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Configurations in abelian categories. III. Stability conditions and identities

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Abstract

This is the third in a series on *configurations* in an abelian category $\mathcal{A}$. Given a finite poset $(I, \preccurlyeq)$, an $(I, \preccurlyeq)$-*configuration* (σ, ι, π) is a finite collection of objects $\sigma(J)$ and morphisms $\iota(J, K)$ or $\pi(J, K): \sigma(J) \to \sigma(K)$ in $\mathcal{A}$ satisfying some axioms, where J, K are subsets of I. Configurations describe how an object X in $\mathcal{A}$ decomposes into subobjects.

The first paper defined configurations and studied moduli spaces of configurations in $\mathcal{A}$, using the theory of Artin stacks. It showed well-behaved moduli stacks $\mathfrak{Obj}_{\mathcal{A}}, \mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}$ of objects and configurations in $\mathcal{A}$ exist when $\mathcal{A}$ is the abelian category $\mathrm{coh}(P)$ of coherent sheaves on a projective scheme P, or mod-$\mathbb{K}Q$ of representations of a quiver Q. The second studied algebras of *constructible functions* and *stack functions* on $\mathfrak{Obj}_{\mathcal{A}}$.

This paper introduces (*weak*) *stability conditions* $(\tau, T, \leqslant)$ on $\mathcal{A}$. We show the moduli spaces $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}$, $\mathrm{Obj}_{\mathrm{si}}^{\alpha}$, $\mathrm{Obj}_{\mathrm{st}}^{\alpha}(\tau)$ of τ-semistable, indecomposable τ-semistable and τ-stable objects in class α are *constructible sets* in $\mathfrak{Obj}_{\mathcal{A}}$, and some associated configuration moduli spaces $\mathcal{M}_{\mathrm{ss}}, \mathcal{M}_{\mathrm{si}}, \mathcal{M}_{\mathrm{st}}, \mathcal{M}_{\mathrm{ss}}^{\mathrm{b}}, \mathcal{M}_{\mathrm{si}}^{\mathrm{b}}$, $\mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau)_{\mathcal{A}}$ constructible in $\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}$, so their characteristic functions $\delta_{\mathrm{ss}}^{\alpha}, \delta_{\mathrm{si}}^{\alpha}, \delta_{\mathrm{st}}^{\alpha}(\tau)$ and $\delta_{\mathrm{ss}}, \ldots$, $\delta_{\mathrm{st}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau)$ are constructible.

We prove many identities relating these constructible functions, and their stack function analogues, under pushforwards. We introduce interesting algebras $\mathcal{H}_{\tau}^{\mathrm{pa}}, \mathcal{H}_{\tau}^{\mathrm{to}}, \overline{\mathcal{H}}_{\tau}^{\mathrm{pa}}, \overline{\mathcal{H}}_{\tau}^{\mathrm{to}}$ of constructible and stack functions, and study their structure. In the fourth paper we show $\mathcal{H}_{\tau}^{\mathrm{pa}}, \ldots, \overline{\mathcal{H}}_{\tau}^{\mathrm{to}}$ are independent of $(\tau, T, \leqslant)$, and construct *invariants* of $\mathcal{A}$, $(\tau, T, \leqslant)$.

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1. Introduction

This is the third in a series of papers [9–11] on *configurations*. Given an abelian category $\mathcal{A}$ and a finite partially ordered set (poset) $(I, \preccurlyeq)$, we define an $(I, \preccurlyeq)$-*configuration* (σ, ι, π) in $\mathcal{A}$ to be a collection of objects $\sigma(J)$ and morphisms $\iota(J, K)$ or $\pi(J, K): \sigma(J) \to \sigma(K)$ in $\mathcal{A}$ satisfying certain axioms, for $J, K \subseteq I$.

The first paper [9] defined configurations, developed their basic properties, and studied moduli spaces of configurations in $\mathcal{A}$, using the theory of Artin stacks. It proved well-behaved moduli stacks $\mathfrak{Obj}_{\mathcal{A}}, \mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}$ of objects and configurations exist when $\mathcal{A}$ is the abelian category $\mathrm{coh}(P)$ of coherent sheaves on a projective $\mathbb{K}$-scheme P, or mod-$\mathbb{K}Q$ of representations of a quiver Q. The second [10] defined and studied infinite-dimensional algebras of *constructible functions* and *stack functions* on $\mathfrak{Obj}_{\mathcal{A}}$, motivated by *Ringel–Hall algebras*.

Configurations are a tool for describing how an object X in $\mathcal{A}$ decomposes into subobjects. They are especially useful for studying *stability conditions* on $\mathcal{A}$, which are the subject of this paper. Given a stability condition $(\tau, T, \leqslant)$ on $\mathcal{A}$, objects X in $\mathcal{A}$ are called τ-*semistable*, τ-*stable* or τ-*unstable* according to whether subobjects $S \subset X$ with $S \neq 0, X$ have $\tau([S]) \leqslant \tau([X])$, $\tau([S]) < \tau([X])$, or $\tau([S]) > \tau([X])$. Examples of stability conditions include slope functions, and Gieseker stability of coherent sheaves.

We also define *weak stability conditions*, which include μ-stability and purity for coherent sheaves. When $(\tau, T, \leqslant)$ is a weak stability condition each $X \in \mathcal{A}$ has a unique *Harder–Narasimhan filtration* by subobjects $0 = A_0 \subset \cdots \subset A_n = X$ whose factors $S_k = A_k/A_{k-1}$ are τ-semistable with $\tau([S_1]) > \cdots > \tau([S_n])$. If $(\tau, T, \leqslant)$ is also a stability condition each τ-semistable X has a (nonunique) filtration with (unique) τ-stable factors S_k with $\tau([S_k]) = \tau([X])$. Thus, τ-stability is well-behaved for stability conditions but badly behaved for weak stability conditions, though τ-semistability is well-behaved for both.

We form moduli spaces $\mathrm{Obj}^{\alpha}_{\mathrm{ss}}, \mathrm{Obj}^{\alpha}_{\mathrm{si}}, \mathrm{Obj}^{\alpha}_{\mathrm{st}}(\tau)$ of τ-semistable, τ-semistable-indecomposable and τ-stable objects in class α in $K(\mathcal{A})$, and moduli spaces $\mathcal{M}_{\mathrm{ss}}, \mathcal{M}_{\mathrm{si}}, \mathcal{M}_{\mathrm{st}}, \mathcal{M}^{\mathrm{b}}_{\mathrm{ss}}, \mathcal{M}^{\mathrm{b}}_{\mathrm{si}}, \mathcal{M}^{\mathrm{b}}_{\mathrm{st}}(I, \preccurlyeq, \kappa, \tau)_{\mathcal{A}}$ of $(I, \preccurlyeq)$-configurations (σ, ι, π) in which the smallest objects $\sigma(\{i\})$ for $i \in I$ lie in $\mathrm{Obj}^{\kappa(i)}_{\mathrm{ss}}, \mathrm{Obj}^{\kappa(i)}_{\mathrm{si}}, \mathrm{Obj}^{\kappa(i)}_{\mathrm{st}}(\tau)$, and (σ, ι, π) is *best* for $\mathcal{M}^{\mathrm{b}}_{*}(\cdots)_{\mathcal{A}}$. It is a central, and unconventional, feature of our approach that we regard these not as spaces in their own right, but as *constructible sets* in the stacks $\mathfrak{Obj}_{\mathcal{A}}, \mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}$, so their characteristic functions $\delta^{\alpha}_{\mathrm{ss}}, \delta^{\alpha}_{\mathrm{si}}, \delta^{\alpha}_{\mathrm{st}}(\tau)$ and $\delta_{\mathrm{ss}}, \ldots, \delta^{\mathrm{b}}_{\mathrm{st}}(I, \preccurlyeq, \kappa, \tau)$ are *constructible functions*.

This has a number of ramifications. Firstly, our approach is helpful for comparing moduli spaces, and especially for understanding how $\mathrm{Obj}^{\alpha}_{\mathrm{ss}}(\tau)$ changes when we vary $(\tau, T, \leqslant)$, as we are not comparing two different varieties, but two subsets of the same stack $\mathfrak{Obj}_{\mathcal{A}}$. Secondly, $\mathrm{Obj}^{\alpha}_{\mathrm{ss}}(\tau)$ is a set of isomorphism classes, not of S-equivalence classes. This is better for studying the family of ways a τ-semistable X may be broken into τ-stable factors. But it means $\mathrm{Obj}^{\alpha}_{\mathrm{ss}}(\tau)$ is *not a well-behaved topological space*, as it may not be Hausdorff, for instance. Because of this, in [11] we focus on 'motivic' invariants of constructible sets such as Euler characteristics and virtual Poincaré polynomials.

We begin in Section 2 with background on abelian categories, constructible sets and functions, and *stack functions* on Artin $\mathbb{K}$-stacks, following [7,8]. Stack functions are a universal generalization of constructible functions, containing more information. Section 3 reviews the previous papers [9,10], and Section 4 defines (*weak*) *stability conditions* $(\tau, T, \leqslant)$ on $\mathcal{A}$. If $(\tau, T, \leqslant)$ is *permissible* $\mathrm{Obj}_{\mathrm{ss}}^\alpha, \mathrm{Obj}_{\mathrm{si}}^\alpha, \mathrm{Obj}_{\mathrm{st}}^\alpha(\tau)$ and $\mathcal{M}_{\mathrm{ss}}, \ldots, \mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau)_{\mathcal{A}}$ are *constructible sets*. We give examples of permissible (weak) stability conditions on $\mathcal{A} = \text{mod-}\mathbb{K}Q$ and $\mathcal{A} = \mathrm{coh}(P)$.

Sections 5 and 6 prove identities relating the six families of constructible functions $\delta_{\mathrm{ss}}, \delta_{\mathrm{si}}, \delta_{\mathrm{st}}, \delta_{\mathrm{ss}}^{\mathrm{b}}, \delta_{\mathrm{si}}^{\mathrm{b}}, \delta_{\mathrm{st}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau)$. These depend on theorems on the Euler characteristics of parts of moduli spaces, and encode facts about the family of ways of decomposing a τ-semistable object into τ-stable factors, and so on. One conclusion is that each of the six families determines the other five.

Section 7 studies the *algebras of constructible functions* $\mathcal{H}_\tau^{\mathrm{pa}}, \mathcal{H}_\tau^{\mathrm{to}}$ on $\mathfrak{Obj}_{\mathcal{A}}$ generated by $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(I))\delta_{\mathrm{ss}}(I, \preccurlyeq, \kappa, \tau)$, $\delta_{\mathrm{ss}}^\alpha(\tau)$ respectively, for all $(I, \preccurlyeq, \kappa), \alpha$. Defining Lie algebras $\mathcal{L}_\tau^{\mathrm{pa}}, \mathcal{L}_\tau^{\mathrm{to}}$ to be the intersections of $\mathcal{H}_\tau^{\mathrm{pa}}, \mathcal{H}_\tau^{\mathrm{to}}$ with the Lie subalgebra $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}}) \subset \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ supported on indecomposables in $\mathcal{A}$, we construct generators of $\mathcal{H}_\tau^{\mathrm{pa}}, \mathcal{H}_\tau^{\mathrm{to}}$ lying in $\mathcal{L}_\tau^{\mathrm{pa}}, \mathcal{L}_\tau^{\mathrm{to}}$, and so show $\mathcal{H}_\tau^{\mathrm{pa}}, \mathcal{H}_\tau^{\mathrm{to}}$ are the *universal enveloping algebras* of $\mathcal{L}_\tau^{\mathrm{pa}}, \mathcal{L}_\tau^{\mathrm{to}}$.

Finally, Section 8 generalizes the results of Section 5–Section 7 from constructible functions to the *stack functions* of [8], giving *stack* (*Lie*) *algebras* $\overline{\mathcal{H}}_\tau^{\mathrm{pa}}, \overline{\mathcal{H}}_\tau^{\mathrm{to}}, \overline{\mathcal{L}}_\tau^{\mathrm{pa}}, \overline{\mathcal{L}}_\tau^{\mathrm{to}}$. The sequel [11] will show the (Lie) algebras $\mathcal{H}_\tau^{\mathrm{pa}}, \ldots, \overline{\mathcal{L}}_\tau^{\mathrm{to}}$ are *independent of* $(\tau, T, \leqslant)$, so that many of our identities here and in [11] can be regarded as *change of basis formulae* in $\mathcal{H}_\tau^{\mathrm{pa}}, \ldots, \overline{\mathcal{L}}_\tau^{\mathrm{to}}$. It also discusses systems of invariants of $\mathcal{A}, (\tau, T, \leqslant)$ 'counting' τ-semistable objects and configurations, and their identities and transformation laws. These can often be interpreted using morphisms from $\overline{\mathcal{H}}_\tau^{\mathrm{pa}}, \ldots, \overline{\mathcal{L}}_\tau^{\mathrm{to}}$ to an explicit (Lie) algebra, as in [10, §6].

A subsequent paper [12] explains how to encode some of the invariants of [11] into *holomorphic generating functions* on the complex manifold of stability conditions. These satisfy an interesting p.d.e., that can be interpreted as the flatness of a connection. The material of Section 7 will be important in [12].

2. Background material

We begin with some background material on abelian categories in Section 2.1, and Artin stacks, constructible functions and stack functions in Sections 2.2–2.4.

2.1. Abelian categories

Here is the definition of abelian category, taken from [2, §II.5].

Definition 2.1. A category $\mathcal{A}$ is called *abelian* if

(i) $\mathrm{Hom}(X, Y)$ is an abelian group for all $X, Y \in \mathcal{A}$, and composition of morphisms is biadditive.
(ii) There exists a zero object $0 \in \mathcal{A}$ such that $\mathrm{Hom}(0, 0) = 0$.
(iii) For any $X, Y \in \mathcal{A}$ there exists $Z \in \mathcal{A}$ and morphisms $\iota_X : X \to Z$, $\iota_Y : Y \to Z$, $\pi_X : Z \to X$, $\pi_Y : Z \to Y$ with $\pi_X \circ \iota_X = \mathrm{id}_X$, $\pi_Y \circ \iota_Y = \mathrm{id}_Y$, $\iota_X \circ \pi_X + \iota_Y \circ \pi_Y = \mathrm{id}_Z$ and $\pi_X \circ \iota_Y = \pi_Y \circ \iota_X = 0$. We write $Z = X \oplus Y$, the *direct sum* of X and Y.
(iv) For any morphism $f : X \to Y$ there is a sequence $K \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} C$ in $\mathcal{A}$ such that $j \circ i = f$, and K is the kernel of f, and C the cokernel of f, and I is both the cokernel of k and the kernel of c.

In an abelian category we can define *exact sequences* as in [2, §II.6]. A short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\mathcal{A}$ is called *split* if there exists a compatible isomorphism $h : X \oplus Z \to Y$. The *Grothendieck group* $K_0(\mathcal{A})$ of $\mathcal{A}$ is the abelian group generated by $\mathrm{Obj}(\mathcal{A})$, with a relation $[Y] = [X] + [Z]$ for each short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$. Throughout the paper $K(\mathcal{A})$ will mean *the quotient of* $K_0(\mathcal{A})$ *by some fixed subgroup*. *Subobjects* of objects in $\mathcal{A}$ are analogous to subgroups of an abelian group.

Definition 2.2. Let $\mathcal{A}$ be an abelian category and $X \in \mathcal{A}$. Two injective morphisms $i : S \to X$, $i' : S' \to X$ are called equivalent if there exists an isomorphism $h : S \to S'$ with $i = i' \circ h$. A *subobject* of X is an equivalence class of injective $i : S \to X$. Usually we refer to S as the subobject, taking i and the equivalence class to be implicitly given, and write $S \subset X$ to mean S is a subobject of X. If $S, T \subset X$ are represented by $i : S \to X$ and $j : T \to X$, we write $S \subset T \subset X$ if there exists $a : S \to T$ with $i = j \circ a$.

We call $\mathcal{A}$ *artinian* if for all $X \in \mathcal{A}$, all descending chains of subobjects $\cdots \subset A_2 \subset A_1 \subset X$ stabilize, that is, $A_{n+1} = A_n$ for all $n \gg 0$. We call $\mathcal{A}$ *noetherian* if all ascending chains of subobjects $A_1 \subset A_2 \subset \cdots \subset X$ stabilize.

2.2. *Introduction to Artin* $\mathbb{K}$*-stacks*

Fix an algebraically closed field $\mathbb{K}$ throughout. There are four main classes of 'spaces' over $\mathbb{K}$ used in algebraic geometry, in increasing order of generality:

$$\mathbb{K}\text{-varieties} \subset \mathbb{K}\text{-schemes} \subset \text{algebraic } \mathbb{K}\text{-spaces} \subset \text{algebraic } \mathbb{K}\text{-stacks}.$$

Algebraic stacks (also known as Artin stacks) were introduced by Artin, generalizing *Deligne–Mumford stacks*. For a good introduction to algebraic stacks see Gómez [4], and for a thorough treatment see Laumon and Moret-Bailly [15]. We make the convention that all algebraic $\mathbb{K}$-stacks in this paper are *locally of finite type*, and $\mathbb{K}$-substacks are *locally closed*.

Algebraic $\mathbb{K}$-stacks form a 2-*category*. That is, we have *objects* which are $\mathbb{K}$-stacks $\mathfrak{F}, \mathfrak{G}$, and also two kinds of morphisms, 1-*morphisms* $\phi, \psi : \mathfrak{F} \to \mathfrak{G}$ between $\mathbb{K}$-stacks, and 2-*morphisms* $A : \phi \to \psi$ between 1-morphisms. An analogy to keep in mind is a 2-category of categories, where objects are categories, 1-morphisms are functors between the categories, and 2-morphisms are isomorphisms (natural transformations) between functors.

We define the set of $\mathbb{K}$-*points* of a stack.

Definition 2.3. Let $\mathfrak{F}$ be a $\mathbb{K}$-stack. Write $\mathfrak{F}(\mathbb{K})$ for the set of 2-isomorphism classes $[x]$ of 1-morphisms $x : \operatorname{Spec}\mathbb{K} \to \mathfrak{F}$. Elements of $\mathfrak{F}(\mathbb{K})$ are called $\mathbb{K}$-*points*, or *geometric points*, of $\mathfrak{F}$. If $\phi : \mathfrak{F} \to \mathfrak{G}$ is a 1-morphism then composition with ϕ induces a map of sets $\phi_* : \mathfrak{F}(\mathbb{K}) \to \mathfrak{G}(\mathbb{K})$.

For a 1-morphism $x : \operatorname{Spec}\mathbb{K} \to \mathfrak{F}$, the *stabilizer group* $\mathrm{Iso}_{\mathbb{K}}(x)$ is the group of 2-morphisms $x \to x$. When $\mathfrak{F}$ is an algebraic $\mathbb{K}$-stack, $\mathrm{Iso}_{\mathbb{K}}(x)$ is an *algebraic* $\mathbb{K}$*-group*. We say that $\mathfrak{F}$ *has affine geometric stabilizers* if $\mathrm{Iso}_{\mathbb{K}}(x)$ is an affine algebraic $\mathbb{K}$-group for all 1-morphisms $x : \operatorname{Spec}\mathbb{K} \to \mathfrak{F}$.

As an algebraic $\mathbb{K}$-group up to isomorphism, $\mathrm{Iso}_{\mathbb{K}}(x)$ depends only on the isomorphism class $[x] \in \mathfrak{F}(\mathbb{K})$ of x in $\mathrm{Hom}(\operatorname{Spec}\mathbb{K}, \mathfrak{F})$. If $\phi : \mathfrak{F} \to \mathfrak{G}$ is a 1-morphism, composition induces a morphism of algebraic $\mathbb{K}$-groups $\phi_* : \mathrm{Iso}_{\mathbb{K}}([x]) \to \mathrm{Iso}_{\mathbb{K}}(\phi_*([x]))$, for $[x] \in \mathfrak{F}(\mathbb{K})$.

One important difference in working with 2-categories rather than ordinary categories is that in diagram-chasing one only requires 1-morphisms to be 2-*isomorphic* rather than *equal*. The simplest kind of *commutative diagram* is:

$$\begin{array}{ccccc} & & \mathfrak{G} & & \\ & \overset{\phi}{\nearrow} & \Downarrow F & \overset{\psi}{\searrow} & \\ \mathfrak{F} & & \xrightarrow[\chi]{} & & \mathfrak{H}, \end{array}$$

by which we mean that $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$ are $\mathbb{K}$-stacks, ϕ, ψ, χ are 1-morphisms, and $F:\psi\circ\phi\to\chi$ is a 2-isomorphism. Usually we omit F, and mean that $\psi\circ\phi\cong\chi$.

Definition 2.4. Let $\phi:\mathfrak{F}\to\mathfrak{H}$, $\psi:\mathfrak{G}\to\mathfrak{H}$ be 1-morphisms of $\mathbb{K}$-stacks. Then one can define the *fiber product stack* $\mathfrak{F}\times_{\phi,\mathfrak{H},\psi}\mathfrak{G}$, or $\mathfrak{F}\times_{\mathfrak{H}}\mathfrak{G}$ for short, with 1-morphisms $\pi_{\mathfrak{F}},\pi_{\mathfrak{G}}$ fitting into a commutative diagram:

$$\begin{array}{ccccc} & \overset{\pi_{\mathfrak{F}}}{\nearrow} & \mathfrak{F} & \overset{\phi}{\searrow} & \\ \mathfrak{F}\times_{\mathfrak{H}}\mathfrak{G} & & \Downarrow & & \mathfrak{H}. \\ & \underset{\pi_{\mathfrak{G}}}{\searrow} & \mathfrak{G} & \underset{\psi}{\nearrow} & \end{array} \tag{1}$$

A commutative diagram

$$\begin{array}{ccccc} & \overset{\theta}{\nearrow} & \mathfrak{F} & \overset{\phi}{\searrow} & \\ \mathfrak{E} & & \Downarrow & & \mathfrak{H} \\ & \underset{\eta}{\searrow} & \mathfrak{G} & \underset{\psi}{\nearrow} & \end{array}$$

is a *Cartesian square* if it is isomorphic to (1), so there is a 1-isomorphism $\mathfrak{E}\cong\mathfrak{F}\times_{\mathfrak{H}}\mathfrak{G}$. Cartesian squares may also be characterized by a universal property.

2.3. Constructible functions on stacks

Next we discuss *constructible functions* on $\mathbb{K}$-stacks, following [7]. For this section we need $\mathbb{K}$ to have *characteristic zero*.

Definition 2.5. Let $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack. We call $C\subseteq\mathfrak{F}(\mathbb{K})$ *constructible* if $C=\bigcup_{i\in I}\mathfrak{F}_i(\mathbb{K})$, where $\{\mathfrak{F}_i: i\in I\}$ is a finite collection of finite type algebraic $\mathbb{K}$-substacks $\mathfrak{F}_i$ of $\mathfrak{F}$. We call $S\subseteq\mathfrak{F}(\mathbb{K})$ *locally constructible* if $S\cap C$ is constructible for all constructible $C\subseteq\mathfrak{F}(\mathbb{K})$.

A function $f:\mathfrak{F}(\mathbb{K})\to\mathbb{Q}$ is called *constructible* if $f(\mathfrak{F}(\mathbb{K}))$ is finite and $f^{-1}(c)$ is a constructible set in $\mathfrak{F}(\mathbb{K})$ for each $c\in f(\mathfrak{F}(\mathbb{K}))\setminus\{0\}$. A function $f:\mathfrak{F}(\mathbb{K})\to\mathbb{Q}$ is called *locally constructible* if $f\cdot\delta_C$ is constructible for all constructible $C\subseteq\mathfrak{F}(\mathbb{K})$, where δ_C is the characteristic function of C. Write $\mathrm{CF}(\mathfrak{F})$ and $\mathrm{LCF}(\mathfrak{F})$ for the $\mathbb{Q}$-vector spaces of $\mathbb{Q}$-valued constructible and locally constructible functions on $\mathfrak{F}$.

Here [7, §4] are some important properties of constructible sets.

Proposition 2.6. *Let $\mathfrak{F},\mathfrak{G}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers, $\phi:\mathfrak{F}\to\mathfrak{G}$ a 1-morphism, and $A,B\subseteq\mathfrak{F}(\mathbb{K})$ constructible. Then $A\cup B$, $A\cap B$ and $A\setminus B$ are constructible in $\mathfrak{F}(\mathbb{K})$, and $\phi_*(A)$ is constructible in $\mathfrak{G}(\mathbb{K})$.*

Following [7, Definitions 4.8, 5.1 and 5.5] we define *pushforwards* and *pullbacks* of constructible functions along 1-morphisms.

Definition 2.7. In [7, §3.3] we define the *Euler characteristic* $\chi(\cdots)$ of constructible subsets in $\mathbb{K}$-schemes. In Section 5 we use the fact [7, Theorem 3.10(vi)] that

$$\chi\big(\mathbb{K}^m\big)=1 \quad \text{and} \quad \chi\big(\mathbb{KP}^m\big)=m+1 \quad \text{for all } m\geqslant 0. \tag{2}$$

Let $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack with affine geometric stabilizers, and $C\subseteq\mathfrak{F}(\mathbb{K})$ a constructible subset. Then [7, Definition 4.8] defines the *naïve Euler characteristic* $\chi^{\mathrm{na}}(C)$ of C. It is called *naïve* as it takes no account of stabilizer groups. For $f\in\mathrm{CF}(\mathfrak{F})$, define $\chi^{\mathrm{na}}(\mathfrak{F},f)$ in $\mathbb{Q}$ by $\chi^{\mathrm{na}}(\mathfrak{F},f)=\sum_{c\in f(\mathfrak{F}(\mathbb{K}))\setminus\{0\}}c\chi^{\mathrm{na}}(f^{-1}(c))$.

Let $\mathfrak{F}$, $\mathfrak{G}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers, and $\phi:\mathfrak{F}\to\mathfrak{G}$ a representable 1-morphism. Then for any $x\in\mathfrak{F}(\mathbb{K})$ we have an injective morphism $\phi_*:\mathrm{Iso}_{\mathbb{K}}(x)\to\mathrm{Iso}_{\mathbb{K}}(\phi_*(x))$ of affine algebraic $\mathbb{K}$-groups. The image $\phi_*(\mathrm{Iso}_{\mathbb{K}}(x))$ is an affine algebraic $\mathbb{K}$-group closed in $\mathrm{Iso}_{\mathbb{K}}(\phi_*(x))$, so the quotient $\mathrm{Iso}_{\mathbb{K}}(\phi_*(x))/\phi_*(\mathrm{Iso}_{\mathbb{K}}(x))$ exists as a quasiprojective $\mathbb{K}$-variety. Define a function $m_\phi:\mathfrak{F}(\mathbb{K})\to\mathbb{Z}$ by $m_\phi(x)=\chi(\mathrm{Iso}_{\mathbb{K}}(\phi_*(x))/\phi_*(\mathrm{Iso}_{\mathbb{K}}(x)))$ for $x\in\mathfrak{F}(\mathbb{K})$.

For $f\in\mathrm{CF}(\mathfrak{F})$, define $\mathrm{CF}^{\mathrm{stk}}(\phi)f:\mathfrak{G}(\mathbb{K})\to\mathbb{Q}$ by

$$\mathrm{CF}^{\mathrm{stk}}(\phi)f(y)=\chi^{\mathrm{na}}\big(\mathfrak{F},m_\phi\cdot f\cdot\delta_{\phi_*^{-1}(y)}\big) \quad \text{for } y\in\mathfrak{G}(\mathbb{K}),$$

where $\delta_{\phi_*^{-1}(y)}$ is the characteristic function of $\phi_*^{-1}(\{y\})\subseteq\mathfrak{G}(\mathbb{K})$ on $\mathfrak{G}(\mathbb{K})$. Then $\mathrm{CF}^{\mathrm{stk}}(\phi):\mathrm{CF}(\mathfrak{F})\to\mathrm{CF}(\mathfrak{G})$ is a $\mathbb{Q}$-linear map called the *stack pushforward*.

Let $\theta:\mathfrak{F}\to\mathfrak{G}$ be a finite type 1-morphism. If $C\subseteq\mathfrak{G}(\mathbb{K})$ is constructible then so is $\theta_*^{-1}(C)\subseteq\mathfrak{F}(\mathbb{K})$. It follows that if $f\in\mathrm{CF}(\mathfrak{G})$ then $f\circ\theta_*$ lies in $\mathrm{CF}(\mathfrak{F})$. Define the *pullback* $\theta^*:\mathrm{CF}(\mathfrak{G})\to\mathrm{CF}(\mathfrak{F})$ by $\theta^*(f)=f\circ\theta_*$. It is a linear map.

Here [7, Theorems 5.4 and 5.6 and Definition 5.5] are some properties of these.

Theorem 2.8. *Let $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers, and $\beta:\mathfrak{F}\to\mathfrak{G}$, $\gamma:\mathfrak{G}\to\mathfrak{H}$ be 1-morphisms. Then*

$$\mathrm{CF}^{\mathrm{stk}}(\gamma\circ\beta)=\mathrm{CF}^{\mathrm{stk}}(\gamma)\circ\mathrm{CF}^{\mathrm{stk}}(\beta):\mathrm{CF}(\mathfrak{F})\to\mathrm{CF}(\mathfrak{H}), \tag{3}$$

$$(\gamma\circ\beta)^*=\beta^*\circ\gamma^*:\mathrm{CF}(\mathfrak{H})\to\mathrm{CF}(\mathfrak{F}), \tag{4}$$

supposing β,γ representable in (3)*, and of finite type in* (4)*. If*

$$\begin{array}{ccc} \mathfrak{E} & \xrightarrow{\eta} & \mathfrak{G} \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\psi} \\ \mathfrak{F} & \xrightarrow{\phi} & \mathfrak{H} \end{array} \quad \begin{array}{l}\textit{is a Cartesian square with}\\ \eta,\phi \textit{ representable and}\\ \theta,\psi \textit{ of finite type, then}\\ \textit{the following commutes:}\end{array} \quad \begin{array}{ccc} \mathrm{CF}(\mathfrak{E}) & \xrightarrow{\mathrm{CF}^{\mathrm{stk}}(\eta)} & \mathrm{CF}(\mathfrak{G}) \\ \uparrow{\scriptstyle\theta^*} & & \uparrow{\scriptstyle\psi^*} \\ \mathrm{CF}(\mathfrak{F}) & \xrightarrow{\mathrm{CF}^{\mathrm{stk}}(\phi)} & \mathrm{CF}(\mathfrak{H}). \end{array} \tag{5}$$

As discussed in [7, §3.3] for the $\mathbb{K}$-scheme case, Eq. (3) is *false* for algebraically closed fields $\mathbb{K}$ of characteristic $p>0$. The definitions and results above all have analogues for locally constructible functions, [7, §5.3].

2.4. Stack functions

Stack functions are a universal generalization of constructible functions introduced in [8]. Here [8, Definition 3.1] is the basic definition. Throughout $\mathbb{K}$ is algebraically closed of arbitrary characteristic, except when we specify $\operatorname{char}\mathbb{K}=0$.

Definition 2.9. Let $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack with affine geometric stabilizers. Consider pairs $(\mathfrak{R},\rho)$, where $\mathfrak{R}$ is a finite type algebraic $\mathbb{K}$-stack with affine geometric stabilizers and $\rho:\mathfrak{R}\to\mathfrak{F}$ is a representable 1-morphism. We call two pairs $(\mathfrak{R},\rho)$, $(\mathfrak{R}',\rho')$ *equivalent* if there exists a 1-isomorphism $\iota:\mathfrak{R}\to\mathfrak{R}'$ such that $\rho'\circ\iota$ and ρ are 2-isomorphic 1-morphisms $\mathfrak{R}\to\mathfrak{F}$. Write $[(\mathfrak{R},\rho)]$ for the equivalence class of $(\mathfrak{R},\rho)$. If $(\mathfrak{R},\rho)$ is such a pair and $\mathfrak{S}$ is a closed $\mathbb{K}$-substack of $\mathfrak{R}$ then $(\mathfrak{S},\rho|_{\mathfrak{S}})$, $(\mathfrak{R}\setminus\mathfrak{S},\rho|_{\mathfrak{R}\setminus\mathfrak{S}})$ are pairs of the same kind. Define $\mathrm{SF}(\mathfrak{F})$ to be the $\mathbb{Q}$-vector space generated by equivalence classes $[(\mathfrak{R},\rho)]$ as above, with for each closed $\mathbb{K}$-substack $\mathfrak{S}$ of $\mathfrak{R}$ a relation

$$\big[(\mathfrak{R},\rho)\big]=\big[(\mathfrak{S},\rho|_{\mathfrak{S}})\big]+\big[(\mathfrak{R}\setminus\mathfrak{S},\rho|_{\mathfrak{R}\setminus\mathfrak{S}})\big].$$

In [8, Definition 3.2] we relate $\mathrm{CF}(\mathfrak{F})$ and $\mathrm{SF}(\mathfrak{F})$.

Definition 2.10. Let $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack with affine geometric stabilizers and $C\subseteq\mathfrak{F}(\mathbb{K})$ be constructible. Then $C=\coprod_{i=1}^n\mathfrak{R}_i(\mathbb{K})$, for $\mathfrak{R}_1,\dots,\mathfrak{R}_n$ finite type $\mathbb{K}$-substacks of $\mathfrak{F}$. Let $\rho_i:\mathfrak{R}_i\to\mathfrak{F}$ be the inclusion 1-morphism. Then $[(\mathfrak{R}_i,\rho_i)]\in\mathrm{SF}(\mathfrak{F})$. Define $\bar\delta_C=\sum_{i=1}^n[(\mathfrak{R}_i,\rho_i)]\in\mathrm{SF}(\mathfrak{F})$. We think of this stack function as the analogue of the characteristic function $\delta_C\in\mathrm{CF}(\mathfrak{F})$ of C. Define a $\mathbb{Q}$-linear map $\iota_{\mathfrak{F}}:\mathrm{CF}(\mathfrak{F})\to\mathrm{SF}(\mathfrak{F})$ by $\iota_{\mathfrak{F}}(f)=\sum_{0\neq c\in f(\mathfrak{F}(\mathbb{K}))}c\cdot\bar\delta_{f^{-1}(c)}$. For $\mathbb{K}$ of characteristic zero, define a $\mathbb{Q}$-linear map $\pi^{\mathrm{stk}}_{\mathfrak{F}}:\mathrm{SF}(\mathfrak{F})\to\mathrm{CF}(\mathfrak{F})$ by

$$\pi^{\mathrm{stk}}_{\mathfrak{F}}\left(\sum_{i=1}^n c_i\big[(\mathfrak{R}_i,\rho_i)\big]\right)=\sum_{i=1}^n c_i\,\mathrm{CF}^{\mathrm{stk}}(\rho_i)1_{\mathfrak{R}_i},$$

where $1_{\mathfrak{R}_i}$ is the function 1 in $\mathrm{CF}(\mathfrak{R}_i)$. Then [8, Proposition 3.3] shows $\pi^{\mathrm{stk}}_{\mathfrak{F}}\circ\iota_{\mathfrak{F}}$ is the identity on $\mathrm{CF}(\mathfrak{F})$. Thus, $\iota_{\mathfrak{F}}$ is injective and $\pi^{\mathrm{stk}}_{\mathfrak{F}}$ is surjective. In general $\iota_{\mathfrak{F}}$ is far from being surjective, and $\mathrm{SF}(\mathfrak{F})$ is much larger than $\mathrm{CF}(\mathfrak{F})$.

In [8, Definition 3.4] we define *pushforwards*, *pullbacks* and *tensor products*.

Definition 2.11. Let $\phi:\mathfrak{F}\to\mathfrak{G}$ be a 1-morphism of algebraic $\mathbb{K}$-stacks with affine geometric stabilizers. For ϕ representable, define the *pushforward* $\phi_*:\mathrm{SF}(\mathfrak{F})\to\mathrm{SF}(\mathfrak{G})$ by $\phi_*:\sum_{i=1}^n c_i[(\mathfrak{R}_i,\rho_i)]\mapsto\sum_{i=1}^n c_i[(\mathfrak{R}_i,\phi\circ\rho_i)]$. For ϕ of finite type, define the *pullback* $\phi^*:\mathrm{SF}(\mathfrak{G})\to\mathrm{SF}(\mathfrak{F})$ by

$$\phi^*:\sum_{i=1}^n c_i\big[(\mathfrak{R}_i,\rho_i)\big]\mapsto\sum_{i=1}^n c_i\big[(\mathfrak{R}_i\times_{\rho_i,\mathfrak{G},\phi}\mathfrak{F},\pi_{\mathfrak{F}})\big].$$

The *tensor product* $\otimes : \mathrm{SF}(\mathfrak{F}) \times \mathrm{SF}(\mathfrak{G}) \to \mathrm{SF}(\mathfrak{F} \times \mathfrak{G})$ is

$$\left(\sum_{i=1}^{m} c_i\big[(\mathfrak{R}_i, \rho_i)\big]\right) \otimes \left(\sum_{j=1}^{n} d_j\big[(\mathfrak{S}_j, \sigma_j)\big]\right) = \sum_{i,j} c_i d_j\big[(\mathfrak{R}_i \times \mathfrak{S}_j, \rho_i \times \sigma_j)\big].$$

Here [8, Theorem 3.5] is the analogue of Theorem 2.8.

Theorem 2.12. *Let $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers, and $\beta : \mathfrak{F} \to \mathfrak{G}$, $\gamma : \mathfrak{G} \to \mathfrak{H}$ be* 1*-morphisms. Then*

$$(\gamma \circ \beta)_* = \gamma_* \circ \beta_* : \mathrm{SF}(\mathfrak{F}) \to \mathrm{SF}(\mathfrak{H}), \qquad (\gamma \circ \beta)^* = \beta^* \circ \gamma^* : \mathrm{SF}(\mathfrak{H}) \to \mathrm{SF}(\mathfrak{F}),$$

for β, γ representable in the first equation, and of finite type in the second. If

$$\begin{array}{ccc} \mathfrak{E} & \xrightarrow{\eta} & \mathfrak{G} \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\psi} \\ \mathfrak{F} & \xrightarrow{\phi} & \mathfrak{H} \end{array}$$

is a Cartesian square with η, ϕ representable and θ, ψ of finite type, then the following commutes:

$$\begin{array}{ccc} \mathrm{SF}(\mathfrak{E}) & \xrightarrow{\eta_*} & \mathrm{SF}(\mathfrak{G}) \\ \uparrow{\scriptstyle\theta^*} & & \uparrow{\scriptstyle\psi^*} \\ \mathrm{SF}(\mathfrak{F}) & \xrightarrow{\phi_*} & \mathrm{SF}(\mathfrak{H}). \end{array}$$

In [8, Proposition 3.7 and Theorem 3.8] we relate pushforwards and pullbacks of stack and constructible functions using $\iota_{\mathfrak{F}}, \pi^{\mathrm{stk}}_{\mathfrak{F}}$.

Theorem 2.13. *Let $\mathbb{K}$ have characteristic zero, $\mathfrak{F}, \mathfrak{G}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers, and $\phi : \mathfrak{F} \to \mathfrak{G}$ be a* 1*-morphism. Then*

(a) $\phi^* \circ \iota_{\mathfrak{G}} = \iota_{\mathfrak{F}} \circ \phi^* : \mathrm{CF}(\mathfrak{G}) \to \mathrm{SF}(\mathfrak{F})$ *if ϕ is of finite type*;
(b) $\pi^{\mathrm{stk}}_{\mathfrak{G}} \circ \phi_* = \mathrm{CF}^{\mathrm{stk}}(\phi) \circ \pi^{\mathrm{stk}}_{\mathfrak{F}} : \mathrm{SF}(\mathfrak{F}) \to \mathrm{CF}(\mathfrak{G})$ *if ϕ is representable*; *and*
(c) $\pi^{\mathrm{stk}}_{\mathfrak{F}} \circ \phi^* = \phi^* \circ \pi^{\mathrm{stk}}_{\mathfrak{G}} : \mathrm{SF}(\mathfrak{G}) \to \mathrm{CF}(\mathfrak{F})$ *if ϕ is of finite type.*

In [8, §5.2] we define *projections* $\Pi^{\mathrm{vi}}_n : \mathrm{SF}(\mathfrak{F}) \to \mathrm{SF}(\mathfrak{F})$ which project to stack functions whose stabilizer groups have 'virtual rank' n.

In [8, §3] we define *local stack functions* $\mathrm{LSF}(\mathfrak{F})$, the analogue of locally constructible functions. Analogues of Definitions 2.10–2.11 and Theorems 2.12–2.13 hold for $\mathrm{LSF}(\mathfrak{F})$, with differences in which 1-morphisms are required to be of finite type. We also study enlarged versions $\underline{\mathrm{SF}}(\mathfrak{F}), \underline{\mathrm{LSF}}(\mathfrak{F})$ of $\mathrm{SF}(\mathfrak{F}), \mathrm{LSF}(\mathfrak{F})$ in which the 1-morphisms ρ of Definition 2.9 are not supposed representable.

In [8, §4–§6] we define other classes of stack functions $\underline{\mathrm{SF}}$, $\overline{\underline{\mathrm{SF}}}$, $\overline{\mathrm{SF}}(\mathfrak{F}, \Upsilon, \Lambda)$, $\underline{\mathrm{SF}}$, $\overline{\mathrm{SF}}(\mathfrak{F}, \Upsilon, \Lambda^\circ)$, $\underline{\mathrm{SF}}$, $\overline{\mathrm{SF}}(\mathfrak{F}, \Theta, \Omega)$ 'twisted' by a *motivic invariant* Υ or Θ of $\mathbb{K}$-varieties, taking values in a $\mathbb{Q}$-algebra Λ, Λ° or Ω; the basic facts are explained in [10, §2.4–§2.5]. All the above material on $\mathrm{SF}(\cdots)$ applies to these spaces, except that $\pi^{\mathrm{stk}}_{\mathfrak{F}}$, Π^{vi}_n are not always defined. For the purposes of this paper the differences between these spaces are unimportant, so we shall not explain them.

3. Background on configurations from [9,10]

We now recall in Sections 3.1 and 3.2 the main definitions and results from [9] on $(I,\preccurlyeq)$-configurations and their moduli stacks that we will need later, and in Section 3.3 some facts about algebras of constructible and stack functions from [10].

3.1. Basic definitions

Here is some notation for *finite posets*, taken from [9, Definitions 3.2, 4.1 and 6.1].

Definition 3.1. A *finite partially ordered set* or *finite poset* $(I,\preccurlyeq)$ is a finite set I with a partial order I. Define $J\subseteq I$ to be an *f-set* if $i\in I$ and $h,j\in J$ and $h\preccurlyeq i\preccurlyeq j$ implies $i\in J$. Define $\mathcal{F}_{(I,\preccurlyeq)}$ to be the set of f-sets of I. Define $\mathcal{G}_{(I,\preccurlyeq)}$ to be the subset of $(J,K)\in\mathcal{F}_{(I,\preccurlyeq)}\times\mathcal{F}_{(I,\preccurlyeq)}$ such that $J\subseteq K$, and if $j\in J$ and $k\in K$ with $k\preccurlyeq j$, then $k\in J$. Define $\mathcal{H}_{(I,\preccurlyeq)}$ to be the subset of $(J,K)\in\mathcal{F}_{(I,\preccurlyeq)}\times\mathcal{F}_{(I,\preccurlyeq)}$ such that $K\subseteq J$, and if $j\in J$ and $k\in K$ with $k\preccurlyeq j$, then $j\in K$.

Let I be a finite set and $\preccurlyeq$, $\trianglelefteq$ partial orders on I such that if $i\preccurlyeq j$ then $i\trianglelefteq j$ for $i,j\in I$. Then we say that $\trianglelefteq$ *dominates* $\preccurlyeq$. Let s be the number of pairs $(i,j)\in I\times I$ with $i\trianglelefteq j$ but $i\not\preccurlyeq j$. Then we say that $\trianglelefteq$ *dominates* $\preccurlyeq$ *by s steps*.

A partial order $\trianglelefteq$ on I is called a *total order* if $i\trianglelefteq j$ or $j\trianglelefteq i$ for all $i,j\in I$. Then $(I,\trianglelefteq)$ is canonically isomorphic to $(\{1,\dots,n\},\leqslant)$ for $n=|I|$. Every partial order $\preccurlyeq$ on I is dominated by a total order $\trianglelefteq$.

We define $(I,\preccurlyeq)$-*configurations*, [9, Definition 4.1].

Definition 3.2. Let $(I,\preccurlyeq)$ be a finite poset, and use the notation of Definition 3.1. Define an $(I,\preccurlyeq)$-*configuration* (σ,ι,π) in an abelian category $\mathcal{A}$ to be maps $\sigma:\mathcal{F}_{(I,\preccurlyeq)}\to\mathrm{Obj}(\mathcal{A})$, $\iota:\mathcal{G}_{(I,\preccurlyeq)}\to\mathrm{Mor}(\mathcal{A})$, and $\pi:\mathcal{H}_{(I,\preccurlyeq)}\to\mathrm{Mor}(\mathcal{A})$, where

(i) $\sigma(J)$ is an object in $\mathcal{A}$ for $J\in\mathcal{F}_{(I,\preccurlyeq)}$, with $\sigma(\emptyset)=0$.
(ii) $\iota(J,K):\sigma(J)\to\sigma(K)$ is injective for $(J,K)\in\mathcal{G}_{(I,\preccurlyeq)}$, and $\iota(J,J)=\mathrm{id}_{\sigma(J)}$.
(iii) $\pi(J,K):\sigma(J)\to\sigma(K)$ is surjective for $(J,K)\in\mathcal{H}_{(I,\preccurlyeq)}$, and $\pi(J,J)=\mathrm{id}_{\sigma(J)}$.

These should satisfy the conditions:

(A) Let $(J,K)\in\mathcal{G}_{(I,\preccurlyeq)}$ and set $L=K\setminus J$. Then the following is exact in $\mathcal{A}$:

$$0\to\sigma(J)\xrightarrow{\iota(J,K)}\sigma(K)\xrightarrow{\pi(K,L)}\sigma(L)\to 0.$$

(B) If $(J,K)\in\mathcal{G}_{(I,\preccurlyeq)}$ and $(K,L)\in\mathcal{G}_{(I,\preccurlyeq)}$ then $\iota(J,L)=\iota(K,L)\circ\iota(J,K)$.
(C) If $(J,K)\in\mathcal{H}_{(I,\preccurlyeq)}$ and $(K,L)\in\mathcal{H}_{(I,\preccurlyeq)}$ then $\pi(J,L)=\pi(K,L)\circ\pi(J,K)$.
(D) If $(J,K)\in\mathcal{G}_{(I,\preccurlyeq)}$ and $(K,L)\in\mathcal{H}_{(I,\preccurlyeq)}$ then

$$\pi(K,L)\circ\iota(J,K)=\iota(J\cap L,L)\circ\pi(J,J\cap L).$$

A *morphism* $\alpha:(\sigma,\iota,\pi)\to(\sigma',\iota',\pi')$ of $(I,\preccurlyeq)$-configurations in $\mathcal{A}$ is a collection of morphisms $\alpha(J):\sigma(J)\to\sigma'(J)$ for each $J\in\mathcal{F}_{(I,\preccurlyeq)}$ satisfying

$$\alpha(K)\circ\iota(J,K)=\iota'(J,K)\circ\alpha(J)\quad\text{for all }(J,K)\in\mathcal{G}_{(I,\preccurlyeq)},\quad\text{and}$$
$$\alpha(K)\circ\pi(J,K)=\pi'(J,K)\circ\alpha(J)\quad\text{for all }(J,K)\in\mathcal{H}_{(I,\preccurlyeq)}.$$

It is an *isomorphism* if $\alpha(J)$ is an isomorphism for all $J\in\mathcal{F}_{(I,\preccurlyeq)}$.

In [9, Proposition 4.7] we relate the classes $[\sigma(J)]$ in $K_0(\mathcal{A})$.

Proposition 3.3. *Let (σ,ι,π) be an $(I,\preccurlyeq)$-configuration in an abelian category $\mathcal{A}$. Then there exists a unique map $\kappa:I\to K_0(\mathcal{A})$ such that $[\sigma(J)]=\sum_{j\in J}\kappa(j)$ in $K_0(\mathcal{A})$ for all f-sets $J\subseteq I$.*

Here [9, Definitions 5.1, 5.2] are two ways to construct new configurations.

Definition 3.4. Let $(I,\preccurlyeq)$ be a finite poset and $J\in\mathcal{F}_{(I,\preccurlyeq)}$. Then $(J,\preccurlyeq)$ is a finite poset, and $\mathcal{F}_{(J,\preccurlyeq)},\mathcal{G}_{(J,\preccurlyeq)},\mathcal{H}_{(J,\preccurlyeq)}\subseteq\mathcal{F}_{(I,\preccurlyeq)},\mathcal{G}_{(I,\preccurlyeq)},\mathcal{H}_{(I,\preccurlyeq)}$. Let (σ,ι,π) be an $(I,\preccurlyeq)$-configuration in an abelian category $\mathcal{A}$. The $(J,\preccurlyeq)$-*subconfiguration* (σ',ι',π') of (σ,ι,π) is given by $\sigma'=\sigma|_{\mathcal{F}_{(J,\preccurlyeq)}}$, $\iota'=\iota|_{\mathcal{G}_{(J,\preccurlyeq)}}$ and $\pi'=\pi|_{\mathcal{H}_{(J,\preccurlyeq)}}$.

Let $(I,\preccurlyeq),(K,\trianglelefteq)$ be finite posets, and $\phi:I\to K$ be surjective with $i\preccurlyeq j$ implies $\phi(i)\trianglelefteq\phi(j)$. Then ϕ^{-1} maps $\mathcal{F}_{(K,\trianglelefteq)},\mathcal{G}_{(K,\trianglelefteq)},\mathcal{H}_{(K,\trianglelefteq)}\to\mathcal{F}_{(I,\preccurlyeq)},\mathcal{G}_{(I,\preccurlyeq)},\mathcal{H}_{(I,\preccurlyeq)}$. Let (σ,ι,π) be an $(I,\preccurlyeq)$-configuration in an abelian category $\mathcal{A}$. Define the *quotient* $(K,\trianglelefteq)$-*configuration* $(\tilde\sigma,\tilde\iota,\tilde\pi)$ by $\tilde\sigma(A)=\sigma(\phi^{-1}(A))$ for $A\in\mathcal{F}_{(K,\trianglelefteq)}$, $\tilde\iota(A,B)=\iota(\phi^{-1}(A),\phi^{-1}(B))$ for $(A,B)\in\mathcal{G}_{(K,\trianglelefteq)}$, and $\tilde\pi(A,B)=\pi(\phi^{-1}(A),\phi^{-1}(B))$ for $(A,B)\in\mathcal{H}_{(K,\trianglelefteq)}$. We call (σ,ι,π) a *refinement* of $(\tilde\sigma,\tilde\iota,\tilde\pi)$.

Following [9, Definition 6.1] we define *improvements* and *best configurations*.

Definition 3.5. Let $(I,\preccurlyeq)$ be a finite poset and $\trianglelefteq$ a partial order on I dominating $\preccurlyeq$, as in Definition 3.1. Let $\mathcal{A}$ be an abelian category. For each $(I,\preccurlyeq)$-configuration (σ,ι,π) in $\mathcal{A}$ we have a quotient $(I,\trianglelefteq)$-configuration $(\tilde\sigma,\tilde\iota,\tilde\pi)$, as in Definition 3.4 with $\phi=\mathrm{id}:I\to I$. We call (σ,ι,π) an *improvement* or an $(I,\preccurlyeq)$-*improvement* of $(\tilde\sigma,\tilde\iota,\tilde\pi)$, and a *strict improvement* if $\preccurlyeq,\trianglelefteq$ are distinct. If $\trianglelefteq$ dominates $\preccurlyeq$ by s steps we also call (σ,ι,π) an *s step improvement* of $(\tilde\sigma,\tilde\iota,\tilde\pi)$. We call an $(I,\trianglelefteq)$-configuration $(\tilde\sigma,\tilde\iota,\tilde\pi)$ *best* if there exists no strict improvement (σ,ι,π) of $(\tilde\sigma,\tilde\iota,\tilde\pi)$. Note that improvements are a special kind of *refinement*.

In [9, Proposition 6.9 and Theorem 6.10] we classify one step improvements and prove a criterion for best $(I,\trianglelefteq)$-configurations. Recall that a short exact sequence $0\to X\to Y\to Z\to 0$ in $\mathcal{A}$ is *split* if there is a compatible isomorphism $Y\cong X\oplus Z$.

Theorem 3.6. *Let $(I,\trianglelefteq)$ be a finite poset. Call $i,j\in I$* consecutive *if $i\trianglelefteq j$ with $i\neq j$, but there exists no $k\in I$ with $i\neq k\neq j$ and $i\trianglelefteq k\trianglelefteq j$. That is, i,j are distinct with $i\trianglelefteq j$, and no other $k\in I$ lies between i,j in the order $\trianglelefteq$.*

An $(I,\trianglelefteq)$-configuration (σ,ι,π) in an abelian category $\mathcal{A}$ is best if and only if for all consecutive i,j in I, the following short exact sequence is not split:

$$0\to\sigma(\{i\})\xrightarrow{\iota(\{i\},\{i,j\})}\sigma(\{i,j\})\xrightarrow{\pi(\{i,j\},\{j\})}\sigma(\{j\})\to 0.\tag{6}$$

Suppose i, j are consecutive, and (6) *is split. Define $\precsim$ on I by $a \precsim b$ if $a \trianglelefteq b$ and $a \neq i$, $b \neq j$, so that $\trianglelefteq$ dominates $\precsim$ by one step. Then the $(I, \precsim)$-improvements of (σ, ι, π) are in* 1-1 *correspondence with* $\mathrm{Hom}(\sigma(\{j\}), \sigma(\{i\}))$.

3.2. Moduli stacks of configurations

Here [9, Assumptions 7.1 and 8.1] is the data we require.

Assumption 3.7. Let $\mathbb{K}$ be an algebraically closed field and $\mathcal{A}$ a $\mathbb{K}$-linear noetherian abelian category with $\mathrm{Ext}^i(X, Y)$ finite-dimensional vector spaces over $\mathbb{K}$ for all $X, Y \in \mathcal{A}$ and $i \geqslant 0$. Let $K(\mathcal{A})$ be the quotient of the Grothendieck group $K_0(\mathcal{A})$ by some fixed subgroup. Suppose that if $X \in \mathcal{A}$ with $[X] = 0$ in $K(\mathcal{A})$ then $X \cong 0$.

To define moduli stacks of objects or configurations in $\mathcal{A}$, we need some *extra data*, to tell us about algebraic families of objects and morphisms in $\mathcal{A}$, parametrized by a base scheme U. We encode this extra data as a *stack in exact categories* $\mathfrak{F}_{\mathcal{A}}$ on the *category of* $\mathbb{K}$-*schemes* $\mathrm{Sch}_{\mathbb{K}}$, made into a *site* with the *étale topology*. The $\mathbb{K}$, $\mathcal{A}$, $K(\mathcal{A})$, $\mathfrak{F}_{\mathcal{A}}$ must satisfy some complex additional conditions [9, Assumptions 7.1 and 8.1], which we do not give.

Note that [9,10] did not assume $\mathcal{A}$ *noetherian*, but we need this to make τ-semistability well-behaved, so we suppose it from the outset. All the examples of [9, §9–§10] have $\mathcal{A}$ noetherian. Here is some new notation.

Definition 3.8. We work in the situation of Assumption 3.7. Define

$$C(\mathcal{A}) = \big\{[X] \in K(\mathcal{A}):\ X \in \mathcal{A},\ X \ncong 0\big\} \subset K(\mathcal{A}). \tag{7}$$

That is, $C(\mathcal{A})$ is the collection of classes in $K(\mathcal{A})$ of *nonzero objects* $X \in \mathcal{A}$. Note that $C(\mathcal{A})$ is *closed under addition*, as $[X \oplus Y] = [X] + [Y]$. Note also that $0 \notin C(\mathcal{A})$, as by Assumption 3.7 if $X \ncong 0$ then $[X] \neq 0$ in $K(\mathcal{A})$.

In [9,10] we worked mostly with $\overline{C}(\mathcal{A}) = C(\mathcal{A}) \cup \{0\}$, the collection of classes in $K(\mathcal{A})$ of all objects $X \in \mathcal{A}$. But here and in [11] we find $C(\mathcal{A})$ more useful, as stability conditions will be defined only on nonzero objects. We think of $C(\mathcal{A})$ as the 'positive cone' and $\overline{C}(\mathcal{A})$ as the 'closed positive cone' in $K(\mathcal{A})$.

Define a set of $\mathcal{A}$-*data* to be a triple $(I, \precsim, \kappa)$ such that $(I, \precsim)$ is a finite poset and $\kappa : I \to C(\mathcal{A})$ a map. We *extend κ to the set of subsets of I* by defining $\kappa(J) = \sum_{j \in J} \kappa(j)$. Then $\kappa(J) \in C(\mathcal{A})$ for all $\emptyset \neq J \subseteq I$, as $C(\mathcal{A})$ is closed under addition. Define an $(I, \precsim, \kappa)$-*configuration* to be an $(I, \precsim)$-configuration (σ, ι, π) in $\mathcal{A}$ with $[\sigma(\{i\})] = \kappa(i)$ in $K(\mathcal{A})$ for all $i \in I$. Then $[\sigma(J)] = \kappa(J)$ for all $J \in \mathcal{F}_{(I,\precsim)}$, by Proposition 3.3.

In the situation above, we define the following $\mathbb{K}$-stacks [9, Definitions 7.2 and 7.4]:

- The *moduli stacks* $\mathfrak{Obj}_{\mathcal{A}}$ of *objects in* $\mathcal{A}$, and $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}$ of *objects in $\mathcal{A}$ with class α in $K(\mathcal{A})$*, for each $\alpha \in \overline{C}(\mathcal{A})$. They are algebraic $\mathbb{K}$-stacks, locally of finite type, with $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}$ an open and closed $\mathbb{K}$-substack of $\mathfrak{Obj}_{\mathcal{A}}$. The underlying geometric spaces $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$, $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K})$ are the sets of isomorphism classes of objects X in $\mathcal{A}$, with $[X] = \alpha$ for $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K})$.

- The *moduli stacks* $\mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}}$ of $(I,\preccurlyeq)$-*configurations* and $\mathfrak{M}(I,\preccurlyeq,\kappa)_{\mathcal{A}}$ of $(I,\preccurlyeq,\kappa)$-*configurations in* $\mathcal{A}$, for all finite posets $(I,\preccurlyeq)$ and $\kappa: I \to \overline{C}(\mathcal{A})$. They are algebraic $\mathbb{K}$-stacks, locally of finite type, with $\mathfrak{M}(I,\preccurlyeq,\kappa)_{\mathcal{A}}$ an open and closed $\mathbb{K}$-substack of $\mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}}$. Write $\mathcal{M}(I,\preccurlyeq)_{\mathcal{A}}$, $\mathcal{M}(I,\preccurlyeq,\kappa)_{\mathcal{A}}$ for the underlying geometric spaces $\mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}}(\mathbb{K})$, $\mathfrak{M}(I,\preccurlyeq,\kappa)_{\mathcal{A}}(\mathbb{K})$. Then $\mathcal{M}(I,\preccurlyeq)_{\mathcal{A}}$ and $\mathcal{M}(I,\preccurlyeq,\kappa)_{\mathcal{A}}$ are the *sets of isomorphism classes of* $(I,\preccurlyeq)$- *and* $(I,\preccurlyeq,\kappa)$-*configurations in* $\mathcal{A}$, by [9, Proposition 7.6].

Each stabilizer group $\mathrm{Iso}_{\mathbb{K}}([X])$ or $\mathrm{Iso}_{\mathbb{K}}([(\sigma,\iota,\pi)])$ in $\mathfrak{Obj}_{\mathcal{A}}$ or $\mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}}$ is the group of invertible elements in the finite-dimensional $\mathbb{K}$-algebra $\mathrm{End}(X)$ or $\mathrm{End}((\sigma,\iota,\pi))$. Thus $\mathfrak{Obj}_{\mathcal{A}}, \mathfrak{Obj}^{\alpha}_{\mathcal{A}}, \mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}}, \mathfrak{M}(I,\preccurlyeq,\kappa)_{\mathcal{A}}$ have *affine geometric stabilizers*, which is required to use the results of Sections 2.3–2.4.

In [9, Definition 7.7 and Proposition 7.8] we define 1-morphisms of $\mathbb{K}$-stacks, as follows:

- For $(I,\preccurlyeq)$ a finite poset, $\kappa: I\to\overline{C}(\mathcal{A})$ and $J\in\mathcal{F}_{(I,\preccurlyeq)}$, we define $\boldsymbol{\sigma}(J):\mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}}\to\mathfrak{Obj}_{\mathcal{A}}$ or $\boldsymbol{\sigma}(J):\mathfrak{M}(I,\preccurlyeq,\kappa)_{\mathcal{A}}\to\mathfrak{Obj}^{\kappa(J)}_{\mathcal{A}}$. The induced maps $\boldsymbol{\sigma}(J)_*:\mathcal{M}(I,\preccurlyeq)_{\mathcal{A}}\to\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ or $\mathcal{M}(I,\preccurlyeq,\kappa)_{\mathcal{A}}\to\mathfrak{Obj}^{\kappa(J)}_{\mathcal{A}}(\mathbb{K})$ act by $\boldsymbol{\sigma}(J)_*:[(\sigma,\iota,\pi)]\mapsto[\sigma(J)]$.
- For $(I,\preccurlyeq)$ a finite poset, $\kappa: I\to\overline{C}(\mathcal{A})$ and $J\in\mathcal{F}_{(I,\preccurlyeq)}$, we define the $(J,\preccurlyeq)$-*subconfiguration* 1-*morphism* $S(I,\preccurlyeq,J):\mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}}\to\mathfrak{M}(J,\preccurlyeq)_{\mathcal{A}}$ or $S(I,\preccurlyeq,J):\mathfrak{M}(I,\preccurlyeq,\kappa)_{\mathcal{A}}\to\mathfrak{M}(J,\preccurlyeq,\kappa|_J)_{\mathcal{A}}$. The induced maps $S(I,\preccurlyeq,J)_*$ act by $S(I,\preccurlyeq,J)_*:[(\sigma,\iota,\pi)]\mapsto[(\sigma',\iota',\pi')]$, where (σ,ι,π) is an $(I,\preccurlyeq)$-configuration in $\mathcal{A}$, and (σ',ι',π') its $(J,\preccurlyeq)$-subconfiguration.
- Let $(I,\preccurlyeq)$, $(K,\trianglelefteq)$ be finite posets, $\kappa: I\to\overline{C}(\mathcal{A})$, and $\phi: I\to K$ be surjective with $i\preccurlyeq j$ implies $\phi(i)\trianglelefteq\phi(j)$ for $i,j\in I$. Define $\mu: K\to\overline{C}(\mathcal{A})$ by $\mu(k)=\kappa(\phi^{-1}(k))$. The *quotient* $(K,\trianglelefteq)$-*configuration* 1-*morphisms* are

$$Q(I,\preccurlyeq,K,\trianglelefteq,\phi):\mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}}\to\mathfrak{M}(K,\trianglelefteq)_{\mathcal{A}}, \tag{8}$$

$$Q(I,\preccurlyeq,K,\trianglelefteq,\phi):\mathfrak{M}(I,\preccurlyeq,\kappa)_{\mathcal{A}}\to\mathfrak{M}(K,\trianglelefteq,\mu)_{\mathcal{A}}. \tag{9}$$

  The induced maps $Q(I,\preccurlyeq,K,\trianglelefteq,\phi)_*$ act by $Q(I,\preccurlyeq,K,\trianglelefteq,\phi)_*:[(\sigma,\iota,\pi)]\mapsto[(\tilde\sigma,\tilde\iota,\tilde\pi)]$, where (σ,ι,π) is an $(I,\preccurlyeq)$-configuration in $\mathcal{A}$, and $(\tilde\sigma,\tilde\iota,\tilde\pi)$ its quotient $(K,\trianglelefteq)$-configuration from ϕ. When $I=K$ and $\phi: I\to I$ is the identity id_I, write $Q(I,\preccurlyeq,\trianglelefteq)=Q(I,\preccurlyeq,I,\trianglelefteq,\mathrm{id}_I)$. Then $\mu=\kappa$, so that

$$Q(I,\preccurlyeq,\trianglelefteq):\mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}}\to\mathfrak{M}(I,\trianglelefteq)_{\mathcal{A}}, \tag{10}$$

$$Q(I,\preccurlyeq,\trianglelefteq):\mathfrak{M}(I,\preccurlyeq,\kappa)_{\mathcal{A}}\to\mathfrak{M}(I,\trianglelefteq,\kappa)_{\mathcal{A}}. \tag{11}$$

Here [9, Theorem 8.4] are some properties of these 1-morphisms:

Theorem 3.9.

(a) $Q(I,\preccurlyeq,K,\trianglelefteq,\phi)$, $Q(I,\preccurlyeq,\trianglelefteq)$ *in* (8)–(11) *are representable, and* (9), (11) *are of finite type.*

(b) $\boldsymbol{\sigma}(I):\mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}}\to\mathfrak{Obj}_{\mathcal{A}}$ *is representable, and* $\boldsymbol{\sigma}(I):\mathfrak{M}(I,\preccurlyeq,\kappa)_{\mathcal{A}}\to\mathfrak{Obj}^{\kappa(I)}_{\mathcal{A}}$ *is representable and of finite type.*

(c) $\prod_{i\in I}\boldsymbol{\sigma}(\{i\}):\mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}}\to\prod_{i\in I}\mathfrak{Obj}_{\mathcal{A}}$ *and* $\prod_{i\in I}\boldsymbol{\sigma}(\{i\}):\mathfrak{M}(I,\preccurlyeq,\kappa)_{\mathcal{A}}\to\prod_{i\in I}\mathfrak{Obj}^{\kappa(i)}_{\mathcal{A}}$ *are of finite type.*

We also define some more moduli spaces $\mathfrak{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}}$ in [9, Definition 8.5], for which $\sigma(I)$ is a fixed object $X \in \mathcal{A}$.

Definition 3.10. In the situation above, let $X \in \mathcal{A}$. Then X corresponds to a 1-morphism $X : \operatorname{Spec}\mathbb{K} \to \mathfrak{Obj}_{\mathcal{A}}^{[X]}$. For $\mathcal{A}$-data $(I, \preccurlyeq, \kappa)$ with $\kappa(I) = [X]$ in $K(\mathcal{A})$, define an algebraic $\mathbb{K}$-stack

$$\mathfrak{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}} = \mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}} \times_{\boldsymbol{\sigma}(I), \mathfrak{Obj}_{\mathcal{A}}^{\kappa(I)}, X} \operatorname{Spec}\mathbb{K}.$$

Theorem 3.9(b) implies $\mathfrak{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}}$ is represented by a *finite type algebraic* $\mathbb{K}$*-space*. Write $\Pi_X : \mathfrak{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}} \to \mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$ for projection 1-morphism. It is of finite type. Write $\mathcal{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}} = \mathfrak{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}}(\mathbb{K})$ for the underlying geometric space. Then [9, Proposition 8.6] identifies $\mathcal{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}}$ with the set of isomorphism classes of $(I, \preccurlyeq, \kappa)$-configurations (σ, ι, π) in $\mathcal{A}$ with $\sigma(I) = X$, modulo isomorphisms $\alpha : (\sigma, \iota, \pi) \to (\sigma', \iota', \pi')$ of $(I, \preccurlyeq)$-configurations with $\alpha(I) = \mathrm{id}_X$.

The 1-morphisms $Q(I, \preccurlyeq, K, \trianglelefteq, \phi)$, $Q(I, \preccurlyeq, \trianglelefteq)$ above on $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$ have analogues for $\mathfrak{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}}$, denoted the same way.

In [9, §9–§10] we define the data $\mathcal{A}$, $K(\mathcal{A})$, $\mathfrak{F}_{\mathcal{A}}$ in some large classes of examples, and prove Assumption 3.7 holds in each case.

3.3. Algebras of constructible and stack functions

Next we summarize parts of [10], which define and study associative multiplications $*$ on $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ and $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$, based on *Ringel–Hall algebras*.

Definition 3.11. Let Assumption 3.7 hold with $\mathbb{K}$ of characteristic zero. Write $\delta_{[0]} \in \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ for the characteristic function of $[0] \in \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$. Following [10, Definition 4.1], using the diagrams of 1-morphisms and pullbacks, pushforwards

$$\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}} \xleftarrow{\boldsymbol{\sigma}(\{1\})\times\boldsymbol{\sigma}(\{2\})} \mathfrak{M}(\{1,2\}, \leqslant)_{\mathcal{A}} \xrightarrow{\boldsymbol{\sigma}(\{1,2\})} \mathfrak{Obj}_{\mathcal{A}},$$

$$\begin{array}{ccccc}
\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \times \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) & \xrightarrow{(\boldsymbol{\sigma}(\{1\}))^*\cdot(\boldsymbol{\sigma}(\{2\}))^*} & & & \\
\downarrow \otimes & & & & \\
\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}}) & \xrightarrow{(\boldsymbol{\sigma}(\{1\})\times\boldsymbol{\sigma}(\{2\}))^*} & \mathrm{CF}(\mathfrak{M}(\{1,2\}, \leqslant)_{\mathcal{A}}) & \xrightarrow{\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(\{1,2\}))} & \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}),
\end{array}$$

define a bilinear operation $* : \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \times \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \to \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ by

$$f * g = \mathrm{CF}^{\mathrm{stk}}\big(\boldsymbol{\sigma}\big(\{1,2\}\big)\big)\big[\boldsymbol{\sigma}\big(\{1\}\big)^*(f) \cdot \boldsymbol{\sigma}\big(\{2\}\big)^*(g)\big]. \tag{12}$$

Then [10, Theorem 4.3] shows $*$ is *associative*, and $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ is a $\mathbb{Q}$-*algebra*, with identity $\delta_{[0]}$ and multiplication $*$.

Following [10, Definition 4.8], write $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ for the vector subspace of f in $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ *supported on indecomposables*, that is, $f([X]) \neq 0$ implies $0 \ncong X$ is indecomposable. Define a bilinear bracket $[\,,\,] : \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \times \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \to \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ by $[f, g] = f * g - g * f$. Since $*$

is associative, [,] satisfies the *Jacobi identity*, and makes $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ into a $\mathbb{Q}$-*Lie algebra*. Then [10, Theorem 4.9] shows $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ is closed under [,], and so is also a $\mathbb{Q}$-Lie algebra.

The next result follows from [10, Definition 4.13 and Proposition 4.14]. The important point is that Φ is an *isomorphism*, not just a homomorphism.

Proposition 3.12. *Suppose Assumption* 3.7 *holds with* $\mathbb{K}$ *of characteristic zero, and use the notation of Definition* 3.11. *Let* $\mathcal{L}$ *be a* $\mathbb{Q}$*-Lie subalgebra of* $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$*, and* $\mathcal{H}_{\mathcal{L}}$ *the* $\mathbb{Q}$*-subalgebra of* $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ *with identity generated by* $\mathcal{L}$*. Write* $U(\mathcal{L})$ *for the universal enveloping algebra of* $\mathcal{L}$*. Then the inclusion* $\mathcal{L}\subseteq\mathcal{H}_{\mathcal{L}}$ *induces a unique* $\mathbb{Q}$*-algebra isomorphism* $\Phi: U(\mathcal{L})\to\mathcal{H}_{\mathcal{L}}$ *with* $\Phi(1)=\delta_{[0]}$ *and* $\Phi(f_1\cdots f_n)=f_1*\cdots*f_n$ *for* $f_1,\ldots,f_n\in\mathcal{L}$*.*

In [10, §5] we extend much of the above to *stack functions*, as in Section 2.4. Here are a few of the basic definitions and results.

Definition 3.13. Suppose Assumption 3.7 holds. If $[(\mathfrak{R},\rho)]\in\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ and $r\in\mathfrak{R}(\mathbb{K})$ with $\rho_*(r)=[X]\in\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ for $X\in\mathcal{A}$, then ρ induces an injective morphism of stabilizer $\mathbb{K}$-groups $\rho_*:\mathrm{Iso}_{\mathbb{K}}(r)\to\mathrm{Iso}_{\mathbb{K}}([X])\cong\mathrm{Aut}(X)$, which induces an isomorphism of $\mathrm{Iso}_{\mathbb{K}}(r)$ with a $\mathbb{K}$-subgroup of $\mathrm{Aut}(X)$. Now $\mathrm{Aut}(X)$ is the $\mathbb{K}$-group of invertible elements in the $\mathbb{K}$-algebra $\mathrm{End}(X)=\mathrm{Hom}(X,X)$.

As in [10, Definition 5.5] define $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ to be the subspace of $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ spanned by $[(\mathfrak{R},\rho)]$ such that for all $r\in\mathfrak{R}(\mathbb{K})$ with $\rho_*(r)=[X]$, the $\mathbb{K}$-subgroup $\rho_*(\mathrm{Iso}_{\mathbb{K}}(r))$ in $\mathrm{Aut}(X)$ is the $\mathbb{K}$-group of invertible elements in a $\mathbb{K}$-subalgebra of $\mathrm{End}(X)$. Then $\iota_{\mathfrak{Obj}_{\mathcal{A}}}$ in Definition 2.10 maps $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})\to\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$.

By analogy with (12), using $\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}$ define [10, Definition 5.1] a bilinear operation $*:\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})\times\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})\to\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ by

$$f*g=\boldsymbol{\sigma}\big(\{1,2\}\big)_*\big[\big(\boldsymbol{\sigma}\big(\{1\}\big)\times\boldsymbol{\sigma}\big(\{2\}\big)\big)^*(f\otimes g)\big]. \tag{13}$$

Write $\bar{\delta}_{[0]}\in\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ for $\bar{\delta}_C$ in Definition 2.10 with $C=\{[0]\}$.

Then [10, Theorem 5.2 and Proposition 5.6] show that $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ is a $\mathbb{Q}$-algebra with associative multiplication $*$ and identity $\bar{\delta}_{[0]}$, and $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ is closed under $*$ and so is a $\mathbb{Q}$-subalgebra. When $\mathbb{K}$ has characteristic zero,

$$\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}:\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})\to\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \tag{14}$$

is a $\mathbb{Q}$-algebra morphism, where $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ is an algebra as in Definition 3.11.

Definition 3.14. Let Assumption 3.7 hold. Following [10, Definition 5.13], define $\mathrm{SF}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ to be the subspace of $f\in\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ with $\Pi^{\mathrm{vi}}_1(f)=f$, where Π^{vi}_1 is the operator of [8, §5.2], interpreted as projecting to stack functions 'supported on virtual indecomposables.' Write $[f,g]=f*g-g*f$ for $f,g\in\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$. As $*$ is associative [,] satisfies the *Jacobi identity*, and makes $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ into a $\mathbb{Q}$-*Lie algebra*. Then [10, Theorem 5.17] shows $\mathrm{SF}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ is closed under [,], and is a *Lie subalgebra*. When $\mathrm{char}\,\mathbb{K}=0$, (14) restricts to a Lie algebra morphism

$$\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}:\mathrm{SF}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})\to\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}}). \tag{15}$$

The above material also works for the other stack function spaces on $\mathfrak{Obj}_{\mathcal{A}}$, in particular for $\overline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$, $\overline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda^\circ)$ and $\overline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Theta, \Omega)$, giving algebras $\overline{\mathrm{SF}}, \overline{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$ and Lie algebras $\overline{\mathrm{SF}}{}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$.

4. Stability conditions

We now introduce our concepts of (*weak*) *stability condition* $(\tau, T, \leqslant)$ on $\mathcal{A}$, which are based on the stability conditions of Rudakov [16]. Perhaps their most important properties are Theorems 4.4 and 4.5 below. These show that for a weak stability condition $(\tau, T, \leqslant)$ with $\mathcal{A}$ noetherian and τ-artinian, each $X \in \mathcal{A}$ may be decomposed into τ-semistable factors S_k in a unique way, and if $(\tau, T, \leqslant)$ is a stability condition the S_k can be further split into τ-stable pieces. One moral of this is that *τ-stability is well-behaved for stability conditions, but badly behaved for weak stability conditions*.

4.1. Definitions and basic properties

Here is our notion of (*weak*) *stability condition*, generalizing Rudakov [16].

Definition 4.1. Let $\mathcal{A}$ be an abelian category, $K(\mathcal{A})$ be the quotient of $K_0(\mathcal{A})$ by some fixed subgroup, and $C(\mathcal{A})$ as in (7). Suppose $(T, \leqslant)$ is a totally ordered set, and $\tau : C(\mathcal{A}) \to T$ a map. We call $(\tau, T, \leqslant)$ a *stability condition* on $\mathcal{A}$ if whenever $\alpha, \beta, \gamma \in C(\mathcal{A})$ with $\beta = \alpha + \gamma$ then either $\tau(\alpha) < \tau(\beta) < \tau(\gamma)$, or $\tau(\alpha) > \tau(\beta) > \tau(\gamma)$, or $\tau(\alpha) = \tau(\beta) = \tau(\gamma)$. We call $(\tau, T, \leqslant)$ a *weak stability condition* on $\mathcal{A}$ if whenever $\alpha, \beta, \gamma \in C(\mathcal{A})$ with $\beta = \alpha + \gamma$ then either $\tau(\alpha) \leqslant \tau(\beta) \leqslant \tau(\gamma)$, or $\tau(\alpha) \geqslant \tau(\beta) \geqslant \tau(\gamma)$. Clearly, a stability condition is a weak stability condition, but not necessarily vice versa.

Our stability conditions are motivated by, and more-or-less equivalent to, Rudakov's [16, Definition 1.1]. The difference is that Rudakov's stability conditions are *preorders* on the nonzero objects of $\mathcal{A}$. In effect our definition requires Rudakov's preorder to factor through the map $\mathrm{Obj}(\mathcal{A}) \to K(\mathcal{A})$, $X \mapsto [X]$, and so amounts to a preorder on $C(\mathcal{A})$. Rudakov calls the trichotomy $\tau(\alpha) < \tau(\beta) < \tau(\gamma)$ or $\tau(\alpha) > \tau(\beta) > \tau(\gamma)$ or $\tau(\alpha) = \tau(\beta) = \tau(\gamma)$ the *seesaw inequality*.

In the same way, we call the alternative $\tau(\alpha) \leqslant \tau(\beta) \leqslant \tau(\gamma)$ or $\tau(\alpha) \geqslant \tau(\beta) \geqslant \tau(\gamma)$ the *weak seesaw inequality*. As far as I know this abstract idea of weak stability condition is new. I believe it is a useful innovation, since as we shall see in Section 4.4 important concepts such as the torsion filtration and μ-(semi)stability of sheaves are examples of weak stability conditions which are not stability conditions. Also, to transform between two stability conditions in [11] we will need to go via a weak stability condition.

We use many ordered sets in the paper: finite posets $(I, \preccurlyeq)$, $(J, \lesssim)$, $(K, \trianglelefteq)$ for $(I, \preccurlyeq)$-configurations, and now total orders $(T, \leqslant)$ for stability conditions. As the number of order symbols is limited, here and in [11] we will always use '$\leqslant$' for the total order, so that $(\tau, T, \leqslant)$, $(\tilde\tau, \tilde T, \leqslant)$ may denote two different stability conditions, with two *different* total orders on $T, \tilde T$ both denoted by '$\leqslant$.'

We define τ-*semistable*, τ-*stable* and τ-*unstable* objects.

Definition 4.2. Let $(\tau, T, \leqslant)$ be a weak stability condition on $\mathcal{A}$, $K(\mathcal{A})$ as above. Then we say that a nonzero object X in $\mathcal{A}$ is

(i) τ-*semistable* if for all $S \subset X$ with $S \ncong 0, X$ we have $\tau([S]) \leqslant \tau([X/S])$;
(ii) τ-*stable* if for all $S \subset X$ with $S \ncong 0, X$ we have $\tau([S]) < \tau([X/S])$; and
(iii) τ-*unstable* if it is not τ-semistable.

If $S \subset X$ is a subobject with $S \neq 0, X$ then $[S], [X], [X/S] \in C(\mathcal{A})$ with $[X] = [S] + [X/S]$. Thus, if $(\tau, T, \leqslant)$ is a *stability condition* then $\tau([S]) \leqslant \tau([X/S])$ in (i) is equivalent to $\tau([S]) \leqslant \tau([X])$ and to $\tau([X]) \leqslant \tau([X/S])$, and $\tau([S]) < \tau([X/S])$ in (ii) is equivalent to $\tau([S]) < \tau([X])$ and to $\tau([X]) < \tau([X/S])$.

We will need the following weakening of $\mathcal{A}$ *artinian* in Definition 2.2.

Definition 4.3. Let $(\tau, T, \leqslant)$ be a weak stability condition on $\mathcal{A}$, $K(\mathcal{A})$. We say $\mathcal{A}$ is τ-*artinian* if there exist no infinite chains of subobjects $\cdots \subset A_2 \subset A_1 \subset X$ in $\mathcal{A}$ with $A_{n+1} \neq A_n$ and $\tau([A_{n+1}]) \geqslant \tau([A_n/A_{n+1}])$ for all n. If $(\tau, T, \leqslant)$ is a *stability condition* $\tau([A_{n+1}]) \geqslant \tau([A_n/A_{n+1}])$ is equivalent to $\tau([A_{n+1}]) \geqslant \tau([A_n])$, and the definition reduces to [16, Definition 1.7].

In the next theorem we call $0 = A_0 \subset A_1 \subset \cdots \subset A_n = X$ a *Harder–Narasimhan filtration*, as it generalizes the filtrations constructed by Harder and Narasimhan [5] for vector bundles over algebraic curves. The proof is adapted from Rudakov [16, Theorem 2], which implies the result for stability conditions.

Theorem 4.4. *Let $(\tau, T, \leqslant)$ be a weak stability condition on an abelian category $\mathcal{A}$. Suppose $\mathcal{A}$ is noetherian and τ-artinian. Then each $X \in \mathcal{A}$ admits a unique filtration $0 = A_0 \subset \cdots \subset A_n = X$ for $n \geqslant 0$, such that $S_k = A_k/A_{k-1}$ is τ-semistable for $k = 1, \ldots, n$, and $\tau([S_1]) > \tau([S_2]) > \cdots > \tau([S_n])$.*

Proof. For $X \cong 0$ the result is trivial with $n = 0$, so fix $X \in \mathcal{A}$ with $X \ncong 0$. We divide the proof into the following seven steps:

Step 1. Given $0 \neq B \subset X$, there exists $0 \neq A \subset B \subset X$ with A τ-semistable and $\tau([A]) \geqslant \tau([B])$.

Step 2. Suppose $0 \neq A, B \subseteq X$ with A τ-semistable and $\tau([A]) \geqslant \tau([B])$. Then $\tau([A + B]) \geqslant \tau([B])$.

Step 3. Call $0 \neq C \subset X$ *greedy in* X if $0 \neq A \subset X$ with A τ-semistable and $\tau([A]) \geqslant \tau([C])$ implies $A \subset C$. Then for any $0 \neq B \subset X$ there exists $C \subset X$ greedy in X with $\tau([C]) \geqslant \tau([B])$.

Step 4. There exist unique $\tau^{\max} \in T$ and (not necessarily unique) $0 \neq B \subset X$ with $\tau([B]) = \tau^{\max}$, such that if $0 \neq A \subset X$ with A τ-semistable then $\tau([A]) \leqslant \tau^{\max}$. We can choose B τ-semistable.

Step 5. If $0 \neq A, B \subset X$ are τ-semistable with $\tau([A]) = \tau([B]) = \tau^{\max}$, then $A + B \subset X$ is τ-semistable with $\tau([A + B]) = \tau^{\max}$.

Step 6. There exists a unique τ-semistable $0 \neq S_1 \subset X$ with $\tau([S_1]) = \tau^{\max}$, such that if $A \subset X$ is τ-semistable with $\tau([A]) = \tau^{\max}$ then $A \subset S_1$.

Step 7. Complete the proof.

Step 1. Suppose for a contradiction there exists no such A. Set $B_1 = B$, and construct by induction a sequence $\cdots B_2 \subset B_1 \subset X$ with $B_{j+1} \neq 0, B_j$ and $\tau([B_{j+1}]) \geqslant \tau([B_j/B_{j+1}])$ as follows. Having chosen B_j, if $j > 1$ then $\tau([B_j]) \geqslant \tau([B_{j-1}/B_j])$ implies $\tau([B_j]) \geqslant \tau([B_{j-1}])$ by the weak seesaw inequality. So $\tau([B_j]) \geqslant \cdots \geqslant \tau([B_1]) = \tau([B])$. As $A = B_j$ will not do, B_j cannot be τ-semistable. Thus B_{j+1} exists as we want by Definition 4.2(i). But the sequence $\cdots B_2 \subset B_1 \subset X$ contradicts $\mathcal{A}$ *τ-artinian* in Definition 4.3.

Step 2. Let A, B be as above. If $A \subset B$ then $A + B = B$ and $\tau([A + B]) = \tau([B])$, so suppose $A \not\subset B$. Then $A \cap B$ is a proper subobject of A, so $A/(A \cap B) \ncong 0$, and A τ-semistable implies $\tau([A/(A\cap B)]) \geqslant \tau([A])$. But $(A+B)/B \cong A/(A\cap B)$ by properties of subobjects in an abelian category. Thus $\tau([(A+B)/B]) \geqslant \tau([A]) \geqslant \tau([B])$, so $\tau([A+B]) \geqslant \tau([B])$ by the weak seesaw inequality.

Step 3. Suppose for a contradiction there exists no such C. Construct by induction a sequence $B = B_1 \subset B_2 \subset \cdots \subset X$ with $B_j \neq B_{j+1}$ and $\tau([B_{j+1}]) \geqslant \tau([B_j])$, as follows. Having chosen B_j, as $\tau([B_j]) \geqslant \cdots \geqslant \tau([B_1]) = \tau([B])$, and $C = B_j$ will not do, B_j cannot be greedy. Thus there exists τ-semistable $A \subset X$ with $\tau([A]) \geqslant \tau([B_j])$ but $A \not\subset B_j$. Define $B_{j+1} = A + B_j$. Then $B_{j+1} \neq B_j$ as $A \not\subset B_j$, and $\tau([B_{j+1}]) \geqslant \tau([B_j])$ by Step 2, completing the induction. But $B_1 \subset B_2 \subset \cdots \subset X$ contradicts $\mathcal{A}$ *noetherian* in Definition 2.2.

Step 4. Suppose for a contradiction that no such (not yet unique) $\tau^{\max}$ and (not necessarily τ-semistable) B exist. Construct by induction a sequence $\cdots \subset C_2 \subset C_1 \subset X$ with C_j greedy and $\tau([C_{j+1}]) > \tau([C_j])$ for all j, as follows. Set $C_1 = X$, which is greedy. Having chosen C_j, as $\tau^{\max} = \tau([C_j])$ and $B = C_j$ will not do, there exists a τ-semistable $A \subset X$ with $\tau([A]) > \tau([C_j])$.

Then $A \subset C_j$, as C_j is greedy. By Step 3 with C_j in place of X, there exists $C_{j+1} \subset C_j$ greedy in C_j with $\tau([C_{j+1}]) \geqslant \tau([A]) > \tau([C_j])$. Suppose $0 \neq A' \subset X$ is τ-semistable with $\tau([A']) \geqslant \tau([C_{j+1}])$. Then $\tau([A']) \geqslant \tau([C_j])$, so $A' \subset C_j$ as C_j is greedy in X, and thus $A' \subset C_{j+1}$ as C_{j+1} is greedy in C_j. Hence C_{j+1} is greedy in X, completing the inductive step.

But $\tau([C_{j+1}]) > \tau([C_j])$ implies $\tau([C_{j+1}]) > \tau([C_j/C_{j+1}])$ by the weak seesaw inequality, so $\cdots \subset C_2 \subset C_1 \subset X$ contradicts $\mathcal{A}$ τ-artinian. Thus $\tau^{\max}$, B exist. Step 1 shows there exists $0 \neq A \subset B \subset X$ with A τ-semistable and $\tau([A]) \geqslant \tau([B]) = \tau^{\max}$. But by definition $\tau([A]) \leqslant \tau^{\max}$, so $\tau([A]) = \tau^{\max}$. Therefore $\tau^{\max}$ is the maximum value in T of $\tau([A])$ for τ-semistable $0 \neq A \subset X$, so $\tau^{\max}$ is unique, and replacing B by A, we can choose B τ-semistable.

Step 5. Suppose $S \subset A + B$ with $S \neq A + B$. Properties of subobjects in abelian categories give isomorphisms $(S + A)/S \cong A/(S \cap A)$ and $(A + B)/(S + A) \cong B/((S + A) \cap B)$. Thus from the exact sequence $0 \to (S + A)/S \to (A + B)/S \to (A + B)/(S + A) \to 0$ we obtain an exact sequence

$$0 \to A/(S \cap A) \to (A + B)/S \to B/\big((S + A) \cap B\big) \to 0. \tag{16}$$

The weak seesaw and A, B τ-semistable give $\tau([A/(S\cap A)]) \geqslant \tau([A]) = \tau^{\max}$ if $A/(S\cap A) \neq 0$, and $\tau([B/((S + A) \cap B)]) \geqslant \tau([B]) = \tau^{\max}$ if $B/((S + A) \cap B) \neq 0$. From (16) and the weak seesaw we deduce $\tau([(A + B)/S]) \geqslant \tau^{\max}$.

In particular, for $S = 0$ we have $\tau([A + B]) \geqslant \tau^{\max}$. But $S \subset X$ and $A + B \subset X$, so by Steps 1 and 4 we see that $\tau([S]) \leqslant \tau^{\max}$ if $S \neq 0$, and $\tau([A + B]) \leqslant \tau^{\max}$. Hence $\tau([A + B]) = \tau^{\max}$, as we want, and if $S \neq 0$ then $\tau([S]) \leqslant \tau^{\max} \leqslant \tau([(A+B)/S])$, which implies $A+B$ is τ-semistable.

Step 6. Suppose for a contradiction no such S_1 exists. Construct by induction a sequence $B_1 \subset B_2 \subset \cdots \subset X$ with $B_j \neq B_{j+1}$ and B_j τ-semistable with $\tau([B_j]) = \tau^{\max}$, as follows. Set $B_1 = B$ from Step 4, chosen τ-semistable. Having chosen B_j, as $S_1 = B_j$ will not do there exists τ-semistable $A \subset X$ with $\tau([A]) = \tau^{\max}$ and $A \not\subset B_j$.

Define $B_{j+1} = A + B_j$. Then B_{j+1} is τ-semistable with $\tau([B_{j+1}]) = \tau^{\max}$ by Step 5, and $B_{j+1} \neq B_j$ as $A \not\subset B_j$, completing the inductive step. But $B_1 \subset B_2 \subset \cdots \subset X$ contradicts $\mathcal{A}$ noetherian, so S_1 exists. If S_1, S_1' satisfy the conditions then $S_1 \subset S_1'$ and $S_1' \subset S_1$, so $S_1 = S_1'$ and S_1 is unique.

Step 7. By induction construct a sequence $0 = A_0 \subset A_1 \subset \cdots \subset A_n = X$ with $0 \neq S_j = A_j/A_{j-1}$ τ-semistable, as follows. Set $A_0 = 0$ and $A_1 \subset X$ to be S_1 from Step 6. Then $S_1 = A_1/A_0$ is τ-semistable. Having constructed A_j, if $A_j = X$ then set $n = j$ and finish. Otherwise define A_{j+1} such that $A_j \subset A_{j+1} \subset X$ and $A_{j+1}/A_j \subset X/A_j$ is the subobject S_1 given by Step 6 with X/A_j in place of X. Then $S_{j+1} = A_{j+1}/A_j$ is nonzero and τ-semistable.

As $A_{j+1} \neq A_j$ and $\mathcal{A}$ is noetherian the sequence must terminate at some n, so $0 = A_0 \subset \cdots \subset A_n = X$ is well-defined. Suppose for a contradiction that $\tau([S_j]) \leqslant \tau([S_{j+1}])$. Then we have subobjects $S_j = A_j/A_{j-1} \subset X/A_{j-1}$ and $A_{j+1}/A_{j-1} \subset X/A_{j-1}$, with $(A_{j+1}/A_{j-1})/S_j \cong S_{j+1}$. Write $\tau_j^{\max} = \tau([S_j])$. Then $\tau([S_{j+1}]) \geqslant \tau_j^{\max}$, so the weak seesaw implies $\tau([A_{j+1}/A_{j-1}]) \geqslant \tau_j^{\max}$, and an argument similar to Step 5 shows A_{j+1}/A_{j-1} is τ-semistable. Hence $A_{j+1}/A_{j-1} \subset S_j$ by definition of S_j, giving $S_{j+1} = 0$, a contradiction.

Therefore $\tau([S_1]) > \tau([S_2]) > \cdots > \tau([S_n])$, as we want. It remains only to prove $0 = A_0 \subset \cdots \subset A_n = X$ is unique. But it is easy to show that for a filtration satisfying the conditions of the theorem, the subobject $S_j \subset X/A_{j-1}$ satisfies the conditions of S_1 in Step 6 with X/A_{j-1} in place of X. Thus, having chosen A_{j-1}, uniqueness in Step 6 implies $S_j = A_j/A_{j-1}$ and A_j are uniquely determined, so uniqueness follows by induction on j. □

Theorem 4.4 justifies the weak case in Definition 4.1, as it shows that *τ-semistability is well-behaved for weak stability conditions*. However the next result, which follows from Rudakov [16, Theorem 3], is *false* for weak stability conditions $(\tau, T, \leqslant)$, as one can show by example. One moral is that *τ-stability is well-behaved for stability conditions, but badly behaved for weak stability conditions*. Therefore in Sections 5 and 6 below, which deal with τ-stability, we will consider only stability conditions, not weak stability conditions.

Theorem 4.5. *Let $\mathcal{A}$ be an abelian category, and $(\tau, T, \leqslant)$ a stability condition on $\mathcal{A}$, $K(\mathcal{A})$. Suppose $\mathcal{A}$ is noetherian and τ-artinian. Then each τ-semistable $X \in \mathcal{A}$ admits a filtration $0 = A_0 \subset A_1 \subset \cdots \subset A_n = X$ for $n \geqslant 1$, such that $S_k = A_k/A_{k-1}$ is τ-stable for $1 \leqslant k \leqslant n$, with $\tau([S_1]) = \cdots = \tau([S_n]) = \tau([X])$. Suppose $0 = A_0 \subset \cdots \subset A_n = X$ and $0 = B_0 \subset \cdots \subset B_m = X$ are two such filtrations with τ-stable factors $S_k = A_k/A_{k-1}$ and $T_k = B_k/B_{k-1}$. Then $n = m$, and for some permutation σ of $1, \ldots, n$ we have $S_k \cong T_{\sigma(k)}$ for $1 \leqslant k \leqslant n$.*

The restriction to *noetherian* $\mathcal{A}$ in these two theorems is unnecessarily strong. Rudakov only assumes $\mathcal{A}$ is 'weakly noetherian' [16, Definition 1.12]. But Rudakov's condition seems unsatisfactory to the author, so we shall not use it.

4.2. Permissible stability conditions

The following notation will be used throughout the rest of the paper.

Definition 4.6. Let Assumption 3.7 hold and $(\tau, T, \leqslant)$ be a weak stability condition on $\mathcal{A}$. Then $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}$ is an algebraic $\mathbb{K}$-stack for $\alpha \in C(\mathcal{A})$, with $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K})$ the set of isomorphism classes of $X \in \mathcal{A}$ with class α in $K(\mathcal{A})$. Define

$$\begin{aligned}
\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau) &= \{[X] \in \mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K})\colon\ X \text{ is } \tau\text{-semistable}\},\\
\mathrm{Obj}_{\mathrm{si}}^{\alpha}(\tau) &= \{[X] \in \mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K})\colon\ X \text{ is } \tau\text{-semistable and indecomposable}\},\\
\mathrm{Obj}_{\mathrm{st}}^{\alpha}(\tau) &= \{[X] \in \mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K})\colon\ X \text{ is } \tau\text{-stable}\}.
\end{aligned} \tag{17}$$

Let $(I, \preccurlyeq, \kappa)$ be $\mathcal{A}$-data, as in Definition 3.8, and $X \in \mathcal{A}$ with $[X] = \kappa(I)$. From Section 3.2 we have algebraic $\mathbb{K}$-stacks $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}, \mathfrak{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}}$ such that $\mathcal{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}} = \mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}(\mathbb{K})$, $\mathcal{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}} = \mathfrak{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}}(\mathbb{K})$ are sets of isomorphism classes $[(\sigma, \iota, \pi)]$ of $(I, \preccurlyeq, \kappa)$-configurations (σ, ι, π) in $\mathcal{A}$, with $\sigma(I) = X$ in the second case. Define an $(I, \preccurlyeq, \kappa)$-configuration (σ, ι, π) to be τ*-semistable* if $\sigma(\{i\})$ is τ-semistable, τ*-semistable-indecomposable* if $\sigma(\{i\})$ is τ-semistable and indecomposable, and τ*-stable* if $\sigma(\{i\})$ is τ-stable, for all $i \in I$. Define

$$\begin{aligned}
&\mathcal{M}_{\mathrm{ss}}, \mathcal{M}_{\mathrm{si}}, \mathcal{M}_{\mathrm{st}}, \mathcal{M}_{\mathrm{ss}}^{\mathrm{b}}, \mathcal{M}_{\mathrm{si}}^{\mathrm{b}}, \mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau)_{\mathcal{A}} \subseteq \mathcal{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}} \quad \text{and}\\
&\mathcal{M}_{\mathrm{ss}}, \mathcal{M}_{\mathrm{si}}, \mathcal{M}_{\mathrm{st}}, \mathcal{M}_{\mathrm{ss}}^{\mathrm{b}}, \mathcal{M}_{\mathrm{si}}^{\mathrm{b}}, \mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(X, I, \preccurlyeq, \kappa, \tau)_{\mathcal{A}} \subseteq \mathcal{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}}
\end{aligned} \tag{18}$$

to be the subsets of $[(\sigma, \iota, \pi)]$ with (σ, ι, π) τ*-semistable* in the $\mathcal{M}_{\mathrm{ss}}, \mathcal{M}_{\mathrm{ss}}^{\mathrm{b}}(\cdots)_{\mathcal{A}}$ cases, and τ*-semistable-indecomposable* in the $\mathcal{M}_{\mathrm{si}}, \mathcal{M}_{\mathrm{si}}^{\mathrm{b}}(\cdots)_{\mathcal{A}}$ cases, and τ*-stable* in the $\mathcal{M}_{\mathrm{st}}, \mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(\cdots)_{\mathcal{A}}$ cases, and *best* in the $\mathcal{M}_{\mathrm{ss}}^{\mathrm{b}}$, $\mathcal{M}_{\mathrm{si}}^{\mathrm{b}}$, $\mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(\cdots)_{\mathcal{A}}$ cases, as in Definition 3.5. Write $\delta_{\mathrm{ss}}^{\alpha}$, $\delta_{\mathrm{si}}^{\alpha}$, $\delta_{\mathrm{st}}^{\alpha}(\tau) : \mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K})$ or $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K}) \to \{0, 1\}$ for the *characteristic functions* of $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}$, $\mathrm{Obj}_{\mathrm{si}}^{\alpha}$, $\mathrm{Obj}_{\mathrm{st}}^{\alpha}(\tau)$. Write $\delta_{\mathrm{ss}}, \delta_{\mathrm{si}}, \delta_{\mathrm{st}}, \delta_{\mathrm{ss}}^{\mathrm{b}}, \delta_{\mathrm{si}}^{\mathrm{b}}, \delta_{\mathrm{st}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau) : \mathcal{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}} \to \{0, 1\}$ for the characteristic functions of $\mathcal{M}_{\mathrm{ss}}$, $\mathcal{M}_{\mathrm{si}}$, $\mathcal{M}_{\mathrm{st}}$, $\mathcal{M}_{\mathrm{ss}}^{\mathrm{b}}$, $\mathcal{M}_{\mathrm{si}}^{\mathrm{b}}$, $\mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau)_{\mathcal{A}}$, and $\delta_{\mathrm{ss}}, \ldots, \delta_{\mathrm{st}}^{\mathrm{b}}(X, I, \preccurlyeq, \kappa, \tau) : \mathcal{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}} \to \{0, 1\}$ for those of $\mathcal{M}_{\mathrm{ss}}, \ldots, \mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(X, I, \preccurlyeq, \kappa, \tau)_{\mathcal{A}}$.

Using [9, Assumption 7.1(iii)] we see $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}$, $\mathrm{Obj}_{\mathrm{si}}^{\alpha}$, $\mathrm{Obj}_{\mathrm{st}}^{\alpha}(\tau)$ are *open* in the natural topology on $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$, and so are *locally constructible*. Being best is also an open condition on configurations. Therefore $\mathcal{M}_{\mathrm{ss}}, \ldots, \mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau)_{\mathcal{A}}$ and $\mathcal{M}_{\mathrm{ss}}, \ldots, \mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(X, I, \preccurlyeq, \kappa, \tau)_{\mathcal{A}}$ are locally constructible, so that

$$\begin{aligned}
&\delta_{\mathrm{ss}}^{\alpha}, \delta_{\mathrm{si}}^{\alpha}, \delta_{\mathrm{st}}^{\alpha}(\tau) \in \mathrm{LCF}(\mathfrak{Obj}_{\mathcal{A}}), \qquad \delta_{\mathrm{ss}}, \ldots, \delta_{\mathrm{st}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau) \in \mathrm{LCF}\big(\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}\big),\\
&\text{and} \quad \delta_{\mathrm{ss}}, \delta_{\mathrm{si}}, \delta_{\mathrm{st}}, \delta_{\mathrm{ss}}^{\mathrm{b}}, \delta_{\mathrm{si}}^{\mathrm{b}}, \delta_{\mathrm{st}}^{\mathrm{b}}(X, I, \preccurlyeq, \kappa, \tau) \in \mathrm{LCF}\big(\mathfrak{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}}\big).
\end{aligned} \tag{19}$$

We want (18) and (19) to be constructible sets, so that (20) are constructible functions, as in Section 2.3. To do this we must impose some assumptions on $(\tau, T, \leqslant)$.

Definition 4.7. Let Assumption 3.7 hold and $(\tau, T, \leqslant)$ be a weak stability condition on $\mathcal{A}$. We call $(\tau, T, \leqslant)$ *permissible* if:

(i) $\mathcal{A}$ is τ-artinian, in the sense of Definition 4.3, and
(ii) $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$ is a constructible subset in $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}$ for all $\alpha \in C(\mathcal{A})$.

Theorem 4.8. *Let Assumption* 3.7 *hold and* $(\tau, T, \leqslant)$ *be a permissible weak stability condition on* $\mathcal{A}$. *Then* $\mathrm{Obj}_{\mathrm{si}}^{\alpha}$, $\mathrm{Obj}_{\mathrm{st}}^{\alpha}(\tau)$ *are constructible sets in* $\mathfrak{Obj}_{\mathcal{A}}$ *for all* $\alpha \in C(\mathcal{A})$. *Suppose* $(I, \preccurlyeq, \kappa)$ *is* $\mathcal{A}$*-data and* $X \in \mathcal{A}$ *with* $[X] = \kappa(I)$ *in* $K(\mathcal{A})$. *Then* $\mathcal{M}_{\mathrm{ss}}$, $\mathcal{M}_{\mathrm{si}}$, $\mathcal{M}_{\mathrm{st}}$, $\mathcal{M}_{\mathrm{ss}}^{\mathrm{b}}$, $\mathcal{M}_{\mathrm{si}}^{\mathrm{b}}$, $\mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau)_{\mathcal{A}}$ *are constructible in* $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$, *and* $\mathcal{M}_{\mathrm{ss}}, \ldots, \mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(X, I, \preccurlyeq, \kappa, \tau)_{\mathcal{A}}$ *in* $\mathfrak{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}}$. *Hence*

$$\begin{aligned} &\delta_{\mathrm{ss}}^{\alpha}, \delta_{\mathrm{si}}^{\alpha}, \delta_{\mathrm{st}}^{\alpha}(\tau) \in \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}), \qquad \delta_{\mathrm{ss}}, \ldots, \delta_{\mathrm{st}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau) \in \mathrm{CF}\big(\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}\big), \\ &\textit{and} \quad \delta_{\mathrm{ss}}, \delta_{\mathrm{si}}, \delta_{\mathrm{st}}, \delta_{\mathrm{ss}}^{\mathrm{b}}, \delta_{\mathrm{si}}^{\mathrm{b}}, \delta_{\mathrm{st}}^{\mathrm{b}}(X, I, \preccurlyeq, \kappa, \tau) \in \mathrm{CF}\big(\mathfrak{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}}\big). \end{aligned} \tag{20}$$

Proof. $\prod_{i \in I} \mathrm{Obj}_{\mathrm{ss}}^{\kappa(i)}(\tau)$ is constructible in $\prod_{i \in I} \mathfrak{Obj}_{\mathcal{A}}^{\kappa(i)}$ by Definition 4.7(ii). But $\prod_{i \in I} \boldsymbol{\sigma}(\{i\})$: $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}} \to \prod_{i \in I} \mathfrak{Obj}_{\mathcal{A}}^{\kappa(i)}$ is finite type by Theorem 3.9(c), and pulls back constructible sets to constructible sets. Thus $\mathcal{M}_{\mathrm{ss}}(I, \preccurlyeq, \kappa, \tau)_{\mathcal{A}} = (\prod_{i \in I} \boldsymbol{\sigma}(\{i\}))_*^{-1}(\prod_{i \in I} \mathrm{Obj}_{\mathrm{ss}}^{\kappa(i)}(\tau))$ is constructible in $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$. By Definition 4.6, $\mathrm{Obj}_{\mathrm{si}}^{\alpha}$, $\mathrm{Obj}_{\mathrm{st}}^{\alpha}(\tau)$ are locally constructible subsets of $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$, which is constructible by Definition 4.7(ii), and $\mathcal{M}_{\mathrm{si}}$, $\mathcal{M}_{\mathrm{st}}$, $\mathcal{M}_{\mathrm{ss}}^{\mathrm{b}}$, $\mathcal{M}_{\mathrm{si}}^{\mathrm{b}}$, $\mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau)_{\mathcal{A}}$ are locally constructible subsets of $\mathcal{M}_{\mathrm{ss}}(I, \preccurlyeq, \kappa, \tau)_{\mathcal{A}}$, which is constructible from above, so all these sets are constructible. As Π_X in Definition 3.10 is finite type and $\mathcal{M}_{\mathrm{ss}}, \ldots, \mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(X, I, \preccurlyeq, \kappa, \tau)_{\mathcal{A}}$ are Π_X^* of $\mathcal{M}_{\mathrm{ss}}, \ldots, \mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau)_{\mathcal{A}}$, they too are constructible. Equation (21) is immediate. □

Here is a useful finiteness property of permissible stability conditions.

Proposition 4.9. *In the situation above, let* $(\tau, T, \leqslant)$ *be permissible. Then for each* $\alpha \in C(\mathcal{A})$, *there are only finitely many pairs* $\beta, \gamma \in C(\mathcal{A})$ *with* $\alpha = \beta + \gamma$, $\tau(\alpha) = \tau(\beta) = \tau(\gamma)$ *and* $\mathrm{Obj}_{\mathrm{ss}}^{\beta}(\tau) \neq \emptyset \neq \mathrm{Obj}_{\mathrm{ss}}^{\gamma}(\tau)$.

Proof. Let $\alpha \in C(\mathcal{A})$ and $X \in \mathcal{A}$ with $[X] = \alpha$. Then as $\mathrm{Hom}(X, X)$ is a finite-dimensional $\mathbb{K}$-algebra by Assumption 3.7, general properties of abelian categories imply $X \cong X_1 \oplus \cdots \oplus X_n$, where the $0 \ncong X_i \in \mathcal{A}$ are *indecomposable*, and are unique up to order and isomorphism. Consider $\{[X_1], \ldots, [X_n]\}$ as a subset of $C(\mathcal{A})$ *with multiplicity*, that is, we remember how many times each element of $C(\mathcal{A})$ is repeated in $[X_1], \ldots, [X_n]$. Then $\{[X_1], \ldots, [X_n]\}$ depends only on the isomorphism class of X, that is, on $[X] \in \mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K})$.

Form the map $[X] \mapsto \{[X_1], \ldots, [X_n]\}$ from $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K})$ to the set of finite subsets of $C(\mathcal{A})$ with multiplicity. Using [9, Assumption 7.1(iii)], it is not difficult to see this map is *locally constructible*. As $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$ is constructible by Definition 4.7(ii), it follows that this map takes *only finitely many values* on $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$.

Suppose $\beta, \gamma \in C(\mathcal{A})$ with $\alpha = \beta + \gamma$, $\tau(\alpha) = \tau(\beta) = \tau(\gamma)$ and $\mathrm{Obj}_{\mathrm{ss}}^{\beta}(\tau) \neq \emptyset \neq \mathrm{Obj}_{\mathrm{ss}}^{\gamma}(\tau)$. Pick $[Y] \in \mathrm{Obj}_{\mathrm{ss}}^{\beta}(\tau)$ and $[Z] \in \mathrm{Obj}_{\mathrm{ss}}^{\gamma}(\tau)$, and set $X = Y \oplus Z$. Then X is τ-semistable with $[X] = \alpha$, so $[X] \in \mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$. Let $Y \cong X_1 \oplus \cdots \oplus X_k$ and $Z \cong X_{k+1} \oplus \cdots \oplus X_n$ with all $0 \ncong X_i \in \mathcal{A}$ indecomposable. Then $X \cong X_1 \oplus \cdots \oplus X_n$ splits X into indecomposables. Hence there are only finitely many possibilities for $\{[X_1], \ldots, [X_n]\}$, as a subset of $C(\mathcal{A})$ with multiplicity. But $\beta = [X_1] + \cdots + [X_k]$ and $\gamma = [X_{k+1}] + \cdots + [X_n]$, so we see there are only finitely many possibilities for β, γ. □

In [11] we will need the following notion.

Definition 4.10. Let $(\tau, T, \leqslant)$ and $(\tilde{\tau}, \tilde{T}, \leqslant)$ be weak stability conditions on an abelian category $\mathcal{A}$, with the same $K(\mathcal{A})$. We say $(\tilde{\tau}, \tilde{T}, \leqslant)$ *dominates* $(\tau, T, \leqslant)$ if $\tau(\alpha) \leqslant \tau(\beta)$ implies $\tilde{\tau}(\alpha) \leqslant \tilde{\tau}(\beta)$ for all $\alpha, \beta \in C(\mathcal{A})$.

Many examples of this arise through the following construction: if $(\tau, T, \leqslant)$ is a weak stability condition, $(\tilde{T}, \leqslant)$ a total order, and $\pi : T \to \tilde{T}$ a map with $t \leqslant t'$ implies $\pi(t) \leqslant \pi(t')$, then setting $\tilde{\tau} = \pi \circ \tau$ we find $(\tilde{\tau}, \tilde{T}, \leqslant)$ is a weak stability condition dominating $(\tau, T, \leqslant)$. The next lemma is elementary.

Lemma 4.11. *Let $(\tilde{\tau}, \tilde{T}, \leqslant)$ dominate $(\tau, T, \leqslant)$ on $\mathcal{A}$. Then X $\tilde{\tau}$-stable implies X τ-stable implies X τ-semistable implies X $\tilde{\tau}$-semistable for $X \in \mathcal{A}$. Also $\mathcal{A}$ $\tilde{\tau}$-artinian implies $\mathcal{A}$ τ-artinian, and if Assumption* 3.7 *holds then $(\tilde{\tau}, \tilde{T}, \leqslant)$ permissible implies $(\tau, T, \leqslant)$ permissible.*

4.3. Stability conditions on quiver representations

We give examples of permissible stability conditions for the data $\mathcal{A}$, $K(\mathcal{A})$, $\mathfrak{F}_{\mathcal{A}}$ of [9, §10]. Here is a criterion for weak stability conditions to be permissible.

Proposition 4.12. *If Assumption* 3.7 *holds and $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}$ is of finite type for all $\alpha \in C(\mathcal{A})$ then all weak stability conditions $(\tau, T, \leqslant)$ on $\mathcal{A}$ are permissible.*

Proof. Suppose $\cdots \subset A_2 \subset A_1 \subset X$ is an infinite chain of subobjects in $\mathcal{A}$ with $A_{n+1} \neq A_n$ for all n. Set $\alpha = [X]$ in $C(\mathcal{A})$. Consider the function $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K}) \to \mathbb{N}$ taking $[Y] \mapsto n$, where $Y \cong Y_1 \oplus \cdots \oplus Y_n$ has n indecomposable factors $0 \ncong Y_1, \ldots, Y_n$. This function is locally constructible, and so takes only finitely many values on $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K})$ as $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}$ is of finite type. Thus it has a maximum value n^{α}. However, $Y = (X/A_2) \oplus (A_2/A_3) \oplus \cdots \oplus (A_{n^{\alpha}}/A_{n^{\alpha}+1}) \oplus A_{n^{\alpha}+1}$ has at least $n^{\alpha} + 1$ indecomposable factors and $[Y] = [X] = \alpha$, a contradiction.

Thus there exist no such infinite chains $\cdots \subset A_2 \subset A_1 \subset X$, so $\mathcal{A}$ is *artinian*, and therefore τ-*artinian* for any $(\tau, T, \leqslant)$, proving Definition 4.7(i). For (ii), as $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$ is locally constructible by Definition 4.6 and a subset of $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K})$ which is constructible as $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}$ is of finite type, $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$ is constructible. □

In [9, Examples 10.5–10.9] we define data $\mathcal{A}$, $K(\mathcal{A})$, $\mathfrak{F}_{\mathcal{A}}$ satisfying Assumption 3.7 with $\mathcal{A} = \text{mod-}\mathbb{K}Q$ or $\text{nil-}\mathbb{K}Q$ for $Q = (Q_0, Q_1, b, e)$ a *quiver*, and $\mathcal{A} = \text{mod-}\mathbb{K}Q/I$ or $\text{nil-}\mathbb{K}Q/I$ for (Q, I) a *quiver with relations*, and $\mathcal{A} = \text{mod-}A$ for A a *finite-dimensional* $\mathbb{K}$-*algebra*. For all of these $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}$ is of finite type by [9, Theorem 10.11], so Proposition 4.12 gives:

Corollary 4.13. *For the data $\mathcal{A}$, $K(\mathcal{A})$, $\mathfrak{F}_{\mathcal{A}}$ defined using quivers in* [9, *Examples* 10.5–10.9], *all weak stability conditions $(\tau, T, \leqslant)$ on $\mathcal{A}$ are permissible.*

Stability conditions on categories of quiver representations were first considered by King [13], who proved the existence of coarse moduli schemes of semistable representations. His definition of stability [13, Definition 1.1] is not of our type, though it gives the same notions of (semi)stable object. Instead, we define stability using *slope functions* following [16, §3], based on much older ideas on slope stability for vector bundles and coherent sheaves.

Example 4.14. Let $\mathbb{K}$ be an algebraically closed field, and $\mathcal{A}$, $K(\mathcal{A})$, $\mathfrak{F}_{\mathcal{A}}$ be as in one of [9, Examples 10.5–10.9]. In each case there is an isomorphism $\dim: K(\mathcal{A}) \to \mathbb{Z}^{Q_0}$, where Q_0 is the finite set of vertices of a quiver Q. If $X \in \mathcal{A}$ then $\dim[X] \in \mathbb{N}^{Q_0} \subset \mathbb{Z}^{Q_0}$ is the *dimension vector* of X, so $\dim C(\mathcal{A}) = \mathbb{N}^{Q_0} \setminus \{0\}$.

Let $c, r: K(\mathcal{A}) \to \mathbb{R}$ be group homomorphisms with $r(\alpha) > 0$ for all $\alpha \in C(\mathcal{A})$. Using $\dim: K(\mathcal{A}) \to \mathbb{Z}^{Q_0}$ we see c, r may be uniquely written

$$c(\alpha) = \sum_{v \in Q_0} c_v(\dim\alpha)(v) \quad \text{and} \quad r(\alpha) = \sum_{v \in Q_0} r_v(\dim\alpha)(v),$$

where $c_v, r_v \in \mathbb{R}$ for $v \in Q_0$, and $r_v > 0$ for all $v \in Q_0$. It is common to take $r_v = 1$ for all v, so that $r(\alpha)$ is the *total dimension* of α. Define $\mu: C(\mathcal{A}) \to \mathbb{R}$ by $\mu(\alpha) = c(\alpha)/r(\alpha)$ for $\alpha \in C(\mathcal{A})$. Then μ is called a *slope function* on $K(\mathcal{A})$, as $\mu(\alpha)$ is the *slope* of the vector $(r(\alpha), c(\alpha))$ in $\mathbb{R}^2$. It is easy to verify $(\mu, \mathbb{R}, \leqslant)$ is a *stability condition* on $\mathcal{A}$, which is *permissible* by Corollary 4.13.

4.4. (Weak) stability conditions on coherent sheaves

Next we define (weak) stability conditions $(\tau, T, \leqslant)$ for the examples of [9, §9], in which $\mathcal{A} = \operatorname{coh}(P)$ is the abelian category of *coherent sheaves* on a projective $\mathbb{K}$-scheme P. Our first example is *Gieseker stability*, introduced by Gieseker [3] for vector bundles on algebraic surfaces, and studied in [6]. We define some total orders $(G_m, \leqslant)$ on sets of *monic polynomials*.

Definition 4.15. Let $m \geqslant 0$ be an integer, and define

$$G_m = \bigl\{p(t) = t^d + a_{d-1}t^{d-1} + \cdots + a_0\colon\ 0 \leqslant d \leqslant m,\ a_0, \ldots, a_{d-1} \in \mathbb{R}\bigr\}. \tag{21}$$

That is, G_m is the set of *monic real polynomials p of degree at most m*. Here 'monic' means *with leading coefficient* 1.

Define a *total order* '$\leqslant$' on G_m by $p \leqslant q$ for $p, q \in G_m$ if either

(a) $\deg p > \deg q$, or
(b) $\deg p = \deg q$ and $p(t) \leqslant q(t)$ for all $t \gg 0$.

Explicitly, if $p(t) = t^d + a_{d-1}t^{d-1} + \cdots + a_0$ and $q(t) = t^e + b_{e-1}t^{e-1} + \cdots + b_0$, we have $p \leqslant q$ if either (a) $d > e$, or (b) $d = e$, and either $p = q$ or for some $k = 0, \ldots, d-1$ we have $a_k < b_k$ and $a_l = b_l$ for $k < l < d$.

Note that (a) and (b) are *not* related in the way one might expect. For if $\deg p > \deg q$ as in (a) then $p(t) > q(t)$ for all $t \gg 0$, which is the opposite of $p(t) \leqslant q(t)$ for all $t \gg 0$ in (b).

We define Gieseker stability conditions on $\operatorname{coh}(P)$, following [16, §2].

Example 4.16. Let $\mathbb{K}$ be an algebraically closed field, P a projective $\mathbb{K}$-scheme of dimension m, $\mathcal{A} = \operatorname{coh}(P)$ the abelian category of *coherent sheaves* on P, and $K(\mathcal{A})$, $\mathfrak{F}_{\mathcal{A}}$ as in [9, Example 9.1 or Example 9.2], supposing P *smooth* in [9, Example 9.1].

Let E be an ample line bundle (invertible sheaf) on P. For $X \in \operatorname{coh}(P)$, following [6, §1.2] define the *Hilbert polynomial* p_X computed using E by

$$p_X(n) = \sum_{i=0}^{m} (-1)^i \dim_{\mathbb{K}} H^k\bigl(P, X \otimes E^n\bigr) \quad \text{for } n \in \mathbb{Z}, \tag{22}$$

where $H^*(P, \cdot)$ is *sheaf cohomology* on P. Then

$$p_X(n) = \sum_{i=0}^{m} b_i n^i / i! \quad \text{for } b_0, \dots, b_m \in \mathbb{Z}, \tag{23}$$

by [6, p. 10]. So $p_X(t)$ is a polynomial with rational coefficients, written $p_X(t) \in \mathbb{Q}[t]$, with degree no more than m. It depends only on the class $[X]$ in $K(\mathcal{A})$, so that $p_X = \Pi([X])$ for a unique group homomorphism $\Pi : K(\mathcal{A}) \to \mathbb{Q}[t]$.

If $X \ncong 0$ then the degree of p_X is the dimension of the support of X, and the leading coefficient of p_X is positive. Hence by (23),

$$\Pi\bigl(C(\mathcal{A})\bigr) \subseteq \left\{ p(t) = \sum_{i=0}^{k} b_i t^i / i!\colon\ 0 \leqslant k \leqslant m,\ b_0, \dots, b_k \in \mathbb{Z},\ b_k > 0 \right\}.$$

Let $(G_m, \leqslant)$ be as in Definition 4.15, and define $\gamma : C(\mathcal{A}) \to G_m$ by

$$\gamma(\alpha) = \sum_{i=0}^{k} \frac{k! b_i}{i! b_k} t^i \quad \text{when } \Pi(\alpha) = \sum_{i=0}^{k} \frac{b_i}{i!} t^i,\ b_k > 0.$$

That is, $\gamma(\alpha)$ is $\Pi(\alpha)$ divided by the leading coefficient $b^k/k!$ to make it *monic*, as in (21). So γ does map $C(\mathcal{A}) \to G_m$.

By Rudakov [16, Lemma 2.5], $(\gamma, G_m, \leqslant)$ is a *stability condition*, in the sense of Definition 4.1. It is *permissible* by Theorem 4.20 below. By construction, γ-(semi)stability coincides with the definition of Gieseker (semi)stability in [6, Definition 1.2.4], which refers to it just as (semi)stability. Note that the restriction in [6, Definition 1.2.4] that (semi)stable sheaves must be *pure* follows automatically from Definitions 4.2 and 4.15(a).

Huybrechts and Lehn also define μ-*(semi)stability* of coherent sheaves [6, Definition 1.2.12]. We can express this as a *weak* stability condition $(\mu, M_m, \leqslant)$ on $\operatorname{coh}(P)$, by *truncating* $p_X(t)$ at the second term.

Example 4.17. In the situation of Example 4.16, define

$$M_m = \bigl\{ p(t) = t^d + a_{d-1} t^{d-1}\colon\ 0 \leqslant d \leqslant m,\ a_{d-1} \in \mathbb{R} \bigr\} \subseteq G_m, \tag{24}$$

and restrict the total order $\leqslant$ on G_m to M_m. Define $\pi_M : G_m \to M_m$ by $\pi_M : t^d + a_{d-1} t^{d-1} + \cdots + a_0 \mapsto t^d + a_{d-1} t^{d-1}$. Define $\mu : C(\operatorname{coh}(P)) \to M_m$ by $\mu = \pi_M \circ \gamma$. Then $p \leqslant q$ implies $\pi(p) \leqslant \pi(q)$ for all $p, q \in G_m$, so as $(\gamma, G_m, \leqslant)$ is a stability condition on $\operatorname{coh}(P)$, the remark after Definition 4.10 shows $(\mu, M_m, \leqslant)$ is a *weak stability condition* on $\mathrm{coh}(P)$, which dominates $(\gamma, G_m, \leqslant)$. It is *permissible* by Theorem 4.20 below.

It is easy to show that $X \in \mathrm{coh}(P)$ is μ-(semi)stable in our sense if and only if X is *pure* and μ-(semi)stable in the sense of [6, Definition 1.2.12]. Note that Huybrechts and Lehn do not require μ-semistable sheaves X to be pure, only that torsion subsheaves of X have codimension at least two.

When $m = \dim P \geqslant 2$ we can find $\alpha, \beta, \gamma \in C(\mathcal{A})$ with $\beta = \alpha + \gamma$ and $\Pi(\alpha) = t^2$, $\Pi(\beta) = t^2 + 1$, $\Pi(\gamma) = 1$ for Π as in Example 4.16. Then $\mu(\alpha) = \mu(\beta) = t^2$ but $\mu(\gamma) = 1$, so that $\mu(\alpha) = \mu(\beta) < \mu(\gamma)$, which violates the seesaw inequality. Therefore $(\mu, M_m, \leqslant)$ is *not* a stability condition.

We defined $(\mu, M_m, \leqslant)$ by truncating Hilbert polynomials $p_X(t)$ at the second term. Truncating after any number of terms also gives a weak stability condition. In particular, we may truncate after one term, which is related to *pure sheaves* [6, Definition 1.1.2] and the *torsion filtration* [6, Definition 1.1.4].

Example 4.18. In the situation of Examples 4.16 and 4.17, define

$$D_m = \{p(t) = t^d\colon 0 \leqslant d \leqslant m\} \subseteq M_m \subseteq G_m,$$

and restrict $\leqslant$ on G_m to D_m, so that $t^d \leqslant t^e$ if and only if $d \geqslant e$. Define $\pi_D : G_m \to D_m$ by $\pi_D : t^d + a_{d-1}t^{d-1} + \cdots + a_0 \mapsto t^d$. Define $\delta : C(\mathrm{coh}(P)) \to M_m$ by $\delta = \pi_D \circ \gamma$. Then $(\delta, D_m, \leqslant)$ is a *weak stability condition* on $\mathrm{coh}(P)$ as in Example 4.17, which dominates $(\gamma, G_m, \leqslant)$ and $(\mu, M_m, \leqslant)$. It is easy to show $X \in \mathrm{coh}(P)$ is δ-semistable if and only if X is *pure*. Note that $\delta([X]) = t^{\dim X}$ for $X \in \mathrm{coh}(P)$, so $(\delta, D_m, \leqslant)$ is independent of choice of ample line bundle E.

We show below that $\mathrm{coh}(P)$ is δ-*artinian*. Thus Theorem 4.4 shows every $X \in \mathrm{coh}(P)$ has a unique filtration $0 = A_0 \subset \cdots \subset A_n = X$ with $S_k = A_k/A_{k-1}$ pure of strictly increasing dimension. This is the *torsion filtration* of X, with repeated terms omitted. Again, $(\delta, D_m, \leqslant)$ is not a stability condition for $m \geqslant 1$. These examples suggest weak stability conditions are a useful idea.

Lemma 4.19. $\mathrm{coh}(P)$ *is δ-artinian in Example* 4.18.

Proof. Suppose for a contradiction that there exists $\cdots \subset A_2 \subset A_1 \subset X$ in $\mathrm{coh}(P)$ with $A_{n+1} \neq A_n$ and $\delta([A_{n+1}]) \geqslant \delta([A_n/A_{n+1}])$ for all n. Then $\delta([A_{n+1}]) \geqslant \delta([A_n])$, so as $\delta([A_n]) = t^{\deg \Pi([A_n])}$ we see $(\deg \Pi([A_n]))_{n\geqslant 1}$ is a decreasing sequence of nonnegative integers. Thus $\deg \Pi([A_n]) = d$ for some N and all $n \geqslant N$. For $n \geqslant N$ we have $\Pi([A_n]) = a_{n,d}t^d/d! + \cdots + a_{n,0}$, and $\delta([A_{n+1}]) \geqslant \delta([A_n/A_{n+1}])$ implies $\delta([A_n/A_{n+1}])$ also has degree d, which forces $a_{n+1,d} < a_{n,d}$. Hence $(a_{n,d})_{n\geqslant N}$ is a strictly decreasing sequence of positive integers, a contradiction. □

Theorem 4.20. $(\gamma, G_m, \leqslant)$ *and* $(\mu, M_m, \leqslant)$ *above are permissible.*

Proof. As $(\delta, D_m, \leqslant)$ dominates $(\gamma, G_m, \leqslant)$ and $(\mu, M_m, \leqslant)$, Lemmas 4.11 and 4.19 imply $\mathrm{coh}(P)$ is γ- and μ-artinian, proving Definition 4.7(i). For (ii), as $\mathfrak{Obj}^\alpha_{\mathrm{ss}}(\gamma)$, $\mathfrak{Obj}^\alpha_{\mathrm{ss}}(\mu)$ are locally constructible by Definition 4.6, they are constructible if they are contained in a constructible set. This is equivalent to the families of γ- and μ-semistable sheaves in class α being *bounded* in the sense of [6, Definition 1.7.5]. This is proved for $\mathbb{K}$ of characteristic zero by Huybrechts and Lehn [6, Theorem 3.3.7], and for arbitrary characteristic by Langer [14, Theorem 4.2]. □

Note that $(\delta, D_m, \leqslant)$ in Example 4.18 is *not* permissible when $m = \dim P \geqslant 1$, as the pure sheaves in a class α of nonzero degree are not bounded.

5. Identities relating the $\delta_{\rm ss}, \delta_{\rm si}, \delta_{\rm st}, \delta_{\rm ss}^{\rm b}, \delta_{\rm si}^{\rm b}, \delta_{\rm st}^{\rm b}(*, \tau)$

Here and in Section 6 we will derive *universal identities* relating the six families of constructible functions $\delta_{\rm ss}$, $\delta_{\rm si}$, $\delta_{\rm st}$, $\delta_{\rm ss}^{\rm b}$, $\delta_{\rm si}^{\rm b}$, $\delta_{\rm st}^{\rm b}(*, \tau)$. This section works using constructible function techniques, mostly involving computing Euler characteristics of pieces of moduli spaces. Section 6 then uses combinatorial methods to invert the identities of this section. As we are working with constructible functions, we assume $\mathbb{K}$ has *characteristic zero* here and in Section 6.

In Sections 5.1–5.2, which relate configurations to best configurations and semistables to semistable-indecomposables, we work with a permissible *weak* stability condition $(\tau, T, \leqslant)$. But in Sections 5.3–5.4, which relate τ-stability and τ-semistability, we take $(\tau, T, \leqslant)$ to be a stability condition, so that Theorem 4.5 applies. Our results show that to express $\delta_{\rm ss}^{\alpha}(\tau)$ in terms of $\delta_{\rm st}^{\beta}(\tau)$ and vice versa, we have to use configuration moduli stacks $\mathfrak{M}(I, \precsim)_{\mathcal{A}}$ for all finite posets $(I, \precsim)$. This is some justification for the work of developing the configurations formalism.

5.1. Counting best improvements

Our first theorem says, in effect, that the family of all *best improvements* of an $(I, \trianglelefteq)$-configuration (σ, ι, π) has Euler characteristic 1.

Theorem 5.1. *Let Assumption* 3.7 *hold and* $(\tau, T, \leqslant)$ *be a permissible weak stability condition on* $\mathcal{A}$. *Suppose* $(I, \trianglelefteq, \kappa)$ *is* $\mathcal{A}$*-data, as in Definition* 3.8*, and* $X \in \mathcal{A}$ *with* $[X] = \kappa(I)$. *Then*

$$\sum_{\substack{\text{p.o.s } \precsim \text{ on } I:\\ \trianglelefteq \text{ dominates } \precsim}} \mathrm{CF}^{\rm stk}\bigl(Q(I, \precsim, \trianglelefteq)\bigr)\delta_{\rm ss}^{\rm b}(X, I, \precsim, \kappa, \tau) = \delta_{\rm ss}(X, I, \trianglelefteq, \kappa, \tau), \tag{25}$$

$$\sum_{\substack{\text{p.o.s } \precsim \text{ on } I:\\ \trianglelefteq \text{ dominates } \precsim}} \mathrm{CF}^{\rm stk}\bigl(Q(I, \precsim, \trianglelefteq)\bigr)\delta_{\rm si}^{\rm b}(X, I, \precsim, \kappa, \tau) = \delta_{\rm si}(X, I, \trianglelefteq, \kappa, \tau), \tag{26}$$

and

$$\sum_{\substack{\text{p.o.s } \precsim \text{ on } I:\\ \trianglelefteq \text{ dominates } \precsim}} \mathrm{CF}^{\rm stk}\bigl(Q(I, \precsim, \trianglelefteq)\bigr)\delta_{\rm st}^{\rm b}(X, I, \precsim, \kappa, \tau) = \delta_{\rm st}(X, I, \trianglelefteq, \kappa, \tau). \tag{27}$$

Proof. Define $S = \{(i, j) \in I \times I\colon i \neq j \text{ and } i \trianglelefteq j\}$, and let $s = |S|$. Choose some arbitrary *total order* $\leqslant$ on S. Define a finite type algebraic $\mathbb{K}$-space $\mathfrak{G}$ by

$$\mathfrak{G} = \coprod_{\substack{\text{p.o.s } \precsim \text{ on } I:\\ \trianglelefteq \text{ dominates } \precsim}} \mathfrak{M}(X, I, \precsim, \kappa)_{\mathcal{A}}.$$

Define a 1-morphism $\phi_r : \mathfrak{G} \to \mathfrak{G}$ for $r = 1, \ldots, s$ by

$$\phi_r|_{\mathfrak{M}(X,I,\preccurlyeq,\kappa)_{\mathcal{A}}} = \begin{cases} \mathrm{id} : \mathfrak{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}} \to \mathfrak{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}}, & m \neq r, \\ Q(I, \preccurlyeq, \lesssim) : \mathfrak{M}(X, I, \preccurlyeq, \kappa)_{\mathcal{A}} \to \mathfrak{M}(X, I, \lesssim, \kappa)_{\mathcal{A}}, & m = r, \end{cases}$$

if $\trianglelefteqslant$ dominates $\preccurlyeq$ by m steps, where $\lesssim$ is defined as follows: let $(i, j) \in S$ be $\leqslant$-least such that (a) $i \not\preccurlyeq j$, (b) if $i \neq k \in I$ with $i \preccurlyeq k$ then $j \preccurlyeq k$, and (c) if $j \neq k \in I$ with $k \preccurlyeq i$, then $k \preccurlyeq j$. Then set $a \lesssim b$ if either $a \preccurlyeq b$ or $a = i$, $b = j$.

By [9, Lemma 6.4] and (a)–(c), $\lesssim$ is a partial order and dominates $\preccurlyeq$ by one step, and $(i, j) \in S$ gives $i \trianglelefteqslant j$, so that $\trianglelefteqslant$ dominates $\lesssim$. Conversely, if $\trianglelefteqslant$ dominates $\lesssim$ dominates $\preccurlyeq$ by one step then it arises in this way for a unique $(i, j) \in S$. As $r \geqslant 1$ there is at least one $\lesssim$ with $\trianglelefteqslant$ dominates $\lesssim$ dominates $\preccurlyeq$ by one step, by [9, Proposition 6.5]. Thus the set of $(i, j) \in S$ which from which we choose the $\leqslant$-least element is nonempty, and ϕ_r is well-defined.

If $\trianglelefteqslant$ dominates $\preccurlyeq$ by m steps then ϕ_r fixes $\preccurlyeq$ if $m \neq r$, and takes $\preccurlyeq$ to $\lesssim$ if $m = r$, where $\trianglelefteqslant$ dominates $\lesssim$ by $r - 1$ steps. So by induction $\phi_r \circ \phi_{r+1} \circ \cdots \circ \phi_s$ takes each $\preccurlyeq$ to some $\lesssim$, where $\trianglelefteqslant$ dominates $\lesssim$ by less than r steps. When $r = 1$ we have $\lesssim \,=\, \trianglelefteqslant$, as $\trianglelefteqslant$ dominates $\lesssim$ by 0 steps. It follows easily that

$$\phi_1 \circ \phi_2 \circ \cdots \circ \phi_s|_{\mathfrak{M}(X,I,\preccurlyeq,\kappa)_{\mathcal{A}}} = Q(I, \preccurlyeq, \trianglelefteqslant). \tag{28}$$

Define

$$\mathcal{C}_s = \coprod_{\substack{\text{p.o.s } \preccurlyeq \text{ on } I: \\ \trianglelefteqslant \text{ dominates } \preccurlyeq}} \mathcal{M}_{\mathrm{ss}}^{\mathrm{b}}(X, I, \preccurlyeq, \kappa, \tau)_{\mathcal{A}} \subseteq \mathfrak{G}(\mathbb{K}). \tag{29}$$

Then $\mathcal{C}_s$ is a constructible set in $\mathfrak{G}$ by Theorem 4.8. For $r = s, s-1, \ldots, 1$ define $\mathcal{C}_{r-1} = (\phi_r)_*(\mathcal{C}_r)$. As ϕ_r is a 1-morphism, Proposition 2.6 shows that $\mathcal{C}_r$ is also constructible for $r = s, s-1, \ldots, 0$. Equation (28) gives

$$\mathcal{C}_0 = \coprod_{\substack{\text{p.o.s } \preccurlyeq \text{ on } I: \\ \trianglelefteqslant \text{ dominates } \preccurlyeq}} Q(I, \preccurlyeq, \trianglelefteqslant)_*\big(\mathcal{M}_{\mathrm{ss}}^{\mathrm{b}}(X, I, \preccurlyeq, \kappa, \tau)_{\mathcal{A}}\big) = \mathcal{M}_{\mathrm{ss}}(X, I, \trianglelefteqslant, \kappa, \tau)_{\mathcal{A}}, \tag{30}$$

as every $(I, \trianglelefteqslant, \kappa)$-configuration admits a best improvement by [9, Lemma 6.2].

Suppose $[(\sigma, \iota, \pi)] \in \mathcal{C}_{r-1}$ for $r \leqslant s$, with (σ, ι, π) an $(I, \lesssim)$-configuration. We shall determine $(\phi_r)_*^{-1}([(\sigma, \iota, \pi)])$ in $\mathcal{C}_r$. If (σ, ι, π) is not best then by Theorem 3.6 there are $i \neq j \in I$ with $i \lesssim j$ but there exists no $k \in I$ with $i \neq k \neq j$ and $i \lesssim k \lesssim j$, such that (6) is split.

Now $i \trianglelefteqslant j$ as $\trianglelefteqslant$ dominates $\lesssim$, so $(i, j) \in S$. Let (i, j) be greatest in the total order $\leqslant$ on S satisfying these conditions. Define $\preccurlyeq$ by $a \preccurlyeq b$ if $a \lesssim b$ and $a \neq i$ or $b \neq j$. Then $\preccurlyeq$ is a partial order on I and $\lesssim$ dominates $\preccurlyeq$ by one step. Furthermore, Theorem 3.6 and the construction of the $\mathcal{C}_r, \phi_r$ imply that $(\phi_r)_*^{-1}([(\sigma, \iota, \pi)])$ is exactly the set of isomorphism classes $[(\sigma', \iota', \pi')]$ of $(I, \preccurlyeq)$-improvements (σ', ι', π') of (σ, ι, π), which are in 1–1 correspondence with $\mathrm{Hom}(\sigma(\{j\}), \sigma(\{i\}))$.

Regard $\mathrm{Hom}(\sigma(\{j\}), \sigma(\{i\})) \cong \mathbb{K}^l$ as an affine $\mathbb{K}$-variety. Using [9, §6.2 and Assumption 7.1(iv)] one can construct a $\mathbb{K}$-subvariety V of $\mathfrak{G}$ isomorphic to $\mathbb{K}^l$, such that $V(\mathbb{K}) = (\phi_r)_*^{-1}([(\sigma, \iota, \pi)])$. Hence $\chi^{\mathrm{na}}((\phi_r)_*^{-1}([(\sigma, \iota, \pi)])) = \chi(V) = \chi(\mathbb{K}^l) = 1$, by (2). If (σ, ι, π) is best then $(\phi_r)_*^{-1}([(\sigma, \iota, \pi)]) = \{[(\sigma, \iota, \pi)]\}$, so again $\chi^{\mathrm{na}}((\phi_r)_*^{-1}([(\sigma, \iota, \pi)])) = 1$.

Write $\delta_{\mathcal{C}_r}$ for the characteristic function of $\mathcal{C}_r$. Then $\delta_{\mathcal{C}_r}$ is a constructible function, as $\mathcal{C}_r$ is a constructible set. Since $\chi^{\mathrm{na}}((\phi_r)_*^{-1}(x)) = 1$ for all $x \in \mathcal{C}_r$ and $m_{\phi_r} \equiv 1$ in Definition 2.7 as $\mathfrak{M}(X, I, \preceq, \kappa)_{\mathcal{A}}$ is an algebraic $\mathbb{K}$-space with trivial stabilizer groups, we see that $\mathrm{CF}^{\mathrm{stk}}(\phi_r)\delta_{\mathcal{C}_r} = \delta_{\mathcal{C}_{r-1}}$ for all r. Hence $\mathrm{CF}^{\mathrm{stk}}(\phi_1 \circ \cdots \circ \phi_s)\delta_{\mathcal{C}_s} = \delta_{\mathcal{C}_0}$, by (3). Equation (25) then follows from (28)–(30). To prove (26) and (27) we proceed in the same way, but define $\mathcal{C}_s$ in (29) using $\mathcal{M}_{\mathrm{si}}^{\mathrm{b}}, \mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(X, I, \preceq, \kappa, \tau)_{\mathcal{A}}$ rather than $\mathcal{M}_{\mathrm{ss}}^{\mathrm{b}}(X, I, \preceq, \kappa, \tau)_{\mathcal{A}}$. □

Here are analogues of (25)–(27) for $\mathfrak{M}(I, \preceq, \kappa)_{\mathcal{A}}$ rather than $\mathfrak{M}(X, I, \preceq, \kappa)_{\mathcal{A}}$.

Theorem 5.2. *Let Assumption* 3.7 *hold,* $(\tau, T, \leqslant)$ *be a permissible weak stability condition on* $\mathcal{A}$*, and* $(I, \trianglelefteq, \kappa)$ *be* $\mathcal{A}$*-data. Then*

$$\sum_{\substack{\text{p.o.s } \preceq \text{ on } I:\\ \trianglelefteq \text{ dominates } \preceq}} \mathrm{CF}^{\mathrm{stk}}\big(Q(I, \preceq, \trianglelefteq)\big)\delta_{\mathrm{ss}}^{\mathrm{b}}(I, \preceq, \kappa, \tau) = \delta_{\mathrm{ss}}(I, \trianglelefteq, \kappa, \tau), \tag{31}$$

$$\sum_{\substack{\text{p.o.s } \preceq \text{ on } I:\\ \trianglelefteq \text{ dominates} \preceq}} \mathrm{CF}^{\mathrm{stk}}\big(Q(I, \preceq, \trianglelefteq)\big)\delta_{\mathrm{si}}^{\mathrm{b}}(I, \preceq, \kappa, \tau) = \delta_{\mathrm{si}}(I, \trianglelefteq, \kappa, \tau), \tag{32}$$

and

$$\sum_{\substack{\text{p.o.s } \preceq \text{ on } I:\\ \trianglelefteq \text{ dominates } \preceq}} \mathrm{CF}^{\mathrm{stk}}\big(Q(I, \preceq, \trianglelefteq)\big)\delta_{\mathrm{st}}^{\mathrm{b}}(I, \preceq, \kappa, \tau) = \delta_{\mathrm{st}}(I, \trianglelefteq, \kappa, \tau). \tag{33}$$

Proof. Let $X \in \mathcal{A}$ with $[X] = \kappa(I)$ in $K(\mathcal{A})$, and $\preceq$ be a partial order on I dominated by $\trianglelefteq$. Consider the Cartesian square

$$\begin{array}{ccc}
\mathfrak{M}(X, I, \preceq, \kappa)_{\mathcal{A}} & \xrightarrow{Q(I,\preceq,\trianglelefteq)} & \mathfrak{M}(X, I, \trianglelefteq, \kappa)_{\mathcal{A}} \\
\downarrow \Pi_X & & \downarrow \Pi_X \\
\mathfrak{M}(I, \preceq, \kappa)_{\mathcal{A}} & \xrightarrow{Q(I,\preceq,\trianglelefteq)} & \mathfrak{M}(I, \trianglelefteq, \kappa)_{\mathcal{A}}.
\end{array}$$

The $Q(I, \preceq, \trianglelefteq)$ are representable by Theorem 3.9(a), and the Π_X of finite type by Definition 3.10. Thus Theorem 2.8 shows the following commutes:

$$\begin{array}{ccc}
\mathrm{CF}(\mathfrak{M}(X, I, \preceq, \kappa)_{\mathcal{A}}) & \xrightarrow{\mathrm{CF}^{\mathrm{stk}}(Q(I,\preceq,\trianglelefteq))} & \mathrm{CF}(\mathfrak{M}(X, I, \trianglelefteq, \kappa)_{\mathcal{A}}) \\
\uparrow \Pi_X^* & & \uparrow \Pi_X^* \\
\mathrm{CF}(\mathfrak{M}(I, \preceq, \kappa)_{\mathcal{A}}) & \xrightarrow{\mathrm{CF}^{\mathrm{stk}}(Q(I,\preceq,\trianglelefteq))} & \mathrm{CF}(\mathfrak{M}(I, \trianglelefteq, \kappa)_{\mathcal{A}}).
\end{array} \tag{34}$$

Using (25), commutativity of (34), $\Pi_X^*(\delta_{\mathrm{ss}}(I, \preceq, \kappa, \tau)) = \delta_{\mathrm{ss}}(X, I, \preceq, \kappa, \tau)$ and $\Pi_X^*(\delta_{\mathrm{ss}}(I, \trianglelefteq, \kappa, \tau)) = \delta_{\mathrm{ss}}(X, I, \trianglelefteq, \kappa, \tau)$ shows that

$$\begin{aligned}
&\Pi_X^*\Bigg[\sum_{\substack{\text{p.o.s } \precsim \text{ on } I:\\ \trianglelefteq \text{ dominates } \precsim}} \mathrm{CF}^{\mathrm{stk}}\big(Q(I,\precsim,\trianglelefteq)\big)\delta_{\mathrm{ss}}^{\mathrm{b}}(I,\precsim,\kappa,\tau)\Bigg]\\
&\quad=\sum_{\substack{\text{p.o.s } \precsim \text{ on } I:\\ \trianglelefteq \text{ dominates } \precsim}} \mathrm{CF}^{\mathrm{stk}}\big(Q(I,\precsim,\trianglelefteq)\big)\circ\Pi_X^*\big(\delta_{\mathrm{ss}}^{\mathrm{b}}(I,\precsim,\kappa,\tau)\big)\\
&\quad=\sum_{\substack{\text{p.o.s } \precsim \text{ on } I:\\ \trianglelefteq \text{ dominates } \precsim}} \mathrm{CF}^{\mathrm{stk}}\big(Q(I,\precsim,\trianglelefteq)\big)\delta_{\mathrm{ss}}^{\mathrm{b}}(X,I,\precsim,\kappa,\tau)=\delta_{\mathrm{ss}}(X,I,\trianglelefteq,\kappa,\tau)\\
&\quad=\Pi_X^*\big(\delta_{\mathrm{ss}}(I,\trianglelefteq,\kappa,\tau)\big).
\end{aligned}$$

This implies that (31) holds at all $[(\sigma,\iota,\pi)]$ in $\mathcal{M}(I,\trianglelefteq,\kappa)_{\mathcal{A}}$ with $\sigma(I)=X$. Since this is true for all $X\in\mathcal{A}$ with $[X]=\kappa(I)$, we have proved (31). Equations (32) and (33) follow from (26) and (27) in the same way. □

5.2. Relating semistables and semistable-indecomposables

Next we shall write $\delta_{\mathrm{ss}}^{\alpha}(\tau)$ and $\delta_{\mathrm{ss}}(K,\trianglelefteq,\mu,\tau)$ in terms of the $\delta_{\mathrm{si}}(I,\precsim,\kappa,\tau)$.

Theorem 5.3. *Let Assumption* 3.7 *hold,* $(\tau,T,\leqslant)$ *be a permissible weak stability condition on* $\mathcal{A}$*, and* $\alpha\in C(\mathcal{A})$*. Then*

$$\sum_{n=1}^{\infty}\frac{1}{n!}\cdot\sum_{\substack{\kappa:\{1,\dots,n\}\to C(\mathcal{A}):\\ \kappa(\{1,\dots,n\})=\alpha,\ \tau\circ\kappa\equiv\tau(\alpha)}}\mathrm{CF}^{\mathrm{stk}}\big(\boldsymbol{\sigma}(\{1,\dots,n\})\big)\delta_{\mathrm{si}}\big(\{1,\dots,n\},\bullet,\kappa,\tau\big)=\delta_{\mathrm{ss}}^{\alpha}(\tau),\tag{35}$$

where $\bullet$ *is the partial order on* $\{1,\dots,n\}$ *with* $i\bullet j$ *if and only if* $i=j$*. Only finitely many functions* $\delta_{\mathrm{si}}(\{1,\dots,n\},\bullet,\kappa,\tau)$ *in this sum are nonzero.*

Proof. Suppose n,κ are as in (35) with $\delta_{\mathrm{si}}(\{1,\dots,n\},\bullet,\kappa,\tau)\neq 0$. If $n=1$ the only possibility is $\kappa(1)=\alpha$, so let $n>1$. Pick $1\leqslant i<n$, and set $\beta=\kappa(\{1,\dots,i\})$ and $\gamma=\kappa(\{i+1,\dots,n\})$. Then $\beta,\gamma\in C(\mathcal{A})$ with $\alpha=\beta+\gamma$, and $\tau\circ\kappa\equiv\tau(\alpha)$ implies $\tau(\alpha)=\tau(\beta)=\tau(\gamma)$, and $\delta_{\mathrm{si}}(\{1,\dots,n\},\bullet,\kappa,\tau)\neq 0$ implies that $\mathrm{Obj}_{\mathrm{ss}}^{\beta}(\tau)\neq\emptyset\neq\mathrm{Obj}_{\mathrm{ss}}^{\gamma}(\tau)$. Hence there are only finitely many possibilities for β,γ, by Proposition 4.9, and it quickly follows that there are only finitely many nonzero terms in (35).

Fix $0\not\cong X\in\mathcal{A}$ with $[X]=\alpha\in C(\mathcal{A})$. Let the pairwise-nonisomorphic indecomposable factors of X be $S_1,\dots,S_k$, with multiplicities $m_1,\dots,m_k\geqslant 1$, so that $X\cong\bigoplus_{a=1}^{k}\bigoplus^{m_a}S_a$. It is easy to see that X is τ-semistable if and only if each S_a is also τ-semistable with $\tau([S_a])=\tau(\alpha)$.

Let $[(\sigma,\iota,\pi)]\in\mathcal{M}_{\mathrm{si}}(\{1,\dots,n\},\bullet,\kappa,\tau)_{\mathcal{A}}$ with $\sigma(\{1,\dots,n\})\cong X$, for n,κ as in (35). Then by definition and properties of configurations, $\sigma(\{i\})$ is τ-semistable and indecomposable for all $i=1,\dots,n$ with $X\cong\bigoplus_{i=1}^{n}\sigma(\{i\})$. So by definition of the S_a, m_a, there must exist a unique, surjective map $\phi:\{1,\dots,n\}\to\{1,\dots,k\}$ with $|\phi^{-1}(\{a\})|=m_a$, such that $\sigma(\{i\})\cong S_{\phi(i)}$ for all $i=1,\dots,n$. This forces $n=\sum_{a=1}^{k}m_a$. It also implies each S_a is also τ-semistable with $\tau([S_a])=\tau(\alpha)$, so X is τ-semistable from above.

Thus, if X is *not* τ-semistable, there exist no such n,κ and $[(\sigma,\iota,\pi)]$, so both sides of (35) are zero at $[X]$. Suppose X is τ-semistable, and consider the set of possible choices n,κ and

$[(\sigma,\iota,\pi)]$ with $\sigma(\{1,\dots,n\})\cong X$. From above $n=\sum_{a=1}^k m_a$, and it is easy to show the possible choices of κ and $[(\sigma,\iota,\pi)]$ are in 1–1 correspondence with maps $\phi:\{1,\dots,n\}\to\{1,\dots,k\}$ with $|\phi^{-1}(\{a\})|=m_a$ for all a. There are exactly $n!/m_1!\cdots m_k!$ such maps ϕ, and in each case we have $\mathrm{Aut}(\sigma,\iota,\pi)\cong\bigotimes_{a=1}^k \mathrm{Aut}(S_a)^{m_a}$. So by definition of $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(\{1,\dots,n\}))$, we see that the left-hand side of (35) at $[X]$ is

$$\frac{1}{n!}\cdot\frac{n!}{m_1!\cdots m_k!}\cdot\chi\left(\frac{\mathrm{Aut}(\bigoplus_{a=1}^k\bigoplus^{m_a}S_a)}{\prod_{a=1}^k\mathrm{Aut}(S_a)^{m_a}}\right). \tag{36}$$

Using elementary facts about finite-dimensional algebras taken from Benson [1, §1] applied to the $\mathbb{K}$-algebras $\mathrm{End}(S_a)$ and $\mathrm{End}(X)$, we find that

$$\mathrm{Aut}(S_a)\cong\mathbb{K}^\times\ltimes J_a\quad\text{and}\quad\mathrm{Aut}\left(\bigoplus_{a=1}^k\bigoplus^{m_a}S_a\right)\cong\left(\prod_{a=1}^k\mathrm{GL}(m_a,\mathbb{K})\right)\ltimes J_X,$$

where J_a and J_X are the *Jacobson radicals* of $\mathrm{End}(S_a)$ and $\mathrm{End}(X)$, which are nilpotent $\mathbb{K}$-groups isomorphic as $\mathbb{K}$-varieties to finite-dimensional vector spaces $\mathbb{K}^{l_a}$, $\mathbb{K}^{l_X}$. Using these isomorphisms we construct a natural fibration

$$\Pi:\mathrm{Aut}\left(\bigoplus_{a=1}^k\bigoplus^{m_a}S_a\right)\Big/\prod_{a=1}^k\mathrm{Aut}(S_a)^{m_a}\to\prod_{a=1}^k\mathrm{GL}(m_a,\mathbb{K})/\big(\mathbb{K}^\times\big)^{m_a},$$

where $(\mathbb{K}^\times)^{m_a}$ is the maximal torus of diagonal matrices in $\mathrm{GL}(m_a,\mathbb{K})$.

The fiber of Π is the quotient of nilpotent groups $J_X/(\prod_{a=1}^k J_a^{m_a})$, which is isomorphic as a $\mathbb{K}$-variety to $\mathbb{K}^{l_X-m_1l_1-\cdots-m_kl_k}$. Therefore every fiber of Π has Euler characteristic 1, so by properties of χ we have

$$\chi\left(\frac{\mathrm{Aut}(\bigoplus_{a=1}^k\bigoplus^{m_a}S_a)}{\prod_{a=1}^k\mathrm{Aut}(S_a)^{m_a}}\right)=\chi\left(\prod_{a=1}^k\frac{\mathrm{GL}(m_a,\mathbb{K})}{(\mathbb{K}^\times)^{m_a}}\right)=\prod_{a=1}^k m_a!. \tag{37}$$

Combining (36) and (37) shows the left-hand side of (35) at $[X]$ is 1, the same as the right-hand side. This proves (35), and the theorem. □

Theorem 5.4. *Let Assumption* 3.7 *hold, and* $(\tau,T,\leqslant)$ *be a permissible weak stability condition on* $\mathcal{A}$. *Then for all* $\mathcal{A}$*-data* $(K,\trianglelefteq,\mu)$ *we have*

$$\begin{aligned}&\sum_{\substack{\text{iso. classes}\\ \text{of finite}\\ \text{sets } I}}\frac{1}{|I|!}\cdot\sum_{\substack{\kappa:I\to C(\mathcal{A}),\ \text{surjective }\phi:I\to K:\\ \kappa(\phi^{-1}(k))=\mu(k)\text{ for }k\in K,\\ \tau\circ\mu\circ\phi\equiv\tau\circ\kappa:I\to T.\\ \text{Define }\preccurlyeq\text{ on }I\text{ by }i\preccurlyeq j\text{ if }i=j\\ \text{or }\phi(i)\neq\phi(j)\text{ and }\phi(i)\trianglelefteq\phi(j)}}\mathrm{CF}^{\mathrm{stk}}\big(Q(I,\preccurlyeq,K,\trianglelefteq,\phi)\big)\delta_{\mathrm{si}}(I,\preccurlyeq,\kappa,\tau)\\&=\delta_{\mathrm{ss}}(K,\trianglelefteq,\mu,\tau).\end{aligned} \tag{38}$$

Only finitely many functions $\delta_{\mathrm{si}}(I,\preccurlyeq,\kappa,\tau)$ *in this sum are nonzero.*

Proof. First we prove only finitely many $\delta_{\rm si}(I,\preccurlyeq,\kappa,\tau)$ in (38) are nonzero. Let I,κ,ϕ be as in (38), fix $k\in K$, set $I_k=\phi^{-1}(\{k\})$, $\alpha=\mu(k)$ and $n=|I_k|$, choose a bijection $\iota:\{1,\dots,n\}\to I_k$, and write $\kappa'=\kappa\circ\iota$. Then α,n,κ' are as in (35), so Theorem 5.3 shows there are only finitely many n,κ' with $\delta_{\rm si}(\{1,\dots,n\},\bullet,\kappa',\tau)\not\equiv 0$. But $\delta_{\rm si}(I,\preccurlyeq,\kappa,\tau)\not\equiv 0$ in (38) implies $\delta_{\rm si}(\{1,\dots,n\},\bullet,\kappa',\tau)\not\equiv 0$. So there are finitely many possibilities for $I_k,\kappa|_{I_k}$ up to isomorphism for each $k\in K$, and thus only finitely many for I,κ,ϕ.

For each $k\in K$, let I_k be a finite set and $\kappa_k:I_k\to C(\mathcal{A})$ a map with $\kappa_k(I_k)=\mu(k)$ and $\tau\circ\kappa_k\equiv\tau\circ\mu(k)$. Define $I=\coprod_{k\in K}I_k$ and $\phi:I\to K$ by $\phi(i)=k$ if $i\in I_k$. Define a partial order $\preccurlyeq$ on I using K,μ,ϕ as in (38). Now by applying the proof of [9, Theorem 7.10] $|K|$ times, we can show that the following commutative diagram of 1-morphisms of stacks is a Cartesian square:

$$\begin{array}{ccc} \mathfrak{M}(I,\preccurlyeq,\kappa)_{\mathcal{A}} & \xrightarrow{\quad Q(I,\preccurlyeq,K,\trianglelefteq,\phi)\quad} & \mathfrak{M}(K,\trianglelefteq,\mu)_{\mathcal{A}} \\ \Big\downarrow{\scriptstyle \prod_{k\in K}S(I,\preccurlyeq,I_k)} & & {\scriptstyle \prod_{k\in K}\boldsymbol{\sigma}(\{k\})}\Big\downarrow \\ \prod_{k\in K}\mathfrak{M}(I_k,\bullet,\kappa_k)_{\mathcal{A}} & \xrightarrow{\quad \prod_{k\in K}\boldsymbol{\sigma}(I_k)\quad} & \prod_{k\in K}\mathfrak{Obj}^{\mu(k)}_{\mathcal{A}}. \end{array} \tag{39}$$

Theorem 3.9 shows the rows are representable, and the right 1-morphism finite type. As (39) is Cartesian the left 1-morphism is finite type. Applying Theorem 2.8 to (39) and $\prod_{k\in K}\delta_{\rm si}(I_k,\bullet,\kappa_k,\tau)\in{\rm CF}(\prod_{k\in K}\mathfrak{M}(I_k,\bullet,\kappa_k)_{\mathcal{A}})$ yields

$$\begin{aligned} &{\rm CF}^{\rm stk}\big(Q(I,\preccurlyeq,K,\trianglelefteq,\phi)\big)\delta_{\rm si}(I,\preccurlyeq,\kappa,\tau)\\ &\quad={\rm CF}^{\rm stk}\big(Q(I,\preccurlyeq,K,\trianglelefteq,\phi)\big)\Big[\prod_{k\in K}S(I,\preccurlyeq,I_k)^*\delta_{\rm si}(I_k,\bullet,\kappa_k,\tau)\Big]\\ &\quad=\prod_{k\in K}\boldsymbol{\sigma}\big(\{k\}\big)^*\big[{\rm CF}^{\rm stk}\big(\boldsymbol{\sigma}(I_k)\big)\delta_{\rm si}(I_k,\bullet,\kappa_k,\tau)\big]. \end{aligned} \tag{40}$$

Now $\delta_{\rm ss}(K,\trianglelefteq,\mu,\tau)=\prod_{k\in K}\boldsymbol{\sigma}(\{k\})^*[\delta_{\rm ss}^{\mu(k)}(\tau)]$. Use (35) with $I_k,\mu(k)$ in place of $\{1,\dots,n\}$, α to substitute for $\delta_{\rm ss}^{\mu(k)}(\tau)$, taking the product in ${\rm CF}(\mathfrak{M}(K,\trianglelefteq,\mu)_{\mathcal{A}})$ of $|K|$ copies of (35) pulled back by $\boldsymbol{\sigma}(\{k\})^*$. Using (40) then yields (38), *except that* rather than summing over isomorphism classes of sets I and maps ϕ we sum over isomorphism classes of sets I_k for $k\in K$ (here the sum over sets I_k replaces the sum over n in (35), with $|I_k|=n$), and instead of the factor $1/|I|!$ we have $1/\prod_{k\in K}|I_k|!$.

The sums over I,ϕ and over I_k, $k\in K$ are related as follows: given I,ϕ we set $I_k=\phi^{-1}(\{k\})$ for $k\in K$, and given I_k for $k\in K$ we define $I=\coprod_{k\in K}I_k$ and $\phi:I\to K$ by $\phi|_{I_k}\equiv k$. But this is not a 1–1 correspondence: fixing I_k for $k\in K$ up to isomorphism forces $|I|=\sum_{k\in K}|I_k|$, which fixes I up to isomorphism; but there are $|I|!/\prod_{k\in K}|I_k|!$ choices of $\phi:I\to K$ with $|\phi^{-1}(\{k\})|=|I_k|$ for $k\in K$. This exactly cancels the difference in the combinatorial factors $1/|I|!$ and $1/\prod_{k\in K}|I_k|!$, proving (38). $\square$

5.3. Counting best τ-stable configurations

Now let $X\in\mathcal{A}$ be τ-semistable. If $[(\sigma,\iota,\pi)]\in\mathcal{M}_{\rm st}(X,I,\preccurlyeq,\kappa,\tau)_{\mathcal{A}}$ with $\tau\circ\kappa(i)=\tau([X])$ for all $i\in I$ then $\sigma(\{i\})$ is τ-stable for all $i\in I$, and we call (σ,ι,π) a *τ-stable configuration.* From Theorem 4.5 we find that $\sigma(\{i\})$ for $i\in I$ are the τ-stable factors of X, and up to isomorphism depend only on X. So $|I|$ also depends only on X. We shall calculate the Euler characteristic of the family of all best τ-stable configurations for X up to isomorphism, the union of $\mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(X,I,\preccurlyeq,\kappa,\tau)_{\mathcal{A}}$ over isomorphism classes of $\mathcal{A}$-data $(I,\preccurlyeq,\kappa)$ with $\tau\circ\kappa\equiv\tau([X])$. Consider the following situation.

Definition 5.5. Let Assumption 3.7 hold, $(\tau,T,\leqslant)$ be a permissible stability condition on $\mathcal{A}$, and $X\in\mathcal{A}$ be τ-semistable. Then Theorem 4.5 decomposes X into τ-stable factors with the same τ-value as X, uniquely up to isomorphism and order. Let X have nonisomorphic τ-stable factors $S_1,\dots,S_n$ with multiplicities $l_1,\dots,l_n>0$.

For any τ-stable $(I,\preccurlyeq,\kappa)$-configuration (σ,ι,π) with $\sigma(I)=X$ and $\tau\circ\kappa\equiv\tau([X])$, the $\sigma(\{i\})$ for $i\in I$ are isomorphic to S_m with multiplicities l_m for $m=1,\dots,n$. Thus $|I|=\sum_{m=1}^n l_m$. Fix an indexing set I with $|I|=\sum_{m=1}^n l_m$. For $m=1,\dots,n$ define $k_m=\dim\mathrm{Hom}(S_m,X)$. Then $\bigoplus^{k_m}S_m\cong S_m\otimes\mathrm{Hom}(S_m,X)\subset X$, so $0\leqslant k_m\leqslant l_m$.

Fix $a\in I$, and set $J=I\setminus\{a\}$. Let $\lesssim$ be a partial order on J, and define $\trianglelefteq$ on I by $i\trianglelefteq j$ for $i,j\in I$ if either $i,j\in J$ and $i\lesssim j$, or $i=a$. Define $\phi:I\to\{1,2\}$ by $\phi(a)=1$ and $\phi(j)=2$ for $j\in J$. Let $(\tilde\sigma,\tilde\iota,\tilde\pi)$ be a $(\{1,2\},\leqslant)$-configuration with $\tilde\sigma(\{1,2\})=X$ and $\tilde\sigma(\{1\})$ τ-stable with $\tau([\tilde\sigma(\{1\})])=\tau([X])$. Then $\tilde\sigma(\{1\})$ is (isomorphic to) one of the τ-stable factors of X. Define $Y=\tilde\sigma(\{2\})$.

Choose $\kappa:I\to K(\mathcal{A})$ such that $(I,\trianglelefteq,\kappa)$ is $\mathcal{A}$-data, $\tau\circ\kappa\equiv\tau([X])$, $\kappa(a)=[\tilde\sigma(\{1\})]$, and $[X]=\kappa(I)$. Then $(J,\lesssim,\kappa|_J)$ is also $\mathcal{A}$-data, and $[Y]=\kappa(J)$. Define $\mu:\{1,2\}\to K(\mathcal{A})$ by $\mu(1)=\kappa(a)$, $\mu(2)=\kappa(J)$. Consider the diagram of 1-morphisms

$$\begin{array}{ccccc}
\mathfrak{M}(I,\trianglelefteq,\kappa)_{\mathcal{A}} & \xrightarrow[Q(I,\trianglelefteq,\{1,2\},\leqslant,\phi)]{} & \mathfrak{M}(\{1,2\},\leqslant,\mu)_{\mathcal{A}} & \xleftarrow{(\tilde\sigma,\tilde\iota,\tilde\pi)} & \\
\Big\downarrow{\scriptstyle S(I,\trianglelefteq,J)} & & \Big\downarrow{\scriptstyle \boldsymbol{\sigma}(\{2\})} & & \mathrm{Spec}\,\mathbb{K}. \\
\mathfrak{M}(J,\lesssim,\kappa)_{\mathcal{A}} & \xrightarrow{\boldsymbol{\sigma}(J)} & \mathfrak{Obj}_{\mathcal{A}}^{\kappa(J)} & \xleftarrow[Y]{} &
\end{array}$$

By [9, Theorem 7.10], the left-hand side is a Cartesian square. And as $\tilde\sigma(\{2\})=Y$, the right-hand side commutes. Therefore $S(I,\trianglelefteq,J)$ induces a 1-isomorphism

$$\begin{aligned}
S(I,\trianglelefteq,J)_*:{}&\mathfrak{M}(I,\trianglelefteq,\kappa)_{\mathcal{A}}\times_{Q(I,\trianglelefteq,\{1,2\},\leqslant,\phi),\mathfrak{M}(\{1,2\},\leqslant,\mu)_{\mathcal{A}},(\tilde\sigma,\tilde\iota,\tilde\pi)}\mathrm{Spec}\,\mathbb{K}\\
&\to\mathfrak{M}(J,\lesssim,\kappa)_{\mathcal{A}}\times_{\boldsymbol{\sigma}(J),\mathfrak{Obj}_{\mathcal{A}}^{\kappa(J)},Y}\mathrm{Spec}\,\mathbb{K}=\mathfrak{M}(Y,J,\lesssim,\kappa)_{\mathcal{A}}.
\end{aligned}\tag{41}$$

But as $\tilde\sigma(\{1,2\})=X$ we have a commutative diagram

$$\begin{array}{ccccc}
\mathfrak{M}(I,\trianglelefteq,\kappa)_{\mathcal{A}} & \xrightarrow[Q(I,\trianglelefteq,\{1,2\},\leqslant,\phi)]{} & \mathfrak{M}(\{1,2\},\leqslant,\mu)_{\mathcal{A}} & \xleftarrow[(\tilde\sigma,\tilde\iota,\tilde\pi)]{} & \mathrm{Spec}\,\mathbb{K} \\
& {\scriptstyle\boldsymbol{\sigma}(I)}\searrow & \Big\downarrow{\scriptstyle \boldsymbol{\sigma}(\{1,2\})} & \swarrow{\scriptstyle X} & \\
& & \mathfrak{Obj}_{\mathcal{A}}^{\kappa(I)}. & &
\end{array}$$

Therefore $\boldsymbol{\sigma}(\{1,2\})$ induces a 1-morphism

$$\begin{aligned}
\boldsymbol{\sigma}\big(\{1,2\}\big)_*:{}&\mathfrak{M}(I,\trianglelefteq,\kappa)_{\mathcal{A}}\times_{Q(I,\trianglelefteq,\{1,2\},\leqslant,\phi),\mathfrak{M}(\{1,2\},\leqslant,\mu)_{\mathcal{A}},(\tilde\sigma,\tilde\iota,\tilde\pi)}\mathrm{Spec}\,\mathbb{K}\\
&\to\mathfrak{M}(I,\trianglelefteq,\kappa)_{\mathcal{A}}\times_{\boldsymbol{\sigma}(I),\mathfrak{Obj}_{\mathcal{A}}^{\kappa(J)},X}\mathrm{Spec}\,\mathbb{K}=\mathfrak{M}(X,I,\trianglelefteq,\kappa)_{\mathcal{A}}.
\end{aligned}\tag{42}$$

As (41) is a 1-isomorphism it is invertible, so (41) and (42) give a 1-morphism

$$\boldsymbol{\sigma}\bigl(\{1,2\}\bigr)_* \circ S(I,\trianglelefteq,J)_*^{-1} : \mathfrak{M}(Y,J,\lesssim,\kappa)_{\mathcal{A}} \to \mathfrak{M}(X,I,\trianglelefteq,\kappa)_{\mathcal{A}}.$$

On the underlying geometric spaces the 1-isomorphism (41) gives a bijection, and (42) an injective map with image $Q(I,\trianglelefteq,\{1,2\},\leqslant,\phi)_*^{-1}(\{[(\tilde\sigma,\tilde\iota,\tilde\pi)]\})$. Thus we have a 1–1 correspondence

$$\begin{aligned}&\bigl(\boldsymbol{\sigma}\bigl(\{1,2\}\bigr)_* \circ S(I,\trianglelefteq,J)_*^{-1}\bigr)_* : \mathcal{M}(Y,J,\lesssim,\kappa)_{\mathcal{A}}\\ &\quad\to Q\bigl(I,\trianglelefteq,\{1,2\},\leqslant,\phi\bigr)_*^{-1}\bigl(\bigl\{\bigl[(\tilde\sigma,\tilde\iota,\tilde\pi)\bigr]\bigr\}\bigr) \subseteq \mathcal{M}(X,I,\trianglelefteq,\kappa)_{\mathcal{A}}.\end{aligned} \tag{43}$$

Here is how to understand (43): it maps $[(\sigma',\iota',\pi')] \mapsto [(\sigma,\iota,\pi)]$, for (σ',ι',π') a $(J,\lesssim,\kappa)$-configuration with $\sigma'(J)=Y$, and (σ,ι,π) the $(I,\trianglelefteq,\kappa)$-configuration constructed by *substituting* (σ',ι',π') into $(\tilde\sigma,\tilde\iota,\tilde\pi)$ at 2, using [9, Definition 5.7].

As $\tilde\sigma(\{1\})$ is τ-stable, (43) is a 1–1 correspondence between τ-stable configurations in its domain and range. Hence

$$\begin{aligned}&\mathrm{CF}^{\mathrm{stk}}\bigl(\boldsymbol{\sigma}\bigl(\{1,2\}\bigr)_* \circ S(I,\trianglelefteq,J)_*^{-1}\bigr)\delta_{\mathrm{st}}(Y,J,\lesssim,\kappa,\tau)\\ &\quad= \delta_{\mathrm{st}}(X,I,\trianglelefteq,\kappa,\tau)\cdot\delta_{Q^{-1}([(\tilde\sigma,\tilde\iota,\tilde\pi)])},\end{aligned} \tag{44}$$

writing $Q^{-1}([(\tilde\sigma,\tilde\iota,\tilde\pi)])$ as a shorthand for $Q(I,\trianglelefteq,\{1,2\},\leqslant,\phi)^{-1}([(\tilde\sigma,\tilde\iota,\tilde\pi)])$, and $\delta_{Q^{-1}([(\tilde\sigma,\tilde\iota,\tilde\pi)])}$ for its characteristic function.

We now apply (27) to rewrite $\delta_{\mathrm{st}}(X,I,\trianglelefteq,\kappa,\tau)\cdot\delta_{Q^{-1}([(\tilde\sigma,\tilde\iota,\tilde\pi)])}$ as a sum over partial orders $\preccurlyeq$ on I dominated by $\trianglelefteq$. The operators $\mathrm{CF}^{\mathrm{stk}}(Q(I,\preccurlyeq,\trianglelefteq))$ commute with multiplication by $\delta_{Q^{-1}([(\sigma,\iota,\pi)])}$. Substituting this into (44) gives

$$\begin{aligned}&\sum_{\substack{\text{p.o.s }\preccurlyeq\text{ on }I:\\ \trianglelefteq\text{ dominates }\preccurlyeq}} \mathrm{CF}^{\mathrm{stk}}\bigl(Q(I,\preccurlyeq,\trianglelefteq)\bigr)\bigl(\delta_{\mathrm{st}}^{\mathrm{b}}(X,I,\preccurlyeq,\kappa,\tau)\cdot\delta_{Q^{-1}([(\tilde\sigma,\tilde\iota,\tilde\pi)])}\bigr)\\ &\quad= \mathrm{CF}^{\mathrm{stk}}\bigl(\boldsymbol{\sigma}\bigl(\{1,2\}\bigr)_* \circ S(I,\trianglelefteq,J)_*^{-1}\bigr)\delta_{\mathrm{st}}(Y,J,\lesssim,\kappa,\tau).\end{aligned} \tag{45}$$

One can show using Theorem 3.6 that the image under $Q(I,\preccurlyeq,\trianglelefteq)_*$ of a best $(I,\preccurlyeq)$-configuration in $Q^{-1}([(\tilde\sigma,\tilde\iota,\tilde\pi)])$ is the image under $(\boldsymbol{\sigma}(\{1,2\})_* \circ S(I,\trianglelefteq,J)_*^{-1})_*$ of a best $(J,\lesssim)$-configuration if and only if $\preccurlyeq|_J = \lesssim$. So restricting (45) to $\preccurlyeq$ with $\preccurlyeq|_J = \lesssim$ gives

$$\begin{aligned}&\sum_{\substack{\text{p.o.s }\preccurlyeq\text{ on }I:\ \preccurlyeq|_J=\lesssim\\ \text{and } j\not\preccurlyeq a\text{ for all } j\in J}} \mathrm{CF}^{\mathrm{stk}}\bigl(Q(I,\preccurlyeq,\trianglelefteq)\bigr)\bigl(\delta_{\mathrm{st}}^{\mathrm{b}}(X,I,\preccurlyeq,\kappa,\tau)\cdot\delta_{Q^{-1}([(\tilde\sigma,\tilde\iota,\tilde\pi)])}\bigr)\\ &\quad= \mathrm{CF}^{\mathrm{stk}}\bigl(\boldsymbol{\sigma}\bigl(\{1,2\}\bigr)_* \circ S(I,\trianglelefteq,J)_*^{-1}\bigr)\delta_{\mathrm{st}}^{\mathrm{b}}(Y,J,\lesssim,\kappa,\tau).\end{aligned}$$

Taking weighted Euler characteristics of both sides, using (3), and summing over all $\lesssim$, κ with $\tau\circ\kappa \equiv \tau([X])$ proves:

Proposition 5.6. *Let X, I be as above, $a \in I$ and $J = I \setminus \{a\}$. Define $\phi : I \to \{1,2\}$ by $\phi(a)=1$ and $\phi(j)=2$ for $j\in J$. Let $(\tilde\sigma,\tilde\iota,\tilde\pi)$ be a $(\{1,2\},\leqslant)$-configuration with $\tilde\sigma(\{1,2\})=X$ and $\tilde\sigma(\{1\})$ τ-stable with $\tau([\tilde\sigma(\{1\})])=\tau([X])$. Define $Y=\tilde\sigma(\{2\})$. Then*

$$\sum_{\substack{\preccurlyeq,\kappa:\ (I,\preccurlyeq,\kappa)\text{ is }\mathcal{A}\text{-data},\\ a\text{ is }\preccurlyeq\text{-minimal, }[\kappa(a)]=[\tilde\sigma(\{1\})],\\ \kappa(I)=[X],\ \tau\circ\kappa\equiv\tau([X])}} \chi^{\mathrm{na}}\bigl(\mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(X,I,\preccurlyeq,\kappa,\tau)_{\mathcal{A}}\cap Q\bigl(I,\preccurlyeq,\{1,2\},\leqslant,\phi\bigr)_*^{-1}\bigl(\bigl\{\bigl[(\tilde\sigma,\tilde\iota,\tilde\pi)\bigr]\bigr\}\bigr)\bigr)$$
$$= \sum_{\substack{\lesssim,\lambda:\ (J,\lesssim,\lambda)\text{ is }\mathcal{A}\text{-data},\\ \lambda(J)=[Y],\ \tau\circ\lambda\equiv\tau([X])}} \chi^{\mathrm{na}}\bigl(\mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(Y,J,\lesssim,\lambda,\tau)_{\mathcal{A}}\bigr). \tag{46}$$

Only finitely many terms in each sum are nonzero.

We have not yet verified only finitely many terms in (46) are nonzero. Set $[X]=\alpha$, and suppose $\preccurlyeq,\kappa$ are as on the left-hand side of (46) with $\mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(X,I,\preccurlyeq,\kappa,\tau)_{\mathcal{A}}\neq\emptyset$. Let $K\neq\emptyset, I$ be an $(I,\preccurlyeq)$ s-set, and set $\beta=\kappa(K)$, $\gamma=\kappa(I\setminus K)$. Then $\beta,\gamma\in C(\mathcal{A})$ with $\alpha=\beta+\gamma$, and $\tau(\alpha)=\tau(\beta)=\tau(\gamma)$ as $\tau\circ\kappa\equiv\tau([X])$. If $[(\sigma,\iota,\pi)]\in\mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(X,I,\preccurlyeq,\kappa,\tau)_{\mathcal{A}}$ then $[\sigma(K)]\in\mathrm{Obj}_{\mathrm{ss}}^{\beta}(\tau)$ and $[\sigma(I\setminus K)]\in\mathrm{Obj}_{\mathrm{ss}}^{\gamma}(\tau)$, so $\mathrm{Obj}_{\mathrm{ss}}^{\beta}(\tau)\neq\emptyset\neq\mathrm{Obj}_{\mathrm{ss}}^{\gamma}(\tau)$.

Hence Proposition 4.9 implies there are only finitely many possibilities for $\kappa(K),\kappa(I\setminus K)$. As this holds for all $(I,\preccurlyeq)$ s-sets $K\neq\emptyset, I$, and there are only finitely many choices for $\preccurlyeq$, there are also only finitely many choices for κ. So only finitely many terms on the left-hand side of (46) are nonzero. The proof for the right-hand side is the same. We can easily extend this proof to fix not just one $\preccurlyeq$-minimal element $a\in I$, but a *minimal subset* $A\subseteq I$.

Proposition 5.7. *Let X, I be as above, $A\subseteq I$ and $J=I\setminus A$. Let $b\notin A$ and set $B=A\cup\{b\}$. Define a partial order $\precsim$ on B by $r\precsim s$ if either $r=s$ or $s=b$. Define $\phi: I\to B$ by $\phi(a)=a$ for $a\in A$ and $\phi(i)=b$ for $i\in I\setminus A$. Let $(\tilde\sigma,\tilde\iota,\tilde\pi)$ be a $(B,\precsim)$-configuration with $\tilde\sigma(B)=X$ and $\tilde\sigma(\{a\})$ τ-stable for all $a\in A$ with $\tau([\tilde\sigma(\{a\})])=\tau([X])$. Define $Y=\tilde\sigma(\{b\})$. Then*

$$\sum_{\substack{\preccurlyeq,\kappa:\ (I,\preccurlyeq,\kappa)\text{ is }\mathcal{A}\text{-data},\\ \text{each }a\in A\text{ is }\preccurlyeq\text{-minimal},\\ [\kappa(a)]=[\tilde\sigma(\{a\})],\ \kappa(I)=[X],\\ \tau\circ\kappa\equiv\tau([X])}} \chi^{\mathrm{na}}\bigl(\mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(X,I,\preccurlyeq,\kappa,\tau)_{\mathcal{A}}\cap Q(I,\preccurlyeq,B,\precsim,\phi)_*^{-1}\bigl(\bigl\{\bigl[(\tilde\sigma,\tilde\iota,\tilde\pi)\bigr]\bigr\}\bigr)\bigr)$$
$$= \sum_{\substack{\lesssim,\lambda:\ (J,\lesssim,\lambda)\text{ is }\mathcal{A}\text{-data},\\ \lambda(J)=[Y],\ \tau\circ\lambda\equiv\tau([X])}} \chi^{\mathrm{na}}\bigl(\mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(Y,J,\lesssim,\lambda,\tau)_{\mathcal{A}}\bigr). \tag{47}$$

Only finitely many terms in each sum are nonzero.

We now calculate the Euler characteristic of the set of all $[(\tilde\sigma,\tilde\iota,\tilde\pi)]$ satisfying the conditions in Proposition 5.7.

Proposition 5.8. *Let X, I and S_m, k_m, l_m for $m=1,\dots,n$ be as in Definition 5.5, and set $k=\sum_{m=1}^n k_m$. For $A\subseteq I$ with $|A|\leqslant k$, define $(B,\precsim)$ as in Proposition 5.7, and define*

$$\mathcal{M}_A=\Biggl\{\bigl[(\tilde\sigma,\tilde\iota,\tilde\pi)\bigr]\in\coprod_{\substack{\mu:\ (B,\precsim,\mu)\text{ is }\mathcal{A}\text{-data},\\ \mu(B)=[X],\\ \tau\circ\mu\equiv\tau([X])}}\mathcal{M}(X,B,\precsim,\mu)_{\mathcal{A}}:\ \tilde\sigma\bigl(\{a\}\bigr)\ \text{is }\tau\text{-stable for all }a\in A\Biggr\}. \tag{48}$$

Then $\mathcal{M}_A$ is constructible with $\chi^{\mathrm{na}}(\mathcal{M}_A)=k!/(k-|A|)!$.

Proof. Write $P(\mathrm{Hom}(S_m, X))$ for the projective space of $\mathrm{Hom}(S_m, X)$. Then $P(\mathrm{Hom}(S_m, X)) \cong \mathbb{KP}^{k_m-1}$ as $\mathrm{Hom}(S_m, X) \cong \mathbb{K}^{k_m}$. Regard $P(\mathrm{Hom}(S_m, X))$ as (the set of geometric points of) a projective $\mathbb{K}$-variety. Define

$$\mathcal{N}_A = \Biggl\{\psi : A \to \coprod_{m=1}^{n} P\bigl(\mathrm{Hom}(S_m, X)\bigr):\ \psi \text{ is injective and} \\ \psi(A) \cap P\bigl(\mathrm{Hom}(S_m, X)\bigr) \text{ is linearly independent for all } m\Biggr\}. \tag{49}$$

Here a finite subset S of a projective space $P(V)$ is *linearly independent* if there exists no linear subspace $U \subseteq V$ with $S \subseteq P(U)$ and $\dim U < |S|$. Then $\mathcal{N}_A$ is an open set in the projective $\mathbb{K}$-scheme $\prod_{a\in A} \coprod_{m=1}^{n} P(\mathrm{Hom}(S_m, X))$, so it is (the set of geometric points of) a quasiprojective $\mathbb{K}$-scheme.

Define a map $\Phi : \mathcal{M}_A \to \mathcal{N}_A$ as follows. If $[(\tilde\sigma, \tilde\iota, \tilde\pi)] \in \mathcal{M}_A$ and $a \in A$ then $\tilde\sigma(\{a\})$ is τ-stable with $\tau([\tilde\sigma(\{a\})]) = \tau([X])$, so it follows that $\tilde\sigma(\{a\})$ is isomorphic to one of the τ*-stable factors* of X. Thus there exists an isomorphism $i : S_m \to \tilde\sigma(\{a\})$ for some unique $m = 1, \ldots, n$. As S_m is τ-stable $\mathrm{End}(S_m) = \mathbb{K}$, so i is unique up to multiplication by a nonzero element of $\mathbb{K}$.

As $i, \tilde\iota(\{a\}, B)$ are injective we have $0 \neq \tilde\iota(\{a\}, B) \circ i \in \mathrm{Hom}(S_m, X)$, and the class $[\tilde\iota(\{a\}, B) \circ i] \in P(\mathrm{Hom}(S_m, X))$ is independent of choice of i. Define $\psi(a) = [\tilde\iota(\{a\}, B) \circ i]$. This defines a map $\psi : A \to \coprod_{m=1}^{n} P(\mathrm{Hom}(S_m, X))$. Define $\Phi([(\tilde\sigma, \tilde\iota, \tilde\pi)]) = \psi$.

Now Φ essentially maps $[(\tilde\sigma, \tilde\iota, \tilde\pi)]$ in $\mathcal{M}_A$ to a set of stable subobjects in X parametrized by A. Using [9, Theorems 4.2 and 4.5] we deduce necessary and sufficient conditions for such a set of subobjects to come from a $(B, \precsim)$-configuration $(\tilde\sigma, \tilde\iota, \tilde\pi)$ with $\tilde\sigma(B) = X$, and they turn out to be that ψ is injective and $\psi(A) \cap P(\mathrm{Hom}(S_m, X))$ is linearly independent for all m. It follows that Φ maps to $\mathcal{N}_A$, and is a 1–1 correspondence.

By [9, Assumption 7.1(iv)] and general facts from [7] and [9], it is not difficult to see that Φ is a *pseudoisomorphism*, in the sense of [7, §4.2]. The point of invoking [9, Assumption 7.1(iv)] is that it gives a *tautological morphism* $\theta_{S_m,X}$, a family of morphisms $S_m \to X$ parametrized by the base $\mathbb{K}$-scheme $\mathrm{Hom}(S_m, X)$. Using this it is easy, for instance, to construct a $\mathbb{K}$-substack P_m of $\mathfrak{M}(X, \{1, 2\}, \leqslant, \mu)_{\mathcal{A}}$ isomorphic to $P(\mathrm{Hom}(S_m, X))$, where $\mu(1) = [S_m]$ and $\mu(2) = [X] - [S_m]$, with

$$P_m(\mathbb{K}) = \bigl\{\bigl[(\tilde\sigma, \tilde\iota, \tilde\pi)\bigr] \in \mathcal{M}\bigl(X, \{1, 2\}, \leqslant, \mu\bigr)_{\mathcal{A}}:\ \tilde\sigma\bigl(\{1\}\bigr) \cong S_m\bigr\}.$$

When $|A| = 1$ we have $\mathcal{M}_A \cong \coprod_{m=1}^{n} P_m(\mathbb{K})$ and $\mathcal{N}_A \cong \coprod_{m=1}^{n} P(\mathrm{Hom}(S_m, X))$, and the result follows. The case $|A| > 1$ is a straightforward generalization.

From [7, Definition 4.8] we see that $\chi^{\mathrm{na}}(\mathcal{M}_A) = \chi(\mathcal{N}_A)$. Thus the proposition follows from $\chi(\mathcal{N}_A) = k!/(k - |A|)!$. One can prove this using $P(\mathrm{Hom}(S_m, X)) \cong \mathbb{KP}^{k_m-1}$, $k = \sum_{m=1}^{n} k_m$, (49), and properties of χ including (2), by a long but elementary calculation that we leave as an exercise. □

In the next theorem, note that the set of $\preccurlyeq$-minimal elements in I *contains* A in (50), and is *equal to* A in (51).

Theorem 5.9. *Let X, I and S_m, k_m, l_m for $m = 1, \ldots, n$ be as in Definition 5.5, and set $k = \sum_{m=1}^{n} k_m$. Then for each $A \subseteq I$ with $|A| \leqslant k$ we have*

$$\sum_{\substack{\preccurlyeq,\kappa:\ (I,\preccurlyeq,\kappa)\text{ is }\mathcal{A}\text{-data},\\ \text{each }a\in A\text{ is }\preccurlyeq\text{-minimal},\\ \kappa(I)=[X],\ \tau\circ\kappa\equiv\tau([X])}} \chi^{\mathrm{na}}\big(\mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(X,I,\preccurlyeq,\kappa,\tau)_{\mathcal{A}}\big)=\frac{(|I|-|A|)!k!}{(k-|A|)!},\quad \textit{and} \tag{50}$$

$$\sum_{\substack{\preccurlyeq,\kappa:\ (I,\preccurlyeq,\kappa)\text{ is }\mathcal{A}\text{-data},\\ A\text{ is the }\preccurlyeq\text{-minimal set},\\ \kappa(I)=[X],\ \tau\circ\kappa\equiv\tau([X])}} \chi^{\mathrm{na}}\big(\mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(X,I,\preccurlyeq,\kappa,\tau)_{\mathcal{A}}\big)=\begin{cases}0, & |A|<k,\\ (|I|-k)!k!, & |A|=k.\end{cases} \tag{51}$$

Only finitely many terms in each sum are nonzero.

Proof. The argument after Proposition 5.6 shows only finitely many terms in (50) and (51) are nonzero. Note that the $\preccurlyeq$-minimal set A in I always has $1\leqslant|A|\leqslant k$ by definition of k, k_m, as X has only k linearly independent τ-stable subobjects. First we show (50) and (51) are equivalent.

Suppose (51) holds. Then letting the $\preccurlyeq$-minimal set in (50) be A', for $A\subseteq A$ with $|A|\leqslant k$ summing (51) with A' in place of A over all $A'\subseteq A$ and using a simple combinatorial argument proves (50). Now (51) holds trivially when $|A|=0$ as both sides are zero. Hence, (51) for $1\leqslant|A|\leqslant k$ implies (50). By a more complicated argument we find (50) for $1\leqslant|A|\leqslant k$ implies (51). Hence, if (50) holds when $1\leqslant|A|\leqslant k$, then both (50) and (51) hold for $|A|\leqslant k$.

We can now prove the theorem by induction on $|I|$. The result is trivial when $|I|=1$, giving the first step. Suppose by induction that (50) and (51) hold whenever $|I|\leqslant m$, and let $|I|=m+1$. Let $A\subseteq I$ with $1\leqslant|A|\leqslant k$, and set $J=I\setminus A$. Let $\mathcal{M}_A$ be as in Proposition 5.8. Define $\mathfrak{G}$ and $T\subseteq\mathfrak{G}(\mathbb{K})$ by

$$\mathfrak{G}=\coprod_{\substack{\preccurlyeq,\kappa:\ (I,\preccurlyeq,\kappa)\text{ is }\mathcal{A}\text{-data},\\ \text{each }a\in A\text{ is }\preccurlyeq\text{-minimal},\\ \kappa(I)=[X],\ \tau\circ\kappa\equiv\tau([X])}}\mathfrak{M}(X,I,\preccurlyeq,\kappa)_{\mathcal{A}},\ T=\coprod_{\substack{\preccurlyeq,\kappa:\ (I,\preccurlyeq,\kappa)\text{ is }\mathcal{A}\text{-data},\\ \text{each }a\in A\text{ is }\preccurlyeq\text{-minimal},\\ \kappa(I)=[X],\ \tau\circ\kappa\equiv\tau([X])}}\mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(X,I,\preccurlyeq,\kappa,\tau)_{\mathcal{A}}. \tag{52}$$

Let B, $\precsim$, ϕ be as in Proposition 5.7. For each $\preccurlyeq,\kappa$ in the definition of $\mathfrak{G}$ in (52), define $\mu:B\to C(\mathcal{A})$ by $\mu(c)=\kappa(\phi^{-1}(\{c\}))$. Then we have a 1-morphism $Q(I,\preccurlyeq,B,\precsim,\phi):\mathfrak{M}(X,I,\preccurlyeq,\kappa)_{\mathcal{A}}\to\mathfrak{M}(X,B,\precsim,\mu)_{\mathcal{A}}$. Define a 1-morphism

$$\psi:\mathfrak{G}\to\coprod_{\substack{\mu:\ (B,\precsim,\mu)\text{ is }\mathcal{A}\text{-data},\\ \mu(B)=[X],\\ \tau\circ\mu\equiv\tau([X])}}\mathfrak{M}(X,B,\precsim,\mu)_{\mathcal{A}}\quad\text{by}$$

$$\psi=\coprod_{\substack{\preccurlyeq,\kappa:\ (I,\preccurlyeq,\kappa)\text{ is }\mathcal{A}\text{-data},\\ \text{each }a\in A\text{ is }\preccurlyeq\text{-minimal},\\ \kappa(I)=[X],\ \tau\circ\kappa\equiv\tau([X])}}Q(I,\preccurlyeq,B,\precsim,\phi). \tag{53}$$

Comparing (48), (53) and the definition of ϕ shows that ψ_* maps $T\to\mathcal{M}_A$. Let $[(\tilde\sigma,\tilde\iota,\tilde\pi)]\in\mathcal{M}_A$, and define $Y=\tilde\sigma(\{b\})$. Then (47) gives an expression for $\chi^{\mathrm{na}}(T\cap\psi_*^{-1}([(\tilde\sigma,\tilde\iota,\tilde\pi)]))$. Now the right-hand side of (47) is the left-hand side of (50) with Y in place of X, J in place of I, and $\emptyset$ in place of A.

Since $|I|=m+1$, $|A|\geqslant 1$ and $J=I\setminus A$ we have $|J|\leqslant m$. Hence by the inductive hypothesis, (50) holds for $Y, J, \emptyset$. So for all $[(\tilde\sigma,\tilde\iota,\tilde\pi)]\in\mathcal{M}_A$ we have

$$\chi^{\mathrm{na}}\big(T\cap\psi_*^{-1}\big(\big[(\tilde\sigma,\tilde\iota,\tilde\pi)\big]\big)\big)=|J|!=\big(|I|-|A|\big)!.$$

Proposition 5.8 and general properties of χ^{na} now imply that

$$\sum_{\substack{\preccurlyeq,\kappa:\ (I,\preccurlyeq,\kappa)\text{ is }\mathcal{A}\text{-data,}\\ \text{each } a\in A\text{ is }\preccurlyeq\text{-minimal,}\\ \kappa(I)=[X],\ \tau\circ\kappa\equiv\tau([X])}}\chi^{\mathrm{na}}\big(\mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(X,I,\preccurlyeq,\kappa,\tau)_{\mathcal{A}}\big)=\chi^{\mathrm{na}}(T)=\big(|I|-|A|\big)!\chi^{\mathrm{na}}(\mathcal{M}_A)$$
$$=\big(|I|-|A|\big)!k!/\big(k-|A|\big)!.$$

Hence (50) holds for $1\leqslant|A|\leqslant k$ with this fixed I, and so (50) and (51) hold for $|A|\leqslant k$ with this I from above. This completes the inductive step. □

Equation (50) with $A=\emptyset$ calculates the Euler characteristic of the family of all best τ-stable configurations for X.

Corollary 5.10. *Let Assumption* 3.7 *hold,* $(\tau,T,\leqslant)$ *be a permissible stability condition on* $\mathcal{A}$*, and* $X\in\mathcal{A}$ *be* τ*-semistable. Fix a finite set* I *such that* X *has* $|I|$ τ*-stable factors in Theorem* 4.5*, counted with multiplicity. Then*

$$\sum_{\substack{\preccurlyeq,\kappa:\ (I,\preccurlyeq,\kappa)\text{ is }\mathcal{A}\text{-data,}\\ \kappa(I)=[X],\ \tau\circ\kappa\equiv\tau([X])}}\chi^{\mathrm{na}}\big(\mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(X,I,\preccurlyeq,\kappa,\tau)_{\mathcal{A}}\big)=|I|!. \tag{54}$$

Only finitely many $\mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(X,I,\preccurlyeq,\kappa,\tau)_{\mathcal{A}}$ *in this sum are nonempty.*

We turn this into an identity on constructible functions:

Theorem 5.11. *Let Assumption* 3.7 *hold,* $(\tau,T,\leqslant)$ *be a permissible stability condition on* $\mathcal{A}$*, and* $\alpha\in C(\mathcal{A})$*. Then*

$$\sum_{\substack{\text{iso. classes}\\ \text{of finite}\\ \text{sets } I}}\frac{1}{|I|!}\cdot\sum_{\substack{\preccurlyeq,\kappa:\ (I,\preccurlyeq,\kappa)\text{ is }\mathcal{A}\text{-data,}\\ \kappa(I)=\alpha,\ \tau\circ\kappa\equiv\tau(\alpha)}}\mathrm{CF}^{\mathrm{stk}}\big(\boldsymbol{\sigma}(I)\big)\delta_{\mathrm{st}}^{\mathrm{b}}(I,\preccurlyeq,\kappa,\tau)=\delta_{\mathrm{ss}}^{\alpha}(\tau). \tag{55}$$

Only finitely many functions $\delta_{\mathrm{st}}^{\mathrm{b}}(I,\preccurlyeq,\kappa,\tau)$ *in this sum are nonzero.*

Proof. A similar proof to that in Section 5.2 showing (35) has only finitely many nonzero terms proves that only finitely many $\delta_{\mathrm{st}}^{\mathrm{b}}(I,\preccurlyeq,\kappa,\tau)$ are nonzero in (55). Let $(I,\preccurlyeq,\kappa)$ be as in (55) and $[(\sigma,\iota,\pi)]\in\mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(I,\preccurlyeq,\kappa,\tau)_{\mathcal{A}}$. Then $\tau\circ\kappa\equiv\tau(\alpha)$ implies $\sigma(I)$ is τ-semistable, so $[\sigma(I)]\in\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$. Hence both sides of (55) are zero outside $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$. But if $X\in\mathcal{A}$ is τ-semistable with $[X]=\alpha$ in $K(\mathcal{A})$ then (54) and the definitions of $\delta_{\mathrm{st}}^{\mathrm{b}}(I,\preccurlyeq,\kappa,\tau)$ and $\mathrm{CF}^{\mathrm{stk}}$ imply that both sides of (55) are equal at $[X]\in\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$, by an argument similar to Theorem 5.2. □

5.4. Counting best τ-stable refinements

Our next result in effect computes the Euler characteristic of the family of all *best τ-stable refinements* of a τ-semistable $(K,\trianglelefteq)$-configuration (σ,ι,π).

Theorem 5.12. *Let Assumption 3.7 hold, and $(\tau,T,\leqslant)$ be a permissible stability condition on $\mathcal{A}$. Then for all $\mathcal{A}$-data $(K,\trianglelefteq,\mu)$ we have*

$$\sum_{\substack{\text{iso. classes}\\ \text{of finite}\\ \text{sets } I}} \frac{1}{|I|!}\cdot \sum_{\substack{\preccurlyeq,\kappa,\phi:\ (I,\preccurlyeq,\kappa) \text{ is } \mathcal{A}\text{-data},\\ \phi: I\to K \text{ is surjective},\\ i\preccurlyeq j \text{ implies } \phi(i)\trianglelefteq\phi(j),\\ \kappa(\phi^{-1}(k))=\mu(k) \text{ for } k\in K,\\ \tau\circ\mu\circ\phi\equiv\tau\circ\kappa: I\to T}} \mathrm{CF}^{\mathrm{stk}}\big(Q(I,\preccurlyeq,K,\trianglelefteq,\phi)\big)\delta_{\mathrm{st}}^{\mathrm{b}}(I,\preccurlyeq,\kappa,\tau)$$
$$= \delta_{\mathrm{ss}}(K,\trianglelefteq,\mu,\tau). \tag{56}$$

Only finitely many functions $\delta_{\mathrm{st}}^{\mathrm{b}}(I,\preccurlyeq,\kappa,\tau)$ in this sum are nonzero.

Proof. A similar proof to that in Section 5.2 showing (38) has only finitely many nonzero terms proves that only finitely many $\delta_{\mathrm{st}}^{\mathrm{b}}(I,\preccurlyeq,\kappa,\tau)$ are nonzero in (56), as (55) has only finitely many nonzero terms. For each $k\in K$, let $(I_k,\lesssim_k,\kappa_k)$ be $\mathcal{A}$-data with $\kappa_k(I_k)=\mu(k)$ and $\tau\circ\kappa_k\equiv\tau\circ\mu(k)$. Define $I=\coprod_{k\in K} I_k$ and $\phi: I\to K$ by $\phi(i)=k$ if $i\in I_k$. Define $\lesssim$ on I by $i\lesssim j$ for $i,j\in I$ if either (a) $\phi(i)\trianglelefteq\phi(j)$ and $\phi(i)\neq\phi(j)$, or (b) $\phi(i)=\phi(j)=k$ and $i\lesssim_k j$. Then $i\lesssim j$ implies $\phi(i)\trianglelefteq\phi(j)$. Define $\kappa: I\to K(\mathcal{A})$ by $\kappa|_{I_k}=\kappa_k$. Then $(I,\lesssim,\kappa)$ is $\mathcal{A}$-data with $\kappa(\phi^{-1}(k))=\mu(k)$ for $k\in K$ and $\tau\circ\mu\circ\phi\equiv\tau\circ\kappa$, as in (56).

As for (39), the following commutative diagram is a Cartesian square

$$\begin{array}{ccc}
\mathfrak{M}(I,\lesssim,\kappa)_{\mathcal{A}} & \xrightarrow{\quad Q(I,\lesssim,K,\trianglelefteq,\phi)\quad} & \mathfrak{M}(K,\trianglelefteq,\mu)_{\mathcal{A}} \\
\downarrow{\scriptstyle \prod_{k\in K} S(I,\lesssim,I_k)} & & \downarrow{\scriptstyle \prod_{k\in K}\boldsymbol{\sigma}(\{k\})} \\
\prod_{k\in K}\mathfrak{M}(I_k,\lesssim_k,\kappa_k)_{\mathcal{A}} & \xrightarrow{\quad \prod_{k\in K}\boldsymbol{\sigma}(I_k)\quad} & \prod_{k\in K}\mathfrak{Obj}_{\mathcal{A}}^{\mu(k)},
\end{array} \tag{57}$$

with representable rows and finite type columns. Since $\delta_{\mathrm{st}}^{\mathrm{b}}(I_k,\lesssim_k,\kappa_k,\tau)\in \mathrm{CF}(\mathfrak{M}(I_k,\lesssim_k,\kappa_k)_{\mathcal{A}})$ by Theorem 4.8, we may apply Theorem 2.8 to (57) and the function $\prod_{k\in K}\delta_{\mathrm{st}}^{\mathrm{b}}(I_k,\lesssim_k,\kappa_k,\tau)\in \mathrm{CF}(\prod_{k\in K}\mathfrak{M}(I_k,\lesssim_k,\kappa_k)_{\mathcal{A}})$. This yields

$$\mathrm{CF}^{\mathrm{stk}}\big(Q(I,\lesssim,K,\trianglelefteq,\phi)\big)\Big[\prod_{k\in K} S(I,\lesssim,I_k)^*\delta_{\mathrm{st}}^{\mathrm{b}}(I_k,\lesssim_k,\kappa_k,\tau)\Big]$$
$$=\prod_{k\in K}\boldsymbol{\sigma}\big(\{k\}\big)^*\big[\mathrm{CF}^{\mathrm{stk}}\big(\boldsymbol{\sigma}(I_k)\big)\delta_{\mathrm{st}}^{\mathrm{b}}(I_k,\lesssim_k,\kappa_k,\tau)\big]. \tag{58}$$

Using Theorem 5.2, the definition of $\delta_{\mathrm{st}}(I,\lesssim,\kappa,\tau)$ and $I=\coprod_{k\in K} I_k$ gives

$$\sum_{\substack{\text{p.o.s} \preccurlyeq \text{ on } I:\\ \lesssim \text{ dominates } \preccurlyeq}} \mathrm{CF}^{\mathrm{stk}}\big(Q(I,\preccurlyeq,\lesssim)\big)\delta_{\mathrm{st}}^{\mathrm{b}}(I,\preccurlyeq,\kappa,\tau)$$
$$=\delta_{\mathrm{st}}(I,\lesssim,\kappa,\tau)=\prod_{k\in K} S(I,\lesssim,I_k)^*\delta_{\mathrm{st}}(I_k,\lesssim_k,\kappa_k,\tau). \tag{59}$$

One can show using Theorem 3.6 that the image of a best configuration under $S(I,\lesssim,I_k)_*\circ Q(I,\lesssim,\preccurlyeq)_*$ is best if and only if $\preccurlyeq|_{I_k}=\lesssim_k$. So restricting (59) to $\preccurlyeq$ with $\preccurlyeq|_{I_k}=\lesssim_k$ for all k proves that

$$\sum_{\substack{\text{p.o.s } \preccurlyeq \text{ on } I:\\ \preccurlyeq|_{I_k}=\lesssim_k,\ k\in K,\\ i\preccurlyeq j \text{ implies } \phi(i)\trianglelefteq\phi(j)}} \mathrm{CF}^{\mathrm{stk}}\bigl(Q(I,\preccurlyeq,\lesssim)\bigr)\delta_{\mathrm{st}}^{\mathrm{b}}(I,\preccurlyeq,\kappa,\tau)=\prod_{k\in K} S(I,\lesssim,I_k)^*\delta_{\mathrm{st}}^{\mathrm{b}}(I_k,\lesssim_k,\kappa_k,\tau).$$

Applying $\mathrm{CF}^{\mathrm{stk}}(Q(I,\lesssim,K,\trianglelefteq,\phi))$ to this equation, using (3) and (58) and noting that $Q(I,\lesssim,K,\trianglelefteq,\phi)\circ Q(I,\preccurlyeq,\lesssim)=Q(I,\preccurlyeq,K,\trianglelefteq,\phi)$ gives

$$\begin{aligned}\sum_{\substack{\text{p.o.s } \preccurlyeq \text{ on } I:\\ \preccurlyeq|_{I_k}=\lesssim_k,\ k\in K,\\ i\preccurlyeq j \text{ implies } \phi(i)\trianglelefteq\phi(j)}} &\mathrm{CF}^{\mathrm{stk}}\bigl(Q(I,\preccurlyeq,K,\trianglelefteq,\phi)\bigr)\delta_{\mathrm{st}}^{\mathrm{b}}(I,\preccurlyeq,\kappa,\tau)\\ &=\prod_{k\in K}\boldsymbol{\sigma}\bigl(\{k\}\bigr)^*\bigl[\mathrm{CF}^{\mathrm{stk}}\bigl(\boldsymbol{\sigma}(I_k)\bigr)\delta_{\mathrm{st}}^{\mathrm{b}}(I_k,\lesssim_k,\kappa_k,\tau)\bigr].\end{aligned}\tag{60}$$

Now using Theorem 5.11 to rewrite $\delta_{\mathrm{ss}}^{\mu(k)}(\tau)$ for each $k\in K$ yields

$$\begin{aligned}&\delta_{\mathrm{ss}}(K,\trianglelefteq,\mu,\tau)\\ &\quad=\prod_{k\in K}\boldsymbol{\sigma}\bigl(\{k\}\bigr)^*\bigl(\delta_{\mathrm{ss}}^{\mu(k)}(\tau)\bigr)\\ &\quad=\prod_{k\in K}\sum_{\substack{\text{iso. classes}\\ \text{of finite}\\ \text{sets } I_k}}\frac{1}{|I_k|!}\cdot\sum_{\substack{\lesssim_k,\kappa_k:\ (I_k,\lesssim_k,\kappa_k)\text{ is }\mathcal{A}\text{-data,}\\ \kappa_k(I_k)=\mu(k),\ \tau\circ\kappa_k\equiv\tau(\mu(k))}}\boldsymbol{\sigma}\bigl(\{k\}\bigr)^*\bigl[\mathrm{CF}^{\mathrm{stk}}\bigl(\boldsymbol{\sigma}(I_k)\bigr)\delta_{\mathrm{st}}^{\mathrm{b}}(I_k,\lesssim_k,\kappa_k,\tau)\bigr]\\ &\quad=\sum_{\substack{\text{iso. classes}\\ \text{of finite sets}\\ I_k,\text{ all } k\in K}}\biggl[\prod_{k\in K}\frac{1}{|I_k|!}\biggr]\cdot\sum_{\substack{\lesssim_k,\kappa_k \text{ for all } k\in K:\\ (I_k,\lesssim_k,\kappa_k)\text{ is }\mathcal{A}\text{-data,}\\ \kappa_k(I_k)=\mu(k),\ \tau\circ\kappa_k\equiv\tau(\mu(k))}}\\ &\qquad\times\sum_{\substack{\text{p.o.s } \preccurlyeq \text{ on } I=\coprod_{k\in K}I_k:\\ \preccurlyeq|_{I_k}=\lesssim_k,\ k\in K,\\ i\preccurlyeq j \text{ implies } \phi(i)\trianglelefteq\phi(j)}}\mathrm{CF}^{\mathrm{stk}}\bigl(Q(I,\preccurlyeq,K,\trianglelefteq,\phi)\bigr)\delta_{\mathrm{st}}^{\mathrm{b}}(I,\preccurlyeq,\kappa,\tau),\end{aligned}\tag{61}$$

substituting in (60) at the last step.

The sums over $I_k,\lesssim_k,\kappa_k$ for all $k\in K$ and $\preccurlyeq$ in (61) are equivalent to the sums over $I,\preccurlyeq,\kappa$ and ϕ in (56), with the following proviso. If we choose sets I_k for $k\in K$ in (61), then in (56) the first sum fixes a unique set I with $|I|=\sum_{k\in K}|I_k|$, and there are then $|I|!/\prod_{k\in K}|I_k|!$ possible surjective maps $\phi:I\rightarrow K$ with $|\phi^{-1}(\{k\})|=|I_k|$ for all $k\in K$. Thus, for each choice of data I_k in (61), there are $|I|!/\prod_{k\in K}|I_k|!$ corresponding choices of data I,ϕ in (56). This exactly cancels the difference between the factors $\prod_{k\in K}1/|I_k|!$ in (61) and $1/|I|!$ in (56). So (61) and (56) are equivalent, completing the proof. □

6. Combinatorial inversion of the identities of Section 5

Next we prove some more identities in pushforwards of the characteristic functions $\delta_{ss}, \delta_{si}, \delta_{st}, \delta_{ss}^{b}, \delta_{si}^{b}, \delta_{st}^{b}(I, \preccurlyeq, \kappa, \tau)$ under 1-morphisms $Q(I, \preccurlyeq, K, \trianglelefteq, \phi)$. Equations (31), (32), (33), (38) and (56) above are of this type. By inverting these explicitly we find six further identities, (64), (65), (66), (71), (72) and (75) below. These mean that given the $\mathfrak{M}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$ and $Q(I, \preccurlyeq, K, \trianglelefteq, \phi)$, any one of the six families $\delta_{ss}, \delta_{si}, \delta_{st}, \delta_{ss}^{b}, \delta_{si}^{b}, \delta_{st}^{b}(*, \tau)$ determines the other five.

In contrast to Section 5, the arguments of this section are all *combinatorial* in nature. Our principal techniques are substituting one complicated sum inside another, and rearranging the order of summation. We continue to suppose $\mathbb{K}$ has characteristic zero.

6.1. Inverting identities (31)–(33)

In (31)–(33) we wrote $\delta_{ss}, \delta_{si}, \delta_{st}(I, \trianglelefteq, \kappa, \tau)$ in terms of $\delta_{ss}^{b}, \delta_{si}^{b}, \delta_{st}^{b}(I, \preccurlyeq, \kappa, \tau)$. We now invert these. We shall need some integers $n(I, \preccurlyeq, \trianglelefteq)$.

Definition 6.1. Let I be a finite set, and $\lesssim, \trianglelefteq$ partial orders on I, where $\trianglelefteq$ dominates $\lesssim$. Define an integer

$$n(I, \lesssim, \trianglelefteq) = \sum_{\substack{n\geqslant 0,\ \lesssim=\preccurlyeq_0,\preccurlyeq_1,\dots,\preccurlyeq_n=\trianglelefteq:\\ \preccurlyeq_m \text{ is a partial order on } I, 0\leqslant m\leqslant n,\\ \preccurlyeq_m \text{ strictly dominates } \preccurlyeq_{m-1}, 1\leqslant m\leqslant n}} (-1)^n. \tag{62}$$

If $\trianglelefteq$ dominates $\lesssim$ by l steps, as in Definition 3.1, then $0 \leqslant n \leqslant l$ in (62), so the sum (62) is finite. The $n(I, \lesssim, \trianglelefteq)$ satisfy the following equation:

Proposition 6.2. *Let I be a finite set and $\lesssim, \trianglelefteq$ partial orders on I, where $\trianglelefteq$ dominates $\lesssim$. Then*

$$\sum_{\substack{\text{partial orders } \preccurlyeq \text{ on } I:\\ \trianglelefteq \text{ dominates } \preccurlyeq \text{ dominates } \lesssim}} n(I, \preccurlyeq, \trianglelefteq) = \begin{cases} 1, & \lesssim = \trianglelefteq, \\ 0, & \lesssim \neq \trianglelefteq. \end{cases} \tag{63}$$

Also, the same equation holds with $n(I, \preccurlyeq, \trianglelefteq)$ replaced by $n(I, \lesssim, \preccurlyeq)$.

Proof. If $\lesssim = \trianglelefteq$ then in (62) there is only one possibility, $n = 0$ and $\lesssim = \preccurlyeq_0 = \trianglelefteq$, so $n(I, \lesssim, \trianglelefteq) = 1$. Also in (63) we have $\preccurlyeq = \lesssim = \trianglelefteq$, so the top line of (63) is immediate. Suppose $\lesssim \neq \trianglelefteq$. Then every term in (62) has $n \geqslant 1$, and by setting $\preccurlyeq = \preccurlyeq_1$, replacing n by $n - 1$ and $\preccurlyeq_m$ by $\preccurlyeq_{m+1}$ we rewrite (62) as

$$\begin{aligned} n(I, \lesssim, \trianglelefteq) &= \sum_{\substack{\text{p.o.s } \preccurlyeq \text{ on } I:\\ \trianglelefteq \text{ dominates } \preccurlyeq\\ \preccurlyeq \text{ strictly dominates } \lesssim}} \ \sum_{\substack{n\geqslant 0,\ \preccurlyeq=\preccurlyeq_0,\dots,\preccurlyeq_n=\trianglelefteq:\\ \preccurlyeq_m \text{ is a p.o. on } I,\\ \preccurlyeq_m \text{ strictly dominates } \preccurlyeq_{m-1}}} (-1)^{n+1} \\ &= -\sum_{\substack{\text{p.o.s } \preccurlyeq \text{ on } I:\\ \trianglelefteq \text{ dominates } \preccurlyeq\\ \preccurlyeq \text{ strictly dominates } \lesssim}} n(I, \preccurlyeq, \trianglelefteq). \end{aligned}$$

The bottom line of (63) follows immediately. We prove (63) with $n(I,\preccurlyeq,\trianglelefteq)$ replaced by $n(I,\lesssim,\preccurlyeq)$ in a similar way, writing $\preccurlyeq$ for $\preccurlyeq_{n-1}$ in (62). □

Here are the inverses of the identities of Theorem 5.2.

Theorem 6.3. *Let Assumption* 3.7 *hold,* $(\tau,T,\leqslant)$ *be a permissible weak stability condition on* $\mathcal{A}$, *and* $(I,\trianglelefteq,\kappa)$ *be* $\mathcal{A}$*-data, as in Definition* 3.8. *Then*

$$\sum_{\substack{\text{p.o.s } \preccurlyeq \text{ on } I:\\ \trianglelefteq \text{ dominates } \preccurlyeq}} n(I,\preccurlyeq,\trianglelefteq)\,\mathrm{CF}^{\mathrm{stk}}\big(Q(I,\preccurlyeq,\trianglelefteq)\big)\delta_{\mathrm{ss}}(I,\preccurlyeq,\kappa,\tau)=\delta^{\mathrm{b}}_{\mathrm{ss}}(I,\trianglelefteq,\kappa,\tau), \tag{64}$$

$$\sum_{\substack{\text{p.o.s } \preccurlyeq \text{ on } I:\\ \trianglelefteq \text{ dominates } \preccurlyeq}} n(I,\preccurlyeq,\trianglelefteq)\,\mathrm{CF}^{\mathrm{stk}}\big(Q(I,\preccurlyeq,\trianglelefteq)\big)\delta_{\mathrm{si}}(I,\preccurlyeq,\kappa,\tau)=\delta^{\mathrm{b}}_{\mathrm{si}}(I,\trianglelefteq,\kappa,\tau), \tag{65}$$

$$\sum_{\substack{\text{p.o.s } \preccurlyeq \text{ on } I:\\ \trianglelefteq \text{ dominates } \preccurlyeq}} n(I,\preccurlyeq,\trianglelefteq)\,\mathrm{CF}^{\mathrm{stk}}\big(Q(I,\preccurlyeq,\trianglelefteq)\big)\delta_{\mathrm{st}}(I,\preccurlyeq,\kappa,\tau)=\delta^{\mathrm{b}}_{\mathrm{st}}(I,\trianglelefteq,\kappa,\tau). \tag{66}$$

Proof. Substituting (31) into the left-hand side of (64) gives

$$\begin{aligned}
&\sum_{\substack{\text{p.o.s } \preccurlyeq \text{ on } I:\\ \trianglelefteq \text{ dominates } \preccurlyeq}} n(I,\preccurlyeq,\trianglelefteq)\,\mathrm{CF}^{\mathrm{stk}}\big(Q(I,\preccurlyeq,\trianglelefteq)\big)\delta_{\mathrm{ss}}(I,\preccurlyeq,\kappa,\tau)\\
&\quad=\sum_{\substack{\text{p.o.s } \preccurlyeq \text{ on } I:\\ \trianglelefteq \text{ dominates } \preccurlyeq}} n(I,\preccurlyeq,\trianglelefteq)\sum_{\substack{\text{p.o.s } \lesssim \text{ on } I:\\ \preccurlyeq \text{ dominates } \lesssim}}\mathrm{CF}^{\mathrm{stk}}\big(Q(I,\preccurlyeq,\trianglelefteq)\big)\big[\mathrm{CF}^{\mathrm{stk}}\big(Q(I,\lesssim,\preccurlyeq)\big)\delta^{\mathrm{b}}_{\mathrm{ss}}(I,\lesssim,\kappa,\tau)\big]\\
&\quad=\sum_{\substack{\text{p.o.s } \lesssim \text{ on } I:\\ \trianglelefteq \text{ dominates } \lesssim}}\Bigg[\sum_{\substack{\text{p.o.s } \preccurlyeq \text{ on } I:\\ \trianglelefteq \text{ dominates } \preccurlyeq,\\ \preccurlyeq \text{ dominates } \lesssim}} n(I,\preccurlyeq,\trianglelefteq)\Bigg]\mathrm{CF}^{\mathrm{stk}}\big(Q(I,\lesssim,\trianglelefteq)\big)\delta^{\mathrm{b}}_{\mathrm{ss}}(I,\lesssim,\kappa,\tau),
\end{aligned}$$

exchanging sums over $\preccurlyeq,\lesssim$ and using $Q(I,\preccurlyeq,\trianglelefteq)\circ Q(I,\lesssim,\preccurlyeq)=Q(I,\lesssim,\trianglelefteq)$ and (3). By (63) the bracketed sum on the last line is 0 unless $\lesssim\,=\,\preccurlyeq\,=\,\trianglelefteq$, when it is 1. But then $Q(I,\lesssim,\trianglelefteq)$ is the identity, so the final line reduces to $\delta^{\mathrm{b}}_{\mathrm{ss}}(I,\trianglelefteq,\kappa,\tau)$, giving (64). The proofs of (65)–(66) from (32)–(33) are the same. □

6.2. *Inverting* (35) *and* (38)

We invert (35) to write $\delta^{\beta}_{\mathrm{si}}(\tau)$ in terms of $\delta^{\kappa(i)}_{\mathrm{ss}}(\tau)$.

Theorem 6.4. *Let Assumption* 3.7 *hold,* $(\tau,T,\leqslant)$ *be a permissible weak stability condition on* $\mathcal{A}$, *and* $\beta\in C(\mathcal{A})$. *Then*

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\cdot\sum_{\substack{\kappa:\{1,\dots,n\}\to C(\mathcal{A}):\\ \kappa(\{1,\dots,n\})=\beta,\ \tau\circ\kappa\equiv\tau(\beta)}}\mathrm{CF}^{\mathrm{stk}}\big(\boldsymbol{\sigma}\big(\{1,\dots,n\}\big)\big)\delta_{\mathrm{ss}}\big(\{1,\dots,n\},\bullet,\kappa,\tau\big)\\
&\quad=\delta^{\beta}_{\mathrm{si}}(\tau),
\end{aligned}\tag{67}$$

where $\bullet$ *is the partial order on* $\{1,\dots,n\}$ *with* $i \bullet j$ *if and only if* $i=j$. *Only finitely many functions* $\delta_{\rm ss}(\{1,\dots,n\},\bullet,\kappa,\tau)$ *in this sum are nonzero.*

Proof. We could give a straight combinatorial proof of (67), but the author finds the following infinite series proof more attractive, and we will also reuse the method in Theorem 7.7. The motivation is that Eq. (35) looks like an exponential series, so its inverse (67) should look like a log. To make (35) look more like an exponential, define $\Box:\mathrm{CF}(\mathfrak{Obj}_{\mathcal A})\times\mathrm{CF}(\mathfrak{Obj}_{\mathcal A})\to\mathrm{CF}(\mathfrak{Obj}_{\mathcal A})$ by $f\,\Box\,g=P_{(\{1,2\},\bullet)}(f,g)$. Then as in [10, §4.8], $\Box$ is an *associative, commutative multiplication* on $\mathrm{CF}(\mathfrak{Obj}_{\mathcal A})$. Also in (35) we have

$$\begin{aligned}&\mathrm{CF}^{\rm stk}\big(\boldsymbol\sigma\big(\{1,\dots,n\}\big)\big)\delta_{\rm si}\big(\{1,\dots,n\},\bullet,\kappa,\tau\big)\\&\quad=P_{(\{1,\dots,n\},\bullet)}\big(\delta_{\rm si}^{\kappa(i)}(\tau)\colon i=1,\dots,n\big)=\Box_{i=1}^n\delta_{\rm si}^{\kappa(i)}(\tau),\end{aligned}$$

where $\Box_{i=1}^n$ is the product over $i=1,\dots,n$ using $\Box$. So (35) becomes

$$\sum_{n=1}^{\infty}\frac{1}{n!}\cdot\sum_{\substack{\kappa:\{1,\dots,n\}\to C(\mathcal A):\\ \kappa(\{1,\dots,n\})=\alpha,\ \tau\circ\kappa\equiv\tau(\alpha)}}\Box_{i=1}^n\delta_{\rm si}^{\kappa(i)}(\tau)=\delta_{\rm ss}^{\alpha}(\tau).\tag{68}$$

To prove (67), fix $t\in T$, and consider the following identity in $\mathrm{LCF}(\mathfrak{Obj}_{\mathcal A})$:

$$\delta_{[0]}+\sum_{\alpha\in C(\mathcal A):\ \tau(\alpha)=t}\delta_{\rm ss}^{\alpha}(\tau)=\delta_{[0]}+\sum_{n\geqslant1}\frac{1}{n!}\Bigg[\sum_{\beta\in C(\mathcal A):\ \tau(\beta)=t}\delta_{\rm si}^{\beta}(\tau)\Bigg]^{\Box^n},\tag{69}$$

where $f^{\Box^n}$ for means $f\,\Box\,f\,\Box\cdots\Box\,f$ with f occurring n times. All three sums in (69) are infinite, so we must explain what they mean.

One way to interpret (69) is as a formal sum which packages up finite identities in $\mathrm{CF}(\mathfrak{Obj}_{\mathcal A}^{\beta})$ for each $\beta\in C(\mathcal A)$. It is easy to see that if $\tau(\beta)\neq t$ then all terms in (69) are zero on $\mathfrak{Obj}_{\mathcal A}^{\beta}(\mathbb K)$, and if $\tau(\beta)=t$ then the restriction of (69) to $\mathfrak{Obj}_{\mathcal A}^{\beta}(\mathbb K)$ is exactly (68), which has finitely many nonzero terms by Theorem 5.3. This proves (69) makes sense, and is true, as such a finite formal sum. Another way to make sense of (69) is to use the ideas of [10, §4.2].

Now $\exp(x)=1+\sum_{n\geqslant1}x^n/n!$, so (69) may be rewritten

$$\delta_{[0]}+\sum_{\alpha\in C(\mathcal A):\ \tau(\alpha)=t}\delta_{\rm ss}^{\alpha}(\tau)=\exp\Bigg[\sum_{\beta\in C(\mathcal A):\ \tau(\beta)=t}\delta_{\rm si}^{\beta}(\tau)\Bigg].$$

Formally taking logs and using $\log(1+x)=\sum_{n\geqslant1}(-1)^{n-1}x^n/n$ gives

$$\sum_{\beta\in C(\mathcal A):\ \tau(\beta)=t}\delta_{\rm si}^{\beta}(\tau)=\sum_{n\geqslant1}\frac{(-1)^n}{n}\Bigg[\sum_{\alpha\in C(\mathcal A):\ \tau(\alpha)=t}\delta_{\rm ss}^{\alpha}(\tau)\Bigg]^{\Box^n}.\tag{70}$$

Here (70) is interpreted in the same way as (69). It follows from (69) and $\log\circ\exp x=x$ as an identity in formal power series. If $\tau(\beta)\neq t$ then all terms in (70) are zero on $\mathfrak{Obj}_{\mathcal A}^{\beta}(\mathbb K)$, and if $\tau(\beta)=t$ then the restriction of (70) to $\mathfrak{Obj}_{\mathcal{A}}^{\beta}(\mathbb{K})$ is (67). So taking $t=\tau(\beta)$ proves (67). The proof for (35) in Theorem 5.3 shows there are only finitely many nonzero terms in (67). □

Following the proof of Theorem 5.4, but starting from (67) rather than (35), gives the following formula for $\delta_{\mathrm{si}}(K,\trianglelefteq,\mu,\tau)$. The only differences are in exchanging $\delta_{\mathrm{si}}(\cdots)$, $\delta_{\mathrm{ss}}(\cdots)$, and the combinatorial factors in the last part.

Theorem 6.5. *Let Assumption* 3.7 *hold, and* $(\tau,T,\leqslant)$ *be a permissible weak stability condition on* $\mathcal{A}$. *Then for all* $\mathcal{A}$*-data* $(K,\trianglelefteq,\mu)$ *we have*

$$\sum_{\substack{\text{iso.}\\ \text{classes}\\ \text{of finite}\\ \text{sets } I}} \frac{(-1)^{|I|-|K|}}{|I|!}\cdot \sum_{\substack{\kappa:I\to C(\mathcal{A}),\ \text{surjective } \phi:I\to K:\\ \kappa(\phi^{-1}(k))=\mu(k)\ \text{for } k\in K,\\ \tau\circ\mu\circ\phi\equiv\tau\circ\kappa:I\to T.\\ \text{Define } \precsim \text{ on } I \text{ by } i\precsim j \text{ if } i=j\\ \text{or } \phi(i)\neq\phi(j) \text{ and } \phi(i)\trianglelefteq\phi(j)}} \prod_{k\in K}\bigl(\bigl|\phi^{-1}(\{k\})\bigr|-1\bigr)!$$
$$\cdot\,\mathrm{CF}^{\mathrm{stk}}\bigl(Q(I,\precsim,K,\trianglelefteq,\phi)\bigr)\delta_{\mathrm{ss}}(I,\precsim,\kappa,\tau)=\delta_{\mathrm{si}}(K,\trianglelefteq,\mu,\tau). \tag{71}$$

Only finitely many functions $\delta_{\mathrm{ss}}(I,\precsim,\kappa,\tau)$ *in this sum are nonzero.*

6.3. *Writing* $\delta_{\mathrm{ss}}^{\mathrm{b}}(*,\tau)$ *in terms of* $\delta_{\mathrm{st}}^{\mathrm{b}}(*,\tau)$

Definition 6.6. Let $(I,\precsim)$ be a finite poset, K a finite set, and $\phi:I\to K$ a surjective map. We call $(I,\precsim,K,\phi)$ *allowable* if there exists a partial order $\trianglelefteq$ on K such that $i\precsim j$ implies $\phi(i)\trianglelefteq\phi(j)$. For $(I,\precsim,K,\phi)$ allowable, define a partial order $\lesssim$ on K by $k\lesssim l$ for $k,l\in K$ if there exist $b\geqslant 0$ and $i_0,\dots,i_b$, $j_0,\dots,j_b$ in I with $\phi(i_0)=k$, $\phi(j_b)=l$, and $i_a\precsim j_a$ for $a=0,\dots,b$, and $\phi(i_a)=\phi(j_{a-1})$ for $a=1,\dots,b$. Write $\mathcal{P}(I,\precsim,K,\phi)=\lesssim$. It has the property that if $\trianglelefteq$ is a partial order on K, then $i\precsim j$ implies $\phi(i)\trianglelefteq\phi(j)$ if and only if $\trianglelefteq$ dominates $\mathcal{P}(I,\precsim,K,\phi)$.

Here is a transitivity property of allowable quadruples. The proof is elementary, and left as an exercise.

Lemma 6.7. *Suppose* $(I,\precsim,J,\psi)$ *is allowable with* $\lesssim\,=\mathcal{P}(I,\precsim,J,\psi)$, *and* $\xi:J\to K$ *is a surjective map. Then* $(J,\lesssim,K,\xi)$ *is allowable if and only if* $(I,\precsim,K,\xi\circ\psi)$ *is allowable, and when they are* $\mathcal{P}(J,\lesssim,K,\xi)=\mathcal{P}(I,\precsim,K,\xi\circ\psi)$.

We can now write $\delta_{\mathrm{ss}}^{\mathrm{b}}(*,\tau)$ in terms of $\delta_{\mathrm{st}}^{\mathrm{b}}(*,\tau)$.

Theorem 6.8. *Let Assumption* 3.7 *hold,* $(\tau,T,\leqslant)$ *be a permissible stability condition on* $\mathcal{A}$, *and* $(J,\lesssim,\lambda)$ *be* $\mathcal{A}$*-data. Then*

$$\sum_{\substack{\text{iso. classes}\\ \text{of finite}\\ \text{sets } I}} \frac{1}{|I|!}\cdot \sum_{\substack{\precsim,\kappa,\psi:\ (I,\precsim,\kappa) \text{ is } \mathcal{A}\text{-data},\\ (I,\precsim,J,\psi) \text{ is allowable},\\ \lesssim=\mathcal{P}(I,\precsim,J,\psi),\\ \kappa(\psi^{-1}(j))=\lambda(j) \text{ for } j\in J,\\ \tau\circ\lambda\circ\psi\equiv\tau\circ\kappa:I\to T}} \mathrm{CF}^{\mathrm{stk}}\bigl(Q(I,\precsim,J,\lesssim,\psi)\bigr)\delta_{\mathrm{st}}^{\mathrm{b}}(I,\precsim,\kappa,\tau)$$
$$=\delta_{\mathrm{ss}}^{\mathrm{b}}(J,\lesssim,\lambda,\tau). \tag{72}$$

Only finitely many functions $\delta_{\mathrm{st}}^{\mathrm{b}}(I,\precsim,\kappa,\tau)$ *in this sum are nonzero.*

Proof. Substituting (56) with J, λ in place of K, μ into (64) with J, λ, $\lesssim$, $\trianglelefteq$ in place of I, κ, $\trianglelefteq$, $\preccurlyeq$ and using $Q(J, \trianglelefteq, \lesssim) \circ Q(I, \preccurlyeq, J, \trianglelefteq, \psi) = Q(I, \preccurlyeq, J, \lesssim, \psi)$, Theorems 2.8, 5.12 and 6.3 and Definition 6.6 gives

$$\sum_{\substack{\text{iso. classes}\\ \text{of finite}\\ \text{sets } I}} \frac{1}{|I|!} \cdot \sum_{\substack{\preccurlyeq,\kappa,\psi:\ (I,\preccurlyeq,\kappa) \text{ is } \mathcal{A}\text{-data},\\ (I,\preccurlyeq,J,\psi) \text{ is allowable},\\ \kappa(\psi^{-1}(j))=\lambda(j) \text{ for } j\in J,\\ \tau\circ\lambda\circ\psi\equiv\tau\circ\kappa:\, I\to T}} \left[\sum_{\substack{\text{partial orders } \trianglelefteq \text{ on } J:\\ \lesssim \text{ dominates } \trianglelefteq,\\ \trianglelefteq \text{ dominates } \mathcal{P}(I,\preccurlyeq,J,\psi)}} n(J, \trianglelefteq, \lesssim) \right]$$
$$\cdot \operatorname{CF}^{\mathrm{stk}}\big(Q(I, \preccurlyeq, J, \lesssim, \psi)\big)\delta_{\mathrm{st}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau) = \delta_{\mathrm{ss}}^{\mathrm{b}}(J, \lesssim, \lambda, \tau),$$

with only finitely many $\delta_{\mathrm{st}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau)$ nonzero. By (63) the bracketed sum is 1 if $\lesssim = \mathcal{P}(I, \preccurlyeq, J, \psi)$ and 0 otherwise, and (72) follows. □

6.4. *Inverting* (72)

Next we invert (72). We will need the following notation.

Definition 6.9. Let I be a finite set. Then equivalence relations $\sim$ on I are in 1–1 correspondence with subsets $S = \{(i, j) \in I \times I\colon i \sim j\}$ of $I \times I$ satisfying the properties (i) $(i, i) \in S$ for all $i \in I$,

(ii) $(i, j) \in S$ implies $(j, i) \in S$, and
(iii) $(i, j) \in S$ and $(j, k) \in S$ imply $(i, k) \in S$.

Given $S \subseteq I \times I$ satisfying (i)–(iii), define an equivalence relation $\sim_S$ on I by $i \sim_S j$ if $(i, j) \in S$. Write $[i]_S$ for the $\sim_S$-equivalence class of i, set $I_S = \{[i]_S\colon i \in I\}$, and define $\psi_S : I \to I_S$ by $\psi_S(i) = [i]_S$.

Now let $(I, \preccurlyeq)$ be a finite poset, and define

$$\mathcal{U}(I, \preccurlyeq) = \big\{S \subseteq I \times I\colon\ S \text{ satisfies (i)–(iii)}, (I, \preccurlyeq, I_S, \psi_S) \text{ is allowable}\big\}.$$

Suppose $(I, \preccurlyeq, K, \phi)$ is allowable, and define $S = \{(i, j) \in I \times I\colon\ \phi(i) = \phi(j)\}$. Then it is easy to see that $S \in \mathcal{U}(I, \preccurlyeq)$, and there is a unique 1–1 correspondence $\iota : I_S \to K$ with $\iota([i]_S) = \phi(i)$ for $i \in I$ such that $\phi = \iota \circ \psi_S$. So $\mathcal{U}(I, \preccurlyeq)$ classifies isomorphism classes of K, ϕ such that $(I, \preccurlyeq, K, \phi)$ is allowable. Define

$$N(I, \preccurlyeq) = \sum_{\substack{n\geqslant 0,\ S_0,\dots,S_n\in\mathcal{U}(I,\preccurlyeq):\\ S_{m-1}\subset S_m,\ S_{m-1}\neq S_m,\ 1\leqslant m\leqslant n\\ S_0=\{(i,i):\ i\in I\},\ S_n=I\times I}} (-1)^n.$$

Now let $(I, \preccurlyeq, K, \phi)$ be allowable, and define

$$N(I, \preccurlyeq, K, \phi) = \sum_{\substack{n\geqslant 0,\ S_0,\dots,S_n\in\mathcal{U}(I,\preccurlyeq):\\ S_{m-1}\subset S_m,\ S_{m-1}\neq S_m,\ 1\leqslant m\leqslant n\\ S_0=\{(i,i):\ i\in I\},\\ S_n=\{(i,j)\in I\times I:\ \phi(i)=\phi(j)\}}} (-1)^n.$$

By a similar proof to Proposition 6.2, using Lemma 6.7, we can show:

Proposition 6.10. *Let $(I,\preccurlyeq,K,\phi)$ be allowable. Then*

$$\sum_{\substack{\text{iso. classes}\\ \text{of finite}\\ \text{sets } J}} \frac{1}{|J|!}\cdot \sum_{\substack{\psi:I\to J,\ \xi:J\to K\\ \text{surjective, } \phi=\xi\circ\psi:\\ (I,\preccurlyeq,J,\psi) \text{ allowable}}} N(I,\preccurlyeq,J,\psi) = \begin{cases} 1, & \phi \text{ is a bijection,}\\ 0, & \text{otherwise.}\end{cases} \tag{73}$$

This also holds with $N(I,\preccurlyeq,J,\psi)$ replaced by $N(J,\mathcal{P}(I,\preccurlyeq,J,\psi),K,\xi)$.

Here is a product formula for $N(I,\preccurlyeq,K,\phi)$. We leave the proof as an exercise; one possible starting point is to note that both sides of (74) satisfy (73).

Proposition 6.11. *Let $(I,\preccurlyeq,K,\phi)$ be allowable. Then*

$$N(I,\preccurlyeq,K,\phi) = \prod_{k\in K} N\big(\phi^{-1}(k),\preccurlyeq|_{\phi^{-1}(k)}\big). \tag{74}$$

Next we invert the identity of Theorem 6.8.

Theorem 6.12. *Let Assumption* 3.7 *hold, $(\tau,T,\leqslant)$ be a permissible stability condition on $\mathcal{A}$, and $(K,\trianglelefteq,\mu)$ be $\mathcal{A}$-data. Then*

$$\sum_{\substack{\text{iso. classes}\\ \text{of finite}\\ \text{sets } J}} \frac{1}{|J|!}\cdot \sum_{\substack{\lesssim,\lambda,\chi:\ (J,\lesssim,\lambda) \text{ is } \mathcal{A}\text{-data},\\ (J,\lesssim,K,\chi) \text{ is allowable},\\ \trianglelefteq=\mathcal{P}(J,\lesssim,K,\chi),\\ \lambda(\chi^{-1}(k))=\mu(k) \text{ for } k\in K,\\ \tau\circ\mu\circ\chi\equiv\tau\circ\lambda:J\to T}} N(J,\lesssim,K,\chi)\,\mathrm{CF}^{\mathrm{stk}}\big(Q(J,\lesssim,K,\trianglelefteq,\chi)\big)\delta_{\mathrm{ss}}^{\mathrm{b}}(J,\lesssim,\lambda,\tau)$$
$$= \delta_{\mathrm{st}}^{\mathrm{b}}(K,\trianglelefteq,\mu,\tau). \tag{75}$$

Only finitely many functions $\delta_{\mathrm{ss}}^{\mathrm{b}}(J,\lesssim,\lambda,\tau)$ in this sum are nonzero.

Proof. Using the proof in Section 5.2 that only finitely many $\delta_{\mathrm{si}}(I,\preccurlyeq,\kappa,\tau)$ in (38) are nonzero, we find that only finitely many $\delta_{\mathrm{ss}}^{\mathrm{b}}(J,\lesssim,\lambda,\tau)$ in (75) are nonzero. Substituting (72) into the left-hand side of (75) gives

$$\sum_{\substack{\text{iso. classes}\\ \text{of finite}\\ \text{sets } J}} \frac{1}{|J|!}\cdot \sum_{\substack{\lesssim,\lambda,\chi:\ (J,\lesssim,\lambda) \text{ is } \mathcal{A}\text{-data},\\ (J,\lesssim,K,\chi) \text{ is allowable},\\ \trianglelefteq=\mathcal{P}(J,\lesssim,K,\chi),\\ \lambda(\chi^{-1}(k))=\mu(k) \text{ for } k\in K,\\ \tau\circ\mu\circ\chi\equiv\tau\circ\lambda:J\to T}} N(J,\lesssim,K,\chi)\cdot \sum_{\substack{\text{iso. classes}\\ \text{of finite}\\ \text{sets } I}} \frac{1}{|I|!}$$
$$\cdot \sum_{\substack{\preccurlyeq,\kappa,\psi:\ (I,\preccurlyeq,\kappa) \text{ is } \mathcal{A}\text{-data},\\ (I,\preccurlyeq,J,\psi) \text{ is allowable},\\ \lesssim=\mathcal{P}(I,\preccurlyeq,J,\psi),\\ \kappa(\psi^{-1}(j))=\lambda(j) \text{ for } j\in J,\\ \tau\circ\lambda\circ\psi\equiv\tau\circ\kappa:I\to T}} \mathrm{CF}^{\mathrm{stk}}\big(Q(J,\lesssim,K,\trianglelefteq,\chi)\big)\big[\mathrm{CF}^{\mathrm{stk}}\big(Q(I,\preccurlyeq,J,\lesssim,\psi)\big)\delta_{\mathrm{st}}^{\mathrm{b}}(I,\preccurlyeq,\kappa,\tau)\big]$$

$$= \sum_{\substack{\text{iso. classes}\\ \text{of finite}\\ \text{sets } I}} \frac{1}{|I|!} \cdot \sum_{\substack{\preceq,\kappa,\phi:\ (I,\preceq,\kappa) \text{ is } \mathcal{A}\text{-data},\\ (I,\preceq,K,\phi) \text{ is allowable},\\ \trianglelefteq=\mathcal{P}(I,\preceq,K,\phi),\\ \kappa(\phi^{-1}(k))=\mu(k) \text{ for } k\in K,\\ \tau\circ\mu\circ\phi\equiv\tau\circ\kappa:\, I\to T}} \mathrm{CF}^{\mathrm{stk}}\big(Q(I,\preceq,K,\trianglelefteq,\phi)\big)\delta_{\mathrm{st}}^{\mathrm{b}}(I,\preceq,\kappa,\tau)$$

$$\cdot\Bigg[\sum_{\substack{\text{iso.classes}\\ \text{of finite}\\ \text{sets } J}} \frac{1}{|J|!} \cdot \sum_{\substack{\psi:\, I\to J,\ \chi:\, J\to K\\ \text{surjective, } \phi=\chi\circ\psi:\\ (I,\preceq,J,\psi) \text{ allowable},\\ \lesssim=\mathcal{P}(I,\preceq,J,\psi)}} N(J,\lesssim,K,\chi)\Bigg], \tag{76}$$

setting $\phi=\chi\circ\psi$ and using $\mathrm{CF}^{\mathrm{stk}}(Q(I,\preceq,K,\trianglelefteq,\phi))=\mathrm{CF}^{\mathrm{stk}}(Q(J,\lesssim,K,\trianglelefteq,\chi))\circ\mathrm{CF}^{\mathrm{stk}}(Q(I,\preceq, J,\lesssim,\psi))$ in the third line.

Here, given $(I,\preceq,J,\psi)$ allowable and $\lesssim=\mathcal{P}(I,\preceq,J,\psi)$, Lemma 6.7 shows that $(J,\lesssim, K,\chi)$ allowable and $\trianglelefteq=\mathcal{P}(J,\lesssim,K,\chi)$ in the first line of (76) is equivalent to $(I,\preceq,K,\phi)$ allowable and $\trianglelefteq=\mathcal{P}(I,\preceq,K,\phi)$ in the third line. Also, $\tau\circ\mu\circ\chi\equiv\tau\circ\lambda$ and $\tau\circ\lambda\circ\psi\equiv\tau\circ\kappa$ in the first and second lines of (76) are equivalent to $\tau\circ\mu\circ\phi\equiv\tau\circ\kappa$ in the third, as $\phi=\chi\circ\psi$.

Now Proposition 6.10 shows that the bracketed term on the last line of (76) is 1 if ϕ is a bijection, and 0 otherwise. When ϕ is a bijection $|I|=|K|$. The first sum on the third line in (76) fixes a unique I with $|I|=|K|$. Then in the second sum there are $|I|!$ bijections $\phi: I\to K$. So by dropping the factor $1/|I|!$ on the third line we may take $I=K$ and $\phi=\mathrm{id}_K$. Then $\trianglelefteq=\preceq$, $\kappa=\mu$, and $\mathrm{CF}^{\mathrm{stk}}(Q(K,\trianglelefteq,K,\trianglelefteq,\mathrm{id}_K))$ is the identity. Thus, the last two lines of (76) reduce to $\delta_{\mathrm{st}}^{\mathrm{b}}(K,\trianglelefteq,\mu,\tau)$, the right-hand side of (75). This completes the proof. □

7. (Lie) algebras of constructible functions

We now define and study some interesting subalgebras $\mathcal{H}_\tau^{\mathrm{pa}}$, $\mathcal{H}_\tau^{\mathrm{to}}$ of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$, for $(\tau,T,\leqslant)$ a permissible weak stability condition. These encode information about the moduli spaces $\mathrm{Obj}_{\mathrm{ss}}^\alpha$, $\mathrm{Obj}_{\mathrm{si}}^\alpha$, $\mathrm{Obj}_{\mathrm{st}}^\alpha(\tau)$ for all $\alpha\in C(\mathcal{A})$. We will see in [11] that these subalgebras are essentially *independent of choice of* $(\tau,T,\leqslant)$, and that changing weak stability conditions amounts to changing bases in $\mathcal{H}_\tau^{\mathrm{pa}}$, $\mathcal{H}_\tau^{\mathrm{to}}$. We suppose $\mathbb{K}$ has characteristic zero throughout this section.

7.1. The algebras $\mathcal{H}_\tau^{\mathrm{pa}}$, $\mathcal{H}_\tau^{\mathrm{to}}$ and Lie algebras $\mathcal{L}_\tau^{\mathrm{pa}}$, $\mathcal{L}_\tau^{\mathrm{to}}$

From Section 3.3, $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ is a $\mathbb{Q}$-algebra with multiplication $*$ and identity $\delta_{[0]}$. Given a permissible weak stability condition $(\tau,T,\leqslant)$ we define two interesting subalgebras $\mathcal{H}_\tau^{\mathrm{pa}}$, $\mathcal{H}_\tau^{\mathrm{to}}$ of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$.

Definition 7.1. Let Assumption 3.7 hold, and $(\tau,T,\leqslant)$ be a permissible weak stability condition on $\mathcal{A}$. Define $\mathbb{Q}$-vector subspaces $\mathcal{H}_\tau^{\mathrm{pa}}$, $\mathcal{H}_\tau^{\mathrm{to}}$ in $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ by

$$\mathcal{H}_\tau^{\mathrm{pa}}=\big\langle \mathrm{CF}^{\mathrm{stk}}\big(\boldsymbol{\sigma}(I)\big)\delta_{\mathrm{ss}}(I,\preceq,\kappa,\tau):\ (I,\preceq,\kappa) \text{ is } \mathcal{A}\text{-data}\big\rangle_{\mathbb{Q}}, \tag{77}$$

$$\mathcal{H}_\tau^{\mathrm{to}}=\big\langle \delta_{[0]},\delta_{\mathrm{ss}}^{\alpha_1}(\tau)*\cdots*\delta_{\mathrm{ss}}^{\alpha_n}(\tau):\ \alpha_1,\dots,\alpha_n\in C(\mathcal{A})\big\rangle_{\mathbb{Q}}. \tag{78}$$

Here $\langle\cdots\rangle_{\mathbb{Q}}$ is the set of all finite $\mathbb{Q}$-linear combinations of the elements '$\cdots$,' and $\mathcal{A}$-data is defined in Definition 3.8. Define $\mathcal{L}_\tau^{\mathrm{pa}}=\mathcal{H}_\tau^{\mathrm{pa}}\cap\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ and $\mathcal{L}_\tau^{\mathrm{to}}=\mathcal{H}_\tau^{\mathrm{to}}\cap\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$.

In [11] we will show that if $(\tau, T, \leqslant)$ and $(\tilde{\tau}, \tilde{T}, \leqslant)$ are permissible weak stability conditions on $\mathcal{A}$, then (under some finiteness conditions) $\mathcal{H}_\tau^{\rm pa} = \mathcal{H}_{\tilde{\tau}}^{\rm pa}$ and $\mathcal{H}_\tau^{\rm to} = \mathcal{H}_{\tilde{\tau}}^{\rm to}$, so that $\mathcal{H}_\tau^{\rm pa}$, $\mathcal{H}_\tau^{\rm to}$ are *independent of the choice of* $(\tau, T, \leqslant)$. To relate $\mathcal{H}_\tau^{\rm to}$ and $\mathcal{H}_\tau^{\rm pa}$, let $(\{1,\ldots,n\}, \leqslant, \kappa)$ be $\mathcal{A}$-data. Then

$$\delta_{\rm ss}\bigl(\{1,\ldots,n\}, \leqslant, \kappa, \tau\bigr) = \prod_{i=1}^{n}\bigl(\boldsymbol{\sigma}\bigl(\{i\}\bigr)^*\bigl(\delta_{\rm ss}^{\kappa(i)}(\tau)\bigr)\bigr). \tag{79}$$

Generalizing the argument of [10, Theorem 4.3] we then find that

$$\delta_{\rm ss}^{\kappa(1)}(\tau) * \cdots * \delta_{\rm ss}^{\kappa(n)}(\tau) = \mathrm{CF}^{\rm stk}\bigl(\boldsymbol{\sigma}\bigl(\{1,\ldots,n\}\bigr)\bigr)\delta_{\rm ss}\bigl(\{1,\ldots,n\}, \leqslant, \kappa, \tau\bigr). \tag{80}$$

Thus $\mathcal{H}_\tau^{\rm pa}$ is the span of $\mathrm{CF}^{\rm stk}(\boldsymbol{\sigma}(I))\delta_{\rm ss}(I, \preccurlyeq, \kappa, \tau)$ for $\mathcal{A}$-data $(I, \preccurlyeq, \kappa)$ with $\preccurlyeq$ a *partial order*, and $\mathcal{H}_\tau^{\rm to}$ the span with $\preccurlyeq$ a *total order*. This explains the notation.

Now in [10, §4.8] we defined multilinear operations $P_{(I,\preccurlyeq)}$ on $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ for $(I, \preccurlyeq)$ a finite poset, and generalizing (79) shows that

$$\mathrm{CF}^{\rm stk}\bigl(\boldsymbol{\sigma}(I)\bigr)\delta_{\rm ss}(I, \preccurlyeq, \kappa, \tau) = P_{(I,\preccurlyeq)}\bigl(\delta_{\rm ss}^{\kappa(i)}(\tau)\colon i \in I\bigr). \tag{81}$$

Thus an alternative expression for $\mathcal{H}_\tau^{\rm pa}$ is

$$\mathcal{H}_\tau^{\rm pa} = \bigl\langle P_{(I,\preccurlyeq)}\bigl(\delta_{\rm ss}^{\kappa(i)}(\tau)\colon i \in I\bigr)\colon (I, \preccurlyeq, \kappa) \text{ is } \mathcal{A}\text{-data}\bigr\rangle_{\mathbb{Q}}. \tag{82}$$

It follows from [10, Theorem 4.22] that $\mathcal{H}_\tau^{\rm pa}$ is closed under the operations $P_{(I,\preccurlyeq)}$.

Proposition 7.2. *In Definition* 7.1, *$\mathcal{H}_\tau^{\rm pa}$, $\mathcal{H}_\tau^{\rm to}$ are subalgebras of* $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ *and $\mathcal{L}_\tau^{\rm pa}$, $\mathcal{L}_\tau^{\rm to}$ Lie subalgebras of* $\mathrm{CF}^{\rm ind}(\mathfrak{Obj}_{\mathcal{A}})$*, with $\mathcal{H}_\tau^{\rm to} \subseteq \mathcal{H}_\tau^{\rm pa}$ and $\mathcal{L}_\tau^{\rm to} \subseteq \mathcal{L}_\tau^{\rm pa}$.*

Proof. Clearly $\mathcal{H}_\tau^{\rm to}$ is the subalgebra of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ generated by the $\delta_{\rm ss}^\alpha(\tau)$ for all $\alpha \in C(\mathcal{A})$. As $\mathcal{H}_\tau^{\rm pa}$ is closed under the operations $P_{(I,\preccurlyeq)}$ from above, it is closed under $* = P_{(\{1,2\},\leqslant)}$. Writing $(\emptyset, \emptyset, \emptyset)$ for the trivial $\mathcal{A}$-data we have $\mathrm{CF}^{\rm stk}(\boldsymbol{\sigma}(\emptyset))\delta_{\rm ss}(\emptyset, \emptyset, \emptyset, \tau) = \delta_{[0]}$, so $\mathcal{H}_\tau^{\rm pa}$ contains the identity $\delta_{[0]}$, and is a subalgebra of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$. Therefore $\mathcal{L}_\tau^{\rm pa}$, $\mathcal{L}_\tau^{\rm to}$ are intersections of Lie subalgebras $\mathcal{H}_\tau^{\rm pa}$, $\mathcal{H}_\tau^{\rm to}$ and $\mathrm{CF}^{\rm ind}(\mathfrak{Obj}_{\mathcal{A}})$ of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$, so they are Lie subalgebras. The inclusion $\mathcal{H}_\tau^{\rm to} \subseteq \mathcal{H}_\tau^{\rm pa}$ is obvious from (80), and this implies $\mathcal{L}_\tau^{\rm to} \subseteq \mathcal{L}_\tau^{\rm pa}$. □

We now apply the work of Sections 5–6 to study $\mathcal{H}_\tau^{\rm pa}$. There we constructed eleven transformations between the six families $\delta_{\rm ss}$, $\delta_{\rm si}$, $\delta_{\rm st}$, $\delta_{\rm ss}^{\rm b}$, $\delta_{\rm si}^{\rm b}$, $\delta_{\rm st}^{\rm b}(*, \tau)$. Their equation numbers are displayed below,

$$\delta_{\rm si}^{\rm b}(*,\tau) \underset{(65)}{\overset{(32)}{\rightleftarrows}} \delta_{\rm si}(*,\tau) \underset{(71)}{\overset{(38)}{\rightleftarrows}} \delta_{\rm ss}(*,\tau) \underset{(31)}{\overset{(64)}{\rightleftarrows}} \delta_{\rm ss}^{\rm b}(*,\tau) \underset{(72)}{\overset{(75)}{\rightleftarrows}} \delta_{\rm st}^{\rm b}(*,\tau) \underset{(66)}{\overset{(33)}{\rightleftarrows}} \delta_{\rm st}(*,\tau), \qquad \delta_{\rm st}^{\rm b}(*,\tau) \xrightarrow{(56)} \delta_{\rm ss}(*,\tau). \tag{83}$$

Note that the identities involving $\delta_{\rm st}$, $\delta_{\rm st}^{\rm b}(*, \tau)$ require $(\tau, T, \leqslant)$ to be a stability condition, but the other identities work for $(\tau, T, \leqslant)$ a *weak* stability condition. Combining these, we can write any of the six families $\delta_{\rm ss}, \delta_{\rm si}, \delta_{\rm st}, \delta_{\rm ss}^{\rm b}, \delta_{\rm si}^{\rm b}, \delta_{\rm st}^{\rm b}(*, \tau)$ in terms of any of the others. Applying $\mathrm{CF}^{\rm stk}(\boldsymbol\sigma(I))$ to (31), noting that $\boldsymbol\sigma(I) \circ Q(I, \preccurlyeq, \trianglelefteq) = \boldsymbol\sigma(I)$ and using (3) yields

$$\sum_{\substack{\text{p.o.s } \preccurlyeq \text{ on } I:\\ \trianglelefteq \text{ dominates } \preccurlyeq}} \mathrm{CF}^{\rm stk}\big(\boldsymbol\sigma(I)\big)\delta_{\rm ss}^{\rm b}(I, \preccurlyeq, \kappa, \tau) = \mathrm{CF}^{\rm stk}\big(\boldsymbol\sigma(I)\big)\delta_{\rm ss}(I, \trianglelefteq, \kappa, \tau). \tag{84}$$

Similarly, all eleven transformations (83) imply transformations between the six families $\mathrm{CF}^{\rm stk}(\boldsymbol\sigma(I))\delta_{\rm ss}, \ldots, \delta_{\rm st}^{\rm b}(*, \tau)$ in $\mathrm{CF}(\mathfrak{Obj}_{\mathcal A})$. Thus we deduce:

Corollary 7.3. *In Definition* 7.1 *we have*

$$\begin{aligned}
\mathcal{H}_\tau^{\rm pa} &= \big\langle \mathrm{CF}^{\rm stk}\big(\boldsymbol\sigma(I)\big)\delta_{\rm ss}^{\rm b}(I, \preccurlyeq, \kappa, \tau)\colon\ (I, \preccurlyeq, \kappa) \text{ is } \mathcal{A}\text{-data}\big\rangle_{\mathbb Q}\\
&= \big\langle \mathrm{CF}^{\rm stk}\big(\boldsymbol\sigma(I)\big)\delta_{\rm si}(I, \preccurlyeq, \kappa, \tau)\colon\ (I, \preccurlyeq, \kappa) \text{ is } \mathcal{A}\text{-data}\big\rangle_{\mathbb Q}\\
&= \big\langle \mathrm{CF}^{\rm stk}\big(\boldsymbol\sigma(I)\big)\delta_{\rm si}^{\rm b}(I, \preccurlyeq, \kappa, \tau)\colon\ (I, \preccurlyeq, \kappa) \text{ is } \mathcal{A}\text{-data}\big\rangle_{\mathbb Q}\\
&= \big\langle \mathrm{CF}^{\rm stk}\big(\boldsymbol\sigma(I)\big)\delta_{\rm st}(I, \preccurlyeq, \kappa, \tau)\colon\ (I, \preccurlyeq, \kappa) \text{ is } \mathcal{A}\text{-data}\big\rangle_{\mathbb Q}\\
&= \big\langle \mathrm{CF}^{\rm stk}\big(\boldsymbol\sigma(I)\big)\delta_{\rm st}^{\rm b}(I, \preccurlyeq, \kappa, \tau)\colon\ (I, \preccurlyeq, \kappa) \text{ is } \mathcal{A}\text{-data}\big\rangle_{\mathbb Q},
\end{aligned} \tag{85}$$

supposing $(\tau, T, \leqslant)$ *is a stability condition in the last two lines.*

The material of Sections 5–6, and other identities in [11], can therefore be interpreted as giving *basis change formulae* in the infinite-dimensional algebra $\mathcal{H}_\tau^{\rm pa}$. In particular, $\mathcal{H}_\tau^{\rm pa}$ contains $\delta_{\rm ss}^\alpha$, $\delta_{\rm si}^\alpha$, $\delta_{\rm st}^\alpha(\tau)$ for all $\alpha \in C(\mathcal{A})$. We can interpret this as saying that $\mathcal{H}_\tau^{\rm pa}$ contains information about τ-semistability, τ-semistable indecomposables, and τ-stability, but $\mathcal{H}_\tau^{\rm to}$ only information about τ-semistability.

We can write down the multiplication relations in $\mathcal{H}_\tau^{\rm pa}$ explicitly for the six spanning sets $\mathrm{CF}^{\rm stk}(\boldsymbol\sigma(I))\delta_{\rm ss}, \ldots, \delta_{\rm st}^{\rm b}(I, \preccurlyeq, \kappa, \tau)$. Let $(I, \preccurlyeq, \kappa)$ and $(J, \lesssim, \lambda)$ be $\mathcal{A}$-data with $I \cap J = \emptyset$. Define $\mathcal{A}$-data $(K, \trianglelefteq, \mu)$ by $K = I \amalg J$, $\mu|_I = \kappa$, $\mu|_J = \lambda$, and $k \trianglelefteq l$ if either $k, l \in I$ and $k \preccurlyeq l$, or $k, l \in J$ and $k \lesssim l$, or $k \in I$ and $l \in J$. Then from [10, Theorem 4.22] we deduce that

$$\begin{aligned}
&\big(\mathrm{CF}^{\rm stk}\big(\boldsymbol\sigma(I)\big)\delta_{\rm ss}(I, \preccurlyeq, \kappa, \tau)\big) * \big(\mathrm{CF}^{\rm stk}\big(\boldsymbol\sigma(J)\big)\delta_{\rm ss}(J, \lesssim, \lambda, \tau)\big)\\
&\quad = \mathrm{CF}^{\rm stk}\big(\boldsymbol\sigma(K)\big)\delta_{\rm ss}(K, \trianglelefteq, \mu, \tau).
\end{aligned} \tag{86}$$

The same holds with $\delta_{\rm si}(*)$ or $\delta_{\rm st}(*)$ in place of $\delta_{\rm ss}(*)$. Using (31)–(33) and (64)–(66) we can now deduce the multiplication relations for the $\mathrm{CF}^{\rm stk}(\boldsymbol\sigma(I))\delta_{\rm ss}^{\rm b}, \delta_{\rm si}^{\rm b}, \delta_{\rm st}^{\rm b}(I, \preccurlyeq, \kappa, \tau)$, and the answer turns out as follows. Let $(I, \preccurlyeq, \kappa)$, $(J, \lesssim, \lambda)$, K and μ be as above, but do not define $\trianglelefteq$. Then

$$\begin{aligned}
&\big(\mathrm{CF}^{\rm stk}\big(\boldsymbol\sigma(I)\big)\delta_{\rm ss}^{\rm b}(I, \preccurlyeq, \kappa, \tau)\big) * \big(\mathrm{CF}^{\rm stk}\big(\boldsymbol\sigma(J)\big)\delta_{\rm ss}^{\rm b}(J, \lesssim, \lambda, \tau)\big)\\
&\quad = \sum_{\substack{\text{p.o.s } \trianglelefteq \text{ on } K:\ \trianglelefteq|_I = \preccurlyeq,\ \trianglelefteq|_J = \lesssim\\ \text{and } i \in I,\ j \in J \text{ implies } j \ntrianglelefteq i}} \mathrm{CF}^{\rm stk}\big(\boldsymbol\sigma(K)\big)\delta_{\rm ss}^{\rm b}(K, \trianglelefteq, \mu, \tau).
\end{aligned} \tag{87}$$

The same holds with $\delta_{\rm si}^{\rm b}(*)$ or $\delta_{\rm st}^{\rm b}(*)$ in place of $\delta_{\rm ss}^{\rm b}(*)$.

7.2. The structure of the Lie algebra $\mathcal{L}_\tau^{\mathrm{pa}}$

If $(I, \preccurlyeq)$ is a finite poset, let $\approx$ be the equivalence relation on I generated by $i \approx j$ if $i \preccurlyeq j$ or $j \preccurlyeq i$, and define the *connected components* of $(I, \preccurlyeq)$ to be the $\approx$-equivalence classes. Equivalently, if Γ is the directed graph with vertices I and edges $\overset{i}{\bullet} \to \overset{j}{\bullet}$ for $i, j \in I$ with $i \preccurlyeq j$, then the connected components of $(I, \preccurlyeq)$ are the sets of vertices of connected components of Γ. We call $(I, \preccurlyeq)$ *connected* if it has exactly one connected component. Then we prove:

Proposition 7.4. *Let Assumption* 3.7 *hold,* $(\tau, T, \leqslant)$ *be a permissible weak stability condition on* $\mathcal{A}$*, and* $(I, \preccurlyeq, \kappa)$ *be* $\mathcal{A}$*-data. If* $(I, \preccurlyeq)$ *has k connected components, then* $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(I))\delta_{\mathrm{si}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau)$ *and* $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(I))\delta_{\mathrm{st}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau)$ *are supported on points* $[X_1 \oplus \cdots \oplus X_k] \in \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$*, with all* X_a *indecomposable.*

Proof. Let $I_1, \ldots, I_k \subseteq I$ be the connected components of $(I, \preccurlyeq)$, so that $I = I_1 \amalg \cdots \amalg I_k$. Suppose $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(I))\delta_{\mathrm{si}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau)$ is nonzero on $[X] \in \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$. Then there exists $[(\sigma, \iota, \pi)] \in \mathcal{M}_{\mathrm{si}}^{\mathrm{b}}(I, \preccurlyeq, \kappa)_{\mathcal{A}}$ with $\sigma(I) = X$ making a nonzero contribution to $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(I))\delta_{\mathrm{si}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau)$ at $[X]$. The $I_1, \ldots, I_k$ are s-sets in I, so setting $X_a = \sigma(I_a)$ we find $\iota(I_a, I): X_a \to X$ defines a subobject $X_a \subset X$, with $X = X_1 \oplus \cdots \oplus X_k$. We shall prove X_a is indecomposable for $a = 1, \ldots, k$. Write $(\sigma_a, \iota_a, \pi_a)$ for the $(I_a, \preccurlyeq)$-subconfiguration of (σ, ι, π). Then $\sigma_a(I_a) = X_a$, and as (σ, ι, π) is best Theorem 3.6 implies $(\sigma_a, \iota_a, \pi_a)$ is best.

Let T_a be a maximal torus in $\mathrm{Aut}(\sigma_a, \iota_a, \pi_a)$ containing $\{\lambda\, \mathrm{id}_{(\sigma_a, \iota_a, \pi_a)}\colon \lambda \in \mathbb{K}^\times\}$. Then $\boldsymbol{\sigma}(\{i\})_*(T_a)$ is a $\mathbb{K}$-subtorus of $\mathrm{Aut}(\sigma_a(\{i\}))$ containing $\{\lambda\, \mathrm{id}_{\sigma_a(\{i\})}\colon 0 \neq \lambda \in \mathbb{K}\}$. But since $\sigma_a(\{i\}) = \sigma(\{i\})$ is *indecomposable* for $i \in I_a$, $\mathrm{Aut}(\sigma_a(\{i\}))$ has *rank one*, so $\{\lambda\, \mathrm{id}_{\sigma_a(\{i\})}\colon 0 \neq \lambda \in \mathbb{K}\}$ is a maximal torus of $\mathrm{Aut}(\sigma_a(\{i\}))$. Since $\boldsymbol{\sigma}(\{i\})_*(T_a)$ must be contained in a maximal torus of $\mathrm{Aut}(\sigma_a(\{i\}))$, we see that

$$\boldsymbol{\sigma}\big(\{i\}\big)_*(T_a) = \big\{\lambda\, \mathrm{id}_{\sigma_a(\{i\})}\colon 0 \neq \lambda \in \mathbb{K}\big\}. \tag{88}$$

We claim that

$$\bigg(\prod_{i \in I_a} \boldsymbol{\sigma}\big(\{i\}\big)_*\bigg)(T_a) = \bigg\{\prod_{i \in I_a} \lambda\, \mathrm{id}_{\sigma(\{i\})}\colon 0 \neq \lambda \in \mathbb{K}\bigg\} \cong \mathbb{K}^\times, \quad \text{where} \tag{89}$$

$$\prod_{i \in I_a} \boldsymbol{\sigma}\big(\{i\}\big)_*\colon \mathrm{Aut}(\sigma_a, \iota_a, \pi_a) \to \prod_{i \in I_a} \mathrm{Aut}\big(\sigma_a\big(\{i\}\big)\big). \tag{90}$$

The right-hand side of (89) is the image of $\{\lambda\, \mathrm{id}_{(\sigma_a, \iota_a, \pi_a)}\colon \lambda \in \mathbb{K}^\times\} \subseteq T_a$, so the left-hand side of (89) contains the right. To prove the opposite inclusion, let $\alpha \in T_a$. Then for each $i \in I_a$, Eq. (88) gives $\alpha(\{i\}) = \lambda_i\, \mathrm{id}_{\sigma(\{i\})}$ for some $\lambda_i \in \mathbb{K}^\times$. We must show $\lambda_i = \lambda$ for some $\lambda \in \mathbb{K}^\times$ and all $i \in I_a$.

Suppose $i \neq j \in I_a$ with $i \preccurlyeq j$ but there is no $k \in I_a$ with $i \neq k \neq j$ and $i \preccurlyeq k \preccurlyeq j$. By Theorem 3.6 the short exact sequence (6) is not split, and so corresponds to a *nonzero* $\gamma_{ij} \in \mathrm{Ext}^1(\sigma(\{j\}), \sigma(\{i\}))$. But α induces an automorphism of (6), so $\gamma_{ij} \circ \lambda_j\, \mathrm{id}_{\sigma(\{j\})} = \lambda_i\, \mathrm{id}_{\sigma(\{i\})} \circ \gamma_{ij}$, giving $\lambda_i = \lambda_j$ as $\gamma_{ij} \neq 0$. Since $(I_a, \preccurlyeq)$ is *connected* there are enough such pairs i, j to force $\lambda_i = \lambda$ for all $i \in I_a$. This proves (89).

If (σ', ι', π') is a $(\{1,2\}, \leqslant)$-configuration then the kernel of $\boldsymbol{\sigma}(\{1\}) \times \boldsymbol{\sigma}(\{2\}) : \mathrm{Aut}(\sigma', \iota', \pi') \to \mathrm{Aut}(\sigma'(\{1\})) \times \mathrm{Aut}(\sigma'(\{2\}))$ is $\mathrm{Hom}(\sigma'(\{2\}), \sigma'(\{1\}))$. Generalizing this, one can show by induction on $|I_a|$ that the kernel of (90) is a *nilpotent* $\mathbb{K}$-group. Thus, (90) is injective on the maximal torus T_a, and (89) implies that $T_a = \{\lambda\, \mathrm{id}_{(\sigma_a, \iota_a, \pi_a)} \colon \lambda \in \mathbb{K}^\times\}$, so $\mathrm{Aut}(\sigma_a, \iota_a, \pi_a)$ has rank one.

The contribution of $[(\sigma, \iota, \pi)]$ to $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(I))\delta_{\mathrm{si}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau)$ at $[X]$, which is nonzero by assumption, is

$$\prod_{a=1}^{k} \chi\bigl(\mathrm{Aut}(X_a)/\boldsymbol{\sigma}(I_a)_*\bigl(\mathrm{Aut}(\sigma_a, \iota_a, \pi_a)\bigr)\bigr). \tag{91}$$

Suppose $\mathrm{Aut}(X_a)$ has rank greater than one, and consider the action of a maximal torus of $\mathrm{Aut}(X_a)$ on $\mathrm{Aut}(X_a)/\boldsymbol{\sigma}(I_a)_*(\mathrm{Aut}(\sigma_a, \iota_a, \pi_a))$. Since $\mathrm{Aut}(\sigma_a, \iota_a, \pi_a)$ has rank one, the orbits of this action are all of the form $(\mathbb{K}^\times)^l$ for $l \geqslant 1$, which implies that the Euler characteristic in (91) is zero, a contradiction. Thus $\mathrm{Aut}(X_a)$ has rank one, and X_a is indecomposable for $a = 1, \ldots, k$, as we have to prove. Since τ-stable objects are indecomposable, the same proof works for $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(I))\delta_{\mathrm{st}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau)$. □

We can now deduce an alternative description of $\mathcal{L}_\tau^{\mathrm{pa}}$.

Proposition 7.5. *In Definition* 7.1 *we have*

$$\begin{aligned} \mathcal{L}_\tau^{\mathrm{pa}} &= \bigl\langle \mathrm{CF}^{\mathrm{stk}}\bigl(\boldsymbol{\sigma}(I)\bigr)\delta_{\mathrm{si}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau) \colon (I, \preccurlyeq, \kappa) \ \mathcal{A}\text{-}data,\ (I, \preccurlyeq)\ connected \bigr\rangle_{\mathbb{Q}} \\ &= \bigl\langle \mathrm{CF}^{\mathrm{stk}}\bigl(\boldsymbol{\sigma}(I)\bigr)\delta_{\mathrm{st}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau) \colon (I, \preccurlyeq, \kappa) \ \mathcal{A}\text{-}data,\ (I, \preccurlyeq)\ connected \bigr\rangle_{\mathbb{Q}}, \end{aligned} \tag{92}$$

supposing $(\tau, T, \leqslant)$ *is a stability condition in the second line. There is a natural* $\mathbb{Q}$*-algebra isomorphism* $\Phi_\tau^{\mathrm{pa}} : U(\mathcal{L}_\tau^{\mathrm{pa}}) \to \mathcal{H}_\tau^{\mathrm{pa}}$*, where* $U(\mathcal{L}_\tau^{\mathrm{pa}})$ *is the universal enveloping algebra of* $\mathcal{L}_\tau^{\mathrm{pa}}$*.*

Proof. Equation (92) follows from Definition 7.1, (85) and Proposition 7.4. From above, the multiplication relations for the $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(I))\delta_{\mathrm{si}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau)$ are given by (87) with $\delta_{\mathrm{si}}^{\mathrm{b}}(*)$ in place of $\delta_{\mathrm{ss}}^{\mathrm{b}}(*)$. From this it is easy to see that if $(I, \preccurlyeq)$ has connected components $I_1, \ldots, I_k$ then

$$\begin{aligned} &\mathrm{CF}^{\mathrm{stk}}\bigl(\boldsymbol{\sigma}(I)\bigr)\delta_{\mathrm{si}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau) \\ &\quad = \bigl(\mathrm{CF}^{\mathrm{stk}}\bigl(\boldsymbol{\sigma}(I_1)\bigr)\delta_{\mathrm{si}}^{\mathrm{b}}(I_1, \preccurlyeq, \kappa, \tau)\bigr) * \cdots * \bigl(\mathrm{CF}^{\mathrm{stk}}\bigl(\boldsymbol{\sigma}(I_k)\bigr)\delta_{\mathrm{si}}^{\mathrm{b}}(I_k, \preccurlyeq, \kappa, \tau)\bigr) \\ &\qquad + \bigl(\mathbb{Q}\text{-linear combination of } \mathrm{CF}^{\mathrm{stk}}\bigl(\boldsymbol{\sigma}(J)\bigr)\delta_{\mathrm{si}}^{\mathrm{b}}(J, \lesssim, \lambda, \tau) \text{ for } (J, \lesssim) \\ &\qquad \text{with} < k \text{ connected components}\bigr). \end{aligned}$$

Then (92) and induction on k shows that $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(I))\delta_{\mathrm{si}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau)$ is contained in the algebra generated by $\mathcal{L}_\tau^{\mathrm{pa}}$ for all $(I, \preccurlyeq, \kappa)$, so $\mathcal{H}_\tau^{\mathrm{pa}}$ is generated by $\mathcal{L}_\tau^{\mathrm{pa}}$ by (85). The isomorphism Φ_τ^{pa} follows using Proposition 3.12. □

7.3. Functions $\epsilon^\alpha(\tau)$ and the Lie algebra $\mathcal{L}_\tau^{\rm to}$

We would like to prove an analogue of Proposition 7.5 for the Lie algebra $\mathcal{L}_\tau^{\rm to}$. The methods of Sections 7.1–7.2 do not really help, as the restriction to total orders $(I,\preccurlyeq)$ in the spanning set $\delta_{\rm ss}(I,\preccurlyeq,\kappa,\tau)$ does not translate to nice restrictions in the other spanning sets such as $\delta_{\rm si}^{\rm b}(*,\tau)$. Instead we introduce alternative generators $\epsilon^\alpha(\tau)$, $\alpha\in C(\mathcal{A})$, for the algebra $\mathcal{H}_\tau^{\rm to}$. These will be important in the author's paper [12] on holomorphic generating functions for invariants counting τ-semistable objects.

Definition 7.6. Let Assumption 3.7 hold, and $(\tau,T,\leqslant)$ be a permissible weak stability condition on $\mathcal{A}$. For $\alpha\in C(\mathcal{A})$, define $\epsilon^\alpha(\tau)$ in $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ by

$$\epsilon^\alpha(\tau)=\sum_{\substack{\mathcal{A}\text{-data }(\{1,\dots,n\},\leqslant,\kappa):\\ \kappa(\{1,\dots,n\})=\alpha,\ \tau\circ\kappa\equiv\tau(\alpha)}}\frac{(-1)^{n-1}}{n}\delta_{\rm ss}^{\kappa(1)}(\tau)*\delta_{\rm ss}^{\kappa(2)}(\tau)*\cdots*\delta_{\rm ss}^{\kappa(n)}(\tau). \tag{93}$$

If n,κ give a nonzero term in (93) and $1\leqslant i<n$ then $\beta=\kappa(\{1,\dots,i\})$, $\gamma=\kappa(\{i+1,\dots,n\})$ lie in $C(\mathcal{A})$ with $\alpha=\beta+\gamma$, $\tau(\alpha)=\tau(\beta)=\tau(\gamma)$ and $\mathrm{Obj}_{\rm ss}^\beta(\tau)\neq\emptyset\neq\mathrm{Obj}_{\rm ss}^\gamma(\tau)$. There are only finitely many such β,γ by Proposition 4.9, and so only finitely many nonzero terms in (93). Thus $\epsilon^\alpha(\tau)$ is well-defined.

Here is the *inverse* of (93). The proof follows that of Theorem 6.4 closely, but using the associative multiplication $*$ on $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ rather than $\square$, and exchanging the rôles of exp and log.

Theorem 7.7. *Let Assumption* 3.7 *hold,* $(\tau,T,\leqslant)$ *be a permissible weak stability condition on* $\mathcal{A}$*, and* $\beta\in C(\mathcal{A})$*. Then*

$$\delta_{\rm ss}^\beta(\tau)=\sum_{\substack{\mathcal{A}\text{-data }(\{1,\dots,m\},\leqslant,\lambda):\\ \lambda(\{1,\dots,m\})=\beta,\ \tau\circ\lambda\equiv\tau(\beta)}}\frac{1}{m!}\epsilon^{\lambda(1)}(\tau)*\epsilon^{\lambda(2)}(\tau)*\cdots*\epsilon^{\lambda(m)}(\tau). \tag{94}$$

There are only finitely many nonzero terms in (94)*.*

Equations (93)–(94) show that the $\epsilon^\alpha(\tau)$ lie in the subalgebra of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ generated by the $\delta_{\rm ss}^\alpha(\tau)$ and vice versa, so they generate the same subalgebra, which is $\mathcal{H}_\tau^{\rm to}$ by (78). Therefore

$$\mathcal{H}_\tau^{\rm to}=\big\langle\delta_{[0]},\epsilon^{\alpha_1}(\tau)*\cdots*\epsilon^{\alpha_n}(\tau):\ \alpha_1,\dots,\alpha_n\in C(\mathcal{A})\big\rangle_{\mathbb{Q}}. \tag{95}$$

Here is an important property of the $\epsilon^\alpha(\tau)$, which the coefficient $(-1)^{n-1}/n$ in (93) was chosen to achieve.

Theorem 7.8. *In Definition* 7.6 *we have* $\epsilon^\alpha(\tau)\in\mathrm{CF}^{\rm ind}(\mathfrak{Obj}_{\mathcal{A}})$.

Proof. Let $\alpha \in C(\mathcal{A})$, $X \in \mathcal{A}$ with $[X] = \alpha$, and $(\{1,\dots,n\},\leqslant,\kappa)$ be $\mathcal{A}$-data with $\kappa(\{1,\dots,n\}) = \alpha$. By Definition 3.10 we have a Cartesian square

$$\begin{array}{ccc} \mathfrak{M}(X,\{1,\dots,n\},\leqslant,\kappa)_{\mathcal{A}} & \xrightarrow{\boldsymbol{\sigma}(\{1,\dots,n\})} & \operatorname{Spec}\mathbb{K} \\ \downarrow \Pi_X & & X \downarrow \\ \mathfrak{M}(\{1,\dots,n\},\leqslant,\kappa)_{\mathcal{A}} & \xrightarrow{\boldsymbol{\sigma}(\{1,\dots,n\})} & \mathfrak{Obj}_{\mathcal{A}}. \end{array}$$

Applying (5) to this and using (80) and $\mathrm{CF}(\operatorname{Spec}\mathbb{K}) = \mathbb{Q}$ we have

$$\begin{aligned} &\big(\delta_{\mathrm{ss}}^{\kappa(1)}(\tau) * \cdots * \delta_{\mathrm{ss}}^{\kappa(n)}(\tau)\big)\big([X]\big) \\ &\quad = X^* \circ \mathrm{CF}^{\mathrm{stk}}\big(\boldsymbol{\sigma}\big(\{1,\dots,n\}\big)\big)\delta_{\mathrm{ss}}\big(\{1,\dots,n\},\leqslant,\kappa,\tau\big) \\ &\quad = \mathrm{CF}^{\mathrm{stk}}\big(\boldsymbol{\sigma}\big(\{1,\dots,n\}\big)\big) \circ \Pi_X^*\big(\delta_{\mathrm{ss}}\big(\{1,\dots,n\},\leqslant,\kappa,\tau\big)\big) \\ &\quad = \mathrm{CF}^{\mathrm{stk}}\big(\boldsymbol{\sigma}\big(\{1,\dots,n\}\big)\big)\delta_{\mathrm{ss}}\big(X,\{1,\dots,n\},\leqslant,\kappa,\tau\big) \\ &\quad = \chi^{\mathrm{na}}\big(\mathcal{M}_{\mathrm{ss}}\big(X,\{1,\dots,n\},\leqslant,\kappa,\tau\big)_{\mathcal{A}}\big). \end{aligned} \tag{96}$$

To prove Theorem 7.8 it is sufficient to show that if $X = Y \oplus Z$ with $0 \ncong Y, Z$ then $\epsilon^{\alpha}(\tau)([X]) = 0$. By (93) and (96) this is equivalent to

$$\sum_{\substack{\mathcal{A}\text{-data } (\{1,\dots,n\},\leqslant,\kappa):\\ \kappa(\{1,\dots,n\})=\alpha,\ \tau\circ\kappa\equiv\tau(\alpha)}} \frac{(-1)^{n-1}}{n}\chi^{\mathrm{na}}\big(\mathcal{M}_{\mathrm{ss}}\big(Y\oplus Z,\{1,\dots,n\},\leqslant,\kappa,\tau\big)_{\mathcal{A}}\big) = 0. \tag{97}$$

Now $\mathrm{Aut}(Y \oplus Z)$ acts naturally on $\mathcal{M}_{\mathrm{ss}}(Y \oplus Z,\{1,\dots,n\},\leqslant,\kappa,\tau)_{\mathcal{A}}$. Define G to be the subgroup $\{\mathrm{id}_Y + \gamma\,\mathrm{id}_Z\colon 0 \neq \gamma \in \mathbb{K}\}$ of $\mathrm{Aut}(Y \oplus Z)$, so that $G \cong \mathbb{K}^{\times}$. Then each orbit of G on $\mathcal{M}_{\mathrm{ss}}(Y \oplus Z,\{1,\dots,n\},\leqslant,\kappa,\tau)_{\mathcal{A}}$ is either a single point or free. Since $\chi(\mathbb{K}^{\times}) = 0$, by properties of the Euler characteristic we have

$$\chi^{\mathrm{na}}\big(\mathcal{M}_{\mathrm{ss}}\big(Y\oplus Z,\{1,\dots,n\},\leqslant,\kappa,\tau\big)_{\mathcal{A}}\big) = \chi^{\mathrm{na}}\big(\mathcal{M}_{\mathrm{ss}}\big(Y\oplus Z,\{1,\dots,n\},\leqslant,\kappa,\tau\big)_{\mathcal{A}}^{G}\big), \tag{98}$$

where $(\cdots)^G$ is the fixed points of G, as the free orbits contribute zero.

By [9, Corollary 4.4] there is a 1–1 correspondence between $[(\sigma,\iota,\pi)] \in \mathcal{M}_{\mathrm{ss}}(Y \oplus Z,\{1,\dots,n\},\leqslant,\kappa,\tau)_{\mathcal{A}}$ and filtrations $0 = A_0 \subset \cdots \subset A_n = Y \oplus Z$ with $S_i = A_i/A_{i-1}$ τ-semistable with $\tau([S_i]) = \kappa(i)$. The condition for $[(\sigma,\iota,\pi)]$ to be G-invariant turns out to be $A_i = B_i \oplus C_i$ for all i as subobjects of $Y \oplus Z$, where $B_i = A_i \cap Y$ and $C_i = A_i \cap Z$. Then $0 = B_0 \subset \cdots \subset B_n = Y$ and $0 = C_0 \subset \cdots \subset C_n = Z$.

Let $0 = B'_0 \subset \cdots \subset B'_l = Y$ and $0 = C'_0 \subset \cdots \subset C'_m = Z$ be the filtrations obtained by omitting repetitions, that is, omit B_i if $B_i = B_{i-1}$ and so on. There are unique maps $\phi:\{0,\dots,n\} \to \{0,\dots,l\}$ and $\psi:\{0,\dots,n\} \to \{0,\dots,m\}$ with $B_i = B'_{\phi(i)}$ and $C_i = C'_{\psi(i)}$ for all i. They are surjective, with $i \leqslant j$ implies $\phi(i) \leqslant \phi(j)$ and $\psi(i) \leqslant \psi(j)$. Also, the condition that $A_i \neq A_{i-1}$ implies that either $\phi(i-1) \neq \phi(i)$ or $\psi(i-1) \neq \psi(i)$ for all $i = 1,\dots,n$.

Conversely, if we fix $l, m > 0$ and filtrations $0 = B'_0 \subset \cdots \subset B'_l = Y$ and $0 = C'_0 \subset \cdots \subset C'_m = Z$ such that $T'_i = B'_i/B'_{i-1}$ and $U'_i = C'_i/C'_{i-1}$ are τ-semistable with $\tau([T'_i]) = \tau([U'_i]) = \tau(\alpha)$, the possible n, κ and $0 = A_0 \subset \cdots \subset A_n = X$ coming from $[(\sigma,\iota,\pi)] \in \mathcal{M}_{\mathrm{ss}}(Y \oplus$

$Z, \{1, \ldots, n\}, \leqslant, \kappa, \tau)_{\mathcal{A}}^G$ and yielding these k, l, B_i', C_i' from the construction above are classified by such ϕ, ψ. Therefore the contribution to (97) from such $[(\sigma, \iota, \pi)]$ is

$$\sum_{n=\max(l,m)}^{l+m} \sum_{\substack{\text{surjective } \phi:\{0,\ldots,n\}\to\{0,\ldots,l\} \\ \text{and } \psi:\{0,\ldots,n\}\to\{0,\ldots,m\}: \\ i\leqslant j \text{ implies } \phi(i)\leqslant\phi(j) \text{ and } \psi(i)\leqslant\psi(j), \\ \phi(i-1)\neq\phi(i) \text{ or } \psi(i-1)\neq\psi(i) \text{ for } 1\leqslant i\leqslant n}} \frac{(-1)^{n-1}}{n}. \tag{99}$$

We shall show (99) is zero. Integrating this over all l, m, B_i', C_i' and using (98) and properties of Euler characteristics proves (97), and Theorem 7.8.

For n, ϕ, ψ as in (99), define $E = \{i \in \{1, \ldots, n\}: \phi(i-1) = \phi(i) \text{ and } \psi(i-1) \neq \psi(i)\}$ and $F = \{i \in \{1, \ldots, n\}: \phi(i-1) \neq \phi(i) \text{ and } \psi(i-1) = \psi(i)\}$. Then E, F are disjoint subsets of $\{1, \ldots, n\}$ with $|E| = n - l$, $|F| = n - m$, and any such E, F determine unique ϕ, ψ. Thus for fixed n the number of ϕ, ψ in (99) is $n!/(n-l)!(n-m)!(m+l-n)!$, and (99) reduces to

$$\sum_{n=\max(l,m)}^{l+m} \frac{(-1)^{n-1}}{n} \cdot \frac{n!}{(n-l)!(n-m)!(m+l-n)!}. \tag{100}$$

Fixing $l > 0$, multiplying (100) by t^m and summing over $m = 0, 1, 2, \ldots$ gives

$$\begin{aligned} &\frac{(-1)^{l-1}}{l} \sum_{n=l}^{\infty} \frac{(-1)^{n-l}(n-1)!t^{n-l}}{(l-1)!(n-l)!} \sum_{m=n-l}^{n} \frac{l!t^{m+l-n}}{(n-m)!(m+l-n)!} \\ &\quad = \frac{(-1)^{l-1}}{l} \sum_{a=0}^{\infty} \frac{(-1)^a(l-1+a)!t^a}{(l-1)!a!} \sum_{b=0}^{l} \frac{l!t^b}{(l-b)!b!} = \frac{(-1)^{l-1}}{l}(1+t)^{-l}(1+t)^l = \frac{(-1)^{l-1}}{l}, \end{aligned}$$

using $a = n - l$, $b = m + l - n$ and the binomial theorem. Equating coefficients of t^m, (100) is zero when $m > 0$, so (99) is zero. This completes the proof. □

Let $[X] \in \mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K})$. For τ-stable X, the only nonzero term at $[X]$ in (93) is $n = 1$, $\kappa(1) = \alpha$. If any term in (93) is nonzero at $[X]$ then X has a filtration $0 = A_0 \subset \cdots \subset A_n = X$ with $S_i = A_i/A_{i-1}$ τ-semistable and $\tau([S_i]) = \tau([X])$ for all i, so X is τ-semistable. Hence by Theorem 7.8 we have

$$\epsilon^{\alpha}(\tau)\big([X]\big) = \begin{cases} 1, & X \text{ is } \tau\text{-stable}, \\ \text{in } \mathbb{Q}, & X \text{ is strictly } \tau\text{-semistable and indecomposable}, \\ 0, & X \text{ is } \tau\text{-unstable or decomposable}. \end{cases}$$

Thus $\epsilon^{\alpha}(\tau)$ interpolates between $\delta_{\mathrm{si}}^{\alpha}(\tau)$ and $\delta_{\mathrm{st}}^{\alpha}(\tau)$.

We can now prove an analogue of Proposition 7.5 for $\mathcal{L}_{\tau}^{\mathrm{to}}, \mathcal{H}_{\tau}^{\mathrm{to}}$.

Corollary 7.9. *Let Assumption* 3.7 *hold and* $(\tau, T, \leqslant)$ *be a permissible weak stability condition on* $\mathcal{A}$, *and use the notation of Sections* 3.3 *and* 7.1. *Then* $\mathcal{L}_{\tau}^{\mathrm{to}}$ *is the Lie subalgebra of* $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ *generated by the* $\epsilon^{\alpha}(\tau)$ *for* $\alpha \in C(\mathcal{A})$. *There is a natural* $\mathbb{Q}$*-algebra isomorphism* $\Phi_{\tau}^{\mathrm{to}}: U(\mathcal{L}_{\tau}^{\mathrm{to}}) \to \mathcal{H}_{\tau}^{\mathrm{to}}$, *where* $U(\mathcal{L}_{\tau}^{\mathrm{to}})$ *is the universal enveloping algebra of* $\mathcal{L}_{\tau}^{\mathrm{to}}$.

Proof. Write $\mathcal{L}'$ for the Lie subalgebra of $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ generated by the $\epsilon^\alpha(\tau)$ for all $\alpha \in C(\mathcal{A})$; this makes sense by Theorem 7.8. By (95) $\mathcal{H}_\tau^{\mathrm{to}}$ is generated by the $\epsilon^\alpha(\tau)$, and so by $\mathcal{L}'$. But $\mathcal{L}' \subseteq \mathcal{L}_\tau^{\mathrm{to}} = \mathcal{H}_\tau^{\mathrm{to}} \cap \mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$, so $\mathcal{H}_\tau^{\mathrm{to}}$ is also generated by $\mathcal{L}_\tau^{\mathrm{to}}$. Thus Proposition 3.12 gives $\mathbb{Q}$-algebra isomorphisms $\Phi' : U(\mathcal{L}') \to \mathcal{H}_\tau^{\mathrm{to}}$ and $\Phi_\tau^{\mathrm{to}} : U(\mathcal{L}_\tau^{\mathrm{to}}) \to \mathcal{H}_\tau^{\mathrm{to}}$. As $\mathcal{L}' \subseteq \mathcal{L}_\tau^{\mathrm{to}}$ we have $U(\mathcal{L}') \subseteq U(\mathcal{L}_\tau^{\mathrm{to}})$, with $\Phi_\tau^{\mathrm{to}}|_{U(\mathcal{L}')} = \Phi'$. Since Φ', Φ_τ^{to} are isomorphisms this forces $\mathcal{L}' = \mathcal{L}_\tau^{\mathrm{to}}$, so $\mathcal{L}_\tau^{\mathrm{to}}$ is the Lie subalgebra of $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ generated by the $\epsilon^\alpha(\tau)$. □

7.4. The $\delta_{\mathrm{ss}}(*, \tau), \ldots$ have no universal linear relations

The identities of Sections 5–6 given in (83), and their projections to $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ as in (84), are *universal linear relations* between the δ_{ss}, δ_{si}, δ_{st}, $\delta_{\mathrm{ss}}^{\mathrm{b}}$, $\delta_{\mathrm{si}}^{\mathrm{b}}$, $\delta_{\mathrm{st}}^{\mathrm{b}}(*, \tau)$. By this we mean that they hold for all choices of $\mathcal{A}$, $\mathfrak{F}_{\mathcal{A}}$, $K(\mathcal{A})$, $(\tau, T, \leqslant)$ and auxiliary $\mathcal{A}$-data $(I, \preccurlyeq, \kappa), \ldots$. Note also that each of these relations expresses one of the families $\delta_{\mathrm{ss}}, \ldots, \delta_{\mathrm{st}}^{\mathrm{b}}(*, \tau)$ in terms of another; they can be thought of as basis change formulae between six different bases of some universal algebra.

We claim that, in contrast, there are *no nontrivial universal linear relations* involving just one of the families $\delta_{\mathrm{ss}}, \ldots, \delta_{\mathrm{st}}^{\mathrm{b}}(*, \tau)$. That is, the $\delta_{\mathrm{ss}}(I, \preccurlyeq, \kappa, \tau)$ over all isomorphism classes of $\mathcal{A}$-data $(I, \preccurlyeq, \kappa)$ should have a kind of *universal linear independence*: there are no systematic relations on them that hold for all $\mathcal{A}$, $\mathfrak{F}_{\mathcal{A}}$, $K(\mathcal{A})$, $(\tau, T, \leqslant)$, only particular relations in each example. Before proving our general result Theorem 7.12, we study an example and prove linear independence of some collections of functions in $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$.

Example 7.10. Fix a nonempty finite set I. Define a *quiver* $Q = (Q_0, Q_1, b, e)$ to have vertices I and an edge $\overset{i}{\bullet} \to \overset{j}{\bullet}$ for all $i, j \in I$, including $i = j$. That is, take $Q_0 = I$, $Q_1 = I \times I$, $b : (i, j) \mapsto i$ and $e : (i, j) \mapsto j$. Set $\mathbb{K} = \mathbb{C}$ and consider the abelian category $\mathcal{A} = \text{nil-}\mathbb{C}Q$ of nilpotent $\mathbb{C}$-representations of Q, with data $K(\mathcal{A}), \mathfrak{F}_{\mathcal{A}}$ satisfying Assumption 3.7 as in [9, Example 10.6]. Then $K(\mathcal{A}) = \mathbb{Z}^I$, with elements of $K(\mathcal{A})$ written as maps $\alpha : I \to \mathbb{Z}$, and $C(\mathcal{A})$ is $\mathbb{N}^I \setminus \{0\}$. For $i \in I$ define $e_i \in C(\mathcal{A})$ by $e_i(j) = 1$ if $j = i$ and $e_i(j) = 0$ otherwise. Then $\sum_{i \in I} e_i = 1$, where $1 \in C(\mathcal{A})$ maps $i \mapsto 1$ for all $i \in I$. Let $(\tau, T, \leqslant)$ be any (weak) stability condition on $\mathcal{A}$, such as one defined using a *slope function* in Example 4.14. Then $(\tau, T, \leqslant)$ is *permissible* by Corollary 4.13.

For $i \in I$ define $\mathbf{V}^i = (V^i, \rho^i)$ in $\mathcal{A}$ by $V_i^i = \mathbb{C}$, $V_j^i = 0$ for $i \neq j \in I$ and $\rho(e) = 0$ for all edges e in Q. Then $[\mathbf{V}^i] = e_i$ in $C(\mathcal{A})$, and $\mathfrak{Obj}_{\mathcal{A}}^i(\mathbb{C}) = \{[\mathbf{V}^i]\}$. Also $\mathbf{V}^i$ is simple, so it is automatically τ-stable, and we see that

$$\delta_{\mathrm{ss}}^{e_i}(\tau) = \delta_{\mathrm{si}}^{e_i}(\tau) = \delta_{\mathrm{st}}^{e_i}(\tau) = \epsilon^{e_i}(\tau) = \delta_{[\mathbf{V}^i]}. \tag{101}$$

Define $\alpha = 1 \in C(\mathcal{A})$ and $\kappa : I \to C(\mathcal{A})$ by $\kappa(i) = e_i$.

Proposition 7.11. *In the situation of Example* 7.10 *we have*:

(a) *There exists no $\mathcal{A}$-data $(J, \lesssim, \lambda)$ with $|J| > |I|$ and $\lambda(J) = \alpha$. If $(J, \lesssim, \lambda)$ is $\mathcal{A}$-data with $|J| = |I|$ and $\lambda(J) = \alpha$, then there is a unique bijection $\iota : J \to I$ with $\lambda = \kappa \circ \iota$.*
(b) *The functions $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(I))\delta_{\mathrm{ss}}(I, \preccurlyeq, \kappa, \tau)$ for all partial orders $\preccurlyeq$ on I are linearly independent in $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$. The same applies with $\delta_{\mathrm{ss}}(\cdots)$ replaced by δ_{si}, δ_{st}, $\delta_{\mathrm{ss}}^{\mathrm{b}}$, $\delta_{\mathrm{si}}^{\mathrm{b}}(\cdots)$ or $\delta_{\mathrm{st}}^{\mathrm{b}}(\cdots)$.*

(c) *The subalgebra of* $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ *generated by* $\delta_{\mathrm{ss}}^{e_i}(\tau)$ *for* $i \in I$ *is freely generated, that is, there are no polynomial relations in* $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ *on the* $\delta_{\mathrm{ss}}^{e_i}(\tau)$ *for* $i \in I$. *The same holds for the* $\delta_{\mathrm{si}}^{e_i}(\tau)$, $\delta_{\mathrm{st}}^{e_i}(\tau)$, *and* $\epsilon^{e_i}(\tau)$.

Proof. For (a), if $(J, \lesssim, \lambda)$ is $\mathcal{A}$-data with $\lambda(J) = \alpha = 1$ then there must exist $|J|$ elements $\lambda(j)$ of $\mathbb{N}^I \setminus \{0\}$ adding up to 1. This is clearly impossible if $|J| > |I|$, and if $|J| = |I|$ the elements $\lambda(j)$ for $j \in J$ must be the set of all e_i, so there is a unique bijection $\iota : J \to I$ with $\lambda(i) = e_i$ for all $i \in I$.

For (b), let $\trianglelefteqslant$ be a partial order on I, and define $(V, \rho) \in \mathcal{A}$ by

$$V_i = \mathbb{C} \quad \text{for all } i \in I \quad \text{and} \quad \rho\big(\overset{i}{\bullet} \to \overset{j}{\bullet}\big) = \begin{cases} 0, & i \trianglelefteqslant j, \\ 1, & i \ntrianglelefteq j. \end{cases}$$

Now $\mathcal{A} = \text{nil-}\mathbb{C}Q$ is an abelian category of finite length, so by the *Jordan–Hölder Theorem* the object (V, ρ) in $\mathcal{A}$ has a composition series into *simple factors*, which are unique up to order and isomorphism. By construction these simple factors are exactly $\mathbf{V}^i$ for $i \in I$.

As the simple factors of (V, ρ) are *pairwise nonisomorphic*, we can apply the work of [9, §3–§4]. These construct a unique partial order $\lesssim$ on the set I indexing the simple factors of (V, ρ), and a *best* $(I, \lesssim)$*-configuration* (σ, ι, π) with $\sigma(I) = (V, \rho)$ and $\sigma(\{i\}) \cong \mathbf{V}^i$ for $i \in I$, which is unique up to canonical isomorphism. Furthermore the $(I, \leqslant)$ s-sets J correspond to subobjects S^J of (V, ρ) induced by $\iota(J, I) : \sigma(J) \to (V, \rho)$.

Now it is not difficult to show that the subobjects of (V, ρ) are given by vector subspaces $V^J \leqslant V$ of the form $V^J = \bigoplus_{j \in J} V_j$ for $J \subseteq I$ an $(I, \trianglelefteqslant)$ s-set. Hence $\lesssim = \trianglelefteqslant$, and by (101) we see that for partial orders $\preccurlyeq$ on I we have

$$\mathcal{M}_{\mathrm{ss}}^{\mathrm{b}}, \mathcal{M}_{\mathrm{si}}^{\mathrm{b}}, \mathcal{M}_{\mathrm{st}}^{\mathrm{b}}\big((V, \rho), I, \preccurlyeq, \kappa, \tau\big)_{\mathcal{A}} = \begin{cases} \{[\sigma, \iota, \pi]\}, & \preccurlyeq = \trianglelefteqslant, \\ \emptyset, & \preccurlyeq \neq \trianglelefteqslant. \end{cases}$$

As (V, ρ) determines (σ, ι, π) up to canonical isomorphism, $\boldsymbol{\sigma}(I)_* : \mathrm{Aut}(\sigma, \iota, \pi) \to \mathrm{Aut}(V, \rho)$ is an isomorphism, so this implies that

$$\mathrm{CF}^{\mathrm{stk}}\big(\boldsymbol{\sigma}(I)\big)\delta_{\mathrm{ss}}^{\mathrm{b}}, \delta_{\mathrm{st}}^{\mathrm{b}}, \delta_{\mathrm{si}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau)\big(\big[(V, \rho)\big]\big) = \begin{cases} 1, & \preccurlyeq = \trianglelefteqslant, \\ 0, & \preccurlyeq \neq \trianglelefteqslant. \end{cases} \tag{102}$$

Since we can find such (V, ρ) for each partial order $\trianglelefteqslant$ on I, (102) implies the $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(I)) \times \delta_{\mathrm{ss}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau)$ for all $\preccurlyeq$ on I are linearly independent, and similarly for $\delta_{\mathrm{si}}^{\mathrm{b}}, \delta_{\mathrm{st}}^{\mathrm{b}}(\cdots)$. But applying $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(I))$ to (31) and (64) show that the $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(I))\delta_{\mathrm{ss}}^{\mathrm{b}}(I, \preccurlyeq, \kappa, \tau)$ and $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(I))\delta_{\mathrm{ss}}(I, \preccurlyeq, \kappa, \tau)$ over all $\preccurlyeq$ span the same subspace of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$, with dimension the number of partial orders on I, so the $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(I))\delta_{\mathrm{ss}}(I, \preccurlyeq, \kappa, \tau)$ over all $\preccurlyeq$ must also be linearly independent. The same holds for $\delta_{\mathrm{si}}, \delta_{\mathrm{st}}(\cdots)$, using (32)–(33) and (65)–(66). This proves (b).

For (c), let $(i_1, \ldots, i_n)$ be an ordered sequence in I, allowing repeated elements. Then using similar techniques we can construct (V', ρ') in $\mathcal{A} = \text{nil-}\mathbb{C}Q$ and a $(\{1, \ldots, n\}, \leqslant)$-configuration (σ', ι', π'), unique up to canonical isomorphism, with $\sigma'(\{1, \ldots, n\}) = (V', \rho')$ and $\sigma'(\{a\}) \cong \mathbf{V}^{i_a}$ for $a = 1, \ldots, n$, such that there exists no such configuration for any other sequence $(j_1, \ldots, j_m)$ in I. It follows that $\delta_{[\mathbf{V}^{j_1}]} * \cdots * \delta_{[\mathbf{V}^{j_m}]}([(V', \rho')])$ is 1 if $(j_1, \ldots, j_m) = (i_1, \ldots, i_n)$ and 0 otherwise. So the $\delta_{[\mathbf{V}^{i_1}]} * \cdots * \delta_{[\mathbf{V}^{i_n}]}$ for all sequences $(i_1, \ldots, i_n)$ are linearly independent in $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$. Part (c) now follows from (101). □

To prove no universal linear relations exist on the $\delta_{\mathrm{ss}}(*,\tau)$, say, we need to explain just what we mean by a universal linear relation, which is not very obvious. In our next result we adopt a rather restrictive definition (103), which includes the identities of Sections 5–6 and is sufficient for the applications below. But the author expects the same principle to hold for other universal forms.

Theorem 7.12. *There exist no universal linear relations of the form*

$$\sum_{\substack{\text{iso. classes of } \mathcal{A}\text{-data}(J,\lesssim,\lambda)\\ \text{and surjective } \psi: J\to K:\\ i\lesssim j\Rightarrow\psi(i)\trianglelefteq\psi(j),\ \lambda(\psi^{-1}(k))=\mu(k)\\ \text{for } k\in K,\ \tau\circ\mu\circ\psi\equiv\tau\circ\lambda}} C_{J,\lesssim,K,\trianglelefteq,\psi}\,\mathrm{CF}^{\mathrm{stk}}\big(\boldsymbol{\sigma}(J)\big)\delta_{\mathrm{ss}}(J,\lesssim,\lambda,\tau)=0 \tag{103}$$

in $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$*, which hold for all choices of* $\mathcal{A}$*,* $\mathfrak{F}_{\mathcal{A}}$*,* $K(\mathcal{A})$ *satisfying Assumption* 3.7*, permissible stability conditions or weak stability conditions* $(\tau,T,\leqslant)$ *on* $\mathcal{A}$*, and* $\mathcal{A}$*-data* $(K,\trianglelefteq,\mu)$*, where* $C_{J,\lesssim,K,\trianglelefteq,\psi}\in\mathbb{Q}$ *depends only on* J*,* $\lesssim$*,* K*,* $\trianglelefteq$*,* ψ *up to isomorphism and is nonzero for at least one choice of* $J,\dots,\psi$*. The same applies with* $\delta_{\mathrm{ss}}(\cdots)$ *replaced by* δ_{si}*,* δ_{st}*,* $\delta_{\mathrm{ss}}^{\mathrm{b}}$*,* $\delta_{\mathrm{si}}^{\mathrm{b}}(\cdots)$ *or* $\delta_{\mathrm{st}}^{\mathrm{b}}(\cdots)$*.*

Proof. Suppose for a contradiction that some such universal linear relation exists. Choose I, $\preccurlyeq$, K, $\trianglelefteq$, ϕ with $|I|$ *minimal* such that $C_{I,\preccurlyeq,K,\trianglelefteq,\phi}\neq 0$. Apply Example 7.10 with this I, to get $\mathcal{A}$, $\mathfrak{F}_{\mathcal{A}}$, $K(\mathcal{A})$ and $\kappa: I\to C(\mathcal{A})$. Let $(\tau,T,\leqslant)$ be the trivial stability condition $T=\{0\}$, $\tau\equiv 0$. Define $\mu: K\to C(\mathcal{A})$ by $\kappa(\phi^{-1}(k))=\mu(k)$ for $k\in K$. Then $\tau\circ\mu\circ\phi\equiv\tau\circ\kappa$ by choice of T.

Consider Eq. (103) with this data. Suppose $(J,\lesssim,\lambda)$, ψ gives a nonzero term. We cannot have $|J|<|I|$, since then $C_{J,\lesssim,K,\trianglelefteq,\psi}=0$ by choice of I. We cannot have $|J|>|I|$, as $\tau\circ\mu\circ\psi\equiv\tau\circ\lambda$ implies $\lambda(J)=\mu(K)=\alpha$, contradicting Proposition 7.11(a). Thus $|J|=|I|$, and Proposition 7.11(a) gives a bijection $\iota: J\to I$ with $\lambda=\kappa\circ\iota$. Since the $\kappa(i)$ for $i\in I$ are linearly independent in $C(\mathcal{A})$, and $\tau\circ\mu\circ\psi\equiv\tau\circ\lambda$ we see that $\psi=\phi\circ\iota$. Thus $(J,\lesssim,\lambda)$, ψ are isomorphic to $(I,\precsim,\kappa)$, ϕ for $\precsim=\iota_*(\lesssim)$, and (103) reduces to

$$\sum_{\text{partial orders }\precsim\text{ on } I} C_{I,\precsim,K,\trianglelefteq,\phi}\,\mathrm{CF}^{\mathrm{stk}}\big(\boldsymbol{\sigma}(I)\big)\delta_{\mathrm{ss}}(I,\precsim,\kappa,\tau)=0.$$

But the $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(I))\delta_{\mathrm{ss}}(I,\precsim,\kappa,\tau)$ for all $\precsim$ are linearly independent by Proposition 7.11(b), and $C_{I,\preccurlyeq,K,\trianglelefteq,\phi}\neq 0$, a contradiction. The proof for $\delta_{\mathrm{si}},\dots,\delta_{\mathrm{st}}^{\mathrm{b}}(\cdots)$ is the same. □

Here are some remarks on this:

- This implies a second result on nonexistence of universal linear relations in $\mathrm{CF}(\mathfrak{M}(K,\trianglelefteq,\mu)_{\mathcal{A}})$ with $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(J))\delta_{\mathrm{ss}}(J,\lesssim,\lambda,\tau)$ in (103) replaced by $\mathrm{CF}^{\mathrm{stk}}(Q(J,\lesssim,K,\trianglelefteq,\phi))\delta_{\mathrm{ss}}(J,\lesssim,\lambda,\tau)$, since applying $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(K))$ to such a relation would yield one of the form (103). The identities of (83) are of this form, though mixing different families $\delta_{\mathrm{ss}},\dots,\delta_{\mathrm{st}}^{\mathrm{b}}(\cdots)$.
- This second result shows that the identities of (83) are *unique* as universal linear relations. So, for instance, (31) is the *only* universal way to write $\delta_{\mathrm{ss}}(I,\preccurlyeq,\kappa,\tau)$ in terms of the $\delta_{\mathrm{ss}}^{\mathrm{b}}(*,\tau)$, at least in the form (103), since if there was another way we could take the difference with (31) to get a universal relation on the $\delta_{\mathrm{ss}}^{\mathrm{b}}(*,\tau)$, contradicting Theorem 7.12.

- We can use a similar method of proof with Proposition 7.11(c) to show there are *no universal polynomial relations* in the $\delta_{\rm ss}^\alpha(\tau)$ for $\alpha \in C(\mathcal{A})$, and similarly for the $\delta_{\rm si}^\alpha$, $\delta_{\rm st}^\alpha(\tau)$, and $\epsilon^\alpha(\tau)$. Effectively this shows that the universal model for $\mathcal{H}_\tau^{\rm to}$ is the free associative $\mathbb{Q}$-algebra generated by $\delta_{\rm ss}^\alpha(\tau)$ for $\alpha \in C(\mathcal{A})$, or equivalently by $\epsilon^\alpha(\tau)$ for $\alpha \in C(\mathcal{A})$.
- The theorem is evidence that the configurations framework is a good one, and in particular, that partial orders are a good choice of combinatorial data to keep track of collections of objects and morphisms. For we know by closure of $\mathcal{H}_\tau^{\rm to}$ under $*$ and other operations that there are not too few partial orders to do everything we want, and the theorem tells us there is no redundant information, and so not too many partial orders.

8. Generalization to stack functions

Finally we discuss the best way to generalize the constructible functions material of Sections 5–7 to stack functions. We would like to define stack function versions $\bar\delta_{\rm ss}^\alpha$, $\bar\delta_{\rm si}^\alpha$, $\bar\delta_{\rm st}^\alpha(\tau)$ of $\delta_{\rm ss}^\alpha$, $\delta_{\rm si}^\alpha$, $\delta_{\rm st}^\alpha(\tau)$, and $\bar\delta_{\rm ss},\dots,\bar\delta_{\rm st}^{\rm b}(I,\preccurlyeq,\kappa,\tau)$ of $\delta_{\rm ss},\dots,\delta_{\rm st}^{\rm b}(I,\preccurlyeq,\kappa,\tau)$, that satisfy analogues of the identities of Sections 5–6 and the (Lie) algebra ideas of Section 7; also, we want the transformation laws between stability conditions $(\tau,T,\leqslant)$ and $(\tilde\tau,\tilde T,\leqslant)$ studied in [11] for these stack functions to be well-behaved.

The most obvious way to define these stack functions is $\bar\delta_{\rm ss}^\alpha(\tau)=\bar\delta_{{\rm Obj}_{\rm ss}^\alpha(\tau)}$, $\bar\delta_{\rm si}^\alpha(\tau)=\bar\delta_{{\rm Obj}_{\rm si}^\alpha(\tau)}$, $\dots,\bar\delta_{\rm st}^{\rm b}(I,\preccurlyeq,\kappa,\tau)=\bar\delta_{\mathcal{M}_{\rm st}^{\rm b}(I,\preccurlyeq,\kappa,\tau)}$, following Definition 4.6. However, investigation shows that this is *not* a helpful definition: none of the identities of Sections 5–6 would then hold, and much of the (Lie) algebra material of Section 7 would not generalize either.

There are two main reasons for this. The first is that constructible function pushforwards $\mathrm{CF}^{\rm stk}(\cdots)$ use Euler characteristics, and many of the identities of Sections 5–6 make essential use of $\chi(\mathbb{K}^m)=1$, and so will not work for general stack function pushforwards. We could get round this by using the stack function spaces $\overline{\mathrm{SF}}(\mathfrak{F},\Theta,\Omega)$ of [8, §6], which also set $[\mathbb{K}^m]=1$.

The second is the idea of *virtual rank* introduced in [8, §5], and the corresponding idea of *virtual indecomposable* in [10, §5]. The point here is that experience shows that the best analogue of constructible functions $\mathrm{CF}^{\rm ind}(\mathfrak{Obj}_{\mathcal{A}})$ supported on indecomposables is not stack functions supported on indecomposables, but stack functions $\mathrm{SF}_{\rm al}^{\rm ind}(\mathfrak{Obj}_{\mathcal{A}})$ 'supported on virtual indecomposables,' which can have nontrivial components over decomposable objects.

Unfortunately these notions of virtual rank and virtual indecomposable are technical and difficult to explain, but here is the rough idea. On the stack functions $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ (or $\mathrm{SF}_{\rm al}(\mathfrak{Obj}_{\mathcal{A}}),\dots$) we define linear maps $\Pi_n^{\rm vi}:\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})\to\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ for $n=0,1,2,\dots$, the projections to stack functions of 'virtual rank n.' These satisfy $(\Pi_n^{\rm vi})^2=\Pi_n^{\rm vi}$ and $\Pi_m^{\rm vi}\Pi_n^{\rm vi}=0$ for $m\neq n$. If $[(\mathfrak{R},\rho)]\in\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ and $\mathfrak{R}$ is a $\mathbb{K}$-stack whose stabilizer groups are all *abelian* algebraic $\mathbb{K}$-groups, then $\Pi_n^{\rm vi}([(\mathfrak{R},\rho)])=[(\mathfrak{R}_n,\rho)]$, where $\mathfrak{R}_n$ is the locally closed $\mathbb{K}$-substack of $\mathfrak{R}$ of points whose stabilizer groups have rank exactly n.

If $[(\mathfrak{R},\rho)]\in\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ and $\mathfrak{R}$ is a $\mathbb{K}$-stack whose stabilizer groups are nonabelian, then $\Pi_n^{\rm vi}([(\mathfrak{R},\rho)])$ replaces each point $x\in\mathfrak{R}(\mathbb{K})$ with stabilizer group $\mathrm{Aut}_{\mathbb{K}}(x)=G$ by a finite $\mathbb{Q}$-linear combination of points with stabilizer groups $C_G(T)$, the centralizer of T in G, for certain subgroups T of the maximal torus T^G of G. It is like regarding a nonabelian stabilizer group G as a formal $\mathbb{Q}$-linear combination of torus stabilizer groups $(\mathbb{C}^\times)^k$ for $\mathrm{rk}\,Z(G)\leqslant k\leqslant\mathrm{rk}\,G$, where $Z(G)$ is the center of G, and then $\Pi_n^{\rm vi}$ selects the $(\mathbb{C}^\times)^n$ components.

An object $X\in\mathcal{A}$ is indecomposable if and only if $\mathrm{Aut}(X)$ has rank 1. By analogy, a stack function $f\in\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ is said to be *supported on virtual indecomposables* if it has virtual rank 1, that is, $\Pi_1^{\rm vi}(f)=f$. We write $\mathrm{SF}_{\rm al}^{\rm ind}(\mathfrak{Obj}_{\mathcal{A}})$ for the subspace of $f\in\mathrm{SF}_{\rm al}(\mathfrak{Obj}_{\mathcal{A}})$ supported on virtual indecomposables. The importance of these ideas for us is that there is a deep compatibility between the projections Π_n^{vi} and multiplication $*$ in $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}}), \mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}), \ldots,$ explored in [10, §5]. This implies, for instance, that $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ is a Lie algebra, that is, it is closed under the Lie bracket $[f,g]=f*g-g*f$. In contrast, the subspace of $f\in\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ supported on (actual, nonvirtual) indecomposable objects is not closed under $[\,,\,]$.

These ideas suggest that the best definition for $\bar\delta_{\mathrm{si}}^{\alpha}(\tau)$ is not $\bar\delta_{\mathrm{Obj}_{\mathrm{si}}^{\alpha}(\tau)}$, but rather a 'characteristic function' of 'τ-semistable virtual indecomposables,' perhaps $\Pi_1^{\mathrm{vi}}(\bar\delta_{\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)})$ in the notation of [8, §5], as in Theorem 8.6 below. Following similar reasoning, one can argue there should be stack function ideas of 'virtual τ-stables' and 'virtual best configurations,' which can have nonzero components over strictly τ-semistable objects and nonbest configurations. However, there does not seem to be a stack function idea of 'virtual τ-semistable': the appropriate notion is just τ-semistable in the usual sense.

Thus the approach we choose is to first set $\bar\delta_{\mathrm{ss}}^{\alpha}(\tau)=\bar\delta_{\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)}$ and $\bar\delta_{\mathrm{ss}}(I,\precsim,\kappa,\tau)=\bar\delta_{\mathcal{M}_{\mathrm{ss}}(I,\precsim,\kappa,\tau)}$, and then define $\bar\delta_{\mathrm{si}}^{\alpha}$, $\bar\delta_{\mathrm{st}}^{\alpha}(\tau)$ and $\bar\delta_{\mathrm{si}},\ldots,\bar\delta_{\mathrm{st}}^{\mathrm{b}}(I,\precsim,\kappa,\tau)$ uniquely such that the analogues of the identities of Sections 5–6 hold. Of course, the meaning of $\bar\delta_{\mathrm{si}}^{\alpha}$, $\bar\delta_{\mathrm{st}}^{\alpha}(\tau)$ and $\bar\delta_{\mathrm{si}},\ldots,\bar\delta_{\mathrm{st}}^{\mathrm{b}}(I,\precsim,\kappa,\tau)$ is then unclear, and we discuss this after Theorem 8.6. The justification for this approach is that nearly all of the (Lie) algebra material of Section 7 generalizes very neatly, as we shall see below, and it fits nicely with the ideas on changing stability conditions in [11].

For simplicity we work throughout with the spaces $\mathrm{SF}(\mathfrak{F})$, but the material below works equally well in the spaces $\overline{\mathrm{SF}}(\mathfrak{F},\Upsilon,\Lambda)$, $\overline{\mathrm{SF}}(\mathfrak{F},\Upsilon,\Lambda^{\circ})$ or $\overline{\mathrm{SF}}(\mathfrak{F},\Theta,\Omega)$ of [8], and much of it also in $\underline{\mathrm{SF}}(\mathfrak{F},\Upsilon,\Lambda)$.

Definition 8.1. Let Assumption 3.7 hold, $(\tau,T,\leqslant)$ be a permissible weak stability condition on $\mathcal{A}$, $\alpha\in C(\mathcal{A})$, and $(I,\trianglelefteq,\kappa)$ be $\mathcal{A}$-data, as in Definition 3.8. Define

$$\begin{gathered}\bar\delta_{\mathrm{ss}}^{\alpha}(\tau)=\bar\delta_{\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)}\in\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})\quad\text{or}\quad\mathrm{SF}\big(\mathfrak{Obj}_{\mathcal{A}}^{\alpha}\big)\quad\text{and}\\ \bar\delta_{\mathrm{ss}}(I,\trianglelefteq,\kappa,\tau)=\bar\delta_{\mathcal{M}_{\mathrm{ss}}(I,\trianglelefteq,\kappa,\tau)}\in\mathrm{SF}\big(\mathfrak{M}(I,\trianglelefteq,\kappa)_{\mathcal{A}}\big).\end{gathered}\tag{104}$$

Since $\mathcal{M}_{\mathrm{ss}}(I,\trianglelefteq,\kappa,\tau)=(\prod_{i\in I}\boldsymbol{\sigma}(\{i\}))_*^{-1}(\prod_{i\in I}\mathrm{Obj}_{\mathrm{ss}}^{\kappa(i)}(\tau))$ we see that

$$\bar\delta_{\mathrm{ss}}(I,\trianglelefteq,\kappa,\tau)=\Big(\prod_{i\in I}\boldsymbol{\sigma}\big(\{i\}\big)\Big)^*\Big(\bigotimes_{i\in I}\bar\delta_{\mathrm{ss}}^{\kappa(i)}(\tau)\Big).\tag{105}$$

By analogy with (64), for $\mathcal{A}$-data $(I,\trianglelefteq,\kappa)$ define

$$\bar\delta_{\mathrm{ss}}^{\mathrm{b}}(I,\trianglelefteq,\kappa,\tau)=\sum_{\substack{\text{p.o.s }\precsim\text{ on }I:\\ \trianglelefteq\text{ dominates }\precsim}}n(I,\precsim,\trianglelefteq)Q(I,\precsim,\trianglelefteq)_*\bar\delta_{\mathrm{ss}}(I,\precsim,\kappa,\tau).\tag{106}$$

By analogy with (67), setting $i\bullet j$ if and only if $i=j$, for $\alpha\in C(\mathcal{A})$ define

$$\bar\delta_{\mathrm{si}}^{\alpha}(\tau)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\cdot\sum_{\substack{\kappa:\,\{1,\ldots,n\}\to C(\mathcal{A}):\\ \kappa(\{1,\ldots,n\})=\alpha,\ \tau\circ\kappa\equiv\tau(\alpha)}}\boldsymbol{\sigma}\big(\{1,\ldots,n\}\big)_*\bar\delta_{\mathrm{ss}}\big(\{1,\ldots,n\},\bullet,\kappa,\tau\big).\tag{107}$$

By analogy with (105) and (65), define

$$\begin{aligned} \bar{\delta}_{\rm si}(I, \trianglelefteq, \kappa, \tau) &= \Bigl(\prod_{i \in I} \boldsymbol{\sigma}\bigl(\{i\}\bigr)\Bigr)^* \Bigl(\bigotimes_{i \in I} \bar{\delta}_{\rm si}^{\kappa(i)}(\tau)\Bigr), \\ \bar{\delta}_{\rm si}^{\rm b}(I, \trianglelefteq, \kappa, \tau) &= \sum_{\substack{\text{p.o.s } \precsim \text{ on } I: \\ \trianglelefteq \text{ dominates } \precsim}} n(I, \precsim, \trianglelefteq) Q(I, \precsim, \trianglelefteq)_* \bar{\delta}_{\rm si}(I, \precsim, \kappa, \tau). \end{aligned} \tag{108}$$

Now let $(\tau, T, \leqslant)$ be a stability condition (not just a weak one). By analogy with the case $K = \{k\}$ in (75), using (74) to simplify the $N(J, \lesssim, K, \chi)$, define

$$\bar{\delta}_{\rm st}^{\alpha}(\tau) = \sum_{\substack{\text{iso. classes} \\ \text{of finite sets } I}} \frac{1}{|I|!} \cdot \sum_{\substack{\precsim, \kappa: \ (I, \precsim, \kappa) \text{ is } \mathcal{A}\text{-data}, \\ \kappa(I) = \alpha, \ \tau \circ \kappa \equiv \tau(\alpha)}} N(I, \precsim) \boldsymbol{\sigma}(I)_* \bar{\delta}_{\rm ss}^{\rm b}(I, \precsim, \kappa, \tau). \tag{109}$$

By analogy with (105) and (66), define

$$\bar{\delta}_{\rm st}(I, \trianglelefteq, \kappa, \tau) = \Bigl(\prod_{i \in I} \boldsymbol{\sigma}\bigl(\{i\}\bigr)\Bigr)^* \Bigl(\bigotimes_{i \in I} \bar{\delta}_{\rm st}^{\kappa(i)}(\tau)\Bigr), \tag{110}$$

$$\bar{\delta}_{\rm st}^{\rm b}(I, \trianglelefteq, \kappa, \tau) = \sum_{\substack{\text{p.o.s } \precsim \text{ on } I: \\ \trianglelefteq \text{ dominates } \precsim}} n(I, \precsim, \trianglelefteq) Q(I, \precsim, \trianglelefteq)_* \bar{\delta}_{\rm st}(I, \precsim, \kappa, \tau). \tag{111}$$

By analogy with (93), for $\alpha \in C(\mathcal{A})$ define

$$\bar{\epsilon}^{\alpha}(\tau) = \sum_{\substack{\mathcal{A}\text{-data } (\{1, \dots, n\}, \leqslant, \kappa): \\ \kappa(\{1, \dots, n\}) = \alpha, \ \tau \circ \kappa \equiv \tau(\alpha)}} \frac{(-1)^{n-1}}{n} \bar{\delta}_{\rm ss}^{\kappa(1)}(\tau) * \bar{\delta}_{\rm ss}^{\kappa(2)}(\tau) * \cdots * \bar{\delta}_{\rm ss}^{\kappa(n)}(\tau). \tag{112}$$

By the proofs in Sections 5–7 there are only finitely many nonzero terms in each equation, so they are all well-defined. It is easy to show $\bar{\delta}_{\rm ss}^{\alpha}(\tau), \bar{\delta}_{\rm si}^{\alpha}(\tau), \bar{\delta}_{\rm st}^{\alpha}(\tau), \bar{\epsilon}^{\alpha}(\tau)$ are supported on $\mathrm{Obj}_{\rm ss}^{\alpha}(\tau)$, and $\bar{\delta}_{\rm ss}, \dots, \bar{\delta}_{\rm st}^{\rm b}(I, \trianglelefteq, \kappa, \tau)$ on $\mathcal{M}_{\rm ss}(I, \trianglelefteq, \kappa, \tau)_{\mathcal{A}}$.

Here are the analogues of the remaining eight identities in (83), that is, (31), (32), (33), (38), (56), (71), (72) and (75) respectively.

Theorem 8.2. *For all $\mathcal{A}$-data $(K, \trianglelefteq, \mu)$ and $\alpha \in C(\mathcal{A})$ we have*

$$\sum_{\substack{\text{p.o.s } \precsim \text{ on } K: \\ \trianglelefteq \text{ dominates } \precsim}} Q(K, \precsim, \trianglelefteq)_* \bar{\delta}_{\rm ss}^{\rm b}(K, \precsim, \mu, \tau) = \bar{\delta}_{\rm ss}(K, \trianglelefteq, \mu, \tau), \tag{113}$$

$$\sum_{\substack{\text{p.o.s } \precsim \text{ on } K: \\ \trianglelefteq \text{ dominates } \precsim}} Q(K, \precsim, \trianglelefteq)_* \bar{\delta}_{\rm si}^{\rm b}(K, \precsim, \mu, \tau) = \bar{\delta}_{\rm si}(K, \trianglelefteq, \mu, \tau), \tag{114}$$

$$\sum_{\substack{\text{p.o.s } \precsim \text{ on } K: \\ \trianglelefteq \text{ dominates } \precsim}} Q(K, \precsim, \trianglelefteq)_* \bar{\delta}_{\rm st}^{\rm b}(K, \precsim, \mu, \tau) = \bar{\delta}_{\rm st}(K, \trianglelefteq, \mu, \tau), \tag{115}$$

$$\sum_{\substack{\text{iso. classes}\\ \text{of finite}\\ \text{sets} I}} \frac{1}{|I|!}\cdot \sum_{\substack{\kappa:I\to C(\mathcal{A}),\ \text{surjective } \phi:I\to K:\\ \kappa(\phi^{-1}(k))=\mu(k) \text{ for } k\in K,\\ \tau\circ\mu\circ\phi\equiv\tau\circ\kappa:I\to T.\\ \text{Define } \precsim \text{ on } I \text{ by } i\precsim j \text{ if } i=j\\ \text{or } \phi(i)\neq\phi(j) \text{ and } \phi(i)\trianglelefteq\phi(j)}} Q(I,\precsim,K,\trianglelefteq,\phi)_*\bar{\delta}_{\mathrm{si}}(I,\precsim,\kappa,\tau)$$

$$=\bar{\delta}_{\mathrm{ss}}(K,\trianglelefteq,\mu,\tau), \tag{116}$$

$$\sum_{\substack{\text{iso. classes}\\ \text{of finite}\\ \text{sets } I}} \frac{1}{|I|!}\cdot \sum_{\substack{\precsim,\kappa,\phi:\ (I,\precsim,\kappa) \text{ is } \mathcal{A}\text{-data},\\ \phi:I\to K \text{ is surjective},\\ i\precsim j \text{ implies } \phi(i)\trianglelefteq\phi(j),\\ \kappa(\phi^{-1}(k))=\mu(k) \text{ for } k\in K,\\ \tau\circ\mu\circ\phi\equiv\tau\circ\kappa:I\to T}} Q(I,\precsim,K,\trianglelefteq,\phi)_*\bar{\delta}^{\mathrm{b}}_{\mathrm{st}}(I,\precsim,\kappa,\tau)$$

$$=\bar{\delta}_{\mathrm{ss}}(K,\trianglelefteq,\mu,\tau), \tag{117}$$

$$\sum_{\substack{\text{iso.}\\ \text{classes}\\ \text{of finite}\\ \text{sets } I}} \frac{(-1)^{|I|-|K|}}{|I|!}\cdot \sum_{\substack{\kappa:I\to C(\mathcal{A}),\ \text{surjective } \phi:I\to K:\\ \kappa(\phi^{-1}(k))=\mu(k) \text{ for } k\in K,\\ \tau\circ\mu\circ\phi\equiv\tau\circ\kappa:I\to T.\\ \text{Define } \precsim \text{ on } I \text{ by } i\precsim j \text{ if } i=j\\ \text{or } \phi(i)\neq\phi(j) \text{ and } \phi(i)\trianglelefteq\phi(j)}} \prod_{k\in K}\big(\big|\phi^{-1}(\{k\})\big|-1\big)!$$

$$\cdot\, Q(I,\precsim,K,\trianglelefteq,\phi)_*\bar{\delta}_{\mathrm{ss}}(I,\precsim,\kappa,\tau)=\bar{\delta}_{\mathrm{si}}(K,\trianglelefteq,\mu,\tau), \tag{118}$$

$$\sum_{\substack{\text{iso. classes}\\ \text{of finite}\\ \text{sets } I}} \frac{1}{|I|!}\cdot \sum_{\substack{\precsim,\kappa,\phi:\ (I,\precsim,\kappa) \text{ is } \mathcal{A}\text{-data},\\ (I,\precsim,K,\phi) \text{ is allowable},\\ \trianglelefteq=\mathcal{P}(I,\precsim,K,\phi),\\ \kappa(\phi^{-1}(k))=\mu(k) \text{ for } k\in K,\\ \tau\circ\mu\circ\phi\equiv\tau\circ\kappa:I\to T}} Q(I,\precsim,K,\trianglelefteq,\phi)_*\bar{\delta}^{\mathrm{b}}_{\mathrm{st}}(I,\precsim,\kappa,\tau)$$

$$=\bar{\delta}^{\mathrm{b}}_{\mathrm{ss}}(K,\trianglelefteq,\mu,\tau), \tag{119}$$

$$\sum_{\substack{\text{iso. classes}\\ \text{of finite}\\ \text{sets } I}} \frac{1}{|I|!}\cdot \sum_{\substack{\precsim,\kappa,\phi:\ (I,\precsim,\kappa) \text{ is } \mathcal{A}\text{-data},\\ (I,\precsim,K,\phi) \text{ is allowable},\\ \trianglelefteq=\mathcal{P}(I,\precsim,K,\phi),\\ \kappa(\phi^{-1}(k))=\mu(k) \text{ for } k\in K,\\ \tau\circ\mu\circ\phi\equiv\tau\circ\kappa:I\to T}} N(I,\precsim,K,\phi)Q(I,\precsim,K,\trianglelefteq,\phi)_*\bar{\delta}^{\mathrm{b}}_{\mathrm{ss}}(I,\precsim,\kappa,\tau)$$

$$=\bar{\delta}^{\mathrm{b}}_{\mathrm{st}}(K,\trianglelefteq,\mu,\tau), \tag{120}$$

$$\bar{\delta}^{\alpha}_{\mathrm{ss}}(\tau)=\sum_{\substack{\mathcal{A}\text{-data } (\{1,\dots,n\},\leqslant,\kappa):\\ \kappa(\{1,\dots,n\})=\alpha,\ \tau\circ\kappa\equiv\tau(\alpha)}} \frac{1}{n!}\bar{\epsilon}^{\kappa(1)}(\tau)*\bar{\epsilon}^{\kappa(2)}(\tau)*\cdots*\bar{\epsilon}^{\kappa(n)}(\tau), \tag{121}$$

supposing $(\tau,T,\leqslant)$ *is a stability condition in* (115), (117), (119) *and* (120). *There are only finitely many nonzero terms in each equation.*

Proof. The proofs in Sections 5–6 imply there are only finitely many nonzero terms in each equation. Equations (113)–(115) are the inverses of (106), (109), (111) respectively, and follow from them by the reverse of the argument in Section 6.1. The argument used to prove (71) from (67) proves (118) from (107), using (105) along the way. Equation (116) then follows from (118) as it is its combinatorial inverse, reversing the argument in Section 6.2 that (71) is the inverse of (38).

Combining (106), (115) and (120) gives an identity writing $\bar{\delta}_{\mathrm{st}}(K, \trianglelefteq, \mu, \tau)$ as a linear combination of $Q(I, \preccurlyeq, K, \trianglelefteq, \phi)_* \bar{\delta}_{\mathrm{ss}}(I, \preccurlyeq, \kappa, \tau)$. When $K = \{k\}$ this is equivalent to the combination of (106) and (109), and so holds. The general case of the identity follows from the case $K = \{k\}$ by (110), using (105) along the way. We can then recover (120) from this identity as we already know (106), (115) and their inverses (113), (111). Equation (119) follows from (120) as it is its combinatorial inverse, reversing the argument in Section 6.4 that (75) is the inverse of (72). We obtain (117) by substituting (119) into (113). Finally, (121) is proved from (112) in the same way as (94) from (93). □

Corollary 8.3. *If* $\operatorname{char}\mathbb{K} = 0$, $\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}$ *takes* $\bar{\delta}^{\alpha}_{\mathrm{ss}}$, $\bar{\delta}^{\alpha}_{\mathrm{si}}$, $\bar{\delta}^{\alpha}_{\mathrm{st}}$, $\bar{\epsilon}^{\alpha}(\tau)$ *to* $\delta^{\alpha}_{\mathrm{ss}}$, $\delta^{\alpha}_{\mathrm{si}}$, $\delta^{\alpha}_{\mathrm{st}}$, $\epsilon^{\alpha}(\tau)$, *and* $\pi^{\mathrm{stk}}_{\mathfrak{M}(I,\trianglelefteq,\kappa)_{\mathcal{A}}}$ *takes* $\bar{\delta}_{\mathrm{ss}}, \dots, \bar{\delta}^{\mathrm{b}}_{\mathrm{st}}(I, \trianglelefteq, \kappa, \tau)$ *to* $\delta_{\mathrm{ss}}, \dots, \delta^{\mathrm{b}}_{\mathrm{st}}(I, \trianglelefteq, \kappa, \tau)$, *and* $\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}$ *takes* $\boldsymbol{\sigma}(I)_* \bar{\delta}_{\mathrm{ss}}, \dots, \bar{\delta}^{\mathrm{b}}_{\mathrm{st}}(I, \trianglelefteq, \kappa, \tau)$ *to* $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(I))\delta_{\mathrm{ss}}, \dots, \delta^{\mathrm{b}}_{\mathrm{st}}(I, \trianglelefteq, \kappa, \tau)$, *supposing* $(\tau, T, \leqslant)$ *is a stability condition for the* $\bar{\delta}^{\alpha}_{\mathrm{st}}(\tau)$ *and* $\bar{\delta}_{\mathrm{st}}$, $\bar{\delta}^{\mathrm{b}}_{\mathrm{st}}(\cdots)$ *cases.*

Proof. By definition $\bar{\delta}^{\alpha}_{\mathrm{ss}}(\tau) = \iota_{\mathfrak{Obj}_{\mathcal{A}}}(\delta^{\alpha}_{\mathrm{ss}}(\tau))$, so $\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}(\bar{\delta}^{\alpha}_{\mathrm{ss}}(\tau)) = \delta^{\alpha}_{\mathrm{ss}}(\tau)$ as $\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}} \circ \iota_{\mathfrak{Obj}_{\mathcal{A}}}$ is the identity. Similarly $\pi^{\mathrm{stk}}_{\mathfrak{M}(I,\trianglelefteq,\kappa)_{\mathcal{A}}}(\bar{\delta}_{\mathrm{ss}}(I, \trianglelefteq, \kappa, \tau)) = \delta_{\mathrm{ss}}(I, \trianglelefteq, \kappa, \tau)$. Now the identities of Definition 8.1 and Theorem 8.2 are all analogues of identities on $\delta^{\alpha}_{\mathrm{ss}}, \dots, \epsilon^{\alpha}(\tau)$ or $\delta_{\mathrm{ss}}, \dots, \delta^{\mathrm{b}}_{\mathrm{st}}(I, \trianglelefteq, \kappa, \tau)$ in Sections 5–7. So applying $\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}, \pi^{\mathrm{stk}}_{\mathfrak{M}(I,\trianglelefteq,\kappa)_{\mathcal{A}}}$ or $\pi^{\mathrm{stk}}_{\mathfrak{M}(K,\trianglelefteq,\mu)_{\mathcal{A}}}$ to these identities and using Theorem 2.13(b), we see that the identities of Sections 5–7 hold with $\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}(\bar{\delta}^{\alpha}_{\mathrm{ss}}(\tau))$ in place of $\delta^{\alpha}_{\mathrm{ss}}(\tau)$, and so on.

But from (83) we see the $\delta_{\mathrm{ss}}(*, \tau)$ determine the δ_{si}, δ_{st}, $\delta^{\mathrm{b}}_{\mathrm{ss}}$, $\delta^{\mathrm{b}}_{\mathrm{si}}$, $\delta^{\mathrm{b}}_{\mathrm{st}}(*, \tau)$, so as $\pi^{\mathrm{stk}}_{\mathfrak{M}(I,\trianglelefteq,\kappa)_{\mathcal{A}}}(\bar{\delta}_{\mathrm{ss}}(I, \trianglelefteq, \kappa, \tau)) = \delta_{\mathrm{ss}}(I, \trianglelefteq, \kappa, \tau)$ we see that $\pi^{\mathrm{stk}}_{\mathfrak{M}(I,\trianglelefteq,\kappa)_{\mathcal{A}}}$ takes $\bar{\delta}_{\mathrm{si}}, \dots, \bar{\delta}^{\mathrm{b}}_{\mathrm{st}}(I, \trianglelefteq, \kappa, \tau)$ to $\delta_{\mathrm{si}}, \dots, \delta^{\mathrm{b}}_{\mathrm{st}}(I, \trianglelefteq, \kappa, \tau)$. The claim for $\boldsymbol{\sigma}(I)_* \bar{\delta}_{\mathrm{ss}}, \dots, \bar{\delta}^{\mathrm{b}}_{\mathrm{st}}(I, \trianglelefteq, \kappa, \tau)$ now follows from Theorem 2.13(b), and for $\bar{\delta}^{\alpha}_{\mathrm{si}}$, $\bar{\delta}^{\alpha}_{\mathrm{st}}$, $\bar{\epsilon}^{\alpha}(\tau)$ from the corresponding identities. □

Here are stack function analogues of material in Sections 7.1 and 7.3.

Definition 8.4. Let Assumption 3.7 hold, and $(\tau, T, \leqslant)$ be a permissible weak stability condition on $\mathcal{A}$. Define $\mathbb{Q}$-vector subspaces $\overline{\mathcal{H}}^{\mathrm{pa}}_{\tau}, \overline{\mathcal{H}}^{\mathrm{to}}_{\tau}$ in $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ by

$$\overline{\mathcal{H}}^{\mathrm{pa}}_{\tau} = \big\langle \boldsymbol{\sigma}(I)_* \bar{\delta}_{\mathrm{ss}}(I, \preccurlyeq, \kappa, \tau)\colon\ (I, \preccurlyeq, \kappa) \text{ is } \mathcal{A}\text{-data} \big\rangle_{\mathbb{Q}},$$
$$\overline{\mathcal{H}}^{\mathrm{to}}_{\tau} = \big\langle \bar{\delta}_{[0]}, \bar{\delta}^{\alpha_1}_{\mathrm{ss}}(\tau) * \cdots * \bar{\delta}^{\alpha_n}_{\mathrm{ss}}(\tau)\colon\ \alpha_1, \dots, \alpha_n \in C(\mathcal{A}) \big\rangle_{\mathbb{Q}}.$$

Here $\langle \cdots \rangle_{\mathbb{Q}}$ is the set of all finite $\mathbb{Q}$-linear combinations of the elements '$\cdots$.' From (105) we see that $\boldsymbol{\sigma}(I)_* \bar{\delta}_{\mathrm{ss}}(I, \preccurlyeq, \kappa, \tau) = P_{(I,\preccurlyeq)}(\bar{\delta}^{\kappa(i)}_{\mathrm{ss}}(\tau)\colon\ i \in I)$, giving

$$\overline{\mathcal{H}}^{\mathrm{pa}}_{\tau} = \big\langle P_{(I,\preccurlyeq)}\big(\bar{\delta}^{\kappa(i)}_{\mathrm{ss}}(\tau)\colon\ i \in I\big)\colon\ (I, \preccurlyeq, \kappa) \text{ is } \mathcal{A}\text{-data} \big\rangle_{\mathbb{Q}}.$$

It follows from [10, Theorem 5.4] that $\overline{\mathcal{H}}^{\mathrm{pa}}_{\tau}$ is closed under the operations $P_{(I,\preccurlyeq)}$.

If we were to work instead in $\overline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$, for instance, it might be better to define $\overline{\mathcal{H}}^{\mathrm{pa}}_{\tau,\Upsilon,\Lambda}$, $\overline{\mathcal{H}}^{\mathrm{to}}_{\tau,\Upsilon,\Lambda}$ to be the Λ-submodules with the above generators, and then $\overline{\mathcal{H}}^{\mathrm{pa}}_{\tau,\Upsilon,\Lambda}$, $\overline{\mathcal{H}}^{\mathrm{to}}_{\tau,\Upsilon,\Lambda}$ will be Λ-algebras rather than $\mathbb{Q}$-algebras.

In [11] we will show that if $(\tau, T, \leqslant)$ and $(\tilde\tau, \tilde T, \leqslant)$ are permissible weak stability conditions on $\mathcal{A}$, then (under some finiteness conditions) we have $\overline{\mathcal{H}}_\tau^{\rm pa} = \overline{\mathcal{H}}_{\tilde\tau}^{\rm pa}$ and $\overline{\mathcal{H}}_\tau^{\rm to} = \overline{\mathcal{H}}_{\tilde\tau}^{\rm to}$, so that $\overline{\mathcal{H}}_\tau^{\rm pa}, \overline{\mathcal{H}}_\tau^{\rm to}$ are *independent of the choice of weak stability condition* $(\tau, T, \leqslant)$. We generalize (85) and (95).

Theorem 8.5. *$\overline{\mathcal{H}}_\tau^{\rm pa}, \overline{\mathcal{H}}_\tau^{\rm to}$ are subalgebras of $\mathrm{SF}_{\rm al}(\mathfrak{Obj}_{\mathcal{A}})$ with $\overline{\mathcal{H}}_\tau^{\rm to} \subseteq \overline{\mathcal{H}}_\tau^{\rm pa}$, and*

$$\begin{aligned} \overline{\mathcal{H}}_\tau^{\rm pa} &= \big\langle \boldsymbol{\sigma}(I)_* \bar\delta_{\rm ss}^{\rm b}(I, \preccurlyeq, \kappa, \tau):\ (I, \preccurlyeq, \kappa) \text{ is } \mathcal{A}\text{-data}\big\rangle_{\mathbb{Q}} \\ &= \big\langle \boldsymbol{\sigma}(I)_* \bar\delta_{\rm si}(I, \preccurlyeq, \kappa, \tau):\ (I, \preccurlyeq, \kappa) \text{ is } \mathcal{A}\text{-data}\big\rangle_{\mathbb{Q}} \\ &= \big\langle \boldsymbol{\sigma}(I)_* \bar\delta_{\rm si}^{\rm b}(I, \preccurlyeq, \kappa, \tau):\ (I, \preccurlyeq, \kappa) \text{ is } \mathcal{A}\text{-data}\big\rangle_{\mathbb{Q}} \\ &= \big\langle \boldsymbol{\sigma}(I)_* \bar\delta_{\rm st}(I, \preccurlyeq, \kappa, \tau):\ (I, \preccurlyeq, \kappa) \text{ is } \mathcal{A}\text{-data}\big\rangle_{\mathbb{Q}} \\ &= \big\langle \boldsymbol{\sigma}(I)_* \bar\delta_{\rm st}^{\rm b}(I, \preccurlyeq, \kappa, \tau):\ (I, \preccurlyeq, \kappa) \text{ is } \mathcal{A}\text{-data}\big\rangle_{\mathbb{Q}}, \end{aligned} \tag{122}$$

$$\overline{\mathcal{H}}_\tau^{\rm to} = \big\langle \bar\delta_{[0]}, \bar\epsilon^{\alpha_1}(\tau) * \cdots * \bar\epsilon^{\alpha_n}(\tau):\ \alpha_1, \dots, \alpha_n \in C(\mathcal{A})\big\rangle_{\mathbb{Q}}, \tag{123}$$

supposing $(\tau, T, \leqslant)$ is a stability condition in the last two lines of (122). When $\mathbb{K}$ has characteristic zero $\pi_{\mathfrak{Obj}_{\mathcal{A}}}^{\rm stk}: \overline{\mathcal{H}}_\tau^{\rm pa} \to \mathcal{H}_\tau^{\rm pa}$ and $\pi_{\mathfrak{Obj}_{\mathcal{A}}}^{\rm stk}: \overline{\mathcal{H}}_\tau^{\rm to} \to \mathcal{H}_\tau^{\rm to}$ are surjective $\mathbb{Q}$-algebra morphisms.

Proof. Clearly $\overline{\mathcal{H}}_\tau^{\rm to}$ is the subalgebra of $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ generated by the $\bar\delta_{\rm ss}^\alpha(\tau)$ for all $\alpha \in C(\mathcal{A})$. The analogue of (80) implies that $\overline{\mathcal{H}}_\tau^{\rm to} \subseteq \overline{\mathcal{H}}_\tau^{\rm pa}$. We have $\bar\delta_{\rm ss}^\alpha(\tau) = \iota_{\mathfrak{Obj}_{\mathcal{A}}}(\delta_{\rm ss}^\alpha(\tau))$, so $\bar\delta_{\rm ss}^\alpha(\tau) \in \mathrm{SF}_{\rm al}(\mathfrak{Obj}_{\mathcal{A}})$ as $\iota_{\mathfrak{Obj}_{\mathcal{A}}}$ maps $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \to \mathrm{SF}_{\rm al}(\mathfrak{Obj}_{\mathcal{A}})$ by [10, Definition 5.5]. Also $\boldsymbol{\sigma}(I)_* \bar\delta_{\rm ss}(I, \preccurlyeq, \kappa, \tau) = P_{(I,\preccurlyeq)}(\bar\delta_{\rm ss}^{\kappa(i)}(\tau):\ i \in I)$ by (105), so $\boldsymbol{\sigma}(I)_* \bar\delta_{\rm ss}(I, \preccurlyeq, \kappa, \tau) \in \mathrm{SF}_{\rm al}(\mathfrak{Obj}_{\mathcal{A}})$ by [10, Proposition 5.6], and $\overline{\mathcal{H}}_\tau^{\rm to} \subseteq \overline{\mathcal{H}}_\tau^{\rm pa} \subseteq \mathrm{SF}_{\rm al}(\mathfrak{Obj}_{\mathcal{A}})$. Since $\overline{\mathcal{H}}_\tau^{\rm to}$ is closed under the $P_{(I,\preccurlyeq)}$ and $* = P_{(\{1,2\},\leqslant)}$, it is closed under $*$, and is a subalgebra of $\mathrm{SF}_{\rm al}(\mathfrak{Obj}_{\mathcal{A}})$. Equation (122) follows from applying $\boldsymbol{\sigma}(I)_*$ or $\boldsymbol{\sigma}(K)_*$ to (106), (109) and (111)–(120), and (123) from (112) and (121), as for (95). Finally, Corollary 8.3 implies $\pi_{\mathfrak{Obj}_{\mathcal{A}}}^{\rm stk}$ induces surjective maps $\overline{\mathcal{H}}_\tau^{\rm pa} \to \mathcal{H}_\tau^{\rm pa}$ and $\overline{\mathcal{H}}_\tau^{\rm to} \to \mathcal{H}_\tau^{\rm to}$, which are $\mathbb{Q}$-algebra morphisms as (14) is. □

The multiplication relations in $\overline{\mathcal{H}}_\tau^{\rm pa}$ for the six spanning sets $\boldsymbol{\sigma}(I)_* \bar\delta_{\rm ss}, \dots, \bar\delta_{\rm st}^{\rm b}(I, \preccurlyeq, \kappa, \tau)$ are given by the analogues of (86) and (87). That is, for $(I, \preccurlyeq, \kappa)$, $(J, \lesssim, \lambda)$, $(K, \trianglelefteq, \mu)$ as defined before (86), using [10, Theorem 5.4] in place of [10, Theorem 4.22] shows the analogue of (86) holds:

$$\big(\boldsymbol{\sigma}(I)_* \bar\delta_{\rm ss}(I, \preccurlyeq, \kappa, \tau)\big) * \big(\boldsymbol{\sigma}(J)_* \bar\delta_{\rm ss}(J, \lesssim, \lambda, \tau)\big) = \boldsymbol{\sigma}(K)_* \bar\delta_{\rm ss}(K, \trianglelefteq, \mu, \tau).$$

From this and identities (106), (109), (111) and (113)–(120) we can deduce multiplication relations for the $\boldsymbol{\sigma}(I)_* \bar\delta_{\rm si}, \dots, \bar\delta_{\rm st}^{\rm b}(*)$. But as (106)–(120) are analogues of constructible functions identities these relations are exactly the analogues of the constructible function relations (86)–(87).

Next we extend the Lie algebra material of Section 7. The following will be a key tool in proving elements of $\overline{\mathcal{H}}_\tau^{\rm pa}, \overline{\mathcal{H}}_\tau^{\rm to}$ lie in the Lie algebra $\mathrm{SF}_{\rm al}^{\rm ind}(\mathfrak{Obj}_{\mathcal{A}})$.

Theorem 8.6. *In Definition* 8.1 *we have* $\bar{\delta}_{\mathrm{si}}^{\alpha}(\tau)=\Pi_1^{\mathrm{vi}}(\bar{\delta}_{\mathrm{ss}}^{\alpha}(\tau))$.

Proof. We shall combine (105) with the definition of Π_1^{vi} in [8, §5.2], and show that the resulting formula for $\Pi_1^{\mathrm{vi}}(\bar{\delta}_{\mathrm{ss}}^{\alpha}(\tau))$ agrees term-by-term with the definition of $\bar{\delta}_{\mathrm{si}}^{\alpha}(\tau)$ in (107). Apply [10, Proposition 5.7] with the constructible set $S\subseteq\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ equal to $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$. This gives a finite decomposition $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)=\coprod_{l\in L}\mathfrak{F}_l(\mathbb{K})$ and 1-isomorphisms $\mathfrak{F}_l\cong[U_l/A_l^{\times}]$, for U_l a quasiprojective $\mathbb{K}$-variety and A_l a finite-dimensional $\mathbb{K}$-algebra, such that if $u\in U_l(\mathbb{K})$ projects to $[X]\in\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ then there exists a subalgebra B_u of A_l with $\mathrm{Stab}_{A_l^{\times}}(u)=B_u^{\times}$ and an isomorphism $B_u\cong\mathrm{End}(X)$ compatible with $\mathrm{Stab}_{A_l^{\times}}(u)\cong\mathrm{Aut}(X)$.

Write $\rho_l:[U_l/A_l^{\times}]\to\mathfrak{Obj}_{\mathcal{A}}$ for the composition of $\mathfrak{F}_l\cong[U_l/A_l^{\times}]$ and the inclusion $\mathfrak{F}_l\to\mathfrak{Obj}_{\mathcal{A}}$. Then the definition [8, Definition 3.2] of $\bar{\delta}_C$ implies that

$$\bar{\delta}_{\mathrm{ss}}^{\alpha}(\tau)=\sum_{l\in L}\big[\big([U_l/A_l^{\times}],\rho_l\big)\big].\tag{124}$$

There exists a subalgebra C_l of A_l isomorphic as an algebra to $\mathbb{K}^{r_l}$, where $r_l=\mathrm{rk}\,A_l^{\times}$, and $C_l^{\times}\cong(\mathbb{K}^{\times})^{r_l}$ is a maximal torus of $A_l^{\times}$. If $u\in U_l(\mathbb{K})$ then $\mathrm{Stab}_{A_l^{\times}}(u)\cap C_l^{\times}=D_u^{\times}$, where $D_u=B_u\cap C_l$ is a subalgebra of C_l, for B_u as above. It is now easy to see, in the notation of [8, §5.2], that

$$\mathcal{P}\big(U_l,C_l^{\times}\big),\mathcal{Q}\big(A_l^{\times},C_l^{\times}\big),\mathcal{R}\big(U_l,A_l^{\times},C_l^{\times}\big)\subseteq\big\{D^{\times}:\ D\subseteq C_l\text{ a subalgebra}\big\}.\tag{125}$$

It is a consequence of the proof in [8, §5] that the definition of Π_1^{vi} is independent of choices, that in defining Π_1^{vi} we can replace $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}(\cdots)$ by larger sets of $\mathbb{K}$-subgroups of $C_l^{\times}$ closed under intersection. So, we can define $\Pi_1^{\mathrm{vi}}(\bar{\delta}_{\mathrm{ss}}^{\alpha}(\tau))$ using the representation (124) and replacing the left-hand side of (125) by the right-hand side of (125). This involves a sum over $l\in L$ and P, Q, R in the right-hand side of (125) with $R\subseteq P\cap Q$ and $\dim R=1$ of a term with coefficient $M_{A_l^{\times}}^{U_l}(P,Q,R)$.

We can simplify this sum in four ways. Firstly, the only R in the right-hand side of (125) with $\dim R=1$ is $\{\lambda\,\mathrm{id}_{C_l}:\ \lambda\in\mathbb{K}^{\times}\}$, so we fix R to be this. Secondly, by [8, Lemma 5.9] if $M_{A_l^{\times}}^{U_l}(P,Q,R)\neq 0$ then P, Q are the smallest elements of their sets containing $P\cap Q$, so as P, Q take values in the same set we can restrict to $P=Q=D^{\times}$. Thirdly, if $D\subseteq C_l\cong\mathbb{K}^{r_l}$ is a subalgebra with $\dim D=n$ then $D\cong\mathbb{K}^n$ and explicit calculation with the definitions of [8, §5.2] shows that

$$M_{A_l^{\times}}^{U_l}\big(D^{\times},D^{\times},R\big)=\left|\frac{N_{A_l^{\times}}(C_l^{\times})}{C_{A_l^{\times}}(D^{\times})\cap N_{A_l^{\times}}(C_l^{\times})}\right|^{-1}\cdot(-1)^n(n-1)!,$$

computing $M_{A_l^{\times}}^{U_l}(\cdots)$ with the right-hand side of (125) in place of $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}(\cdots)$. Fourthly, we choose an algebra isomorphism $\mu:\mathbb{K}^n\to D$. The number of such isomorphisms is $n!$, so to compensate we divide by $n!$, which together with the factor $(-1)^n(n-1)!$ above yields $(-1)^n/n$. Combining these simplifications yields:

$$\Pi_1^{\mathrm{vi}}\big(\bar\delta_{\mathrm{ss}}^\alpha(\tau)\big)=\sum_{n\geqslant 1}\frac{(-1)^n}{n}\cdot\Bigg[\sum_{l\in L}\sum_{\substack{\text{injective algebra}\\ \text{morphisms}\\ \mu:\mathbb{K}^n\to C_l}}\big|N_{A_l^\times}(C_l^\times)/C_{A_l^\times}\big(\mu\big((\mathbb{K}^\times)^n\big)\big)\cap N_{A_l^\times}(C_l^\times)\big|^{-1}\cdot\Big[\Big(\Big[U_l^{\mu((\mathbb{K}^\times)^n)}/C_{A_l^\times}\big(\mu\big(\mathbb{K}^\times\big)^n\big)\Big],\rho_l\circ\iota^{\mu((\mathbb{K}^\times)^n)}\Big)\Big]\Bigg].\tag{126}$$

Let n, l, μ be as in (126) and $u\in U_l^{\mu((\mathbb{K}^\times)^n)}$ project to $[X]\in\mathfrak{F}_l(\mathbb{K})\subseteq\mathfrak{Obj}_{\mathcal A}(\mathbb{K})$. Then the morphism $\mu:(\mathbb{K}^\times)^n\to\mathrm{Stab}_{A_l^\times}(u)\cong\mathrm{Aut}(X)$ induces a splitting $X\cong X_1\oplus\cdots\oplus X_n$, with $\mu(\gamma_1,\dots,\gamma_n)\cong\gamma_1\,\mathrm{id}_{X_1}+\cdots+\gamma_n\,\mathrm{id}_{X_n}$, and $X_i\ncong 0$. Conversely, one can show that any $[X]\in\mathfrak{F}_l(\mathbb{K})$ and splitting $X\cong X_1\oplus\cdots\oplus X_n$ with $X_i\ncong 0$ come from such μ,u, and the possible choices of μ are all conjugate under the Weyl group $W_{A_l^\times}$ of $A_l^\times$, and having chosen μ the possible choices of u form a $C_{A_l^\times}(\mu((\mathbb{K}^\times)^n))$-orbit in $U_l(\mathbb{K})$. The orbit of μ under $W_{A_l^\times}$ is finite and isomorphic to $N_{A_l^\times}(C_l^\times)/C_{A_l^\times}(\mu((\mathbb{K}^\times)^n))\cap N_{A_l^\times}(C_l^\times)$.

Now a splitting $X\cong X_1\oplus\cdots\oplus X_n$ is equivalent up to canonical isomorphism to a $(\{1,\dots,n\},\bullet)$-configuration (σ,ι,π) with $\sigma(\{1,\dots,n\})=X$ and $\sigma(\{i\})=X_i$. Thus we see that the bottom line $[\cdots]$ of (126) is equal as a stack function to $[(\mathfrak{G}_n^\alpha,\boldsymbol\sigma(\{1,\dots,n\}))]$, where $\mathfrak{G}_n^\alpha$ is the open $\mathbb{K}$-substack of points $[(\sigma,\iota,\pi)]$ in $\mathfrak{M}(\{1,\dots,n\},\bullet)_{\mathcal A}$ with $[\sigma(\{1,\dots,n\})]\in\mathrm{Obj}_{\mathrm{ss}}^\alpha(\tau)$ and $\sigma(\{i\})\ncong 0$ for all $i=1,\dots,n$. The factor $|N_{A_l^\times}(C_l^\times)/C_{A_l^\times}(\mu((\mathbb{K}^\times)^n))\cap N_{A_l^\times}(C_l^\times)|^{-1}$ exactly cancels the multiplicity of choices of μ to make this true.

Let $[(\sigma,\iota,\pi)]\in\mathcal{M}(\{1,\dots,n\},\bullet)_{\mathcal A}$ with $\sigma(\{i\})\ncong 0$ for all $i=1,\dots,n$. Define $\kappa:\{1,\dots,n\}\to C(\mathcal A)$ by $\kappa(i)=[\sigma(\{i\})]$, so that (σ,ι,π) is an $(\{1,\dots,n\},\bullet,\kappa)$-configuration. As $\sigma(\{1,\dots,n\})\cong\sigma(\{1\})\oplus\cdots\oplus\sigma(\{n\})$, it is easy to show that $[\sigma(\{1,\dots,n\})]\in\mathrm{Obj}_{\mathrm{ss}}^\alpha(\tau)$ if and only if $\kappa(\{1,\dots,n\})=\alpha$, $\tau\circ\kappa\equiv\tau(\alpha)$ and $\sigma(\{i\})$ is τ-semistable for all i, that is, $[(\sigma,\iota,\pi)]$ lies in $\mathcal{M}_{\mathrm{ss}}(\{1,\dots,n\},\bullet,\kappa,\tau)_{\mathcal A}$. Thus

$$\mathfrak{G}_n^\alpha(\mathbb{K})=\coprod_{\substack{\kappa:\{1,\dots,n\}\to C(\mathcal A):\\ \kappa(\{1,\dots,n\})=\alpha,\ \tau\circ\kappa\equiv\tau(\alpha)}}\mathcal{M}_{\mathrm{ss}}\big(\{1,\dots,n\},\bullet,\kappa,\tau\big)_{\mathcal A}.$$

Hence $[(\mathfrak{G}_n^\alpha,\boldsymbol\sigma(\{1,\dots,n\}))]$ equals the second sum in (107). But it also equals the bottom line of (126), so comparing (107), (126) completes the proof. □

The theorem enables us to interpret the stack functions $\bar\delta_{\mathrm{si}}^\alpha(\tau),\bar\delta_{\mathrm{si}}(I,\trianglelefteq,\kappa,\tau)$. Since $\bar\delta_{\mathrm{ss}}^\alpha(\tau)$ is the 'characteristic function' of $\mathrm{Obj}_{\mathrm{ss}}^\alpha(\tau)$ and Π_1^{vi} is the projection to stack functions 'supported on virtual indecomposables,' we should understand $\bar\delta_{\mathrm{si}}^\alpha(\tau)$ as the 'characteristic function of τ-semistable virtual indecomposables in class α,' and $\bar\delta_{\mathrm{si}}(I,\trianglelefteq,\kappa,\tau)$ as the 'characteristic function of $(I,\trianglelefteq,\kappa)$-configurations $[(\sigma,\iota,\pi)]$ with each $\sigma(\{i\})$ τ-semistable and virtual indecomposable.' Note that because 'virtual indecomposable' stack functions can have nonzero components over decomposable objects, $\bar\delta_{\mathrm{si}}^\alpha(\tau),\bar\delta_{\mathrm{si}}(I,\trianglelefteq,\kappa,\tau)$ will generally not be supported on $\mathrm{Obj}_{\mathrm{si}}^\alpha(\tau)$, $\mathcal{M}_{\mathrm{si}}(I,\trianglelefteq,\kappa,\tau)_{\mathcal A}$.

It remains to interpret $\bar\delta_{\mathrm{st}}^\alpha(\tau)$ and $\bar\delta_{\mathrm{st}},\bar\delta_{\mathrm{ss}}^{\mathrm b},\bar\delta_{\mathrm{si}}^{\mathrm b},\bar\delta_{\mathrm{st}}^{\mathrm b}(I,\trianglelefteq,\kappa,\tau)$. These are all defined by analogues of constructible functions equations in Sections 5–6 that were proved using $\chi(\mathbb{K}^m)=1$. Since the spaces $\mathrm{SF}(\cdots)$ do not set $[\mathbb{K}^m]=1$, the $\bar\delta^\alpha_{\mathrm{st}}(\tau)$ and $\bar\delta_{\mathrm{st}},\ldots,\bar\delta^{\mathrm{b}}_{\mathrm{st}}(I,\trianglelefteq,\kappa,\tau)$ do *not* have a nice interpretation in $\mathrm{SF}(\cdots)$.

However, in the spaces $\overline{\mathrm{SF}}(\mathfrak{F},\Theta,\Omega)$ the relations do set $[\mathbb{K}^m]=1$, so here the identities have the same interpretations as their constructible function analogues, but using ideas of 'virtual τ-stable' and 'virtual best configuration.' Thus, we interpret $\bar\delta^\alpha_{\mathrm{st}}(\tau)$ in $\overline{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},\Theta,\Omega)$ as the 'characteristic function of virtual τ-stables in class α,' and $\bar\delta^{\mathrm{b}}_{\mathrm{ss}}(I,\trianglelefteq,\kappa,\tau)$ in $\overline{\mathrm{SF}}(\mathfrak{M}(I,\trianglelefteq,\kappa)_{\mathcal{A}},\Theta,\Omega)$ as the 'characteristic function of virtual best $(I,\trianglelefteq,\kappa)$-configurations $[(\sigma,\iota,\pi)]$ with each $\sigma(\{i\})$ τ-semistable,' and so on.

This suggests that if we wish to define invariants 'counting τ-stables in class α' we should apply some linear map to $\bar\delta^\alpha_{\mathrm{st}}(\tau)$ in $\overline{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},\Theta,\Omega)$, but we should not work in larger spaces such as $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$, as the result might not mean what we want it to mean. The same applies to 'counting best configurations.'

Combining Theorem 8.6 with the ideas of Section 7.4 we prove:

Theorem 8.7. *In Definition* 8.1, *for all $k\geqslant 0$ we have*

$$\begin{aligned}&\Pi^{\mathrm{vi}}_k\big(\boldsymbol{\sigma}(I)_*\bar\delta^{\mathrm{b}}_{\mathrm{si}}(I,\trianglelefteq,\kappa,\tau)\big)\\&\quad=\begin{cases}\boldsymbol{\sigma}(I)_*\bar\delta^{\mathrm{b}}_{\mathrm{si}}(I,\trianglelefteq,\kappa,\tau), & (I,\trianglelefteq)\text{ has }k\text{ connected components},\\ 0, & \text{otherwise},\end{cases}\end{aligned}\tag{127}$$

$$\begin{aligned}&\Pi^{\mathrm{vi}}_k\big(\boldsymbol{\sigma}(I)_*\bar\delta^{\mathrm{b}}_{\mathrm{st}}(I,\trianglelefteq,\kappa,\tau)\big)\\&\quad=\begin{cases}\boldsymbol{\sigma}(I)_*\bar\delta^{\mathrm{b}}_{\mathrm{st}}(I,\trianglelefteq,\kappa,\tau), & (I,\trianglelefteq)\text{ has }k\text{ connected components},\\ 0, & \text{otherwise},\end{cases}\end{aligned}\tag{128}$$

supposing $(\tau,T,\leqslant)$ is a stability condition in (128). *Also $\bar\epsilon^\alpha(\tau)\in\mathrm{SF}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$.*

Proof. Make the convention that the constants $C_{\ldots},D_{\ldots},E_{\ldots},F_{\ldots}$ below lie in $\mathbb{Q}$ and depend only on their subscripts up to isomorphism. In [10, Theorem 5.16], if $(I,\trianglelefteq)$ is a finite poset and $f_i\in\mathrm{SF}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ for $i\in I$, we write $\Pi^{\mathrm{vi}}_k(P_{(I,\trianglelefteq)}(f_i\colon i\in I))$ as a $\mathbb{Q}$-linear combination of $P_{(I,\precsim)}(f_i\colon i\in I)$ over partial orders $\precsim$ on I dominated by $\trianglelefteq$. Since $\bar\delta_{\mathrm{si}}(I,\trianglelefteq,\kappa,\tau)=P_{(I,\trianglelefteq)}(\bar\delta^{\kappa(i)}_{\mathrm{si}}(\tau)\colon i\in I)$ and $\bar\delta^{\kappa(i)}_{\mathrm{si}}(\tau)\in\mathrm{SF}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ by Theorem 8.6, this implies a universal formula

$$\sum_{\substack{\text{p.o.s }\precsim\text{ on }I:\\ \trianglelefteq\text{ dominates }\precsim}}C_{\trianglelefteq,\precsim,k}\cdot\boldsymbol{\sigma}(I)_*\bar\delta_{\mathrm{si}}(I,\precsim,\kappa,\tau)=\Pi^{\mathrm{vi}}_k\big(\boldsymbol{\sigma}(I)_*\bar\delta_{\mathrm{si}}(I,\trianglelefteq,\kappa,\tau)\big).$$

Combining this with (109) and (114) gives

$$\sum_{\substack{\text{p.o.s }\precsim\text{ on }I:\\ \trianglelefteq\text{ dominates }\precsim}}D_{\trianglelefteq,\precsim,k}\cdot\boldsymbol{\sigma}(I)_*\bar\delta^{\mathrm{b}}_{\mathrm{si}}(I,\precsim,\kappa,\tau)=\Pi^{\mathrm{vi}}_k\big(\boldsymbol{\sigma}(I)_*\bar\delta^{\mathrm{b}}_{\mathrm{si}}(I,\trianglelefteq,\kappa,\tau)\big).\tag{129}$$

In [10, Theorem 5.17] we show that if $f\in\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ with $\Pi^{\mathrm{vi}}_1(f)=f$ and $\operatorname{char}\mathbb{K}=0$ then $\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}(f)$ is supported on points $[X]$ for $0\ncong X$ indecomposable. A generalization of the same proof shows that if $\Pi^{\mathrm{vi}}_k(f)=f$ then $\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}(f)$ is supported on points $[X_1\oplus\cdots\oplus X_k]$ for $0\ncong X_a$

indecomposable. Since $(\Pi_k^{\rm vi})^2=\Pi_k^{\rm vi}$, we see that for any $f\in{\rm SF}_{\rm al}(\mathfrak{Obj}_{\mathcal A})$, $\pi^{\rm stk}_{\mathfrak{Obj}_{\mathcal A}}(\Pi_k^{\rm vi}(f))$ is the component of $\pi^{\rm stk}_{\mathfrak{Obj}_{\mathcal A}}(f)$ supported on points $[X_1\oplus\cdots\oplus X_k]$ for $0\not\cong X_a$ indecomposable.

Applying this to $f=\boldsymbol\sigma(I)_*\bar\delta^{\rm b}_{\rm si}(I,\trianglelefteq,\kappa,\tau)$, so that $\pi^{\rm stk}_{\mathfrak{Obj}_{\mathcal A}}(f)={\rm CF}^{\rm stk}(\boldsymbol\sigma(I))\delta^{\rm b}_{\rm si}(I,\trianglelefteq,\kappa,\tau)$ by Corollary 8.3, and using Proposition 7.4 and (129) shows that

$$\begin{aligned}&\sum_{\text{p.o.s }\preceq\text{ on }I:\ \trianglelefteq\text{ dominates }\preceq} D_{\trianglelefteq,\preceq,k}\cdot{\rm CF}^{\rm stk}\bigl(\boldsymbol\sigma(I)\bigr)\delta^{\rm b}_{\rm si}(I,\preceq,\kappa,\tau)\\&\quad=\pi^{\rm stk}_{\mathfrak{Obj}_{\mathcal A}}\bigl[\Pi_k^{\rm vi}\bigl(\boldsymbol\sigma(I)_*\bar\delta^{\rm b}_{\rm si}(I,\trianglelefteq,\kappa,\tau)\bigr)\bigr]\\&\quad=\begin{cases}{\rm CF}^{\rm stk}(\boldsymbol\sigma(I))\delta^{\rm b}_{\rm si}(I,\trianglelefteq,\kappa,\tau), & (I,\trianglelefteq)\text{ has }k\text{ connected components},\\ 0, & \text{otherwise}.\end{cases}\end{aligned}\tag{130}$$

Now the difference between the top and bottom lines of (130) is a *universal linear relation* on the ${\rm CF}^{\rm stk}(\boldsymbol\sigma(I))\delta^{\rm b}_{\rm si}(I,\preceq,\kappa,\tau)$. Theorem 7.12 shows that there exist no such universal linear relations with nonzero coefficients. Therefore $D_{\trianglelefteq,\preceq,k}$ is 1 when $\trianglelefteq=\preceq$ and $(I,\trianglelefteq)$ has k connected components, and 0 otherwise. Equation (127) now follows from (129).

Next we prove (128). Substituting (32) into (38) into (64) into (75) and applying ${\rm CF}^{\rm stk}(\boldsymbol\sigma(I))$ gives a universal formula

$$\begin{aligned}&\sum_{\substack{\text{iso. classes of }\mathcal A\text{-data }(J,\lesssim,\lambda)\text{ and}\\ \text{surjective }\phi:J\to I:\ i\lesssim j\text{ implies }\phi(i)\trianglelefteq\phi(j),\\ \lambda(\phi^{-1}(i))=\kappa(k)\text{ for }i\in I,\ \tau\circ\kappa\circ\phi\equiv\tau\circ\lambda}} E_{J,\lesssim,I,\trianglelefteq,\phi}\,{\rm CF}^{\rm stk}\bigl(\boldsymbol\sigma(J)\bigr)\delta^{\rm b}_{\rm si}(J,\lesssim,\lambda,\tau)\\&\quad={\rm CF}^{\rm stk}\bigl(\boldsymbol\sigma(I)\bigr)\delta^{\rm b}_{\rm st}(I,\trianglelefteq,\kappa,\tau).\end{aligned}\tag{131}$$

Fix $(I,\trianglelefteq,\kappa)$ and $(J,\lesssim,\mu)$ in (131), and let $(I,\trianglelefteq)$, $(J,\lesssim)$ have k,l connected components. Then by Proposition 7.4, the terms on the right- and left-hand sides of (131) are supported on points $[X_1\oplus\cdots\oplus X_k]$ and $[Y_1\oplus\cdots\oplus Y_l]$ in $\mathfrak{Obj}_{\mathcal A}(\mathbb K)$, respectively, with all X_a,Y_b indecomposable. So, for fixed $k\neq l$, consider the sum of all terms on the left-hand side of (131) in which $(J,\lesssim)$ has l connected components. This is simply the component of (131) supported on $[Y_1\oplus\cdots\oplus Y_l]$ for Y_b indecomposable, and as $k\neq l$ the right-hand side of (131) is zero on such points. Thus restricting to $(J,\lesssim)$ with l connected components gives a universal identity of the form (103). Theorem 7.12 therefore shows that $E_{J,\lesssim,I,\trianglelefteq,\phi}=0$ if $k\neq l$.

Similarly, substituting (114) into (116) into (106) into (120) and applying $\boldsymbol\sigma(I)_*$ gives the stack function analogue of (131), with the same $E_{J,\lesssim,I,\trianglelefteq,\phi}$. This writes $\boldsymbol\sigma(I)_*\bar\delta^{\rm b}_{\rm st}(I,\trianglelefteq,\kappa,\tau)$ as a linear combination of $\boldsymbol\sigma(J)_*\bar\delta^{\rm b}_{\rm si}(J,\lesssim,\lambda,\tau)$, over $(J,\lesssim)$ with the same number of connected components as $(I,\trianglelefteq)$. But (127) shows $\Pi_k^{\rm vi}$ is the identity on these terms if this number of connected components is k, and 0 otherwise. Equation (128) follows.

Finally, substituting (32) into (38) into (80) into (93) gives an identity

$$\sum_{\substack{\text{iso. classes of }\mathcal A\text{-data }(I,\preceq,\kappa):\\ \kappa(I)=\alpha,\ \tau\circ\kappa\equiv\tau(\alpha)}} F_{I,\preceq}\,{\rm CF}^{\rm stk}\bigl(\boldsymbol\sigma(I)\bigr)\delta^{\rm b}_{\rm si}(I,\preceq,\kappa,\tau)=\epsilon^\alpha(\tau).\tag{132}$$

Using Proposition 7.4 and Theorem 7.8, the same method shows $F_{I,\preceq}=0$ unless $(I,\preceq)$ is connected. Substituting (114) into (116) into the analogue of (80) into (112) gives the stack function version of (132), writing $\bar{\epsilon}^\alpha(\tau)$ as a linear combination of $\boldsymbol{\sigma}(I)_*\bar{\delta}_{\mathrm{si}}^{\mathrm{b}}(I,\preccurlyeq,\kappa,\tau)$ with $(I,\preccurlyeq)$ connected. By (127), Π_1^{vi} is the identity on each term, so $\Pi_1^{\mathrm{vi}}(\bar{\epsilon}^\alpha(\tau))=\bar{\epsilon}^\alpha(\tau)$, and $\bar{\epsilon}^\alpha(\tau)\in \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$. $\square$

By (122) and (127) $\overline{\mathcal{H}}_\tau^{\mathrm{pa}}$ is spanned by eigenvectors of Π_k^{vi}, proving:

Corollary 8.8. *In Definition* 8.4, $\overline{\mathcal{H}}_\tau^{\mathrm{pa}}$ *is closed under* Π_k^{vi} *for all* $k\geqslant 0$.

In general $\overline{\mathcal{H}}_\tau^{\mathrm{to}}$ is not closed under Π_k^{vi} for $k>0$. We can now define and study Lie algebras $\overline{\mathcal{L}}_\tau^{\mathrm{pa}}$, $\overline{\mathcal{L}}_\tau^{\mathrm{to}}$, the analogues of $\mathcal{L}_\tau^{\mathrm{pa}}$, $\mathcal{L}_\tau^{\mathrm{to}}$.

Definition 8.9. Let Assumption 3.7 hold, and $(\tau,T,\leqslant)$ be a permissible weak stability condition on $\mathcal{A}$. Define $\overline{\mathcal{L}}_\tau^{\mathrm{pa}}=\overline{\mathcal{H}}_\tau^{\mathrm{pa}}\cap \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$. Then $\overline{\mathcal{L}}_\tau^{\mathrm{pa}}$ is a *Lie subalgebra* of $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$, since Theorem 8.5 implies $\overline{\mathcal{H}}_\tau^{\mathrm{pa}}$ is a Lie algebra. From (122), (127) and (128) we see that

$$\begin{aligned}\overline{\mathcal{L}}_\tau^{\mathrm{pa}}&=\big\langle \boldsymbol{\sigma}(I)_*\bar{\delta}_{\mathrm{si}}^{\mathrm{b}}(I,\preccurlyeq,\kappa,\tau)\colon (I,\preccurlyeq,\kappa)\ \mathcal{A}\text{-data},\ (I,\preccurlyeq)\text{ connected}\big\rangle_{\mathbb{Q}}\\&=\big\langle \boldsymbol{\sigma}(I)_*\bar{\delta}_{\mathrm{st}}^{\mathrm{b}}(I,\preccurlyeq,\kappa,\tau)\colon (I,\preccurlyeq,\kappa)\ \mathcal{A}\text{-data},\ (I,\preccurlyeq)\text{ connected}\big\rangle_{\mathbb{Q}},\end{aligned}\tag{133}$$

supposing $(\tau,T,\leqslant)$ is a stability condition in the second line. Using (92), (133), Corollary 8.3 and (15) a Lie algebra morphism, we find $\pi_{\mathfrak{Obj}_{\mathcal{A}}}^{\mathrm{stk}}:\overline{\mathcal{L}}_\tau^{\mathrm{pa}}\to\mathcal{L}_\tau^{\mathrm{pa}}$ is a surjective Lie algebra morphism when $\operatorname{char}\mathbb{K}=0$. Also $\overline{\mathcal{L}}_\tau^{\mathrm{pa}}$ generates $\overline{\mathcal{H}}_\tau^{\mathrm{pa}}$ as in Proposition 7.4, so there is a natural, surjective $\mathbb{Q}$-algebra morphism $\overline{\Phi}_\tau^{\mathrm{pa}}:U(\overline{\mathcal{L}}_\tau^{\mathrm{pa}})\to\overline{\mathcal{H}}_\tau^{\mathrm{pa}}$. As we have no analogue of Proposition 3.12 we cannot show $\overline{\Phi}_\tau^{\mathrm{pa}}$ is an isomorphism, but the ideas of Section 7.4 imply there is no nontrivial 'universal' kernel of $\overline{\Phi}_\tau^{\mathrm{pa}}$ generated by universal multiplicative relations on $\overline{\mathcal{L}}_\tau^{\mathrm{pa}}$.

Motivated by Corollary 7.9, and using Theorem 8.7, define $\overline{\mathcal{L}}_\tau^{\mathrm{to}}$ to be the Lie subalgebra of $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ generated by the $\bar{\epsilon}^\alpha(\tau)$ for all $\alpha\in C(\mathcal{A})$. Then $\overline{\mathcal{L}}_\tau^{\mathrm{to}}\subseteq\overline{\mathcal{L}}_\tau^{\mathrm{pa}}$. Using Corollaries 7.9 and 8.3 and (15) a Lie algebra morphism, we see that $\pi_{\mathfrak{Obj}_{\mathcal{A}}}^{\mathrm{stk}}:\overline{\mathcal{L}}_\tau^{\mathrm{to}}\to\mathcal{L}_\tau^{\mathrm{to}}$ is a surjective Lie algebra morphism. Equation (123) implies $\overline{\mathcal{L}}_\tau^{\mathrm{to}}$ generates $\overline{\mathcal{H}}_\tau^{\mathrm{to}}$, so there is a natural, surjective $\mathbb{Q}$-algebra morphism $\overline{\Phi}_\tau^{\mathrm{to}}:U(\overline{\mathcal{L}}_\tau^{\mathrm{to}})\to\overline{\mathcal{H}}_\tau^{\mathrm{to}}$, but as above we cannot prove $\overline{\Phi}_\tau^{\mathrm{to}}$ is an isomorphism. As we have no stack function analogue of Proposition 3.12, and $\overline{\mathcal{H}}_\tau^{\mathrm{to}}$ may not be closed under Π_k^{vi}, we also cannot prove that $\overline{\mathcal{L}}_\tau^{\mathrm{to}}=\overline{\mathcal{H}}_\tau^{\mathrm{to}}\cap\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$.

In [11] we will apply these ideas as follows. Under extra assumptions on $\mathcal{A}$, in [10, §6] we defined (Lie) algebra morphisms $\Phi^\Lambda\circ\Pi_{\mathfrak{Obj}_{\mathcal{A}}}^{\Upsilon,\Lambda},\dots$ from $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ or $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ to some explicit algebras $A(\mathcal{A},\Lambda,\chi),\dots,C(\mathcal{A},\Omega,\chi)$. Restricting these yields (Lie) algebra morphisms from $\overline{\mathcal{H}}_\tau^{\mathrm{pa}}$, $\overline{\mathcal{H}}_\tau^{\mathrm{to}}$ or $\overline{\mathcal{L}}_\tau^{\mathrm{pa}}$, $\overline{\mathcal{L}}_\tau^{\mathrm{to}}$.

We shall regard these maps $\overline{\mathcal{H}}_\tau^{\mathrm{pa}}\to A(\mathcal{A},\Lambda,\chi),\dots$ as encoding *systems of invariants* that 'count' τ-(semi)stable objects and configurations. The fact that the maps are morphisms implies *multiplicative relations* upon these invariants, and also that the map is determined by its values on a generating set for the (Lie) algebra, such as the $\bar{\epsilon}^\alpha(\tau)$ for $\overline{\mathcal{H}}_\tau^{\mathrm{to}}$ or $\overline{\mathcal{L}}_\tau^{\mathrm{to}}$. The identities of Sections 5–7 imply identities on the invariants, and the results of [11] yield *transformation laws* for the invariants between different stability conditions $(\tau,T,\leqslant)$, $(\tilde{\tau},\tilde{T},\leqslant)$.

In particular, if P is a *Calabi–Yau* 3*-fold* and $\mathcal{A}=\mathrm{coh}(P)$, then [10, §6.6] defined a Lie algebra morphism $\Psi^\Omega\circ\overline{\Pi}_{\mathfrak{Obj}_{\mathcal{A}}}^{\Theta,\Omega}:\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})\to C(\mathcal{A},\Omega,\frac{1}{2}\bar{\chi})$. Restricting this to $\overline{\mathcal{L}}_\tau^{\mathrm{pa}}$ and $\overline{\mathcal{L}}_\tau^{\mathrm{to}}$

yields interesting invariants 'counting' τ-semistable sheaves on Calabi–Yau 3-folds, with attractive transformation laws, which may be related to *Donaldson–Thomas invariants*. This is one reward for the work we put in to construct $\overline{\mathcal{L}}_\tau^{\mathrm{pa}}$, $\overline{\mathcal{L}}_\tau^{\mathrm{to}}$ and show they lie in $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$.

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References

[1] D.J. Benson, Representations and Cohomology: I, Cambridge Stud. Adv. Math., vol. 30, Cambridge University Press, Cambridge, 1995.
[2] S.I. Gelfand, Y.I. Manin, Methods of Homological Algebra, second ed., Springer Monogr. Math., Springer-Verlag, Berlin, 2003.
[3] D. Gieseker, On the moduli of vector bundles on an algebraic surface, Ann. of Math. 106 (1977) 45–60.
[4] T.L. Gómez, Algebraic stacks, Proc. Indian Acad. Sci. Math. Sci. 111 (2001) 1–31, math.AG/9911199.
[5] G. Harder, M.S. Narasimhan, On the cohomology groups of moduli spaces of vector bundles on curves, Math. Ann. 212 (1975) 215–248.
[6] D. Huybrechts, M. Lehn, The Geometry of Moduli Spaces of Sheaves, Aspects Math., vol. E31, Vieweg, Braunschweig, 1997.
[7] D.D. Joyce, Constructible functions on Artin stacks, J. London Math. Soc. 74 (2006) 583–606, math.AG/0403305.
[8] D.D. Joyce, Motivic invariants of Artin stacks and 'stack functions', math.AG/0509722, 2005, Q. J. Math., doi: 10.1093/qmath/ham019, in press.
[9] D.D. Joyce, Configurations in abelian categories. I. Basic properties and moduli stacks, Adv. Math. 203 (2006) 194–255, math.AG/0312190.
[10] D.D. Joyce, Configurations in abelian categories. II. Ringel–Hall algebras, Adv. Math. 210 (2007) 635–706, math.AG/0503029.
[11] D.D. Joyce, Configurations in abelian categories. IV. Invariants and changing stability conditions, math.AG/0410268, 2007, version 5.
[12] D.D. Joyce, Holomorphic generating functions for invariants counting coherent sheaves on Calabi–Yau 3-folds, hep-th/0607039, 2006, Geom. Topol., in press.
[13] A.D. King, Moduli of representations of finite-dimensional algebras, Q. J. Math. 45 (1994) 515–530.
[14] A. Langer, Semistable sheaves in positive characteristic, Ann. Math. 159 (2004) 251–276.
[15] G. Laumon, L. Moret-Bailly, Champs algébriques, Ergeb. Math. Grenzgeb., vol. 39, Springer-Verlag, Berlin, 2000.
[16] A. Rudakov, Stability for an abelian category, J. Algebra 197 (1997) 231–245.