

# Portfolio Optimization under a Quantile Hedging Constraint <sup>\*†</sup>

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## Abstract

We study a problem of portfolio optimization under a European quantile hedging constraint. More precisely, we consider a class of Markovian optimal stochastic control problems in which two controlled processes must meet a probabilistic shortfall constraint at some terminal date. We denote by  $V$  the corresponding value function. Following the arguments introduced in the literature on stochastic target problems, we convert this problem into a state constraint one in which the constraint is defined by means of an auxiliary value function  $v$  characterizing the reachable set. This set is therefore not given a priori but is naturally integrated in  $v$  solving, in a viscosity sense, a nonlinear parabolic partial differential equation (PDE). Relying on the existing literature, we derive, in the interior of the domain, a Hamilton-Jacobi-Bellman characterization of  $V$ . However,  $v$  involves an additional controlled state variable coming from the diffusion of the probability of reaching the target and belonging to the compact set  $[0, 1]$ . This leads to non-trivial boundaries for  $V$  that must be discussed. Our main result is thus the characterization of  $V$  at those boundaries. We also provide examples for which comparison results exist for the PDE solved by  $V$  on the interior of the domain.

**Keywords:** Quantile Hedging Constraints; Optimal Control; Viscosity Solutions.

*AMS 2010 Subject Classification* Primary, 3E20, 49L25; Secondary, 60J60.

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<sup>\*</sup>Part of this work has been written during the author's PhD at Imperial College London sponsored by the Natixis Foundation for Quantitative Finance and during the author's visit to the Department of Economics of the University of Verona in February 2018 under the YITP research prize.

<sup>†</sup>The final publication is available at <https://doi.org/10.1142/S0219024918500486>.

<sup>‡</sup>The author is grateful to Prof Jean-François Chassagneux and Prof. Bruno Bouchard for their informed comments on this project. All the remaining errors are the author's ones.

# 1 Introduction

On a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$  and for  $(t, z) \in [0, T] \times \mathcal{O}_+^d \times \mathbb{R}$ , with  $\mathcal{O}_+^d := (0, \infty)^d$  and  $z := (x, y)$ , we are given  $Z_s^{t,z,\nu} := (X_s^{t,x,\nu}, Y_s^{t,z,\nu})$ ,  $t \leq s \leq T$ , valued in  $\mathcal{O}_+^d \times \mathbb{R}$  with initial conditions  $(t, z)$ . Both processes are controlled by some  $\nu \in \mathcal{U}$ , with  $\mathcal{U}$  the set of  $\mathbb{R}^d$ -valued, square integrable and progressively measurable processes, and are strong solutions to

$$X_s^{t,x,\nu} = x + \int_t^s \mu_X(r, X_r^{t,x,\nu}, \nu_r) dr + \int_t^s \sigma_X(r, X_r^{t,x,\nu}, \nu_r) dW_r \text{ on } \mathcal{O}_+^d, \quad (1.1)$$

$$Y_s^{t,z,\nu} = y + \int_t^s \mu_Y(r, Z_r^{t,z,\nu}, \nu_r) dr + \int_t^s \sigma_Y^\top(r, Z_r^{t,z,\nu}, \nu_r) dW_r \text{ on } \mathbb{R}, \quad (1.2)$$

for some  $d$ -dimensional Brownian motion  $W$ . As usual  $X^{t,x,\nu}$  stands for the price of some risky assets while  $Y^{t,z,\nu}$  is the portfolio process. We allow the hedging strategy  $\nu$  to impact the price process  $X^{t,x,\nu}$  as it may arise in a context of large trading movements.

We intend to study a stochastic control problem under a quantile hedging constraint which frequently arises in finance and insurance (see e.g. [El Karoui et al. \(2005\)](#) and the references therein). This problem is defined by

$$V(t, z, p) := \sup_{\nu \in \mathcal{U}_{t,z,p}} \mathbb{E}[f(Z_T^{t,z,\nu})], \quad (1.3)$$

with  $p \in [0, 1]$  and where

$$\mathcal{U}_{t,z,p} := \left\{ \nu \in \mathcal{U}_{t,z} : \mathbb{P}[Y_T^{t,z,\nu} \geq g(T, X_T^{t,x,\nu})] \geq p \right\}, \quad (1.4)$$

with

$$\mathcal{U}_{t,z} := \{ \nu \in \mathcal{U} : \text{s.t. } Y^{t,z,\nu} \geq 0 \text{ on } [t, T] \}, \quad (1.5)$$

and where  $(f, g)$  are two locally bounded Borel-measurable functions with  $f$  satisfying a polynomial growth, for the above expectation to be well defined for any  $\nu \in \mathcal{U}$ . The level given by  $g(\cdot)$  can be viewed as a solvency constraint coming from an outside party, a minimal requirement for a fund manager, or the willingness to avoid a huge dis-utility. It will be designated as a stochastic target/benchmark. Moreover the function  $f$  can be seen as a utility function.

There exists an important literature on state constraint problems that relies on the control theory. In particular, for deterministic control problems, we refer the reader to [Soner \(1986\)](#), [Capuzzo-Dolcetta & Lions \(1990\)](#), and [Ishii & Koike \(1996\)](#); while for stochastic control problems, we refer to [Lasry & Lions \(1989\)](#), [Katsoulakis \(1994\)](#), [Barles & Burdeau \(1995\)](#), [Ishii & Loreti \(2002\)](#), and [Federico \(2008\)](#).

On the contrary to the references quoted above, the problem described in (1.3) is, with the terminology in Bouchard et al. (2010), *non-standard* as the constraint does not hold almost surely over time. Following the observation in Bouchard et al. (2010), we first try to convert this *non-standard* problem into a more *classical* one. We therefore build on the original idea in Bouchard et al. (2010) and introduce an auxiliary value function  $v$  defined by

$$v(t, x, p) := \inf \left\{ y \geq 0 : \mathcal{U}_{t,z,p} \neq \emptyset \right\}, \quad (1.6)$$

characterizing (the closure) of the *viability domain*  $\mathcal{C}$  (with the terminology in Aubin & Cellina (1984)) defined as

$$\mathcal{C} := \left\{ (t, z, p) \in [0, T] \times \mathcal{O}_+^d \times \mathbb{R}^+ \times [0, 1] \text{ s.t. } \mathcal{U}_{t,z,p} \neq \emptyset \right\}. \quad (1.7)$$

Then, similarly to what has been done by Bouchard et al. (2009), we introduce an additional controlled state variable  $P^{t,p,\alpha}$ , for some admissible  $\alpha$ , coming from the diffusion of the probability of reaching the target. More precisely  $\alpha$  is an  $\mathbb{R}^d$ -valued square integrable process being such that  $P^{t,p,\alpha} \in [0, 1]$   $\mathbb{P}$ -a.s. We thus prove, assuming that the infimum in (1.6) is reached and appealing to the so-called geometric dynamic programming (GDP) principle introduced by Soner & Touzi (2002a,b), that the initial problem in (1.3) is equivalent to

$$V(t, z, p) := \sup_{\nu \in \mathcal{D}} \mathbb{E} \left[ f(Y_T^{t,z,\nu}) \right], \quad (1.8)$$

where

$$\mathcal{D} := \left\{ \nu \in \mathcal{U} : Y_s^{t,z,\nu} \geq v(s, X_s^{t,x,\nu}, P_s^{t,p,\alpha}) \text{ } \mathbb{P}\text{-a.s. } \forall s \in [t, T], \text{ for some admissible } \alpha \right\}. \quad (1.9)$$

On the contrary to standard state-space constraints and similarly to the case studied by Bouchard et al. (2010), the *viability domain* is not given a priori but is implicitly determined by the auxiliary value function  $v$  defining a stochastic target problem. The paper is thus an extension of the work done by Bouchard et al. (2010) to the case where there is both an obstacle constraint holding almost surely (i.e.  $Y^{t,z,\nu} \geq 0$  on  $[t, T]$ ) and a European constraint in probability.

The aim of the present work is to characterize  $V$  and the results build on the following observations (we refer the reader to Section 3 for a detailed informal discussion on the characterization of  $V$ ). We first observe that for  $p = 1$  the constraint at the terminal date holds almost surely while for  $p \leq p_{\min}(\cdot)$  with  $p_{\min}(\cdot) := \sup\{p \in$

$[0, 1]$  s.t.  $v(\cdot, p) = v_2(\cdot)$ , where  $v_2(\cdot) := \inf\{y \geq 0 : \mathcal{U}_{\cdot, y} \neq \emptyset\}$ , the initial problem reduces to a problem of maximization under the constraint that  $Y^{t, z, \nu} \geq 0$   $\mathbb{P}$ -a.s. on  $[t, T]$ . To avoid degenerate cases, we assume that  $p_{\min}(t, \cdot) < 1, t < T$ . We remark that  $p_{\min}(T, \cdot) = \{0, 1\}$  which may raise additional discontinuity problems of the value function when  $t \rightarrow T$ . We then observe that if  $(t, z, p) \in \{(t, z, p) \in [0, T) \times \mathcal{O}_+^d \times \mathbb{R}^+ \times [0, 1] : p_{\min}(t, x) < p < 1, y > v(t, x, p)\}$ , the constraint is not binding and  $V$  should be a (discontinuous) viscosity solution of the standard Hamilton-Jacobi-Bellman equation. On the contrary, assuming that  $v$  is smooth on its domain and that  $(t, z, p) \in \{(t, z, p) \in [0, T) \times \mathcal{O}_+^d \times \mathbb{R}^+ \times [0, 1] : p_{\min}(t, x) < p < 1, y = v(t, x, p)\}$ , any admissible control should satisfy  $dY^{t, z, \nu} \geq dv(\cdot, X^{t, x, \nu}, P^{t, p, \alpha})$  for the processes not to exit the domain  $\mathcal{D}$ . The value function  $V$  is therefore expected to solve in a (discontinuous) viscosity sense a Hamilton-Jacobi-Bellman equation where the controls are constrained to satisfy the previous condition. Interestingly and similarly to [Bouchard et al. \(2010\)](#), we will see that no *a priori* assumptions have to hold on the coefficients at the boundary for the process  $(Z, P)$  to stay within the domain  $\mathcal{D}$ . This condition is already integrated into the auxiliary value function  $v$  via its characterization derived by [Soner & Touzi \(2002b\)](#) and [Bouchard et al. \(2009\)](#) (see Theorem 3.1). This is the main difference with the papers quoted previously.

At this stage we can underline the technical difficulties we are facing.

- (1) We are dealing with a pair of controls that are not valued in a compact set. Therefore the operators involved in the characterization of  $V$  are discontinuous and we will work with their lower and upper semi-continuous envelopes (see [Crandall et al. \(1992\)](#)). Moreover, for technical reasons related to the proof, we will have to impose some regularity conditions on  $v$  and some additional assumptions on the coefficients of the diffusions.
- (2) The introduction of the variable  $P^{t, p, \alpha}$  leads to boundary conditions at  $p = 1$  and  $p \leq p_{\min}(\cdot)$ . However there may be discontinuities at those points and we shall derive those conditions in a viscosity sense which is not trivial (see the Discussion in Section 3).

The characterization of  $V$  for  $p_{\min}(\cdot) < p < 1$  will follow from the arguments in [Bouchard et al. \(2010\)](#). However, as mentioned above, the characterization of  $V$  for  $p \leq p_{\min}(\cdot)$  and  $p = 1$  is not trivial and is therefore our main result. This answers, in particular, a question raised by [Bouchard et al. \(2010\)](#) (see Remark 5.2). We also provide examples for which comparison results exist for the partial differential equation (PDE) solved by  $V$  on the interior of the domain. The last point is critical for

the construction of a numerical scheme and is not trivial as the operators involved in the PDE characterization are nonlinear and not continuous (see (1) above). A deeper analysis of the PDEs involved (existence of regular solutions, comparison results under a more general framework, ...) is however left for further research. This paper therefore provides new conceptual insights for the characterization of stochastic control problems under *weak* stochastic target constraints.

We would like to add that a few authors worked in a non-Markovian setting on utility maximization problems under a risk constraint. For instance, [Boyle & Tian \(2007\)](#) considered a complete financial market and derived the optimal solution for an investor who wanted to maximize his expected utility under a probabilistic constraint. In their setting, the utility function satisfies some regularity assumptions which do not hold in our case. Moreover, the threshold they use (given in our case by the function  $g$ ) is independent of  $(t, x)$  and positive, ensuring the continuity of  $t \in [0, T] \mapsto p_{\min}(t, x), x \in \mathcal{O}_+^d$ . Then, [Gundel & Weber \(2007\)](#) considered, in an incomplete market framework, a similar problem under an expected shortfall constraint including all coherent risk measures. [De Franco & Tankov \(2011\)](#) extended the previous result to the case where the utility function is applied to positive gains only while the risk measure is applied to negative shortfall. They provided a full solution under a complete market setting. This paper actually extends the previous results to the Markovian case and, similarly to [Boyle & Tian \(2007\)](#), considers a risk measure involving an irregular non-convex loss function opening the way to the study of different types of expected shortfall constraints as stated below.

Several extensions can be studied. Indeed (1.6)-(1.9) open the door to the study of any type of constraints as long as they can be incorporated in an auxiliary value function satisfying some regularity assumptions, as it will be made clear in this paper. For instance, our results can be extended to the case of a moment constraint prevailing on a set of discrete times (see [Bouchard et al. \(2016\)](#)), by applying the American version of the dynamic programming principle (see [Bouchard & Vu \(2010\)](#)). Finally one can consider extending the results to the case where the controls are valued in a compact subset  $U$  of  $\mathbb{R}^d$ . In the latter case the difficulty will arise from the possible emptiness of the admissible set of controls allowing the processes to revert inside the continuation region once they reach the boundary.

The rest of the paper is organized as follows: in Section 2 we formally state the problem and rigourously prove (1.8), while in Section 3 we derive the PDE characterization of  $V$  on its domain.

**Notations.** We let  $d \geq 1$  be an integer. Any vector  $x$  of  $\mathbb{R}^d$  is viewed as a column

vector unless otherwise stated. We denote by  $|x|$  the Euclidean norm of  $x$ , and by  $x^\top$  its transpose. We define  $|x|_1^k := \sum_{i=1}^d |x_i|^k$ ,  $k \geq 1$ . The notation  $\mathbb{M}^d$  (resp.  $\mathbb{M}^{d*}$ ) denotes the set of  $d$ -dimensional square matrices whose each element belongs to  $\mathbb{R}$  (resp.  $\mathbb{R} \setminus \{0\}$ ), and  $\mathbb{S}^d$  is the subset of elements of  $\mathbb{M}^d$  that are symmetric. We set  $M^\top$  the transpose of  $M \in \mathbb{M}^d$ , while  $|M|$  and  $\text{Tr}[M]$  are respectively its Euclidean norm and trace. Similarly we define  $|M|_1^k := \sum_{i,j=1}^d |M_{i,j}|^k$ ,  $k \geq 1$ . We fix a finite time horizon  $T > 0$ . We consider a smooth function  $\psi : (t, x, p) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \mapsto \psi(t, x, p)$ . We denote by  $\partial_t \psi$  its derivative with respect to  $t$  and by  $D\psi$  its Jacobian matrix with respect to the space variables whose rows are given by  $D_x \psi$  and  $D_p \psi$ , i.e. the derivative with respect to  $x$  and  $p$ . The Hessian matrix with respect to the space variables is  $D^2 \psi$  whose elements are given by  $D_{xx} \psi, D_{pp} \psi, D_{xp} \psi, D_{px} \psi$ , i.e. the second derivative with respect to  $x$  and  $p$  and the cross derivatives. We also denote by  $B_r(t, x)$  (resp.  $B_r(x)$ ) the open ball of radius  $r > 0$  centered at  $(t, x) \in [0, T] \times \mathbb{R}^d$  (resp.  $x \in \mathbb{R}^d$ ). We write  $\partial B_r(x)$  the spatial boundary of  $B_r(x)$  and  $\text{cl}(B_r(x))$  its closure. Besides we denote  $\mathcal{C} := [0, T] \times \mathcal{O}_+^d \times \mathbb{R}^+ \times [0, 1]$  and we define  $\text{cl}(\mathcal{C}) := [0, T] \times \mathcal{O}_+^d \times \mathbb{R}^+ \times [0, 1]$  and  $\partial_T(\text{cl}(\mathcal{C})) := \{T\} \times \mathcal{O}_+^d \times \mathbb{R}^+ \times [0, 1]$ . Moreover we also write  $\mathcal{D} := [0, T] \times \mathcal{O}_+^d \times [0, 1]$  and similarly  $\text{cl}(\mathcal{D}) := [0, T] \times \mathcal{O}_+^d \times [0, 1]$  and  $\partial_T(\text{cl}(\mathcal{D})) := \{T\} \times \mathcal{O}_+^d \times [0, 1]$ . Finally, we denote by  $\mathcal{S}_d$  the sphere of  $\mathbb{R}^d$  of radius one and by  $\mathcal{D}_d$  the set of vectors  $\beta \in \mathcal{S}_d$  such that their first component  $\beta^1 = 0$ . As usual, the abbreviation 's.t.' stands for 'such that'. In this paper, the constant  $C > 0$  is generic unless otherwise stated. All over the paper, inequalities between random variables have to be understood in the  $\mathbb{P}$ -a.s. sense.

## 2 Problem Statement and Problem Reduction

### 2.1 Problem statement

In the sequel we consider  $\Omega$ , the space of  $\mathbb{R}^d$ -valued continuous functions  $(\omega_t)_{t \leq T}$  on  $[0, T]$ ,  $d \geq 1$ , endowed with the Wiener measure  $\mathbb{P}$ . We define  $W$  the coordinate mapping, i.e.  $(W(\omega)_t)_{t \leq T}$  for  $\omega \in \Omega$  so that  $W$  is a  $d$ -dimensional Brownian motion on the canonical filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ . In the latter  $\mathcal{F}$  is the Borel tribe of  $\Omega$  and  $\mathbb{F} := \{\mathcal{F}_t, 0 \leq t \leq T\}$  is the  $\mathbb{P}$ -augmentation of the filtration generated by  $W$ . We let  $\mathcal{U}$  be the set of progressively measurable processes in  $L^2([0, T] \times \Omega)$  with values in  $\mathbb{R}^d$ . For  $t \in [0, T]$ ,  $z := (x, y) \in \mathcal{O}_+^d \times \mathbb{R}$ , with  $\mathcal{O}_+^d := (0, \infty)^d$ , and for  $\nu \in \mathcal{U}$ , the controlled process  $Z^{t,z,\nu} := (X^{t,x,\nu}, Y^{t,z,\nu})$  is the strong solution to the following stochastic differential equations (SDEs)

$$X_s^{t,x,\nu} = x + \int_t^s \mu_X(r, X_r^{t,x,\nu}, \nu_r) dr + \int_t^s \sigma_X(r, X_r^{t,x,\nu}, \nu_r) dW_r \text{ on } \mathcal{O}_+^d, \quad (2.1)$$

$$Y_s^{t,z,\nu} = y + \int_t^s \mu_Y(r, Z_r^{t,z,\nu}, \nu_r) dr + \int_t^s \sigma_Y^\top(r, Z_r^{t,z,\nu}, \nu_r) dW_r \text{ on } \mathbb{R}, \quad (2.2)$$

where  $(\mu_X, \sigma_X) : (t, x, u) \in [0, T] \times \mathcal{O}_+^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{M}^d$ ,  $(\mu_Y, \sigma_Y) : (t, z, u) \in [0, T] \times \mathcal{O}_+^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R} \times \mathbb{R}^d$  are, in particular, Lipschitz continuous functions. The coefficients  $\mu_X$  and  $\sigma_X$  are supposed to be such that  $X_s^{t,x,\nu} \in \mathcal{O}_+^d$ ,  $s \in [t, T]$  whenever the original data lies in  $\mathcal{O}_+^d$ . In what follows we consider  $\mu_Z : [0, T] \times \mathcal{O}_+^d \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}^{d+1}$  and  $\sigma_Z : [0, T] \times \mathcal{O}_+^d \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{M}^{d+1,d}$ , where  $\mathbb{M}^{d+1,d}$  is the matrix of size  $(d+1) \times d$ , given by

$$\mu_Z(t, z, u) := \begin{pmatrix} \mu_X(t, x, u) \\ \mu_Y(t, z, u) \end{pmatrix}, \quad \sigma_Z(t, z, u) := \begin{pmatrix} \sigma_X(t, x, u) \\ \sigma_Y^\top(t, z, u) \end{pmatrix}. \quad (2.3)$$

For  $(t, p) \in [0, T] \times [0, 1]$ , we will appeal to  $\mathcal{A}_{t,p}$ , the set of  $\mathbb{R}^d$ -valued  $\mathbb{F}$ -progressively measurable and square integrable processes  $\alpha$  such that

$$P_T^{t,p,\alpha} := p + \int_t^T \alpha_s^\top dW_s \in [0, 1]. \quad (2.4)$$

Moreover, for  $(t, z) \in [0, T] \times \mathcal{O}_+^d \times \mathbb{R}^+$ , we will refer to the set

$$\mathcal{U}_{t,z} := \{\nu \in \mathcal{U} : \text{s.t. } Y^{t,z,\nu} \geq 0 \text{ on } [t, T]\}. \quad (2.5)$$

We now introduce two locally bounded Borel-measurable maps  $f : \mathcal{O}_+^d \times \mathbb{R} \mapsto \mathbb{R}$  and  $g : [0, T] \times \mathcal{O}_+^d \mapsto \mathbb{R}^+$ . More precisely we assume that  $f$  satisfies a polynomial growth (for the below expectation to be well defined for any  $\nu \in \mathcal{U}$ ) and that the function  $y \mapsto f(x, y)$  is non-decreasing. The objective of the fund manager on  $\text{cl}(\mathcal{E})$  is to solve

$$V(t, z, p) := \sup_{\nu \in \mathcal{U}_{t,z,p}} \mathbb{E}[f(Z_T^{t,z,\nu})], \quad (2.6)$$

where

$$\mathcal{U}_{t,z,p} := \left\{ \nu \in \mathcal{U}_{t,z} : \mathbb{P}[Y_T^{t,z,\nu} \geq g(T, X_T^{t,x,\nu})] \geq p \right\}. \quad (2.7)$$

**Remark 2.1.** It follows from (2.7) that for  $y \geq y'$ ,  $\mathcal{U}_{t,x,y,p} \supset \mathcal{U}_{t,x,y',p}$ .

## 2.2 Problem reduction

We will prove here that (2.6) can be formulated with a constraint holding almost surely over time.

Similarly to Bouchard et al. (2009) or Bouchard et al. (2016) one can prove that for all  $(t, z, p) \in \text{cl}(\mathcal{C})$ ,  $\mathcal{U}_{t,z,p} = \bar{\mathcal{U}}_{t,z,p}$  with

$$\bar{\mathcal{U}}_{t,z,p} := \left\{ (\nu, \alpha) \in \mathcal{U}_{t,z} \times \mathcal{A}_{t,p} \text{ s.t. } Y_T^{t,z,\nu} \geq g(T, X_T^{t,x,\nu}) \mathbf{1}_{\{P_T^{t,p,\alpha} > 0\}} \right\}, \quad (2.8)$$

with  $\mathcal{A}_{t,p}$  and  $P^{t,p,\alpha}$  defined in (2.4). Note that in particular for  $(t, z, p) \in \partial_T(\text{cl}(\mathcal{C}))$ ,  $V(T, z, p) = f(z) \mathbf{1}_{\{y \geq g(T, x) \mathbf{1}_{\{p > 0\}}\}} - \infty \mathbf{1}_{\{y < g(T, x) \mathbf{1}_{\{p > 0\}}\}}$ .

Let us introduce the set

$$\mathcal{C} := \{(t, z, p) \in \text{cl}(\mathcal{C}) \text{ s.t. } \mathcal{U}_{t,z,p} \neq \emptyset\}. \quad (2.9)$$

We also define for all  $(t, x, p) \in \text{cl}(\mathcal{D})$  the auxiliary value function

$$v(t, x, p) := \inf \{y \geq 0 \text{ s.t. } (t, z, p) \in \mathcal{C}\}, \quad (2.10)$$

characterizing the closure of  $\mathcal{C}$  by Remark 2.1. The following theorem holds.

**Theorem 2.1.** *For any  $(t, z, p) \in \mathcal{C}$ ,  $(\nu, \alpha) \in \mathcal{U} \times \mathcal{A}_{t,p}$ , and  $[t, T]$ -valued stopping time  $\theta$ , we have:*

- (1) *if  $Y_\theta^{t,z,\nu} > v(\theta, X_\theta^{t,x,\nu}, P_\theta^{t,p,\alpha})$   $\mathbb{P}$ -a.s. then there exists a control  $(\tilde{\nu}, \tilde{\alpha}) \in \bar{\mathcal{U}}_{t,z,p}$  such that  $\nu = \tilde{\nu}$  and  $\alpha = \tilde{\alpha}$  on  $[t, \theta]$ ;*
- (2) *if there exists a control  $(\tilde{\nu}, \tilde{\alpha}) \in \bar{\mathcal{U}}_{t,z,p}$  such that  $\nu = \tilde{\nu}$  and  $\alpha = \tilde{\alpha}$  on  $[t, \theta]$ , then  $Y_\theta^{t,z,\nu} \geq v(\theta, X_\theta^{t,x,\nu}, P_\theta^{t,p,\alpha})$   $\mathbb{P}$ -a.s.*

**Proof.** Appealing to the arguments in Soner & Touzi (2002a), one can prove that for any  $(t, z, p) \in \mathcal{C}$ ,  $(\nu, \alpha) \in \mathcal{U} \times \mathcal{A}_{t,p}$  and  $[t, T]$ -valued stopping time  $\theta$ , we have the subsequent equivalence

$$\exists (\tilde{\nu}, \tilde{\alpha}) \in \bar{\mathcal{U}}_{t,z,p} \text{ s.t. } \nu = \tilde{\nu} \text{ and } \alpha = \tilde{\alpha} \text{ on } [t, \theta] \Leftrightarrow (X_\theta^{t,x,\nu}, Y_\theta^{t,z,\nu}, P_\theta^{t,p,\alpha}) \in \mathcal{C} \text{ } \mathbb{P} - \text{a.s.} \quad (2.11)$$

Theorem 2.1 is thus a direct consequence of what precedes.

**Remark 2.2.** *In the condition (1) of the above theorem a strict inequality appears as we cannot ensure that the infimum is always achieved. However in the setting described in Remark 2.6 (2) below we have  $v \neq \infty$  on  $\text{cl}(\mathcal{D})$ .*



We define on  $[0, T] \times \mathcal{O}_+^d$ ,

$$p_{\min}(t, x) := \sup \{p \in [0, 1] : v(t, x, p) = v_2(t, x)\} , \quad (2.12)$$

where

$$v_2(t, x) := \inf \{y \geq 0 \text{ s.t. } \mathcal{U}_{t,z} \neq \emptyset\} . \quad (2.13)$$

**Remark 2.3.** *As already noticed by [Bouchard et al. \(2016\)](#) we have on  $[0, T] \times \mathcal{O}_+^d \times \mathbb{R}^+$  and for  $p \leq p_{\min}(\cdot)$ ,  $V(\cdot, p) = V(\cdot, p_{\min}(\cdot))$ .*

**Standing Assumption 1** On  $(t, x) \in [0, T) \times \mathcal{O}_+^d$ , we assume that  $p_{\min}$  is continuous and that  $p_{\min}(\cdot) < 1$ .

**Remark 2.4.** *The previous assumption is satisfied, for example, in the setting of Assumption 3.1.2 below and when  $g$  is the payoff function of a straddle option (in this case  $p_{\min} \equiv 0$  on  $[0, T) \times \mathcal{O}_+^d$ ).*

**Remark 2.5.** *Note that here  $p_{\min}(T, \cdot)$  equals 0 or 1 by definition. Therefore the function  $t \mapsto p_{\min}(t, x), x \in \mathcal{O}_+^d$  may not be continuous at  $T$ .*

From the epigraph of  $p_{\min}$ ,  $\mathcal{E}(p_{\min}) := \{(t, x, p) \in \mathcal{D} \text{ s.t. } p_{\min}(t, x) \leq p\}$  we derive different sets

$$\begin{cases} \mathcal{E}^{\text{int}}(p_{\min}) := \{(t, x, p) \in \mathcal{D} \text{ s.t. } p_{\min}(t, x) < p < 1\} \\ \mathcal{E}_T^{\text{int}}(p_{\min}) := \{(t, x, p) \in \partial_T(\text{cl}(\mathcal{D})) \text{ s.t. } p_{\min}(t, x) < p < 1\} . \\ \mathcal{E}^c(p_{\min}) := \{(t, x, p) \in \text{cl}(\mathcal{D}) \text{ s.t. } p \leq p_{\min}(t, x), p < 1\} \end{cases} \quad (2.14)$$

Observe that  $\mathcal{E}_T^{\text{int}}(p_{\min}) = \emptyset$  when  $p_{\min}(T, \cdot) = 1$ . We shall also work under the following standing assumption.

**Standing Assumption 2** The auxiliary value function  $v$  is strictly convex in its  $p$ -variable on  $\mathcal{E}^{\text{int}}(p_{\min})$ , finite and continuous on  $\text{cl}(\mathcal{D})$ . We also assume that  $v$  is  $C^{1,2,2}(\mathcal{E}^{\text{int}}(p_{\min}))$ .

**Remark 2.6.** (1) *The previous assumption is key to characterize  $V$  on the space boundary of the domain (see Remark 3.7).*

(2) *This assumption holds in the Black-Scholes setting in [Föllmer & Leukert \(1999\)](#) and when  $g$  is, for example, the payoff function of a straddle option. Indeed appealing to the techniques developed by [Föllmer & Leukert \(1999\)](#) one can prove that the value function writes as a combination of call and put options as well as binary options. We can therefore obtain a closed-form solution for  $v$ .*

A direct consequence of Theorem 2.1, the finiteness of  $v$  and its continuity is the following corollary.

**Corollary 2.1.** *For any  $(t, z, p) \in \text{cl}(\mathcal{C})$ ,*

$$V(t, z, p) := \sup_{\nu \in \tilde{\mathcal{U}}_{t,z,p}} \mathbb{E}[f(Z_T^{t,z,\nu})], \quad (2.15)$$

with

$$\tilde{\mathcal{U}}_{t,z,p} := \{(\nu, \alpha) \in \mathcal{U} \times \mathcal{A}_{t,p} \text{ s.t. } Y_s^{t,z,\nu} \geq v(s, X_s^{t,x,\nu}, P_s^{t,p,\alpha}) \mathbb{P} - a.s. \forall s \in [t, T]\}. \quad (2.16)$$

As already explained in the introduction, the main difference with standard state-space constraint problems comes from the set  $\mathcal{C}$  which is not provided a priori but defined implicitly via an auxiliary value function defining a stochastic target problem.

It follows from the continuity of  $v$  on its domain and the continuity of  $p_{\min}(\cdot)$  on  $[0, T] \times \mathcal{O}_+^d$ , that the closure of  $\mathcal{C}$ , denoted  $\text{cl}(\mathcal{C})$ , reads

$$\text{cl}(\mathcal{C}) := \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3, \quad (2.17)$$

where

$$\mathcal{D}_1 := \text{int}(\mathcal{D}_1) \cup \partial_Z \mathcal{D}_1 \cup \partial_T \mathcal{D}_1, \quad (2.18)$$

and

$$\left\{ \begin{array}{l} \text{int}(\mathcal{D}_1) := \{(t, z, p) \text{ s.t. } (t, x, p) \in \mathcal{E}^{\text{int}}(p_{\min}) : y > v(t, x, p)\} \\ \partial_Z \mathcal{D}_1 := \{(t, z, p) \text{ s.t. } (t, x, p) \in \mathcal{E}^{\text{int}}(p_{\min}) : y = v(t, x, p)\} \\ \partial_T \mathcal{D}_1 := \{(t, z, p) \text{ s.t. } (t, x, p) \in \mathcal{E}_T^{\text{int}}(p_{\min}) : y \geq v(t, x, p)\} \\ \mathcal{D}_2 := \{(t, z, p) \text{ s.t. } (t, x, p) \in \mathcal{E}^c(p_{\min}) : y \geq v(t, x, p)\} \\ \mathcal{D}_3 := \{(t, z, p) \in [0, T] \times \mathcal{O}_+^d \times \mathbb{R}^+ \times \{1\} : y \geq v(t, x, p)\} \end{array} \right., \quad (2.19)$$

with the definitions (2.14). Observe that, as  $y \geq v$  in the above sets, we obviously have  $y \in \mathbb{R}^+$ . Moreover we have  $\partial_T \mathcal{D}_1 = \emptyset$  if  $p_{\min}(T, \cdot) = 1$ .

The aim of the subsequent subsections is to characterize  $V$ . The construction of a numerical scheme requires at least some comparison results which are not trivial here as the operators involved in the PDE characterization are nonlinear and not continuous. As mentioned earlier, we can provide examples for which comparison results exist for the PDE solved by  $V$  on the interior of the domain (see Theorem 3.2). More generally, the results below imply that we can first solve (3.23) to compute  $v$  and thus  $\mathcal{C}$ , before

solving the Hamilton-Jacobi-Bellman equations (3.24)-(3.27) on  $\mathcal{D}_1$  with the boundary conditions  $V(t, z, p) = V_2(t, z)$  on  $\mathcal{D}_2$  and  $V(t, z, 1) = V_1(t, z)$  on  $\mathcal{D}_3$  (see Section 3.2.1 and Section 3.2.2 and the definitions in (3.14) and (3.16)). Note that  $V_i(\cdot)$ ,  $i = \{1, 2\}$  has already been characterized by Bouchard et al. (2010).

### 3 Viscosity Characterization of the Value Function

We start this section with a notation that will be used in the rest of this paper.

We denote for a smooth function  $\varphi(t, z, p)$  defined on  $[0, T] \times \mathcal{O}_+^d \times \mathbb{R} \times [0, 1]$ , the Dynkin operator in  $(Z, P)$  for the control  $(u, a) \in \mathbb{R}^{2d}$  as,

$$\begin{aligned} \mathcal{L}_{(Z,P)}^{(u,a)}(\cdot, \partial_t \varphi(\cdot, p), D\varphi(\cdot, p), D^2\varphi(\cdot, p))(t, z) \\ := \partial_t \varphi(t, z, p) + \mu_Z^\top(t, z, u) D_z \varphi(t, z, p) + \frac{1}{2} \text{Tr}[\sigma_Z \sigma_Z^\top(t, z, u) D_{zz} \varphi(t, z, p)] \\ + \frac{1}{2} |a|^2 D_{pp} \varphi(t, z, p) + a^\top \sigma_Z^\top(t, z, u) D_{zp} \varphi(t, z, p). \end{aligned} \quad (3.1)$$

We will write  $\mathcal{L}_{(Z,P)}^{(u,a)} \varphi(t, z, p)$  for  $\mathcal{L}_{(Z,P)}^{(u,a)}(\cdot, \partial_t \varphi(\cdot, p), D\varphi(\cdot, p), D^2\varphi(\cdot, p))(t, z)$ . We define accordingly  $\mathcal{L}_Z^u \varphi(t, z, p)$ . Moreover, for a smooth function  $\psi(t, x, p)$ ,  $\varphi(t, x)$ , or  $\phi(t, z)$ , respectively defined on  $[0, T] \times \mathcal{O}_+^d \times [0, 1]$ ,  $[0, T] \times \mathcal{O}_+^d$  and  $[0, T] \times \mathcal{O}_+^d \times \mathbb{R}$ , the Dynkin operators  $\mathcal{L}_{(X,P)}^{(u,a)} \psi(t, x, p)$ ,  $\mathcal{L}_X^u \varphi(t, x)$  and  $\mathcal{L}_Z^u \phi(t, z)$  are defined similarly.

Then we note that as the value function  $V$  may not be smooth we will provide a PDE characterization in a viscosity sense and use the following relaxation

$$V_*(t, z, p) := \liminf_{(t', z', p') \in \text{int}(\mathcal{D}_1) \rightarrow (t, z, p)} V(t', z', p'), \quad (3.2)$$

$$\text{and } V^*(t, z, p) := \limsup_{(t', z', p') \in \text{int}(\mathcal{D}_1) \rightarrow (t, z, p)} V(t', z', p'). \quad (3.3)$$

We define similarly the upper/lower semi-continuous envelopes of  $f$  and  $V_i$ ,  $i \in \{1, 2\}$ , defined in (3.14) and (3.16) below.

In the same spirit as the discussion led by Bouchard et al. (2010), we start an informal description of what happens on each part of the domain.

#### Discussion

**On  $\text{int}(\mathcal{D}_1)$ .**

Theorem 2.1 implies that the constraint is not binding on  $\text{int}(\mathcal{D}_1)$ . The function  $V$  should solve in a viscosity sense the standard Hamilton-Jacobi-Bellman equation

$$-\partial_t \varphi(t, z, p) + H(t, z, D\varphi(t, z, p), D^2\varphi(t, z, p)) = 0, \quad (3.4)$$

where, for  $(t, z, q) \in [0, T] \times \mathcal{O}_+^d \times \mathbb{R} \times \mathbb{R}^{d+2}$ , with  $q := (q^z, q^p)^\top$  where  $(q^z, q^p) \in \mathbb{R}^{d+1} \times \mathbb{R}$ , and for  $A \in \mathbb{S}^{d+2}$ , with  $A := \begin{pmatrix} A^{zz} & A^{zp} \\ A^{zp^\top} & A^{pp} \end{pmatrix}$  where  $A^{zz} \in \mathbb{S}^{d+1}$  and  $(A^{zp}, A^{pp}) \in \mathbb{R}^{d+1} \times \mathbb{R}$ , we have

$$H(t, z, q, A) := \inf_{(u, a) \in \mathbb{R}^{2d}} H^{(u, a)}(t, z, q, A), \quad (3.5)$$

with

$$\begin{aligned} H^{(u, a)}(t, z, q, A) &:= -\mu_Z^\top(t, z, u)q^z - \frac{1}{2} \text{Tr}[\sigma_Z \sigma_Z^\top(t, z, u)A^{zz}] \\ &\quad - \frac{1}{2}|a|^2 A^{pp} - a^\top \sigma_Z^\top(t, z, u)A^{zp}. \end{aligned} \quad (3.6)$$

However as  $(u, a) \in \mathbb{R}^{2d}$ , the operator  $H$  may not be continuous and we may have to use the upper/lower semi-continuous version of the above operator defined as

$$H^*(t, z, q, A) = \limsup_{(t', z', q', A') \rightarrow (t, z, q, A)} \inf_{(u, a) \in \mathbb{R}^{2d}} H^{(u, a)}(t', z', q', A'), \quad (3.7)$$

for the upper semi-continuous envelope, the lower semi-continuous one being defined similarly.

**On  $\partial_T \mathcal{D}_1$  if  $\partial_T \mathcal{D}_1 \neq \emptyset$ .**

One could naturally presume, from Theorem A.1, that  $V_*(T, z, p) \geq f_*(z)$  and  $V^*(T, z, p) \leq f^*(z)$ . We will however have to deal with the possibly unbounded set of controls.

**On  $\partial_Z \mathcal{D}_1$ .**

From Theorem 2.1 we know that the process  $Z^{t, z, \nu}$  should not cross  $\partial_Z \mathcal{D}_1$  as long as  $\nu \in \mathcal{U}_{t, z, p}$ . Therefore, considering (2.19), when  $y \rightarrow v(t, x, p)$  any admissible controls should be such that  $dY^{t, z, \nu} \geq dv(\cdot, X^{t, x, \nu}, P^{t, p, \alpha})$ . Thus using Itô's formula and a standard comparison result we deduce that the admissible controls  $(\nu, \alpha)$  should satisfy

$$N^{(\nu_t, \alpha_t)}(t, z, Dv(t, x, p)) = 0 \quad \text{and} \quad \mu_Y(t, z, \nu_t) - \mathcal{L}_{(X, P)}^{(\nu_t, \alpha_t)} v(t, x, p) \geq 0, \quad (3.8)$$

where for  $(t, z, q) \in [0, T] \times \mathcal{O}_+^d \times \mathbb{R} \times \mathbb{R}^{d+1}$ ,  $q := (q^x, q^p)^\top$ , and  $(u, a) \in \mathbb{R}^{2d}$ ,

$$N^{(u, a)}(t, z, q) := \sigma_Y(t, z, u) - \sigma_X^\top(t, x, u)q^x - aq^p. \quad (3.9)$$

As a result,  $V$  should satisfy

$$-\partial_t \varphi(t, z, p) + H_{\text{int}}(t, z, p, D\varphi(t, z, p), D^2\varphi(t, z, p)) = 0, \quad (3.10)$$

where for  $(t, z, p, q) \in [0, T] \times \mathcal{O}_+^d \times \mathbb{R} \times [0, 1] \times \mathbb{R}^{d+2}$ , and  $A \in \mathbb{S}^{d+2}$ ,

$$H_{\text{int}}(t, z, p, q, A) := \inf_{(u, a) \in \hat{U}_{\text{int}}(t, z, p)} H^{(u, a)}(t, z, q, A), \quad (3.11)$$

with

$$\hat{U}_{\text{int}}(t, z, p) := \left\{ (u, a) \in \mathbb{R}^{2d} : \begin{array}{l} N^{(u, a)}(t, z, Dv(t, x, p)) = 0 \\ \text{and } \mu_Y(t, z, u) - \mathcal{L}_{(X, P)}^{(u, a)} v(t, x, p) \geq 0 \end{array} \right\}. \quad (3.12)$$

The set  $\hat{U}_{\text{int}}$  corresponds to controls preventing the process from exiting the domain.

**Remark 3.1.** (1) Obviously  $H_{\text{int}} \geq H$  since  $\hat{U}_{\text{int}} \subset \mathbb{R}^{2d}$ .

(2) Condition (3.8) can be explained as follows. Let us define the process  $\Delta_s := Y_s^{t, z, \nu} - v(s, X_s^{t, x, \nu}, P_s^{t, p, \alpha})$  on  $[0, T]$  for some  $(t, z, p, \nu, \alpha) \in [0, T] \times \mathcal{O}_+^d \times \mathbb{R}^+ \times (0, 1) \times \mathcal{U} \times \mathcal{A}_{t, p}$ . Using Itô's Lemma we obtain

$$\begin{aligned} \Delta_s &= \Delta_t + \int_t^s \left\{ \mu_Y(r, Z_r^{t, z, \nu}, \nu_r) - \mathcal{L}_{(X, P)}^{(\nu, \alpha)} v(r, X_r^{t, x, \nu}, P_r^{t, p, \alpha}) \right\} dr \\ &\quad + \int_t^s N^{(\nu, \alpha)}(r, Z_r^{t, z, \nu}, Dv(r, X_r^{t, x, \nu}, P_r^{t, p, \alpha})) dW_r. \end{aligned} \quad (3.13)$$

Now we define a stopping time  $\tau := \inf\{s \geq t : \Delta_s = 0\}$  together with  $\tau_\varepsilon := \inf\{s \geq t : \Delta_s = -\varepsilon\}$  for a fixed  $\varepsilon > 0$ . We have  $\tau < \tau_\varepsilon$ . The condition  $\Delta_\tau = 0$  together with (3.8) imply that  $\Delta_{\tau_\varepsilon} \geq 0$ . As a consequence  $\tau_\varepsilon = \infty$ . The latter combined with the arbitrariness of  $\varepsilon$  leads to  $\Delta_t \geq 0, \forall t \in [0, T]$ .

On  $\mathcal{D}_2 \cup \mathcal{D}_3$ .

We define on  $[0, T] \times \mathcal{O}_+^d \times \mathbb{R}^+$

$$V_1(t, z) := \sup_{\nu \in \mathcal{U}_{t, z}^1} \mathbb{E}[f(Z_T^{t, z, \nu})], \quad (3.14)$$

where

$$\mathcal{U}_{t, z}^1 := \left\{ \nu \in \mathcal{U}_{t, z}, Y_T^{t, z, \nu} \geq g(T, X_T^{t, x, \nu}) \mathbb{P} - \text{a.s.} \right\}, \quad (3.15)$$

and

$$V_2(t, z) = \sup_{\nu \in \mathcal{U}_{t, z}} \mathbb{E}[f(Z_T^{t, z, \nu})]. \quad (3.16)$$

Following Remark 2.3 we have on  $[0, T] \times \mathcal{O}_+^d \times \mathbb{R}^+$ ,

$$V(\cdot, p) = V_2(\cdot), \forall 0 \leq p \leq p_{\min}(\cdot), \quad (3.17)$$

and for all  $p \in [p_{\min}(\cdot), 1]$  we obtain using the non-increasing property of  $V$  in  $p$ ,

$$V_1(\cdot) = V(\cdot, 1) \leq V(\cdot, p) \leq V(\cdot, p_{\min}(\cdot)) = V_2(\cdot). \quad (3.18)$$

Hence

$$V_{1*}(\cdot) \leq V_*(\cdot, 1) \leq V^*(\cdot, p_{\min}(\cdot)) \leq V_2^*(\cdot), \quad (3.19)$$

and one can expect

$$V_*(\cdot, 1) = V_{1*}(\cdot) \quad \text{and} \quad V^*(\cdot, p_{\min}(\cdot)) = V_2^*(\cdot). \quad (3.20)$$

However the function  $V$  may have discontinuities at  $p = p_{\min}(\cdot)$  and  $p = 1$  and the boundary has to be stated in a weak form (see Section 3.2). This corresponds to classical state-space constraint problems (see e.g. Barles (1994) and Fleming & Soner (2006)).

It follows from the previous discussion that we need to characterize  $V$  on  $\mathcal{D}_1$  together with  $V^*(\cdot, p_{\min}(\cdot))$  and  $V_*(\cdot, 1)$ . While the characterization of  $V$  on  $\mathcal{D}_1$  will be a consequence of the work done by Bouchard et al. (2010) (see Theorem 3.1, Remark 6.2 & Lemma 6.3), the characterization of  $V^*(\cdot, p_{\min}(\cdot))$  and  $V_*(\cdot, 1)$  will be more involved and will be our main result. We also provide a comparison result for the PDE solved by  $V$  on  $\text{int}(\mathcal{D}_1)$ .

To alleviate the notations we will write on  $\text{cl}(\mathcal{C})$ ,

$$\mathbf{H}^* \varphi(t, z, p) \quad \text{for} \quad \mathbf{H}^*(t, z, \mathbf{D}\varphi(t, z, p), \mathbf{D}^2\varphi(t, z, p)), \quad (3.21)$$

and similarly for the operators  $\mathbf{H}_*$ ,  $\mathbf{F}^\beta$  defined in the appendix in (2.1), as well as for  $\mathbf{H}_{\text{int}}\varphi$ ,  $\bar{\mathbf{H}}^*\varphi$  and  $\bar{\mathbf{H}}_*\varphi$  defined below in (3.40).

### 3.1 Viscosity characterization of the value function on $\mathcal{D}_1$

We intend in this section to prove the characterization of  $V$  on  $\mathcal{D}_1$ . As already mentioned before, we will see that it is actually a consequence of the work done by Bouchard et al. (2010) (see Theorem 3.1, Remark 6.2 & Lemma 6.3). However we need the following standing assumption.

**Standing Assumption 3** There exists a locally Lipschitz map  $\check{u} : [0, T] \times \mathcal{O}_+^d \times (0, \infty) \times \mathbb{R}^{d+1} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  such that

$$\mathcal{N}(t, x, y, q) := \{(u, a) \in \mathbb{R}^{2d} : \mathbf{N}^{(u, a)}(t, x, y, q) = 0\} = \{(\check{u}(t, x, y, q, a), a), a \in \mathbb{R}^d\}, \quad (3.22)$$

(recall (3.9)).

**Remark 3.2.** *This assumption is satisfied in the setting of Assumption 3.1.2 below when  $\sigma^{-1}$  is Lipschitz continuous in  $x$ .*

Now it is proved by both [Soner & Touzi \(2002b\)](#) and [Bouchard et al. \(2009\)](#) that  $v$  is a solution to (3.23) below, as a consequence of the GDP principle derived by [Soner & Touzi \(2002a,b\)](#). More precisely it is based on Theorem 2.1. Therefore  $\hat{U}_{\text{int}}(t, z, p) \neq \emptyset$  and the conditions in (3.8) are satisfied by  $v$ . The latter result is key to characterize  $V$  on  $\partial_Z \mathcal{D}_1$  (recall the discussion above) and is thus involved in the proof of Proposition 3.1 below. We recall that, by assumption,  $v$  is smooth on  $\mathcal{E}^{\text{int}}(p_{\min})$ .

**Theorem 3.1** (see [Soner & Touzi \(2002b\)](#), [Bouchard et al. \(2009\)](#)). *The function  $v$  satisfies on  $\mathcal{E}^{\text{int}}(p_{\min})$ ,*

$$\sup_{(u,a) \in \mathcal{N}(\cdot, v, Dv)} \left\{ \mu_Y(\cdot, v, u) - \mathcal{L}_{(X,P)}^{(u,a)} v \right\} = \mu_Y(\cdot, v, \hat{u}) - \mathcal{L}_{(X,P)}^{(\hat{u}, \hat{a})} v = 0, \quad (3.23)$$

where  $\hat{a} := \hat{a}(t, x, p) := \check{a}(\cdot, v(\cdot, p), Dv(\cdot, p), D^2v(\cdot, p))(t, x)$  and with  $\hat{u} := \hat{u}(t, x, p) := \check{u}(\cdot, v(\cdot, p), Dv(\cdot, p), \hat{a}(\cdot, p))(t, x)$  defined in (3.22). In particular  $\hat{a}$  is unique thanks to the strict convexity of  $v$  on the domain.

**Remark 3.3.** (1) *Observe that  $v > 0$  on  $\mathcal{E}^{\text{int}}(p_{\min})$  (see (2.12)). This condition is important to prove that the sub-solution property holds in the above theorem (see [Bouchard et al. \(2009\)](#)).*

(2) *The characterization (3.23) implies that the process  $(Z, P)$  actually stays at the boundary of the viability domain given by  $\mathcal{C}$ .*

The subsequent assumption is employed in the proof of (3.24) on  $\partial_Z \mathcal{D}_1$ .

**Assumption 3.1.1.** We assume that  $\check{a}(\cdot)$  defined in Theorem 3.1 is locally Lipschitz on its domain.

**Remark 3.4.** *The previous assumption holds in the setting of Assumption 3.1.2 below when  $\sigma^{-1}$  is Lipschitz continuous in  $x$ .*

The next proposition states the PDE characterization of  $V$  on  $\mathcal{D}_1$ . We will only state the result as the proof follows the same arguments as the ones in [Bouchard et al. \(2010\)](#), Theorem 3.1, Remark 6.2 & Lemma 6.3.

**Proposition 3.1.** *The following characterization holds.*

*Consider  $(\hat{u}, \hat{a})$  defined in Theorem 3.1 and let Assumption 3.1.1 hold. Then the function  $V_*$  is a viscosity super-solution of*

$$\begin{cases} (-\partial_t \varphi + H^* \varphi)(t, z, p) \geq 0 & \text{if } (t, z, p) \in \text{int}(\mathcal{D}_1) \\ (-\mathcal{L}_{(Z,P)}^{(\hat{u}, \hat{a})} \varphi)(t, z, p) \geq 0 & \text{if } (t, z, p) \in \partial_Z \mathcal{D}_1 \end{cases}. \quad (3.24)$$

Moreover for  $(t, z, p) \in \partial_T \mathcal{D}_1 \neq \emptyset$  and  $y > v(T, x, p)$  (resp.  $y = v(T, x, p)$ ),

$$\varphi(z, p) \geq f_*(z) \text{ if } H^* \varphi(z, p) < \infty, \quad (3.25)$$

(resp.

$$\varphi(z, p) \geq f_*(z) \text{ if } \limsup_{(t', x', p') \rightarrow (T, x, p), t' < T} \max \{ |\hat{u}(t', x', p')|, |\hat{a}(t', x', p')| \} < \infty. \quad (3.26)$$

)

The function  $V^*$  is a viscosity sub-solution of

$$\begin{cases} (-\partial_t \varphi + H_* \varphi)(t, z, p) \leq 0 & \text{if } (t, z, p) \in \text{int}(\mathcal{D}_1) \cup \partial_Z \mathcal{D}_1 \\ \varphi(z, p) \leq f^*(z) & \text{if } (t, z, p) \in \partial_T \mathcal{D}_1 \neq \emptyset, H_* \varphi(z, p) > -\infty \end{cases}. \quad (3.27)$$

**Remark 3.5.** (1) Observe that  $\partial_T \mathcal{D}_1 \neq \emptyset$  when  $p_{\min}(T, \cdot) = 0$ .

(2) Assume that the admissible controls  $\nu$  are valued in a compact subset  $U \in \mathbb{R}^d$ . Then, using the Lipschitz continuity of  $\mu_Z$  and  $\sigma_Z$ , we can verify that for all sequences  $(t_n, z_n, p_n, \nu_n)_n \in \text{int}(\mathcal{D}_1) \times \mathcal{U}_{t_n, z_n, p_n}$  such that  $(t_n, z_n, p_n)_n \rightarrow (T, z, p) \in \partial_T \mathcal{D}_1 \neq \emptyset$ , we have  $Z_T^{t_n, z_n, \nu_n} \rightarrow z$  in  $L^2(\Omega)$ . Appealing to the definition of  $V$  and the polynomial growth of  $f$ , we finally obtain that  $\liminf_{n \rightarrow \infty} V(t_n, z_n, p_n) \geq f_*(z)$ .

**Remark 3.6.** Similarly to [Bouchard et al. \(2010\)](#), Section 3.1, the sub-solution property is expressed above with  $H_*$  on  $\partial_Z \mathcal{D}_1$  while one would expect to have it stated with  $H_{\text{int}}$  as for the super-solution. This is because as  $H_{\text{int}} \geq H_*$  the operator

$$\mathcal{H}\varphi(t, z, p) := \begin{cases} H_* \varphi(t, z, p) & \text{if } (t, z, p) \in \text{int}(\mathcal{D}_1) \\ H_{\text{int}} \varphi(t, z, p) = (-\mathcal{L}_{(Z, P)}^{(\hat{u}, \hat{a})} \varphi)(t, z, p) & \text{if } (t, z, p) \in \partial_Z \mathcal{D}_1 \end{cases}, \quad (3.28)$$

may not be lower semi-continuous. However, according to [Crandall et al. \(1992\)](#) the sub-solution property must be expressed in terms of the lower semi-continuous envelope of  $\mathcal{H}$  implying that it cannot be a priori stated in terms of  $H_{\text{int}}$ .

**Remark 3.7.** If  $v$  were not smooth (but only continuous on  $\text{cl}(\mathcal{D})$ ) one would expect to obtain a characterization using a relaxed version of the operator  $H_{\text{int}}$  on  $\partial_Z \mathcal{D}_1$  (recall the definition of  $H_{\text{int}}$  in (3.11)). This would be needed to ensure that the required property on the admissible controls holds in a neighborhood of the optimum. For the super-solution property the latter would be given, for  $(t, z, p, q, A) \in \partial_Z \mathcal{D}_1 \times \mathbb{R}^{d+2} \times \mathbb{S}^{d+2}$  such that  $\min_{\text{cl}(\mathcal{C})} (V_* - \varphi)(t, z, p) = (V_* - \varphi)(t, z, p) = 0$ , with  $\varphi$  a smooth function on  $\text{cl}(\mathcal{C})$ , and



for  $\phi$  a smooth function on  $\text{cl}(\mathcal{D})$  such that  $\max_{\text{cl}(\mathcal{D})}(v - \phi)(t, x, p) = (v - \phi)(t, x, p) = 0$ , by

$$F^{\phi*}(t, z, p, q, A) := \limsup_{\substack{(t', z', p', q', A') \rightarrow (t, z, p, q, A) \\ (\delta', \gamma') \downarrow 0}} F_{\delta', \gamma'}^{\phi}(t', z', p', q', A'), \quad (3.29)$$

with

$$F_{\delta, \gamma}^{\phi}(t, z, p, q, A) := \inf_{(u, a) \in \hat{U}_{\delta, \gamma}^{\phi}(t, z, p)} H^{(u, a)}(t, z, q, A), \quad (3.30)$$

where

$$\hat{U}_{\delta, \gamma}^{\phi}(t, z, p) := \left\{ (u, a) \in \mathcal{N}_{\delta}(\cdot, y, D\phi(\cdot, p))(t, x) : \mu_Y(t, z, u) - \mathcal{L}_{(X, P)}^{(u, a)}\phi(t, x, p) \geq \gamma \right\}, \quad (3.31)$$

with

$$\mathcal{N}_{\delta}(\cdot, y, D\phi(\cdot, p))(t, x) := \left\{ (u, a) : |\mathbf{N}^{(u, a)}(\cdot, y, D\phi(\cdot, p))(t, x)| \leq \delta \right\}. \quad (3.32)$$

However we cannot prove the existence of a test function satisfying (3.31). Indeed, for instance, we observe with Theorem 3.1 that the conditions in (3.31) do not hold in the smooth case.

We can now state a comparison result that holds under the following assumption and which is proved in B.

**Assumption 3.1.2.** We consider the existence of two Lipschitz continuous functions  $\mu : x \in \mathcal{O}_+^d \mapsto \mu(x) \in \mathbb{R}^d$  and  $\sigma : x \in \mathcal{O}_+^d \mapsto \sigma(x) \in \mathbb{R} \setminus \{0\}$  such that the coefficients for the diffusion of the process  $Z$  are given by:  $\mu_X : (t, x, u) \in [0, T] \times \mathcal{O}_+^d \times \mathbb{R}^d \mapsto \text{diag}[x]\mu(x) \in \mathbb{R}^d$ ,  $\sigma_X : (t, x, u) \in [0, T] \times \mathcal{O}_+^d \times \mathbb{R}^d \mapsto \sigma(x)\text{diag}[x] \in \mathbb{M}^{d*}$ ,  $\mu_Y : (t, z, u) \in [0, T] \times \mathcal{O}_+^d \times \mathbb{R} \times \mathbb{R}^d \mapsto yu^\top \mu(x) \in \mathbb{R}$ , and  $\sigma_Y : (t, z, u) \in [0, T] \times \mathcal{O}_+^d \times \mathbb{R} \times \mathbb{R}^d \mapsto y\sigma(x)u \in \mathbb{R}^d$ . In particular, for all  $x \in \mathcal{O}_+^d$ ,

$$\underline{\Lambda}^\sigma \leq |\sigma(x)| \leq \bar{\Lambda}^\sigma \text{ and } \underline{\Lambda}^\mu \leq |\mu(x)|_1 \leq \bar{\Lambda}^\mu, \quad (3.33)$$

with  $\underline{\Lambda}^\sigma > 0$  and  $\underline{\Lambda}^\mu \geq 0$  and  $\bar{\Lambda}^\mu \leq (\underline{\Lambda}^\sigma)^2$ .

**Remark 3.8.** We observe that the Black-Scholes setting in Föllmer & Leukert (1999) (and therefore the setting described in Remark 2.6 (2)) is a particular case of the above assumption when the coefficients  $\mu$  and  $\sigma$  are constant.

**Theorem 3.2.** *Let Assumption 3.1.2 hold. Let  $V$  (resp.  $U$ ) be a lower semi-continuous (resp. upper semi-continuous) map satisfying a polynomial growth of order  $k \geq 1$  in the  $x$ -variable and of order  $0 < \bar{m} < m$ , with  $m$  defined in Lemma B.1, in the  $y$ -variable on  $[0, T] \times \mathcal{O}_+^d \times \mathbb{R}^+ \times [0, 1]$ . Moreover assume that,*

(1) *on  $\text{int}(\mathcal{D}_1)$ ,  $U$  is a viscosity sub-solution of (3.27) and  $V$  is a viscosity super-solution of (3.24),*

(2) *on  $\partial_Z \mathcal{D}_1 \cup \partial_T \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$ ,  $V \geq U$ ,*

*then  $V \geq U$  on  $\text{cl}(\mathcal{C})$ .*

**Remark 3.9.** *Under the assumptions of Theorem 3.5 and Theorem 3.6 we have  $V_* \geq V^*$  on  $\partial_T \mathcal{D}_1 \cup \mathcal{D}_2 \setminus \{(t, z, p) \in \{T\} \times \mathcal{O}_+^d \times \mathbb{R}^+ \times [0, 1], p \leq p_{\min}(t, x), p < 1 \text{ and } y \geq v(t, x, p)\} \cup \mathcal{D}_3$ .*

### 3.2 Viscosity characterization of the value function on $\mathcal{D}_2 \cup \mathcal{D}_3$

We recall  $V_1$  and  $V_2$  defined in (3.14) and (3.16). The main results of this section (see Theorem 3.5 and Theorem 3.6) show that the natural boundary conditions in (3.20) indeed hold true whenever Assumption 3.2.1 and Assumption 3.2.2 are in force and under some additional assumptions.

#### 3.2.1 Characterization of $V(\cdot, 1)$

The proof of Theorem 3.5 uses the characterization of  $V_1$  already derived by Bouchard et al. (2010). The first part of this section is thus devoted to recall this result.

#### Characterization of $V_1$ : summary of the existing results

We introduce the set

$$\mathcal{P}_1 := \left\{ (t, z) \in [0, T] \times \mathcal{O}_+^d \times \mathbb{R}^+ \text{ s.t. } \mathcal{U}_{t,z}^1 \neq \emptyset \right\}, \quad (3.34)$$

with  $\mathcal{U}_{t,z}^1$  defined in (3.15), and we define for all  $(t, x) \in [0, T] \times \mathcal{O}_+^d$  the auxiliary value function

$$v_1(t, x) := \inf \{ y \geq 0 \text{ s.t. } (t, z) \in \mathcal{P}_1 \}, \quad (3.35)$$

which characterizes the closure of  $\mathcal{P}_1$ .

We state the following standing assumption.

**Standing Assumption 4** The auxiliary value function  $v_1$  is finite and continuous on  $[0, T] \times \mathcal{O}_+^d$  and  $C^{1,2}([0, T] \times \mathcal{O}_+^d)$ .

**Remark 3.10.** (1) *This assumption holds in the setting described in Remark 2.6 (2).*

(2) *Observe that  $v_1 > 0$  on  $[0, T) \times \mathcal{O}_+^d$  as, by assumption,  $p_{\min}(\cdot) < 1$  on  $[0, T) \times \mathcal{O}_+^d$ .*

It is proved by both [Soner & Touzi \(2002b\)](#) and [Bouchard et al. \(2009\)](#) that  $v_1$  is a solution of (3.36), as a consequence of the GDP principle in [Soner & Touzi \(2002a,b\)](#). This is an important result that is involved in the proof of Theorem 3.5.

**Theorem 3.3** (see [Soner & Touzi \(2002b\)](#), [Bouchard et al. \(2009\)](#)). *The function  $v_1$  satisfies on  $[0, T) \times \mathcal{O}_+^d$ ,*

$$\mu_Y(\cdot, v_1, \hat{u}_1) - \mathcal{L}_X^{\hat{u}_1} v_1 = 0, \quad (3.36)$$

where  $\hat{u}_1 := \check{u}(\cdot, v_1, Dv_1, 0)(t, x)$  (recall (3.22)).

In particular  $\hat{v}_{1_s} := \check{u}(\cdot, v_1, Dv_1, 0)(s, X_s^{t,x,\hat{v}_1})$ ,  $s \geq t$ , corresponds to the control preventing the process from exiting the domain as it respectively cancels out the stochastic term of  $Y^{t,x,v_1,\nu} - v_1(\cdot, X^{t,x,\nu})$ ,  $(t, x) \in [0, T) \times \mathcal{O}_+^d$  while making the corresponding drift term non-negative (see Theorem 3.3).

We then define the closure of  $\mathcal{P}_1$ , denoted  $\text{cl}(\mathcal{P}_1)$ , as  $\text{int}(\mathcal{P}_1) \cup \partial_Z \mathcal{P}_1 \cup \partial_T \mathcal{P}_1$ , with

$$\begin{cases} \text{int}(\mathcal{P}_1) := \{(t, z) \in [0, T) \times \mathcal{O}_+^d \times \mathbb{R}^+ : y > v_1(t, x)\} \\ \partial_Z \mathcal{P}_1 := \{(t, z) \in [0, T) \times \mathcal{O}_+^d \times \mathbb{R}^+ : y = v_1(t, x)\} \\ \partial_T \mathcal{P}_1 := \{(t, z) \in \{T\} \times \mathcal{O}_+^d \times \mathbb{R}^+ : y \geq v_1(t, x)\} \end{cases} \quad (3.37)$$

We now introduce, for  $(t, z, q) \in [0, T] \times \mathcal{O}_+^d \times \mathbb{R} \times \mathbb{R}^{d+1}$  and  $A \in \mathbb{S}^{d+1}$ , the following operators

$$\bar{H}(t, z, q, A) := \inf_{u \in \mathbb{R}^d} \bar{H}^u(t, z, q, A), \quad (3.38)$$

and

$$\bar{H}^u(t, z, q, A) := -\mu_Z^\top(t, z, u)q - \frac{1}{2} \text{Tr}[\sigma_Z \sigma_Z^\top(t, z, u)A], \quad (3.39)$$

together with

$$\bar{H}^*(t, z, q, A) = \limsup_{(t', z', q', A') \rightarrow (t, z, q, A)} \inf_{u \in \mathbb{R}^d} \bar{H}^u(t', z', q', A'). \quad (3.40)$$

The lower semi-continuous envelope is defined similarly. [Bouchard et al. \(2010\)](#) proved the following PDE characterization on  $\text{cl}(\mathcal{P}_1)$  of  $V_1$ .

**Theorem 3.4.** (Bouchard et al. 2010, Theorem 3.1, Lemma 6.3) *The function  $V_{1*}$  is a viscosity super-solution on  $\text{cl}(\mathcal{P}_1)$  of*

$$\begin{cases} (-\partial_t \varphi + \bar{H}^* \varphi)(t, z) \geq 0 & \text{if } (t, z) \in \text{int}(\mathcal{P}_1) \\ (-\mathcal{L}_Z^{\hat{u}_1} \varphi)(t, z) \geq 0 & \text{if } (t, z) \in \partial_Z \mathcal{P}_1 \\ \varphi(z) \geq f_*(z) & \text{if } (t, z) \in \partial_T \mathcal{P}_1, \bar{H}^* \varphi(z) < \infty, \\ & \text{and if } \limsup_{(t', x') \rightarrow (T, x), t' < T} |\hat{u}_1(t', x')| < \infty \end{cases}, \quad (3.41)$$

where  $\hat{u}_1$  is defined in Theorem 3.3.

The function  $V_1^*$  is a viscosity sub-solution on  $\text{cl}(\mathcal{P}_1)$  of

$$\begin{cases} (-\partial_t \varphi + \bar{H}_* \varphi)(t, z) \leq 0 & \text{if } (t, z) \in \text{int}(\mathcal{P}_1) \cup \partial_Z \mathcal{P}_1 \\ \varphi(z) \leq f^*(z) & \text{if } (t, z) \in \partial_T \mathcal{P}_1, \bar{H}_* \varphi(z) > -\infty \end{cases}. \quad (3.42)$$

We can now work towards the PDE characterization of  $V(\cdot, 1)$ .

**Characterization of  $V(\cdot, 1)$**  We make the subsequent comparison assumption that will be used to link  $V^*(\cdot, 1)$  to  $V_1$ .

*Assumption 3.2.1.* There is a class of functions  $\mathcal{K}_1$  containing all functions lower bounded by  $V_{1*}(\cdot)$  such that for every

- (1)  $k_1 \in \mathcal{K}_1$ , lower semi-continuous viscosity super-solution of the system (3.41) on  $\text{cl}(\mathcal{P}_1)$ ,
- (2)  $k_2 \in \mathcal{K}_1$ , upper semi-continuous viscosity sub-solution of the system (3.42) on  $\text{cl}(\mathcal{P}_1)$ ,

we have  $k_1 \geq k_2$  on  $\text{cl}(\mathcal{P}_1)$ .

**Remark 3.11.** *This assumption holds in the setting of Assumption 3.1.2 (see B and Bouchard et al. (2010), Corollary 3.1) and when the function  $f$  is continuous and non-negative.*

**Theorem 3.5** (Main Result - Characterization of  $V(\cdot, 1)$ ). *Assume that  $V^*$  has a polynomial growth. Then the function  $V^*(\cdot, 1)$  is a viscosity sub-solution on  $\text{cl}(\mathcal{P}_1)$  of (3.42). If in addition Assumption 3.2.1 holds, then  $V_*(\cdot, 1) = V^*(\cdot, 1) = V_{1*}(\cdot) = V_1^*(\cdot)$  on  $\text{cl}(\mathcal{P}_1)$ .*

**Proof.** As argued by Bouchard et al. (2009), in order to prove that  $V^*(\cdot, 1)$  is a viscosity sub-solution of (3.42) we prove that  $V^*(\cdot, 1)$  is a viscosity sub-solution of

$$\min \{V^*(\cdot, 1) - V_1^*(\cdot), -\partial_t V^*(\cdot, 1) + \bar{H}_* V^*(\cdot, 1)\} \leq 0 \text{ on } \text{int}(\mathcal{P}_1) \cup \partial_Z \mathcal{P}_1, \quad (3.43)$$

and that  $V^*(T, \cdot, 1)$  is a viscosity sub-solution of

$$\min \left\{ V^*(T, \cdot, 1) - V_1^*(T, \cdot), (V^*(T, \cdot, 1) - f^*(\cdot)) \mathbf{1}_{\{\bar{H}_* V^*(T, \cdot, 1) > -\infty\}} \right\} \leq 0 \text{ on } \partial_T \mathcal{P}_1. \quad (3.44)$$

Indeed, we can prove, for example, that (3.43) implies the first inequality in (3.42). Let  $(t_0, z_0)$  be a local maximizer of  $V^*(\cdot, 1) - \varphi$ , with  $\varphi$  a smooth function. Then:

- (1) either  $V^*(\cdot, 1) > V_1^*(\cdot)$  and the first inequality in (3.42) is valid for  $\varphi$  at  $(t_0, z_0)$ ,
- (2) or  $V^*(\cdot, 1) = V_1^*(\cdot)$ , implying that  $(t_0, z_0)$  is a local maximizer of  $V_1^*(\cdot) - \varphi$ , and the first inequality in (3.42) is valid for  $\varphi$  at  $(t_0, z_0)$  by Theorem 3.4.

**Step 1.** We first prove that for any smooth function  $\varphi$  and  $(t_0, z_0, 1) \in \mathcal{D}_3$ ,  $t_0 < T$  such that

$$(\text{strict}) \max_{\text{cl}(\mathcal{C})} (V^* - \varphi) = (V^* - \varphi)(t_0, z_0, 1) = 0, \quad (3.45)$$

we have

$$\min \{ V^*(t_0, z_0, 1) - V_1^*(t_0, z_0), (-\partial_t \varphi + H_* \varphi)(t_0, z_0, 1) \} \leq 0. \quad (3.46)$$

We appeal to the arguments in Bouchard et al. (2010) (see Section 6.2.2), except that we have to handle the state constraint for  $p = 1$ . We report the entire argument. Indeed we assume on the contrary that

$$\min \{ V^*(t_0, z_0, 1) - V_1^*(t_0, z_0), (-\partial_t \varphi + H_* \varphi)(t_0, z_0, 1) \} > 0. \quad (3.47)$$

Since the coefficients  $\mu_Z$  and  $\sigma_Z$  are continuous we can find  $\varepsilon > 0$  such that for all  $(u, a) \in \mathbb{R}^{2d}$ ,

$$\min \left\{ \varphi(t, z, p) - V_1(t, z), -\mathcal{L}_{(Z, P)}^{(u, a)} \varphi(t, z, p) \right\} \geq 0 \text{ on } \mathcal{O}, \quad (3.48)$$

where

$$\mathcal{O} := \left\{ (t, z, p) \in [0, T) \times \mathcal{O}_+^d \times \mathbb{R}^+ \times (0, 1] \text{ s.t. } (t, z, p) \in B_\varepsilon(t_0, z_0) \times [1 - \varepsilon, 1] \right\}. \quad (3.49)$$

Let  $(t_n, z_n, p_n)_n$  be a sequence in  $\mathcal{O} \cap \text{int}(\mathcal{D}_1)$  such that  $V(t_n, z_n, p_n) \rightarrow V^*(t_0, z_0, 1)$  and  $(t_n, z_n, p_n) \rightarrow (t_0, z_0, 1)$ . For each  $n$ , as  $(t_n, z_n, p_n) \in \text{int}(\mathcal{D}_1) \subset \mathcal{C}$ , we can find a control  $(\nu_n, \alpha_n) \in \bar{\mathcal{U}}_{t_n, z_n, p_n}$ . We set  $(Z^n, P^n) := (Z^{t_n, z_n, \nu_n}, P^{t_n, p_n, \alpha_n})$  and we define

$$\theta_n := \inf \{ s \geq t_n : (s, Z_s^n, P_s^n) \notin \mathcal{O} \cap (\text{int}(\mathcal{D}_1) \cup \partial_Z \mathcal{D}_1) \}. \quad (3.50)$$

Thanks to the definition of  $\bar{\mathcal{U}}_{t_n, z_n, p_n}$  and Theorem 2.1 we have  $(s, Z_s^n, P_s^n) \in (\text{int}(\mathcal{D}_1) \cup \partial_Z \mathcal{D}_1)$  on  $[t_n, T)$  and thus  $(\theta_n, Z_{\theta_n}^n, P_{\theta_n}^n) \in \partial_p \mathcal{O}$ , where

$$\partial_p \mathcal{O} := (\partial_p B_\varepsilon(t_0, z_0) \times [1 - \varepsilon, 1]) \cup (B_\varepsilon(t_0, z_0) \times \{1 - \varepsilon, 1\}), \quad (3.51)$$

with  $\partial_p B_\varepsilon(t_0, z_0) := (\{t_0 + \varepsilon\} \times \text{cl}(B_\varepsilon(z_0))) \cup ([t_0, t_0 + \varepsilon) \times \partial B_\varepsilon(z_0))$ , the parabolic boundary of  $B_\varepsilon(t_0, z_0)$ . Applying Itô's Lemma and using (3.48) we therefore obtain that for  $n$  large enough

$$\varphi(t_n, z_n, p_n) \geq \mathbb{E}[\varphi(\theta_n, Z_{\theta_n}^n, P_{\theta_n}^n)] \geq \mathbb{E}[V^*(\theta_n, Z_{\theta_n}^n, P_{\theta_n}^n)] + \xi, \quad (3.52)$$

where, from (3.45) and (3.48),  $\xi$  is such that

$$-\xi := \sup_{\partial_p \mathcal{O}} (V^* - \varphi) < 0. \quad (3.53)$$

Finally as  $(V - \varphi)(t_n, z_n, p_n) \rightarrow (V^* - \varphi)(t_0, z_0, 1) = 0$  as  $n \rightarrow \infty$  we obtain a contradiction to Theorem A.1 for  $n$  large enough.

**Step 2.** We now prove that for any smooth function  $\varphi$  and  $(t_0, z_0) \in \text{int}(\mathcal{P}_1) \cup \partial_Z \mathcal{P}_1$  such that

$$(\text{strict}) \max_{\text{cl}(\mathcal{P}_1)} (V^*(\cdot, 1) - \varphi) = (V^*(\cdot, 1) - \varphi)(t_0, z_0) = 0, \quad (3.54)$$

we have

$$\min\{V^*(t_0, z_0, 1) - V_1^*(t_0, z_0), (-\partial_t \varphi + \bar{H}_* \varphi)(t_0, z_0)\} \leq 0. \quad (3.55)$$

To this aim, similarly to Bouchard et al. (2009) (see Section 6), we introduce on  $\text{cl}(\mathcal{C})$ , for every  $k > 0$ , a new test function

$$\varphi_k(t, z, p) := \varphi(t, z) + |x - x_0|^{2m} + |y - y_0|^{2m} + (t - t_0)^2 + kh(p), \quad (3.56)$$

for some  $m > 0$  to be chosen later and where

$$h(p) := \rho \int_p^1 \frac{1}{e^2 - e^r} dr, \quad \rho > 0. \quad (3.57)$$

Since  $V^*$  has a polynomial growth, for  $m \geq 2$  large enough,  $V^* - \varphi_k$  admits a local maximizer  $(t_k, z_k, p_k)$  on  $\text{cl}(\mathcal{C})$ . We remark that by definition of  $(t_k, z_k, p_k)$  and  $(t_0, z_0)$  we obtain

$$\begin{aligned} 0 &= (V^*(\cdot, 1) - \varphi)(t_0, z_0) \\ &= (V^* - \varphi_k)(t_0, z_0, 1) \\ &\leq (V^* - \varphi_k)(t_k, z_k, p_k) \\ &= (V^*(\cdot, p_k) - \varphi)(t_k, z_k) - |x_k - x_0|^{2m} - |y_k - y_0|^{2m} - (t_k - t_0)^2 - kh(p_k) \\ &\leq (V^*(\cdot, p_k) - \varphi)(t_k, z_k) - |x_k - x_0|^{2m} - |y_k - y_0|^{2m} - (t_k - t_0)^2 - k\bar{h}(k), \end{aligned} \quad (3.58)$$

where the last inequality follows from

$$h(p_k) = - \int_{p_k}^1 D_p h(r) dr = \int_{p_k}^1 \frac{\rho}{e^2 - e^p} dr \geq \int_{p_k}^1 \frac{\rho}{e^2 - 1} dr = \frac{\rho(1 - p_k)}{e^2 - 1} =: \bar{h}(k). \quad (3.59)$$

Since  $V^*$  has a polynomial growth this implies that the sequence  $(t_k, z_k, p_k)$  is bounded and thus converges to some  $(t_*, z_*, p_*)$  up to a subsequence. Obviously  $p_* = 1$  or else  $k(1 - p_k) \rightarrow \infty$ . We thus have by definition of  $(t_0, z_0)$ ,

$$\begin{aligned} 0 &= (V^*(\cdot, 1) - \varphi)(t_0, z_0) \\ &\leq \limsup_{k \rightarrow \infty} (V^* - \varphi_k)(t_k, z_k, p_k) \\ &= (V^*(\cdot, 1) - \varphi)(t_*, z_*) - |x_* - x_0|^{2m} - |y_* - y_0|^{2m} - (t_* - t_0)^2 + \limsup_{k \rightarrow \infty} (-k\bar{h}(k)) \\ &\leq (V^*(\cdot, 1) - \varphi)(t_0, z_0) - |x_* - x_0|^{2m} - |y_* - y_0|^{2m} - (t_* - t_0)^2 + \limsup_{k \rightarrow \infty} (-k\bar{h}(k)) \\ &\leq (V^*(\cdot, 1) - \varphi)(t_0, z_0). \end{aligned} \quad (3.60)$$

After possibly passing to a subsequence we thus obtain

$$(t_k, z_k, p_k) \rightarrow (t_0, z_0, 1), \quad k(1 - p_k) \rightarrow 0, \quad V^*(t_k, z_k, p_k) \rightarrow V^*(t_0, z_0, 1), \quad (3.61)$$

and  $t_k < T$ ,  $p_k > p_{\min}(t_k, x_k)$ . We assume

$$V^*(t_0, z_0, 1) - V_1^*(t_0, z_0) > 0, \quad (3.62)$$

and we aim at proving that

$$(-\partial_t \varphi + \bar{H}_* \varphi)(t_0, z_0) \leq 0. \quad (3.63)$$

We deduce from (3.61)-(3.62) that the sequence  $(t_k, z_k, p_k)_k$  of maximizer of  $V^* - \varphi_k$  is such that  $V^*(t_k, z_k, p_k) - V_1^*(t_k, z_k) > 0$ , after possibly passing to a subsequence. Thanks to Step 1. and (3.27) we conclude that

$$(-\partial_t \varphi_k + H_* \varphi_k)(t_k, z_k, p_k) \leq 0. \quad (3.64)$$

Moreover it follows from (3.61) that

$$(\partial_t \varphi_k, D_z \varphi_k, D_{zz} \varphi_k)(t_k, z_k, p_k) \rightarrow (\partial_t \varphi, D\varphi, D^2 \varphi)(t_0, z_0) \text{ as } k \rightarrow \infty, \quad (3.65)$$

$$(D_p \varphi_k, D_{zp} \varphi_k, D_{pp} \varphi_k)(t_k, z_k, p_k) = (kD_p h(p_k), 0, kD_{pp} h(p_k)) \text{ for every } k > 1. \quad (3.66)$$

Thanks to the definition of  $H_*$  we can find some sequences  $(t_l^k, z_l^k, p_l^k) \in [0, T] \times \mathcal{O}_+^d \times \mathbb{R}^+ \times (0, 1]$  where  $p_l^k > p_{\min}(t_l^k, x_l^k)$ , a vector  $q_l^k := (q_l^{k,z}, q_l^{k,p})^\top$  where  $(q_l^{k,z}, q_l^{k,p}) \in \mathbb{R}^{d+2}$ ,

a symmetric matrix  $A_l^k \in \mathbb{S}^{d+2}$  with  $A_l^k := \begin{pmatrix} A_l^{k,zz} & A_l^{k,zp} \\ A_l^{k,zp^\top} & A_l^{k,pp} \end{pmatrix}$  where  $A_l^{k,zz} \in \mathbb{S}^{d+1}$  and  $(A_l^{k,zp}, A_l^{k,pp}) \in \mathbb{R}^{d+1} \times \mathbb{R}$ , and a minimizing sequence  $(u_l^k, a_l^k) \in \mathbb{R}^{2d}$  such that

$$(t_l^k, z_l^k, p_l^k) \rightarrow (t_k, z_k, p_k) \quad \text{and} \quad \left| (q_l^k, A_l^k) - (D\varphi_k, D^2\varphi_k)(t_k, z_k, p_k) \right|_1 \leq l^{-1}, \quad (3.67)$$

and

$$-\partial_t \varphi_k(t_k, z_k, p_k) + H^{(u_l^k, a_l^k)}(t_l^k, z_l^k, q_l^k, A_l^k) \leq 2l^{-1}. \quad (3.68)$$

Now using (3.66)-(3.67) we can find  $C > 0$  such that

$$\begin{aligned} & -\mathcal{L}_Z^{u_l^k}(t_l^k, z_l^k, \partial_t \varphi_k(\cdot), D\varphi_k(\cdot), D^2\varphi_k(\cdot))(t_k, z_k, p_k) \\ & \leq Cl^{-1}(1 + |\mu_Z(t_l^k, z_l^k, u_l^k)|_1 + |\sigma_Z(t_l^k, z_l^k, u_l^k)|^2 + |a_l^k|^2) + k\frac{1}{2}|a_l^k|^2 D_{pp}h(p_k). \end{aligned} \quad (3.69)$$

Taking the infimum over  $u \in \mathbb{R}^d$  on both sides we obtain

$$\begin{aligned} & \inf_{u \in \mathbb{R}^d} \left\{ -\mathcal{L}_Z^u(t_l^k, z_l^k, \partial_t \varphi_k(\cdot), D\varphi_k(\cdot), D^2\varphi_k(\cdot))(t_k, z_k, p_k) \right\} \\ & \leq \mathfrak{A} + Cl^{-1} + |a_l^k|^2(Cl^{-1} + k\frac{1}{2}D_{pp}h(p_k)), \end{aligned} \quad (3.70)$$

with  $\mathfrak{A} := Cl^{-1} \times \inf_{u \in \mathbb{R}^d} \{ |\mu_Z(t_l^k, z_l^k, u)|_1 + |\sigma_Z(t_l^k, z_l^k, u)|^2 \}$ .

As  $kD_{pp}h(p) = \frac{-k\rho e^p}{[e^p - e^2]^2} \rightarrow -\infty$  when  $k \rightarrow \infty$ , we obtain that for  $k$  large enough

$$-\partial_t \varphi_k(t_k, z_k, p_k) + \lim_{l \rightarrow \infty} \inf_{u \in \mathbb{R}^d} \left\{ \begin{aligned} & -\mu_Z^\top(t_l^k, z_l^k, u) D_z \varphi_k(t_k, z_k, p_k) \\ & -\frac{1}{2} \text{Tr}[\sigma_Z \sigma_Z^\top(t_l^k, z_l^k, u) D_{zz} \varphi_k(t_k, z_k, p_k)] \end{aligned} \right\} \leq 0, \quad (3.71)$$

leading, with (3.65), to

$$-\partial_t \varphi(t_0, z_0) + \lim_{(l,k) \rightarrow \infty} \inf_{u \in \mathbb{R}^d} \left\{ \begin{aligned} & -\mu_Z^\top(t_l^k, z_l^k, u) D_z \varphi_k(t_k, z_k, p_k) \\ & -\frac{1}{2} \text{Tr}[\sigma_Z \sigma_Z^\top(t_l^k, z_l^k, u) D_{zz} \varphi_k(t_k, z_k, p_k)] \end{aligned} \right\} \leq 0. \quad (3.72)$$

Therefore, after using (3.61), (3.65) and (3.67), we finally get

$$(-\partial_t \varphi + \bar{H}_* \varphi)(t_0, z_0) \leq 0. \quad (3.73)$$

**Step 3.** We now prove that for any smooth function  $\varphi$  and  $(T, z_0, 1) \in \mathcal{D}_3$  where  $(z_0, 1)$  is such that

$$(\text{strict}) \max_{(z,p) \text{ s.t. } (T,z,p) \in \text{cl}(\mathcal{C})} (V^*(T, \cdot) - \varphi) = (V^*(T, \cdot) - \varphi)(z_0, 1) = 0, \quad (3.74)$$



we have

$$\min \{V^*(T, z_0, 1) - V_1^*(T, z_0), (V^*(T, z_0, 1) - f^*(z_0))\mathbf{1}_{\{H_*\varphi(z_0, 1) > -\infty\}}\} \leq 0. \quad (3.75)$$

The proof is standard but we provide it for the sake of completeness (see e.g. [Bouchard et al. \(2010\)](#), Section 6.2.3). We argue by contradiction and assume that for  $V^*(T, z_0, 1) > V_1^*(T, z_0)$  and  $H_*\varphi(z_0, 1) > -\infty$  we have  $V^*(T, z_0, 1) > f^*(z_0)$ . As a consequence we can find  $(r, \eta) > 0$  such that

$$\varphi \geq f^* + \eta \text{ on } \{T\} \times \mathcal{O}, \quad (3.76)$$

with  $\mathcal{O} := B_r(z_0) \times [1 - r, 1]$ . Moreover as  $(z_0, 1)$  is a strict maximizer we have  $-2\xi := \max_{\partial_p \mathcal{O}} (V^*(T, \cdot) - \varphi)(z, p) < 0$ , with  $\partial_p \mathcal{O} := (\partial B_r(z_0) \times [1 - r, 1]) \cup (B_r(z_0) \times \{1 - r, 1\})$ . As a consequence, after possibly modifying  $r > 0$ , we have

$$V^* - \varphi \leq -\xi < 0 \text{ on } [T - r, T] \times \partial_p \mathcal{O}. \quad (3.77)$$

We define  $\mathcal{B} := [T - r, T] \times \mathcal{O}$  and consider  $(t_n, z_n, p_n)_n$ , a sequence in  $\mathcal{B} \cap \text{int}(\mathcal{D}_1)$  such that  $V(t_n, z_n, p_n) \rightarrow V^*(T, z_0, 1)$  and  $(t_n, z_n, p_n) \rightarrow (T, z_0, 1)$ . We now introduce a modified test function  $\tilde{\varphi}(t, z, p) := \varphi(z, p) + (T - t)^{\frac{1}{2}}$ . As  $H_*\varphi(z_0, 1) > -\infty$ , by assumption, and  $-\partial_t \tilde{\varphi}(t, z, p) = \frac{1}{2}(T - t)^{-\frac{1}{2}} \rightarrow \infty$  as  $t \rightarrow T$ , we have for all  $(u, a) \in \mathbb{R}^{2d}$ , after possibly changing  $r$ ,

$$-\mathcal{L}_{(Z, P)}^{(u, a)} \tilde{\varphi} \geq 0 \text{ on } \mathcal{B}. \quad (3.78)$$

For each  $n$ , since  $(t_n, z_n, p_n) \in \text{int}(\mathcal{D}_1) \subset \mathcal{C}$  there exists a control  $(\nu_n, \alpha_n) \in \bar{\mathcal{U}}_{t_n, z_n, p_n}$ . We set  $(Z^n, P^n) := (Z^{t_n, z_n, \nu_n}, P^{t_n, p_n, \alpha_n})$  and we define

$$\theta_n := \inf \{s \geq t_n : (s, Z_s^n, P_s^n) \notin \mathcal{B} \cap (\text{int}(\mathcal{D}_1) \cup \partial_Z \mathcal{D}_1)\}. \quad (3.79)$$

Thanks to the definition of  $\bar{\mathcal{U}}_{t_n, z_n, p_n}$  and Theorem 2.1 we know that  $(s, Z_s^n, P_s^n) \in (\text{int}(\mathcal{D}_1) \cup \partial_Z \mathcal{D}_1)$  on  $[t_n, T)$  and thus  $(\theta_n, Z_{\theta_n}^n, P_{\theta_n}^n) \in \partial_p \mathcal{B}$ , where  $\partial_p \mathcal{B}$  denotes the parabolic boundary of  $\mathcal{B}$ , i.e.

$$\partial_p \mathcal{B} := (\{T\} \times \text{cl}(\mathcal{O})) \cup ([T - r, T] \times \partial_p \mathcal{O}), \quad (3.80)$$

with  $\text{cl}(\mathcal{O})$  the closure of  $\mathcal{O}$ . Using (3.76)-(3.78) we thus obtain that for sufficiently small  $r > 0$ ,

$$\tilde{\varphi}(t_n, z_n, p_n) \geq \mathbb{E} [f^*(Z^n(\theta_n))\mathbf{1}_{\{\theta_n = T\}} + V^*(\theta_n, Z^n(\theta_n), P^n(\theta_n))\mathbf{1}_{\{\theta_n < T\}}] + \eta \wedge \xi. \quad (3.81)$$

Finally as  $(V - \tilde{\varphi})(t_n, z_n, p_n) \rightarrow 0$  as  $n \rightarrow \infty$  we obtain a contradiction to Theorem A.1 for  $n$  large enough.

**Step 4.** We now prove that for any smooth function  $\varphi$  and  $(T, z_0) \in \partial_T \mathcal{P}_1$  where  $z_0$  is such that

$$(\text{strict}) \max_{z \text{ s.t. } (T, z) \in \text{cl}(\mathcal{P}_1)} (V^*(T, \cdot, 1) - \varphi) = (V^*(T, \cdot, 1) - \varphi)(z_0) = 0, \quad (3.82)$$

we have

$$\min\{V^*(T, z_0, 1) - V_1^*(T, z_0), (V^*(T, z_0, 1) - f^*(z_0))\mathbf{1}_{\{\bar{H}_*\varphi(z_0) > -\infty\}}\} \leq 0. \quad (3.83)$$

To this aim we assume that  $V^*(T, z_0, 1) > V_1^*(T, z_0)$  and  $\bar{H}_*\varphi(z_0) > -\infty$  and we proceed as in Step 2. More precisely, we introduce on  $\mathcal{O}_+^d \times \mathbb{R}^+ \times [0, 1]$ , for every  $k > 0$ , the test function

$$\varphi_k(z, p) := \varphi(z) + |x - x_0|^{2m} + |y - y_0|^{2m} + kh(p), \quad (3.84)$$

for some  $m \geq 2$  large enough and with the function  $h$  defined in Step 2. We prove that the difference  $V^*(T, \cdot) - \varphi_k$  has a local maximizer  $(z_k, p_k)$  such that

$$(z_k, p_k) \rightarrow (z_0, 1), \quad k(1 - p_k) \rightarrow 0, \quad V^*(T, z_k, p_k) \rightarrow V^*(T, z_0, 1). \quad (3.85)$$

In particular as  $\bar{H}_*\varphi(z_k) > -\infty$  we have  $H_*\varphi_k(z_k, p_k) > -\infty$  for  $k$  large enough. Hence using (3.27), and (3.75) we deduce that  $V^*(T, z_k, p_k) \leq f^*(z_k)$ . We conclude sending  $k \rightarrow \infty$  and using (3.85).

**Step 5.** We know by definition and from (3.19) that  $V_{1*}(\cdot) \leq V_*(\cdot, 1) \leq V^*(\cdot, 1)$ . Moreover, it follows from the previous steps and Assumption 3.2.1 that  $V^*(\cdot, 1) \leq V_{1*}(\cdot)$ . We thus obtain  $V_{1*}(\cdot) = V_*(\cdot, 1) = V^*(\cdot, 1) = V_1^*(\cdot)$  appealing again to Assumption 3.2.1.

### 3.2.2 Characterization of $V(\cdot, p_{\min}(\cdot))$

Similarly to Section 3.2.1, we introduce the set

$$\mathcal{P}_2 := \left\{ (t, z) \in [0, T] \times \mathcal{O}_+^d \times \mathbb{R}^+ \text{ s.t. } \mathcal{U}_{t,z} \neq \emptyset \right\}. \quad (3.86)$$

For all  $(t, x) \in [0, T] \times \mathcal{O}_+^d$ , the auxiliary value function  $v_2$  defined in (2.13) reads

$$v_2(t, x) := \inf \{y \geq 0 \text{ s.t. } (t, z) \in \mathcal{P}_2\}, \quad (3.87)$$

and characterizes the closure of  $\mathcal{P}_2$ . To characterize  $V_*(\cdot, p_{\min}(\cdot))$  we introduce Assumption 3.2.2 which is used to link  $V_*(\cdot, p_{\min}(\cdot))$  to  $V_2$ .

**Assumption 3.2.2.** (1) The set  $\mathcal{U}_{t,x,0} \neq \emptyset$  for all  $(t, x) \in [0, T] \times \mathcal{O}_+^d$ , and either the admissible controls  $\nu$  satisfy  $\sigma_Y(t, x, 0, \nu) = 0$  and  $\mu_Y(t, x, 0, \nu) = 0$ , or  $\mathcal{U}_{t,x,0} = \mathcal{U}$  (i.e. zero is an absorbing state for the process  $Y$  for all  $\nu \in \mathcal{U}$ ).

(2) The function  $(t, z) \in \text{int}(\mathcal{P}_2) \cup \partial_Z \mathcal{P}_2 \mapsto V_2(t, z)$  is continuous.

**Remark 3.12.** (1) Assumption 3.2.2 (1) is satisfied, for example, in the setting of Assumption 3.1.2.

(2) Under Assumption 3.2.2 (1)  $v_2 \equiv 0$ .

(3) We know from Assumption 3.2.2 (1) that the admissible controls  $(\nu, \alpha)$  such that  $(\nu, \alpha) \in \tilde{\mathcal{U}}_{t,x,0,p}$ ,  $p \leq p_{\min}(t, x)$ , are those for which  $Y^{t,x,0,\nu} = 0$  and  $P^{t,p,\alpha} \leq p_{\min}(\cdot, X^{t,x,\nu})$  as, in this case,  $Y^{t,x,0,\nu} \geq v(\cdot, X^{t,x,\nu}, P^{t,p,\alpha}) = 0$ .

(4) Remark 3.11 holds for  $V_2$  as well since, as for  $V_1$ , its characterization follows from Bouchard et al. (2010), Theorem 3.1, Lemma 6.3. This therefore provides an example for which Assumption 3.2.2 (2) holds.

We then define the closure of  $\mathcal{P}_2$ , denoted  $\text{cl}(\mathcal{P}_2)$ , as  $\text{int}(\mathcal{P}_2) \cup \partial_Z \mathcal{P}_2 \cup \partial_T \mathcal{P}_2$ , with

$$\begin{cases} \text{int}(\mathcal{P}_2) := \{(t, z) \in [0, T] \times \mathcal{O}_+^d \times \mathbb{R}^+ : y > v_2(t, x)\} \\ \partial_Z \mathcal{P}_2 := \{(t, z) \in [0, T] \times \mathcal{O}_+^d \times \mathbb{R}^+ : y = v_2(t, x)\} \\ \partial_T \mathcal{P}_2 := \{(t, z) \in \{T\} \times \mathcal{O}_+^d \times \mathbb{R}^+ : y \geq v_2(t, x)\} \end{cases} \quad (3.88)$$

We now introduce the following standing assumption allowing to obtain specific properties when limit arguments are required, as  $\nu$  is not valued in a compact set (see e.g. Proposition 3.2 below).

**Standing Assumption 5** We assume that the process  $X$  does not depend on  $\nu$  and that the function  $f$  is continuous on its domain.

**Remark 3.13.** (1) The previous assumption on  $X$  holds in the setting of Assumption 3.1.2.

(2) The previous assumption implies, in particular, that for all  $(t, x) \in [0, T] \times \mathcal{O}_+^d$ ,

$$p_{\min}(t, x) = \mathbb{E} \left[ \mathbb{E} \left[ \mathbf{1}_{\{g(T, X_T^{t,x})=0\}} \middle| \mathcal{F}_s \right] \right] = \mathbb{E}[p_{\min}(s, X_s^{t,x})]. \quad (3.89)$$

We start by proving a proposition that will be used in the proof of Theorem 3.6.

**Proposition 3.2.** Let Assumption 3.2.2 hold. Then for all  $(t, x) \in [0, T] \times \mathcal{O}_+^d$ ,

$$V_*(t, x, 0, p_{\min}(t, x)) = V_{2*}(t, x, 0). \quad (3.90)$$

**Proof.** Fix  $(t, x) \in [0, T] \times \mathcal{O}_+^d$ . We let  $(t_n, x_n, y_n, p_n)_n$  be a sequence in  $\text{int}(\mathcal{D}_1)$  such that  $V(t_n, x_n, y_n, p_n) \rightarrow V_*(t, x, 0, p_{\min}(t, x))$  and  $(t_n, x_n, y_n, p_n) \rightarrow (t, x, 0, p_{\min}(t, x))$ . The definition of  $V$  (see (2.6)) and Assumption 3.2.2 (1) imply that  $V(t, x, 0, p_{\min}(t, x)) = \mathbb{E}[f(X_T^{t,x}, 0)]$ . Moreover as  $(t_n, x_n, y_n, p_n)_n \in \text{int}(\mathcal{D}_1)$  we have  $y_n > v(t_n, x_n, p_n)$  and thus appealing to the GDP principle (see Soner & Touzi (2002a,b)) we obtain that for all  $\theta_n \leq T$  there exists  $(\nu^n, \alpha^n) \in \mathcal{U} \times \mathcal{A}_{t_n, p_n}$  such that  $Y_{\theta_n}^{t_n, x_n, y_n, \nu^n} \geq v(\theta_n, X_{\theta_n}^{t_n, x_n}, P_{\theta_n}^{t_n, p_n, \alpha^n})$ . As a consequence  $\tilde{\mathcal{U}}_{t_n, x_n, y_n, p_n} \neq \emptyset$  and we obtain by definition of  $V$  (see (2.15)) that  $V(t_n, x_n, y_n, p_n) \geq \mathbb{E}[f(X_T^{t_n, x_n}, Y_T^{t_n, x_n, y_n, \nu^n})]$ . Moreover, appealing to the non-decreasing property of  $y \mapsto f(x, y)$  we deduce that  $V(t_n, x_n, y_n, p_n) \geq \mathbb{E}[f(X_T^{t_n, x_n}, 0)]$ . Therefore as  $f$  and  $X$  are continuous and  $f$  has a polynomial growth we pass to the limit and apply the Dominated Convergence Theorem to obtain  $V_*(t, x, 0, p_{\min}(t, x)) \geq V(t, x, 0, p_{\min}(t, x))$ . We thus conclude using (3.18) that

$$V_*(t, x, 0, p_{\min}(t, x)) \geq V(t, x, 0, p_{\min}(t, x)) = \mathbb{E}[f(X_T^{t,x}, 0)] = V_2(t, x, 0) \geq V_{2*}(t, x, 0), \quad (3.91)$$

which in view of (3.19) and Assumption 3.2.2 (2) gives the result.

We shall work under the following standing assumption (recall Remark 2.4).

**Standing Assumption 6** The function  $(t, x) \in [0, T] \times \mathcal{O}_+^d \mapsto p_{\min}(t, x) \equiv p_{\min} < 1$ , i.e. the function  $p_{\min}(\cdot)$  is independent of  $(t, x)$  on  $[0, T] \times \mathcal{O}_+^d$ .

**Theorem 3.6** (Main Result - Characterization of  $V(\cdot, p_{\min})$ ). *Let Assumption 3.2.2 hold and assume that  $V_*$  has a polynomial growth. Then  $V_*(\cdot, p_{\min}) = V^*(\cdot, p_{\min}) = V_{2*}(\cdot) = V_2^*(\cdot)$  on  $\text{int}(\mathcal{P}_2) \cup \partial_Z \mathcal{P}_2$ .*

**Proof.** Let us prove that

$$V_*(\cdot, p_{\min}) = V^*(\cdot, p_{\min}) = V_{2*}(\cdot) = V_2^*(\cdot) \text{ on } \text{int}(\mathcal{P}_2) \cup \partial_Z \mathcal{P}_2. \quad (3.92)$$

**Step 1.** We first prove that for any smooth function  $\varphi$  and  $(t_0, z_0, p_{\min}) \in \mathcal{D}_2$ ,  $t_0 < T$  such that

$$(\text{strict}) \min_{\text{cl}(\mathcal{G})} (V_* - \varphi) = (V_* - \varphi)(t_0, z_0, p_{\min}) = 0, \quad (3.93)$$

with  $\text{cl}(\mathcal{G}) := \{(t, z, p) \in [0, T] \times \mathcal{O}_+^d \times \mathbb{R}^+ \times [p_{\min}, 1] : y \geq v(t, x, p)\}$ , the function  $V_*$  is a viscosity super-solution of

$$\max \left\{ V_*(t_0, z_0, p_{\min}) - V_{2*}(t_0, z_0), -\partial_t \varphi(t_0, z_0, p_{\min}) + H^* \varphi(t_0, z_0, p_{\min}) \right\} \geq 0, \text{ if } y_0 > 0. \quad (3.94)$$

In this case we have  $y_0 > v(t_0, x_0, p_{\min}) = 0$ . We appeal to the arguments in [Bouchard et al. \(2010\)](#) (see Section 6.2.1), except that we have to handle the state constraint for  $p = p_{\min}$ . We report the entire argument. We assume that

$$\max \left\{ V_*(t_0, z_0, p_{\min}) - V_{2*}(t_0, z_0), -\partial_t \varphi(t_0, z_0, p_{\min}) + H^* \varphi(t_0, z_0, p_{\min}) \right\} < 0. \quad (3.95)$$

By continuity of the coefficients we can find a closed bounded neighborhood  $\mathcal{O}$  of  $(t_0, z_0, p_{\min})$  such that  $\mathcal{O} \subset \mathcal{C}$  and  $p \geq p_{\min}$ ,  $y \geq v(t, x, p) + r$ ,  $r > 0$  and  $t < T$  on  $\mathcal{O}$ , and we can find  $(\hat{u}, \hat{a}) \in \mathbb{R}^{2d}$  such that

$$\max \left\{ \varphi(t, z, p) - V_2(t, z), -\mathcal{L}_{(Z, P)}^{(\hat{u}, \hat{a})} \varphi(t, z, p) \right\} \leq 0 \text{ on } \mathcal{O}. \quad (3.96)$$

We let  $(t_n, z_n, p_n)$  be a sequence in  $\mathcal{O} \cap \text{int}(\mathcal{D}_1)$  such that  $V(t_n, z_n, p_n) \rightarrow V_*(t_0, z_0, p_{\min})$  and  $(t_n, z_n, p_n) \rightarrow (t_0, z_0, p_{\min})$ . We let  $(\hat{Z}^n, \hat{P}^n) := (Z^{t_n, z_n, \hat{u}}, P^{t_n, z_n, \hat{a}})$  denote the solution to (2.1), (2.2) and (2.4), for the control  $\hat{u}$  and  $\hat{a}$  viewed as constant controls in  $\mathcal{U} \times \mathcal{A}_{t_n, p_n}$ . We then define  $\theta_n := \inf\{s \geq t_n : (s, \hat{Z}_s^n, \hat{P}_s^n) \notin \mathcal{O}\}$ . We thus have  $(\theta_n, \hat{Z}_{\theta_n}^n, \hat{P}_{\theta_n}^n) \in \partial_p \mathcal{O} \subset \mathcal{C}$ , the parabolic boundary of  $\mathcal{O}$ , and thus appealing to Theorem 2.1 we obtain the existence of  $(\nu^n, \alpha^n) \in \tilde{\mathcal{U}}_{t_n, z_n, p_n}$  such that  $\nu^n = \hat{u}$  and  $\alpha^n = \hat{a}$  on  $[t_n, \theta_n]$ . As a consequence  $(Z^n, P^n) := (Z^{t_n, z_n, \nu^n}, P^{t_n, p_n, \alpha^n}) = (\hat{Z}^n, \hat{P}^n)$  on  $[t_n, \theta_n]$  by continuity of both processes. Applying Itô's Lemma and using (3.96) we therefore obtain that for  $n$  large enough

$$\varphi(t_n, z_n, p_n) \leq \mathbb{E}[\varphi(\theta_n, Z_{\theta_n}^n, P_{\theta_n}^n)] \leq \mathbb{E}[V_*(\theta_n, Z_{\theta_n}^n, P_{\theta_n}^n)] - \xi, \quad (3.97)$$

where, from (3.93) and (3.96),  $\xi$  is such that

$$\xi := \inf_{\partial_p \mathcal{O}} (V_* - \varphi) > 0. \quad (3.98)$$

Finally as  $(V - \varphi)(t_n, z_n, p_n) \rightarrow (V_* - \varphi)(t_0, z_0, p_{\min}) = 0$  as  $n \rightarrow \infty$  we obtain a contradiction to Theorem A.1 for  $n$  large enough.

**Step 2.** We prove (3.92). We let  $\varphi$  be a smooth function and  $(t_0, z_0) \in \text{int}(\mathcal{P}_2) \cup \partial_Z \mathcal{P}_2$  be such that

$$(\text{strict}) \min_{\text{cl}(\mathcal{P}_2)} (V_*(\cdot, p_{\min}) - \varphi) = (V_*(\cdot, p_{\min}) - \varphi)(t_0, z_0) = 0. \quad (3.99)$$

By definition we have  $V_*(t_0, z_0, p_{\min}) \leq V_{2*}(t_0, z_0)$ . Let us assume that

$$V_*(t_0, z_0, p_{\min}) < V_{2*}(t_0, z_0). \quad (3.100)$$

It follows from Proposition 3.2 that, in this case,  $(t_0, z_0) \notin \partial_Z \mathcal{P}_2$ . Similarly to [Bouchard et al. \(2009\)](#) (see Section 6), we now introduce on  $\text{cl}(\mathcal{C})$ , for every  $k > 0$ , a new test function

$$\varphi_k(t, z, p) := \varphi(t, z) - |x - x_0|^{2m} - |y - y_0|^{2m} - (t - t_0)^2 - kh(1 - p + p_{\min}), \quad (3.101)$$

for some  $m \geq 2$  large enough and  $h$  defined in (3.57). Arguing as in Step 2. of the proof of Theorem 3.5, we obtain the existence of  $(t_k, z_k, p_k)$ , a local minimizer of  $V_* - \varphi_k$  on  $\text{cl}(\mathcal{G})$  being such that, after possibly passing to a subsequence,

$$(t_k, z_k, p_k) \rightarrow (t_0, z_0, p_{\min}), \quad k(p_k - p_{\min}) \rightarrow 0, \quad V_*(t_k, z_k, p_k) \rightarrow V_*(t_0, z_0, p_{\min}), \quad (3.102)$$

and  $t_k < T$ ,  $p_k < 1$ . Moreover as  $y_0 > 0 = v(t_0, x_0, p_{\min})$  we can assume after possibly passing to a subsequence that  $y_k > v(t_k, x_k, p_k)$ . Besides since, by assumption,

$$V_*(t_0, z_0, p_{\min}) - V_{2*}(t_0, z_0) < 0, \quad (3.103)$$

we obtain with (3.102) and after possibly passing to a subsequence that,

$$V_*(t_k, z_k, p_k) - V_{2*}(t_k, z_k) < 0. \quad (3.104)$$

Hence using (3.24), (3.94) and (3.104), we obtain

$$(-\partial_t \varphi_k + H^* \varphi_k)(t_k, z_k, p_k) \geq 0. \quad (3.105)$$

Moreover it follows from (3.102) that

$$(\partial_t \varphi_k, D_z \varphi_k, D_{zz} \varphi_k)(t_k, z_k, p_k) \rightarrow (\partial_t \varphi, D \varphi, D^2 \varphi)(t_0, z_0) \text{ as } k \rightarrow \infty, \quad (3.106)$$

$$(D_p \varphi_k, D_{zp} \varphi_k, D_{pp} \varphi_k)(t_k, z_k, p_k) = (-k D_p h(u_k), 0, -k D_{pp} h(u_k)) \text{ for every } k > 1, \quad (3.107)$$

with  $u_k := 1 - p_k + p_{\min}$ .

Thanks to the definition of  $H^*$  we can find some sequences  $(t_l^k, z_l^k, p_l^k) \in [0, T) \times \mathcal{O}_+^d \times \mathbb{R}^+ \times [p_{\min}, 1)$  where  $y_l^k > v(t_l^k, x_l^k, p_l^k)$ , a vector  $q_l^k := (q_l^{k,z}, q_l^{k,p})^\top$  where  $(q_l^{k,z}, q_l^{k,p}) \in \mathbb{R}^{d+2}$ , and a symmetric matrix  $A_l^k \in \mathbb{S}^{d+2}$  with  $A_l^k := \begin{pmatrix} A_l^{k,zz} & A_l^{k,zp} \\ A_l^{k,zp^\top} & A_l^{k,pp} \end{pmatrix}$  where  $A_l^{k,zz} \in \mathbb{S}^{d+1}$  and  $(A_l^{k,zp}, A_l^{k,pp}) \in \mathbb{R}^{d+1} \times \mathbb{R}$ , such that

$$(t_l^k, z_l^k, p_l^k) \rightarrow (t_k, z_k, p_k) \quad \text{and} \quad |(q_l^k, A_l^k) - (D \varphi_k, D^2 \varphi_k)(t_k, z_k, p_k)| \leq l^{-1}, \quad (3.108)$$

and where for all  $(u, a) \in \mathbb{R}^{2d}$ ,

$$-\partial_t \varphi_k(t_k, z_k, p_k) + H^{(u,a)}(t_l^k, z_l^k, q_l^k, A_l^k) \geq -l^{-1}. \quad (3.109)$$

As the latter inequality holds for all  $(u, a) \in \mathbb{R}^{2d}$  we obtain a contradiction after sending  $(l, k) \rightarrow \infty$ , appealing to (3.102), (3.106)-(3.108) and after noticing that  $k D_{pp} h(1 - p_k + p_{\min}) \rightarrow -\infty$ . Therefore  $V_*(t_0, z_0, p_{\min}) = V_{2*}(t_0, z_0)$  on  $\text{int}(\mathcal{P}_2) \cup \partial_Z \mathcal{P}_2$ . We conclude appealing to Assumption 3.2.2 (2) and (3.19).

## A Proof of the Dynamic Programming Principle

We now prove the dynamic programming principle that is involved throughout the proofs of the main results.

**Theorem A.1.** *Fix  $(t, z, p) \in \text{int}(\mathcal{D}_1)$  and let  $\theta$  be a stopping time with values in  $[t, T]$ . Then*

$$V(t, z, p) \leq \sup_{(\nu, \alpha) \in \bar{\mathcal{U}}_{t,z,p}} \mathbb{E} \left[ f^*(Z_\theta^{t,z,\nu}) \mathbf{1}_{\{\theta=T\}} + V^*(\theta, Z_\theta^{t,z,\nu}, P_\theta^{t,p,\alpha}) \mathbf{1}_{\{\theta < T\}} \right], \quad (1.1)$$

$$V(t, z, p) \geq \sup_{(\nu, \alpha) \in \bar{\mathcal{U}}_{t,z,p}} \mathbb{E} \left[ f_*(Z_\theta^{t,z,\nu}) \mathbf{1}_{\{\theta=T\}} + V_*(\theta, Z_\theta^{t,z,\nu}, P_\theta^{t,p,\alpha}) \mathbf{1}_{\{\theta < T\}} \right]. \quad (1.2)$$

**Proof.** The first inequality is standard (see e.g. [Bouchard et al. \(2010\)](#), Proof of Theorem 6.1, 1.). Indeed using the Flow property and the definition of  $V$  in (2.6) we obtain that for any  $(\nu, \alpha) \in \bar{\mathcal{U}}_{t,z,p}$  (recall (2.8))

$$\mathbb{E}[f(Z_T^{t,z,\nu})] \leq \mathbb{E} \left[ f(Z_\theta^{t,z,\nu}) \mathbf{1}_{\{\theta=T\}} + V^*(\theta, Z_\theta^{t,z,\nu}, P_\theta^{t,p,\alpha}) \mathbf{1}_{\{\theta < T\}} \right]. \quad (1.3)$$

We conclude taking the supremum over  $\bar{\mathcal{U}}_{t,z,p}$ .

We prove the second one. Fix  $(\nu_1, \alpha_1) \in \bar{\mathcal{U}}_{t,z,p}$  for some  $(t, z, p) \in \text{int}(\mathcal{D}_1)$ . We consider  $m$ , the measure induced by  $(\theta, \xi, \zeta)$  on  $\text{cl}(\mathcal{C})$  where  $(\xi, \zeta) := (Z_\theta^{t,z,\nu_1}, P_\theta^{t,p,\alpha_1})$ . We appeal to Proposition 7.50 & Lemma 7.27 in [Bertsekas & Shreve \(1978\)](#) to prove, after noticing that  $(\theta, \xi, \zeta) \in \mathcal{C}$  by Theorem 2.1, that, for each  $\varepsilon > 0$ , we can build two Borel-measurable maps  $\hat{\nu}_m^\varepsilon$  and  $\hat{\alpha}_m^\varepsilon$  such that

$$(\hat{\nu}_m^\varepsilon, \hat{\alpha}_m^\varepsilon)(\theta, \xi, \zeta) \in \bar{\mathcal{U}}_{\theta,\xi,\zeta} \quad \text{and} \quad \mathbb{E} \left[ f(Z_T^{\theta,\xi,\hat{\nu}_m^\varepsilon}) \middle| (\theta, \xi, \zeta) \right] \geq V_*(\theta, \xi, \zeta) - \varepsilon. \quad (1.4)$$

We now use Lemma 2.1 in [Soner & Touzi \(2002a\)](#) to obtain  $\nu_2^\varepsilon$  and  $\alpha_2^\varepsilon$  such that

$$\nu_2^\varepsilon \mathbf{1}_{[\theta,T]} = \hat{\nu}_m^\varepsilon(\theta, \xi, \zeta) \mathbf{1}_{[\theta,T]} dt \times d\mathbb{P}\text{-a.e.}, \quad (1.5)$$

$$\alpha_2^\varepsilon \mathbf{1}_{[\theta,T]} = \hat{\alpha}_m^\varepsilon(\theta, \xi, \zeta) \mathbf{1}_{[\theta,T]} dt \times d\mathbb{P}\text{-a.e.}, \quad (1.6)$$

implying that  $\nu^\varepsilon := \nu_1 \mathbf{1}_{[t,\theta)} + \nu_2^\varepsilon \mathbf{1}_{[\theta,T]}$  and  $\alpha^\varepsilon := \alpha_1 \mathbf{1}_{[t,\theta)} + \alpha_2^\varepsilon \mathbf{1}_{[\theta,T]}$  belong to  $\bar{\mathcal{U}}_{t,z,p}$  and

$$\mathbb{E} \left[ f(Z_T^{\theta,\xi,\nu^\varepsilon}) \middle| (\theta, \xi, \zeta) \right] \geq V_*(\theta, \xi, \zeta) - \varepsilon. \quad (1.7)$$

We thus conclude taking the expectation on both sides and using the arbitrariness of  $(\nu_1, \alpha_1) \in \bar{\mathcal{U}}_{t,z,p}$  and  $\varepsilon$ .

## B Proof of a Comparison Principle for the PDE Solved by $V$ on $\text{int}(\mathcal{D}_1)$

This section is an adaptation of the arguments presented by [Bouveret & Chassagneux \(2017\)](#). We work under the following standing assumptions.

**Standing Assumption Appendix B.1** Assumption [3.1.2](#) holds.

We now introduce some notations that will be used throughout this section.

For a vector  $\beta \in \mathcal{S}_{2d+1} \setminus \mathcal{D}_{2d+1}$ , we denote  $\beta^b := \frac{1}{\beta^1}(\beta^2, \dots, \beta^{d+1}) \in \mathbb{R}^d$  and  $\beta^\sharp := \frac{1}{\beta^1}(\beta^{d+2}, \dots, \beta^{2d+1}) \in \mathbb{R}^d$ . Moreover we define for  $\Theta := (t, z, b, q, A) \in [0, T] \times \mathcal{O}_+^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d+2} \times \mathbb{S}^{d+2}$  where  $q := (q^z, q^p)^\top$  with  $(q^z, q^p) \in \mathbb{R}^{d+1} \times \mathbb{R}$ , and  $A := \begin{pmatrix} A^{zz} & A^{zp} \\ A^{zp^\top} & A^{pp} \end{pmatrix}$  with  $A^{zz} \in \mathbb{S}^{d+1}$  and  $(A^{zp}, A^{pp}) \in \mathbb{R}^{d+1} \times \mathbb{R}$ , and for  $\beta \in \mathcal{S}_{2d+1}$ , the following operator

$$F^\beta(\Theta) := \begin{cases} (\beta^1)^2 \left( -b + H^{\beta^b, \beta^\sharp}(t, z, q, A) \right), & \beta \in \mathcal{S}_{2d+1} \setminus \mathcal{D}_{2d+1} \\ -\frac{1}{2}y^2\sigma(x)^2|\bar{\beta}^b|^2A^{yy} - y\sigma(x)\bar{\beta}^b{}^\top \bar{\beta}^\sharp A^{yp} - \frac{1}{2}|\bar{\beta}^\sharp|^2A^{pp}, & \beta \in \mathcal{D}_{2d+1} \end{cases}, \quad (2.1)$$

(recall [\(3.6\)](#)) where  $\bar{\beta}^b := (\beta^2, \dots, \beta^{d+1})$  and  $\bar{\beta}^\sharp := (\beta^{d+2}, \dots, \beta^{2d+1})$ .

**Remark B.1.** The operator  $\beta \mapsto F^\beta$  is continuous on  $\mathcal{S}_{2d+1}$ . In particular,

$$\inf_{\beta \in \mathcal{S}_{2d+1}} F^\beta(\Theta) = \inf_{\beta \in \mathcal{S}_{2d+1} \setminus \mathcal{D}_{2d+1}} F^\beta(\Theta). \quad (2.2)$$

As in the above quoted papers we first provide an alternative PDE characterization of  $V$  on  $\text{int}(\mathcal{D}_1)$  involving a continuous operator.

**Theorem B.1.** On  $\text{int}(\mathcal{D}_1)$ ,  $V_*$  (resp.  $V^*$ ) is a viscosity super-solution (resp. sub-solution) of

$$\inf_{\beta \in \mathcal{S}_{2d+1}} F^\beta \varphi = 0. \quad (2.3)$$

**Proof.** We only prove the sub-solution property as the super-solution one is an easy adaptation of the arguments developed by [Bouveret & Chassagneux \(2017\)](#), (see the proof of Theorem 3.1). Let  $\varphi$  be a smooth function such that  $\max_{\text{cl}(C)}(V^* - \varphi)(t, z, p) = \max_{\text{int}(\mathcal{D}_1)}(V^* - \varphi)(t, z, p) = (V^* - \varphi)(t_0, z_0, p_0) = 0$ . We verify that

$$\inf_{\beta \in \mathcal{S}_{2d+1}} F^\beta \varphi(t_0, z_0, p_0) \leq 0. \quad (2.4)$$



According to (3.27) and by definition of  $H_*$  we can find sequences  $(t_k, z_k, p_k) \in \text{int}(\mathcal{D}_1)$ ,  $q_k := (q_k^z, q_k^p)^\top$  where  $(q_k^z, q_k^p) \in \mathbb{R}^{d+2}$ , a symmetric matrix  $A_k \in \mathbb{S}^{d+2}$  and two maximizing sequences  $(u_k, a_k) \in \mathbb{R}^{2d}$  such that

$$(t_k, z_k, p_k) \rightarrow (t_0, z_0, p_0) \quad \text{and} \quad |(q_k, A_k) - (D\varphi, D^2\varphi)(t_0, z_0, p_0)| \leq k^{-1}, \quad (2.5)$$

and

$$-\partial_t \varphi(t_0, z_0, p_0) + H^{u_k, a_k}(t_k, z_k, q_k, A_k) \leq 2k^{-1}. \quad (2.6)$$

Taking  $\beta_k := (\frac{1}{\sqrt{1+|a_k|^2+|u_k|^2}}, \frac{u_k}{\sqrt{1+|a_k|^2+|u_k|^2}}, \frac{a_k}{\sqrt{1+|a_k|^2+|u_k|^2}}) \in \mathcal{S}_{2d+1} \setminus \mathcal{D}_{2d+1}$ , we have

$$(\beta_k^1)^2 \left( -\partial_t \varphi(t_0, z_0, p_0) + H^{\beta_k^b, \beta_k^\sharp}(t_k, z_k, q_k, A_k) \right) \leq 2k^{-1}(\beta_k^1)^2. \quad (2.7)$$

Thanks to the relative compactness of the set  $\mathcal{S}_{2d+1} \setminus \mathcal{D}_{2d+1}$  we obtain the existence of a subsequence such that  $\lim_{k' \rightarrow \infty} \beta_{k'}' = \bar{\beta}$  with  $\bar{\beta} \in \mathcal{S}_{2d+1}$ . We finally appeal to (2.5) and the assumptions on the coefficients of the diffusion of  $Z$  to conclude

$$\inf_{\beta \in \mathcal{S}_{2d+1}} F^\beta \varphi(t_0, z_0, p_0) \leq F^{\bar{\beta}} \varphi(t_0, z_0, p_0) \leq 0, \quad (2.8)$$

which completes the proof.

As usual the proof of a comparison principle will appeal to a super-solution argument (see for instance, Ishii & Lions (1990) and Cheridito et al. (2005)).

**Lemma B.1** (Super-solution property). *We define on  $[0, T] \times \mathcal{O}_+^d \times \mathbb{R}^+ \times [0, 1]$  the smooth functions  $\phi(t, p) := e^{\kappa(T-t)}(\theta - \frac{e^{-4p}}{2})$ ,  $h(t, x) := e^{\kappa(T-t)}(|x|_1^{2k} + |x|_1^{-2})$ ,  $\ell(t, y) := e^{\kappa(T-t)} \frac{y^m}{m}$  and*

$$f(t, z, p) := \phi(t, p) + h(t, x) + \ell(t, y) > 0, \quad (2.9)$$

for some  $\kappa, \theta \geq 1$ ,  $k \geq 1$  and  $0 < m \leq 1$ .

Let  $V$  be a lower semi-continuous super-solution of (2.3). Then, there exists  $\kappa$  and  $\theta$  big enough, and a  $m$  such that the function  $V + \xi f$ ,  $\xi > 0$ , is a viscosity super-solution of (2.3) on  $\text{int}(\mathcal{D}_1)$ . More precisely, given a smooth function  $\varphi$  such that  $\min_{\text{cl}(\mathcal{C})}((V + \xi f) - \varphi)(t, z, p) = \min_{\text{int}(\mathcal{D}_1)}((V + \xi f) - \varphi)(t, z, p) = ((V + \xi f) - \varphi)(t_0, z_0, p_0) = 0$ , one has

$$\inf_{\beta \in \mathcal{S}_{2d+1}} F^\beta \varphi(t_0, z_0, p_0) \geq \xi \left( \frac{\kappa \wedge 4}{e^4} \wedge \frac{\bar{\Lambda}^\mu}{2} y_0^m \right), \quad (2.10)$$

with  $\bar{\Lambda}^\mu$  defined in Assumption 3.1.2.

**Proof.** Let  $\varphi$  be a smooth function such that  $\min_{\text{cl}(\mathcal{C})}((V + \xi f) - \varphi)(t, z, p) = \min_{\text{int}(\mathcal{D}_1)}((V + \xi f) - \varphi)(t, z, p) = ((V + \xi f) - \varphi)(t_0, z_0, p_0) = 0$ ,  $\xi > 0$ . As  $f$  is a smooth function, the function  $\psi := \varphi - \xi f$  is a test function for  $V$  at  $(t_0, z_0, p_0)$ . Recalling the definition of  $F^\beta$ , we obtain

$$F^\beta \varphi(t_0, z_0, p_0) \geq F^\beta \psi(t_0, z_0, p_0) + \mathfrak{A} + \mathfrak{B} + \mathfrak{C}, \quad (2.11)$$

where

$$\mathfrak{A} = \xi e^{\kappa(T-t_0)} (\beta^1)^2 \left( \kappa \left[ \theta - \frac{e^{-4p_0}}{2} \right] + 4|\beta^\sharp|^2 e^{-4p_0} \right), \quad (2.12)$$

$$\mathfrak{B} = \xi e^{\kappa(T-t_0)} (\beta^1)^2 \left( \kappa - 2(k+1)\bar{\Lambda}^\mu - \bar{\Lambda}^{\sigma^2}(k(2k-1)+3) \right) \left[ |x_0|_1^{2k} + |x_0|_1^{-2} \right], \quad (2.13)$$

$$\mathfrak{C} = \xi e^{\kappa(T-t_0)} \left( (\beta^1)^2 \left[ \frac{\kappa}{m} - \frac{1}{2}\bar{\Lambda}^\mu \right] + (\beta^1)^2 |\beta^\flat|^2 \frac{1}{2} \left[ (1-m)\underline{\Lambda}^{\sigma^2} - \bar{\Lambda}^\mu \right] \right) y_0^m, \quad (2.14)$$

with  $\bar{\Lambda}^\mu$ ,  $\bar{\Lambda}^\sigma$  and  $\underline{\Lambda}^\sigma$  defined in Assumption 3.1.2. We will now provide a lower bound for each term.

1. For  $\theta = 2e^{-4p_0} + 1$  we have

$$\mathfrak{A} \geq \xi \frac{\kappa \wedge 4}{e^4} \left[ (\beta^1)^2 + (\beta^1)^2 |\beta^\sharp|^2 \right]. \quad (2.15)$$

2. For the second term, we observe that we can find  $\kappa \geq 1$  such that  $\mathfrak{B} \geq 0$ .

3. For the third term, we can find  $\kappa \geq 1$  and  $0 < m < 1 - \frac{\bar{\Lambda}^\mu}{\underline{\Lambda}^{\sigma^2}}$  such that

$$\mathfrak{C} \geq \xi \frac{\bar{\Lambda}^\mu}{2} (\beta^1)^2 |\beta^\flat|^2 y_0^m. \quad (2.16)$$

We therefore obtain

$$F^\beta \varphi(t_0, z_0, p_0) \geq F^\beta \psi(t_0, z_0, p_0) + \xi \left( \frac{\kappa \wedge 4}{e^4} \wedge \frac{\bar{\Lambda}^\mu}{2} y_0^m \right). \quad (2.17)$$

We conclude by taking the supremum over  $\mathcal{S}_{2d+1}$  and using the super-solution property of  $\psi$ , recall Remark B.1.

**Lemma B.2** (Modulus of continuity). *Let  $(t, b, x, r, y, l, p, q) \in [0, T] \times \mathbb{R} \times (\mathcal{O}_+^d)^2 \times (\mathbb{R}^+)^2 \times [0, 1]^2$ . Moreover, for  $\varepsilon > 0$ , let  $\mathcal{X}$  and  $\mathcal{R} \in \mathbb{S}^{d+2} \times \mathbb{S}^{d+2}$  being such that*

$$\begin{pmatrix} \mathcal{X} & 0 \\ 0 & -\mathcal{R} \end{pmatrix} \leq \frac{10}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \quad (2.18)$$

with  $I$  the  $d+2$ -dimensional identity matrix. Setting  $\delta = \frac{2}{\varepsilon} \begin{pmatrix} x-r \\ y-l \\ p-q \end{pmatrix}$ ,  $\Theta = (t, r, l, b, \delta, \mathcal{R})$ ,

$\Theta' = (t, x, y, b, \delta, \mathcal{X})$  we have

$$\inf_{\beta \in \mathcal{S}_{2d+1}} F^\beta(\Theta) - \inf_{\beta \in \mathcal{S}_{2d+1}} F^\beta(\Theta') \leq \frac{C}{\varepsilon} (1 + |x|^2 + |r|^2 + |y|^2 + |l|^2) (|x-r|^2 + |y-l|^2), \quad (2.19)$$

for some constant  $C > 0$ .

**Proof.** We consider  $\Theta$  and  $\Theta'$  defined in the theorem. We first notice that

$$\inf_{\beta \in \mathcal{S}_{2d+1}} F^\beta(\Theta) - \inf_{\beta \in \mathcal{S}_{2d+1}} F^\beta(\Theta') \leq \sup_{\beta \in \mathcal{S}_{2d+1} \setminus \mathcal{D}_{2d+1}} \left\{ F^\beta(\Theta) - F^\beta(\Theta') \right\}, \quad (2.20)$$

(recall Remark B.1).

For  $\beta \in \mathcal{S}_{2d+1} \setminus \mathcal{D}_{2d+1}$ , appealing to the definition of  $F^\beta$ , the Lipschitz property of  $\mu$  and Young's inequality, we obtain

$$\begin{aligned} & F^\beta(\Theta) - F^\beta(\Theta') \\ & \leq (\beta^1)^2 \left( \frac{C}{\varepsilon} \left( 1 + |x|_1 + |r|_1 + |\beta^b|_1 (1 + |y| + |l|) \right) (|x-r|^2 + |y-l|^2) + \mathfrak{B} \right), \end{aligned} \quad (2.21)$$

with  $C > 0$  depending on  $\bar{\Lambda}^\mu$  defined in Assumption 3.1.2, and

$$\mathfrak{B} = -\frac{1}{2} \text{Tr} \left[ \bar{\sigma} \bar{\sigma}^\top (t, r, l, \beta^b, \beta^\sharp) \mathcal{R} \right] + \frac{1}{2} \text{Tr} \left[ \bar{\sigma} \bar{\sigma}^\top (t, x, y, \beta^b, \beta^\sharp) \mathcal{X} \right], \quad (2.22)$$

where for  $(t, x, y) \in [0, T) \times \mathcal{O}_+^d \times \mathbb{R}^+$  and  $\beta \in \mathcal{S}_{2d+1} \setminus \mathcal{D}_{2d+1}$ ,

$$\bar{\sigma}(t, x, y, \beta^b, \beta^\sharp) := \begin{pmatrix} \sigma(x) \text{diag}[x] \\ y \sigma(x) (\beta^b)^\top \\ (\beta^\sharp)^\top \end{pmatrix}. \quad (2.23)$$

For the second-order term  $\mathfrak{B}$ , (2.18), the Lipschitz property of  $\sigma$  and Young's inequality, give the existence of  $C > 0$  (depending on  $\bar{\Lambda}^\sigma$  defined in Assumption 3.1.2) such that

$$\mathfrak{B} \leq (\beta^1)^2 \left( \frac{C}{\varepsilon} \left( 1 + |x|^2 + |r|^2 + |\beta^b|^2 (1 + |y|^2 + |l|^2) \right) (|x-r|^2 + |y-l|^2) \right). \quad (2.24)$$

We thus have a constant  $C > 0$  such that

$$F^\beta(\Theta) - F^\beta(\Theta') \leq \frac{C}{\varepsilon} (\beta^1)^2 (1 + |\beta^b|^2) (1 + |x|^2 + |r|^2 + |y|^2 + |l|^2) (|x-r|^2 + |y-l|^2). \quad (2.25)$$

The proof is concluded by observing that  $(\beta^1)^2 \leq 1$  and  $(\beta^1)^2 |\beta^b|^2 \leq 1$ .

We can now prove Theorem 3.2.

**Proof of Theorem 3.2** We first introduce on  $(0, T] \times \mathcal{O}_+^d \times \mathbb{R}^+ \times [0, 1]$  the following non-negative auxiliary function

$$V_\xi(t, z, p) := (V + \xi f)(t, z, p) + \frac{\xi}{t}, \quad (2.26)$$

with  $f$  defined in Lemma B.1. Using Lemma B.1, one can easily check that  $V_\xi$  is a super-solution of (2.3) and satisfies (2.10).

We also define  $U_\xi(t, z, p) = U(t, z, p) - \xi(h(t, x) + \ell(t, y))$ , recall Lemma B.1. Adapting the proof of Lemma B.1, we prove that  $U_\xi$  remains a sub-solution to (2.3). In this proof we introduce  $\text{cl}(\bar{\mathcal{C}}) := \{(t, z, p) \in (0, T] \times \mathcal{O}_+^d \times \mathbb{R}^+ \times [0, 1] \text{ s.t. } y \geq v(t, x, p)\}$  and define similarly  $\bar{\mathcal{D}}_2, \bar{\mathcal{D}}_3, \text{int}(\bar{\mathcal{D}}_1), \partial_Z \bar{\mathcal{D}}_1$ , and  $\partial_T \bar{\mathcal{D}}_1$ . We will prove that  $U - V \leq 0$  on  $\text{cl}(\bar{\mathcal{C}})$ . To this aim we will first argue by contradiction and show that for all  $\xi > 0$  we have  $U_\xi - V_\xi \leq 0$ . The result will then be obtained sending  $\xi$  to zero.

**Step 1.** We argue by contradiction and assume that there exists  $\xi > 0$  such that

$$M := \sup_{\text{cl}(\bar{\mathcal{C}})} (U_\xi - V_\xi)(t, z, p) = \gamma > 0. \quad (2.27)$$

For  $\varepsilon > 0$ , we introduce on  $(0, T] \times (\mathcal{O}_+^d)^2 \times (\mathbb{R}^+)^2 \times [0, 1]^2$ ,

$$\Psi_\varepsilon(t, x, r, y, l, p, q) := U_\xi(t, x, y, p) - V_\xi(t, r, l, q) - \frac{1}{\varepsilon} (|x - r|^2 + |y - l|^2 + |p - q|^2). \quad (2.28)$$

Appealing to the growth conditions and semi-continuity of  $U$  and  $V$ , we obtain that for  $\varepsilon > 0$  the function  $\Psi_\varepsilon$  admits a maximum  $M_\varepsilon$  at  $(t_\varepsilon, x_\varepsilon, r_\varepsilon, y_\varepsilon, l_\varepsilon, p_\varepsilon, q_\varepsilon)$  on  $(0, T] \times (\mathcal{O}_+^d)^2 \times (\mathbb{R}^+)^2 \times ([0, 1])^2$  under the constraint  $y \geq v(t, x, p)$  and  $l \geq v(t, r, q)$ . Using standard arguments (see for instance, Pham (2009), Theorem 4.4.4, and Crandall et al. (1992), Lemma 3.1), one can prove that there exists  $(\hat{t}, \hat{z}, \hat{p}) \in \text{cl}(\bar{\mathcal{C}})$  such that

$$\begin{cases} \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} (|x_\varepsilon - r_\varepsilon|^2 + |y_\varepsilon - l_\varepsilon|^2 + |p_\varepsilon - q_\varepsilon|^2) = 0, \\ M = \lim_{\varepsilon \downarrow 0} M_\varepsilon = (U_\xi - V_\xi)(\hat{t}, \hat{z}, \hat{p}). \end{cases} \quad (2.29)$$

If  $(\hat{t}, \hat{z}, \hat{p}) \in \partial_Z \bar{\mathcal{D}}_1 \cup \partial_T \bar{\mathcal{D}}_1 \cup \bar{\mathcal{D}}_2 \cup \bar{\mathcal{D}}_3$ , we obtain a contradiction with the assumptions made on  $V$  and  $U$  on these boundaries. We therefore assume that  $(\hat{t}, \hat{z}, \hat{p}) \in \text{int}(\bar{\mathcal{D}}_1)$  and therefore that, up to a subsequence,  $(t_\varepsilon, x_\varepsilon, y_\varepsilon, p_\varepsilon) \in \text{int}(\bar{\mathcal{D}}_1)$  and  $(t_\varepsilon, r_\varepsilon, l_\varepsilon, q_\varepsilon) \in \text{int}(\bar{\mathcal{D}}_1)$ .

**Step 2.** From Ishii's Lemma (see Crandall et al. (1992), Theorem 8.3) we obtain the existence of real coefficients  $b_\varepsilon^1, b_\varepsilon^2$ , a vector  $d_\varepsilon$  and two symmetric matrices  $\mathcal{X}_\varepsilon$  and  $\mathcal{R}_\varepsilon$  being such that

$$(b_\varepsilon^1, d_\varepsilon, \mathcal{X}_\varepsilon) \in \bar{\mathcal{J}}_{\bar{\mathcal{O}}}^+ U_\xi(t_\varepsilon, x_\varepsilon, y_\varepsilon, p_\varepsilon) \quad \text{and} \quad (-b_\varepsilon^2, d_\varepsilon, \mathcal{R}_\varepsilon) \in \bar{\mathcal{J}}_{\bar{\mathcal{O}}}^- V_\xi(t_\varepsilon, r_\varepsilon, l_\varepsilon, q_\varepsilon), \quad (2.30)$$

with  $\bar{\mathcal{O}} := \text{int}(\bar{\mathcal{D}}_1)$  and  $\bar{\mathcal{J}}^+$  (resp.  $\bar{\mathcal{J}}^-$ ) the limiting second-order super-jet (resp. sub-jet) of  $U_\xi$  (resp.  $V_\xi$ ) at  $(t_\varepsilon, x_\varepsilon, y_\varepsilon, p_\varepsilon) \in \bar{\mathcal{O}}$  (resp.  $(t_\varepsilon, r_\varepsilon, l_\varepsilon, q_\varepsilon) \in \bar{\mathcal{O}}$ ) and where

$$b_\varepsilon^1 + b_\varepsilon^2 := 0, \quad (2.31a)$$

$$d_\varepsilon := \frac{2}{\varepsilon} \begin{pmatrix} x_\varepsilon - r_\varepsilon \\ y_\varepsilon - l_\varepsilon \\ p_\varepsilon - q_\varepsilon \end{pmatrix}, \quad (2.31b)$$

$$\begin{pmatrix} \mathcal{X}_\varepsilon & 0 \\ 0 & -\mathcal{R}_\varepsilon \end{pmatrix} \leq \frac{10}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \quad (2.31c)$$

with  $I$  the  $d+2$ -dimensional identity matrix. Thanks to the definition of  $U_\xi$  and  $V_\xi$  we know that they are respectively sub-/super-solution of (2.3). As a result, Lemma B.1 gives

$$\inf_{\beta \in \mathcal{S}_{2d+1}} F^\beta(t_\varepsilon, x_\varepsilon, y_\varepsilon, b_\varepsilon^1, d_\varepsilon, \mathcal{X}_\varepsilon) \leq 0, \quad (2.32)$$

$$\inf_{\beta \in \mathcal{S}_{2d+1}} F^\beta(t_\varepsilon, r_\varepsilon, l_\varepsilon, -b_\varepsilon^2, d_\varepsilon, \mathcal{R}_\varepsilon) \geq \xi \rho(l_\varepsilon) > 0, \quad (2.33)$$

where  $\rho(l_\varepsilon) := \left( \frac{\kappa \wedge 4}{\varepsilon^4} \wedge \frac{\bar{\Lambda}^\mu}{2} (l_\varepsilon)^m \right)$ . Hence

$$\inf_{\beta \in \mathcal{S}_{2d+1}} F^\beta(t_\varepsilon, r_\varepsilon, l_\varepsilon, -b_\varepsilon^2, d_\varepsilon, \mathcal{R}_\varepsilon) - \inf_{\beta \in \mathcal{S}_{2d+1}} F^\beta(t_\varepsilon, x_\varepsilon, y_\varepsilon, b_\varepsilon^1, d_\varepsilon, \mathcal{X}_\varepsilon) \geq \xi \rho(l_\varepsilon). \quad (2.34)$$

**Step 3.** On the other hand, Lemma B.2 and (2.31) provide the existence of  $C > 0$  such that

$$\begin{aligned} & \inf_{\beta \in \mathcal{S}_{2d+1}} F^\beta(t_\varepsilon, r_\varepsilon, l_\varepsilon, -b_\varepsilon^2, d_\varepsilon, \mathcal{R}_\varepsilon) - \inf_{\beta \in \mathcal{S}_{2d+1}} F^\beta(t_\varepsilon, x_\varepsilon, y_\varepsilon, b_\varepsilon^1, d_\varepsilon, \mathcal{X}_\varepsilon) \\ & \leq \frac{C}{\varepsilon} (1 + |x_\varepsilon|^2 + |r_\varepsilon|^2 + |y_\varepsilon|^2 + |l_\varepsilon|^2) (|x_\varepsilon - r_\varepsilon|^2 + |y_\varepsilon - l_\varepsilon|^2). \end{aligned} \quad (2.35)$$

Finally, sending  $\varepsilon$  to zero and using (2.29) we find that the last inequality is non positive which contradicts (2.34) as  $l_\varepsilon \rightarrow \hat{y} > 0$  (recall that  $(\hat{t}, \hat{z}, \hat{p}) \in \text{int}(\bar{\mathcal{D}}_1)$  and then  $\hat{y} > v(\hat{t}, \hat{x}, \hat{p}) \geq 0$ ). Therefore  $U_\xi \leq V_\xi$  for all  $\xi > 0$  on  $\text{cl}(\bar{\mathcal{C}})$  and we conclude sending  $\xi$  to zero.  $\square$

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