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Obvious representations[☆,☆☆]

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ABSTRACT

Under standard assumptions, the respect of obvious dominance characterizes the existence of what we call *obvious representations*. We show that various conditions imposed directly on these representations pin down general monotonic, invariant bi-separable, general Choquet, convex Choquet, neo-additive Choquet, and subjective expected utility preferences. Among other insights, the alternative axiomatizations which we thus provide facilitate comparisons between models, as well as the identification of their parameters. Co-minimal and co-maximal acts — a natural generalization of co-monotonic acts, the relevance of which is further confirmed by our results — play a significant role in our analysis. Our results further testify of the fundamental importance of obvious dominance, and they offer a proof of concept that obvious representations constitute a fruitful format for axiomatic investigations.

1. Introduction

Obvious Dominance is a minimal and intuitive rationality condition that has recently attracted considerable attention in various areas of economics. While this may have started with mechanism design (via the celebrated notion of "obvious strategy-proofness"), this has by now certainly extended to decision theory as well. There, the condition has progressively emerged as perhaps the greatest common denominator — save, of course, the weak order properties — to the models of decision-making under uncertainty most widely used in theory or applications. In this paper, we show that this fundamental status of Obvious Dominance can be further exploited to obtain new insights, some of which could prove experimentally useful, into the structure of the most popular models of decision-making under uncertainty.

To see this, say that a preference relation has an *obvious representation* if it can be represented by

$$V(f) = \alpha(f) \max_{s \in S} u(f(s)) + (1 - \alpha(f)) \min_{s \in S} u(f(s)),$$

where f is a finitely-valued act, u a utility function over the possible consequences of the act, and the α function ranges over $[0, 1]$.[1] One axiomatization for the class of preferences admitting such a representation is offered in Zhang and Levin (2017). Our proposed terminology for this representation is explained by the fact that the crucial property in that axiomatization is the respect of *Obvious Dominance*. This is simply the requirement that every act be at least as good as its worst consequence and at most as good as its best consequence (further see Li (2017) or, under different names such as "Internality" or "Uniform Monotonicity", Blavatskyy (2007), Nascimento and Riella (2010) and Chambers and Echenique (2016)). Clearly this is an extremely minimal rationality condition. Indeed, it is satisfied by the most standard models in the literature, and the above representation can be seen to nest most representations of interest. While this fact appears to be well appreciated, to the best of our knowledge it has not hitherto been leveraged to further investigate the foundations of the traditional models of decision-making under uncertainty.

In this paper, we make first steps in this direction by characterizing several standard models directly in terms of properties of the above α function. In particular we show that various conditions on α pin down general monotonic; invariant bi-separable (IB) preferences (Ghirardato et al., 2004); general Choquet Expected Utility (CEU); its special cases featuring convex (Schmeidler, 1989) and neo-additive (Chateauneuf

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[1] The differences between decision-making under uncertainty and decision-making under ignorance notwithstanding, this can be interpreted as an act-dependent generalization of the so-called Hurwicz criterion (Hurwicz, 1951).

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et al., 2007) capacities; and the benchmark Subjective Expected Utility (SEU) model (Savage, 1954; Anscombe and Aumann, 1963). Like alternative axiomatizations, ours facilitate inter-model comparisons; for instance, our characterizations of neo-additive (Theorem 2) and convex (Theorem 3) CEU compare more easily to each other than extant ones. Our results also facilitate the identification of relevant model parameters; see in particular, once again, our characterization of neo-additive CEU (Theorem 2). Most interestingly perhaps, our findings reveal that in several cases, the behavior of the α function over a remarkably small subset of acts suffices to constrain it globally in distinctive ways; see especially our characterizations of general monotonic (Lemma 1) and independent (Lemma 2) preferences, respectively. As we shall see, co-minimal and co-maximal acts — a natural generalization of co-monotonic acts, already considered under a different name in Chateauneuf et al. (2007) — often play a significant role in this regard. Overall, our results confirm the fundamental importance of Obvious Dominance, which delivers the α function on which our analysis relies. They also offer a proof of concept that obvious representations occupy an interesting intermediary position that — as we explain: for theoretical and experimental reasons alike — is well worth exploring between the primitive level of preferences and the traditional models, such as the ones previously listed, of decision-making under uncertainty.

The rest of the paper is organized as follows. Section 2 gathers various preliminaries. Section 3 contains all the results. A brief conclusion is offered in Section 4. All proofs are relegated to the Appendix.

2. Preliminaries

2.1. Framework

We consider the framework of Anscombe and Aumann (1963). Let S be a *state space* and Σ a σ-algebra on S. Elements of Σ are called *events*. $\Delta(S,\Sigma)$ denotes the set of Σ-measurable probability distribution over S. X denotes the set of *consequences*, which we assume to be the set of all lotteries on some final set of prizes. *Acts* are finite-valued, Σ-measurable mappings from S to X, the set of which we denote by $\mathcal{F}$. With the usual abuse of notation, X also denotes the set of *constant* acts. For $x, y \in X$ and $E \in \Sigma$, $x_E y$ denotes the act which results in x on E and y on $E^c := S\backslash E$. For $f, g \in F$ and $\lambda \in [0,1]$, $\lambda f + (1-\lambda)g$ denotes the act which results in $\lambda f(s)+(1-\lambda)g(s) \in X$ for all $s \in S$. Our primitive is a binary relation $\succcurlyeq$ over $\mathcal{F}$, interpreted as the *preferences* of a decision-maker among acts. Its asymmetric and symmetric parts are denoted $\succ$ and $\sim$, respectively.

We also need the following definitions. An act $f \in \mathcal{F}$ is called *preferentially constant* if $f(s) \sim f(s') \; \forall s \in S$. The set of preferentially constant acts is denoted by $X/\sim$. (Note that $X \subseteq X/\sim$.) Two acts $f, g \in \mathcal{F}$ are called *preferentially complementary* (or complementary for short) if $\frac{1}{2}f + \frac{1}{2}g \in X/\sim$. This notion of complementarity was introduced by Siniscalchi (2009), who highlights the convenient fact that for representable preferences, complementary acts have utility profiles that are "mirror images" (p. 810).[2] For an act $f \in \mathcal{F}$, $\overline{f}$ denotes a (weakly) best consequence and $\underline{f}$ denotes a (weakly) worst consequence of f.[3] Two acts $f, g \in \mathcal{F}$ are called *co-maximal* if they result in a (weakly) best consequence in the same state, i.e. if $\arg\max_{s\in S}\{f(s) \succcurlyeq f(s') \; \forall s' \in S\} \cap \arg\max_{s\in S}\{g(s) \succcurlyeq g(s') \; \forall s' \in S\} \neq \emptyset$. They are called *co-minimal* if they obtain a (weakly) worst consequence in the same state, i.e. if $\arg\min_{s\in S}\{f(s') \succcurlyeq f(s) \; \forall s' \in S\} \cap \arg\min_{s\in S}\{g(s') \succcurlyeq g(s) \; \forall s' \in S\} \neq \emptyset$.[4] Co-maximal and co-minimal acts play a fundamental role in our analysis and, building on Chateauneuf et al. (2007), it is one of our overall purposes to highlight their significance. Acts are called *co-extreme* if they are both co-maximal and co-minimal. Two acts $f, g \in \mathcal{F}$ are called *co-monotonic* if there do not exist states $s, s' \in S$ such that $f(s) \succ f(s')$ and $g(s') \succ g(s)$. Note that if two acts are co-monotonic then they are also co-extreme but the converse does not hold. Indeed co-extremality, co-maximality, and co-minimality are naturally interpreted as generalizations, the relevance of which our results will confirm, of the better known notion of co-monotonicity.

A functional $V : \mathcal{F} \to \mathbb{R}$ is a *representation functional* for $\succcurlyeq$ if for all $f, g \in \mathcal{F}$, $f \succcurlyeq g$ if and only if $V(f) \geq V(g)$. Preferences have a *Choquet expected utility* (CEU) representation if there exists a utility function $u : X \to \mathbb{R}$ and a *capacity* (i.e., a normalized and monotonic set function) $\nu : \Sigma \to [0,1]$ such that $\succcurlyeq$ is represented by

$$V(f) = \int u(f)d\nu,$$

where the integral denotes a Choquet integral (Choquet, 1954). A *non-extreme outcome (NEO)-additive* capacity is a capacity of the form $\nu(E) = \delta\gamma + (1-\delta)\pi(E)$ for all $E \in \Sigma\backslash\{\emptyset, S\}$, with $\delta, \gamma \in [0,1]$ and $\pi \in \Delta(S,\Sigma)$.[5] A capacity ν is called *convex* if $\nu(E) + \nu(F) \leq \nu(E \cup F) + \nu(E \cap F)$ for all $E, F \in \Sigma$. Finally, preferences have a *subjective expected utility* (SEU) representation if there exists a utility function $u : X \to \mathbb{R}$ and a probability distribution $\pi \in \Delta(S,\Sigma)$ such that $\succcurlyeq$ is represented by

$$V(f) = \int u(f)d\pi,$$

where the integral denotes the standard Riemann integral. Throughout the paper, we follow the standard approach in the Anscombe–Aumann framework in axiomatically restricting our attention to the cases where the function $u : X \to \mathbb{R}$ featured in any of the models above is *affine*. This means that over constant acts, that are lotteries, u is linear in probability, as in the benchmark (von Neumann–Morgenstern) model of expected utility under risk. While that restriction will not always be repeated, it will always hold.

2.2. Obvious representations, elicitation of α, and the range

The axiom of Obvious Dominance, stated next, is the requirement that every act be at most as good as its best consequence and at least as good as its worst consequence.

Axiom 1 (*Obvious Dominance*)**.** For all $f \in \mathcal{F}$, $\overline{f} \succcurlyeq f \succcurlyeq \underline{f}$.

In the Anscombe–Aumann framework, Zhang and Levin (2017) show that given some standard assumptions,[6] Obvious Dominance is equivalent to preferences being represented by the very general functional

$$V(f) = \alpha(f)u(\overline{f}) + (1-\alpha(f))u(\underline{f}), \tag{1}$$

with affine $u : X \to \mathbb{R}$ and $\alpha : \mathcal{F} \to [0,1]$. As usual, u is cardinally unique. Furthermore, α is unique on $\mathcal{F}\backslash(X/\sim)$.[7] Without loss of generality, one may adopt the convention $\alpha(f) = \frac{1}{2}$ for all $f \in X/\sim$, thus making α absolutely unique. Henceforth, we refer to preferences

[2] This idea has wide applicability. For instance, in Hartmann (2023), it is demonstrated that complementary pairs can play a crucial role in axiomatizing α-maxmin expected utility preferences.

[3] Throughout the paper, if two or more consequences are tied as best or worst, it is irrelevant which of them is considered.

[4] Note that co-maximality (resp. co-minimality) of two acts merely requires that the maximizing (resp. minimizing) states intersect. It does not imply that indifference obtains amongst the best (resp. worst) consequences of the two acts.

[5] In most of the literature the parameter γ is denoted by the variable α. We change notation to avoid any confusion in the context of our paper, where the symbol α is already used.

[6] These axioms are Weak Order, a Continuity condition, Risk Independence (i.e., the classic von Neumann - Morgenstern Independence axiom from decision theory under risk), and Non-Degeneracy.

[7] For $f \in X/\sim$, any $\alpha \in [0,1]$ results in the same evaluation.

with a representation as in (1) as preferences admitting an *obvious representation* (OR).

It should be emphasized that once the affine utility function u is known (which may be achieved in the usual ways), the function α of OR preferences is easily elicited. Indeed consider an act $f \in \mathcal{F}$ with $\overline{f} \succ \underline{f}$. Then $\alpha(f)$ simply coincides with the number $\alpha \in [0,1]$ such that $f \sim \alpha\overline{f} + (1-\alpha)\underline{f}$, which is well-defined and unique. That is to say,

$$\alpha(f) = \frac{V(f) - u(\underline{f})}{u(\overline{f}) - u(\underline{f})}.$$

This is reminiscent of the standard — specifically: the so-called "standard gamble method" — construction of the utility function in the von Neumann - Morgenstern theorem (von Neumann and Morgenstern, 1947).

Formally, the function α defines a probability. But given the special properties of that probability function — especially the fact that it is act-dependent and that, for each act, its support is restricted to the best and worst consequences of the act — one should be wary of interpreting it naively as encoding the beliefs of the decision-maker about the respective likelihoods of the states of nature. That being said, α nevertheless corresponds, for each act, to the *matching probability* between the best and the worst consequence of that act. More precisely, one usually seeks matching probabilities for binary acts (see esp. Dimmock et al. (2016) and Baillon et al. (2025)). This is done by looking for the lottery that is defined over the two outcomes of the act (that are, trivially, its best and worst consequences) and that is indifferent to that act. By contrast, one here seeks matching probabilities for all acts, non-binary ones included; this is done by looking for the indifferent lottery defined over the best and the worst consequences of that act. Thus, as soon as a model is characterized in terms of the properties of its α function, the familiar matching probability approach provides a simple and tractable methodology for testing or indeed calibrating that model. Crucially, instead of testing the model of interest by verifying whether the standard axioms of classical representation results hold (which can be fairly complicated, and vary in format across models), one can simply use matching probabilities to test whether the characterizing properties of α hold (which seems a good deal simpler, and constitutes an homogeneous approach across models). This is why the theoretical results in our paper — that are precisely, as envisaged above, characterizations of classical decision models in terms of properties of the α function — could prove useful for experimental work as well.

Example 1. For a concrete illustration, we may refer by way of anticipation to our characterization of SEU (Theorem 1.c). That characterization will feature three conditions on the α function, to be introduced shortly in Definitions 2, 5 and 6, respectively. To appreciate the simplicity of the observational implications of these conditions, take e.g. the state space $S = \{s_1, s_2, s_3\}$, assume for simplicity that acts directly map to utilities, and consider the four acts in Table 1.

Table 1
f dominates h, it is co-minimal to g, and it is complementary to f^c.

	s_1	s_2	s_3
f	0	6	4
h	0	5	4
g	2	2	8
f^c	6	0	2

With reference to the above acts, the three characterizing conditions in Theorem 1.c necessitate checking simple equations such as:[8]

$$\alpha(f) \geq \frac{5}{6}\alpha(h), \tag{2}$$

$$\frac{5}{3}\alpha\left(\frac{1}{2}f + \frac{1}{2}g\right) = \alpha(f) + \alpha(g), \tag{3}$$

and

$$\alpha(f) + \alpha(f^c) = 1, \tag{4}$$

respectively. This is easily verified or falsified through matching probabilities.

[8] The first and the last equations exactly correspond to the corresponding characterizing conditions. By contrast, for concreteness, the second equation corresponds to the special case of Definition 5 where $\lambda = \frac{1}{2}$. (Naturally, the condition would need to hold also when $\lambda \neq \frac{1}{2}$).

A definition needed throughout the paper is that of the *range* of an act. We here understand that term as in elementary statistics, where it denotes the difference between the highest and the lowest values in a given data set.

Definition 1 (*Range*)**.** Let $\succcurlyeq$ be an OR preference relation characterized by $u : X \to \mathbb{R}$ and $\alpha : \mathcal{F} \to [0,1]$. The function $r^u_f : \mathcal{F} \to \mathbb{R}$, defined by

$$r^u_f = u(\overline{f}) - u(\underline{f}), \tag{5}$$

is called the range of f.

Even though the range clearly depends on the utility function u, the exact choice of u will prove irrelevant for our purposes in the sense that (since they pertain to induced orders) our results are invariant under cardinal transformations of u. Accordingly, to make notation lighter, henceforth we drop the superscript u and simply refer to the range of f as r_f.

Notice that the range permits an equivalent representation of OR preferences, namely $V(f) = \alpha(f)r_f + u(\underline{f})$. In words, the decision-maker evaluates an act by its minimum value plus an act-specific discount of the increment between its minimal and maximal values. That equivalent representation is particularly intuitive, and it will be used in our proofs wherever convenient.

3. Results

We consider a wide family of models. Fig. 1 specifies these models and their logical relationship. An arrow points towards a more restrictive model.

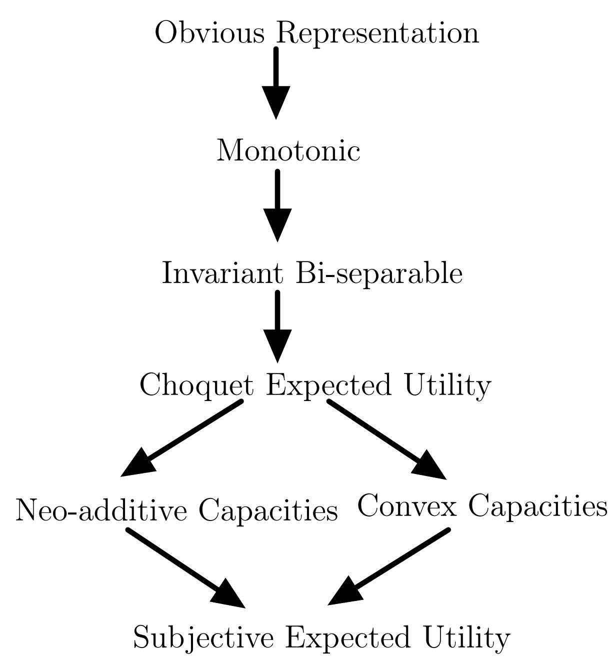

Fig. 1. Overview of the models covered.

To work our way up to the main results regarding these models, stated in Theorem 1 below, we first establish, in two separate sub-sections, two lemmas of independent interest.

3.1. Monotonic preferences

We start by considering the classic Monotonicity axiom.

Axiom 2 (*Monotonicity*)**.** For all $f, g \in \mathcal{F}$,

$$\text{if } f(s) \succcurlyeq g(s) \text{ for all } s \in S, \text{ then } f \succcurlyeq g.$$

To analyze Monotonicity, we introduce the following property. It constrains the value of α over an extremely small class of acts, namely those pairs of acts that not only (i) are among those over which Monotonicity has a bite, but also (ii) have equally good best or worst consequences.

Definition 2 (*Range-Dominance*)**.** The function $\alpha : \mathcal{F} \to \mathbb{R}$ is called range-dominant if for all $f, g \in \mathcal{F}$ such that $f(s) \succcurlyeq g(s)$ for all $s \in S$:

- $\underline{f} \sim \underline{g}$ implies $\alpha(f) r_f \geq \alpha(g) r_g$.
- $\overline{g} \sim \overline{f}$ implies $(1 - \alpha(g)) r_g \geq (1 - \alpha(f)) r_f$.

Lemma 1 states that the Range-Dominance of α is equivalent to the Monotonicity of $\succcurlyeq$.

Lemma 1 (*Monotonicity*)**.** *Let $\succcurlyeq$ be an OR preference relation on $\mathcal{F}$ characterized by $\alpha : \mathcal{F} \to [0, 1]$. Then the following are equivalent:*

- *α is range-dominant.*
- *$\succcurlyeq$ is monotonic.*

What may be most noteworthy in Lemma 1 is that Range-Dominance is not only necessary but also sufficient for preferences to satisfy Monotonicity. Indeed, when $f(s) \succcurlyeq g(s)$ for all $s \in S$ but neither $\underline{f} \sim \underline{g}$ nor $\overline{f} \sim \overline{g}$ holds, Range-Dominance still happens to imply that $f \succcurlyeq g$. Thus, to evaluate whether Monotonicity holds, surprisingly, attention can be restricted to acts such that either $\underline{f} \sim \underline{g}$ or $\overline{f} \sim \overline{g}$.

3.2. Independence conditions

Going beyond the mere requirement of Monotonicity, the majority of the axiomatic results in the literature on decision-making under uncertainty traditionally refer to independence axioms. The most important ones are listed next, in order of increasing logical strength.

Axiom 3 (*Certainty Independence*)**.** For all $f, g \in \mathcal{F}$, $x \in X$ and $\lambda \in (0, 1)$,

$$f \succcurlyeq g \iff \lambda f + (1 - \lambda) x \succcurlyeq \lambda g + (1 - \lambda) x.$$

Axiom 4 (*Co-Monotonic Independence*)**.** For all pairwise co-monotonic $f, g, h \in \mathcal{F}$ and $\lambda \in (0, 1)$,

$$f \succcurlyeq g \iff \lambda f + (1 - \lambda) h \succcurlyeq \lambda g + (1 - \lambda) h.$$

Axiom 5 (*Independence*)**.** For all $f, g, h \in \mathcal{F}$ and $\lambda \in (0, 1)$,

$$f \succcurlyeq g \iff \lambda f + (1 - \lambda) h \succcurlyeq \lambda g + (1 - \lambda) h.$$

We now show that the above independence axioms are equivalent to various additivity conditions placed on the function α. Accordingly, testing these conditions is equivalent to testing the axioms. To formulate these conditions, we introduce two new definitions.

The definition that will play the biggest role is the first one, *Range-Additivity*. The definition is schematic in that it refers to an additivity property holding at certain pairs of acts (f, g), with the relevant pair — e.g., co-monotonic or co-minimal acts — to vary along our analysis. Schematically, given any acts $f, g \in \mathcal{F}$, we will call a function $\alpha : \mathcal{F} \to [0, 1]$ *range-additive at (f, g)* if for all $\lambda \in [0, 1]$,

$$\alpha(\lambda f + (1 - \lambda) g)\, r_{\lambda f + (1-\lambda) g} = \lambda \alpha(f) r_f + (1 - \lambda) \alpha(g) r_g. \tag{6}$$

To better understand this idea, assume for non-triviality that $r_{\lambda f + (1-\lambda) g} > 0$. Then (6) implies that

$$\alpha(\lambda f + (1 - \lambda) g) = \lambda \frac{r_f}{r_{\lambda f + (1-\lambda) g}} \alpha(f) + (1 - \lambda) \frac{r_g}{r_{\lambda f + (1-\lambda) g}} \alpha(g).$$

Thus, range-additivity at (f, g) requires that the α-value of any convex combination of f and g coincides with a weighted average of the α-values of f and g.[9] This is a consistency condition of sorts—the stringency of which naturally depends on the richness of the class of acts over which it is imposed.

The following definitions have the range-additivity condition hold over increasingly more general pairs of acts.

Definition 3 (*Constant Range-Additivity*)**.** A function $\alpha : \mathcal{F} \to [0, 1]$ is called constant range-additive if for all $f \in \mathcal{F}$, $x \in X$ and $\lambda \in [0, 1]$,

$$\alpha(\lambda f + (1 - \lambda) x)\, r_{\lambda f + (1-\lambda) x} = \lambda \alpha(f) r_f + (1 - \lambda) \alpha(x) r_x. \tag{7}$$

Remark 1. Note that $r_x = 0$ and $r_{\lambda f + (1-\lambda) x} = \lambda r_f$, so that (7) is equivalent to $\alpha(\lambda f + (1 - \lambda) x) = \alpha(f)$.

Definition 4 (*Co-Monotonic Range-Additivity*)**.** A function $\alpha : \mathcal{F} \to [0, 1]$ is called co-monotonic range-additive if for all co-monotonic $f, g \in \mathcal{F}$ and $\lambda \in [0, 1]$,

$$\alpha(\lambda f + (1 - \lambda) g)\, r_{\lambda f + (1-\lambda) g} = \lambda \alpha(f) r_f + (1 - \lambda) \alpha(g) r_g. \tag{8}$$

Definition 5 (*Co-Minimal Range-Additivity*)**.** A function $\alpha : \mathcal{F} \to [0, 1]$ is called co-minimal range-additive if for all co-minimal $f, g \in \mathcal{F}$ and $\lambda \in [0, 1]$,[10]

$$\alpha(\lambda f + (1 - \lambda) g)\, r_{\lambda f + (1-\lambda) g} = \lambda \alpha(f) r_f + (1 - \lambda) \alpha(g) r_g. \tag{9}$$

The second definition we need is another kind of additivity condition, one which concerns preferentially complementary acts exclusively.

Definition 6 (*Complementary Additivity*)**.** A function $\alpha : \mathcal{F} \to [0, 1]$ is called complementary additive if for all preferentially complementary pairs (f, f'), $\alpha(f) + \alpha(f') = 1$.[11]

Both of our range-additivity and our complementary additivity properties will be later, when necessary, generalized to sub- and super-additivity properties.

Our next Lemma states equivalences that are crucial for our main results.

Lemma 2. *Let $\succcurlyeq$ be an OR preference relation on $\mathcal{F}$ characterized by $\alpha : \mathcal{F} \to [0, 1]$. Then :*

(a) $\succcurlyeq$ satisfies Certainty Independence if and only if α is Constant Range-Additive.
(b) $\succcurlyeq$ satisfies Co-Monotonic Independence if and only if α is Co-Monotonic Range-Additive.
(c) $\succcurlyeq$ satisfies Independence if and only if α is Co-Minimal Range-Additive and Complementary Additive.

[9] It should be noted however that these weights need not sum up to 1. Indeed, while $\lambda \frac{r_f}{r_{\lambda f + (1-\lambda) g}} + (1 - \lambda) \frac{r_g}{r_{\lambda f + (1-\lambda) g}} \geq 1$ always holds (simply because so does $r_{\lambda f + (1-\lambda) g} \leq \lambda r_f + (1 - \lambda) r_g$), equality is satisfied if and only if f and g are co-extreme.

[10] Some readers might expect us to similarly define Co-Maximal Range-Additivity; on which see Remark 2 below.

[11] Recall that we have conventionally set $\alpha(f) = \frac{1}{2}$ for all $f \in X/\sim$.

Lemma 2 thus establishes that the α function of OR preferences can efficiently encode the various independence conditions of decision-making under uncertainty. Beyond the general proof of concept across the three cases, the equivalence in (c) may be the most remarkable result in the Lemma. This is because whereas Independence ranges over all acts, the equivalent properties of the OR representation pose restrictions over co-minimal and complementary acts only. Thus, considering this very specific kind of acts suffices to check whether the classical Independence axiom holds for all acts.[12]

Remark 2. Note that, given Complementary Additivity, Co-Minimal Range-Additivity is equivalent to a similar condition involving, instead, co-maximal acts; see Lemma 7 in the Appendix.

3.3. The OR representation of models

The combination of Lemmas 2 and 1 are the crucial insights behind the model characterizations which we now provide. The results also provide unsurprising but appreciably simple identification methods for the parameters of CEU and SEU. Following Fig. 1, we start by a characterization of *invariant biseparable* (IB) preferences (Ghirardato et al., 2004), i.e. preferences satisfying Monotonicity as well as Certainty Independence.[13]

Theorem 1 (*IB, CEU, SEU*). *Let $\succcurlyeq$ be an OR preference relation on $\mathcal{F}$ characterized by $\alpha : \mathcal{F} \to [0,1]$ Then:*

(a) $\succcurlyeq$ is IB if and only if α is range-dominant and constant range-additive.
(b) $\succcurlyeq$ is CEU if and only if α is range-dominant and co-monotonic range-additive.
Furthermore, for all events $E \notin \{\emptyset, S\}$, $v(E) = \alpha(x_E y)$ for $x \succ y$.
(c) $\succcurlyeq$ is SEU if and only if α is range-dominant, co-minimal range-additive, and complementary additive.
Furthermore, for all events $E \notin \{\emptyset, S\}$, $\pi(E) = \alpha(x_E y)$ for $x \succ y$.

Remark 3. In Theorem 1.c, Range-Dominance could be omitted. This is because as shown in Lemma 8 in the Appendix, Co-Minimal Range-Additivity together with Complementary Additivity implies Range-Dominance, thus making this last property redundant in the context of this theorem.[14]

3.4. Two special cases of CEU

We now extend our exercise to the characterization of two important subclasses of the CEU model that are still more general than the SEU model.

3.4.1. Neo-additive capacities

Preferences which can be represented by a Choquet integral with respect to a neo-additive capacity are introduced and axiomatized in Chateauneuf et al. (2007). The authors show that such preferences are represented by the functional

$$V(f) = \delta(\gamma u(\overline{f}) + (1-\gamma)u(\underline{f})) + (1-\delta)\int u(f)d\pi.$$

Chateauneuf et al. interpret the parameter π as the probabilistic belief of the decision maker; δ as the perceived ambiguity (alternatively $1-\delta$ as the confidence in π); and γ as the ambiguity attitude. Note that in the neo-additive model, these parameters do not depend on the act considered.

Adapted to our terminology, the characterizing axiom from Chateauneuf et al. (2007) is the following.

Axiom 6 (*Extreme Events Sensitivity*). For all $f, g, h \in \mathcal{F}$, when $f \sim g$ and g, h are co-extreme, then for all $\lambda \in [0,1]$,

$$\text{if } f \text{ and } h \text{ are co-minimal, then } \lambda f + (1-\lambda)h \preccurlyeq \lambda g + (1-\lambda)h \tag{10}$$

and

$$\text{if } f \text{ and } h \text{ are co-maximal, then } \lambda f + (1-\lambda)h \succcurlyeq \lambda g + (1-\lambda)h. \tag{11}$$

Since the model is a special case of CEU, to achieve a characterization of these preferences in term of their OR representation, the Co-Monotonic Range-Additivity condition featured in Lemma 2(b) must be strengthened. Indeed we show that it must be strengthened in two directions, viz. one for co-maximal acts and one for co-minimal acts.[15] We also show that the OR representation offers a simple identification method for the parameters π, δ, and γ.

Definition 7 (*Co-Minimal Range-Subadditivity*). A function $\alpha : \mathcal{F} \to [0,1]$ is called co-minimal range-subadditive if for all co-minimal $f, g \in \mathcal{F}$ and $\lambda \in [0,1]$,

$$\alpha(\lambda f + (1-\lambda)g)\, r_{\lambda f+(1-\lambda)g} \leq \alpha(f)\lambda r_f + \alpha(g)(1-\lambda)r_g. \tag{12}$$

Note that contrary to the other conditions, Definition 8 is stated next, for convenience, in terms of $(1-\alpha)$ rather than α. Accordingly, we say that it pertains to a *reverse* form of additivity.

Definition 8 (*Reverse Co-Maximal Range-Subadditivity*). A function $\alpha : \mathcal{F} \to [0,1]$ is called reverse co-maximal range-subadditive if for all co-maximal $f, g \in \mathcal{F}$ and $\lambda \in [0,1]$,

$$(1-\alpha(\lambda f + (1-\lambda)g))r_{\lambda f+(1-\lambda)g} \leq (1-\alpha(f))\lambda r_f + (1-\alpha(g))(1-\lambda)r_g. \tag{13}$$

Theorem 2 (*NEO*). *Let $\succcurlyeq$ be an OR preference relation characterized by $\alpha : \mathcal{F} \to [0,1]$. Then the following are equivalent:*

- *α is range-dominant, co-monotonic range-additive, co-minimal range-subadditive, and reverse co-maximal range-subadditive.*
- *$\succcurlyeq$ has a CEU representation with respect to a neo-additive capacity.*

Furthermore, the three parameters of the capacity can be elicited as follows: for an arbitrary partition $\{E_1, E_2, E_3\}$ of S such that $E_i \neq \emptyset$ for all i and arbitrary $x, y \in X$ such that $x \succ y$,

$$\delta = 1 + \sum_{i=1}^{3} \alpha(x_{E_i} y) - \sum_{\substack{i,j\in\{1,2,3\}\\ i\neq j}} \alpha(x_{E_i \cup E_j} y). \tag{14}$$

When $\delta > 0$,

$$\gamma = \frac{\alpha(x_{E_1} y) + \alpha(x_{E_2} y) - \alpha(x_{E_1\cup E_2} y)}{\delta}. \tag{15}$$

When $\delta = 0$, γ plays no role. When $\delta \in (0,1)$, for all $E \in \Sigma \backslash \{\emptyset, S\}$,

$$\pi(E) = \frac{\alpha(x_E y) - \alpha(x_{E_1} y) - \alpha(x_{E_2} y) + \alpha(x_{E_1\cup E_2} y)}{1-\delta}. \tag{16}$$

When $\delta = 0$, (16) reduces to $\pi(E) = \alpha(x_E y)$ for all $E \in \Sigma\backslash\{\emptyset, S\}$. When $\delta = 1$, π plays no role.

[12] In light of (a) and (b), some readers may expect in (c) the stronger requirement of range-additivity for *all* pairs of acts. However, this would be too strong; i.e., this is not implied by the Independence axiom. To see this, consider e.g. the case where $f \notin X/\sim$, g is complementary to f, and $\lambda = \frac{1}{2}$. Then (6) directly implies $\alpha(f) = \alpha(g) = 0$, thus contradicting Complementary Additivity, while that axiom is implied by Independence.

[13] For traditional characterizations of IB preferences, see Ghirardato et al. (2004), Amarante (2009) and Chandrasekher et al. (2022).

[14] This complements what was established in Baccelli and Hartmann (2023) (see esp. their Cor. 3.b), namely that in the Anscombe–Aumann axiomatization of SEU, Monotonicity can be replaced by the much weaker assumption of Obvious Dominance.

[15] This is unsurprising inasmuch as the characterizing axiom Extreme Events Sensitivity also has two components, one for co-minimal and the other for co-maximal acts.

Remark 4. In Theorem 2, Co-Monotonic Range-Additivity could be omitted. It is implied by Co-Minimal Range-Subadditivity together with Reverse Co-Maximal Range-Subadditivity.

3.4.2. Convex capacities

As shown by Schmeidler (1989), a capacity is convex if and only if the following uncertainty aversion condition holds.

Axiom 7 (*Uncertainty Aversion*)**.** For all $f, g \in \mathcal{F}$ and $\lambda \in [0,1]$, $f \sim g$ implies $\lambda f + (1-\lambda)g \succcurlyeq g$.

To characterize convex capacities, we need two new properties.

Definition 9 (*Co-Minimal Range-Superadditivity*)**.** A function $\alpha : \mathcal{F} \to [0,1]$ is called co-minimal range-superadditive if for all co-minimal $f, g \in \mathcal{F}$ and $\lambda \in [0,1]$,

$$\alpha(\lambda f + (1-\lambda)g)\, r_{\lambda f + (1-\lambda)g} \geq \alpha(f)\lambda r_f + \alpha(g)(1-\lambda) r_g. \tag{17}$$

Definition 10 (*Complementary Subadditivity*)**.** A function $\alpha : \mathcal{F} \to [0,1]$ is called complementary subadditive if for all complementary pairs (f, f'), $\alpha(f) + \alpha(f') \leq 1$.

Theorem 3 (*Convex Capacities*)**.** *Let $\succcurlyeq$ be an OR preference relation characterized by $\alpha : \mathcal{F} \to [0,1]$. Then the following are equivalent:*

- *α is range-dominant, co-monotonic range-additive, reverse co-maximal range-subadditive, co-minimal range-superadditive, and complementary subadditive.*
- *$\succcurlyeq$ has a CEU representation with respect to a convex capacity.*

Most notably, Theorems 2 and 3 allow an inter-model comparison of the behavior associated with neo-additive and convex capacities. Indeed, our results indicate that to transition from neo-additive to convex capacities (which, the reader will recall, are not comparable from a logical point of view), one needs to switch co-minimal range subadditivity to its superadditive counterpart, and to add complementary subadditivity. Preexisting axiomatizations of these models can be seen to make this specific inter-model comparison less transparent.

4. Conclusion

In this paper, we hope to have established that obvious representations are a useful format for further investigating the foundations of decision-making under uncertainty. Specifically, we have shown that various conditions on the function α characteristic of such representations pin down, facilitate comparisons between, and simplify the measurement of general monotonic, invariant bi-separable, general Choquet, convex Choquet, neo-additive Choquet, and subjective expected utility preferences. To finish, and to stress that these are but first steps in a broader new direction, we single out two questions for future research. First, as we noted along the way, it would be interesting to find an α-characterization of uncertainty aversion that would, unlike the one previously provided in this paper, extend beyond the special case of CEU preferences. Second, it would also be interesting to identify the properties of α that are equivalent to ambiguity aversion in the classical sense of Ghirardato and Marinacci (2002). The associated comparative notion "being more ambiguity averse than" is easily thus characterized, but axiomatizing the absolute notion appears to constitute a more challenging and hence a more interesting problem.

CRediT authorship contribution statement

Lorenz Hartmann: Writing – review & editing, Writing – original draft, Conceptualization, Formal analysis. **Jean Baccelli:** Writing – original draft, Formal analysis, Writing – review & editing, Conceptualization.

Declaration of competing interest

The authors have no competing interests of any kind.

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Appendix

Throughout the appendix, $\succcurlyeq$ is an OR preference relation and $V : \mathcal{F} \to \mathbb{R}$ its OR functional, that is $V(f) = \alpha(f)u(\overline{f}) + (1-\alpha(f))u(\underline{f})$.

Proof of Lemma 1. For brevity, henceforth we write $f \geq_{SD} g$ — with SD standing for "standard dominance" — whenever $f(s) \succcurlyeq g(s)$ for all $s \in S$.

Assume first that $\succcurlyeq$ is monotonic, which implies that V is monotonic. Consider $f, g \in \mathcal{F}$ with $f \geq_{SD} g$. Monotonicity of V implies $V(f) \geq V(g)$, which implies both

$$\alpha(f) r_f + u(\underline{f}) \geq \alpha(g) r_g + u(\underline{g}) \tag{18}$$

and

$$(\alpha(f) - 1) r_f + u(\overline{f}) \geq (\alpha(g) - 1) r_g + u(\overline{g}). \tag{19}$$

First assume that $\underline{f} \sim \underline{g}$, which implies $u(\underline{f}) = u(\underline{g})$. It follows from (18) that

$$\alpha(f) r_f \geq \alpha(g) r_g.$$

Now assume $\overline{f} \sim \overline{g}$, which implies $u(\overline{f}) = u(\overline{g})$. It follows from (19) that

$$(1 - \alpha(f)) r_f \leq (1 - \alpha(g)) r_g.$$

For the other direction we can restrict attention to acts that only differ in one state, as general monotonicity is then implied through what may be intuitively called a staircase argument.[16] So consider $f, g \in \mathcal{F}$ with $f \geq_{SD} g$ such that f, g differ only in one state $s^* \in S$. First assume that $\underline{f} \sim \underline{g}$. Then

$$V(f) = \alpha(f) r_f + u(\underline{f}) \geq \alpha(g) r_g + u(\underline{g}) = V(g).$$

Now assume that $\overline{f} \sim \overline{g}$. Then

$$V(f) = (\alpha(f) - 1) r_f + u(\overline{f}) \geq (\alpha(g) - 1) r_g + u(\overline{g}) = V(g).$$

Now assume that neither $\underline{f} \sim \underline{g}$ nor $\overline{f} \sim \overline{g}$. Then it must hold that $f(s^*) \sim \overline{f}$ and $g(s^*) \sim \underline{g}$. The former must hold, since if $\overline{f} \succ f(s^*)$ were the case, $\overline{g} \sim \overline{f}$ would hold since f, g differ in only one state. Similarly, the latter must hold, since if $g(s^*) \succ \underline{g}$ were the case, $\underline{g} \sim \underline{f}$ would hold since f, g differ in only one state. Consider the act $h \in \mathcal{F}$ resulting in $\underline{f}$ on s^* and f otherwise. Note that $\overline{h} = \overline{g}$, $\underline{h} = \underline{f}$ and furthermore $f \geq_{SD} h \geq_{SD} g$. Thus the two previous cases imply $V(f) \geq V(h) \geq V(g)$ and thus $V(f) \geq V(g)$. □

We continue with a few facts. The simple proofs are omitted.

Lemma 3. *For all $f \in \mathcal{F}$,*

$$V(f) = \alpha(f) r_f + u(\underline{f}) \tag{20}$$

and

$$V(f) = (\alpha(f) - 1) r_f + u(\overline{f}). \tag{21}$$

For all $f, g \in \mathcal{F}$ and $\lambda \in [0,1]$,

$$u(\underline{\lambda f + (1-\lambda) g}) \geq \lambda u(\underline{f}) + (1-\lambda) u(\underline{g}) \tag{22}$$

[16] To illustrate: if $f \geq_{SD} g$ and f, g differ in more than one state, then there always exist acts $h_1, \ldots, h_n$ such that $f \geq_{SD} h_1 \geq_{SD} \cdots \geq_{SD} h_n \geq_{SD} g$ such that the acts only differ in one state in each step.

and

$$u(\overline{\lambda f + (1-\lambda)g}) \leq \lambda u(\overline{f}) + (1-\lambda)u(\overline{g}). \tag{23}$$

Furthermore,

$$r_{\lambda f+(1-\lambda)g} \leq \lambda r_f + (1-\lambda) r_g. \tag{24}$$

f,g are co-minimal if and only if

$$u(\underline{\lambda f + (1-\lambda)g}) = \lambda u(\underline{f}) + (1-\lambda)u(\underline{g}) \tag{25}$$

f,g are co-maximal if and only if

$$u(\overline{\lambda f + (1-\lambda)g}) = \lambda u(\overline{f}) + (1-\lambda)u(\overline{g}). \tag{26}$$

f,g are co-extreme if and only if

$$r_{\lambda f+(1-\lambda)g} = \lambda r_f + (1-\lambda) r_g. \tag{27}$$

If $g \in X/\sim$, then

$$r_{\lambda f+(1-\lambda)g} = \lambda r_f. \tag{28}$$

Before we prove Lemma 2, we prove two lemmas whose statements are well-known to hold under stronger conditions, e.g. for monotonic preferences, but not (as far as we know) for general OR preferences.

Lemma 4. *The following are equivalent:*

- *$\succcurlyeq$ satisfies Certainty Independence.*
- *$V(\lambda f + (1-\lambda)x) = \lambda V(f) + (1-\lambda)V(x)$ for all $f \in \mathcal{F}$, $x \in X$ and $\lambda \in [0,1]$.*

Proof. Consider $f, g \in \mathcal{F}$, $x \in X$ and $\lambda \in (0,1)$. We first show the necessity of Certainty Independence:

$$\begin{aligned} f \succcurlyeq g &\iff V(f) \geq V(g) \\ &\iff \lambda V(f) + (1-\lambda)V(x) \geq \lambda V(g) + (1-\lambda)V(x) \\ &\iff V(\lambda f + (1-\lambda)x) \geq V(\lambda g + (1-\lambda)x) \\ &\iff \lambda f + (1-\lambda)x \succcurlyeq \lambda g + (1-\lambda)x. \end{aligned}$$

For the other direction assume that Certainty Independence holds. Let $y \in X$ be such that $f \sim y$. Such y exists, choose for instance $y = \alpha(f)\overline{f} + (1-\alpha(f))\underline{f}$. Note that Certainty Independence implies in particular $\lambda f + (1-\lambda)x \sim \lambda y + (1-\lambda)x$. Note that the functions V and u coincide on X. Thus

$$\begin{aligned} V(\lambda f + (1-\lambda)x) &= V(\lambda y + (1-\lambda)x) \\ &= u(\lambda y + (1-\lambda)x) \\ &\overset{(*)}{=} \lambda u(y) + (1-\lambda)u(x) \\ &= \lambda V(f) + (1-\lambda)V(x), \end{aligned}$$

where $(*)$ follows from the fact that u is affine. $\square$

Lemma 5. *The following are equivalent:*

- *$\succcurlyeq$ satisfies Co-Monotonic Independence.*
- *$V(\lambda f + (1-\lambda)g) = \lambda V(f) + (1-\lambda)V(g)$ for all co-monotonic $f, g \in \mathcal{F}$ and $\lambda \in [0,1]$.*

Proof. Consider pairwise co-monotonic $f, g, h \in \mathcal{F}$ and $\lambda \in (0,1)$. We first show the necessity of Co-Monotonic Independence:

$$\begin{aligned} f \succcurlyeq g &\iff V(f) \geq V(g) \\ &\iff \lambda V(f) + (1-\lambda)V(h) \geq \lambda V(g) + (1-\lambda)V(h) \\ &\iff V(\lambda f + (1-\lambda)h) \geq V(\lambda g + (1-\lambda)h) \\ &\iff \lambda f + (1-\lambda)h \succcurlyeq \lambda g + (1-\lambda)h. \end{aligned}$$

For the other direction assume that Co-Monotonic Independence holds. Let $x, y \in X$ be such that $f \sim x$ and $g \sim y$. Again, as explained in the proof of Lemma 4, such x, y exist. Note that constant acts are co-monotonic to all acts. Thus Co-Monotonic Independence implies in particular $\lambda f + (1-\lambda)g \sim \lambda x + (1-\lambda)g$ as well as $\lambda x + (1-\lambda)g \sim \lambda x + (1-\lambda)y$. Note once again that the functions V and u coincide on X. Thus

$$\begin{aligned} V(\lambda f + (1-\lambda)g) &= V(\lambda x + (1-\lambda)g) \\ &= V(\lambda x + (1-\lambda)y) \\ &= u(\lambda x + (1-\lambda)y) \\ &\overset{(*)}{=} \lambda u(x) + (1-\lambda)u(y) \\ &= \lambda V(f) + (1-\lambda)V(g), \end{aligned}$$

where $(*)$ follows from the fact that u is affine. $\square$

Proof of Lemma 2(a). Assume first that $\succcurlyeq$ satisfies Certainty Independence. Consider $f \in \mathcal{F}$, $x \in X$ and $\lambda \in [0,1]$. Then

$$\begin{aligned} \alpha(\lambda f + (1-\lambda)x) r_{\lambda f+(1-\lambda)x} &\overset{(20)}{=} V(\lambda f + (1-\lambda)x) - u(\underline{\lambda f + (1-\lambda)x}) \\ &\overset{(*)}{=} \lambda V(f) + (1-\lambda)V(x) - u(\underline{\lambda f + (1-\lambda)x}) \\ &\overset{(25)}{=} \lambda V(f) + (1-\lambda)V(x) - \lambda u(\underline{f}) - (1-\lambda)u(\underline{x}) \\ &= \lambda \alpha(f) r_f + (1-\lambda)\alpha(x) r_x, \end{aligned}$$

where $(*)$ follows from Lemma 4. Note also that $r_x = 0$, thus (28) implies $\alpha(\lambda f + (1-\lambda)x) = \alpha(f)$.

Now assume that α satisfies the required range-additivity condition. Consider $f, g \in \mathcal{F}$, $x \in X$ and $\lambda \in [0,1]$. Then

$$\begin{aligned} f \succcurlyeq g &\iff V(f) \geq V(g) \\ &\overset{(20)}{\iff} \alpha(f) r_f + u(\underline{f}) \geq \alpha(g) r_g + u(\underline{g}) \\ &\iff \lambda(\alpha(f) r_f + u(\underline{f})) + (1-\lambda)\alpha(x) r_x + u(\underline{x}) \\ &\qquad \geq \lambda(\alpha(g) r_g + u(\underline{g})) + (1-\lambda)\alpha(x) r_x + u(\underline{x}) \\ &\overset{(*)}{\iff} \alpha(\lambda f + (1-\lambda)x) r_{\lambda f+(1-\lambda)x} + u(\underline{\lambda f + (1-\lambda)x}) \\ &\qquad \geq \alpha(\lambda g + (1-\lambda)x) r_{\lambda g+(1-\lambda)x} + u(\underline{\lambda g + (1-\lambda)x}) \\ &\overset{(20)}{\iff} V(\lambda f + (1-\lambda)x) \geq V(\lambda g + (1-\lambda)x) \\ &\iff \lambda f + (1-\lambda)x \succcurlyeq \lambda g + (1-\lambda)x, \end{aligned}$$

where $(*)$ follows from the assumed range-additivity as well as (25). $\square$

Proof of Lemma 2(b). Assume first that $\succcurlyeq$ satisfies Co-Monotonic Independence. Consider $f, g \in \mathcal{F}$ co-monotonic and $\lambda \in [0,1]$. Then

$$\begin{aligned} \alpha(\lambda f + (1-\lambda)g) r_{\lambda f+(1-\lambda)g} &\overset{(20)}{=} V(\lambda f + (1-\lambda)g) - u(\underline{\lambda f + (1-\lambda)g}) \\ &\overset{(*)}{=} \lambda V(f) + (1-\lambda)V(g) - u(\underline{\lambda f + (1-\lambda)g}) \\ &\overset{(25)}{=} \lambda V(f) + (1-\lambda)V(g) - \lambda u(\underline{f}) - (1-\lambda)u(\underline{g}) \\ &\overset{(20)}{=} \lambda \alpha(f) r_f + (1-\lambda)\alpha(g) r_g, \end{aligned}$$

where $(*)$ follows from Lemma 5.

Now assume that α satisfies Co-Monotonic Range-Additivity. Consider $f, g, h \in \mathcal{F}$ pairwise co-monotonic and $\lambda \in [0,1]$. Then

$$\begin{aligned} f \succcurlyeq g &\iff V(f) \geq V(g) \\ &\overset{(20)}{\iff} \alpha(f) r_f + u(\underline{f}) \geq \alpha(g) r_g + u(\underline{g}) \\ &\iff \lambda(\alpha(f) r_f + u(\underline{f})) + (1-\lambda)\alpha(h) r_h + u(\underline{h}) \\ &\qquad \geq \lambda(\alpha(g) r_g + u(\underline{g})) + (1-\lambda)\alpha(h) r_h + u(\underline{h}) \\ &\overset{(*)}{\iff} \alpha(\lambda f + (1-\lambda)h) r_{\lambda f+(1-\lambda)h} + \lambda u(\underline{f}) + (1-\lambda)u(\underline{h}) \\ &\qquad \geq \alpha(\lambda g + (1-\lambda)h) r_{\lambda g+(1-\lambda)h} + \lambda u(\underline{g}) + (1-\lambda)u(\underline{h}) \\ &\overset{(25)}{\iff} \alpha(\lambda f + (1-\lambda)h) r_{\lambda f+(1-\lambda)h} + u(\underline{\lambda f + (1-\lambda)h}) \\ &\qquad \geq \alpha(\lambda g + (1-\lambda)h) r_{\lambda g+(1-\lambda)h} + u(\underline{\lambda g + (1-\lambda)h}) \end{aligned}$$

$$\overset{(20)}{\iff} V(\lambda f+(1-\lambda)h) \geq V(\lambda g+(1-\lambda)h)$$
$$\iff \lambda f+(1-\lambda)h \succcurlyeq \lambda g+(1-\lambda)h,$$

where $(*)$ follows from Co-Monotonic Range-Additivity. □

Before we prove Lemma 2(c), we prove some other results.

Lemma 6. *For every act $f \in \mathcal{F}$ there exists an act f' which is preferentially complementary to f.*

Proof. Given f, for all $s \in S$ there exists a unique $\lambda^f(s) \in [0,1]$ such that $f(s) \sim \lambda^f(s)\overline{f}+(1-\lambda^f(s))\underline{f}$. Define $f'(s) = (1-\lambda^f(s))\overline{f}+\lambda^f(s)\underline{f}$. Then $\frac{1}{2}u(f(s)) + \frac{1}{2}u(f'(s)) = \frac{1}{2}u(\overline{f}) + \frac{1}{2}u(\underline{f})$ for all $s \in S$, i.e. it is independent of s. Thus f and f' are preferentially complementary. □

Definition 11 (*Reverse Co-Maximal Range-Additivity*)**.** A function $\alpha : \mathcal{F} \to [0,1]$ is called *reverse co-maximal range-additive if for all co-maximal* $f, g \in \mathcal{F}$ and $\lambda \in [0,1]$,

$$(1-\alpha(\lambda f+(1-\lambda)g))r_{\lambda f+(1-\lambda)g} = (1-\alpha(f))\lambda r_f + (1-\alpha(g))(1-\lambda)r_g. \quad (29)$$

Lemma 7. *Under Complementary Additivity, Co-Minimal Range-Additivity is equivalent to Reverse Co-Maximal Range-Additivity.*

Proof. Assume Co-Minimal Range-Additivity. Consider co-maximal $f, g \in \mathcal{F}$. Let $f', g' \in \mathcal{F}$ be complementary to f and g, respectively, where these complements exist due to Lemma 6. Note that in general, the event in which an act has its best consequence is the same as the one in which the complementary act has its worst consequence. This implies that f', g' are co-minimal. Co-Minimal Range-Additivity implies

$$\alpha(\lambda f'+(1-\lambda)g')r_{\lambda f'+(1-\lambda)g'} = \alpha(f)\lambda r_{f'} + \alpha(g)(1-\lambda)r_{g'}.$$

Note that $\lambda f+(1-\lambda)g$ and $\lambda f'+(1-\lambda)g'$ are also complementary. Furthermore note that the ranges of complementary acts are the same, because for complementary h and h' we have $u(\overline{h})+u(\underline{h'}) = u(\overline{h'})+u(\underline{h})$ and thus $r_h = u(\overline{h}) - u(\underline{h}) = u(\overline{h'}) - u(\underline{h'}) = r_{h'}$. Thus Complementary Additivity implies

$$1-\alpha(\lambda f+(1-\lambda)g)r_{\lambda f+(1-\lambda)g} = (1-\alpha(f))\lambda r_f + (1-\alpha(g))(1-\lambda)r_g.$$

Thus Reverse Co-Maximal Range-Additivity holds. The other direction is shown analogously. □

Lemma 8. *Co-Minimal Range-Additivity and Complementary Additivity implies Range-Dominance.*

Proof. Consider $f, g \in \mathcal{F}$ with $f \geq_{SD} g$. First assume that $\underline{f} \sim \underline{g}$. We need to show $\alpha(f)r_f \geq \alpha(g)r_g$. Let $x \in X$ be such that $u(x) = 0$ (rescaling u to this effect if necessary) and $h \in \mathcal{F}$ be such that $u(h(s)) = u(f(s)) - u(g(s))$ for all $s \in S$. Note that h and g are co-minimal. This is because $f \geq_{SD} g$ implies that $u(h(s)) \geq 0$ for all $s \in S$, furthermore $\underline{f} \sim \underline{g}$ implies that h takes its minimum also in $\arg\{s \in S | g(s') \succcurlyeq g(s)\ \forall\ s' \in S\}$. Furthermore note that $\frac{1}{2}f + \frac{1}{2}x$ and $\frac{1}{2}g + \frac{1}{2}h$ are statewise indifferent, i.e. $\frac{1}{2}u(f(s)) + \frac{1}{2}u(x(s)) = \frac{1}{2}u(g(s)) + \frac{1}{2}u(h(s))$ for all $s \in S$. Due to Lemma 6 there exists an act $h' \in \mathcal{F}$ which is preferentially complementary to $\frac{1}{2}f + \frac{1}{2}x$ and thus also to $\frac{1}{2}g + \frac{1}{2}h$. Complementary Additivity implies $\alpha(\frac{1}{2}f+\frac{1}{2}x)+\alpha(h') = 1 = \alpha(\frac{1}{2}g+\frac{1}{2}h)+\alpha(h')$, thus $\alpha(\frac{1}{2}f+\frac{1}{2}x) = \alpha(\frac{1}{2}g+\frac{1}{2}h)$.

Co-Minimal Range-Additivity implies

$$\begin{aligned}
\alpha(f)r_f &\overset{(*)}{=} 2\alpha\left(\frac{1}{2}f+\frac{1}{2}x\right) r_{\frac{1}{2}f+\frac{1}{2}x} \\
&= 2\alpha\left(\frac{1}{2}g+\frac{1}{2}h\right) r_{\frac{1}{2}g+\frac{1}{2}h} \\
&= 2\left(\frac{1}{2}\alpha(g)r_g + \frac{1}{2}\alpha(h)r_h\right) \\
&\geq \alpha(g)r_g,
\end{aligned}$$

where $(*)$ follows from Lemma 2(a) as well as (28).

For the second part we need to show that $f \geq_{SD}$ and $\overline{f} \sim \overline{g}$ implies $(1-\alpha(g))r_g \geq (1-\alpha(f))r_f$. Lemma 7 implies that we have Reverse Co-Maximal Range-Additivity.

Assume that $\overline{f} \sim \overline{g}$. We need to show $(1-\alpha(f))r_f \leq (1-\alpha(g))r_g$. Let $x \in X$ be such that $u(x) = 0$ as above. Let $h \in \mathcal{F}$ be such that $u(h(s)) = u(g(s)) - u(f(s))$ for all $s \in S$. Then h and f are co-maximal. This is because $f \geq_{SD} g$ implies that $u(h(s)) \leq 0$ for all $s \in S$, furthermore $\overline{f} \sim \overline{g}$ implies that h takes its maximum also in $\arg\{s \in S | f(s) \succcurlyeq f(s')\ \forall\ s' \in S\}$. The acts $\frac{1}{2}f + \frac{1}{2}h$ and $\frac{1}{2}g + \frac{1}{2}x$ are statewise indifferent. Thus as above $\alpha(\frac{1}{2}f + \frac{1}{2}h) = \alpha(\frac{1}{2}g + \frac{1}{2}x)$.

Reverse Co-Maximal Range-Additivity implies

$$\begin{aligned}
(1-\alpha(g))r_g &\overset{(*)}{=} 2(1-\alpha)\left(\frac{1}{2}g+\frac{1}{2}x\right) r_{\frac{1}{2}g+\frac{1}{2}x} \\
&= 2\left(1-\alpha\left(\frac{1}{2}f+\frac{1}{2}h\right)\right) r_{\frac{1}{2}f+\frac{1}{2}h} \\
&= 2\left(\frac{1}{2}(1-\alpha(f))r_f + \frac{1}{2}(1-\alpha(h))r_h\right) \\
&\geq (1-\alpha(f))r_f,
\end{aligned}$$

where $(*)$ follows from Lemma 2(a) as well as (28). □

Proof of Lemma 2(c). Assume first that $\succcurlyeq$ satisfies independence. Then from Corollary 3.b in Baccelli and Hartmann (2023) we know that $\succcurlyeq$ respects not only Obvious Dominance but also Monotonicity, and thus that it has an SEU representation. In particular,

$$V(\lambda f+(1-\lambda)g) = \lambda V(f) + (1-\lambda)V(g) \quad (30)$$

for all $f, g \in \mathcal{F}$ and $\lambda \in [0,1]$.

Now assume that f, g are co-minimal. Then

$$\begin{aligned}
\alpha(\lambda f+(1-\lambda)g)r_{\lambda f+(1-\lambda)g} &\overset{(20)}{=} V(\lambda f+(1-\lambda)g) - u(\underline{\lambda f+(1-\lambda)g}) \\
&\overset{(30)}{=} \lambda V(f) + (1-\lambda)V(g) - u(\underline{\lambda f+(1-\lambda)g}) \\
&\overset{(25)}{=} \lambda V(f) + (1-\lambda)V(g) - \lambda u(\underline{f}) - (1-\lambda)u(\underline{g}) \\
&\overset{(20)}{=} \lambda\alpha(f)r_f + (1-\lambda)\alpha(g)r_g.
\end{aligned}$$

Now assume that f, g are complementary. Note that this implies the existence of some $x \in X$ such that $\frac{1}{2}f(s) + \frac{1}{2}g(s) = u(x)$ for all $s \in S$. Further note that $r_f = r_g$. We have

$$\begin{aligned}
u(x) &= V\left(\frac{1}{2}f+\frac{1}{2}g\right) \\
&\overset{(30)}{=} \frac{1}{2}V(f) + \frac{1}{2}V(g) \\
&\overset{(20)}{=} \frac{1}{2}\left(\alpha(f)r_f + u(\underline{f}) + \alpha(g)r_g + u(\underline{g})\right) \\
&\overset{(*)}{=} \frac{1}{2}\left(\alpha(f)r_f + u(\underline{f}) + \alpha(g)r_g + 2u(x) - u(\overline{f})\right) \\
&= \frac{1}{2}\left(\alpha(f)+\alpha(g)-1\right)r_f + u(x),
\end{aligned}$$

where $(*)$ follows from the fact that $\frac{1}{2}u(\overline{f}) + \frac{1}{2}u(\underline{g}) = u(x)$ and thus $u(\underline{g}) = 2u(x) - u(\overline{f})$. If $r_f = 0$, then f and g are preferentially constant and therefore $\alpha(f) = \alpha(g) = \frac{1}{2}$. If $r_f > 0$, then we must have $\alpha(f)+\alpha(g)-1 = 0$. Thus we have shown the required $\alpha(f)+\alpha(g) = 1$.

Now assume that α satisfies Co-Minimal Range-Additivity as well as Complementary Additivity. Lemma 8 guarantees that α also satisfies Range-Dominance. Thus by Lemma 1 we know that $\succcurlyeq$ is monotonic. Furthermore, Co-Minimal Range-Additivity implies Co-Monotonic Range-Additivity, since co-monotonic acts are always co-minimal. Thus Lemma 2(b) implies that $\succcurlyeq$ has a CEU representation. To show Independence it suffices to show that the representing capacity is additive, since this implies that $\succcurlyeq$ is SEU and thus satisfies Independence.

So let $\nu : \Sigma \to [0,1]$ be this capacity and $u : X \to \mathbb{R}$ the utility function. Throughout we consider $x, y \in X$ such that $u(x) = 1$ and $u(y) = 0$. Such consequences can always be found by rescaling u if necessary. We abuse notation and denote by 1_E0 the act x_Ey, i.e. the act which results in x on E and y on $E^c := S\backslash E$. For all $E, F \in \Sigma$ we must show that $v(E)+v(F) = v(E\cup F)+v(E\cap F)$. We show this by considering four mutually exclusive cases regarding the relationship between E and F. It suffices to consider $E, F \in \Sigma\backslash\{\emptyset, S\}$ since the required equality trivially holds otherwise. Note that $v(E) = V(1_E0) = \alpha(1_E0)$ for all $E \in \Sigma\backslash\{\emptyset, S\}$.

Case 1: $F = E^c$. This case follows immediately from Complementary Additivity of α. Note that 1_E0 and $1_{E^c}0$ are complementary.

$$v(E) + v(F) = V(1_E0) + V(1_{E^c}0) = \alpha(1_E0) + \alpha(1_{E^c}0) = 1 = v(E \cup F).$$

Case 2: $F \cap E = \emptyset$, $E \cup F \neq S$. Here we need co-minimal range-additivity, where we note that 1_E0 and 1_F0 are co-minimal. Denote by $\frac{1}{2}{}_{E\cup F}0 = \frac{1}{2}1_E0 + \frac{1}{2}1_F0$ the act which results in $\frac{1}{2}x + \frac{1}{2}y$ on $E \cup F$ and y otherwise. Note that $r_{\frac{1}{2}{}_{E\cup F}0} = \frac{1}{2}$.

$$\begin{aligned}
v(E) + v(F) =& \alpha(1_E0) + \alpha(1_F0)\\
=& 2\left(\frac{1}{2}\alpha(1_E0)r_{1_E0} + \frac{1}{2}\alpha(1_F0)r_{1_F0}\right)\\
\overset{(*)}{=}& 2\left(\alpha\left(\frac{1}{2}{}_{E\cup F}0\right)r_{\frac{1}{2}{}_{E\cup F}0}\right)\\
\overset{(28)}{=}& \alpha\left(\frac{1}{2}{}_{E\cup F}0\right)\\
\overset{(\dagger)}{=}& \alpha(1_{E\cup F}0)\\
=& V(1_{E\cup F}0)\\
=& v(E \cup F),
\end{aligned}$$

where $(*)$ follows from range-additivity of co-minimal acts and $(\dagger)$ follows from the fact shown in Lemma 2(a) that $\alpha(\lambda f + (1-\lambda)x) = \alpha(f)$ for all $f \in \mathcal{F}$, $x \in X$ and $\lambda \in [0, 1]$.

Case 3: $E \cap F \neq \emptyset$, $E \cup F = S$. This case follows from cases 1 and 2. Note that E^c and F^c fall into case 2.

$$\begin{aligned}
v(E) + v(F) =& 1 - v(E^c) + 1 - v(F^c)\\
=& 2 - v(\underbrace{E^c \cup F^c}_{(E\cap F)^c}) - v(\underbrace{E^c \cap F^c}_{=\emptyset})\\
=& 2 - (1 - v(E \cap F))\\
=& v(E \cup F) + v(E \cap F).
\end{aligned}$$

Case 4: $E \cap F \neq \emptyset$, $E \cup F \neq S$. This case follows by using case 2 twice.

$$\begin{aligned}
v(E) + v(F) =& v(E \cap F) + v(E\backslash F) + v(F)\\
=& v(E \cup F) + v(E \cap F).
\end{aligned}$$

This covers all the possible cases. Thus v is additive. □

We can now prove our main theorem.

Proof of Theorem 1. For (a), it follows from Ghirardato et al. (2004) that an OR preference is IB if and only if it satisfies Monotonicity and Certainty Independence. Lemmas 1 and 2(a) thus imply the required equivalence.

For (b), it follows from Schmeidler (1989) that an OR preferences is CEU if and only if it satisfies Monotonicity and Co-Monotonic Independence. Lemmas 1 and 2(b) thus imply the required equivalence. It remains to be shown that for $x \succ y$, $v(E) = \alpha(x_E y)$ for all $E \in \Sigma\backslash\{\emptyset, S\}$. This follows immediately from $v(E)u(x) + (1 - v(E))u(y) = V(x_E y) = \alpha(x_E y)u(x) + (1 - \alpha(x_E y))u(y)$ and the fact that $u(x) > u(y)$.

For (c), it follows directly from Anscombe and Aumann (1963) that an OR preference is SEU if and only if it satisfies Monotonicity and Independence. Corollary 3.b in Baccelli and Hartmann (2023) implies that Monotoncity is redundant here since it is implied by the OR axioms and Independence. Thus Lemma 2(c) implies the required equivalence. It remains to be shown that for $x \succ y$, $\pi(E) = \alpha(x_E y)$ for all $E \in \Sigma\backslash\{\emptyset, S\}$. This follows immediately from $\pi(E)u(x) + (1 - \pi(E))u(y) = V(x_E y) = \alpha(x_E y)u(x) + (1 - \alpha(x_E y))u(y)$ and the fact that $u(x) > u(y)$. □

Proof of Theorem 2. We first show that the CEU axioms plus Extreme Events Sensitivity imply the required α-properties. Since CEU preferences are monotonic, Lemma 1 implies Range-Dominance. For the other two properties, consider $f, h \in \mathcal{F}$ and $x \in X$ such that $f \sim x$. Note that g and x are co-extreme. Consider $\lambda \in [0, 1]$. First assume that f and h are co-minimal. Then

$$\begin{aligned}
\alpha(\lambda f + (1-\lambda)h)r_{\lambda f+(1-\lambda)h} &\overset{(20)}{=} V(\lambda f + (1-\lambda)h) - u(\underline{\lambda f + (1-\lambda)h})\\
&\overset{(10)}{\leq} V(\lambda x + (1-\lambda)h) - u(\underline{\lambda f + (1-\lambda)h})\\
&\overset{(*)}{=} \lambda V(x) + (1-\lambda)V(h) - u(\underline{\lambda f + (1-\lambda)h})\\
&\overset{(25)}{=} \lambda V(f) + (1-\lambda)V(h) - \lambda u(\underline{f}) - (1-\lambda)u(\underline{h})\\
&\overset{(20)}{=} \lambda\alpha(f)r_f + (1-\lambda)\alpha(h)f_h,
\end{aligned}$$

where $(*)$ follows from Co-Monotonic Range-Additivity. Thus Co-Minimal Range-Subadditivity holds.

Now assume that f and h are co-maximal.

$$\begin{aligned}
(1 - \alpha(\lambda f + (1-\lambda)h))r_{\lambda f+(1-\lambda)h} &\overset{(21)}{=} -V(\lambda f + (1-\lambda)h) + u(\overline{\lambda f + (1-\lambda)h})\\
&\overset{(11)}{=} -V(\lambda x + (1-\lambda)h) + u(\overline{\lambda f + (1-\lambda)h})\\
&\overset{(*)}{=} -\lambda V(x) - (1-\lambda)V(h)\\
&\quad + u(\overline{\lambda f + (1-\lambda)h})\\
&\overset{(26)}{=} -\lambda V(f) - (1-\lambda)V(h)\\
&\quad + \lambda u(\overline{f}) + (1-\lambda)u(\overline{h})\\
&\overset{(20)}{=} \lambda(1 - \alpha(f))r_f + (1-\lambda)(1 - \alpha(h))f_h,
\end{aligned}$$

where $(*)$ follows from Co-Monotonic Range-Additivity. Thus Reverse Co-Maximal Range-Subadditivity holds. This finishes the first part of the proof.

Now assume that α satisfies Range-Dominance, Co-Minimal Range-Subadditivity and Reverse Co-Maximal Range-Subadditivity. Lemma 1 guarantees that $\succcurlyeq$ is monotonic. Consider co-extreme $f, g \in \mathcal{F}$. Note that these acts are also co-minimal and co-maximal. Thus we have

$$\alpha\left(\lambda f + (1-\lambda)g\right)r_{\lambda f+(1-\lambda)g} \overset{(12)}{\leq} \alpha(f)\lambda r_f + \alpha(g)(1-\lambda)r_g \tag{31}$$

as well as

$$(1-\alpha\left(\lambda f + (1-\lambda)g\right))r_{\lambda f+(1-\lambda)g} \overset{(13)}{\leq} (1-\alpha(f))\lambda r_f + (1-\alpha(g))(1-\lambda)r_g. \tag{32}$$

These two equations imply range-additivity for co-extreme acts. In particular we have Co-Monotonic Range-Additivity, since co-monotonic acts are co-extreme. Thus Theorem 1(b) implies that $\succcurlyeq$ has a CEU representation. It thus remains to be shown that Extreme Events Sensitivity holds.

Consider $f, g, h \in \mathcal{F}$ such that $f \sim g$, g and h co-extreme, and $\lambda \in [0, 1]$. Assume first that f and h are co-minimal.

$$\begin{aligned}
V(\lambda f + (1-\lambda)h) &\overset{(20)}{=} \alpha(\lambda f + (1-\lambda)h)r_{\lambda f+(1-\lambda)h} + u(\underline{\lambda f + (1-\lambda)h})\\
&\overset{(12)}{\leq} \lambda\alpha(f)r_f + (1-\lambda)\alpha(h)r_h + u(\underline{\lambda f + (1-\lambda)h})\\
&\overset{(25)}{=} \lambda\alpha(f)r_f + (1-\lambda)\alpha(h)r_h + \lambda u(\underline{f}) + (1-\lambda)u\underline{h}\\
&\overset{(20)}{=} \lambda V(f) + (1-\lambda)V(h)\\
&= \lambda V(g) + (1-\lambda)V(h)\\
&\overset{(*)}{=} V(\lambda g + (1-\lambda)h),
\end{aligned}$$

where $(*)$ follows from the above demonstrated range-additivity for co-extreme acts.

Now assume that f and h are co-maximal.

$$\begin{aligned}
V(\lambda f + (1-\lambda)h) &\overset{(21)}{=} -(1 - \alpha(\lambda f + (1-\lambda)h))r_{\lambda f+(1-\lambda)h}\\
&\quad + u(\overline{\lambda f + (1-\lambda)h})
\end{aligned}$$

$$\begin{aligned}
&\overset{(13)}{\geq} -\lambda(1-\alpha(f))r_f-(1-\lambda)(1-\alpha(h))r_h\\
&\quad +u(\overline{\lambda f+(1-\lambda)h})\\
&\overset{(26)}{=} -\lambda(1-\alpha(f))r_f-(1-\lambda)(1-\alpha(h))r_h\\
&\quad +\lambda u(\overline{f})+(1-\lambda)u\overline{h}\\
&\overset{(21)}{=}\lambda V(f)+(1-\lambda)V(h)\\
&=\lambda V(g)+(1-\lambda)V(h)\\
&\overset{(*)}{=}V(\lambda g+(1-\lambda)h),
\end{aligned}$$

where $(*)$ follows from the above demonstrated range-additivity for co-extreme acts.

It remains to be shown that the formulas for the parameters of the capacity are correct. Let $\{E_1,E_2,E_3\}$ be an arbitrary partition of the state space such that $E_i\neq\emptyset$ for $i\in\{1,2,3\}$. Let $x,y\in X$ be arbitrary consequences such that $x\succ y$. Without loss of generality we can assume that $u(x)=1$ and $u(y)=0$. For any event $E\in\Sigma\backslash\{\emptyset,S\}$ we have $V(x_E y)=v(E)=\delta\gamma+(1-\delta)\pi(E)$. Furthermore $V(x_E y)=\alpha(x_E y)$. Thus we have

$$\alpha(x_E y)=\delta\gamma+(1-\delta)\pi(E). \tag{33}$$

Using (33) it follows that

$$\sum_{i=1}^{3}\alpha(x_{E_i}y)=3\delta\gamma+(1-\delta)\underbrace{\sum_{i=1}^{3}\pi(E_i)}_{=1}$$

as well as

$$\sum_{\substack{i,j\in\{1,2,3\}\\ i\neq j}}\alpha(x_{E_i\cup E_j}y)=3\delta\gamma+(1-\delta)\underbrace{\sum_{\substack{i,j\in\{1,2,3\}\\ i\neq j}}\pi(E_i\cup E_j)}_{=2}.$$

Solving both equations for γ and equating them leads to

$$\frac{\sum_{i=1}^{3}\alpha(x_{E_i}y)-1+\delta}{3\delta}=\frac{\sum_{\substack{i,j\in\{1,2,3\}\\ i\neq j}}\alpha(x_{E_i\cup E_j}y)-2+2\delta}{3\delta}.$$

Solving for δ results in the required $\delta=1+\sum_{i=1}^{3}\alpha(x_{E_i}y)-\sum_{\substack{i,j\in\{1,2,3\}\\ i\neq j}}\alpha(x_{E_i\cup E_j}y)$. Note that $\delta=0$ obtains if and only if $\alpha(x_{E_i}y)+\alpha(x_{E_j}y)=\alpha(x_{E_i\cup E_j}y)$ for all $i,j\in\{1,2,3\}$, $i\neq j$.

Once again using (33), whenever $\delta>0$ we have $\alpha(x_{E_1}y)+\alpha(x_{E_2}y)-\alpha(x_{E_1\cup E_2}y)=\delta\gamma$, thus obtaining the required $\gamma=\frac{\alpha(x_{E_1}y)+\alpha(x_{E_2}y)-\alpha(x_{E_1\cup E_2}y)}{\delta}$. For $E\in\Sigma\backslash\{\emptyset,S\}$, rearranging (33) we get

$$\pi(E)=\frac{\alpha(x_E y)-\delta\gamma}{1-\delta}=\frac{\alpha(x_E y)-\alpha(x_{E_1}y)-\alpha(x_{E_2}y)+\alpha(x_{E_1\cup E_2}y)}{1-\delta}$$

When $\delta=0$, we have $\pi(E)=\frac{\alpha(x_E y)-\delta\gamma}{1-\delta}=\alpha(x_E y)$. □

Proof of Theorem 3. We first show that a CEU representation with respect to a convex capacity induces the required properties of α.

Range-dominance follows from Lemma 1. The other properties follow from Uncertainty Aversion, which is well known to imply that V is concave, i.e.

$$V(\lambda f+(1-\lambda)g)\geq\lambda V(f)+(1-\lambda)V(g) \tag{34}$$

for all $f,g\in\mathcal{F}$ and $\lambda\in[0,1]$. First, let $f,g\in\mathcal{F}$ be complementary. That is $\frac{1}{2}f(s)+\frac{1}{2}g(s)=u(x)$ for all $s\in S$ for some $x\in X$. This implies $r_f=r_g$ and $u(\underline{g})=2u(x)-u(\overline{f})$.

If $r_f=0$, then $\alpha(f)=\alpha(g)=\frac{1}{2}$ by definition, so in particular $\alpha(f)+\alpha(g)\leq 1$. Thus assume $r_f>0$. We have

$$\begin{aligned}
u(x)&=V(\frac{1}{2}f+\frac{1}{2}g)\\
&\overset{(34)}{\leq}\frac{1}{2}V(f)+\frac{1}{2}V(g)\\
&\overset{(20)}{=}\frac{1}{2}\left(\alpha(f)r_f+u(\underline{f})+\alpha(g)r_g+u(\underline{g})\right)\\
&=\frac{1}{2}(\alpha(f)r_f+\alpha(g)r_f+2u(x)+\underbrace{u(\underline{f})-u(\overline{f})}_{=-r_f})\\
&=\frac{1}{2}(\alpha(f)+\alpha(g)-1)r_f+u(x).
\end{aligned}$$

This implies $\alpha(f)+\alpha(g)\leq 1$. Thus Complementary Subadditivity holds.

Now, consider co-minimal $f,g\in\mathcal{F}$ and $\lambda\in[0,1]$. We have

$$\begin{aligned}
\alpha(\lambda f+(1-\lambda)g)\,r_{\lambda f+(1-\lambda)g}&\overset{(20)}{=}V(\lambda f+(1-\lambda)g)-u(\underline{\lambda f+(1-\lambda)g})\\
&\overset{(34)}{\geq}\lambda V(f)+(1-\lambda)V(g)-u(\underline{\lambda f+(1-\lambda)g})\\
&\overset{(25)}{=}\lambda V(f)+(1-\lambda)V(g)-\lambda u(\underline{f})-(1-\lambda)u(\underline{g})\\
&\overset{(20)}{=}\lambda\alpha(f)r_f+(1-\lambda)\alpha(g)r_g.
\end{aligned}$$

We have thus shown Co-Minimal Range-Superadditivity.

Finally, consider co-maximal $f,g\in\mathcal{F}$ and $\lambda\in[0,1]$. We have

$$\begin{aligned}
(1-\alpha(\lambda f+(1-\lambda)g))r_{\lambda f+(1-\lambda)g}&\overset{(21)}{=}-V(\lambda f+(1-\lambda)g)\\
&\quad+u(\overline{\lambda f+(1-\lambda)g})\\
&\overset{(34)}{\leq}-\lambda V(f)-(1-\lambda)V(g)\\
&\quad+u(\overline{\lambda f+(1-\lambda)g})\\
&\overset{(26)}{=}-\lambda V(f)-(1-\lambda)V(g)+\lambda u(\overline{f})\\
&\quad+(1-\lambda)u(\overline{g})\\
&\overset{(21)}{=}\lambda(1-\alpha(f))r_f+(1-\lambda)(1-\alpha(g))r_g.
\end{aligned}$$

We have thus shown Reverse Co-Maximal Range-Subadditivity.

We now show that the stated properties of α imply that $\succcurlyeq$ has a CEU representation with respect to a convex capacity. Theorem 1(b) implies that $\succcurlyeq$ has a CEU representation. Let $v:\Sigma\to[0,1]$ be the capacity and $u:X\to\mathbb{R}$ the utility function. As in Lemma 2(c), we consider $x,y\in X$ such that $u(x)=1$ and $u(y)=0$. Such consequences can always be found by rescaling u if necessary. We abuse notation and denote by 1_E0 the act $x_E y$, i.e. the act which results in x on E and y on $E^c:=S\backslash E$. Note that $v(E)=V(1_E0)=\alpha(1_E0)$ for all $E\in\Sigma\backslash\{\emptyset,S\}$.

To show that the capacity v is convex we must show that $v(E)+v(F)\leq v(E\cap F)+v(E\cup F)$ for all $E,F\in\Sigma$. Again we can assume that $E,F\notin\{\emptyset,S\}$ and consider four cases.

Case 1: $F=E^c$. Noting that the acts 1_E0 and $1_{E^c}0$ are complementary, this case follows immediately from Complementary Subadditivity of α.

$$v(E)+v(F)=V(1_E0)+V(1_{E^c}0)=\alpha(1_E0)+\alpha(1_{E^c}0)\leq 1=v(E\cup F).$$

Case 2: $F\cap E=\emptyset$, $E\cup F\neq S$. Here we use Co-Minimal Range-Superadditivity, where we note that 1_E0 and 1_F0 are co-minimal. Denote by $\frac{1}{2}{}_{E\cup F}0=\frac{1}{2}1_E0+\frac{1}{2}1_F0$ the act which results in $\frac{1}{2}x+\frac{1}{2}y$ on $E\cup F$ and y otherwise. Note that $r_{\frac{1}{2}{}_{E\cup F}0}=\frac{1}{2}$.

$$\begin{aligned}
v(E)+v(F)&=\alpha(1_E0)+\alpha(1_F0)\\
&=2\left(\frac{1}{2}\alpha(1_E0)r_{1_E0}+\frac{1}{2}\alpha(1_F0)r_{1_F0}\right)\\
&\overset{(*)}{\leq}2\left(\alpha\left(\frac{1}{2}{}_{E\cup F}0\right)r_{\frac{1}{2}{}_{E\cup F}0}\right)\\
&=\alpha\left(\frac{1}{2}{}_{E\cup F}0\right)\\
&\overset{(\dagger)}{=}\alpha(1_{E\cup F}0)\\
&=V(1_{E\cup F}0)\\
&=v(E\cup F),
\end{aligned}$$

where $(*)$ follows from range-superadditivity of co-minimal acts and $(\dagger)$ follows from the fact shown in Lemma 2(a) that $\alpha(\lambda f+(1-\lambda)x)=\alpha(f)$ for all $f\in\mathcal{F}$, $x\in X$ and $\lambda\in[0,1]$.

Case 3: $F \cap E \neq \emptyset$, $E \cup F = S$. Here we need Reverse Co-Maximal Range-Subadditivity, where we note that 1_E0 and 1_F0 are co-maximal. Denote by $1_{E\cap F}\frac{1}{2} = \frac{1}{2}1_E0 + \frac{1}{2}1_F0$ the act which results in x on $E \cup F$ and in $\frac{1}{2}x + \frac{1}{2}y$ otherwise. Note that $r_{1_{E\cap F}\frac{1}{2}} = \frac{1}{2}$.

$$\begin{aligned}
v(E) + v(F) =& \alpha(1_E0) + \alpha(1_F0) \\
=& 2 - 2\left(\frac{1}{2}(1-\alpha(1_E0))r_{1_E0} + \frac{1}{2}(1-\alpha(1_F0))r_{1_F0}\right) \\
\overset{(*)}{\leq}& 2 - 2\left(\left(1-\alpha\left(1_{E\cup F}\frac{1}{2}\right)\right) r_{1_{E\cup F}\frac{1}{2}}\right) \\
=& 1 + 2\alpha\left(1_{E\cap F}\frac{1}{2}\right) r_{1_{E\cap F}\frac{1}{2}} \\
\overset{(\dagger)}{=}& 1 + \alpha(1_{E\cap F}0) \\
=& v(E\cup F) + v(E\cap F),
\end{aligned}$$

where $(*)$ follows from Reverse Co-Maximal Range-Subadditivity and $(\dagger)$ follows from the fact, shown in Lemma 2(a), that $\alpha(\lambda f + (1-\lambda)x) = \alpha(f)$ for all $f \in \mathcal{F}$, $x \in X$, and $\lambda \in [0,1]$.

Case 4: $F \cap E \neq \emptyset$, $E \cup F = S$. Note that the acts 1_E0 and 1_F0 are co-minimal. Using somewhat unorthodox notation for brevity, denote by $1_{E\cap F}\frac{1}{2}_{E\cup F}0 = \frac{1}{2}1_E0 + \frac{1}{2}1_F0$ the act which results in x on $E \cap F$, $\frac{1}{2}x + \frac{1}{2}y$ on $(E\cup F)\backslash(E\cap F)$ and y otherwise. Note that $V(1_{E\cap F}\frac{1}{2}_{E\cup F}0) = v(E\cap F) + \frac{1}{2}(v(E\cup F) - v(E\cap F))$.

$$\begin{aligned}
v(E) + v(F) =& \alpha(1_E0) + \alpha(1_F0) \\
=& 2\left(\frac{1}{2}\alpha(1_E0)r_{1_E0} + \frac{1}{2}\alpha(1_F0)r_{1_F0}\right) \\
\overset{(*)}{\leq}& 2\left(\alpha\left(1_{E\cap F}\frac{1}{2}_{E\cup F}0\right) r_{1_{E\cap F}\frac{1}{2}_{E\cup F}0}\right) \\
=& 2V\left(1_{E\cap F}\frac{1}{2}_{E\cup F}0\right) - \underbrace{u\left(\underline{1_{E\cap F}\frac{1}{2}_{E\cup F}0}\right)}_{=0} \\
=& 2\left(v(E\cap F) + \frac{1}{2}(v(E\cup F) - v(E\cap F))\right) \\
=& v(E\cup F) + v(E\cap F),
\end{aligned}$$

This covers all the possible cases. Thus v is convex. □

Data availability

No data was used for the research described in the article.

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