Quantifiers on languages and codensity monads

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***Abstract*—This paper contributes to the techniques of topo-algebraic recognition for languages beyond the regular setting as they relate to logic on words. In particular, we provide a general construction on recognisers corresponding to adding one layer of various kinds of quantifiers and prove a related Reutenauer-type theorem. Our main tools are codensity monads and duality theory. Our construction yields, in particular, a new characterisation of the profinite monad of the free S-semimodule monad for finite and commutative semirings S, which generalises our earlier insight that the Vietoris monad on Boolean spaces is the codensity monad of the finite powerset functor.**

I. Introduction

It is well known that the combinatorial property of a language of being given by a star-free regular expression can be described both by algebraic and by logical means. Indeed, on the algebraic side, star-free languages are exactly those languages whose syntactic monoids contain no nontrivial subgroups. On the logical side, properties of words can be expressed in predicate logic by considering variables as positions in the word, relation symbols asserting that a position in a word has a certain letter of the alphabet, and possibly additional predicates on positions (known as numerical predicates). As shown by McNaughton and Papert, the class of languages definable by first-order sentences over the numerical predicate $<$ consists precisely of the star-free ones.

A fundamental tool in studying the connection between algebra and logic in the regular setting is the availability of constructions on monoids which mirror the action of quantifiers. That is, given the syntactic monoid for a language defined by a formula with a free first-order variable, one constructs a monoid recognising the quantified language. Constructions of this type abound, and are all versions of semidirect products, with the block product playing a central rôle as it allows to obtain recognisers for many different quantifiers [1].

The purpose of this paper is to expand these techniques and provide the algebraic characterisation of adding one layer of various kinds of quantifiers, beyond the regular setting. A first step was our previous paper [2], where a) we introduce a topological notion of recogniser, that will be motivated in Section I-A, and b) we give a notion of unary Schützenberger product that corresponds, on the recogniser side, to the existential quantifier for arbitrary languages of words.

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No.670624). The third author also acknowledges financial support from Sorbonne Paris Cité (PhD agreement USPC IDEX – REGGI15RDXMTSPC1GEHRKE).

978-1-5090-3018-7/17/$31.00

Our overall goal consists in expanding beyond the regular languages the sophisticated tools of algebraic automata theory in order to search for solutions to open problems (as well as new solutions to settled ones) in Boolean circuit complexity [3]. Here, several problems about separation of complexity classes (which contain non-regular languages) are equivalent to questions about the expressive power of extensions of first-order logic by means of additional quantifiers, e.g. the modular quantifiers $\exists_{p \text{ mod } q}$ (see [4] for a discussion; we recall that a word satisfies $\exists_{p \text{ mod } q} x.\phi(x)$ provided the number of positions x for which $\phi(x)$ holds is congruent to p modulo q).

In Section I-A we provide a gentle introduction and motivate the duality-theoretic approach to language recognition. In Section I-B we present codensity monads, our tool of choice for systematically obtaining the relevant topological constructions, and we briefly discuss related work. Finally, in Section I-C we present the main contributions of this paper.

A. *Duality for language recognition*

The fact that the spaces underlying profinite completions are Boolean Stone spaces was observed by Birkhoff already in 1937, and a first, but isolated, application of this in language theory may be found in [5]. Only recently, starting with the papers [6], [7], has the deeper connection between Stone duality and formal language theory started to emerge. In these papers a new notion of language recognition, based on topological methods, was proposed for the setting of non-regular languages. Moreover, the duality-theoretic nature of the celebrated Eilenberg-Reiterman theorems, which establish a correspondence between varieties of languages, pseudo-varieties of finite algebras, and profinite equations, was identified. This has led to an active research area, where categorical and duality-theoretic methods are used to encompass notions of language recognition for various automata models, see for example the monadic approach to language recognition put forward by Bojańczyk [8], or the long series of papers on a category-theoretic approach to Eilenberg-Reiterman theory, see [9] and the references therein.

Let us illustrate the interplay between duality theory and the theory of regular languages by explaining the duality between the syntactic monoid of a regular language L on a finite alphabet A, and the Boolean subalgebra $\mathcal{B} \hookrightarrow \mathcal{P}(A^*)$ generated by the quotients of L, i.e. by the sets

$$w^{-1}Lv^{-1} = \{u \in A^* \mid wuv \in L\}, \text{ for } w, v \in A^*.$$

In this setting one makes use only of the duality between the categories of finite Boolean algebras and of finite sets which, at the level of objects, asserts that each finite Boolean algebra is isomorphic to the powerset of its atoms. Since the language L is regular, it has only finitely many quotients, say

$$\{w_1^{-1}Lv_1^{-1}, \dots, w_n^{-1}Lv_n^{-1}\}.$$

The finite Boolean algebra generated by this set has as atoms the non-empty subsets of A^* of the form

$$\bigcap_{i\in I} w_i^{-1}Lv_i^{-1} \cap \bigcap_{j\in J}(w_j^{-1}Lv_j^{-1})^c$$

for some partition $I \cup J$ of $\{1,\dots,n\}$. We clearly see that such atoms are in one-to-one correspondence with the equivalence classes of the Myhill syntactic congruence $\sim_L$, and thus with the elements of the syntactic monoid $A^*/{\sim_L}$ of L.

However, the more interesting aspect of this approach is that one can also explain the monoid structure of $A^*/{\sim_L}$ and the syntactic morphism in duality-theoretic terms. For this, we have to recall first the duality between the category of sets and the category of complete atomic Boolean algebras. At the level of objects, every complete atomic Boolean algebra is isomorphic to the powerset of its atoms. So the dual of A^* is $\mathcal{P}(A^*)$, but the duality also tells us that quotients on one side are turned into embeddings on the other. Thus, we have the duality between the following morphisms

$$\mathcal{B} \hookrightarrow \mathcal{P}(A^*) \qquad \Big| \qquad A^* \twoheadrightarrow A^*/{\sim_L}.$$

Also, the left action of A^* on itself given by appending a word w on the left corresponds, on the dual side, to a left quotient operation (which is technically a right action):

$$\begin{array}{ccc} \mathcal{P}(A^*) & \xrightarrow{\Lambda_w} & \mathcal{P}(A^*) \\ U & \longmapsto & w^{-1}U \end{array} \qquad \Big| \qquad \begin{array}{ccc} A^* & \xrightarrow{l_w} & A^*. \\ v & \longmapsto & wv \end{array}$$

Since the Boolean algebra $\mathcal{B}$ is closed under quotients, and thus we have commuting squares as the left one in diagram (1), by duality we obtain a left action of A^* on $A^*/{\sim_L}$. By an analogous argument, one also obtains a right action of A^* on $A^*/{\sim_L}$ and the two actions commute. It is a simple lemma, see [2], that since $A^*/{\sim_L}$ is a quotient of A^* and it is equipped with commuting left and right A^*-actions (called in loc. cit. an A^*-biaction), then one can uniquely define a monoid multiplication on $A^*/{\sim_L}$ so that the quotient $A^* \twoheadrightarrow A^*/{\sim_L}$ is a monoid morphism.

$$\begin{array}{ccc} \mathcal{B} & \hookrightarrow & \mathcal{P}(A^*) \\ {\scriptstyle\Lambda_w}\downarrow & & \downarrow{\scriptstyle\Lambda_w} \\ \mathcal{B} & \hookrightarrow & \mathcal{P}(A^*) \end{array} \qquad \Big| \qquad \begin{array}{ccc} A^* & \twoheadrightarrow & A^*/{\sim_L} \\ {\scriptstyle l_w}\downarrow & & \downarrow{\scriptstyle l_w} \\ A^* & \twoheadrightarrow & A^*/{\sim_L} \end{array} \tag{1}$$

This approach paves the way for understanding a right notion of recogniser and syntactic object for a non-regular L. In this case, the Boolean algebra $\mathcal{B}$ spanned by the quotients of L is no longer finite, so the finite duality theorems we have employed previously are no longer applicable. Instead, we use the full power of Stone duality, which establishes the dual equivalence between the category of Boolean algebras and the category BStone of Boolean (Stone) spaces, that is, compact Hausdorff topological spaces that are zero-dimensional. In this setting, the dual of $\mathcal{P}(A^*)$ is the Stone-Čech compactification $\beta(A^*)$ of the discrete space A^*. The embedding of $\mathcal{B}$ into $\mathcal{P}(A^*)$ is turned by the duality theorem into a quotient of topological spaces as displayed below, where we denote the dual of $\mathcal{B}$ by X.

$$\mathcal{B} \hookrightarrow \mathcal{P}(A^*) \qquad \Big| \qquad \beta(A^*) \twoheadrightarrow X$$

The syntactic monoid of the language L, now infinite, can be seen as a dense subset of X, and is indeed the image of the composite map $A^* \hookrightarrow \beta(A^*) \twoheadrightarrow X$ where the first arrow is the embedding of A^* in its Stone-Čech compactification. We thus obtain a commuting diagram as follows.

$$\begin{array}{ccc} \beta(A^*) & \twoheadrightarrow & X \\ \uparrow & & \uparrow \\ A^* & \twoheadrightarrow & A^*/{\sim_L} \end{array}$$

Furthermore, one can show that the syntactic monoid acts (continuously) on X both on the left and right, and these actions commute. This situation has lead us to axiomatise, in [2], the definition of a *Boolean space with an internal monoid* (BiM) as a suitable notion for language recognition beyond the regular setting. We recall this (in fact a small variation of it) in Definition II.1 below.

B. *Profinite monads*

Profinite methods have a long tradition in language theory, see for example [10]. To accommodate these tools in his monadic approach to language recognition, Bojańczyk [8] has recently introduced a construction transforming a monad T on Set (the category of sets and functions) into a so-called *profinite monad*, again on the category of sets, which allowed him to study in this generic framework the profinite version of the objects modelled by T, such as profinite words, profinite countable chains, and profinite trees.

A very much related construction of a *profinite monad of* T was introduced in [11], this time as a monad on the category of Boolean spaces, obtained as a so-called *codensity monad* for a functor from the category of finitely carried T-algebras to Boolean spaces, that we describe in the next section.

The codensity monad is a very standard construction in category theory, going back to the work of Kock in the 60s. It is well known that any right adjoint functor G induces a monad obtained by composition with its left adjoint, and this is exactly the codensity monad of G. In general, the codensity monad of a functor which is not necessarily right adjoint, provided it exists, is the best approximation to this phenomenon. For example the codensity monad of the forgetful functor $|-|\colon \mathsf{BStone} \to \mathsf{Set}$ on Boolean spaces is the ultrafilter monad on Set obtained by composition with its left adjoint $\beta\colon \mathsf{Set} \to \mathsf{BStone}$. The same monad has yet another description as a codensity monad, this time for the inclusion of the category Set_f of finite sets into Set, a fact proved in [12] and recently revisited in the elegant paper [13].

The departing point of the present paper is the observation that the unary Schützenberger product $(\Diamond X, \Diamond M)$ of a BiM (X, M) from our paper [2] hinges, at a deeper level, on the fact that the *Vietoris monad* $\mathcal{V}$ on the category of Boolean spaces (which is heavily featured in that construction) is the profinite monad of the finite powerset monad $\mathcal{P}_f$ on Set. Recall that any Boolean space X is the cofiltered (or inverse) limit of its finite quotients X_i. Then one can check that the Vietoris space $\mathcal{V}X$ is the cofiltered limit of the finite sets $\mathcal{P}_f X_i$.

In order to find suitable recognisers for languages quantified by, e.g., modular quantifiers, we need a slightly different construction than $(\Diamond X, \Diamond M)$ of [2]. Specifically, we observe that the semantics of these quantifiers can be modelled, at least at the level of finite monoids, by the free S-semimodule monad $\mathcal{S}$, for a suitable choice of the semiring S. It should be noted that $\mathcal{P}_f$ is also an instance of the free S-semimodule monad, for the Boolean semiring 2. To obtain corresponding constructions at the level of Boolean spaces with internal monoids, one needs to understand the analogue of the Vietoris construction for the monad $\mathcal{S}$. And the obvious candidate, from a category-theoretic perspective, is the codensity monad of $\mathcal{S}$.

C. Contributions

This paper contributes to the connection between the topological approach to language recognition and logical formalisms beyond the setting of regular languages, and furthers, along the way, the study of profinite monads.

The main result of Section III allows one to extend finitary commutative Set-monads to the category of Boolean spaces with internal monoids. An instance of this result is presented in Section IV where duality-theoretic insights are used to provide a concrete description of the constructions involved in terms of measure-like functions. In Section V we develop a generic approach for mirroring operations on languages, such as modular quantifiers, associating to a BiM (X, M) a new BiM $(\Diamond X, \Diamond M)$, while Section VI explains how these constructions are indeed canonical and provides a Reutenauer-like result characterising the Boolean algebra closed under quotients generated by the languages recognised by $(\Diamond X, \Diamond M)$.

The interested reader is referred to the extended version [14] of this paper for additional details and proofs.

II. Preliminaries

A. Logic on words

Fix an arbitrary finite set A, and write A^* for the free monoid over A. In the logical approach to language theory a word w over the alphabet A (an A-word, for short), i.e. an element of A^*, is regarded as a (relational) structure on the set $\{1, \ldots, |w|\}$, where $|w|$ denotes the length of the word, equipped with a unary relation P_a for each $a \in A$ which singles out the positions in the word where the letter a appears. If φ is a sentence (i.e. a formula in which every variable is in the scope of a quantifier) in a language interpretable over words, we denote by L_φ the set of words satisfying φ.

Assume now $\varphi(x)$ is a formula with a free first-order variable x (intuitively this means that $\varphi(x)$ can talk about positions in the word). In order to be able to interpret the free variable, we consider an extended alphabet $A \times 2$ which we think of as consisting of two copies of A, that is, we identify $A \times 2$ with the set $A \cup \{a' \mid a \in A\}$, and we call the elements of the second copy of A *marked letters*. Assuming $w = a_1 \ldots a_n$ and $1 \leq i \leq |w|$, we write $w^{(i)}$ for the word $a_1 \ldots a_{i-1} a'_i a_{i+1} \ldots a_n$, i.e. for the word in $(A \times 2)^*$ having the same shape as w but with the letter in position i marked, and w^0 for the word $a_1 \ldots a_n$ seen as a word in $(A \times 2)^*$. Then we define $L_{\varphi(x)}$ as the set of all words in the alphabet $A \times 2$ *with only one marked letter* such that the underlying word in the alphabet A satisfies φ when the variable x points at the marked letter.

Now, given $L \subseteq (A \times 2)^*$, denote by $L_\exists$ the language consisting of those words $w = a_1 \ldots a_n$ over A such that there exists $1 \leq i \leq |w|$ with $a_1 \ldots a_{i-1} a'_i a_{i+1} \ldots a_n \in L$. Observe that $L = L_{\varphi(x)}$ entails $L_\exists = L_{\exists x. \varphi(x)}$, thus recovering the usual existential quantification.

Among the generalizations of the existential quantifier are the *modular quantifiers*. Consider the ring $\mathbb{Z}_q$ of integers modulo q, and pick $p \in \mathbb{Z}_q$. We say that an A-word w satisfies the sentence $\exists_{p \text{ mod } q} x. \varphi(x)$ if there exist p modulo q positions in w for which the formula $\varphi(x)$ holds. Also, for an arbitrary language $L \subseteq (A \times 2)^*$, we define $L_{\exists_{p \text{ mod } q}}$ as the set of A-words $w = a_1 \ldots a_n$ such that the cardinality of the set

$$\{1 \leq i \leq |w| \mid a_1 \ldots a_{i-1} a'_i a_{i+1} \ldots a_n \in L\} \tag{2}$$

is congruent to p modulo q. Clearly, if the language L is defined by the formula $\varphi(x)$, then $L_{\exists_{p \text{ mod } q}}$ is defined by the formula $\exists_{p \text{ mod } q} x. \varphi(x)$.

Finally, generalising the preceding situations, we can consider an arbitrary semiring $(S, +, \cdot, 0_S, 1_S)$ and an element $k \in S$. For $L \subseteq (A \times 2)^*$, an A-word $w = a_1 \ldots a_n$ belongs to the quantified language, denoted by $\mathcal{Q}_k(L)$, provided that

$$\underbrace{1_S + \cdots + 1_S}_{m \text{ times}} = k,$$

where m is the cardinality of the set in (2).

B. Stone duality and the Vietoris hyperspace

Stone duality for Boolean algebras [15] establishes a categorical equivalence between the category of Boolean algebras and their homomorphisms, and the opposite of the category BStone of Boolean (Stone) spaces and continuous maps.

A *Boolean space* is a compact Hausdorff space that admits a basis of *clopen* (=simultaneously closed and open) subsets. There is an obvious forgetful functor $|-| \colon \mathsf{BStone} \to \mathsf{Set}$. When clear from the context, we will omit writing $|-|$.

The dual of the Boolean space X is the Boolean algebra $\mathit{Clop}(X)$ of its clopen subsets, equipped with set-theoretic operations. Conversely, given a Boolean algebra B, the dual space X may be taken either as the set of ultrafilters on B (i.e. those proper filters F satisfying $a \in F$ or $\neg a \in F$ for every $a \in B$) or as the Boolean algebra homomorphisms $h \colon B \to 2$ equipped with the topology generated by the sets

$$\widehat{a} := \{F \mid a \in F\} \cong \{h \mid h(a) = 1\}, \text{ for } a \in B.$$

An example of Boolean space, central to our treatment, is the *Stone-Čech compactification* of an arbitrary set K. This is the dual space of the Boolean algebra $\mathcal{P}K$, and is denoted by βK. It is well known that the assignment $K \mapsto \beta K$ induces a functor $\beta\colon \mathsf{Set} \to \mathsf{BStone}$ that is left adjoint to the forgetful functor $|-|\colon \mathsf{BStone} \to \mathsf{Set}$. Another functor, which played a key rôle in [2] and will serve here as a leading example, is the *Vietoris functor* $\mathcal{V}\colon \mathsf{BStone} \to \mathsf{BStone}$. Given a Boolean space X, consider the collection $\mathcal{V}X$ of all closed subsets of X equipped with the topology generated by the clopen subbasis

$$\{\Diamond V \mid V \in \mathit{Clop}(X)\} \cup \{(\Diamond V)^c \mid V \in \mathit{Clop}(X)\},$$

where $\Diamond V := \{K \in \mathcal{V}X \mid K \cap V \neq \emptyset\}$. The resulting space is called the Vietoris (hyper)space of X, and is again a Boolean space. Further, if $f\colon X \to Y$ is a morphism in BStone, then so is the direct image function $\mathcal{V}X \to \mathcal{V}Y,\ K \mapsto f[K]$. In fact, it is well known that this is the functor part of a monad $\mathcal{V}$ on BStone. The Vietoris hyperspace of an arbitrary topological space was first introduced by Vietoris in 1923; for a complete account, including results stated here without proof, see [16].

C. Boolean spaces with internal monoids

In this section we give the definition of a *Boolean space with an internal monoid*, or BiM for short (see Definition II.1 below), a topological recogniser well-suited for dealing with non-regular languages. In [2] a *Boolean space with an internal monoid* was defined as a pair (X, M) consisting of a Boolean space X, a dense subspace M equipped with a monoid structure, and a *biaction* (=pair of compatible left and right actions) of M on X with continuous components extending the obvious biaction of M on itself. Here we use a small variation and simplification of this notion. Instead of imposing that the monoid is a dense subset of the space, we require a map from the monoid to the space with dense image.

For spaces X and Y, we write $[X, Y]$ for the set of continuous functions from X to Y. Note that $[X, X]$ comes with a monoid operation given by composition. Given a monoid M, we will denote by $r\colon M \to M^M$ and $l\colon M \to M^M$ the two maps induced from the monoid operation by currying, which correspond to the right, respectively left, action of M on itself.

Definition II.1. *A* Boolean space with an internal monoid*, or a* BiM*, is a tuple (X, M, h, ρ, λ), where X is a Boolean space, M is a monoid, $h\colon M \to X$, $\lambda\colon M \to [X, X]$ and $\rho\colon M \to [X, X]$ are functions such that h has a dense image and for all $m \in M$ the following diagrams commute in* Set.

$$\begin{array}{ccc} M & \xrightarrow{h} & X \\ {\scriptstyle r(m)}\downarrow & & \downarrow{\scriptstyle \rho(m)} \\ M & \xrightarrow{h} & X \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{h} & X \\ {\scriptstyle l(m)}\downarrow & & \downarrow{\scriptstyle \lambda(m)} \\ M & \xrightarrow{h} & X \end{array} \tag{3}$$

If no confusion arises, we write (X, M), or even just X, for the BiM (X, M, h, ρ, λ)*. A* morphism *between two* BiM*s (X, M) and (X', M') is a pair $(\tilde{\psi}, \psi)$ where $\tilde{\psi}\colon X \to X'$ is a continuous map and $\psi\colon M \to M'$ is a monoid morphism such that $\tilde{\psi}h = h'\psi$. Since the image of h is dense in X, given ψ, $\tilde{\psi}$ is uniquely determined if it exists. Accordingly, we will sometimes just write ψ to indicate the pair as well as each of its components. We denote the category of* BiM*s by* BiM.

Remark II.2. *From the above definition one can* deduce *that ρ and λ induce commuting right and left M-actions on X, so that h is an M-biaction morphism. Thus $(X, \mathsf{Im}(h))$ is a Boolean space with an internal monoid as defined in [2].*

The topology on X specifies the subsets of M that can be used for language recognition, a notion that we recall next.

Definition II.3. *Let A be a finite alphabet, and $L \in \mathcal{P}(A^*)$. A morphism of* BiM*s $(\tilde{\psi}, \psi)\colon (\beta(A^*), A^*) \to (X, M)$* recognises *the language L if there is a clopen $C \subseteq X$ such that $L = \psi^{-1}(h^{-1}(C))$, or, equivalently, $\tilde{\psi}^{-1}(C)$ is the clopen $\widehat{L}$ of $\beta(A^*)$ that corresponds via Stone duality to the language $L \in \mathcal{P}(A^*)$. Moreover, we say that the* BiM *(X, M) recognises the language L if there exists a* BiM *morphism $(\beta(A^*), A^*) \to (X, M)$ recognising L. Finally, if $\mathcal{B} \hookrightarrow \mathcal{P}(A^*)$ is a Boolean subalgebra, the* BiM *(X, M) is said to recognise $\mathcal{B}$ provided that it recognises each $L \in \mathcal{B}$.*

D. Monads and algebras

We assume familiarity with the basics of category theory including monads as a categorical approach to general algebra (for background see e.g. [17]).

Consider a monad (T, η, μ) on a category $\mathcal{C}$. Recall that an *Eilenberg-Moore algebra* for T (or T*-algebra*, for short) is a pair (X, h) where X is an object of $\mathcal{C}$ and $h\colon TX \to X$ is a morphism in $\mathcal{C}$ so that $h \circ \eta_X = \mathsf{id}_X$ and $h \circ Th = h \circ \mu_X$. A *morphism* of T-algebras $(X_1, h_1) \to (X_2, h_2)$ is a morphism $f\colon X_1 \to X_2$ in $\mathcal{C}$ satisfying $f \circ h_1 = h_2 \circ Tf$. Let $\mathcal{C}^T$ denote the category of T-algebras. When T is a monad on the category Set of sets and functions, categories of the form Set^T are, up to equivalence, precisely the varieties of (possibly infinite arity) algebras. This correspondence restricts to categories of Eilenberg-Moore algebras for *finitary* monads (=monads preserving filtered colimits) and varieties of algebras in types consisting of finite arity operations. A T-algebra (X, h) is said to be *finitely carried* provided X is finite. We write Set^T_f for the full subcategory of Set^T on the finitely carried objects. The forgetful functor $\mathsf{Set}^T \to \mathsf{Set}$ that sends (X, h) to X restricts to the finitely carried algebras, and gives rise to a functor $\mathsf{Set}^T_f \to \mathsf{Set}_f$.

In Section V-B we shall see how several logical quantifiers can be modelled by considering modules over a semiring and the appropriate profinite monad. Recall that a *semiring* is a tuple $(S, +, \cdot, 0, 1)$ such that $(S, +, 0)$ is a commutative monoid, $(S, \cdot, 1)$ is a monoid, the operation $\cdot$ distributes over $+$, and $0 \cdot s = 0 = s \cdot 0$ for all $s \in S$. If no confusion arises, we will denote the semiring by S only.

Example II.4. *A semiring S induces a functor $\mathcal{S}\colon \mathsf{Set} \to \mathsf{Set}$ which associates to a set X the set of all functions $X \to S$ with* finite support*, that is*

$$\mathcal{S}X := \{f\colon X \to S \mid f(x) = 0 \text{ for all but finitely many } x \in X\}.$$

If $\psi\colon X \to Y$ is any function, define $\mathcal{S}\psi\colon \mathcal{S}X \to \mathcal{S}Y$ as $f \mapsto (y \mapsto \sum_{\psi(x)=y} f(x))$. Any element $f \in \mathcal{S}X$ can be represented as a formal sum $\sum_{i=1}^{n} s_i x_i$, where $\{x_1, \ldots, x_n\}$ is the support of f and $s_i = f(x_i)$ for each i. The functor $\mathcal{S}$ is part of a monad $(\mathcal{S}, \eta, \mu)$ on Set*, called the* free S-semimodule monad*, whose unit is*

$$\eta_X\colon X \to \mathcal{S}X,\ \eta_X(x)(x') = 1 \textit{ if } x' = x \textit{ and } 0 \textit{ otherwise},$$

and whose multiplication is

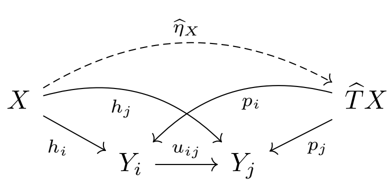
$$\mu_X\colon \mathcal{S}\mathcal{S}X \to \mathcal{S}X,\ \ \sum_{i=1}^{n} s_i f_i \mapsto \left(x \mapsto \sum_{i=1}^{n} s_i f_i(x)\right).$$

The category $\mathsf{Set}^{\mathcal{S}}$ is the category of modules over the semiring S. E.g., if S is the Boolean semiring 2 then $\mathcal{S} = \mathcal{P}_f$ (the finite powerset monad), whose Eilenberg-Moore algebras are join semilattices. If S is $\mathbb{N}$ or $\mathbb{Z}$, then the algebras for the monad $\mathcal{S}$ are, respectively, Abelian monoids and Abelian groups.

E. Profinite monads

Throughout this subsection we fix a monad T on Set. We begin by recalling the definition of the associated profinite monad $\widehat{T}$ on the category of Boolean spaces, following [11]. First we provide an intuitive idea of the construction, and then we give the formal definition.

Given a Boolean space X, one considers all continuous maps $h_i\colon X \to Y_i$ where the Y_i's are finite sets equipped with Eilenberg-Moore algebra structures $\alpha_i\colon TY_i \to Y_i$, as well as all the algebra morphisms $u_{ij}\colon Y_i \to Y_j$ satisfying $u_{ij} \circ h_i = h_j$. Equipping the finite sets Y_i with the discrete topology, one obtains a cofiltered diagram (or inverse limit system) $\mathcal{D}_X$ in BStone, and we set $\widehat{T}X$ to be its limit. It turns out that $\widehat{T}$ is the underlying functor of a monad $(\widehat{T}, \widehat{\eta}, \widehat{\mu})$ on BStone, called the *profinite monad* associated to T. For example, it is very easy to see how to obtain its unit $\widehat{\eta}$ from the universal property of the limit, as in the following diagram.



To give the formal definition of $\widehat{T}$, we define the functor $G\colon \mathsf{Set}_f^T \to \mathsf{BStone}$ as the composition of the forgetful functor to Set_f with the embedding of Set_f into BStone:

$$\mathsf{Set}_f^T \longrightarrow \mathsf{Set}_f \longrightarrow \mathsf{BStone}.$$

The shape of the diagram we constructed above for a Boolean space X is the comma category $X \downarrow G$ whose objects are essentially the maps $h_i\colon X \to G(Y_i, \alpha_i)$, and whose arrows are the maps u_{ij} as above. The diagram $\mathcal{D}_X$ is then given by the composition

$$X \downarrow G \xrightarrow{\ cod\ } \mathsf{Set}_f^T \xrightarrow{\ G\ } \mathsf{BStone}$$

of the codomain functor $X \downarrow G \to \mathsf{Set}_f^T$, mapping $h_i\colon X \to G(Y_i, \alpha_i)$ to (Y_i, α_i), and $G\colon \mathsf{Set}_f^T \to \mathsf{BStone}$. Formally, for an arbitrary Boolean space X, we have $\widehat{T}X := \lim \mathcal{D}_X$.

Notice that this is just the pointwise limit computation of the right Kan extension (see [17]) of G along itself, that is, using standard category-theoretic notation, $\widehat{T} = \mathrm{Ran}_G\, G$. It is well known (see for example [13]) that the right Kan extension of a functor G along itself, when it exists, is the functor part of a monad, called the codensity monad for G.

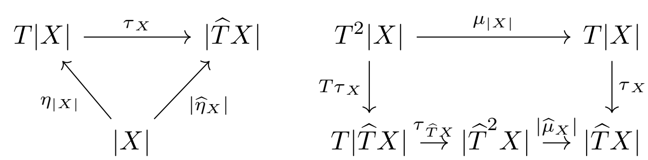
Example II.5. *The profinite monad of the finite powerset monad $\mathcal{P}_f$ on* Set *is the Vietoris monad $\mathcal{V}$ on* BStone *introduced in Section II-B. Clearly, the statement remains true if we replace $\mathcal{P}_f$ with the full powerset monad $\mathcal{P}$.*

The universal property of the right Kan extension, along with the fact that the underlying-set functor $|-|\colon \mathsf{BStone} \to \mathsf{Set}$ is a right adjoint and thus preserves right Kan extensions, allows one to define a natural transformation

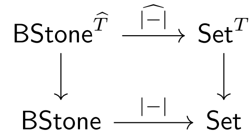
$$\tau_X\colon T|X| \to |\widehat{T}X| \tag{4}$$

which was also used in [11]. Here we give a presentation based on the limit computation of $\widehat{T}X$. Notice that the maps $|h_i|\colon |X| \to |Y_i|$ are functions into the carrier sets of the Eilenberg-Moore algebras $\alpha_i\colon TY_i \to Y_i$, and thus, by the universal property of the free algebra $T|X|$, we can extend the maps $|h_i|$ to algebra morphisms $h_i^{\#}$ from $T|X|$ to (Y_i, α_i). The functions $h_i^{\#}$ form a cone for the diagram $|-| \circ \mathcal{D}_X$ in Set whose limit is $|\widehat{T}X|$, by virtue of the fact that the forgetful functor $|-|\colon \mathsf{BStone} \to \mathsf{Set}$ preserves limits. By the universal property of the limit, this yields a unique map τ_X as in (4).

The natural transformation τ behaves well with respect to the units and multiplications of the monads T and $\widehat{T}$, in the sense that the next two diagrams commute, see [18, Proposition B.7]. Thus the pair $(|-|, \tau)$ is a *monad morphism*, or *monad functor* in the terminology of [19].



The fact that $(|-|, \tau)$ is a monad functor entails that the functor $|-|$ lifts to a functor $\widehat{|-|}$ between the categories of Eilenberg-Moore algebras for the monads $\widehat{T}$ and T, as below.



(5)

It follows at once that the set $|\widehat{T}X|$ admits a T-algebra structure, a result also used in [18] for finite algebras. This structure is essentially the one obtained by applying the functor $\widehat{|-|}$ to the free $\widehat{T}$-algebra $(\widehat{T}X, \widehat{\mu}_X)$. In more detail,

Lemma II.6. *Given a Boolean space X, the composite map*

$$T|\widehat{T}X| \xrightarrow{\tau_{\widehat{T}X}} |\widehat{T}^2X| \xrightarrow{|\widehat{\mu}_X|} |\widehat{T}X|$$

is a T-algebra structure on $|\widehat{T}X|$. With respect to this structure, $\tau_X : T|X| \to |\widehat{T}X|$ is morphism of T-algebras.

While in the proofs it is essential to keep track of the forgetful functor, we will sometimes omit it in what follows, and simply write $\tau_X : TX \to \widehat{T}X$. We will need some further properties of τ: for example we extend the result of [18, Proposition B.7(b)] when X is not finite and T is a Set-monad.

Lemma II.7. *For every Boolean space X, the image of the map $\tau_X : TX \to \widehat{T}X$ is dense in $\widehat{T}X$.*

Remark II.8. *For an arbitrary monad T, the components of the natural transformation τ do not have to be injective. However, that is the case if T is finitary and restricts to finite sets, e.g. if T is the finite powerset monad $\mathcal{P}_f$ on* Set.

We conclude with a consequence of Lemma II.7 that will be needed for the main result of the next section.

Corollary II.9. *For every Boolean space X and for every function $h\colon M \to X$ with dense image, the composite function $TM \xrightarrow{Th} TX \xrightarrow{\tau_X} \widehat{T}X$ has dense image.*

III. Extending Set monads to BiMs

Let us fix a monad T on Set. The main result of the present section is Theorem III.1 which provides sufficient conditions for T to be extended in a canonical way to the category BiM. In Section II-E we have seen a canonical way of extending any Set-monad T to Boolean spaces. We will consider next ways of lifting T to the category of monoids.

It is well known that there are two "canonical" natural transformations of bifunctors $\otimes, \otimes' \colon TX \times TY \to T(X \times Y)$, defined intuitively as follows. Think of elements in TX as terms $t(x_1, \ldots, x_n)$. Then $t(x_1, \ldots, x_n) \otimes s(y_1, \ldots, y_m)$ is defined as

$$t(s((x_1, y_1), \ldots, (x_1, y_m)), \ldots, s((x_n, y_1), \ldots, (x_n, y_m))),$$

whereas $t(x_1, \ldots, x_n) \otimes' s(y_1, \ldots, y_m)$ is defined as

$$s(t((x_1, y_1), \ldots, (x_n, y_1)), \ldots, t((x_1, y_m), \ldots, (x_n, y_m))).$$

In general $\otimes$ and $\otimes'$ do not coincide, and when they do the monad is called *commutative*, a notion due to Kock [20].

Given a monoid $(M, \cdot, 1)$, one has two possibly different "canonical" ways of defining a binary operation on TM, obtained as either of the two composites

$$TM \times TM \underset{\otimes'}{\overset{\otimes}{\rightrightarrows}} T(M \times M) \xrightarrow{T(\cdot)} TM.$$

If $e\colon 1 \to M$ denotes the map selecting the unit of the monoid, we can also define a map $1 \to TM$ obtained as the composite $Te \circ \eta_1$. That these data (with either of the two binary operations) give rise to monoid structures on TM is a consequence of [20, Theorem 2.1], where the level of generality is beyond the purposes of our present paper. In the next theorem we shall assume that the monad T is commutative, thus the two monoid structures on TM coincide.

Theorem III.1. *Any finitary commutative* Set*-monad T can be extended to a monad on* BiM *mapping (X, M, h, ρ, λ) to a* BiM $(\widehat{T}X, TM, \widehat{h}, \widehat{\rho}, \widehat{\lambda})$.

Proof Sketch. We define the function $\widehat{h}\colon TM \to \widehat{T}X$ as the composite $TM \xrightarrow{Th} TX \xrightarrow{\tau_X} \widehat{T}X$. By Corollary II.9, this function has dense image.

Consider the composite of the following two maps, where $\widehat{T}_{X,X}$ is given by the application of the functor $\widehat{T}$ to a continuous function in $[X, X]$,

$$M \xrightarrow{\rho} [X, X] \xrightarrow{\widehat{T}_{X,X}} [\widehat{T}X, \widehat{T}X]. \tag{6}$$

We notice that $[\widehat{T}X, \widehat{T}X]$ is a T-algebra (for a proof see [14, Lemma 3.2]), hence one can uniquely extend the map in (6) to a T-algebra morphism $\widehat{\rho}\colon TM \to [\widehat{T}X, \widehat{T}X]$. The function $\widehat{\lambda}$ is defined similarly, as the unique T-algebra morphism extending $\widehat{T}_{X,X} \circ \lambda$. It remains to prove that the functions $\widehat{h}$, $\widehat{\rho}$ and $\widehat{\lambda}$ make the diagrams in Definition II.1 commute. Equivalently, we have to prove that the next square and the analoguous one (with $\widehat{\rho}$ replaced by $\widehat{\lambda}$, and $\widehat{r}$ by $\widehat{l}$) commute,

$$\begin{array}{ccc} [\widehat{T}X, \widehat{T}X] & \xrightarrow{-\circ \widehat{h}} & \widehat{T}X^{TM} \\ {\scriptstyle \widehat{\rho}}\uparrow & & \uparrow{\scriptstyle \widehat{h}\circ -} \\ TM & \xrightarrow{\widehat{r}} & TM^{TM} \end{array} \tag{7}$$

where $\widehat{r}$ and $\widehat{l}$ denote the right and left action, respectively, of TM on itself. To this end, notice that the following diagram commutes. The two trapezoids are easily seen to be commutative using the definition of $\widehat{h}$, whereas the inner square is a reformulation of the left commuting square in (3).

$$\begin{array}{ccccc} [\widehat{T}X, \widehat{T}X] & & \xrightarrow{\quad -\circ \widehat{h} \quad} & & \widehat{T}X^{TM} \\ {\scriptstyle \widehat{T}_{X,X}}\uparrow & & & \nearrow{\scriptstyle \tau_X \circ T-} & \\ [X, X] & \xrightarrow{-\circ h} & X^M & & \uparrow{\scriptstyle \widehat{h}\circ -} \\ {\scriptstyle \rho}\uparrow & & \uparrow{\scriptstyle h\circ -} & & \\ M & \xrightarrow{r} & M^M & \xrightarrow{T_{M,M}} & TM^{TM} \end{array}$$

We derive the commutativity of (7) using the universal property of the free T-algebra on M and by observing that a) in the outer square above, the top horizontal and the right vertical arrows are morphisms of T-algebras; b) the map $\widehat{\rho}$ was defined as the unique extension of $\widehat{T}_{X,X} \circ \rho$ to the free algebra TM; and, c) the map $\widehat{r}$ is the unique algebra morphism extending $T_{M,M} \circ r$ to TM. It is now a straightforward computation to check that the assignment $(X, M) \mapsto (\widehat{T}X, TM)$ gives the functor part of a monad on the category of BiMs. ☐

Remark III.2. *The commutativity of the monad T is used to show that $(\widehat{T}X, TM)$ is a well defined* BiM*, and that we have indeed obtained a monad. Assume T is not commutative and* *we attempt to use in the above proof the monoid multiplication given by $\otimes$. All is fine for the right action, and indeed the right action $\widehat{r}$ of TM on itself is the unique extension of $T \circ r$. However, this is not the case for the left action.*

IV. Extending the free semimodule monad to BiMs

In Theorem III.1 we showed how to lift any finitary commutative monad on Set to a monad on BiM. The purpose of the present section is then twofold. On the one hand we provide an example of a family of Set-monads to which this result applies, and on the other hand we describe explicitly the various objects, maps and actions of the associated monads on BiM. This will be essential for our further work on recognisers.

Given a semiring S, recall from Example II.4 the free S-semimodule monad $\mathcal{S}$ on Set. Notice that $\mathcal{S}$ is a commutative monad if, and only if, S is a commutative semiring, i.e. the multiplication $\cdot$ is a commutative operation. Indeed, for a monoid M, the two monoid operations one can define on $\mathcal{S}M$ are given as follows. If $f, f' \in \mathcal{S}M$, then one can define $ff'(x)$ either by

$$\sum_{mm'=x} f(m) \cdot f'(m') \quad \text{or} \quad \sum_{m'm=x} f'(m') \cdot f(m),$$

and the two coincide precisely when the semiring is commutative. For this reason, for the rest of the paper we will only consider commutative semirings S. We also consider the associated Set-monad $\mathcal{S}$, along with the profinite monad $\widehat{\mathcal{S}}$ on BStone (cf. Section II-E).

Let X be a Boolean space, and denote by B its dual algebra. We shall see in Section IV-A that the Boolean space $\widehat{\mathcal{S}}X$ admits a description in terms of *measures* on X, provided S is commutative and finite. Some of these measures can be equivalently regarded as continuous functions from X to the semiring. This will be particularly important for BiMs, and in Section IV-B we identify the concrete nature of the algebraic structure for the lifting of the monads for finite commutative semirings.

Finally, when S is also idempotent, and thus a semilattice, we obtain a representation of $\widehat{\mathcal{S}}X$ as the set of all continuous functions from X into the lattice S equipped with the topology of all downsets. In particular, this description extends the well-known characterisation of the Vietoris construction (which is the codensity monad for $S = 2$, the two-element lattice) in terms of continuous functions into the Sierpiński space.

A. $\widehat{\mathcal{S}}X$ as the space of measures on X

Let us fix a finite and commutative semiring S.

Definition IV.1. *Let X be a Boolean space and B the dual algebra of clopens. An S-*valued measure *(or just a* measure *when the semiring is clear from the context) on X is a function $\mu \colon B \to S$ which is finitely additive, that is*

1) *$\mu(0) = 0$, and*
2) *$\mu(K \vee L) = \mu(K) + \mu(L)$ whenever $K, L \in B$ are disjoint.*

Note that in 1) the first 0 is the bottom of the Boolean algebra, while the second 0 is in S. Also, one can express 2) without reference to disjointness:

2)' *$\mu(K \vee L) + \mu(K \wedge L) = \mu(K) + \mu(L)$ for all $K, L \in B$.*

Notice that our notion of measure is not standard. We only require finite additivity. Also, the measure is only defined on the clopens of the space X. Finally, it takes values in a (finite and commutative) semiring.

We start by describing the Boolean algebra $\widehat{B}$ dual to $\widehat{\mathcal{S}}X$. For this purpose, the following notation is useful.

Notation IV.2. *Let X be a set and $f \colon X \to S$ a function. If $Y \subseteq X$ is a subset such that the sum $\sum_{x \in Y} f(x)$ exists in S, then we write*

$$\int_Y f := \sum_{x \in Y} f(x).$$

If $B \subseteq \mathcal{P}X$ is such that $\int_Y f$ exists for each $Y \in B$, then we denote by $\int f \colon B \to S$ the function taking Y to $\int_Y f$.

Assume now X is a Boolean space and $B \subseteq \mathcal{P}X$ is its dual Boolean algebra. We call $f \colon X \to S$ a density *on X provided $\int_Y f$ exists for each $Y \in B$. Then it is not hard to see that, when f is a density on X, $\int f \colon B \to S$ is a measure on X. In particular, each element of $\mathcal{S}X$ is a density on X.*

Lemma IV.3. *Suppose X is a Boolean space and B is its dual algebra. The Boolean algebra $\widehat{B}$ dual to $\widehat{\mathcal{S}}X$ is isomorphic to the subalgebra of $\mathcal{P}(\mathcal{S}X)$ generated by the elements of the form*

$$[L, k] := \{f \in \mathcal{S}X \mid \int_L f = k\},$$

for some $L \in B$ and $k \in S$.

Now, regarding the elements of $\widehat{\mathcal{S}}X$ as Boolean algebra homomorphisms $\widehat{B} \to 2$, define the function

$$\widehat{\mathcal{S}}X \longrightarrow \mathsf{Set}(B, S), \quad \varphi \mapsto \mu_\varphi \tag{8}$$

where μ_φ sends $L \in B$ to the unique $k \in S$ such that $\varphi[L, k] = 1$. Since the $[L, k]$'s generate $\widehat{B}$ this correspondence is injective. We observe that the μ_φ are measures on X: since $\int_0 f = 0$ for any f, one has $\mu_\varphi(0) = 0$. For the additivity, suppose $L_1, L_2 \in B$ satisfy $L_1 \wedge L_2 = 0$. Set $k_1 := \mu_\varphi(L_1)$ and $k_2 := \mu_\varphi(L_2)$. Then $[L_1, k_1] \cap [L_2, k_2] \subseteq [L_1 \cup L_2, k_1 + k_2]$, and therefore $\varphi[L_1, k_1] = 1 = \varphi[L_2, k_2]$ entails $\varphi[L_1 \cup L_2, k_1 + k_2] = 1$. We conclude $\mu_\varphi(L_1 \cup L_2) = k_1 + k_2 = \mu_\varphi(L_1) + \mu_\varphi(L_2)$.

On the other hand,

Lemma IV.4. *The image of the embedding* (8) *is exactly the set of all measures on X.*

Proof Sketch. Suppose μ is a measure on X, and define the set $F := \{[L, \mu(L)] \mid L \in B\} \subseteq \mathcal{P}(\mathcal{S}X)$. The empty set does not belong to F and, using additivity, one can show that F generates a proper filter. Further, this filter is an ultrafilter since, for each $[L, k]$, either $\mu(L) = k$ so that $[L, k] \in F$, or $\mu(L) \neq k$ and then $[L, \mu(L)] \subseteq [L, k]^c$, showing that $[L, k]^c$ is in the filter generated by F. Hence $\mu = \mu_\varphi$ where φ is the characteristic function of the filter generated by F. □

Notation IV.5. *Let $L \in B$ and $k \in S$. Notice that under the bijection between $\widehat{\mathcal{S}}X$ and the set of all measures on X, the* *clopen corresponding to $[L,k]$ is sent to the set of all measures μ such that $\mu(L)=k$, which we henceforth denote by $\widehat{[L,k]}$.*

Thus we have proved the following

Theorem IV.6. *Let S be a finite and commutative semiring. For any Boolean space X, $\widehat{\mathcal{S}}X$ is homeomorphic to the space of S-valued measures on X equipped with the topology generated by the sets*

$$\widehat{[L,k]} := \{\mu \in \widehat{\mathcal{S}}X \mid \mu(L)=k\},$$

where L is a clopen of X and $k \in S$.

Henceforth we identify $\widehat{\mathcal{S}}X$ with its homeomorphic image. We conclude this subsection with the following observation which corresponds to Lemma II.7 (cf. also Remark II.8).

Proposition IV.7. *If X is a Boolean space with dual algebra B, then*

$$\tau_X \colon \mathcal{S}X \to \widehat{\mathcal{S}}X,\ f \mapsto \int f$$

embeds $\mathcal{S}X$ in $\widehat{\mathcal{S}}X$ as a dense subspace. Further, $\widehat{[L,k]}$ is the topological closure of $[L,k]$ for any clopen $L \in B$ and $k \in S$.

B. The algebraic structure on $\widehat{\mathcal{S}}X$

Like in the previous subsection, we work with a fixed finite and commutative semiring S. As follows by the general results in Sections II-E and III, respectively, $\widehat{\mathcal{S}}X$ is a module over the semiring S and it is a Boolean space with an internal monoid. Here we identify the concrete nature of this structure relative to the incarnation of $\widehat{\mathcal{S}}X$ as the space of measures on X.

Lemma IV.8. *Let X be a Boolean space and $\mu,\nu\in\widehat{\mathcal{S}}X$. Then*

$$\mu+\nu \colon K \mapsto \mu(K)+\nu(K)$$

is again a measure on X and the ensuing binary operation on $\widehat{\mathcal{S}}X$ is continuous. Further, for $k \in S$

$$k\mu \colon K \mapsto k\cdot\mu(K)$$

is again a measure on X and the ensuing unary operation on $\widehat{\mathcal{S}}X$ is continuous.

This accounts for the S-semimodule structure on $\widehat{\mathcal{S}}X$. Now assume that X is not just a Boolean space but a BiM. To improve readability, we assume $h\colon M \to X$ is injective and identify M with its image. Firstly, we observe that $\mathcal{S}M$ sits as a dense subspace of $\widehat{\mathcal{S}}X$ by composing the map $\mathcal{S}h\colon \mathcal{S}M \to \mathcal{S}X$ with the integration map of Proposition IV.7. This is an instance of Corollary II.9 in the case of the monad $\mathcal{S}$.

Lemma IV.9. *Let (X,M) be a* BiM. *Then*

$$\mathcal{S}M \to \widehat{\mathcal{S}}X,\ f \mapsto \int f$$

is the map $\widehat{h}$ (cf. Theorem III.1) transporting $\mathcal{S}M$ into a dense subspace of $\widehat{\mathcal{S}}X$.

To exhibit the BiM structure of $\widehat{\mathcal{S}}X$, we start by identifying the actions of M on $\widehat{\mathcal{S}}X$.

Lemma IV.10. *Suppose (X,M) is a* BiM, *$\mu \in \widehat{\mathcal{S}}X$ and $m \in M$. Then*

$$m\mu \colon K \mapsto \mu(m^{-1}K),$$

where $m^{-1}K = \{x \in X \mid mx \in K\}$ whenever $K \subseteq X$ is clopen, is again a measure on X. This defines a left action of M on $\widehat{\mathcal{S}}X$ with continuous components. Similarly,

$$\mu m \colon K \mapsto \mu(Km^{-1})$$

defines a right action of M on $\widehat{\mathcal{S}}X$ with continuous components, and these actions are compatible in the sense that $(m\mu)n = m(\mu n)$.

Using the S-semimodule structure of $\widehat{\mathcal{S}}X$ along with the biaction of M on $\widehat{\mathcal{S}}X$, we obtain the biaction of $\mathcal{S}M$ on $\widehat{\mathcal{S}}X$.

Lemma IV.11. *Let (X,M) be a* BiM. *The map*

$$\mathcal{S}M \times \widehat{\mathcal{S}}X \to \widehat{\mathcal{S}}X,\ (f,\mu) \mapsto f\mu := \sum_{m\in M} f(m)\cdot m\mu$$

is a left action of $\mathcal{S}M$ on $\widehat{\mathcal{S}}X$ with continuous components. A right action with continuous components may be defined similarly. Finally, the two actions are compatible and provide the BiM *structure on $\widehat{\mathcal{S}}X$.*

Finally, we consider a restriction of the above action of $\mathcal{S}M$ on $\widehat{\mathcal{S}}X$ which we will need for the construction of the space $\Diamond X$ in Section V. This is given by precomposing with the unit of the monad $\widehat{\mathcal{S}}$:

$$\widehat{\eta}_X \colon X \to \widehat{\mathcal{S}}X,\ x \mapsto \mu_x$$

where $\mu_x = \int \chi_x$ and χ_x is the characteristic function of $\{x\}$ into S. That is, $\mu_x(K) = 1$ if, and only if, $x \in K$ and it is 0 otherwise. It is immediate that this map embeds X as a (closed) subspace of $\widehat{\mathcal{S}}X$. Thus we obtain an "action"

$$\mathcal{S}M \times X \to \widehat{\mathcal{S}}X,\ (f,x) \mapsto f\mu_x.$$

Lemma IV.12. *Let (X,M) be a Boolean space with an internal monoid. Consider the map*

$$\mathcal{S}M \times X \to \mathcal{S}X,\ (f,x) \mapsto fx,$$

where $fx(y) := \sum_{mx=y} f(m)$. Then we have

$$f\mu_x = \int fx.$$

Also, for each $f\in\mathcal{S}M$, the assignment $x \mapsto \int fx$ is continuous.

C. The idempotent case and a generalisation of Vietoris

In this subsection we assume S is a finite, commutative, and *idempotent* semiring. Under these assumptions, the S-valued measures on a Boolean space X can be equivalently described as the functions $X \to S$ which are continuous with respect to the appropriate topology on S. As an idempotent semiring, S is equipped with a natural partial order defined by $x \le y$ if, and only if, $x+y=y$. Denote by $S^{\downarrow}$ the set S equipped with the topology of all downsets of the poset $(S,\le)$ (this is the Alexandrov topology of the reverse order), and equip the set $[X, S^{\downarrow}]$ of all the continuous functions $X \to S^{\downarrow}$ with the topology whose subbasic clopens are of the form

$$\{f \in [X, S] \mid \int_L f = k\},$$

for some clopen subset L of X and $k \in S$. Then one can prove the following theorem.

Theorem IV.13. *For any finite, commutative, and idempotent semiring S and any Boolean space X, $\widehat{S}X$ is homeomorphic to the space $[X, S^{\downarrow}]$ of continuous $S^{\downarrow}$-valued functions on X.*

Example IV.14. *Notice that, in the case of the Boolean semiring S=2, Theorem IV.13 yields the well-known representation of the Vietoris space $\mathcal{V}X$ as the space of continuous functions from X to the Sierpiński space with the 'hit-or-miss' topology.*

V. Recognisers and S-valued transductions

In this section we show how to use the extension of a Set-monad T to BiMs from Section III to generate recognisers for languages obtained by applying an operation modelled by the monad T, specifically by a map $R\colon A^* \to T(B^*)$. If T is the powerset monad, then R is just a transduction, and it is a standard result that transductions can be used to model operations on languages, see [21]. In Section V-A we present the blueprint of our approach, using the additional assumption that the map R is a monoid morphism. In Section V-B we instantiate T to the free S-semimodule monads for commutative semirings S, we adapt the general theory developed previously to the case when R is not a monoid morphism, and we see how existential and modular quantifiers fit into this framework.

A. Recognising operations modelled by a monad T

Let T be an arbitrary commutative and finitary monad on Set, and let A, B be finite sets. We start by observing that a map $R\colon A^* \to T(B^*)$ could be used to transform languages in the alphabet B into languages in the alphabet A. Assume that $L = \phi^{-1}(P)$ for some monoid morphism $\phi\colon B^* \to M$ and some $P \subseteq M$. We consider the function $A^* \xrightarrow{R} TB^* \xrightarrow{T\phi} TM$. Since T is a commutative monad, we know that it lifts to the category of monoids, and hence $T\phi$ is a monoid morphism. If R is also a monoid morphism, and we will assume this only in this subsection, then so is $T\phi \circ R$, and it could be used for language recognition in the standard way. Assuming that we have a way of turning the recognising sets in M into recognising sets in TM, i.e., that we have a predicate transformer $\mathcal{P}M \to \mathcal{P}TM$ mapping P to $\widetilde{P}$, we obtain a language $\widetilde{L}$ in A^* as the preimage of $\widetilde{P}$ under the morphism $T\phi \circ R$.

Given a BiM (X, M, h, ρ, λ), we consider languages recognised by a BiM morphism

$$(\tilde{\phi}, \phi)\colon (\beta(B^*), B^*) \to (X, M)\,. \tag{9}$$

By Theorem III.1, we know that $(\widehat{T}X, TM)$ is a BiM and we use it for recognising A-languages, by constructing another BiM morphism $(\beta(A^*), A^*) \to (\widehat{T}X, TM)$. To proceed in a systematic manner, notice that the maps of the form $R\colon A^* \to T(B^*)$ correspond to morphisms between the free algebras $T(A^*)$ and $T(B^*)$. Formally, R is a morphism from A^* to B^* in the Kleisli category of T, see [17]. In Lemma V.1, we need to "lift" the map $R\colon A^* \to T(B^*)$ to a Kleisli map for the monad $\widehat{T}$ from $\beta(A^*)$ to $\beta(B^*)$. To this end, akin to the lifting (5) of the functor $|-|$ to the Eilenberg-Moore categories, its left adjoint $\beta\colon \mathsf{Set} \to \mathsf{BStone}$ can be lifted to a functor $\widehat{\beta}$ between the Kleisli categories Set_T and $\mathsf{BStone}_{\widehat{T}}$ for the monads T, respectively $\widehat{T}$. In the next commutative diagram, the vertical functors are the free algebra functors to the Kleisli categories.

$$\begin{array}{ccc} \mathsf{Set}_T & \xrightarrow{\widehat{\beta}} & \mathsf{BStone}_{\widehat{T}} \\ \uparrow & & \uparrow \\ \mathsf{Set} & \xrightarrow{\beta} & \mathsf{BStone} \end{array} \tag{10}$$

By [19] such a lifting corresponds to a natural transformation $\tau^{\#}\colon \beta T \to \widehat{T}\beta$ that behaves well with respect to the units and multiplications of the monads – called a *monad opfunctor* in [19]. The natural transformation $\tau^{\#}$ can be obtained from τ using the unit ι and counit ϵ of the adjunction $\beta \dashv |-|$:

$$\beta T \xrightarrow{\beta T \iota} \beta T|-|\beta \xrightarrow{\beta \tau \beta} \beta|-|\widehat{T}\beta \xrightarrow{\epsilon \widehat{T}\beta} \widehat{T}\beta\,.$$

Lemma V.1. *If the pair* $(\tilde{\phi}, \phi)$ *from* (9) *is a morphism of* BiM*s, then so is the pair $(\widehat{T}\tilde{\phi} \circ \widehat{R}, T\phi \circ R)$ described in the next diagram, where $\widehat{R}$ is the $\widehat{T}$-Kleisli map obtained by applying the functor $\widehat{\beta}$ of* (10) *to R.*

$$\begin{array}{ccccc} \beta(A^*) & \xrightarrow{\widehat{R}} & \widehat{T}\beta(B^*) & \xrightarrow{\widehat{T}\tilde{\phi}} & \widehat{T}X \\ \uparrow & & & & \uparrow{\scriptstyle \tau_X \circ Th} \\ A^* & \xrightarrow{R} & T(B^*) & \xrightarrow{T\phi} & TM \end{array}$$

B. Recognising quantified languages via S-transductions

Here we show how to construct BiMs recognising quantified languages. We point out that the contents of this subsection could be easily adapted to arbitrary Kleisli maps for the monads of the form $\widehat{\mathcal{S}}$ for a commutative semiring S.

We start with a language L in the extended alphabet $(A \times 2)^*$ which is recognised by the next BiM morphism.

$$\begin{array}{ccc} \beta((A \times 2)^*) & \xrightarrow{\tilde{\phi}} & X \\ \uparrow & & \uparrow{\scriptstyle h} \\ (A \times 2)^* & \xrightarrow{\phi} & M \end{array}$$

In other words, there exists a clopen C in X such that $L = \phi^{-1}(h^{-1}(C))$. The aim of this subsection is to construct recognisers for the quantified languages $L_{\exists}$ and $L_{\exists_{p \bmod q}}$, as defined in Section II-A. To this end, using the formal sum notation in the definition of the monad $\mathcal{S}$, we consider the map $R\colon A^* \to \mathcal{S}((A \times 2)^*)$ defined by

$$w \mapsto \sum_{i=1}^{|w|} 1_S \cdot w^{(i)}.$$

If S is the Boolean semiring 2, then R associates to a word w the set of all words in $(A \times 2)^*$ with the same shape as w

and with exactly one marked letter. The framework developed in Section V-A does not immediately apply, since R is not a monoid morphism. The first step is to obtain a monoid morphism from R, which will then be used to construct BiM recognisers for quantified languages. Upon viewing R as an S-transduction (see [22]), we observe that it is realised by the rational S-transducer $\mathcal{T}_R$ pictured in Figure 1, in which we have drawn transition maps only for a generic letter $a \in A$.

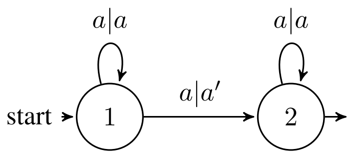


Fig. 1. The S-transducer $\mathcal{T}_R$ realising R. All the transitions have weights 1_S, and thus the transducer outputs value 1_S for all pairs of the form $(w, w^{(i)})$, with $w \in A^*$ and $1 \le i \le |w|$.

This transducer provides the following representation of R in terms of a monoid morphism

$$\mathsf{R}\colon A^* \to \mathcal{M}_2(\mathcal{S}((A \times 2)^*)), \tag{11}$$

where $\mathcal{M}_n(\mathcal{S}((A \times 2)^*))$ denotes the set of $n \times n$-matrices over the semiring $\mathcal{S}((A \times 2)^*)$. For a word $w \in A^*$, the matrix $\mathsf{R}(w)$ has at position (i, j) the formal sum of output words obtained from the transducer $\mathcal{T}_R$ by going from state i to state j while reading input word w. That is, R is given by

$$w \mapsto \begin{pmatrix} 1_S \cdot w^0 & \sum_i 1_S \cdot w^{(i)} \\ 0_S & 1_S \cdot w^0 \end{pmatrix}.$$

The next two examples provide the motivation for considering the particular transduction R in the first place.

Example V.2. *Assume S is the Boolean semiring 2, thus $\mathcal{S} = \mathcal{P}_f$, and $R(w) = \{w^{(i)} \mid 1 \le i \le |w|\}$. The language $L_\exists \subseteq A^*$ is recognised by the following composite monoid morphism, that will be denoted by $\phi_\exists$.*

$$A^* \xrightarrow{\ \mathsf{R}\ } \mathcal{M}_2(\mathcal{P}_f((A \times 2)^*)) \xrightarrow{\ \mathcal{M}_2(\mathcal{P}_f\phi)\ } \mathcal{M}_2(\mathcal{P}_f M)$$

Indeed, if $L = \phi^{-1}(P)$ for $P \subseteq M$, then $L_\exists = \phi_\exists^{-1}(\widetilde{P})$, where $\widetilde{P}$ is the set of matrices in $\mathcal{M}_2(\mathcal{P}_f M)$ such that the finite set in position $(1, 2)$ intersects P.

Example V.3. *Assume S is the semiring $\mathbb{Z}_q$. The language $L_{\exists_{p \bmod q}} \subseteq A^*$ is recognised by the following composite monoid morphism, that will be denoted by $\phi_{\exists_{p \bmod q}}$.*

$$A^* \xrightarrow{\ \mathsf{R}\ } \mathcal{M}_2(\mathcal{S}((A \times 2)^*)) \xrightarrow{\ \mathcal{M}_2(\mathcal{S}\phi)\ } \mathcal{M}_2(\mathcal{S}M)$$

Indeed, if $L{=}\phi^{-1}(P)$ for $P \subseteq M$, then $L_{\exists_{p \bmod q}}{=}\phi^{-1}_{\exists_{p \bmod q}}(\widetilde{P})$, where $\widetilde{P}$ is the set of matrices in $\mathcal{M}_2(\mathcal{S}M)$ such that the finitely supported function $f\colon \mathbb{Z}_q \to M$ in position $(1, 2)$ has the property that $\int_P f = p$ in $\mathbb{Z}_q$.

In view of Theorem III.1, we know that whenever (X, M) is a BiM, then so is $(\widehat{\mathcal{S}}X, \mathcal{S}M)$ with the actions of the internal monoid as in Lemma IV.11. Using this fact, as an intermediate step, we can prove the following general lemma.

Lemma V.4. *If (X, M) is a Boolean space with an internal monoid, then so is $(\mathcal{M}_n(\widehat{\mathcal{S}}X), \mathcal{M}_n(\mathcal{S}M))$ for any $n \ge 1$.*

Proof Sketch. Notice that $\mathcal{M}_n(\widehat{\mathcal{S}}X)$ is a Boolean space with respect to the product topology of $n{\times}n$ copies of $\widehat{\mathcal{S}}X$. The actions of $\mathcal{M}_n(\mathcal{S}M)$ on $\mathcal{M}_n(\widehat{\mathcal{S}}X)$ are given using the actions of $\mathcal{S}M$ on $\widehat{\mathcal{S}}X$ via matrix multiplication, and the S-semimodule structure of $\widehat{\mathcal{S}}X$. E.g., the left action of $(f_{ij})_{i,j}{\in}\mathcal{M}_n(\mathcal{S}M)$ on $(\mu_{ij})_{i,j}{\in}\mathcal{M}_n(\widehat{\mathcal{S}}X)$ yields a matrix of measures in $\widehat{\mathcal{S}}X$ having at position (i, j) the measure $\sum_{k=1}^{n} f_{ik}\mu_{kj}$. ☐

An immediate consequence of the next lemma is that the monoid morphisms $\phi_\exists$ and $\phi_{\exists_{p \bmod q}}$ constructed in Examples V.2 and V.3 can be extended to BiM morphisms recognising $L_\exists$ and $L_{\exists_{p \bmod q}}$, respectively.

Lemma V.5. *If the pair $(\tilde{\phi}, \phi)$ from (9) is a morphism of* BiM*s and $\mathsf{R}\colon A^* \to \mathcal{M}_n(\mathcal{S}(B^*))$ is a monoid morphism, then so is the pair $(\mathcal{M}_n(\widehat{T}\tilde{\phi}) \circ \widehat{\mathsf{R}}, \mathcal{M}_n(T\phi) \circ \mathsf{R})$ described in the next diagram,*

$$\begin{array}{ccccc} \beta(A^*) & \xrightarrow{\ \widehat{\mathsf{R}}\ } & \mathcal{M}_n(\widehat{\mathcal{S}}\beta(B^*)) & \xrightarrow{\ \mathcal{M}_n(\widehat{\mathcal{S}}\tilde{\phi})\ } & \mathcal{M}_n(\widehat{\mathcal{S}}X) \\ \uparrow & & & & \uparrow{\scriptstyle \mathcal{M}_n(\tau_X \circ \mathcal{S}h)} \\ A^* & \xrightarrow{\ \mathsf{R}\ } & \mathcal{M}_n(\mathcal{S}(B^*)) & \xrightarrow{\ \mathcal{M}_n(\mathcal{S}\phi)\ } & \mathcal{M}_n(\mathcal{S}M) \end{array}$$

where $\widehat{\mathsf{R}}$ is the map obtained as follows:

$$A^* \xrightarrow{\ \mathsf{R}\ } \mathcal{M}_n(\mathcal{S}(B^*)) \xrightarrow{\ \mathcal{M}_n(\eta)\ } \mathcal{M}_n(\beta\mathcal{S}(B^*)) \xrightarrow{\ \mathcal{M}_n(\tau^\#)\ } \mathcal{M}_n(\widehat{\mathcal{S}}\beta(B^*)).$$

When we apply this lemma to the monoid morphism R of (11) we obtain the BiM $(\mathcal{M}_2(\widehat{\mathcal{S}}X), \mathcal{M}_2(\mathcal{S}M))$ which, when instantiated with the appropriate semiring S, recognises e.g. the quantified languages $L_\exists$ and $L_{\exists_{p \bmod q}}$.

For instance, suppose $S = \mathbb{Z}_q$. If L is recognised by a clopen $C \subseteq X$ then, upon recalling that clopens of $\widehat{\mathcal{S}}X$ are of the form $\widehat{[K, k]}$ for K a clopen of X and $k \in S$, one can easily prove that the quantified language $L_{\exists_{p \bmod q}}$ is recognised by the clopen subset of $\mathcal{M}_2(\widehat{\mathcal{S}}X)$ given by the product $\widehat{\mathcal{S}}X \times \widehat{[C, p]} \times \widehat{\mathcal{S}}X \times \widehat{\mathcal{S}}X$, where the elements of the clopen $\widehat{[C, p]}$ should appear in position $(1, 2)$ in the matrix view of the space.

However, notice that the image of the morphism $\mathcal{M}_2(\widehat{\mathcal{S}}\tilde{\phi}) \circ \widehat{\mathsf{R}}$, denoted hereafter by ϕ_{R}, is contained in the subspace of $\mathcal{M}_2(\widehat{\mathcal{S}}X)$ which can be represented by the matrix

$$\begin{pmatrix} X & \widehat{\mathcal{S}}X \\ 0 & X \end{pmatrix}.$$

As a consequence, we can use for the same recognition purpose a smaller BiM, through which the morphism ϕ_{R} factors. We denote this BiM morphism by

$$\Diamond\phi\colon (\beta(A^*), A^*) \to (\Diamond X, \Diamond M),$$

where $\Diamond M := \mathcal{S}M \times M$ and $\Diamond X := \widehat{\mathcal{S}}X \times X$, with monoid structure and biactions defined by identifying the products above with upper triangular matrices, and then using the matrix multiplication and the concrete description of several monoid actions from Lemmas IV.10 and IV.12. Using the notations described in these lemmas, the left action can be described by

$$\begin{pmatrix} m & f \\ 0 & m \end{pmatrix} \begin{pmatrix} x & \mu \\ 0 & x \end{pmatrix} = \begin{pmatrix} mx & m\mu + \int fx \\ 0 & mx \end{pmatrix}. \tag{12}$$

Recall from Section II-A that the language $\mathcal{Q}_k(L) \subseteq A^*$ is obtained by quantifying $L \subseteq (A \times 2)^*$ with respect to the quantifier associated to a semiring S and $k \in S$. We summarize the preceding observations in the following theorem.

Theorem V.6. *Let S be a commutative semiring, and $k \in S$. Suppose a language $L \subseteq (A \times 2)^*$ is recognised by the* BiM *morphism $\phi\colon (\beta((A \times 2)^*), (A \times 2)^*) \to (X, M)$. Then the quantified language $\mathcal{Q}_k(L) \subseteq A^*$ is recognised by the* BiM *morphism $\Diamond\phi\colon (\beta(A^*), A^*) \to (\Diamond X, \Diamond M)$.*

Taking $S = 2$ the Boolean semiring and $k = 1$, we recover the result [2, Proposition 13] on existential quantification.

VI. Duality-theoretic account of the construction

Consider a finite and commutative semiring S and a BiM (X, M). As usual, let B be the dual algebra of X. Further, let $\phi\colon (\beta((A \times 2)^*), (A \times 2)^*) \to (X, M)$ be a BiM morphism. We denote by $\mathcal{B}$ the preimage under ϕ of B. That is, $\mathcal{B}$ is the Boolean algebra, closed under quotients in $\mathcal{P}((A \times 2)^*)$, of languages recognised by the BiM morphism ϕ.

In Section V-B we introduced the map $\Diamond\phi$ as a recogniser for the quantified languages obtained from the languages in $\mathcal{B}$. Here we will see that $\Diamond\phi$ is in fact the dual of a certain morphism of Boolean algebras with quotient operations whose image $\mathcal{B}'$ is generated as a Boolean algebra closed under quotients by the languages obtained either by quantification of languages from $\mathcal{B}$, or by direct recognition via the composition of ϕ with the embedding $(\ )^0\colon \beta(A^*) \to \beta((A \times 2)^*), w \mapsto w^0$.

Crucially, one can show that the actions of $\Diamond M$ on $\Diamond X$, given by matrix multiplication in Section V-B, arise by duality from the quotient operations on $\mathcal{B}'$.

Finally, we conclude with a Reutenauer-type theorem, showing that the Boolean algebra closed under quotients generated by the languages in A^* recognised by length preserving morphisms into $\Diamond X$ is precisely the Boolean algebra generated by those recognised by X directly, and those obtained by quantification from languages in $(A \times 2)^*$ recognised by X.

Notice that the continuous map

$$\phi_{\mathcal{Q}}\colon \beta(A^*) \xrightarrow{\widehat{R}} \widehat{\mathcal{S}}\beta((A \times 2)^*) \xrightarrow{\widehat{\mathcal{S}}\phi} \widehat{\mathcal{S}}X,$$

which is given for $w \in A^*$ by

$$\phi_{\mathcal{Q}}(w) = \int f_w \text{ where } f_w := \sum_{1 \le i \le |w|} 1_S \cdot \phi(w^{(i)}),$$

recognises the quantified languages $\mathcal{Q}_k(L)$ for $k \in S$ and $L \in \mathcal{B}$. In fact, the clopen in $\beta(A^*)$ corresponding to $\mathcal{Q}_k(L)$ is ${\phi_{\mathcal{Q}}}^{-1}(\widehat{[K,k]})$, if $K \subseteq X$ is the clopen recognising L via ϕ. By Theorem IV.6, the clopens of $\widehat{\mathcal{S}}X$ are generated by the sets of the form $\widehat{[K,k]}$ with $k \in S$ and $K \subseteq X$ clopen, therefore:

Proposition VI.1. *The Boolean algebra $\mathcal{QB}$ of languages over A recognised by $\phi_{\mathcal{Q}}$ is generated by the quantified languages $\mathcal{Q}_k(L)$ for $k \in S$ and $L \in \mathcal{B}$.*

Note that $\mathcal{QB}$ is *not* closed under quotients. This is the reason we had to make an adjustment between Sections V-A and V-B above.

We denote by $\mathcal{B}_0$ the Boolean algebra of languages closed under quotients which is recognised by the BiM morphism

$$\phi_0\colon (\beta(A^*), A^*) \xrightarrow{(\ )^0} (\beta(A \times 2)^*, (A \times 2)^*) \xrightarrow{\phi} (X, M).$$

Note that $\mathcal{B}_0$ consists of all languages $L_0 := {\phi_0}^{-1}(K)$ obtained as the preimage under $(\ )^0$ of languages $L = \phi^{-1}(K)$ in $\mathcal{B}$. Taking the product map, it now follows that

$$\Diamond\phi = \phi_{\mathcal{Q}} \times \phi_0\colon \beta(A^*) \to \widehat{\mathcal{S}}X \times X,$$

viewed just as a map of Boolean spaces, recognises the Boolean algebra $<\mathcal{QB} \cup \mathcal{B}_0>_{BA}$ generated by $\mathcal{QB} \cup \mathcal{B}_0$. However, since $\mathcal{QB}$ is *not* closed under quotients, a priori, neither is $<\mathcal{QB} \cup \mathcal{B}_0>_{BA}$.

The Boolean algebra $\mathcal{B}'$ that we are interested in, is the closure under quotients of $<\mathcal{QB} \cup \mathcal{B}_0>_{BA}$. The important observation is that $<\mathcal{QB} \cup \mathcal{B}_0>_{BA}$ *is already closed under the quotient operations*, thus explaining why $\widehat{\mathcal{S}}X \times X$, along with the above product map, is the right recogniser space-wise.

Proposition VI.2. *The Boolean algebra generated by $\mathcal{QB} \cup \mathcal{B}_0$ is closed under quotients. That is,*

$$\mathcal{B}' = <\mathcal{Q}_k(L), L_0 \mid L \in \mathcal{B} \text{ and } k \in S>_{BA}.$$

Corollary VI.3. *The dual space of $\mathcal{B}'$ is a closed subspace of $\widehat{\mathcal{S}}X \times X$. In particular, $\mathcal{B}'$ is recognised as a Boolean algebra by $\widehat{\mathcal{S}}X \times X$.*

Now, in order to understand why the actions on $\Diamond X$ are as in (12), we consider the homomorphism dual to $\Diamond\phi$:

$$\varphi\colon \widehat{B} + B \to \mathcal{P}(A^*), \widehat{[K,k]} \mapsto {\phi_{\mathcal{Q}}}^{-1}(\widehat{[K,k]}), K \mapsto {\phi_0}^{-1}(K).$$

Each internal monoid element $w \in A^*$ yields an internal monoid element $(f_w, \phi(w^0))$ in $\mathcal{S}M \times M$. We want to define an action of $(f_w, \phi(w^0))$ on $\widehat{B} + B$ so that φ becomes a homomorphism sending the action of $(f_w, \phi(w^0))$ to the action of the quotient operation $w^{-1}(\ )$ on $\mathcal{P}(A^*)$. The computation in the proof of Proposition VI.2 yields the components of the required action.

Proposition VI.4. *The map $\varphi\colon \widehat{B} + B \to \mathcal{P}(A^*)$ is a homomorphism of Boolean algebras with actions when we define the left quotient operation $\Lambda(f, m)$ of $\widehat{B} + B$ on $\widehat{B}$ by*

$$\Lambda(f, m)\colon \widehat{[K,k]} \mapsto \bigvee_{k_1 + k_2 = k} (\Lambda_{11}(\widehat{[K,k_1]}) \wedge \Lambda_{12}(\widehat{[K,k_2]}))$$

where $\Lambda_{11}(f, m)\colon \widehat{B} \to \widehat{B}$ is defined as $\widehat{[K,k]} \mapsto \widehat{[m^{-1}K,k]}$ and $\Lambda_{12}(f, m)\colon \widehat{B} \to B$ is given by

$$\widehat{[K,k]} \mapsto \bigcup_{\substack{I \subseteq \mathrm{Sup}(f) \\ \int_I f = k}} \left([\bigcap_{n \in I} n^{-1}K] \cap [\bigcap_{n \in I^c} n^{-1}K^c] \right),$$

and we define $\Lambda(f,m)$ *on* B *by* $\Lambda(f,m)\colon K \mapsto m^{-1}K$.

It is now an easy verification that the maps Λ_{11} and Λ_{12} are dual to the summands of the first component of the action of (f,m) on $\Diamond X$. As a consequence, we have

Theorem VI.5. *Let* $\phi\colon (\beta((A \times 2)^*), (A \times 2)^*) \to (X, M)$ *be a* BiM *morphism. Then the* BiM *morphism*

$$\Diamond\phi\colon (\beta(A^*), A^*) \to (\Diamond X, \Diamond M)$$

derived in Section V-B is dual to the homomorphism of Boolean algebras with biactions

$$\varphi\colon \widehat{B}+B \to \mathcal{P}(A^*),\ \widehat{[K,k]} \mapsto {\phi_{\mathcal{Q}}}^{-1}(\widehat{[K,k]}),\ K \mapsto {\phi_0}^{-1}(K)$$

when $\widehat{B}+B$ *is equipped with the biaction of Proposition VI.4.*

We conclude with a Reutenauer-type theorem which relies on an extension of the usual notion of length preserving monoid morphism.

Definition VI.6. *We call a* BiM *morphism* $\psi\colon \beta(A^*) \to \Diamond X$ length preserving *provided, for each* $a \in A$, $\pi_1 \circ \psi(a)\colon M \to S$ *is the characteristic function* χ_{m_a} *for some single* $m_a \in M$. *That is,* $\pi_1 \circ \psi(a)(m_a) = 1$ *and* $\pi_1 \circ \psi(a)(m) = 0$ *for all* $m \in M$ *with* $m \neq m_a$.

Recall that, given any BiM morphism $\phi\colon \beta((A \times 2)^*) \to X$, we obtain a BiM morphism

$$\Diamond\phi\colon \beta(A^*) \to \widehat{\mathcal{S}}X \times X, \quad w \mapsto (\int f_w, \phi(w^0)).$$

Note that $f_a := \pi_1 \circ \Diamond\phi(a) = \chi_{m_a}$ where $m_a = \phi(a,1)$, so that $\Diamond\phi$ is length preserving. It is now a matter of a straightforward computation to prove the following proposition.

Proposition VI.7. *Let* X *be a* BiM. *Every length preserving* BiM *morphism* $\beta(A^*) \to \Diamond X$ *is of the form* $\Diamond\phi$ *for some* $\phi\colon \beta((A \times 2)^*) \to X$.

We thus obtain the following Reutenauer-type theorem.

Theorem VI.8. *Let* X *be a* BiM *and let* A *be a finite alphabet. The Boolean subalgebra closed under quotients of* $\mathcal{P}(A^*)$ *generated by all languages over* A *which are recognised by a length preserving* BiM *morphism into* $\Diamond X$ *is generated as a Boolean algebra by the languages over* A *recognised by* X*, and the languages* $\mathcal{Q}_k(L)$ *for* L *a language over* $A \times 2$ *recognised by* X.

VII. Conclusion

In this paper we provide a general construction for recognisers which captures the action of quantifier-like operations on languages. We also contribute a new characterisation of the profinite monad of the free semimodule monad for finite commutative semirings.

This paper is a stepping stone towards finding ultrafilter equations for logically defined classes of non-regular languages. Thus the next step is to discover the effect on equations of the constructions introduced here. Furthermore, we remark that the approach of Section V may be extended to encompass a wider range of operations modelled by rational transducers which, by the Kleene-Schützenberger theorem (see e.g. [22]), admit a matrix representation.

Finally, it would be interesting to understand a common framework for our contributions and the recent work [23].

Acknowledgements: We would like to thank Jean-Éric Pin for suggesting the transducer approach, Thomas Colcombet for sharing his `knowledge` package, and the DuaLL team for helpful discussions. We would also like to thank the Simons Institute for the Theory of Computing for the opportunity to present this work.

References

[1] P. Tesson and D. Thérien, "Logic meets algebra: the case of regular languages," *Logical Methods in Computer Science*, vol. 3, no. 1, 2007.
[2] M. Gehrke, D. Petrişan, and L. Reggio, "The Schützenberger product for syntactic spaces," in *ICALP*, ser. LIPIcs, vol. 55. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2016, pp. 112:1–112:14.
[3] M. Gehrke and A. Krebs, "Stone duality for languages and complexity," *SIGLOG News*, vol. 4, no. 2, pp. 28–53, 2017.
[4] H. Straubing and D. Thérien, "Modular quantifiers," in *Logic and Automata*, ser. Texts in Logic and Games, vol. 2. Amsterdam University Press, 2008, pp. 613–628.
[5] N. Pippenger, "Regular languages and Stone duality," *Theory of Computing Systems*, vol. 30, no. 2, pp. 121–134, 1997.
[6] M. Gehrke, S. Grigorieff, and J. Pin, "A topological approach to recognition," in *ICALP (2)*, ser. Lecture Notes in Computer Science, vol. 6199. Springer, 2010, pp. 151–162.
[7] ——, "Duality and equational theory of regular languages," in *ICALP (2)*, ser. LNCS, vol. 5126. Springer, 2008, pp. 246–257.
[8] M. Bojańczyk, "Recognisable languages over monads," in *DLT*, ser. LNCS, vol. 9168. Springer, 2015, pp. 1–13.
[9] J. Adámek, R. Myers, H. Urbat, and S. Milius, "Varieties of languages in a category," in *LICS*. IEEE Computer Society, 2015, pp. 414–425.
[10] J. Almeida and P. Weil, "Profinite categories and semidirect products," *Journal of Pure and Applied Algebra*, vol. 123, no. 1, pp. 1 – 50, 1998.
[11] L. Chen, J. Adámek, S. Milius, and H. Urbat, "Profinite monads, profinite equations, and Reiterman's theorem," in *FoSSaCS*, ser. Lecture Notes in Computer Science, vol. 9634. Springer, 2016, pp. 531–547.
[12] J. Kennison and D. Gildenhuys, "Equational completion, model induced triples and pro-objects," *Journal of Pure and Applied Algebra*, vol. 1, no. 4, pp. 317 – 346, 1971.
[13] T. Leinster, "Codensity and the ultrafilter monad," *Theory and Applications of Categories*, pp. 332–370, 2013.
[14] M. Gehrke, D. Petrişan, and L. Reggio, "Quantifiers on languages and codensity monads," *CoRR*, vol. abs/1702.08841, 2017.
[15] M. H. Stone, "The theory of representations for Boolean algebras," *Trans. Amer. Math. Soc.*, vol. 40, no. 1, pp. 37–111, 1936.
[16] E. Michael, "Topologies on spaces of subsets," *Trans Amer. Math. Soc.*, vol. 71, pp. 152–182, 1951.
[17] S. Mac Lane, *Categories for the Working Mathematician*. New York, NY: Springer New York, 1978.
[18] L. Chen, J. Adámek, S. Milius, and H. Urbat, "Profinite monads, profinite equations and Reiterman's theorem," *CoRR*, vol. abs/1511.02147, 2015.
[19] R. Street, "The formal theory of monads," *Journal of Pure and Applied Algebra*, vol. 2, no. 2, pp. 149 – 168, 1972.
[20] A. Kock, "Monads on symmetric monoidal closed categories," *Journal of The Australian Mathematical Society*, vol. 21, pp. 1–10, 1970.
[21] J. Pin and J. Sakarovitch, "Some operations and transductions that preserve rationality," in *Proceedings of the 6th GI-Conference on Theor. Comput. Sci.* London, UK, UK: Springer-Verlag, 1982, pp. 277–288.
[22] J. Sakarovitch, *Elements of Automata Theory*. New York, NY, USA: Cambridge University Press, 2009.
[23] L. Chen and H. Urbat, "Schützenberger products in a category," in *DLT*, ser. LNCS, vol. 9840. Springer, 2016, pp. 89–101.