

# A Fractional Variational Approach for Modelling Dissipative Mechanical Systems: Continuous and Discrete Settings

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**Abstract:** Employing a phase space which includes the (Riemann-Liouville) fractional derivative of curves evolving on real space, we develop a restricted variational principle for Lagrangian systems yielding the so-called restricted fractional Euler-Lagrange equations (both in the continuous and discrete settings), which, as we show, are invariant under linear change of variables. This principle relies on a particular restriction upon the admissible variation of the curves. In the case of the half-derivative and mechanical Lagrangians, i.e. kinetic minus potential energy, the restricted fractional Euler-Lagrange equations model a dissipative system in both directions of time, summing up to a set of equations that is invariant under time reversal. Finally, we show that the discrete equations are a meaningful discretisation of the continuous ones.

*Keywords:* Variational analysis, Mechanical systems, Lagrangian mechanics, Damping, Fractional derivatives, Discretisation, Variational integrators

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## 1. INTRODUCTION

Variational principles are powerful tools for the modelling and simulation of mechanical systems. As it is well-known, the fulfilment of a variational principle leads to the Euler-Lagrange equations of motion describing the dynamics of such systems. On the other hand, a variational discretisation directly yields unified numerical schemes with powerful structure-preserving properties. Using these variational or symplectic integrators (VIs, Marsden and West (2001)) (integration schemes which themselves satisfy a variational principle), the geometric structure is not destroyed and characteristic properties of the dynamics are inherited by the discrete approximation such as preservation of energy and momentum maps in the presence of symmetries. Continuous and discrete Lagrangian mechanics have been primarily used to model and to simulate conservative systems, for which the excellent long-time energy behaviour of VIs is explained by the fact that the discrete solution is the exact solution of a nearby Lagrangian system, which is a result obtained by backward error analysis (BEA) techniques (Hairer et al. (2002)). However, because most classical processes observed in the engineering world are non-conservative, it is important to be able to apply the power of variational methods to these cases.

There have been several attempts to provide a general method of dealing with non-conservative forces in classical mechanics (Riewe (1996); Goldstein (1980), pg. 21; Celeghini et al. (1989); Bloch (2003); Kane et al. (2000); Ober-Blöbaum et al. (2011)). However, for all approaches in the literature, either the physical meaning of the induced

quantities is not physically meaningful or the proposed principles are not purely variational in the usual sense.

A promising purely variational approach was formulated in Riewe (1996): it is shown that a simple system with linear damping can be modelled using a variational principle involving fractional derivatives and a Euler-Lagrange equation of the form

$$\sum_{\alpha} (-1)^{\alpha} \frac{d^{\alpha}}{dt^{\alpha}} \left( \frac{\partial L}{\partial q^{(\alpha)}} \right) = 0,$$

where  $\alpha \in \mathbb{Q}$  the set of rational numbers. Several definitions of a fractional derivative have been proposed (for an overview see e.g. Samko et al. (1993) and Agrawal (2002)), among which we pick Riemann-Liouville's. Most importantly to mention is that the fractional derivative of a function is not determined by the value of the function at a single value  $t$ , but depends on the values over an entire interval.

While Riewe introduced fractional Euler-Lagrange equations for dissipative systems based on fractional Lagrangian functionals using left and right Riemann-Liouville derivatives,  $D_{-}^{\alpha}$  and  $D_{+}^{\alpha}$ , it is shown in Cresson and Inizan (2012) that critical points of a fractional Lagrangian functional and solutions of a dissipative equation are not equivalent. The reason is that due to fractional integration by parts a mixing of left and right fractional derivatives occurs and in general

$$D_{-}^{1/2} D_{+}^{1/2} x \neq \frac{dx}{dt}.$$

To overcome this problem, in Cresson and Inizan (2012) so called *asymmetric fractional embeddings* are introduced. By artificially doubling the state space and by restricting the set of variations, fractional Euler-Lagrange equations

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are derived whose solutions are equivalent to solutions of dissipative equations. However, the geometric structure of such enlarged phase space is not provided and the meaning of the restricted variations is not clear and apparently *ad hoc* chosen.

In this paper, we also consider an enlarged fractional phase space and clarify its geometric structure in terms of vector spaces and bundle products (section 3). We develop a restricted variational principle and obtain a sufficient condition for the extremals providing the so-called *restricted fractional Euler-Lagrange equations* (theorem 4.6) (see Agrawal (2002) for other approaches to obtain fractional Euler-Lagrange equations). This sufficient condition relies on a restriction upon the admissible variations, allowing different varied curves for the extra variables with the same variations (definition 4.4). We show as well that the restricted fractional Euler-Lagrange equations are invariant under linear change of variables (theorem 4.7). In the  $\alpha = 1/2$  case and considering mechanical Lagrangians, i.e. kinetic minus potential energy, the obtained equations describe a dissipative mechanical system, whereas the dynamics of the extra variables can be interpreted as the reversed time dynamics (corollary 4.8). As a whole, the system of equations is invariant under time reversal. In section 5, we develop the discrete counterpart of the previous results. We obtain the *restricted discrete fractional Euler-Lagrange equations* (theorem 5.3) and prove a new property for double discrete fractional derivatives (lemma 5.4) which allows to derive a meaningful discretisation of the dissipative mechanical equations in the  $\alpha = 1/2$  mechanical case.

## 2. RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVES

Let  $\alpha \in [0, 1] \subset \mathbb{R}$  and  $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  a smooth function. The Riemann-Liouville fractional derivatives are defined by

$$D_-^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-\tau)^{-\alpha} f(\tau) d\tau, \quad (1a)$$

$$D_+^\alpha f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b (\tau-t)^{-\alpha} f(\tau) d\tau, \quad (1b)$$

for  $t \in [a, b]$ , and  $\Gamma(z)$  the gamma function (Samko et al. (1993)). As it is well-known, the fractional derivatives are non-local operators: in the sequel  $-$  and  $+$  will denote the retarded and advanced cases, respectively. Let us consider two smooth functions  $f, g$ . The fractional integration by parts rule is given by

$$\int_a^b f(t) D_\sigma^\alpha g(t) dt = \int_a^b (D_{-\sigma}^\alpha f(t)) g(t) dt, \quad (2)$$

where  $\sigma$  stands either for  $-$  or  $+$ . An important feature of fractional integrals is, when  $\alpha = 1/2$ :

$$D_-^{1/2} D_-^{1/2} f(t) = \frac{d}{dt} f(t), \quad D_+^{1/2} D_+^{1/2} f(t) = -\frac{d}{dt} f(t). \quad (3)$$

See Samko et al. (1993) for more details. According to the above definitions, the fractional derivatives are  $\mathbb{R}$ -valued.

## 3. FRACTIONAL PHASE SPACE

Consider a smooth curve  $\gamma : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}^d$  for  $d \in \mathbb{N}$ . The local representation of the curve is given by  $\gamma(t) =$

$(x^1(t), \dots, x^d(t))$ ,  $t \in [a, b]$ . For the set of all smooth curves  $C^\infty([a, b], \mathbb{R}^d)$ , let us define the *fractional tangent vector* of the curve  $\gamma$  by means of the following application

$$X_\sigma^\alpha : C^\infty([a, b], \mathbb{R}^d) \rightarrow \mathbb{R}^d, \\ \gamma \mapsto X_\sigma^\alpha \gamma,$$

where the fractional tangent vector is defined by

$$X_\sigma^\alpha \gamma := D_\sigma^\alpha \gamma(t) = (D_\sigma^\alpha x^1(t), \dots, D_\sigma^\alpha x^d(t)), \quad t \in [a, b].$$

In the sequel we shall omit the  $t$ -dependence in curves and fractional tangent vectors.

*Definition 3.1.* We define the fractional tangent space of the curve  $\gamma$  as

$$V_\sigma^\alpha \mathbb{R}^d = \{X_\sigma^\alpha \gamma \mid \text{for } \gamma \in C^\infty([a, b], \mathbb{R}^d), t \in [a, b]\}.$$

Taking into account the vector structure of  $C^\infty([a, b], \mathbb{R}^d)$ , which is defined pointwise for  $t \in [a, b]$ , it is easy to prove that  $V_\sigma^\alpha \mathbb{R}^d$  is a  $d$ -dimensional vector space.

With these elements, we define the *fractional tangent bundle* for curves  $\gamma$  as follows.

*Definition 3.2.* Define the fractional tangent bundle for  $\gamma$  and  $t \in [a, b]$  by

$$T_\sigma^\alpha \mathbb{R}^d = \{(\gamma, X_\sigma^\alpha \gamma) \in \mathbb{R}^d \times V_\sigma^\alpha \mathbb{R}^d \mid \gamma \in C^\infty([a, b], \mathbb{R}^d)\}.$$

We shall consider  $\mathbb{R}^d$  as the base space and  $V_\sigma^\alpha \mathbb{R}^d$  the fiber. Denoting  $\mathcal{V}_{(\gamma, \sigma)}^\alpha := (\gamma, X_\sigma^\alpha \gamma) \in T_\sigma^\alpha \mathbb{R}^d$  its local representation is given by  $\mathcal{V}_{(\gamma, \sigma)}^\alpha = (x^1, \dots, x^d, D_\sigma^\alpha x^1, \dots, D_\sigma^\alpha x^d)$ .

The bundle projection  $\tau_\sigma^\alpha : T_\sigma^\alpha \mathbb{R}^d \rightarrow \mathbb{R}^d$  is locally given by

$$\tau_\sigma^\alpha(\mathcal{V}_{(\gamma, \sigma)}^\alpha) = (x^1, \dots, x^d).$$

According to this definition, we see that  $(\tau_\sigma^\alpha)^{-1}(\mathbb{R}^d) = V_\sigma^\alpha \mathbb{R}^d$ . It is easy to see that at each time  $t$  this leads to the local isomorphism  $T_\sigma^\alpha \mathbb{R}^d \cong \mathbb{R}^d \times \mathbb{R}^d$ . As it is apparent, we have constructed the fractional tangent bundle in a similar way as the usual tangent bundle  $T\mathbb{R}^d$  for curves in  $\mathbb{R}^d$  (Abraham and Marsden (1978)).

Consider now two smooth curves  $\gamma_x, \gamma_y : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}^d$ , forming the new curve  $\tilde{\gamma} : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}^d \times \mathbb{R}^d$  by  $\tilde{\gamma} = (\gamma_x, \gamma_y)$ . Locally it is expressed by  $\tilde{\gamma} = (x^1, \dots, x^d, y^1, \dots, y^d)$ .

*Definition 3.3.* Define the double fractional tangent bundle for curves  $\tilde{\gamma}$  and  $t \in [a, b]$  by

$$T^\alpha \mathbb{R}^d = \{((\gamma_x, \gamma_y), (X_-^\alpha \gamma_x, X_+^\alpha \gamma_y)) \\ \in (\mathbb{R}^d \times \mathbb{R}^d) \times (V_-^\alpha \mathbb{R}^d \times V_+^\alpha \mathbb{R}^d) \mid \gamma_x, \gamma_y \in C^\infty([a, b], \mathbb{R}^d)\},$$

where now  $\mathbb{R}^d \times \mathbb{R}^d$  is the base space and  $V_-^\alpha \mathbb{R}^d \times V_+^\alpha \mathbb{R}^d$  the fiber. Denoting  $\mathcal{V}_{\tilde{\gamma}}^\alpha := ((\gamma_x, \gamma_y), (X_-^\alpha \gamma_x, X_+^\alpha \gamma_y)) \in T^\alpha \mathbb{R}^d$ , its local representation is given by  $\mathcal{V}_{\tilde{\gamma}}^\alpha = (x^i, y^i, D_-^\alpha x^i, D_+^\alpha y^i)$ ,  $i = 1, \dots, d$ ; while the bundle projection  $\tau^\alpha : T^\alpha \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$  is

$$\tau^\alpha(\mathcal{V}_{\tilde{\gamma}}^\alpha) = (x, y).$$

From now on we shall drop the  $i$  superindex for sake of simplicity.

*Remark 3.4.* The vector structure of  $(\tau^\alpha)^{-1}(\mathbb{R}^d \times \mathbb{R}^d) = V_-^\alpha \mathbb{R}^d \times V_+^\alpha \mathbb{R}^d$  follows directly from the vector structure of each  $V_\sigma^\alpha \mathbb{R}^d$  and its Cartesian product. Moreover, we have that  $\dim(V_-^\alpha \mathbb{R}^d \times V_+^\alpha \mathbb{R}^d) = 2d$ . Locally we have the isomorphism

$$T^\alpha \mathbb{R}^d \cong T_-^\alpha \mathbb{R}^d \times T_+^\alpha \mathbb{R}^d \cong \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d.$$

Analogously to definition 3.3, we can define the *double tangent bundle* by

$$\mathbb{TR}^d = \{ ((\gamma_x, \gamma_y), (X_- \gamma_x, X_+ \gamma_y)) \in (\mathbb{R}^d \times \mathbb{R}^d) \times (V_- \mathbb{R}^d \times V_+ \mathbb{R}^d) \mid \gamma_x, \gamma_y \in C^\infty([a, b], \mathbb{R}^d) \},$$

where now we are considering the usual tangent bundle of  $\mathbb{R}^d$  instead of the fractional one. For a curve  $\gamma \in C^\infty([a, b], \mathbb{R}^d)$ , a vector  $X_\sigma \gamma \in V_\sigma \mathbb{R}^d$  shall be defined by its time derivative, i.e.  $X_\sigma \gamma := \dot{\gamma}$ . Defining  $\mathcal{V}_{\tilde{\gamma}} := ((\gamma_x, \gamma_y), (X_- \gamma_x, X_+ \gamma_y))$ , we have the local expression

$$\mathcal{V}_{\tilde{\gamma}} = (x, y, \dot{x}, \dot{y}).$$

The bundle projection is  $\tau : \mathbb{TR}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ ,  $\tau(\mathcal{V}_{\tilde{\gamma}}) = (x, y)$ .

*Remark 3.5.*  $V_\sigma \mathbb{R}^d$  and  $V_- \mathbb{R}^d \times V_+ \mathbb{R}^d$  are vector spaces. Moreover, locally

$$\mathbb{TR}^d \cong T_- \mathbb{R}^d \times T_+ \mathbb{R}^d \cong \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d.$$

In the forthcoming sections we are going to consider Lagrangian functions defined on a particular phase space that we describe now.

*Definition 3.6.* Consider the double fractional tangent bundle  $\mathbb{T}^\alpha \mathbb{R}^d$  and the double tangent bundle  $\mathbb{TR}^d$ . We define the fractional phase space as the bundle product of them over  $\mathbb{R}^d \times \mathbb{R}^d$ , i.e.

$$\mathfrak{TR}^d := \mathbb{TR}^d \otimes_{\mathbb{R}^d \times \mathbb{R}^d} \mathbb{T}^\alpha \mathbb{R}^d.$$

Thus,  $\mathcal{V}_{\tilde{\gamma}} := (\gamma_x, \gamma_y, X_- \gamma_x, X_+ \gamma_y, X_-^\alpha \gamma_x, X_+^\alpha \gamma_y) \in \mathfrak{TR}^d$  is locally described by

$$\mathcal{V}_{\tilde{\gamma}} = (x, y, \dot{x}, \dot{y}, D_-^\alpha x, D_+^\alpha y). \quad (4)$$

The bundle projection  $\mathcal{T} : \mathfrak{TR}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$  is defined by  $\mathcal{T}(\mathcal{V}_{\tilde{\gamma}}) = (x, y)$ .

For more details on bundle products we refer to Abraham and Marsden (1978). According to the previous development, we conclude the following (the proofs are straightforward): the fiber  $\mathcal{T}^{-1}(\mathbb{R}^d \times \mathbb{R}^d)$  is a vector space with dimension  $4d$ ; and  $\mathfrak{TR}^d$  is locally the Cartesian product of  $6$  copies of  $\mathbb{R}^d$ .

#### 4. RESTRICTED CONTINUOUS VARIATIONAL PRINCIPLE

Consider a smooth curve  $\tilde{\gamma} = (\gamma_x, \gamma_y)$ , for  $\gamma_x, \gamma_y \in C^\infty([a, b], \mathbb{R}^d)$  with fixed endpoints  $x_a = \gamma_x(a) = \gamma_y(b)$  and  $x_b = \gamma_x(b) = \gamma_y(a)$  in  $\mathbb{R}^d$ .

*Definition 4.1.* Define the path space by

$$C^\infty(x_a, x_b; \mathbb{R}^d \times \mathbb{R}^d) = \{ \tilde{\gamma} \in C^\infty([a, b], \mathbb{R}^d \times \mathbb{R}^d) \mid \tilde{\gamma}(a) = (x_a, x_b), \tilde{\gamma}(b) = (x_b, x_a) \}$$

*Definition 4.2.* Given a curve  $\tilde{\gamma} \in C^\infty(x_a, x_b; \mathbb{R}^d \times \mathbb{R}^d)$ , a varied curve is a map  $\Gamma : \mathbb{R} \times [a, b] \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ ,

$$\Gamma(\epsilon, t) := \tilde{\gamma}(t) + \epsilon \delta \tilde{\gamma}(t)$$

for  $\delta \tilde{\gamma} : [a, b] \rightarrow \mathbb{R}^d \times \mathbb{R}^d$  a variation, which is smooth and defined such that  $\delta \tilde{\gamma}(a) = \delta \tilde{\gamma}(b) = 0$ .

*Remark 4.3.* Note that we are establishing that  $\gamma_y(a) = x_b$  and  $\gamma_y(b) = x_a$ . This will make sense afterwards since we shall interpret  $\gamma_y$  as  $\gamma_x$  for reversed time (see Remark 4.9).

From definition 4.2 it follows that  $\delta \gamma_x(a) = \delta \gamma_y(b) = \delta \gamma_x(a) = \delta \gamma_y(b) = 0$ . Locally

$$\delta x(a) = \delta y(a) = \delta x(b) = \delta y(b) = 0. \quad (5)$$

The variations  $\delta \tilde{\gamma}$  induce the map  $\delta \hat{\tilde{\gamma}} : [a, b] \rightarrow \mathfrak{TR}^d$ . Using coordinates (4) and definition 4.2, it follows that at each  $t \in [a, b]$

$$\delta \hat{\tilde{\gamma}} = (\delta x, \delta y, \delta \dot{x}, \delta \dot{y}, D_-^\alpha x, D_+^\alpha y). \quad (6)$$

Next, we define the set of restricted varied curves.

*Definition 4.4.* Define the set of restricted varied curves as  $\Gamma_\eta(\epsilon, t) := \tilde{\gamma}(t) + \epsilon \eta(t)$ , where  $\delta \tilde{\gamma} = \eta(t) = (\delta \gamma_x(t), \delta \gamma_y(t))$ . In other words, we impose  $\delta \gamma_x(t) = \delta \gamma_y(t)$  to the variations, which locally is expressed by  $\delta x = \delta y$ .

*Remark 4.5.* Observe that under definition 4.4 we are allowing different varied curves, i.e.  $\tilde{\gamma}_x \neq \tilde{\gamma}_y$ , while restricting the variations, i.e.  $\delta \gamma_x = \delta \gamma_y$ .

Now, define a  $C^2$  Lagrangian function  $L : \mathfrak{TR}^d \rightarrow \mathbb{R}$  and the action functional  $\mathcal{A} : C^\infty(x_a, x_b; \mathbb{R}^d \times \mathbb{R}^d) \rightarrow \mathbb{R}$  given by

$$\mathcal{A}(\tilde{\gamma}) := \int_a^b L(\mathcal{V}_{\tilde{\gamma}}) dt. \quad (7)$$

We have already all the ingredients to establish a restricted variational principle:

*Theorem 4.6.* A curve  $\tilde{\gamma} : [a, b] \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ , subject to restricted variations  $\eta(t)$  in definition 4.4, is an extremal of the action  $\mathcal{A} : C^\infty(x_a, x_b; \mathbb{R}^d \times \mathbb{R}^d) \rightarrow \mathbb{R}$  defined in (7) if it satisfies the restricted fractional Euler-Lagrange equations:

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^i} \right) + D_-^\alpha \left( \frac{\partial L}{\partial D_+^\alpha y^i} \right) = 0, \quad (8a)$$

$$\frac{\partial L}{\partial y^i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}^i} \right) + D_+^\alpha \left( \frac{\partial L}{\partial D_-^\alpha x^i} \right) = 0, \quad (8b)$$

for  $i = 1, \dots, d$ .

*Proof.* To find the extremals of  $\mathcal{A}$  for restricted varied curves  $\Gamma_\eta(\epsilon, t)$  we impose the usual critical condition, i.e.  $\delta \mathcal{A} := \frac{d}{d\epsilon} \mathcal{A}(\Gamma_\eta)|_{\epsilon=0} = 0$ . Using the expression (6) we obtain:

$$\begin{aligned} \delta \mathcal{A} &= \int_a^b \langle \nabla L(\mathcal{V}_{\tilde{\gamma}}), \delta \hat{\tilde{\gamma}} \rangle dt = \int_a^b \left\{ \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} + \frac{\partial L}{\partial D_-^\alpha x} D_-^\alpha \delta x \right. \\ &\quad \left. + \frac{\partial L}{\partial y} \delta y + \frac{\partial L}{\partial \dot{y}} \delta \dot{y} + \frac{\partial L}{\partial D_+^\alpha y} D_+^\alpha \delta y \right\} dt \\ &= \int_a^b \left\{ \frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) + D_+^\alpha \left( \frac{\partial L}{\partial D_-^\alpha x} \right) \right. \\ &\quad \left. + \frac{\partial L}{\partial y} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}} \right) + D_-^\alpha \left( \frac{\partial L}{\partial D_+^\alpha y} \right) \right\} \delta x dt + \frac{\partial L}{\partial \dot{x}} \delta x \Big|_a^b + \frac{\partial L}{\partial \dot{y}} \delta y \Big|_a^b, \end{aligned}$$

where in the second equality we have employed the constraints  $\delta x = \delta y$ , while in the third equality we have used integration by parts with respect to the total and fractional derivatives (2). According to the endpoint conditions (5), all the border terms vanish leading to

$$\begin{aligned} \delta \mathcal{A} &= \int_a^b \left\{ \frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) + D_+^\alpha \left( \frac{\partial L}{\partial D_-^\alpha x} \right) \right. \\ &\quad \left. + \frac{\partial L}{\partial y} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}} \right) + D_-^\alpha \left( \frac{\partial L}{\partial D_+^\alpha y} \right) \right\} \delta x dt \\ &= \int_a^b \left\{ \left[ \frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) + D_-^\alpha \left( \frac{\partial L}{\partial D_+^\alpha y} \right) \right] \delta x \right. \\ &\quad \left. + \left[ \frac{\partial L}{\partial y} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}} \right) + D_+^\alpha \left( \frac{\partial L}{\partial D_-^\alpha x} \right) \right] \delta x \right\} dt. \quad (9) \end{aligned}$$

Finally, from this last expression of  $\delta\mathcal{A}$  is easy to see that the restricted fractional Euler-Lagrange equations (8) are a sufficient condition for  $\delta\mathcal{A} = 0$ ; and the claim holds.  $\square$

As mentioned above, equations (8) are only sufficient conditions for the extremal curves.

*Theorem 4.7. The restricted fractional Euler-Lagrange equations (8) are invariant under linear change of variables  $x = \Lambda z$  and  $y = \Lambda \tilde{z}$ , where  $\Lambda \in \mathbb{M}^{d \times d}(\mathbb{R})$  is full rank.*

*Proof.* Since the fractional derivative operators (1) are linear (so are total time derivatives), we have that

$$\begin{aligned} \dot{x} &= \Lambda \dot{z}, & D_-^\alpha x &= \Lambda D_-^\alpha z, \\ \dot{y} &= \Lambda \dot{\tilde{z}}, & D_+^\alpha y &= \Lambda D_+^\alpha \tilde{z}. \end{aligned}$$

Taking this into account and using the chain rule, it is easy to prove that  $\delta\mathcal{A}$  in its (9) form can be rewritten as

$$\begin{aligned} \delta\mathcal{A} &= \int_a^b \left\{ \left[ \frac{\partial L}{\partial z} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{z}} \right) + D_-^\alpha \left( \frac{\partial L}{\partial D_-^\alpha \tilde{z}} \right) \right] \Lambda^{-1} \delta x \right. \\ &\quad \left. + \left[ \frac{\partial L}{\partial \tilde{z}} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\tilde{z}}} \right) + D_+^\alpha \left( \frac{\partial L}{\partial D_+^\alpha z} \right) \right] \Lambda^{-1} \delta x \right\} dt, \end{aligned}$$

where we have used that  $\Lambda$  is full rank. Again  $\delta x$  is arbitrary, and therefore a sufficient condition for the extremal curves is

$$\begin{aligned} \left[ \frac{\partial L}{\partial z} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{z}} \right) + D_-^\alpha \left( \frac{\partial L}{\partial D_-^\alpha \tilde{z}} \right) \right] \Lambda^{-1} &= 0, \\ \left[ \frac{\partial L}{\partial \tilde{z}} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\tilde{z}}} \right) + D_+^\alpha \left( \frac{\partial L}{\partial D_+^\alpha z} \right) \right] \Lambda^{-1} &= 0. \end{aligned}$$

Using again that  $\Lambda$  is full rank, we obtain the restricted fractional Euler-Lagrange equations (8) in the new coordinates  $(z, \tilde{z}, \dot{z}, \dot{\tilde{z}}, D_-^\alpha z, D_+^\alpha \tilde{z})$  and the claim holds.  $\square$

Define now the Lagrangian  $L(x, y, \dot{x}, \dot{y}, D_-^\alpha x, D_+^\alpha y)$  by

$$L(x, y, \dot{x}, \dot{y}, D_-^\alpha x, D_+^\alpha y) := L_x(x, \dot{x}) + L_y(y, \dot{y}) - \llbracket D_-^\alpha x, D_+^\alpha y \rrbracket_R, \quad (10)$$

where  $L_x : T_- \mathbb{R}^d \rightarrow \mathbb{R}$  and  $L_y : T_+ \mathbb{R}^d \rightarrow \mathbb{R}$  are  $C^2$  functions, and  $\llbracket \cdot, \cdot \rrbracket_R : V_-^\alpha \mathbb{R}^d \times V_+^\alpha \mathbb{R}^d \rightarrow \mathbb{R}$  is a symmetric bilinear form defined by

$$\llbracket D_-^\alpha x, D_+^\alpha y \rrbracket_R := (D_-^\alpha x)^T R D_+^\alpha y, \quad (11)$$

where  $R = \text{diag}(\rho_1, \dots, \rho_d) \in \mathbb{M}^{d \times d}(\mathbb{R}^+)$ . With this new Lagrangian we obtain the following corollary to theorem 4.6.

*Corollary 4.8. If  $L$  is given by (10),  $\alpha = 1/2$  and  $L_x(x, \dot{x}) = \frac{1}{2} \dot{x}^T M \dot{x} - U(x)$ ,  $L_y(y, \dot{y}) = \frac{1}{2} \dot{y}^T M \dot{y} - U(y)$ , where  $M = \text{diag}(m_1, \dots, m_d) \in \mathbb{M}^{d \times d}(\mathbb{R}^+)$  and  $U : \mathbb{R}^d \rightarrow \mathbb{R}$  is a smooth function, then (8) are*

$$M \ddot{x} + R \dot{x} + \nabla U(x) = 0, \quad (12a)$$

$$M \ddot{y} - R \dot{y} + \nabla U(y) = 0. \quad (12b)$$

The proof is straightforward after the mentioned substitutions and considering (3).

*Remark 4.9. We observe that if we interpret that  $\gamma_y(t)$  is  $\gamma_x(t)$  in reversed time, i.e.  $\gamma_y(t) = \gamma_x(a+b-t)$ , the system of second order differential equations (12) is invariant under time reversal, this is  $t \rightarrow a+b-t$ .*

## 5. RESTRICTED DISCRETE VARIATIONAL PRINCIPLE

Let us consider the increasing sequence of times  $\{t_k = hk \mid k = 0, \dots, N\} \subset \mathbb{R}$  where  $h$  is the fixed time step determined by  $h = (b-a)/N$ . Define a discrete curve as a collection of points in  $\mathbb{R}^d$  i.e.  $\gamma_d := \{x_0, x_1, \dots, x_{N-1}, x_N\} = \{x_k\}_{0:N} \in \mathbb{R}^{(N+1)d}$ . As usual, we will consider these points as an approximation of the continuous curve at time  $t_k$ , namely  $x_k \simeq x(t_k)$ . Given  $\{z_k\}_{0:N}$  (later on we shall particularise in  $\{x_k\}_{0:N}$  and  $\{y_k\}_{0:N}$ ) define the following sequences:

$$\{S z_k\}_{0:N-1}, \quad S z_k := \frac{z_k + z_{k+1}}{2}, \quad (13a)$$

$$\{\Delta_- z_k\}_{1:N}, \quad \Delta_- z_k := \frac{z_k - z_{k-1}}{h}, \quad \Delta_- z_0 := 0, \quad (13b)$$

$$\{\Delta_+ z_k\}_{0:N-1}, \quad \Delta_+ z_k := -\frac{z_{k+1} - z_k}{h}, \quad (13c)$$

$$\{\Delta_-^\alpha z_k\}_{0:N}, \quad \Delta_-^\alpha z_k := \frac{1}{h^\alpha} \sum_{n=0}^k \alpha_n z_{k-n}, \quad (13d)$$

$$\{\Delta_+^\alpha z_k\}_{0:N}, \quad \Delta_+^\alpha z_k := \frac{1}{h^\alpha} \sum_{n=0}^{N-k} \alpha_n z_{k+n}, \quad (13e)$$

where

$$\alpha_n := \frac{-\alpha(1-\alpha)(2-\alpha) \cdots (n-1-\alpha)}{n!}; \quad \alpha_0 := 1. \quad (14)$$

Naturally, we shall consider  $\Delta_- x_k$  (resp.  $\Delta_+ y_k$ ) as an approximation of  $\dot{x}(t_k)$  (resp.  $\dot{y}(t_k)$ ) and  $\Delta_-^\alpha x_k$  (resp.  $\Delta_+^\alpha y_k$ ) as an approximation of  $D_-^\alpha x(t_k)$  (resp.  $D_+^\alpha y(t_k)$ ). For more details on the approximation of fractional derivatives we refer to (Cresson (2014), Chapter 5).

*Remark 5.1. We observe that (13d), (13e) are well-defined for  $k = 0$  and  $k = N$ . Namely, straightforward computations lead to  $\Delta_-^\alpha x_0 = \alpha_0 x_0 / h^\alpha$  and  $\Delta_+^\alpha y_N = \alpha_0 y_N / h^\alpha$ .*

Given two sequences  $\{F_k\}_{0:N}, \{G_k\}_{0:N}$ , the discrete derivatives (13b), (13c), and discrete fractional derivatives (13d), (13e), obey the following discrete integration by parts relationships:

$$\begin{aligned} \sum_{k=0}^{N-1} F_k (\Delta_+ G_k) &= \sum_{k=1}^N (\Delta_- F_k) G_k + \frac{1}{h} F_0 G_0 - \frac{1}{h} F_N G_N, \\ \sum_{k=0}^{N-1} F_k (\Delta_+^\alpha G_k) &= \sum_{k=1}^N (\Delta_-^\alpha F_k) G_k + \frac{1}{h^\alpha} F_0 G_0 - \frac{1}{h^\alpha} F_N G_N. \end{aligned}$$

See Cresson (2014) for proofs. Define now the sets of discrete curves

$$\begin{aligned} C_d^x &= \left\{ \gamma_d^x = \{x_k\}_{0:N} \in \mathbb{R}^{(N+1)d} \mid x_0 = x_a, x_N = x_a \right\}, \\ C_d^y &= \left\{ \gamma_d^y = \{y_k\}_{0:N} \in \mathbb{R}^{(N+1)d} \mid y_0 = x_b, y_N = x_a \right\}, \end{aligned}$$

and the action sum  $\mathcal{A}_d : C_d^x \times C_d^y \rightarrow \mathbb{R}$ ,

$$\mathcal{A}_d(\gamma_d^x, \gamma_d^y) := h \sum_{k=0}^{N-1} L(S x_k, S y_k, \Delta_- x_k, \Delta_+ y_k, \Delta_-^\alpha x_k, \Delta_+^\alpha y_k), \quad (15)$$

where we consider a Lagrangian function  $L : \mathfrak{T} \mathbb{R}^d \rightarrow \mathbb{R}$  as defined in section 4. We observe that the evaluation of such a Lagrangian function at  $(S x_k, S y_k, \Delta_- x_k, \Delta_+ y_k, \Delta_-^\alpha x_k, \Delta_+^\alpha y_k)$  for sequences (13) makes sense since  $\mathfrak{T} \mathbb{R}^d \cong \mathbb{R}^d \times$

$\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ . We define now the variation of discrete curves.

*Definition 5.2.* Given a discrete curve  $\tilde{\gamma}_d = (\gamma_d^x, \gamma_d^y) \in C_d^x \times C_d^y$ , we define the set of varied discrete curves by

$$\Gamma_\epsilon^x := \gamma_d^x + \epsilon \delta \gamma_d^x, \quad \Gamma_\epsilon^y := \gamma_d^y + \epsilon \delta \gamma_d^y.$$

where  $\delta \gamma_d^x := \{\delta x_k\}_{0:N}$ ,  $\delta \gamma_d^y := \{\delta y_k\}_{0:N}$  are the discrete variations, defined such that

$$\delta x_0 = \delta x_N = 0, \quad \delta y_0 = \delta y_N = 0. \quad (16)$$

We define the set of restricted varied discrete curves, by

$$\Gamma_{(\epsilon, \eta)}^x := \gamma_d^x + \epsilon \eta_d, \quad \Gamma_{(\epsilon, \eta)}^y := \gamma_d^y + \epsilon \eta_d,$$

where we establish  $\eta_d = \delta \gamma_d^x = \delta \gamma_d^y$ . Locally this means that  $\delta x_k = \delta y_k$  for  $k = 1, \dots, N-1$ .

In the following, we shall set

$$L(k) := L(S x_k, S y_k, \Delta_- x_k, \Delta_+ y_k, \Delta_-^\alpha x_k, \Delta_+^\alpha y_k)$$

for simplicity.

*Theorem 5.3.* A discrete curve  $\tilde{\gamma}_d = (\gamma_d^x, \gamma_d^y) \in C_d^x \times C_d^y$ , subject to restricted discrete variations  $\eta_d$  in definition 5.2, is an extremal of the action sum  $\mathcal{A}_d$  defined in (15) if it satisfies the restricted discrete fractional Euler-Lagrange equations:

$$\frac{1}{2} (D_1 L(k) + D_1 L(k-1)) + \Delta_+ D_3 L(k) + \Delta_-^\alpha D_6 L(k) = 0, \quad (17a)$$

$$\frac{1}{2} (D_2 L(k) + D_2 L(k-1)) + \Delta_- D_4 L(k) + \Delta_+^\alpha D_5 L(k) = 0, \quad (17b)$$

for  $k = 1, \dots, N-1$ .  $D_m L := \frac{\partial L}{\partial z_m^i}$ , where  $z_m$  is the  $m$ -th variable in (15),  $m=1, \dots, 6$ , and  $i = 1, \dots, d$ .

*Proof.* The extremals will satisfy the condition  $\delta \mathcal{A}_d := \frac{d}{d\epsilon} \mathcal{A}_d(\Gamma_{(\eta, \epsilon)}^x, \Gamma_{(\eta, \epsilon)}^y) \big|_{\epsilon=0} = 0$ .

$$\delta \mathcal{A}_d = h \sum_{k=0}^{N-1} \left\{ D_1 L(k) S \delta x_k + D_2 L(k) S \delta x_k + D_3 L(k) \Delta_- \delta x_k + D_4 L(k) \Delta_+ \delta x_k + D_5 L(k) \Delta_-^\alpha \delta x_k + D_6 L(k) \Delta_+^\alpha \delta x_k \right\}$$

where we have imposed the restricted variations  $\delta x_k = \delta y_k$  and have taken into account that  $\delta(S x_k) = S \delta x_k$ ,  $\delta(\Delta_- x_k) = \Delta_- \delta x_k$ ,  $\delta(\Delta_+ x_k) = \Delta_+ \delta x_k$ ,  $\delta(\Delta_-^\alpha x_k) = \Delta_-^\alpha \delta x_k$ ,  $\delta(\Delta_+^\alpha x_k) = \Delta_+^\alpha \delta x_k$  according to (13) and (16). We display the treatment of terms 1,3,5 (for terms 2,4,6 we use equivalent techniques).

$$\begin{aligned} \sum_{k=0}^{N-1} D_1 L(k) S \delta x_k &= \sum_{k=0}^{N-1} \frac{1}{2} D_1 L(k) (\delta x_k + \delta x_{k+1}) \\ &= \sum_{k=1}^{N-1} \frac{1}{2} (D_1 L(k) + D_1 L(k-1)) \delta x_k; \end{aligned}$$

where in the second equality we have rearranged the summation index and taken into account the endpoint conditions  $\delta x_0 = \delta x_N = 0$ .

$$\begin{aligned} \sum_{k=0}^{N-1} D_3 L(k) \Delta_- \delta x_k &= \sum_{k=1}^N D_3 L(k) \Delta_- \delta x_k \\ &= \sum_{k=0}^{N-1} \Delta_+ D_3 L(k) \delta x_k = \sum_{k=1}^{N-1} \Delta_+ D_3 L(k) \delta x_k; \end{aligned}$$

in the first equality we have used  $\delta x_0 = \delta x_N = 0$  and (13b), in the second equality employed the discrete integration by parts and, in the third,  $\delta x_0 = 0$ .

$$\begin{aligned} \sum_{k=0}^{N-1} D_5 L(k) \Delta_-^\alpha \delta x_k &= \sum_{k=1}^N D_5 L(k) \Delta_-^\alpha \delta x_k \\ &= \sum_{k=0}^{N-1} \Delta_+^\alpha D_5 L(k) \delta x_k = \sum_{k=1}^{N-1} \Delta_+^\alpha D_5 L(k) \delta x_k; \end{aligned}$$

in the first equality we have employed  $\delta x_0 = \delta x_N = 0$ , in the second used the discrete fractional integration by parts, and in the third  $\delta x_0 = 0$ . Gathering all terms together and ordering them conveniently we arrive to

$$\begin{aligned} \delta \mathcal{A}_d &= h \sum_{k=1}^{N-1} \left( \left\{ \frac{1}{2} (D_1 L(k) + D_1 L(k-1)) + \Delta_+ D_3 L(k) + \Delta_-^\alpha D_6 L(k) \right\} \delta x_k \right. \\ &\quad \left. + \left\{ \frac{1}{2} (D_2 L(k) + D_2 L(k-1)) + \Delta_- D_4 L(k) + \Delta_+^\alpha D_5 L(k) \right\} \delta x_k \right). \end{aligned}$$

Taking into account that  $\delta x_k$  are arbitrary for  $k = 1, \dots, N-1$  we observe that the discrete fractional Euler-Lagrange equations (17) are sufficient conditions for the extremal discrete curves.  $\square$

Let us consider the Lagrangian function (10) and  $L_x, L_y$  given in corollary 4.8, i.e.

$$\begin{aligned} L(k) &= \frac{1}{2} (\Delta_- x_k)^T M \Delta_- x_k + \frac{1}{2} (\Delta_+ y_k)^T M \Delta_+ y_k \\ &\quad - U(S x_k) - U(S y_k) - \llbracket \Delta_-^\alpha x_k, \Delta_+^\alpha y_k \rrbracket_R. \end{aligned} \quad (18)$$

Straightforward computations provide:

$$\begin{aligned} D_1 L(k) &= -\nabla U(S x_k), \quad D_3 L(k) = M \Delta_- x_k, \quad D_5 L(k) = -R \Delta_+^\alpha y_k, \\ D_2 L(k) &= -\nabla U(S y_k), \quad D_4 L(k) = M \Delta_+ y_k, \quad D_6 L(k) = -R \Delta_-^\alpha x_k. \end{aligned}$$

Plugging these expressions into (17) we obtain

$$\begin{aligned} -\frac{1}{2} \nabla U(S x_k) - \frac{1}{2} \nabla U(S x_{k-1}) + M \Delta_+ \Delta_- x_k - R \Delta_-^\alpha \Delta_+^\alpha x_k &= 0, \\ -\frac{1}{2} \nabla U(S y_k) - \frac{1}{2} \nabla U(S y_{k-1}) + M \Delta_- \Delta_+ y_k - R \Delta_+^\alpha \Delta_-^\alpha y_k &= 0, \end{aligned}$$

for  $k = 1, \dots, N-1$ . Employing (13), changing signs and reordering terms these equations are

$$\begin{aligned} M \frac{x_{k+1} - 2x_k + x_{k-1}}{h^2} + R \frac{1}{h^{2\alpha}} \sum_{n=0}^k \alpha_n \sum_{p=0}^{k-n} \alpha_p x_{k-n-p} \\ + \frac{1}{2} \nabla U \left( \frac{x_{k+1} + x_k}{2} \right) + \frac{1}{2} \nabla U \left( \frac{x_k + x_{k-1}}{2} \right) &= 0, \end{aligned} \quad (19a)$$

$$\begin{aligned} M \frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} + R \frac{1}{h^{2\alpha}} \sum_{n=0}^{N-k} \alpha_n \sum_{p=0}^{N-k-n} \alpha_p y_{k+n+p} \\ + \frac{1}{2} \nabla U \left( \frac{y_{k+1} + y_k}{2} \right) + \frac{1}{2} \nabla U \left( \frac{y_k + y_{k-1}}{2} \right) &= 0. \end{aligned} \quad (19b)$$

The following lemma will help interpret equations (19).

*Lemma 5.4.* For  $\alpha = 1/2$  and  $k = 1, \dots, N-1$ ,

$$\sum_{n=0}^k \alpha_n \sum_{p=0}^{k-n} \alpha_p x_{k-n-p} = x_k - x_{k-1}.$$

*Proof* According to (14) we have that  $\alpha_0 = 1$  and  $\alpha_1 = -1/2$  for  $\alpha = 1/2$ . For a fixed  $k = \tilde{k}$  and after some manipulations we arrive at

$$\sum_{n=0}^{\tilde{k}} \alpha_n \sum_{p=0}^{\tilde{k}-n} \alpha_p x_{\tilde{k}-n-p} = x_{\tilde{k}} - x_{\tilde{k}-1} \\ + \beta_0^2 x_{\tilde{k}-2} + \beta_0^3 x_{\tilde{k}-3} + \cdots + \beta_0^{\tilde{k}-2} x_2 + \beta_0^{\tilde{k}-1} x_1 + \beta_0^{\tilde{k}} x_0,$$

where

$$\beta_l^j = 2\alpha_{l+j} + \sum_{i=1}^{j-1} \alpha_i \alpha_{l+j-i}. \quad (20)$$

According to this, it is enough to prove that  $\beta_0^j = 0$  for any  $j$ , for which we proceed by induction. From (20) and (14), it follows that  $\beta_0^2 = 2\alpha_2 + \alpha_1\alpha_1$ , which vanishes for  $\alpha = 1/2$ . Taking this as the first induction step, it is enough to prove that  $\beta_0^{j+1} = 0$  assuming that  $\beta_0^j = 0$ .

$$\beta_0^{j+1} = 2\alpha_{j+1} + \sum_{i=1}^j \alpha_i \alpha_{j+1-i} = 2\alpha_{j+1} + \sum_{i=1}^{r-1} \alpha_i \alpha_{r-i} \\ = 2\alpha_{j+1} - 2\alpha_r + 2\alpha_r + \sum_{i=1}^{r-1} \alpha_i \alpha_{r-i} = 2\alpha_{j+1} - 2\alpha_r = 0,$$

where we have set  $r = j + 1$ . Hence the claim follows.  $\square$

Using similar arguments, one can prove that

$$\sum_{n=0}^{N-k} \alpha_n \sum_{p=0}^{N-k-n} \alpha_p y_{k+n+p} = -(y_{k+1} - y_k),$$

for  $k = 1, \dots, N-1$ . Therefore, in the case  $\alpha = 1/2$  we observe that (19) becomes

$$M \frac{x_{k+1} - 2x_k + x_{k-1}}{h^2} + R \frac{x_k - x_{k-1}}{h} \\ + \frac{1}{2} \nabla U \left( \frac{x_{k+1} + x_k}{2} \right) + \frac{1}{2} \nabla U \left( \frac{x_k + x_{k-1}}{2} \right) = 0, \\ M \frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} - R \frac{y_{k+1} - y_k}{h} \\ + \frac{1}{2} \nabla U \left( \frac{y_{k+1} + y_k}{2} \right) + \frac{1}{2} \nabla U \left( \frac{y_k + y_{k-1}}{2} \right) = 0.$$

These equations provide a classical midpoint rule variational integrator (see e.g. Marsden and West (2001)) for the conservative part of equations (12), i.e. for kinetic and potential energy terms, whereas the velocity-dependent damping term is approximated by backward and forward difference operators for the equations in forward and backward time, respectively. Equivalent equations may be obtained from the discretisation of the Lagrange-d'Alembert principle (see Marsden and West (2001); Ober-Blöbaum et al. (2011)). However, this principle implies equating the variation of the action to the work done by the external forces, which is not the point of our approach since we obtain such external forces from the restricted variational principle.

## 6. CONCLUSIONS

In summary, we have obtained the so-called *restricted fractional Euler-Lagrange equations* (both continuous and discrete) from a restricted variational principle defined on a  $\alpha$ -fractional phase space and curves evolving on real space. In the case of  $\alpha = 1/2$  and mechanical Lagrangian functions, i.e. kinetic minus potential energy,

these equations model a dissipative mechanical system, and present time symmetry and invariance under linear change of variables. As one could expect, the discrete equations are a meaningful discretisation of the continuous ones, giving rise to the construction and application of variational integrators for non-conservative mechanical systems.

This work establishes solid grounds for future developments. We are in position to construct a canonical transformation providing meaningful fractional Hamilton equations. Moreover, these Hamilton equations would facilitate the introduction of controlled external forces and the development of a *Fractional Pontryagin Maximum Principle* leading to necessary optimality conditions for dissipative mechanical systems. Needless to say, the numerical behaviour of the obtained schemes shall be tested by means of proper implementations, as well as compared to state-of-the-art numerical integrators for dissipative systems.

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