

# Extensions of the Katznelson-Tzafriri Theorem for Operator Semigroups

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A thesis submitted for the degree of  
*Doctor of Philosophy*

Hilary Term 2014

## Abstract

This thesis is concerned with extensions and refinements of the Katznelson-Tzafriri theorem, a cornerstone of the asymptotic theory of operator semigroups which recently has received renewed interest in the context of damped wave equations. The thesis comprises three main parts. The key results in the first part are a version of the Katznelson-Tzafriri theorem for bounded  $C_0$ -semigroups in which a certain function appearing in the original statement of the result is allowed more generally to be a bounded Borel measure, and bounds on the rate of decay in an important special case. The second part deals with the discrete version of the Katznelson-Tzafriri theorem and establishes upper and lower bounds on the rate of decay in this setting too. In an important special case these general bounds are then shown to be optimal for general Banach spaces but not on Hilbert space. The third main part, finally, turns to general operator semigroups. It contains a version of the Katznelson-Tzafriri theorem in the Hilbert space setting which relaxes the main assumption of the original result. Various applications and extensions of this general result are also presented.

## Acknowledgements

I am grateful to my supervisor, Professor Charles J.K. Batty, for all his generous help and advice relating to this thesis. My thanks go also to Professor Yuri Tomilov for the instructive discussions we had during his visit to Oxford in autumn 2012.

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**Part I**

**Preliminaries**

# Chapter 1

## Introduction

This introductory chapter offers some motivation for investigating the problems addressed in the later parts of this thesis and provides an overview of the chapters that follow. More detailed introductory remarks, including surveys of the relevant literature, are included in the chapters themselves.

### 1.1 Background and motivation

Many problems arising in physics and other sciences describe the evolution of some quantity in time and space. Typically these problems can be formulated as a so-called *abstract Cauchy problem* of the form

$$\begin{cases} \dot{u}(t) = Au(t), & t \geq 0, \\ u(0) = x. \end{cases} \quad (1.1.1)$$

Here  $u : \mathbb{R}_+ \rightarrow X$  takes values in some complex Banach space  $X$ ,  $A$  is a suitable linear operator on  $X$  and  $x \in X$ . The problem is said to be *well-posed* if the operator  $A$  is the generator of a  $C_0$ -semigroup  $T$ , and in this case the unique solution of (1.1.1) is given by the orbit  $u(t) = T(t)x$  for all  $t \geq 0$ ; see [6, Chapter 3] and [45] for background material on  $C_0$ -semigroups. Since these orbits are continuous but not in general differentiable, the function  $u$  is a solution only in a certain mild sense unless the initial value  $x$  lies in the domain  $D(A)$  of the generator  $A$ , in which case the solution is said to be a *classical solution*.

It is this interpretation of semigroup orbits as solutions of an associated Cauchy problem that provides one of the main motivations for studying the

properties of  $C_0$ -semigroups. In most examples, though, there is little hope of obtaining an explicit formula for the operators  $T(t)$  for  $t > 0$ . The main task therefore is to deduce properties of the semigroup  $T$ , concerning for instance its regularity or long-time asymptotic behaviour, from the properties of the generator  $A$ , in particular its spectral properties. Perhaps one of the most important results of this kind is the Katznelson-Tzafriri theorem, which gives a sufficient condition for certain orbits to decay to zero as  $t \rightarrow \infty$ . Given a function  $a \in L^1(\mathbb{R})$ , define the Fourier-Laplace transform  $\widehat{a}$  for  $s \in \mathbb{R}$  by

$$\widehat{a}(is) = \int_{\mathbb{R}} a(t)e^{ist} dt$$

and, given a bounded  $C_0$ -semigroup  $T$  on a complex Banach space  $X$  and a function  $a \in L^1(\mathbb{R}_+)$ , define the operator  $\widehat{a}(T)$  for  $x \in X$  by

$$\widehat{a}(T)x = \int_{\mathbb{R}_+} a(t)T(t)x dt.$$

The Katznelson-Tzafriri theorem can now be stated as follows; see [48], [104].

**Theorem 1.1.1.** *Let  $X$  be a complex Banach space and let  $T$  be a bounded  $C_0$ -semigroup on  $X$  with generator  $A$ . Suppose that  $a \in L^1(\mathbb{R}_+)$  is of spectral synthesis with respect to  $\sigma(A) \cap i\mathbb{R}$ . Then  $\|T(t)\widehat{a}(T)\| \rightarrow 0$  as  $t \rightarrow \infty$ .*

Here a function  $a \in L^1(\mathbb{R}_+)$  is said to be of *spectral synthesis* with respect to a closed subset  $\Lambda$  of  $i\mathbb{R}$  if  $a$  can be approximated in  $L^1$ -norm by elements of  $L^1(\mathbb{R})$  whose Fourier-Laplace transforms vanish in a neighbourhood of  $\Lambda$ . One particularly striking application of the Katznelson-Tzafriri theorem is contained in [48], where it is used to give a new proof of the well-known Arendt-Batty-Lyubich-Vũ theorem stating that, for a bounded  $C_0$ -semigroup  $T$  on a complex Banach space  $X$  such that the unitary spectrum  $\sigma(A) \cap i\mathbb{R}$  of its generator  $A$  is at most countable and contains no eigenvalues of the adjoint operator  $A'$  of  $A$ ,  $\|T(t)x\| \rightarrow 0$  as  $t \rightarrow \infty$  for all  $x \in X$ .

A rather simpler consequence of Theorem 1.1.1 is the observation that

$$\lim_{t \rightarrow \infty} \|T(t)R(1, A)\| = 0 \tag{1.1.2}$$

provided  $\sigma(A) \cap i\mathbb{R} = \emptyset$ , which follows by applying the result to the function to the function  $a$  given by  $a(t) = e^{-t}$  for  $t \geq 0$ . Note that (1.1.2) is equivalent to

$$\lim_{t \rightarrow \infty} \sup \{ \|T(t)x\| : x \in D(A), \|x\|_A = 1 \} = 0, \tag{1.1.3}$$

where  $\|\cdot\|_A$  denotes the graph norm given, for  $x \in D(A)$ , by  $\|x\|_A = \|x\| + \|Ax\|$ . Problems in which the generator  $A$  of a bounded  $C_0$ -semigroup  $T$  has empty unitary spectrum arise for instance in the study of damped wave equations, and in this context (1.1.3) can be interpreted as saying that the total energy of solutions decays to zero over time, uniformly for initial values  $x \in D(A)$ . Indeed, consider the damped wave equation

$$\frac{\partial^2 v}{\partial t^2} - \Delta v + a(x) \frac{\partial v}{\partial t} = 0, \quad x \in \Omega, t > 0,$$

to be solved subject to the boundary condition  $v(x, t) = 0$  for  $x \in \partial\Omega$ ,  $t > 0$  and with initial data  $v(\cdot, 0) = v_0 \in H_0^1(\Omega)$ ,  $v_t(\cdot, 0) = w_0 \in L^2(\Omega)$ . Here  $\Omega$  denotes a bounded and sufficiently smooth domain in  $\mathbb{R}^n$  for some  $n \geq 1$  and  $a : \Omega \rightarrow [0, \infty)$  is a continuous function measuring the strength of the damping at different points in  $\Omega$ . The *energy* of the solution is defined as

$$E(t) = \frac{1}{2} \int_{\Omega} \left( \|\nabla v\|^2 + \left| \frac{\partial v}{\partial t} \right|^2 \right) dx.$$

A straightforward calculation shows that the energy is a non-increasing function of time, and it is known that except in degenerate cases the energy decays to zero. The standard approach to turning this damped wave equation into an abstract Cauchy problem of the same form as (1.1.1) is to rewrite the equation as a first-order system for  $u = (v, v_t)$  on the Hilbert space  $X = H_0^1(\Omega) \times L^2(\Omega)$ . Let  $A$  denote the generator of the resulting Cauchy problem and write  $T$  for the bounded  $C_0$ -semigroup generated by  $A$ . Then the energy can be rewritten simply as  $E(t) = \frac{1}{2} \|T(t)x\|^2$ , where  $x = (u_0, w_0) \in X$  now denotes the initial data. Since  $A$  can be shown by PDE methods to satisfy  $\sigma(A) \cap i\mathbb{R} = \emptyset$ , energy decay for all initial data, uniform for initial data chosen from  $D(A) = (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$ , follows immediately from the above discussion.

The real question here, however, is whether it is possible to say anything about the *rate* at which the energy decays. As is shown in [10], the rate of decay is necessarily at least exponential when the domain of damping  $\{x \in \Omega : a(x) > 0\}$  satisfies a certain *geometric optics condition*. In other cases, however, the rate of decay can be polynomial or logarithmic, uniformly for all classical solutions. The interest more generally, therefore, is in the rate of decay of  $\|T(t)R(1, A)\|$  as  $t \rightarrow \infty$ . As it turns out, this rate of decay is determined by the rate at which the resolvent  $\|R(is, A)\|$  grows as  $|s| \rightarrow \infty$ . General results in this direction are obtained in [11], [15], [27], [31], [70] and [74]. It follows from the results in [27], for instance, that in the important case where

$\|R(is, A)\| = O(|s|^\alpha)$  for some  $\alpha > 0$  as  $|s| \rightarrow \infty$  the quantity  $\|T(t)R(1, A)\|$  decays no more slowly than  $t^{-1/\alpha}$  as  $t \rightarrow \infty$ . When  $X$  is no longer assumed to be a Hilbert space, the results in [15] lead to a rate of decay which is slower by a logarithmic factor, and it is shown in [27] that this factor cannot in general be dropped. The focus in the first of the three main parts of this thesis will be on questions that arise naturally out of these previously known results.

While much of the recent interest in the Katznelson-Tzafriri theorem has been directed at the version stated in Theorem 1.1.1, which applies to  $C_0$ -semigroups, the original result proved by Katznelson and Tzafriri in [62] deals with a single operator and its powers. In its most basic form it states the following.

**Theorem 1.1.2.** *Let  $X$  be a complex Banach space and  $T \in \mathcal{B}(X)$  a power-bounded operator. Then*

$$\lim_{n \rightarrow \infty} \|T^n(I - T)\| = 0 \tag{1.1.4}$$

*if and only if  $\sigma(T) \cap \mathbb{T} \subset \{1\}$ .*

This result and a number of its variants have interesting applications for instance in the theory of iterative methods (see [86]), although the main motivation in [62] comes from applications to so-called *zero-two laws* for certain stochastic processes. Given the importance of Theorem 1.1.2 in applications and the interest in rates of decay for the continuous version of the Katznelson-Tzafriri theorem described above, it is natural to ask whether anything of interest can be said about the rate at which decay takes place in (1.1.4), and this will form the subject of the second main part of the thesis. One important conclusion that follows from the general results established in this context is that, if  $\sigma(T) \cap \mathbb{T} = \{1\}$  and if there exists  $\alpha \geq 1$  such that  $\|R(e^{i\theta}, T)\| \asymp |\theta|^{-\alpha}$  as  $\theta \rightarrow 0$ , then there exist constants  $c, C > 0$  such that

$$\frac{c}{n^{1/\alpha}} \leq \|T^n(I - T)\| \leq C \left( \frac{\log n}{n} \right)^{1/\alpha} \tag{1.1.5}$$

for all sufficiently large  $n \geq 0$ . These bounds on the rate of decay mirror those obtained in the continuous setting discussed above, and one of the main issues addressed in the relevant chapter of this thesis is the question whether, as in the continuous case, the logarithmic factor (1.1.5) is needed. The answer again turns out to depend on the space  $X$ .

Even though the connection between Theorems 1.1.1 and 1.1.2 may not be immediately obvious, it is possible, as was first demonstrated in [19], to unify these continuous and discrete versions of the Katznelson-Tzafriri theorem by moving to the more general framework of representations of locally compact abelian semigroups. Indeed, if the  $C_0$ -semigroup  $T$  in Theorem 1.1.1 is replaced by a bounded representation  $T$  of a locally compact semigroup  $S$  and if the unitary spectrum  $\sigma(A) \cap i\mathbb{R}$  of the generator  $A$  is replaced by the unitary spectrum  $\text{Sp}_u(T)$  of the representation  $T$ , then the Katznelson-Tzafriri theorem can be stated in the following general form.

**Theorem 1.1.3.** *Let  $X$  be a complex Banach space and let  $T$  be a bounded representation of a locally compact semigroup  $S$  on  $X$ . Suppose that  $a \in L^1(S)$  is of spectral synthesis with respect to  $\text{Sp}_u(T)$ . Then  $\|T(s)\widehat{a}(T)\| \rightarrow 0$  as  $s \rightarrow \infty$ .*

Some structure is lost by moving to this general setting. In particular, there is less scope here than in the continuous and discrete frameworks to use tools from complex analysis, and it is no longer obvious even how to formulate questions about rates of decay. Nevertheless, the fact that any improvement of Theorem 1.1.3 immediately gives a result in both the continuous and the discrete settings, as well as in many other interesting special cases, more than outweighs this drawback, and therefore it is precisely this more general approach that is taken in the last of the three main parts of this thesis.

## 1.2 Overview of thesis

The thesis is divided into four parts. The first of these, Part I, comprises the present chapter and the next, which introduces the necessary notation and preliminary results.

The first main part, Part II, is concerned with continuous operator semigroups, or  $C_0$ -semigroups, and is made up of two chapters. Chapter 3 first establishes an extension of Theorem 1.1.1 to the case in which the function  $a \in L^1(\mathbb{R})$  is replaced by a bounded Borel measure on  $\mathbb{R}_+$ , and in an important special case goes on to give an upper bound on the rate of decay in terms of the behaviour of the resolvent. Chapter 4 is concerned with the so-called *non-analytic growth bound*  $\zeta(T)$  of a  $C_0$ -semigroup  $T$ , which appears in the statement of the main results of Chapter 3. The first half of the chapter is concerned with obtaining a better understanding of the relationship between

$\zeta(T)$  and the resolvent of the semigroup generator. The chapter then defines a discrete analogue of the non-analytic growth bound for a single operator and goes on to show that it has a simple spectral characterisation.

The second half of Chapter 4, turning as it does to the behaviour of a single operator  $T$  and its powers, leads naturally into Part III, which is concerned with the discrete version of the Katznelson-Tzafriri theorem stated in Theorem 1.1.2 and more specifically with the rate of decay in (1.1.4). Chapter 5 establishes both upper and lower bounds on the rate of decay which depend on the growth of  $\|R(e^{i\theta}, A)\|$  as  $|\theta| \rightarrow 0$ , before Chapter 6 turns to the important special case in which the growth of the resolvent is polynomial. It is shown that on the one hand the results obtained in Chapter 5 for this case cannot be improved upon for general Banach spaces, but that a sharper result holds when  $X$  is a Hilbert space.

The last part of this thesis, Part IV, turns to general operator semigroups. The main result in Chapter 7 is a general version of the Katznelson-Tzafriri theorem for operator semigroups acting on Hilbert space in which the assumption of spectral synthesis is replaced by a weaker condition that is also necessary. Chapter 8, finally, presents a series of extensions and applications of the ideas contained in Chapter 7. These include a version of the Katznelson-Tzafriri theorem for individual orbits, a quantified version of the Katznelson-Tzafriri theorem for contractive operator semigroups and some results relating to the invertibility of contractive representations, in each case assuming that the underlying space is a Hilbert space.

## Chapter 2

# Notation and preliminary results

### 2.1 Basic notation

This section introduces various important pieces of notation. Several more ad-hoc definitions and conventions are supplied as the need arises throughout the main chapters.

**Banach and Hilbert spaces.** Banach and Hilbert spaces will typically be denoted by  $X$ , sometimes by  $Y$  or  $Z$ , with appropriate sub- and superscripts when there is risk of confusion. Throughout, all Banach spaces will be assumed to be complex and non-trivial unless specified otherwise. The dual space of a Banach space  $X$  is denoted by  $X^*$  and, if  $X$  is a Hilbert space, its inner product by  $(\cdot, \cdot) : X \times X \rightarrow \mathbb{C}$ . Given a subset  $Y$  of a Banach space  $X$ , write  $Y^\perp$  for the annihilator  $\{\phi \in X^* : \phi(y) = 0 \text{ for all } y \in Y\}$  in  $X^*$  of  $Y$ . In the Hilbert space setting, the same notation denotes the orthogonal complement  $\{x \in X : (x, y) = 0 \text{ for all } y \in Y\}$  in  $X$  of  $Y$ .

**Operators.** Given a Banach space  $X$ , let  $\mathcal{B}(X)$  denote the Banach algebra of all bounded linear operators on  $X$ . Elements of  $\mathcal{B}(X)$  will usually be denoted by letters from near the end of the alphabet, such as  $Q$  or, most often,  $T$ . For any closed (but not necessarily bounded) operator  $A$  on  $X$ , write  $D(A)$  for the domain of  $A$ , and  $\text{Ran } A$  and  $\text{Ker } A$  for its range and kernel, respectively. Note that a closed linear operator  $A$  lies in  $\mathcal{B}(X)$  if and only if  $D(A) = X$ , by the closed graph theorem. Denote by  $\text{Fix}(A)$  the set  $\{x \in D(A) : Ax = x\}$

of fixed points of  $A$ . Furthermore, let  $\sigma(A)$  ( $\sigma_p(A)$ ,  $\sigma_a(A)$  and  $\sigma_r(A)$ ) be the (respectively, point, approximate point and residual) spectrum of  $A$  and let  $\rho(A) = \mathbb{C} \setminus \sigma(A)$  be the resolvent set of  $A$ . Given  $\lambda \in \rho(A)$ , write  $R(\lambda, A) = (\lambda - A)^{-1}$  for the resolvent operator and, if  $A$  is densely defined, let  $A'$  be the Banach adjoint of  $A$ . If  $X$  is a Hilbert space, write  $A^*$  for the Hilbert adjoint of  $A$ .

**Banach algebras.** Given a commutative Banach algebra  $\mathcal{A} \subset \mathcal{B}(X)$  for some Banach space  $X$ , let  $\Delta(\mathcal{A})$  denote the space of all characters (i.e. non-zero complex homomorphisms) on  $\mathcal{A}$  and recall that every element of  $\Delta(\mathcal{A})$  is in fact a bounded linear functional on  $\mathcal{A}$  with norm at most 1 and that  $\Delta(\mathcal{A})$  is compact in the inherited weak\* topology precisely when  $\mathcal{A}$  is unital; see [81, Section 1.3]. Recall also that the Gelfand transform  $\Phi : \mathcal{A} \rightarrow C_0(\Delta(\mathcal{A}))$ , defined for  $A \in \mathcal{A}$  and  $\chi \in \Delta(\mathcal{A})$  by  $\Phi(A)(\chi) = \chi(A)$ , is an algebra homomorphism which satisfies  $\|\Phi(A)\| = r(A)$  and becomes an isometric \*-isomorphism when  $\mathcal{A}$  is a commutative  $C^*$ -algebra; see [81, Theorem 2.1.10]. Here

$$r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$$

denotes the spectral radius of  $A \in \mathcal{A}$  and  $C_0(\Delta(\mathcal{A}))$  denotes the algebra of continuous functions on  $\Delta(\mathcal{A})$  that vanish at infinity.

**Function spaces.** If  $X$  is a Banach space,  $(\Omega, \nu)$  is any measure space and  $1 \leq p \leq \infty$ , let  $L^p(\Omega; X)$  be the shorthand notation for the space  $L^p(\Omega, \nu; X)$  of  $X$ -valued (equivalence classes of) functions that are  $p$ -integrable with respect to  $\nu$ . Furthermore, for any locally compact space  $\Lambda$ , let  $C(\Lambda; X)$  denote the space of  $X$ -valued continuous functions, and write  $C_0(\Lambda; X)$  for the set of continuous functions  $\phi : \Lambda \rightarrow X$  which *vanish at infinity* in the sense that, given any  $\varepsilon > 0$ , there exists a compact subset  $K$  of  $\Lambda$  such that  $\|\phi(\lambda)\| < \varepsilon$  for all  $\lambda \in \Lambda \setminus K$ . For  $k \in \mathbb{Z}_+ \cup \{\infty\}$ , let  $C^k(\mathbb{R}; X)$  denote the space of  $X$ -valued functions on  $\mathbb{R}$  that possess  $k$  continuous derivatives, and define  $C_c^\infty(\mathbb{R}; X) = \{\phi \in C^\infty(\mathbb{R}; X) : \phi \text{ has compact support}\}$ . In all cases, the target space  $X$  will be suppressed if  $X = \mathbb{C}$ . Finally, the Schwartz space  $\mathcal{S}(\mathbb{R})$  of rapidly decreasing functions on  $\mathbb{R}$  is defined as

$$\mathcal{S}(\mathbb{R}) = \left\{ \phi \in C^\infty(\mathbb{R}) : \sup_{t \in \mathbb{R}} |t^m \phi^{(n)}(t)| < \infty \text{ for all } m, n \geq 0 \right\}.$$

**Asymptotic behaviour.** Given a subset  $\Omega$  of  $\mathbb{C}$  and functions  $f, g : \Omega \rightarrow \mathbb{R}_+$ , the notation  $f(\omega) = O(g(\omega))$  (as  $\omega$  tends to some limit or to  $\infty$ ) indicates that there exists a constant  $C > 0$  such that  $f(\omega) \leq Cg(\omega)$  eventually, i.e. for all  $\omega$  sufficiently close to the limit or sufficiently large, respectively. To say that  $f(\omega) \asymp g(\omega)$  for a given limiting behaviour of  $\omega$  means that both  $f(\omega) = O(g(\omega))$  and  $g(\omega) = O(f(\omega))$ . Furthermore, if  $g(\omega) > 0$  for all relevant values of  $\omega$ , let  $f(\omega) \sim g(\omega)$  (again as  $\omega$  tends to some limit or to  $\infty$ ) indicate that  $f(\omega)/g(\omega) \rightarrow 1$ . The notation  $f(\omega) = o(g(\omega))$  means that, given any  $\varepsilon > 0$ ,  $f(\omega) \leq \varepsilon g(\omega)$  eventually.

**Miscellaneous.** The symbols  $\mathbb{N}$ ,  $\mathbb{Z}_+$  and  $\mathbb{R}_+$  will be used to denote, respectively, the sets  $\{1, 2, \dots\}$ ,  $\{0, 1, 2, \dots\}$  and  $[0, \infty)$ . The open unit disc  $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$  will be denoted by  $\mathbb{D}$  and its boundary, the unit circle, by  $\mathbb{T}$ . For  $\theta > 0$ , let  $\Sigma_\theta$  denote the open sector  $\Sigma_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \theta\}$  and, for  $\omega \in \mathbb{R}$ , let  $H_\omega = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > \omega\}$ . Given a subset  $\Omega$  of  $\mathbb{C}$ , write  $\bar{\Omega}$  for the closure of  $\Omega$  and  $\partial\Omega$  for its topological boundary.

## 2.2 Geometric properties of Banach spaces

Given  $p \in [1, 2]$ , a Banach space  $X$  is said to have *type  $p$*  if there exists a constant  $C \geq 0$  such that, for any  $n \in \mathbb{N}$  and arbitrary  $x_1, \dots, x_n \in X$ ,

$$\left( \int_0^1 \left\| \sum_{k=1}^n r_k(t)x_k \right\|^2 dt \right)^{1/2} \leq C \left( \sum_{k=1}^n \|x_k\|^p \right)^{1/p},$$

where the *Rademacher functions*  $r_k : (0, 1) \rightarrow \mathbb{R}$  are defined by

$$r_k(t) = \operatorname{sgn} \sin(2^k \pi t).$$

Likewise, for  $q \in [2, \infty]$ ,  $X$  is said to have *cotype  $q$*  if there exists a constant  $C \geq 0$  such that, in the same setting as above,

$$\left( \sum_{k=1}^n \|x_k\|^q \right)^{1/q} \leq C \left( \int_0^1 \left\| \sum_{k=1}^n r_k(t)x_k \right\|^2 dt \right)^{1/2}$$

if  $q < \infty$  and with the left-hand side replaced by  $\max\{\|x_k\| : 1 \leq k \leq n\}$  when  $q = \infty$ . Every Banach space  $X$  has type 1 and cotype  $\infty$ , and  $X$  has type and cotype 2 if and only if it is isomorphic to a Hilbert space; see [67]. Moreover, if  $X$  has type  $p$ , then  $X^*$  has cotype  $p^*$  satisfying

$$\frac{1}{p} + \frac{1}{p^*} = 1; \tag{2.2.1}$$

see [102, Section 4] for further details.

Given  $p \in [1, 2]$ , a Banach space  $X$  is said to have *Fourier type  $p$*  if the operator  $\mathcal{F}$ , which is defined in terms of the Fourier-Laplace transform for  $a \in L^1(\mathbb{R}; X)$  and  $t \in \mathbb{R}$  by

$$(\mathcal{F}a)(t) = \widehat{a}(-it) = \int_{\mathbb{R}} e^{-it} a(t) dt$$

extends from the dense subspace  $C_c^\infty(\mathbb{R}; X)$  to a bounded linear operator from  $L^p(\mathbb{R}; X)$  into  $L^{p^*}(\mathbb{R}; X)$ . Any Banach space  $X$  has Fourier type 1, and  $X$  has Fourier type 2 if and only if it is isomorphic to a Hilbert space; see [67] again. Furthermore,  $X$  has Fourier type  $p$  if and only if  $X^*$  does, and any uniformly convex Banach space has Fourier type strictly exceeding 1; see [66, Note 4.6].

If the (Fourier) type of a Banach space  $X$  is strictly greater than 1,  $X$  is also said to have *non-trivial* (Fourier) type. It is known that  $X$  has non-trivial Fourier type if and only if it has non-trivial type (see [28]), and spaces with this property are also referred to as being *B-convex*. The most important examples of *B-convex* spaces for the purposes of this thesis are the so-called UMD-spaces. A Banach space  $X$  is said to be a *UMD-space* whenever the *Hilbert transform*  $H$ , which is defined, for  $a \in C_c^\infty(\mathbb{R}; X)$  and  $t \in \mathbb{R}$ , by

$$(Ha)(t) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\pi} \int_{|s-t| > \varepsilon} \frac{a(s)}{t-s} ds,$$

extends to an element of  $\mathcal{B}(L^p(\mathbb{R}; X))$  for some (or, equivalently, all)  $p \in (1, \infty)$ ; see [29] for a proof that this definition is equivalent to  $X$  satisfying the eponymous *unconditional martingale difference* property. Examples of UMD-spaces include all Hilbert spaces as well as the Lebesgue spaces  $L^p(\mathbb{R}; X)$  for  $p \in (1, \infty)$  and any other UMD-space  $X$ , and it is known that the UMD-property is preserved under taking duals, closed subspaces and quotients by closed subspaces; see [66, Note 3.15] for further details.

The definition of the UMD-property has a natural reformulation in terms of Fourier multipliers. Recall that a bounded measurable function  $M : \mathbb{R} \rightarrow \mathcal{B}(X)$  is said to be a *Fourier multiplier* on  $L^p(\mathbb{R}; X)$ , where  $1 \leq p < \infty$ , if the operator

$$T_M : C_c^\infty(\mathbb{R}; X) \rightarrow L^p(\mathbb{R}; X)$$

given by

$$T_M a = \mathcal{F}^{-1}(M \cdot \mathcal{F}a)$$

extends to an element of  $\mathcal{B}(L^p(\mathbb{R}; X))$ . Here

$$(\mathcal{F}^{-1}a)(t) = \frac{(\mathcal{F}a)(-t)}{2\pi} = \frac{\widehat{a}(it)}{2\pi} = \frac{1}{2\pi} \int_{\mathbb{R}} e^{it} a(t) dt$$

for  $a \in L^1(\mathbb{R}; X)$  and  $t \in \mathbb{R}$ , and multiplication in the expression  $M \cdot \mathcal{F}a$  is to be understood in the pointwise sense. Note that the Hilbert transform  $H$  coincides with the operator  $T_\phi$ , where  $\phi(t) = -i \operatorname{sgn}(t)$  for  $t \in \mathbb{R}$ . It follows that  $X$  is a UMD-space if and only if  $\phi$  is a Fourier multiplier on  $L^p(\mathbb{R}; X)$  for some (or, equivalently, all)  $p \in (1, \infty)$ .

## 2.3 Sectorial operators and fractional powers

Let  $X$  be a complex Banach space. An operator  $T \in \mathcal{B}(X)$  is said to be *sectorial* if there exists  $\theta \in (0, \pi]$  such that  $-\Sigma_\theta \subset \rho(T)$  and if  $\sup\{\|\lambda R(\lambda, T)\| : \lambda \in -\Sigma_\vartheta\} < \infty$  for all  $\vartheta \in (0, \theta)$ . It follows from the theory of functional calculus for sectorial operators that one can form the fractional power  $T^\alpha \in \mathcal{B}(X)$  of  $T$  for any  $\alpha \geq 0$ , which is defined as

$$T^\alpha = \frac{1}{2\pi i} \oint_{\Gamma} \lambda^\alpha R(\lambda, T) d\lambda,$$

where  $\Gamma$  is any contour that contains the origin and otherwise encloses  $\sigma(T)$  without touching it, and where the complex plane is cut along the negative real axis. Fractional powers coincide with the usual ones whenever  $\alpha \in \mathbb{N}$ , and moreover  $T^{\alpha+\beta} = T^\alpha T^\beta$  for all  $\alpha, \beta \geq 0$ . If  $T$  is sectorial and injective, it is possible to define the negative fractional power  $T^{-\alpha}$  for  $\alpha > 0$  as a closed operator on  $\operatorname{Ran} T^\alpha$  by taking it to be the inverse of  $T^\alpha$ . For details on sectorial operators and fractional powers, see for instance [55].

Suppose now that  $T \in \mathcal{B}(X)$  is a power-bounded operator with  $M = \sup\{\|T^n\| : n \geq 0\}$ . Since

$$\|R(\lambda, T)\| \leq \frac{M}{|\lambda| - 1}$$

for  $|\lambda| > 1$ , and hence  $\|R(\lambda, I - T)\| \leq M|\operatorname{Re} \lambda|^{-1}$  for  $\operatorname{Re} \lambda < 0$ , the operator  $I - T$  is sectorial. Sectorial operators of this form will play an important role in what follows.

A fundamental result in the context of fractional powers of sectorial operators is the following *moment inequality*; see for instance [54, Corollary 7.2] for a relatively direct proof.

**Proposition 2.3.1.** *Let  $X$  be a complex Banach space and let  $T \in \mathcal{B}(X)$  be a sectorial operator. Suppose that  $\alpha > \beta > \gamma \geq 0$  and let  $\theta = \frac{\beta-\gamma}{\alpha-\gamma}$ . Then there exists a constant  $C > 0$  such that*

$$\|T^\beta x\| \leq C \|T^\alpha x\|^\theta \|T^\gamma x\|^{1-\theta}$$

for all  $x \in X$ .

## 2.4 Elements of abstract harmonic analysis and representation theory

### 2.4.1 Locally compact abelian groups and semigroups

This section introduces various concepts relating to topological groups and semigroups. Whenever either is introduced in a later section without further specification, it will implicitly be assumed to fall into the framework set out below.

**Groups and semigroups.** Let  $G$  be a locally compact abelian group and let  $S$  be a subsemigroup of  $G$ , i.e. a subset of  $G$  satisfying  $S + S \subset S$ . Denote by  $\mu$  the Haar measure on  $G$ , and assume that  $S$  is measurable and hence is itself a measure space with respect to the restriction of  $\mu$ . Assume further that  $S$ , in the topology induced by  $G$ , has non-empty interior  $S^\circ$  and note that, replacing  $G$  by a smaller group if necessary, there is no loss of generality in assuming that  $S$  generates  $G$ , which is to say that  $G = S - S$ . The semigroup  $S$  defines a relation  $\succeq$  on  $G$  via  $s \succeq t$  if and only if  $s - t \in S \cup \{0\}$ . This turns  $G$  into a directed set and hence makes it possible to speak of limits as  $s \rightarrow \infty$ . When  $s - t \in S \setminus \{0\}$  the notation  $s \succ t$  will occasionally be used.

**The dual group.** Let  $\Gamma$  denote the dual group of  $G$ , consisting of all continuous characters of  $G$ , and let  $S^*$  stand for the set of all continuous bounded characters on  $S$ . Since  $S$  spans the group  $G$ , the *unitary part* of  $S^*$ , which consists of all  $\chi \in S^*$  such that  $|\chi(s)| = 1$  for all  $s \in S$ , can be identified with  $\Gamma$ . This will be done throughout without further comment.

**The Fourier-Laplace transform.** Given  $a \in L^1(S)$ , the *Fourier(-Laplace) transform*  $\hat{a} : S^* \rightarrow \mathbb{C}$  of  $a$  is defined by

$$\hat{a}(\chi) = \int_S a(s)\chi(s) \, d\mu(s). \quad (2.4.1)$$

When  $S$  is replaced by  $G$ , this formula defines  $\widehat{a}(\chi)$  for  $a \in L^1(G)$  and  $\chi \in \Gamma$ . The final assumption on  $S$  is that the set  $\{\widehat{a} : a \in L^1(S)\}$  of Fourier-Laplace transforms separates points, both from each other and from zero; i.e. for each  $\chi \in S^*$  there exists  $a \in L^1(S)$  such that  $\widehat{a}(\chi) \neq 0$  and, given any two distinct elements  $\chi$  and  $\xi$  of  $S^*$ , there exist  $a \in L^1(S)$  satisfying  $\widehat{a}(\chi - \xi) \neq 0$ . While satisfied in all natural examples of interest, this property does not follow from the previous assumptions, as is shown in [52, Example 1.1.4]. Nevertheless, the assumption is not excessively restrictive and is satisfied for instance whenever the interior  $S^\circ$  is dense in  $S$ ; see [52, page 2].

**Translation, approximation and convolution.** For  $a \in L^1(S)$ ,  $s \in G$  and  $t \in S$ , set

$$a_s(t) = \begin{cases} a(t-s) & \text{if } t-s \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Then  $a_s \in L^1(S)$  and, for each  $s \in S$  and any  $\chi \in \Gamma$ ,  $\chi(s)\widehat{a}(\chi) = \widehat{a}_s(\chi)$ . Note that, in the case where  $S = G$ ,  $a_s$  is simply the ordinary (untruncated) translate of  $a$ . Moreover, for each  $a \in L^1(G)$ , the translation map  $\tau_a : G \rightarrow L^1(G)$  given by  $\tau_a(s) = a_s$  is uniformly continuous (see [95, Section 1.1.5]) and restricts, for  $a \in L^1(S)$ , to a uniformly continuous map  $S \rightarrow L^1(S)$ . Hence, given  $a \in L^1(S)$ ,  $\varepsilon > 0$  and  $s_0 \in S^\circ$ , there exists an open neighbourhood  $U$  of  $s_0$  in  $S$  such that  $\|a_s - a_{s_0}\|_1 < \varepsilon$  for all  $s \in U$ . As in [95, Section 1.1.8], it follows that, if  $b \in L^1(S)$  is any non-negative function of unit norm whose support is contained in  $U$ , then by an application of Fubini's theorem

$$\|a * b - a_{s_0}\|_1 = \int_U b(s) \|a_s - a_{s_0}\|_1 d\mu(s) < \varepsilon.$$

Here an element  $b$  of  $L^p(S)$ , for  $p \in [1, \infty]$ , is said to be *non-negative* if  $b(s) \geq 0$  for  $\mu$ -almost all  $s \in S$ , and  $*$  denotes the convolution operation on  $L^1(S)$ , which is defined, for  $a, b \in L^1(S)$  and  $s \in S$ , by

$$(a * b)(s) = \int_S a(t)b_t(s) d\mu(t).$$

In particular,  $a * b = b * a \in L^1(S)$  for all  $a, b \in L^1(S)$ , which shows that, as a Banach algebra under convolution,  $L^1(S)$  is commutative. Note finally that given any  $a \in L^1(S)$  and any  $\varepsilon > 0$ , there exists a compact subset  $K$  of  $S$  such that

$$\int_{S \setminus K} |a(s)| d\mu(s) < \varepsilon.$$

**Følner nets and amenability.** Given a net  $(\Omega_\alpha)$  of non-null compact Borel subsets of  $S$ ,  $(\Omega_\alpha)$  is said to be a *Følner net* for  $S$  if

$$\lim_{\alpha \rightarrow \infty} \frac{\mu(\Omega_\alpha \triangle (\Omega_\alpha + s))}{\mu(\Omega_\alpha)} = 0,$$

uniformly for  $s$  in compact subsets of  $S$ , where  $A \triangle B$  denotes the symmetric difference of two sets  $A, B$ . The notion of a *Følner sequence* is defined analogously. It follows from the assumptions on  $S$  and the results in [90, Chapter 4] that there exists a Følner net for  $S$  and that the net may be taken to be a sequence whenever  $S$  is  $\sigma$ -compact; see also [19, Section 2].

If  $(\Omega_\alpha)$  is any Følner net for  $S$ , consider the net  $(\phi_\alpha)$  in  $L^\infty(S)^*$ , where, for  $a \in L^\infty(S)$ ,

$$\phi_\alpha(a) = \frac{1}{\mu(\Omega_\alpha)} \int_{\Omega_\alpha} a(s) \, d\mu(s).$$

Then  $\|\phi_\alpha\| = 1$  for all  $\alpha$  and, if  $\phi$  denotes any weak\* limit point, then  $\phi$  is left-translation invariant in the sense that  $\phi(a) = \phi(a_{-s})$  for any  $a \in L^\infty(S)$  and any  $s \in S$ . Furthermore  $\phi(\mathbb{1}_S) = \|\phi\| = 1$ , where  $\mathbb{1}_\Omega$  denotes the characteristic function of a measurable subset  $\Omega$  of  $S$ . In addition,  $\phi(a) \geq 0$  whenever  $a \in L^\infty(S)$  is non-negative, and

$$\operatorname{ess-lim\,inf}_{s \rightarrow \infty} a(s) \leq \phi(a) \leq \operatorname{ess-lim\,sup}_{s \rightarrow \infty} a(s),$$

where

$$\operatorname{ess-lim\,inf}_{s \rightarrow \infty} a(s) = \lim_{s \rightarrow \infty} \operatorname{ess-inf}\{a(t) : t \succeq s\}$$

with

$$\operatorname{ess-inf}\{a(t) : t \succeq s\} = \sup \{c \in \mathbb{R} : \mu(\{t \succeq s : a(t) < c\}) = 0\}$$

and  $\operatorname{ess-lim\,sup}_{s \rightarrow \infty} a(s)$  is defined analogously. Writing

$$\operatorname{ess-lim}_{s \rightarrow \infty} a(s) = \ell$$

if  $a \in L^\infty(S)$  and  $\ell \in \mathbb{R}$  are such that, given any  $\varepsilon > 0$ , there exists  $s \in S$  with  $\mu(\{t \succeq s : |a(t) - \ell| < \varepsilon\}) = 0$ , it follows in particular that

$$\phi(a) = \operatorname{ess-lim}_{s \rightarrow \infty} a(s)$$

whenever the essential limit exists. Elements  $\phi$  of  $L^\infty(S)^*$  with the above properties are known as *Banach limits*. Given any Banach limit  $\phi$ , it is possible to define a finitely additive and left-translation invariant probability measure  $\nu_\phi$  on  $S$  by setting  $\nu_\phi(\Omega) = \phi(\mathbb{1}_\Omega)$  for any Borel subset  $\Omega$  of  $S$ . Whenever such a measure (known sometimes as a *mean*) exists, the semigroup is said to be *amenable*.

### 2.4.2 Semigroup representations

Given a locally compact abelian semigroup  $S$  and a complex Banach space  $X$ , suppose a map  $T : S \rightarrow \mathcal{B}(X)$  is a homomorphism, so that  $T(s+t) = T(s)T(t)$  for all  $s, t \in S$ , and furthermore, if  $0 \in S$ , satisfies  $T(0) = I$ . Then  $T$  is said to be a *representation* of  $S$  on  $X$  if the map  $s \mapsto T(s)$  is strongly continuous on  $S$ . A representation  $T$  of a locally compact abelian semigroup  $S$  is said to be *bounded* if  $\sup\{\|T(s)\| : s \in S\} < \infty$ . In this case, given any  $a \in L^1(S)$ , the operator  $\widehat{a}(T) \in \mathcal{B}(X)$  is defined, for each  $x \in X$ , by

$$\widehat{a}(T)x = \int_S a(s)T(s)x \, d\mu(s). \quad (2.4.2)$$

Note that, for all  $s \in S$  and all  $a \in L^1(S)$ ,  $T(s)\widehat{a}(T) = \widehat{a_s}(T)$ . Any representation  $T$  of a semigroup  $S$  on a Banach space  $X$  induces a homomorphism  $T' : S \rightarrow \mathcal{B}(X^*)$  given by  $T'(s) = T(s)'$ . The action of  $T'$  may, however, need to be restricted to a suitable subspace of  $X^*$  in order for it to induce a representation of  $S$ .

An important example of a bounded representation of a locally compact abelian semigroup  $S$  on the Banach space  $L^1(S)$  is the *shift semigroup*  $T_\tau$  given, for  $s \in S$  and  $a \in L^1(S)$ , by  $T_\tau(s)a = \tau_a(s)$ .

### 2.4.3 The spectrum of a representation

Given a locally compact abelian group  $G$  and a closed subset  $\Lambda$  of the dual group  $\Gamma$ , let

$$J_\Lambda(G) = \{a \in L^1(G) : \text{supp}(\widehat{a}) \cap \Lambda = \emptyset\},$$

where  $\text{supp}(\phi)$  denotes the support of  $\phi \in C(\Lambda)$ , and define

$$K_\Lambda(G) = \{a \in L^1(G) : \widehat{a}(\chi) = 0 \text{ for all } \chi \in \Lambda\}.$$

Thus  $K_\Lambda(G)$  is a closed ideal of  $L^1(G)$  containing  $J_\Lambda(G)$ . An element of  $K_\Lambda(G)$  is said to be of *spectral synthesis* with respect to  $\Lambda$  if it lies in the closure of  $J_\Lambda(G)$ , while the set  $\Lambda$  itself is said to be of *spectral synthesis* whenever this closure coincides with the whole of  $K_\Lambda(G)$ . If  $T$  is a bounded representation of  $G$  on a complex Banach space  $X$ , then the *spectral subspace*  $M(\Lambda)$  of  $X$  is defined, again for any closed subset  $\Lambda$  of  $\Gamma$ , by

$$M(\Lambda) = \bigcap \{ \text{Ker } \widehat{a}(T) : a \in J_\Lambda(G) \}.$$

Thus  $M(\Lambda)$  is a closed subspace of  $X$  satisfying  $M(\emptyset) = \{0\}$ ,  $M(\Gamma) = X$  and  $M(\Lambda_1) \subset M(\Lambda_2)$  whenever  $\Lambda_1 \subset \Lambda_2$ ; see [93, Theorem 8.1.4]. Since moreover

$$\bigcap_{\alpha} M(\Lambda_{\alpha}) = M\left(\bigcap_{\alpha} \Lambda_{\alpha}\right)$$

for any collection  $\{\Lambda_{\alpha}\}$  of closed subsets of  $\Gamma$  (again see [93, Theorem 8.1.4]), it makes sense to speak of the *smallest* closed subset  $\Lambda$  of  $\Gamma$  for which  $M(\Lambda) = X$ . This set is denoted by  $\text{Sp}(T)$  (or  $\text{Sp}(T; G)$  if there is risk of confusion) and is given by

$$\text{Sp}(T) = \bigcap \{ \Lambda \subset \Gamma : \Lambda \text{ is closed and } M(\Lambda) = X \}.$$

It is known as the (*Arveson*) *spectrum* of  $T$  and coincides with the *finite  $L$ -spectrum* of [75, Section 5.2]. It can be shown (see [93, Proposition 8.1.9]) that

$$\text{Sp}(T) = \{ \chi \in \Gamma : |\widehat{a}(\chi)| \leq \|\widehat{a}(T)\| \text{ for all } a \in L^1(G) \},$$

that  $\text{Sp}(T)$  is non-empty and that  $\text{Sp}(T)$  is compact if and only if  $T$  is norm continuous; see [93, Theorem 8.1.12]. Given a bounded representation  $T$  of a locally compact abelian group  $G$  on a Banach space  $X$ , define the set  $\Lambda(T)$  by

$$\Lambda(T) = \{ \chi \in \Gamma : \widehat{a}(\chi) = 0 \text{ for all } a \in L^1(G) \text{ s.t. } \widehat{a}(T) = 0 \}.$$

The following result provides a further alternative characterisation of  $\text{Sp}(T)$ ; see [93, Proposition 8.1.9] for a proof.

**Theorem 2.4.1.** *Let  $T$  be a bounded representation of a locally compact abelian group  $G$  on a complex Banach space  $X$ . Then  $\text{Sp}(T) = \Lambda(T)$ .*

If  $X$  is a Hilbert space and  $T$  is a representation of  $G$  on  $X$  by unitary operators, then there exists a (unique) *spectral measure*  $E$  which associates with any Borel subset of the dual group  $\Gamma$  of  $G$  an element of  $\mathcal{B}(X)$  and which is such that, for each  $s \in G$ ,

$$T(s) = \int_{\Gamma} \chi(s) \, dE(\chi),$$

the integral here being taken in the weak sense. Moreover, the *spectral projections*  $E(\Lambda)$  are orthogonal and satisfy  $\text{Ran } E(\Lambda) = M(\Lambda)$  whenever  $\Lambda$  is a closed subset of  $\Gamma$ ; see [93, Theorem 8.3.2].

When  $T$  is a bounded representation  $T$  of a locally compact abelian semi-group  $S$  on a Banach space  $X$ , define the *spectrum*  $\text{Sp}(T)$  of  $T$  by

$$\text{Sp}(T) = \{ \chi \in S^* : |\widehat{a}(\chi)| \leq \|\widehat{a}(T)\| \text{ for all } f \in L^1(S) \},$$

and let  $\mathrm{Sp}_u(T) = \mathrm{Sp}(T) \cap \Gamma$  (denoted also by  $\mathrm{Sp}_u(T; S)$  if there is risk of confusion) be the *unitary spectrum*. The following result is taken from [52, Lemma 1.3.7].

**Lemma 2.4.2.** *Let  $G$  be a locally compact abelian group and  $S$  a generating subsemigroup of  $G$ . If  $T$  is a representation of  $G$  by isometries on a complex Banach space  $X$ , then  $\mathrm{Sp}(T; G) = \mathrm{Sp}_u(T; S)$ .*

To see how the unitary spectrum  $\mathrm{Sp}_u(T)$  of a general bounded representation  $T$  of a locally compact abelian semigroup  $S$  on a Banach space  $X$  relates to other notions of spectrum, recall first that  $\chi \in S^*$  is said to be an *eigenvalue* of  $T$  if there exists a non-zero vector  $x \in X$  such that  $T(s)x = \chi(s)x$  for all  $s \in S$ . Denote by  $\mathrm{PSP}(T)$  the *point spectrum* of  $T$ , consisting of all its eigenvalues, and write  $\mathrm{PSP}_u(T)$  for the set  $\mathrm{PSP}(T) \cap \Gamma$  of *unitary eigenvalues* of  $T$ . A character  $\chi \in S^*$  is said to be an *approximate eigenvalue* of  $T$  if there exists a net  $(x_\alpha)$  of unit vectors in  $X$  such that

$$\lim_{\alpha \rightarrow \infty} \|T(s)x_\alpha - \chi(s)x_\alpha\| = 0,$$

uniformly for  $s$  in compact subsets of  $S$ . The *approximate point spectrum*, which consists of all approximate eigenvalues, is denoted by  $\mathrm{ASp}(T)$  and, furthermore,  $\mathrm{ASp}_u(T)$  stands for the set  $\mathrm{ASp}(T) \cap \Gamma$  of unitary approximate eigenvalues of  $T$ . The sets  $\mathrm{PSP}(T')$ ,  $\mathrm{PSP}_u(T')$ ,  $\mathrm{ASp}(T')$  and  $\mathrm{ASp}_u(T')$  are defined analogously. The following result is taken from [19, Proposition 2.2].

**Proposition 2.4.3.** *Let  $T$  be a bounded representation of a locally compact abelian semigroup  $S$  on a complex Banach space  $X$ . Then  $\mathrm{PSP}(T) \cup \mathrm{PSP}(T') \subset \mathrm{Sp}(T)$  and  $\mathrm{Sp}_u(T) = \mathrm{ASp}_u(T)$ .*

Finally, given a bounded representation  $T$  of a locally compact abelian semigroup  $S$ , consider the algebra homomorphism  $\pi_T : L^1(S) \rightarrow \mathcal{B}(X)$  given by  $\pi_T(a) = \widehat{a}(T)$ , and let  $\mathcal{A}_T$  denote the norm closure of  $\mathrm{Ran} \pi_T$ . Note that each  $\chi \in \mathrm{Sp}(T)$  induces a character  $\xi_\chi$  on  $\mathcal{A}_T$  defined, for  $a \in L^1(S)$ , by  $\xi_\chi(\pi_T(a)) = \widehat{a}(\chi)$  and then extended continuously. The following result is taken from [19, Proposition 2.4].

**Proposition 2.4.4.** *Let  $T$  be a bounded representation of a locally compact abelian semigroup  $S$  on a complex Banach space  $X$ . Then the map  $\chi \mapsto \xi_\chi$  maps  $\mathrm{Sp}(T)$  bijectively onto  $\Delta(\mathcal{A}_T)$ .*

## 2.5 Discrete and continuous operator semigroups

**Discrete operator semigroups** The first important special instance of the general framework set out in Section 2.4 is the case of what here will be referred to as *discrete (operator) semigroups*, namely  $S = \mathbb{Z}_+^n$  and  $G = \mathbb{Z}^n$  with the counting measure for some  $n \geq 1$ . In this case the character space  $S^*$  of  $S$  can be identified with the closed unit  $n$ -disc  $\overline{\mathbb{D}}^n$ , via the map that sends  $\lambda = (\lambda_1, \dots, \lambda_n) \in \overline{\mathbb{D}}^n$  to the element  $\chi_\lambda$  of  $S^*$  given, for  $s = (s_1, \dots, s_n) \in S$ , by

$$\chi_\lambda(s) = \prod_{k=1}^n \lambda_k^{s_k},$$

with the understanding that  $\lambda_k^{s_k} = 1$  when  $\lambda_k = s_k = 0$ . Moreover,  $\Gamma = \mathbb{T}^n$ . A natural example of a Følner sequence in this case is  $(\Omega_k)$  where, for  $k \geq 1$ ,  $\Omega_k = \{1, \dots, k\}^n$ .

A (bounded) representation of the discrete semigroup  $S = \mathbb{Z}_+^n$  corresponds to an  $n$ -tuple of commuting (power-bounded) operators  $T_1, \dots, T_n \in \mathcal{B}(X)$ . In what follows, the term ‘discrete operator semigroup’ refers in most cases to a bounded representation of the semigroup  $S = \mathbb{Z}_+$  and hence can be identified with a power-bounded operator  $T \in \mathcal{B}(X)$ . In this case, the dual operator  $T'$  gives rise to a representation of  $\mathbb{Z}_+$  on the whole of  $X^*$ . Furthermore,  $\sigma(T) \subset \text{Sp}(T)$  and  $\text{Sp}_u(T) = \sigma(T) \cap \mathbb{T}$ ; see for instance [20, Example 5.1(i)].

**Continuous operator semigroups.** The second important special case is that of so-called *continuous (operator) semigroups*, where, for some  $n \geq 1$ ,  $S = \mathbb{R}_+^n$  and  $G = \mathbb{R}^n$ , with Lebesgue measure. The character space  $S^*$  of  $S$  can in this case be identified with the region  $\mathbb{C}_-^n$  via the map sending  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}_-^n$  to the element  $\chi_\lambda$  of  $S^*$  given, for  $s = (s_1, \dots, s_n) \in S$ , by

$$\chi_\lambda(s) = \exp\left(\sum_{k=1}^n s_k \lambda_k\right).$$

Here  $\mathbb{C}_- = \{\lambda \in \mathbb{C} : \text{Re } \lambda \leq 0\}$ , and hence  $\Gamma = i\mathbb{R}^n$  under this identification. A natural example of a Følner net in this case is  $(\Omega_\alpha)$  where, for  $\alpha > 0$ ,  $\Omega_\alpha = [0, \alpha]^n$ .

A (bounded) representation of the discrete semigroup  $S = \mathbb{R}_+^n$  corresponds to an  $n$ -tuple of commuting (bounded)  $C_0$ -semigroups  $T_1, \dots, T_n$  on  $X$ . In what follows, the term ‘continuous operator semigroup’ refers in most cases to a bounded representation of the semigroup  $S = \mathbb{R}_+$  and hence is synonymous

with a bounded  $C_0$ -semigroup  $T$  on  $X$ . For any, not necessarily bounded  $C_0$ -semigroup  $T$  on a complex Banach space  $X$ , it follows from the semigroup property and the uniform boundedness theorem that there exist  $M \geq 1$  and  $\omega \in \mathbb{R}$  such that  $\|T(t)\| \leq Me^{\omega t}$  for all  $t \geq 0$ . Thus the *growth bound*  $\omega_0(T)$  of  $T$ , defined as

$$\omega_0(T) = \inf \{ \omega \in \mathbb{R} : \exists M \geq 1 \text{ s.t. } \|T(t)\| \leq Me^{\omega t} \text{ for all } t \geq 0 \}, \quad (2.5.1)$$

satisfies  $\omega_0(T) < \infty$ . Furthermore, associated with any  $C_0$ -semigroup  $T$  is a closed operator  $A$ , known as the *generator* of  $T$ , which is defined for  $x$  in its domain

$$D(A) = \{ x \in X : \lim_{t \rightarrow 0^+} t^{-1}(T(t)x - x) \text{ exists} \},$$

which is dense in  $X$ , by

$$Ax = \lim_{t \rightarrow 0^+} t^{-1}(T(t)x - x).$$

Furthermore, letting

$$s_0(A) = \inf \{ \omega \in \mathbb{R} : H_\omega \subset \rho(A) \text{ and } \sup \{ \|R(\lambda, A)\| : \lambda \in H_\omega \} < \infty \},$$

it follows from the relation

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x \, dt \quad (2.5.2)$$

for  $\operatorname{Re} \lambda > \omega_0(T)$  and  $x \in X$  that  $s_0(A) \leq \omega_0(T)$ .

$C_0$ -semigroups arise naturally in the study of abstract Cauchy problems of the form

$$\begin{cases} \dot{u}(t) = Au(t), & t \geq 0, \\ u(0) = x. \end{cases} \quad (2.5.3)$$

A function  $u : \mathbb{R}_+ \rightarrow X$  is said to be a *classical solution* of (2.5.3) if  $u \in C^1(\mathbb{R}_+; X)$ ,  $u(t) \in D(A)$  for all  $t \geq 0$  and (2.5.3) is satisfied. If a classical solution exists, then in particular  $x \in D(A)$ , which suggests the need for a weaker notion of solution allowing for arbitrary initial values  $x \in X$ . Thus a continuous function  $u : \mathbb{R}_+ \rightarrow X$  is said to be a *mild solution* of (2.5.3) if, for all  $t \geq 0$ ,  $\int_0^t u(s) \, ds \in D(A)$  and

$$A \int_0^t u(s) \, ds = u(t) - x. \quad (2.5.4)$$

If the operator  $A$  appearing in (2.5.3) generates a  $C_0$ -semigroup  $T$ , then the unique mild solution of (2.5.3) is given, for any initial value  $x \in X$  and all

$t \geq 0$ , by  $u(t) = T(t)x$ , and the solution is a classical solution if and only if  $x \in D(A)$ . In this case (2.5.4) becomes

$$A \int_0^t T(s)x \, ds = T(t)x - x,$$

which for  $x \in D(A)$  can be written as

$$\int_0^t T(s)Ax \, ds = T(t)x - x. \quad (2.5.5)$$

For proofs of the above results and further details on  $C_0$ -semigroups and Cauchy problems, see for instance [6, Chapter 3] and [45].

When  $T$  is a  $C_0$ -semigroup with generator  $A$  on a complex Banach space  $X$ , it is well known that  $T'$  is strongly continuous on the subspace  $X^\odot = \overline{D(A')}$  of  $X^*$  and that the generator  $A^\odot$  of the restricted  $C_0$ -semigroup, known as the *sun dual* of  $T$  and denoted by  $T^\odot$ , is the part of  $A'$  in  $X^\odot$ ; see [6, Proposition 3.3.14]. As is shown in [19, Corollary 2.3], it follows from Proposition 2.4.3 and standard results in spectral theory that  $\sigma(A) \subset \text{Sp}(T)$  and that  $\text{Sp}_u(T) = \sigma(A) \cap i\mathbb{R}$ . See [19, Example 2.1], however, for a class of  $C_0$ -semigroups satisfying  $\sigma(A) \neq \text{Sp}(T)$ .

## Part II

# Continuous operator semigroups

## Chapter 3

# A Katznelson-Tzafriri theorem for measures

### 3.1 Introduction

The purpose of this chapter is to establish a version of the Katznelson-Tzafriri theorem for bounded  $C_0$ -semigroups, Theorem 1.1.1, in which the function  $a \in L^1(\mathbb{R}_+)$  is replaced by a bounded Borel measure  $\mu$  on  $\mathbb{R}_+$  and, for an important special choice of  $\mu$ , to estimate the rate at which decay takes place. The starting point for these investigations is the following result; see [78, Proposition 3.1].

**Theorem 3.1.1.** *Let  $X$  be a complex Banach space and let  $T$  be a bounded  $C_0$ -semigroup on  $X$  with generator  $A$ . Suppose that  $\sigma(A) \cap i\mathbb{R} = \{0\}$  and, for  $1 < s \leq 1$  and  $s \geq 1$ , respectively, let*

$$m(s) = \sup \{ \|R(ir, A)\| : s \leq |r| \leq 1 \}$$

and

$$M(s) = \sup \{ \|R(ir, A)\| : 1 \leq |r| \leq s \}.$$

Furthermore, let  $m_{\log}^{-1}$  and  $M_{\log}-1$  denote, respectively, the inverse maps of the functions  $m_{\log}$  and  $M_{\log}$  given, for  $0 < s \leq 1$  and  $s \geq 1$ , by

$$m_{\log}(s) = m(s) \log \left( \frac{1 + m(s)}{s} \right)$$

and

$$M_{\log}(s) = M(s) \log ((1 + M(s))(1 + s)).$$

Then

$$\lim_{t \rightarrow \infty} \|T(t)AR(1, A)^2\| = 0$$

and in fact there exists  $c \in (0, 1)$  such that

$$\|T(t)AR(1, A)^2\| = O\left(m_{\log}^{-1}(ct) + \frac{1}{M_{\log}^{-1}(ct)}\right)$$

as  $t \rightarrow \infty$ .

Notice that the  $t^{-1}$ -term appearing in [78, Proposition 3.1] can be omitted in this context since  $m^{-1}(t) \geq t^{-1}$  and hence  $m_{\log}^{-1}(t) \geq t^{-1}$  by the elementary theory of resolvents (see also [14, Section 8]), and notice furthermore that the first conclusion, namely that  $\|T(t)AR(1, A)^2\| \rightarrow 0$  as  $t \rightarrow \infty$  is a consequence of Theorem 1.1.1. For a related result in the discrete setting, see Theorem 5.2.12.

As discussed in Chapter 1, problems in which  $\sigma(A) \cap i\mathbb{R} = \emptyset$  and the resolvent grows along the imaginary axis, arise in connection with damped wave equations. Partial differential equations leading to  $\sigma(A) \cap i\mathbb{R} = \{0\}$  arise just as naturally. Consider for instance the problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \Delta u &= 0, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial n} + \int_{-\infty}^t a(t-s) \frac{\partial u}{\partial t}(x, s) ds &= 0, & x \in \partial\Omega, t > 0, \end{aligned}$$

where  $\Omega$  is a bounded open subset of  $\mathbb{R}^n$  for some  $n \geq 1$  with sufficiently smooth boundary,  $\frac{\partial u}{\partial n}$  denotes the outward normal derivative and  $a$  denotes a sufficiently well-behaved function defined on  $\mathbb{R}_+$ . If  $u$  is interpreted as acoustic pressure, the equation can be understood as modelling the evolution of sound in a compressible medium with viscoelastic surface. It is shown in [35] and [36] that the associated semigroup generator  $A$  of this problem has unitary spectrum  $\sigma(A) \cap i\mathbb{R} \subset \{0\}$  and is in general non-empty; see also [1].

In the problem of the damped wave equation, a rate of decay for the quantity  $\|T(t)R(1, A)\|$  as  $t \rightarrow \infty$  has the interpretation of a uniform bound on the rate of decay for differentiable orbits, or in other words of classical solutions, which are those solutions of the associated Cauchy problem whose initial values lie in  $D(A)$ . Alternatively, since studying the asymptotic behaviour of  $\|T(t)R(1, A)\|$  as  $t \rightarrow \infty$  is equivalent to studying the asymptotic behaviour of  $\|T(t)A^{-1}\|$  when  $\sigma(A) \cap i\mathbb{R} = \emptyset$  and since the map  $t \mapsto T(t)A^{-1}x$  is the primitive of  $t \mapsto T(t)x$  for any  $x \in X$ , this setting can be interpreted as describing

the asymptotics of primitives of general semigroup orbits. On the other hand, since  $\text{Ran } AR(1, A)^2 = \text{Ran}(A) \cap D(A)$  by Lemma 3.2.1 below, Theorem 3.1.1 can be interpreted as providing a uniform bound on the rate of decay for orbits that are the derivative of an orbit whose initial value lies in  $D(A^2)$ . It is natural to ask, then, whether a similar result can be obtained for orbits that are the derivatives of orbits with initial value in  $D(A)$ . By Lemma 3.2.1 again, this case can be treated by investigating the asymptotic behaviour of  $\|T(t)AR(1, A)\|$  as  $t \rightarrow \infty$ . However, whereas the unquantified part of Theorem 3.1.1 followed from Theorem 1.1.1, this connection is now lost since there is no function  $a \in L^1(\mathbb{R}_+)$  such that  $\widehat{a}(T) = AR(1, A)$ . This raises the need for a version of the Katznelson-Tzafriri theorem in which the function  $a \in L^1(\mathbb{R}_+)$  is replaced by a bounded Borel measure  $\mu$  on  $\mathbb{R}_+$ , and such a result is given in Theorem 3.3.8 below. The crucial additional assumption required for this result is connected with the degree to which  $T$  can be approximated asymptotically by certain analytic operator-valued functions, which is encoded in the so-called *non-analytic growth bound* of  $T$ . With transforms involving bounded Borel measures being defined analogously to the case of integrable functions, Corollary 3.3.4 extends Theorem 3.3.8 in the important example of the measure  $\mu$  for which  $\widehat{\mu}(T) = AR(1, A)$  to give upper and lower bounds on the rate of decay depending on the growth of the resolvent operator near zero. The upper bound turns out to be optimal in some cases but unsatisfactory in others. The present chapter, which is based on [96], is closely related to the material contained in [14, Section 6], and indeed the main objective here is to extend the results given there from the Hilbert space setting to the Banach space setting. All the main results will be presented in Section 3.3, once the required preliminary results have been outlined in Section 3.2.

## 3.2 Background and preliminary results

The following elementary lemma makes it possible to interpret the conclusions of various special cases of Theorem 1.1.1 and its generalisations in terms of certain kinds of orbits of the associated Cauchy problem. The result itself is part of the more general [14, Proposition 3.10(a)].

**Lemma 3.2.1.** *Let  $X$  be a complex Banach space and let  $A$  be the generator of a bounded  $C_0$ -semigroup on  $X$ . Then, for  $k = 1, 2$ ,  $\text{Ran } AR(1, A)^k = \text{Ran}(A) \cap D(A^{k-1})$ .*

PROOF. Since  $\text{Ran}(A) = \text{Ran } AR(1, A)$  and  $D(A^{k-1}) = \text{Ran } R(1, A)^{k-1}$ ,

$$\text{Ran } AR(1, A)^k \subset \text{Ran } AR(1, A) \cap \text{Ran } R(1, A)^{k-1} = \text{Ran}(A) \cap D(A^{k-1}).$$

Conversely, given  $x \in \text{Ran}(A) \cap D(A^{k-1})$  let  $y \in D(A)$  and  $z \in X$  be such that  $x = Ay$  and  $x = R(1, A)^{k-1}z$ . Then  $Ay = R(1, A)^{k-1}z$ , so  $y \in D(A^k) = D((I - A)^k)$ . Letting  $w = (I - A)^ky$ , it follows that  $x = AR(1, A)^kw$  and hence that  $\text{Ran}(A) \cap D(A^{k-1}) \subset \text{Ran } AR(1, A)^k$ , as required.  $\square$

Thus, while Theorem 3.1.1 provides a bound on the rate of decay of orbits starting from elements in  $\text{Ran}(A) \cap D(A)$  that is uniform in a natural sense, uniform rates of decay for orbits starting in  $\text{Ran } A$  may be studied by considering the asymptotic behaviour of  $\|T(t)AR(1, A)\|$  as  $t \rightarrow \infty$ . Indeed, the property  $\|T(t)AR(1, A)\| \rightarrow 0$  as  $t \rightarrow \infty$  is equivalent to

$$\limsup_{t \rightarrow \infty} \{ \|T(t)Ax\| : x \in D(A), \|x\|_A = 1 \} = 0,$$

where  $\|\cdot\|_A$  denotes the graph norm given, for  $x \in D(A)$ , by  $\|x\|_A = \|x\| + \|Ax\|$ . Note that if the supremum is instead taken over the unit sphere of  $X$ , then the semigroup must be eventually differentiable, and then the following result applies; see [7, Theorem 3.10].

**Theorem 3.2.2.** *Let  $X$  be a complex Banach space and let  $T$  be a bounded and eventually differentiable  $C_0$ -semigroup on  $X$  with generator  $A$ . Then*

$$\lim_{t \rightarrow \infty} \|AT(t)\| = 0$$

*if and only if  $\sigma(A) \cap i\mathbb{R} \subset \{0\}$ .*

PROOF. Note first that, since  $T$  is eventually differentiable, there exists  $t_0 > 0$  such that  $T(t_0)X \subset D(A)$ . Then  $T(2t_0)X \subset D(A^2)$  and hence, by the closed graph theorem,  $(I - A)^2T(2t_0) \in \mathcal{B}(X)$ .

Suppose first that  $\sigma(A) \cap i\mathbb{R} \subset \{0\}$ . Since

$$AT(t + 2t_0) = (I - A)^2T(2t_0)T(t)AR(1, A)^2$$

for all  $t \geq 0$ , it follows from Theorem 3.1.1 that  $\|AT(t)\| \rightarrow 0$  as  $t \rightarrow \infty$ .

Conversely, assume that  $\|AT(t)\| \rightarrow 0$  as  $t \rightarrow \infty$  and suppose that  $s \in \mathbb{R}$  is such that  $is \in \sigma(A) \cap i\mathbb{R}$ . Since  $T$  is bounded,  $\sigma(A) \subset \{\lambda \in \mathbb{C} : \text{Re } \lambda \leq 0\}$  and hence  $is \in \sigma_{\text{ap}}(A)$ . It follows that there exists a sequence  $(x_n)_{n \geq 1}$  in  $D(A)$

such that  $\|x_n\| = 1$  for each  $n \geq 1$  and  $\|(A - is)x_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . By (2.5.5) applied to the  $C_0$ -semigroup generated by  $A - is$ ,

$$AT(t)x_n - ise^{ist}x_n = T(t)(A - is)x_n + is \int_0^t e^{-is(r-t)}T(r)(A - is)x_n dr$$

for each  $n \geq 1$  and all  $t \geq 0$ , and hence  $\|AT(t)x_n - ise^{ist}x_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . Thus  $ise^{ist} \in \sigma(AT(t))$  and consequently  $|s| = |ise^{ist}| \leq \|AT(t)\|$  for all  $t \geq t_0$ , where  $t_0 > 0$  is as above. Since  $\|AT(t)\| \rightarrow 0$  as  $t \rightarrow \infty$  by assumption, it follows that  $s = 0$  and therefore that  $\sigma(A) \cap i\mathbb{R} \subset \{0\}$ .  $\square$

**Remark 3.2.3.** The above proof can also be found in [14, Remark 6.3] and is different from the original one given in [7]. Note the argument could be modified to give a quantified result similar to Theorem 3.1.1.

The method used in the second half of the above proof carries over to the setting of interest here and leads to the following simple observation; see [14, Proposition 6.4] for a slightly stronger result.

**Lemma 3.2.4.** *Let  $X$  be a complex Banach space and let  $T$  be a bounded  $C_0$ -semigroup on  $X$  with generator  $A$ . Suppose that*

$$\lim_{t \rightarrow \infty} \|T(t)AR(1, A)\| = 0.$$

*Then  $\sigma(A) \cap i\mathbb{R} \subset \{0\}$ .*

PROOF. Suppose that  $s \in \mathbb{R}$  is such that  $is \in \sigma(A) \cap i\mathbb{R}$ . Then  $is \in \sigma_{\text{ap}}(A)$ , as in the proof of Theorem 3.2.2, so there exists a sequence  $(x_n)_{n \geq 1}$  in  $D(A)$  such that  $\|x_n\| = 1$  for each  $n \geq 1$  and  $\|(A - is)x_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . By (2.5.5),

$$\begin{aligned} ise^{ist}x &= isT(t)x - is \int_0^t e^{-is(r-t)}T(r)(A - is)x dr \\ &= T(t)R(1, A)((1 - is)Ax - (A - is)x) \\ &\quad - is \int_0^t e^{-is(r-t)}T(r)(A - is)x dr \end{aligned} \tag{3.2.1}$$

for all  $x \in D(A)$  and  $t \geq 0$ , and hence

$$\begin{aligned} T(t)AR(1, A)x - \frac{ise^{ist}}{1 - is}x &= \frac{1}{1 - is}T(t)R(1, A)(A - is)x \\ &\quad + \frac{is}{1 - is} \int_0^t e^{-is(r-t)}T(r)(A - is)x dr. \end{aligned}$$

It follows that

$$\left\| T(t)AR(1, A)x_n - \frac{\text{ise}^{\text{ist}}}{1 - \text{is}}x_n \right\| \rightarrow 0$$

for all  $t \geq 0$  as  $n \rightarrow \infty$ . Thus  $(1 - \text{is})^{-1}\text{ise}^{\text{ist}} \in \sigma(T(t)AR(1, A))$  and, in particular,

$$|s| \leq \left| \frac{\text{ise}^{\text{ist}}}{1 - \text{is}} \right| \leq \|T(t)AR(1, A)\|$$

for all  $t \geq 0$ . Letting  $t \rightarrow \infty$  gives  $s = 0$  and hence  $\sigma(A) \cap i\mathbb{R} \subset \{0\}$ .  $\square$

Thus, as in Theorem 3.2.2, the condition  $\sigma(A) \cap i\mathbb{R} \subset \{0\}$  is necessary for  $\|T(t)AR(1, A)\|$  to decay to zero as  $t \rightarrow \infty$ . Sufficiency of this spectral condition is far from clear, however. For while the operator  $AR(1, A)^2$  appearing in the proof of Theorem 3.2.2 can be expressed as  $\widehat{a}(T)$  for the function  $a \in L^1(\mathbb{R}_+)$  given by  $a(t) = (t - 1)e^{-t}$ , so that  $\|T(t)AR(1, A)^2\| \rightarrow 0$  as  $t \rightarrow \infty$  by a simple application of Theorem 1.1.1, the operator  $AR(1, A)$  has no such representation. On the other hand,  $AR(1, A) = \widehat{\mu}(T)$  where  $\mu$  is the bounded Borel measure on  $\mathbb{R}_+$  given by  $\mu = e - \delta_0$ . Here  $\delta_0$  denotes the Dirac mass at 0 and the function  $e$  is defined, for  $t \geq 0$ , by  $e(t) = e^{-t}$ . This observation motivates the search of a version of Theorem 1.1.1 in which the function  $a \in L^1(\mathbb{R}_+)$  is replaced by a bounded Borel measure  $\mu$  on  $\mathbb{R}_+$ . Such a result will be obtained in Theorem 3.3.8 below, whose proof requires an additional assumption involving the so-called *non-analytic growth bound* of the semigroup  $T$ .

Given a complex Banach space  $X$  and a set  $\Omega \subset \mathbb{C}$ , a function  $S : \Omega \rightarrow \mathcal{B}(X)$  will be said to be *exponentially bounded* if there exist constants  $C \geq 0$  and  $\omega \in \mathbb{R}$  such that  $\|S(\lambda)\| \leq Ce^{\omega|\lambda|}$  for all  $\lambda \in \Omega$  and, suppressing the dependence on the Banach space  $X$ , the space of all exponentially bounded holomorphic  $\mathcal{B}(X)$ -valued functions on  $\Omega$  will be denoted by  $H(\Omega)$ . The *non-analytic growth bound*  $\zeta(T)$  of  $T$  is defined as

$$\zeta(T) = \inf \{ \omega_0(T - S) : S \in H(\Sigma_\theta) \text{ for some } \theta > 0 \},$$

where  $S - T$  is understood to be defined on  $(0, \infty)$  and  $\omega_0(T - S)$  is defined as in (2.5.1). Thus the non-analytic growth bound measures the degree to which  $T$  can, or rather *cannot*, be approximated asymptotically by exponentially bounded analytic functions defined on certain sectors. It is clear that  $\zeta(T) \leq \omega_0(T)$  for any  $C_0$ -semigroup  $T$  and that  $\zeta(T) = -\infty$  when  $T$  is analytic. It is shown in [12, Theorem 5.7] that  $\zeta(T) = -\infty$  also if  $T$  is eventually differentiable or if  $T$  has an  $L^p$ -resolvent for some  $p \in (1, \infty)$ , in the sense that

there exist  $\alpha \in \mathbb{R}$  and  $\beta \geq 0$  such that  $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq \alpha, |\operatorname{Im} \lambda| \geq \beta\} \subset \rho(A)$  and

$$\int_{|s| \geq \beta} \|R(\alpha + is, A)\|^p ds < \infty.$$

For further discussion of the non-analytic growth bound, see also Chapter 4.

Given  $\alpha \in \mathbb{R}$  and  $\beta \geq 0$ , let  $Q_{\alpha, \beta} = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq \alpha, |\operatorname{Im} \lambda| \geq \beta\}$  and, for any semigroup generator  $A$ , let

$$s_0^\infty(A) = \inf \left\{ \alpha \in \mathbb{R} : Q_{\alpha, \beta} \subset \rho(A) \text{ and } \|R(\lambda, A)\| \text{ is uniformly bounded on } Q_{\alpha, \beta} \text{ for some } \beta \geq 0 \right\}.$$

It is shown in [12, Proposition 2.4] that  $s_0^\infty(A) \leq \zeta(T)$ . When  $X$  is a Hilbert space, the following non-analytic analogue of the Gearhart-Prüss theorem holds, see [17, Example 3.12].

**Theorem 3.2.5.** *Let  $X$  be a complex Hilbert space and let  $T$  be a  $C_0$ -semigroup on  $X$  with generator  $A$ . Then  $s_0^\infty(A) = \zeta(T)$ .*

The proof relies on Plancherel's theorem and the following result, which will be crucial in what follows; see [17, Theorem 3.6] for a proof. Here  $L_s^1(\mathbb{R}; \mathcal{B}(X))$  denotes the space of maps  $S : \mathbb{R} \rightarrow \mathcal{B}(X)$  such that  $t \mapsto S(t)x$  is Bochner measurable for all  $x \in X$  and such that there exists  $g \in L^1(\mathbb{R})$  with  $\|S(t)\| \leq g(t)$  for almost all  $t \in \mathbb{R}$ . Furthermore, given a semigroup generator  $A$  satisfying  $s_0^\infty(A) < 0$ , a smooth function  $\phi \in C^\infty(\mathbb{R})$  will be said to be *compatible with  $A$*  if there exists a bounded open interval  $I \subset \mathbb{R}$  satisfying  $\{s \in \mathbb{R} : is \in \sigma(A)\} \subset I$  and if  $\phi(s) = 0$  for all  $s \in I$  and  $\phi(s) = 1$  when  $|s|$  is sufficiently large.

**Theorem 3.2.6.** *Let  $X$  be a complex Banach space and let  $T$  be a  $C_0$ -semigroup on  $X$  with generator  $A$ . Suppose that  $1 \leq p < \infty$ . Then the following are equivalent:*

- (i)  $\zeta(T) < 0$ ;
- (ii)  $s_0^\infty(A) < 0$  and, for any compatible function  $\phi \in C^\infty(\mathbb{R})$ , there exists a map  $S \in L_s^1(\mathbb{R}; \mathcal{B}(X))$  such that  $\phi(s)R(is, A) = (\mathcal{F}S)(s)$  for all  $s \in \mathbb{R}$ ;
- (iii)  $s_0^\infty(A) < 0$  and, for any compatible function  $\phi \in C^\infty(\mathbb{R})$ , the map  $s \mapsto \phi(s)R(is, A)$  is a Fourier multiplier on  $L^p(\mathbb{R}; X)$ .

Here  $\phi(s)R(is, A)$  is taken to be zero whenever  $\phi(s) = 0$  and the Fourier transform  $\mathcal{F}$  is taken in the strong operator topology, so that, given  $S \in$

$L^1_{\mathbb{S}}(\mathbb{R}; \mathcal{B}(X))$  and  $s \in \mathbb{R}$ ,

$$(\mathcal{F}S)(s)x = \int_{\mathbb{R}} e^{-ist} S(t)x \, dt$$

for all  $x \in X$ . Note also that, even though the maps  $s \mapsto \phi(s)R(is, A)$  depend on the choice of  $\phi$ , the question whether  $s \mapsto \phi(s)R(is, A)$  is a Fourier multiplier on  $L^p(\mathbb{R}; X)$  is independent of this choice; see [17, Remark 2.2].

### 3.3 Main results

Let  $T$  be a bounded  $C_0$ -semigroup with generator  $A$  on a Banach space  $X$ . The main aims of this section are to establish a sufficient condition for the quantity  $\|T(t)AR(1, A)\|$  to converge to zero as  $t \rightarrow \infty$  and to study the possible rates at which this decay takes place. By Lemma 3.2.4, it suffices to restrict attention to the case where  $\sigma(A) \cap i\mathbb{R} \subset \{0\}$ . In fact, the only case of real interest is when  $\sigma(A) \cap i\mathbb{R} = \{0\}$ , as will be explained in Remark 3.3.13 below. Given  $C_0$ -semigroups  $T$  whose generator  $A$  satisfies  $\sigma(A) \cap i\mathbb{R} = \{0\}$ , it will be convenient to have in place a few non-standard pieces of notation. Thus a decreasing function  $m : (0, 1] \rightarrow (0, \infty)$  such that  $\|R(is, A)\| \leq m(|s|)$  for  $s$  with  $0 < |s| \leq 1$  will be said to be a *dominating function (for the resolvent of  $A$ )*. Likewise a decreasing function  $\omega : \mathbb{R}_+ \rightarrow (0, \infty)$  such that  $\|T(t)AR(1, A)\| \leq \omega(t)$  for all  $t \in \mathbb{R}_+$  will be said to be a *dominating function (for  $T$ )*. The *minimal* dominating functions are given, for  $s \in (0, 1]$  and  $t \geq 0$ , by

$$\begin{aligned} m(s) &= \sup \{ \|R(ir, a)\| : s \leq |r| \leq 1 \}, \\ \omega(t) &= \sup \{ \|T(s)AR(1, A)\| : s \geq t \}, \end{aligned} \tag{3.3.1}$$

respectively. The function  $m$  defined in (3.3.1) is continuous and in what follows the same will be assumed to be true of any dominating function  $m$ . In particular, any such dominating function  $m$  possesses a right-inverse  $m^{-1}$  defined on the range of  $m$ . On the other hand, given a dominating function  $\omega$  for  $T$  such that  $\omega(t) \rightarrow 0$  as  $t \rightarrow \infty$ , define the function  $\omega^* : (0, \infty) \rightarrow \mathbb{R}_+$  by

$$\omega^*(s) = \min \{ t \geq 0 : \omega(t) \leq s \}. \tag{3.3.2}$$

Then  $\omega(\omega^*(s)) \leq s$  for all  $s > 0$ , with equality for all  $s$  in the range of  $\omega$ .

Given a bounded  $C_0$ -semigroup  $T$  on a Banach space  $X$  whose generator  $A$  satisfies  $\sigma(A) \cap i\mathbb{R} = \{0\}$ , it follows from the elementary properties of resolvent operators that  $m(s) \geq s^{-1}$  for any dominating function  $m$  for the resolvent of  $A$ .

As is shown in [14, Example 6.7] by means of a simple direct sum construction, dominating functions  $\omega$  for  $T$  need not satisfy any such lower bound and indeed can decay arbitrarily slowly, even for semigroups of normal operators on Hilbert space. The following result, on the other hand, shows that cases in which a dominating function  $\omega$  for  $T$  decays faster than  $t^{-1}$  must be precisely of this type involving a direct sum. The proof, which relies on a simple spectral splitting argument combined with a result from [61], can be found in [14, Theorem 6.9]; for an analogous result in the discrete setting, see Theorem 5.2.1.

**Theorem 3.3.1.** *Let  $X$  be a complex Banach space and let  $T$  be a bounded  $C_0$ -semigroup on  $X$  with generator  $A$ . Suppose that  $\sigma(A) \cap i\mathbb{R} = \{0\}$ . Then either*

$$\limsup_{t \rightarrow \infty} t \|T(t)AR(1, A)\| > 0$$

*or there exist closed  $T$ -invariant subspaces  $X_0$  and  $X_1$  of  $X$  such that  $X_0 \subset \text{Fix}(T)$ , the restriction  $A_1$  of  $A$  to  $X_1$  is invertible, and  $X = X_0 \oplus X_1$ .*

The following result provides an estimate on the growth of  $\|R(is, A)\|$  for small and large values of  $s$ . It is taken from [14, Theorem 6.10].

**Theorem 3.3.2.** *Let  $X$  be a complex Banach space and let  $T$  be a bounded  $C_0$ -semigroup on  $X$  with generator  $A$ . Suppose  $\omega$  is a dominating function for  $T$  such that  $\omega(t) \rightarrow 0$  as  $t \rightarrow \infty$ , and let  $\omega^*$  be defined as in (3.3.2). Then  $\sigma(A) \cap i\mathbb{R} \subset \{0\}$ ,  $\sup\{\|R(is, A)\| : |s| \geq 1\} < \infty$  and, for any  $c \in (0, 1)$ ,*

$$\|R(is, A)\| = O\left(\frac{1}{|s|} + \omega^*(cs)\right)$$

as  $s \rightarrow 0$ .

PROOF. The fact that  $\sigma(A) \cap i\mathbb{R} \subset \{0\}$  follows from Lemma 3.2.4. Let  $s \in \mathbb{R} \setminus \{0\}$ , so that  $is \in \rho(A)$ . Since  $\|T(t)R(1, A)\| \leq M$  for all  $t \geq 0$ , where  $M = \sup\{\|T(t)\| : t \geq 0\}$ , taking norms in (3.2.1) gives

$$|s|\|x\| \leq |1 - is|\omega(t)\|x\| + M(1 + |s|t)\|(is - A)x\| \quad (3.3.3)$$

for all  $x \in D(A)$  and all  $t \geq 0$ . Since  $\omega(t) \rightarrow 0$  as  $t \rightarrow \infty$ , there exists  $t_0 > 0$  such that  $\omega(t_0) \leq \frac{1}{3}$ . For  $|s| \geq 1$ , setting  $t = t_0$  in (3.3.3) shows that

$$|s|\|x\| \leq \frac{2}{3}|s|\|x\| + M(1 + |s|t_0)\|(is - A)x\|,$$

and hence  $\|x\| \leq 3M(1 + t_0)\|(is - A)x\|$  for all  $x \in D(A)$ . It follows that  $\sup\{\|R(is, A)\| : |s| \geq 1\} < \infty$ . On the other hand, given  $c \in (0, 1)$ , let  $t = \omega^*(c|s|)$ . Since  $|1 - is|c < 1$  when  $|s|$  is sufficiently small, (3.3.3) gives

$$\|x\| \leq \frac{M(1 + |s|\omega^*(c|s|))}{|s|(1 - |1 - is|c)} \|(is - A)x\|$$

for all such values of  $s$  and all  $x \in D(A)$ . Thus, given any  $K > M$ ,

$$\|R(is, A)\| \leq \frac{K}{1 - c} \left( \frac{1}{|s|} + \omega^*(c|s|) \right)$$

whenever  $|s|$  is sufficiently small, and the result follows.  $\square$

**Remark 3.3.3.** A similar argument shows that, given any constant  $K > M$ , where  $M$  is as above, there exists  $c \in (0, 1)$  such that

$$\|R(is, A)\| \leq K \left( \frac{1}{|s|} + \omega^*(c|s|) \right) \quad (3.3.4)$$

whenever  $|s|$  is sufficiently small.

Theorem 3.3.2 is an analogue of [15, Proposition 1.3] and provides the following bound on how fast the quantity  $\|T(t)AR(1, A)\|$  can decay as  $t \rightarrow \infty$ , which is taken from [14, Corollary 6.11].

**Corollary 3.3.4.** *Let  $X$  be a complex Banach space and let  $T$  be a bounded  $C_0$ -semigroup on  $X$  with generator  $A$ . Suppose that  $\sigma(A) \cap i\mathbb{R} = \{0\}$  and that*

$$\lim_{s \rightarrow 0} \max \{ \|sR(is, A)\|, \|sR(-is, A)\| \} = \infty, \quad (3.3.5)$$

and let  $m$  be the minimal dominating function for the resolvent of  $A$  defined in (3.3.1). Then, given any right-inverse  $m^{-1}$  of  $m$ , there exist constants  $c, C > 0$  such that

$$\|T(t)AR(1, A)\| \geq cm^{-1}(Ct) \quad (3.3.6)$$

for all sufficiently large  $t \geq 0$ .

PROOF. The result is elementary unless  $\|T(t)AR(1, A)\| \rightarrow 0$  as  $t \rightarrow \infty$ . Assume, therefore, that this holds and let  $\omega$  be as defined in (3.3.1). Then  $\omega(t) \rightarrow 0$  as  $t \rightarrow \infty$  and it follows from Theorem 3.3.2 that there exists  $B > 0$  such that

$$m(s) \leq B \left( \frac{1}{s} + \omega^*(s/2) \right),$$

and hence

$$\omega^*(s/2) \geq m(s) \left( \frac{1}{B} - \frac{1}{sm(s)} \right)$$

for all  $s \in (0, 1]$ . Letting  $C = 2B$ , it follows from (3.3.5) that  $\omega^*(s/2) > C^{-1}m(s)$  for all sufficiently small  $s > 0$ . Let  $s = 2\omega(t)$ . Then  $\omega^*(s/2) \leq t$  and hence  $m(s) < Ct$  for all sufficiently large  $t \geq 0$ . It follows that  $m(m^{-1}(Ct)) > m(s)$  so, as  $m$  is decreasing,  $2\omega(t) > m^{-1}(Ct)$  provided  $t \geq 0$  is sufficiently large. Since  $\omega(t) \leq M\|T(t)AR(1, A)\|$ , where  $M = \sup\{\|T(t)\| : t \geq 0\}$ , the result follows with  $c = (2M)^{-1}$ .  $\square$

**Remark 3.3.5.** A similar argument using Remark 3.3.3 instead of Theorem 3.3.2 shows that the conclusion (3.3.4) remains true if (3.3.5) is replaced by the weaker condition that  $L > M$ , where  $M$  is as above and

$$L = \liminf_{s \rightarrow 0} \max \{ \|sR(is, A)\|, \|sR(-is, A)\| \}.$$

Taking  $A = 0$  shows that the conclusion can become false when  $L = M$ .

In view of (3.3.6), it is natural to ask whether there exists a corresponding upper bound in terms of  $m^{-1}$  for the quantity  $\|T(t)AR(1, A)\|$  as  $t \rightarrow \infty$ . The next result shows that it is not necessarily possible even for semigroups of normal operators on a Hilbert space; see [14, Theorem 6.13] for a proof and Proposition 5.2.8 below for an analogous result in the discrete case.

**Theorem 3.3.6.** *Let  $X$  be a complex Hilbert space and let  $T$  be a bounded  $C_0$ -semigroup of normal operators on  $X$  with generator  $A$ . Suppose that  $\sigma(A) \cap i\mathbb{R} = \{0\}$  and that  $\sup\{\|R(is, A)\| : |s| \geq 1\} < \infty$ . Furthermore, let  $m$  be the minimal dominating function for the resolvent of  $A$  and let  $m^{-1}$  be any right-inverse of  $m$ . Then, given any constant  $c > 0$ ,*

$$\|T(t)AR(1, A)\| \leq O(m^{-1}(ct))$$

as  $t \rightarrow \infty$  if and only if there exists a constant  $C > 0$  such that

$$\frac{m(s)}{m(t)} \geq c \log \left( \frac{t}{s} \right) - C$$

for  $0 < s, t \leq 1$ .

An upper bound for  $\|T(t)AR(1, A)\|$  as  $t \rightarrow \infty$  in a rather more general setting will be obtained in Theorem 3.3.11. This result is a refinement of the Theorem 3.3.8 below, whose proof relies on the following lemma.

**Lemma 3.3.7.** *Let  $X$  be a complex Banach space and let  $S \in L^1(\mathbb{R}; X)$ .*

(a) *If  $\mu$  is a bounded Borel measure on  $\mathbb{R}$ , then  $\mu * S \in L^1(\mathbb{R}; X)$ .*

(b) *If*

$$\int_{\mathbb{R}} \rho(t) S(t) dt = 0 \tag{3.3.7}$$

*for all  $\rho \in \mathcal{S}(\mathbb{R})$ , then  $S(t) = 0$  for almost all  $t \in \mathbb{R}$ .*

PROOF. Part (a) is a special case of the vector-valued version of Fubini's theorem; see for instance [6, Theorem 1.1.9] and [39, Chapter III, Section 11, Theorem 9]. For a direct proof, let the map  $F : \mathbb{R}^2 \rightarrow X$  be given by  $F(s, t) = S(t-s)$  and note that, by Pettis' theorem (see [6, Theorem 1.1.1]), there is no loss of generality in assuming that  $X$  is separable. Moreover, the map  $F$  is measurable with respect to the product measure of  $\mu$  and the Lebesgue measure on  $\mathbb{R}$ , as can be seen for instance by another application of Pettis' theorem and a simple approximation argument based on the fact that, given any Lebesgue measurable subset  $E$  of  $\mathbb{R}$ , there exists a Borel measurable set  $E' \subset \mathbb{R}$  such that the symmetric difference  $E \Delta E'$  is null. Since

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \|F(s, t)\| dt d|\mu|(s) < \infty, \tag{3.3.8}$$

it follows from the scalar-valued versions of Tonelli's and Fubini's theorems that  $(s, t) \mapsto \|F(s, t)\|$  is integrable over  $\mathbb{R}^2$  with respect to the product measure and that  $\int_{\mathbb{R}} \|F(s, t)\| d\mu(s)$  exists for almost all  $t \in \mathbb{R}$ . Hence so does  $\int_{\mathbb{R}} F(s, t) d\mu(s)$ . Furthermore, for each  $\phi \in X^*$ , the map

$$t \mapsto \int_{\mathbb{R}} \phi(F(s, t)) d\mu(s) = \phi \left( \int_{\mathbb{R}} F(s, t) d\mu(s) \right)$$

is measurable on  $\mathbb{R}$ . By Pettis' theorem, the map  $\mu * S : t \mapsto \int_{\mathbb{R}} F(s, t) d\mu(s)$  is Bochner measurable on  $\mathbb{R}$ , and the result now follows, since

$$\int_{\mathbb{R}} \|(\mu * S)(t)\| dt \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \|F(s, t)\| d|\mu|(s) dt < \infty$$

by the scalar-valued version of Fubini's theorem and (3.3.8).

The proof of (b) runs along similar lines. Once again, by Pettis' theorem, it is possible to assume that  $X$  is separable. Let  $\{x_n : n \geq 1\}$  be a dense subset of  $X$  and, for each  $n \geq 1$ , let  $\phi_n \in X^*$  be a Hahn-Banach functional

corresponding to  $x_n$ , so that  $\|\phi_n\| = 1$  and  $\phi_n(x_n) = \|x_n\|$ . Applying  $\phi_n$  to both sides of (3.3.7) shows that

$$\int_{\mathbb{R}} \rho(t) \phi_n(S(t)) dt = 0$$

for all  $\rho \in \mathcal{S}(\mathbb{R})$ , and hence, for each  $n \geq 1$ , there exists a null subset  $E_n$  of  $\mathbb{R}$  such that  $\phi_n(S(t)) = 0$  for all  $t \in \mathbb{R} \setminus E_n$ . Let  $E = \bigcup_{n \geq 1} E_n$ , noting that  $E$  itself is null, and suppose that  $t \in \mathbb{R} \setminus E$ . Given  $\varepsilon > 0$ , there exists  $n \geq 1$  such that  $\|x_n - S(t)\| < \varepsilon/2$ . Thus

$$\|S(t)\| \leq \|x_n\| + \|x_n - S(t)\| < \phi_n(x_n - S(t)) + \frac{\varepsilon}{2} < \varepsilon.$$

Since  $\varepsilon > 0$  was arbitrary, it follows that  $S(t) = 0$  for all  $t \in \mathbb{R} \setminus E$ .  $\square$

The next result is an unquantified version of the Katznelson-Tzafriri theorem for measures which generalises [14, Theorem 6.14] to the Banach space setting. The main new idea is to use Theorem 3.2.6. Recall that, in the notation of Section 2.4.3, a closed subset  $\Lambda$  of  $i\mathbb{R}$  is said to be of *spectral synthesis* if  $K_\Lambda(\mathbb{R}) = J_\Lambda(\mathbb{R})$ .

**Theorem 3.3.8.** *Let  $X$  be a complex Banach space and let  $T$  be a bounded  $C_0$ -semigroup on  $X$  with generator  $A$ . Suppose that  $\sigma(A) \cap i\mathbb{R}$  is of spectral synthesis and that  $\zeta(T) < 0$ , and let  $\mu$  be any bounded Borel measure on  $\mathbb{R}_+$  such that  $\hat{\mu}(\lambda) = 0$  for all  $\lambda \in \sigma(A) \cap i\mathbb{R}$ . Then*

$$\lim_{t \rightarrow \infty} \|T(t)\hat{\mu}(T)\| = 0.$$

PROOF. Since  $s_0^\infty(A) \leq \zeta(T)$ , the assumption  $\zeta(T) < 0$  implies that  $\{is : |s| \geq R\} \subset \rho(A)$  and that the set  $\{R(is, A) : |s| \geq R\}$  is uniformly norm-bounded for sufficiently large  $R > 0$ . Fix a function  $\varphi \in C_c^\infty(\mathbb{R})$  such that  $\varphi(s) = 1$  for  $|s| \leq R$  and let  $\psi = \mathcal{F}^{-1}\varphi$ . Then  $\psi \in \mathcal{S}(\mathbb{R})$ , since  $C_c^\infty(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$  and  $\mathcal{F}$  maps  $\mathcal{S}(\mathbb{R})$  bijectively onto itself. Furthermore, let the bounded Borel measures  $\nu$  and  $\xi$  on  $\mathbb{R}$  be defined by  $\nu = \mu * \psi$  and  $\xi = \mu * (\delta_0 - \psi)$ , so that  $\mu = \nu + \xi$ , and let the functions  $F, G : \mathbb{R} \rightarrow \mathcal{B}(X)$  be given by

$$F(t) = \int_{\mathbb{R}} T(s+t) d\nu(s) \quad \text{and} \quad G(t) = \int_{\mathbb{R}} T(s+t) d\xi(s).$$

Here the semigroup has been extended by zero to  $(-\infty, 0)$  and the integrals are to be understood in the strong sense. It is clear that  $T(t)\hat{\mu}(T) = F(t) + G(t)$  for all  $t \geq 0$ .

Since the measure  $\nu$  is absolutely continuous with respect to Lebesgue measure on  $\mathbb{R}$ , there exists a function  $a \in L^1(\mathbb{R})$  such that  $d\nu(t) = a(t)dt$ , and hence  $(\mathcal{F}a)(s) = (\mathcal{F}\mu)(s)\varphi(s)$  for all  $s \in \mathbb{R}$ . Since  $(\mathcal{F}\mu)(s) = \widehat{\mu}(-is)$  for all  $s \in \mathbb{R}$ ,  $\mathcal{F}a$  vanishes on  $i\sigma(A) \cap \mathbb{R}$ . By assumption this set is of spectral synthesis, and hence there exist functions  $a_n \in L^1(\mathbb{R})$ ,  $n \geq 1$ , such that  $\mathcal{F}a_n$  vanishes in an open neighbourhood of  $i\sigma(A) \cap \mathbb{R}$  for each  $n \geq 1$  and  $\|a_n - a\|_1 \rightarrow 0$  as  $n \rightarrow \infty$ . Since functions with compactly supported Fourier transform are dense in  $L^1(\mathbb{R})$ , there is no loss of generality in assuming that each  $\mathcal{F}a_n$  has compact support. By the dominated convergence theorem and Parseval's identity,

$$\begin{aligned} \int_{\mathbb{R}} a_n(s)T(t+s) ds &= \lim_{\alpha \rightarrow 0^+} \int_{\mathbb{R}} a_n(s-t)e^{-\alpha s}T(s) ds \\ &= \lim_{\alpha \rightarrow 0^+} \frac{1}{2\pi} \int_{\mathbb{R}} e^{ist}(\mathcal{F}a_n)(-s)R(\alpha + is, A) ds \quad (3.3.9) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{ist}(\mathcal{F}a_n)(-s)R(is, A) ds \end{aligned}$$

for all  $t \in \mathbb{R}$  and  $n \geq 1$ . Since the last integral exists as a Bochner integral in  $\mathcal{B}(X)$ ,

$$\left\| \int_{\mathbb{R}} a_n(s)T(t+s) ds \right\| \rightarrow 0$$

as  $t \rightarrow \infty$  by the Riemann-Lebesgue Lemma. Now

$$\left\| F(t) - \int_{\mathbb{R}} a_n(s)T(t+s) ds \right\| \leq M\|a - a_n\|_1$$

for all  $t \in \mathbb{R}$ , and hence

$$\left\| F(t) - \int_{\mathbb{R}} a_n(s)T(t+s) ds \right\| \rightarrow 0 \quad (3.3.10)$$

as  $n \rightarrow \infty$ , uniformly for  $t \in \mathbb{R}$ . It follows that  $\|F(t)\| \rightarrow 0$  as  $t \rightarrow \infty$ .

Now consider the function  $G$ . Since the map  $\phi \in C^\infty(\mathbb{R})$  given by  $\phi(s) = 1 - \varphi(-s)$  is compatible with  $A$ , it follows from Theorem 3.2.6 that there exists a map  $S \in L_s^1(\mathbb{R}; \mathcal{B}(X))$  such that  $\phi(s)R(is, A) = (\mathcal{F}S)(s)$  for all  $s \in \mathbb{R}$ . Writing  $\mu'$  for the bounded Borel measure on  $(-\infty, 0]$  satisfying  $\mu'(E) = \mu(-E)$  for any Borel subset  $E$  of  $(-\infty, 0]$ , part (a) of Lemma 3.3.7 shows that  $\mu' * S \in L_s^1(\mathbb{R}; \mathcal{B}(X))$ . For  $\rho \in \mathcal{S}(\mathbb{R})$  and  $x \in X$ , it follows by Fubini's theorem, the

dominated convergence theorem and Parseval's identity that

$$\begin{aligned}
\int_{\mathbb{R}} \rho(t)G(t)x \, dt &= \int_{\mathbb{R}} \rho(t) \int_{\mathbb{R}} T(s+t)x \, d\xi(s) \, dt \\
&= \lim_{\alpha \rightarrow 0^+} \int_{\mathbb{R}} \int_{\mathbb{R}} \rho(t-s)e^{-\alpha t}T(t)x \, dt \, d\xi(s) \\
&= \lim_{\alpha \rightarrow 0^+} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ist}(\mathcal{F}^{-1}\rho)(t)R(\alpha+it, A)x \, dt \, d\xi(s) \\
&= \lim_{\alpha \rightarrow 0^+} \int_{\mathbb{R}} (\mathcal{F}^{-1}\rho)(t)(\mathcal{F}\mu)(-t)(1-\varphi(-t))R(\alpha+it, A)x \, dt \\
&= \int_{\mathbb{R}} (\mathcal{F}^{-1}\rho)(t)(\mathcal{F}\mu)(-t)(\mathcal{F}S)(t)x \, dt \\
&= \int_{\mathbb{R}} \rho(t)S_{\mu}(t)x \, dt,
\end{aligned}$$

where  $S_{\mu} = \mu' * S$ . Fix  $x \in X$  and, for  $t \in \mathbb{R}$ , let  $S_x \in L^1(\mathbb{R}; X)$  be given by  $S_x(t) = G(t)x - S_{\mu}(t)x$ . Then  $S_x(t) = 0$  for almost all  $t \in \mathbb{R}$  by part (b) of Lemma 3.3.7, and hence  $G(\cdot)x \in L^1(\mathbb{R}; X)$ . Since the map  $\Phi : X \rightarrow L^1(\mathbb{R}; X)$  given by  $\Phi(x) = G(\cdot)x$  has closed graph, it follows from the closed graph theorem that, for some constant  $C > 0$ ,

$$\int_{\mathbb{R}} \|G(t)x\| \, dt \leq C\|x\| \quad (3.3.11)$$

for all  $x \in X$ . Now, given any  $x \in X$  and  $t \geq 0$ ,

$$\begin{aligned}
\int_0^t T(t-s)G(s)x \, ds &= \int_0^t \left( \int_{\mathbb{R}} - \int_{[-t, -s]} \right) T(t+r)x \, d\xi(r) \, ds \\
&= tG(t)x + \int_{[-t, 0)} rT(t+r)x \, d\xi(r)
\end{aligned}$$

and hence, for  $t > 0$ ,

$$G(t)x = \frac{1}{t} \left( \int_0^t T(t-s)G(s)x \, ds - \int_{[-t, 0)} rT(t+r)x \, d\xi(r) \right). \quad (3.3.12)$$

By (3.3.11),

$$\left\| \int_0^t T(t-s)G(s) \, ds \right\| \leq MC, \quad (3.3.13)$$

where  $M = \sup\{\|T(t)\| : t \geq 0\}$ , and the dominated convergence theorem gives

$$\left\| \int_{[-t, 0)} \frac{r}{t} T(t+r) \, d\xi(r) \right\| \rightarrow 0$$

as  $t \rightarrow \infty$ . Thus  $\|G(t)\| \rightarrow 0$  as  $t \rightarrow \infty$  and the result follows.  $\square$

**Remark 3.3.9.** As observed in [17, Remark 3.8], the map  $S \in L_s^1(\mathbb{R}; \mathcal{B}(X))$  in the above proof can be written down explicitly. Indeed, given  $x \in X$ ,

$$S(t)x = T(t)x - T_R(t)x - \frac{1}{2\pi i} \int_{|s| \geq R} e^{ist} \varphi(-s) R(is, A)x \, ds$$

for almost all  $t \in \mathbb{R}$ . Here  $\varphi$  and  $R$  are as above, the semigroup  $T$  has again been extended by zero to  $(-\infty, 0)$  and

$$T_R(t) = \frac{1}{2\pi i} \int_{\Gamma_R} e^{\lambda t} R(\lambda, A) \, d\lambda$$

for any path  $\Gamma_R$  that connects the points  $\pm iR$  and otherwise lies in  $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0\}$ . By Cauchy's theorem, the definition of  $T_R$  is independent of the precise choice of the contour  $\Gamma_R$ .

As discussed in Section 3.2, the condition  $\zeta(T) < 0$  is satisfied automatically in a number of important instances. The first part of the following result corresponds to [14, Theorem 6.14].

**Corollary 3.3.10.** *Let  $X$  be a complex Banach space and let  $T$  be a bounded  $C_0$ -semigroup  $X$  with generator  $A$ . Suppose that  $\sigma(A) \cap i\mathbb{R}$  is of spectral synthesis, and let  $\mu$  be any bounded Borel measure on  $\mathbb{R}_+$  such that  $\widehat{\mu}(\lambda) = 0$  for all  $\lambda \in \sigma(A) \cap i\mathbb{R}$ . Then*

$$\lim_{t \rightarrow \infty} \|T(t)\widehat{\mu}(T)\| = 0$$

provided one of the following conditions is satisfied:

- (i)  $X$  is a Hilbert space and  $\sup\{\|R(is, A)\| : |s| \geq R\} < \infty$  for sufficiently large  $R > 0$ ;
- (ii) There exists  $t > 0$  and a compact operator  $K \in \mathcal{B}(X)$  such that  $\|T(t) - K\| < 1$ ;
- (iii)  $T$  has  $L^p$ -resolvent for some  $p \in (1, \infty)$ ;
- (iv)  $T$  is eventually differentiable.

PROOF. The first statement follows from Theorem 3.2.5. Indeed, since the assumption on the resolvent implies that  $s_0^\infty(A) < 0$  by a simple Neumann series argument, the result is immediate. The second statement follows immediately from [12, Proposition 5.3], and the final two statements are consequences of the observations made in Section 3.2.  $\square$

The final theorem of this chapter is a quantified version of Theorem 3.3.8 in the special case where  $\widehat{\mu}(T) = AR(1, A)$ . This result is a generalisation to the Banach space setting of [14, Theorem 6.15] and provides a counterpart to Corollary 3.3.4.

**Theorem 3.3.11.** *Let  $X$  be a complex Banach space and let  $T$  be a bounded  $C_0$ -semigroup  $X$  with generator  $A$ . Suppose that  $\sigma(A) \cap i\mathbb{R} = \{0\}$  and that  $\zeta(T) < 0$ . Then*

$$\lim_{t \rightarrow \infty} \|T(t)AR(1, A)\| = 0. \quad (3.3.14)$$

*In fact, given any dominating function  $m$  for the resolvent of  $A$ , any right-inverse  $m^{-1}$  of  $m$  and any  $\varepsilon \in (0, 1)$ ,*

$$\|T(t)AR(1, A)\| = O(m^{-1}(t^{1-\varepsilon})) \quad (3.3.15)$$

as  $t \rightarrow \infty$ .

PROOF. Since the set  $\{0\}$  is of spectral synthesis, the first statement follows immediately from Theorem 3.3.8 applied to the bounded Borel measure on  $\mathbb{R}_+$  given by  $\mu = e - \delta_0$ , whose Fourier-Laplace transform satisfies  $\widehat{\mu}(\lambda) = \lambda(1-\lambda)^{-1}$  for  $\operatorname{Re} \lambda \leq 0$ .

The quantified statement follows from a modification of the argument given in the proof of Theorem 3.3.8. Fix a function  $\varphi \in C_c^\infty(\mathbb{R})$  such that  $\varphi(s) = 1$  for  $|s| \leq 2$  and let  $\psi \in \mathcal{S}(\mathbb{R})$  be given by  $\psi = \mathcal{F}^{-1}\varphi$ . As in the proof of Theorem 3.3.8, let the bounded Borel measures  $\nu$  and  $\xi$  on  $\mathbb{R}$  be defined by  $\nu = \mu * \psi$  and  $\xi = \mu * (\delta_0 - \psi)$ . Then  $\mu = \nu + \xi$  and  $T(t)\widehat{\mu}(T) = F(t) + G(t)$  for all  $t \geq 0$ , where  $F, G : \mathbb{R} \rightarrow \mathcal{B}(X)$  are given by

$$F(t) = \int_{\mathbb{R}} T(s+t) d\nu(s) \quad \text{and} \quad G(t) = \int_{\mathbb{R}} T(s+t) d\xi(s).$$

As before, the measure  $\nu$  is absolutely continuous with respect to Lebesgue measure on  $\mathbb{R}$  and hence there exists a function  $a \in L^1(\mathbb{R})$  such that  $d\nu(t) = a(t)dt$ . In fact,

$$a(t) = \int_0^\infty \psi(t-s)e^{-s} ds - \psi(t)$$

for almost all  $t \in \mathbb{R}$ . Since  $\mu$  is supported in  $\mathbb{R}_+$ ,  $\xi$  coincides with  $-\nu$  on

$(-\infty, 0)$ . In particular,

$$\begin{aligned} \sup_{t \geq 0} \left\| \int_{[-t, 0)} rT(t+r) d\xi(r) \right\| &\leq M \int_{(-\infty, 0)} |r| d|\xi|(r) \\ &= M \int_{-\infty}^0 |r| \left| \psi(r) - \int_0^\infty \psi(r-s)e^{-s} ds \right| dr \\ &< \infty, \end{aligned}$$

where  $M = \sup\{\|T(t)\| : t \geq 0\}$ . By (3.3.12) and (3.3.13), which are consequences Theorem 3.2.6 and the assumption that  $\zeta(T) < 0$ ,

$$\|G(t)\| = O(t^{-1}) \quad (3.3.16)$$

as  $t \rightarrow \infty$ . To complete the proof it remains to obtain a rate of decay for  $\|F(t)\|$  as  $t \rightarrow \infty$ .

Note first that

$$(\mathcal{F}a)(s) = -\frac{is}{1+is}\varphi(s) \quad (3.3.17)$$

for all  $s \in \mathbb{R}$ . Now let  $\rho \in C_c^\infty(\mathbb{R})$  be such that  $0 \leq \rho(s) \leq 1$  for all  $t \in \mathbb{R}$ ,  $\rho(s) = 1$  for  $|s| \leq 1$  and  $\rho(s) = 0$  for  $|s| \geq 2$ . For  $0 < r \leq 1$ , let  $\rho_r \in C_c^\infty(\mathbb{R})$  be given by  $\rho_r(s) = \rho(s/r)$  and let  $\phi_r \in \mathcal{S}(\mathbb{R})$  be such that  $\mathcal{F}\phi_r = (\mathcal{F}a)\rho_r$ . Since  $\varphi(s) = 1$  for all  $s \in \text{supp}(\rho_r)$ , it follows from (3.3.17) that

$$(\mathcal{F}\phi_r)(s) = -\frac{is}{1+is}\rho_r(s)$$

for all  $s \in \mathbb{R}$ . Thus  $|(\mathcal{F}\phi_r)(s)| \leq |s|$  for  $|s| \leq r$ . Since  $(\mathcal{F}\phi_r)(s) = 0$  for  $|s| \geq r$ , it follows that  $\|\mathcal{F}\phi_r\|_1 \leq r^2$ . Moreover, for  $|s| \leq r$  and  $k \geq 1$ ,

$$(\mathcal{F}\phi_r)^{(k)}(s) = \sum_{j=0}^{k-1} \frac{k!(-i)^{k-j}}{j!r^{k-j}(1+is)^{k-j+1}}\rho^{(j)}\left(\frac{s}{r}\right) - \frac{is}{r^k(1+is)}\rho^{(k)}\left(\frac{s}{r}\right).$$

Thus for each  $k \geq 1$  there exists a constant  $B_k > 0$  such that

$$|(\mathcal{F}\phi_r)^{(k)}(s)| \leq \frac{B_k}{r^{k-1}} \left(1 + \frac{|s|}{r}\right)$$

for  $|s| \leq r$  and  $\|(\mathcal{F}\phi_r)^{(k)}\|_1 \leq B_k r^{2-k}$ . It follows from the inversion formula for the Fourier transform that

$$|\phi_r(t)| = \left| \frac{1}{2\pi} \int_{-r}^r e^{ist} (\mathcal{F}\phi_r)(s) ds \right| \leq \frac{1}{2\pi} \int_{-r}^r |s| ds \leq \frac{r^2}{2\pi}$$

and

$$|t^2 \phi_r(t)| = \left| \frac{1}{2\pi} \int_{-r}^r e^{ist} (\mathcal{F}\phi_r)''(s) ds \right| \leq C$$

for all  $t \in \mathbb{R}$ . Here and in the remainder of this proof the letter  $C$  denotes a positive constant which is not assumed to be the same at each occurrence. From these last two estimates it follows that

$$\|\phi_r\|_1 \leq \int_{|t| \leq r^{-1}} \frac{r^2}{2\pi} dt + \int_{|t| \geq r^{-1}} \frac{C}{|t|^2} dt \leq Cr.$$

Now let  $a_r = a - \phi_r$ . Then  $a_r \in L^1(\mathbb{R})$  and  $(\mathcal{F}a_r)(s) = 0$  for  $|s| \leq r$ . Furthermore,  $\|a - a_r\|_1 = \|\phi_r\|_1 \leq Cr$ . By (3.3.17) and the above estimates,  $\|(\mathcal{F}a_r)^{(k)}\|_1 \leq C$  for  $k = 0, 1$  and, making the constants  $B_k$  larger if necessary,  $\|(\mathcal{F}a_r)^{(k)}\|_1 \leq B_k r^{2-k}$  for  $k \geq 2$ . For  $0 < r \leq 1$ , let the map  $F_r : \mathbb{R} \rightarrow \mathcal{B}(X)$  be given by

$$F_r(t) = \int_{\mathbb{R}} a_r(t) T(s+t) dt,$$

where the semigroup has again been extended by zero to  $(-\infty, 0)$  and the integral is to be understood in the strong sense. Applying the dominated convergence theorem and Parseval's identity as in (3.3.9) gives

$$F_r(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ist} (\mathcal{F}a_r)(-s) R(is, A) ds$$

for all  $t \in \mathbb{R}$ , and integrating by parts  $k$  times it follows that, for  $t > 0$ ,

$$F_r(t) = \frac{1}{2\pi t^k} \int_{\mathbb{R}} e^{ist} \sum_{j=0}^k \frac{k!}{j!} i^j (\mathcal{F}a_r)^{(j)}(-s) R(is, A)^{k-j+1} ds.$$

Hence there exists a constant  $C_k > 0$  such that

$$\|F_r(t)\| \leq \frac{C_k}{t^k} \left( m(r)^{k+1} + m(r)^k + \sum_{j=2}^k \frac{m(r)^{k-j+1}}{r^{j-2}} \right).$$

Since  $m(r) \geq r^{-1} \geq 1$ , it follows on making  $C_k$  larger that

$$\|F_r(t)\| \leq \frac{C_k m(r)^{k+1}}{t^k}$$

for all  $t > 0$ . Thus

$$\|F(t)\| \leq \|F_r(t)\| + M \|a - a_r\|_1 \leq \frac{C_k m(r)^{k+1}}{t^k} + Cr \quad (3.3.18)$$

for all  $t > 0$ . Given  $\varepsilon \in (0, 1)$ , let  $r = m^{-1}(t^{1-\varepsilon})$  for sufficiently large  $t > 0$ . Combining (3.3.18) with (3.3.16) shows that, for each  $k \geq 1$ ,

$$\|T(t)AR(1, A)\| = O\left(\frac{1}{t^{\varepsilon(k+1)-1}} + \frac{1}{t} + m^{-1}(t^{1-\varepsilon})\right)$$

as  $t \rightarrow \infty$ . Now choose  $k \geq 1$  such that  $\varepsilon(k+1) \geq 2$ . Then the first term on the right-hand side is dominated by the second and, since  $m^{-1}(t^{1-\varepsilon}) \geq t^{\varepsilon-1} \geq t^{-1}$  for  $t \geq 1$ , it follows that (3.3.15) holds.  $\square$

**Remark 3.3.12.** By Theorem 3.3.2 and a simple Neumann series argument, (3.3.14) implies that  $s_0^\infty(A) < 0$ . If  $X$  is a Hilbert space, it follows from Theorem 3.2.5 that  $\zeta(T) < 0$ . It remains an open question whether on general Banach spaces too this condition is necessary for (3.3.14) to hold.

**Remark 3.3.13.** It is possible to formulate versions of Theorem 3.3.11 in which the assumption  $\zeta(T) < 0$  is replaced by one of conditions (i)–(iv) of Corollary 3.3.10. Note also that from the point of view of decay rates the case  $\sigma(A) \cap i\mathbb{R} = \emptyset$  would be of no real interest in Theorem 3.3.11. Indeed, [12, Proposition 2.4] shows that  $\omega_0(T) = \max\{\zeta(T), s(A)\}$ , and hence the rate of decay when  $\sigma(A) \cap i\mathbb{R} = \emptyset$  is necessarily at least exponential.

Theorem 3.3.11 yields an asymptotic upper bound on  $\|T(t)AR(1, A)\|$  as  $t \rightarrow \infty$  in terms of a modification of the function  $m^{-1}$  appearing in the corresponding lower bound (3.3.6). This phenomenon is unavoidable, as Theorem 3.3.6 shows, and can be observed also in Theorem 3.1.1 and in [15, Theorem 1.5]. The following example, which is the same as [14, Example 6.16], shows that in cases where  $m(s)$  grows very rapidly as  $s \rightarrow 0+$  the estimate (3.3.15) may be optimal but that it is unsatisfactory in general.

**Example 3.3.14.** Consider a bounded  $C_0$ -semigroup  $T$  with generator  $A$  on a complex Banach space  $X$ . Suppose that  $\sigma(A) \cap i\mathbb{R} = \{0\}$  and that  $\zeta(T) < 0$ , and let  $m : (0, 1] \rightarrow (0, \infty)$  be a dominating function for the resolvent of  $A$ .

(a) If  $m(s) = e^{\alpha/s}$  for some constant  $\alpha > 0$ , then

$$m^{-1}(t^{1-\varepsilon}) = \frac{\alpha}{(1-\varepsilon)\log t}$$

for  $t > 1$  and  $\varepsilon \in (0, 1)$ . It follows from Theorem 3.3.11 that

$$\|T(t)AR(1, A)\| = O((\log t)^{-1}) = O(m^{-1}(t))$$

as  $t \rightarrow \infty$  and, by Corollary 3.3.4, this estimate is optimal.

- (a) Now suppose that  $m(s) = s^{-\alpha}$  for some  $\alpha \geq 1$ . In this case Theorem 3.3.8 shows that, for any  $\beta < \alpha^{-1}$ ,

$$\|T(t)AR(1, A)\| = O(t^{-\beta})$$

as  $t \rightarrow \infty$ . It is shown in [14, Theorem 7.6] by a completely different method that this holds also for  $\beta = \alpha^{-1}$  when  $X$  is a Hilbert space, and Theorem 3.1.1 and [15, Theorem 1.5] suggest that even on Banach space only a logarithmic correction factor should be needed.

## Chapter 4

# Some results relating to the non-analytic growth bound

### 4.1 Introduction

The main results of Chapter 3, Theorems 3.3.8 and 3.3.11, involved the assumption that the  $C_0$ -semigroup  $T$  under consideration should satisfy  $\zeta(T) < 0$ . In general, however, there is no easy way of checking whether this condition holds unless  $T$  satisfies one of the properties mentioned in Section 3.2 or the underlying space is a Hilbert space. This chapter contains a number of observations that shed further light on the issue. The main result here, Theorem 4.3.5, is a strengthening of Theorem 3.2.6. Section 4.4 then turns to the study of a natural discrete analogue of the non-analytic growth bound and thus establishes a link with Part III of this thesis. First of all, Section 4.2 presents the necessary preliminary material.

### 4.2 Preliminaries

The aim in Section 4.3 is to establish a purely spectral characterisation of the non-analytic growth bound  $\zeta(T)$  of a  $C_0$ -semigroup  $T$ . A crucial notion in this undertaking is that of *R-boundedness*. Given a complex Banach space  $X$ , a family  $\mathcal{B} \subset \mathcal{B}(X)$  is said to be *R-bounded* if there exists a constant  $C > 0$  such that, given any  $n \in \mathbb{N}$ , any  $T_1, \dots, T_n \in \mathcal{B}$  and arbitrary  $x_1, \dots, x_n \in X$ ,

$$\int_0^1 \left\| \sum_{k=1}^n r_k(t) T_k x_k \right\|^2 dt \leq C \int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\|^2 dt,$$

where the *Rademacher functions*  $r_k : (0, 1) \rightarrow \mathbb{R}$  are defined as in Section 2.2. For details on the notion of  $R$ -boundedness, see for instance [66, Section 2]. It is easy to see that  $R$ -boundedness is a stronger condition than uniform boundedness in operator norm, but the two notions coincide when  $X$  is a Hilbert space; see [66, Section 1.9]. The following proposition collects some important techniques for showing that a given family of bounded linear operators is  $R$ -bounded; see [66, Corollary 2.17], [66, Example 2.19] and [100, Proposition 2.8], respectively.

**Proposition 4.2.1.** *Let  $X$  be a complex Banach space.*

(a) *Let  $S \in L_s^1(\mathbb{R}; \mathcal{B}(X))$  and there exists a constant  $C > 0$  such that*

$$\int_{\mathbb{R}} \|S(t)x\| dt \leq C\|x\|$$

*for all  $x \in X$ . Furthermore, given  $h \in L^\infty(\mathbb{R})$ , define the operator  $T_h \in \mathcal{B}(X)$  by*

$$T_h x = \int_{\mathbb{R}} h(t)S(t)x dt.$$

*Then the family  $\{T_h : \|h\|_\infty \leq 1\}$  is  $R$ -bounded.*

(b) *Let  $\Omega$  be an open subset of  $\mathbb{C}$  and let  $F : \Omega \rightarrow \mathcal{B}(X)$  be a holomorphic function. Then, for any compact subset  $K$  of  $\Omega$ , the family  $\{F(\lambda) : \lambda \in K\}$  is  $R$ -bounded.*

(c) *Suppose that  $\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{B}(X)$  are two  $R$ -bounded families of operators. Then the families  $\mathcal{B}_1 + \mathcal{B}_2$  and  $\mathcal{B}_1 \cup \mathcal{B}_2$  are also  $R$ -bounded.*

In Section 4.4 the focus shifts to the connection between  $C_0$ -semigroups and the properties of individual bounded operators. This material provides a natural link to the next part of the thesis but it arises directly out of the search for an alternative, more easily verifiable formulation of the condition  $\zeta(T) < 0$ . To see this, define the quantity  $\delta(T)$  for any  $C_0$ -semigroup  $T$  with generator  $A$  on a complex Banach space  $X$  by

$$\delta(T) = \inf \{ \omega_0(T - S) : S : (0, \infty) \rightarrow \mathcal{B}(X) \text{ is norm continuous} \}.$$

It is clear from the definitions that  $\delta(T) \leq \zeta(T)$ . When  $X$  is a Hilbert space,  $\delta(T) = \zeta(T) = s_0^\infty(A)$ . Other cases in which  $\delta(T)$  and  $\zeta(T)$  coincide are presented in [12, Section 5], but it remains open whether this is always the case. The following result is proved in [24, Theorem 3.6]; see also [4] and [83, Proposition 4.3].

**Theorem 4.2.2.** *Let  $X$  be a complex Banach space and let  $T$  be a  $C_0$ -semigroup on  $X$  with generator  $A$ . Then, for all  $t > 0$ ,*

$$\sigma(T(t)) \cap \{\lambda \in \mathbb{C} : |\lambda| > e^{\delta(T)t}\} = \exp(t\sigma(A)) \cap \{\lambda \in \mathbb{C} : |\lambda| > e^{\delta(T)t}\}.$$

With  $Q_{\alpha,\beta}$  defined, for  $\alpha \in \mathbb{R}$  and  $\beta \geq 0$ , as in Section 3.2 and given a semigroup generator  $A$ , let

$$s^\infty(A) = \inf \{\alpha \in \mathbb{R} : Q_{\alpha,\beta} \subset \rho(A) \text{ for some } \beta \geq 0\}.$$

Then  $s^\infty(A) \leq s_0^\infty(A) \leq \zeta(T)$  and it follows from Theorem 4.2.2 that  $-1 \in \rho(T(t))$  for all sufficiently small  $t > 0$  when  $\zeta(T) < 0$ , since then  $\delta(T) < 0$  as well. The following partial converse is proved in [23, Sections 4.3 and 4.5].

**Theorem 4.2.3.** *Let  $X$  be a complex Banach space and let  $T$  be a bounded  $C_0$ -semigroup on  $X$  with generator  $A$ . Suppose there exists a non-null set  $I \subset \mathbb{R}_+$  such that  $-1 \in \rho(T(t))$  for all  $t \in I$ . Then  $s_0^\infty(A) < 0$ .*

It follows that, for a bounded  $C_0$ -semigroup  $T$  on a complex Hilbert space,  $\zeta(T) < 0$  if and only if  $-1 \in \rho(T(t))$  for all sufficiently small  $t > 0$ . On a general Banach space, no such characterisation is known. Nevertheless, Theorems 4.2.2 and 4.2.3 suggest at least loosely that the property  $\zeta(T) < 0$  is connected with some notion of regularity of the operators  $T(t)$  for small  $t > 0$  relating to the location of the spectrum of these operators. The aim in Section 4.4 is to formulate this idea rigorously for a given single operator and to show that it describes precisely the degree to which the powers of this operator can be approximated by operator-valued functions which, in a sense to be made precise, are analytic.

### 4.3 The non-analytic growth bound and Fourier multipliers

Suppose that  $T$  is a  $C_0$ -semigroup with generator  $A$  on some complex Banach space  $X$ . By Theorem 3.2.5,  $\zeta(T) = s_0^\infty(A)$  when  $X$  is a Hilbert space. This follows from Plancherel's theorem combined with Theorem 3.2.6. The purpose in this section is to gain a deeper insight into the situation on general Banach spaces by slightly extending Theorem 3.2.6, which is done in Theorem 4.3.5 below.

Given a strongly measurable function  $S : (0, \infty) \rightarrow \mathcal{B}(X)$  that is exponentially bounded in the sense of Section 3.2, define the *Laplace transform*  $\mathcal{L}S$ , for  $\operatorname{Re} \lambda > \omega_0(T)$  and  $x \in X$ , by

$$(\mathcal{L}S)(\lambda)x = \int_0^\infty e^{-\lambda t} S(t)x \, dt.$$

By [6, Theorem 3.1.7],  $(\mathcal{L}T)(\lambda) = R(\lambda, A)$  for  $\operatorname{Re} \lambda > \omega_0(T)$ , and in particular  $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > \omega_0(T)\} \subset \rho(A)$ . Moreover,  $\{R(\lambda, A) : \operatorname{Re} \lambda > \omega\}$  is uniformly norm-bounded for every  $\omega > \omega_0(T)$ , and hence  $s_0(A) \leq \omega_0(T)$ . The following simple lemma gives analogous statements involving the notion of  $R$ -boundedness. Here

$$s_R(A) = \inf \left\{ \omega \in \mathbb{R} : H_\omega \subset \rho(A) \text{ and } \{R(\lambda, A) : \lambda \in H_\omega\} \text{ is } R\text{-bounded} \right\}.$$

**Lemma 4.3.1.** *Let  $X$  be a complex Banach space and let  $S : (0, \infty) \rightarrow \mathcal{B}(X)$  be an exponentially bounded and strongly measurable function. Then  $\{(\mathcal{L}S)(\lambda) : \lambda \in H_\omega\}$  is  $R$ -bounded for every  $\omega > \omega_0(S)$ . In particular, if  $T$  is a  $C_0$ -semigroup on  $X$  with generator  $A$ , then  $s_R(A) \leq \omega_0(T)$ .*

PROOF. Let  $\omega > \omega_0(S)$  and, given  $\lambda \in H_\omega$ , write  $(\mathcal{L}S)(\lambda)$  as

$$(\mathcal{L}S)(\lambda) = \int_0^\infty h_\lambda(t) S_\omega(t) \, dt,$$

where  $h_\lambda(t) = e^{-(\lambda-\omega)t}$  and  $S_\omega(t) = e^{-\omega t} S(t)$  for all  $t > 0$ . Since  $h_\lambda \in L^\infty(0, \infty)$  with  $\|h_\lambda\|_\infty \leq 1$  for all  $\lambda \in H_\omega$ , the first part follows from Proposition 4.2.1(a). The second statement is a consequence of the relation (2.5.2) and the definition of  $s_R(A)$ .  $\square$

The aim now is to obtain a non-analytic version of the statement  $s_R(A) \leq \omega_0(T)$ , that is to say a version in which  $\omega_0(T)$  is replaced by the non-analytic growth bound  $\zeta(T)$  and  $s_R(A)$  is replaced by the quantity  $s_R^\infty(A)$  defined by

$$s_R^\infty(A) = \inf \left\{ \alpha \in \mathbb{R} : Q_{\alpha, \beta} \subset \rho(A) \text{ and } \{R(\lambda, A) : \lambda \in Q_{\alpha, \beta}\} \right. \\ \left. \text{is } R\text{-bounded for some } \beta \geq 0 \right\}.$$

Here, as before, given  $\alpha \in \mathbb{R}$  and  $\beta \geq 0$ ,  $Q_{\alpha, \beta} = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq \alpha, |\operatorname{Im} \lambda| \geq \beta\}$ . The desired result will follow from Proposition 4.3.3 below, which in turn requires the following simple lemma taken from [12, Section 2]. Given a suitable

function  $S : (0, \infty) \rightarrow \mathcal{B}(X)$  and  $\alpha \in \mathbb{R}$ ,  $\beta \geq 0$  such that  $\mathcal{L}S$  extends holomorphically to a region containing  $Q_{\alpha, \beta}$ , let

$$\tilde{S}_{\alpha, \beta}(t) = \frac{1}{2\pi i} \int_{\Gamma_{\alpha, \beta}} e^{t\lambda} (\mathcal{L}S)(\lambda) d\lambda$$

for all  $t \geq 0$ , where  $\Gamma_{\alpha, \beta}$  is any sufficiently smooth path from  $\alpha - i\beta$  to  $\alpha + i\beta$  contained in a region to which  $\mathcal{L}S$  can be extended holomorphically. By Cauchy's theorem, this definition is independent of the choice of  $\Gamma_{\alpha, \beta}$ . In particular, if  $T$  is a  $C_0$ -semigroup with generator  $A$  and if  $\operatorname{Re} \lambda > s(A)$ , then

$$\tilde{T}_{\alpha, \beta}(t) = \frac{1}{2\pi i} \int_{\Gamma_{\alpha, \beta}} e^{t\lambda} R(\lambda, A) d\lambda \quad (4.3.1)$$

for all  $t \geq 0$ , where  $\Gamma_{\alpha, \beta}$  is a suitable path in  $\rho(A)$ .

**Lemma 4.3.2.** *Let  $X$  be a complex Banach space and let  $T$  be a  $C_0$ -semigroup on  $X$  with generator  $A$ . Furthermore, let  $\alpha \in \mathbb{R}$  and  $\beta \geq 0$  be such that  $Q_{\alpha, \beta} \subset \rho(A)$ . Then  $\tilde{T}_{\alpha, \beta}$  is exponentially bounded and*

$$\omega_0(\tilde{T}_{\alpha, \beta}) \leq \max\{\alpha, s(A)\}.$$

PROOF. Fix  $\omega > \max\{\alpha, s(A)\}$  and let  $\Gamma_{\alpha, \beta}$  consist of line segments joining  $\alpha - i\beta$ ,  $\omega - i\beta$ ,  $\omega + i\beta$  and  $\alpha + i\beta$  successively. By (4.3.1) and continuity of the resolvent, there exists  $C > 0$  such that  $\|\tilde{T}_{\alpha, \beta}(t)\| \leq Ce^{\omega t}$  for all  $t \geq 0$ , and the result follows.  $\square$

The next result extends [12, Proposition 2.1] by introducing the notion of  $R$ -boundedness. Its most important consequence for the present purposes is stated in Corollary 4.3.4 below, but the result itself may be of independent interest. Here  $H(\Omega)$  is defined for an open subset  $\Omega$  of  $\mathbb{C}$  as in Section 3.2.

**Proposition 4.3.3.** *Let  $X$  be a complex Banach space and let  $T$  be a  $C_0$ -semigroup on  $X$  with generator  $A$ . Furthermore, let  $\omega \in \mathbb{R}$ . The following are equivalent:*

- (i) *There exists  $S \in H(\Sigma_\theta)$  for some  $\theta > 0$  such that  $\omega_0(T - S) < \omega$ .*
- (ii) *There exist  $\alpha < \omega$  and  $\beta \geq 0$  such that  $Q_{\alpha, \beta} \subset \rho(A)$ ,  $\{R(\lambda, A) : \lambda \in Q_{\alpha, \beta}\}$  is  $R$ -bounded and*

$$\omega_0(T - \tilde{T}_{\alpha, \beta}) < \omega. \quad (4.3.2)$$

- (ii)' *There exist  $\alpha < \omega$  and  $\beta \geq 0$  such that  $Q_{\alpha, \beta} \subset \rho(A)$  and (4.3.2) holds.*

(iii) *There exists an entire exponentially bounded function  $S : \mathbb{C} \rightarrow \mathcal{B}(X)$  such that  $\omega_0(T - S) < \omega$  and  $\omega_0(S) \leq \max\{\omega, s(A)\}$ .*

Furthermore, when these conditions hold, then (4.3.2) holds for all  $\alpha < \omega$  and  $\beta \geq 0$  such that  $Q_{\alpha, \beta} \subset \rho(A)$ .

PROOF. Apart from the main step (i)  $\implies$  (ii) the proof is identical to that of [12, Proposition 2.1]. The implications (ii)  $\implies$  (ii)' and (iii)  $\implies$  (i) are trivial. Throughout this proof, the letter  $C$  will be used to denote a positive constant which need not be the same at each occurrence.

(i)  $\implies$  (ii): Suppose that  $\theta > 0$  and  $S \in H(\Sigma_\theta)$  are as in (i). Since  $\theta$  may be taken to lie in  $(0, \pi/2)$  without loss of generality, there exist  $\sigma \in \mathbb{R}$  and  $C > 0$  be such that  $\|S(\lambda)\| \leq Ce^{\sigma \operatorname{Re} \lambda}$  for all  $\lambda \in \Sigma_\theta$ . In particular,  $\omega_0(S) \leq \sigma$ . Note also that there is no loss of generality in assuming that  $\sigma \geq \max\{0, \omega\}$ . Furthermore, fix  $\theta', \theta'' \in (0, \theta)$  with  $\theta' < \theta''$  and let  $\alpha, \beta$  be such that  $\omega_0(T - S) < \alpha < \omega$  and  $\beta = (\sigma - \alpha) \cot \theta'$ ; see Figure 6.2. For  $t > 0$ , let  $U(t) = T(t) - S(t)$ . Then by Lemma 4.3.1  $\{(\mathcal{L}U)(\lambda) : \lambda \in Q_{\alpha, \beta}\}$  is  $R$ -bounded. Now

$$R(\cdot, A) = \mathcal{L}(S + U) = \mathcal{L}S + \mathcal{L}U$$

by linearity of  $\mathcal{L}$ . Thus by Proposition 4.2.1(c) to prove the first part of (ii) it suffices to show that  $\mathcal{L}S$  extends analytically to  $Q_{\alpha, \beta}$  and that  $\{(\mathcal{L}S)(\lambda) : \lambda \in Q_{\alpha, \beta}\}$  is  $R$ -bounded.

Mimicking the proof of [6, Theorem 2.6.1], let  $\Gamma_\pm$  denote the rays  $\{re^{\mp i\theta''} : r \in \mathbb{R}_+\}$ . It follows from Cauchy's theorem that, for  $\operatorname{Re} \lambda > \sigma$ ,

$$(\mathcal{L}S)(\lambda) = \int_{\Gamma_\pm} e^{-\lambda z} S(z) dz,$$

with either choice of sign. Hence

$$(\mathcal{L}S)(\lambda) = e^{\mp i\theta''} \int_0^\infty \exp(-\lambda r e^{\mp i\theta''}) S(r e^{\mp i\theta''}) dr. \quad (4.3.3)$$

Fix  $\varepsilon$  such that  $0 < \varepsilon < \min\{\frac{\pi}{2} - \theta'', \theta'' - \theta'\}$  and let

$$\Sigma_\pm = \left\{ \lambda \in \mathbb{C} \setminus \{\sigma\} : |\arg(\lambda - \sigma) \mp \theta''| < \frac{\pi}{2} - \varepsilon \right\},$$

so that  $\Sigma_\pm = \sigma + e^{\pm i\theta''} \Sigma_{\frac{\pi}{2} - \varepsilon}$ . Then

$$\operatorname{Re}((\lambda - \sigma)e^{\mp i\theta''}) \geq |\lambda - \sigma| \sin \varepsilon$$

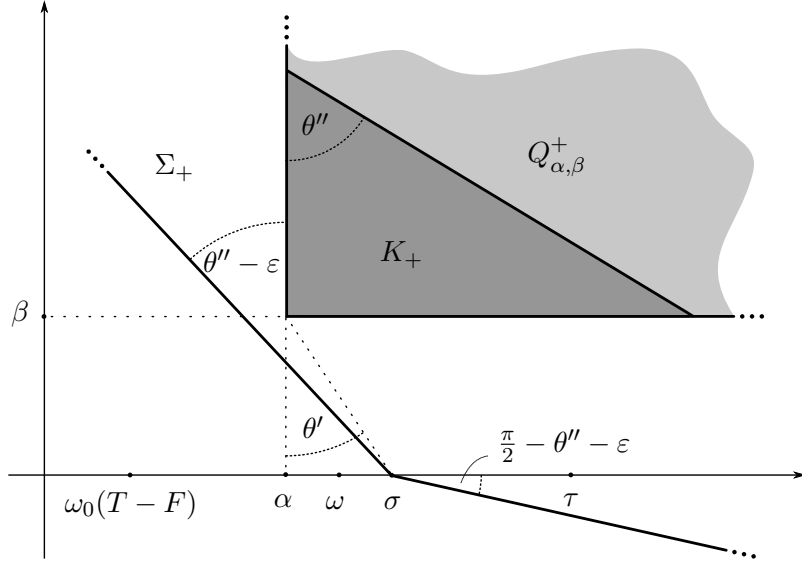


Figure 4.1: The setting in the upper half-plane. Here the slanted side of the triangular region  $K_+$  is distance  $\tau$  from the origin.

for all  $\lambda \in \Sigma_{\pm}$ , respectively, and hence

$$\|\exp(-\lambda r e^{\mp i\theta''})S(re^{\mp i\theta''})\| \leq C \exp(-r|\lambda - \sigma| \sin \varepsilon) \quad (4.3.4)$$

according as  $\lambda \in \Sigma_{\pm}$ . It follows that the integrals in (4.3.3) are absolutely convergent for  $\lambda \in \Sigma_{\pm}$  and hence, by an application of Morera's and Fubini's theorems, define holomorphic functions on these regions. Now let

$$Q_{\alpha, \beta}^{\pm} = Q_{\alpha, \beta} \cap \{z \in \mathbb{C} : \text{Im } z \geq 0\},$$

so that  $Q_{\alpha, \beta}^{\pm} \subset \Sigma_{\pm}$  by the choice of  $\varepsilon$ . For  $\tau > \sigma$  fixed, the sets

$$K_{\pm} = \{\lambda \in Q_{\alpha, \beta}^{\pm} : \text{Re}(e^{\mp i\theta''} \lambda) \leq \tau\}$$

are compact and hence the families  $\{(\mathcal{L}S)(\lambda) : \lambda \in K_{\pm}\}$  are  $R$ -bounded by Proposition 4.2.1(b). For  $\lambda \in Q_{\alpha, \beta}^+ \setminus K_+$ , (4.3.3) can be written as

$$(\mathcal{L}S)(\lambda) = \int_0^{\infty} h_{\lambda}(r) S_{\tau}(r) dr,$$

where  $h_{\lambda}(r) = \exp(-i\theta'' - r(e^{-i\theta''} \lambda - \tau))$  and  $S_{\tau}(r) = e^{-\tau r} S(re^{-i\theta''})$  for all  $r > 0$ . Since  $\|S_{\tau}(r)\| \leq C e^{-r(\tau - \sigma)}$  and  $|h_{\lambda}(r)| < 1$  for all  $r > 0$  and all  $\lambda \in Q_{\alpha, \beta}^+ \setminus K_+$ , it follows from Proposition 4.2.1(a) that  $\{(\mathcal{L}S)(\lambda) : \lambda \in Q_{\alpha, \beta}^+ \setminus K_+\}$

is  $R$ -bounded. Likewise  $\{(\mathcal{L}S)(\lambda) : \lambda \in Q_{\alpha,\beta}^- \setminus K_-\}$  is  $R$ -bounded and hence by Proposition 4.2.1(c)  $\{(\mathcal{L}S)(\lambda) : \lambda \in Q_{\alpha,\beta}\}$ , too, is  $R$ -bounded, as required.

In order to prove the second part of (ii), note that by the inversion formula for Laplace transforms (see [6, Section 2.3])

$$S(t) = \frac{1}{2\pi i} \int_{\Gamma \cup \Gamma_{\alpha,\beta}} e^{t\lambda} (\mathcal{L}S)(\lambda) d\lambda$$

for all  $t > 0$ . Here  $\Gamma_{\alpha,\beta}$  is any path from  $\alpha - i\beta$  to  $\alpha + i\beta$  contained in a region to which  $\mathcal{L}S$  can be extended holomorphically and  $\Gamma$  consists of two rays emanating from  $\alpha \pm i\beta$  at angles  $\pm(\frac{\pi}{2} + \theta')$ , respectively. Equations (4.3.3) and (4.3.4) imply that  $\|(\mathcal{L}S)(\lambda)\| \leq C|\lambda - \sigma|^{-1}$  for all  $\lambda \in \Sigma_{\theta'}$  and hence, for  $r_0 = (\sigma - \alpha) \operatorname{cosec} \theta'$  and  $t \geq 1$ ,

$$\begin{aligned} \|S(t) - \tilde{S}_{\alpha,\beta}(t)\| &= \left\| \frac{1}{2\pi i} \int_{\Gamma} e^{t\lambda} (\mathcal{L}S)(\lambda) d\lambda \right\| \\ &\leq C e^{t\alpha} \int_{r_0}^{\infty} \exp(-t(r - r_0) \sin \theta') \frac{dr}{r} \\ &\leq C e^{t\alpha}. \end{aligned}$$

It follows that  $\omega_0(S - \tilde{S}_{\alpha,\beta}) \leq \alpha$ . Recall that  $\omega_0(U) < \alpha$ . Choosing for  $\Gamma_{\alpha,\beta}$  the vertical straight-line segment from  $\alpha - i\beta$  to  $\alpha + i\beta$ , it follows that

$$\|\tilde{U}_{\alpha,\beta}(t)\| = \left\| \frac{1}{2\pi i} \int_{\Gamma_{\alpha,\beta}} e^{t\lambda} (\mathcal{L}U)(\lambda) d\lambda \right\| \leq C e^{\alpha t}$$

and hence  $\omega_0(\tilde{U}_{\alpha,\beta}) \leq \alpha$ . Since  $\tilde{U}_{\alpha,\beta} = \tilde{T}_{\alpha,\beta} - \tilde{S}_{\alpha,\beta}$ ,

$$\omega_0(T - \tilde{T}_{\alpha,\beta}) \leq \max \{ \omega_0(T - S), \omega_0(S - \tilde{S}_{\alpha,\beta}), \omega_0(\tilde{U}_{\alpha,\beta}) \} \leq \alpha,$$

which proves (ii).

(ii)'  $\implies$  (iii): For  $\lambda \in \mathbb{C}$  set

$$S(\lambda) = \frac{1}{2\pi i} \int_{\Gamma_{\alpha,\beta}} e^{\lambda z} R(z, A) dz,$$

where  $\Gamma_{\alpha,\beta}$  is any path from  $\alpha - i\beta$  to  $\alpha + i\beta$  lying in  $\rho(A)$ . Then  $S$  is an exponentially bounded entire function which agrees with  $\tilde{T}_{\alpha,\beta}$  on  $(0, \infty)$ , so (iii) follows from Lemma 4.3.2.

For the final statement, suppose that  $Q_{\alpha,\beta}, Q_{\alpha',\beta'} \subset \rho(A)$  for some  $\alpha, \alpha' \in$

$(-\infty, \omega)$  and some  $\beta, \beta' \geq 0$  and let  $\alpha_0 = \max\{\alpha, \alpha'\}$ . Then by Cauchy's theorem

$$\tilde{T}_{\alpha, \beta}(t) - \tilde{T}_{\alpha', \beta'}(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{t\lambda} R(\lambda, A) d\lambda,$$

where  $\Gamma$  consists of two paths in  $\{\lambda \in \rho(A) : \operatorname{Re} \lambda \leq \alpha_0\}$  connecting  $\alpha \pm i\beta$  to  $\alpha' \pm i\beta'$ , respectively. It follows that

$$\|\tilde{T}_{\alpha, \beta}(t) - \tilde{T}_{\alpha', \beta'}(t)\| \leq Ce^{\alpha_0 t}$$

for all  $t \geq 0$  and hence  $\omega_0(\tilde{T}_{\alpha, \beta} - \tilde{T}_{\alpha', \beta'}) \leq \alpha_0$ . Since

$$\omega_0(T - \tilde{T}_{\alpha', \beta'}) \leq \max\{\omega_0(T - \tilde{T}_{\alpha, \beta}), \omega_0(\tilde{T}_{\alpha, \beta} - \tilde{T}_{\alpha', \beta'})\},$$

the result follows.  $\square$

The following consequence of Proposition 4.3.3 is an extension of [12, Proposition 2.4 a)] in the context of  $C_0$ -semigroups.

**Corollary 4.3.4.** *Let  $X$  be a complex Banach space and let  $T$  be a  $C_0$ -semigroup on  $X$  with generator  $A$ . Then  $s_R^\infty(A) \leq \zeta(T)$ .*

PROOF. Let  $\omega > \zeta(T)$ . Then there exists  $S \in H(\Sigma_\theta)$  for some  $\theta > 0$  satisfying  $\omega_0(T - S) < \omega$  and by Proposition 4.3.3 there exist  $\alpha < \omega$  and  $\beta \geq 0$  such that  $Q_{\alpha, \beta} \subset \rho(A)$  and  $\{R(\lambda, A) : \lambda \in Q_{\alpha, \beta}\}$  is  $R$ -bounded. Hence  $s_R^\infty(A) \leq \alpha$  and the result follows.  $\square$

The following result is a strengthening of Theorem 3.2.6. The proof is identical to the one given in [17, Theorem 3.6] except that the statement  $s_0^\infty(A) \leq \zeta(T)$  is replaced by the conclusion of Corollary 4.3.4 in proving the implication (i)  $\implies$  (ii). The notation here is as in Section 3.2.

**Theorem 4.3.5.** *Let  $X$  be a complex Banach space and  $T$  a  $C_0$ -semigroup on  $X$  with generator  $A$ . Suppose that  $1 \leq p < \infty$ . Then the following are equivalent:*

- (i)  $\zeta(T) < 0$ ;
- (ii)  $s_R^\infty(A) < 0$  and, for any compatible function  $\phi \in C^\infty(\mathbb{R})$ , there exists a map  $S \in L_s^1(\mathbb{R}; \mathcal{B}(X))$  such that  $\phi(s)R(is, A) = (\mathcal{F}S)(s)$  for all  $s \in \mathbb{R}$ ;
- (iii)  $s_R^\infty(A) < 0$  and, for any compatible function  $\phi \in C^\infty(\mathbb{R})$ , the map  $s \mapsto \phi(s)R(is, A)$  is a Fourier multiplier on  $L^p(\mathbb{R}; X)$ .

The final result in this section is analogous to [17, Corollary 3.10] and gives a characterisation of  $\zeta(T)$  in terms of a Fourier multiplier condition.

**Corollary 4.3.6.** *Let  $X$  be a complex Banach space, let  $T$  be a  $C_0$ -semigroup on  $X$  with generator  $A$  and let  $1 < p < \infty$ . Then*

$$\zeta(T) = \inf \{ \alpha > s_R^\infty(A) : s \mapsto \phi(s)R(\alpha + is, A) \text{ is a Fourier multiplier on } L^p(\mathbb{R}; X) \text{ for all } \phi \in C^\infty(\mathbb{R}) \text{ that are compatible with } A - \alpha \}.$$

PROOF. Let  $\alpha \in \mathbb{R}$ . By Theorem 4.3.5 applied to the rescaled semigroup  $T_\alpha$  given, for  $t \geq 0$ , by  $T_\alpha(t) = e^{-\alpha t}T(t)$ ,  $\zeta(T_\alpha) < 0$  if and only if  $s_R^\infty(A) < \alpha$  and  $s \mapsto \phi(s)R(\alpha + is, A)$  is a Fourier multiplier on  $L^p(\mathbb{R}; X)$  for all  $\phi \in C^\infty(\mathbb{R})$  that are compatible with  $A - \alpha$ . The result follows.  $\square$

## 4.4 Analytic approximation for a single operator

The discussion at the end of Section 4.2 suggested a special importance of power-bounded operators  $T$  on a Banach space  $X$  satisfying  $-1 \in \rho(T)$ . A natural quantity to introduce, then, is

$$r_\infty(T) = \inf \{ r > 0 : \sigma(T) \cap \mathbb{T} \subset 1 + r\mathbb{D} \}.$$

Thus  $0 \leq r_\infty(T) \leq 2$  for any power-bounded operator  $T$ , and the results obtained at the end of Section 4.2 by combining Theorems 4.2.2 and 4.2.3 can be restated as  $\zeta(T) < 0$  for a bounded  $C_0$ -semigroup  $T$  on a Hilbert space if and only if  $r_\infty(T(t)) < 2$  for all sufficiently small  $t > 0$ . Note also that  $r_\infty(T) = 0$  precisely when  $\sigma(T) \cap \mathbb{T} \subset \{1\}$ . Operators with this property will be the focus of the next part of this thesis. The purpose here is to show that the quantity  $r_\infty(T)$  for a power-bounded operator  $T$  is connected to the degree to which the powers of  $T$  can be approximated by certain *analytic* maps  $Q : \mathbb{Z}_+ \rightarrow \mathcal{B}(X)$ .

Given a complex Banach space  $X$ , a function  $Q : \mathbb{Z}_+ \rightarrow \mathcal{B}(X)$  will be said to be *exponentially bounded* if, for some constants  $C, r > 0$ ,  $\|Q(n)\| \leq Cr^n$  for all  $n \geq 0$ . If  $Q : \mathbb{Z}_+ \rightarrow \mathcal{B}(X)$  is exponentially bounded and  $C, r > 0$  are as above, let

$$F_Q(\lambda) = \sum_{n=0}^{\infty} \frac{Q(n)}{\lambda^{n+1}}$$

for all  $\lambda \in \mathbb{C}$  with  $|\lambda| > r$  and note that  $F_Q$  is holomorphic in this region. For  $\theta \in (0, \pi/2]$ , let  $\Omega_\theta$  denote the closed convex hull of the point 1 and the set  $\{\lambda \in \mathbb{C} : |\lambda| \leq \cos \theta\}$ ; see Figure 4.2. An exponentially bounded map

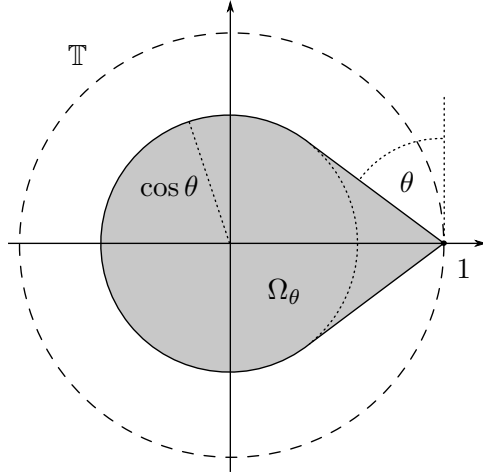


Figure 4.2: The region  $\Omega_\theta$  for a typical value of  $\theta \in (0, \pi/2]$ .

$Q : \mathbb{Z}_+ \rightarrow \mathcal{B}(X)$  will be said to be *analytic* if there exist  $\theta \in (0, \pi/2]$  and  $C > 0$  such that  $F_Q$  extends to the complement of  $\Omega_\theta$  and

$$\|F_Q(\lambda)\| \leq \frac{C}{|1 - \lambda|} \quad (4.4.1)$$

for all  $\lambda \in \mathbb{C} \setminus \Omega_\theta$ .

The next result is an analogue of [87, Theorem 2.1] and provides two alternative characterisations of analyticity for exponentially bounded maps  $Q : \mathbb{Z}_+ \rightarrow \mathcal{B}(X)$ . Given such a map, let  $S_Q : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$  be given by

$$S_Q(t) = e^{-t} \sum_{n=0}^{\infty} Q(n) \frac{t^n}{n!}.$$

**Theorem 4.4.1.** *Let  $X$  be a complex Banach space and let  $Q : \mathbb{Z}_+ \rightarrow \mathcal{B}(X)$  be exponentially bounded. Then the following are equivalent:*

- (i)  $Q$  is analytic;
- (ii)  $F_Q$  extends analytically to  $\{\lambda \in \mathbb{C} \setminus \{1\} : |\lambda| \geq 1\}$  and there exists  $C > 0$  such that

$$\left\| \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} Q(n+j) \right\| \leq C^{1+k} \binom{n+k}{k}^{-1} \quad (4.4.2)$$

for all  $k, n \geq 0$ ;

- (iii)  $F_Q$  extends analytically to  $\{\lambda \in \mathbb{C} \setminus \{1\} : |\lambda| \geq 1\}$  and there exists  $\theta \in (0, \pi/2]$  such that  $S_Q$  has a bounded analytic extension to the sector  $\Sigma_\theta$ .

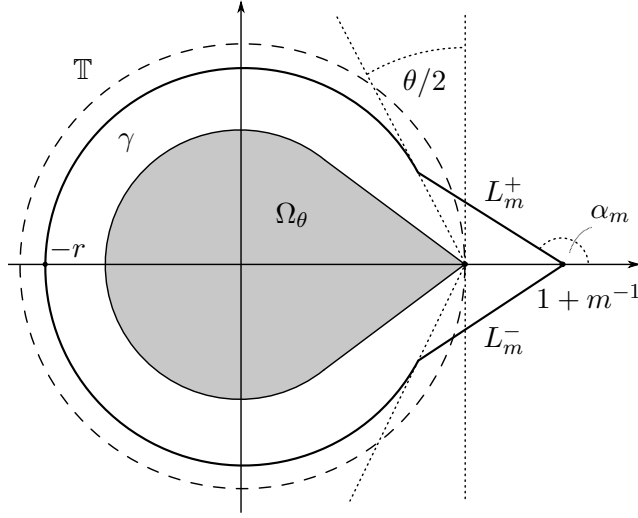


Figure 4.3: The contour  $\Gamma_m = \gamma \cup L_m^- \cup L_m^+$ .

PROOF. Suppose that (i) holds and note that, for  $n \geq 0$ ,

$$Q(n) = \frac{1}{2\pi i} \oint_{\Gamma} \lambda^n F_Q(\lambda) d\lambda,$$

where  $\Gamma$  is any contour which contains  $\Omega_\theta$  in its interior. In particular,

$$\begin{aligned} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} Q(n+j) &= \frac{1}{2\pi i} \oint_{\Gamma} \sum_{j=0}^k \binom{k}{j} \lambda^{n+j} (-1)^{k-j} F_Q(\lambda) d\lambda \\ &= \frac{1}{2\pi i} \oint_{\Gamma} \lambda^n (\lambda-1)^k F_Q(\lambda) d\lambda \end{aligned} \quad (4.4.3)$$

for all  $n, k \geq 0$ . The aim is to show that (ii) holds, which requires establishing (4.4.2). If  $n = k = 0$  there is nothing to prove, so assume that  $n + k > 0$ . Now fix  $r \in (0, 1)$  such that  $-r \notin \Omega_\theta$  and choose for  $\Gamma$  the contour  $\Gamma_{n+k}$ , where, for  $m \geq 1$ ,  $\Gamma_m$  consists of the arc  $\gamma = r\mathbb{T} \setminus (1 + \Sigma_{(\pi+\theta)/2})$  together with the rays  $L_m^\pm$  connecting the point  $1 + m^{-1}$  to the end points of  $\gamma$  lying in the upper and lower half-plane, respectively; see Figure 4.3. By modifying the values of  $r$  and  $\theta$  if necessary, it is possible to assume that  $|\lambda - 1| \leq \frac{1}{2}$  when  $\lambda$  is either of the endpoints of  $\gamma$ . Letting the letter  $C$  denote a positive constant whose value may vary from occurrence to occurrence, it follows from the fact that  $F_Q$  is bounded along  $\gamma$  that

$$\left\| \frac{1}{2\pi i} \int_{\gamma} \lambda^n (\lambda-1)^k F_Q(\lambda) d\lambda \right\| \leq C r^n (1+r)^k \leq C^{1+k} r^n.$$

Keeping  $r \in (0, 1)$  fixed, an elementary calculation shows that  $r^x (\frac{x}{k} + 1)^k \leq C^k$  for all  $x \geq 0$  and  $k > 0$ , and hence  $r^n \leq C^k (1 + \frac{n}{k})^{-k}$  for all  $n \geq 0$  and  $k > 0$ .

Since

$$\binom{n+k}{k} \leq e^k \left(1 + \frac{n}{k}\right)^k, \quad (4.4.4)$$

for all  $n \geq 0$  and  $k > 0$ , it follows that

$$\left\| \frac{1}{2\pi i} \int_{\gamma} \lambda^n (\lambda - 1)^k F_Q(\lambda) d\lambda \right\| \leq C^{1+k} \binom{n+k}{k}^{-1} \quad (4.4.5)$$

for all  $n, k \geq 0$  with  $n+k > 0$ , the claim being obvious if  $k = 0$ . It remains to estimate the contributions along the rays  $L_{n+k}^{\pm}$ . Suppose first that  $k > n$ . If  $n = 0$  and  $k = 1$ , the estimate (4.4.5) with  $\gamma$  replaced by  $L_{n+k}^{\pm}$  can be achieved simply by allowing the constant  $C$  to be sufficiently large. In all other cases,  $n+k > 2$  and it follows from the choice of  $r$  and  $\theta$  that  $|\lambda - 1| \leq \frac{1}{2}$  for all  $\lambda \in L_{n+k}^{\pm}$ . Since moreover  $|\lambda^n| \leq (1 + \frac{1}{n+k})^n \leq e$  for  $k > n \geq 0$  and all  $\lambda \in L_{n+k}^{\pm}$ , the estimates (4.4.1) and  $1 + \frac{n}{k} \leq 2$  show that

$$\left\| \frac{1}{2\pi i} \int_{L_{n+k}^{\pm}} \lambda^n (\lambda - 1)^k F_Q(\lambda) d\lambda \right\| \leq C 2^{1-k} \leq C \left(1 + \frac{n}{k}\right)^{-k}$$

in this case, and it follows from (4.4.4) that (4.4.5) holds again with  $\gamma$  replaced by  $L_{n+k}^{\pm}$ . Now suppose that  $0 \leq k \leq n$ . Note, as in the proof of [86, Theorem 4.9.3], that the rays  $L_m^{\pm}$  are subsets of the lines parameterised by  $\lambda(s) = 1 + m^{-1} + se^{\pm i\alpha_m}$ , where  $s \geq 0$  and  $\alpha_m \in (\frac{\pi}{2}, \pi)$  is chosen as shown in Figure 4.3. Hence there exists  $c > 0$  such that  $|\lambda(s)| \leq (1 + m^{-1})e^{-cs}$  and consequently  $|\lambda(s)|^m \leq e^{1-cms}$  for all  $s \geq 0$  and all  $m \geq 1$ . Consider first the case  $k = 0$ . Since  $\alpha_n \leq \alpha_1 < \pi$  for all  $n \geq 1$ ,

$$\left| \frac{1}{n} + se^{\pm i\alpha_n} \right|^2 \geq \frac{1}{n^2} ((1 - ns)^2 + 2ns(1 + \cos \alpha_1))$$

and hence  $|n^{-1} + se^{\pm i\alpha_n}| \geq (Cn)^{-1}$  for all  $n \geq 1$ . It follows from (4.4.1) that

$$\begin{aligned} \left\| \frac{1}{2\pi i} \int_{L_n^{\pm}} \lambda^n F_Q(\lambda) d\lambda \right\| &\leq C \int_0^{\infty} e^{-cns} \left( \frac{1}{n} + se^{\pm i\alpha_n} \right)^{-1} ds \\ &\leq Cn \int_0^{\infty} e^{-cns} ds, \end{aligned}$$

so that

$$\left\| \frac{1}{2\pi i} \int_{L_n^{\pm}} \lambda^n F_Q(\lambda) d\lambda \right\| \leq C$$

for all  $n \geq 1$ , which is the required estimate when  $k = 0$ . For  $0 < k \leq n$ , it follows from (4.4.1) and the estimate  $|m^{-1} + se^{\pm i\alpha_m}| \leq m^{-1} + s$  for  $s \geq 0$  that

$$\left\| \frac{1}{2\pi i} \int_{L_{n+k}^\pm} \lambda^n (\lambda - 1)^k F_Q(\lambda) d\lambda \right\| \leq C^{1+k} \int_0^\infty e^{-cns} \left( \frac{1}{n+k} + s \right)^{k-1} ds$$

and hence

$$\left\| \frac{1}{2\pi i} \int_{L_{n+k}^\pm} \lambda^n (\lambda - 1)^k F_Q(\lambda) d\lambda \right\| \leq \frac{C^{1+k}}{n^k} \int_0^\infty e^{-s} s^{k-1} ds = C^{1+k} \frac{(k-1)!}{n^k}.$$

Since  $\frac{k}{n} \leq 2(1 + \frac{n}{k})^{-1}$  in this case, it follows that

$$\left\| \frac{1}{2\pi i} \int_{L_{n+k}^\pm} \lambda^n (\lambda - 1)^k F_Q(\lambda) d\lambda \right\| \leq C^{1+k} \left( 1 + \frac{n}{k} \right)^{-k}.$$

Using (4.4.3) and (4.4.4), these estimates together show that (ii) holds.

Now suppose that (ii) holds and note that, for  $k \geq 0$  and  $t \geq 0$ ,

$$S_Q^{(k)}(t) = e^{-t} \sum_{n=0}^\infty \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} Q(n+j) \frac{t^n}{n!}.$$

It follows from (4.4.2) that there exists  $C > 0$  such that

$$\frac{\|S_Q^{(k)}(t)\|}{k!} \leq e^{-t} \sum_{n=0}^\infty \frac{C^{1+k} t^n}{(n+k)!} \leq \frac{C^{1+k}}{t^k},$$

and hence the Taylor expansion

$$S_Q(t + \lambda) = \sum_{k=0}^\infty S_Q^{(k)}(t) \frac{\lambda^k}{k!}$$

is valid for  $|\lambda| < t/C$ . Thus  $F_Q$  has a bounded holomorphic extension to  $\Sigma_\theta$  for  $0 < \theta < \sin^{-1}(1/C)$ , so (iii) holds.

Suppose finally that (iii) is satisfied and let  $U_Q(t) = e^t S_Q(t)$ , so that

$$U_Q(t) = \sum_{n=0}^\infty Q(n) \frac{t^n}{n!}$$

for all  $t \geq 0$ . By [6, Theorem 2.6.1] the Laplace transform

$$(\mathcal{L}U_Q)(\lambda) = \int_0^\infty e^{-\lambda t} U_Q(t) dt$$

extends holomorphically to the shifted sector  $1 + \Sigma_{(\pi+\theta)/2}$  and there exists  $C > 0$  such that

$$\|(\mathcal{L}U_Q)(\lambda)\| \leq \frac{C}{|1 - \lambda|}$$

for all  $\lambda \in 1 + \Sigma_{(\pi+\theta)/2}$ . Since  $Q$  is exponentially bounded, it follows from Fubini's theorem that the Laplace transform  $\mathcal{L}U_Q$  of  $U_Q$  satisfies

$$(\mathcal{L}U_Q)(\lambda) = \int_0^\infty \sum_{n=0}^\infty Q(n) \frac{e^{-\lambda t} t^n}{n!} dt = \sum_{n=0}^\infty \frac{Q(n)}{\lambda^{n+1}} = F_Q(\lambda)$$

for  $\operatorname{Re} \lambda$  sufficiently large. Hence by the identity theorem  $F_Q$  extends to  $1 + \Sigma_{(\pi+\theta)/2}$  and satisfies the estimate (4.4.1) on this region. Since  $F_Q$  extends to an open set containing  $\mathbb{T} \setminus (1 + \Sigma_{(\pi+\theta)/2})$  and is bounded on compact subsets of its domain, it follows that  $Q$  is analytic.  $\square$

Given  $T \in \mathcal{B}(X)$ , let  $Q_T(n) = T^n$  for  $n \geq 0$ . Then, writing  $F_T$  instead of  $F_{Q_T}$ ,  $F_T(\lambda) = R(\lambda, T)$  for  $|\lambda| > r(T)$ . The following lemma shows that in this special case (4.4.2) can be replaced by a simpler condition when  $T$  is power-bounded.

**Lemma 4.4.2.** *Let  $X$  be a complex Banach space and let  $T \in \mathcal{B}(X)$  be a power-bounded operator satisfying  $\sigma(T) \cap \mathbb{T} \subset \{1\}$ . Then  $Q_T$  is analytic if and only if there exists  $B > 0$  such that*

$$\|T^n(I - T)\| \leq \frac{B}{1 + n} \quad (4.4.6)$$

for all  $n \geq 0$ .

PROOF. It is clear that the first part of condition (ii) in Theorem 4.4.1 is equivalent to  $\sigma(T) \cap \mathbb{T} \subset \{1\}$  in this case. If  $Q_T$  is analytic, then (4.4.2) holds by Theorem 4.4.1 and the estimate (4.4.6) follows on setting  $k = 1$ .

Suppose conversely that (4.4.6) holds. For  $k = 0$ , (4.4.2) is satisfied by power-boundedness of  $T$ . Suppose therefore that  $k \geq 1$ . If  $n \leq k$ , then  $1 + \frac{n}{k} \leq 2$  and hence

$$\|T^n(I - T)^k\| \leq M \|I - T\|^k \leq C^{1+k} \left(1 + \frac{n}{k}\right)^{-k}, \quad (4.4.7)$$

where  $M = \sup\{\|T^n\| : n \geq 0\}$  and  $C = 2(1 + M)$ . If  $n > k$ , on the other hand, let  $n = mk + j$  for some  $m \geq 1$  and  $0 \leq j < k$ . By (4.4.6),

$$\|T^n(I - T)^k\| \leq M \|T^m(I - T)\|^k \leq MB^k \left(1 + \frac{n-j}{k}\right)^{-k}.$$

Since  $1 + \frac{n}{k} \leq 2\left(1 + \frac{n-j}{k}\right)$ , (4.4.7) holds for  $C = \max\{M, 2B\}$  in this case, and the result follows from (4.4.4) and Theorem 4.4.1.  $\square$

In fact, given a power-bounded operator  $T \in \mathcal{B}(X)$  satisfying  $\sigma(T) \cap \mathbb{T} \subset \{1\}$ , it follows by a simple Neumann series argument that analyticity of the map  $Q_T$  is equivalent to the existence of a constant  $C > 0$  such that

$$\|R(\lambda, T)\| \leq \frac{C}{|1 - \lambda|}$$

just for  $|\lambda| \geq 1$ ; see [76], [85], [86, Theorem 4.5.4] and also Theorem 5.2.3 in the next chapter. By Theorem 4.4.1, (4.4.6) is equivalent not only to the map  $Q_T$  being analytic but also to the  $C_0$ -semigroup  $S_T$  generated by  $T - I$  being a bounded analytic semigroup; see [87, Theorem 2.1]. An operator  $T$  satisfying these equivalent conditions is said to be a *Ritt operator*; various interesting results on Ritt operators may be found for instance in [8], [25], [26], [40], [42], [43], [44], [68], [79], [80] and [103].

This connection between bounded analytic  $C_0$ -semigroups and Ritt operators motivates the following definition of a discrete analogue of the non-analytic growth bound. Given an exponentially bounded function  $Q : \mathbb{Z}_+ \rightarrow \mathcal{B}(X)$ , let

$$A_Q(n) = \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} Q(j)$$

for all  $n \geq 0$  and write  $A_T$  instead of  $A_{Q_T}$ . Note that, for all  $t \geq 0$ ,

$$S_Q(t) = \sum_{n=0}^{\infty} A_Q(n) \frac{t^n}{n!}$$

and that  $S_T(t) = S_{Q_T}(t) = \exp(t(T - I))$  for  $T \in \mathcal{B}(X)$ . Furthermore, let

$$\omega_0(Q) = \inf \{r > 0 : \text{there exists } C > 0 \text{ s.t. } \|Q(n)\| \leq Cr^n \text{ for all } n \geq 0\}.$$

The *non-analytic growth bound*  $\eta(T)$  of a power-bounded operator  $T \in \mathcal{B}(X)$  is defined as

$$\eta(T) = \inf \{\omega_0(A_T - A_Q) : Q \text{ is analytic}\}.$$

Note that  $0 \leq \eta(T) \leq 2$  for any power-bounded operator  $T$ . The next result is a discrete analogue of [12, Proposition 2.1].

**Proposition 4.4.3.** *Let  $X$  be a complex Banach space and let  $T \in \mathcal{B}(X)$  be a power-bounded operator. Furthermore, let  $r \in (0, 2]$ . The following are equivalent:*

- (i) *There exists an analytic map  $Q : \mathbb{Z}_+ \rightarrow \mathcal{B}(X)$  such that  $\omega_0(A_T - A_Q) < r$ ;*

(ii) There exists  $s \in (0, r)$  such that  $\sigma(T) \cap \mathbb{T} \subset \{\lambda \in \mathbb{C} : |\lambda - 1| < s\}$  and  $\omega_0(A_T - A_{Q_s}) < r$ , where

$$Q_s(n) = \frac{1}{2\pi i} \int_{C_s} \lambda^n R(\lambda, T) d\lambda \quad (4.4.8)$$

for all  $n \geq 0$ ,  $C_s$  being any contour lying in  $\rho(T) \cap \mathbb{D}$  such that  $\sigma(T)$  can be enclosed by adding to  $C_s$  an arc contained in  $\{\lambda \in \mathbb{C} : |\lambda - 1| = s\}$ .

Moreover, when these conditions are satisfied then in fact  $\omega_0(A_T - A_{Q_s}) < r$  for all  $s \in (0, r)$  such that  $\sigma(T) \cap \mathbb{T} \subset \{\lambda \in \mathbb{C} : |\lambda - 1| < s\}$ .

PROOF. Suppose (i) holds and let  $\omega_0(A_T - A_Q) < s_0 < s < r$ . For  $n \geq 0$  let  $B(n) = A_T(n) - A_Q(n)$ . By assumption there exists  $C > 0$  such that  $\|B(n)\| \leq Cs_0^n$  for all  $n \geq 0$  and hence  $F_B$  extends analytically to  $\{\lambda \in \mathbb{C} : |\lambda| > s_0\}$ . Moreover, using the notation and also the Laplace transform argument in the proof of Theorem 4.4.1,

$$F_{A_Q}(\lambda) = (\mathcal{L}S_Q)(\lambda) = (\mathcal{L}U_Q)(\lambda + 1) = F_Q(\lambda + 1)$$

when  $\operatorname{Re} \lambda$  is sufficiently large. Since  $Q$  is analytic,  $F_{A_Q}$  extends analytically to the complement of  $\Omega_\theta - 1$  for some  $\theta > 0$ . Since  $A_T = B + A_Q$  and

$$F_{A_T}(\lambda) = \sum_{n=0}^{\infty} \frac{(T - I)^n}{\lambda^{n+1}} = R(\lambda + 1, T)$$

when  $|\lambda|$  is sufficiently large, it follows from the identity theorem that the resolvent set of  $T$  contains the complement of the set  $\Omega_\theta \cup \{\lambda \in \mathbb{C} : |\lambda - 1| \leq s_0\}$ . In particular,  $\sigma(T) \cap \mathbb{T} \subset \{\lambda \in \mathbb{C} : |\lambda - 1| < s\}$ . Furthermore,

$$A_T(n) = (T - I)^n = \frac{1}{2\pi i} \oint_{\Gamma} (\lambda - 1)^n R(\lambda, T) d\lambda$$

for any contour  $\Gamma$  that lies in the exterior of  $\Omega_\theta \cup \{\lambda \in \mathbb{C} : |\lambda - 1| \leq s_0\}$  and encloses  $\sigma(T)$ . Choose  $\Gamma = C_s \cup \gamma_s$ , where  $C_s$  is as in the statement of the result and  $\gamma_s$  is a suitable arc contained in  $\{\lambda \in \mathbb{C} : |\lambda - 1| = s\}$ . For  $Q_s$  as in (4.4.8) it then follows that there exists  $C > 0$  such that

$$\|A_T(n) - A_{Q_s}(n)\| = \left\| \frac{1}{2\pi i} \oint_{\gamma_s} (\lambda - 1)^n R(\lambda, T) d\lambda \right\| \leq Cs^n$$

for all  $n \geq 0$ , and hence  $\omega_0(A_T - A_{Q_s}) \leq s$ . Thus (ii) holds.

To show conversely that (ii)  $\implies$  (i) it suffices to note that, by Fubini's theorem,

$$S_{Q_s}(t) = \sum_{n=0}^{\infty} \frac{e^{-t}}{2\pi i} \int_{C_s} \frac{\lambda^n t^n}{n!} R(\lambda, T) d\lambda = \frac{1}{2\pi i} \int_{C_s} e^{(\lambda-1)t} R(\lambda, T) d\lambda$$

for all  $t \geq 0$  and

$$F_{Q_s}(\lambda) = \frac{1}{2\pi i} \int_{C_s} \frac{R(z, T)}{\lambda - z} dz$$

for  $|\lambda| > 1$ . Thus  $S_{Q_s}$  has a bounded analytic extension to any sector  $\Sigma_\theta$  with  $0 < \theta \leq \pi$  and  $F_{Q_s}$  extends analytically to the complement of  $C_s$ , thus in particular to  $\{\lambda \in \mathbb{C} \setminus \{1\} : |\lambda| \geq 1\}$ . It follows from Theorem 4.4.1 that  $Q_s$  is analytic.

To prove the final statement, suppose that  $s, s' \in (0, r)$  are such that  $\sigma(T) \cap \mathbb{T} \subset \{\lambda \in \mathbb{C} : |\lambda - 1| < \min\{s, s'\}\}$ . By Cauchy's theorem

$$A_{Q_s}(n) - A_{Q_{s'}}(n) = \frac{1}{2\pi i} \int_{\Gamma_{s,s'}} (\lambda - 1)^n R(\lambda, T) d\lambda$$

for all  $n \geq 0$ , where  $\Gamma_{s,s'}$  consists of any two curves lying in  $\rho(T) \cap \mathbb{D}$  that connect points in  $\{\lambda \in \mathbb{C} : |\lambda - 1| = s\}$  and  $\{\lambda \in \mathbb{C} : |\lambda - 1| = s'\}$  that lie in the upper and lower half-plane, respectively. Letting  $r_0 = \max\{s, s'\}$ , it is possible to choose  $\Gamma_{s,s'}$  to lie entirely within  $\{\lambda \in \mathbb{D} : |\lambda - 1| \leq r_0\}$ . It follows that there exists  $C > 0$  such that  $\|A_{Q_s}(n) - A_{Q_{s'}}(n)\| \leq Cr_0^n$  for all  $n \geq 0$ , and hence  $\omega_0(A_{Q_s} - A_{Q_{s'}}) \leq r_0$ . Since

$$\omega_0(A_T - A_{Q_{s'}}) \leq \max\{\omega_0(A_T - A_{Q_s}), \omega_0(A_{Q_s} - A_{Q_{s'}})\},$$

the claim follows.  $\square$

The following result shows that  $\eta(T)$  and  $r_\infty(T)$  not only take values in the same range for any power-bounded operator  $T \in \mathcal{B}(X)$  but in fact coincide. The result is a discrete analogue of Theorem 3.2.5 but does not require  $X$  to be a Hilbert space.

**Theorem 4.4.4.** *Let  $X$  be a complex Banach space and let  $T \in \mathcal{B}(X)$  be a power-bounded operator. Then  $\eta(T) = r_\infty(T)$ .*

PROOF. Suppose that  $r > r_\infty(T)$  and let  $s \in (r_\infty(T), r)$  be such that  $\sigma(T) \cap \mathbb{T} \subset \{\lambda \in \mathbb{C} : |\lambda - 1| < s\}$ . Letting  $Q_s$  denote the analytic map defined in (4.4.8), it follows by the same argument as in the proof of Proposition 4.4.3

that  $\omega_0(A_T - A_{Q_s}) \leq s$ . Hence  $\eta(T) < r$ , so  $\eta(T) \leq r_\infty(T)$ . Now suppose that  $r > \eta(T)$ . By Proposition 4.4.3 there exists  $s \in (0, r)$  such that  $\sigma(T) \cap \mathbb{T} \subset \{\lambda \in \mathbb{C} : |\lambda - 1| < s\}$ . Hence  $r_\infty(T) \leq s$ , so  $r_\infty(T) \leq \eta(T)$ . It follows that  $\eta(T) = r_\infty(T)$ .  $\square$

**Remark 4.4.5.** Suppose that  $T$  is a power-bounded operator on some complex Banach space  $X$ . Given  $r > \eta(T)$ , let  $Q : \mathbb{Z}_+ \rightarrow \mathcal{B}(X)$  be an analytic map such that  $\|A_T(n) - A_Q(n)\| \leq Cr^n$  for some constant  $C > 0$  and all  $n \geq 0$ . Then  $\|S_T(t) - S_Q(t)\| \leq Ce^{rt}$  for all  $t \geq 0$ , and it follows that  $\zeta(S_T) \leq \eta(T)$ . This bound is of no interest, however, as the semigroup  $S_T$  generated by  $T - I$  is bounded and hence satisfies the stronger estimate  $\zeta(S_T) \leq \omega_0(S_T) \leq 0$ . Suppose on the other hand that  $T$  is a bounded  $C_0$ -semigroup on a complex Banach space  $X$ . It follows from Theorem 4.2.2 and the discussion following it that  $\eta(T(t)) < 2$  for all sufficiently small  $t > 0$  whenever  $\zeta(T) < 0$ , and by Theorems 3.2.5 and 4.2.3 the two statements are equivalent when  $X$  is a Hilbert space. Determining the precise relationship between the discrete and the continuous non-analytic growth bounds remains a problem for future research.

## Part III

# Discrete operator semigroups

## Chapter 5

# Rates of decay in the classical Katznelson-Tzafriri theorem

### 5.1 Introduction

The Katznelson-Tzafriri theorem in its original and most basic form states the following; see [62, Theorem 1] and also [46].

**Theorem 5.1.1.** *Let  $X$  be a complex Banach space and  $T \in \mathcal{B}(X)$  a power-bounded operator. Then*

$$\lim_{n \rightarrow \infty} \|T^n(I - T)\| = 0 \tag{5.1.1}$$

*if and only if  $\sigma(T) \cap \mathbb{T} \subset \{1\}$ .*

This result, which has applications both in the theory of iterative methods and in the context of zero-two laws for Markov processes (see [86] and [62], [89], respectively), is a cornerstone of the asymptotic theory of operator semigroups and is shown in [33, Section 4] to be optimal in various ways. One aspect which so far has been investigated only in special cases is the *rate* at which the decay takes place in (5.1.1). The purpose of this chapter, then, is to relate this rate of decay to the growth of the resolvent  $\|R(e^{i\theta}, T)\|$  as  $\theta \rightarrow 0$ ; see [37], [41], [86, Chapter 4], [87] and [88] for previous results in this area.

The main results of this chapter are Corollary 5.2.6 and Theorem 5.2.12. The first establishes a rate of decay which, under certain conditions, is the fastest that can occur, while the latter adapts a contour integral argument used in [15] and [78] to yield a slowest possible rate of decay in (5.1.1). The

question of whether Theorem 5.2.12 is optimal, at least in an important special case, will form the main subject of Chapter 6. The material in this chapter and the next is based on [97].

## 5.2 Main results

Let  $X$  be a complex Banach space and let  $T \in \mathcal{B}(X)$  be a power-bounded operator. Since rate of decay in (5.1.1) is necessarily exponential when  $\sigma(T) \cap \mathbb{T} = \emptyset$ , the real interest lies in the case when  $\sigma(T) \cap \mathbb{T} = \{1\}$ . In order to obtain estimates on the rate of decay in this situation, it will be convenient to have in place a few additional pieces of notation. Thus, given an operator  $T$  as above, a decreasing function  $m : (0, \pi] \rightarrow (0, \infty)$  such that  $\|R(e^{i\theta}, T)\| \leq m(|\theta|)$  for all  $\theta$  with  $0 < |\theta| \leq \pi$  will be said to be a *dominating function (for the resolvent of  $T$ )*. Likewise a decreasing function  $\omega : \mathbb{Z}_+ \rightarrow (0, \infty)$  such that  $\|T^n(I - T)\| \leq \omega(n)$  for all  $n \in \mathbb{Z}_+$  will be said to be a *dominating function (for  $T$ )*. The *minimal* dominating functions are given, for  $\theta \in (0, \pi]$  and  $n \geq 0$ , by

$$\begin{aligned} m(\theta) &= \sup \{ \|R(e^{i\vartheta}, T)\| : \theta \leq |\vartheta| \leq \pi \}, \\ \omega(n) &= \sup \{ \|T^k(I - T)\| : k \geq n \}, \end{aligned} \tag{5.2.1}$$

respectively. Thus, for the minimal dominating function  $\omega$  of  $T$ ,  $\omega(n) \rightarrow 0$  as  $n \rightarrow \infty$  precisely when (5.1.1) holds. Note also that the function  $m$  defined in (5.2.1) is continuous. In what follows, the same will be assumed to be true of any dominating function  $m$  for the resolvent of  $T$ . In particular, any such dominating function  $m$  possesses a right-inverse  $m^{-1}$  defined on the range of  $m$ . On the other hand, given a dominating function  $\omega$  for  $T$  which satisfies  $\omega(n) \rightarrow 0$  as  $n \rightarrow \infty$ , define the function  $\omega^* : (0, \infty) \rightarrow \mathbb{Z}_+$  by

$$\omega^*(s) = \min \{ n \in \mathbb{Z}_+ : \omega(n) \leq s \}. \tag{5.2.2}$$

Then  $\omega(\omega^*(s)) \leq s$  for all  $s > 0$ , with equality for all  $s$  in the range of  $\omega$ .

### 5.2.1 An upper bound on the rate of decay

Recall that, given any complex Banach space  $X$  and any  $T \in \mathcal{B}(X)$ ,

$$\|R(\lambda, T)\| \geq \frac{1}{\text{dist}(\lambda, \sigma(T))} \tag{5.2.3}$$

for all  $\lambda \in \rho(T)$ . Thus if  $1 \in \sigma(T)$  then  $m(\theta) \geq \theta^{-1}$  for all  $\theta \in (0, \pi]$ , so there is a minimal rate at which the resolvent of any operator  $T$  as above must blow up

near its singularity. This might suggest that there should exist a corresponding minimal rate, independent of the operator  $T$ , at which decay takes place in (5.1.1). This is false, however, as Corollary 5.2.6 below will show; see also [3, Theorem 4.2] for a related observation in the context of Banach algebras. The next result, on the other hand, shows that instances in which the decay is faster than that of  $n^{-1}$  are of a very special nature. It is a direct analogue of [14, Theorem 6.9]; see also [87, Remarks 2.3 and 2.4].

**Theorem 5.2.1.** *Let  $X$  be a complex Banach space and let  $T \in \mathcal{B}(X)$  be such that  $\sigma(T) \cap \mathbb{T} = \{1\}$ . Then either*

$$\limsup_{n \rightarrow \infty} n \|T^n(I - T)\| > 0 \tag{5.2.4}$$

*or there exist closed  $T$ -invariant subspaces  $X_0$  and  $X_1$  of  $X$  such that  $X_0 \subset \text{Fix}(T)$ , the restriction  $T_1$  of  $T$  to  $X_1$  satisfies  $r(T_1) < 1$ , and  $X = X_0 \oplus X_1$ .*

PROOF. Supposing first that 1 is a limit point of  $\sigma(T)$ , let  $\lambda_j \in \sigma(T) \setminus \{1\}$  be such that  $\lambda_j \rightarrow 1$  as  $j \rightarrow \infty$  and set  $n_j = \lfloor |1 - \lambda_j|^{-1} \rfloor$ . Since the spectral radius  $r(T^{n_j}(I - T))$  of  $T^{n_j}(I - T)$  is bounded by the norm of this operator for each  $n_j \geq 0$ , it follows that

$$\limsup_{n \rightarrow \infty} n \|T^n(I - T)\| \geq \lim_{j \rightarrow \infty} \frac{n_j}{n_j + 1} \left(1 - \frac{1}{n_j}\right)^{n_j} = e^{-1},$$

and hence (5.2.4) holds.

If 1 is an isolated point of  $\sigma(T)$ , on the other hand, then a standard spectral decomposition argument (see for instance [6, Proposition B.9]) shows that there exist closed  $T$ -invariant subspaces  $X_0$  and  $X_1$  of  $X$  and a bounded projection  $P$  of  $X$  onto  $X_1$  along  $X_0$  which commutes with  $T$ . In particular,  $X = X_0 \oplus X_1$ . Moreover, the restrictions  $T_0$  and  $T_1$  of  $T$  to  $X_0$  and  $X_1$  satisfy  $\sigma(T_0) = \{1\}$  and  $\sigma(T_1) = \sigma(T) \setminus \{1\}$ , respectively. Now, if (5.2.4) fails, then

$$\liminf_{n \rightarrow \infty} n \|T_0^n(I - T_0)\| = 0$$

and it follows from [61, Theorem 2.2] that  $Tx = x$  for all  $x \in X_0$ . □

**Remark 5.2.2.** It is easy to see that, if  $X$  splits, then in fact  $X_0 = \text{Fix}(T)$  and  $X_1 = \text{Ran}(I - T)$ . In particular,  $\text{Ran}(I - T)$  is closed; see also [86, Theorem 4.4.2].

Thus the rate of decay as  $n \rightarrow \infty$  in (5.1.1) is either at least exponential or no faster than  $n^{-1}$ . The case of decay at this borderline rate is connected with the class of Ritt operators introduced in Section 4.4. Ritt operators will be treated as part of a more general framework in the following chapter. The reason why they are important in the present context is that a power-bounded operator  $T$  satisfying  $\sigma(T) \cap \mathbb{T} = \{1\}$  is a Ritt operator if and only if  $\|R(e^{i\theta}, T)\| = O(|\theta|^{-1})$  as  $|\theta| \rightarrow 0$ ; see for instance the proof of Lemma 6.4.1 below. The following result shows that these operators are precisely those for which the rate of decay in (5.1.1) is no slower than  $n^{-1}$ . This characterisation was obtained independently in [76], [85] and [86, Theorem 4.5.4].

**Theorem 5.2.3.** *Let  $X$  be a complex Banach space. An operator  $T \in \mathcal{B}(X)$  is a Ritt operator if and only if it is power-bounded and*

$$\|T^n(I - T)\| = O(n^{-1})$$

as  $n \rightarrow \infty$ .

Thus decay in (5.1.1) at a rate no slower than  $n^{-1}$  already implies a strong condition on the growth of the resolvent near its singularity at 1. The next result establishes a corresponding resolvent bound in a rather more general situation; see [14, Theorem 6.10] for an analogous result in the setting of  $C_0$ -semigroups.

**Theorem 5.2.4.** *Let  $X$  be a complex Banach space and let  $T \in \mathcal{B}(X)$  be a power-bounded operator. Suppose that  $\omega$  is a dominating function for  $T$  such that  $\omega(n) \rightarrow 0$  as  $n \rightarrow \infty$  and let  $\omega^*$  be as defined in (5.2.2). Then  $\sigma(T) \cap \mathbb{T} \subset \{1\}$  and, for any  $c \in (0, 1)$ ,*

$$\|R(e^{i\theta}, T)\| = O\left(\frac{1}{|\theta|} + \omega^*(c|\theta|)\right) \quad (5.2.5)$$

as  $|\theta| \rightarrow 0$ .

PROOF. Suppose that  $\lambda \in \sigma(T) \cap \mathbb{T}$ . By the spectral mapping theorem for polynomials,  $\lambda^n(1 - \lambda) \in \sigma(T^n(I - T))$  and hence

$$|1 - \lambda| = |\lambda^n(1 - \lambda)| \leq \omega(n)$$

for all  $n \geq 0$ . Letting  $n \rightarrow \infty$ , it follows that  $\lambda = 1$ , so  $\sigma(T) \cap \mathbb{T} \subset \{1\}$ .

Now let  $\lambda \in \mathbb{T} \setminus \{1\}$ . Then, for  $n \geq 0$ ,

$$\lambda^n(1 - \lambda) - T^n(I - T) = (1 - \lambda)\lambda^{n-1} \sum_{k=0}^{n-1} \lambda^{-k} T^k(\lambda - T) - T^n(\lambda - T)$$

and hence, letting  $M = \sup\{\|T^n\| : n \geq 0\}$ ,

$$|1 - \lambda|\|R(\lambda, T)x\| \leq \omega(n)\|R(\lambda, T)x\| + M(1 + n|1 - \lambda|)\|x\|$$

for all  $x \in X$ . Fix  $b \in (c, 1)$  and let  $n = \omega^*(b|1 - \lambda|)$ . Then

$$\|R(\lambda, T)\| \leq \frac{M}{1 - b} \left( \frac{1}{|1 - \lambda|} + \omega^*(b|1 - \lambda|) \right)$$

and, since  $b|1 - \lambda| \geq c|\theta|$  for  $\lambda = e^{i\theta}$  with sufficiently small  $\theta \in (-\pi, \pi) \setminus \{0\}$ , the result follows.  $\square$

**Remark 5.2.5.** A similar argument shows that, given any constant  $K > M$ , where  $M$  is as above, there exists  $c \in (0, 1)$  such that

$$\|R(e^{i\theta}, T)\| \leq K \left( \frac{1}{|\theta|} + \omega^*(c|\theta|) \right) \quad (5.2.6)$$

whenever  $|\theta|$  is sufficiently small. Note also that, by (5.2.3), the  $|\theta|^{-1}$  term in (5.2.5) and (5.2.6) cannot in general be omitted.

In analogy with [14, Corollary 6.11], these observations can be used to obtain a lower bound on the quantity  $\|T^n(I - T)\|$  when  $n \geq 0$  is large, thus giving a fastest possible rate of decay in (5.1.1).

**Corollary 5.2.6.** *Let  $X$  be a complex Banach space, let  $T \in \mathcal{B}(X)$  be a power-bounded operator such that  $\sigma(T) \cap \mathbb{T} = \{1\}$  and let  $m$  be the minimal dominating function for the resolvent of  $T$  defined in (5.2.1). Suppose that*

$$\lim_{\theta \rightarrow 0} \max \{ \|\theta R(e^{i\theta}, T)\|, \|\theta R(e^{-i\theta}, T)\| \} = \infty. \quad (5.2.7)$$

*Then, given any right-inverse  $m^{-1}$  of  $m$ , there exist constants  $c, C > 0$  such that*

$$\|T^n(I - T)\| \geq cm^{-1}(Cn) \quad (5.2.8)$$

*for all sufficiently large  $n \geq 0$ .*

PROOF. Let  $\omega$  be as defined in (5.2.1). Since  $\omega(n) \rightarrow 0$  as  $n \rightarrow \infty$  by Theorem 5.1.1, it follows from Theorem 5.2.4 that there exists  $B > 0$  such that

$$m(\theta) \leq B \left( \frac{1}{\theta} + \omega^*(\theta/2) \right)$$

for all sufficiently small  $\theta \in (0, \pi]$ , and hence

$$\omega^*(\theta/2) \geq m(\theta) \left( \frac{1}{B} - \frac{1}{\theta m(\theta)} \right) \quad (5.2.9)$$

for all such values of  $\theta$ . Let  $C = 2B$  and, for  $n \geq 0$ , let  $\theta_n = 2\omega(n)$ . By (5.2.7),  $\theta_n m(\theta_n) > C$  for all sufficiently large  $n \geq 0$ , so (5.2.9) implies that  $\omega^*(\theta_n/2) > C^{-1}m(\theta_n)$  for each such  $n \geq 0$ . Since  $\omega^*(\theta_n/2) \leq n$  and therefore

$$m(m^{-1}(Cn)) = Cn \geq C\omega^*(\theta_n/2) > m(\theta_n),$$

it follows that  $m^{-1}(Cn) < \theta_n$  for all sufficiently large  $n \geq 0$ . Moreover,  $\theta_n \leq 2M\|T^n(I - T)\|$  for all  $n \geq 0$ , where  $M = \sup\{\|T^n\| : n \geq 0\}$ . Hence (5.2.8) holds for  $c = (2M)^{-1}$ .  $\square$

**Remark 5.2.7.** A similar argument using Remark 5.2.5 instead of Theorem 5.2.4 shows that the conclusion (5.2.8) remains true if (5.2.7) is replaced by the weaker condition that  $L > M$ , where  $M$  is as above and

$$L = \liminf_{\theta \rightarrow 0} \max \{ \|\theta R(e^{i\theta}, T)\|, \|\theta R(e^{-i\theta}, T)\| \}.$$

Taking  $T$  to be the identity operator shows that the conclusion can become false when  $L = M$ .

### 5.2.2 A lower bound on the rate of decay

Suppose that  $T$  is a power-bounded operator such that  $\sigma(T) \cap \mathbb{T} = \{1\}$  and let  $m$  be the minimal dominating function for the resolvent of  $T$  defined in (5.2.1). If  $T$  is a Ritt operator, then it follows from Theorem 5.2.3 that, for any  $c \in (0, 1)$ ,

$$\|T^n(I - T)\| = O(m^{-1}(cn)) \quad (5.2.10)$$

as  $n \rightarrow \infty$  and, in view of Corollary 5.2.6, this type of upper bound is in general the best possible. The next result describes the class of functions  $m$  for which such an upper bound is satisfied in the case of a normal operator on a Hilbert space; see also [14, Proposition 6.13].

**Proposition 5.2.8.** *Let  $X$  be a complex Hilbert space and let  $T \in \mathcal{B}(X)$  be a normal operator such that  $\sigma(T) \subset \mathbb{D} \cup \{1\}$ . Furthermore, let  $m$  be the minimal dominating function for the resolvent of  $T$  defined in (5.2.1), let  $m^{-1}$  be any right-inverse of  $m$  and let  $S \subset \mathbb{N}$ .*

1. Suppose there exist constants  $c, C > 0$  such that

$$\|T^n(I - T)\| \leq Cm^{-1}(cn) \quad (5.2.11)$$

for all  $n \in S$ . Then, for any  $b \in (0, c)$ , there exists a constant  $B > 0$  such that

$$\frac{m(\theta)}{m(\vartheta)} \geq b \log \frac{\vartheta}{\theta} - B, \quad (5.2.12)$$

for all  $\theta \in (0, \pi]$  of the form  $\theta = m^{-1}(cn)$  with  $n \in S$  and for all sufficiently small  $\vartheta \in (0, \pi]$ .

2. Conversely, if there exist constants  $b, B > 0$  such that (5.2.12) holds for all  $\theta \in (0, \pi]$  of the form  $\theta = m^{-1}(bn)$  with  $n \in S$  and all  $\vartheta \in (0, \pi]$ , then there exists a constant  $C > 0$  such that (5.2.11) holds with  $c = b$ .

PROOF. Note first that, for  $\theta \in (0, \pi]$ ,

$$m(\theta)^{-1} = \min \{ |\lambda - e^{i\varphi}| : \lambda \in \sigma(T), \theta \leq |\varphi| \leq \pi \},$$

and that (5.2.11) is equivalent to having

$$n \log \frac{1}{|\lambda|} \geq \log \frac{|1 - \lambda|}{Cm^{-1}(cn)} \quad (5.2.13)$$

for all  $\lambda \in \sigma(T) \setminus \{1\}$  and all  $n \in S$ .

Suppose this holds and let  $\theta = m^{-1}(cn)$  for some  $n \in S$ . Then

$$m(\theta) \geq \frac{c}{\log \frac{1}{|\lambda|}} \log \frac{|1 - \lambda|}{C\theta}$$

for all  $\lambda \in \sigma(T) \setminus \{1\}$ . Define the function  $g : (0, 1) \rightarrow \mathbb{R}$  by

$$g(s) = \frac{s - 1}{\log s}.$$

Then  $g$  is a continuous increasing function satisfying  $g(s) \rightarrow 1$  as  $s \rightarrow 1$ , and in fact

$$g(s) = \inf \left\{ \frac{r - 1}{\log r} : s < r < 1 \right\}$$

for all  $s \in (0, 1)$ . Thus, given any  $b \in (0, c)$ , there exists  $s_0 \in (0, 1)$  such that  $cg(s_0) > b$ . Now suppose that  $\vartheta \in (0, 1 - s_0)$ , let  $\lambda \in \sigma(T)$  be such that  $m(\vartheta) = |\lambda - e^{i\varphi}|^{-1}$  for some  $\varphi \in (0, \pi]$  with  $|\varphi| \geq \vartheta$ , and let  $r = |\lambda|$ . Since

$m(\vartheta) \geq \vartheta^{-1}$ , it follows from the estimate  $1 - r \leq |\lambda - e^{i\varphi}|$  that  $r > s_0$ . Thus, if  $|1 - \lambda| \geq \frac{\vartheta}{2}$ , then

$$\frac{m(\theta)}{m(\vartheta)} \geq \frac{c|\lambda - e^{i\varphi}|}{\log \frac{1}{r}} \log \frac{|1 - \lambda|}{C\theta} \geq b \log \frac{\vartheta}{2C\theta},$$

which gives (5.2.12) with  $B = b \log 2C$ . If  $|1 - \lambda| < \frac{\vartheta}{2}$ , on the other hand, then  $|\lambda - e^{i\varphi}| \geq \frac{\vartheta}{3}$  and hence

$$\frac{m(\theta)}{m(\vartheta)} \geq \frac{\vartheta}{3\theta} \geq b \log \left( \frac{\theta}{3b\vartheta} \right),$$

which gives (5.2.12) with  $B = b \log 3b$ . This proves the first statement with  $B = b \max\{\log 2C, \log 3b\}$ .

Now suppose, conversely, that (5.2.12) holds for some constants  $b, B > 0$ , all  $\theta \in (0, \pi]$  of the form  $\theta = m^{-1}(cn)$  with  $n \in S$  and all  $\vartheta \in (0, \pi]$ . Let  $\lambda = re^{i\phi} \in \sigma(T) \setminus \{1\}$ , and set  $\vartheta = |\phi|$ , so that

$$\log \frac{1}{r} \geq 1 - r = |e^{i\phi} - \lambda| \geq \frac{1}{m(\vartheta)}.$$

Hence, if  $\vartheta \geq \frac{1}{2}|1 - \lambda|$ , then (5.2.12) gives

$$n \log \frac{1}{r} \geq \frac{1}{b} \frac{m(m^{-1}(bn))}{m(\vartheta)} \geq \log \left( \frac{\vartheta}{m^{-1}(bn)} \right) - \frac{B}{b} \geq \log \left( \frac{|1 - \lambda|}{2m^{-1}(bn)} \right) - \frac{B}{b},$$

thus establishing (5.2.13) with  $c = b$  and  $C = 2e^{B/b}$ . On the other hand, if  $\vartheta < \frac{1}{2}|1 - \lambda|$ , then

$$1 - r \geq |1 - \lambda| - |e^{i\phi} - 1| \geq |1 - \lambda| - \vartheta > \frac{|1 - \lambda|}{2}$$

and consequently

$$n \log \frac{1}{r} \geq n(1 - r) \geq \frac{|1 - \lambda|}{2b} m(m^{-1}(bn)) \geq \frac{|1 - \lambda|}{2bm^{-1}(bn)} \geq \log \left( \frac{|1 - \lambda|}{2bm^{-1}(bn)} \right),$$

which gives (5.2.13) with  $c = b$  and  $C = 2b$ . Thus taking  $C = 2 \max\{e^{B/b}, b\}$  completes the proof.  $\square$

**Remark 5.2.9.** The conclusion of Proposition 5.2.8 remains true, with the same proof, for any complex Banach space  $X$  and any power-bounded operator  $T \in \mathcal{B}(X)$  satisfying

$$\|f(T)\| = \sup \{ |f(\lambda)| : \lambda \in \sigma(T) \}$$

for all functions  $f$  of the form  $f(\lambda) = \lambda^n(1 - \lambda)$  with  $n \geq 0$  or  $f(\lambda) = (\mu - \lambda)^{-1}$  with  $\mu \in \rho(T)$ . This includes, in particular, the class of multiplication operators on any of the classical function or sequence spaces. Note also that the second of the two implications holds more generally when  $m$  is an arbitrary dominating function for the resolvent of  $T$ .

Thus (5.2.11) holds for a normal operator  $T$  if and only if the minimal dominating function  $m(\theta)$  for the resolvent of  $T$  grows in a fairly regular way as  $\theta \rightarrow 0$ . The following example, which is an adaptation of [6, Example 4.4.15], exhibits a class of normal operators for which this is not the case.

**Example 5.2.10.** Let  $X = \ell^2(\mathbb{N})$ . Given a strictly increasing sequence  $(r_k)$  of positive terms such that  $r_k \rightarrow 1$  as  $k \rightarrow \infty$ , let  $\lambda_k = r_k e^{i/k}$  and consider the operator  $T \in \mathcal{B}(X)$  given by  $Tx = (\lambda_k x_k)$ . Then  $T$  is a normal contraction with  $\sigma(T) = \{\lambda_k : k \geq 1\} \cup \{1\}$ . Moreover, if

$$r_k > 1 - \frac{1}{2(1+k)^2}$$

for all  $k \geq 1$ , then, for  $j \neq k$ ,

$$|e^{i/k} - \lambda_j| \geq |e^{i/k} - \lambda_{k+1}| \geq \frac{1}{2} \left( \frac{1}{k} - \frac{1}{k+1} \right) > 1 - r_k = |e^{i/k} - \lambda_k|$$

and hence

$$m(k^{-1}) = \|R(e^{i/k}, T)\| = \frac{1}{1 - r_k}$$

for all  $k \geq 1$ , where  $m$  is the minimal dominating function for the resolvent of  $T$  defined in (5.2.1). Suppose furthermore that

$$\log r_{k!+1} \geq 2 \log r_{(k+1)!}$$

for all  $k \geq 1$  and, given  $c > 0$ , let  $b \in (0, c)$  and

$$n_k = \left\lceil \frac{-1}{b \log r_{(k+1)!}} \right\rceil.$$

Then

$$m(((k+1)!)^{-1}) = \frac{1}{1 - r_{(k+1)!}} \sim -\frac{1}{\log r_{(k+1)!}} \sim bn_k$$

as  $k \rightarrow \infty$ , and hence

$$m^{-1}(cn_k) \leq \frac{1}{(k+1)!}$$

for all sufficiently large  $k \geq 1$ . Since

$$|1 - \lambda_{k!+1}| \geq \frac{1}{3k!}$$

and  $|\lambda_{k!+1}^{n_k}| \geq e^{-2/b}$  for all  $k \geq 1$ , it follows that

$$\|T^{n_k}(I - T)\| \geq |\lambda_{k!+1}^{n_k}(1 - \lambda_{k!+1})| \geq \frac{1}{3e^{2/b}k!} \geq \frac{k}{3e^{2/b}} m^{-1}(cn_k)$$

when  $k \geq 1$  is sufficiently large. In particular, (5.2.11) fails for every  $c > 0$ .

Thus, given a power-bounded operator  $T$  such that  $\sigma(T) \cap \mathbb{T} = \{1\}$ , a right-inverse  $m^{-1}$  of some dominating function  $m$  for the resolvent of  $T$  and a constant  $c \in (0, 1)$ , the asymptotic upper bound (5.2.10) fails to hold in general. The next result shows that it is nevertheless possible to obtain an upper bound of this kind provided the function  $m$  is modified appropriately. Indeed, given an operator  $T$  as above and a dominating function  $m$  for the resolvent of  $T$ , define the function  $m_{\log} : (0, \pi] \rightarrow (0, \infty)$  by

$$m_{\log}(\theta) = m(\theta) \log \left( 1 + \frac{m(\theta)}{\theta} \right), \quad (5.2.14)$$

noting that this function is strictly decreasing and hence possesses a well-defined inverse  $m_{\log}^{-1}$  defined on the range of  $m_{\log}$ . As Theorem 5.2.12 below shows, (5.2.10) is valid when  $m^{-1}$  is replaced by  $m_{\log}^{-1}$ . This raises the question by how much the asymptotic behaviour of these two functions differs in particular instances; see also [15, Example 1.4] and [78, Section 2].

**Example 5.2.11.** (a) If  $m(\theta) = Ce^{\alpha/\theta}$ , where  $C, \alpha > 0$  are constants, then

$$m_{\log}^{-1}(s) \sim \frac{\alpha}{\log s}$$

as  $s \rightarrow \infty$ , so  $m_{\log}^{-1}$  has the same asymptotic behaviour as  $m^{-1}$  in this case.

(b) If  $m(\theta) = C\theta^{-\alpha}$  for some constants  $C > 0$  and  $\alpha \geq 1$ , then

$$m_{\log}^{-1}(s) \sim \left( \frac{\log s}{s} \right)^{1/\alpha}$$

as  $s \rightarrow \infty$ , so  $m_{\log}^{-1}$  differs from  $m^{-1}$  by a logarithmic factor.

The following theorem is the main result of this chapter. It is a discrete analogue of [14, Theorem 6.15] and [78, Proposition 3.1], which in turn are variants of [15, Theorem 1.5]; see also [49, Chapter VI], where similar techniques are discussed in the context of combinatorial problems.

**Theorem 5.2.12.** *Let  $X$  be a complex Banach space and let  $T \in \mathcal{B}(X)$  be a power-bounded operator such that  $\sigma(T) \cap \mathbb{T} = \{1\}$ . Furthermore, let  $m$  be a dominating function for the resolvent of  $T$  and let  $m_{\log}$  be as defined in (5.2.14). Then, for any  $c \in (0, 1)$ ,*

$$\|T^n(I - T)\| = O(m_{\log}^{-1}(cn))$$

as  $n \rightarrow \infty$ .

PROOF. Throughout this proof, the letter  $C$  will stand for a positive constant which will typically be thought of as being large and which need not be the same at each occurrence.

Having fixed a dominating function  $m$  and a constant  $c \in (0, 1)$ , let  $\Omega$  denote the closure of the set

$$\left\{ re^{i\theta} \in \mathbb{C} : 0 \leq r \leq 1 - \frac{c}{m(|\theta|)}, 0 < |\theta| \leq \pi \right\}.$$

Noting that  $\sigma(T) \subset \Omega$  by a standard Neumann series argument, define the function  $F_n : \mathbb{C} \setminus \Omega \rightarrow \mathcal{B}(X)$  by

$$F_n(\lambda) = T^n(2 - T)(I - (\lambda - 1)R(\lambda, T)).$$

It then follows from the resolvent identity that

$$F_n(\lambda) = T^n(2 - T)^2 R(\lambda, T)(I - R(2, T)), \quad (5.2.15)$$

and hence  $F_n(2) = T^n(I - T)$ . Thus, by Cauchy's integral formula,

$$T^n(I - T) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{h(\lambda)}{\lambda - 2} F_n(\lambda) d\lambda,$$

where  $\Gamma$  is any contour outside  $\Omega$  around the point 2 and where  $h$  is any function that is holomorphic in the relevant region and satisfies  $h(2) = 1$ . In what follows, it will be convenient to take  $\Gamma = \Gamma_{\text{in}} \cup \Gamma_{\text{out}}$  to consist of an outer contour  $\Gamma_{\text{out}}$ , which encloses both the point 2 and the set  $\Omega$ , and an inner contour  $\Gamma_{\text{in}}$ , which lies in the interior of  $\Gamma_{\text{out}}$  and encloses  $\Omega$  but not the point

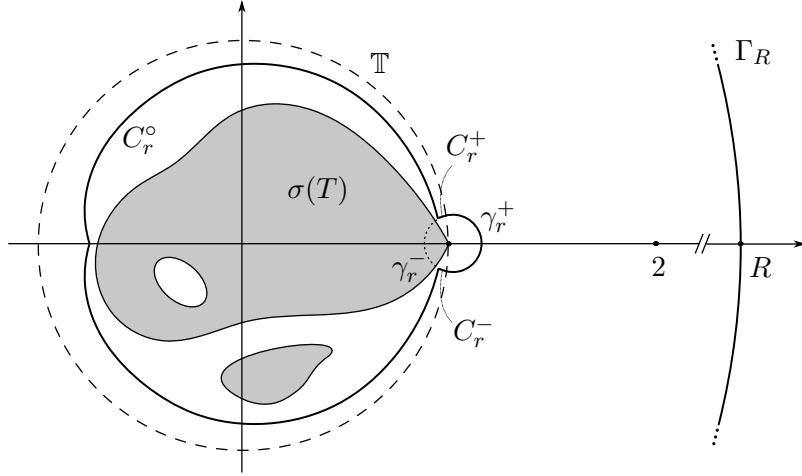


Figure 5.1: The contour  $\Gamma = \Gamma_{\text{in}} \cup \Gamma_{\text{out}}$ , where  $\Gamma_{\text{out}} = \Gamma_R$  and  $\Gamma_{\text{in}} = C_r \cup \gamma_r^+$  with  $C_r = C_r^\circ \cup C_r^+ \cup C_r^-$ .

$\lambda = 2$ . Such a contour can be thought of as being closed by inserting a cut from any point on  $\Gamma_{\text{in}}$  to any point on  $\Gamma_{\text{out}}$ , the contributions along which cancel out.

Let  $\varphi$  be the Cayley transform defined by

$$\varphi(\lambda) = \frac{1 - \lambda}{1 + \lambda}$$

and, for  $r \in (0, 1)$  and  $R > 0$ , let

$$\gamma_r = \left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{1+r^2}{1-r^2} \right| = \frac{2r}{1-r^2} \right\} \quad (5.2.16)$$

and  $\Gamma_R = \{\lambda \in \mathbb{C} : |\lambda - 1| = R\}$ , noting that  $\varphi$  maps  $\gamma_r$  onto  $r\mathbb{T}$ , the real line onto itself and the unit circle  $\mathbb{T}$  onto the imaginary axis. Now suppose that  $r \in (0, \frac{1}{3})$  and  $R > 2$ , and let  $\Gamma_{\text{out}} = \Gamma_R$  and  $\Gamma_{\text{in}} = C_r \cup \gamma_r^+$ , where  $\gamma_r^+$  denotes the part of  $\gamma_r$  that lies outside  $\mathbb{D}$  and where  $C_r$  is any suitable path in  $\mathbb{D} \setminus \Omega$  connecting the endpoints of  $\gamma_r^+$ . Furthermore, choose for  $h$  the map  $h_r$  given by

$$h_r(\lambda) = \frac{1}{1 + 9r^2} \left( 1 + \frac{r^2}{\varphi(\lambda)^2} \right),$$

so that  $h_r$  is holomorphic away from the point  $\lambda = 1$ . Now, letting  $M = \sup\{\|T^n\| : n \geq 0\}$ , it follows from the series expansion of the resolvent that

$$\|R(\lambda, T)\| \leq \frac{M}{|\lambda| - 1} \quad (5.2.17)$$

whenever  $|\lambda| > 1$  and hence, by (5.2.15) and the fact that  $T$  is power-bounded,

$$\|F_n\| \leq \frac{C}{|\lambda| - 1}$$

for all such  $\lambda$ , where  $C$  is independent of  $n \geq 0$ . Since  $h_r$  is bounded above in modulus independently of  $r$  along  $\Gamma_R$ , it follows that

$$\left\| \oint_{\Gamma_R} \frac{h_r(\lambda)}{\lambda - 2} F_n(\lambda) d\lambda \right\| \leq \frac{C}{R},$$

where  $C$  is independent of  $n \geq 0$ , and hence, appealing to Cauchy's theorem and allowing  $R \rightarrow \infty$ , this contribution can be neglected.

Next note that, for  $\lambda \in r\mathbb{T}$ , the function  $g_r$  defined by

$$g_r(\lambda) = 1 + \frac{r^2}{\lambda^2}$$

satisfies  $|g_r(\lambda)| = 2r^{-1}|\operatorname{Re} \lambda|$ . Moreover, an elementary calculation shows that, for  $\lambda \in \gamma_r$ ,

$$1 - |\lambda|^2 = \frac{4 \operatorname{Re} \varphi(\lambda)}{1 + 2 \operatorname{Re} \varphi(\lambda) + r^2}. \quad (5.2.18)$$

Since

$$h_r = \frac{g_r \circ \varphi}{1 + 9r^2}$$

and  $\varphi(\gamma_r) = r\mathbb{T}$ , it follows that

$$|h_r(\lambda)| \leq C \frac{|\operatorname{Re} \varphi(\lambda)|}{r} \leq C \frac{||\lambda| - 1|}{r} \quad (5.2.19)$$

for all  $\lambda \in \gamma_r$ . But for each  $\lambda \in \gamma_r$ ,  $|\lambda| - 1 \leq Cr$  and  $|1 - \lambda| \leq Cr$  so, by (5.2.17) and the definition of  $F_n$ ,

$$\|F_n(\lambda)\| \leq C \|I - (\lambda - 1)R(\lambda, T)\| \leq \frac{Cr}{|\lambda| - 1}$$

for all  $\lambda \in \gamma_r^+$ . Hence

$$\left\| \int_{\gamma_r^+} \frac{h_r(\lambda)}{\lambda - 2} F_n(\lambda) d\lambda \right\| \leq Cr,$$

where  $C$  is independent of  $n \geq 0$ , and it remains to control only the contribution along  $C_r$ .

Let  $\theta_r \in (0, \frac{\pi}{2})$  denote the argument of the point at which  $\gamma_r$  meets  $\mathbb{T}$  in the upper half-plane and define the curve  $C_r^\circ$ , for  $\theta_r \leq |\theta| \leq \pi$ , by

$$C_r^\circ(\theta) = \left( 1 - \frac{c}{m(|\theta|)} \right) e^{i\theta}.$$

Furthermore, let  $C_r^\pm$  denote the rays given, for  $1 - cm(\theta_r)^{-1} \leq s \leq 1$ , by  $C_r^\pm(s) = se^{\pm i\theta_r}$  and set  $C_r = C_r^\circ \cup C_r^+ \cup C_r^-$ . Defining

$$p_n(\lambda) = \sum_{k=0}^{n-1} \lambda^{n-k-1} T^k$$

for  $\lambda \in \mathbb{C} \setminus \Omega$ , it follows from the resolvent identity, the relation

$$p_n(\lambda) = (\lambda^n - T^n)R(\lambda, T)$$

and some elementary manipulations that

$$F_n(\lambda) = \frac{1}{\lambda - 2} (2 - T)^2 \left( (\lambda - 1)(\lambda^n R(\lambda, T) - p_n(\lambda)) - T^n R(2, T) \right)$$

for all  $\lambda \in \mathbb{C} \setminus \Omega$  with  $\lambda \neq 2$ ; see also [78, Lemma 2.2]. Hence Cauchy's theorem gives

$$\begin{aligned} R(2, T)^2 \int_{C_r} \frac{h_r(\lambda)}{\lambda - 2} F_n(\lambda) d\lambda &= \int_{C_r} \frac{h_r(\lambda)(\lambda - 1)\lambda^n}{(\lambda - 2)^2} R(\lambda, T) d\lambda \\ &\quad - \int_{\gamma_r^-} \frac{h_r(\lambda)}{(\lambda - 2)^2} ((\lambda - 1)p_n(\lambda) + T^n R(2, T)) d\lambda, \end{aligned}$$

where  $\gamma_r^- = \gamma_r \cap \mathbb{D}$ . To estimate the first integral on the right-hand side, note first that, by a standard Neumann series argument,

$$\|R(\lambda, T)\| \leq \frac{m(\theta_r)}{1 - c}$$

for all  $\lambda \in C_r^\circ$ . Since  $h_r$  is uniformly bounded independently of  $r$  along  $C_r^\circ$ , it follows that

$$\left\| \int_{C_r^\circ} \frac{h_r(\lambda)(\lambda - 1)\lambda^n}{(\lambda - 2)^2} R(\lambda, T) d\lambda \right\| \leq Cm(\theta_r) \left( 1 - \frac{c}{m(\theta_r)} \right)^n.$$

Similarly,

$$\left\| \int_{C_r^\pm} \frac{h_r(\lambda)(\lambda - 1)\lambda^n}{(\lambda - 2)^2} R(\lambda, T) d\lambda \right\| \leq C \int_{1 - cm(\theta_r)^{-1}}^1 s^n ds \leq \frac{C}{n + 1}.$$

To bound the integral along  $\gamma_r^-$ , note that

$$\|(\lambda - 1)p_n(\lambda)\| \leq \frac{Cr}{1 - |\lambda|}$$

for all  $\lambda \in \gamma_r^-$ . Thus by (5.2.19) both  $h_r(\lambda)$  and  $h_r(\lambda)(\lambda-1)p_n(\lambda)$  are uniformly bounded, independently of  $r$  and  $n$ , as  $\lambda$  ranges over  $\gamma_r^-$ , and it follows that

$$\left\| \int_{\gamma_r^-} \frac{h_r(\lambda)}{(\lambda-2)^2} ((\lambda-1)p_n(\lambda) + T^n R(2, T)) d\lambda \right\| \leq Cr,$$

where  $C$  is independent of  $n \geq 0$ .

Since  $C^{-1}r \leq \theta_r \leq Cr$ , combining these bounds gives

$$\|T^n(I - T)\| \leq C \left( \theta_r + \frac{1}{n+1} + m(\theta_r) \left( 1 - \frac{c}{m(\theta_r)} \right)^n \right),$$

where  $C$  is independent of  $n \geq 0$  and  $r \in (0, \frac{1}{3})$ . Now, if  $n \geq 0$  is sufficiently large, choosing  $r \in (0, \frac{1}{3})$  so as to satisfy  $\theta_r = m_{\log}^{-1}(cn)$  gives

$$\exp\left(\frac{cn}{m(\theta_r)}\right) = 1 + \theta_r^{-1}m(\theta_r)$$

and hence

$$m(\theta_r) \left( 1 - \frac{c}{m(\theta_r)} \right)^n \leq m(\theta_r) \exp\left(-\frac{cn}{m(\theta_r)}\right) \leq \theta_r.$$

Since  $(n+1)^{-1} \leq Cm_{\log}^{-1}(cn)$  for all  $n \geq 0$ , the proof is complete.  $\square$

**Remark 5.2.13.** Applying the above arguments to the rotated operator  $T_\theta = e^{-i\theta}T$ , it is clear that the condition  $\sigma(T) \cap \mathbb{T} = \{1\}$  can be replaced, in the previous result and elsewhere in this chapter, by the condition  $\sigma(T) \cap \mathbb{T} = \{e^{i\theta}\}$  for some  $\theta \in [0, 2\pi)$ . In this case the conclusion of Theorem 5.2.12 becomes that, for any  $c \in (0, 1)$ ,

$$\|T^n(e^{i\theta} - T)\| = O(m_{\theta, \log}^{-1}(cn))$$

as  $n \rightarrow \infty$ , where the function  $m_{\theta, \log} : (0, \pi] \rightarrow (0, \infty)$  is defined, in analogy with (5.2.14), in terms of a given dominating function  $m_\theta$  for the resolvent of  $T_\theta$ . In fact, by an argument analogous to the one outlined in [78, Section 3.2], it is possible to extend Theorem 5.2.12 to certain cases with finite unitary spectrum. Supposing that  $\sigma(T) \cap \mathbb{T} = \Lambda$ , where  $\Lambda = \{e^{i\theta_j} : 1 \leq j \leq k\}$  for some  $k \in \mathbb{N}$  and  $0 \leq \theta_1 \leq \dots \leq \theta_k < 2\pi$ , let

$$\theta_0 = \min \left( \left\{ \frac{\theta_{j+1} - \theta_j}{2} : 1 \leq j \leq k-1 \right\} \cup \left\{ \pi + \frac{\theta_1 - \theta_k}{2} \right\} \right)$$

and define the decreasing function  $m_\Lambda : (0, \theta_0] \rightarrow (0, \infty)$  by

$$m_\Lambda(\theta) = \sup \left\{ \|R(e^{i\vartheta}, T)\| : \vartheta \in [0, 2\pi) \text{ with } |\vartheta - \theta_j| \geq \theta \text{ for all } j \right\}.$$

Then, given any  $c \in (0, 1)$ ,

$$\|T^n p(T)\| = O(m_{\Lambda, \log}^{-1}(cn))$$

as  $n \rightarrow \infty$ , where

$$p_{\Lambda}(z) = \prod_{\lambda \in \Lambda} (\lambda - z)$$

and  $m_{\Lambda, \log} : (0, \theta_0] \rightarrow (0, \infty)$  is defined in terms of  $m_{\Lambda}$  as in (5.2.14).

## Chapter 6

# Rates of decay for polynomial resolvent growth

### 6.1 Introduction

The purpose of this chapter is to investigate more closely the general results of Chapter 5 in the special case where the resolvent growth is at most polynomial. Given a power-bounded operator  $T \in \mathcal{B}(X)$  on a complex Banach space with  $\sigma(T) \cap \mathbb{T} = \{1\}$ , the resolvent growth is said to be polynomial if the resolvent of  $T$  admits a dominating function of the form  $m(\theta) = C\theta^{-\alpha}$  for some constants  $C > 0$  and  $\alpha \geq 1$ . Here the restriction on the parameter  $\alpha$  is a consequence of (5.2.3). If the resolvent  $R(e^{i\theta}, T)$  of  $T$  happens also to grow no more slowly than  $|\theta|^{-\alpha}$  as  $\theta \rightarrow 0$ , Corollary 5.2.6 and Theorem 5.2.12 combine to give the following result, which describes the range of decay rates that are possible in this situation.

**Corollary 6.1.1.** *Let  $X$  be a complex Banach space and let  $T \in \mathcal{B}(X)$  be a power-bounded operator such that  $\sigma(T) \cap \mathbb{T} = \{1\}$ . Suppose that, for some  $\alpha \geq 1$ ,*

$$\|R(e^{i\theta}, T)\| \asymp |\theta|^{-\alpha}$$

*as  $\theta \rightarrow 0$ . Then there exist constants  $c, C > 0$  such that*

$$\frac{c}{n^{1/\alpha}} \leq \|T^n(I - T)\| \leq C \left( \frac{\log n}{n} \right)^{1/\alpha} \quad (6.1.1)$$

*for all sufficiently large  $n \geq 0$ .*

The remainder of this chapter is concerned with the question whether the logarithmic factor on the right-hand side of (6.1.1) is in fact needed. In Section 6.2, the question is first investigated in the case of a simple but illustrative example involving a Toeplitz operator. Here the logarithmic factor turns out not to be required, but Section 6.3 then shows that this is not true in general. Finally, Section 6.4 turns to the Hilbert space setting, showing both that the logarithmic factor can be omitted in this case (Theorem 6.4.2) and that this result is robust under perturbations by certain finite-rank operators (Theorem 6.4.7).

## 6.2 An example

Let  $X = \ell^p(\mathbb{Z}_+)$ ,  $1 \leq p \leq \infty$ , and write  $S$  for the right-shift operator on  $X$  given by  $(Sx)_k = x_{k+1}$ . Define the operator  $T \in \mathcal{B}(X)$  as  $T = \frac{1}{4}(I + S)^2$ . Then  $T$  is a non-normal Toeplitz operator which has unit norm and spectrum

$$\sigma(T) = \left\{ re^{i\theta} \in \mathbb{C} : -\pi \leq \theta \leq \pi \text{ and } 0 \leq r \leq \frac{1 + \cos \theta}{2} \right\}.$$

In particular,  $\sigma(T) \cap \mathbb{T} = \{1\}$ ; see Figure 6.1.

A simple calculation using the estimate (5.2.3) shows that  $\|R(e^{i\theta}, T)\| \geq 4|\theta|^{-2}$  as  $\theta \rightarrow 0$ , which by Corollary 5.2.6 implies that the decay of  $\|T^n(I - T)\|$  as  $n \rightarrow \infty$  can be no faster than that of  $n^{-1/2}$ .

Finding an expression for the resolvent operator  $R(\lambda, T)$  for  $\lambda \in \rho(T)$ , and hence an upper bound on its norm as  $\lambda$  approaches 1 through values in  $\mathbb{T}$ , requires solving the recurrence relations

$$\left(\lambda - \frac{1}{4}\right)y_k - \frac{1}{2}y_{k+1} - \frac{1}{4}y_{k+2} = x_k$$

for  $k \geq 0$  and a fixed  $x \in X$ . Iterating this equation and letting  $\mu = \lambda - \frac{1}{4}$  shows that the components of the solution  $y = R(\lambda, T)x$  are given by

$$y_k = \sum_{n=0}^{\infty} \frac{p_n(\mu)}{2^n \mu^{n+1}} x_{k+n}, \quad (6.2.1)$$

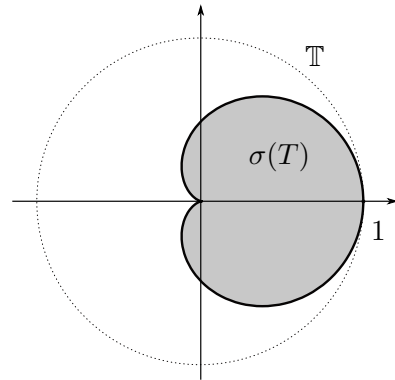


Figure 6.1: The spectrum  $\sigma(T)$  of  $T$  is enclosed by a cardioid.

where  $p_0, p_1, \dots$  are polynomials satisfying the recurrence relation

$$p_{n+2}(z) = p_{n+1}(z) + zp_n(z)$$

for all  $n \geq 0$  and all  $z \in \mathbb{C}$ . Solving this yields

$$p_n(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} z^k$$

or, alternatively,

$$p_n(z) = \frac{(1 + \sqrt{1+4z})^{n+1} - (1 - \sqrt{1+4z})^{n+1}}{2^{n+1}\sqrt{1+4z}},$$

where the complex square root is defined with the plane being cut along  $(-\infty, -\frac{1}{4}]$  in the  $z$ -plane. Using this in (6.2.1) gives

$$y_k = \frac{1}{\sqrt{\lambda}} \sum_{n=0}^{\infty} (-1)^{n+1} \left( \frac{1}{(1-2\sqrt{\lambda})^{n+1}} - \frac{1}{(1+2\sqrt{\lambda})^{n+1}} \right) x_{k+n},$$

and a simple estimate now shows that, for  $p \in \{1, \infty\}$ ,

$$\|R(\lambda, T)\| = O\left(\frac{1}{|1-2\sqrt{\lambda}| - 1}\right)$$

as  $\lambda \rightarrow 1$  through  $\rho(T)$ . By interpolation, the statement remains true for  $1 \leq p \leq \infty$ . In particular, therefore,  $\|R(e^{i\theta}, T)\| = O(|\theta|^{-2})$  as  $\theta \rightarrow 0$  and hence, by Example 5.2.11(b) and Theorem 5.2.12,

$$\|T^n(I - T)\| = O\left(\left(\frac{\log n}{n}\right)^{1/2}\right).$$

In order to obtain the actual rate of decay in this example, and hence to determine whether this upper bound is in fact optimal, recall first that, for  $p = 1$  and  $p = \infty$ , the norm of an operator on  $X$  is simply the supremum of the absolute column or row sums, respectively, of the matrix representing the operator with respect to the standard basis. It follows in this example that, for each  $n \geq 0$ , the norms  $\|T^n(I - T)\|$  are the same for  $p = 1$  and for  $p = \infty$ . In fact, for  $p = 1$

$$\|T^n(I - T)\| = \|T^n(I - T)e_0\|_1,$$

where  $e_0 = (1, 0, 0, \dots)$ , and a simple calculation shows that for  $p = \infty$

$$\|T^n(I - T)\| = \|T^n(I - T)x^{(n)}\|_{\infty},$$

where

$$x_k^{(n)} = \begin{cases} -1 & \text{for } 0 \leq k \leq z^{(n)} \\ +1 & \text{for } z^{(n)} < k \leq 2n \\ 0 & \text{for } k > 2n \end{cases}$$

with

$$z^{(n)} = \frac{1}{2} \left( 3 + 4n - \sqrt{4n^2 + 6n + 3} \right).$$

Since  $z^{(n)} = n + \frac{3}{4} + O(n^{-1})$  as  $n \rightarrow \infty$ , it follows that for  $p \in \{1, \infty\}$

$$\begin{aligned} \|T^n(I - T)\| &= (T^n(I - T)x^{(n)})_{2n} \\ &= \frac{1}{4^{n+1}} \sum_{j=0}^{2n} \binom{2n}{j} (3x_j^{(n)} - 2x_{j+1}^{(n)} - x_{j+2}^{(n)}) \\ &= \frac{1}{4^{n+1}} \left( 2 \binom{2n}{n-1} + 6 \binom{2n}{n} \right) \end{aligned}$$

when  $n$  is sufficiently large and hence, by Stirling's formula,

$$\|T^n(I - T)\| \sim \frac{2}{\sqrt{\pi n}}$$

as  $n \rightarrow \infty$ . By interpolation

$$\|T^n(I - T)\| \asymp n^{-1/2}$$

as  $n \rightarrow \infty$  for  $1 \leq p \leq \infty$ , so the logarithmic factor in Corollary 6.1.1 is not required in this case.

**Remark 6.2.1.** In this example the lower bound on the rate of decay follows straightforwardly from Corollary 5.2.6 and the geometry of  $\sigma(T)$ . However, asymptotic behaviour of type  $\|T^n(I - T)\| \asymp n^{-1/\alpha}$  as  $n \rightarrow \infty$  for some  $\alpha \geq 1$  can arise even when  $\sigma(T) = \{1\}$ , the smallest possible spectrum of interest here. Indeed, it is shown in [73] that for each  $\alpha \in (1, 2)$  there exists a contraction  $T$  on a Hilbert space such that  $\sigma(T) = \{1\}$  and  $\|T^n(I - T)\| \asymp n^{-1/\alpha}$  as  $n \rightarrow \infty$ . This answered a question raised in [107]; see also [77].

### 6.3 Optimality of Theorem 5.2.12

The purpose of this section is to exhibit a family of Banach spaces and an operator on each of these spaces for which the logarithmic factor in Corollary 6.1.1

cannot be omitted. This is achieved in Theorem 6.3.4 by considering the restriction of the left-shift operator to a certain Banach space continuously embedded in  $\ell^\infty(\mathbb{N})$ . As Theorem 6.4.2 in the following section will show, the situation changes if the underlying space is required to be a Hilbert space. All of these results are discrete analogues of statements contained in [27]. The proof of Theorem 6.3.4 requires two lemmas. The first is a variant of [6, Lemma 4.6.6], which itself is a special form of Levinson's log-log theorem; see for instance [65, VII D7].

**Lemma 6.3.1.** *Let  $X$  be a complex Banach space, let  $\theta \in (-\pi, \pi]$  and let  $\Omega$  be a neighbourhood of the point  $e^{i\theta} \in \mathbb{T}$ . Furthermore, given  $r \in (0, 1)$ , let*

$$\Omega_{r,\theta} = \left\{ \lambda \in \mathbb{C} : \left| \lambda - e^{i\theta} \frac{1+r^2}{1-r^2} \right| \leq \frac{2r}{1-r^2} \right\}. \quad (6.3.1)$$

*Then, for some constant  $C > 0$ , the following holds: If  $r \in (0, \frac{1}{4})$  is such that  $\Omega_{2r,\theta} \subset \Omega$  and if  $F : \Omega \rightarrow X$  is a holomorphic function such that, for some constant  $B > 0$ ,*

$$\|F(\lambda)\| \leq \frac{B}{|1-|\lambda||}$$

*for all  $\lambda \in \partial\Omega_{2r,\theta} \setminus \mathbb{T}$ , then  $\|F(\lambda)\| \leq BCr^{-1}$  for all  $\lambda \in \Omega_{r,\theta}$ .*

PROOF. Assume without loss of generality that  $\theta = 0$  and, as in the proof of Theorem 5.2.12, let  $\varphi$  denote the Möbius transformation defined by

$$\varphi(\lambda) = \frac{1-\lambda}{1+\lambda},$$

so that  $\varphi$  maps the circle  $\gamma_r = \partial\Omega_{r,0}$  onto  $r\mathbb{T}$  for each  $r \in (0, 1)$ . Moreover, by (5.2.18) with  $r$  replaced by  $2r$ , there exists a constant  $C' > 0$  which is independent of  $r \in (0, \frac{1}{4})$  and such that  $|\operatorname{Re} \varphi(\lambda)| \leq C' \left| |\lambda| - 1 \right|$  for all  $\lambda \in \gamma_{2r}$ . Consider the function  $G : \Omega \rightarrow X$  defined by

$$G(\lambda) = \left( 1 + \frac{\varphi(\lambda)^2}{4r^2} \right) F(\lambda).$$

For  $\lambda \in \gamma_{2r}$ ,

$$\left| 1 + \frac{\varphi(\lambda)^2}{4r^2} \right| = \frac{|\operatorname{Re} \varphi(\lambda)|}{r}$$

and hence, by the assumption on  $F$ ,  $\|G(\lambda)\| \leq BC'r^{-1}$  for all such  $\lambda$ . Since  $\Omega_{r,0} \subset \Omega_{2r,0}$ , it follows from the maximum principle that  $\|G(\lambda)\| \leq BC'r^{-1}$  for all  $\lambda \in \Omega_{r,0}$ . But if  $\lambda \in \Omega_{r,0}$ , then  $|\varphi(\lambda)| \leq r$  and hence  $\|G(\lambda)\| \geq \frac{3}{4}\|F(\lambda)\|$ , which gives the result with  $C = \frac{4}{3}C'$ .  $\square$

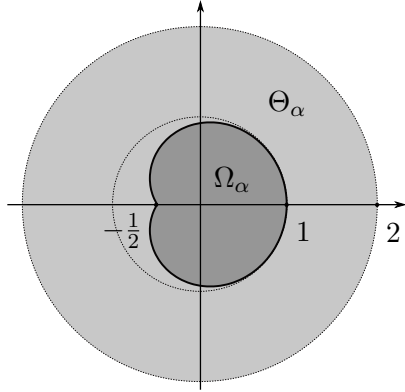


Figure 6.2: The regions  $\Omega_\alpha$  and  $\Theta_\alpha$  for a typical value of  $\alpha > 1$ .

The second auxiliary result is a technical one and analogous to [27, Lemma 3.9]. Given  $\alpha \geq 1$  and  $\lambda \in \mathbb{C} \setminus \{0\}$ , let

$$K_\alpha(\lambda) = \frac{|\arg \lambda|^\alpha}{2\pi^\alpha}, \quad (6.3.2)$$

where the argument of a complex number is taken to lie in  $(-\pi, \pi]$ , and define the regions  $\Omega_\alpha, \Theta_\alpha \subset \mathbb{C}$  by

$$\begin{aligned} \Omega_\alpha &= \{\lambda \in \mathbb{C} \setminus \{0\} : |\lambda| \leq 1 - K_\alpha(\lambda)\} \cup \{0\}, \\ \Theta_\alpha &= \{\lambda \in \mathbb{C} \setminus \{0\} : 1 - K_\alpha(\lambda) < |\lambda| < 2\}, \end{aligned} \quad (6.3.3)$$

respectively, so that  $\Theta_\alpha = 2\mathbb{D} \setminus \Omega_\alpha$ ; see Figure 6.2. Furthermore, given a complex measure  $\mu$  whose support is contained in  $\Omega_\alpha$ , define the transforms  $\mathcal{C}_\alpha\mu$ ,  $\mathcal{L}_\alpha\mu$  and  $\mathcal{D}_\alpha\mu$ , for  $\lambda \in \Theta_\alpha$ ,  $k \geq 1$  and  $n \geq 0$ , respectively, by

$$\begin{aligned} (\mathcal{C}_\alpha\mu)(\lambda) &= \int_{\Omega_\alpha} \frac{d\mu(z)}{\lambda - z}, \\ (\mathcal{L}_\alpha\mu)(k) &= \int_{\Omega_\alpha} z^{k-1} d\mu(z), \\ (\mathcal{D}_\alpha\mu)(n) &= \int_{\Omega_\alpha} z^n (1 - z) d\mu(z). \end{aligned} \quad (6.3.4)$$

**Lemma 6.3.2.** *Suppose that  $\alpha > 2$  and let the function  $K_\alpha$ , the regions  $\Omega_\alpha$  and  $\Theta_\alpha$ , and the transforms  $\mathcal{C}_\alpha$ ,  $\mathcal{L}_\alpha$  and  $\mathcal{D}_\alpha$  be defined as in (6.3.2), (6.3.3) and (6.3.4), respectively. Then there exists a constant  $C > 0$  with the following property: Given any  $n_0 \in \mathbb{N}$ , there exists a complex measure  $\mu$  whose support is contained in  $\Omega_\alpha$  and for which*

- (i)  $K_\alpha(\lambda)|(\mathcal{C}_\alpha\mu)(\lambda)| \leq C$  for all  $\lambda \in \Theta_\alpha$ ;

(ii)  $|(\mathcal{L}_\alpha\mu)(k)| \leq C$  for all  $k \geq 1$ ;

(iii)  $|(\mathcal{D}_\alpha\mu)(n_1)|^\alpha \geq (Cn_1)^{-1} \log n_1$  for some  $n_1 > n_0$ .

PROOF. Throughout this proof, the letter  $C$ , if used without having been introduced explicitly, will stand for a positive constants, which will be thought of as being large and which need not be the same at each occurrence. Choose  $\theta \in (0, \frac{1}{2})$  and  $\beta \in (\frac{\alpha}{32}, \frac{\alpha}{16})$  in such a way that  $\ell = -\beta\theta^{-\alpha} \log \theta$  is an integer satisfying  $\ell > \frac{n_0}{2} + 2$  and that  $\theta^{-(\alpha-2)} > 2\alpha\beta^{-1} + 1$ . Now let

$$\mu = \frac{B_\ell^{\ell-1}}{\ell^{1/2}} \sum_{r=0}^{\ell-1} \zeta_\ell^r \left(1 + \frac{\zeta_\ell^r}{2B_\ell\lambda_0}\right) \delta_{\lambda_0 + \frac{\zeta_\ell^r}{2B_\ell}},$$

where  $B_\ell = 2\ell \log_2 \ell$ ,  $\zeta_\ell = e^{2\pi i/\ell}$  and  $\lambda_0 = \frac{1}{2}e^{i\theta}$ , and where  $\delta_\lambda$  denotes the Dirac measure concentrated at  $\lambda$ .

Then, for any  $\lambda \in \Theta_\alpha$ ,

$$(\mathcal{C}_\alpha\mu)(\lambda) = \frac{B_\ell^{\ell-1}}{\ell^{1/2}} \sum_{r=0}^{\ell-1} \left( \frac{2B_\ell\zeta_\ell^r}{2B_\ell(\lambda - \lambda_0) - \zeta_\ell^r} + \frac{1}{\lambda_0} \frac{\zeta_\ell^{2r}}{2B_\ell(\lambda - \lambda_0) - \zeta_\ell^r} \right).$$

However, for  $1 \leq j \leq \ell$  and  $\lambda \in \mathbb{C}$  such that  $\lambda^\ell \neq 1$ ,

$$\sum_{r=0}^{\ell-1} \frac{\zeta_\ell^{jr}}{\lambda - \zeta_\ell^r} = \frac{\ell\lambda^{j-1}}{\lambda^\ell - 1}$$

(see also the proof of [27, Lemma 3.9]), and applying this with  $j = 1, 2$  gives

$$(\mathcal{C}_\alpha\mu)(\lambda) = \frac{\lambda}{\lambda_0} \frac{2\ell^{1/2}}{2^\ell(\lambda - \lambda_0)^\ell - B_\ell^{-\ell}} \quad (6.3.5)$$

for all  $\lambda \in \Theta_\alpha$ . Since  $B_\ell^{-1} \leq |\lambda - \lambda_0|$  and  $|\lambda| \leq 2$  for all  $\lambda \in \Theta_\alpha$ , this in turn becomes

$$|(\mathcal{C}_\alpha\mu)(\lambda)| \leq \frac{C\ell^{1/2}}{2^\ell|\lambda - \lambda_0|^\ell}. \quad (6.3.6)$$

Let  $\lambda \in \Theta_\alpha$  be given. If  $K_\alpha(\lambda) > \theta^\alpha$ , then  $|\arg \lambda| > \pi\theta$  and an elementary geometric argument involving the assumption that  $\alpha > 2$  gives

$$|\lambda - \lambda_0| \geq \frac{1}{2} + \frac{1}{2}(1 - \cos((\pi - 1)\theta) - K_\alpha(e^{i\pi\theta})) \geq \frac{1}{2}(1 + \theta^2 - \theta^\alpha).$$

Since  $1 + \theta^2 - \theta^\alpha \geq e^{\frac{1}{2}(\theta^2 - \theta^\alpha)}$  for all  $\theta \in (0, \frac{1}{2})$ , it follows from (6.3.6) that

$$|(\mathcal{C}_\alpha\mu)(\lambda)| \leq C\ell^{1/2}e^{-\frac{\ell}{2}(\theta^2 - \theta^\alpha)} = C(-\beta \log \theta)^{1/2}\theta^{-\frac{\alpha}{2} + \frac{\beta}{2}(\theta^{-(\alpha-2)} - 1)}.$$

Now the choices of  $\theta$  and  $\beta$  ensure that the exponent of  $\theta$  on the right-hand side of this expression is strictly greater than  $\frac{\alpha}{2}$ , and hence  $|(\mathcal{C}_\alpha\mu)(\lambda)|$  and consequently  $K_\alpha(\lambda)|(\mathcal{C}_\alpha\mu)(\lambda)|$  are uniformly bounded, independently of  $\theta$  and  $\beta$ , for all  $\lambda \in \Theta_\alpha$  satisfying  $K_\alpha(\lambda) > \theta^\alpha$ . If  $K_\alpha(\lambda) \leq \theta^\alpha$ , on the other hand, then  $|\lambda - \lambda_0| \geq \frac{1}{2}(1 - 2\theta^\alpha)$  and, using the fact that  $1 - 2\theta^\alpha \geq e^{-4\theta^\alpha}$  for all  $\theta \in (0, \frac{1}{2})$ , (6.3.6) gives

$$|(\mathcal{C}_\alpha\mu)(\lambda)| \leq C\ell^{1/2}e^{4\ell\theta^\alpha} = C(-\beta \log \theta)^{1/2}\theta^{-(\frac{\alpha}{2}+4\beta)}.$$

Since the choice of  $\beta$  ensures that  $\frac{\alpha}{2} + 4\beta < \frac{3\alpha}{4}$ ,  $K_\alpha(\lambda)|(\mathcal{C}_\alpha\mu)(\lambda)|$  is uniformly bounded, again independently of  $\theta$  and  $\beta$ , also for all  $\lambda \in \Theta_\alpha$  with  $K_\alpha(\lambda) \leq \theta^\alpha$ . This establishes (i) for  $C = C_1$ , where  $C_1 > 0$  is some suitably large constant.

Next observe that, for each  $k \geq 1$ ,

$$(\mathcal{L}_\alpha\mu)(k) = \frac{B_\ell^{\ell-1}\lambda_0^{k-1}}{\ell^{1/2}} \sum_{r=0}^{\ell-1} \zeta_\ell^r \left(1 + \frac{\zeta_\ell^r}{2B_\ell\lambda_0}\right)^k. \quad (6.3.7)$$

Expanding and using the fact that, for any integer  $s \geq 0$ ,

$$\sum_{r=0}^{\ell-1} \zeta_\ell^{r(s+1)} = \begin{cases} \ell & \text{if } s+1 = 0 \pmod{\ell}, \\ 0 & \text{otherwise,} \end{cases}$$

this becomes

$$\begin{aligned} (\mathcal{L}_\alpha\mu)(k) &= \frac{B_\ell^{\ell-1}\lambda_0^{k-1}}{\ell^{1/2}} \sum_{s=0}^k \sum_{r=0}^{\ell-1} \binom{k}{s} \frac{\omega_\ell^{r(s+1)}}{(2B_\ell\lambda_0)^s} \\ &= \ell^{1/2} B_\ell^{\ell-1} \lambda_0^{k-1} \sum_{r=1}^{\lfloor \frac{k+1}{\ell} \rfloor} \frac{\binom{k}{r\ell-1}}{(2B_\ell\lambda_0)^{r\ell-1}}. \end{aligned} \quad (6.3.8)$$

Next note that, for  $1 \leq r \leq \lfloor \frac{k+1}{\ell} \rfloor$ ,

$$\binom{k}{r\ell-1} \leq \binom{k}{\ell-1} \frac{(\ell-1)!}{(r\ell-1)!} k^{(r-1)\ell}.$$

Thus, for  $1 \leq k < B_\ell$ ,

$$|(\mathcal{L}_\alpha\mu)(k)| \leq \frac{\ell^{1/2}}{2^{k-1}} \binom{k}{\ell-1} \sum_{r=1}^{\lfloor \frac{k+1}{\ell} \rfloor} \frac{(\ell-1)!}{(r\ell-1)!} \leq C \frac{\ell^{1/2}}{2^k} \binom{k}{\ell-1},$$

where  $C$  is independent of  $\ell$ . Now, if  $k \leq 2\ell - 3$ , then  $\binom{k}{\ell-1} \leq \frac{1}{2} \binom{k+1}{\ell-1}$  and, if  $k \geq 2\ell - 2$ , then  $\binom{k}{\ell-1} \leq 2 \binom{k-1}{\ell-1}$ , so in either case

$$\frac{1}{2^k} \binom{k}{\ell-1} \leq \frac{1}{2^{2\ell-2}} \binom{2\ell-2}{\ell-1}.$$

Hence

$$|(\mathcal{L}_\alpha\mu)(k)| \leq C \frac{\ell^{1/2}}{4^{\ell-1}} \binom{2\ell-2}{\ell-1}, \quad (6.3.9)$$

which by Stirling's formula is bounded above independently of  $\ell$ . If  $k \geq B_\ell$ , on the other hand, then (6.3.7) and the fact that  $1 + B_\ell^{-1} \leq e^{1/B_\ell}$  give

$$|(\mathcal{L}_\alpha\mu)(k)| \leq \frac{\ell^{1/2} B_\ell^{\ell-1}}{2^{k-1}} \left(1 + \frac{1}{B_\ell}\right)^k \leq 2\ell^{1/2} B_\ell^{\ell-1} e^{-k(\log 2 - B_\ell^{-1})} \leq C \frac{\ell^{1/2} B_\ell^{\ell-1}}{2^{B_\ell}}$$

and, by the definition of  $B_\ell$ , the right-hand side is again bounded above independently of  $\ell$ . Thus (ii) holds with  $C = C_2$  for some sufficiently large  $C_2 > 0$ .

In order to establish (iii), note first that, for each  $n \geq 0$ ,

$$(\mathcal{D}_\alpha\mu)(n) = \frac{B_\ell^{\ell-1} \lambda_0^n}{\ell^{1/2}} \sum_{r=0}^{\ell-1} \zeta_\ell^r \left(1 + \frac{\zeta_\ell^r}{2B_\ell\lambda_0}\right)^{n+1} \left(1 - \lambda_0 \left(1 + \frac{\zeta_\ell^r}{2B_\ell\lambda_0}\right)\right).$$

Thus, if  $n \geq 0$  is such that  $\lfloor \frac{n+2}{\ell} \rfloor = \lfloor \frac{n+3}{\ell} \rfloor$ , proceeding as in (6.3.8) gives

$$(\mathcal{D}_\alpha\mu)(n) = \ell^{1/2} B_\ell^{\ell-1} \lambda_0^n \sum_{r=1}^{\lfloor \frac{n+2}{\ell} \rfloor} \frac{\binom{n+1}{r\ell-1} - \lambda_0 \binom{n+2}{r\ell-1}}{(2B_\ell\lambda_0)^{r\ell-1}}.$$

Now let  $n_1 = 2\ell - 4$ , so that  $n_1 > n_0$  and  $\lfloor \frac{n_1+2}{\ell} \rfloor = \lfloor \frac{n_1+3}{\ell} \rfloor = 1$ . Then

$$|(\mathcal{D}_\alpha\mu)(n_1)| = \frac{\ell^{1/2}}{4^{\ell-2}} \binom{2\ell-2}{\ell-1} \left| \frac{1}{2} - \lambda_0 \right|,$$

and hence, by another application of Stirling's formula,  $|(\mathcal{D}_\alpha\mu)(n_1)| \geq c\theta$ , where  $c$  is independent of  $\theta$ . Since the definition of  $\ell$  implies that  $\theta^\alpha \geq cn_1^{-1} \log n_1$ , it follows that (iii) holds for  $C = C_3$ , where  $C_3 > 0$  is another suitably large constant. Setting  $C = \max\{C_1, C_2, C_3\}$  now completes the proof.  $\square$

**Remark 6.3.3.** The estimates leading to (6.3.9) may be viewed from a probabilistic point of view. Indeed, given  $\ell \geq 1$ , let  $Y_\ell$  be the random variable counting the number of tosses of a fair coin required in order to obtain a total of exactly  $\ell$  heads, so that  $Y_\ell$  has the negative binomial distribution with

$$\mathbb{P}(Y_\ell = k) = \frac{1}{2^k} \binom{k-1}{\ell-1}$$

for each  $k \geq 1$ . Then the aforementioned estimates amount to the observation that  $Y_\ell$  has mode  $2\ell - 1$ . This probabilistic interpretation will reappear in Remark 6.3.5 below.

The following result, which is an analogue of [27, Theorem 4.1], shows that the logarithmic factor in Corollary 6.1.1 cannot in general be omitted.

**Theorem 6.3.4.** *Given any  $\alpha > 2$ , there exists a complex Banach space  $X_\alpha$  and a power-bounded operator  $T \in \mathcal{B}(X_\alpha)$  such that  $\sigma(T) \cap \mathbb{T} = \{1\}$  and*

$$\|R(e^{i\theta}, T)\| = O(|\theta|^{-\alpha})$$

as  $\theta \rightarrow 0$ , and for which

$$\limsup_{n \rightarrow \infty} \|T^n(I - T)\| \left( \frac{n}{\log n} \right)^{1/\alpha} > 0. \quad (6.3.10)$$

PROOF. Throughout this proof, the letters  $c$  and  $C$  will be used to denote positive constants which will be thought of, respectively, as being small and large and which need not be the same at each occurrence.

Given any sequence  $x \in \ell^\infty(\mathbb{N})$ , define the function  $F_x$ , for  $|\lambda| > 1$ , by

$$F_x(\lambda) = \sum_{k=1}^{\infty} \frac{x_k}{\lambda^k}.$$

Now, with  $K_\alpha$ ,  $\Omega_\alpha$  and  $\Theta_\alpha$  as defined in (6.3.2) and (6.3.3), let  $X_\alpha$  denote the subspace of  $\ell^\infty(\mathbb{N})$  consisting of sequences  $x$  for which  $F_x$  extends analytically to  $\Theta_\alpha$  and satisfies

$$\sup \{K_\alpha(\lambda)|F_x(\lambda)| : \lambda \in \Theta_\alpha\} < \infty.$$

This space is non-trivial, containing for instance any finitely supported sequence as well as the constant sequence  $(1, 1, 1, \dots)$ , and, by an application of Vitali's theorem, it is complete under the norm  $\|\cdot\|_{X_\alpha}$  given by  $\|x\|_{X_\alpha} = \|x\|_\infty + \|x\|_\alpha$ , where

$$\|x\|_\alpha = \sup \{K_\alpha(\lambda)|F_x(\lambda)| : \lambda \in \Theta_\alpha\}.$$

Consider the restriction  $T = S|_{X_\alpha}$  to  $X_\alpha$  of the left-shift operator  $S \in \mathcal{B}(\ell^\infty)$ .

Given  $x \in X_\alpha$  and  $|\lambda| > 1$ ,

$$F_{Tx}(\lambda) = \lambda F_x(\lambda) - x_1,$$

so  $F_{Tx}$  extends analytically to  $\Theta_\alpha$  and

$$\|Tx\|_\alpha \leq \frac{1}{2}\|x\|_\infty + 2\|x\|_\alpha.$$

Thus  $T$  maps  $X_\alpha$  into itself and defines an element of  $\mathcal{B}(X_\alpha)$  with norm  $\|T\| \leq 2$ . More generally, having fixed some  $x \in X_\alpha$  and given  $n \geq 0$ , let  $F_n = F_{T^n x}$ . Then  $F_n$  extends analytically to  $\Theta_\alpha$  and is given, for  $|\lambda| > 1$ , by

$$F_n(\lambda) = \sum_{k=1}^{\infty} \frac{x_{n+k}}{\lambda^k} = \lambda^n F_x(\lambda) - \sum_{k=1}^n \lambda^{n-k} x_k. \quad (6.3.11)$$

Writing  $\mathbb{A}$  for the annulus  $\{\lambda \in \mathbb{C} : 1 < |\lambda| < 2\}$ , it follows that

$$|F_n(\lambda)| \leq \begin{cases} \frac{\|x\|_\infty}{|1-|\lambda||} & \text{if } \lambda \in \mathbb{A}, \\ \frac{\|x\|_\infty}{|1-|\lambda||} + |F_x(\lambda)| & \text{if } \lambda \in \Theta_\alpha \cap \mathbb{D}, \end{cases}$$

and, in particular,

$$|F_n(\lambda)| \leq \frac{\|x\|_{X_\alpha}}{|1-|\lambda||}$$

for all  $\lambda \in \Theta_\alpha \setminus \mathbb{T}$ . Let  $\lambda \in \Theta_\alpha$  be given. If  $\lambda \in (1, 2)$ , then  $K_\alpha(\lambda)|F_n(\lambda)| = 0$ . Suppose therefore that  $\theta = \arg \lambda$  satisfies  $0 < |\theta| \leq \pi$ , and note that  $\Omega_{2r_\lambda, \theta} \subset \Theta_\alpha$ , where  $r_\lambda = \frac{1}{4}K_\alpha(\lambda)$  and  $\Omega_{r, \theta}$  is defined, for  $r \in (0, 1)$ , as in (6.3.1). Now either  $|1 - |\lambda|| > r_\lambda$ , in which case  $K_\alpha(\lambda)|F_n(\lambda)| \leq C\|x\|_\alpha$  for some constant  $C$  which is independent of  $x \in X_\alpha$ ,  $n \geq 0$  and  $\lambda$ , or  $|1 - |\lambda|| \leq r_\lambda$ . In the latter case  $\lambda \in \Omega_{r_\lambda, \theta}$ , so the same estimate follows from Lemma 6.3.1 applied to the function  $F_n$  on the disc  $\Omega_{2r_\lambda, \theta}$ . Thus  $\|T^n x\|_\alpha \leq C\|x\|_{X_\alpha}$  for some constant  $C$  which is independent of  $x \in X_\alpha$  and  $n \geq 0$ . Since  $\|T^n x\|_\infty \leq \|x\|_\infty$  for all  $n \geq 0$ , it follows that  $T$  is power-bounded.

Now fix  $x \in X_\alpha$  and let  $\Omega = \{\lambda \in \mathbb{C} : 1 < |\lambda| < \frac{3}{2}\}$ . Then  $\Omega \subset \rho(T)$  and  $(R(\lambda, T)x)_{n+1} = F_n(\lambda)$  for all  $\lambda \in \Omega$  and all  $n \geq 0$ , where  $F_n$  is as above. In particular, (6.3.11) remains true for each  $n \geq 0$  when the left-hand side is replaced by  $(R(\lambda, T)x)_{n+1}$ , so the argument in the previous paragraph shows that

$$K_\alpha(\lambda)\|R(\lambda, T)x\|_\infty \leq C\|x\|_{X_\alpha}, \quad (6.3.12)$$

where  $C$  is independent of both  $x \in X_\alpha$  and  $\lambda \in \Omega$ . The aim now is to show that

$$K_\alpha(\lambda)\|R(\lambda, T)x\|_\alpha \leq C\|x\|_\alpha$$

for all  $\lambda \in \Omega$ , from which it will follow that the norm of the resolvent of  $T$  grows at most polynomially. Since the estimate holds trivially when  $\lambda$  is real, assume that  $\lambda \in \Omega$  satisfies  $0 < |\arg \lambda| \leq \pi$  and let  $F_\lambda = F_{R(\lambda, T)x}$ . Then, for  $|\mu| \geq \frac{3}{2}$ ,

$$F_\lambda(\mu) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{x_{k+n-1}}{\lambda^k \mu^n} = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{\mu^n} \left( F_x(\lambda) - \sum_{k=1}^{n-1} \frac{x_k}{\lambda^k} \right) = -\frac{F_x(\lambda) - F_x(\mu)}{\lambda - \mu},$$

so  $F_\lambda$  extends analytically to  $\Theta_\alpha$ , taking the values

$$F_\lambda(\mu) = \begin{cases} -\frac{F_x(\lambda) - F_x(\mu)}{\lambda - \mu} & \text{if } \mu \in \Theta_\alpha \setminus \{\lambda\}, \\ -F'_x(\lambda) & \text{if } \mu = \lambda. \end{cases} \quad (6.3.13)$$

Now let  $\mu \in \Theta_\alpha$  and set

$$M_\alpha(\lambda, \mu) = \frac{1}{4} \max \{K_\alpha(\lambda), K_\alpha(\mu)\},$$

which is positive by the assumption on  $\arg \lambda$ . If  $|\lambda - \mu| > M_\alpha(\lambda, \mu)$ , then by (6.3.13)

$$K_\alpha(\lambda)K_\alpha(\mu) |F_\lambda(\mu)| \leq \frac{K_\alpha(\lambda) + K_\alpha(\mu)}{M_\alpha(\lambda, \mu)} \|x\|_\alpha \leq C \|x\|_\alpha, \quad (6.3.14)$$

where  $C$  is independent of  $\lambda$  and  $\mu$ . Now suppose that  $|\lambda - \mu| \leq M_\alpha(\lambda, \mu)$ . By Cauchy's formula,

$$F_\lambda(\mu) = \frac{1}{2\pi i} \oint_\Gamma \frac{F_x(\lambda) - F_x(z)}{(\lambda - z)(z - \mu)} dz, \quad (6.3.15)$$

where  $\Gamma$  is any contour in  $\Theta_\alpha$  whose interior contains the point  $\mu$  and is itself contained in  $\Theta_\alpha$ . If  $|\lambda - \mu| \leq \frac{1}{4}K_\alpha(\lambda)$ , choose  $\Gamma$  to be the circle with centre  $\lambda$  and radius  $\frac{1}{2}K_\alpha(\lambda)$ , so that  $\Gamma \subset \Theta_\alpha$  by the definitions of  $\Theta_\alpha$  and  $\Omega$ . Elementary estimates show that, for any  $z \in \Theta_\alpha$  satisfying  $|\lambda - z| \leq \frac{1}{2}K_\alpha(\lambda)$ ,

$$c|\arg \lambda| \leq |\arg z| \leq C|\arg \lambda|$$

and hence

$$cK_\alpha(\lambda) \leq K_\alpha(z) \leq CK_\alpha(\lambda),$$

where  $c$  and  $C$  are independent of  $\lambda$  and  $z$ . This applies in particular to all  $z \in \Gamma$  and also to  $z = \mu$ . Since  $|\mu - z| \geq \frac{1}{4}K_\alpha(\lambda)$  for all  $z \in \Gamma$ , it follows from (6.3.15) that

$$K_\alpha(\lambda)K_\alpha(\mu) |F_\lambda(\mu)| \leq \frac{C\|x\|_\alpha}{K_\alpha(\lambda)} \oint_\Gamma \left( \frac{K_\alpha(\mu)}{K_\alpha(\lambda)} + \frac{K_\alpha(\mu)}{K_\alpha(z)} \right) |dz| \leq C\|x\|_\alpha, \quad (6.3.16)$$

where  $C$  depends neither on  $\lambda$  nor on  $\mu$ . A similar argument applies when

$$\frac{1}{4}K_\alpha(\lambda) < |\lambda - \mu| \leq \frac{1}{4}K_\alpha(\mu),$$

this time taking  $\Gamma$  to be the circle with centre  $\lambda$  and radius  $\frac{1}{2}K_\alpha(\mu)$ . Then  $|\mu - z| \leq \frac{3}{4}K_\alpha(\mu)$  for all  $z \in \Gamma$ , so  $\Gamma \subset \Theta_\alpha$  as before. Moreover,  $K_\alpha(\mu) \leq CK_\alpha(z)$  for all  $z \in \Gamma$ , where  $C$  is independent of  $\lambda$  and  $\mu$ , and  $K_\alpha(\lambda) < K_\alpha(\mu)$ , giving

$$K_\alpha(\lambda)K_\alpha(\mu)|F_\lambda(\mu)| \leq \frac{C\|x\|_\alpha}{K_\alpha(\mu)} \oint_\Gamma \left(1 + \frac{K_\alpha(\lambda)}{K_\alpha(z)}\right) |dz| \leq C\|x\|_\alpha. \quad (6.3.17)$$

Combining (6.3.14), (6.3.16) and (6.3.17) shows that

$$K_\alpha(\lambda)\|R(\lambda, T)x\|_\alpha \leq C\|x\|_\alpha$$

for all  $\lambda \in \Omega$ . Together with (6.3.12), this gives

$$K_\alpha(\lambda)\|R(\lambda, T)x\|_{X_\alpha} \leq C\|x\|_{X_\alpha},$$

where  $C$  is independent of  $x \in X_\alpha$  and  $\lambda \in \Omega$ , and hence

$$\sup \{K_\alpha(\lambda)\|R(\lambda, T)\| : \lambda \in \Omega\} < \infty.$$

In particular, it follows from by (5.2.3) that  $\sigma(T) \cap \mathbb{T} \subset \{1\}$ , and a simple approximation argument shows that  $\|R(e^{i\theta}, T)\| = O(|\theta|^{-\alpha})$  as  $\theta \rightarrow 0$ . Furthermore, since  $(1, 1, 1, \dots)$  is a fixed point of  $T$ ,  $1 \in \sigma(T)$ .

Finally, let the transforms  $\mathcal{C}_\alpha$ ,  $\mathcal{L}_\alpha$  and  $\mathcal{D}_\alpha$  be as defined in (6.3.4) and note that, given any complex measure  $\mu$  whose support is contained in  $\Omega_\alpha$  and for which

$$\sup \{ |(\mathcal{L}_\alpha\mu)(k)| : k \geq 1 \} < \infty,$$

there exists an associated sequence  $x^\mu \in \ell^\infty(\mathbb{N})$  whose entries are given, for each  $k \geq 1$ , by  $x_k^\mu = (\mathcal{L}_\alpha\mu)(k)$ . By Fubini's theorem,

$$F_{x^\mu}(\lambda) = \sum_{k=1}^{\infty} \int_{\Omega_\alpha} \frac{z^{k-1}}{\lambda^k} d\mu(z) = \int_{\Omega_\alpha} \frac{d\mu(z)}{\lambda - z} = (\mathcal{C}_\alpha\mu)(\lambda)$$

whenever  $|\lambda| > 1$ , so  $x^\mu \in X_\alpha$  provided

$$\sup \{ K_\alpha(\lambda)|(\mathcal{C}_\alpha\mu)(\lambda)| : \lambda \in \Theta_\alpha \} < \infty.$$

Note also that, for each  $n \geq 0$ ,  $(\mathcal{D}_\alpha\mu)(n) = x_{n+1}^\mu - x_{n+2}^\mu$ , which coincides with the first entry of  $T^n(I - T)x^\mu$ . Now, by Lemma 6.3.2, it is possible to find an increasing sequence  $(n_j)$  of integers, with  $n_j \rightarrow \infty$  as  $j \rightarrow \infty$ , and associated measures  $\mu_j$  such that  $\{x^{\mu_j} : j \geq 1\}$  is a bounded subset of  $X_\alpha$  and moreover

$$|(\mathcal{D}_\alpha\mu_j)(n_j)|^\alpha \geq c \frac{\log n_j}{n_j}$$

for each  $j \geq 1$ . By rescaling if necessary, there is no loss of generality in assuming that  $\|x^{\mu_j}\|_{X_\alpha} \leq 1$  for all  $j \geq 1$ , so that

$$\|T^{n_j}(I - T)\| \geq \|T^{n_j}(I - T)x^{\mu_j}\|_\infty \geq |(\mathcal{D}_\alpha \mu_j)(n_j)|.$$

Hence (6.3.10) holds and the proof is complete.  $\square$

**Remark 6.3.5.** It is possible to replace Lemma 6.3.2, which here gives rise to the sequences  $x^\mu \in X_\alpha$  used to establish (6.3.10), by a simpler, more ad-hoc construction. Indeed, using the notation introduced in the proof of that result, let

$$x_k = \frac{\ell^{1/2} \lambda_0^{k-\ell}}{2^{\ell-1}} \binom{k}{\ell-1}$$

for each  $k \geq 1$ , so that  $x_k$  equals the first term of the final sum in (6.3.8) which defines  $x_k^\mu$  in the above proof. In the notation of Remark 6.3.3, this becomes

$$x_k = 4\ell^{1/2} (2\lambda_0)^{k-\ell} \mathbb{P}(Y_\ell = k+1),$$

so the formula for the probability generating function of  $Y_\ell$  (see for instance [53, Section 4.2]) gives

$$F_x(\lambda) = \frac{\lambda}{\lambda_0} \frac{2\ell^{1/2}}{2^\ell (\lambda - \lambda_0)^\ell}$$

whenever  $|\lambda| > 1$ , which should be compared with the right-hand sides of (6.3.5) and (6.3.6). Since the estimates for  $x^\mu$  established in Lemma 6.3.2 apply equally to  $x$ , it follows that (6.3.10) may also be obtained using sequences of this simpler form in the final paragraph of the above proof.

**Remark 6.3.6.** It is unclear whether Theorem 6.3.4 can be extended, for instance by modifying the construction in Lemma 6.3.2, to the range  $1 < \alpha \leq 2$ . Note however that, by Theorem 5.2.3, the case  $\alpha = 1$  is necessarily excluded. For a result relating specifically to the case  $\alpha = 2$ , see [41, Theorem 1.2].

## 6.4 The Hilbert space case

### 6.4.1 Non-optimality of Theorem 5.2.12

While Theorem 6.3.4 shows that on general Banach spaces the logarithmic factor in Corollary 6.1.1 cannot in general be omitted, thus proving Theorem 5.2.12 to be essentially optimal in the case of polynomial resolvent growth, Theorem 6.4.2 below will show that the situation is different on Hilbert space.

The ideas and results in this section are discrete analogues of statements contained in [14] and [27]. In particular, the following lemma is analogous to [27, Lemma 2.3]; see also [11, Lemma 1.1], [14, Theorem 5.5] and [69, Lemma 3.2].

**Lemma 6.4.1.** *Let  $X$  be a complex Banach space and let  $T \in \mathcal{B}(X)$  be a power-bounded operator such that  $\sigma(T) \cap \mathbb{T} = \{1\}$ . Furthermore, let  $\alpha \geq 1$  and suppose that*

$$\|R(e^{i\theta}, T)\| = O(|\theta|^{-\alpha}) \quad (6.4.1)$$

as  $\theta \rightarrow 0$ . Then

$$\sup \{ \|(I - T)^\alpha R(\lambda, T)\| : |\lambda| > 1 \} < \infty.$$

PROOF. Let  $M = \sup\{\|T^n\| : n \geq 0\}$  and, throughout the proof, let  $C$  denote various positive constants, which need not be the same at each occurrence. Since

$$\|R(\lambda, T)\| \leq \frac{M}{|\lambda| - 1} \quad (6.4.2)$$

for  $|\lambda| > 1$ , it suffices to prove that

$$\sup \{ \|(I - T)^\alpha R(\lambda, T)\| : \lambda \in \mathbb{A} \} < \infty,$$

where  $\mathbb{A} = \{\lambda \in \mathbb{C} : 1 < |\lambda| < 2\}$ . A first step towards this result is to establish that, under the above assumptions,

$$\sup \{ \|(1 - \lambda)^\alpha R(\lambda, T)\| : \lambda \in \mathbb{A} \} < \infty.$$

Thus, given  $r \in (0, 1)$ , let

$$\Omega_r = \left\{ \lambda \in \mathbb{C} : 1 \leq |\lambda| \leq 2 \text{ and } \left| \frac{1 + r^2}{1 - r^2} - \lambda \right| \geq \frac{2r}{1 - r^2} \right\}$$

(see Figure 6.3) and let the map  $H_r : \Omega_r \rightarrow \mathcal{B}(X)$  be defined by

$$H_r(\lambda) = \left( 1 + \frac{r^2}{\varphi(\lambda)^2} \right) (1 - \lambda)^\alpha R(\lambda, T),$$

so that  $H_r(\lambda) = h_r(\lambda)(1 - \lambda)^\alpha R(\lambda, T)$ , where  $h_r = g_r \circ \varphi$  with  $g_r$  and  $\varphi$  as in the proof of Theorem 5.2.12. Then

$$\sup \{ |h_r(\lambda)| : r \in (0, 1/2), \lambda \in \Omega_r \} < \infty$$

and, by the argument leading to equation (5.2.19),

$$|h_r(\lambda)| \leq C \frac{|\lambda| - 1}{r}$$

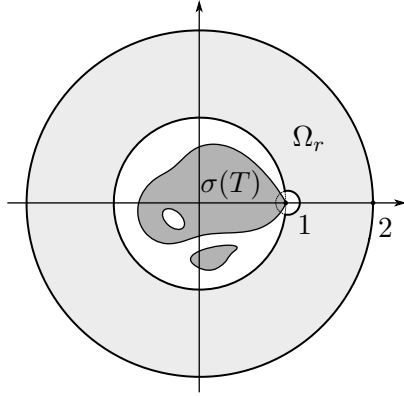


Figure 6.3: The region  $\Omega_r$  for a typical value of  $r \in (0, 1)$ .

for all  $\lambda \in \partial\Omega_r \cap \gamma_r$ , where  $\gamma_r$  is the circle defined in (5.2.16). Note also that  $|1 - \lambda| \leq Cr$  for all  $\lambda \in \partial\Omega_r \cap \gamma_r$ . Thus, by (6.4.1) and (6.4.2),  $\|H_r(\lambda)\|$  is uniformly bounded, independently of  $r$ , for all  $\lambda \in \partial\Omega_r$  and hence by the maximum principle

$$\sup \{ \|H_r(\lambda)\| : r \in (0, 1/2), \lambda \in \Omega_r \} < \infty.$$

Since, for any  $\lambda \in \mathbb{A}$ , there exists  $r \in (0, \frac{1}{2})$  such that  $\lambda \in \Omega_r$  and  $|h_r(\lambda)| \geq \frac{1}{2}$ , the claim follows.

Now let  $n \in \mathbb{N}$  and  $\beta \in [0, 1)$  be such that  $\alpha = n + \beta$ , and note that, for any  $k \geq 0$  and  $\lambda \in \rho(T)$ ,

$$\|(I - T)^k R(\lambda, T)\| \leq |1 - \lambda|^k \|R(\lambda, T)\| + \sum_{j=0}^{k-1} \binom{k}{j} |1 - \lambda|^j \|\lambda - T\|^{k-j-1}.$$

Setting  $k = n - 1$  and  $k = n$ , this shows, respectively, that

$$\|(I - T)^{n-1} R(\lambda, T)\| \leq \frac{C}{|1 - \lambda|^{1+\beta}} \quad \text{and} \quad \|(I - T)^n R(\lambda, T)\| \leq \frac{C}{|1 - \lambda|^\beta}$$

for all  $\lambda \in \mathbb{A}$ . In particular, if  $\beta = 0$ , the proof is complete. If  $\beta \neq 0$ , the moment inequality (see Proposition 2.3.1) gives

$$\|(I - T)^{\alpha-1} R(\lambda, T)\| \leq C \|(I - T)^{n-1} R(\lambda, T)\|^{1-\beta} \|(I - T)^n R(\lambda, T)\|^\beta,$$

and hence

$$\|(I - T)^{\alpha-1} R(\lambda, T)\| \leq \frac{C}{|1 - \lambda|}$$

for all  $\lambda \in \mathbb{A}$ . Since

$$\|R(\lambda, T)(I - T)^\alpha\| \leq |1 - \lambda| \|R(\lambda, T)(I - T)^{\alpha-1}\| + \|(I - T)^{\alpha-1}\|$$

for all  $\lambda \in \rho(T)$ , the result follows.  $\square$

The final result shows that the phenomenon described in Theorem 6.3.4 cannot arise on Hilbert space. For analogous results in the continuous-time setting see [14, Theorem 7.6] and [27, Theorem 2.4]; compare also with [88, Theorem 9].

**Theorem 6.4.2.** *Let  $X$  be a complex Hilbert space and let  $T \in \mathcal{B}(X)$  be a power-bounded operator such that  $\sigma(T) \cap \mathbb{T} = \{1\}$ . Furthermore, let  $\alpha \geq 1$ . Then*

$$\|R(e^{i\theta}, T)\| = O(|\theta|^{-\alpha}) \quad (6.4.3)$$

as  $\theta \rightarrow 0$  if and only if

$$\|T^n(I - T)\| = O(n^{-1/\alpha}) \quad (6.4.4)$$

as  $n \rightarrow \infty$ .

PROOF. The fact that (6.4.4) implies (6.4.3) is an immediate consequence of Theorem 5.2.4. Suppose therefore that (6.4.3) holds, so that by Lemma 6.4.1

$$\sup \{ \|(I - T)^\alpha R(\lambda, T)\| : |\lambda| > 1 \} < \infty.$$

For  $n \geq 0$  and  $|\lambda| > 1$ , let

$$F_n(\lambda) = \lambda R(\lambda, T) \sum_{k=0}^n \lambda^{-k} T^k.$$

Then

$$\begin{aligned} F_n(\lambda) &= \sum_{k=0}^{\infty} \left( \sum_{j=0}^k \mathbb{1}_{\{1, \dots, n\}}(k-j) \lambda^{-j} T^j \right) \lambda^{-k} T^k \\ &= \sum_{k=0}^{\infty} (\min\{k, n\} + 1) \lambda^{-k} T^k, \end{aligned} \quad (6.4.5)$$

where  $\mathbb{1}_\Omega$  denotes the indicator function of a set  $\Omega$ . Hence, by Parseval's identity,

$$\sum_{k=0}^{\infty} (\min\{k, n\} + 1)^2 \frac{\|T^k x\|^2}{r^{2k}} = \frac{1}{2\pi} \int_0^{2\pi} \|F_n(re^{i\theta})x\|^2 d\theta$$

for all  $n \geq 0$ ,  $x \in X$  and  $r > 1$ . Replacing  $x$  with  $(I - T)^\alpha x$  and letting

$$B = \sup \{ \|(I - T)^\alpha R(\lambda, T)\| : |\lambda| > 1 \},$$

it follows from the definition of  $F_n$  that

$$\sum_{k=0}^n \frac{(k+1)^2}{r^{2k}} \|T^k (I - T)^\alpha x\|^2 \leq \frac{B^2 r^2}{2\pi} \int_0^{2\pi} \left\| \sum_{k=0}^n r^{-k} e^{-ik\theta} T^k x \right\|^2 d\theta,$$

or indeed, by another application of Parseval's identity, that

$$\sum_{k=0}^n \frac{(k+1)^2}{r^{2n}} \|T^k(I-T)^\alpha x\|^2 \leq B^2 r^2 \sum_{k=0}^n \frac{\|T^k x\|^2}{r^{2k}}.$$

Letting  $r \rightarrow 1+$ , this gives

$$\sum_{k=0}^n (k+1)^2 \|T^k(I-T)^\alpha x\|^2 \leq M^2 B^2 (n+1) \|x\|^2, \quad (6.4.6)$$

where  $M = \sup\{\|T^n\| : n \geq 0\}$ . Now, for  $y \in X$  and  $n \geq 0$ ,

$$((n+2)T^n(I-T)^\alpha x, y) = \frac{2}{n+1} \sum_{k=0}^n ((k+1)T^k(I-T)^\alpha x, (T^*)^{n-k} y).$$

By (6.4.6) and Cauchy's inequality, the right-hand side is bounded above in modulus by  $2M^2 B \|x\| \|y\|$ , and hence

$$\|T^n(I-T)^\alpha\| \leq \frac{2M^2 B}{n+2} \quad (6.4.7)$$

for all  $n \geq 0$ . Thus the proof is complete in the case  $\alpha = 1$ . If  $\alpha > 1$ , on the other hand, the moment inequality gives

$$\|T^n(I-T)\| \leq C \|T^n\|^{(\alpha-1)/\alpha} \|T^n(I-T)^\alpha\|^{1/\alpha}$$

for all  $n \geq 0$ , and the result now follows from (6.4.7) and the fact that  $T$  is power-bounded.  $\square$

**Remark 6.4.3.** From the proof of [14, Theorem 4.7], which establishes the first part of the above argument in the continuous setting, it is clear that the calculation in (6.4.5) could be reformulated in terms of convolutions.

The above proof uses the ideas in [14], but there is an alternative proof analogous to the method used in [27]. This alternative proof requires the following preparatory result, which is contained in more or less the same form in [50].

**Proposition 6.4.4.** *Let  $X$  be a complex Hilbert space and let  $T \in \mathcal{B}(X)$ . Then  $T$  is power-bounded if and only if  $r(T) \leq 1$  and*

$$\sup_{r>1} (r-1) \int_0^{2\pi} \left( \|R(re^{i\theta}, T)x\|^2 + \|R(re^{i\theta}, T^*)x\|^2 \right) d\theta < \infty \quad (6.4.8)$$

for every  $x \in X$ .

PROOF. Suppose first that  $T$  is power-bounded and let  $M = \sup\{\|T^n\| : n \geq 0\}$ . Then certainly

$$r(T) = \inf \{ \|T^n\|^{1/n} : n \geq 0 \} \leq 1.$$

Given  $x \in X$  and  $r > 1$ , define the function  $f \in L^2(0, 2\pi; X)$  by

$$f(\theta) = re^{i\theta}R(re^{i\theta}, T)x,$$

and note that

$$f(\theta) = \sum_{n=0}^{\infty} \frac{T^n x}{r^n} e^{-in\theta}.$$

It follows from Parseval's identity that

$$\sum_{n=0}^{\infty} \frac{\|T^n x\|^2}{r^{2n}} = \frac{1}{2\pi} \int_0^{2\pi} \|f(\theta)\|^2 d\theta.$$

But

$$\sum_{n=0}^{\infty} \frac{\|T^n x\|^2}{r^{2n}} \leq \frac{r^2 M^2}{r^2 - 1} \|x\|^2$$

and hence

$$(r - 1) \int_0^{2\pi} \|R(re^{i\theta}, T)x\|^2 d\theta \leq \frac{2\pi M^2}{r + 1} \|x\|^2.$$

Repeating the argument with  $T$  replaced by  $T^*$  proves that (6.4.8) holds.

Now suppose, conversely, that  $r(T) \leq 1$  and that (6.4.8) is satisfied. By the closed graph theorem, there exists a constant  $C > 0$  such that

$$\sup_{r>1} (r - 1) \int_0^{2\pi} \left( \|R(re^{i\theta}, T)x\|^2 + \|R(re^{i\theta}, T^*)x\|^2 \right) d\theta \leq C \|x\|^2$$

for every  $x \in X$ . Now for  $x, y \in X$  and  $n \geq 0$ , integration by parts gives

$$\begin{aligned} (T^n x, y) &= \frac{1}{2\pi i} \oint_{\Gamma_n} \lambda^n (R(\lambda, T)x, y) d\lambda \\ &= \frac{1}{2\pi i} \oint_{\Gamma_n} \frac{\lambda^{n+1}}{n+1} (R(\lambda, T)^2 x, y) d\lambda, \end{aligned}$$

where  $\Gamma_n = r_n \mathbb{T}$  with  $r_n = \frac{n+2}{n+1}$ . By (6.4.8) and the Cauchy-Schwarz inequality,

applied first in  $X$  and then in  $L^2(0, 2\pi; X)$ , it follows that

$$\begin{aligned} |(T^n x, y)| &\leq \frac{r_n^{n+2}}{2\pi(n+1)} \int_0^{2\pi} |(R(r_n e^{i\theta}, T)x, R(r_n e^{-i\theta}, T^*)y)| d\theta \\ &\leq \frac{e}{\pi} \left( (r_n - 1) \int_0^{2\pi} \|R(r_n e^{i\theta}, T)x\|^2 d\theta \right)^{1/2} \\ &\quad \cdot \left( (r_n - 1) \int_0^{2\pi} \|R(r_n e^{-i\theta}, T^*)y\|^2 d\theta \right)^{1/2} \\ &\leq \frac{Ce}{\pi} \|x\| \|y\|, \end{aligned}$$

and hence  $\|T^n\| \leq Ce/\pi$  for all  $n \geq 0$ . Thus  $T$  is power-bounded.  $\square$

This result leads to the following alternative proof of Theorem 6.4.2.

PROOF. Since the other implication follows from Theorem 5.2.4, suppose that (6.4.3) holds. As before, it suffices to show, by the moment inequality, that

$$\|T^n(I - T)^\alpha\| = O(n^{-1})$$

as  $n \rightarrow \infty$ . Consider the operator  $Q \in \mathcal{B}(X \times X)$  defined by

$$Q = \begin{pmatrix} T & T(I - T)^\alpha \\ 0 & T \end{pmatrix}.$$

Then

$$Q^n = \begin{pmatrix} T^n & nT^n(I - T)^\alpha \\ 0 & T^n \end{pmatrix}$$

for all  $n \geq 0$  and  $\rho(Q) = \rho(T)$ . Moreover, for  $\lambda \in \rho(Q)$ ,

$$R(\lambda, Q) = \begin{pmatrix} R(\lambda, T) & R(\lambda, T)^2 T(I - T)^\alpha \\ 0 & R(\lambda, T) \end{pmatrix}.$$

By Lemma 6.4.1,

$$\sup \{ \|(I - T)^\alpha R(\lambda, T)\| : |\lambda| > 1 \} < \infty$$

and hence there exists  $C > 0$  such that, for any  $z = (x, y) \in X \times X$  and any  $\lambda \in \mathbb{C}$  with  $|\lambda| > 1$ ,

$$\|R(\lambda, Q)z\|^2 \leq C(\|R(\lambda, T)x\|^2 + \|R(\lambda, T)y\|^2)$$

and the same estimate holds with  $Q$  and  $T$  replaced by  $Q^*$  and  $T^*$ , respectively. Since  $T$  is power-bounded, this estimate together with Proposition 6.4.4 implies that

$$\sup_{r>1} (r-1) \int_0^{2\pi} \left( \|R(re^{i\theta}, Q)z\|^2 + \|R(re^{i\theta}, Q^*)z\|^2 \right) d\theta < \infty$$

for all  $z \in X \times X$ . By Proposition 6.4.4,  $Q$  is power-bounded and, since  $T$  itself is power-bounded, it follows that  $\|T^n(I-T)^\alpha\| = O(n^{-1})$  as  $n \rightarrow \infty$ , which completes the proof.  $\square$

The final result of this section is an extension of Theorem 6.4.2 in the spirit of [27, Theorem 2.4]. It shows that the two equivalent conditions of the theorem are in turn equivalent to having a certain rate of decay for individual orbits.

**Theorem 6.4.5.** *Let  $X$  be a complex Hilbert space and let  $T \in \mathcal{B}(X)$  be a power-bounded operator such that  $\sigma(T) \cap \mathbb{T} = \{1\}$ . Furthermore, let  $\alpha \geq 1$ . Then the following are equivalent:*

- (i)  $\|R(e^{i\theta}, T)\| = O(|\theta|^{-\alpha})$  as  $\theta \rightarrow 0$ ;
- (ii)  $\|T^n(I-T)\| = O(n^{-1/\alpha})$  as  $n \rightarrow \infty$ ;
- (iii) for each  $x \in X$ ,  $\|T^n(I-T)x\| = o(n^{-1/\alpha})$  as  $n \rightarrow \infty$ .

PROOF. Equivalence of (i) and (ii) is established in Theorem 6.4.2. By the moment inequality, (ii) and (iii) are equivalent, respectively, to

$$nT^n(I-T)^\alpha x = o(1)$$

for all  $x \in X$  and to

$$nT^n(I-T)^\alpha = O(1)$$

as  $n \rightarrow \infty$ . Thus the implication (iii)  $\implies$  (ii) follows at once from the uniform boundedness theorem and it remains only to show that (ii)  $\implies$  (iii).

Suppose therefore that

$$\sup \{n\|T^n(I-T)^\alpha\| : n \geq 0\} < \infty.$$

Then the operator  $Q \in \mathcal{B}(X \times X)$  introduced in the previous proof is power-bounded and  $\sigma(Q) = \sigma(T)$ . In particular,  $\sigma(Q) \cap \mathbb{T} = \{1\}$ . Letting  $P_T$  and  $P_Q$  denote the projections onto  $\text{Fix}(T)$  along the closure of  $\text{Ran}(I-T)$  and onto  $\text{Fix}(Q)$  along the closure of  $\text{Ran}(I-Q)$ , respectively, it follows from [9,

Theorem 4.1] that  $T^n \rightarrow P_T$  and  $Q^n \rightarrow P_Q$  strongly as  $n \rightarrow \infty$ . Now, given  $z = (x, y) \in X \times X$ ,

$$Qz = (Tx + T(I - T)^\alpha y, Ty).$$

Suppose that  $Qz = z$ . Then  $y \in \text{Fix}(T)$  and hence, as  $\alpha \geq 1$ ,  $(I - T)^\alpha y = 0$ . Thus  $x \in \text{Fix}(T)$ , and it follows that  $\text{Fix}(Q) = \text{Fix}(T) \times \text{Fix}(T)$  and  $P_Q z = (P_T x, P_T y)$ . Since  $Q^n \rightarrow P_Q$  strongly as  $n \rightarrow \infty$ , it follows from the matrix representation of  $Q^n$  that

$$T^n x + nT^n(I - T)^\alpha y \rightarrow P_T x$$

as  $n \rightarrow \infty$  for all  $x, y \in X$ . But since  $T^n x \rightarrow P_T x$  for all  $x \in X$  as  $n \rightarrow \infty$ , this implies that  $nT^n(I - T)^\alpha \rightarrow 0$  strongly as  $n \rightarrow \infty$ . Thus (iii) holds and the proof is complete.  $\square$

#### 6.4.2 Robustness under finite-rank perturbations

The purpose of this final section is to study the robustness of polynomial decay under perturbations. Theorem 6.4.7 below combines ideas from the preceding sections with adaptations of results contained in [92] to show that decay of the form described in (6.4.4) is stable under certain finite-rank perturbations; see also [91]. This result relies on the following simple observation, known sometimes as the Sherman-Morrison-Woodbury formula, which may be verified by direct calculation.

**Lemma 6.4.6.** *Let  $X$  and  $Y$  be complex Banach spaces and let  $T \in \mathcal{B}(X)$ ,  $U \in \mathcal{B}(Y, X)$  and  $V \in \mathcal{B}(X, Y)$ . Suppose that  $\lambda \in \rho(T)$  and that  $1 \in \rho(VR(\lambda, T)U)$ , and let  $S = T + UV$ . Then  $\lambda \in \rho(S)$  and*

$$R(\lambda, S) = R(\lambda, T) + R(\lambda, T)UR(1, VR(\lambda, T)U)VR(\lambda, T). \quad (6.4.9)$$

The following is the main result of this section. Recall that the assumption  $\text{Fix} T = \{0\}$  for a power-bounded operator  $T$  on a Banach space  $X$  makes it possible to define the negative fractional powers of  $I - T$  on suitable subspaces of  $X$ .

**Theorem 6.4.7.** *Let  $X$  be a complex Hilbert space and suppose that  $T \in \mathcal{B}(X)$  is a power-bounded operator such that  $\text{Fix}(T) = \{0\}$  and, for some  $\alpha \geq 1$ ,*

$$\|T^n(I - T)\| = O(n^{-1/\alpha})$$

as  $n \rightarrow \infty$ . Furthermore let  $\beta, \gamma \geq 0$  be such that  $\beta + \gamma \geq \alpha$  and, for some  $m \in \mathbb{N}$ , let  $U \in \mathcal{B}(\mathbb{C}^m, X)$  and  $V \in \mathcal{B}(X, \mathbb{C}^m)$ , and suppose that

$$\text{Ran } U \subset \text{Ran}(I - T)^\beta \quad \text{and} \quad \text{Ran } V^* \subset \text{Ran}(I - T^*)^\gamma. \quad (6.4.10)$$

Let  $S = T + UV$ . Then there exists  $\delta > 0$  such that

$$\|S^n(I - S)\| = O(n^{-1/\alpha}) \quad (6.4.11)$$

as  $n \rightarrow \infty$  provided  $\|(I - T)^{-\beta}U\| \|(I - T^*)^{-\gamma}V^*\| < \delta$ .

PROOF. Throughout the proof, the letter  $C$  will denote various positive constants which need not be the same at each occurrence. There is no loss of generality in assuming that  $\beta + \gamma = \alpha$ .

Let  $\Omega = \{\lambda \in \mathbb{C} : |\lambda| \geq 1, \lambda \neq 1\}$ . Then by Theorem 5.2.4,  $\Omega \subset \rho(T)$  and, for  $\lambda \in \Omega$ ,

$$\begin{aligned} \|VR(\lambda, T)U\| &= \sup_{\|x\|, \|y\|=1} |(VR(\lambda, T)Ux, y)| \\ &= \sup_{\|x\|, \|y\|=1} |((I - T)^{\beta+\gamma}R(\lambda, T)(I - T)^{-\beta}Ux, (I - T^*)^{-\gamma}V^*y)| \\ &\leq \|(I - T)^\alpha R(\lambda, T)\| \|(I - T)^{-\beta}U\| \|(I - T^*)^{-\gamma}V^*\|. \end{aligned}$$

Since

$$\sup \{ \|(I - T)^\alpha R(\lambda, T)\| : \lambda \in \Omega \} < \infty$$

by Lemma 6.4.1, there exists  $\delta > 0$  such that  $\|VR(\lambda, T)U\| \leq \frac{1}{2}$  for all  $\lambda \in \Omega$  provided  $\|(I - T)^{-\beta}U\| \|(I - T^*)^{-\gamma}V^*\| < \delta$ . In this case  $1 \in \rho(VR(\lambda, T)U)$  for all  $\lambda \in \Omega$  and

$$\sup \{ \|R(1, VR(\lambda, T)U)\| : \lambda \in \Omega \} < \infty.$$

By Lemma 6.4.6,  $\Omega \subset \rho(S)$  and (6.4.9) holds for all  $\lambda \in \Omega$ .

Suppose that  $\gamma = 0$ . Then, for  $\lambda \in \Omega$  and  $x \in X$ ,

$$\|R(\lambda, S)x\| \leq (1 + \|UR(1, VR(\lambda, T)U)VR(\lambda, T)\|) \|R(\lambda, T)x\|.$$

Letting

$$F(\lambda) = \|UR(1, VR(\lambda, T)U)VR(\lambda, T)\|,$$

it follows that

$$F(\lambda) \leq \|(I - T)^{-\alpha}U\| \|R(1, VR(\lambda, T)U)\| \|V\| \|(I - T)^\alpha R(\lambda, T)\| \leq C$$

and hence that

$$\|R(\lambda, S)x\| \leq C\|R(\lambda, T)x\| \quad (6.4.12)$$

for all  $\lambda \in \Omega$  and all  $x \in X$ . Similarly

$$\|R(\lambda, S^*)x\| \leq C\|R(\lambda, T^*)x\|$$

for all  $\lambda \in \Omega$  and  $x \in X$ . Since  $T$  is power-bounded, Proposition 6.4.4 implies that

$$\sup_{r>1} (r-1) \int_0^{2\pi} \left( \|R(re^{i\theta}, T)x\|^2 + \|R(re^{i\theta}, T^*)x\|^2 \right) d\theta < \infty \quad (6.4.13)$$

for all  $x \in X$  and, by the above estimates, the same condition holds when  $T$  is replaced by  $S$ . Hence  $S$  is power-bounded by Proposition 6.4.4. Furthermore,

$$\|R(e^{i\theta}, T)\| = O(|\theta|^{-\alpha})$$

as  $\theta \rightarrow 0$  by Theorem 6.4.2, so by (6.4.12)

$$\|R(e^{i\theta}, S)\| = O(|\theta|^{-\alpha})$$

as  $\theta \rightarrow 0$ . It follows from Theorem 6.4.2 that (6.4.11) holds. An analogous argument applies when  $\beta = 0$ .

Suppose now that  $\beta, \gamma > 0$ . It follows from (6.4.6) and the elementary estimate  $(a+b)^2 \leq 2(a^2+b^2)$  for real numbers  $a, b$  that, for  $\lambda \in \Omega$  and  $x \in X$ ,

$$\|R(\lambda, S)x\|^2 \leq 2(\|R(\lambda, T)x\|^2 + C\|R(\lambda, T)U\|^2\|VR(\lambda, T)\|^2\|x\|^2). \quad (6.4.14)$$

In order to control the term  $\|R(\lambda, T)U\|\|VR(\lambda, T)\|$ , let the operators  $U_\beta, V_\gamma \in \mathcal{B}(\mathbb{C}^m, X)$  be given by  $U_\beta = (I - T)^{-\beta}U$  and  $V_\gamma = (I - T^*)^{-\gamma}V^*$ , respectively. By the moment inequality,

$$\begin{aligned} \|R(\lambda, T)U\| &= \|(I - T)^\beta R(\lambda, T)U_\beta\| \\ &\leq C\|R(\lambda, T)U_\beta\|^{1-\beta/\alpha} \|(I - T)^\alpha R(\lambda, T)U_\beta\|^{\beta/\alpha} \\ &\leq C\|R(\lambda, T)U_\beta\|^{1-\beta/\alpha} \end{aligned}$$

and similarly

$$\|VR(\lambda, T)\| = \|R(\lambda, T^*)V^*\| \leq C\|R(\lambda, T^*)V_\gamma\|^{1-\gamma/\alpha}$$

for all  $\lambda \in \Omega$ . It follows that

$$\|R(\lambda, T)U\|\|VR(\lambda, T)\| \leq C\|R(\lambda, T)U_\beta\|^{1-\beta/\alpha} \|R(\lambda, T^*)V_\gamma\|^{1-\gamma/\alpha}. \quad (6.4.15)$$

Now, since  $U_\beta$  has finite rank, there exist  $x_1, \dots, x_m \in X$  such that

$$U_\beta(v) = \sum_{k=1}^m v_k x_k$$

for all  $v = (v_1, \dots, v_m) \in \mathbb{C}^m$ . A simple application of the Cauchy-Schwarz inequality shows that

$$\|R(\lambda, T)U_\beta\|^2 \leq \sum_{k=1}^m \|R(\lambda, T)x_k\|^2$$

for all  $\lambda \in \Omega$  and hence, using (6.4.13),

$$\sup_{r>1} (r-1) \int_0^{2\pi} \|R(re^{i\theta}, T)U_\beta\|^2 d\theta \leq \sum_{k=1}^m \sup_{r>1} (r-1) \int_0^{2\pi} \|R(re^{i\theta}, T)x_k\|^2 d\theta < \infty.$$

An analogous argument shows that

$$\sup_{r>1} (r-1) \int_0^{2\pi} \|R(re^{i\theta}, T)^* V_\gamma\|^2 d\theta < \infty.$$

Letting  $p = (1 - \beta/\alpha)^{-1}$  and  $q = (1 - \gamma/\alpha)^{-1}$ , so that  $p^{-1} + q^{-1} = 1$ , it follows from (6.4.15) and Hölder's inequality that

$$\sup_{r>1} (r-1) \int_0^{2\pi} \|R(re^{i\theta}, T)U\|^2 \|VR(re^{i\theta}, T)\|^2 d\theta < \infty.$$

Thus (6.4.13) and (6.4.14) show that

$$\sup_{r>1} (r-1) \int_0^{2\pi} \|R(re^{i\theta}, S)x\|^2 d\theta < \infty$$

for all  $x \in X$ , and a similar argument gives

$$\sup_{r>1} (r-1) \int_0^{2\pi} \|R(re^{i\theta}, S^*)x\|^2 d\theta < \infty$$

for all  $x \in X$ . It follows from Proposition 6.4.4 that  $S$  is power-bounded. Moreover, (6.4.15) implies that

$$\|R(\lambda, T)U\| \|VR(\lambda, T)\| \leq C \|R(\lambda, T)\|$$

for all  $\lambda \in \Omega$ , so by (6.4.14)

$$\|R(e^{i\theta}, S)\| = O(|\theta|^{-\alpha})$$

as  $\theta \rightarrow 0$ . As before, the result now follows from Theorem 6.4.2.  $\square$

**Remark 6.4.8.** Note that the assumption that  $\text{Fix } T = \{0\}$  could be relaxed. Indeed, what is really needed here is merely that there exist operators  $T_U, T_V$ , such that  $(I - T)^\beta T_U U = U$  and  $(I - T^*)^\gamma T_V^* V^* = V^*$ , and this follows from (6.4.10) and the assumption that  $U, V$  have finite rank. Moreover, the condition on the size of  $\|T_U U\| \|T_V^* V^*\|$  can now at least in principle be replaced by conditions involving only finite-dimensional vector spaces, though this is unlikely to be very insightful unless the ranks of the operators in question are very small indeed.

**Remark 6.4.9.** Note also that the condition  $\text{Fix } T = \{0\}$  in the reflexive setting is equivalent to  $1 \notin \sigma_p(T^*)$  and hence implies that  $T$  in the above result satisfies  $\|T^n x\| \rightarrow 0$  as  $n \rightarrow \infty$  for all  $x \in X$ ; see [5]. This property of strong stability is inherited by  $S$  when  $\delta$  is sufficiently small. Indeed, suppose that  $x \in X$  is such that  $Sx = x$  and let  $\beta' \in [0, \beta], \gamma' \in [0, \gamma]$  be such that  $\beta' + \gamma' = 1$ . Then, choosing  $\delta > 0$  so small as to ensure that

$$\|(I - T)^{-\beta'} UV(I - T)^{-\gamma'}\| < 1,$$

it follows that

$$0 = (I - S)x = (I - T)^{\beta'} (I - (I - T)^{-\beta'} UV(I - T)^{-\gamma'}) (I - T)^{\gamma'} x,$$

and hence  $x = 0$ , as all three operators on the right-hand side are injective. Thus  $\sigma(S) \cap \mathbb{T} \subset \{1\}$  and  $\sigma_p(S) \cap \mathbb{T} = \sigma_p(S^*) \cap \mathbb{T} = \emptyset$ , and therefore  $\|S^n x\| \rightarrow 0$  as  $n \rightarrow \infty$  for all  $x \in X$ . See [92] for a discussion of robustness of strong stability in the case of  $C_0$ -semigroups.

## Part IV

# General operator semigroups

## Chapter 7

# The Katznelson-Tzafriri theorem on Hilbert space

### 7.1 Introduction

Parts II and III of this thesis have dealt with extensions of the Katznelson-Tzafriri theorem in the case of continuous and discrete operator semigroups, respectively. Both results can be viewed as special cases of the following general theorem for bounded representations of general locally compact abelian semigroups, which was obtained in [19, Theorem 4.3] and [104]. Here and in what follows, all groups and semigroups that appear will implicitly be assumed to satisfy the assumptions set out in Section 2.4.

**Theorem 7.1.1.** *Let  $X$  be a complex Banach space and let  $T$  be a bounded representation of a locally compact abelian semigroup  $S$  on  $X$ . Suppose that  $a \in L^1(S)$  is of spectral synthesis with respect to  $\text{Sp}_u(T)$ . Then  $\|T(s)\hat{a}(T)\| \rightarrow 0$  as  $s \rightarrow \infty$ .*

Note that the condition on the function  $a$  could alternatively have been stated simply as  $a \in L^1(S) \cap J_\Lambda(G)$ , where  $\Lambda = \text{Sp}_u(T)$ . The Katznelson-Tzafriri theorem for  $C_0$ -semigroups, or continuous operator semigroups, discussed in Part II corresponds to the choice  $S = \mathbb{R}_+$  and likewise the original Katznelson-Tzafriri theorem for discrete operator semigroups [62, Theorem 5], a special case of which was discussed in Part III, corresponds to the choice  $S = \mathbb{Z}_+$ .

As is shown in [33, Section 5], the Katznelson-Tzafriri theorem is optimal in various important ways. Nevertheless stronger statements are true in special

cases, for instance when  $X$  is a Hilbert space. Note first that in the case  $S = \mathbb{Z}_+$  the mapping  $\widehat{a} \mapsto \widehat{a}(T)$  can be viewed as a functional calculus from  $A^+(\mathbb{D})$  into  $\mathcal{B}(X)$ , where  $A^+(\mathbb{D})$  is the set of maps  $g : \mathbb{D} \rightarrow \mathbb{C}$  such that  $g = \widehat{a}$  for some  $a \in \ell^1(\mathbb{Z}_+)$ . If  $X$  is a Hilbert space and  $T \in \mathcal{B}(X)$  is a contraction, it follows from the validity of von Neumann's inequality that this functional calculus can in fact be extended from  $A^+(\mathbb{D})$  to the entire disc algebra  $A(\mathbb{D}) = H(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ , where  $H(\mathbb{D})$  denotes the set of all holomorphic functions  $g : \mathbb{D} \rightarrow \mathbb{C}$ . It is shown in [47, Corollary 2.12] that the conclusion of the Katznelson-Tzafriri theorem, namely that  $\|T^n g(T)\| \rightarrow 0$  as  $n \rightarrow \infty$ , remains valid in this case under the weaker (and necessary) condition that  $g \in A(\mathbb{D})$  satisfies  $g(\lambda) = 0$  for all  $\lambda \in \sigma(T) \cap \mathbb{T}$ ; see [63, Proposition 1.6] for a slightly more general result. In the classical setting of functions  $g = \widehat{a}$  with  $a \in L^1(S)$  this result in turn has recently been improved, in [106] to the cases of two commuting contractions and of  $C_0$ -semigroups of contractions, and in [71] to the case of a single operator  $T$  which is merely power-bounded rather than contractive.

The purpose of this chapter is to extend both of these improvements to bounded representations  $T$  of general locally compact abelian semigroups  $S$  on a Hilbert space, by showing that the condition  $a \in L^1(S) \cap J_\Lambda(G)$  in Theorem 7.1.1 can be replaced by  $a \in L^1(S) \cap K_\Lambda(G)$  in this case. In fact, the main result of this chapter, Theorem 7.3.1, shows not only that this spectral condition is necessary and sufficient for the conclusion  $\|T(s)\widehat{a}(T)\| \rightarrow 0$  as  $s \rightarrow \infty$  to hold but also that both are equivalent to a certain ergodic condition. The argument is a generalisation of the one given in [71] for the case  $S = \mathbb{Z}_+$ . Before the main result is presented in Section 7.3, Section 7.2 develops the three important constructions that are needed in the proof of Theorem 7.3.1. The chapter concludes with an alternative, more direct proof of the equivalence of the aforementioned spectral and asymptotic statements, this time without going through the ergodic condition. This argument is presented in Section 7.4. The present chapter, and also parts of the next, are based on [98].

## 7.2 Preliminary results

This section contains three constructions that will be needed in the later parts of the chapter. Given two representations  $T_1$  and  $T_2$  of a locally compact abelian semigroup  $S$  on Banach spaces  $X_1$  and  $X_2$ , respectively, a bounded linear map  $\pi : X_1 \rightarrow X_2$  is said to *intertwine* the representations  $T_1$  and  $T_2$  if  $\pi T_1(s) = T_2(s)\pi$  for all  $s \in S$ . In this case it follows from the continuity of  $\pi$

that, for  $a \in L^1(S)$  and  $x \in X_1$ ,

$$\pi \widehat{a}(T_1)x = \int_S a(s) \pi T_1(s)x \, d\mu(s) = \int_S a(s) T_2(s) \pi(x) \, d\mu(s) = \widehat{a}(T_2)\pi(x),$$

and hence  $\pi \widehat{a}(T_1) = \widehat{a}(T_2)\pi$ .

The first construction is a powerful tool allowing one to obtain results for the operator norm from corresponding statements that hold for individual orbits. It is similar to a construction appearing in the proof of [19, Theorem 4.3] (see also [104, Note added in proof]) but differs in that it preserves the inner product structure of the underlying space; see [57], [84] and [94] for similar constructions.

**Proposition 7.2.1.** *Let  $X$  be a complex Hilbert space, let  $T$  be a bounded representation of a locally compact abelian semigroup  $S$  on  $X$  and let  $A$  be any directed set. Then there exist:*

- (a) a Banach space  $X_A^\infty \subset \ell^\infty(A; X)$ , containing all nets of the form  $(\widehat{a}(T)x_\alpha)$  for  $a \in L^1(S)$  and  $(x_\alpha) \in \ell^\infty(A; X)$ , a bounded representation  $T_A^\infty$  of  $S$  on  $X_A^\infty$  with  $\text{Sp}(T_A^\infty) = \text{Sp}(T)$  and, for each  $\alpha \in A$ , a surjective intertwining operator  $\pi_\alpha : X_A^\infty \rightarrow X$  of unit norm;
- (b) a Hilbert space  $X_A$  which is a quotient of  $X_A^\infty$ , a bounded representation  $T_A$  of  $S$  on  $X_A$  with  $\text{Sp}(T_A) \subset \text{Sp}(T_A^\infty)$  and a surjective intertwining operator  $\pi_A : X_A^\infty \rightarrow X_A$  which is contractive and such that

$$\|\pi_A(x_\alpha)\| \geq \liminf_{\alpha \rightarrow \infty} \|x_\alpha\| \tag{7.2.1}$$

for all  $(x_\alpha) \in X_A^\infty$ .

In particular,  $\text{Sp}(T_A) \subset \text{Sp}(T)$ .

PROOF. Denote by  $T_A^\infty : S \rightarrow \mathcal{B}(\ell^\infty(A; X))$  the bounded homomorphism given, for  $s \in S$  and  $(x_\alpha) \in \ell^\infty(A; X)$ , by

$$T_A^\infty(s)(x_\alpha) = (T(s)x_\alpha),$$

and let  $X_A^\infty$  be the maximal closed subspace of  $\ell^\infty(A; X)$  on which  $T_A^\infty$  is strongly continuous. Let  $M = \sup\{\|T(s)\| : s \in S\}$  and note that  $\sup\{\|T_A^\infty(s)\| : s \in S\} \leq M$ . Now, for  $a \in L^1(S)$ ,  $x \in X$  and any  $s, t \in S$ ,

$$\|(T(s) - T(t))\widehat{a}(T)x\| = \|(\pi_T(a_s - a_t))x\| \leq M\|\tau_a(s) - \tau_a(t)\|_1\|x\|$$

and hence, by continuity of the shift maps  $\tau_\alpha$ ,  $X_A^\infty$  contains all nets of the form  $(\widehat{a}(T)x_\alpha)$  with  $a \in L^1(S)$  and  $(x_\alpha) \in \ell^\infty(A; X)$ . For  $\alpha \in A$ , define the map  $\pi_\alpha : X_A^\infty \rightarrow X$  by  $\pi_\alpha(x_\beta) = x_\alpha$ . Then each  $\pi_\alpha$  is a surjection of unit norm which intertwines  $T_A^\infty$  with  $T$ . It follows from the fact that  $\pi_\alpha \widehat{a}(T_A^\infty) = \widehat{a}(T)\pi_\alpha$  for all  $a \in L^1(S)$  and  $\alpha \in A$  that  $\|\widehat{a}(T_A^\infty)\| = \|\widehat{a}(T)\|$  and hence that  $\text{Sp}(T_A^\infty) = \text{Sp}(T)$ . This proves (a).

Letting  $\mathcal{F}$  be the filter on  $A$  generated by the sets  $A_\alpha = \{\beta \in A : \beta \succeq \alpha\}$  with  $\alpha \in A$ , and  $\mathcal{U}$  any ultrafilter on  $A$  that contains  $\mathcal{F}$ , define

$$X_A = X_A^\infty / c_0(A; X; \mathcal{U}) \cap X_A^\infty,$$

where  $c_0(A; X; \mathcal{U})$  denotes the subspace of  $\ell^\infty(A; X)$  consisting of nets that converge to zero along the ultrafilter  $\mathcal{U}$ . Since  $X_A$  is a closed subspace of the ultrapower

$$X_{\mathcal{U}} = \ell^\infty(A; X) / c_0(A; X; \mathcal{U}),$$

which is a Hilbert space (see [64] for details),  $X_A$  is itself a Hilbert space. Moreover, by invariance of  $c_0(A; X; \mathcal{U})$  under  $T_A^\infty(s)$  for each  $s \in S$ ,  $T_A^\infty$  descends to a bounded representation  $T_A$  of  $S$  on  $X_A$ . Let  $\pi_A : X_A^\infty \rightarrow X_A$  denote the canonical surjection. Then  $\pi_A$  is a linear contraction which intertwines  $T_A^\infty$  with  $T_A$  and, if  $\ell(x_\alpha)$  denotes the limit of the bounded real net  $(\|x_\alpha\|)$  along the ultrafilter  $\mathcal{U}$  for a given  $(x_\alpha) \in X_A^\infty$ , then

$$\|\pi_A(x_\alpha)\| = \ell(x_\alpha)$$

for all  $(x_\alpha) \in X_A^\infty$ . Supposing that for some fixed  $(x_\alpha) \in X_A^\infty$  there exists  $c \in \mathbb{R}$  with

$$\ell(x_\alpha) < c < \liminf_{\alpha \rightarrow \infty} \|x_\alpha\|,$$

let  $\alpha_0 \in A$  be such that  $\inf\{\|x_\alpha\| : \alpha \succeq \alpha_0\} > c$ . Then

$$A_c = \{\alpha \in A : \|\|x_\alpha\| - \ell(x_\alpha)\| < c - \ell(x_\alpha)\}$$

is an element of  $\mathcal{U}$  satisfying  $A_{\alpha_0} \cap A_c = \emptyset$ . Since  $\mathcal{U}$  is an ultrafilter, this is a contradiction and hence proves (7.2.1). Finally, given  $a \in L^1(S)$ ,

$$\begin{aligned} \|\widehat{a}(T_A)\| &\leq \sup \{ \|\widehat{a}(T_A)\pi_A(x_\alpha)\| : \|(x_\alpha)\|_\infty \leq 1 \} \\ &= \sup \{ \|\pi_A \widehat{a}(T_A^\infty)(x_\alpha)\| : \|(x_\alpha)\|_\infty \leq 1 \} \\ &\leq \sup \{ \|\widehat{a}(T_A^\infty)(x_\alpha)\| : \|(x_\alpha)\|_\infty \leq 1 \} \\ &= \|\widehat{a}(T_A^\infty)\|, \end{aligned}$$

from which it follows that  $\text{Sp}(T_A) \subset \text{Sp}(T_A^\infty)$ , thus proving (b).  $\square$

The remaining results of this section follow well-known patterns. The first is a Hilbert space version of [19, Proposition 3.1] and is similar to the construction given in [63, Section 1]. Here, given a bounded representation  $T$  of a locally compact abelian semigroup  $S$  on a Banach space  $X$ , the space  $X_0(T)$  is defined as

$$X_0(T) = \left\{ x \in X : \lim_{s \rightarrow \infty} \|T(s)x\| = 0 \right\}.$$

**Proposition 7.2.2.** *Let  $X$  be a complex Hilbert space, let  $T$  be a bounded representation of a locally compact abelian semigroup  $S$  on  $X$  with  $M = \sup\{\|T(s)\| : s \in S\}$  and let  $\phi$  be any Banach limit on  $L^\infty(S)$ . Then there exist a Hilbert space  $X_\phi$ , a representation  $T_\phi$  of  $S$  by isometries on  $X_\phi$  with  $\text{Sp}(T_\phi) \subset \text{Sp}(T)$  and an intertwining operator  $\pi_\phi : X \rightarrow X_\phi$  with  $\|\pi_\phi\| \leq M$  which has dense range and satisfies  $\text{Ker } \pi_\phi = X_0(T)$ .*

PROOF. Let  $\phi$  be a Banach limit on  $L^\infty(S)$  and let  $(\cdot, \cdot)_\phi : X \times X \rightarrow \mathbb{C}$  denote the semi-inner product given, for  $x, y \in X$ , by

$$(x, y)_\phi = \phi((T(\cdot)x, T(\cdot)y)).$$

Define  $K_\phi = \{x \in X : (x, x)_\phi = 0\}$  and write  $X_\phi$  for the Hilbert space obtained by completing the quotient space  $X/K_\phi$  with respect to the norm  $\|\cdot\|_\phi$  induced by the inner product

$$(x + K_\phi, y + K_\phi)_\phi = (x, y)_\phi.$$

Since  $\phi$  is translation invariant,  $T(s)$  descends, for each  $s \in S$ , to a well-defined isometry on  $X/K_\phi$  given by  $x + K_\phi \mapsto T(s)x + K_\phi$ , and hence, by continuous extension, to an isometry  $T_\phi(s)$  on  $X_\phi$ . Let  $\pi_\phi : X \rightarrow X_\phi$  denote the canonical surjection onto  $X/K_\phi$ , so that  $\pi_\phi$  has dense range in  $X_\phi$ ,  $\pi_\phi$  is bounded with  $\|\pi_\phi\| \leq M$  and  $\pi_\phi T(s) = T_\phi(s)\pi_\phi$  for all  $s \in S$ . Since, for  $s, t \in S$  and any  $x \in X$ ,

$$\|(T_\phi(s) - T_\phi(t))\pi_\phi(x)\|_\phi = \|\pi_\phi(T(s) - T(t))x\|_\phi \leq M\|(T(s) - T(t))x\|,$$

it follows from the density of  $X/K_\phi$  in  $X_\phi$  that the homomorphism  $T_\phi : S \rightarrow \mathcal{B}(X_\phi)$  is strongly continuous and hence a bounded representation of  $S$  on  $X_\phi$ .

Note further that  $x \in \text{Ker } \pi_\phi$  if and only if  $\phi(\|T(\cdot)x\|^2) = 0$ . Hence  $\|T(s)x\| \rightarrow 0$  as  $s \rightarrow \infty$  certainly implies that  $x \in \text{Ker } \pi_\phi$ . Suppose conversely that  $x \in \text{Ker } \pi_\phi$ . Since

$$\inf \{ \|T(s)x\|^2 : s \in S \} \leq \text{ess-lim inf}_{s \rightarrow \infty} \|T(\cdot)x\|^2 \leq \phi(\|T(\cdot)x\|^2),$$

it follows that

$$\inf \{ \|T(s)x\|^2 : s \in S \} = 0.$$

Suppose there exist  $\varepsilon > 0$  and a net  $(s_\alpha)$  in  $S$  with  $s_\alpha \rightarrow \infty$  as  $\alpha \rightarrow \infty$  and such that  $\|T(s_\alpha)x\| \geq \varepsilon$  for each  $\alpha$ . Let  $s_0 \in S$  be such that  $\|T(s_0)x\| < \varepsilon \min\{1, M^{-1}\}$ . Then  $s_0 \neq s_\alpha$  for all  $\alpha$  and, given  $\alpha$  such that  $s_\alpha \succ s_0$ ,

$$\varepsilon \leq \|T(s_\alpha)x\| = \|T(s_\alpha - s_0)T(s_0)x\| \leq M\|T(s_0)x\| < \varepsilon,$$

which is absurd. Hence  $\|T(s)x\| \rightarrow 0$  as  $s \rightarrow \infty$ , giving  $\text{Ker } \pi_\phi = X_0(T)$ .

Finally, using the fact that  $\pi_\phi \widehat{a}(T) = \widehat{a}(T_\phi)\pi_\phi$ , it is easy to show that

$$\|\widehat{a}(T_\phi)\pi_\phi(x)\| \leq \|\widehat{a}(T)\| \|\pi_\phi(x)\|$$

for all  $a \in L^1(S)$  and all  $x \in X$ . By density of  $\text{Ran } \pi_\phi$  in  $X_\phi$  it follows that  $\|\widehat{a}(T_\phi)\| \leq \|\widehat{a}(T)\|$  for all  $a \in L^1(S)$  and hence that  $\text{Sp}(T_\phi) \subset \text{Sp}(T)$ , as required.  $\square$

Proposition 7.2.2 is particularly powerful in conjunction with the following result, which is contained in various forms in [16, Proposition 2.1], [19, Proposition 3.2], [20], [38] and [58].

**Proposition 7.2.3.** *Let  $S$  be a locally compact abelian semigroup and let  $G$  be the group generated by  $S$ . Furthermore, let  $X$  be a complex Hilbert space and let  $T$  be a representation of  $S$  by isometries on  $X$ . Then there exist a Hilbert space  $Y$ , a representation  $U$  of  $G$  by unitary operators on  $Y$  with  $\text{Sp}(U) = \text{Sp}_u(T)$  and an isometric intertwining operator  $\pi_Y : X \rightarrow Y$  such that the set  $\{U(-s)\pi_Y(x) : s \in S, x \in X\}$  is dense in  $Y$ .*

PROOF. Define the complex vector space  $Z$  by

$$Z = \{ \phi \in X^S : \exists s_\phi \in S \text{ s.t. } \phi(s + s_\phi) = T(s)\phi(s_\phi) \text{ for all } s \in S \},$$

where  $X^S$  denotes the space of all maps  $\phi : S \rightarrow X$ , and note that  $Z$  admits a semi-inner product  $(\cdot, \cdot)_T$  defined, for  $\phi, \varphi \in Z$ , by

$$(\phi, \varphi)_T = \lim_{s \rightarrow \infty} (\phi(s), \varphi(s)).$$

Given  $s \in S$ , consider the map  $T_Z(s)$  on  $Z$  given, for  $\phi \in Z$  and  $t \in S$ , by

$$(T_Z(s)\phi)(t) = T(s)\phi(t).$$

Then  $T_Z(s)$  maps  $Z$  into itself, since

$$(T_Z(s)\phi)(t + s_\phi) = T(t)(T_Z(s)\phi)(s_\phi)$$

for all  $\phi \in Z$  and all  $t \in S$ , and, writing  $\|\cdot\|_T$  for the semi-norm induced by  $(\cdot, \cdot)_T$ ,

$$\|T_Z(s)\phi\|_T = \lim_{t \rightarrow \infty} \|T(s)\phi(t)\| = \|\phi\|_T$$

by isometricity of  $T$ . Since the space  $Z_0 = \{\phi \in Z : \|\phi\|_T = 0\}$  is invariant under  $T_Z(s)$  for all  $s \in S$ , each of these operators descends to an isometry on the normed vector space  $Z/Z_0$ . Noting, moreover, that

$$\|(T(s) - T(t))\phi(r + s_\phi)\| = \|(T(s) - T(t))\phi(s_\phi)\|$$

for all  $\phi \in Z$  and all  $r, s, t \in S$ , it follows that  $T_Z$  induces an isometric representation  $U$  of  $S$  on the Hilbert space  $Y$  obtained from  $Z/Z_0$  by completion with respect to the quotient norm. Now, given any  $s_0 \in S$  and any  $\phi \in Z$ , let  $\varphi_0 \in X^S$  be defined by

$$\varphi_0(s) = \begin{cases} \phi(s - s_0) & \text{if } s \succeq s_0 + s_\phi, \\ 0 & \text{otherwise.} \end{cases}$$

Then  $\varphi_0 \in Z$  with  $s_{\varphi_0} = s_0 + s_\phi$  and

$$U(s_0)(\varphi_0 + Z) = \phi + Z,$$

since

$$(T_Z(s_0)\varphi_0 - \phi)(s) = 0$$

for all  $s \succeq s_0 + s_\phi$ . By density of  $Z/Z_0$  in  $Y$ , therefore,  $U(s)$  is in fact a surjection for each  $s \in S$ . Defining

$$U(s - t) = U(s)U(t)^{-1}$$

for all  $s, t \in S$ , it is easy to see that  $U$  extends to a representation of the group  $G = S - S$  by unitary operators on  $Y$ .

Let  $\pi_Z : X \rightarrow Z$  denote the map given, for  $x \in X$  and  $s \in S$ , by

$$(\pi_Z(x))(s) = T(s)x,$$

and write  $\pi_0 : Z \rightarrow Y$  for the canonical surjection onto  $Z/Z_0$ . Then the map  $\pi_Y : X \rightarrow Y$  given by the composition  $\pi_Y = \pi_0 \circ \pi_Z$  is isometric and intertwines  $T$  with  $U$ . Given  $\phi \in Z$ , let  $x_\phi = \phi(s_\phi)$ . Then

$$(T_Z(s_\phi)\phi - \pi_Z(x_\phi))(s) = 0$$

for all  $s \succcurlyeq s_\phi$ , and hence

$$\pi_0(\phi) = U(-s_\phi)\pi_Y(x_\phi),$$

which by density of  $\text{Ran } \pi_0$  in  $Y$  implies that  $\{U(-s)\pi_Y(x) : s \in S, x \in X\}$  is dense in  $Y$ . From this it follows in turn that

$$\begin{aligned} \|\widehat{a}(U)\| &= \sup \{ \|\widehat{a}(U)U(-s)\pi_Y(x)\| : s \in S, \|x\| \leq 1 \} \\ &= \sup \{ \|\pi_Y\widehat{a}(T)x\| : \|x\| \leq 1 \} \\ &= \|\widehat{a}(T)\| \end{aligned}$$

for all  $a \in L^1(S)$ . Hence  $\text{Sp}(U; S) = \text{Sp}(T)$  so, by Lemma 2.4.2,  $\text{Sp}(U) = \text{Sp}_u(T)$ .  $\square$

### 7.3 The main result

The aim of this section is to prove the following generalisation of [71, Theorem 2.1].

**Theorem 7.3.1.** *Let  $X$  be a complex Hilbert space, let  $T$  be a bounded representation of a locally compact abelian semigroup  $S$  on  $X$  and let  $a \in L^1(S)$ . Then the following are equivalent:*

- (i)  $\widehat{a}(\chi) = 0$  for every  $\chi \in \text{Sp}_u(T)$ ;
- (ii) Given any Følner net  $(\Omega_\alpha)$  for  $S$  and any  $\chi \in \text{Sp}_u(T)$ ,

$$\lim_{\alpha \rightarrow \infty} \frac{1}{\mu(\Omega_\alpha)} \left\| \int_{\Omega_\alpha} \overline{\chi(s)} T(s) \widehat{a}(T) \, d\mu(s) \right\| = 0; \quad (7.3.1)$$

- (iii)  $\|T(s)\widehat{a}(T)\| \rightarrow 0$  as  $s \rightarrow \infty$ .

**Remark 7.3.2.** In [71, Theorem 2.1], conditions (ii) and (iii) above are presented in a slightly more general form, with the operator  $\widehat{a}(T)$  replaced by an arbitrary  $Q \in \mathcal{B}(X)$  that commutes with the representation. The presentation here is restricted to the case  $Q = \widehat{a}(T)$  purely for the sake of clarity.

The proof of this result will be broken up into a number of separate steps, all of which correspond to some part of the proof of [71, Theorem 2.1] but typically involve some modifications to accommodate the more general setting in which the representation need not be norm continuous. The following lemma constitutes the main step towards proving that (i)  $\implies$  (ii); it corresponds to [71, Lemma 2.2]. Note that the Hilbert space assumption is not required for this part of the argument.

**Lemma 7.3.3.** *Let  $X$  be a complex Banach space, let  $T$  be a bounded representation of a locally compact abelian semigroup  $S$  on  $X$  and let  $a \in L^1(S)$ . Then, for all  $\chi \in \Gamma$ ,*

$$\lim_{\alpha \rightarrow \infty} \frac{1}{\mu(\Omega_\alpha)} \left\| \int_{\Omega_\alpha} \overline{\chi(s)} T(s) (\widehat{a}(T) - \widehat{a}(\chi)) \, d\mu(s) \right\| = 0,$$

where  $(\Omega_\alpha)$  is any Følner net for  $S$  and the integral is taken in the strong sense.

PROOF. With  $a \in L^1(S)$  and  $\chi \in \Gamma$  fixed, let  $x \in X$  be of unit norm and set

$$I_x(\alpha) = \frac{1}{\mu(\Omega_\alpha)} \left\| \int_{\Omega_\alpha} \overline{\chi(s)} T(s) (\widehat{a}(T)x - \widehat{a}(\chi)x) \, d\mu(s) \right\|.$$

Then, by a simple application of Fubini's theorem,

$$I_x(\alpha) \leq M \int_S \frac{\mu(\Omega_\alpha \Delta (\Omega_\alpha + s))}{\mu(\Omega_\alpha)} |a(s)| \, d\mu(s),$$

where  $M = \sup\{\|T(s)\| : s \in S\}$ . Let  $\varepsilon > 0$ . Since  $a \in L^1(S)$ , there exists a compact subset  $K$  of  $S$  such that  $\int_{S \setminus K} |a(s)| \, d\mu(s) < \varepsilon/4M$ . Defining

$$\delta_K(\alpha) = \sup \left\{ \frac{\mu(\Omega_\alpha \Delta (\Omega_\alpha + s))}{\mu(\Omega_\alpha)} : s \in K \right\},$$

it follows from the definition of a Følner net that  $\delta_K(\alpha) \rightarrow 0$  as  $\alpha \rightarrow \infty$ . Since

$$I_x(\alpha) \leq M \|a\|_1 \delta_K(\alpha) + 2M \int_{S \setminus K} |a(s)| \, d\mu(s),$$

$I_x(\alpha) < \varepsilon$  for all sufficiently large  $\alpha$ , and the result follows.  $\square$

**Corollary 7.3.4.** *Let  $X$  be a complex Banach space, let  $T$  be a bounded representation of a locally compact abelian semigroup  $S$  on  $X$  and let  $\chi \in \Gamma$ . Suppose that  $a \in L^1(S)$  is such that  $\widehat{a}(\chi) = 0$ . Then (7.3.1) holds for any Følner net  $(\Omega_\alpha)$  for  $S$ .*

The next result is crucial in establishing the implication (ii)  $\implies$  (iii) in Theorem 7.3.1 and should be compared with [71, Lemma 2.4].

**Proposition 7.3.5.** *Let  $S$  be a locally compact abelian semigroup and let  $G$  be the group generated by  $S$ . Furthermore, let  $X$  be a complex Hilbert space, let  $T$  be a representation of  $G$  by unitary operators on  $X$  and let  $a \in L^1(G)$ . Suppose that, for each  $\chi \in \text{Sp}(T)$ ,*

$$\lim_{\alpha \rightarrow \infty} \frac{1}{\mu(\Omega_\alpha)} \left\| \int_{\Omega_\alpha} \overline{\chi(s)} T(s) \widehat{a}(T) \, d\mu(s) \right\| = 0,$$

where  $(\Omega_\alpha)$  is any Følner net for  $S$ . Then  $\widehat{a}(T) = 0$ .

PROOF. Let  $E$  denote the spectral measure associated with  $T$  (see [93, Theorem 8.3.2]) and, for  $s \in G$  and any Borel subset  $\Lambda$  of  $\Gamma$ , let  $T_\Lambda(s) = T(s)E(\Lambda)$ . Then

$$T_\Lambda(s) = \int_\Lambda \chi(s) dE(\chi),$$

the integral being taken in the weak sense, and, by Fubini's theorem,

$$\widehat{b}(T_\Lambda) = \int_\Lambda \widehat{b}(\chi) dE(\chi) \quad (7.3.2)$$

for all  $b \in L^1(G)$ . Thus if  $\Lambda$  is a closed subset of  $\Gamma$  and  $b \in J_\Lambda(G)$ , then  $\widehat{b}(T_\Lambda) = 0$ , and it follows that  $M_{T_\Lambda}(\Lambda) = X$ , so that  $\text{Sp}(T_\Lambda) \subset \Lambda$ . Choosing  $\Lambda$  to be a compact subset of  $\Gamma$  ensures that the representation  $T_\Lambda$  of  $G$  on  $X$  is norm continuous.

Set  $Q = \widehat{a}(T)$  and, for a given compact subset  $\Lambda$  of  $\text{Sp}(T)$ , define  $Q_\Lambda = QE(\Lambda)$ , noting that  $Q_\Lambda$  is normal and that  $Q_\Lambda \rightarrow Q$  in the strong operator topology as  $\Lambda$  approaches  $\text{Sp}(T)$  through compact subsets. Furthermore, let  $\mathcal{A}_\Lambda$  denote the commutative unital  $C^*$ -algebra generated by  $\{Q_\Lambda, Q_\Lambda^*\} \cup \{T_\Lambda(s) : s \in G\}$ , and let  $\Delta(\mathcal{A}_\Lambda)$  denote its character space. Write  $\Phi_\Lambda : \mathcal{A}_\Lambda \rightarrow C(\Delta(\mathcal{A}_\Lambda))$  for the Gelfand transform of  $\mathcal{A}_\Lambda$ , which is an isometric  $*$ -isomorphism, and consider the map  $\chi_\xi : G \rightarrow \mathbb{C} \setminus \{0\}$  given, for  $\xi \in \Delta(\mathcal{A}_\Lambda)$  and  $s \in G$ , by  $\chi_\xi(s) = \Phi_\Lambda(T_\Lambda(s))(\xi)$ . Since the representation  $T_\Lambda$  is norm continuous,  $\chi_\xi$  is a continuous group homomorphism, and the fact that each  $\xi \in \Delta(\mathcal{A}_\Lambda)$  is a bounded linear functional on  $\mathcal{A}_\Lambda$  with  $\|\xi\| = |\xi(E(\Lambda))| = 1$  implies that  $|\chi_\xi(s)| \leq 1$  for all  $s \in G$ . Hence  $\chi_\xi \in \Gamma$ . Moreover, if  $b \in L^1(G)$ , then

$$|\widehat{b}(\chi_\xi)| = \left| \xi \left( \int_G b(s) T_\Lambda(s) d\mu(s) \right) \right| \leq \|\widehat{b}(T_\Lambda)\|,$$

which is to say that  $\chi_\xi \in \text{Sp}(T_\Lambda)$ , and hence  $\chi_\xi \in \text{Sp}(T)$ . Let  $g_\Lambda = \Phi_\Lambda(Q_\Lambda)$ . Then

$$\begin{aligned} |g_\Lambda(\xi)| &= \frac{1}{\mu(\Omega_\alpha)} \left| \int_{\Omega_\alpha} |\chi_\xi(s)|^2 g_\Lambda(\xi) d\mu(s) \right| \\ &\leq \frac{1}{\mu(\Omega_\alpha)} \left\| \Phi_\Lambda \left( \int_{\Omega_\alpha} \overline{\chi_\xi(s)} T_\Lambda(s) Q_\Lambda d\mu(s) \right) \right\|_\infty \\ &= \frac{1}{\mu(\Omega_\alpha)} \left\| \int_{\Omega_\alpha} \overline{\chi_\xi(s)} T_\Lambda(s) Q_\Lambda d\mu(s) \right\| \\ &\leq \frac{1}{\mu(\Omega_\alpha)} \left\| \int_{\Omega_\alpha} \overline{\chi_\xi(s)} T(s) Q d\mu(s) \right\|, \end{aligned}$$

for any  $\xi \in \Delta(\mathcal{A}_\Lambda)$  and letting  $\alpha \rightarrow \infty$  shows that  $g_\Lambda = 0$ . Since  $\Phi_\Lambda$  is an isometry, it follows that  $Q_\Lambda = 0$ , and allowing  $\Lambda$  to approach  $\text{Sp}(T)$  through compact subsets gives  $Q = 0$ , as required.  $\square$

**Remark 7.3.6.** The result remains true when  $\widehat{a}(T)$  is replaced by any  $Q \in \mathcal{B}(X)$  which commutes with  $T$ . If  $Q$  is normal, this follows from the same argument as above, and the general case can then be obtained by considering the operator  $Q^*Q$ ; see also [71, Lemma 2.4].

Propositions 7.3.7 and 7.3.8 below correspond in essence to the two main stages in the proof of [71, Theorem 2.1] and show, via an intermediate result for individual orbits, that (ii)  $\implies$  (iii) in Theorem 7.3.1.

**Proposition 7.3.7.** *Let  $X$  be a complex Hilbert space, let  $T$  be a bounded representation of a locally compact abelian semigroup  $S$  on  $X$  and let  $a \in L^1(S)$ . Suppose that, for some Følner net  $(\Omega_\alpha)$  for  $S$ , (7.3.1) holds for all  $\chi \in \text{Sp}_u(T)$ . Then, for each  $x \in X$ ,  $\|T(s)\widehat{a}(T)x\| \rightarrow 0$  as  $s \rightarrow \infty$ .*

PROOF. Fix a Banach limit  $\phi$  on  $L^\infty(S)$  and let  $X_\phi, T_\phi$  and  $\pi_\phi$  be as in Proposition 7.2.2, noting in particular that, for any operator  $Q \in \mathcal{B}(X)$  that commutes with  $T$ , the operator  $Q_\phi \in \mathcal{B}(X_\phi)$  defined by  $\pi_\phi Q = Q_\phi \pi_\phi$  satisfies  $\|Q_\phi\| \leq \|Q\|$ . By Proposition 7.2.3, there exist a further Hilbert space  $Y_\phi$ , a representation  $U_\phi$  of the group  $G = S - S$  by unitary operators on  $Y_\phi$  with  $\text{Sp}(U_\phi) = \text{Sp}_u(T_\phi)$  and an isometric intertwining operator  $\pi_{Y_\phi} : X_\phi \rightarrow Y_\phi$  such that  $\{U_\phi(-s)\pi_{Y_\phi}(x) : s \in S, x \in X_\phi\}$  is dense in  $Y_\phi$ . The latter implies, in particular, that  $\|Q_{Y_\phi}\| = \|Q_\phi\|$  for all  $Q_\phi \in \mathcal{B}(X_\phi)$  and all  $Q_{Y_\phi} \in \mathcal{B}(Y_\phi)$  which commute with  $U_\phi$  and satisfy  $\pi_{Y_\phi} Q_\phi = Q_{Y_\phi} \pi_{Y_\phi}$ . Thus it is possible to assume, dropping only the density condition on the range of the intertwining operator, that  $T_\phi$  itself is in fact a representation of  $G$  by unitary operators on  $X_\phi$ .

Now, given  $\chi \in \text{Sp}(T_\phi)$ , define operators  $Q_\alpha \in \mathcal{B}(X)$  and  $Q_{\phi,\alpha} \in \mathcal{B}(X_\phi)$  as

$$Q_\alpha = \frac{1}{\mu(\Omega_\alpha)} \int_{\Omega_\alpha} \overline{\chi(s)} T(s) \widehat{a}(T) \, d\mu(s) \quad (7.3.3)$$

and

$$Q_{\phi,\alpha} = \frac{1}{\mu(\Omega_\alpha)} \int_{\Omega_\alpha} \overline{\chi(s)} T_\phi(s) \widehat{a}(T_\phi) \, d\mu(s).$$

Then  $\pi_\phi Q_\alpha = Q_{\phi,\alpha} \pi_\phi$ , from which it follows that  $\|Q_{\phi,\alpha}\| \leq \|Q_\alpha\|$ . In particular,  $\|Q_{\phi,\alpha}\| \rightarrow 0$  as  $\alpha \rightarrow \infty$ . Identifying  $L^1(S)$  in the natural way with a subset of  $L^1(G)$ , Proposition 7.3.5 applied to  $T_\phi, X_\phi$  and  $a$  implies that  $\widehat{a}(T_\phi) = 0$ . Thus  $\pi_\phi(\widehat{a}(T)x) = \widehat{a}(T_\phi)\pi_\phi(x) = 0$  for any  $x \in X$ , so the result follows from the fact that  $\text{Ker } \pi_\phi = X_0(T)$ .  $\square$

**Proposition 7.3.8.** *Let  $X$  be a complex Hilbert space, let  $T$  be a bounded representation of a locally compact abelian semigroup  $S$  on  $X$  and let  $a \in L^1(S)$ .*

Suppose that, for some Følner net  $(\Omega_\alpha)$  for  $S$ , equation (7.3.1) is satisfied for all  $\chi \in \text{Sp}_u(T)$ . Then  $\|T(s)\widehat{a}(T)\| \rightarrow 0$  as  $s \rightarrow \infty$ .

PROOF. Suppose, for the sake of contradiction, that (7.3.1) holds for all  $\chi \in \text{Sp}_u(T)$  and some Følner net  $(\Omega_\alpha)$  but that there exist  $\varepsilon > 0$  and a net  $(s_\beta)$  in  $S$ , with indexing set  $A$  say, such that  $s_\beta \rightarrow \infty$  as  $\beta \rightarrow \infty$  and, for some suitable sequence  $(y_\beta)$  of unit vectors in  $X$ ,  $\|T(s_\beta)\widehat{a}(T)y_\beta\| \geq \varepsilon$  for all  $\beta \in A$ . Letting  $M = \sup\{\|T(s)\| : s \in S\}$ , it follows that  $\|T(s)\widehat{a}(T)y_\beta\| \geq \varepsilon M^{-1}$  whenever  $s_\beta - s \in S$ . Fix  $t \in S^\circ$  and let  $b \in L^1(S)$  satisfy  $\|b\|_1 = 1$  and  $\|a * b - a_t\|_1 < \varepsilon/2M^3$ , where  $a_t \in L^1(S)$  is given by

$$a_t(s) = \begin{cases} a(s-t) & \text{if } s-t \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Let  $X_A, T_A, X_A^\infty, T_A^\infty$  and  $\pi_A$  be as in Proposition 7.2.1 and consider the element  $(x_\beta)$  of  $X_A^\infty$ , where  $x_\beta = \widehat{b}(T)y_\beta$ . Then, writing  $c = a * b - a_t$ ,

$$\begin{aligned} \|T_A(s)\widehat{a}(T_A)\pi_A(x_\beta)\| &= \|\pi_A T_A^\infty(s)\widehat{a}(T_A^\infty)(x_\beta)\| \\ &= \|\pi_A(T(s)\widehat{a} * \widehat{b}(T)y_\beta)\| \\ &\geq \|\pi_A(T(s+t)\widehat{a}(T)y_\beta)\| - \|\pi_A(T(s)\widehat{c}(T)y_\beta)\| \\ &\geq \liminf_{\beta \rightarrow \infty} \|T(s+t)\widehat{a}(T)y_\beta\| - M^2\|c\|_1 \end{aligned}$$

for all  $s \in S$ , where the last line follows from the definition of the norm on  $X_A$ . Thus  $\|T_A(s)\widehat{a}(T_A)\pi_A(x_\beta)\| \geq \varepsilon/2M$  for all  $s \in S$ .

Fix  $\chi \in \text{Sp}_u(T_A)$  and define the operators  $Q_{A,\alpha}^\infty \in \mathcal{B}(X_A^\infty)$  and  $Q_{A,\alpha} \in \mathcal{B}(X_A)$  as

$$Q_{A,\alpha}^\infty = \frac{1}{\mu(\Omega_\alpha)} \int_{\Omega_\alpha} \overline{\chi(s)} T_A^\infty(s) \widehat{a}(T_A^\infty) d\mu(s)$$

and

$$Q_{A,\alpha} = \frac{1}{\mu(\Omega_\alpha)} \int_{\Omega_\alpha} \overline{\chi(s)} T_A(s) \widehat{a}(T_A) d\mu(s).$$

Then  $\pi_A Q_{A,\alpha}^\infty = Q_{A,\alpha} \pi_A$ , so the properties of  $\pi_A$  and the fact that  $Q_{A,\alpha}^\infty$  acts on  $X_A^\infty$  by entrywise application of the operator  $Q_\alpha$  defined in equation (7.3.3) imply that  $\|Q_{A,\alpha}\| \leq \|Q_\alpha\|$ . Hence  $\|Q_{A,\alpha}\| \rightarrow 0$  as  $\alpha \rightarrow \infty$ , and Proposition 7.3.7 applied to  $T_A$  and  $X_A$  leads to the required contradiction.  $\square$

Corollary 7.3.4 and Proposition 7.3.8 together prove the implications (i)  $\implies$  (ii)  $\implies$  (iii) of Theorem 7.3.1. The following simple lemma, which follows immediately from the definition of the spectrum of a semigroup representation

$T$  along with the observation that  $\widehat{a}_s(T) = T(s)\widehat{a}(T)$  for all  $a \in L^1(S)$  and  $s \in S$ , shows that (iii)  $\implies$  (i), thus completing the proof of the main result.

**Lemma 7.3.9.** *Let  $X$  be a complex Banach space, let  $T$  be a bounded representation of a locally compact abelian semigroup  $S$  on  $X$  and let  $a \in L^1(S)$ . Then  $|\widehat{a}(\chi)| \leq \|T(s)\widehat{a}(T)\|$  for all  $\chi \in \text{Sp}_u(T)$  and all  $s \in S$ .*

**Remark 7.3.10.** There is a direct proof of the implication (iii)  $\implies$  (ii) in Theorem 7.3.1. Indeed, if  $T$  is a bounded representation of a locally compact abelian semigroup  $S$  on any Banach space  $X$ , if  $a \in L^1(S)$  and if  $(\Omega_\alpha)$  is any Følner net for  $S$ , then

$$\|Q_\alpha\| \leq \sup \{ \|T(s)\widehat{a}(T)\| : s \succeq t \} + M^2 \|a\|_1 \frac{\mu(\Omega_\alpha \Delta (\Omega_\alpha + t))}{\mu(\Omega_\alpha)}$$

for any  $t \in S$ , where  $Q_\alpha$  is as in (7.3.3) and  $M = \sup\{\|T(s)\| : s \in S\}$ . Hence (iii)  $\implies$  (ii) by definition of a Følner net. Moreover, it is possible, at least in special cases, to show directly that (ii)  $\implies$  (i). When  $S = \mathbb{Z}_+$ , this follows from Corollary 7.3.4 and the uniform ergodic theorem (see [71, Corollary 2.3]), and a similar argument works when  $S = \mathbb{R}_+$ .

## 7.4 A direct argument

There is an alternative argument of independent interest which establishes more directly the equivalence of statements (i) and (iii) in Theorem 7.3.1. Recall the classical fact that, given a representation  $T$  of a locally compact abelian group  $G$  by isometries on a Banach space  $X$ , one has  $\widehat{a}(T) = 0$  for all  $a \in L^1(G)$  which are of spectral synthesis with respect to  $\text{Sp}(T)$ . This is a simple consequence of the definition of  $\text{Sp}(T)$  (see also [34, Chapter 8], [75, Chapter 5] and [101, Lemma 2.4.3]) and is used in [19, Theorem 4.3] along with constructions analogous to those in Section 7.2, to obtain Theorem 7.1.1. Corollary 7.4.2 below, which is an improved version of this classical result in the case where  $X$  is a Hilbert space, makes it possible to obtain the implication (i)  $\implies$  (iii) of Theorem 7.3.1 by an analogous argument that bypasses Proposition 7.3.5. It is a special case of the following more general statement.

**Proposition 7.4.1.** *Let  $X$  be a complex Hilbert space, let  $T$  be a representation of a locally compact abelian group  $G$  by unitary operators on  $X$  and let  $a \in L^1(G)$ . Then*

$$\|\widehat{a}(T)\| = \sup\{|\widehat{a}(\chi)| : \chi \in \text{Sp}(T)\}.$$

PROOF. Let  $\mathcal{A}_T$  denote the norm closure in  $\mathcal{B}(X)$  of  $\{\widehat{b}(T) : g \in L^1(G)\}$ . Then  $\mathcal{A}_T$  is a commutative  $C^*$ -algebra and hence, writing  $\Delta(\mathcal{A}_T)$  for the character space of  $\mathcal{A}_T$ , the Gelfand transform  $\Phi : \mathcal{A}_T \rightarrow C_0(\Delta(\mathcal{A}_T))$  is an isometric  $*$ -isomorphism. By [19, Proposition 2.4], the map sending a character  $\chi \in \text{Sp}(T)$  to the character  $\xi_\chi$  on  $\mathcal{A}_T$  defined, on the dense subspace  $\{\widehat{b}(T) : g \in L^1(G)\}$ , by  $\xi_\chi(\widehat{b}(T)) = \widehat{b}(\chi)$  is a bijection, and hence  $\|\widehat{a}(T)\| = \|\Phi(\widehat{a}(T))\|_\infty = \sup\{|\widehat{a}(\chi)| : \chi \in \text{Sp}(T)\}$ , as required.  $\square$

**Corollary 7.4.2.** *Let  $X$  be a complex Hilbert space, let  $T$  be a representation of a locally compact abelian group  $G$  by unitary operators on  $X$  and let  $a \in L^1(G)$ . Then  $\widehat{a}(T) = 0$  if and only if  $\widehat{a}(\chi) = 0$  for all  $\chi \in \text{Sp}_u(T)$ .*

**Remark 7.4.3.** Proposition 7.4.1 follows also from (7.3.2) with  $\Lambda = \text{Sp}(T)$ , while Corollary 7.4.2 has an alternative proof using Corollary 7.3.4 and Proposition 7.3.5 together with Lemma 7.3.9.

## Chapter 8

# Extensions and applications

### 8.1 Overview

The present chapter contains a number of extensions and applications of the results and ideas presented in Chapter 7. Consider a bounded representation  $T$  of a locally compact abelian semigroup  $S$  on a Hilbert space  $X$ . The first extension, described in Section 8.2, is a Katznelson-Tzafriri type result for individual orbits. Section 8.3 then establishes a certain quantified form of the Katznelson-Tzafriri theorem for contractive representations and Section 7.4, finally, addresses the problem of invertibility of contractive representations.

### 8.2 A Katznelson-Tzafriri theorem for orbits

#### 8.2.1 Introduction

The first natural extension of the results of the previous chapter is to individual orbits of a given bounded representation, leading to so-called ‘local’ results. Local results have been studied for instance in [18], [20, Section 4], [63] and [105, Chapter 6] and are of particular interest in the context of  $C_0$ -semigroups, where orbits correspond to solutions of the associated abstract Cauchy problem; see Section 1.1. In general, the aim is to obtain stability results for individual orbits under assumptions weaker than those used to obtain the corresponding ‘global’ result valid in the operator norm. The purpose of this section is to obtain a local version of Theorem 7.3.1 using the appropriate local analogue of the unitary spectrum.

### 8.2.2 The local spectrum

Given a power-bounded operator  $T$  on a Banach space  $X$  and a vector  $x \in X$ , one reasonably natural definition of a local spectrum  $\sigma(T, x)$  of  $T$  at  $x$  is as the set of numbers  $\mu \in \mathbb{C}$  such that there does not exist a neighbourhood  $\Omega$  of  $\mu$  and a holomorphic function  $g : \Omega \rightarrow X$  satisfying  $(\lambda - T)g(\lambda) = x$  for all  $\lambda \in \Omega$ . Similarly, given a bounded  $C_0$ -semigroup  $T$  with generator  $A$ , the local spectrum  $\sigma(A, x)$  of  $T$  at  $x$  can be defined as the set of numbers  $\mu \in \mathbb{C}$  such that there does not exist a neighbourhood  $\Omega$  of  $\lambda$  and a holomorphic function  $g : \Omega \rightarrow X$  satisfying  $g(\lambda) \in D(A)$  and  $(\lambda - A)g(\lambda) = x$  for all  $\lambda \in \Omega$ . The analogue for general semigroup representations that turns out to be appropriate in the present context is the (*Albrecht*) *local spectrum*; see [2], [105, Chapter 6].

Consider a bounded representation  $T$  of a locally compact abelian semigroup  $S$  on some Banach space  $X$  and let  $x \in X$  be given. A character  $\chi \in S^*$  will be said to be *locally regular at  $x$*  if there exist  $n \in \mathbb{N}$ ,  $a_1, \dots, a_n \in L^1(S)$ , a neighbourhood  $\Omega$  of the point  $\lambda_0 = (\widehat{a}_1(\chi), \dots, \widehat{a}_n(\chi))$  in  $\mathbb{C}^n$  and holomorphic functions  $g_1, \dots, g_n : \Omega \rightarrow X$  such that

$$\sum_{k=0}^n (\lambda_k - \widehat{a}_k(T))g_k(\lambda) = x \quad (8.2.1)$$

for all  $\lambda = (\lambda_1, \dots, \lambda_n) \in \Omega$ . The *local (Albrecht) spectrum*  $\text{Sp}(T, x)$  of  $T$  at  $x$  is then defined to be the set of all characters  $\chi \in S^*$  which fail to be locally regular at  $x$ , and the *unitary local (Albrecht) spectrum* of  $T$  at  $x$  is given by  $\text{Sp}_u(T, x) = \text{Sp}(T, x) \cap \Gamma$ . The following simple observation shows why spectral conditions on the unitary local spectrum are typically weaker than the conditions imposed in Chapter 7.

**Lemma 8.2.1.** *Let  $X$  be a complex Banach space and let  $T$  be a bounded representation of a locally compact abelian semigroup  $S$  on  $X$ . Then, for any  $x \in X$ ,  $\text{Sp}(T, x) \subset \text{Sp}(T)$ .*

PROOF. By definition of the spectrum, if  $\chi \notin \text{Sp}(T)$  then there exists  $a \in L^1(S)$  such that  $|\widehat{a}(\chi)| > \|\widehat{a}(T)\|$ . Let  $Q = \widehat{a}(T)$  and  $\lambda_0 = \widehat{a}(\chi)$ . Then  $\lambda_0 \in \rho(Q)$ , and taking  $n = 1$ ,  $\Omega = \rho(Q)$  and  $g(\lambda) = R(\lambda, Q)x$  for  $\lambda \in \Omega$ , it follows that  $\chi$  is locally regular at  $x$ .  $\square$

A further set that will be of interest in what follows is

$$\Lambda(T, x) = \{\chi \in S^* : \widehat{a}(\chi) = 0 \text{ for all } a \in L^1(S) \text{ s.t. } \widehat{a}(T)x = 0\}.$$

The importance of this set arises from the following fact; see [105, Theorem 6.2.7].

**Theorem 8.2.2.** *Let  $X$  be a complex Banach space and let  $T$  be a bounded representation of a locally compact abelian group  $G$  on  $X$ . Then, for any  $x \in X$ ,  $\text{Sp}(T, x) = \Lambda(T, x)$ .*

As is shown in [105, Proposition 6.2.5], it follows from this result that if  $T$  is a bounded representation of  $\mathbb{Z}$  or  $\mathbb{R}$ , then  $\text{Sp}(T, x) = \sigma(T, x)$  (respectively,  $\text{Sp}(T, x) = \sigma(A, x)$ ). Moreover, for bounded representations of the corresponding semigroups  $\mathbb{Z}_+$  and  $\mathbb{R}_+$ ,  $\text{Sp}(T, x) \subset \sigma(T, x)$  (respectively,  $\text{Sp}(T, x) \subset \sigma(A, x)$ ) by [105, Propositions 6.2.9 and 6.2.10], but it remains an open problem whether the two sets necessarily coincide in this case; see [105, Chapter 4] for some partial results.

### 8.2.3 The main result

The following lemma plays an important part in proving a local analogue of the Katznelson-Tzafriri theorem; it is a combination of [105, Lemmas 6.4.1 and 6.4.2]. Recall that, by Proposition 7.2.2, given a bounded representation  $T$  of a locally compact abelian semigroup  $S$  on a Hilbert space  $X$  and any Banach limit  $\phi$  on  $L^\infty(S)$ , there exist a Hilbert space  $X_\phi$ , a representation  $T_\phi$  of  $S$  by isometries on  $X_\phi$  which satisfies  $\text{Sp}(T_\phi) \subset \text{Sp}(T)$  and an intertwining operator  $\pi_\phi : X \rightarrow X_\phi$  with  $\text{Ker } \pi_\phi = X_0(T)$ , where

$$X_0(T) = \{x \in X : \lim_{s \rightarrow \infty} \|T(s)x\| = 0\}$$

as in Section 7.2. Moreover, by Proposition 7.2.3, there exist a further Hilbert space  $Y_\phi$ , a unitary representation  $U_\phi$  on  $Y_\phi$  of the group  $G = S - S$ , and an isometric intertwining operator  $\pi_{Y_\phi} : X_\phi \rightarrow Y_\phi$ . Fix  $\phi$  and let  $\pi$  denote the composition  $\pi_{Y_\phi} \circ \pi_\phi$ . Given  $z \in Y_\phi$ , finally, let  $Y_z$  denote the closed linear span of the set  $\{U_\phi(s)z : s \in G\}$  and write  $U_z$  for the representation obtained by restricting  $U_\phi$  to  $Y_z$ .

**Lemma 8.2.3.** *Let  $X$  be a complex Hilbert space,  $T$  a bounded representation of a locally compact abelian semigroup  $S$  on  $X$  and let  $x \in X$ . Furthermore, let  $\phi$  be a Banach limit on  $L^\infty(S)$  and let  $z = \pi(x)$ . Then, with  $U_z$  as above,  $\text{Sp}(U_z) \subset \text{Sp}_u(T, x)$ .*

PROOF. By Theorem 2.4.1,  $\text{Sp}(U_z) = \Lambda(U_z)$ . But for  $a \in L^1(G)$ ,  $\widehat{a}(U_z) = 0$  if and only if  $\widehat{a}(U_\phi)z = 0$  and hence  $\Lambda(U_z) = \Lambda(U_\phi; z)$ . Thus, by Theorem 8.2.2,

it remains to show only that  $\text{Sp}(U_\phi; z) \subset \text{Sp}_u(T, x)$ . Suppose therefore that  $\chi \in \Gamma \setminus \text{Sp}_u(T, x)$ . Then there exist  $n \in \mathbb{N}$ ,  $a_1, \dots, a_n \in L^1(S)$ , a neighbourhood  $\Omega$  of the point  $\lambda_0 = (\widehat{a}_1(\chi), \dots, \widehat{a}_n(\chi))$  in  $\mathbb{C}^n$  and holomorphic functions  $g_1, \dots, g_n : \Omega \rightarrow X$  such that (8.2.1) holds. Applying the map  $\pi$  to both sides gives

$$\sum_{k=0}^n (\lambda_k - \widehat{a}_k(U_\phi)) h_k(\lambda) = z,$$

where  $h_k$ , for  $1 \leq k \leq n$ , is the holomorphic function  $\pi \circ g_k : \Omega \rightarrow Y_\phi$ . Since  $L^1(S) \subset L^1(G)$ , the result follows.  $\square$

**Remark 8.2.4.** The result remains true (with the same proof) in the Banach space setting; see [105, Section 6.4]. A slightly more careful argument in the second half of the proof shows that in fact  $\text{Sp}(U_\phi; z) \subset \text{Sp}_u(T_\phi; y) \subset \text{Sp}_u(T, x)$ , where  $y = \pi_\phi(x)$ .

The main result of this section is the following theorem, which extends and improves the local result [18, Theorem 5.1] in the Hilbert space setting.

**Theorem 8.2.5.** *Let  $X$  be a complex Hilbert space,  $T$  a bounded representation of a locally compact abelian semigroup  $S$  on  $X$  and let  $x \in X$ . Suppose that  $a \in L^1(S)$  is such that  $\widehat{a}(\chi) = 0$  for all  $\chi \in \text{Sp}_u(T, x)$ . Then  $\|T(s)\widehat{a}(T)x\| \rightarrow 0$  as  $s \rightarrow \infty$ .*

PROOF. Using the same notation as before, Lemma 8.2.3 gives the inclusion  $\text{Sp}(U_z) \subset \text{Sp}_u(T, x)$ , where  $z = \pi(x)$ . Hence  $\widehat{a}(\chi) = 0$  for all  $\chi \in \text{Sp}(U_z)$ , and it follows from Corollary 7.4.2 that  $\widehat{a}(U_z) = 0$ . Thus  $\pi(\widehat{a}(T)x) = \widehat{a}(U_\phi)z = 0$ , and the fact that  $\pi_{Y_\phi}$  is isometric implies that  $\widehat{a}(T)x \in \text{Ker } \pi_\phi$ . Since  $\text{Ker } \pi_\phi = X_0(T)$ , this completes the proof.  $\square$

## 8.2.4 Unbounded representations

It is natural to ask whether the results of the previous section carry over to the case of unbounded orbits, and indeed the answer is affirmative provided the growth is sufficiently slow and regular. A measurable function  $w : S \rightarrow [1, \infty)$  which is bounded on compact subsets of  $S$  and satisfies

$$w(s+t) \leq w(s)w(t)$$

for all  $s, t \in S$  is said to be a *weight*, and a representation  $T$  of  $S$  on a Banach space  $X$  is said to be *dominated* by the weight  $w$  if  $\|T(s)\| \leq w(s)$  for all  $s \in S$ .

Given a semigroup  $S$  and a weight  $w$  on  $S$ , let  $L_w^1(S)$  denote the so-called *Beurling algebra* of measurable functions  $a : S \rightarrow \mathbb{C}$  for which

$$\int_S |a(s)|w(s) \, d\mu(s) < \infty,$$

and let the superscript  $w$  on a spectrum (as e.g. in  $\mathrm{Sp}_u^w(T, x)$  for  $x \in X$ ) indicate that any occurrence of  $L^1(S)$  is replaced in the definition by the Beurling algebra  $L_w^1(S)$ . For more details on weights and representations dominated by weights, see [20], [56] and [75] and [105, Section 2.8].

The main result of this section is the following generalisation of Theorem 8.2.5 to representations which are dominated by sufficiently well-behaved, but potentially unbounded, weights.

**Theorem 8.2.6.** *Let  $X$  be a complex Hilbert space and let  $T$  be a representation of a locally compact abelian semigroup  $S$  on  $X$  which is dominated by a weight  $w$  satisfying*

$$\lim_{t \rightarrow \infty} \frac{w(s+t)}{w(t)} = 1 \tag{8.2.2}$$

for all  $s \in S$ . Furthermore, let  $x \in X$  and suppose that  $a \in L_w^1(S)$  is such that  $\widehat{a}(\chi) = 0$  for all  $\chi \in \mathrm{Sp}_u^w(T, x)$ . Then

$$\lim_{s \rightarrow \infty} \frac{\|T(s)\widehat{a}(T)x\|}{w(s)} = 0.$$

PROOF. The proof follows the same lines as that of Theorem 8.2.5 except that Proposition 7.2.2 is replaced by a version allowing for non-constant weights; see [105, Section 5.1]. In particular, the regularity assumption (8.2.2) on the weight  $w$  ensures that the limit representation of [105, Section 5.1] is isometric, and hence can be extended as before to a representation of  $G$  by unitary operators on some Hilbert space. Using the notation of the previous section, the spectral inclusion  $\mathrm{Sp}(U_z) \subset \mathrm{Sp}_u^w(T, x)$  remains valid (see [105, Lemma 6.4.2]), so the result follows almost exactly as in the proof of Theorem 8.2.5.  $\square$

**Remark 8.2.7.** As before, the first part of the proof is valid in the more general Banach space setting. It remains open whether the argument in the proof of Proposition 7.3.8, which converts a result for individual orbits to a result valid for the operator norm, can be adapted to the case of representations that are bounded by a weight of the form considered here; for a result of this type in the Banach space setting see [105, Theorem 5.2.2]. It is also unclear whether Theorems 8.2.5 and 8.2.6 can be established when the boundedness condition

(involving a constant weight or otherwise) is imposed only on the orbit in question instead of the representation itself.

## 8.3 A quantified Katznelson-Tzafriri type theorem for contractive representations

### 8.3.1 Introduction

Given a bounded representation  $T$  of a locally compact abelian semigroup  $S$  on a complex Hilbert space  $X$  and a function  $a \in L^1(S)$ , Theorem 7.3.1 gives a necessary and sufficient condition in terms of the unitary spectrum  $\mathrm{Sp}_u(T)$  for the limit  $\lim_{s \rightarrow \infty} \|T(s)\widehat{a}(T)\|$  to vanish. However, even in the case of a contractive representation  $T$ , for which this limit exists without any assumptions on  $a$ , no conclusion is possible if  $\widehat{a}$  fails to vanish at all points of  $\mathrm{Sp}_u(T)$ . It was recently shown in [106] that for contractive representations of  $S = \mathbb{Z}_+^n$  with  $n \geq 1$  or  $S = \mathbb{R}_+$  and any  $a \in L^1(S)$

$$\lim_{s \rightarrow \infty} \|T(s)\widehat{a}(T)\| = \sup\{|\widehat{a}(\chi)| : \chi \in \mathrm{Sp}_u(T)\}, \quad (8.3.1)$$

and that, for  $S = \mathbb{Z}_+^n$  with  $n = 1, 2$ , the conclusion remains valid when  $\widehat{a}$  is replaced by elements of larger algebras of functions. A number of related results may be found in [3], [13], [21], [22] and [82]. The intention here is to examine to what extent the validity of (8.3.1) can be extended to contractive representations of more general semigroups on a Hilbert space. Section 8.3.2 sets the scene with a few preliminary observations before the main results are presented in Section 8.3.3.

### 8.3.2 Preliminaries

The following simple lemma shows that, for the present purpose of studying the asymptotics of contractive representations  $T$  of a locally compact abelian semigroup  $S$ , nothing is lost by assuming that in fact  $\|T(s)\| = 1$  for all  $s \in S$ .

**Lemma 8.3.1.** *Let  $X$  be a complex Banach space and let  $T$  be a bounded representation of a locally compact abelian semigroup  $S$  on  $X$ . Suppose that there exists some  $s_0 \in S$  such that  $\|T(s_0)\| < 1$ . Then  $\mathrm{Sp}_u(T) = \emptyset$  and  $\|T(s)\| \rightarrow 0$  as  $s \rightarrow \infty$ .*

PROOF. Let  $\chi \in \text{ASp}(T)$ . Then there exists a net  $(x_\alpha)$  of unit vectors in  $X$  with

$$\lim_{\alpha \rightarrow \infty} \|T(s)x_\alpha - \chi(s)x_\alpha\| \rightarrow 0,$$

uniformly for  $s$  in compact subsets of  $S$ . In particular,  $\chi(s_0) \in \sigma_a(T(s_0))$  and it follows that

$$|\chi(s_0)| \leq r(T(s_0)) \leq \|T(s_0)\| < 1,$$

so that  $\chi \notin \Gamma$ . Thus  $\text{ASp}_u(T) = \emptyset$ , which by Proposition 2.4.3 implies the first part of the claim.

For the second part, suppose there exists  $\varepsilon > 0$  such that, for some net  $(s_\alpha)$  in  $S$  with  $s_\alpha \rightarrow \infty$  as  $\alpha \rightarrow \infty$ ,  $\|T(s_\alpha)\| \geq \varepsilon$  for each  $\alpha$ . Let  $n_0 \in \mathbb{N}$  be such that  $\|T(s_0)\|^{n_0} < \varepsilon \min\{1, M^{-1}\}$ , where  $M = \sup\{\|T(s)\| : s \in S\}$ . Thus  $n_0 s_0 \neq s_\alpha$  for all  $\alpha$  and, if  $\alpha$  is such that  $s_\alpha \succ n_0 s_0$ , then

$$\varepsilon \leq \|T(s_\alpha)\| \leq \|T(s_0)\|^{n_0} \|T(s_\alpha - n_0 s_0)\| < \varepsilon,$$

which gives the required contradiction.  $\square$

**Remark 8.3.2.** If  $T$  is a  $C_0$ -semigroup, the result follows immediately from the inequality  $s(A) \leq \omega_0(T)$  together with the fact that, for every  $t > 0$ ,

$$\omega_0(T) = \frac{\log r(T(t))}{t} \leq \frac{\log \|T(t)\|}{t}.$$

Now, given a locally compact abelian semigroup  $S$ , let  $G$  be the group generated by  $S$  and let  $\Lambda$  be any closed subset of the dual group  $\Gamma$ . Consider the contractive algebra homomorphism  $W_\Lambda : L^1(G) \rightarrow C_0(\Lambda)$  defined by  $W_\Lambda(a) = \widehat{a}|_\Lambda$ . Then  $\text{Ker } W_\Lambda = K_\Lambda(G)$  and, by the Stone-Weierstrass theorem,  $\text{Ran } W_\Lambda$  is dense in  $C_0(\Lambda)$ ; see [95, Theorem 1.2.4]. Since  $K_\Lambda(G)$  is a closed ideal of  $L^1(G)$ , the quotient  $L^1(G)/K_\Lambda(G)$  is a Banach algebra under the quotient norm

$$\|a + K_\Lambda(G)\| = \inf \{\|a - b\|_1 : b \in K_\Lambda(G)\}.$$

Moreover,  $W_\Lambda$  induces a well-defined injective algebra homomorphism

$$U_\Lambda : L^1(G)/K_\Lambda(G) \rightarrow C_0(\Lambda)$$

given, for  $a \in L^1(G)$ , by  $U_\Lambda(a + K_\Lambda(G)) = W_\Lambda(a)$ . This homomorphism is bounded with  $\|U_\Lambda\| = \|W_\Lambda\| \leq 1$  so that, in particular,

$$\sup\{|\widehat{a}(\chi)| : \chi \in \Lambda\} \leq \|a + K_\Lambda(G)\| \tag{8.3.2}$$

for all  $a \in L^1(G)$  and all closed subsets  $\Lambda$  of  $\Gamma$ .

### 8.3.3 Main results

Let  $T$  be a contractive representation of a locally compact abelian semigroup  $S$  on a complex Hilbert space  $X$ . Then, by a simple application of Lemma 7.3.9,

$$\sup\{|\widehat{a}(\chi)| : \chi \in \Lambda\} \leq \lim_{s \rightarrow \infty} \|T(s)\widehat{a}(T)\| \quad (8.3.3)$$

whenever  $\Lambda \subset \text{Sp}_u(T)$ . This raises the question whether it is possible to compare the right-hand side of (8.3.2) with that of (8.3.3). Proposition 8.3.4 below, which again follows from the corresponding result for individual orbits established in Proposition 8.3.3, answers this in the affirmative when  $\Lambda = \text{Sp}_u(T)$ . The argument is similar to that of [13, Proposition 5.5], but the result is sharper since there exist locally compact abelian groups  $G$  whose dual groups contain subsets  $\Lambda$  which are not of spectral synthesis; see [95, Chapter 7].

**Proposition 8.3.3.** *Let  $X$  be a complex Hilbert space, let  $T$  be a contractive representation of a locally compact abelian semigroup  $S$  on  $X$  and let  $\Lambda = \text{Sp}_u(T)$ . Furthermore, let  $G$  be the group generated by  $S$  and suppose that  $a \in L^1(S)$ . Then, for all  $x \in X$ ,*

$$\lim_{s \rightarrow \infty} \|T(s)\widehat{a}(T)x\| \leq \|a + K_\Lambda(G)\| \|x\|$$

PROOF. Fix a Banach limit  $\phi$  on  $L^\infty(S)$  and let  $X_\phi$ ,  $T_\phi$  and  $\pi_\phi$  be as in Proposition 7.2.2, noting that, since  $T$  is contractive,

$$\|\pi_\phi(x)\|_\phi = \lim_{s \rightarrow \infty} \|T(s)x\|$$

for all  $x \in X$ . As in the proof of Proposition 7.3.7, it is possible by Proposition 7.2.3 to assume that  $T_\phi$  is in fact a representation of the group  $G$  by unitary operators on  $X_\phi$ . It follows from Corollary 7.4.2 that  $\widehat{b}(T_\phi) = 0$  for all  $b \in K_\Lambda(G)$  and hence that

$$\|\widehat{a}(T_\phi)\| \leq \|\widehat{a}(T_\phi) - \widehat{b}(T_\phi)\| \leq \|a - b\|_1$$

for any such  $b$ . Thus  $\|\widehat{a}(T_\phi)\| \leq \|a + K_\Lambda(G)\|$  and hence, given any  $x \in X$ ,

$$\begin{aligned} \lim_{s \rightarrow \infty} \|T(s)\widehat{a}(T)x\| &= \|\pi_\phi(\widehat{a}(T)x)\|_\phi \\ &= \|\widehat{a}(T_\phi)\pi_\phi(x)\|_\phi \\ &\leq \|a + K_\Lambda(G)\| \|\pi_\phi(x)\|_\phi. \end{aligned}$$

The result now follows since  $\|\pi_\phi\| \leq \sup\{\|T(s)\| : s \in S\} = 1$ .  $\square$

**Proposition 8.3.4.** *Let  $X$  be a complex Hilbert space, let  $T$  be a contractive representation of a locally compact abelian semigroup  $S$  on  $X$  and let  $\Lambda = \text{Sp}_u(T)$ . Furthermore, let  $G$  be the group generated by  $S$  and suppose that  $a \in L^1(S)$ . Then*

$$\lim_{s \rightarrow \infty} \|T(s)\widehat{a}(T)\| \leq \|a + K_\Lambda(G)\|.$$

PROOF. Suppose not. Then there exist  $\varepsilon > 0$  and a net  $(s_\beta)$  in  $S$ , with indexing set  $A$ , say, and satisfying  $s_\beta \rightarrow \infty$  as  $\beta \rightarrow \infty$ , as well as a net of unit vectors  $(y_\beta)$  in  $X$  such that

$$\|T(s_\beta)\widehat{a}(T)y_\beta\| \geq \|a + K_\Lambda(G)\| + \varepsilon$$

for all  $\beta \in A$ . In particular,

$$\|T(s)\widehat{a}(T)y_\beta\| \geq \|a + K_\Lambda(G)\| + \varepsilon$$

whenever  $s_\beta \succ s$ .

Let  $X_A, X_A^\infty, T_A$  and  $\pi_A$  be as in Proposition 7.2.1 and, for a fixed  $t \in S^\circ$ , choose  $b \in L^1(S)$  such that  $\|b\|_1 = 1$  and  $\|a * b - a_t\|_1 < \varepsilon/2$ , and define  $(x_\beta) \in X_A^\infty$  by setting  $x_\beta = \widehat{b}(T)y_\beta$ . It then follows from a calculation analogous to the one in the proof of Proposition 7.3.8 that

$$\|T_A(s)\widehat{a}(T_A)\pi_A(x_\beta)\| > \|a + K_\Lambda(G)\| + \frac{\varepsilon}{2}$$

for all  $s \in S$ . However, Proposition 8.3.3 applied to the contractive representation  $T_A$  of  $S$  on  $X_A$  with unitary spectrum  $\Lambda_A \subset \Lambda$  shows that

$$\lim_{s \rightarrow \infty} \|T_A(s)\widehat{a}(T_A)\pi_A(x_\beta)\| \leq \|a + K_{\Lambda_A}(G)\| \|\pi_A(x_\beta)\| \leq \|a + K_\Lambda(G)\|,$$

since  $\|\pi_A(x_\beta)\| \leq \|\widehat{b}(T)\| \leq 1$ . This gives the required contradiction.  $\square$

Remaining in the setting of this result, it turns out that in order to obtain an upper bound on the limit  $\lim_{s \rightarrow \infty} \|T(s)\widehat{a}(T)\|$  in terms of  $\sup\{|\widehat{a}(\chi)| : \chi \in \Lambda\}$  rather than  $\|a + K_\Lambda(G)\|$  it is necessary to require the unitary spectrum  $\Lambda$  to be of a special type. Note first that, by the Inverse Mapping Theorem, the operator  $U_\Lambda$  is an isomorphism precisely when it is surjective. Since its range is dense in  $C_0(\Lambda)$ , this is the case if and only if the map is an isomorphic embedding, which in turn is equivalent to the dual operator  $U'_\Lambda : M_0(\Lambda) \rightarrow K_\Lambda(G)^\perp$  being either a surjection or an isomorphic embedding; see for instance [95, Appendix C11]. Here  $M_0(\Lambda)$  denotes the space of bounded Radon measures on  $\Lambda$  and the annihilator  $K_\Lambda(G)^\perp$  of  $K_\Lambda(G)$  is a subspace of  $L^\infty(G)$ . A set  $\Lambda$  for which these conditions are satisfied is said to be a *Helson set*, and the quantity  $\alpha(\Lambda) = \|U_\Lambda^{-1}\|$  is then known as its *Helson constant*.

**Remark 8.3.5.** Note that, for any Helson set  $\Lambda$ ,  $\alpha(\Lambda) \geq \|U_\Lambda\|^{-1} \geq 1$ , and that, by an application of Fubini's theorem,  $U'_\Lambda(\nu) = \widehat{\nu}$ , where

$$\widehat{\nu}(s) = \int_\Lambda \chi(s) d\nu(\chi)$$

for  $\nu \in M_0(\Lambda)$  and  $s \in G$ . In particular, it follows from [95, Section 1.3.3] that every element  $K_\Lambda(G)^\perp$  is uniformly continuous.

The following result provides an upper bound for representations whose spectrum is a Helson set; it should be compared with the corresponding results in [106].

**Proposition 8.3.6.** *Let  $X$  be a complex Hilbert space, let  $T$  be a contractive representation of a locally compact abelian semigroup  $S$  on  $X$  and let  $a \in L^1(S)$ . Furthermore, let  $\Lambda = \text{Sp}_u(T)$  and suppose that  $\Lambda$  is a Helson set. Then*

$$\lim_{s \rightarrow \infty} \|T(s)\widehat{a}(T)\| \leq \alpha(\Lambda) \sup\{|\widehat{a}(\chi)| : \chi \in \Lambda\}. \quad (8.3.4)$$

PROOF. By Proposition 8.3.4 and the definition of the Helson constant,

$$\lim_{s \rightarrow \infty} \|T(s)\widehat{a}(T)\| \leq \|U_\Lambda^{-1}(W_\Lambda(a))\| \leq \alpha(\Lambda) \|W_\Lambda(a)\|_\infty,$$

as required. □

Combining this result with equation (8.3.3) for  $\Lambda = \text{Sp}_u(T)$  gives the following quantified result, which combines aspects of [106, Theorems 3.1 and 5.4]. The first of these applies only when  $S = \mathbb{R}_+$  but involves no assumptions on the unitary spectrum, whereas the second imposes an even stronger assumption on the unitary spectrum but does not require  $X$  to be a Hilbert space.

**Corollary 8.3.7.** *Let  $X$  be a complex Hilbert space, let  $T$  be a contractive representation of a locally compact abelian semigroup  $S$  on  $X$  and let  $a \in L^1(S)$ . Furthermore, let  $\Lambda = \text{Sp}_u(T)$  and suppose that  $\Lambda$  is a Helson set. Then*

$$\sup\{|\widehat{a}(\chi)| : \chi \in \Lambda\} \leq \lim_{s \rightarrow \infty} \|T(s)\widehat{a}(T)\| \leq \alpha(\Lambda) \sup\{|\widehat{a}(\chi)| : \chi \in \Lambda\}.$$

**Remark 8.3.8.** Note that the Hilbert space assumption on  $X$  in Proposition 8.3.6 and Corollary 8.3.7 cannot be dropped, at least not completely. Indeed, given any closed subset  $\Lambda$  of  $\Gamma$ , if (8.3.4) were to hold for all contractive representations  $T$  of  $S$  with  $\text{Sp}_u(T) = \Lambda$  on any Banach space  $X$  (or even just for the shift semigroup on any quotient of  $L^1(G)$  by a closed ideal), then  $\Lambda$

would necessarily be not only a Helson set but moreover of spectral synthesis; see [106, Theorem 5.4]. Since there exist Helson sets which are not of spectral synthesis (see [51, Section 4.6]), some condition on the geometry of  $X$  must be retained. It is unclear, however, whether a weaker condition on  $X$ , such as (super)reflexivity (see also [72, Remark 4.2.6]) or  $B$ -convexity, is sufficient.

**Example 8.3.9.** 1. If  $T$  is a contractive representation of a locally compact abelian semigroup  $S$  on a complex Hilbert space  $X$  such that  $\Lambda = \text{Sp}_u(T)$  is finite, then

$$\lim_{s \rightarrow \infty} \|T(s)\widehat{a}(T)\| \leq |\Lambda|^{1/2} \sup\{|\widehat{a}(\chi)| : \chi \in \Lambda\}$$

for any  $a \in L^1(S)$ . This follows from the fact that any finite set  $\Lambda$  is a Helson set with Helson constant  $\alpha(\Lambda) \leq |\Lambda|^{1/2}$ ; see [51, Proposition 1.1.6]. In particular, if  $\Lambda = \{\chi\}$  then

$$\lim_{s \rightarrow \infty} \|T(s)\widehat{a}(T)\| = |\widehat{a}(\chi)|.$$

2. If  $T$  is a contractive  $C_0$ -semigroup on a complex Hilbert space and  $\Lambda = \sigma(A) \cap i\mathbb{R}$  is a Helson set, then

$$\frac{1}{\text{dist}(1, \Lambda)} \leq \lim_{t \rightarrow \infty} \|T(t)R(1, A)\| \leq \frac{\alpha(\Lambda)}{\text{dist}(1, \Lambda)}.$$

This follows from Corollary 8.3.7 with the choice  $a(t) = e^{-t}$  for  $t \geq 0$  and extends [101, Corollary 5.2.6] in the Hilbert space case.

3. If  $T$  is a contraction on a Hilbert space and  $\Lambda = \sigma(T) \cap \mathbb{T}$  is a Helson set, then

$$\text{dist}(1, \Lambda) \leq \lim_{n \rightarrow \infty} \|T^n(I - T)\| \leq \alpha(\Lambda) \text{dist}(1, \Lambda).$$

This follows from Corollary 8.3.7 with the choice  $a = (1, -1, 0, 0, \dots)$ , and is a special case of the more general result [106, Theorem 3.2].

## 8.4 Invertibility of contractive representations

### 8.4.1 Introduction

The purpose of this section is to apply the results of Chapter 7 to the study of invertibility of contractive representations on Hilbert space. The objective is to find conditions on the spectrum which ensure that a representation of a

locally compact abelian semigroup  $S$  extends to a representation of the group  $G = S - S$  generated by  $S$ ; for previous results in this area, see for instance [13] and [99]. Section 8.4.2 contains the necessary preliminary observations for the main result, Theorem 8.4.6, which is presented in Section 8.4.3. In Section 8.4.4 this result is then used to obtain a quantified stability result for individual orbits of a representation.

### 8.4.2 Preliminaries

Let  $S$  be a locally compact abelian semigroup and let  $G = S - S$  be the group generated by  $S$ . Note that, for any closed subset  $\Lambda$  of the dual group  $\Gamma$  of  $G$ , there exists a contractive embedding

$$V_\Lambda : L^1(S)/K_\Lambda(S) \rightarrow L^1(G)/K_\Lambda(G),$$

where  $K_\Lambda(S) = K_\Lambda(G) \cap L^1(S)$ . A closed subset  $\Lambda$  of  $\Gamma$  will be said to be *thin* if the map  $V_\Lambda$  is an isometric isomorphism. Making the usual identifications, this is equivalent to the restriction mapping

$$V'_\Lambda : K_\Lambda(G)^\perp \rightarrow K_\Lambda(S)^\perp$$

given by  $V'_\Lambda(\phi) = \phi|_S$  being an isometry; see [95, Appendix C11]. Noting that if  $\phi \in K_\Lambda(G)^\perp$  then  $\phi_s \in K_\Lambda(G)^\perp$  for each  $s \in G$  and  $\|\phi\|_\infty = \|\phi_s\|_\infty$ , this in turn is the same as requiring that

$$\|\phi\|_\infty = \operatorname{ess-lim\,sup}_{s \rightarrow \infty} |\phi(s)| \tag{8.4.1}$$

for all  $\phi \in K_\Lambda(G)^\perp$ .

**Remark 8.4.1.** It is clear that, if  $\Lambda$  is a thin subset of the dual group  $\Gamma$  of a locally compact abelian group  $G$  and  $\Lambda' \subset \Lambda$ , then  $\Lambda'$  is also thin.

It is known that any countable closed subset  $\Lambda$  of  $\Gamma$  is thin (see [13, Proposition 2.1]), and in fact the same argument applies to any scattered closed subset. In general, however, there is no easy way of determining whether a given closed subset  $\Lambda$  of  $\Gamma$  is thin. The following lemma constitutes a small step in this direction, although it is not clear whether it provides any interesting examples, such as sets which fail to be of spectral synthesis. Recall from Remark 8.3.5 that elements of  $K_\Lambda(G)^\perp$  are uniformly continuous whenever  $\Lambda$  is a Helson set, so that the “ess” in (8.4.1) can be dropped in this case.

**Lemma 8.4.2.** *Let  $S$  be a locally compact abelian semigroup and let  $G$  be the group generated by  $S$ . Suppose that  $\Lambda$  is a Helson subset of the dual group  $\Gamma$  of  $G$  and that*

$$\liminf_{s \rightarrow \infty} \sup\{|1 - \chi(s)| : \chi \in \Lambda\} = 0. \quad (8.4.2)$$

*Then  $\Lambda$  is thin.*

PROOF. By definition of a Helson set, the map  $U'_\Lambda : M_0(\Lambda) \rightarrow K_\Lambda(G)^\perp$  introduced in Section 8.3 is surjective. Thus, given  $\phi \in K_\Lambda(G)^\perp$  and  $s, t \in G$ , there exists a measure  $\nu \in M_0(\Lambda)$  such that  $U'_\Lambda(\nu) = \widehat{\nu} = \phi$  and hence

$$|\phi(s) - \phi(t)| = \left| \int_\Lambda (\chi(s) - \chi(t)) d\nu(\chi) \right| \leq \|\nu\| \sup\{|1 - \chi(s - t)| : \chi \in \Lambda\}.$$

Thus the result follows from (8.4.2).  $\square$

**Remark 8.4.3.** The result remains true for  $S = \mathbb{Z}_+$  even when  $\Lambda$  is not assumed to be a Helson set; see for instance [32] and [59]. In this case it follows from this result together with an application of the Dirichlet approximation theorem that any closed subset  $\Lambda$  of  $\mathbb{T}$  is thin provided

$$\liminf_{\varepsilon \rightarrow 0} \frac{N_\varepsilon(\Lambda)}{|\log \varepsilon|} = 0, \quad (8.4.3)$$

where  $N_\varepsilon(\Lambda)$  denotes the least number of arcs of length at most  $\varepsilon$  needed to cover  $\Lambda$ ; see [59]. Using Lemma 8.4.2 it is possible to extend this result for instance to the case  $S = \mathbb{R}_+$  with  $\Lambda$  a Helson subset of the dual group  $i\mathbb{R}$  satisfying (8.4.3) with a suitable interpretation of  $N_\varepsilon(\Lambda)$ . For further details on sets  $\Lambda \subset \mathbb{T}$  satisfying (8.4.3), including a proof that any such set is necessarily of measure zero, see also [60, Chapter VII].

### 8.4.3 Main results

The results presented here are analogous to those contained in [13, Section 4] and [18, Section 7], where the unitary spectrum is assumed to be countable; see also [16] and [52, Section 5.4]. The first corresponds to [13, Lemma 4.4]. Together with Proposition 8.4.5 below it leads to the main result of this section, Theorem 8.4.6, which gives a sufficient condition in terms not of countability but of thinness of the unitary spectrum for a contractive representation of a locally compact abelian semigroup  $S$  on a Hilbert space to extend to a representation of the group  $G = S - S$  by unitary operators on the same Hilbert space. A representation  $T$  of a locally compact abelian semigroup  $S$  on a Banach space  $X$  is said to be *trivially asymptotically stable* if  $X_0(T) = \{0\}$ .

**Lemma 8.4.4.** *Let  $S$  be a locally compact abelian semigroup, let  $G$  be the group generated by  $S$  and let  $X$  be a complex Hilbert space. Suppose that  $T$  is a trivially asymptotically stable contractive representation of  $S$  on  $X$  and let  $\Lambda = \text{Sp}_u(T)$ . Then, for any  $a \in L^1(S)$ ,  $\|\widehat{a}(T)\| \leq \|a + K_\Lambda(S)\|$ . In particular, if  $\Lambda$  is thin, then  $\|\widehat{a}(T)\| \leq \|a + K_\Lambda(G)\|$ .*

PROOF. Let  $b \in K_\Lambda(S)$ . By Theorem 7.3.1,  $\|T(s)\widehat{b}(T)\| \rightarrow 0$  as  $s \rightarrow \infty$  and, in particular,  $\text{Ran } \widehat{b}(T) \subset X_0(T)$ . Since  $T$  is assumed to be trivially asymptotically stable, it follows that  $\widehat{b}(T) = 0$ . Hence

$$\|\widehat{a}(T)\| = \|\widehat{a}(T) - \widehat{b}(T)\| \leq \|a - b\|_1$$

and, taking the infimum over all  $b \in K_\Lambda(S)$ , it follows that  $\|\widehat{a}(T)\| \leq \|a + K_\Lambda(S)\|$ . The final statement follows from the definition of a thin set.  $\square$

The following result corresponds to [13, Proposition 4.5] and [18, Lemma 7.1]. Here, given  $a \in L^1(G)$  and  $s \in G$  for some locally compact abelian group  $G$ , the notation  $a_s$  once again denotes the translate of  $a$  by  $s$ .

**Proposition 8.4.5.** *Let  $S$  be a locally compact abelian semigroup, let  $G$  be the group generated by  $S$  and let  $X$  be a complex Hilbert space. Suppose that  $T$  is a trivially asymptotically stable contractive representation of  $S$  on  $X$ , let  $\Lambda = \text{Sp}_u(T)$  and suppose that  $\Lambda$  is thin. Then there exists a contractive algebra homomorphism  $\pi_\Lambda : L^1(G) \rightarrow \mathcal{B}(X)$  with  $\text{Ker } \pi_\Lambda = K_\Lambda(G)$  such that*

- (i)  $\pi_\Lambda(a) = \widehat{a}(T)$  for all  $a \in L^1(S)$ ;
- (ii)  $\pi_\Lambda(a_s) = T(s)\widehat{a}(T)$  for all  $a \in L^1(G)$  and all  $s \in S$ .

PROOF. Let  $a \in L^1(G)$ . Since  $\Lambda$  is thin, there exists a function  $b \in L^1(S)$  such that  $a - b \in K_\Lambda(G)$ . Define  $\pi_\Lambda(a) = \widehat{b}(T)$ . Then  $\pi_\Lambda$  is well-defined by Lemma 8.4.4 and it is clear that  $\pi_\Lambda$  is an algebra homomorphism satisfying (i). Furthermore,

$$\|\pi_\Lambda(a)\| \leq \|b + K_\Lambda(S)\| = \|a + K_\Lambda(G)\| \leq \|a\|_1$$

and, by Theorem 7.3.1,  $\pi_\Lambda(a) = 0$  if and only if  $b \in K_\Lambda(S)$ , which is equivalent to  $a \in K_\Lambda(G)$ . Since  $a_s - b_s \in K_\Lambda(G)$  for each  $s \in S$ , it follows from the fact that  $\text{Ker } \pi_\Lambda = K_\Lambda(G)$  together with (i) that

$$\pi_\Lambda(a_s) = \widehat{b}_s(T) = T(s)\pi_\Lambda(a),$$

as required.  $\square$

The following is the main result of this section and corresponds to [13, Theorem 4.8]; see also [18, Theorem 7.2]. Suppose that  $S$  is a locally compact abelian semigroup and let  $G$  be the group generated by  $S$ . Given a representation  $T$  of  $S$  on a Banach space  $X$  and  $x \in X$ , a map  $\phi : G \rightarrow X$  is said to be a *complete orbit through  $x$*  if  $\phi(0) = x$  and  $T(s)\phi(t) = \phi(s+t)$  for all  $s \in S$  and all  $t \in G$ . Note that any trivially asymptotically stable representation has at most one complete orbit through any  $x \in X$ .

**Theorem 8.4.6.** *Let  $S$  be a locally compact abelian semigroup, let  $G$  be the group generated by  $S$  and let  $X$  be a complex Hilbert space. Suppose that  $T$  is a trivially asymptotically stable contractive representation of  $S$  on  $X$  such that  $\text{Sp}_u(T)$  is thin. Then  $T$  extends to a representation of  $G$  by unitary operators on  $X$ .*

PROOF. Fix  $x \in X$  and let  $\Lambda = \text{Sp}_u(T)$ . For  $a \in L^1(G)$  such that the support  $K$  of  $\hat{a}$  is compact, define the map  $\phi_{a,x} : G \rightarrow X$  by  $\phi_{a,x}(s) = \pi_\Lambda(a_s)x$ , where the map  $\pi_\Lambda : L^1(G) \rightarrow \mathcal{B}(X)$  is as in Proposition 8.4.5. By [95, Section 2.6.8], it is possible, given any  $\varepsilon > 0$ , to find a function  $b \in L^1(G)$  such that  $\hat{b}(\chi) = 1$  for all  $\chi \in K$  and  $\|b\|_1 \leq 1 + \varepsilon$ . Then  $\hat{a} \cdot \hat{b} = \hat{a}$  and hence, for any  $s \in G$ ,

$$\phi_{a,x}(s) = \pi_\Lambda((a * b)_s)x = \pi_\Lambda(a * b_s)x = \pi_\Lambda(b_s)\pi_\Lambda(a)x.$$

Thus  $\sup\{\|\phi_{a,x}(s)\| : s \in G\} \leq (1 + \varepsilon)\|\pi_\Lambda(a)x\|$ . By part (ii) of Proposition 8.4.5,  $\phi_{a,x}(s) = T(s)\phi_{a,x}(0) = T(s)\pi_\Lambda(a)x$  whenever  $s \in S$ , and hence  $\phi_{a,x}$  is a complete orbit through  $\pi_\Lambda(a)x$ .

Fix  $s_0 \in S^\circ$  and let  $(a^{(\beta)})$  be a net in  $L^1(G)$  whose elements have compactly supported Fourier-Laplace transform and are such that

$$\|T(s_0)x - \pi_\Lambda(a^{(\beta)})x\| \rightarrow 0$$

as  $\beta \rightarrow \infty$ . Such a net exists by density in  $L^1(G)$  of functions whose Fourier-Laplace transform is compactly supported together with an approximation argument based on the result of Section 2.4.1. As in the proof of [18, Theorem 7.2], a standard completeness argument now shows that the map  $\phi_x : G \rightarrow X$  given by

$$\phi_x(s) = \lim_{\beta \rightarrow \infty} \phi_{a^{(\beta)},x}(s)$$

is well-defined and, moreover,

$$\sup\{\|\phi_x(s)\| : s \in G\} \leq \|T(s_0)x\| \leq \|x\|$$

and  $\phi_x$  is the complete orbit through  $T(s_0)x$ . Note also that  $T(s_0)\phi_x(-s_0) = \phi_x(0) = T(s_0)x$  and hence, by injectivity of  $T(s_0)$ ,  $\phi_x(-s_0) = x$ .

Let  $U : G \rightarrow \mathcal{B}(X)$  be given, for  $s \in G$  and  $x \in X$ , by  $U(s)x\phi_x(s - s_0)$ . Then  $U$  is well-defined and satisfies  $\|U(s)\| \leq 1$  for all  $s \in G$ . For  $s \in S$  and  $x \in X$ ,  $U(s)x = \phi_x(s - s_0) = T(s)x$ , giving  $U(s) = T(s)$ . Furthermore,  $U(-s) = U(s)^{-1}$  for all  $s \in G$ , so each of the operators  $U(s)$  is unitary and the result follows.  $\square$

**Remark 8.4.7.** The requirement that the unitary spectrum be thin is in general too strong. For instance, it is shown in [99, Theorem I.3.7 and Proposition II.6.7] and [30, Corollary 2], respectively, that for  $S = \mathbb{Z}_+$  or  $\mathbb{R}_+$  the conclusion Theorem 8.4.6 holds under the milder condition that the unitary spectrum have measure zero; see also [63] and Remark 8.4.3. On the other hand, the example given in [13, Example 4.1] of an operator  $T \in \mathcal{B}(\ell^2(\mathbb{Z}))$  which has unit norm, satisfies  $\sigma(T) \cap \mathbb{T} = \mathbb{T}$  and is trivially asymptotically stable but neither isometric nor invertible shows that some restriction on the unitary spectrum is necessary.

#### 8.4.4 Stability of orbits

The purpose here is to apply Theorem 8.4.6 to the study of stability of individual orbits of a semigroup representation with thin unitary spectrum. The main result, Corollary 8.4.9 below, is a simple consequence of Theorem 8.4.6 and corresponds to the first statement in [13, Corollary 5.3]; see also [18, Theorem 7.4]. It requires the following auxiliary result, which is a Hilbert space version of [13, Proposition 4.3]. Note that if  $T$  is a representation of a locally compact abelian semigroup  $S$  on a Hilbert space  $X$  then, by  $T$ -invariance of  $X_0(T)$ ,  $T$  induces a representation  $T_0$  of  $S$  on  $X_0(T)^\perp$  given, for  $s \in S$ , by  $T_0(s) = PT(s)|_{X_0(T)^\perp}$ , where  $P$  denotes the orthogonal projection of  $X$  onto  $X_0(T)^\perp$ .

**Lemma 8.4.8.** *Let  $X$  be a complex Hilbert space, let  $T$  be a contractive representation of  $S$  on  $X$  and let  $P$  denote the orthogonal projection of  $X$  onto  $X_0(T)^\perp$ . Then, for any  $x \in X$ ,*

$$\lim_{s \rightarrow \infty} \|T(s)x\| = \lim_{s \rightarrow \infty} \|T_0(s)y\|,$$

where  $T_0$  is the representation of  $S$  on  $X_0(T)^\perp$  induced by  $T$  and  $y = Px$ . In particular,  $X_0(T_0) = \{0\}$ .

PROOF. Given  $x \in X$ , fix  $\varepsilon > 0$  and let  $\ell = \lim_{s \rightarrow \infty} \|T_0(s)y\|$ , where  $y = Px$ . Then  $T_0(s)y = PT(s)x$  for all  $s \in S$ , so  $\ell = \lim_{s \rightarrow \infty} \|PT(s)x\|$ . Hence there exist  $s_0 \in S$  and  $z \in X_0(T)$  such that  $\|T(s_0)x - z\| < \ell + \varepsilon$  and therefore

$$\|T(s_0 + t)x\| \leq \|T(s_0 + t)x - T(t)z\| + \|T(t)z\| < \ell + \varepsilon + \|T(t)z\|$$

for any  $t \in S$ . Letting  $t \rightarrow \infty$  gives  $\lim_{s \rightarrow \infty} \|T(s)x\| \leq \ell + \varepsilon$ . Since  $\|T_0(s)y\| \leq \|T(s)x\|$  for all  $s \in S$  and since  $\varepsilon > 0$  was arbitrary, it follows that  $\ell = \lim_{s \rightarrow \infty} \|T(s)x\|$ . In particular,  $\ell = 0$  if and only if  $x \in X_0(T)$ , so  $T_0$  is trivially asymptotically stable, as required.  $\square$

**Corollary 8.4.9.** *Let  $X$  be a complex Hilbert space and let  $T$  be a contractive representation of a locally compact abelian semigroup  $S$  on  $X$  such that  $\text{Sp}_u(T)$  is thin. Then, for any  $x \in X$ ,*

$$\lim_{s \rightarrow \infty} \|T(s)x\| = \inf \{ \|x - z\| : z \in X_0(T) \}.$$

PROOF. By Lemma 8.4.8, the induced representation  $T_0$  of  $S$  on  $X_0(T)^\perp$  is trivially asymptotically stable. Moreover,  $\|\widehat{a}(T_0)\| \leq \|\widehat{a}(T)\|$  for any  $a \in L^1(S)$  and hence  $\text{Sp}(T_0) \subset \text{Sp}(T)$ . In particular,  $\text{Sp}_u(T_0)$  is thin by Remark 8.4.1. Thus Theorem 8.4.6 shows that  $T_0$  extends to a representation by unitary operators on  $X_0(T)^\perp$  of the group  $G = S - S$ . In particular, it follows from Lemma 8.4.8 that, for  $x \in X$  and  $y = Px$ ,

$$\lim_{s \rightarrow \infty} \|T(s)x\| = \lim_{s \rightarrow \infty} \|T_0(s)y\| = \|y\|,$$

as required.  $\square$

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