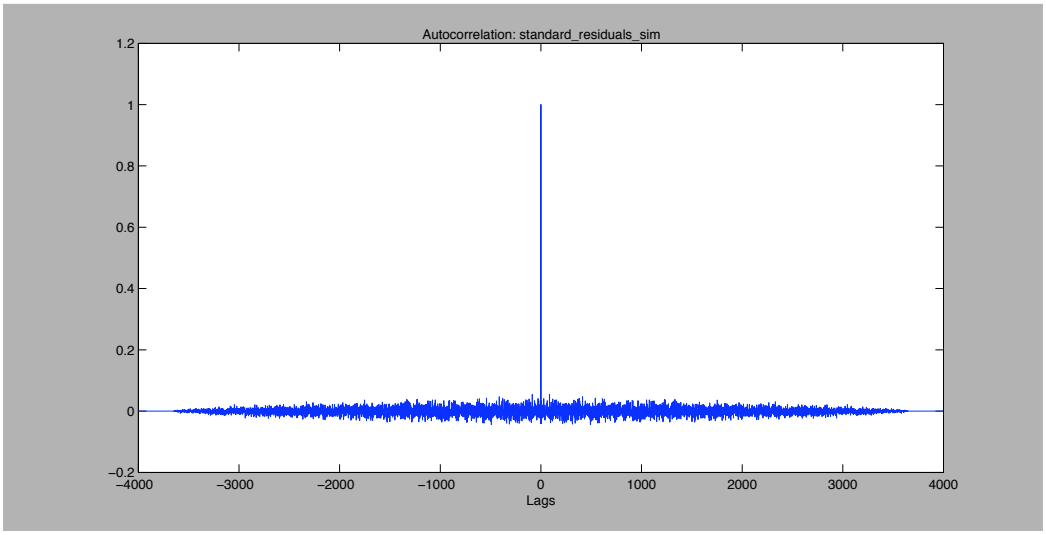


Figure 5: Autocorrelation function of residuals of the simulated data temperatures

4.2 Modeling real temperature data

At this part we will construct a time series consisting of 10 years of daily temperatures (average of maximum and minimum temperature of each day) taken from central England for the period 1988-1997. Denote this time series by T_i for $i = \{1, \dots, 3650\}$. For convenience, 29 of February was removed from the sample in each leap year in order to obtain years of equal size. We create an other time series for the long-term average temperature, obtained by taking the average of the 10 temperatures on the corresponding date and smoothing the series by moving-average smoothing (using 7 days window). Denote this time series by $\bar{T}_i$ for $i = \{1, \dots, 3650\}$. Note that the units are tenths of degrees Celsius

In order to include seasonality in our model, we will alter the temperature model in (50). Consider the following model for the temperature:

$$\mathrm{d}T_t = \mathrm{d}b_t + a(b_t - T_t)\,\mathrm{d}_t + c\,\mathrm{d}W_t \tag{55}$$

$$\Rightarrow \mathrm{d}D_t = -aD_t\,\mathrm{d}t + c\,\mathrm{d}W_t \tag{56}$$

where $D_t := T_t - b_t$. Here b_t essentially represents the long-term average temperature denoted above by $\bar{T}$. As we can see the difference between this model and (50) is that now we let b be time-depended in order to capture the seasonal component that we ignored in our simulations above. As shown in [1] we can specify $b_t \equiv b(t)$ to be of the form :

$$b(t) = p + qt + \sum_{i=1}^{I_1} p_i \sin(2i\pi(t - f_i)/365) + \sum_{j=1}^{J_1} q_j \cos(2j\pi(t - g_j)/365)$$

with the coefficients to be determined. We will now solve equation (56) without worrying for the form of b_t. As before, we apply the Ito formula to $L_t = e^{at}D_t$ to get:

$$\mathrm{d}L_t = ce^{at}\,\mathrm{d}W_t$$

Then integrate this form t to $t+1$ to get:

$$\begin{aligned} D_{t+1} &= e^{at}D_t e^{-a(t+1)} + ce^{-a(t+1)} \int_t^{t+1} e^{au}\,\mathrm{d}W_u \\ &= e^{-a}D_t + ce^{-a} \int_t^{t+1} e^{-a(t-u)}\,\mathrm{d}W_u \end{aligned}$$

But $\int_t^{t+1} e^{-a(t-u)}\,\mathrm{d}W_u \sim N\left(0, \sqrt{\frac{e^{2a}-1}{2a}}\right)$. Hence let $\int_t^{t+1} e^{-a(t-u)}\,\mathrm{d}W_u = \sqrt{\frac{e^{2a}-1}{2a}}\epsilon_t$ where $\epsilon_t \sim N(0,1)$. Therefore we have that:

$$D_{t+1} = e^{-a}D_t + ce^{-a}\sqrt{\frac{e^{2a}-1}{2a}}\epsilon_t$$

and hence we conclude that:

$$D_{t+1} = \alpha D_t + \gamma\epsilon_t \tag{57}$$

where

$$\alpha = e^{-a} \quad \Leftrightarrow \quad a = -\log\alpha \tag{58}$$

$$\gamma = c\sqrt{\frac{1-e^{-2a}}{2a}} \quad \Leftrightarrow \quad c = \frac{\gamma}{\sqrt{\frac{1-e^{-2a}}{2a}}} \tag{59}$$

Back to our time series, we let $D_i = T_i - \bar{T}_i$ to denote the deseasonalised series of temperature. Figure 6 shows the long-term average temperature $\bar{T}_i$ the temperatures T_i and the deseasonalised temperatures D_i for the years 1988-89. Equation (57) suggests to model D_i as a first-order auto regression (AR):

$$D_{i+1} = \alpha D_i + \gamma\epsilon_i \tag{60}$$

where ϵ_i is a sequence of independent Gaussian residuals with unit variance and $D_0 = 0$. The least-squares estimates $\hat{\alpha}, \hat{\gamma}$ of the parameters α, γ are found using the whole data and are quite stable over three-year windows of data:

$$\hat{\alpha} = 0.7899 \qquad \hat{\gamma} = 15.7211 \tag{61}$$

Figure 7 shows the histogram of detrended and deseasonalised temperatures D_i for the ten years period. As we can see a normal distribution is not at all unreasonable, and pretty much expected from the simulations made in the previous subsection. A linear regression analysis follows which results to the above least-square estimates (see Figure 8).

Figure 9 shows a normal probability plot of the residuals ϵ_i. Most of the points lie on the straight line which suggests the residuals are normally distributed. A Jarque-Bera normality test was also carried out, which reject the h_0 hypothesis. This means that there is no sufficient evidence that the data come from a normal distribution,

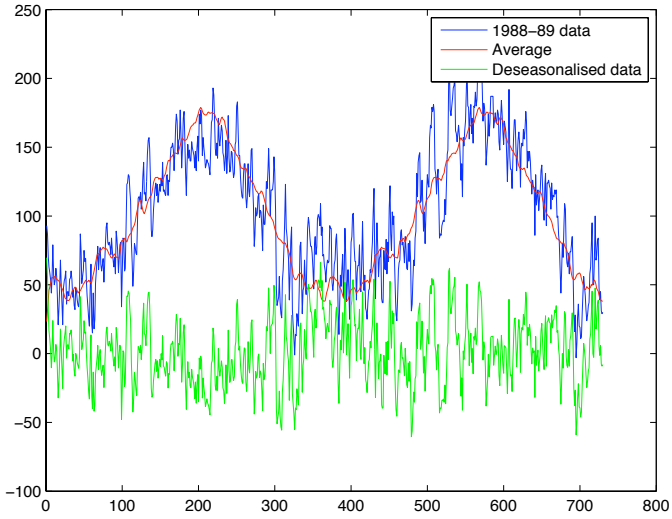


Figure 6: Time plot for the years 1988,1989

but roughly speaking it is not completely unreasonable to assume that the residuals are normally distributed. Figure 10 shows a plot of the autocorrelation function of the residuals. This function lies in a regime less than ± 0.1 which implies that the residuals are roughly independent. If we compare this with Figure 5 we can see that both functions behave almost similarly.

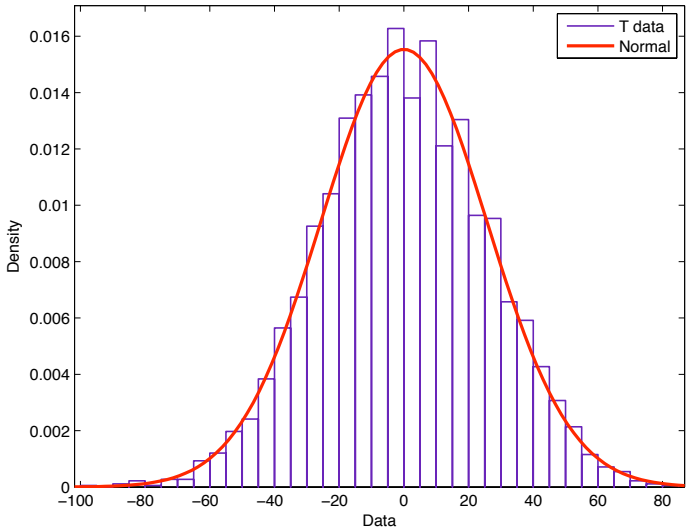


Figure 7: Histogram of detrended and deseasonalised temperatures Di (real temperature data)

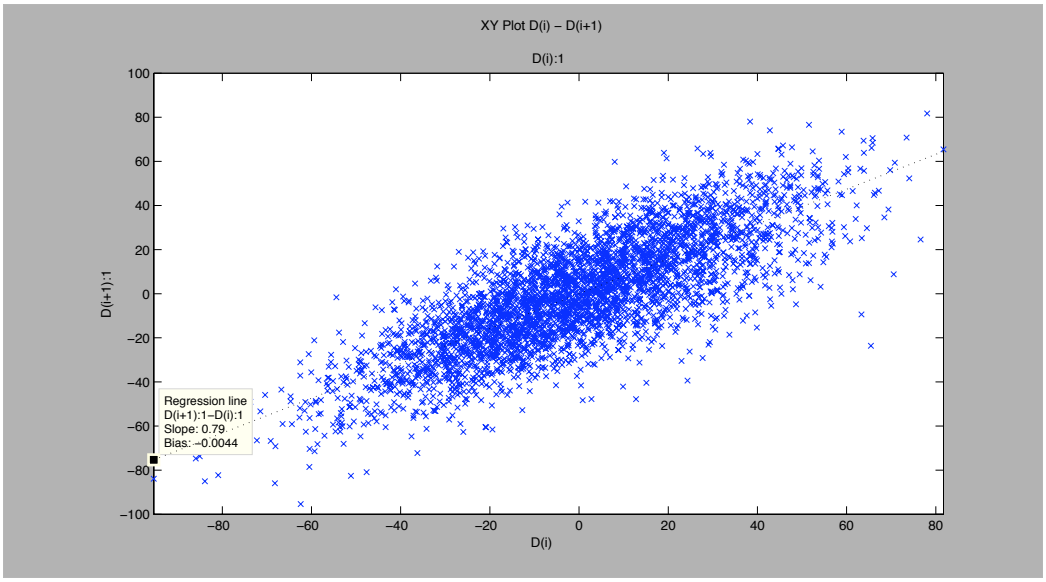


Figure 8: Linear regression of D_i against D_{i+1} (real temperature data)

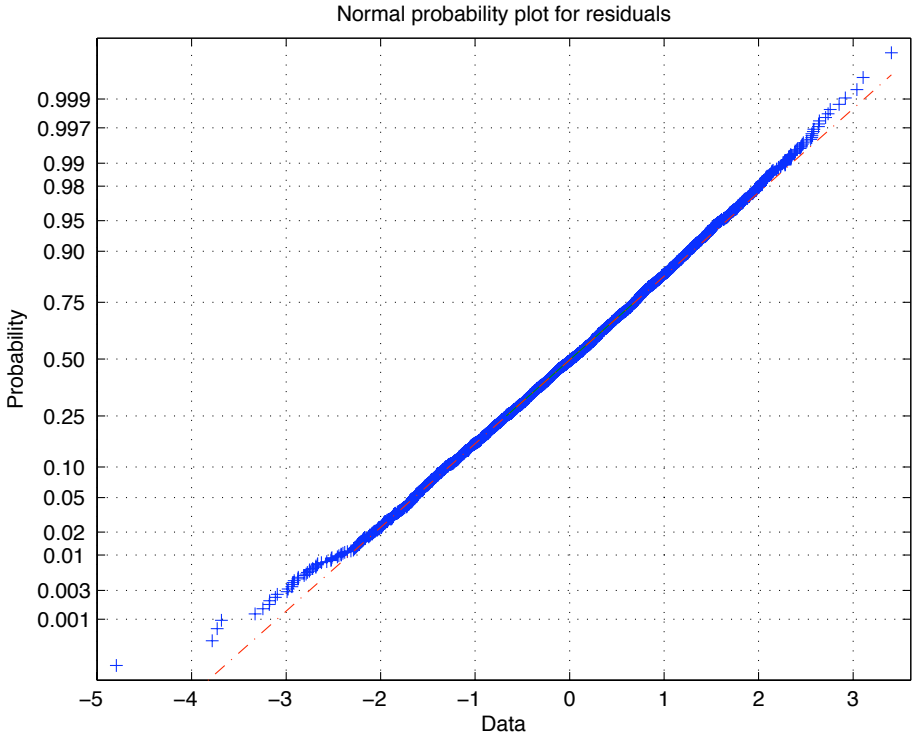


Figure 9: Normal probability plot of residuals

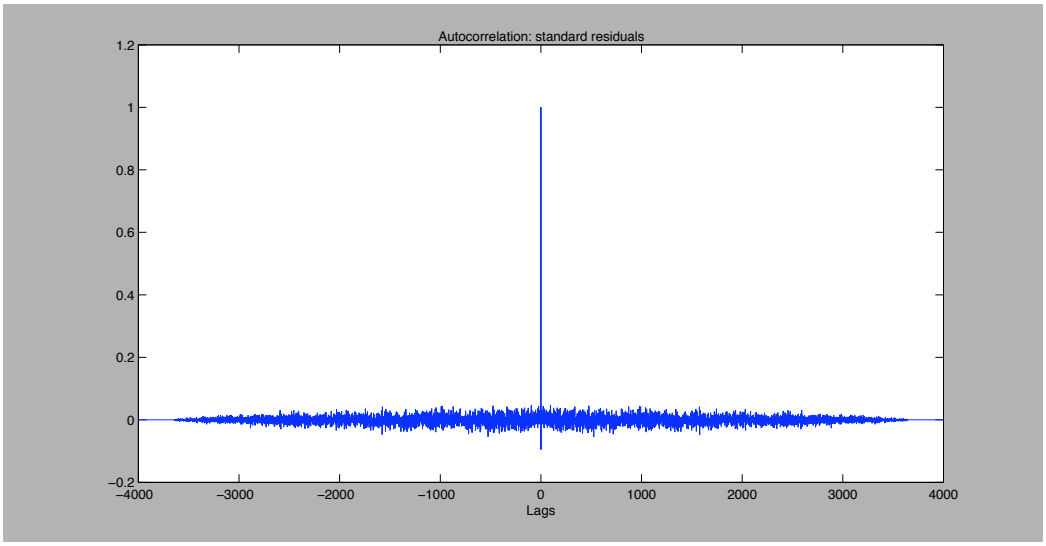


Figure 10: Autocorrelation function of residuals ϵ_i

4.3 HDD and accumulated HDD modeling

Recall that, according to Davis [3], if we let T_i be the average of minimum and maximum temperatures at day i (measured in degrees of Celsius), then the daily number of heating degrees is $\text{HDD}_i = \max(18 - T_i, 0)$. The accumulated 'heating degree days' (HDD) over a month period (31 days) ending at time t is defined as :

$$X_t := \sum_{i=0}^{30} \text{HDD}_{t-i} \tag{62}$$

So X_t is used as the underlying (non-traded) asset on written contracts. In [3] Davis have shown empirically that the distribution of the accumulated HDD on the final day of the month, say, is lognormal. We aim to reproduce this result using simulated and real temperature data.

First we would like to see if its reasonable to use a lognormal distribution for the accumulated HDD, by using simulated values of T_i. Simulate equation (50), where T represents the daily temperature variable, and use the following continuous version of (62):

$$H_S = \frac{1}{S} \int_0^S \max(18 - T_t, 0)\, \mathrm{d}t \tag{63}$$

where in this case S represents a terminal deterministic time. We simulate 10 years of daily temperature data using (50) with parameters:

$$a = 3, \qquad b = 10, \qquad c = 0.25, \quad T_0 = 4.1$$

We suppose that all the days of the year are used for pricing derivatives (and not only 252) since it will be more convenient to compare with the results using real temperature data. We simulate (63) 10,000 times in order to get 10,000 values of H_S. We repeat this procedure ten times in order to check for consistency. Figure 11 shows a normal probability plot of H_S and Figure 12 shows a normal probability plot of $\log(H_S)$. As we can see both suggest that either a normal or lognormal distribution are not unreasonable.

We also perform a Jarque-Bera normality test for H_S and $\log H_S$, at 5% significance level. The h_0 hypothesis[3] was not rejected (therefore accepted) on neither of these, which imply that there is no sufficient evidence that a normal or lognormal distribution is not appropriate. In fact the Jarque-Bera statistic was found to be smaller on average when we were testing for normality of H_S rather than $\log H_S$:

$$JB_{stat}(H_S) = 2.4415 \qquad JB_{stat}(\log H_S) = 2.6951$$

This fact implies that normal distribution is slightly more appropriate since the smaller the Jarque-Bera statistic is, the less is the evidence to reject h_0.

Figure 13 shows a histogram of the simulated data H_S together with normal and lognormal distribution curves. The fact that the normal and lognormal curves coincide shows that H_S may come from normal or lognormal distribution. So generally we can say that it is reasonable to model the accumulated HDD as being lognormally distributed, as Davis did.

Now we will try to replicate what Davis [3] did in his paper. We will simulate D_i from equation (56):

$$\mathrm{d}D_t = -aD_t\,\mathrm{d}t + c\,\mathrm{d}W_t \tag{64}$$

where a and c are given by (58), (59) using the values of $\hat{\alpha}$ and $\hat{\gamma}$ from the regression analysis we did in Section 3.2. Then take the simulated temperature as $T_i = \bar{T}_i + D_i$, and finally find the accumulated HDD for one month (say) using (62).

We simulate values of D_i for a 10,000 years period and find the accumulated HDD for the last day of January over this period. Figure 14 shows the empirical distribution and best normal and lognormal fit for the accumulated HDD over the month of January. As we can see both distributions are not unreasonable at all. Schachermayer and Teichmann [11] wrote in their paper, that Bachelier pricing formulas give very similar prices with those taken from the Black-Merton-Scholes pricing formulas. This result is essentially very similar to what we found above, that is, both normal and

[3] h_0 hypothesis assumes that sample comes from a normal distribution

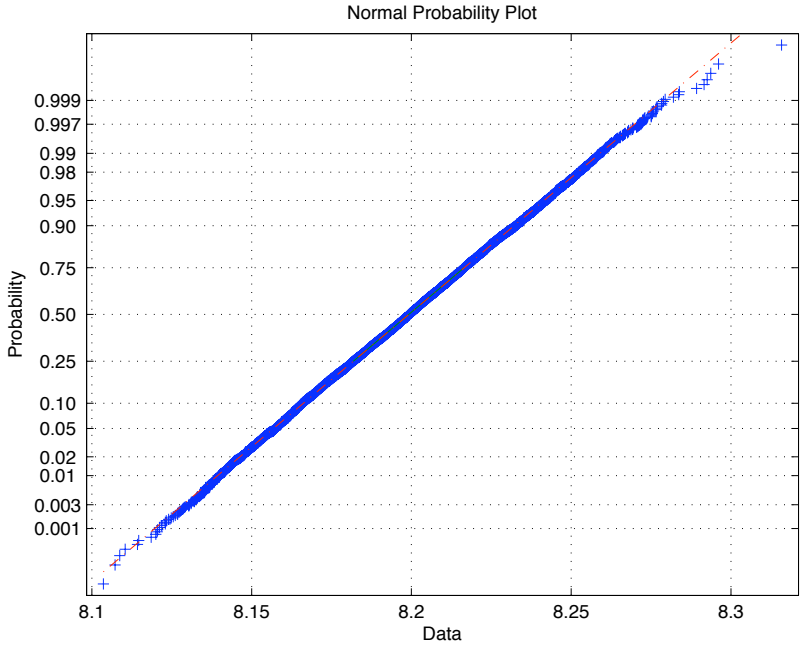


Figure 11: Normal probability plot of simulated H_S

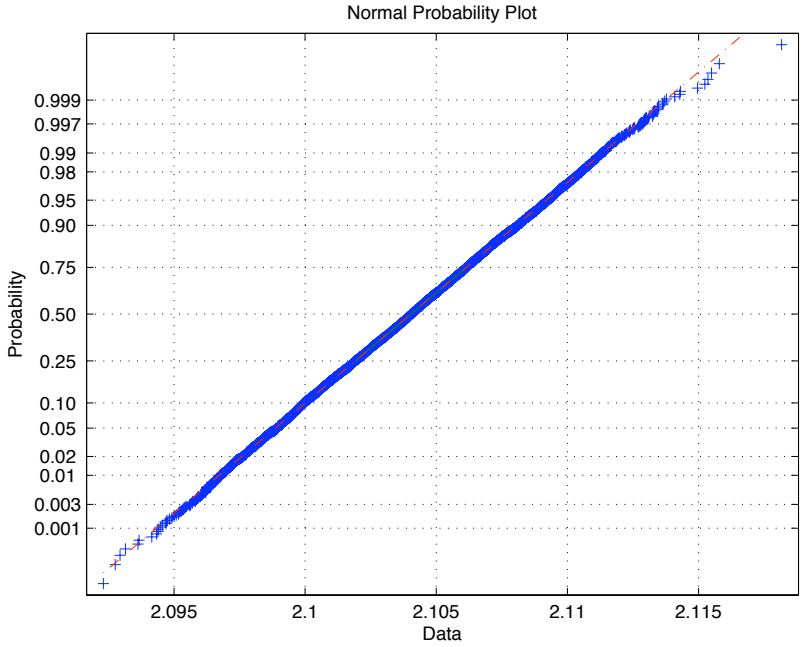


Figure 12: Normal probability plot of simulated $\log H_S$

lognormal distributions are reasonable. But of course lognormal distribution is more appropriate in this case since its positive and we don't have to worry about the fact that we can go negative (Normal distribution) even with a very small probability.

Finally in order to have a more clear view we will use real daily temperature data for a period of 226 years (1772-1997). We will calculate the accumulated HDD for the last day of January (say) for 226 years and plot its empirical distribution. Figure 15 shows the empirical distribution together with the best lognormal and normal fits. As we can see again both distributions are reasonable but the lognormal one is more appropriate for the reasons mentioned above.

5 Conclusion-Epilogus

In the first part of this dissertation we used the utility indifference valuation technique in order to price and hedge a contingent claim on a non-traded asset which follows a mean reverting OU process, under known and constant parameters and exponential utility preferences. We have derived a probabilistic representation for the indifference price of general contingent claims and found analytical pricing formulas in the case that the contingent claim is a put on the underlying non-traded asset. We found the processes that the hedging errors follow under optimal and naive hedging methods. By finding the terminal hedging error distributions we showed that optimal hedging can indeed outperform naive hedging methods based on complete market assumptions.

In the second part of this dissertation we tried to model the daily weather temperature using a mean reverting OU process with constant coefficients. Using time series analysis methods we compared simulated temperature data with real daily temperature data. By removing seasonality and trend from both simulated and real data series we concluded that the de-seasonalised data can be successfully modeled by a first order autoregressive model. We argued that Davis' [3] result on the lognormal distribution of the accumulated HDD is reasonable, but not perfect. We did that by replicating what Davis did, by analyzing pure simulated data and by analyzing real weather data.

As an extension of what we did so far, we can find pricing formulas and an optimal hedging strategy of an option based on the average temperature between two future days, by using the results shown in the first two Sections of the dissertation. We can also introduce uncertainty into our non-traded asset's model parameters and check if the optimal hedge still outperforms the naive one.

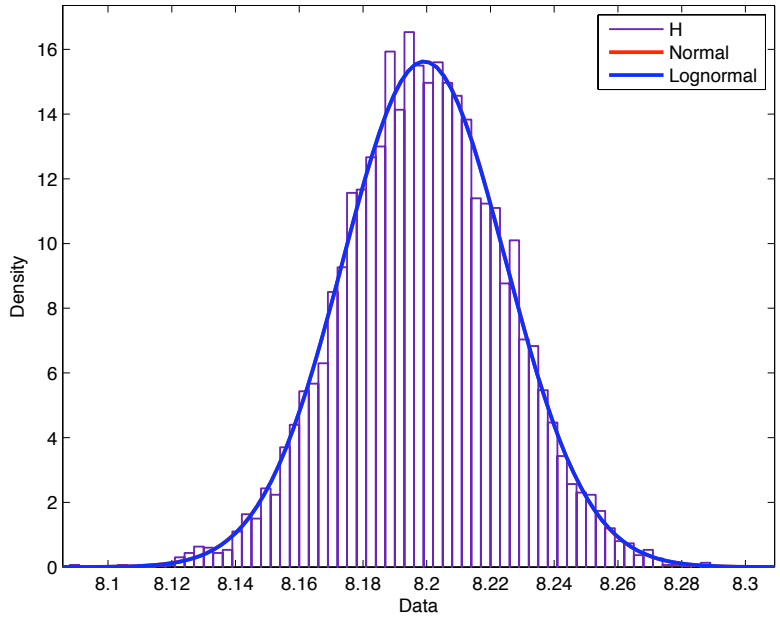


Figure 13: Histogram of H_S

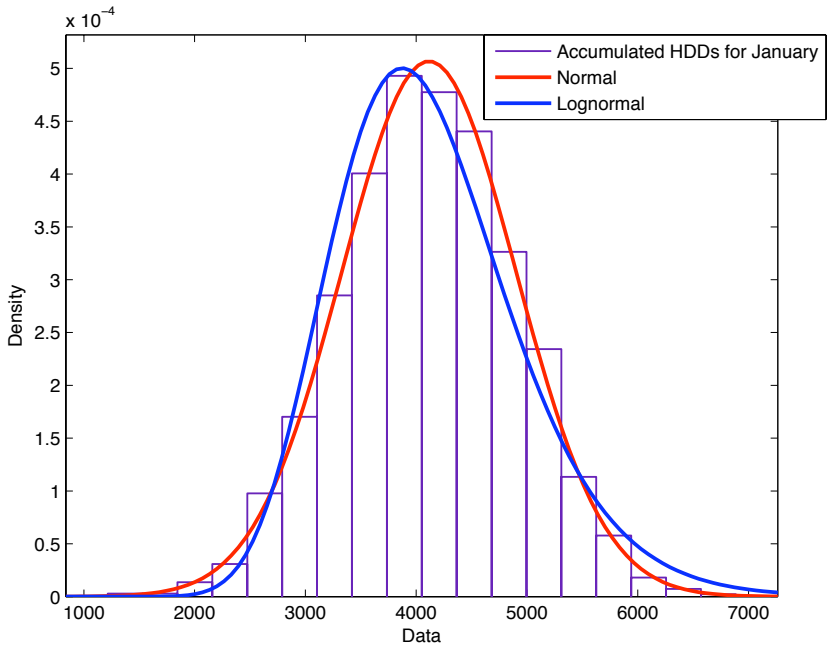


Figure 14: Empirical distribution of simulated accumulated HDD for the month of January

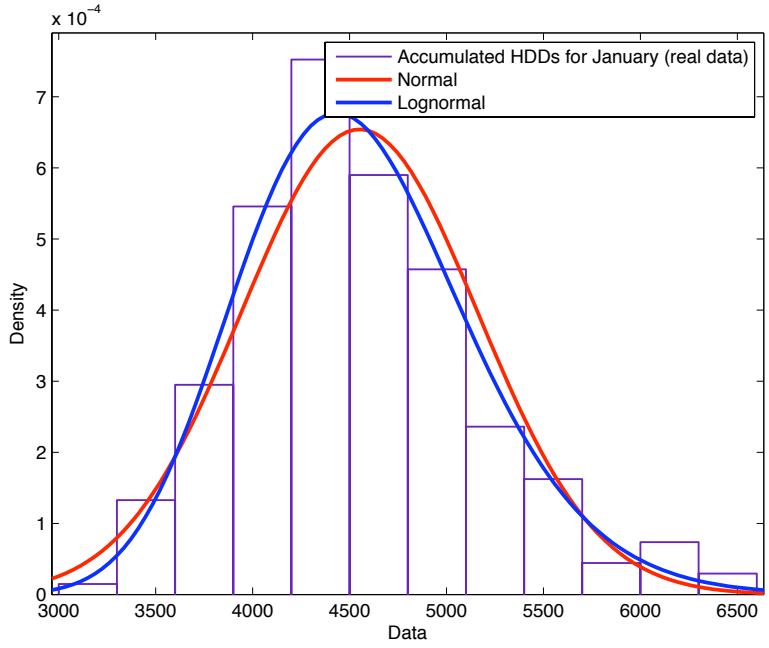


Figure 15: Empirical distribution of accumulated HDD for the month of January

Appendix

This Section will provide the Matlab code used in this dissertation

Code for the function $\mathcal{G}_t$:

```
function XX1 = function_G(l,Y,a,b,c,lambda,rho,K,T)

gamma = exp(-a*T).*Y +(a*b-rho*lambda*c)*(1-exp(-a*T))*(1/a);
V = (c^2/(2*a))*(1-exp(-2*a*T));
q=(K-gamma)./sqrt(V);

Psi=zeros(length(Y),6);

Psi0=normcdf(q, 0,1);
Psi(:,1)=Psi0;
Psi1=-(1/sqrt(2*pi)).*exp(-(q.^2)./2);
Psi(:,2)=Psi1;
Psi2=q.*Psi1+Psi0;
Psi(:,3)=Psi2;
```

```
Psi3=(2+q.^2).*Psi1;
Psi(:,4)=Psi3;
Psi4=(q.^3).*Psi1 + 3.*Psi2;
Psi(:,5)=Psi4;
Psi5= (q.^4).*Psi1+4.*Psi3;
Psi(:,6)=Psi5;

G=zeros(length(Y),l+1);
for k=0:l
    G(:,k+1)=nchoosek(l,k).*((sqrt(V)).^(l-k)).*(gamma.^k).*Psi(:,l-k+1);
end
G=sum(G,2);

XX1=G;
```

Code for the function $\partial \mathcal{G}_t$:

```
function XX2 = function_dG(l,Y,a,b,c,lambda,rho,K,T)

gamma = exp(-a*T).*Y +(a*b-rho*lambda*c)*(1-exp(-a*T))*(1/a);
dgamma=exp(-a*T);
V = (c^2/(2*a))*(1-exp(-2*a*T));
q=(K-gamma)./sqrt(V);

Psi=zeros(length(Y),6);

Psi0=normcdf(q, 0,1);
Psi(:,1)=Psi0;
Psi1=-(1/sqrt(2*pi)).*exp(-(q.^2)./2);
Psi(:,2)=Psi1;
Psi2=q.*Psi1+Psi0;
Psi(:,3)=Psi2;
Psi3=(2+q.^2).*Psi1;
Psi(:,4)=Psi3;
Psi4=(q.^3).*Psi1 + 3.*Psi2;
Psi(:,5)=Psi4;
Psi5= (q.^4).*Psi1+4.*Psi3;
```

```
Psi(:,6)=Psi5;

dPsi=zeros(length(Y),6);

dPsi0=Psi1.*(1./sqrt(V)).*dgamma;
dPsi(:,1)=dPsi0;
dPsi1= q.*dPsi0;
dPsi(:,2)=dPsi1;
dPsi2=-(1./sqrt(V)).*dgamma.*Psi1 + q.*dPsi1 + dPsi0;
dPsi(:,3)=dPsi2;
dPsi3=(2+q.^2).*dPsi1-2.*q.*(1./sqrt(V)).*Psi1.*dgamma;
dPsi(:,4)=dPsi3;
dPsi4=-3*(q.^2).*(1./sqrt(V)).*dgamma.*Psi1 + (q.^3).*dPsi1 + 3.*dPsi2;
dPsi(:,5)=dPsi4;
dPsi5=-4*(q.^3).*(1./sqrt(V)).*dgamma.*Psi1 + (q.^4).*dPsi1 + 4.*dPsi3;
dPsi(:,6)=dPsi5;

dG=zeros(length(Y),l);
for k=0:l
    dG(:,k+1)=nchoosek(l,k).*(sqrt(V)).^(l-k).*k.*(gamma).^(k-1).*dgamma
    .*Psi(l-k+1) + nchoosek(l,k)*(sqrt(V)).^(l-k).*(gamma).^(k).*dPsi(l-k+1);
end
dG=sum(dG,2);

XX2=dG;
```

Code for the function m:

```
function XX3 = function_m(j,Y,a,b,c,lambda,rho,K,T)

m = zeros(length(Y),j+1);

for l = 0:j
    G=function_G(l,Y,a,b,c,lambda,rho,K,T);
    m(:,l+1) = nchoosek(j,l).*(-1)^l.*K.^(j-l).*G(l+1);

end
```

```
m = sum(m,2);

XX3 = m;
```

Code for the function ∂m:

```
function XX4 = function_dm(j,Y,a,b,c,lambda,rho,K,T)

dm = zeros(length(Y),j+1);

for l = 0:j
    dG=function_dG(l,Y,a,b,c,lambda,rho,K,T);
    dm(:,l+1) = nchoosek(j,l).*(-1)^l.*K.^(j-l).*dG(l+1);
end
dm = sum(dm,2);

XX4 = dm;
```

Code for the function ∂p_t:

```
function y = dp(Y,a,b,c,lambda,alpha,rho,K,T,n)

d=-(alpha)*(1-(rho)^2)*n;
gamma = exp(-a*T).*Y +(a*b-rho*lambda*c)*(1-exp(-a*T))*(1/a);
dgamma=exp(-a*T);
V = (c^2/(2*a))*(1-exp(-2*a*T));
q=(K-gamma)./sqrt(V);

m = zeros(length(Y),5);
dm = zeros(length(Y),5);
dk = zeros(length(Y),5);
for i=1:5
    dm(:,i) = function_dm(i,Y,a,b,c,lambda,rho,K,T);
    m(:,i) = function_m(i,Y,a,b,c,lambda,rho,K,T);
end
%
% The terms of the cumulant expansion for the derivative of the
% indifference price is calculated
```

```
%
dk(:,1) = dm(:,1);
dk(:,2) = dm(:,2)-2*m(:,1).*dm(:,1);
dk(:,3) = dm(:,3)-3*(m(:,1).*dm(:,2)+m(:,2).*dm(:,1)-2*(m(:,1).^2).*dm(:,1));
dk(:,4) = dm(:,4)-4*(m(:,1).*dm(:,3)+m(:,3).*dm(:,1))-6*m(:,2).*dm(:,2)+...
12*m(:,1).*(m(:,1).*dm(:,2)+2*m(:,2).*dm(:,1)-2*(m(:,1).^2).*dm(:,1));
dk(:,5) = dm(:,5)-5*(m(:,1).*dm(:,4)+m(:,4).*dm(:,1))-10*(m(:,2).*dm(:,3)
+m(:,3).*dm(:,2))+20*m(:,1).*(m(:,1).*dm(:,3)+2*m(:,3).*dm(:,1))
+30*m(:,2).*(2*m(:,1).*dm(:,2)+m(:,2).*dm(:,1))
-60*(m(:,1).^2).*(m(:,1).*dm(:,2)
+3*m(:,2).*dm(:,1)-2*(m(:,1).^2).*dm(:,1));
%
% The order a^4 expansion for the delta is then
% y=dk1+d*dk2/2+d^2*dk3/3!+ d^3*dk4/4! + d^4*dk5/5!
%
y=dk(:,1) + 0.5*dk(:,2)*d + dk(:,3)*(d^2)/6 + dk(:,4)*(d^3)/24
   + dk(:,5)*(d^4)/120;
```

Code for the calculation of the terminal hedging error:

```
clear all
clc

Y0=100; S0=100; K=110; sig=0.3; c=0.25; b=100; a=2; lambda=0.4; T=1;
rho=0.75; n=-1; alpha=0.01;
N=252;  M=10000;

d=-(alpha)*(1-(rho)^2)*n;
epsilon=sqrt(1-rho^2);
mu=sig*lambda;
h=T/N;

ERR=zeros(M,1);
ERRnaive=zeros(M,1);
```

```
for i=1:M
    i

    E1=randn(N,1);
    VS=(mu-0.5*(sig^2))*h*ones(N,1) + sig*sqrt(h)*E1;
    logS=log(S0)+cumsum(VS);
    logS=[log(S0); logS];
    S=exp(logS);

    E2=randn(N,1);
    E0=rho*E1 + epsilon*E2;

    Y=zeros(N+1,1);
    Y(1)=Y0;
    dW=sqrt(h)*E0;

    for j=2:length(Y)
        Y(j)=Y(j-1) + h*(a*(b-Y(j-1)))+c*dW(j-1);
    end

    % Pre-allocate vectors to store the residual risk processes.

    R=zeros(N+1,1);
    Rnaive=zeros(N+1,1);

    % Now make some relevant time vectors and time to maturity vectors
    % First, the time vector

    tv=[0:h:T]';

    % then, the time to maturity vector, going from T down to dt - we
    % can generate this by flipping tv using "flipud"

    tmat=flipud(tv);
    tm=tmat(1:end-1);
```

```
    % We'll need the Y vector with the last value taken out:

    YY = Y(1:end-1);

    % Computing the prices and deltas in vector form i.e. for all values
    % of YY

    py=dp(YY,a,b,c,lambda,alpha,rho,K,tm,n);

    dBbar=sqrt(h)*E2;
    dR=0.5*(c^2)*(n^2)*alpha*(epsilon^2).*(py.^2)*h + c*n*epsilon.*py.*dBbar;

    % Then R is given by a cumsum:

    R=[R(1); R(1) + cumsum(dR)];

    % The terminal hedging error is the last component of R:

    ERR(i)=R(end);

    %For the naive trading stradegy:

    Py=-normcdf((K-YY)./sqrt(tm*(c^2)),0,1);
    dB=sqrt(h)*E1;
    dRnaive=n.*(a.*(b-YY)-lambda*c).*Py.*h + n*c.*Py.*((rho-1).*dB +
    epsilon.*dBbar);
    % Then Rnaive is given by a cumsum:

    Rnaive=[Rnaive(1); Rnaive(1) + cumsum(dRnaive)];

    % The terminal hedging error is the last component of Rnaive:

    ERRnaive(i)=Rnaive(end);
end

max_error=max(ERR)
```

```
min_error=min(ERR)
mean_error=mean(ERR)
sd_error=std(ERR)
median_error=median(ERR)
EUcalET=mean(-exp(-alpha.*ERR))
max_naive_error=max(ERRnaive)
min_naive_error=min(ERRnaive)
mean_naive_error=mean(ERRnaive)
sd_naive_error=std(ERRnaive)
median_naive_error=median(ERRnaive)
EUcalETnaive=mean(-exp(-alpha.*ERRnaive))
subplot(2,1,1)
EDGES=min(ERR):(range(ERR)/30):max(ERR);
N1=histc(ERR,EDGES);
bar(EDGES,N1)
ylabel('Frequency')
xlabel('Terminal Hedge Error')
legend('Optimal Hedge',0)
title('Hedging Error Distributions')
subplot(2,1,2)
EDGESnaive=min(ERRnaive):(range(ERRnaive)/30):max(ERRnaive);
Nnaive=histc(ERRnaive,EDGESnaive);
bar(EDGESnaive,Nnaive)
ylabel('Frequency')
xlabel('Terminal Hedge Error')
legend('Naive Hedge',0)
beep
```

Code for simulating OU process:

```
function T = Simulation_OU_process(T0, S, N, b, c, a)
    h = S/N;
    T=zeros(N+1,1);
    T(1)=T0;
    dW=sqrt(h)*randn(N,1);

    for j=2:length(T)
```

```
        T(j)=T(j-1) + h*(a*(b-T(j-1)))+c*dW(j-1);
    end
end
```

Code for simulating the integral $H_S = \frac{1}{S}\int_0^S \max(K - T_t, 0)\,\mathrm{d}t$:

```
% OU process:  dT = a(b-T)dt +cdW
% Simulation_OU(T0, S, N, b, c, a)

clear all
clc
%simulation of (1/S)*integral(max(K-T,0)) from 0 to S
T0=4.1; S=10; N=356*10; b=10; c=0.25; a=3; K=18;
h=S/N;
Hl=zeros(1,10000);
for j=1:10
    for i=1:10000
        T = simulation_OU_process(T0, S, N, b, c, a);

        M = max(K-T,0);
        L = sum(M*h);
        Hl(i)=(1/S)*L;
    end
    Hl;
    lnHl=log(Hl);

    figure
    normplot(lnHl)
    figure
    normplot(Hl)

    [h1,p1,jbstat1,critval1] = jbtest(lnHl)

    [h2,p2,jbstat2,critval2] = jbtest(Hl)

    S1(j)=jbstat1;
    S2(j)=jbstat2;
```

```
end
Jbstat_lnHl=mean(S1)
Jbstat_Hl=mean(S2)
```

Code for time-series analysis of 10 years of temperature data and HDD simulation:

```
clc
clear all

Tavg=xlsread('Temps_daily.xls');

yearly_mean=mean(Tavg);
mean_of_10_years=mean(yearly_mean);

Tbar=(mean(Tavg'))';
Tavg_ts=[Tavg(:,1); Tavg(:,2); Tavg(:,3);Tavg(:,4);Tavg(:,5);Tavg(:,6);
Tavg(:,7);Tavg(:,8);Tavg(:,9); Tavg(:,10)];
Tbar_ts=[Tbar;Tbar;Tbar;Tbar;Tbar;Tbar;Tbar;Tbar;Tbar;Tbar];

span = 7; % Size of the averaging window
window = ones(span,1)/span;
Tbar_smooth = convn(Tbar_ts,window,'same'); %moving average smoothing
T=Tavg_ts - Tbar_smooth;
size(T);

Tx=T(1:3649);
Ty=T(2:3650);

stats1 = regstats(Ty,Tx,'linear',{'beta', 'r','standres'});
residuals=stats1.r;
coeffs= stats1.beta;
stand_residuals=stats1.standres;

figure
hist(stand_residuals,30)
[h1,p1,jbstat1,critval1] = jbtest(stand_residuals)
```

```
figure
normplot(stand_residuals)
title('Normal probability plot for residuals')

gamma=sqrt(var(residuals));
alpha=0.7899;%slope by results above

%D(j)=D(j-1) + h*alpha*D(j-1) + gamma* dW(j-1);
%D(j)=alpha*D(j-1) + gamma* dW(j-1);

h = 1/365;
N=365;
AccHDD=zeros(100000,1);
delta=log(alpha); sig=(gamma/alpha);
a=-log(alpha); c=gamma/sqrt((1-exp(-2*a))/(2*a));
for l=1:100000

    D=zeros(N+1,1);
    D(1)=0;
    dW=sqrt(h)*randn(N,1);

    for j=2:length(D)
        D(j)=D(j-1) - h*a*D(j-1) + c*dW(j-1);
    end
    size(D);
    size(Tbar);
    Tt=D(122:152) + Tbar_smooth(121:151);
    AccHDD(l)=sum(max(180-Tt,0));
end

Time=10000;%years
N=365*10000;
h = Time/N;
D=zeros(N+1,1);
D(1)=0;
dW=sqrt(h)*randn(N,1);
```

```
for j=2:length(D)
    D(j)=D(j-1) - h*a*D(j-1) + c*dW(j-1);
end
D=D(2:3650001);
size(D);

Tb=zeros(3650000,1);
a=1;b=3650;

for j=1:1000
    Tb(a:b)=Tbar_smooth;
    a=a+3650;
    b=b+3650;
end

Tsim=D+Tb;
HDDsim = max(180-Tsim,0);
HDDacc=zeros(10000,1);
%a1=182; a2=212;
a1=1; a2=31; %for January

for k=1:10000
    HDDacc(k)=sum(HDDsim(a1:a2));
    a1=a1+365;
    a2=a2+365;
end

figure
hist(HDDacc,20)

figure
tt=1:2*365;
plot(tt,Tavg_ts(1:2*365),'b',tt,Tbar_smooth(1:2*365),'r',tt,T(1:2*365),'g')
legend('1988-89 data','Average', 'Deseasonalised data')
```

Code for time-series analysis of simulated temperature data:

```
clc
clear all
hold off

%linear regression model for simulated OU process Y
%T = Simulation_OU_process(T0, S, N, b, c, a)

T = simulation_OU_process(9, 200, 200*365,10, 0.25, 3);

y=T(2:3651);
y1=y(1:365);y2=y(366:366+365-1); y3=y(731:1095); y4=y(1096:1460);
y5=y(1461:1825);
y6=y(1826:2190); y7=y(2191:2555); y8=y(2556:2920); y9=y(2921:3285);
y10=y(3286:3650);
yy=[y1 y2 y3 y4 y5 y6 y7 y8 y9 y10];

ybar=(mean(yy'))';
ybar_ts=[ybar;ybar;ybar;ybar;ybar;ybar;ybar;ybar;ybar;ybar];
T_d= y - ybar_ts;

D = detrend(T,'constant');

plot(T_d)
hold on
plot(D,'r')
hold off

Dy=D(2:3651);
Dx=D(1:3650);

stats = regstats(Dy,Dx,'linear',{'beta', 'r','standres'});

residuals_sim=stats.r;
standard_residuals_sim=stats.standres;
coeffs_sim=stats.beta
```

```
var_sim=var(residuals_sim)

figure
normplot(standard_residuals_sim)

figure
hist(standard_residuals_sim,40)

figure
hist(D,40)
```

Code for HDD calculation of January using 226 years of real temperature data:

```
clear all
clc

HDD_jan=max(180-T_jan,0);

AccHDD_jan=zeros(226,1);
a=1;b=31;
for j=1:226
    AccHDD_jan(j)=sum(HDD_jan(a:b));
    a=a+31;
    b=b+31;
end

hist(AccHDD_jan,10)

figure
normplot(log(AccHDD_jan))
title('Normal probability plot ')
```

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