

DENSITY OF RATIONAL POINTS ON A QUADRIC BUNDLE IN $\mathbb{P}^3 \times \mathbb{P}^3$

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ABSTRACT. An asymptotic formula is established for the number of rational points of bounded anticanonical height which lie on a certain Zariski dense subset of the biprojective hypersurface

$$x_1y_1^2 + \cdots + x_4y_4^2 = 0$$

in $\mathbb{P}^3 \times \mathbb{P}^3$. This confirms the modified Manin conjecture for this variety, in which the removal of a “thin” set of rational points is allowed.

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1. INTRODUCTION

The main goal of this paper is to study the density of rational points on the biprojective hypersurface $X \subset \mathbb{P}^3 \times \mathbb{P}^3$ cut out by the equation $F(\mathbf{x}; \mathbf{y}) = 0$, where

$$F(\mathbf{x}; \mathbf{y}) = x_1y_1^2 + \cdots + x_4y_4^2.$$

Let $H : X(\mathbb{Q}) \rightarrow \mathbb{R}_{\geq 0}$ be the anticanonical height function and let

$$N(\Omega, B) = \# \{(x, y) \in \Omega : H(x, y) \leq B\}, \tag{1.1}$$

for any subset $\Omega \subset X(\mathbb{Q})$. The variety X defines a smooth hypersurface of bidegree $(1, 2)$ and has Picard group $\text{Pic}(X) \cong \mathbb{Z}^2$. If a point $(x, y) \in X(\mathbb{Q})$ is represented by a vector $(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}_{\text{prim}}^4 \times \mathbb{Z}_{\text{prim}}^4$, then $H(x, y) = |\mathbf{x}|^3 |\mathbf{y}|^2$, where

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$|\cdot| : \mathbb{R}^4 \rightarrow \mathbb{R}_{\geq 0}$ is the sup-norm. In view of the Manin Conjecture [9], one might expect that there is a Zariski open subset $U \subset X$ such that

$$N(U(\mathbb{Q}), B) \sim cB \log B,$$

as $B \rightarrow \infty$, where c is the constant predicted by Peyre [18]. We certainly require U to exclude all subvarieties of the form $x_i = x_j = x_k = y_l = 0$ for $\{i, j, k, l\} = \{1, 2, 3, 4\}$, since the rational points on X which satisfy these constraints are easily seen to contribute $\gg B^{3/2}$ to $N(X(\mathbb{Q}), B)$. Similarly, we get a contribution of the same order of magnitude from rational points for which $\mathbf{x} = (0, 0, 1, -1)$ and $\mathbf{y} = (a, b, c, c)$, for example.

More interestingly, any fibre $X_x = \pi_1^{-1}(x)$ over a point $x \in \mathbb{P}^3(\mathbb{Q})$ such that $X_x \cong \mathbb{P}^1 \times \mathbb{P}^1$ will contribute $\sim c_x B \log B$, as $B \rightarrow \infty$, for an appropriate constant $c_x > 0$ depending on x . It is expected that the total contribution from these rational points will be $\sim aB \log B$, where

$$a = \sum_{\substack{x \in \mathbb{P}^3(\mathbb{Q}) \\ X_x \cong \mathbb{P}^1 \times \mathbb{P}^1}} c_x$$

is a convergent series. Note that if $x = [x_1, \dots, x_4] \in \mathbb{P}^3(\mathbb{Q})$ then the isomorphism $X_x \cong \mathbb{P}^1 \times \mathbb{P}^1$ holds if and only if $X_x(\mathbb{Q}) \neq \emptyset$ and $x_1 \dots x_4$ is a square in \mathbb{Q}^* . In view of this we are led to study (1.1) when Ω is obtained from $X(\mathbb{Q})$ by deleting the set

$$T = \{(x, y) \in X(\mathbb{Q}) : x_1 \dots x_4 = \square\}. \quad (1.2)$$

Our main result is then the following.

Theorem 1.1. *Let $\Omega = X(\mathbb{Q}) \setminus T$. Then*

$$N(\Omega, B) \sim cB \log B$$

as $B \rightarrow \infty$, where

$$c = \frac{\tau_\infty}{4\zeta(3)\zeta(4)} \quad (1.3)$$

is the Peyre constant for the variety X , with

$$\tau_\infty = \int_{-\infty}^{\infty} \int_{[-1,1]^8} e(-\theta F(\mathbf{x}; \mathbf{y})) d\mathbf{x} d\mathbf{y} d\theta. \quad (1.4)$$

This answers a question that was originally raised by Colliot-Thélène, and mentioned in work of Batyrev and Tschinkel [2, Ex. 3.5.3] over 20 years ago. The set T in (1.2) is an example of a “thin” set of rational points, as introduced to the subject by Serre [21, §3.1]. Theorem 1.1 therefore confirms the refined Manin conjecture for X , in which one is allowed to remove a finite number of thin sets. Forthcoming work of Lehmann, Sengupta and Tanimoto [16] develops a geometric framework for identifying the relevant thin sets for

any Fano variety. It would be interesting to determine whether our set T is compatible with their predictions for the quadric bundle X .

Our result adds to the small store of examples in which thin sets have been shown to exert a demonstrable influence on the distribution of rational points on Fano varieties. One of the first examples in this vein was discovered by Batyrev and Tschinkel [1], who showed that the split cubic surfaces in the biprojective hypersurface $\{x_1y_1^3 + \cdots + x_4y_4^3 = 0\} \subset \mathbb{P}^3 \times \mathbb{P}^3$ contribute significantly more than the Manin conjecture would predict for the number of rational points of bounded anticanonical height. More recently, Le Rudulier [17] has investigated Manin's conjecture for the Hilbert schemes $\text{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$ and $\text{Hilb}^2(\mathbb{P}^2)$, with the outcome that a thin set of rational points needs to be removed in order for the associated counting functions to behave as they should.

The basic line of attack in the proof of Theorem 1.1 involves counting points on X as a union of planes when \mathbf{y} is small, and as a union of quadric surfaces when \mathbf{x} is small. In the first case \mathbf{x} lies in a lattice determined by \mathbf{y} , and we will use counting arguments that come from the geometry of numbers. In the second case we can count vectors \mathbf{y} using the circle method, taking care to control the dependence of the error terms on \mathbf{x} . It turns out that we can handle the case $|\mathbf{y}| \leq B^{1/4}$, giving an asymptotic formula, using lattices. Moreover we can deal with the range $B^\delta \leq |\mathbf{x}| \leq B^{1/6-\delta}$ via the circle method, for any fixed $\delta > 0$. In terms of the inequality $|\mathbf{x}|^3|\mathbf{y}|^3 \leq B$, this leaves two small ranges uncovered, and here it will suffice to use an upper bound of the correct order of magnitude. Indeed such an upper bound is also indispensable as an auxiliary tool in the treatment of the lattice point counting problem. The range $B^{1/6-\delta} \leq |\mathbf{x}| \leq B^{1/6}$ contributes $O(\delta B \log B)$ to $N(\Omega, B)$, and this is $o(B \log B)$ when we allow δ to tend to 0. However, we do not obtain an explicit error term, though it would be possible in principle to do so, by examining more closely the dependence on δ in our other estimates.

This paper is naturally arranged in three main parts. We begin by discussing upper bounds in §2. We go on to use these in proving our asymptotic formula for the range $|\mathbf{y}| \leq B^{1/4}$, using lattice point counting in §3. Thirdly, we develop our circle method argument in §4, to deal with values $|\mathbf{x}| \leq B^{1/6-\delta}$. Once all this is in place, it remains in §5 and §6 to combine the various results and consider the overall leading constant that arises.

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2. UPPER BOUNDS

We begin by introducing some notation. Let $\Delta(\mathbf{x}) = x_1x_2x_3x_4$. For $Y \geq 1$ and any fixed $\mathbf{x} \in \mathbb{Z}^4$ we let

$$M_1(\mathbf{x}; Y) = \# \{ \mathbf{y} \in \mathbb{Z}_{\text{prim}}^4 : |\mathbf{y}| \leq Y, F(\mathbf{x}; \mathbf{y}) = 0 \}.$$

We then set

$$M_2(X, Y) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}^4 \\ \Delta(\mathbf{x}) \neq \square, |\mathbf{x}| \leq X}} M_1(\mathbf{x}; Y),$$

so that $M_2(X, Y)$ counts solutions $(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}^4 \times \mathbb{Z}_{\text{prim}}^4$ of $F(\mathbf{x}; \mathbf{y}) = 0$ in the region $|\mathbf{x}| \leq X, |\mathbf{y}| \leq Y$, such that $\Delta(\mathbf{x}) \neq \square$. Similarly, we write

$$M_3(X, Y) = \sum_{\substack{\mathbf{x} \in (\mathbb{Z}_{\neq 0})^4 \\ |\mathbf{x}| \leq X}} M_1(\mathbf{x}; Y).$$

The primary result in this section is the following collection of upper bounds.

Lemma 2.1. *We have*

$$M_2(X, Y) \ll X^3Y^2 + X^5Y^{2/3}$$

and

$$M_3(X, Y) \ll_{\varepsilon} X^3Y^2 + X^5Y^{2/3} + X^{2+\varepsilon}Y^{2+\varepsilon},$$

for any $\varepsilon > 0$. Moreover, if $1 \leq X_1 \leq X$ then

$$\sum_{\substack{\mathbf{x} \in (\mathbb{Z}_{\neq 0})^4 \\ |\mathbf{x}| \leq X, |x_1| \leq X_1}} M_1(\mathbf{x}; Y) \ll_{\varepsilon} (XY)^{\varepsilon} X_1^{3/4} X^{-3/4} \{X^3Y^2 + X^4Y\},$$

for any $\varepsilon > 0$.

The principal tool that we will use is the authors' result [5, Thm. 1.1]. To state this we introduce the arithmetic function

$$\varpi(m) = \prod_{p|m} (1 + p^{-1}), \tag{2.1}$$

along with the notation

$$\Delta_{\text{bad}}(\mathbf{x}) = \prod_{\substack{p^e || x_1 \dots x_4 \\ e \geq 2}} p^e \tag{2.2}$$

for $\mathbf{x} \in (\mathbb{Z}_{\neq 0})^4$. We then have the following.

Lemma 2.2. *Let $\mathbf{x} \in (\mathbb{Z}_{\neq 0})^4$ and $\Delta_{\text{bad}}(\mathbf{x}) \leq Y^{1/20}$. Then*

$$M_1(\mathbf{x}; Y) \ll \varpi(\Delta(\mathbf{x})) \Delta_{\text{bad}}(\mathbf{x})^{1/3} \left(\frac{|\mathbf{x}|^4}{|\Delta(\mathbf{x})|} \right)^{5/8} L(\sigma_Y, \chi) \left(Y^{4/3} + \frac{Y^2}{|\Delta(\mathbf{x})|^{1/4}} \right),$$

where $\sigma_Y = 1 + \frac{1}{\log Y}$ and $\chi(n)$ is the Dirichlet character defined by taking $\chi(2) = 0$ and

$$\chi(p) = \left(\frac{\Delta(\mathbf{x})}{p} \right)$$

for odd primes p . The implied constant in this estimate is absolute.

In order to apply this to Lemma 2.1 we will need to handle vectors with $\Delta_{\text{bad}}(\mathbf{x}) > Y^{1/20}$ via a separate auxiliary bound. Indeed various auxiliary bounds will be used elsewhere in the proof of Theorem 1.1, and it is therefore natural to begin this section by dealing with these.

2.1. Auxiliary upper bounds. We begin by recording a uniform upper bound for the counting function for rational points on quadric surfaces. The following result is due to Heath-Brown [13, Thm. 2].

Lemma 2.3. *For any irreducible quadratic form $Q(\mathbf{y}) \in \mathbb{Z}[y_1, \dots, y_4]$ and any $\varepsilon > 0$ we have*

$$\#\{\mathbf{y} \in \mathbb{Z}^4 : |\mathbf{y}| \leq Y, Q(\mathbf{y}) = 0\} \ll_{\varepsilon} Y^{2+\varepsilon}.$$

We next examine two counting problems involving fewer than 4 terms.

Lemma 2.4. *Let $X, Y \geq 1$. Then*

$$\#\{(x_1, y_1, x_2, y_2) \in (\mathbb{Z}_{\neq 0})^4 : |x_i| \leq X, |y_i| \leq Y, x_1 y_1^2 = x_2 y_2^2\} = O(XY).$$

Proof. We extract common factors $h = (x_1, x_2)$ and $k = (y_1, y_2)$. For each h and k we are left with counting non-zero integers x'_i, y'_i with $|x'_i| \leq X/h$ and $|y'_i| \leq Y/k$, such that $x'_1 = \pm y'_2{}^2$ and $x'_2 = \pm y'_1{}^2$. The number of such integers is

$$\ll \left(\min \left\{ \sqrt{X/h}, Y/k \right\} \right)^2 \leq (\sqrt{X/h})^{4/3} (Y/k)^{2/3}.$$

Summing over $h \leq X$ and $k \leq Y$ gives $O(XY)$ as required. \square

Lemma 2.5. *Let $X, Y, U, V \geq 1$ with $XY^2 = UV^2$. Define $T(X, Y, U, V)$ to be the number of solutions $\mathbf{x}, \mathbf{y} \in (\mathbb{Z}_{\neq 0})^3$ of the equation*

$$x_1 y_1^2 + x_2 y_2^2 + x_3 y_3^2 = 0,$$

with

$$|x_1|, |x_2| \leq X, |x_3| \leq U, |y_1|, |y_2| \leq Y, |y_3| \leq V.$$

Then $T(X, Y, U, V) \ll_{\varepsilon} XUVY^{\varepsilon}$ for any $\varepsilon > 0$.

Proof. For the proof we first estimate the quantity $T^*(X, Y, U, V)$ which counts pairs of vectors \mathbf{x}, \mathbf{y} as for $T(X, Y, U, V)$, but with the added restrictions that

\mathbf{x} and \mathbf{y} should be primitive. According to Heath-Brown [10, Lemma 3] there are

$$\begin{aligned} &\ll 1 + \frac{X^2U}{\max(Xy_1^2, Xy_2^2, Uy_3^2)} \\ &\ll 1 + X^2U(Xy_1^2)^{-1/3}(Xy_2^2)^{-1/3}(Uy_3^2)^{-1/3} \end{aligned}$$

primitive solutions \mathbf{x} for each primitive \mathbf{y} . Summing over \mathbf{y} and using the relation $XY^2 = UV^2$ then yields

$$T^*(X, Y, U, V) \ll Y^2V + XUV. \quad (2.3)$$

In particular $T^*(X, Y, U, V) \ll XUV$ if $Y^2 \leq X$.

Alternatively we may use Corollary 2 of Browning and Heath-Brown [4]. This shows that for any given $\mathbf{x} \in (\mathbb{Z}_{\neq 0})^3$ there are

$$\ll_{\varepsilon} \left\{ 1 + \left(\frac{Y^2V(x_1x_2, x_1x_3, x_2x_3)^{3/2}}{|x_1x_2x_3|} \right)^{1/3} \right\} (XY)^{\varepsilon}$$

corresponding primitive \mathbf{y} . Let $n = |x_1x_2x_3|$ and $d = (x_1x_2, x_1x_3, x_2x_3)$, so that $d^3 \mid n^2$. Summing over the \mathbf{x} then gives an estimate

$$\begin{aligned} T^*(X, Y, U, V) &\ll_{\varepsilon} X^2U(XY)^{\varepsilon} + (Y^2V)^{1/3}(XY)^{2\varepsilon} \sum_{d \leq X^2U} d^{1/2} \sum_{\substack{n \leq X^2U \\ d^3 \mid n^2}} n^{-1/3} \\ &\ll_{\varepsilon} X^2U(XY)^{\varepsilon} + (Y^2V)^{1/3}(XY)^{2\varepsilon} \sum_{d \leq X^2U} d^{1/2} (X^2U)^{2/3} d^{-3/2} \\ &\ll_{\varepsilon} \{X^2U + (Y^2V)^{1/3}(X^2U)^{2/3}\} (XY)^{3\varepsilon} \\ &= \{X^2U + XUV\} (XY)^{3\varepsilon}. \end{aligned}$$

In particular $T^*(X, Y, U, V) \ll \{X^2U + XUV\}Y^{9\varepsilon}$ if $Y^2 \geq X$, and by our remark above the same is true in the alternative case $Y^2 \leq X$ as well.

Comparing this bound with (2.3) shows that

$$T^*(X, Y, U, V) \ll_{\varepsilon} \{\min(X^2U, Y^2V) + XUV\}Y^{9\varepsilon}$$

whether $Y^2 \geq X$ or not. However

$$\min(X^2U, Y^2V) \leq (X^2U)^{2/3}(Y^2V)^{1/3} = XUV,$$

whence

$$T^*(X, Y, U, V) \ll_{\varepsilon} XUVY^{9\varepsilon}.$$

We then find that

$$\begin{aligned} T(X, Y, U, V) &\leq \sum_{d \leq X} \sum_{e \leq Y} T^*(X/d, Y/e, U/d, V/e) \\ &\ll_{\varepsilon} \sum_{d \leq X} \sum_{e \leq Y} XUVY^{9\varepsilon} d^{-2} e^{-1-9\varepsilon} \\ &\ll_{\varepsilon} XUVY^{9\varepsilon}. \end{aligned}$$

The lemma now follows on redefining ε . \square

Lemmas 2.4 and 2.5 have the following corollary.

Lemma 2.6. *Let $X, Y \geq 1$. Then the number of solutions $\mathbf{x} \in \mathbb{Z}^4$, $\mathbf{y} \in \mathbb{Z}_{\text{prim}}^4$ of the equation $F(\mathbf{x}; \mathbf{y}) = 0$, lying in the region $|\mathbf{x}| \leq X$, $|\mathbf{y}| \leq Y$, and satisfying the constraint that $\prod_{i=1}^4 x_i y_i = 0$, is $O_{\varepsilon}(X^3 Y^{1+\varepsilon} + Y^4)$, for any $\varepsilon > 0$.*

Proof. Suppose firstly that exactly one of the products $x_i y_i$ vanishes, $x_4 y_4 = 0$ say. Then there are $O(X + Y)$ choices for x_4 and y_4 , and $O_{\varepsilon}(X^2 Y^{1+\varepsilon})$ choices for the remaining variables, by Lemma 2.5. Thus the total contribution is $O_{\varepsilon}(X^3 Y^{1+\varepsilon} + X^2 Y^{2+\varepsilon})$. This is satisfactory for the lemma after redefining ε , since $X^2 Y^{2+\varepsilon} \leq \max(X^3 Y^{1+2\varepsilon}, Y^4)$.

Suppose next that exactly two of the terms $x_i y_i$ vanish, $x_3 y_3 = x_4 y_4 = 0$, say. There are then $O(X^2 + Y^2)$ choices for x_3, y_3, x_4 and y_4 . Moreover there are $O(XY)$ choices for the remaining variables, by Lemma 2.4. We therefore have a total $O(X^3 Y + XY^3)$, which is satisfactory since $XY^3 \leq \max(X^3 Y, Y^4)$.

It is not possible for exactly three terms $x_i y_i$ to vanish, when $F(\mathbf{x}; \mathbf{y}) = 0$, so the only remaining case is that in which $x_i y_i = 0$ for every index i . Since \mathbf{y} is primitive it cannot vanish, and hence there are $O(X^3 Y + Y^4)$ possibilities in this situation. This completes the proof of the lemma. \square

The final case to consider is that in which the product $x_1 \dots x_4$ is non-zero but has a large square divisor. Let

$$\Delta_{\text{bad}}(\mathbf{x}) = \prod_{\substack{p^e \parallel x_1 \dots x_4 \\ e \geq 2}} p^e,$$

as in (2.2). The following result shows that vectors \mathbf{x} with a large value of $\Delta_{\text{bad}}(\mathbf{x})$ make a small contribution to $M_3(X, Y)$.

Lemma 2.7. *Let $D \geq 1$ and let $\varepsilon > 0$. Then*

$$\sum_{\substack{\mathbf{x} \in (\mathbb{Z}_{\neq 0})^4 \\ |\mathbf{x}| \leq X \\ \Delta_{\text{bad}}(\mathbf{x}) \geq D}} M_1(\mathbf{x}; Y) \ll_{\varepsilon} (XY)^{\varepsilon} \left\{ \frac{X^3 Y^2}{D^{1/24}} + X^4 Y \right\}.$$

In general, if we write $s(n)$ for the largest square-full divisor of n , then $s(uv) \mid s(u)s(v)(u, v)^2$, as one sees by considering the case in which u and v are powers of the same prime. Thus

$$s(x_1x_2x_3x_4) \mid s(x_1)s(x_2)s(x_3)s(x_4)(x_1x_2, x_3x_4)^2(x_1, x_2)^2(x_3, x_4)^2,$$

and since

$$(x_1x_2, x_3x_4) \mid (x_1, x_3)(x_1, x_4)(x_2, x_3)(x_2, x_4)$$

we see that if $\Delta_{\text{bad}}(\mathbf{x}) \geq D$ then either $(x_i, x_j) \geq D^{1/24}$ for some pair of indices $i \neq j$, or $s(x_i) \geq D^{1/8}$ for some i . In the latter case $d^2 \mid x_i$ for some $d \geq D^{1/24}$. Hence

$$\sum_{\substack{\mathbf{x} \in (\mathbb{Z}_{\neq 0})^4 \\ |\mathbf{x}| \leq X \\ \Delta_{\text{bad}}(\mathbf{x}) \geq D}} M_1(\mathbf{x}; Y) \leq \sum_{d \geq D^{1/24}} \sum_{\mathbf{x}} M_1(\mathbf{x}; Y),$$

where the \mathbf{x} -summation is over $\mathbf{x} \in (\mathbb{Z}_{\neq 0})^4$ such that $|\mathbf{x}| \leq X$, with $d^2 \mid x_i$ or $d \mid (x_i, x_j)$, for some choice of distinct indices $i, j \in \{1, 2, 3, 4\}$.

For any $k \in \mathbb{N}$ we write

$$S_k(\alpha) = S_k(\alpha; X) = \sum_{\substack{0 < |x| \leq X \\ k \mid x}} \sum_{|y| \leq Y} e(\alpha xy^2). \quad (2.4)$$

Then

$$\sum_{\mathbf{x}} M_1(\mathbf{x}; Y) \ll I_1(d) + I_2(d),$$

where

$$I_1(d) = \int_0^1 S_d(\alpha)^2 S_1(\alpha)^2 d\alpha \quad \text{and} \quad I_2(d) = \int_0^1 S_{d^2}(\alpha) S_1(\alpha)^3 d\alpha.$$

We note that $I_j(d) = 0$ unless $d^j \leq X$. In estimating these integrals we are led to an auxiliary counting problem, treated in the following result.

Lemma 2.8. *Let $k \in \mathbb{N}$ and let $X, Y \geq 1$. Let $L_k(X, Y)$ denote the number of $(x, y, y', x_1, y_1, x_2, y_2) \in \mathbb{Z}^7$ such that $k \mid x$, with*

$$0 < |x|, |x_1|, |x_2| \leq X, \quad |y|, |y'|, |y_1|, |y_2| \leq Y$$

and

$$x_1 y_1^2 - x_2 y_2^2 = x(y - y')(y + y').$$

Then for any $\varepsilon > 0$ we have

$$L_k(X, Y) \ll_{\varepsilon} (XY)^{\varepsilon} \left\{ \frac{X^2 Y^2 + X^3 Y}{k} + k X^2 \right\}.$$

We will establish this in a moment, but first we show how it may be used to complete the proof of Lemma 2.7. In general we have

$$|S_d(\alpha)|^2 \leq \#\{x \in \mathbb{Z}_{\neq 0} \cap [-X, X] : d \mid x\} \times \sum_{\substack{0 < |x| \leq X \\ d \mid x}} \sum_{|y|, |y'| \leq Y} e(\alpha x(y^2 - y'^2)), \quad (2.5)$$

by Cauchy–Schwarz. We therefore deduce from Lemma 2.8 that

$$\begin{aligned} \sum_{d \geq D^{1/24}} I_1(d) &\leq \sum_{D^{1/24} \leq d \leq X} \frac{2X}{d} L_d(X, Y) \\ &\ll_{\varepsilon} \sum_{D^{1/24} \leq d \leq X} (XY)^{\varepsilon} \left\{ \frac{X^3 Y^2 + X^4 Y}{d^2} + X^3 \right\} \\ &\ll_{\varepsilon} (XY)^{\varepsilon} \left\{ \frac{X^3 Y^2 + X^4 Y}{D^{1/24}} + X^4 \right\}. \end{aligned}$$

This is satisfactory for Lemma 2.7.

To handle $I_2(d)$ we apply the Cauchy–Schwarz inequality, yielding

$$|I_2(d)|^2 \leq \left(\int_0^1 |S_{d^2}(\alpha)|^2 |S_1(\alpha)|^2 d\alpha \right) \left(\int_0^1 |S_1(\alpha)|^4 d\alpha \right). \quad (2.6)$$

We proceed as before, using (2.5) and Lemma 2.8 to deduce that

$$\int_0^1 |S_{d^2}(\alpha)|^2 |S_1(\alpha)|^2 d\alpha \leq \frac{2X}{d^2} L_{d^2}(X, Y) \ll_{\varepsilon} (XY)^{\varepsilon} \left\{ \frac{X^3 Y^2 + X^4 Y}{d^4} + X^3 \right\}.$$

Taking $d = 1$ we see that the second factor in (2.6) is

$$\int_0^1 |S_1(\alpha)|^4 d\alpha \ll_{\varepsilon} (XY)^{\varepsilon} (X^3 Y^2 + X^4 Y). \quad (2.7)$$

Thus

$$I_2(d) \ll_{\varepsilon} (XY)^{\varepsilon} \left\{ \frac{X^3 Y^2 + X^4 Y}{d^2} + X^3 Y + X^{7/2} Y^{1/2} \right\},$$

whence

$$\begin{aligned} \sum_{d \geq D^{1/24}} I_2(d) &\ll_{\varepsilon} \sum_{D^{1/24} \leq d \leq \sqrt{X}} (XY)^{\varepsilon} \left\{ \frac{X^3 Y^2 + X^4 Y}{d^2} + X^3 Y + X^{7/2} Y^{1/2} \right\} \\ &\ll_{\varepsilon} (XY)^{\varepsilon} \left\{ \frac{X^3 Y^2 + X^4 Y}{D^{1/24}} + X^{7/2} Y + X^4 Y^{1/2} \right\}. \end{aligned}$$

This too is satisfactory for Lemma 2.7.

Proof of Lemma 2.8. Clearly $L_k(X, Y) = 0$ unless $k \leq X$, which we now assume. There are $O(k^{-1}XY)$ triples with $x(y - y')(y + y') = 0$, and for each there are $O(XY)$ corresponding quadruples x_1, y_1, x_2, y_2 with $y_1 y_2 \neq 0$, by Lemma 2.4, and there are $O(X^2)$ quadruples with $y_1 y_2 = 0$. This case therefore contributes a total $O(k^{-1}(X^2 Y^2 + X^3 Y))$ to $L_k(X, Y)$.

When $x(y - y')(y + y') \neq 0$, we get

$$\ll \tau_3(|x_1 y_1^2 - x_2 y_2^2|) \ll_\varepsilon (XY)^\varepsilon$$

solutions x, y, y' . It therefore remains to count the number of x_1, y_1, x_2, y_2 for which $x_1 y_1^2 \equiv x_2 y_2^2 \pmod k$. Breaking into residue classes modulo k we deduce that

$$L_k(X, Y) \ll_\varepsilon \frac{X^2 Y^2 + X^3 Y}{k} + (XY)^\varepsilon \frac{X^2}{k^2} \left(1 + \frac{Y}{k}\right)^2 \varrho(k), \quad (2.8)$$

where

$$\varrho(k) = \#\{(x_1, x_2, y_1, y_2) \in (\mathbb{Z}/k\mathbb{Z})^4 : x_1 y_1^2 \equiv x_2 y_2^2 \pmod k\}.$$

Since $\varrho(k)$ is a multiplicative arithmetic function it suffices to estimate $\varrho(p^e)$. When $p \nmid y_1$ the values of x_2, y_1, y_2 determine x_1 , so that there are at most $p^{2e} \varphi(p^e)$ such solutions. The same argument applies when $p \nmid y_2$ so that there are at most $2p^{2e} \varphi(p^e)$ solutions with $p \nmid (y_1, y_2)$. If $e \geq 2$ there are $p^6 \varrho(p^{e-2})$ solutions with $p \mid (y_1, y_2)$, while if $e = 1$ there are p^2 solutions. Hence $\varrho(p) \leq 2p^2(p-1) + p^2 \leq 2p^3$ and

$$\varrho(p^e) \leq 2p^{3e}(1 - p^{-1}) + p^6 \varrho(p^{e-2})$$

for $e \geq 2$. One can now show that $\varrho(p^e) \leq (e+1)p^{3e}$ for all $e \geq 1$, by induction. We then have $\varrho(k) \ll_\varepsilon k^{3+\varepsilon}$ for any $\varepsilon > 0$. We therefore complete the proof of the lemma by inserting this into (2.8) and redefining ε . \square

2.2. Proof of Lemma 2.1. Using Lemma 2.2 we will establish the following result.

Lemma 2.9. *We have*

$$\sum_{\substack{\mathbf{x} \in (\mathbb{Z}_{\neq 0})^4 \\ |\mathbf{x}| \leq X, \Delta(\mathbf{x}) \neq \square \\ \Delta_{\text{bad}}(\mathbf{x}) \leq Y^{1/20}}} M_1(\mathbf{x}; Y) \ll X^3 Y^2 + X^4 Y^{4/3} \ll X^3 Y^2 + X^5 Y^{2/3}.$$

Proof. The second inequality follows since $X^4 Y^{4/3} \leq \max(X^3 Y^2, X^5 Y^{2/3})$. To prove the first inequality, we begin by considering dyadic ranges

$$X_i/2 < |x_i| \leq X_i, \quad \text{for } 1 \leq i \leq 4 \quad (2.9)$$

and denote the corresponding contribution $M(X_1, \dots, X_4; Y)$. Then, on writing $\widehat{X} = X_1 X_2 X_3 X_4$ and

$$c(X_1, \dots, X_4; Y) = \left(\frac{\max X_i^4}{\widehat{X}} \right)^{5/8} \left(Y^{4/3} + \frac{Y^2}{\widehat{X}^{1/4}} \right),$$

it follows from Lemma 2.2 that

$$M(X_1, \dots, X_4; Y) \ll c(X_1, \dots, X_4; Y) \sum_{\mathbf{x}} \varpi(\Delta(\mathbf{x})) \Delta_{\text{bad}}(\mathbf{x})^{1/3} L(\sigma_Y, \chi),$$

where ϖ is given by (2.1) and the sum is for $\mathbf{x} \in \mathbb{Z}^4$ in the region (2.9) such that $\Delta(\mathbf{x}) \neq \square$. Note that we are free to include vectors with $\Delta_{\text{bad}}(\mathbf{x}) > Y^{1/20}$ on the right, since $L(\sigma_Y, \chi) > 0$.

By checking the inequality at prime powers, one easily sees that

$$\varpi(n) \prod_{\substack{p^e \parallel n \\ e \geq 2}} p^{e/3} \leq \sum_{s, t | n} \frac{s^{1/3}}{t},$$

where s and t run over square-full and square-free integers respectively. It follows that

$$\sum_{\mathbf{x}} \varpi(\Delta(\mathbf{x})) \Delta_{\text{bad}}(\mathbf{x})^{1/3} L(\sigma_Y, \chi) \leq \sum_{s, t} \frac{s^{1/3}}{t} \sum_{\substack{d_1, d_2, d_3, d_4 \in \mathbb{N} \\ d_1 d_2 d_3 d_4 = [s, t]}} \sum_{\mathbf{x} \in S} L(\sigma_Y, \chi), \quad (2.10)$$

where s and t run over square-full and square-free integers respectively, and $S = S(X_1, \dots, X_4; d_1, \dots, d_4)$ is the set of $\mathbf{x} \in \mathbb{Z}^4$ in the region (2.9) such that $\Delta(\mathbf{x}) \neq \square$ and $d_i \mid x_i$ for $1 \leq i \leq 4$.

Let $\widehat{d} = d_1 d_2 d_3 d_4$. We claim that

$$\sum_{\mathbf{x} \in S} L(\sigma, \chi) \ll \widehat{X} \widehat{d}^{-7/8}, \quad (2.11)$$

uniformly for $\sigma \geq 1$. We will prove this later, but first we observe that we can now estimate (2.10) as

$$\ll \widehat{X} \sum_{s, t} \frac{s^{1/3}}{t} \tau_4([s, t]) [s, t]^{-7/8}.$$

The infinite sum has an Euler product, with factors of the shape

$$1 + 4p^{-1-7/8} + \sum_{e \geq 2} \sum_{f=0,1} p^{e/3-f} \tau_4(p^e) p^{-7e/8} = 1 + O(p^{-13/12}).$$

The resulting product therefore converges, so that (2.10) is $O(\widehat{X})$. We then see that

$$\begin{aligned} M(X_1, \dots, X_4; Y) &\ll c(X_1, \dots, X_4; Y) \widehat{X} \\ &= (\max X_i)^{5/2} \widehat{X}^{3/8} \left(Y^{4/3} + \frac{Y^2}{\widehat{X}^{1/4}} \right). \end{aligned}$$

On summing over dyadic values for the X_i we obtain Lemma 2.9.

It remains to prove (2.11). Our key tool for the proof is a form of Burgess' estimate [7]. If $\theta > 3/16$ we have

$$\sum_{n \leq N} \psi(n) \ll_{\theta} N^{1/2} q^{\theta},$$

where ψ is any non-principal character of modulus q . We obtain the same bound for the corresponding sum over all integers n such that $|n| \leq N$. For our purposes it will be enough to take $\theta = 1/5$ in these estimates.

The character χ is non-principal, with modulus $O(\widehat{X})$. By the Burgess bound coupled with partial summation, we see that

$$\sum_{n > N} \frac{\chi(n)}{n^{\sigma}} \ll N^{1/2-\sigma} \widehat{X}^{1/5} \ll N^{-1/2} \widehat{X}^{1/5}.$$

It follows that terms with $n > \widehat{X}^{2/5}$ contribute $O(1)$ to $L(\sigma, \chi)$, which is satisfactory since $\widehat{d} \leq \widehat{X}$.

We proceed to consider the terms with $n \leq \widehat{X}^{2/5}$. Suppose that X_i/d_i is largest for $i = 1$, say. If we write $x_1 = d_1 q$ then

$$\sum_{\mathbf{x} \in S} \sum_{n \leq \widehat{X}^{2/5}} \frac{\chi(n)}{n^{\sigma}} = \sum_{x_2, x_3, x_4} \sum_{n \leq \widehat{X}^{2/5}} \frac{1}{n^{\sigma}} \left(\frac{d_1 x_2 x_3 x_4}{n} \right) \sum_q \left(\frac{q}{n} \right),$$

where the sum over q is for integers with $X_1/2d_1 < |q| \leq X_1/d_1$, for which $qd_1 x_2 x_3 x_4$ is a non-square. In general, for any integer k , there are $O(Q^{1/2})$ integers $q \in [-Q, Q]$ for which kq is a square. Thus if we adjust the sum over q to include all integers with $X_1/2d_1 < |q| \leq X_1/d_1$, and then apply the Burgess bound, we find that

$$\sum_q \left(\frac{q}{n} \right) \ll (X_1/d_1)^{1/2} n^{1/5},$$

provided that n is not a square. On the other hand, if n is a square, we have a trivial bound $O(X_1/d_1)$. Thus

$$\begin{aligned} \sum_{n \leq \widehat{X}^{2/5}} \frac{1}{n^\sigma} \left| \sum_q \left(\frac{q}{n} \right) \right| &\ll \sum_{n \leq \widehat{X}^{2/5}} n^{1/5-\sigma} (X_1/d_1)^{1/2} + \sum_{m \leq \widehat{X}^{1/5}} m^{-2\sigma} (X_1/d_1) \\ &\ll \widehat{X}^{2/25} (X_1/d_1)^{1/2} + X_1/d_1. \end{aligned}$$

Since $X_1/d_1 \geq (\widehat{X}/\widehat{d})^{1/4}$ we will have

$$\widehat{X}^{2/25} (X_1/d_1)^{1/2} \leq \widehat{X}^{2/25} (X_1/d_1) (\widehat{X}/\widehat{d})^{-1/8} \leq (X_1/d_1) \widehat{d}^{1/8}.$$

When we sum over x_2, x_3, x_4 we now find that

$$\sum_{\mathbf{x} \in S} \sum_{n \leq \widehat{X}^{2/5}} \frac{\chi(n)}{n^\sigma} \ll \frac{\widehat{X}}{\widehat{d}} \widehat{d}^{1/8} = \widehat{X} \widehat{d}^{-7/8}.$$

This completes the proof of (2.11) and so the proof of the lemma. \square

To finish the proof of Lemma 2.1 we proceed to show how to remove the condition $\Delta_{\text{bad}}(\mathbf{x}) \leq Y^{1/20}$ from Lemma 2.9. It follows from Lemma 2.7 that

$$\sum_{\substack{\mathbf{x} \in (\mathbb{Z}_{\neq 0})^4 \\ |\mathbf{x}| \leq X \\ \Delta_{\text{bad}}(\mathbf{x}) > Y^{1/20}}} M_1(\mathbf{x}; Y) \ll_\varepsilon (XY)^\varepsilon \{X^3 Y^{2-1/480} + X^4 Y\}.$$

When $\varepsilon = 1/800$ we have

$$(XY)^\varepsilon X^3 Y^{2-1/480} = (X^3 Y^2)^{1-1/1600} (X^5 Y^{2/3})^{1/1600} \leq X^3 Y^2 + X^5 Y^{2/3}$$

and

$$(XY)^\varepsilon X^4 Y \leq X^{13/3} Y^{10/9} = (X^3 Y^2)^{1/3} (X^5 Y^{2/3})^{2/3} \leq X^3 Y^2 + X^5 Y^{2/3}.$$

This completes the proof of the first part of Lemma 2.1.

Next, if $|\mathbf{x}| \leq X$ with $\mathbf{x} \in (\mathbb{Z}_{\neq 0})^4$ and $\Delta(\mathbf{x}) = k^2$ say, then $0 < |k| \leq X^2$, and each such k corresponds to at most $8\tau_4(k^2) \ll X^\varepsilon$ vectors \mathbf{x} . For each such \mathbf{x} we use the bound $M_1(\mathbf{x}; Y) = O_\varepsilon(Y^{2+\varepsilon})$, which follows from Lemma 2.3. Hence

$$\sum_{\substack{\mathbf{x} \in (\mathbb{Z}_{\neq 0})^4 \\ |\mathbf{x}| \leq X \\ \Delta(\mathbf{x}) = \square}} M_1(\mathbf{x}; Y) \ll_\varepsilon X^2 Y^2 (XY)^\varepsilon,$$

for any $\varepsilon > 0$, giving us the second part of Lemma 2.1.

Finally, with $S_k(\alpha; X)$ as in (2.4), we have

$$\sum_{\substack{\mathbf{x} \in (\mathbb{Z}_{\neq 0})^4 \\ |\mathbf{x}| \leq X, |x_1| \leq X_1}} M_1(\mathbf{x}; Y) \leq \int_0^1 S_1(\alpha; X)^3 S_1(\alpha; X_1) d\alpha,$$

whence Hölder's inequality yields

$$\sum_{\substack{\mathbf{x} \in (\mathbb{Z}_{\neq 0})^4 \\ |\mathbf{x}| \leq X, |x_1| \leq X_1}} M_1(\mathbf{x}; Y) \leq \left\{ \int_0^1 |S_1(\alpha; X)|^4 d\alpha \right\}^{3/4} \left\{ \int_0^1 |S_1(\alpha; X_1)|^4 d\alpha \right\}^{1/4}.$$

Appealing to (2.7), this is

$$\begin{aligned} &\ll_{\varepsilon} (XY)^{\varepsilon} (X^3 Y^2 + X^4 Y)^{3/4} (X_1^3 Y^2 + X_1^4 Y)^{1/4} \\ &\ll_{\varepsilon} (XY)^{\varepsilon} X_1^{3/4} X^{-3/4} (X^3 Y^2 + X^4 Y), \end{aligned}$$

and the third part of Lemma 2.1 follows.

3. AN ASYMPTOTIC FORMULA USING LATTICE POINT COUNTING

In this section we write

$$M_4(\mathbf{y}; R) = \# \{ \mathbf{x} \in \mathbb{Z}^4 : |\mathbf{x}| \leq R, F(\mathbf{x}; \mathbf{y}) = 0 \} \quad (3.1)$$

and prove an asymptotic formula for

$$\sum_{\substack{\mathbf{y} \in \mathbb{Z}_{\text{prim}}^4 \\ Y < |\mathbf{y}| \leq 2Y}} M_4(\mathbf{y}; (B/|\mathbf{y}|^2)^{1/3}) = N_1(B; Y), \quad (3.2)$$

say.

Theorem 3.1. *Let $Y \geq \frac{1}{2}$. Then*

$$N_1(B; Y) = B \sum_{\substack{\mathbf{y} \in \mathbb{Z}_{\text{prim}}^4 \\ Y < |\mathbf{y}| \leq 2Y}} \frac{\varrho_{\infty}(\mathbf{y})}{|\mathbf{y}|^2} + O(B^{2/3} Y^{4/3}) + O(BY^{-1/3}) + O(Y^4),$$

where

$$\varrho_{\infty}(\mathbf{y}) = \int_{-\infty}^{\infty} \int_{[-1,1]^4} e(-\theta F(\mathbf{x}; \mathbf{y})) d\mathbf{x} d\theta. \quad (3.3)$$

We note that if \mathbf{y} has at least two non-zero components, then

$$\int_{[-1,1]^4} e(-\theta F(\mathbf{x}; \mathbf{y})) d\mathbf{x} = \prod_{j=1}^4 \frac{\sin(2\pi\theta y_j^2)}{\pi\theta y_j^2} \ll_{\mathbf{y}} (1 + |\theta|)^{-2},$$

so that the outer integral in (3.3) is absolutely convergent. On the other hand, if $\mathbf{y} = (1, 0, 0, 0)$, for example, then

$$\int_{[-1,1]^4} e(-\theta F(\mathbf{x}; \mathbf{y})) d\mathbf{x} = 8 \frac{\sin(2\pi\theta)}{\pi\theta},$$

and the outer integral is conditionally convergent, with value 8.

We begin the proof by estimating $M_4(\mathbf{y}; R)$ for an individual vector \mathbf{y} , as follows.

Lemma 3.2. *Let $\mathbf{y} \in \mathbb{Z}_{\text{prim}}^4$ and put $d(\mathbf{y}) = \sqrt{y_1^4 + \cdots + y_4^4}$. Let $V(\mathbf{y})$ be the volume of the intersection of the cube $[-1, 1]^4$ with the hyperplane*

$$\{\mathbf{x} \in \mathbb{R}^4 : F(\mathbf{x}; \mathbf{y}) = 0\}.$$

Then there exists a vector $\mathbf{x}_1 = \mathbf{x}_1(\mathbf{y}) \in \mathbb{Z}_{\text{prim}}^4$ satisfying

$$0 < |\mathbf{x}_1| \ll |\mathbf{y}|^{2/3} \quad \text{and} \quad F(\mathbf{x}_1; \mathbf{y}) = 0, \quad (3.4)$$

such that

$$M_4(\mathbf{y}; R) = \frac{V(\mathbf{y})}{d(\mathbf{y})} R^3 + O\left(\frac{R^2}{|\mathbf{x}_1|^2}\right) + O(1).$$

Proof. When \mathbf{y} is primitive the function $M_4(\mathbf{y}; R)$ counts vectors $\mathbf{x} \in \mathbb{Z}^4$ from a 3-dimensional lattice Λ of determinant $d(\mathbf{y})$, as in [13, Lemma 1(i)], for example. We now claim that

$$M_4(\mathbf{y}; R) = \frac{V(\mathbf{y})}{d(\mathbf{y})} R^3 + O\left(\frac{R^2}{\lambda_1 \lambda_2} + \frac{R}{\lambda_1} + 1\right),$$

where the implied constant is absolute and $\lambda_1 \leq \lambda_2 \leq \lambda_3$ are the successive minima of Λ . If we had been using the L^2 -norm in place of the L^∞ -norm this would have followed directly from Schmidt [20, Lemma 2]. One should note here that, in Schmidt's notation, $\Lambda^{k(l-i)}$ contains the vectors $\mathbf{g}_1, \dots, \mathbf{g}_{k-i}$, and has successive minima $\lambda_1, \dots, \lambda_{k-i}$, so that $d(\Lambda^{k(l-i)}) \gg_k \lambda_1 \dots \lambda_{k-i}$. To handle the L^∞ -norm one needs only trivial modifications to Schmidt's argument, which we leave to the reader. To complete the proof of Lemma 3.2 we note that $R^2/(\lambda_1 \lambda_2) \leq (R/\lambda_1)^2$ and $R/\lambda_1 \leq \max((R/\lambda_1)^2, 1)$. Moreover, by the definition of the successive minima the lattice Λ contains a vector of length λ_1 . Writing \mathbf{x}_1 for this vector we see that \mathbf{x}_1 will be primitive, and the lemma follows, since $\lambda_1 \leq (\lambda_1 \lambda_2 \lambda_3)^{1/3} \ll d(\mathbf{y})^{1/3} \ll |\mathbf{y}|^{2/3}$. \square

We turn now to the proof of Theorem 3.1. In our argument certain “bad” vectors \mathbf{y} will have to be dealt with separately. We denote the set of these by \mathcal{B} , and write \mathcal{G} for the remaining set of good vectors. The definition of the set \mathcal{B} will be given later, since it is hard to motivate at this stage.

For the bad vectors we note that

$$B \frac{V(\mathbf{y})}{|\mathbf{y}|^2 d(\mathbf{y})} \ll BY^{-4},$$

whence

$$M_4(\mathbf{y}; (B/|\mathbf{y}|^2)^{1/3}) = B \frac{V(\mathbf{y})}{|\mathbf{y}|^2 d(\mathbf{y})} + O(BY^{-4}) + O(M_4(\mathbf{y}; (B/|\mathbf{y}|^2)^{1/3}))$$

when $\mathbf{y} \in \mathcal{B}$. It therefore follows from (3.2) along with Lemma 3.2 that

$$\begin{aligned} N_1(B; Y) = B \sum_{\substack{\mathbf{y} \in \mathbb{Z}_{\text{prim}}^4 \\ Y < |\mathbf{y}| \leq 2Y}} \frac{V(\mathbf{y})}{|\mathbf{y}|^2 d(\mathbf{y})} + O(BY^{-4} \# \mathcal{B}) + O(\Sigma_1) \\ + O(B^{2/3} Y^{-4/3} \Sigma_2) + O(Y^4), \end{aligned} \quad (3.5)$$

where

$$\Sigma_1 = \sum_{\mathbf{y} \in \mathcal{B}} M_4(\mathbf{y}; (B/|\mathbf{y}|^2)^{1/3}),$$

and

$$\Sigma_2 = \sum_{\mathbf{y} \in \mathcal{G}} |\mathbf{x}_1(\mathbf{y})|^{-2}.$$

We begin by discussing Σ_2 , since this motivates our choice of the sets \mathcal{G} and \mathcal{B} . We define

$$E(Y) = \sum_{\mathbf{y} \in \mathcal{G}} \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^4 \\ 0 < |\mathbf{x}| \leq c|\mathbf{y}|^{2/3} \\ F(\mathbf{x}; \mathbf{y}) = 0}} \frac{1}{|\mathbf{x}|^2},$$

where c is the implied constant in (3.4). We shall prove the following bound for this sum, which shows that the term Σ_2 in (3.5) makes a satisfactory contribution in Theorem 3.1.

Lemma 3.3. *We have $E(Y) \ll Y^{8/3}$ for any $Y \geq 1$.*

Proof. Our strategy for estimating $E(Y)$ will be to sort the inner sum into dyadic intervals for $|\mathbf{x}|$. When all the components of \mathbf{x} are non-zero we shall be able to invoke the second part of Lemma 2.1, and when exactly three of the components of \mathbf{x} are non-zero we will use Lemma 2.5. Thus the remaining vectors \mathbf{x} are those with at most two non-zero components, and these will correspond to \mathbf{y} being in the bad set \mathcal{B} , which we now proceed to describe.

If \mathbf{x} has exactly one non-zero component the equation $F(\mathbf{x}; \mathbf{y}) = 0$ forces the corresponding component of \mathbf{y} to vanish. We therefore include vectors \mathbf{y} with $\prod y_i = 0$ in the bad set \mathcal{B} .

Suppose on the other hand that exactly two components of \mathbf{x} vanish, say $x_3 = x_4 = 0$, and that $\prod y_i \neq 0$. Then $(x_1, x_2) = 1$, since \mathbf{x} is primitive.

Moreover $x_1 y_1^2 + x_2 y_2^2 = 0$. If we write $h = (y_1, y_2)$ then we must have $x_2 = \pm(y_1/h)^2$ and $x_1 = \mp(y_2/h)^2$. It follows that $|y_1|/h \leq c^{1/2}|\mathbf{y}|^{1/3}$, and similarly $|y_2|/h \leq c^{1/2}|\mathbf{y}|^{1/3}$. We then say that a vector \mathbf{y} is “bad” if either $\prod y_i = 0$ or if there are two components, y_1 and y_2 say, such that $|y_1|, |y_2| \leq c^{1/2}(y_1, y_2)|\mathbf{y}|^{1/3}$. Here c is the implied constant in (3.4). We now define \mathcal{B} to be the set of bad vectors $\mathbf{y} \in \mathbb{Z}_{\text{prim}}^4$ with $Y < |\mathbf{y}| \leq 2Y$. Similarly we write \mathcal{G} for the complement of \mathcal{B} in the set of $\mathbf{y} \in \mathbb{Z}_{\text{prim}}^4$ with $Y < |\mathbf{y}| \leq 2Y$. Thus if \mathbf{y} is in the complementary set \mathcal{G} , any corresponding vector \mathbf{x} has at most one zero entry.

We are now ready to estimate $E(Y)$. Let $S(L, \mathcal{G})$ be the number of pairs \mathbf{x}, \mathbf{y} that arise for which $L/2 < |\mathbf{x}| \leq L$. Then

$$E(Y) \ll \sum_L L^{-2} S(L, \mathcal{G}),$$

the sum over L being for powers of 2 only, with $L \ll Y^{2/3}$. Our definitions ensure that

$$S(L, \mathcal{G}) \ll M_3(L, 2Y) + YT(L, 2Y, L, 2Y),$$

in the notation of Lemmas 2.1 and 2.5, which then yield

$$S(L, \mathcal{G}) \ll_{\varepsilon} L^3 Y^2 + L^5 Y^{2/3} + L^{2+\varepsilon} Y^{2+\varepsilon},$$

for any $\varepsilon > 0$. For $L \ll Y^{2/3}$ and $\varepsilon = 3/10$ this is $\ll L^3 Y^2 + L^2 Y^{5/2}$, whence

$$E(Y) \ll \sum_{2^i \ll Y^{2/3}} (2^i Y^2 + Y^{5/2}) \ll Y^{8/3} + Y^{5/2} \log Y,$$

which is satisfactory for Lemma 3.3. \square

We next estimate $\#\mathcal{B}$. There are $O(Y^3)$ vectors \mathbf{y} with $\prod y_i = 0$. For the remaining bad vectors, if we have $y_1 = h z_1$, for example, then $y_2 = h z_2$ with z_1, z_2 coprime and

$$|z_1|, |z_2| \leq c^{1/2} |\mathbf{y}|^{1/3} \ll Y^{1/3}.$$

There are $O(Y)$ choices for h and $O(Y^{1/3})$ choices for each of z_1 and z_2 , and since there are $O(Y^2)$ possible values for y_3 and y_4 we see that there are $O(Y^{11/3})$ options for \mathbf{y} . Thus $\#\mathcal{B} \ll Y^{11/3}$, so that the corresponding term in (3.5) is satisfactory for Theorem 3.1.

It remains to consider Σ_1 . We begin by disposing of the contribution from solutions \mathbf{x}, \mathbf{y} with $\prod x_i y_i = 0$. We apply Lemma 2.6 with $X \ll B^{1/3} Y^{-2/3}$ obtaining a bound $O_{\varepsilon}(B Y^{\varepsilon-1} + Y^4)$. The corresponding contribution to (3.5) will turn out to be satisfactory for our purposes, as we shall see shortly. In what follows we may assume $\prod x_i y_i \neq 0$.

We now focus on terms for which $y_1 = hz_1$ and $y_2 = hz_2$ where z_1 and z_2 are coprime and $0 < |z_1|, |z_2| \leq c^{1/2}|\mathbf{y}|^{1/3}$, so that

$$(x_1z_1^2 + x_2z_2^2)h^2 + x_3y_3^2 + x_4y_4^2 = 0 \quad (3.6)$$

with non-zero integer variables. We set

$$X = (BY^{-2})^{1/3} \quad \text{and} \quad Z = \max\{|z_1|, |z_2|\},$$

whence $1 \leq h \leq 2Y/Z$. When $x_1z_1^2 + x_2z_2^2 = 0$ we have $x_3y_3^2 + x_4y_4^2 = 0$ as well. There are then $O(Y/Z)$ choices for h , while Lemma 2.4 shows that there are $O(XZ)$ values for x_1, x_2, z_1, z_2 and $O(XY)$ possibilities for x_3, x_4, y_3, y_4 . The case $x_1z_1^2 + x_2z_2^2 = 0$ therefore contributes $O(X^2Y^2) = O(B^{2/3}Y^{2/3})$ to Σ_1 . We shall see in a moment that this makes a suitably small contribution to (3.5).

We count the remaining solutions according to the values taken by z_1 and z_2 . It will be convenient in what follows to write $N(Y; z_1, z_2)$ for the number of solutions to the equation (3.6) in non-zero integers $x_1, \dots, x_4, h, y_3, y_4$ with $x_1z_1^2 + x_2z_2^2 \neq 0$ and

$$|\mathbf{x}| \leq X, \quad 1 \leq h \leq 2Y/Z \quad \text{and} \quad Y < |y_3|, |y_4| \leq 2Y.$$

It follows from our analysis thus far that

$$\Sigma_1 \ll_\varepsilon BY^{\varepsilon-1} + Y^4 + B^{2/3}Y^{2/3} + \sum_{0 < |z_1|, |z_2| \ll Y^{1/3}} N(Y; z_1, z_2). \quad (3.7)$$

We put $t = x_1z_1^2 + x_2z_2^2$, which is assumed to be non-zero. For a given non-zero t (and fixed z_1, z_2) the number of x_1, x_2 such that $x_1z_1^2 + x_2z_2^2 = t$ is $O(1 + X/Z^2)$. Moreover the equation $th^2 + x_3y_3^2 + x_4y_4^2 = 0$ has at most $T(X, 2Y, 2XZ^2, 2Y/Z)$ solutions, in the notation of Lemma 2.5, which then shows that

$$N(Y; z_1, z_2) \ll_\varepsilon (1 + XZ^{-2})X^2Y^{1+\varepsilon}Z$$

for any fixed $\varepsilon > 0$. We insert this estimate into (3.7) and find that

$$\begin{aligned} \Sigma_1 &\ll_\varepsilon BY^{\varepsilon-1} + Y^4 + B^{2/3}Y^{2/3} + (Y^{2/3} + X)X^2Y^{4/3+\varepsilon} \\ &\ll_\varepsilon Y^4 + B^{2/3}Y^{2/3+\varepsilon} + BY^{\varepsilon-2/3}. \end{aligned}$$

Taking $\varepsilon = 1/3$ now gives us a suitable contribution to (3.5).

Returning to (3.5), in order to complete the proof of Theorem 3.1 it remains to show that

$$\frac{V(\mathbf{y})}{\mathbf{d}(\mathbf{y})} = \varrho_\infty(\mathbf{y}), \quad (3.8)$$

for any non-zero vector $\mathbf{y} \in \mathbb{R}^4$, where $\varrho_\infty(\mathbf{y})$ is defined in (3.3). It will be convenient to put $y_i^2 = \mathbf{d}(\mathbf{y})w_i$ for $1 \leq i \leq 4$. Then if $\|\cdot\|_2$ denotes the

L^2 -norm, it follows that $\|\mathbf{w}\|_2 = 1$. Moreover, in this notation we have

$$\varrho_\infty(\mathbf{y})d(\mathbf{y}) = \int_{-\infty}^{\infty} \int_{[-1,1]^4} e(-\theta \mathbf{w} \cdot \mathbf{x}) d\mathbf{x} d\theta.$$

As already noted following the statement of Theorem 3.1, the repeated integral is 8 if \mathbf{w} has a single non-zero component. This suffices for (3.8), since one easily sees that $V(\mathbf{w}) = 8$ in this case.

On the other hand, if \mathbf{w} has at least two non-zero components then, as remarked earlier, the inner integral is $O((1+|\theta|^2)^{-1})$, so that the double integral is

$$\begin{aligned} \lim_{\delta \downarrow 0} \int_{-\infty}^{\infty} \left(\frac{\sin(\pi \delta \theta)}{\pi \delta \theta} \right)^2 \int_{[-1,1]^4} e(-\theta \mathbf{w} \cdot \mathbf{x}) d\mathbf{x} d\theta \\ = \lim_{\delta \downarrow 0} \int_{[-1,1]^4} \int_{-\infty}^{\infty} \left(\frac{\sin(\pi \delta \theta)}{\pi \delta \theta} \right)^2 e(-\theta \mathbf{w} \cdot \mathbf{x}) d\theta d\mathbf{x}. \end{aligned} \quad (3.9)$$

In general if

$$K(u; \delta) = \begin{cases} \delta^{-2}(\delta - |u|), & \text{if } |u| \leq \delta, \\ 0, & \text{if } |u| \geq \delta, \end{cases} \quad (3.10)$$

then

$$K(u; \delta) = \int_{-\infty}^{\infty} e(\theta u) \left(\frac{\sin(\pi \delta \theta)}{\pi \delta \theta} \right)^2 d\theta, \quad (3.11)$$

so that the inner integral on the right of (3.9) is $K(\mathbf{w} \cdot \mathbf{x}; \delta)$.

Since $\|\mathbf{w}\|_2 = 1$ there exists a 4×4 orthogonal matrix $\mathbf{M} \in O_4(\mathbb{R})$ such that $\mathbf{M}\mathbf{w} = (1, 0, 0, 0)$. Then $\mathbf{w} \cdot \mathbf{x} = (\mathbf{M}\mathbf{w})^T \mathbf{M}\mathbf{x}$. Changing variables from \mathbf{x} to $\mathbf{z} = \mathbf{M}\mathbf{x}$, so that \mathbf{z} runs over the set $Z = \mathbf{M}[-1, 1]^4$, we see that

$$\varrho_\infty(\mathbf{y})d(\mathbf{y}) = \lim_{\delta \downarrow 0} \int_Z K(z_1; \delta) d\mathbf{z} = \text{meas}\{\mathbf{z} \in Z : z_1 = 0\} = V(\mathbf{w}) = V(\mathbf{y}),$$

as required. This concludes the proof of Theorem 3.1.

4. COUNTING POINTS ON QUADRICS

In this section we will establish an asymptotic formula for the smoothly weighted counting function

$$N_w(P) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}^4 \\ F(\mathbf{x})=0}} w(P^{-1}\mathbf{x}),$$

as $P \rightarrow \infty$, where $F(\mathbf{x})$ is the non-singular integral diagonal quadratic form

$$F(\mathbf{x}) = A_1 x_1^2 + A_2 x_2^2 + A_3 x_3^2 + A_4 x_4^2.$$

Here $w : \mathbb{R}^4 \rightarrow \mathbb{R}_{\geq 0}$ is a fixed infinitely differentiable weight function of compact support, which vanishes in some neighbourhood of the origin. Our goal is to establish an asymptotic formula even when the coefficients A_i are of size a small power of P , and it will be crucial to our success that the size we are able to handle is sufficiently large.

Our asymptotic formula for $N_w(P)$ is only valid for suitable weights w and “generic” choices of the coefficients A_i . To specify the necessary conditions we define

$$\|F\| = \max_{1 \leq i \leq 4} |A_i|, \quad \Delta_F = A_1 A_2 A_3 A_4 (\neq 0),$$

and

$$\Delta_{\text{bad}} = \prod_{\substack{p^e \parallel \Delta_F \\ e \geq 2}} p^e. \quad (4.1)$$

We then require that

$$w(\mathbf{x}) = 0 \quad \text{for} \quad |\mathbf{x}| \leq \eta, \quad (4.2)$$

that

$$\|F\|^{1-\eta} \leq |A_i| (\leq \|F\|), \quad \text{for } 1 \leq i \leq 4, \quad (4.3)$$

and that

$$\Delta_{\text{bad}} \leq \|F\|^\eta, \quad (4.4)$$

for a small parameter $\eta \in (0, \frac{1}{100})$ at our disposal. The first two conditions imply that

$$|\nabla F(\mathbf{x})| \gg_\eta \|F\|^{1-\eta} \quad \text{for } w(\mathbf{x}) \neq 0, \quad (4.5)$$

while the last condition implies in particular that $\Delta_F \neq \square$ when $\|F\| > 1$.

Our asymptotic formula involves the “singular integral”, defined to be

$$\sigma_\infty(w; F) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^4} w(\mathbf{x}) e(-\theta F(\mathbf{x})) d\mathbf{x} d\theta, \quad (4.6)$$

and the “singular series”

$$\mathfrak{S}(F) = \prod_p \lim_{r \rightarrow \infty} p^{-3r} \# \{ \mathbf{x} \in (\mathbb{Z}/p^r \mathbb{Z})^4 : F(\mathbf{x}) \equiv 0 \pmod{p^r} \}. \quad (4.7)$$

We will see in Lemma 4.10 that this is convergent whenever $\Delta_F \neq \square$. With this notation our principal result in this section is the following.

Theorem 4.1. *When (4.2), (4.3) and (4.4) hold we have*

$$N_w(P) = \sigma_\infty(w; F) \mathfrak{S}(F) P^2 + O_{w,\eta}(\|F\|^{-1/2+7\eta} P^{3/2} + \|F\|^{1/2+2\eta} P),$$

provided that $\|F\| \geq P^\eta$.

The main term here is typically of size around $P^2\|F\|^{-1}$, so that we get an asymptotic formula when $P^\eta \leq \|F\| \leq P^{2/3+O(\eta)}$. For comparison, Browning [3, Prop. 2] shows that

$$N_w(P) = \sigma_\infty(w; F) \mathfrak{S}(F) P^2 + O_{w,\eta}(\|F\|^{3+9\eta} P^{3/2+\eta}),$$

for a special choice of weight function w , under the assumptions that $\Delta_F \neq \square$ and that (4.3) and (4.4) hold. Theorem 4.1 refines this result considerably for forms whose discriminant is close to being square-free, provided that the coefficients of F are not too small compared with P . We should emphasize that in both Theorem 4.1 and the result of Browning [3] the coefficients of η are relatively unimportant. In particular they have no significance for the current application.

The condition that $\|F\| \geq P^\eta$ is somewhat unnatural and deserves further comment. Under this assumption together with the hypotheses (4.2), (4.3) and (4.4) we are able to eliminate certain awkward terms that arise when we apply Poisson summation. This is explained further in §4.5. At this point it is crucial that the quadratic form $F(\mathbf{x})$ is diagonal. We could remove the condition $\|F\| \geq P^\eta$ and handle non-diagonal forms, but this would be at the expense of a worse dependence on $\|F\|$.

4.1. Preliminaries. Our proof of Theorem 4.1 uses the smooth δ -function variant of the circle method introduced by Duke, Friedlander and Iwaniec [8], and later developed by Heath-Brown [12, Thm. 1]. We proceed to review the technical apparatus required.

For any $q \in \mathbb{N}$, any $\mathbf{c} \in \mathbb{Z}^4$ and any $Q \geq 1$, we define the complete exponential sum

$$S_q(\mathbf{c}) = \sum_{a \bmod q}^* \sum_{\mathbf{b} \bmod q} e_q(aF(\mathbf{b}) + \mathbf{b} \cdot \mathbf{c}), \quad (4.8)$$

and the oscillatory integral

$$I_q(\mathbf{c}) = \int_{\mathbb{R}^4} w(P^{-1}\mathbf{x}) h\left(\frac{q}{Q}, \frac{F(\mathbf{x})}{Q^2}\right) e_q(-\mathbf{c} \cdot \mathbf{x}) d\mathbf{x},$$

for a certain function $h : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ described in [12, §3]. All we need to know here is that $h(x, y)$ is infinitely differentiable for $(x, y) \in (0, \infty) \times \mathbb{R}$, and that $h(x, y)$ is non-zero only for $x \leq \max\{1, 2|y|\}$. It then follows from [12, Thm. 2] that

$$N_w(P) = \frac{c_Q}{Q^2} \sum_{\mathbf{c} \in \mathbb{Z}^4} \sum_{q=1}^{\infty} q^{-4} S_q(\mathbf{c}) I_q(\mathbf{c}). \quad (4.9)$$

where the constant c_Q satisfies $c_Q = 1 + O_N(Q^{-N})$, for any $N > 0$.

We shall take $Q = \sqrt{\|F\|P^2}$ in our work. In our proof of Theorem 4.1 we shall often encounter a small positive parameter ε , and for the sake of

convenience we shall allow it to take different values at different stages of the argument, so that $x^\varepsilon \log x \ll x^\varepsilon$, for example. All of our implied constants are allowed to depend on the weight function w and on η and ε , but on nothing else unless specified. Ultimately we will take ε to be fixed in terms of η but much smaller than it, so that the dependence on ε will disappear. As above we assume that w , besides being infinitely differentiable and of compact support, satisfies the condition (4.2).

We now wish to apply the bounds for the exponential integral $I_q(\mathbf{c})$ that were derived in [12, §§7–8]. Unfortunately, the implied constants in each of these estimates is allowed to depend implicitly on the coefficients of F , a deficiency that we shall need to remedy here.

Lemma 4.2. *Let $\mathbf{c} \in \mathbb{Z}^4$ be non-zero. Then the following hold:*

(i) *For any $N \geq 0$ we have*

$$I_q(\mathbf{c}) \ll_N \frac{P^5 \|F\|^{(N+1)/2}}{q |\mathbf{c}|^N}.$$

(ii) *We have*

$$I_q(\mathbf{c}) \ll \frac{q \|F\|^2 P^3}{|\Delta_F|^{1/2} |\mathbf{c}|}.$$

Proof. To begin with we may write

$$I_q(\mathbf{c}) = P^4 \int_{\mathbb{R}^4} w(\mathbf{x}) h(r, G(\mathbf{x})) e(-\mathbf{v} \cdot \mathbf{x}) d\mathbf{x},$$

where $r = q/Q$, $\mathbf{v} = q^{-1}P\mathbf{c}$ and $G = \|F\|^{-1}F$. In particular

$$\frac{\partial^{k_1+\dots+k_4}}{\partial x_1^{k_1} \dots \partial x_4^{k_4}} G(\mathbf{x}) \ll_{k_1, \dots, k_4} 1,$$

for all $\mathbf{x} \in \text{supp}(w)$ and $k_1, \dots, k_4 \in \mathbb{Z}_{\geq 0}$. The function $h(x, y)$ is non-zero only for $x \leq \max\{1, 2|y|\}$ and satisfies $h(x, y) \ll x^{-1}$. In fact, for any $i, j \geq 0$ and any $N \geq 0$, we have

$$\frac{\partial^{i+j} h(x, y)}{\partial^i x \partial^j y} \ll_{i, j, N} x^{-1-i-j} (x^N + \min\{1, (x/|y|)^N\}). \quad (4.10)$$

These facts are all explained in [12, Lemmas 4 and 5]. Repeated integration by parts now establishes part (i).

Turning to part (ii), we see that $I_q(\mathbf{c}) = P^4 r^{-1} I(\mathbf{v})$, in the notation of [12, Lemma 14], with $f = rh$. The argument in the proof of [12, Lemma 17] shows that there exists a smooth weight function $w_1 : \mathbb{R}^4 \rightarrow \mathbb{R}_{\geq 0}$, such that $\text{supp}(w_1) \subseteq \text{supp}(w)$ and a function $p(t) \ll r$ such that

$$I(\mathbf{v}) = \int_{-\infty}^{\infty} p(t) \int_{\mathbb{R}^4} w_1(\mathbf{x}) e(tG(\mathbf{x}) - \mathbf{v} \cdot \mathbf{x}) d\mathbf{x} dt.$$

We may analyse the inner integral J , say, using the smoothly weighted stationary phase bound worked out by Heath-Brown and Pierce in Lemmas 3.1 and 3.2 of [14]. Recall that $G(\mathbf{x}) = \|F\|^{-1}F(\mathbf{x}) \ll 1$ for all $\mathbf{x} \in \text{supp}(w_1)$ and observe that integration by parts gives

$$\int_{\mathbb{R}^4} |\widehat{w}_1(\mathbf{y})| d\mathbf{y} \ll \int_{\mathbb{R}^4} \min\{1, |\mathbf{y}|^{-1}\}^5 d\mathbf{y} \ll 1.$$

This shows that $J = O_M(|\mathbf{v}|^{-M})$ for any $M \geq 0$ if $|\mathbf{v}| \gg |t|$, while we have $J = O(\|F\|^2 |\Delta_F|^{-1/2} t^{-2})$ otherwise. Applying these bounds with $M = 2$, and noting that $|\Delta_F|^{1/2} \leq \|F\|^2$, the statement of part (ii) easily follows. \square

The effect of part (i) is that the sum over \mathbf{c} in (4.9) can be truncated to $|\mathbf{c}| \ll \|F\|^{1/2} Q^\varepsilon$ for any $\varepsilon > 0$, with negligible error. The following estimate allows us to work with $I_q(\mathbf{c})$ when $\mathbf{c} = \mathbf{0}$.

Lemma 4.3. *Assume that (4.2) and (4.3) hold. Then we have*

$$q^k \frac{\partial^k I_q(\mathbf{0})}{\partial q^k} \ll \|F\|^\eta P^4, \quad \text{for } k \in \{0, 1\}.$$

Proof. Let $k \in \{0, 1\}$ and recall the notation $r = q/Q$. Then

$$\begin{aligned} q^k \frac{\partial^k I_q(\mathbf{0})}{\partial q^k} &= r^k P^4 \int_{\mathbb{R}^4} w(\mathbf{x}) \frac{\partial^k h(r, G(\mathbf{x}))}{\partial r^k} d\mathbf{x} \\ &\ll r^{-1} P^4 \int_{\mathbf{x} \in \text{supp}(w)} \left(r^2 + \min \left\{ 1, \frac{r^2}{G(\mathbf{x})^2} \right\} \right) d\mathbf{x}, \end{aligned}$$

on taking $N = 2$ in (4.10). We now appeal to (4.5), which implies that $|\nabla G(\mathbf{x})| \gg \|F\|^{-\eta}$ for all $\mathbf{x} \in \text{supp}(w)$. Thus the measure of the set where $|G(\mathbf{x})| \leq z$ is $O(z\|F\|^\eta)$. The integral is therefore $O(r\|F\|^\eta)$, as in the proof of [12, Lemma 15], and the lemma follows. \square

We conclude this section by considering the integral

$$J(\theta; w) = \int_{\mathbb{R}^4} w(\mathbf{x}) e(-\theta G(\mathbf{x})) d\mathbf{x}. \quad (4.11)$$

Lemma 4.4. *Under the assumptions (4.2) and (4.3) we have*

$$J(\theta; w) \ll_N |\theta|^{-N} \|F\|^{2N\eta}$$

for any non-negative integer N .

Proof. To prove this we use the first derivative bound for smooth exponential integrals, see Heath-Brown [12, Lemma 10]. First however we must reduce the

support of the weight function w by using Lemma 2 of [12]. This shows that if $0 < \delta < 1$ then there is a smooth function w_δ of compact support such that

$$w(\mathbf{x}) = \delta^{-4} \int_{\mathbb{R}^4} w_\delta(\delta^{-1}(\mathbf{x} - \mathbf{y}), \mathbf{y}) d\mathbf{y}.$$

Thus there is some vector $\mathbf{y} = \mathbf{y}(\delta)$ such that

$$J(\theta; w) \ll \delta^{-4} \left| \int_{\mathbb{R}^4} w_\delta(\delta^{-1}(\mathbf{x} - \mathbf{y}), \mathbf{y}) e(-\theta G(\mathbf{x})) d\mathbf{x} \right|.$$

We will choose $\delta = \|F\|^{-1} \min |A_i|$. Thus $\delta \geq \|F\|^{-\eta}$, by (4.3). We now see that

$$J(\theta; w) \ll \left| \int_{\mathbb{R}^4} w_*(\mathbf{u}) e(-\theta G_*(\mathbf{u})) d\mathbf{u} \right|, \quad (4.12)$$

where we have set $w_*(\mathbf{u}) = w_\delta(\mathbf{u}, \mathbf{y})$ and $G_*(\mathbf{u}) = G(\mathbf{y} + \delta\mathbf{u})$. According to [12, Lemma 2], if $w_*(\mathbf{u}) \neq 0$ then $w(\mathbf{y} + \delta\mathbf{u}) \neq 0$, so that $|\mathbf{y} + \delta\mathbf{u}| \geq \eta$, by (4.2). It follows that $|\nabla G_*(\mathbf{u})| \gg_\eta \delta^2$ on the support of w_* . Moreover, the second order derivatives of G_* are $O(\delta^2)$ and the higher derivatives vanish. We then see from [12, Lemma 10] that

$$\int_{\mathbb{R}^4} w_*(\mathbf{u}) e(-\theta G_*(\mathbf{u})) d\mathbf{u} \ll_{N,w,\eta} |\theta|^{-N} \delta^{-2N}$$

for any non-negative integer N . The reader should note that the implied constant is independent of \mathbf{y} and δ by the technical properties of w_δ described in [12, Lemma 2]. The statement now follows from (4.12). \square

4.2. The exponential sums. In this section we summarise what we will need to know about the exponential sums $S_q(\mathbf{c})$ defined in (4.8). As proved in [12, Lemma 23], these satisfy the multiplicativity property $S_{q_1 q_2}(\mathbf{c}) = S_{q_1}(\mathbf{c}) S_{q_2}(\mathbf{c})$ for any coprime integers q_1, q_2 . We begin by establishing the following basic estimate.

Lemma 4.5. *For any $q \in \mathbb{N}$ we have*

$$S_q(\mathbf{c}) \ll q^3 \prod_{1 \leq i \leq 4} (q, A_i, c_i)^{1/2}.$$

Proof. To begin with, for any prime power p^r we have

$$S_{p^r}(\mathbf{c}) = \sum_{a \bmod p^r}^* \prod_{1 \leq i \leq 4} G(a A_i, c_i; p^r), \quad (4.13)$$

where

$$G(b, c; q) = \sum_{x \bmod q} e_q(bx^2 + cx) \quad (4.14)$$

denotes the generalised Gauss sum for any $q \in \mathbb{N}$ and $b, c \in \mathbb{Z}$. For $q = p^r$ we put $\beta = v_p(b)$. Breaking into residue classes modulo $p^{r-\min\{\beta, r\}}$, it is easy

to see that $G(b, c; p^r) = 0$ unless $p^{\min\{\beta, r\}} \mid c$. We conclude that if $S_{p^r}(\mathbf{c}) \neq 0$ then we must have $\min\{v_p(A_i), r\} \leq v_p(c_i)$, for $1 \leq i \leq 4$.

Next, on inspecting the proof of [12, Lemma 25], we find that

$$\begin{aligned} S_{p^r}(\mathbf{c}) &\leq p^{3r} \#\{\mathbf{y} \bmod p^r : p^r \mid 2A_i y_i \text{ for } 1 \leq i \leq 4\}^{1/2} \\ &\leq p^{3r} \prod_{1 \leq i \leq 4} p^{(\min\{v_p(A_i), r\} + v_p(2))/2} \\ &= p^{3r+2v_p(2)} \prod_{1 \leq i \leq 4} p^{\min\{v_p(A_i), r, v_p(c_i)\}/2}. \end{aligned}$$

The statement of the lemma now follows from multiplicativity. \square

Our next result relies on an explicit evaluation of the Gauss sum (4.14). Let $b, c \in \mathbb{Z}$ and let q be an odd integer. Then according to [15, p. 66], we have

$$G(b, c; q) = e_q(-\overline{4b}c^2) \left(\frac{b}{q}\right) \delta_q \sqrt{q}, \quad (4.15)$$

provided that $(b, q) = 1$. Here $\overline{4b}$ denotes the multiplicative inverse of $4b$ modulo q , and

$$\delta_q = \begin{cases} 1 & \text{if } q \equiv 1 \pmod{4}, \\ i & \text{if } q \equiv 3 \pmod{4}. \end{cases}$$

Define the dual form

$$F^*(\mathbf{y}) = A_2 A_3 A_4 y_1^2 + A_1 A_3 A_4 y_2^2 + A_1 A_2 A_4 y_3^2 + A_1 A_2 A_3 y_4^2. \quad (4.16)$$

The following result follows on inserting (4.15) into (4.13).

Lemma 4.6. *Let $p \nmid 2\Delta_F$ be a prime. Then*

$$S_{p^r}(\mathbf{c}) = \left(\frac{\Delta_F}{p^r}\right) p^{2r} c_{p^r}(F^*(\mathbf{c})),$$

where $c_{p^r}(N)$ is the Ramanujan sum.

By standard properties of the Ramanujan sum, this result implies that $S_{p^r}(\mathbf{c}) = 0$ unless $p \mid F^*(\mathbf{c})$, whenever $p \nmid 2\Delta_F$ and $r \geq 2$.

We proceed by using our work so far to study the asymptotic behaviour of the sum

$$\Sigma(x; \mathbf{c}) = \sum_{q \leq x} q^{-3} S_q(\mathbf{c}), \quad (4.17)$$

for suitable vectors \mathbf{c} .

Lemma 4.7. *Let $\varepsilon > 0$ and assume that $F^*(\mathbf{c}) \neq 0$. Then*

$$|\Sigma(x; \mathbf{c})| \leq \sum_{q \leq x} q^{-3} |S_q(\mathbf{c})| \ll_\varepsilon x^\varepsilon |\mathbf{c}|^\varepsilon \|F\|^\varepsilon \prod_{1 \leq i \leq 4} (A_i, c_i)^{1/2}.$$

Proof. Define the non-zero integer $N = 2\Delta_F F^*(\mathbf{c})$. To handle $\Sigma(x; \mathbf{c})$ we sum trivially over q , finding that

$$\sum_{q \leq x} q^{-3} |S_q(\mathbf{c})| = \sum_{\substack{q_2 \leq x \\ q_2 | N^\infty}} q_2^{-3} |S_{q_2}(\mathbf{c})| \sum_{\substack{q_1 \leq x/q_2 \\ (q_1, N)=1}} q_1^{-3} |S_{q_1}(\mathbf{c})|.$$

It follows from Lemma 4.6 that the inner sum is restricted to square-free integers q_1 and that $|S_{q_1}(\mathbf{c})| \leq q_1^2$. Lemma 4.5 yields

$$\sum_{q \leq x} q^{-3} |S_q(\mathbf{c})| \ll \log x \sum_{\substack{q_2 \leq x \\ q_2 | N^\infty}} \frac{|S_{q_2}(\mathbf{c})|}{q_2^3} \ll \log x \sum_{\substack{q_2 \leq x \\ q_2 | N^\infty}} \prod_{1 \leq i \leq 4} (A_i, c_i)^{1/2}.$$

The statement of the lemma follows on noting that there are $O_\varepsilon(N^\varepsilon x^\varepsilon)$ values of q_2 that contribute to the remaining sum. \square

We shall also need to study $\Sigma(x; \mathbf{0})$. First, we recall the definition (4.1) of Δ_{bad} and establish the following result.

Lemma 4.8. *Let $r \geq 1$ and $p \mid \Delta_F$, with $p \nmid 2\Delta_{\text{bad}}$. Then $S_{p^r}(\mathbf{0}) = 0$.*

Proof. We return to (4.13) with $\mathbf{c} = \mathbf{0}$. The assumption $p \nmid 2\Delta_{\text{bad}}$ implies that $v_p(\Delta_F) = 1$. We suppose without loss of generality that $v_p(A_4) = 1$ and $p \nmid A_1 A_2 A_3$. We may evaluate $G(aA_i, 0; p^r)$ using (4.15) for $1 \leq i \leq 3$. Next, on writing $A'_4 = A_4/p$, it is easy to see that

$$G(aA_4, 0; p^r) = \sum_{x \bmod p^r} e_{p^{r-1}}(aA'_4 x^2) = \begin{cases} p & \text{if } r = 1, \\ \left(\frac{aA'_4}{p^{r-1}}\right) \delta_{p^{r-1}} p^{\frac{r+1}{2}} & \text{if } r \geq 2. \end{cases}$$

Hence $S_{p^r}(\mathbf{0}) = 0$ for $r \geq 1$, since the a -sum is $\sum_{a \bmod p^r}^* \left(\frac{a}{p}\right) = 0$. \square

We now have everything in place to study $\Sigma(x; \mathbf{0})$. This time we shall sum non-trivially over q using the Burgess bound for short character sums.

Lemma 4.9. *Let $\varepsilon > 0$ and assume that $\Delta_F \neq \square$. Then*

$$\Sigma(x; \mathbf{0}) \ll_\varepsilon |\Delta_F|^{3/16+\varepsilon} \Delta_{\text{bad}}^{3/8} x^{1/2+\varepsilon}.$$

Proof. Define the non-zero integer $N = 2\Delta_{\text{bad}}$. We have

$$\Sigma(x; \mathbf{0}) = \sum_{\substack{q_2 \leq x \\ q_2 | N^\infty}} q_2^{-3} S_{q_2}(\mathbf{0}) \sum_{\substack{q_1 \leq x/q_2 \\ (q_1, N)=1}} q_1^{-3} S_{q_1}(\mathbf{0}).$$

It follows from Lemma 4.8 that the inner sum is actually only over q_1 which are coprime to $2\Delta_F$. Setting $A = 2\Delta_F$ for convenience, it follows from Lemma 4.6

that

$$\sum_{\substack{q \leq X \\ (q, A)=1}} q^{-3} S_q(\mathbf{0}) = \sum_{\substack{q \leq X \\ (q, A)=1}} \left(\frac{\Delta_F}{q} \right) \varphi^*(q),$$

where $\varphi^* = 1 * h$, with $h(d) = \mu(d)/d$. Opening up φ^* and inverting the order of summation, we conclude that

$$\sum_{\substack{q \leq X \\ (q, A)=1}} q^{-3} S_q(\mathbf{0}) = \sum_{\substack{u \leq X \\ (u, A)=1}} \frac{\mu(u)}{u} \left(\frac{\Delta_F}{u} \right) \sum_{\substack{v \leq X/u \\ (v, A)=1}} \left(\frac{\Delta_F}{v} \right).$$

According to Burgess [6, 7], the inner sum is $O_\varepsilon(|\Delta_F|^{3/16+\varepsilon}(X/u)^{1/2})$, for any $\varepsilon > 0$. But then we have

$$\sum_{\substack{q \leq X \\ (q, A)=1}} q^{-3} S_q(\mathbf{0}) \ll_\varepsilon |\Delta_F|^{3/16+\varepsilon} X^{1/2} \sum_{u \leq X} \frac{1}{u^{3/2}} \ll_\varepsilon |\Delta_F|^{3/16+\varepsilon} X^{1/2}.$$

Applying this with $X = x/q_2$ and returning to the start of the argument, we may now deduce from Lemma 4.5 that

$$\Sigma(x; \mathbf{0}) \ll_\varepsilon |\Delta_F|^{3/16+\varepsilon} x^{1/2} \sum_{\substack{q_2 \leq x \\ q_2 | \Delta_{\text{bad}}^\infty}} \frac{\prod_{1 \leq i \leq 4} (q_2, A_i)^{1/2}}{q_2^{1/2}} \ll_\varepsilon |\Delta_F|^{3/16+\varepsilon} \Delta_{\text{bad}}^{3/8} x^{1/2+\varepsilon},$$

since there are $O_\varepsilon(|\Delta_F|^\varepsilon x^\varepsilon)$ choices for q_2 in this sum. This completes the proof of the lemma. \square

We end this section by considering the singular series (4.7), which we may write as

$$\mathfrak{S}(F) = \prod_p \sum_{r=0}^{\infty} p^{-4r} S_{p^r}(\mathbf{0}).$$

We prove the following upper bound.

Lemma 4.10. *Whenever $\Delta_F \neq \square$ we have*

$$\mathfrak{S}(F) \ll_\varepsilon \Delta_{\text{bad}}^{1/4+\varepsilon} L(1, \chi_F) \ll_\varepsilon \Delta_{\text{bad}}^{1/4} \|F\|^\varepsilon,$$

where χ_F is the quadratic character defined by taking $\chi_F(2) = 0$ and

$$\chi_F(p) = \left(\frac{\Delta_F}{p} \right)$$

for odd primes p .

Proof. When $p \nmid 2\Delta_F$ we have

$$S_{p^r}(\mathbf{0}) = \left(\frac{\Delta_F}{p^r}\right) p^{2r} \varphi(p^r),$$

by Lemma 4.6, whence

$$\prod_{p \mid 2\Delta_F} \sum_{r=0}^{\infty} p^{-4r} S_{p^r}(\mathbf{0}) \ll L(1, \chi_F).$$

The conductor of χ_F is $O(\|F\|^4)$, whence $L(1, \chi_F) \ll \log(2 + \|F\|) \ll_{\varepsilon} \|F\|^{\varepsilon}$.

The factor corresponding to primes for which $p \mid \Delta_F$ and $p \nmid 2\Delta_{\text{bad}}$ is just 1, by Lemma 4.8. It therefore remains to consider primes $p \mid 2\Delta_{\text{bad}}$. Suppose that $p^{f_j} \parallel 2A_j$ with $f_1 \leq f_2 \leq f_3 \leq f_4$, say. Then Lemma 4.5 yields

$$\sum_{r=0}^{\infty} p^{-4r} S_{p^r}(\mathbf{0}) \ll \sum_{r=0}^{\infty} p^{-r} \prod_{1 \leq j \leq 4} \min(p^{r/2}, p^{f_j/2}).$$

If we bound the minimum by $p^{r/2}$ for $j = 4$ we see that

$$\sum_{r=0}^{\infty} p^{-4r} S_{p^r}(\mathbf{0}) \ll \sum_{r=0}^{\infty} p^{-r/2} \min(p^{3r/2}, p^{(f_1+f_2+f_3)/2}) \ll p^{(f_1+f_2+f_3)/3}.$$

If $p^e \parallel \Delta_{\text{bad}}$ this is $O(p^{e/4})$, so that primes which divide $2\Delta_{\text{bad}}$ provide a total $O_{\varepsilon}(\Delta_{\text{bad}}^{1/4+\varepsilon})$ for any $\varepsilon > 0$. The lemma then follows. \square

4.3. The main term. We now collect our estimates together in order to complete the proof of Theorem 4.1. Let $\varepsilon > 0$ and let $C = \|F\|^{1/2} Q^{\varepsilon}$. Returning to (4.9), it follows from part (i) of Lemma 4.2 that

$$N_w(P) = M(P) + E(P) + O(1),$$

where

$$M(P) = \frac{1}{Q^2} \sum_{q=1}^{\infty} q^{-4} S_q(\mathbf{0}) I_q(\mathbf{0}) \quad \text{and} \quad E(P) = \frac{1}{Q^2} \sum_{\substack{\mathbf{c} \in \mathbb{Z}^4 \\ 0 < |\mathbf{c}| \leq C}} \sum_{q=1}^{\infty} q^{-4} S_q(\mathbf{c}) I_q(\mathbf{c}).$$

In both $M(P)$ and $E(P)$ we recall that $I_q(\mathbf{c})$ vanishes for $q \gg Q$, by the properties of the function h .

To handle $M(P)$ our first task is to relate $I_q(\mathbf{0})$ to the singular integral, given by (4.6). To begin with we note that

$$I_q(\mathbf{0}) = P^4 \int_{\mathbb{R}^4} w(\mathbf{x}) h\left(\frac{q}{Q}, G(\mathbf{x})\right) d\mathbf{x},$$

where $G(\mathbf{x}) = \|F\|^{-1} F(\mathbf{x})$. Since w is compactly supported we will have $|G(\mathbf{x})| \leq c_w$ whenever $w(\mathbf{x}) \neq 0$, for some constant c_w depending only on w .

We choose a smooth weight function $w_0 : \mathbb{R} \rightarrow \mathbb{R}$ supported on $[-1 - c_w, 1 + c_w]$ such that $w_0(t) = 1$ for $t \in [-c_w, c_w]$. This choice can be made in an explicit way such that w_0 depends only on w . This allows us to write

$$I_q(\mathbf{0}) = P^4 \int_{\mathbb{R}^4} w(\mathbf{x}) w_0(G(\mathbf{x})) h\left(\frac{q}{Q}, G(\mathbf{x})\right) d\mathbf{x}.$$

The function $f(t) = w_0(t) h(q/Q, t)$ is compactly supported with a continuous second derivative. Recall the definition (3.10) of the function $K(u; \delta)$. The above condition is enough to ensure that

$$\int_{-\infty}^{\infty} f(t) K(t - \tau; \delta) dt \rightarrow f(\tau) \quad \text{as } \delta \downarrow 0,$$

uniformly in τ . As a result one sees that

$$I_q(\mathbf{0}) = P^4 \lim_{\delta \downarrow 0} \int_{\mathbb{R}^4} \int_{-\infty}^{\infty} w(\mathbf{x}) w_0(t) h\left(\frac{q}{Q}, t\right) K(t - G(\mathbf{x}); \delta) dt d\mathbf{x}.$$

Using the equation (3.11) we are now led to the expression

$$I_q(\mathbf{0}) = P^4 \lim_{\delta \downarrow 0} \int_{-\infty}^{\infty} \left(\frac{\sin(\pi \delta \theta)}{\pi \delta \theta} \right)^2 J(\theta; w) L(\theta) d\theta,$$

with $J(\theta; w)$ given by (4.11) and

$$L(\theta) = \int_{-\infty}^{\infty} w_0(t) h\left(\frac{q}{Q}, t\right) e(\theta t) dt.$$

The following result is concerned with estimating this integral.

Lemma 4.11. *Assume that $q \leq Q$. Then*

$$L(\theta) = 1 + O_N((q/Q)^N) + O_N((q/Q)^N |\theta|^N)$$

for any integer $N \geq 1$.

We will prove this at the end of this section, but first we use it to complete our treatment of $I_q(\mathbf{0})$. Combining Lemma 4.4 with the trivial bound $J(\theta; w) \ll 1$ we have $J(\theta; w) \ll_N (1 + |\theta|)^{-2N} \|F\|^{4N\eta}$. The error terms in Lemma 4.11 therefore contribute a total

$$\ll_N P^4 \int_{-\infty}^{\infty} \frac{(q/Q)^N \{1 + |\theta|\}^N}{\{1 + |\theta|\}^{2N}} \|F\|^{4N\eta} d\theta \ll_N P^4 (q/Q)^N \|F\|^{4N\eta},$$

for any $N \geq 2$. Moreover

$$\int_{-\infty}^{\infty} \left(\frac{\sin(\pi \delta \theta)}{\pi \delta \theta} \right)^2 J(\theta; w) d\theta \rightarrow \int_{-\infty}^{\infty} J(\theta; w) d\theta$$

as $\delta \downarrow 0$. On replacing θ by $\|F\|\theta$ we then see that

$$I_q(\mathbf{0}) = P^4 \left\{ \|F\| \sigma_{\infty}(w; F) + O_N((q/Q)^N \|F\|^{4N\eta}) \right\}$$

for any $q \leq Q$ and any $N \geq 2$. In particular, if $q \leq Q\|F\|^{-5\eta}$ then, by taking N suitably large, the error term can be made smaller than any given negative power of P , by virtue of our assumption that $\|F\| \geq P^\eta$.

We will need the following upper bound for the singular integral, in which we do not make either of the assumptions (4.2) or (4.3).

Lemma 4.12. *Suppose that $\Delta_F \neq 0$. Suppose either that w is a smooth weight function of compact support, or that w is the characteristic function of $[-\kappa, \kappa]^4$ for some $\kappa > 0$. Then $\sigma_\infty(w; F) \ll_w |\Delta_F|^{-1/4}$.*

We will prove this in the next section. Taking the lemma as proved, we see that (4.3) implies that

$$\sigma_\infty(w; F) \ll \|F\|^{-1+\eta}. \quad (4.18)$$

We now have

$$M(P) = \frac{\|F\|\sigma_\infty(w; F)P^4}{Q^2} \sum_{q \leq Q\|F\|^{-5\eta}} q^{-4} S_q(\mathbf{0}) + O(|T(Q\|F\|^{-5\eta})| + 1),$$

where

$$T(M) = \frac{1}{Q^2} \sum_{q > M} q^{-4} S_q(\mathbf{0}) I_q(\mathbf{0}).$$

Summation by parts yields

$$T(M) = -\frac{I_M(\mathbf{0})}{Q^2 M} \Sigma(M; \mathbf{0}) - \frac{1}{Q^2} \int_M^\infty \Sigma(x; \mathbf{0}) \frac{\partial}{\partial x} \frac{I_x(\mathbf{0})}{x} dx,$$

where $\Sigma(x; \mathbf{0})$ is given by (4.17). However $I_x(\mathbf{0})$ vanishes identically when $x \gg Q$, whence it follows from Lemmas 4.3 and 4.9 that

$$T(M) \ll \frac{\|F\|^\eta P^4}{Q^2 M} \sup_{M \leq x \leq Q} |\Sigma(x; \mathbf{0})| \ll_\varepsilon \frac{\|F\|^\eta P^4}{Q^2 M} |\Delta_F|^{3/16+\varepsilon} \Delta_{\text{bad}}^{3/8} Q^{1/2+\varepsilon}.$$

Taking $M = Q\|F\|^{-5\eta}$ with $Q = P\|F\|^{1/2}$, and using the bounds $|\Delta_F| \leq \|F\|^4$ and $\Delta_{\text{bad}} \leq \|F\|^\eta$, this yields

$$T(Q\|F\|^{-5\eta}) \ll_\varepsilon \|F\|^{-1/2+51\eta/8+\varepsilon} P^{3/2+\varepsilon}.$$

Our assumption that $\|F\| \geq P^\eta$ enables us to replace P^ε by $\|F\|^{\varepsilon/\eta}$, whence, on choosing ε suitably small, we obtain the bound

$$M(P) = \frac{\|F\|\sigma_\infty(w; F)P^4}{Q^2} \sum_{q \leq Q\|F\|^{-5\eta}} q^{-4} S_q(\mathbf{0}) + O(\|F\|^{-1/2+7\eta} P^{3/2}).$$

A similar analysis shows that

$$\frac{\|F\|P^4}{Q^2} \sum_{q \leq Q\|F\|^{-5\eta}} q^{-4} S_q(\mathbf{0}) = \frac{\|F\|P^4}{Q^2} \sum_{q=1}^\infty q^{-4} S_q(\mathbf{0}) + O(\|F\|^{1/2+6\eta} P^{3/2}).$$

The infinite sum is just $\mathfrak{S}(F)$. Hence, using (4.18), we find that

$$M(P) = \sigma_\infty(w; F) \mathfrak{S}(F) P^2 + O(\|F\|^{-1/2+7\eta} P^{3/2}).$$

This is satisfactory for the statement of Theorem 4.1.

Proof of Lemma 4.11. When $|\theta| \leq 1$ the result is an immediate application of Heath-Brown [12, Lemma 9]. Moreover, taking $N = 2$ in (4.10) yields $h(x, t) \ll x + \min(x^{-1}, xt^{-2})$, whence one trivially has $L(\theta) \ll 1$ for $q \leq Q$. We may therefore assume that $qQ^{-1}|\theta| \leq 1 \leq |\theta|$. In this case we must modify the proof of [12, Lemma 9]. It will be convenient to write $x = q/Q$ and $X = \sqrt{x/|\theta|}$. Since w_0 has compact support we may suppose that $t \ll_w 1$. Thus (4.10) implies that $h(x, t) \ll_N x^{N-1}|t|^{-N}$ for any $N \geq 1$. It then follows that the range $|t| \geq X$ contributes $O_N(x^{N-1}X^{1-N}) = O_N((x|\theta|)^{(N-1)/2})$ to $L(\theta)$. This is satisfactory for the error terms of Lemma 4.11, on redefining N .

When $|t| \leq X$ we use Taylor's theorem to approximate $w_0(t)e(\theta t)$ by a polynomial of degree M , say, together with an error $O_M(X^{M+1}|\theta|^{M+1})$. Since we have $h(x, t) \ll x + \min(x^{-1}, xt^{-2})$, as noted above, this error term contributes $O_M((x|\theta|)^{(M+1)/2})$ to $L(\theta)$, which again is satisfactory if M is large enough. The polynomial produced by Taylor's theorem has terms $c_m t^m$ with $c_m \ll_M |\theta|^m$. When $1 \leq m \leq M$ we apply [12, Lemma 8], producing an overall bound $O_M((X|\theta|)^m (x/X)^M)$ for each value of m . One should note here firstly that the required condition $x \ll \min(1, X)$ holds, by virtue of the condition $x|\theta| \leq 1$, and secondly that $Xx^{M-1} \leq x^M X^{-M}$ for $M \geq 1$, since $x \leq 1 \leq |\theta|$. On considering the possible values for $m \in \{1, \dots, M\}$ we then see that each monomial $c_m t^m$ contributes $O_M((x|\theta|)^M) + O_M((x|\theta|)^{(M+1)/2})$, which is satisfactory when M is taken large enough. Finally, the constant term c_0 is handled analogously using [12, Lemma 6], producing the same error term together with a main term c_0 . However the function w_0 takes the value 1 at the origin, and the lemma follows. \square

4.4. The singular integral. We begin by proving Lemma 4.12. The function

$$J(\theta; w) = \int_{\mathbb{R}^4} w(\mathbf{x}) e(-\theta G(\mathbf{x})) d\mathbf{x}$$

is well-defined and continuous. We claim that $J(\theta; w) \in L^1(\mathbb{R})$ for w as in the lemma. If w is smooth and supported in $[-\kappa, \kappa]^4$ then

$$J(\theta; w) \ll_w \int_{[-\kappa, \kappa]^4} e(-\theta G(\mathbf{x}) + \mathbf{x} \cdot \mathbf{y}) d\mathbf{x}$$

for some $\mathbf{y} \in \mathbb{R}^4$, by Lemma 3.2 of Heath-Brown and Pierce [14]. The integral on the right factors into four 1-dimensional integrals, with

$$\int_{-\kappa}^{\kappa} e(-\theta A_j \|F\|^{-1} x^2 + x y_j) dx \ll_{\kappa} \min \left\{ 1, \sqrt{\frac{\|F\|}{|\theta A_j|}} \right\}.$$

Clearly we get the same estimate when w is the characteristic function of $[-\kappa, \kappa]^4$. It then follows that $J(\theta; w) \ll_{w, G} \min(1, \theta^{-2})$, which is enough for the claim.

We now calculate that

$$\begin{aligned} \int_{-\infty}^{\infty} J(\theta; w) d\theta &= \lim_{\delta \downarrow 0} \int_{-\infty}^{\infty} \left(\frac{\sin(\pi \delta \theta)}{\pi \delta \theta} \right)^2 J(\theta; w) d\theta \\ &= \lim_{\delta \downarrow 0} \int_{-\infty}^{\infty} \int_{\mathbb{R}^4} w(\mathbf{x}) e(-\theta G(\mathbf{x})) \left(\frac{\sin(\pi \delta \theta)}{\pi \delta \theta} \right)^2 d\mathbf{x} d\theta. \end{aligned}$$

The conditions for Fubini's Theorem are satisfied, allowing us to switch the two integrations. The relation (3.11) then shows us that

$$\int_{-\infty}^{\infty} J(\theta; w) d\theta = \lim_{\delta \downarrow 0} \int_{\mathbb{R}^4} w(\mathbf{x}) K(-G(\mathbf{x}); \delta) d\mathbf{x}. \quad (4.19)$$

The integral on the left is $\|F\| \sigma_{\infty}(w; F)$. However if $w(\mathbf{x})$ vanishes for $|\mathbf{x}| \geq \kappa$, say, then

$$\int_{\mathbb{R}^4} w(\mathbf{x}) K(-G(\mathbf{x}); \delta) d\mathbf{x} \ll \int_{[-\kappa, \kappa]^4} K(-G(\mathbf{x}); \delta) d\mathbf{x},$$

since $K(-G(\mathbf{x}); \delta) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^4$. We now write w_* for the characteristic function of $[-\kappa, \kappa]^4$, so that a second application of (4.19) shows that

$$\|F\| \sigma_{\infty}(w; F) \ll \int_{-\infty}^{\infty} J(\theta; w_*) d\theta.$$

We saw above that

$$J(\theta; w_*) \ll_{\kappa} \prod_{j=1}^4 \min \left\{ 1, \sqrt{\frac{\|F\|}{|\theta A_j|}} \right\},$$

whence

$$\int_{-\infty}^{\infty} J(\theta; w_*) d\theta \ll \int_{-\infty}^{\infty} \min \left\{ 1, \frac{\|F\|^2}{\theta^2 |\Delta_F|^{1/2}} \right\} d\theta \ll \frac{\|F\|}{|\Delta_F|^{1/4}}.$$

The statement of Lemma 4.12 then follows.

We now use the machinery developed above to see how to compare $\sigma_{\infty}(w; F)$ for different weights.

Lemma 4.13. *Let $w_0(\mathbf{x})$ be the characteristic function of the region $[-1, 1]^4$. Suppose that $w_1(\mathbf{x})$ (respectively, $w_2(\mathbf{x})$) is supported in the region $\eta \leq |\mathbf{x}| \leq 1$ (respectively, $\eta \leq |\mathbf{x}| \leq 1 + \eta$) and takes values in $[0, 1]$ there. Suppose further that $w_1(\mathbf{x}) = 1$ whenever $2\eta \leq |\mathbf{x}| \leq 1 - \eta$ (respectively, $w_2(\mathbf{x}) = 1$ whenever $2\eta \leq |\mathbf{x}| \leq 1$). Then*

$$\sigma_\infty(w_i; F) = \sigma_\infty(w_0; F) + O(\eta^{1/2} |\Delta_F|^{-1/4}),$$

for $i = 1, 2$, the implied constant being absolute.

Proof. We confine ourselves to the result for w_1 , the function w_2 being treated analogously. It follows from (4.19) that

$$\|F\| \{ \sigma_\infty(w_1; F) - \sigma_\infty(w_0; F) \} = \lim_{\delta \downarrow 0} \int_{\mathbb{R}^4} \{w_1(\mathbf{x}) - w_0(\mathbf{x})\} K(-G(\mathbf{x}); \delta) d\mathbf{x}.$$

We have $|w_1(\mathbf{x}) - w_0(\mathbf{x})| \leq 1$ for all \mathbf{x} , the difference being non-zero only if either $|\mathbf{x}| \leq 2\eta$ or there is some index i for which $1 - \eta \leq |x_i| \leq 1$. Hence

$$|w_1(\mathbf{x}) - w_0(\mathbf{x})| \leq \sum_{n=0}^8 f_n(\mathbf{x}),$$

where each $f_n(\mathbf{x})$ is the characteristic function of a certain box $I_1 \times \dots \times I_4$. For $n = 0$ this is just $[-2\eta, 2\eta]^4$, but otherwise 3 of the intervals I_j have length 2, and the fourth has length η . Thus

$$\|F\| \cdot |\sigma_\infty(w_1; F) - \sigma_\infty(w_0; F)| \leq \sum_{n=0}^8 \lim_{\delta \downarrow 0} \int_{\mathbb{R}^4} f_n(\mathbf{x}) K(-G(\mathbf{x}); \delta) d\mathbf{x}.$$

The expression on the right may be evaluated via a further application of (4.19). As in the proof of Lemma 4.12, we have

$$\int_{I_j} e(\theta A_j \|F\|^{-1} x^2) dx \ll \min \left\{ \text{meas}(I_j), \sqrt{\frac{\|F\|}{|\theta A_j|}} \right\},$$

leading to a bound

$$\begin{aligned} \|F\| \{ \sigma_\infty(w_1; F) - \sigma_\infty(w_0; F) \} &\ll \int_{-\infty}^{\infty} \min \left\{ \eta, \frac{\|F\|^2}{\theta^2 |\Delta_F|^{1/2}} \right\} d\theta \\ &\ll \eta^{1/2} \frac{\|F\|}{|\Delta_F|^{1/4}} \end{aligned}$$

and the lemma follows. \square

It is convenient at this point to record some facts about τ_∞ , as given in Theorem 1.1.

Lemma 4.14. *Let $\varrho_\infty(\mathbf{y})$ be given by (3.3), and define*

$$\sigma_\infty(\mathbf{x}) = \int_{-\infty}^{\infty} \int_{[-1,1]^4} e(-\theta F(\mathbf{x}; \mathbf{y})) d\mathbf{y} d\theta. \quad (4.20)$$

Then

$$\tau_\infty = \int_{[-1,1]^4} \varrho_\infty(\mathbf{y}) d\mathbf{y} = \int_{[-1,1]^4} \sigma_\infty(\mathbf{x}) d\mathbf{x}. \quad (4.21)$$

Proof. In order to verify (4.21) it is enough to confirm that the orders of integration may be suitably changed, and this will be permissible provided both

$$\int_{[-1,1]^4} e(-\theta F(\mathbf{x}; \mathbf{y})) d\mathbf{x} \quad \text{and} \quad \int_{[-1,1]^4} e(-\theta F(\mathbf{x}; \mathbf{y})) d\mathbf{y}$$

are in $L^1(\mathbb{R} \times [-1,1]^4)$. However these are of order

$$\prod_{i=1}^4 \min\{1, |\theta|^{-1} |y_i|^{-2}\} \quad \text{and} \quad \prod_{i=1}^4 \min\{1, |\theta|^{-1/2} |x_i|^{-1/2}\}$$

respectively, and so are both integrable. \square

4.5. Analysis of $E(P)$. The key observation in handling $E(P)$ is that the hypothesis $\|F\| \geq P^\eta$ in Theorem 4.1, along with the assumption (4.4), allow us to restrict to $\mathbf{c} \in \mathbb{Z}^4$ for which $F^*(\mathbf{c}) \neq 0$. Indeed, suppose for a contradiction that $\mathbf{c} \in \mathbb{Z}^4$ is such that $0 < |\mathbf{c}| \ll C$ and $F^*(\mathbf{c}) = 0$, where $F^*(\mathbf{c})$ is given by (4.16). Let us assume, for example, that $c_1 \neq 0$. Now, if we write

$$A_1^b = \prod_{\substack{p \mid A_1 \\ p \nmid A_2 A_3 A_4}} p,$$

then the equation $F^*(\mathbf{c}) = 0$ implies that $A_1^b \mid c_1$. Since $1 \leq |c_1| \ll C$ we would then deduce that $A_1^b \ll C$. However (4.3) and (4.4) yield

$$A_1^b \geq \frac{|A_1|}{\Delta_{\text{bad}}} \geq \|F\|^{1-2\eta}.$$

Since $C = \|F\|^{1/2} Q^\varepsilon$ with $Q = \|F\|^{1/2} P$, we therefore obtain a contradiction if ε is small enough and P is large enough, since $P^\eta \leq \|F\|$.

We now have

$$E(P) = \frac{1}{Q^2} \sum_{\substack{\mathbf{c} \in \mathbb{Z}^4 \\ |\mathbf{c}| \ll C \\ F^*(\mathbf{c}) \neq 0}} \sum_{q=1}^{\infty} q^{-4} S_q(\mathbf{c}) I_q(\mathbf{c}).$$

The analysis of quaternary quadratic forms, in Heath-Brown [12, §12] for example, normally requires one to obtain some cancellation from the summation

over q , but this is no longer necessary because we have been able to remove vectors \mathbf{c} for which $F^*(\mathbf{c}) = 0$.

Noting that $I_q(\mathbf{c})$ is only supported on $q \ll Q$, we deduce from part (ii) of Lemma 4.2 that

$$\sum_{q=1}^{\infty} q^{-4} S_q(\mathbf{c}) I_q(\mathbf{c}) \ll \frac{\|F\|^2 P^3}{|\Delta_F|^{1/2} |\mathbf{c}|} \sum_{q \ll Q} q^{-3} |S_q(\mathbf{c})|.$$

Lemma 4.7 now implies that

$$E(P) \ll \frac{\|F\|^{2+\varepsilon} P^{3+\varepsilon}}{|\Delta_F|^{1/2} Q^2} \sum_{\substack{\mathbf{c} \in \mathbb{Z}^4 \\ 0 < |\mathbf{c}| \ll C}} \frac{\prod_{1 \leq i \leq 4} (A_i, c_i)^{1/2}}{|\mathbf{c}|}.$$

It remains to estimate the \mathbf{c} -sum, which we temporarily denote by K . We plainly have

$$K \leq \sum_{d_i | A_i} \sqrt{d_1 d_2 d_3 d_4} \sum_{\substack{\mathbf{c} \in \mathbb{Z}^4 \setminus \{\mathbf{0}\} \\ c_i \ll C/d_i}} \frac{1}{\max_{1 \leq i \leq 4} |d_i c_i|}.$$

The inner sum is

$$\ll \left(\frac{C}{d_i} + 1 \right) \left(\frac{C}{d_j} + 1 \right) \left(\frac{C}{d_k} + 1 \right) \frac{\log C}{d_l}$$

for some permutation $\{i, j, k, l\}$ of $\{1, 2, 3, 4\}$. Multiplying this by $\sqrt{d_1 d_2 d_3 d_4}$ and recalling that $C = \|F\|^{1/2} Q^\varepsilon$, this gives

$$K \ll \sum_{d_i | A_i} \frac{\log C}{\sqrt{d_l}} \left(C + \sqrt{d_i} \right) \left(C + \sqrt{d_j} \right) \left(C + \sqrt{d_k} \right) \ll \|F\|^{3/2+\varepsilon} P^\varepsilon,$$

on employing the trivial estimate for the divisor function.

Absorbing P^ε into $\|F\|^\varepsilon$, it now follows that

$$E(P) \ll \frac{\|F\|^{5/2+\varepsilon} P}{|\Delta_F|^{1/2}}.$$

Our hypotheses (4.3) implies that $|\Delta_F| \geq \|F\|^{4-3\eta}$, from which one sees that our bound for $E(P)$ is satisfactory for the error term given in Theorem 4.1.

5. COMBINING THE VARIOUS INGREDIENTS

Theorems 3.1 and 4.1 will be our main tools for the proof of Theorem 1.1. In this section we begin by adapting them so as to count only primitive vectors. We then apply Theorem 4.1 to the quadrics in \mathbf{y} given by $F(\mathbf{x}; \mathbf{y}) = 0$, and sum the resulting asymptotic formulae with respect to \mathbf{x} . In doing so we must allow for those vectors \mathbf{x} excluded by the conditions of the theorem. The next stage is to remove the weight w occurring in Theorem 4.1. We then piece

together our two main estimates, and make suitable adjustments so as to cover all primitive \mathbf{x}, \mathbf{y} with $|\mathbf{x}|^3 |\mathbf{y}|^2 \leq B$. In §6 we will show that the main terms have a total asymptotically $cB \log B$, with the constant c given by (1.3).

5.1. Primitive solutions with small \mathbf{y} . In analogy to (3.1) and (3.2) we define

$$M_5(\mathbf{y}; R) = \# \{ \mathbf{x} \in \mathbb{Z}^4 : |\mathbf{x}| \leq R, \Delta(\mathbf{x}) \neq \square, F(\mathbf{x}; \mathbf{y}) = 0 \}$$

and

$$N_2(B; Y) = \sum_{\substack{\mathbf{y} \in \mathbb{Z}_{\text{prim}}^4 \\ Y < |\mathbf{y}| \leq 2Y}} M_5(\mathbf{y}; (B/|\mathbf{y}|^2)^{1/3}),$$

so that

$$N_2(B; Y) \ll B + B^{5/3} Y^{-8/3}, \quad (5.1)$$

by the first part of Lemma 2.1.

We may also estimate $N_2(B; Y)$ by removing solutions with $\Delta(\mathbf{x}) = \square$ from the counting function $N_1(B; Y)$ given by Theorem 3.1. According to Lemma 2.6 there are $O_\varepsilon(BY^{-1+\varepsilon} + Y^4)$ solutions with $\Delta(\mathbf{x}) = 0$. Moreover there are $O(X^2)$ possible square values for $\Delta(\mathbf{x}) \neq 0$ when $|\mathbf{x}| \leq X$, each such value corresponding to $O_\varepsilon(X^\varepsilon)$ vectors \mathbf{x} . Thus Lemma 2.3 shows that solutions with $\Delta(\mathbf{x}) = \square \neq 0$ contribute $O_\varepsilon(B^{2/3+\varepsilon} Y^{2/3})$ to $N_1(B; Y)$. Taking $\varepsilon = \frac{2}{15}$ we deduce that

$$B^{2/3+\varepsilon} Y^{2/3} = (B^{2/3} Y^{4/3})^{3/5} (BY^{-1/3})^{2/5} \leq B^{2/3} Y^{4/3} + BY^{-1/3}.$$

It then follows from Theorem 3.1 that

$$N_2(B; Y) = B \sum_{\substack{\mathbf{y} \in \mathbb{Z}_{\text{prim}}^4 \\ Y < |\mathbf{y}| \leq 2Y}} \frac{\varrho_\infty(\mathbf{y})}{|\mathbf{y}|^2} + O(B^{2/3} Y^{4/3}) + O(BY^{-1/3}) + O(Y^4) \quad (5.2)$$

for $Y \geq \frac{1}{2}$, where $\varrho_\infty(\mathbf{y})$ is given by (3.3).

We now set

$$M_6(\mathbf{y}; R) = \# \{ \mathbf{x} \in \mathbb{Z}_{\text{prim}}^4 : |\mathbf{x}| \leq R, \Delta(\mathbf{x}) \neq \square, F(\mathbf{x}; \mathbf{y}) = 0 \}$$

and

$$N_3(B; Y) = \sum_{\substack{\mathbf{y} \in \mathbb{Z}_{\text{prim}}^4 \\ Y < |\mathbf{y}| \leq 2Y}} M_6(\mathbf{y}; (B/|\mathbf{y}|^2)^{1/3}),$$

whence

$$M_6(\mathbf{y}; R) = \sum_{d \leq R} \mu(d) M_5(\mathbf{y}; R/d).$$

Our goal now is the following estimate.

Lemma 5.1. *If $\frac{1}{2} \leq Y \leq B^{1/4}$ we have*

$$N_3(B; Y) = \frac{B}{\zeta(3)} \sum_{\substack{\mathbf{y} \in \mathbb{Z}_{\text{prim}}^4 \\ Y < |\mathbf{y}| \leq 2Y}} \frac{\varrho_{\infty}(\mathbf{y})}{|\mathbf{y}|^2} + O(B^{2/3}Y^{4/3}) + O(BY^{-1/3}).$$

Proof. We start from the relation

$$N_3(B; Y) = \sum_{d \leq (B/Y^2)^{1/3}} \mu(d) N_2(B/d^3; Y).$$

To estimate this sum we choose a parameter D in the range $1 \leq D \leq B^{1/3}Y^{-2/3}$ and use (5.2) for $d \leq D$ and (5.1) for $d > D$. This yields

$$\begin{aligned} N_3(B; Y) &= B \left(\sum_{d \leq D} \frac{\mu(d)}{d^3} \right) \sum_{\substack{\mathbf{y} \in \mathbb{Z}_{\text{prim}}^4 \\ Y < |\mathbf{y}| \leq 2Y}} \frac{\varrho_{\infty}(\mathbf{y})}{|\mathbf{y}|^2} + O(B^{2/3}Y^{4/3}) + O(BY^{-1/3}) \\ &\quad + O(Y^4 D) + O(BD^{-2}) + O(B^{5/3}D^{-4}Y^{-8/3}). \end{aligned}$$

It follows from (3.8) that

$$\sum_{\substack{\mathbf{y} \in \mathbb{Z}_{\text{prim}}^4 \\ Y < |\mathbf{y}| \leq 2Y}} \frac{\varrho_{\infty}(\mathbf{y})}{|\mathbf{y}|^2} = \sum_{\substack{\mathbf{y} \in \mathbb{Z}_{\text{prim}}^4 \\ Y < |\mathbf{y}| \leq 2Y}} \frac{V(\mathbf{y})}{|\mathbf{y}|^2 d(\mathbf{y})} \ll 1,$$

so that the leading term is

$$\frac{B}{\zeta(3)} \sum_{\substack{\mathbf{y} \in \mathbb{Z}_{\text{prim}}^4 \\ Y < |\mathbf{y}| \leq 2Y}} \frac{\varrho_{\infty}(\mathbf{y})}{|\mathbf{y}|^2} + O(BD^{-2}).$$

Thus if we choose $D = B^{1/3}Y^{-4/3}$ we obtain

$$N_3(B; Y) = \frac{B}{\zeta(3)} \sum_{\substack{\mathbf{y} \in \mathbb{Z}_{\text{prim}}^4 \\ Y < |\mathbf{y}| \leq 2Y}} \frac{\varrho_{\infty}(\mathbf{y})}{|\mathbf{y}|^2} + O(B^{2/3}Y^{4/3}) + O(BY^{-1/3}) + O(B^{1/3}Y^{8/3}).$$

Since $Y \leq B^{1/4}$ the final error term is bounded by the first, as required. \square

5.2. Primitive solutions for typical small \mathbf{x} . We next perform a similar computation for solutions in which \mathbf{x} is small and \mathbf{y} is large, counted via the fibration into quadrics, using Theorem 4.1. We write

$$M_7(\mathbf{x}; P, w) = \sum_{\substack{\mathbf{y} \in \mathbb{Z}^4 \\ F(\mathbf{x}; \mathbf{y})=0}} w(P^{-1}\mathbf{y}),$$

where $w(\mathbf{y})$ is an infinitely differentiable weight function of compact support that vanishes for $|\mathbf{y}| \leq \eta$. Then Theorem 4.1 shows that

$$M_7(\mathbf{x}; P, w) = \sigma_\infty(\mathbf{x}; w) \mathfrak{S}(\mathbf{x}) P^2 + O_{w, \eta}(|\mathbf{x}|^{-1/2+7\eta} P^{3/2} + |\mathbf{x}|^{1/2+2\eta} P)$$

when $|\mathbf{x}| \geq P^\eta$, provided that

$$|\mathbf{x}|^{1-\eta} \leq |x_i| \leq |\mathbf{x}|, \quad \text{for } 1 \leq i \leq 4, \quad (5.3)$$

and that

$$\Delta_{\text{bad}}(\mathbf{x}) \leq |\mathbf{x}|^\eta. \quad (5.4)$$

The singular integral and series are given by

$$\sigma_\infty(\mathbf{x}; w) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^4} w(\mathbf{y}) e(-\theta F(\mathbf{x}; \mathbf{y})) d\mathbf{y} d\theta \quad (5.5)$$

and

$$\mathfrak{S}(\mathbf{x}) = \prod_p \lim_{r \rightarrow \infty} p^{-3r} \# \{ \mathbf{y} \in (\mathbb{Z}/p^r \mathbb{Z})^4 : F(\mathbf{x}; \mathbf{y}) \equiv 0 \pmod{p^r} \}.$$

We now write

$$M_8(\mathbf{x}; P, w) = \sum_{\substack{\mathbf{y} \in \mathbb{Z}_{\text{prim}}^4 \\ F(\mathbf{x}; \mathbf{y})=0}} w(P^{-1} \mathbf{y})$$

and proceed to derive the following estimate.

Lemma 5.2. *Suppose that \mathbf{x} satisfies the conditions (5.3) and (5.4), and that $P^\eta \leq |\mathbf{x}| \leq P^{2/3}$. We then have*

$$M_8(\mathbf{x}; P, w) = \frac{\sigma_\infty(\mathbf{x}; w) \mathfrak{S}(\mathbf{x})}{\zeta(2)} P^2 + O_{w, \eta}(|\mathbf{x}|^{-1/2} P^{5/3+5\eta}). \quad (5.6)$$

Proof. Our starting point is the relation

$$\begin{aligned} M_8(\mathbf{x}; P, w) &= \sum_{d \leq P} \mu(d) M_7(\mathbf{x}; P/d, w) \\ &= \sigma_\infty(\mathbf{x}; w) \mathfrak{S}(\mathbf{x}) P^2 \{ \zeta(2)^{-1} + O(P^{-1}) \} \\ &\quad + O_{w, \eta}(|\mathbf{x}|^{-1/2+7\eta} P^{3/2} + |\mathbf{x}|^{1/2+2\eta} P^{1+\eta}). \end{aligned}$$

If we assume that $|\mathbf{x}| \leq P^{2/3}$ the final error term is $O_{w, \eta}(|\mathbf{x}|^{-1/2} P^{5/3+5\eta})$. We also observe that $\sigma_\infty(\mathbf{x}; w) \ll |\mathbf{x}|^{-1+\eta}$, as in (4.18), and that $\mathfrak{S}(\mathbf{x}) \ll |\mathbf{x}|^\eta$ by Lemma 4.10 and (5.4). The estimate (5.6) then follows. \square

We are now ready to consider the average

$$N_4(B; X, w) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^4, \Delta(\mathbf{x}) \neq \square \\ X < |\mathbf{x}| \leq 2X}} M_8(\mathbf{x}; (B/|\mathbf{x}|^3)^{1/2}, w),$$

for which we have the following estimate.

Lemma 5.3. *Let*

$$B^{2\eta} \leq X \leq B^{1/6-4\eta}. \quad (5.7)$$

Then if η is small enough we will have

$$N_4(B; X, w) = \frac{B}{\zeta(2)} \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^4, \Delta(\mathbf{x}) \neq \square \\ X < |\mathbf{x}| \leq 2X}} \frac{\sigma_\infty(\mathbf{x}; w) \mathfrak{S}(\mathbf{x})}{|\mathbf{x}|^3} + O_{w, \eta}(B^{1-\eta^2/20}).$$

Proof. We would like to apply Lemma 5.2 for those vectors \mathbf{x} which satisfy the conditions (5.3) and (5.4), and for which $(B/|\mathbf{x}|^3)^{\eta/2} \leq |\mathbf{x}| \leq (B/|\mathbf{x}|^3)^{1/3}$. This final constraint holds if $X < |\mathbf{x}| \leq 2X$ with X satisfying (5.7). Moreover, the error term contributes a total $O_{w, \eta}(B^{5/6+3\eta}X)$, which is satisfactory for X in the range (5.7). Thus to complete our treatment of $N_4(B; X, w)$ we must consider vectors \mathbf{x} for which either (5.3) or (5.4) fails.

We begin by considering the number of solutions (\mathbf{x}, \mathbf{y}) for such \mathbf{x} . By the third part of Lemma 2.1 the number of solutions (\mathbf{x}, \mathbf{y}) for which (5.3) fails will be

$$\ll_\varepsilon B^\varepsilon X^{-3\eta/4} \{B + B^{1/2} X^{5/2}\} \ll_\varepsilon B^\varepsilon X^{-3\eta/4} \{B + B^{11/12}\} \ll_\varepsilon B^{1+\varepsilon} X^{-3\eta/4}.$$

This is satisfactory when $B^{2\eta} \leq X \leq B^{1/6-4\eta}$, provided that we take $\varepsilon \leq \eta^2$. Similarly, by Lemma 2.7, the number of solutions (\mathbf{x}, \mathbf{y}) for which (5.4) fails will be

$$\ll_\varepsilon B^\varepsilon \{BX^{-\eta/24} + B^{1/2} X^{5/2}\}$$

for any fixed $\varepsilon > 0$. As before, under the assumption (5.7) this becomes $O(B^{1-\eta^2/20})$ if we choose ε small enough. Thus vectors \mathbf{x} which fail to satisfy either (5.3) or (5.4) will make a suitably small contribution to $N_4(B; X, w)$.

To complete the proof of Lemma 5.3 it remains to prove that

$$\sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^4, \Delta(\mathbf{x}) \neq \square \\ X < |\mathbf{x}| \leq 2X \\ (5.3) \text{ or } (5.4) \text{ fails}}} \frac{\sigma_\infty(\mathbf{x}; w) \mathfrak{S}(\mathbf{x})}{|\mathbf{x}|^3} \ll X^{-\eta/5}, \quad (5.8)$$

since for X in the range (5.7) the right hand side will then be $O(B^{-2\eta^2/5})$, which is satisfactory for Lemma 5.3. According to Lemmas 4.10 and 4.12 we have

$$\frac{\sigma_\infty(\mathbf{x}; w) \mathfrak{S}(\mathbf{x})}{|\mathbf{x}|^3} \ll_\varepsilon X^{-3+\varepsilon} \frac{\Delta_{\text{bad}}(\mathbf{x})^{1/4}}{|x_1 x_2 x_3 x_4|^{1/4}}.$$

Let s run over square-full positive integers, and write $s = \Delta_{\text{bad}}(\mathbf{x})$ and $n = |x_1 x_2 x_3 x_4|$. Then vectors for which (5.3) fails will contribute

$$\begin{aligned} &\ll_{\varepsilon} X^{-3+\varepsilon} \sum_s s^{1/4} \sum_{\substack{n \leq 16X^{4-\eta} \\ s|n}} \tau_4(n) n^{-1/4} \\ &\ll_{\varepsilon} X^{-3+2\varepsilon} \sum_s \sum_{m \leq 16X^{4-\eta}/s} m^{-1/4} \\ &\ll_{\varepsilon} X^{-3+2\varepsilon} \sum_s X^{3-3\eta/4} s^{-3/4}. \end{aligned}$$

However, if s runs over square-full integers the infinite sum $\sum s^{-3/4}$ converges, so that the above will be $O(X^{-\eta/2})$ if we choose ε small enough, which is satisfactory for (5.8). Similarly, vectors for which (5.4) fails will contribute

$$\begin{aligned} &\ll_{\varepsilon} X^{-3+\varepsilon} \sum_{s \geq X^{\eta}} s^{1/4} \sum_{\substack{n \leq 16X^4 \\ s|n}} \tau_4(n) n^{-1/4} \\ &\ll_{\varepsilon} X^{-3+2\varepsilon} \sum_{s \geq X^{\eta}} \sum_{m \leq 16X^4/s} m^{-1/4} \\ &\ll_{\varepsilon} X^{-3+2\varepsilon} \sum_{s \geq X^{\eta}} X^3 s^{-3/4}. \end{aligned}$$

Since

$$\sum_{s \geq S} s^{-3/4} \ll S^{-1/4}$$

for any $S \geq 1$, the above will be $O(X^{-\eta/5})$ for small enough ε , which again produces a satisfactory contribution to (5.8). This completes the proof of the lemma. \square

5.3. Removing the weights. The counting function $N_4(B; X, w)$ involves a weight function w , and our next task is to remove it so as to produce

$$N_5(B; X) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^4, \Delta(\mathbf{x}) \neq \square \\ X < |\mathbf{x}| \leq 2X}} \sum_{\substack{\mathbf{y} \in \mathbb{Z}_{\text{prim}}^4 \\ F(\mathbf{x}; \mathbf{y}) = 0}} w_0((B/|\mathbf{x}|^3)^{-1/2} \mathbf{y}),$$

where w_0 is the characteristic function of $[-1, 1]^4$, as in Lemma 4.13. The result is described in the following estimate.

Lemma 5.4. *If X is in the range (5.7) we have*

$$N_5(B; X) = \frac{B}{\zeta(2)} \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^4, \Delta(\mathbf{x}) \neq \square \\ X < |\mathbf{x}| \leq 2X}} \frac{\sigma_{\infty}(\mathbf{x}) \mathfrak{S}(\mathbf{x})}{|\mathbf{x}|^3} + O(\eta^{1/2} B) + O_{\eta}(B^{1-\eta^2/20}).$$

We remark that $\sigma_\infty(\mathbf{x}) = \sigma_\infty(\mathbf{x}; w_0)$ in view of (4.20) and (5.5).

Proof. Given $\eta \in (0, \frac{1}{100})$ we can construct specific weights w_1, w_2 depending on η alone, and satisfying the conditions of Lemma 4.13. Thus for all \mathbf{u} we have $0 \leq w_1(\mathbf{u}), w_2(\mathbf{u}) \leq 1$. Both functions vanish when $|\mathbf{u}| \leq \eta$. The weight w_1 takes the value 1 for $2\eta \leq |\mathbf{u}| \leq 1 - \eta$ and vanishes for $|\mathbf{u}| \geq 1$; the weight w_2 takes the value 1 for $2\eta \leq |\mathbf{u}| \leq 1$ and vanishes for $|\mathbf{u}| \geq 1 + \eta$. In particular $0 \leq w_1(\mathbf{u}) \leq w_0(\mathbf{u})$ for all \mathbf{u} , so that $N_4(B; X, w_1) \leq N_5(B; X)$. The condition that $|(B/|\mathbf{x}|^3)^{-1/2}\mathbf{y}| \leq 2\eta$ is equivalent to the condition $|(B'/|\mathbf{x}|^3)^{-1/2}\mathbf{y}| \leq 1$ with $B' = 4\eta^2 B$, whence

$$N_5(B; X) - N_5(4\eta^2 B; X) \leq N_4(B; X, w_2).$$

Since the first part of Lemma 2.1 shows that

$$N_5(4\eta^2 B; X) \ll \eta^{2/3} B$$

for $X \leq B^{1/6}$, we see that it will suffice to show that

$$N_4(B; X, w_i) = \frac{B}{\zeta(2)} \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^4, \Delta(\mathbf{x}) \neq \square \\ X < |\mathbf{x}| \leq 2X}} \frac{\sigma_\infty(\mathbf{x}) \mathfrak{S}(\mathbf{x})}{|\mathbf{x}|^3} + O(\eta^{1/2} B) + O_\eta(B^{1-\eta^2/20}).$$

for $i = 1, 2$. However according to Lemma 5.3 we have

$$N_4(B; X, w_i) = \frac{B}{\zeta(2)} \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^4, \Delta(\mathbf{x}) \neq \square \\ X < |\mathbf{x}| \leq 2X}} \frac{\sigma_\infty(\mathbf{x}; w_i) \mathfrak{S}(\mathbf{x})}{|\mathbf{x}|^3} + O_\eta(B^{1-\eta^2/20})$$

for $i = 1, 2$. It needs to be stressed at this point that the implied constant for the error term depends only on η , since the two weight functions are completely fixed once η is chosen. Moreover our two weight functions do indeed vanish on a neighbourhood of the origin as was required at the outset in §4.

We now use Lemma 4.13 to replace $\sigma_\infty(\mathbf{x}; w_i)$ by $\sigma_\infty(\mathbf{x})$, introducing an error $O(\eta^{1/2} BS(X))$ with

$$S(X) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^4, \Delta(\mathbf{x}) \neq \square \\ X < |\mathbf{x}| \leq 2X}} |x_1 x_2 x_3 x_4|^{-1/4} \frac{\mathfrak{S}(\mathbf{x})}{|\mathbf{x}|^3}. \quad (5.9)$$

We therefore deduce that

$$\begin{aligned} N_5(B; X) &= \frac{B}{\zeta(2)} \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^4, \Delta(\mathbf{x}) \neq \square \\ X < |\mathbf{x}| \leq 2X}} \frac{\sigma_\infty(\mathbf{x}) \mathfrak{S}(\mathbf{x})}{|\mathbf{x}|^3} \\ &\quad + O(\eta^{1/2} B) + O(\eta^{1/2} BS(X)) + O_\eta(B^{1-\eta^2/20}). \end{aligned} \quad (5.10)$$

In order to estimate the sum $S(X)$ we apply Lemma 4.10 with $\varepsilon = \frac{1}{20}$, which yields

$$S(X) \ll X^{-3} \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^4, \Delta(\mathbf{x}) \neq \square \\ X < |\mathbf{x}| \leq 2X}} |x_1 x_2 x_3 x_4|^{-1/4} \Delta_{\text{bad}}(\mathbf{x})^{3/10} L(1, \chi).$$

We proceed by mimicking the proof of Lemma 2.9. Let $S(X_1, \dots, X_4; X)$ denote the contribution to the right hand side from the dyadic ranges

$$X_i/2 < |x_i| \leq X_i, \quad \text{for } 1 \leq i \leq 4. \quad (5.11)$$

It will be convenient to put $\widehat{X} = X_1 \dots X_4$. Writing $s = \Delta_{\text{bad}}(\mathbf{x})$, which is a square-full integer, we conclude that

$$S(X_1, \dots, X_4; X) \ll X^{-3} \widehat{X}^{-1/4} \sum_{s \text{ square-full}} s^{3/10} \sum_{\substack{d_1, \dots, d_4 \\ d_1 \dots d_4 = s}} \sum_{\mathbf{x} \in S} L(1, \chi),$$

where S is the set of $\mathbf{x} \in \mathbb{Z}^4$ in the region (5.11) for which with $\Delta(\mathbf{x}) \neq \square$ and $d_i \mid x_i$ for $1 \leq i \leq 4$. Appealing to (2.11), it follows that

$$\begin{aligned} S(X_1, \dots, X_4; X) &\ll X^{-3} \widehat{X}^{3/4} \sum_{s \text{ square-full}} s^{3/10} \sum_{\substack{d_1, \dots, d_4 \\ d_1 \dots d_4 = s}} (d_1 \dots d_4)^{-7/8} \\ &\ll X^{-3} \widehat{X}^{3/4} \sum_{s \text{ square-full}} \tau_4(s) s^{-23/40} \\ &\ll X^{-3} \widehat{X}^{3/4}. \end{aligned}$$

On summing over dyadic values for the X_i subject to $\max X_i \ll X$, we finally conclude that

$$S(X) \ll 1. \quad (5.12)$$

Once inserted into (5.10), this therefore completes the proof of Lemma 5.4. \square

5.4. The counting function $N(\Omega; B)$. Using Lemma 5.1, together with a dyadic subdivision of the range for $|\mathbf{y}|$, we find that

$$\begin{aligned} &\# \left\{ (\mathbf{x}, \mathbf{y}) \in \mathbb{Z}_{\text{prim}}^4 \times \mathbb{Z}_{\text{prim}}^4 : \begin{array}{l} \Delta(\mathbf{x}) \neq \square, F(\mathbf{x}; \mathbf{y}) = 0 \\ |\mathbf{x}|^3 |\mathbf{y}|^2 \leq B, |\mathbf{y}| \leq B^{1/4} \end{array} \right\} \\ &= \frac{B}{\zeta(3)} \sum_{\substack{\mathbf{y} \in \mathbb{Z}_{\text{prim}}^4 \\ |\mathbf{y}| \leq B^{1/4}}} \frac{\varrho_{\infty}(\mathbf{y})}{|\mathbf{y}|^2} + O(B). \end{aligned} \quad (5.13)$$

We would like to handle the range $|\mathbf{x}| \leq B^{1/6}$ similarly, using Lemma 5.4. We claim that

$$\begin{aligned}
& \# \left\{ (\mathbf{x}, \mathbf{y}) \in \mathbb{Z}_{\text{prim}}^4 \times \mathbb{Z}_{\text{prim}}^4 : \begin{array}{l} \Delta(\mathbf{x}) \neq \square, F(\mathbf{x}; \mathbf{y}) = 0 \\ |\mathbf{x}|^3 |\mathbf{y}|^2 \leq B, |\mathbf{x}| \leq B^{1/6} \end{array} \right\} \\
&= \frac{B}{\zeta(2)} \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^4 \\ B^{2\eta} \leq |\mathbf{x}| \leq B^{1/6} \\ \Delta(\mathbf{x}) \neq \square}} \frac{\sigma_{\infty}(\mathbf{x}) \mathfrak{S}(\mathbf{x})}{|\mathbf{x}|^3} \\
&\quad + O(\eta^{1/2} B \log B) + O_{\eta}(B^{1-\eta^2/20} \log B).
\end{aligned} \tag{5.14}$$

In order to prove this we must handle the contribution of the two ranges $|\mathbf{x}| < B^{2\eta}$ and $B^{1/6-4\eta} < |\mathbf{x}| \leq B^{1/6}$, both for the number of solutions to $F(\mathbf{x}; \mathbf{y}) = 0$, and for the second range in respect of the sum of leading terms. Lemma 2.1 shows that $N_5(B; X) \ll B$ when $X \ll B^{1/6}$, so that the two awkward ranges contribute $O(\eta B \log B)$ on the left, which is dominated by the error term $O(\eta^{1/2} B \log B)$ in (5.14). In view of Lemma 4.12, a range $X < |\mathbf{x}| \leq 2X$ contributes $O(BS(X))$ to the main term on the right in (5.14), in the notation of (5.9). Using the bound (5.12), and summing over dyadic values of X in the range $B^{1/6-4\eta} \ll X \ll B^{1/6}$ we obtain a contribution $O(\eta B \log B)$, which again is satisfactory. This establishes the claim in (5.14).

We now combine the estimates (5.13) and (5.14) so as to cover the entire range $|\mathbf{x}|^3 |\mathbf{y}|^2 \leq B$ in the definition (1.1) of the counting function $N(\Omega; B)$, with $\Omega = X(\mathbb{Q}) \setminus T$ and T being given by (1.2). We may remove the points with $|\mathbf{x}| \leq B^{1/6}$ and $|\mathbf{y}| \leq B^{1/4}$ at a cost $O(B)$, using Lemma 2.1. Passing to the affine cone and allowing for multiplication of \mathbf{x} and \mathbf{y} by units, we therefore reach the following conclusion.

Lemma 5.5. *We have*

$$\begin{aligned}
N(\Omega; B) &= \frac{B}{4} \left(\frac{1}{\zeta(3)} M_1(B) + \frac{1}{\zeta(2)} M_2(B) \right) \\
&\quad + O(\eta^{1/2} B \log B) + O_{\eta}(B^{1-\eta^2/20} \log B).
\end{aligned}$$

where

$$M_1(B) = \sum_{\substack{\mathbf{y} \in \mathbb{Z}_{\text{prim}}^4 \\ |\mathbf{y}| \leq B^{1/4}}} \frac{\varrho_{\infty}(\mathbf{y})}{|\mathbf{y}|^2} \quad \text{and} \quad M_2(B) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^4 \\ B^{2\eta} \leq |\mathbf{x}| \leq B^{1/6} \\ \Delta(\mathbf{x}) \neq \square}} \frac{\sigma_{\infty}(\mathbf{x}) \mathfrak{S}(\mathbf{x})}{|\mathbf{x}|^3}.$$

6. THE FINAL RECKONING

In this section we shall produce asymptotic formulae for $M_1(B)$ and $M_2(B)$, as $B \rightarrow \infty$. We shall begin in §6.1 by dealing with $M_1(B)$, which is the easier to handle, before developing the techniques further in §6.2 to handle $M_2(B)$. Finally, in §6.3 we shall confirm that the two contributions combine in a satisfactory manner to complete the proof of Theorem 1.1.

6.1. Analysis of $M_1(B)$. The goal of the present section is the following result.

Lemma 6.1. *We have*

$$M_1(B) = \frac{1}{2\zeta(4)} \tau_\infty \log B + O(1),$$

where τ_∞ is given by (1.4).

We begin by using the Möbius function to detect the primitivity condition, which shows that

$$M_1(B) = \sum_{k \leq B^{1/4}} \frac{\mu(k)}{k^2} \sum_{\mathbf{y} \in \mathbb{Z}^4 \cap T_0} \frac{\varrho_\infty(k\mathbf{y})}{|\mathbf{y}|^2},$$

where

$$T_0 = T_0(k) = \{\mathbf{y} \in \mathbb{R}^4 : 1 \leq |\mathbf{y}| \leq B^{1/4}/k\}. \quad (6.1)$$

One sees from the definition (3.3) that $\varrho_\infty(k\mathbf{y}) = k^{-2} \varrho_\infty(\mathbf{y})$, whence

$$M_1(B) = \sum_{k \leq B^{1/4}} \frac{\mu(k)}{k^4} \sum_{\mathbf{y} \in \mathbb{Z}^4 \cap T_0} \frac{\varrho_\infty(\mathbf{y})}{|\mathbf{y}|^2}. \quad (6.2)$$

We now wish to replace the sum over \mathbf{y} by an integral. The argument will make repeated use of the bound

$$\varrho_\infty(\mathbf{y}) \ll |\mathbf{y}|^{-2}, \quad (6.3)$$

which is immediate from (3.8). We start with the following estimate.

Lemma 6.2. *If $\min_i |y_i| \geq 2$ then*

$$\frac{\varrho_\infty(\mathbf{y})}{|\mathbf{y}|^2} = \int_{[0,1]^4} \frac{\varrho_\infty(\mathbf{y} + \mathbf{t})}{|\mathbf{y} + \mathbf{t}|^2} d\mathbf{t} + O((\min_i |y_i|)^{-1/3} |\Delta(\mathbf{y})|^{-2/3} |\mathbf{y}|^{-2}).$$

Proof. We begin by showing that

$$\nabla \varrho_\infty(\mathbf{u}) \ll (\min_i |u_i|)^{-1/3} |\Delta(\mathbf{u})|^{-2/3}. \quad (6.4)$$

Without loss of generality we may just examine the partial derivative with respect to y_1 . Our definition (3.3) shows that

$$\varrho_\infty(\mathbf{u}) = \int_{-\infty}^{\infty} \prod_{i=1}^4 I(-\theta u_i^2) d\theta,$$

where we write temporarily

$$I(\psi) = \int_{-1}^1 e(\psi x) dx.$$

Then $I(\psi) \ll \min\{1, |\psi|^{-1}\}$ and

$$\frac{\partial}{\partial u_1} I(-\theta u_1^2) = -4\pi i \theta u_1 \int_{-1}^1 x e(-\theta x u_1^2) dx \ll |u_1|^{-1}.$$

Thus

$$\begin{aligned} \frac{\partial}{\partial u_1} \sigma_\infty(\mathbf{u}) &\ll |u_1|^{-1} \int_{-\infty}^{\infty} \min\{1, |\theta|^{-3} |u_2 u_3 u_4|^{-2}\} d\theta \\ &\ll |u_1|^{-1} |u_2 u_3 u_4|^{-2/3} \\ &\ll (\min_i |u_i|)^{-1/3} |\Delta(\mathbf{u})|^{-2/3}, \end{aligned}$$

as required.

We now use the decomposition

$$\frac{\varrho_\infty(\mathbf{y} + \mathbf{t})}{|\mathbf{y} + \mathbf{t}|^2} - \frac{\varrho_\infty(\mathbf{y})}{|\mathbf{y}|^2} = \frac{\varrho_\infty(\mathbf{y} + \mathbf{t}) - \varrho_\infty(\mathbf{y})}{|\mathbf{y} + \mathbf{t}|^2} + \varrho_\infty(\mathbf{y}) \{|\mathbf{y} + \mathbf{t}|^{-2} - |\mathbf{y}|^{-2}\}.$$

If $|\mathbf{t}| \leq 1$ and $\min_i |y_i| \geq 2$ then

$$|y_i + t_i| \geq |y_i| - |t_i| \geq \frac{1}{2} |y_i|,$$

so that $|\mathbf{y} + \mathbf{t}|^{-2} \ll |\mathbf{y}|^{-2}$. Moreover the Mean Value Theorem shows that

$$|\varrho_\infty(\mathbf{y} + \mathbf{t}) - \varrho_\infty(\mathbf{y})| \leq \sup_{0 \leq \xi \leq 1} \left| \frac{\partial}{\partial \xi} \varrho_\infty(\mathbf{y} + \xi \mathbf{t}) \right|.$$

It then follows from (6.4) that

$$\frac{\varrho_\infty(\mathbf{y} + \mathbf{t}) - \varrho_\infty(\mathbf{y})}{|\mathbf{y} + \mathbf{t}|^2} \ll (\min_i |y_i|)^{-1/3} |\Delta(\mathbf{y})|^{-2/3} |\mathbf{y}|^{-2}.$$

We also have

$$|\mathbf{y} + \mathbf{t}|^{-2} - |\mathbf{y}|^{-2} = |\mathbf{y} + \mathbf{t}|^{-2} |\mathbf{y}|^{-2} \{|\mathbf{y} + \mathbf{t}| + |\mathbf{y}|\} \{|\mathbf{y}| - |\mathbf{y} + \mathbf{t}|\}.$$

Assuming as above that $|\mathbf{t}| \leq 1$ and $|\mathbf{y}| \geq 2$ we see that $|\mathbf{y}| \ll |\mathbf{y} + \mathbf{t}| \ll |\mathbf{y}|$ and $|\mathbf{y} + \mathbf{t}| - |\mathbf{y}| \ll 1$, so that

$$\varrho_\infty(\mathbf{y}) \{|\mathbf{y} + \mathbf{t}|^{-2} - |\mathbf{y}|^{-2}\} \ll |\mathbf{y}|^{-3} \varrho_\infty(\mathbf{y}) \ll (\min_i |y_i|)^{-1/3} |\Delta(\mathbf{y})|^{-2/3} |\mathbf{y}|^{-2},$$

by (6.3). We therefore have

$$\frac{\varrho_\infty(\mathbf{y} + \mathbf{t})}{|\mathbf{y} + \mathbf{t}|^2} = \frac{\varrho_\infty(\mathbf{y})}{|\mathbf{y}|^2} + O\left((\min_i |y_i|)^{-1/3} |\Delta(\mathbf{y})|^{-2/3} |\mathbf{y}|^{-2}\right), \quad (6.5)$$

and the lemma follows. \square

Our next result converts the summation over \mathbf{y} in (6.2) into an integral.

Lemma 6.3. *We have*

$$\sum_{\mathbf{y} \in \mathbb{Z}^4 \cap T_0} \frac{\varrho_\infty(\mathbf{y})}{|\mathbf{y}|^2} = J_1(B; k) + O(1),$$

where

$$J_1(B; k) = \int_{T_0(k)} \frac{\varrho_\infty(\mathbf{y})}{|\mathbf{y}|^2} d\mathbf{y}.$$

Proof. We define

$$X = \{\mathbf{y} \in \mathbb{Z}^4 : |\mathbf{y}| \leq B^{1/4}/k - 2, \min |y_i| \geq 2\}$$

and

$$Y = \bigcup_{\mathbf{y} \in X} (\mathbf{y} + (0, 1]^4).$$

The reader should note that these could be empty if k is large enough. The sets $\mathbf{y} + (0, 1]^4$ forming Y are disjoint, and Y lies inside the set T_0 defined in (6.1). Moreover $T_0 \setminus Y$ is a subset of $T_1 \cup T_2$, where

$$T_1 = \{\mathbf{t} \in T_0 : B^{1/4}/k - 3 \leq |\mathbf{t}| \leq B^{1/4}/k\},$$

and

$$T_2 = \{\mathbf{t} \in T_0 : \min |t_i| \leq 3\}.$$

It then follows from Lemma 6.2 and (6.3) that

$$\sum_{\mathbf{y} \in \mathbb{Z}^4 \cap T_0} \frac{\varrho_\infty(\mathbf{y})}{|\mathbf{y}|^2} = J_1(B; k) + O\left(\sum_{i=0}^2 E_i\right),$$

where

$$E_0 = \sum_{\substack{\mathbf{y} \in \mathbb{Z}^4 \cap T_0 \\ \min |y_i| \geq 2}} (\min_i |y_i|)^{-1/3} |\Delta(\mathbf{y})|^{-2/3} |\mathbf{y}|^{-2},$$

and

$$E_i = \sum_{\mathbf{y} \in \mathbb{Z}^4 \cap T_i} |\mathbf{y}|^{-4} + \int_{T_i} |\mathbf{y}|^{-4} d\mathbf{y}$$

for $i = 1, 2$. We readily find that $E_i \ll 1$ for $i = 0, 1, 2$, and the lemma follows. \square

In order to complete our argument we will need the following evaluation of $J_1(B; k)$.

Lemma 6.4. *If $k \leq B^{1/4}$ we have*

$$J_1(B; k) = 2\tau_\infty \log(B^{1/4}/k),$$

with τ_∞ given by (1.4).

Proof. We divide $T_0(k)$ into four (overlapping) pieces according to the index i for which $|\mathbf{y}| = |y_i|$. We observe from (3.3) that $\varrho_\infty(\mathbf{y})$ is unchanged when we permute the coordinates, and that $\varrho_\infty(\mathbf{y}) = |\mathbf{y}|^{-2} \varrho_\infty(t_1, t_2, t_3, 1)$ if $|\mathbf{y}| = |y_4|$ and $t_i = y_i/|\mathbf{y}|$ for $i = 1, 2, 3$. It therefore follows that

$$J_1(B; k) = 8 \int_1^{B^{1/4}/k} \frac{dy_4}{y_4} \int_{[-1,1]^3} \varrho_\infty(t_1, t_2, t_3, 1) d\mathbf{t}.$$

In a precisely similar way Lemma 4.14 yields

$$\tau_\infty = \int_{[-1,1]^4} \varrho_\infty(\mathbf{y}) d\mathbf{y} = 8 \int_0^1 y_4 dy_4 \int_{[-1,1]^3} \varrho_\infty(t_1, t_2, t_3, 1) d\mathbf{t},$$

and the lemma follows. \square

We can now complete the proof of Lemma 6.1. Combining (6.2) with Lemma 6.3 we obtain

$$M_1(B) = \sum_{k \leq B^{1/4}} \frac{\mu(k)}{k^4} J_1(B; k) + O(1).$$

We have

$$\sum_{k \leq B^{1/4}} \frac{\mu(k)}{k^4} \log k \ll 1$$

and

$$\sum_{k \leq B^{1/4}} \frac{\mu(k)}{k^4} = \zeta(4)^{-1} + O(B^{-3/4}),$$

so that Lemma 6.4 yields

$$M_1(B) = \frac{2\tau_\infty}{\zeta(4)} \log B^{1/4} + O(1),$$

and the required estimate follows.

6.2. **Analysis of $M_2(B)$.** We remind the reader that

$$M_2(B) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^4 \\ B^{2\eta} \leq |\mathbf{x}| \leq B^{1/6} \\ \Delta(\mathbf{x}) \neq \square}} \frac{\sigma_\infty(\mathbf{x}) \mathfrak{S}(\mathbf{x})}{|\mathbf{x}|^3}$$

where

$$\sigma_\infty(\mathbf{x}) = \int_{-\infty}^{\infty} \int_{[-1,1]^4} e(-\theta F(\mathbf{x}; \mathbf{y})) d\mathbf{y} d\theta$$

and

$$\mathfrak{S}(\mathbf{x}) = \sum_{q=1}^{\infty} q^{-4} S_q, \quad \text{with } S_q = S_q(\mathbf{x}) = \sum_{a \bmod q}^* \sum_{\mathbf{b} \bmod q} e_q(aF(\mathbf{x}; \mathbf{b})).$$

The goal of the present section is the following result.

Lemma 6.5. *We have*

$$M_2(B) = \frac{1}{2} \cdot \frac{\zeta(2)}{\zeta(3)\zeta(4)} \cdot \tau_\infty \log B + O(\eta \log B) + O_\eta(1),$$

where τ_∞ is given by (1.4).

In order to estimate $M_2(B)$ our plan will begin by showing that the singular series $\mathfrak{S}(\mathbf{x})$ can be replaced by a truncated sum

$$\mathfrak{S}(\mathbf{x}; R) = \sum_{q \leq R} q^{-4} S_q,$$

for suitable R . Using Heath-Brown's large sieve for real characters [11], we shall ultimately succeed in showing that R can be taken an arbitrarily small power of B , with acceptable error. The constraint $\Delta(\mathbf{x}) \neq \square$ can now be replaced by $\Delta(\mathbf{x}) \neq 0$, again with an acceptable error. We then interchange the q and \mathbf{x} summations in $M_2(B)$ and approximate the \mathbf{x} -sum by a 4-fold integral. Lastly, the remaining q -sum will be extended to infinity to get our final asymptotic formula for $M_2(B)$. Throughout this analysis we will use repeatedly the estimate

$$\sigma_\infty(\mathbf{x}) \ll |\Delta(\mathbf{x})|^{-1/4}$$

given by Lemma 4.12.

In order to carry out this plan we first make a crude first analysis of the tail of the singular series $\mathfrak{S}(\mathbf{x})$. Since $\Delta(\mathbf{x}) \neq \square$, it follows from Lemma 4.9 and partial summation that

$$\sum_{q > B} q^{-4} S_q \ll_\varepsilon |\Delta(\mathbf{x})|^{3/16} \Delta_{\text{bad}}(\mathbf{x})^{3/8} B^{-1/2+\varepsilon},$$

for any $\varepsilon > 0$. Since $|\mathbf{x}|^3 \geq |\Delta(\mathbf{x})|^{3/4}$, an application of Lemma 4.12 now shows that the tail of the singular series contributes

$$\ll_{\varepsilon} B^{-1/2+\varepsilon} \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^4 \\ |\mathbf{x}| \leq B^{1/6}}} \frac{\Delta_{\text{bad}}(\mathbf{x})^{3/8}}{|\Delta(\mathbf{x})|^{13/16}}$$

to $M_2(B)$. Writing $s = \Delta_{\text{bad}}(\mathbf{x})$ and $n = |\Delta(\mathbf{x})|$, we see that this is

$$\ll_{\varepsilon} B^{-1/2+\varepsilon} \sum_{\substack{s \leq B^{2/3} \\ s \text{ square-full}}} s^{3/8} \sum_{\substack{n \leq B^{2/3} \\ s|n}} \frac{\tau_4(n)}{n^{13/16}} \ll_{\varepsilon} B^{-3/8+2\varepsilon} \sum_{s \text{ square-full}} \frac{1}{s^{5/8}}.$$

Taking $\varepsilon = \frac{1}{8}$ and noting that the s -sum is convergent, this shows that

$$M_2(B) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^4 \\ B^{2\eta} \leq |\mathbf{x}| \leq B^{1/6} \\ \Delta(\mathbf{x}) \neq \square}} \frac{\sigma_{\infty}(\mathbf{x}) \mathfrak{S}(\mathbf{x}; B)}{|\mathbf{x}|^3} + O(B^{-1/8}).$$

Building on this, we now show that the singular series can be truncated to a much smaller power of B , with acceptable error.

Lemma 6.6. *We have*

$$M_2(B) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^4 \\ B^{2\eta} \leq |\mathbf{x}| \leq B^{1/6} \\ \Delta(\mathbf{x}) \neq \square}} \frac{\sigma_{\infty}(\mathbf{x}) \mathfrak{S}(\mathbf{x}; B^{\eta/8})}{|\mathbf{x}|^3} + O_{\eta}(1).$$

Proof. Since $|\mathbf{x}|^3 \geq |\Delta(\mathbf{x})|^{3/4}$ and $\sigma_{\infty}(\mathbf{x}) \ll |\Delta(\mathbf{x})|^{-1/4}$ it will be enough to show that

$$E(B) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^4 \\ B^{2\eta} \leq |\mathbf{x}| \leq B^{1/6} \\ \Delta(\mathbf{x}) \neq 0}} \frac{1}{|\Delta(\mathbf{x})|} \left| \sum_{B^{\eta/8} < q \leq B} q^{-4} S_q \right| \ll_{\eta} 1.$$

Note that we have relaxed the condition $\Delta(\mathbf{x}) \neq \square$ to require only that $\Delta(\mathbf{x})$ is non-zero. We shall write $s = \Delta_{\text{bad}}(\mathbf{x})$ and $n = |\Delta(\mathbf{x})|$, so that ns^{-1} is square-free. Since $|\Delta(\mathbf{x})| \geq |\mathbf{x}| \geq B^{2\eta}$, we are only interested in integers n in the range $B^{2\eta} \leq |n| \leq B^{2/3}$.

It follows from the multiplicativity of S_q that

$$\left| \sum_{B^{\eta/8} < q \leq B} q^{-4} S_q \right| \leq \sum_{\substack{v \leq B \\ v|(2s)^\infty}} v^{-4} |S_v| \left| \sum_{\substack{B^{\eta/8}/v < u \leq B/v \\ (u, 2s)=1}} u^{-4} S_u \right|.$$

Lemma 4.5 shows that

$$\begin{aligned} S_v &\ll v^3 \prod_{1 \leq i \leq 4} (v, x_i)^{1/2} \\ &\leq v^3 (v^4, x_1 \dots x_4)^{1/2} \\ &\ll v^3 (v^4, s)^{1/2} \\ &\leq v^3 \min(v^4, s)^{1/2} \\ &\leq v^{7/2} s^{3/8} \end{aligned}$$

for $v \mid (2s)^\infty$. Moreover, Lemmas 4.6 and 4.8 imply that

$$S_u = \left(\frac{n}{u}\right) \varphi^*(u) u^3,$$

when $(u, 2s) = 1$, where $\varphi^* = 1 * h$, with $h(d) = \mu(d)/d$. Hence

$$E(B) \ll_\varepsilon B^\varepsilon \sum_{\substack{s \leq B^{2/3} \\ s \text{ square-full}}} \sum_{\substack{v \leq B \\ v|(2s)^\infty}} v^{-1/2} s^{3/8} \sum_{\substack{B^{2\eta} \leq |n| \leq B^{2/3} \\ s \mid n}} \frac{1}{|n|} \left| \sum_{\substack{B^{\eta/8}/v < u \leq B/v \\ (u, 2s)=1}} \left(\frac{n}{u}\right) \frac{\varphi^*(u)}{u} \right|,$$

since the number of \mathbf{x} associated to n is at most $\tau_4(n) = O_\varepsilon(B^\varepsilon)$.

We now write $n = sm$ and split the ranges for m and u into dyadic intervals. This gives us values M and $U \leq U_1 \leq 2U$, with

$$\max\left(1, \frac{B^{2\eta}}{s}\right) \ll M \ll B^{2/3} \quad \text{and} \quad \frac{B^{\eta/8}}{v} \ll U \ll \frac{B}{v}$$

such that

$$E(B) \ll_\varepsilon \frac{B^{2\varepsilon}}{MU} \sum_{\substack{s \leq B^{2/3} \\ s \text{ square-full}}} \sum_{\substack{v \leq B \\ v|(2s)^\infty}} v^{-1/2} s^{-5/8} \sum_{M < m \leq 2M} \left| \sum_{U < u \leq U_1} \left(\frac{m}{u}\right) \alpha_{u,s} \right|,$$

with

$$\alpha_{u,s} = U \frac{\varphi^*(u)}{u} \left(\frac{4s}{u}\right) \ll 1.$$

We now write

$$\sum_{M < m \leq 2M} \left| \sum_{U < u \leq U_1} \left(\frac{m}{u} \right) \alpha_{u,s} \right| = \sum_{M < m \leq 2M} \sum_{U < u \leq U_1} \left(\frac{m}{u} \right) \alpha_{u,s} \beta_m,$$

with $\beta_m = \pm 1$, and apply the large sieve for real characters in the form given by Heath-Brown [11, Cor. 4]. This shows that

$$\sum_{M < m \leq 2M} \sum_{U < u \leq U_1} \left(\frac{m}{u} \right) \alpha_{u,s} \beta_m \ll_{\varepsilon} (MU)^{\varepsilon} \{MU^{1/2} + M^{1/2}U\},$$

whence

$$E(B) \ll_{\varepsilon} B^{4\varepsilon} \sum_{\substack{s \leq B^{2/3} \\ s \text{ square-full}}} \sum_{\substack{v \leq B \\ v|(2s)^{\infty}}} v^{-1/2} s^{-5/8} \{U^{-1/2} + M^{-1/2}\}.$$

However

$$\begin{aligned} U^{-1/2} + M^{-1/2} &\ll v^{1/2} B^{-\eta/16} + \min\{1, s^{1/2} B^{-\eta}\} \\ &\ll v^{1/2} B^{-\eta/16} + 1^{15/16} (s^{1/2} B^{-\eta})^{1/16} \\ &\ll B^{-\eta/16} v^{1/2} s^{1/32}. \end{aligned}$$

We therefore deduce that

$$E(B) \ll_{\varepsilon} B^{-\eta/16+4\varepsilon} \sum_{\substack{s \leq B^{2/3} \\ s \text{ square-full}}} s^{-19/32} \sum_{\substack{v \leq B \\ v|(2s)^{\infty}}} 1 \ll_{\varepsilon} B^{-\eta/16+5\varepsilon},$$

since there are $O_{\varepsilon}((sB)^{\varepsilon})$ possible values for v , and the sum over square-full s is convergent. Taking $\varepsilon = \eta/80$, we therefore conclude the proof of the lemma. \square

Next we wish to show that the condition $\Delta(\mathbf{x}) \neq \square$ can be replaced by $\Delta(\mathbf{x}) \neq 0$ with an acceptable error. We trivially have

$$|\mathfrak{S}(\mathbf{x}; B^{\eta/8})| \leq \sum_{q \leq B^{\eta/8}} q \ll B^{\eta/4}.$$

Moreover Lemma 4.12 shows that $\sigma_{\infty}(\mathbf{x}) \ll |\Delta(\mathbf{x})|^{-1/4}$. We now write $\Delta(\mathbf{x}) = n^2$ so that $|\mathbf{x}| \leq n^2 \leq |\mathbf{x}|^4$. In particular we will have $B^{\eta} \leq n \leq B^{1/3}$ and $|\mathbf{x}|^{-3} \leq n^{-3/2}$. Moreover each value of n corresponds to $O_{\varepsilon}(B^{\varepsilon})$ vectors \mathbf{x} . Thus

$$\sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^4 \\ B^{2\eta} \leq |\mathbf{x}| \leq B^{1/6} \\ \Delta(\mathbf{x}) = \square \neq 0}} \frac{\sigma_{\infty}(\mathbf{x}) \mathfrak{S}(\mathbf{x}; B^{\eta/8})}{|\mathbf{x}|^3} \ll_{\varepsilon} B^{\eta/4+\varepsilon} \sum_{B^{\eta} \leq n \leq B^{1/3}} n^{-2} \ll_{\varepsilon} B^{-3\eta/4+\varepsilon}.$$

This may be absorbed into the error term of Lemma 6.6 on choosing $\varepsilon = 3\eta/4$.

Now that we have truncated the singular series satisfactorily, we may open up the expression for $\mathfrak{S}(\mathbf{x}; B^{\eta/8})$ and interchange the q -sum with the \mathbf{x} -sum, before breaking the latter into congruence classes modulo q . This leads to the expression

$$M_2(B) = \sum_{q \leq B^{\eta/8}} q^{-4} \sum_{\substack{\mathbf{a}, \mathbf{b} \bmod q \\ (q, \mathbf{a})=1}} c_q(F(\mathbf{a}; \mathbf{b})) U(q; \mathbf{a}) + O_\eta(1),$$

where $c_q(\cdot)$ is the Ramanujan sum and

$$U(q; \mathbf{a}) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}_{\text{prim}}^4 \\ B^{2\eta} \leq |\mathbf{x}| \leq B^{1/6} \\ \Delta(\mathbf{x}) \neq 0 \\ \mathbf{x} \equiv \mathbf{a} \bmod q}} \frac{\sigma_\infty(\mathbf{x})}{|\mathbf{x}|^3}.$$

Any \mathbf{x} counted by $U(q; \mathbf{a})$ is automatically coprime to q . Using the Möbius function to detect the residual primitivity of \mathbf{x} we may now write

$$U(q; \mathbf{a}) = \sum_{\substack{k \leq B^{1/6} \\ (k, q)=1}} \frac{\mu(k)}{k^3} \sum_{\substack{\mathbf{x} \in \mathbb{Z}^4 \cap T_0 \\ \mathbf{x} \equiv \bar{k} \mathbf{a} \bmod q}} \frac{\sigma_\infty(k\mathbf{x})}{|\mathbf{x}|^3},$$

where \bar{k} is the multiplicative inverse of k modulo q and

$$T_0 = T_0(k) = \{\mathbf{t} \in (\mathbb{R}_{\neq 0})^4 : B^{2\eta}/k \leq |\mathbf{t}| \leq B^{1/6}/k\}. \quad (6.6)$$

(The reader should note that this is not the same set that is defined in (6.1); we recycle our notation for this and other similar sets.) Since $\sigma_\infty(k\mathbf{x}) = k^{-1}\sigma_\infty(\mathbf{x})$, this simplifies to give

$$U(q; \mathbf{a}) = \sum_{\substack{k \leq B^{1/6} \\ (k, q)=1}} \frac{\mu(k)}{k^4} \sum_{\substack{\mathbf{x} \in \mathbb{Z}^4 \cap T_0 \\ \mathbf{x} \equiv \bar{k} \mathbf{a} \bmod q}} \frac{\sigma_\infty(\mathbf{x})}{|\mathbf{x}|^3}.$$

We therefore obtain the following formula.

Lemma 6.7. *We have*

$$M_2(B) = \sum_{q \leq B^{\eta/8}} q^{-4} \sum_{\substack{k \leq B^{1/6} \\ (k, q)=1}} \frac{\mu(k)}{k^4} \sum_{\substack{\mathbf{a}, \mathbf{b} \bmod q \\ (q, \mathbf{a})=1}} c_q(F(\mathbf{a}; \mathbf{b})) \sum_{\substack{\mathbf{x} \in \mathbb{Z}^4 \cap T_0 \\ \mathbf{x} \equiv \bar{k} \mathbf{a} \bmod q}} \frac{\sigma_\infty(\mathbf{x})}{|\mathbf{x}|^3} + O_\eta(1).$$

The next stage is to compare the \mathbf{x} -sum to an integral. We start with the following estimate, which is an analogue of Lemma 6.2.

Lemma 6.8. *If $\min_i |x_i| \geq 2q$ then*

$$\frac{\sigma_\infty(\mathbf{x})}{|\mathbf{x}|^3} = q^{-4} \int_{[0, q]^4} \frac{\sigma_\infty(\mathbf{x} + \mathbf{t})}{|\mathbf{x} + \mathbf{t}|^3} d\mathbf{t} + O(q(\min_i |x_i|)^{-1} |\Delta(\mathbf{x})|^{-1/4} |\mathbf{x}|^{-3}).$$

Proof. The proof follows the same lines as that of Lemma 6.2. We begin by showing that

$$\nabla \sigma_\infty(\mathbf{x}) \ll (\min_i |x_i|)^{-1} |\Delta(\mathbf{x})|^{-1/4}. \quad (6.7)$$

Without loss of generality it will suffice to examine the partial derivative with respect to y_1 in proving this. Our definition of the singular integral yields

$$\sigma_\infty(\mathbf{x}) = \int_{-\infty}^{\infty} \prod_{i=1}^4 I(-\theta x_i) d\theta,$$

where we write temporarily

$$I(\psi) = \int_{-1}^1 e(\psi y^2) dy.$$

By the standard second derivative test [22, Lemma 4.4] we see that

$$I(\psi) \ll \min\{1, |\psi|^{-1/2}\}.$$

Moreover

$$\frac{\partial}{\partial x_1} I(-\theta x_1) = \frac{1}{2x_1} \int_{-1}^1 y \frac{\partial}{\partial y} e(-\theta x_1 y^2) dy.$$

The integral on the right is uniformly bounded, as one sees on integrating by parts. Thus

$$\frac{\partial}{\partial x_1} I(-\theta x_1) \ll |x_1|^{-1},$$

whence

$$\begin{aligned} \frac{\partial}{\partial x_1} \sigma_\infty(\mathbf{x}) &\ll |x_1|^{-1} \int_{-\infty}^{\infty} \min\{1, |\theta|^{-3/2} |x_2 x_3 x_4|^{-1/2}\} d\theta \\ &\ll |x_1|^{-1} |x_2 x_3 x_4|^{-1/3} \\ &\ll (\min_i |x_i|)^{-1} |\Delta(\mathbf{x})|^{-1/4}, \end{aligned}$$

as required.

We now use the decomposition

$$\begin{aligned} \frac{\sigma_\infty(\mathbf{x} + \mathbf{t})}{|\mathbf{x} + \mathbf{t}|^3} - \frac{\sigma_\infty(\mathbf{x})}{|\mathbf{x}|^3} &= \frac{\sigma_\infty(\mathbf{x} + \mathbf{t}) - \sigma_\infty(\mathbf{x})}{|\mathbf{x} + \mathbf{t}|^3} + \sigma_\infty(\mathbf{x}) \{|\mathbf{x} + \mathbf{t}|^{-3} - |\mathbf{x}|^{-3}\}. \end{aligned}$$

If $|\mathbf{t}| \leq q$ and $\min_i |x_i| \geq 2q$ we may show, as in the proof of (6.5), that

$$\begin{aligned} \frac{\sigma_\infty(\mathbf{x} + \mathbf{t}) - \sigma_\infty(\mathbf{x})}{|\mathbf{x} + \mathbf{t}|^3} &\ll |\mathbf{x}|^{-3} q \sup_{0 \leq \xi \leq 1} \left| \frac{\partial}{\partial \xi} \sigma_\infty(\mathbf{x} + \xi \mathbf{t}) \right| \\ &\ll q (\min_i |x_i|)^{-1} |\Delta(\mathbf{x})|^{-1/4} |\mathbf{x}|^{-3}, \end{aligned}$$

via (6.7). Similarly, also as in the proof of (6.5), we will have

$$|\mathbf{x} + \mathbf{t}|^{-3} - |\mathbf{x}|^{-3} \ll q|\mathbf{x}|^{-4},$$

so that Lemma 4.12 yields

$$\sigma_\infty(\mathbf{x})\{|\mathbf{x} + \mathbf{t}|^{-3} - |\mathbf{x}|^{-3}\} \ll q|\Delta(\mathbf{x})|^{-1/4}|\mathbf{x}|^{-4}.$$

We therefore have

$$\frac{\sigma_\infty(\mathbf{x} + \mathbf{t})}{|\mathbf{x} + \mathbf{t}|^3} = \frac{\sigma_\infty(\mathbf{x})}{|\mathbf{x}|^3} + O(q(\min_i |x_i|)^{-1}|\Delta(\mathbf{x})|^{-1/4}|\mathbf{x}|^{-3}),$$

and the lemma follows. \square

We are now ready to tackle the \mathbf{x} -summation in Lemma 6.7.

Lemma 6.9. *We have*

$$\sum_{\substack{\mathbf{x} \in \mathbb{Z}^4 \cap T_0 \\ \mathbf{x} \equiv \bar{k}\mathbf{a} \pmod{q}}} \frac{\sigma_\infty(\mathbf{x})}{|\mathbf{x}|^3} = q^{-4}J_2(B; k) + O(qkB^{-3\eta/2}),$$

where

$$J_2(B; k) = \int_{T_0(k)} \frac{\sigma_\infty(\mathbf{y})}{|\mathbf{y}|^3} d\mathbf{y}.$$

Proof. We define

$$X = \{\mathbf{x} \in \mathbb{Z}^4 : B^{2\eta}/k + 2q \leq |\mathbf{x}| \leq B^{1/6}/k - 2q, \min |x_i| \geq 2q, \mathbf{x} \equiv \bar{k}\mathbf{a} \pmod{q}\}$$

and

$$Y = \bigcup_{\mathbf{x} \in X} (\mathbf{x} + (0, q]^4).$$

The reader should note that these could be empty if k and q are large enough. The sets $\mathbf{x} + (0, q]^4$ forming Y are disjoint, and both X and Y lie inside the set T_0 defined in (6.6). Moreover $T_0 \setminus Y$ is a subset of $T_1 \cup T_2 \cup T_3$, where

$$T_1 = \{\mathbf{t} \in T_0 : B^{2\eta}/k \leq |\mathbf{t}| \leq B^{2/\eta}/k + 3q\},$$

$$T_2 = \{\mathbf{t} \in T_0 : B^{1/6}/k - 3q \leq |\mathbf{t}| \leq B^{1/6}/k\},$$

and

$$T_3 = \{\mathbf{t} \in T_0 : \min |t_i| \leq 3q\}.$$

It then follows from Lemma 6.8 that

$$\sum_{\substack{\mathbf{x} \in \mathbb{Z}^4 \cap T_0 \\ \mathbf{x} \equiv \bar{k}\mathbf{a} \pmod{q}}} \frac{\sigma_\infty(\mathbf{x})}{|\mathbf{x}|^3} = q^{-4}J_2(B; k) + O\left(\sum_{i=0}^3 E_i\right), \quad (6.8)$$

where

$$E_0 = q \sum_{\substack{\mathbf{x} \in \mathbb{Z}^4 \cap T_0 \\ \min |x_i| \geq 2q}} (\min_i |x_i|)^{-1} |\Delta(\mathbf{x})|^{-1/4} |\mathbf{x}|^{-3},$$

and

$$E_i = \sum_{\mathbf{x} \in \mathbb{Z}^4 \cap T_i} |\Delta(\mathbf{x})|^{-1/4} |\mathbf{x}|^{-3} + q^{-4} \int_{T_i} |\Delta(\mathbf{y})|^{-1/4} |\mathbf{y}|^{-3} d\mathbf{y}$$

for $i = 1, 2, 3$. Note that we have dropped the condition $\mathbf{x} \equiv \bar{k}\mathbf{a} \pmod{q}$ in these error terms. We now find that

$$E_0 \ll q \sum_{\substack{2q \leq x_1 \leq x_2, x_3 \leq x_4 \\ x_4 \geq B^{2\eta}/k}} x_1^{-5/4} (x_2 x_3)^{-1/4} x_4^{-13/4} \ll q^{3/4} \sum_{x_4 \geq B^{2\eta}/k} x_4^{-7/4} \ll (qk B^{-2\eta})^{3/4},$$

for example. This holds whether $k \leq B^{2\eta}$ or not. Similar calculations show that

$$E_1 \ll \sum_{B^{2\eta}/k \leq x_4 \leq B^{2\eta}/k + 3q} x_4^{-1} + \int_{B^{2\eta}/k}^{B^{2\eta}/k + 3q} y_4^{-1} dy_4 \ll qk B^{-2\eta},$$

and

$$E_2 \ll qk B^{-1/6}.$$

For the sum in E_3 we have

$$\begin{aligned} \sum_{\mathbf{x} \in T_3} |\Delta(\mathbf{x})|^{-1/4} |\mathbf{x}|^{-3} &\ll \sum_{\substack{1 \leq x_1 \leq 3q \\ 1 \leq x_2, x_3 \leq x_4 \\ x_4 \geq B^{2\eta}/k}} (x_1 x_2 x_3)^{-1/4} x_4^{-13/4} \\ &\ll q^{3/4} \sum_{x_4 \geq B^{2\eta}/k} x_4^{-7/4} \\ &\ll (qk B^{-2\eta})^{3/4}, \end{aligned}$$

and similarly for the integral. Thus (6.8) becomes

$$\sum_{\substack{\mathbf{x} \in \mathbb{Z}^4 \cap T_0 \\ \mathbf{x} \equiv \bar{k}\mathbf{a} \pmod{q}}} \frac{\sigma_\infty(\mathbf{x})}{|\mathbf{x}|^3} = q^{-4} J_2(B; k) + O(qk B^{-3\eta/2}),$$

as required. \square

Combining Lemma 6.9 with Lemma 6.7 we see that

$$M_2(B) = \sum_{q \leq B^{\eta/8}} q^{-8} \psi(q) \sum_{\substack{k \leq B^{1/6} \\ (k, q) = 1}} \frac{\mu(k)}{k^4} J_2(B; k) + O_\eta(1), \quad (6.9)$$

with

$$\psi(q) = \sum_{\substack{\mathbf{a}, \mathbf{b} \bmod q \\ (q, \mathbf{a})=1}} c_q(F(\mathbf{a}; \mathbf{b})).$$

We therefore need information about the function ψ .

Lemma 6.10. *The function ψ is multiplicative, with*

$$\psi(p^f) = \begin{cases} \varphi(p^f)p^{6f}(1-p^{-4}), & \text{if } 2 \mid f, \\ 0, & \text{if } 2 \nmid f, \end{cases}$$

for every positive integer f .

Proof. The function $\psi(q)$ is clearly multiplicative, and for prime powers we have

$$\psi(p^f) = \sum_{\substack{c \bmod p^f \\ (c,p)=1}} \sum_{\substack{\mathbf{a}, \mathbf{b} \bmod p^f \\ p \nmid \mathbf{a}}} e_{p^f}(cF(\mathbf{a}; \mathbf{b})) = \varphi(p^f) \sum_{\substack{\mathbf{a}, \mathbf{b} \bmod p^f \\ p \nmid \mathbf{a}}} e_{p^f}(F(\mathbf{a}; \mathbf{b})),$$

on replacing $c\mathbf{a}$ by \mathbf{a} . It follows that

$$\psi(p^f) = \varphi(p^f) \sum_{\mathbf{a}, \mathbf{b} \bmod p^f} e_{p^f}(F(\mathbf{a}; \mathbf{b})) - \varphi(p^f) \sum_{\substack{\mathbf{a}, \mathbf{b} \bmod p^f \\ p \mid \mathbf{a}}} e_{p^f}(F(\mathbf{a}; \mathbf{b})).$$

Hence if we write

$$\psi_1(p^f) = \sum_{\mathbf{a}, \mathbf{b} \bmod p^f} e_{p^f}(F(\mathbf{a}; \mathbf{b}))$$

we will have $\psi(p^f) = \varphi(p^f)\psi_1(p^f) - p^4\varphi(p^f)\psi_1(p^{f-1})$. However, on performing the summation over \mathbf{a} we find that

$$\psi_1(p^f) = p^{4f} \#\{\mathbf{b} \bmod p^f : p^f \mid (b_1^2, \dots, b_4^2)\} = p^{4f}(p^{\lfloor f/2 \rfloor})^4.$$

The required formula for $\psi(p^f)$ then follows. \square

We also have the following evaluation of $J_2(B; k)$.

Lemma 6.11. *We have*

$$J_2(B; k) = 3\tau_\infty \log(B^{1/6-2\eta}).$$

Proof. The argument is completely analogous to that used for Lemma 6.4, based on the fact that $\sigma_\infty(\mathbf{y}) = |\mathbf{y}|^{-1}\sigma_\infty(t_1, t_2, t_3, 1)$ if $|\mathbf{y}| = |y_4|$ and $t_i = y_i/|\mathbf{y}|$ for $i = 1, 2, 3$. We find that

$$J_2(B; k) = 8 \int_{B^{2\eta}/k}^{B^{1/6}/k} \frac{dy_4}{y_4} \int_{[-1,1]^3} \sigma_\infty(t_1, t_2, t_3, 1) d\mathbf{t},$$

while Lemma 4.14 yields

$$\tau_\infty = \int_{[-1,1]^4} \sigma_\infty(\mathbf{y}) d\mathbf{y} = 8 \int_0^1 y_4^2 dy_4 \int_{[-1,1]^3} \sigma_\infty(t_1, t_2, t_3, 1) d\mathbf{t}.$$

The lemma follows from these relations. \square

We now have everything in place to complete the proof of Lemma 6.5. According to Lemma 6.11 we have

$$\begin{aligned} \sum_{\substack{k \leq B^{1/6} \\ (k,q)=1}} \frac{\mu(k)}{k^4} J_2(B; k) &= 3\tau_\infty \log(B^{1/6-2\eta}) \sum_{\substack{k \leq B^{1/6} \\ (k,q)=1}} \frac{\mu(k)}{k^4} \\ &= 3\tau_\infty \log(B^{1/6-2\eta}) \sum_{\substack{k=1 \\ (k,q)=1}}^{\infty} \frac{\mu(k)}{k^4} + O(B^{-1/2} \log B) \\ &= \frac{3\tau_\infty \log(B^{1/6-2\eta})}{\zeta(4)} \prod_{p|q} (1 - p^{-4})^{-1} + O(B^{-1/2} \log B). \end{aligned}$$

We can now insert this into (6.9), using Lemma 6.10 to observe that $\psi(q)$ is supported on the squares, with $\psi(r^2) \ll r^{14}$. This leads to the estimate

$$M_2(B) = \frac{3\tau_\infty \log(B^{1/6-2\eta})}{\zeta(4)} \sum_{q=1}^{\infty} q^{-8} \psi(q) \prod_{p|q} (1 - p^{-4})^{-1} + O_\eta(1).$$

Finally we note that

$$\begin{aligned} \sum_{q=1}^{\infty} q^{-8} \psi(q) \prod_{p|q} (1 - p^{-4})^{-1} &= \prod_p \left(1 + \frac{p^{-16} \psi(p^2) + p^{-32} \psi(p^4) + \dots}{1 - p^{-4}} \right) \\ &= \prod_p (1 + (1 - p^{-1}) \{p^{-2} + p^{-4} + \dots\}) \\ &= \prod_p \left(\frac{1 - p^{-3}}{1 - p^{-2}} \right) \\ &= \frac{\zeta(2)}{\zeta(3)}. \end{aligned}$$

We therefore conclude that

$$M_2(B) = \frac{\zeta(2)}{\zeta(3)\zeta(4)} 3\tau_\infty \log(B^{1/6-2\eta}) + O_\eta(1).$$

Lemma 6.5 then follows.

6.3. Conclusion. It is now time to bring Lemmas 6.1 and 6.5 together in Lemma 5.5. This yields

$$N(\Omega; B) = \frac{B \log B}{4\zeta(3)\zeta(4)} \tau_\infty + O(\eta^{1/2} B \log B) + O_\eta(B).$$

This is an asymptotic formula which holds for any $\eta \in (0, \frac{1}{100})$. Suppose that the error terms are $E_1 + E_2$, in which $|E_1| \leq c_1 \eta^{1/2} B \log B$, and $|E_2| \leq c_2(\eta) B$. We claim that the error terms may be replaced by $o(B \log B)$. To show this, we suppose that some small $\varepsilon > 0$ is given, and we proceed to show that there is a $B(\varepsilon)$ such that $|E_1 + E_2| \leq \varepsilon B \log B$ whenever $B \geq B(\varepsilon)$. Let $\eta = \{\varepsilon/(2c_1)\}^2$. Then $|E_1| \leq \frac{1}{2}\varepsilon B \log B$ for every B . With this value of η we then set

$$B(\varepsilon) = \exp\{2c_2(\eta)/\varepsilon\},$$

so that $|E_2| \leq \frac{1}{2}\varepsilon B \log B$ for all $B \geq B(\varepsilon)$. This proves our claim. It therefore follows that

$$N(\Omega; B) \sim c B \log B,$$

as $B \rightarrow \infty$, with

$$c = \frac{\tau_\infty}{4\zeta(3)\zeta(4)}.$$

In order to complete the proof of Theorem 1.1 it remains to check that our leading constant agrees with the prediction by Peyre [18]. According to Schindler [19, §3], the Peyre constant is equal to

$$\frac{1}{4\zeta(2)\zeta(3)} \cdot \tau_\infty \prod_p \lim_{t \rightarrow \infty} p^{-7t} n(p^t), \quad (6.10)$$

where τ_∞ is given by (1.4) and where $n(p^t)$ is the number of $(\mathbf{x}, \mathbf{y}) \in (\mathbb{Z}/p^t\mathbb{Z})^8$ such that $F(\mathbf{x}; \mathbf{y}) \equiv 0 \pmod{p^t}$. If $t \geq 1$ we have

$$\begin{aligned} n(p^t) &= \sum_{j=0}^t \sum_{\substack{\mathbf{y} \pmod{p^t} \\ (\mathbf{y}, p^t) = p^j}} \#\{\mathbf{x} \in (\mathbb{Z}/p^t\mathbb{Z})^4 : F(\mathbf{x}; \mathbf{y}) \equiv 0 \pmod{p^t}\} \\ &= \sum_{j=0}^{\lfloor t/2 \rfloor} \sum_{\substack{\mathbf{u} \pmod{p^{t-j}} \\ (\mathbf{u}, p) = 1}} \#\{\mathbf{x} \in (\mathbb{Z}/p^t\mathbb{Z})^4 : F(\mathbf{x}; \mathbf{u}) \equiv 0 \pmod{p^{t-2j}}\} + O(p^{6t}). \end{aligned}$$

Since $p \nmid \mathbf{u}$ the number of $\mathbf{x} \in (\mathbb{Z}/p^{t-2j})^4$ such that $p^{t-2j} \mid F(\mathbf{x}; \mathbf{u})$ is $p^{3(t-2j)}$. Thus

$$\begin{aligned}
 n(p^t) &= \sum_{j=0}^{[t/2]} \sum_{\substack{\mathbf{u} \bmod p^{t-j} \\ (\mathbf{u}, p)=1}} p^{3t+2j} + O(p^{6t}) \\
 &= \sum_{j=0}^{[t/2]} \{p^{4(t-j)} - p^{4(t-j-1)}\} p^{3t+2j} + O(p^{6t}) \\
 &= \{1 - p^{-4}\} \sum_{j=0}^{[t/2]} p^{7t-2j} + O(p^{6t}) \\
 &= \{1 + p^{-2}\} p^{7t} + O(p^{6t}).
 \end{aligned}$$

It follows that $p^{-7t}n(p^t)$ tends to $1 + p^{-2}$, so that (6.10) is

$$\frac{1}{4\zeta(2)\zeta(3)} \cdot \tau_{\infty} \frac{\zeta(2)}{\zeta(4)}.$$

Thus our leading constant c in Theorem 1.1 agrees with the Peyre constant.

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