

Higher Interpolation and Extension for Persistence Modules*

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Abstract. The use of topological persistence in contemporary data analysis has provided considerable impetus for investigations into the geometric and functional-analytic structure of the space of persistence modules. In this paper, we isolate a coherence criterion which guarantees the extensibility of non-expansive maps into this space across embeddings of the domain to larger ambient metric spaces. Our coherence criterion is category-theoretic, allowing Kan extensions to provide the desired extensions. Our main construction gives an isometric embedding of a metric space into the metric space of persistence modules with values in the spacetime of this metric space. As a consequence of such “higher interpolation,” it becomes possible to compare Vietoris–Rips and Čech complexes built within the space of persistence modules.

Key words. persistent homology, Lipschitz extensions, Kan extensions

AMS subject classifications. 55N99, 46A22, 18A40

DOI. 10.1137/16M1100472

1. Introduction. The combination of rigorously developed foundations [6, 10], efficient computability [21, 25], and stability properties [8, 11] has resulted in the widespread adoption of *topological persistence* [15, 25] as a technique for the analysis of large and complex datasets [7, 16, 22]. The output of this process is a collection of *persistent homology groups*, which are typically represented via a barcode or a persistence diagram. Recent applications of persistence often confront dynamically evolving data [2, 17], and in these cases one requires the ability to make inferences about the dynamics from collections of persistence diagrams. Substantial efforts have been devoted to this end; among the best-known outcomes are *vineyards* [12], *Fréchet means* [24], and *persistence landscapes* [4].

In this work, we provide a new geometric lens with which to view the space of persistence diagrams. Our main result is in fact a statement about the space of (sufficiently tame) *persistence modules*—these consist of vector spaces and linear maps indexed by the real line \mathbb{R} , and their representation theory produces persistence diagrams [23]. The class **Mod** of persistence modules admits an *interleaving metric*, and the *interpolation lemma* from [8] establishes that **Mod** is a path metric space—two modules which are e -interleaved for $0 \leq e < \infty$ can always be connected by a path in **Mod** of length e . This lemma plays a key role in

*Received by the editors October 24, 2016; accepted for publication (in revised form) March 27, 2017; published electronically June 7, 2017.

<http://www.siam.org/journals/siaga/1/M110047.html>

Funding: The first author’s work was partially supported by the AFOSR grant FA9550-13-1-0115. The third author’s work was supported by The Alan Turing Institute under the EPSRC grant number EP/N510129/1.

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the proof of the *stability theorem*, which confirms that if two point-clouds are within Hausdorff distance e of each other, then (their persistence modules are e -interleaved, and hence) their persistence diagrams are also within *bottleneck distance* e of each other [8, 11].

The interpolation lemma provides an affirmative answer to the *Lipschitz extension problem* [3, Chap. 1] encoded in the following commutative diagram of metric spaces (and 1-Lipschitz maps):

$$(1.1) \quad \begin{array}{ccc} \{0, e\} & \xrightarrow{f} & \mathbf{Mod} \\ \downarrow & \nearrow f' & \\ [0, e] & & \end{array}$$

Here $\{0, e\}$ and $[0, e]$ are given the traditional Euclidean metric inherited from \mathbb{R} , and the fact that f is 1-Lipschitz follows immediately from our assumption that $f(0)$ and $f(e)$ are e -interleaved. The existence of an extension f' allows us to assign intermediate persistence modules $f'(x)$ to all x in $[0, e]$ so that f' agrees with f on the endpoints $\{0, e\}$ and the interleaving distance between $f'(x)$ and $f'(y)$ does not exceed $|x - y|$. Similarly, one seeks 1-Lipschitz extensions across more general choices of metric inclusions. Our objective here is to prescribe sufficient categorical conditions on f which guarantee the existence of such extensions. Here is a consequence of our main result.

Theorem 1.1 (higher interpolation and extension). *Let M be any metric space, and let A be a subspace. If a map $f : A \rightarrow \mathbf{Mod}$ is **coherent** (in the sense of Definition 4.1), then it admits three 1-Lipschitz extensions $M \rightarrow \mathbf{Mod}$.*

In order to precisely describe what it takes for $f : A \rightarrow \mathbf{Mod}$ to be coherent, we examine a pair of functors relating **Cat**, the usual category of small categories, and **Met**, the category of metric spaces with 1-Lipschitz maps as morphisms. Although our functors fail to form an adjoint pair in general, there is a distinguished natural transformation η from the identity functor on **Met** to their composite. Coherent maps are precisely those $f : A \rightarrow \mathbf{Mod}$ which factor through this natural transformation. The rest of this paper is organized as follows: In section 2 we use known facts about the metric space of persistence modules to describe a functor **Cat** \rightarrow **Met**, and in section 3 we describe a functor **Met** \rightarrow **Cat**. The proof of the higher interpolation theorem occupies section 4 and some of its consequences are explored in section 5.

2. The geometry of persistence modules. We assume that the reader has prior familiarity with the basics of category theory [1, 20]. We also adopt the following conventions throughout: given a category **C** we write \mathbf{C}_o for its class of objects and $\mathbf{C}(x, y)$ for its set of morphisms from an object x to an object y . For a small category **C**, we will denote the *category of functors* from **C** to **D** by $\mathbf{D}^{\mathbf{C}}$. Although we will survey some relevant definitions and results here, the reader is invited to consult [5, 6, 8, 10, 23] for detailed background material on the categorical and metric aspects of persistence modules.

2.1. Persistence modules as functors. Let **R** denote the category whose objects are the real numbers \mathbb{R} , and which admits a unique morphism $a \rightarrow b$ whenever $a \leq b$. Persistent

homology associates algebraic invariants to *filtered topological spaces*, which are naturally regarded as members of $\mathbf{Top}^{\mathbf{R}}$ —these are functors from \mathbf{R} to the category \mathbf{Top} of topological spaces and continuous maps. In practice, one also encounters filtered spaces indexed by proper subcategories of \mathbf{R} (typically finite sets $\mathbf{n} = \{0, 1, \dots, n\}$, natural numbers \mathbf{N} , or integers \mathbf{Z}). In such cases, a standard dictionary may be used to modulate between indexing subcategories: in the diagram

$$(2.1) \quad \begin{array}{ccccccc} \mathbf{n} & \longrightarrow & \mathbf{N} & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{R} \\ & & & & \searrow & \searrow & \searrow \\ & & & & & & \mathbf{Top} \end{array}$$

horizontal arrows are inclusions and pull-backs are given by restriction. Conversely, one may extend the following:

- $U : \mathbf{n} \rightarrow \mathbf{Top}$ to $U' : \mathbf{N} \rightarrow \mathbf{Top}$ by assigning $U'(k) = U(n)$ for $k > n$;
- $U' : \mathbf{N} \rightarrow \mathbf{Top}$ to $U'' : \mathbf{Z} \rightarrow \mathbf{Top}$ by assigning $U''(k) = \emptyset$ for $k < 0$;
- $U'' : \mathbf{Z} \rightarrow \mathbf{Top}$ to $U''' : \mathbf{R} \rightarrow \mathbf{Top}$ by assigning $U'''(a) = U''(\lfloor a \rfloor)$ for all $a \in \mathbb{R}$

(here $\lfloor \cdot \rfloor$ indicates the floor function). That such constructions are possible in each case depicted above is a pleasant consequence of the fact that \mathbf{Top} admits left Kan extensions.

Since we may pass from one of these functors to another, we will henceforth treat all filtered spaces as functors $\mathbf{R} \rightarrow \mathbf{Top}$. Letting \mathbf{Vect} denote the category of vector spaces and linear maps over a fixed underlying field, we note that any functor $H : \mathbf{Top} \rightarrow \mathbf{Vect}$ (such as singular homology) induces a push-forward from $\mathbf{Top}^{\mathbf{R}}$ to $\mathbf{Vect}^{\mathbf{R}}$ via postcomposition. The resulting structure is a persistence module.

Definition 2.1. *The category \mathbf{Mod} of persistence modules is $\mathbf{Vect}^{\mathbf{R}}$ —its objects are functors $U : \mathbf{R} \rightarrow \mathbf{Vect}$ and morphisms in $\mathbf{Mod}(U, V)$ are natural transformations $U \Rightarrow V$.*

The morphisms from U to V in \mathbf{Mod} admit a convenient pointwise description as collections of linear maps $\{\Phi(a) : U(a) \rightarrow V(a) \mid a \in \mathbb{R}\}$ which satisfy the following property. Across all choices of $a \leq b$ in \mathbb{R} , the following diagram commutes:

$$\begin{array}{ccc} U(a) & \xrightarrow{U(a \leq b)} & U(b) \\ \Phi(a) \downarrow & & \downarrow \Phi(b) \\ V(a) & \xrightarrow{V(a \leq b)} & V(b) \end{array}$$

It is often convenient to pass to a more general setting by considering different choices of target categories. We therefore follow [6] and work with $\mathbf{C}^{\mathbf{R}}$ for an arbitrary category \mathbf{C} , keeping in mind that this functor category specializes to \mathbf{Mod} whenever $\mathbf{C} = \mathbf{Vect}$. We call $\mathbf{C}^{\mathbf{R}}$ the category of *persistence modules with values in \mathbf{C}* .

2.2. The interpolation lemma. For each $e \geq 0$, one has a *translation* functor $T_e : \mathbf{R} \rightarrow \mathbf{R}$ (sending a to $a + e$) as well as a unique natural transformation σ_e from the identity $1_{\mathbf{R}}$ to this functor. It is readily confirmed that every such translation induces an endofunctor on $\mathbf{C}^{\mathbf{R}}$ which

- (1) sends each $U \in \mathbf{C}_o^{\mathbf{R}}$ to UT_e satisfying $UT_e(a) = U(a + e)$ for $a \in \mathbb{R}$, and
- (2) admits a distinguished natural transformation from the identity.

Definition 2.2. Given $e \geq 0$, two functors $U, V \in \mathbf{C}_o^{\mathbf{R}}$ are said to be e -interleaved if there are morphisms $\Phi : U \rightarrow VT_e$ and $\Psi : V \rightarrow UT_e$ in $\mathbf{C}^{\mathbf{R}}$ satisfying $(\Psi T_e)\Phi = U\sigma_{2e}$ and $(\Phi T_e)\Psi = V\sigma_{2e}$, as encoded in commutativity of the following diagrams:

$$\begin{array}{ccc}
 U & \xrightarrow{U\sigma_{2e}} & UT_{2e} \\
 & \searrow \Phi & \nearrow \Psi T_e \\
 & VT_e &
 \end{array}
 \qquad
 \begin{array}{ccc}
 & UT_e & \\
 \Psi \nearrow & & \searrow \Phi T_e \\
 V & \xrightarrow{V\sigma_{2e}} & VT_{2e}
 \end{array}$$

The *interleaving distance* [6, 8, 10, 19] on $\mathbf{C}_o^{\mathbf{R}}$ is defined as follows:

$$d_{\text{Int}}(U, V) = \inf \{e \geq 0 \mid U, V \text{ are } e\text{-interleaved}\},$$

with the understanding that $d_{\text{Int}}(U, V) = \infty$ if no interleaving exists.

We want to say that $\mathbf{C}_o^{\mathbf{R}}$ together with the interleaving distance is a metric space. To make this possible, throughout this paper, we relax the usual requirements for a metric space (M, d) in three ways:

- (1) we allow $d(x, y)$ to attain the value $+\infty$,
- (2) we allow $d(x, y) = 0$ for $x \neq y$ in M , and
- (3) we allow M to be a class rather than a set.

In other words, we work with *symmetric Lawvere metric spaces* as defined in [18].

At times, we will not allow the third generalization. That is, we will need the collection of elements in a metric space to be a set. Let \mathbf{Met} be the category whose objects are metric spaces with a set of elements, and whose morphisms are *nonexpansive* or *1-Lipschitz* maps.¹ Similarly, let \mathbf{Cat} denote the category of small categories and functors.

Theorem 2.3 (see [6]). For each category \mathbf{C} and functor $H : \mathbf{C} \rightarrow \mathbf{D}$,

- (1) The pair $(\mathbf{C}_o^{\mathbf{R}}, d_{\text{Int}})$ is a metric space.
- (2) The map $H^{\mathbf{R}} : \mathbf{C}_o^{\mathbf{R}} \rightarrow \mathbf{D}_o^{\mathbf{R}}$ sending U to HU is 1-Lipschitz with respect to d_{Int} .

Specializing to small categories, these assignments define a functor $\bullet^{\mathbf{R}} : \mathbf{Cat} \rightarrow \mathbf{Met}$.

We call $(\mathbf{C}_o^{\mathbf{R}}, d_{\text{Int}})$ the *metric space of persistence modules with values in \mathbf{C}* and call the functor $\bullet^{\mathbf{R}}$ the *persistence module functor*.

Recall that a metric space (M, d) is a *path metric space* if for each x, y in M the infimum of the lengths of all paths between them equals $d(x, y)$. The following *interpolation lemma* establishes that \mathbf{Mod}_o is a path metric space when endowed with the interleaving distance as a metric.

Lemma 2.4 (see [8]). Given $e \geq 0$ and two e -interleaved persistence modules U_0 and U_e in \mathbf{Mod}_o , there exists a one-parameter family $\{U_a \mid a \in (0, e)\}$ in \mathbf{Mod}_o so that U_a and U_b are $|a - b|$ -interleaved for all $a, b \in [0, e]$.

¹That is, a map $f : (M, d_M) \rightarrow (N, d_N)$ satisfying $d_N(f(x), f(y)) \leq d_M(x, y)$ for all $x, y \in M$.

Note in general that the interpolating family U_t is not unique, and that this lemma need not hold for general categories $\mathbf{C}^{\mathbf{R}}$. (We will describe additional hypotheses on \mathbf{C} under which the interpolation lemma holds for $\mathbf{C}^{\mathbf{R}}$ in Proposition 4.2.)

As mentioned in the introduction, our main result is a higher interpolation lemma. A simple example illustrating that higher interpolations are not always possible may be found in [14], and we will reproduce it here in section 2.3. On the other hand, a pathway towards the desired generalization is provided by the following *sharp interpolation lemma*: not only can one find an interpolating family of modules, but one can also find a compatible family of interleaving maps between them.

Lemma 2.5 (see [10]). *Given persistence modules U_0 and U_e along with morphisms $\Phi : U_0 \rightarrow U_e T_e$ and $\Psi : U_e \rightarrow U_0 T_e$ which realize an e -interleaving, there exist*

- (1) *persistence modules $\{U_a \mid a \in (0, e)\}$, and*
 - (2) *module morphisms $\Phi_a^b : U_a \rightarrow U_b T_{b-a}$ and $\Psi_b^a : U_b \rightarrow U_a T_{b-a}$ for all $a \leq b$ in $[0, e]$,*
- so that*

- (3) $\Phi_0^e = \Phi$ and $\Psi_e^0 = \Psi$,
- (4) Φ_a^b and Ψ_b^a realize a $(b-a)$ -interleaving between U_a and U_b , and
- (5) $(\Phi_b^c T_{b-a})\Phi_a^b = \Phi_a^c$ and $(\Psi_c^b T_{b-a})\Psi_b^a = \Psi_c^a$ hold for all $a \leq b \leq c$ in $[0, e]$.

In general, the intermediate modules U_a and maps Φ_a^b and Ψ_b^a are not uniquely defined.

2.3. Failure of higher interpolation. The main result of [13] asserts that (isomorphism classes of) *tame*² persistence modules are faithfully represented by their *persistence diagrams* [8, 11, 25], which are multisets of points in the upper half-plane.³ Moreover, these diagrams admit a *bottleneck distance* d_{Bot} and the following *isometry theorem* establishes that the assignment dgm which sends a (tame) persistence module to its corresponding diagram preserves distances.

Theorem 2.6 (see [6, 10, 19]). *The equality*

$$d_{\text{Int}}(U, V) = d_{\text{Bot}}(\text{dgm } U, \text{dgm } V)$$

*holds across all pairs U, V of tame persistence modules.*⁴

It was shown in [14] that higher interpolations may fail to exist even for simple choices of $A \hookrightarrow M$.

Example 2.7. Let (A, d) be the three-point metric space $\{x_1, x_2, x_3\}$ with $d(x_i, x_j) = 1$ for $i \neq j$, and let M be A together with x_0 where $d(x_0, x_i) = \frac{1}{2}$ for $i \geq 1$. Let $f : A \rightarrow \mathbf{Mod}_o$ be the function whose images $f(x_j)$ prescribe the three persistence diagrams shown in Figure 1. Growing a ball of radius one (in the L_∞ norm) around points in any one diagram subsumes points in the other two diagrams, whence the pairwise bottleneck distances satisfy

$$d_{\text{Bot}}(\text{dgm } f(x_i), \text{dgm } f(x_j)) = 1 \text{ for } i \neq j.$$

²These are modules $U : \mathbf{R} \rightarrow \mathbf{Vect}$ for which $U(t)$ is finite-dimensional for each t .

³To be precise, one needs to take decorated persistence diagrams [10]. This result extends to the q -tame modules described in [9, 10].

⁴This result also holds for q -tame persistence modules [10].

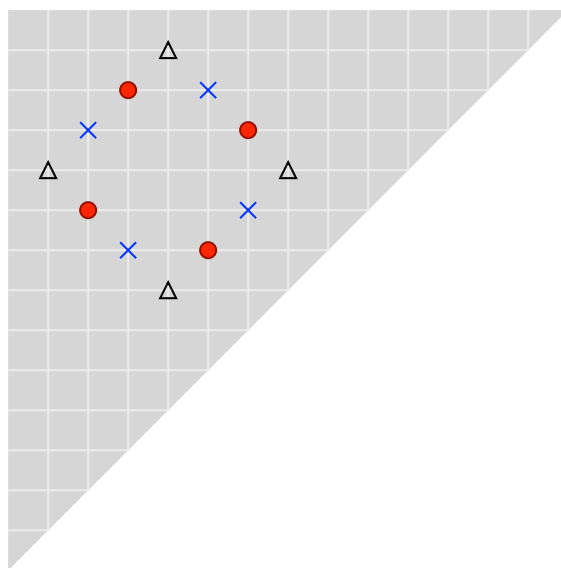


Figure 1. Three overlaid persistence diagrams Δ , \times , and \bullet . The corresponding persistence modules have pairwise interleaving distance 1, but there is no persistence module within distance e of all three for any $e < 1$.

Note that any one of these diagrams lies at distance exactly 1 from the other two. However, no persistence diagram is within distance < 1 of each of the three persistence diagrams in Figure 1. Thus, it follows from the isometry theorem that there is no tame persistence module which may be assigned to x_0 in any extension of f without strictly increasing the Lipschitz constant.

This sharp interpolation lemma suggests the reason for the failure of the higher-order interpolation in Example 2.7: the 1-interleavings do not satisfy the compatibility condition (5).

3. Categories from metric spaces. In Theorem 2.3 we defined the persistence module functor $\bullet^{\mathbf{R}} : \mathbf{Cat} \rightarrow \mathbf{Met}$. The central goal of this section is to describe the construction of a functor $\mathbf{Met} \rightarrow \mathbf{Cat}$, whose interaction with $\bullet^{\mathbf{R}}$ will be of crucial importance in the proof of our main result. To this end, consider $(M, d) \in \mathbf{Met}_o$, and let $M\mathbb{R}$ denote the product $M \times \mathbb{R}$ equipped with the binary relation

$$(x, s) \leq_M (y, t) \text{ if and only if } d(x, y) \leq t - s.$$

Henceforth, we will often drop the subscript and simply write $(x, s) \leq (y, t)$, relying on context for clarity. We call $M\mathbb{R}$ the *spacetime* of M , and illustrate an instance of it in Figure 2. The following result is straightforward.

Proposition 3.1. *The relation \leq induces a preorder on $M\mathbb{R}$. Moreover, if d is a genuine metric in the sense that $d(x, y) = 0$ holds only for $x = y$, then \leq induces a partial order on $M\mathbb{R}$.*

Proof. Since $d(x, x) = 0$, we have reflexivity: $(x, s) \leq (x, s)$. Turning to transitivity, assume $(x, s) \leq (y, t)$ and $(y, t) \leq (z, u)$. By the triangle inequality, we have

$$d(x, z) \leq d(x, y) + d(y, z) \leq (t - s) + (u - t) = u - s.$$

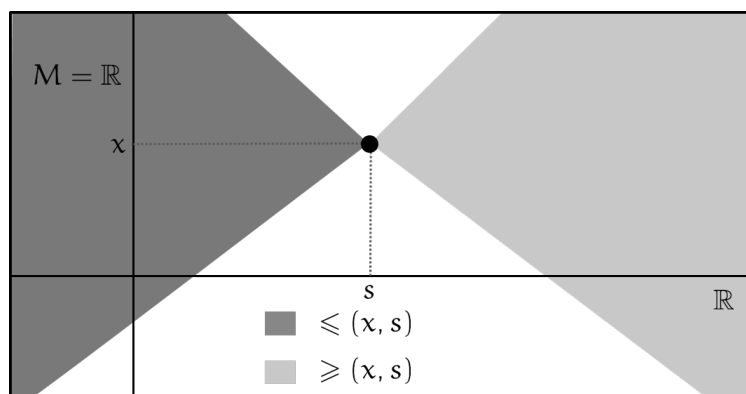


Figure 2. An illustration of the order \leq on $M\mathbb{R}$ in the special case where $M = \mathbb{R}$ with the usual metric. Given $(x, s) \in M\mathbb{R}$, the lighter region consists of the up-set $(y, t) \geq (x, s)$ while the darker region consists of the down-set. The up-set is inside the future light cone and the down-set is inside the past light cone.

Hence, $(x, s) \leq (z, u)$ and thus \leq is transitive. Finally, if $(x, s) \leq (y, t) \leq (x, s)$, then $0 \leq d(x, y) \leq \min(s - t, t - s)$. Thus, $s = t$. and we have antisymmetry only if $d(x, y) = 0$ forces $x = y$ in M . ■

Since $M\mathbb{R}$ is a preordered set, it may be treated as a *thin* category.⁵ Given a 1-Lipschitz map $f \in \mathbf{Met}(M, N)$, define $f\mathbb{R} : M\mathbb{R} \rightarrow N\mathbb{R}$ via the mapping $(x, s) \mapsto (f(x), s)$.

Theorem 3.2. The assignment $\bullet\mathbb{R}$ prescribes a functor $\mathbf{Met} \rightarrow \mathbf{Cat}$, which we call the spacetime functor.

Proof. We first confirm that $f\mathbb{R} : M\mathbb{R} \rightarrow N\mathbb{R}$ is a morphism in \mathbf{Cat} whenever $f : (M, d_M) \rightarrow (N, d_N)$ is 1-Lipschitz. If $(x, s) \leq (y, t)$, we have $d_M(x, y) \leq t - s$. Since f is 1-Lipschitz, we obtain $d_N(f(x), f(y)) \leq d_M(x, y) \leq t - s$. Thus, by definition,

$$f\mathbb{R}(x, s) = (f(x), s) \leq_N (f(y), t) = f\mathbb{R}(y, t).$$

Thus, $f\mathbb{R}$ is order-preserving. It is easy to confirm that $1_{M\mathbb{R}} = 1_{M\mathbb{R}}$, so we turn to the task of establishing functoriality. Consider $M \xrightarrow{f} N \xrightarrow{g} P$ in \mathbf{Met} , and note that

$$g\mathbb{R} \circ f\mathbb{R}(x, s) = g\mathbb{R}(f(x), s) = (gf(x), s) = [(gf)\mathbb{R}](x, s),$$

which concludes the proof. ■

With the existence of the spacetime functor $\bullet\mathbb{R} : \mathbf{Met} \rightarrow \mathbf{Cat}$ established, one might hope for an adjunction with the persistence module functor $\bullet^{\mathbf{R}} : \mathbf{Cat} \rightarrow \mathbf{Met}$ from Theorem 2.3. If such an adjunction existed, then for each metric space $M \in \mathbf{Met}_o$ and category $\mathbf{C} \in \mathbf{Cat}_o$, we would expect a natural bijection of sets between

$$\mathbf{Cat}(M\mathbb{R}, \mathbf{C}) \text{ and } \mathbf{Met}(M, \mathbf{C}^{\mathbf{R}}),$$

i.e., the set of functors from $M\mathbb{R}$ to \mathbf{C} would correspond with 1-Lipschitz maps from M to $\mathbf{C}^{\mathbf{R}}$. Example 2.7 confirms that there is no such bijection in general, whence our functors $\bullet^{\mathbf{R}}$

⁵A category is thin if it admits at most one morphism between any pair of objects.

and $\bullet\mathbb{R}$ do not constitute an adjoint pair. Instead, we seek solace in the existence of a *unit*, as described below.

Let F denote the endofunctor on \mathbf{Met} arising from the following composition:

$$(3.1) \quad F : \mathbf{Met} \xrightarrow{\bullet\mathbb{R}} \mathbf{Cat} \xrightarrow{\bullet\mathbb{R}} \mathbf{Met}.$$

Chasing definitions, one can explicitly describe the effect of F on the objects and morphisms of \mathbf{Met} : each metric space M is mapped to $M\mathbb{R}^{\mathbf{R}}$ (with the interleaving distance), and every 1-Lipschitz $f : M \rightarrow N$ is sent to the map $M\mathbb{R}^{\mathbf{R}} \rightarrow N\mathbb{R}^{\mathbf{R}}$ which takes $U : \mathbf{R} \rightarrow M\mathbb{R}$ to $f\mathbb{R} \circ U : \mathbf{R} \rightarrow N\mathbb{R}$. Note that the objects of $F(M)$ consist of *world lines* in the spacetime of M .

Theorem 3.3. *The functor F admits a natural transformation $\eta : \mathbf{I}_{\mathbf{Met}} \Rightarrow F$ from the identity endofunctor on \mathbf{Met} . Furthermore, for each metric space M , η_M is the isometric embedding of M into the metric space of persistence modules valued in $M\mathbb{R}$ given by the constant world lines in the spacetime $M\mathbb{R}$.*

Proof. For each $(M, d_M) \in \mathbf{Met}_o$, we require a 1-Lipschitz map η_M in $\mathbf{Met}(M, FM)$ which sends points of M to functors $\mathbf{R} \rightarrow M\mathbb{R}$. We provisionally define this map as follows: for each $x \in M$, let $\eta_M(x)$ be the functor which sends $s \in \mathbb{R}$ to (x, s) and $s \leq t$ to $(x, s) \leq (x, t)$. To check that the latter inequality holds in $M\mathbb{R}$, we verify that $d_M(x, x) \leq t - s$. This definition sends identities to identities and respects composition since $M\mathbb{R}$ is a thin category. Thus, $\eta_M(x)$ is indeed a functor.

Next, we confirm that η_M is 1-Lipschitz. Letting x and y be points in M with $e = d_M(x, y)$, it suffices to construct an e -interleaving between $\eta_M(x)$ and $\eta_M(y)$. By definition, $(x, s) \leq (y, t)$ if and only if $d_M(x, y) \leq t - s$. Thus, we have the inequalities

$$(3.2) \quad (x, s) \leq (y, s + e) \text{ for all } s \in \mathbb{R},$$

whose images under $\bullet\mathbf{R}$ yield a morphism $\Phi : \eta_M(x) \rightarrow \eta_M(y)T_e$ in $FM = M\mathbb{R}^{\mathbf{R}}$. And similarly, we have the inequalities

$$(3.3) \quad (y, t) \leq (x, t + e) \text{ for all } t \in \mathbb{R},$$

whose images under $\bullet\mathbf{R}$ assemble into a morphism $\Psi : \eta_M(y) \rightarrow \eta_M(x)T_e$ in FM . One can readily check that Φ and Ψ furnish the desired e -interleaving of $\eta_M(x)$ and $\eta_M(y)$: since $M\mathbb{R}$ is a thin category, the diagrams from Definition 2.2 must commute. Thus, $\eta_M : M \rightarrow FM$ is 1-Lipschitz as desired.

Note that because $(x, s) \leq (y, t)$ if and only if $d_M(x, y) \leq t - s$, there is no smaller value than e for which the inequalities (3.2) and (3.3) hold. Thus $\eta_M(x)$ and $\eta_M(y)$ are not e' -interleaved for any $e' < e$. Therefore, the interleaving distance between $\eta_M(x)$ and $\eta_M(y)$ is e , and η_M is in fact an isometric embedding.

Finally, we check that the assignment $x \mapsto \eta_M(x)$ prescribes a natural transformation.

Given $f : (M, d_M) \rightarrow (N, d_N)$ in **Met**, we must verify that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\eta_M} & FM \\ f \downarrow & & \downarrow Ff \\ N & \xrightarrow{\eta_N} & FN \end{array}$$

Pick any $x \in M$. For all $s \in \mathbb{R}$, we have

$$\begin{aligned} ([Ff \circ \eta_M](x))(s) &= Ff(x, s) \\ &= (f(x), s) = ([\eta_N \circ f](x))(s); \end{aligned}$$

thus, our diagram commutes and η is a natural transformation. ■

4. Coherence and higher interpolation. Throughout this section, we fix a choice of category \mathbf{C} and metric space $A \in \mathbf{Met}$. We also let $\eta : \mathbf{I}_{\mathbf{Met}} \Rightarrow F$ be the natural transformation from the proof of Theorem 3.3. For a functor $G : A\mathbb{R} \rightarrow \mathbf{C}$, define $\theta(G) = G^{\mathbf{R}} \circ \eta_A$:

$$(4.1) \quad A \xrightarrow{\eta_A} FA = A\mathbb{R}^{\mathbf{R}} \xrightarrow{G^{\mathbf{R}}} \mathbf{C}^{\mathbf{R}}.$$

Definition 4.1. The 1-Lipschitz functions $g : A \rightarrow \mathbf{C}_o^{\mathbf{R}}$ which lie in the image of θ are called *coherent*. In the special case where $\mathbf{C} = \mathbf{Vect}$, such functions are called *coherent persistence modules*.

By definition, for every coherent 1-Lipschitz map $g : A \rightarrow \mathbf{C}^{\mathbf{R}}$ there is some functor $G : A\mathbb{R} \rightarrow \mathbf{C}$ satisfying $g = \theta(G)$. The map $G^{\mathbf{R}}$ now serves as an extension of g across $\eta_A : A \rightarrow A\mathbb{R}^{\mathbf{R}}$ because the following diagram commutes by the definition of θ :

$$(4.2) \quad \begin{array}{ccc} A & \xrightarrow{g} & \mathbf{C}^{\mathbf{R}} \\ \eta_A \downarrow & \nearrow G^{\mathbf{R}} & \\ A\mathbb{R}^{\mathbf{R}} & & \end{array}$$

If A isometrically embeds into a larger metric space $M \in \mathbf{Met}$, then it is easy to check that $A\mathbb{R}$ is a full subcategory of $M\mathbb{R}$. Given any functor $G : A\mathbb{R} \rightarrow \mathbf{C}$, one has the following functor extension problem:

$$(4.3) \quad \begin{array}{ccc} A\mathbb{R} & \xrightarrow{G} & \mathbf{C} \\ \downarrow & \nearrow \hat{G} & \\ M\mathbb{R} & & \end{array}$$

Recall that the category \mathbf{C} is (co)complete if it has all (co)limits. The solution to problems such as (4.3) for functors taking values in (co)complete categories is furnished by *Kan extensions* [20, Chap. X].

Proposition 4.2. An extension \hat{G} of G exists under any of the following circumstances:

- If \mathbf{C} is cocomplete, we can take \hat{G} to be the left Kan extension $\text{Lan}G$ of G .
- If \mathbf{C} is complete, we can take \hat{G} to be the right Kan extension $\text{Ran}G$ of G .
- If \mathbf{C} is bicomplete and abelian, we can take \hat{G} to be the image of the universal natural transformation $\text{Lan}G \Rightarrow \text{Ran}G$.

If $\mathbf{C} = \mathbf{Vect}$ (as in the case of persistence modules), then we have all three extensions, but if $\mathbf{C} = \mathbf{Top}$ (as in the case of filtered topological spaces), then we only have the left and right extensions. The following theorem is the main result of this paper.

Theorem 4.3. *Let A be the subspace of a metric space $M \in \mathbf{Met}$, and assume that the 1-Lipschitz map $g : A \rightarrow \mathbf{C}_o^{\mathbf{R}}$ is coherent. If \mathbf{C} is (co)complete, then g admits a coherent 1-Lipschitz extension $\hat{g} : M \rightarrow \mathbf{C}_o^{\mathbf{R}}$.*

Proof. Since g is coherent, it equals $\theta(G)$ for some functor $G : A\mathbb{R} \rightarrow \mathbf{C}$. By Proposition 4.2 and the (co)completeness hypothesis on \mathbf{C} , there is an extension $\hat{G} : M\mathbb{R} \rightarrow \mathbf{C}$ of G as in (4.3). Note that the following diagram of metric spaces and 1-Lipschitz maps commutes:

$$\begin{array}{ccccc} A & \xrightarrow{\eta_A} & A\mathbb{R} & \xrightarrow{G^{\mathbf{R}}} & \mathbf{C}^{\mathbf{R}} \\ \downarrow & & \downarrow & \nearrow \hat{G}^{\mathbf{R}} & \\ M & \xrightarrow{\eta_M} & M\mathbb{R} & & \end{array}$$

The square on the left commutes because η is a natural transformation. The triangle on the right commutes since $G^{\mathbf{R}}(F) = G \circ F = \hat{G} \circ i \circ F = \hat{G}^{\mathbf{R}}(i(F))$, where $i : A\mathbb{R} \hookrightarrow M\mathbb{R}$ and the middle equality is by (4.3). Since the composite in the top row of our diagram equals g , it is immediately seen that the desired extension $\hat{g} : M \rightarrow \mathbf{C}^{\mathbf{R}}$ is given by $\theta(\hat{G}) = \hat{G}^{\mathbf{R}} \circ \eta_M$. ■

The $\mathbf{C} = \mathbf{Vect}$ specialization of Theorem 4.3 yields the higher interpolation lemma promised in the introduction. In this case, for a given G satisfying $\theta(G) = g$ we have at least three possible choices⁶ of \hat{G} arising from Proposition 4.2. Regardless of which \hat{G} is chosen, the map $\theta(\hat{G})$ is itself coherent by construction, and hence admits further extensions to larger metric spaces.

5. Consequences. In this section we describe some applications of Theorem 4.3.

5.1. Discrete and continuous interpolation. Let U_1, \dots, U_n be a collection of $n \geq 1$ persistence modules, and let $e \geq 0$ be a fixed constant. Assume further that U_i and U_j are $2e$ -interleaved for all $i \neq j$. Let $A = \{a_1, \dots, a_n\}$ be the metric space where all nontrivial distances $d(a_i, a_j)$ equal $2e$, and note that we may describe each U_i as the image $g(a_i)$ of a 1-Lipschitz map $g : A \rightarrow \mathbf{Mod}_o$. Recall the translation functor T and the natural transformation σ as defined in section 2.2. The following result provides an easily computable criterion for coherence (compare with Lemma 2.5 as well as [14, Thm. 4.2]).

Proposition 5.1. *The map $g : A \rightarrow \mathbf{Mod}_o$ is coherent if and only if for all distinct i, j in $\{1, \dots, n\}$ there exist morphisms $\Phi_{ij} : U_i \rightarrow U_j T_{2e}$ in \mathbf{Mod} satisfying:*

⁶For explicit calculations and a comparison of all three extensions in the context of the sharp interpolation lemma, consult [10, sect. 3.5] (and particularly Proposition 3.6 therein). The image extension is optimal among the three in the sense that it satisfies two universal properties instead of one.

- (1) $(\Phi_{ji}T_{2e}) \circ \Phi_{ij} = U_i\sigma_{4e}$ for all distinct i, j , and
- (2) $(\Phi_{jk}T_{2e}) \circ \Phi_{ij} = \Phi_{ik}T_{2e}$ for all distinct i, j, k .

Proof. Assume first that g is coherent, so there exists a functor $G : A\mathbb{R} \rightarrow \mathbf{Vect}$ satisfying $g = G^{\mathbf{R}} \circ \eta_A$. To define the desired morphisms $\Phi_{ij} : U_i \rightarrow U_j T_{2e}$ in \mathbf{Mod} , it suffices to construct linear maps $\Phi_{ij}(s) : U_i(s) \rightarrow U_j(s+2e)$ indexed by $s \in \mathbb{R}$ and $i, j \leq n$ (subject to the constraint that diagrams from Definition 2.2 commute). To this end, note that $(a_i, s) \leq (a_j, s+2e)$ in $A\mathbb{R}$ because $d(a_i, a_j) \leq 2e$ in A by assumption. Define $\Phi_{ij}(s)$ to be the image under G of $(a_i, s) \leq (a_j, s+2e)$. Since $A\mathbb{R}$ is a thin category and G is a functor, all the required diagrams commute and the two desired properties follow.

On the other hand, assume the existence of maps Φ_{ij} satisfying the two properties from the statement of this proposition. We will use them to define a functor $G : A\mathbb{R} \rightarrow \mathbf{Vect}$ which renders g coherent by satisfying $\theta(G) = g$. Set $G(a_i, t) = U_i(t)$ for all $i \leq n$ and $t \in \mathbb{R}$. Given $(a_i, s) \leq (a_j, t)$ in $A\mathbb{R}$, we either have $t - s \geq 2e = d(a_i, a_j)$ if $i \neq j$ or simply $t - s \geq 0$ if $i = j$; thus, define

$$G((a_i, s) \leq (a_j, t)) = \begin{cases} \Phi_{ij}(s) \circ T_{(t-s)-2e}, & i \neq j, \\ U_i\sigma_{t-s}, & i = j. \end{cases}$$

Straightforward calculations confirm that G is a functor, and that $\theta(G) = g$. ■

Let M be the metric space which consists of A above, along with an additional point a so that $d(a, a_i) = e$. The *discrete interpolation problem* for persistence modules seeks to extend our 1-Lipschitz map $g : A \rightarrow \mathbf{Mod}_o$ to a 1-Lipschitz map $\hat{g} : M \rightarrow \mathbf{Mod}_o$ across the obvious inclusion $A \hookrightarrow M$. On the other hand, the *continuous interpolation problem* for persistence modules seeks an extension of g across the inclusion of A as vertices of the standard $(n-1)$ -simplex $\Sigma \subset \mathbb{R}^n$, given by

$$\Sigma = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 + \dots + x_n = \sqrt{2}e \text{ and } x_j \geq 0 \right\}.$$

It follows immediately from Theorem 4.3 that the discrete and continuous interpolation problems both admit solutions (in triplicate) whenever the modules U_1, \dots, U_n are connected by morphisms Φ_{ij} which satisfy the two properties from Proposition 5.1. Note that in the latter case, g extends not only to Σ but to \mathbb{R}^n .

5.2. Čech and Rips complexes of persistence modules. Let (M, d_M) be an ambient metric space with a distinguished subspace $A \subset M$. We recall the *Vietoris–Rips* complex \mathcal{V} and the *Čech* complex \mathcal{C} on A . Both are abstract simplicial complexes filtered by a single nonnegative real parameter; their vertices are the points of A , but the construction of higher-dimensional simplices sets them apart. In particular, a collection $[a_0, \dots, a_n]$ of points in A forms an n -simplex

- (1) in $\mathcal{V}(A, e)$ if and only if $d_M(a_i, a_j) \leq e$ for $0 \leq i < j \leq n$, and
- (2) in $\mathcal{C}(A, e)$ if and only if some b in M satisfies $d_M(a_i, b) \leq e$ for $0 \leq i \leq n$.

Any b which is within distance e of all the points $[a_0, \dots, a_n]$ is called an e -witness for those points. It is well-known and immediate from the definitions that for each $e \geq 0$ one always has the following simplicial sandwich:

$$\mathcal{C}(A, e) \hookrightarrow \mathcal{V}(A, 2e) \hookrightarrow \mathcal{C}(A, 2e),$$

where the first inclusion follows directly from the triangle inequality, and the second follows from the fact that any a_i serves as a $2e$ -witness for a simplex $[a_0, \dots, a_n]$ in $\mathcal{V}(A, 2e)$.

We may ask if these inclusions are tight. For the first inclusion, assume that M is a path metric space (e.g., \mathbf{Mod}_o), and that there exist $a_0, a_1 \in A$ with $d_M(a_0, a_1) = 2e + \delta$ for some $\delta > 0$. Then $[a_0, a_1]$ is not a 1-simplex in $\mathcal{V}(A, 2e)$ and there exists a path from a_0 to a_1 with length at most $2e + 2\delta$. The midpoint of this path is a $(e + \delta)$ -witness for $[a_0, a_1]$, which is therefore a 1-simplex in $\mathcal{C}(A, e + \delta)$. Thus $\mathcal{C}(A, e + \delta) \not\hookrightarrow \mathcal{V}(A, 2e)$ for $\delta > 0$. The second inclusion $\mathcal{V}(A, 2e) \hookrightarrow \mathcal{C}(A, 2e)$ might be improved, depending on M . For example, $\mathcal{V}(A, 2e)$ always includes into $\mathcal{C}(A, 2e/\sqrt{3})$ whenever A is a subset of \mathbb{R}^2 with the standard Euclidean metric.

When working within \mathbf{Mod}_o , one typically strengthens the requirements in the definitions of Rips and Čech complexes slightly since the infimum over interleavings may not actually be attained. In particular, a collection $[U_0, \dots, U_n]$ of persistence modules forms an n -simplex

- (1) in $\mathcal{V}(\mathbf{Mod}_o, e)$ if and only if U_i and U_j are e -interleaved for all $0 \leq i < j \leq n$, and
- (2) in $\mathcal{C}(\mathbf{Mod}_o, e)$ if and only if some V is e -interleaved with U_i for all $0 \leq i \leq n$.

It is straightforward to check that $\mathcal{V}(\mathbf{Mod}_o, 2e)$ does not include into $\mathcal{C}(\mathbf{Mod}_o, d)$ for any $d < 2e$ by appealing to Figure 1 and the isometry theorem. On the other hand, the following result from [14] characterizes those simplices of $\mathcal{V}(\mathbf{Mod}_o, 2e)$ which do include into $\mathcal{C}(\mathbf{Mod}_o, e)$.

Theorem 5.2. *Let U_0, \dots, U_n be a collection of persistence modules, and let $e \geq 0$. Then, $[U_0, \dots, U_n]$ is an n -simplex in $\mathcal{C}(\mathbf{Mod}_o, e)$ if and only if there exist morphisms Φ_{ij} for $i \neq j$ which satisfy the conditions of Proposition 5.1.*

Thus, a simplex in $\mathcal{V}(\mathbf{Mod}_o, 2e)$ forms a simplex in $\mathcal{C}(\mathbf{Mod}_o, e)$ if and only if the module morphisms which realize the pairwise $2e$ -interleavings can be chosen to commute (up to factors of the natural transformation σ). From our perspective here, the preceding result is a direct consequence of the discrete interpolation discussed in section 5.1.

Acknowledgment. The authors are indebted to the anonymous referees for their thoughtful comments and suggestions.

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