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On Linearly Ordered Sets and Permutation Groups of Uncountable Degree

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For my parents and Simon

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Abstract

In this thesis a set, Ω, of cardinality $\aleph_\kappa$ and a group acting on Ω, with $\aleph_{\kappa+1}$ orbits on the power set of Ω, is found for every infinite cardinal $\aleph_\kappa$.
Let ω_κ denote the initial ordinal of cardinality $\aleph_\kappa$. Define

$$N := \{\alpha_1\alpha_2\ldots\alpha_n \mid 0 < n < \omega,\ \alpha_j \in \omega_\kappa \text{ for } j = 1,\ldots,n, \\ \alpha_n \text{ a successor ordinal}\}$$

$$R := \{x \in N \mid \text{length}(x) \equiv 1 \bmod 2\}$$

and let these sets be ordered lexicographically.

The order types of N and R are $\mathcal{K}$-*types* (countable unions of scattered types) which have cardinality $\aleph_\kappa$ and do not embed ω_1^*. Each interval in N or R embeds every ordinal of cardinality $\aleph_\kappa$ and every countable converse ordinal. N and R then embed every $\mathcal{K}$-type of cardinality $\aleph_\kappa$ with no uncountable descending chains. Hence any such order type can be written as a countable union of well-ordered types, each of order type smaller than ω_κ^ω. In particular, if α is an ordinal between ω_κ^ω and $\omega_{\kappa+1}$, and A is a set of order type α then

$$A = \bigcup_{n<\omega} A_n$$

where each A_n has order type ω_κ^n.

If X is a subset of N with X and $N - X$ dense in N, then X is order-isomorphic to R, whence any dense subset of R has the same order type as R. If Y is any subset of R then R is (finitely) piece-wise order-preserving isomorphic (PWOP) to $R \,\dot\cup\, Y$. Thus there is only one PWOP equivalence class of $\aleph_\kappa$-dense $\mathcal{K}$-types which have cardinality $\aleph_\kappa$, and which do not embed ω_1^*. There are $\aleph_{\kappa+1}$ PWOP equivalence classes of ordinals of cardinality $\aleph_\kappa$. Hence the PWOP automorphisms of R have $\aleph_{\kappa+1}$ orbits on $\mathcal{P}(R)$. The countably piece-wise order-preserving automorphisms of R have $\aleph_0$ orbits on R if $|\kappa|$ is smaller than ω_1 and $|\kappa|$ if it is not smaller.

Contents

Background, Introduction, Preliminaries and Notation

The primary aim of this work is to provide an answer to the following question in infinite group theory.

Suppose $\aleph_\kappa$ is an infinite cardinal, $\aleph_{\kappa+1}$ is its successor cardinal and Ω is a set of size $\aleph_\kappa$. Does there exist a group $G \leqslant \mathrm{Sym}(\Omega)$ such that G has $\aleph_{\kappa+1}$ orbits on the power set of Ω?

This is trivial unless $\aleph_{\kappa+1} < 2^{\aleph_\kappa}$ so the generalised continuum hypothesis will not be assumed.

Background to the Problem

The question above is just one part of the following group-theoretic problem.

Suppose $\aleph_\kappa$ and $\aleph_\lambda$ are infinite cardinals with $\aleph_\kappa \leqslant \aleph_\lambda \leqslant 2^{\aleph_\kappa}$ and Ω is a set of size $\aleph_\kappa$. Does there exist a group $G \leqslant \mathrm{Sym}(\Omega)$ such that G has $\aleph_\lambda$ orbits on the power set of Ω?

The special case of this question when $\aleph_\kappa = \aleph_0$ is Problem 9.39 in the 1986 edition of the Kourovka Notebook, which has already received two answers. The first of them is in the paper that was the starting point for the work in this thesis – "On Linearly Ordered Sets and Permutation Groups of Countable Degree" by Hans Läuchli and Peter Neumann [5]. Here it is shown that there exists a group acting on a countable set, with $\aleph_1$ orbits on the power set of the set.

This of course only answers a very small part of either of the questions above, since it just deals with the successor cardinal to $\aleph_0$. A pre-print of [5] interested Shelah and Thomas in the problem. They solved it completely in the case when $\aleph_\kappa = \aleph_0$ in "Implausible subgroups of infinite symmetric groups" [12]. In this paper Martin's axiom was needed, whereas the original work was done assuming only the normal axioms of Zermelo-Fraenkel set theory with the axiom of choice and proving that a "natural" subgroup of $\mathrm{Sym}(\Omega)$ had the correct number of orbits on $\mathcal{P}(\Omega)$. In [**12**] it is suggested (I believe correctly) that there are no naturally occurring groups acting on countable sets which have $\aleph_\lambda$ orbits on the power set, for cardinals $\aleph_\lambda$ which are larger than $\aleph_1$. However, in [**5**] a belief is expressed that the same (or a similar) construction as the one that is used there would generalise the result proved and give a group acting on a set of cardinality $\aleph_\kappa > \aleph_0$, and having $\aleph_{\kappa+1}$ orbits on the power set.

Introduction

One group which acts on a countable set and has $\aleph_1$ orbits on the power set of this set, is the group of *piece-wise order preserving permutations* (PWOP permutations for short) of $\mathbb{Q}$, the rational numbers with their usual ordering [**5**]. These, as the name suggests, are elements of Sym($\mathbb{Q}$) for which we can find a finite partition of $\mathbb{Q}$ such that the permutation is order preserving on each member of the partition. (This is a construction due to Stoller [**15**].)

In [**5**] it is shown that there are $\aleph_1$ non-PWOP equivalent countable ordinals (which immediately implies that $\mathbb{Q}$ has $\aleph_1$ non-PWOP isomorphic subsets). To show that there are no more than $\aleph_1$ non-PWOP isomorphic subsets of $\mathbb{Q}$, Läuchli and Neumann use some results concerning the embeddability of order types into one another from "On Fraïssé's Order-type Conjecture" by Laver [**7**]. Their work only requires results about scattered order types (an order type is called scattered if we cannot embed the order type of the rational numbers in it). However [**7**] also contains other interesting and elegant results about order types which are a countable union of scattered order types. My own work relies heavily upon these. All but one of the lemmas from [**7**] needed in this thesis can be proved using a very restricted version of Laver's hypothesis (namely, replacing a general better-quasi-ordered set Q with a one element set). Therefore to make this work as self-contained as possible all the results from [**7**] that are needed are stated and proved in this chapter, apart from the proof where a general set is needed. Laver's work also relies on a deep combinatorial theorem from [**9**] which is beyond the scope of this thesis. For a clear proofs of both these theorems see Chapter 10 on Fraïssé's conjectures in [**11**].

In looking for an answer to the initial question the construction of a group is then suggested in [**5**], namely the PWOP permutations. Therefore, the search is for a linearly ordered set X whose PWOP automorphism group will give the correct number of orbits on the power set of X. The properties that X requires are, of course, that it has $\aleph_{\kappa+1}$ subsets that are not PWOP isomorphic, but no more than this. It is also immediately suggested thatthe set X should retain some of the properties of $\mathbb{Q}$ in higher cardinalities. These properties (not necessary but sufficient for this purpose) are that the set X will embed every ordinal of cardinality $\aleph_\kappa$ while itself only having cardinality $\aleph_\kappa$, is isomorphic to any dense subset of itself, and has no more than $\aleph_{\kappa+1}$ PWOP isomorphism classes of subsets that are neither dense in it nor well-ordered.

It soon becomes clear that subsets of the real line properly containing the rational numbers will not lend themselves very easily to this approach. Since ω_1 is not embeddable into the real line a different set of $\aleph_{\kappa+1}$ non-PWOP isomorphic subsets from the well-ordered ones of cardinality $\aleph_\kappa$, would have to be found. Even a brief survey of the literature shows other difficulties. The two papers "Martin's axiom does not imply that every two $\aleph_1$-dense sets of reals are isomorphic" [**1**] and "All $\aleph_1$-dense sets of reals can be isomorphic" [**2**], suggest that finding a set with the second property would involve work in areas where the results are independent of the normal axioms of Zermelo-Fraenkel set theory with the axiom of choice. Finally a strong result of Sierpinski [**14**], namely that there exists a strictly descending sequence $(\phi_\mu)_{\mu<2^{\aleph_0}}$ of order types which are the order types of subsets of the real line (where strictly descending means that $\mu > \nu$ implies ϕ_μ is embeddable into ϕ_ν but does not take an embedding of ϕ_ν), suggested that the last property could be as difficult to find in subsets of the real line as the first two.

However there are two immediate candidates for the set required in other work in the literature (on linearly ordered sets rather than permutation groups). One is a set constructed in Fraïssé's book [**3**] in the following way. Let ω_κ be the initial ordinal of cardinality $\aleph_\kappa$. Let $\omega_\kappa \times \mathbb{Q}$ be the product of ω_κ copies of the rational numbers, ordered lexicographically. Let Q be the set of words of finite length of elements in $\omega_\kappa \times \mathbb{Q}$. Order Q lexicographically. Then Q takes embeddings of all the ordinals of cardinality $\aleph_\kappa$, while itself having cardinality $\aleph_\kappa$. Whether the other properties hold is not so clear, however. Another candidate is the set L of all finite words with elements in $\omega_\kappa \times \omega^*$ (ordered lexicographically) with a lexicographic ordering on it. This set has order type $\eta_{\omega_1,\omega_{\kappa+1}}$, which is first defined in [7]. Again this set has cardinality $\aleph_\kappa$, all its intervals have cardinality $\aleph_\kappa$ and it takes embeddings of all ordinals of cardinality $\aleph_\kappa$. However, its structure is significantly simpler than that of Q and so answering questions about the number of non-PWOP isomorphic order types it embeds should be easier.

It is immediately obvious that, since words of any particular, or bounded, length in L form a scattered set, its denseness properties come from its ordering and the different lengths of words allowed. The simplest ordered set with the same sort of structure, called M in what follows, is made up simply of finite words of elements in ω_κ, ordered lexicographically. This set also embeds all ordinals of cardinality $\aleph_\kappa$, leading to the surprising result that all such ordinals are the union of countably many order types smaller than ω_κ^ω. This is quite striking even in the case when $\omega_\kappa = \omega$, saying for example, that a set of order type $\omega^{\omega^{\omega^\omega}}$ can be written as the union of sets of order type $\omega, \omega^2, \omega^3, \ldots$, with none of the sets bigger than ω^ω. The set M also satisfies conditions which Laver [7] shows are enough to ensure that it takes an embedding of every set which is a countable union of scattered sets. Hence the order types of all such sets can be given a decomposition into such relatively small, well-ordered sets.

Although the set M has very nice properties when considered simply as a linearly ordered set, the subset of it which consists simply of the elements which have cofinality ω_κ in M, is easier to deal with in a group-theoretical way, since its automorphism group is order 2-transitive. This set, which will be denoted by N, is dense in M so all its elements have cofinality ω_κ in N. In fact N has order type $\eta_{\omega_1,\omega_{\kappa+1}}$ so L and N are order-isomorphic. Since both of the sets M and N take embeddings of all the ordinals of cardinality $\aleph_\kappa$, they both have the first property sought.

When we consider the second property (being isomorphic to any dense subset) we see that any bounded ω_λ-sequence in M has a least upper bound, for all regular ω_λ with $\omega < \omega_\lambda < \omega_\kappa$. Thus M is not isomorphic to the subset of it that we gain when we remove any point with cofinality greater than ω and less than ω_κ, if ω_κ is singular. When ω_κ is regular, every bounded ω_κ-sequence N has a least upper bound. Then since all elements of N have cofinality equal to the cofinality of ω_κ, if one element is removed from N the set that remains has a bounded ω_κ-sequence with no least upper bound, and is dense in N. This means both M (in every case) and N (in the regular case) do not have the second property required.

For the third property, work in [5] using results from [7] to find the number of isomorphism classes of scattered subsets of $\mathbb{Q}$ generalises immediately to isomorphism classes of scattered sets of higher cardinalities. Moreover, Laver's results in [7] give an inductive definition of <u>all</u> order types which are a countable union of scattered types and, therefore, a characterisation of all types which occur as the order type of a subset of M or N. The elements at the smallest level of Laver's hierarchy are initial ordinals, converse initial ordinals and $\aleph_\kappa$-dense sets. The higher levels are order types of sets constructed by replacing points in a set whose order type is in a lower level by more sets with order types from the lower levels (these constructions satisfying other embeddability conditions). Having proved that there are no more than $\aleph_{\kappa+1}$ non-PWOP isomorphic order types in the first level of the hierarchy, it is easy to prove that $\aleph_{\kappa+1}$ is also an upper bound for the number of non-PWOP isomorphic types in the higher levels. To prove this upper bound holds for the $\aleph_\kappa$-dense sets in question is more difficult, at least in the regular case.

To give the first stage of the induction it is sufficient to find a set which is isomorphic to all subsets which are dense in it – a set with the property we sought above, in fact. This suggests constucting a set and using a "back and forth" argument similar to that often used in proofs of Cantor's Theorem. However, when working just in cardinality $\aleph_1$ this argument requires sets which are complete (where every ω-sequence has a limit) and such sets embed the order type of the real line, which brings back in the difficulties outlined above. Even so, it is possible to use the fact that sets which can be written as a countable union of scattered sets can also be written as a countable union of *well-ordered* sets (if they do not embed ω_1^*) to give an inductive definition of an isomorphism between any two dense subsets of N with dense complements in N. In the case when ω_κ is singular a stronger result can be proved in a similar way, namely that any subset of N which is dense in N is isomorphic to N. It is also possible to find an isomorphism between any subset which is dense in Q and Q itself.

Once the existence of one permutation group with $\aleph_{\kappa+1}$ orbits on the power set of a set of size $\aleph_\kappa$ has been proved then it is easy to find $2^{\aleph_\kappa}$ non-isomorphic sets which can be embedded into the original set and which also have PWOP permutation groups with this property. A few of these have nice structures and some of their properties are demonstrated in what follows. Finally a chapter is included on piece-wise order preserving permutation groups, where the number of pieces is allowed to be finite, countable or larger. It is shown that in the countable case, these groups have a very small number of orbits on the ordinals of any particular uncountable cardinality.

Notation and Definitions

I will generally use U, V, X, Y and Z to denote (linearly ordered) sets and u, v, w, x, y and z for their elements. Order types will be denoted by small greek letters, with ϕ and ψ being general order types, α, β and γ being well-ordered (or occasionally conversely well-ordered) ones and ν and μ being elements of index sets (mostly well-ordered ones). I will use A, B, C and W for well-ordered sets and f, g and h for mappings. The symbol $\mathbb{Q}$ will be used for the rational numbers with their usual ordering <u>and</u> the order type of the rational numbers with their usual ordering. In addition to this I will normally adhere to the convention that i, j, k, l, m, n and $p \in \mathbb{Z}$ and that q, r, s and $t \in \mathbb{Q}$. If X is a set then $\mathcal{P}(X)$ denotes the power set (set of subsets) of X. I will use $\mathrm{Sym}(X)$ for the symmetric group on X, that is, the group of all bijective mappings from X to itself.

I will call a one-to-one function between two linearly ordered sets which is onto and preserves order an *order-isomorphism* (or simply an *isomorphism*) and a one-to-one and onto function that reverses order, an *anti-isomorphism.* I will denote the subgroup of $\mathrm{Sym}(X)$ consisting of all order-isomorphisms by $\mathrm{Aut}(X)$. I will say an ordered set X is *order 2-transitive* if $\mathrm{Aut}(X)$ acts order 2-transitively on X, that is, for every set of elements $x_1, x_2, y_1, y_2 \in X$ with $x_1 < x_2$ and $y_1 < y_2$ we can find $f \in \mathrm{Aut}(X)$ with $f(x_1) = y_1$ and $f(x_2) = y_2$. Two sets U and V will be called *order isomorphic* if there is an order-isomorphism between them and I will be write this

$$U \simeq V.$$

I will use $|X|$ or $|\phi|$ to mean the cardinality of a set X or an order-type ϕ.

The *converse* of a set X will be used to mean the set which is anti-isomorphic to X. If ϕ is the order type of X then the converse of ϕ means the order type of the converse of X, written ϕ^*. The notation (X^1, X^2) will mean a Dedekind cut of X. For any element x of X the interval $(x, \infty)_X$ will mean all the elements of X which are greater than x and $(-\infty, x)_X$ all the elements which are smaller than x. If $x, y \in X$ with $x < y$ then $(x, y)_X$ is $(-\infty, x)_X \cap (y, +\infty)_X$. The notation $[x, \infty)_X$ will mean $(x, \infty)_X \cup \{x\}$ and similarly for $(-\infty, x]_X$, $(x, y]_X$, $[x, y)_X$ and $[x, y]_X$. I will omit X if the set under consideration is clear from the context. If $Y \subseteq X$ then the ordering on Y will always be the restriction of $\leqslant$ to Y. Let the *convex closure* of Y in X denote the set

$$\{x \in X \mid \text{ there exist } y, z \in Y \text{ such that } y \leqslant x \leqslant z\}.$$

I will say that Y is *dense in* X if for all $x, y \in X$ with $x < y$ there exists $z \in Y$ with $x < z < y$. Then X is *dense* if it is dense in itself. I will say X is *scattered* if no subset of X is dense. A *scattered order type* is the order type of a scattered set and $\mathcal{S}$ is the class of all scattered order types. I will use $\aleph_\kappa$-*dense* to describe a set X where for every pair of elements x and $y \in X$ with $x < y$ we have that $|(x, y)| = \aleph_\kappa$. Thus dense is the same as $\aleph_0$-dense.

If $X \subseteq Y$ and x is an element of Y define

$$(\leftarrow, x)_X := \{v \in Y \mid v < x \text{ and if } u \in X \text{ with } u < x \text{ then } u < v\}.$$

This definition implies that $(\leftarrow, x)_X = \emptyset$ if we can find a sequence in X which has supremum x when the sequence is considered as a sequence in Y. Dually, define

$$(x, \rightarrow)_X := \{v \in Y \mid v > x \text{ and if } u \in X \text{ with } u > x \text{ then } u > v\}.$$

Again, this implies that $(x, \rightarrow)_X = \emptyset$ if we can find a sequence in X which has infimum x when the sequence is considered as a sequence in Y. The base set being considered will not be specified since this will always be clear from the context. The set $(\leftarrow, x]_X$ is $(\leftarrow, x)_X \cup \{x\}$.

For any ordinal α an *α-sequence* will mean a strictly increasing sequence indexed by α and an *α^*-sequence* is a strictly descending sequence indexed by α. For any ordinals α and β with $\beta \leqslant \alpha$ I will use $\alpha - \beta$ to mean the unique ordinal γ such that $\beta + \gamma = \alpha$. If α is a limit and α_0 is the smallest ordinal for which α is the supremum of an α_0-sequence then α_0 is called the *cofinality* of α, which I shall write as $\operatorname{cof}(\alpha)$. If α is a successor ordinal then $\operatorname{cof}(\alpha)$ is 1. If β is a converse ordinal I will call the converse ordinal α such that $\operatorname{cof}(\beta^*) = \alpha^*$ the *coinitiality* of β. If $x \in X$ then the cofinality of x in X will be the smallest limit ordinal α such that x is the supremum of an α-sequence in X or 1 if x has a predecessor in X. This will be written as $\operatorname{cof}_X(x)$ with X omitted if the set concerned is clear from the context. Dually, the coinitiality of x in X will be the cofinality of x in the converse of X. A set $Y \subseteq X$ is *cofinal* in X if there is no upper bound for Y in X. It is *coinitial* if there is no lower bound for Y in X and *coterminal* if it is both coinitial and cofinal. The set Y is defined to be *closed in X* if for every ascending or descending sequence $(y_\mu)_{\mu \in \beta}$ in Y such that $\sup_X(y_\mu)$ or $\inf_X(y_\mu)$ exists we have that $\sup_X(y_\mu)$ or $\inf_X(y_\mu)$ is an element of Y. The set Y is defined to be *ω_λ-closed in X* if for every non-cofinal ω_λ-sequence $(x_\mu)_{\mu < \omega_\lambda}$ in Y we have that $\sup_X(x_\mu)$ exists and is an element of Y.

I will use $+$ for the ordered sum of any finite number of order types and $\sum$ for infinite (ordered) sums. I will use $\phi \times \psi$ for the product of ϕ and ψ, meaning ϕ copies of ψ, rather than as is usual, ψ copies of ϕ. I will say that an order type ψ is a ϕ-sum if we can write $\psi = \sum_{x \in \phi} \psi_x$ for some family $(\psi_x)_{x \in \phi}$ of order types. For any ordinal we can find a unique decomposition

$$\alpha = m_1 \times \omega^{\alpha_1} + m_2 \times \omega^{\alpha_2} + \ldots + m_r \times \omega^{\alpha_r}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_r$ is a strictly decreasing finite sequence of ordinal numbers and $m_1, m_2, \ldots, m_r$ are positive integers (see [**13**]). If β is another ordinal we can decompose β in the same way but now allowing integers equal to zero (and possibly adding some zero terms into our decomposition of α) to get

$$\beta = n_1 \times \omega^{\alpha_1} + n_2 \times \omega^{\alpha_2} + \ldots + n_s \times \omega^{\alpha_s}.$$

Then we can define the *Hessenberg sum* of α and β, written $\oplus$ by

$$\alpha \oplus \beta := (m_1 + n_1) \times \omega^{\alpha_1} + (m_2 + n_2) \times \omega^{\alpha_2} + \ldots + (m_s + n_s) \times \omega^{\alpha_s}.$$

Obviously this ordinal is well-defined by α and β and the operation $\oplus$ is associative and commutative. For more details about order types and operations on them see [**11**] or [**13**].

Let $\preccurlyeq$ denote the pre-order (transitive, reflexive, binary relation) of embeddability between order types (in the case of ordinals this is just the usual ordering and I will normally use $\leqslant$). So if ϕ, ψ are the order types of U, V respectively then $\phi \preccurlyeq \psi$ means there is an order preserving injective map $U \to V$. I will write $\phi \equiv \psi$ if $\phi \preccurlyeq \psi$ and $\psi \preccurlyeq \phi$ and write $\phi \prec \psi$ if $\phi \preccurlyeq \psi$ and $\psi \not\preccurlyeq \phi$.

A map $f : U \to V$ between linearly ordered sets U, V is said to be *(finitely) piece-wise order preserving* (PWOP for short), if there is a finite covering $U = U_1 \cup \ldots \cup U_n$, such that the restrictions of f to the subsets U_j, for all $j \leqslant n$, are order preserving maps. A composite of two PWOP maps is itself PWOP. If $f : U \to V$ is a PWOP bijection then $f^{-1} : V \to U$ is also PWOP. Thus linearly ordered sets with PWOP maps form a category.

If U is a linearly ordered set with order type ψ and we can write

$$U = U_1 \dot{\cup} \ldots \dot{\cup} U_n$$

where ψ_i is the order type of U_i for $i = 1, \ldots, n$ then I will write

$$\psi \sim \psi_1 \dot{\cup} \ldots \dot{\cup} \psi_n$$

(so $\dot{\cup}$ can be thought of as the symbol for an piecewise sum of order types). I will say that order types ψ, ϕ are PWOP equivalent, writing this $\phi \sim \psi$ when there exist order types $\psi_1, \ldots, \psi_m$ such that $\psi \sim \psi_1 \dot{\cup} \ldots \dot{\cup} \psi_m$ and $\phi \sim \psi_1 \dot{\cup} \ldots \dot{\cup} \psi_m$. Since linearly ordered sets and PWOP maps form a category $\sim$ is an equivalence relation on order types and, in fact, ordertype(U) $\sim$ ordertype(V) if and only if U and V are PWOP isomorphic.

Suppose $\aleph_\kappa$ is any infinite cardinal. The successor cardinal to $\aleph_\kappa$ is $\aleph_{\kappa+1}$. Let ω_κ and $\omega_{\kappa+1}$ denote the initial ordinals of cardinality $\aleph_\kappa$ and $\aleph_{\kappa+1}$ respectively. Let δ_κ be the cofinality of ω_κ. A map $f : U \to V$ between linearly ordered sets U, V is said to be *$\aleph_\kappa$-piece-wise order preserving* ($\aleph_\kappa$-PWOP for short), if there is a covering $U = \bigcup_{\mu<\alpha} U_\mu$, where $|\alpha| = \aleph_\kappa$ and such that the restrictions of f to the subsets U_μ are order preserving maps. Again linearly ordered sets with $\aleph_\kappa$-PWOP maps form a category.

If U is a linearly ordered set with order type ψ and we can write

$$U = \dot{\bigcup_{\mu<\alpha}} U_\mu$$

where ψ_μ is the order type of U_μ for $\mu < \alpha$ and $|\alpha| = \aleph_\kappa$ then as above I will write

$$\psi \sim_\kappa \dot{\bigcup_{\mu<\alpha}} \psi_\mu.$$

I will say that order types ψ, ϕ are $\aleph_\kappa$-PWOP equivalent, writing this $\phi \sim_\kappa \psi$ when there exist order types ψ_μ, for $\mu < \alpha$ such that

$$\psi \sim_\kappa \dot{\bigcup_{\mu<\alpha}} \psi_\mu \qquad \text{and} \qquad \phi \sim_\kappa \dot{\bigcup_{\mu<\alpha}} \psi_\mu.$$

As above $\sim_\kappa$ is an equivalence relation on order types and ordertype$(U) \sim_\kappa$ ordertype(V) if and only if U and V are $\aleph_\kappa$-PWOP isomorphic.

If $\mathcal{U}$ is a set of order types and β is an initial ordinal, or the converse of an initial ordinal, let a $(\mathcal{U},\beta)$*-unbounded sum* be an order type of the form $\sum_{\alpha\in\beta} \psi_\alpha$, where $\mathcal{U} = \{\psi_\alpha \mid \alpha \in \beta\}$ and for all $\alpha \in \beta$, the set $\{\gamma \mid \psi_\alpha \preccurlyeq \psi_\gamma\}$ has cardinality $|\beta|$.

Let X be a linearly ordered set which is a countable union of scattered sets, of cardinality $\aleph_\kappa$, such that X has no uncountable descending chains. Suppose that all intervals of X embed α^* and β for all $\alpha < \omega_1$, $\beta < \omega_{\kappa+1}$. If ϕ is the order type of X, then ϕ will be called an $\eta_{\omega_1,\omega_{\kappa+1}}$*-universal.* Notice that the only η_{ω_1,ω_1}-universals are the order types of open, half-open or closed intervals of $\mathbb{Q}$.

Let $\mathcal{U}$ be a set of order types. An order type of the form $\sum_{x\in X} \phi_x$ is defined to be a $(\mathcal{U},\eta_{\omega_1,\omega_{\kappa+1}})$*-universal* if

(1) X is a set with order type an $\eta_{\omega_1,\omega_{\kappa+1}}$-universal,

(2) $\phi_x \in \mathcal{U}$ for all $x \in X$,

(3) given any sum $\sum_{y\in Y} \psi_y$, with Y any other set with order type a $\eta_{\omega_1,\omega_{\kappa+1}}$-universal and $(\psi_y)_{y\in Y}$ a family of order types such that $\psi_y \in \mathcal{U} \cup \{0\}$ for all $y \in Y$, there is an order preserving function $g : Y \to X$ such that $\psi_y \preccurlyeq \phi_{g(y)}$ for all y in Y. (This notation follows that in [**7**].)

This definition is consistent with that of an $\eta_{\omega_1,\omega_{\kappa+1}}$-universal, in the sense that an $\eta_{\omega_1,\omega_{\kappa+1}}$- universal is a $(\mathcal{U},\eta_{\omega_1,\omega_{\kappa+1}})$-universal where $\mathcal{U} = \{0,1\}$, since for any such universal ξ, Theorem 3.3 of [**7**] (the second corollary to Theorem 0.6 in this chapter) states that $\xi \equiv$ ordertype(X).

Now I will define a class of order types, $\mathcal{K} := \bigcup_{\beta\in\mathrm{On}} \mathcal{K}_\beta$, where

$\mathcal{K}_0 := \{0,1\}$;

$$\mathcal{K}_\beta := \{\phi \mid \phi \text{ is a } (\mathcal{U},\omega^*)\text{-unbounded sum or a } (\mathcal{U},\omega_\kappa)\text{-unbounded sum or a } (\mathcal{U},\eta_{\omega_1,\omega_{\kappa+1}})\text{-universal, for some } \mathcal{U} \subseteq \bigcup_{\delta<\beta} \mathcal{K}_\delta \text{ some initial ordinal } \omega_\kappa\}.$$

Preliminary Lemmas and Theorems

Lemma 0.1. *Suppose X is a dense subset of a linearly ordered set Y. Then if $x \in X$ we have that $\mathrm{cof}_X(x) = \mathrm{cof}_Y(x)$, and that the coinitiality of any element is the same in both sets.*

Proof. Obviously $\mathrm{cof}_Y(x) \leqslant \mathrm{cof}_X(x)$. Let $(y_\gamma)_{\gamma<\delta}$ be a sequence in Y with $x = \sup_Y(y_\gamma)$. Since X is dense there exists $x_\gamma \in (y_\gamma, y_{\gamma+1})$ and then $(x_\gamma)_{\gamma<\delta}$ is a sequence in X with $x = \sup_X(x_\gamma)$. Hence $\mathrm{cof}_X(x) \leqslant \mathrm{cof}_Y(x)$. Considering the converses of X and Y shows the coinitiality part of the lemma is true.

Lemma 0.2. *Suppose X is a dense subset of a linearly ordered set Y and $x = \sup_Y(x_\mu)$ for some sequence (x_μ) contained in Y. Then if $X \subseteq Y - \{x\}$ and $(x_\mu) \subseteq X$ then (x_μ) has no least upper bound in X.*

Proof. If $y = \sup(x_\mu)$ in $Y - \{x\}$ then y must be the successor of x in Y. However Y is dense so none of its elements have a successor.

Lemma 0.3 (Läuchli and Neumann [5]). *Suppose α is an ordinal and $\beta \sim \alpha$. Then β is an ordinal and $\beta < \omega \times \alpha$.*

Proof. Suppose B is a linearly ordered set of order type β. Since β is PWOP equivalent to α we can write B as the disjoint union of finitely many well-ordered sets, $B_1, B_2, \ldots, B_n$. If A is any non-empty subset of B then $A \cap B_i$ are non-empty for some of the i and we may assume it is the first m of them. If a_i is defined to be the least element of $A \cap B_i$ for $i = 1, \ldots, m$ then $\{a_1, a_2, \ldots, a_m\}$ is a finite linearly ordered set. Therefore it has a least element which must also be the least element of A, whence B is well-ordered and β is an ordinal.

If $\alpha \sim \alpha_1 \dot\cup \ldots \dot\cup \alpha_n$ we can obviously find an embedding of α into $\alpha_1 \oplus \ldots \oplus \alpha_n$ so $\alpha \leqslant \alpha_1 \oplus \ldots \oplus \alpha_n$. Consequently, if $\beta \sim \alpha$ then

$$\beta \leqslant \alpha_1 \oplus \ldots \oplus \alpha_n \leqslant \alpha \oplus \ldots \oplus \alpha < \omega \times \alpha$$

as stated.

Lemma 0.4 (Milner and Rado [8]). *Suppose ω_κ is a regular initial ordinal and* $\mathrm{ordertype}(A) = \omega_\kappa^n \times \beta$ *for some set A, some ordinal β and some integer $n \geqslant 1$.*

If

$$A = \dot{\bigcup_{\mu<\delta}} A_\mu$$

where $\delta < \omega_\kappa$ *then* $\omega_\kappa^n \times \beta \leqslant \text{ordertype}(A_\mu)$ *for some* $\mu < \delta$.

Proof. Induction on the order type of A will be used to prove the lemma. Assume that it is true for A' if $\text{ordertype}(A') < \text{ordertype}(A)$. We have that

$$A = \dot{\bigcup_{\mu<\delta}} A_\mu$$

and also, since $n \geqslant 1$ that

$$A = \sum_{\nu<\omega_\kappa} B_\nu$$

where each B_ν has order type $\omega_\kappa^{n-1} \times \beta$. Now let

$$B_{\nu,\mu} = B_\nu \cap A_\mu.$$

For each $\nu < \omega_\kappa$ $\text{ordertype}(B_\nu) = \omega_\kappa^{n-1} \times \beta < \text{ordertype}(A)$ and $B_\nu = \bigcup_{\mu<\delta} B_{\nu,\mu}$. Thus the inductive hypothesis implies that for each $\nu < \omega_\kappa$ there exists $\mu_\nu < \delta$ such that B_{ν,μ_ν} has order type $\omega_\kappa^{n-1} \times \beta$. Since $\delta < \omega_\kappa$ this implies that there is at least one μ' such that $B_{\nu,\mu'}$ has order type $\omega_\kappa^{n-1} \times \beta$ for ω_κ many ν. Then $\omega_\kappa^n \times \beta \leqslant \text{ordertype}(A_{\mu'}) = \sum_{\nu<\omega_\kappa} B_{\nu,\mu'}$ and the lemma is true.

Theorem 0.5 (Hausdorff [4]). *Remember that* $\mathcal{S}$ *is the class of all scattered order types. Then*

$$\mathcal{S} = \bigcup_{\alpha\in\text{Ord}} \mathcal{S}_\alpha,$$

where

$\mathcal{S}_0 := \{0, 1\}$ *and if* $\beta > 0$ *then*

$\mathcal{S}_\beta := \{\phi \mid \phi$ *is a well-ordered or conversely well-ordered sum of order types from* $\bigcup_{\gamma<\beta} \mathcal{S}_\gamma\}$.

Theorem 0.6 (Laver [7]). *Suppose* ϕ *is an* $\eta_{\omega_1,\omega_{\kappa+1}}$*-universal and that* U *is a linearly ordered set of cardinality* $\aleph_\kappa$ *which satisfies*

(1) *U is a countable union of scattered sets;*

(2) $\omega_1^* \not\leqslant \text{ordertype}(U)$.

Then ordertype$(U) \preccurlyeq \phi$.

Proof. Let V be a linearly ordered set of order type ϕ. Notice that every interval of V also has its order type an $\eta_{\omega_1,\omega_\kappa+1}$-universal. For any scattered set U_0 we can find an embedding f of U_0 into V such that for any Dedekind cut (U_0^1, U_0^2) of U_0 there is an interval (x,y) of V such that

$$z \in (x,y),\ u \in U_0^1,\ v \in U_0^2 \text{ implies } f(u) < z < f(v).$$

The proof is by induction on S. Assume by Theorem 0.5 that ordertype(U_0) is the δ-sum of smaller order types for which the lemma holds, with $\delta <$ ordertype(U_0). Then δ can be mapped into V in a way which satisfies the Dedekind cut condition, and then the inductive hypothesis shows that the smaller order types can be mapped into V in an appropriate manner to establish the claim.

Now to embed U into V, write U as a countable union of scattered sets $U = \bigcup_{n<\omega} U_n$. Embed U_0 into V so that the Dedekind cut condition holds, then extend this to map $U_0 \cup U_1$ into V satisfying the Dedekind cut condition (remember every interval of V has an $\eta_{\omega_1,\omega_\kappa+1}$-universal as its order type) and so on to get ordertype$(U) \preccurlyeq \phi$ as required.

Corollary 1(Laver [7]). *If ϕ and ψ are both $\eta_{\omega_1,\omega_\kappa+1}$-universals then $\phi \equiv \psi$.*

Corollary 2. *Let $\mathcal{U} = \{0,1\}$ and suppose ϕ is an $\eta_{\omega_1,\omega_\kappa+1}$-universal. Then ψ is an $(\mathcal{U}, \eta_{\omega_1,\omega_\kappa+1})$-universal if and only if $\phi \equiv \psi$.*

Proof. First assume that ψ is an $(\mathcal{U}, \eta_{\omega_1,\omega_\kappa+1})$-universal. Suppose ψ has order type $\sum_{x\in X} \psi_x$ where each ψ_x is either 0 or 1. Suppose Y is a set of order type ϕ. Then we can consider ϕ as a sum $\phi = \sum_{y\in Y} \phi_y$ where ϕ_y is 1, for all $y \in Y$. Then, by the definition of $(\mathcal{U}, \eta_{\omega_1,\omega_\kappa+1})$-universals we can find a mapping $g: Y \to X$ such that $\phi_y \preccurlyeq \psi_{g(y)}$. The definition of an $(\mathcal{U}, \eta_{\omega_1,\omega_\kappa+1})$-universal implies then that $\psi_{g(y)} = 1$ for all $y \in Y$ and so g is simply an order preserving injective map from Y into X whence $\phi \preccurlyeq \psi$. Since by the first corollary ordertype$(X) \equiv$ ordertype(Y) there is an order preserving injection $f: X \to Y$. Then if we remove the points of X labelled with 0 we get a set of order type ψ and the restriction of f to this set is still an order preserving injection.

Now suppose $\phi \equiv \psi$ and X is a set of order type ψ. Again $\psi = \sum_{x \in X} \psi_x$ with $\phi_x = 1$ for all $x \in X$. Suppose $\rho = \sum_{z \in Z} \rho_z$ where ordertype(Z) is an $\eta_{\omega_1,\omega_{\kappa+1}}$-universal and $\rho_z = 0$ or 1 for all $z \in Z$. Then ordertype(Z) $\preccurlyeq \phi \preccurlyeq \psi$ and if $f : Z \to X$ is the mapping that witnesses this then f also satisfies $\rho_z \preccurlyeq \psi_{f(z)}$ since $\rho_z = 0$ or $1 \preccurlyeq 1 = \psi_{f(z)}$ whence $\rho \preccurlyeq \psi$.

Laver's Order Type $\eta_{w_1,w_{\kappa+1}}$ (Laver [7]). Define $\eta_{w_1,w_{\kappa+1}}$ to be the order type of the set X, where $X := \bigcup_{n<\omega} X_n$ with X_n defined as follows.

(1) X_0 is a set of order type $\omega_\kappa \times \omega^*$;

(2) X_{n+1} is obtained from X_n by inserting into each empty interval of X_n a set with order type $\omega_\kappa \times \omega^*$.

Lemma 0.7 (Laver [7]). *The order type $\eta_{\omega_1,\omega_{\kappa+1}}$ is an $\eta_{\omega_1,\omega_{\kappa+1}}$-universal.*

Proof. Obviously $\omega_1^*, \omega_{\kappa+1} \not\preccurlyeq \omega_\kappa \times \omega^*$. Thus $\omega_1^*, \omega_{\kappa+1} \not\preccurlyeq$ ordertype(X_0) and $\omega_1^*, \omega_{\kappa+1} \not\preccurlyeq$ ordertype(X_n) implies $\omega_1^*, \omega_{\kappa+1} \not\preccurlyeq$ ordertype(X_{n+1}). Since X is a countable union of the sets X_n we have ω_1^*, $\omega_{\kappa+1} \not\preccurlyeq$ ordertype(X) and the first condition for an $\eta_{\omega_1,\omega_{\kappa+1}}$-universal is satisfied. Also by induction X_n is scattered for each n so X is a countable union of scattered sets.

Suppose $\beta < \omega_{\kappa+1}$ and δ can be embedded into every interval of X for all $\delta < \beta$. If $x, y \in X$ with $x < y$ then $x, y \in X_m$ for some $m < \omega$. Since X_{m+1} is obtained from X_m by inserting a set of order type $\omega_\kappa \times \omega^*$ in each empty interval of X_m we know $\omega_\kappa \leqslant$ ordertype$(x, y) \cap X_{m+1}$. Therefore cof(β) $\leqslant$ ordertype(x, y). If cof(β) $< \beta$ then we can write β as the sum of cof(β) smaller ordinals and use the inductive hypothesis to embed these into the appropriate sub-intervals of (x, y). This also shows X is dense, so all its intervals embed every countable reverse ordinal whence the lemma is true.

Theorem 0.8 (Laver [7]). *Define $\mathcal{D} := \{\phi \mid \phi \prec \eta_{\omega_1,\omega_{\kappa+1}}\}$. Then*

(1) *if $\psi \in \mathcal{D}$ then a ψ-sum of order types in $\mathcal{D}$ is in $\mathcal{D}$;*

(2) $\mathcal{D} = \bigcup_{\gamma < \omega_{\kappa+1}} \mathcal{D}_\gamma$ *where*

$\mathcal{D}_0 := \{0,1\}$

$\mathcal{D}_\alpha := \{\phi \mid \phi$ *is an* ω^**-sum or a* β*-sum or an* $\eta_{\omega_1,\beta}$*-sum, for some* $\beta < \omega_{\kappa+1}$ *of elements of* $\bigcup_{\gamma<\alpha} \mathcal{D}_\gamma\}$.

Proof.

(1) Theorem 0.6, together with Lemma 0.7, implies that $\eta^2_{\omega_1,\omega_{\kappa+1}} \equiv \eta_{\omega_1,\omega_{\kappa+1}}$. Therefore if $\psi = \text{ordertype}(Y) \prec \eta_{\omega_1,\omega_{\kappa+1}}$ and $\psi_y \prec \eta_{\omega_1,\omega_{\kappa+1}}$ for all $y \in Y$ then $\sum_{y\in Y} \psi_y \preccurlyeq \eta_{\omega_1,\omega_{\kappa+1}}$. If $\eta_{\omega_1,\omega_{\kappa+1}} \preccurlyeq \sum_{y\in Y} \psi_y$ then either $\eta_{\omega_1,\omega_{\kappa+1}} \preccurlyeq$ ordertype(Y) or some interval of $\eta_{\omega_1,\omega_{\kappa+1}}$ is $\preccurlyeq \psi_y$ for some y. But since $\eta_{\omega_1,\omega_{\kappa+1}}$ can be embedded in all its intervals this would imply $\eta_{\omega_1,\omega_{\kappa+1}} \preccurlyeq \psi_y$. Therefore $\sum_{y\in Y} \psi_y \prec \eta_{\omega_1,\omega_{\kappa+1}}$ and so is in $\mathcal{D}$.

(2) Let $\mathcal{C} := \bigcup_{\gamma<\omega_{\kappa+1}} \mathcal{D}_\gamma$. Since $\alpha \prec \omega_1^*$ implies $\alpha \in \mathcal{D}$ and $\beta \prec \omega_{\kappa+1}$ implies β and $\eta_{\omega_1,\beta} \in \mathcal{D}$ (if $\beta = \omega_{\iota+1}$ for some initial ordinal ω_ι) part (1) of this theorem shows $\mathcal{C} \subseteq \mathcal{D}$. Suppose that $\mathcal{C} \subset \mathcal{D}$ and $\phi \in \mathcal{D}-\mathcal{C}$. If Y is a set of order type ϕ and $x, y \in Y$ with $x < y$ then define

$$x \sim_{\mathcal{C}} y \text{ if ordertype}((x,y)_Y) \in \mathcal{C}.$$

Let $y \sim_{\mathcal{C}} x$ if $x \sim_{\mathcal{C}} y$ to make $\sim_{\mathcal{C}}$ an equivalence relation, moreover, one which partitions Y into intervals. Suppose $X \subseteq Y$ is an equivalence class. Then we can pick a coterminal subset of X of order type $\alpha^* + \beta$ (with $\alpha \leqslant \omega$ and $\beta < \omega_{\kappa+1}$) and write ordertype$(X)$ as an $(\alpha^* + \beta)$-sum of order types from $\mathcal{C}$. Since $\alpha^* + \beta \in \mathcal{C}$ this implies ordertype$(X) \in \mathcal{C}$.

Now let $Y_{\mathcal{C}}$ be a subset of Y obtained by picking one member out of each equivalence class. Each interval $(u,v)_{Y_{\mathcal{C}}}$ of $Y_{\mathcal{C}}$ must have order type ψ_u in $\mathcal{D}-\mathcal{C}$ since otherwise $(u,v)_X$ would be an ψ_u sum of elements that the preceding paragraph tells us are in $\mathcal{C}$, which implies that ordertype$(u,v)_Y \in \mathcal{C}$ and so $u \sim_{\mathcal{C}} v$, contrary to the way we chose $Y_{\mathcal{C}}$.

Obviously $Y_{\mathcal{C}}$ has a dense order type, otherwise it would be in $\mathcal{C}$. Since ordertype$(Y_{\mathcal{C}}) \in \mathcal{D}$ there must be some interval $(x,y)_{Y_{\mathcal{C}}}$ of $Y_{\mathcal{C}}$ which fails to embed some $\beta < \omega_{\kappa+1}$, otherwise $\omega_{\kappa+1} \leqslant$ ordertype$(Y_{\mathcal{C}})$. Assume that every interval of $Y_{\mathcal{C}}$ embeds γ for all $\gamma < \beta$. Since ordertype$((x,y)_{Y_{\mathcal{C}}}) \in \mathcal{D}-\mathcal{C}$ this interval must be dense and therefore embed every countable reverse ordinal. However Theorem 0.6 then shows that ordertype$((x,y)_{Y_{\mathcal{C}}}) \equiv \eta_{\omega_1,\beta}$ whence ordertype$((x,y)_{Y_{\mathcal{C}}}) \in \mathcal{C}$. This contradiction gives the theorem.

Remember that $\mathcal{K} := \bigcup_{\beta \in \mathrm{On}} \mathcal{K}_\beta$, where

$$\mathcal{K}_0 := \{0,1\};$$

$$\mathcal{K}_\beta := \{\phi \mid \phi \text{ is a } (\mathcal{U},\omega^*)\text{-unbounded sum or a } (\mathcal{U},\omega_\kappa)\text{-unbounded sum or a } (\mathcal{U},\eta_{\omega_1,\omega_{\kappa+1}})\text{-universal, for some } \mathcal{U} \subseteq \bigcup_{\delta<\beta} \mathcal{K}_\delta, \text{ some initial ordinal } \omega_\kappa\}.$$

Let *well quasi-ordered* describe a class of order types for which there are no infinite descending chains, (with respect to the quasi-order $\preccurlyeq$) and no infinite sequences of order types $(\phi_i)_{i<\omega}$ such that $\phi_i \not\preccurlyeq \phi_j$ and $\phi_j \not\preccurlyeq \phi_i$ for all $i,j < \omega$ with $i \neq j$.

Theorem 0.9 (Laver [7] and Nash-Williams [9]). *The class $\mathcal{K}$ is well quasi-ordered.*

Theorem 0.10 (Laver [7]). *If U is a countable union of scattered sets with no uncountable descending chains then* ordertype(U) *is a finite sum of elements from $\mathcal{K}$.*

Proof. Suppose $|U| = \aleph_\kappa$ and let $\phi =$ ordertype(U). Assume the theorem holds for all sets which satisfy the hypotheses and have cardinality smaller than $\aleph_\kappa$. By Theorem 0.6 we know $\phi \preccurlyeq \eta_{\omega_1,\omega_{\kappa+1}}$. Suppose first of all that $\phi \prec \eta_{\omega_1,\omega_{\kappa+1}}$ so $\phi \in \mathcal{D}$. The theorem obviously holds if $\phi \in \mathcal{D}_0$; assume that $\gamma \geqslant 1$, $\phi \in \mathcal{D}_\gamma$ and the theorem holds for all $\delta < \gamma$. Then Theorem 0.8 implies that ϕ is either an ω^*-sum or a β-sum or an $\eta_{\omega_1,\beta}$-sum for some $\beta < \omega_{\kappa+1}$, of elements of $\bigcup_{\delta<\gamma} \mathcal{D}_\delta$.

Case 1. Suppose ϕ is a β-sum and the theorem fails for ϕ. Then there exists a least ordinal λ such that for some $\theta \in \mathcal{D}_\gamma$ we can write $\theta = \sum_{\mu<\lambda} \theta_\mu$ where each θ_μ is a finite sum of elements in $\mathcal{K}$ but θ cannot be written as such a finite sum. We can write θ as $\sum_{\mu<\text{cof}(\lambda)} \theta^\mu$ where each order type θ^μ is the sum of less than λ of the θ_μ. Since λ was minimal this implies that $\text{cof}(\lambda) = \lambda$, that is λ is regular. Now since each θ_μ was a finite sum of elements in $\mathcal{K}$ we have that $\theta = \sum_{\mu<\lambda} \theta'_\mu$ where each $\theta'_\mu \in \mathcal{K}$. Then

(1) there is $\mu_0 \in \lambda$ such that, for all $\iota, \nu \in \lambda$, if $\mu_0 \leqslant \iota \leqslant \nu$ then we can find $\sigma \in \lambda$ with $\nu \leqslant \sigma$ and $\theta'_\iota \preccurlyeq \theta'_\sigma$.

This follows because, if there is no such μ_0 then for all $\iota \in \lambda$ we can find $\nu \in \lambda$ such that $\iota < \nu$ and if $\sigma \in \lambda$ and $\nu \leqslant \sigma$ then $\theta'_\iota \not\preccurlyeq \theta'_\sigma$. But then we can find an infinite sequence of order types $(\theta_i)_{i<\omega}$ such that $i < j$ implies $\theta'_i \not\preccurlyeq \theta_j$ contradicting Theorem 0.9. Hence $\sum_{\mu>\mu_0} \theta'_\mu$ is a $(\mathcal{U}, \lambda)$-unbounded sum (with $\mathcal{U} = \{\theta'_\mu \mid \mu_0 < \mu\}$) and so is in $\mathcal{K}$. The minimality of λ implies $\sum_{\mu\leqslant\mu_0} \theta'_\mu$ is a finite sum of elements in $\mathcal{K}$ and so θ is a finite sum of elements in $\mathcal{K}$.

Case 2. The case when ϕ is an ω^*-sum is similar to Case 1.

Case 3. Suppose ϕ is an $\eta_{\omega_1,\beta}$-sum of finite sums of elements in $\mathcal{K}$. Since $\eta^2_{\omega_1,\beta} \equiv \eta_{\omega_1,\beta}$ we can write $\phi = \sum_{\mu\in\xi} \phi_\mu$ where $\xi \equiv \eta_{\omega_1,\beta}$ and $\phi_\mu \in \mathcal{K}$. But then, by the original inductive hypothesis ϕ is a finite sum of elements of $\mathcal{K}$.

Suppose then that $\phi \equiv \eta_{\omega_1,w_{\kappa+1}}$. By the second corollary to Theorem 0.6 this implies that ϕ is an $(\mathcal{U}, \eta_{\omega_1,\omega_{\kappa+1}})$-universal, where $\mathcal{U} = \{0,1\}$. Hence $\phi \in \mathcal{K}$, so ϕ is obviously a finite sum of elements in $\mathcal{K}$ and the theorem is true.

Theorem 0.11 (Laver [7]). *The number of non-$\equiv$-equivalent order types in $\mathcal{K}$ of cardinality $\aleph_\kappa$ is $\aleph_{\kappa+1}$.*

Chapter One

Some Dense Sets and Their Properties

Fix ω_κ. Remember that δ_κ was defined to be the cofinality of ω_κ. The following linearly ordered sets L, M, N, P, Q and $P^{(\lambda)}$ for all regular $\omega_\lambda < \omega_\kappa$, will be defined by specifying that in each case the base set is given a lexicographic ordering. In other words, if $x = \alpha_1\alpha_2\ldots\alpha_n$ and $y = \beta_1\beta_2\ldots\beta_m$ then $x \leqslant y$ if there exists $j \leqslant n$ with $\alpha_j < \beta_j$ and $\alpha_i = \beta_i$ for $i < j$ or $\alpha_i = \beta_i$ for $i \leqslant n$ and $n \leqslant m$.

The first set to be defined is

$$L := \{\alpha_1 i_1 \ldots \alpha_n i_n |\ 0 < n < \omega,\ \alpha_j \in \omega_\kappa,\ i_j \in \omega^* \text{ for } j = 1,\ldots,n,\ i_j \neq 0, \text{ if } j < n\}.$$

This set was chosen to give a concrete presentation of the order type $\eta_{\omega_1,\omega_{\kappa+1}}$ defined by Laver in [6] (see the preliminary lemmas and theorems). The proof that this set has the required order type is given later in this chapter. This is not as obvious as it may seem. If, for instance, we define L' to be the set of all finite words of elements in $\omega_\kappa \times \omega^*$, where all converse ordinals in the word (not just the last one) are allowed to take the value zero then we have a set which seems to have been constructed as Laver specified. Indeed it seems that we can define X_0 to be the words consisting of a single element of $\omega_\kappa \times \omega^*$ only and X_n to be the words containing $\leqslant n$ elements. Obviously $L' = \bigcup\limits_{n<\omega} X_n$ and it seems as if X_{n+1} is X_n with a copy of $\omega_\kappa \times \omega^*$ put into each interval in X_n. In fact we are also putting sets of order type $\omega_\kappa \times \omega^*$ into Dedekind cuts of X_n and thus L' does not have order type $\eta_{\omega_1,\omega_{\kappa+1}}$. Although we will see that L does have order type $\eta_{\omega_1,\omega_{\kappa+1}}$ actually there is a much simpler presentation of this order type, namely N given below.

The set L is dense although it is a countable union of scattered sets. A simpler construction that retains all the important properties of L is

$$M := \{\alpha_1\alpha_2\ldots\alpha_n |\ 0 < n < \omega,\ \alpha_j \in \omega_\kappa \text{ for } j = 1,\ldots,n,\ \alpha_n \neq 0\}.$$

This is of course a set which is a countable union of *well-ordered* sets which is dense (in fact $\aleph_\kappa$-dense). Laver's theorems on $\aleph_\kappa$-dense sets show that

$$\text{ordertype}(M) \equiv \text{ordertype}(L)$$

but M contains elements of cofinality β for every regular ordinal $\beta \leqslant \omega_\kappa$ (whereas all the elements of L have cofinality δ_κ). Obviously then M cannot be order 2-transitive. However it has a subset which is dense in it which is, namely

$$N := \{\alpha_1\alpha_2 \ldots \alpha_n | \; 0 < n < \omega, \; \alpha_j \in \omega_\kappa \text{ for } j = 1, \ldots, n, \; \alpha_n \text{ a successor ordinal}\}.$$

This set has several very nice properties – as mentioned above, it is in some ways the simplest presentation of $\eta_{\omega_1,\omega_\kappa+1}$. All its elements have cofinality δ_κ. When ω_κ is regular this says that all the elements of N are the least upper bound for an ω_κ-sequence in N. Conversely, when ω_κ is regular, every bounded ω_κ-sequence in N has a least upper bound. N is also order 2-transitive and any of its subsets have their order type determined if they are dense in N with dense complement in N. One such subset, which will be used to give a concrete presentation of the order type of a subset dense in N, with dense complement in N is R, defined as follows.

$$R := \{x \in N \mid \text{length}(x) \equiv 1 \bmod 2\}.$$

Let $\zeta := \text{ordertype}(R)$. All the elements of R have cofinality δ_κ. However its most useful property is that, rather than every bounded ω_κ-sequence having a least upper bound, in every interval we can find an ω_κ-sequence with no supremum. In this way R imitates a characteristic of $\mathbb{Q}$ in higher cardinalities which is sufficient for R to also share with $\mathbb{Q}$ the characteristic of being order-isomorphic to all subsets which are dense in it.

The sets whose definitions follow are formed of the elements that remain in $M - N$.

$$P^{(\lambda)} := \{\alpha_1\alpha_2 \ldots \alpha_n |\ 0 < n < \omega,\ \alpha_j \in \omega_\kappa \text{ for } j = 1, \ldots, n,\ \alpha_n \text{ an } \omega_\lambda\text{–limit ordinal}\}$$

(where $\omega_0 \leqslant \omega_\lambda < \omega_\kappa$ and ω_λ is regular).

$$P := \bigcup_{\omega_\lambda < \omega_\kappa} P^{(\lambda)}.$$

These sets then contain all the elements of M with cofinality smaller than ω_κ and show how some of the properties of N depend on the cofinality of the elements – for instance if $\omega_\lambda > \omega$ then every ω_λ-sequence in $P^{(\lambda)}$ (or in P) has a least upper bound. Notice that

$$M = N \cup P.$$

Finally, the following is a set due to Fraïssé, defined in [**3**];

$$Q := \{\alpha_1 q_1 \ldots \alpha_n q_n |\ 0 < n < \omega,\ \alpha_j \in \omega_\kappa,\ q_j \in \mathbb{Q} \text{ for } j = 1, \ldots, n\}.$$

Fraïssé's purpose of defining this set was to prove that there exists an $\aleph_\kappa$-dense set of cardinality $\aleph_\kappa$, for all infinite cardinals. In fact the sets above all provide much simpler examples of such a set. Q is also a countable union of scattered sets with no uncountable descending chains but it does not immediately provide the insights into other sets of this form that the sets above give.

Now let $u = \alpha_1 i_1 \ldots \alpha_n i_n \in L, v = \beta_1\beta_2 \ldots \beta_m \in M, w = \gamma_1 s_1 \ldots \gamma_l s_l \in Q$ and define

$$\text{length}(u) := n,$$

$$\text{length}(v) := m,$$

$$\text{length}(w) := l,$$

$$\text{first }(u) := \alpha_1 i_1, \qquad \text{last }(u) := \alpha_n i_n,$$

$$\text{first }(v) := \beta_1, \qquad \text{last }(v) := \beta_m,$$

$$\text{first }(w) := \gamma_1 s_1, \qquad \text{last }(w) := \gamma_l s_l.$$

Let $j < w$ and X be any of the sets above and define

$$X_j := \{x \in X \mid \text{length}(x) = j\}.$$

If $u, v \in X$ and the letters of u tagged $1, \ldots, n$ are the same as the letters of v tagged $1, \ldots, n$ but the $(n+1)$th letters are not the same then $\text{cis}(u, v)$ is the word consisting of the letters numbered $1, \ldots, n$, (that is, their common initial segment) and the empty word if their first elements are different.

The first half of this chapter is an examination of the order-types of L, M, N, P, R, Q and $P^{(\lambda)}$ for $\omega_\lambda < \omega_\kappa$. The properties of these sets to notice in particular are the cofinalities of elements and the form (with or without a supremum) that bounded ω_λ or ω_κ-sequences take, for any regular $\omega_\lambda < \omega_\kappa$. We will also look at the denseness of the sets in each other and whether simple denseness in each of these sets is enough to determine order type. The main results are as follows.

Theorem 1.1. *For all $\omega_\lambda < \omega_\kappa$*

(1) *If $\omega_\kappa = \omega$ then all the sets $L, M, N, P, P^{(\lambda)}, Q$ and R are isomorphic to $\mathbb{Q}$.*

(2) *If $\omega_\kappa > \omega$ then the sets $M, N, P, P^{(\lambda)}$ and Q all have different order types, and L and N both have order type $\eta_{\omega_1, \omega_{\kappa+1}}$. If ω_κ is regular then R has a different order type from any of these sets. If ω_κ is singular then N and R are isomorphic.*

(3) *If* $\omega_\kappa > \omega$ *then there exist subsets of* M, P *and* $P^{(\lambda)}$ *which are dense in* M, P *and* $P^{(\lambda)}$ *respectively, but are not isomorphic to them. If* ω_κ *is uncountable and regular the same is true of* N.

(4) *Every subset of* R *which is dense in* R *is isomorphic to* R.

The assertion in (2) about R and N when ω_κ is singular is included for completeness but proved in Chapter Two.

The second half of this chapter is concerned with the proof that each of these sets – apart from M – is order 2-transitive. It will also be shown that the automorphism group of M is as transitive as possible, in that for any two ordered pairs (x_1, x_2) and (y_1, y_2) with $\text{cof}(x_i) = \text{cof}(y_i)$ for $i = 1, 2$ there is an element f of $\text{Aut}(M)$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$.

If X is any of L, M, N, P, R, Q or $P^{(\lambda)}$ for $\omega_\lambda < \omega_\kappa$ then X has cofinality δ_κ (the cofinality of ω_κ) since the words of length 1 form a cofinal set of order type ω_κ for N, M, P, R and $P^{(\lambda)}$ and the words of length 1 with second element 0 form a cofinal set of order type ω_κ in the remaining cases. It is also true that X has coinitiality ω^*. Indeed the set $\{(0)(i) \mid i \in \omega^*\}$ is coinitial in Q and L and if we define

$$M' := \{\underbrace{0\ldots0}_{2n+1\text{ zeros}}1 \mid n < \omega\} \quad \text{and} \quad P' := \{\underbrace{0\ldots0}_{n\text{ zeros}}\omega_\lambda \mid n < \omega,\ \omega_\lambda < \omega_\kappa\}$$

then M' is coinitial in N, M and R and P' is coinitial in P and $P^{(\lambda)}$.

Lemma 1.2. *The set* L *has has order type* $\eta_{\omega_1, \omega_{\kappa+1}}$.

Proof. Let $X_n := \bigcup_{m \leqslant n+1} L_m$. Obviously X_0 has order type $\omega_\kappa \times \omega^*$. Assume inductively that the set X_n has been formed from X_{n-1} by putting a set with order type $\omega_\kappa \times \omega^*$ into each interval of X_{n-1}, so $L_n = X_n - X_{n-1}$ consists of these sets. Thus if $x, y \in X_n$ and y is the successor of x then the inductive assumption implies that $x, y \in X_n - X_{n-1}$, in fact that $\text{length}(x) = \text{length}(y) = n$. Since y is the successor of x it must also be true that $\text{last}(x) \neq \beta 0$ for any $\beta \in \omega_\kappa$, for words in L_n whose last element has this form do not have a successor in L_n.

Now define $I_{x,y} := \{x\alpha i \mid \alpha \in \omega_\kappa,\ i \in \omega^*\}$. Then $I_{x,y}$ has order type $\omega_\kappa \times \omega^*$ and all its elements are between x and y. Moreover $X_{n+1} = X_n \cup L_{n+1}$ and L_{n+1} is the union of the sets $I_{x,y}$ for all pairs $x, y \in L_n$ such that y is the successor to x in L_n. So X_{n+1} is formed from X_n by putting a set of order type $\omega_\kappa \times \omega^*$ into each interval of X_n, whence $L = \bigcup_{n<\omega} X_n$ has order type $\eta_{\omega_1,\omega_\kappa+1}$.

Lemma 1.3. *The sets N, R, P and $P^{(\lambda)}$ are dense in M for all $\omega_\lambda < \omega_\kappa$ and R is also dense in N and $P^{(\lambda)}$ in P.*

Proof. Suppose that $u, v \in M$ with $u < v$ and $u = \alpha_1\alpha_2\ldots\alpha_n, v = \beta_1\beta_2\ldots\beta_m$. If $n > m$ we must have $\alpha_1\alpha_2\ldots\alpha_n 1 \in (u,v)_M$ and if $n \leqslant m$ we must have $\beta_1\beta_2\ldots\beta_{m-1}(\beta_m - 1)1 \in (u,v)_M$. Therefore there exists $z \in (u,v)_M$ with $\text{length}(z) > \text{length}(u), \text{length}(v)$. Suppose $z = \gamma_1\gamma_2\ldots\gamma_p$. Then $\gamma_1\gamma_2\ldots\gamma_p 1 \in (u,v)_M \cap N$ and $\gamma_1\gamma_2\ldots\gamma_p\omega_\lambda \in (u,v)_M \cap P^{(\lambda)}$ and $\gamma_1\gamma_2\ldots\gamma_p\omega_\lambda \in (u,v)_M \cap P$. If $p \equiv 1 \pmod 2$ then $\gamma_1\gamma_2\ldots\gamma_{p-1}(\gamma_p + 1) \in (u,v)_M \cap R$ and if $p \equiv 0 \pmod 2$ then $\gamma_1\gamma_2\ldots\gamma_p 1 \in (u,v)_M \cap R$. If $x < y \in N$ then $x, y \in M$ and the elements just used to demonstrate the denseness of R in M do the same for R in N. The proof that $P^{(\lambda)}$ is dense in P is exactly the same.

Lemma 1.4.

(1) *All elements of L have cofinality δ_κ.*

(2) *There are elements of M of cofinality ω_λ for all regular initial ordinals $\omega_\lambda \leqslant \omega_\kappa$.*

(3) *All elements of N have cofinality δ_κ in N and all elements of R have cofinality δ_κ in R.*

(4) *All elements of $P^{(\lambda)}$ have cofinality ω_λ.*

(5) *All elements of Q have cofinality ω.*

Proof. Let $(\gamma)_{\gamma<\delta_\kappa}$ be a cofinal sequence of successors in ω_κ.

(1) Suppose $u = \alpha_1 i_1 \ldots \alpha_n i_n \in L$. For each $\gamma < \delta_\kappa$ define

$$v_\gamma := \alpha_1 i_1 \ldots \alpha_{n-1} i_{n-1} \alpha_n (i_n - 1)(\gamma)(-1).$$

Then the lemma is demonstrated by the sequence $(v_\gamma)_{\gamma<\delta_\kappa}$. Obviously u is an upper bound for this δ_κ-sequence. Suppose $\beta_1 j_1 \dots \beta_m j_m$ is also an upper bound for the sequence. Then

$$\beta_1 j_1 \dots \beta_m j_m \geqslant \alpha_1 i_1 \dots \alpha_{n-1} i_{n-1} \alpha_n (i_n - 1).$$

Suppose that

$$\alpha_1 i_1 \dots \alpha_{n-1} i_{n-1} \alpha_n (i_n - 1) < \beta_1 j_1 \dots \beta_m j_m < \alpha_1 i_1 \dots \alpha_n i_n.$$

Then

$$\beta_1 j_1 \dots \beta_n j_n = \alpha_1 i_1 \dots \alpha_{n-1} i_{n-1} \alpha_n (i_n - 1).$$

and we must have $m > n$ but then if $\beta_{n+1} = \lambda$ we would have

$$v_{\lambda+1} \geqslant \alpha_1 i_1 \dots \alpha_{n-1} i_{n-1} \alpha_n (i_n - 1)(\lambda + 1)(-1) > \beta_1 j_1 \dots \beta_m j_m$$

which is a contradiction. Hence

$$\beta_1 j_1 \dots \beta_m j_m \geqslant \alpha_1 i_1 \dots \alpha_n i_n = u.$$

Therefore u is the supremum of this δ_κ-sequence, so u has cofinality δ_κ.

(2) Suppose $u \in M$ for some $u = \alpha_1 \alpha_2 \dots \alpha_n$. If α_n is a successor define

$$v_\gamma := \alpha_1 \alpha_2 \dots \alpha_{n-1} (\alpha_n - 1)\gamma.$$

Then u is an upper bound for the δ_κ-sequence $(v_\gamma)_{\gamma<\delta_\kappa}$. Suppose that we have another upper bound $\beta_1 \beta_2 \dots \beta_m$. Suppose that

$$\alpha_1 \alpha_2 \dots \alpha_{n-1} (\alpha_n - 1) < \beta_1 \beta_2 \dots \beta_m < \alpha_1 \alpha_2 \dots \alpha_n$$

in the lexicographic order (for, possibly, $\alpha_1 \alpha_2 \dots \alpha_{n-1} (\alpha_n - 1) \notin M$). Then

$$\beta_1 \beta_2 \dots \beta_n = \alpha_1 \alpha_2 \dots \alpha_{n-1} (\alpha_n - 1)$$

and we must have $m > n$. But then if $\beta_{n+1} = \lambda$ we would have

$$v_{\lambda+1} \geqslant \alpha_1 \alpha_2 \dots \alpha_{n-1} (\alpha_n - 1)(\lambda + 1) > \beta_1 \beta_2 \dots \beta_m$$

which is a contradiction. Hence

$$\beta_1\beta_2\ldots\beta_m \geqslant \alpha_1\alpha_2\ldots\alpha_n = u.$$

Therefore u is the supremum of this δ_κ-sequence, so if the last element of u is a successor then u has cofinality δ_κ.

If α_n is an ω_λ-limit ordinal of, say, the sequence of ordinals $(\delta_\mu)_{\mu<\omega_\lambda}$, where ω_λ is regular then we can define

$$u_\mu := \alpha_1\alpha_2\ldots\alpha_{n-1}\delta_\mu$$

and u is the supremum of $(u_\mu)_{\mu<\omega_\lambda}$. Without question u is an upper bound for this sequence. Suppose $v = \beta_1 r_1\ldots\beta_m r_m$ is also an upper bound. If there exists $l \leqslant n-1$ with $\alpha_i = \beta_i$ for $i = 1,\ldots,l-1$ and $\beta_l > \alpha_l$ then $v \geqslant u$. Suppose not so we have

$$\beta_1\beta_2\ldots\beta_{n-1} = \alpha_1\alpha_2\ldots\alpha_{n-1}$$

and $m \geqslant n$. If $\beta_n < \alpha_n$ then since α_n is the limit of $(\delta_\mu)_{\mu<\omega_\lambda}$ there exists $\delta_\mu < \omega_\lambda$ such that $\beta_n < \delta_\mu$. Then $v < u_\mu$ which is a contradiction. Hence $\beta_n \geqslant \alpha_n$ and so $v \geqslant u$ which means u is the supremum of this sequence, whence it has cofinality ω_λ.

(3) and (4) follow from (2), since N and $P^{(\lambda)}$ were defined to be the subsets of M containing words whose last element is a successor and those whose last element is an ω_λ-limit respectively and by (2) these elements have the stated cofinalities in M. This together with Lemma 1.3 which shows that N and $P^{(\lambda)}$ are dense in M and Lemma 0.1 which shows that $\mathrm{cof}_X(x) = \mathrm{cof}_Y(x)$ if $x \in Y$ and Y is dense in X gives the required result for N and $P^{(\lambda)}$ and since R is dense in N Lemma 0.1 also implies the result for R.

(5) Let $u = \alpha_1 q_1\ldots\alpha_n q_n$ be any element of Q. Let $(t_k)_{k<\omega}$ be a sequence of elements of $\mathbb{Q}$ with supremum q_n. Then if we define

$$u_k := \alpha_1 q_1\ldots\alpha_{n-1}q_{n-1}\alpha_n t_k$$

the sequence $(u_k)_{k<\omega}$ has supremum u. Once again, u is obviously an upper bound for this sequence. Suppose $v = \beta_1 r_1 \ldots \beta_m r_m$ is also an upper bound for the sequence. If

$$\alpha_1 q_1 \ldots \alpha_{n-1} q_{n-1} \alpha_n < \beta_1 r_1 \ldots \beta_m r_m < \alpha_1 q_1 \ldots \alpha_n q_n$$

in the lexicographic ordering we have

$$\beta_1 r_1 \ldots \beta_{n-1} r_{n-1} \beta_n = \alpha_1 q_1 \ldots \alpha_{n-1} q_{n-1} \alpha_n$$

and $r_n < q_n$. Since q_n was the limit of the sequence $(t_k)_{k<\omega}$ we must have $t_k > r_n$ for some k. But then $u_k > v$ which is a contradiction. Hence $\beta_1 r_1 \ldots \beta_m r_m \geqslant \alpha_1 q_1 \ldots \alpha_n q_n = u$. Therefore u is the supremum of this ω-sequence, so u has cofinality ω.

Lemma 1.5. *Suppose $\delta_\kappa > \omega$ and ω_λ is a regular initial ordinal with $\omega < \omega_\lambda < \omega_\kappa$. Then no ω_κ-sequence or ω_λ-sequence in Q has a least upper bound. Every bounded ω_λ-sequence in P or $P^{(\lambda)}$ has a least upper bound but no bounded ω_κ-sequence does. Every bounded ω_κ or ω_λ-sequence in M has a least upper bound and no bounded ω_λ-sequence in N or R does unless $\omega_\lambda = \delta_\kappa$. In any interval of R there are both bounded ω_κ-sequences with suprema and ones without. Suppose ω_κ is regular and uncountable. Then any bounded ω_κ-sequence in L or N has a least upper bound.*

Proof. Assume that $\omega_\kappa > \omega$ and that $\omega < \omega_\lambda < \omega_\kappa$. Since all elements in Q have cofinality ω there cannot be any ω_κ or ω_λ-sequences with suprema in Q, for if $x = \sup(x_\mu)_{\mu<\omega_\kappa}$ then x has cofinality ω_κ. For the same reason there can be no ω_κ-sequences with suprema in $P^{(\lambda)}$, and none in P. All elements of N and R have cofinality δ_κ so there are no ω_λ-sequences with a least upper bound unless $\omega_\lambda = \delta_\kappa$.

Suppose that ω_κ is regular. Let X be L, M or N. Suppose that $(x_\mu)_{\mu<\omega_\kappa}$ is a bounded ω_κ-sequence in X. Let S_i be the set of initial segments (not necessarily proper) of length i, of words in $\{x_\mu \mid \mu < \omega_\kappa\}$. Since ω_κ is regular and uncountable and $\{x_\mu \mid \mu < \omega_\kappa\} \subseteq \bigcup_{i<\omega} S_i$ there exists an integer i such that ordertype(S_i) $= \omega_\kappa$. Let n be the greatest integer such that S_n has order type smaller than ω_κ. Since the sequence is bounded ordertype(S_1) must be smaller than ω_κ so $n > 0$ and thus $S_n \neq \emptyset$. Now S_{n+1} has order type ω_κ and since ordertype(S_n) $< \omega_\kappa$ the regularity of ω_κ implies that there is an element of S_n which is an initial segment of every element of a subset A of S_{n+1}, with ordertype(A) $= \omega_\kappa$. In fact this initial segment, x say, must be the greatest element of S_n otherwise the sequence we specified as an ω_κ-sequence would have order type $\omega_\kappa + \alpha$ where α is some non-zero ordinal.

The paragraph above describes what happens for each of L, M and N. The nature of the element x decides whether the bounded ω_κ-sequence has a least upper bound or not.

If $\{x_\mu \mid \mu < \omega_\kappa\} \subseteq M$ (or N) then $x = \alpha_1\alpha_2\ldots\alpha_n$ with $\alpha_k \in \omega_\kappa$ for $k = 1, \ldots, n$. Then $v := \alpha_1\alpha_2\ldots\alpha_{n-1}(\alpha_n + 1) \in M$ (or N) and this must be the supremum for the given sequence, by the same argument as used in the proof of part (2) of Lemma 1.4.

Suppose we have the situation described above, in L so $x = \alpha_1 i_1 \ldots \alpha_n i_n$ with $i_n \neq 0$, since x is an initial segment of a word in L. The supremum of the sequence is then $\alpha_1 i_1 \ldots \alpha_{n-1} i_{n-1} \alpha_n (i_n + 1)$ and even if $i_n + 1 = 0$ this is an element of L. To see that this *is* the supremum suppose that $y = \beta_1 j_1 \ldots \beta_m j_m \in L$ is any upper bound for the sequence. We must have $y > \alpha_1 i_1 \ldots \alpha_n i_n$. So suppose

$$\alpha_1 i_1 \ldots \alpha_n i_n < y < \alpha_1 i_1 \ldots \alpha_{n-1} i_{n-1} \alpha_n (i_n + 1).$$

Then $y = \alpha_1 i_1 \ldots \alpha_n i_n \beta_{n+1} j_{n+1} \ldots \beta_m j_m$ for some $\beta_{n+1} \in \omega_\kappa$. But there is an element z of A whose $(n+1)$th letter is greater that β_{n+1} since ordertype(A) $= \omega_\kappa$. Then if x_μ is an element of the sequence with z as an initial segment we have $x_\mu > y$, which is a contradiction. Hence $y \geqslant x$.

Now let $(x_\mu)_{\mu < \omega_\lambda}$ be a bounded ω_λ-sequence in M. Since ω_λ is regular when we define S_i to be the set of initial segments of words in $\{x_\mu \mid \mu < \omega_\lambda\}$ and let j be the greatest integer for which ordertype(S_j) $< \omega_\lambda$ then the same reasoning as above shows that there is a word $x = \beta_1\beta_2\ldots\beta_j$ which is an initial segment to every element of a subset A of S_{j+1} with order type ω_λ. Now, however, we may have $j = 0$ in which case x will be the empty word. If we consider $A_0 := \{\alpha_{j+1} \mid \alpha_1\alpha_2\ldots\alpha_j\alpha_{j+1} \in A\}$ then A_0 is a set of ordinals of order type ω_λ so $\sup_{\mathrm{Ord}}(A_0) = \beta$ for some ω_λ-limit ordinal β. It is easy to see that unless $\beta_1\beta_2\ldots\beta_j\beta$ is an upper bound for (x_μ) the sequence would have order type larger than ω_λ. In fact $\beta_1\beta_2\ldots\beta_j\beta$ must be the least upper bound by the same argument as used in (2) of the previous lemma.

If the sequence $(x_\mu)_{\mu<\omega_\lambda}$ is also in $P^{(\lambda)}$ then, since $\beta_1\beta_2\ldots\beta_j\beta$ has cofinality ω_λ in M it is in $P^{(\lambda)}$, and thus is $\sup_P(x_\mu)$.

Since all elements in R have cofinality δ_κ there are some ω_κ-sequences in R with a least upper bound. Define $R' := N - R$. Now suppose $u, v \in R$. Then $u, v \in N$ and there exists $x \in R'$ with $u < x < v$. If $x = \gamma_1\gamma_2\ldots\gamma_n$ with, necessarily, γ_n a successor we can define $x_\lambda := \gamma_1\gamma_2\ldots\gamma_n(\gamma_n - 1)\lambda$ for any successor $\lambda \in \omega_\kappa$. Then $x = \sup_N(x_\mu)_{\mu<\omega_\kappa}$ by the same proof as in Lemma 1.4 so there is $\nu < \omega_\kappa$ with $x_\mu \in (u, v)$ for all $\mu > \nu$. Since $x \in R'$ we know that $\mathrm{length}(x) = n \equiv 0 \pmod 2$ so it must be true that $\mathrm{length}(x_\mu) = n + 1 \equiv 1 \pmod 2$ whence $x_\mu \in R$ for all μ. Hence $(x_\mu)_{\nu<\mu<\omega_\kappa}$ is a sequence contained in $(u, v) \cap R$ which has no supremum in R by Lemma 0.2.

Lemma 1.6. *The set N has order type $\eta_{\omega_1,\omega_\kappa+1}$ (so L and N are order-isomorphic).*

Proof. To prove this it must be shown that N is the union of sets X_m for $m < \omega$ where X_0 has order type $\omega_\kappa \times \omega^*$ and, for all $m > 0$, the set X_m is formed from X_{m-1} by putting a set of order type $\omega_\kappa \times \omega^*$ into each interval of X_{m-1}.

To show this we will prove it is enough to find a subset X_0 of N that has order type $\omega_\kappa \times \omega^*$, such that $N_1 \subset X_0$ and X_0 is coinitial and cofinal in N and for all $m > 0$ to find subsets X_m such that;

(1) if y is the successor of x in X_{m-1} then

$$\mathrm{ordertype}(X_m \cap (x, y)) = \omega_\kappa \times \omega^*,$$

this set is coterminal in the interval and $(x, y) \cap N_{m+1} \subseteq X_m$;

(2) $x \in X_m - X_{m-1}$ implies $(x, \rightarrow)_{X_m} = (x, y)$ for some $y \in X_m$ or $(x, \rightarrow)_{X_m} = \emptyset$;

(3) if $x = \alpha_1\alpha_2 \ldots \alpha_p \in X_m - X_{m-1}$ then $z = \alpha_1\alpha_2 \ldots \alpha_{p-1}(\alpha_p - 1)1 \in X_m$ and $(\leftarrow, x)_{X_m} = (z, x)$.

Define

$$X_0 := N_1 \cup \{\alpha \underbrace{0 \ldots 0}_{i \text{ zeros}} 1 \mid \alpha \in \omega_\kappa,\ i < \omega\}.$$

Notice that

$$A_\alpha := \{(\alpha + 1)\} \cup \{\alpha \underbrace{0 \ldots 0}_{i \text{ zeros}} 1 \mid i < \omega\} \simeq \omega^*$$

for all $\alpha < \omega_\kappa$ so

$$X_0 = \bigcup_{\alpha < \omega_\kappa} A_\alpha \ \simeq\ \omega_\kappa \times \omega^*.$$

Since $N_1 \subset X_0$ this set contains all words of length 1 and is cofinal in N (since N_1 is cofinal in N). The sequence (y_i) where $y_i := \underbrace{0 \ldots 0}_{i \text{ zeros}} 1$ is coinitial in N. If $x \in N_1$ then $(x, \rightarrow)_{X_0} = \emptyset$ since in this case $x = (\alpha + 1)$ for some $\alpha \in \omega_\kappa$ and then $(x_i)_{i<\omega}$ where $x_i := (\alpha + 1) \underbrace{0 \ldots 0}_{i \text{ zeros}} 1$ is a sequence in X_0 with infimum x in N. The immediate predecessor of x in X_0 is $z := \alpha 1$ so $(\leftarrow, x)_{X_0} = (z, x)$. If $x \in X_0 - N_1$ then $x = \alpha \underbrace{0 \ldots 0}_{i \text{ zeros}} 1$. If $i = 0$ then $y := (\alpha + 1)$ is the successor of x in X_0 and if $i > 0$ then

$$y := \alpha \underbrace{0 \ldots 0}_{i-1 \text{ zeros}} 1$$

is the successor of x in X_0, so $(x, \rightarrow)_{X_0} = (x, y)$. In both cases

$$z := \alpha \underbrace{0 \ldots 0}_{i+1 \text{ zeros}} 1$$

is the predecessor of x in X_0 so $(\leftarrow, x)_{X_0} = (z, x)$.

Now assume that $n > 0$ and that the inductive hypothesis is true when $m = n - 1$. If $j \leqslant n - 2$ then $(x, y) \cap X_j$ is coterminal in (x, y) for all elements $x, y \in X_{j-1}$ such that y is the successor of x in X_{j-1}. This implies that $(x, \rightarrow)_{X_{n-1}} = \emptyset$ and that $(\leftarrow, y)_{X_{n-1}} = \emptyset$ for all $x, y \in X_{n-2}$. Suppose u, v are in X_{n-1} with v the successor of u in $X_{n-1} - X_{n-2}$. Then if $v = \alpha_1\alpha_2 \ldots \alpha_p$ we know we must have $u = \alpha_1\alpha_2\ldots\alpha_{p-1}(\alpha_p - 1)1$ by the third part of the inductive hypothesis. Define

$$U_v := \{\alpha_1\alpha_2\ldots\alpha_{p-1}(\alpha_p - 1)\beta \mid \beta \in \omega_\kappa,\ \beta \text{ a successor }\} \tag{a}$$

$$\cup\, \{\alpha_1\alpha_2\ldots\alpha_{p-1}(\alpha_p - 1)\beta\,\underbrace{0\ldots0}_{i \text{ zeros}}1 \mid 0 < \beta < \omega_\kappa,\ i < \omega\}. \tag{b}$$

The set labelled (a) contains all the words of length $p+1$ in (u,v) and is cofinal in (u,v). Also

$$B_\beta := \{\alpha_1\alpha_2\ldots\alpha_{p-1}(\alpha_p-1)(\beta+1)\}\cup\{\alpha_1\alpha_2\ldots\alpha_{p-1}(\alpha_p-1)\beta\,\underbrace{0\ldots0}_{i \text{ zeros}}1 \mid i < \omega\} \simeq \omega^*$$

for all β with $0 < \beta < \omega_\kappa$ so since $U_v = \bigcup\limits_{\beta<\omega_\kappa} B_\beta$ we have

$$U_v \simeq \omega_\kappa \times \omega^*.$$

The set labelled (b) contains the sequence (u_i) defined by

$$u_i := \alpha_1\alpha_2\ldots\alpha_{p-1}(\alpha_p - 1)1\,\underbrace{0\ldots0}_{i \text{ zeros}}1$$

which is coinitial in (u,v). So if we define

$$X_n := X_{n-1} \cup \left(\bigcup_{\substack{v\in X_{n-1}\\ -X_{n-2}}} U_v \right)$$

this set satisfies part (1) of the inductive hypothesis. If $x \in X_n - X_{n-2}$ then there are two possibilities for x. If $x \in X_{n-1}$ then by the inductive hypothesis $(x,\rightarrow)_{X_{n-1}} = \emptyset$ or x has a successor y' in X_{n-1}. In the second case we must have $U_{y'} \subset X_n$ whence the coinitiality of $U_{y'}$ in (x,y') implies that $(x,\rightarrow)_{X_n} = \emptyset$. Also x has a predecessor z' in X_{n-1} and so U_x is defined and is cofinal in (z',x) whence $(\leftarrow,x)_{X_n} = \emptyset$. Suppose then that $x \in X_n - X_{n-1}$, whence $x \in U_v$ for some $v \in X_{n-1}$. Then there are two cases to be considered. Suppose $x = \beta_1\beta_2\ldots\beta_r$ with $\beta_r = \gamma + 1$ for some $\gamma > 0$ (so x is one of the elements in the set tagged (a) above). Then

$$\{\beta_1\beta_2\ldots\beta_{r-1}(\gamma+1)\,\underbrace{0\ldots0}_{i \text{ zeros}}1 \mid i < \omega\} \subset U_v \subset X_n$$

and this set has infimum x in N so $(x, \rightarrow)_{X_n} = \emptyset$. Also x has predecessor $z = \beta_1\beta_2\ldots\beta_{r-1}\gamma 1$ in X_n. The other case is that x is in the set labelled (b) above. Then

$$x = \alpha_1\alpha_2\ldots\alpha_p\underbrace{0\ldots0}_{i\text{ zeros}}1$$

for some $i < \omega$. If $i = 0$ then x has successor $\alpha_1\alpha_2\ldots\alpha_{p-1}(\alpha_p + 1)$ in X_n and if $i > 0$ then x has $y = \alpha_1\alpha_2\ldots\alpha_p \underbrace{0\ldots0}_{i-1\text{ zeros}} 1$ as successor in X_n. In either case the predecessor of x in X_n is $z = \alpha_1\alpha_2\ldots\alpha_{p-1}(\alpha_p - 1)1$.

Finally it must be shown that the inductive hypothesis means that $x \in N$ implies $x \in \bigcup_{m<\omega} X_m$. Suppose, on the contrary, that $y = \gamma_1\gamma_2\ldots\gamma_n$ is a word in N which is not in the second set and that there are no shorter elements of N not in $\bigcup_{m<\omega} X_m$. The first part of the proof shows that $n > 1$. There exists $m < \omega$ such that $z := \gamma_1\gamma_2\ldots\gamma_{n-2}(\gamma_{n-1} + 1) \in X_m$ but $z \notin X_{m-1}$ (if $m > 0$). However part (3) of the inductive hypothesis then implies that $y \in X_m$ (in fact that y is the predecessor of z in X_m). Thus $N = \bigcup_{m<\omega} X_m$ and the lemma is true.

Lemma 1.7. *Suppose $\omega_\kappa > \omega$. There is a subset of $P^{(0)}$ of order type ω_1 which is ω-closed in $P^{(0)}$. There is no ω-closed cofinal ω_1-sequence in Q.*

Proof. Remember that a subset X was defined to be ω_λ-closed in $P^{(0)}$ if for every non-cofinal ω_λ-sequence $(x_i)_{i<\omega_\lambda}$ in X we have that $\sup_{P^{(0)}}(x_i)$ exists and is an element of X. Consider the initial segment A of $P^{(0)}_1$ of order type ω_1. Suppose $(x_i)_{i<\omega}$ is an ω-sequence in A. If we consider this sequence of one element words as a sequence of ordinals then it is one of the defining properties of ordinals that they must have a least upper bound, an ordinal α. But obviously α is a limit ordinal of cofinality ω. Then the one element word $x := \alpha$ is the supremum of the sequence <u>in</u> A and must be an element of A so A is ω-closed in $P^{(0)}$.

Let $(y_\mu)_{\mu<\omega_1}$ be an ω_1-sequence in Q. If $S_i := \{x \mid x \text{ is an initial segment of length } i \text{ of a word in } (y_\mu)\}$ then as in the proof of Lemma 1.5 we have $\alpha_1 q_1 \ldots \alpha_n q_n$ as a (possibly empty) initial segment to every element of a subset

A of Q_{n+1} of order type ω_1. Define $\eta(y)$ as the $(n+1)$th ordinal element of y

A of Q_{n+1} of order type ω_1. Define $\eta(y)$ as the $(n+1)$th ordinal element of y for $y \in A$. Then

$$\{\eta(y) \mid y \in A\} \simeq \omega_1.$$

If we pick $(\alpha_k)_{k<\omega}$ a sub-sequence of this set of order type ω then it has a least upper bound α. Then if for each k we have that y_k is a word in (y_μ) with α_k as last element then any upper bound is $> \alpha_1 q_1 \ldots \alpha_n q_n \alpha$ in the lexicographic ordering. Thus if $z = \alpha_1 q_1 \ldots \alpha_n q_n \alpha r$ is an upper bound for $(y_k)_{k<\omega}$ then $\alpha_1 q_1 \ldots \alpha_n q_n \alpha(r-1)$ is a lower upper bound. Hence there is no supremum in Q for $(y_k)_{k<\omega}$. Therefore Q has no ω-closed sequences of order type ω_1.

Lemma 1.8. *If $\omega_\kappa = \omega$ then all the sets defined are isomorphic to* $\mathbb{Q}$, *(of course there are no initial ordinals smaller than ω so we cannot have $P^{(\lambda)}$ in this case). If ω_κ is uncountable then $M, N, P, P^{(\lambda)}, Q$ and R are pair-wise non-isomorphic, for all $\omega_\lambda < \omega_\kappa$ with the possible exception of N and R when ω_κ is singular.*

Proof. Comparison of the cofinalities of elements of $M, N, P, P^{(\lambda)}$ and Q shows that these sets are pair-wise non-isomorphic, for any particular values of κ and λ, with the possible exception of the pairs $P^{(0)}$ and Q and N and R. Lemma 1.7 shows $P^{(0)} \not\simeq Q$ and Lemma 1.5 shows $N \not\simeq R$ if ω_κ is regular.

Lemma 1.9. *There is a subset of M which is dense in M and is not isomorphic to M.*

Proof. Lemma 1.3 shows that N is dense in M, and Lemma 1.8 shows these two sets are not isomorphic.

Lemma 1.10. *If ω_κ is regular and uncountable there is a subset of N which is dense in N and is not isomorphic to N.*

Proof. If we consider the set $N - \{2\}$ (where 2 signifies the one element word whose only element is 2) then this is obviously dense in N. It contains a bounded ω_κ-sequence without a least upper bound, namely $(1\ \mu)_{\mu<\omega_\kappa}$, so it is not isomorphic to N.

Lemma 1.11. *There are subsets of $P^{(\lambda)}$ and P which are dense in $P^{(\lambda)}$ and P respectively but are not isomorphic to $P^{(\lambda)}$ and P respectively.*

A of Q_{n+1} of order type ω_1. Define $\eta(y)$ as the $(n+1)$th ordinal element of y for $y \in A$. Then

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If we pick $(\alpha_k)_{k<\omega}$ a sub-sequence of this set of order type ω then it has a least upper bound α. Then if for each k we have that y_k is a word in (y_μ) with α_k as last element then any upper bound is $> \alpha_1 q_1 \ldots \alpha_n q_n \alpha$ in the lexicographic ordering. Thus if $z = \alpha_1 q_1 \ldots \alpha_n q_n \alpha r$ is an upper bound for $(y_k)_{k<\omega}$ then $\alpha_1 q_1 \ldots \alpha_n q_n \alpha(r-1)$ is a lower upper bound. Hence there is no supremum in Q for $(y_k)_{k<\omega}$. Therefore Q has no ω-closed sequences of order type ω_1.

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Lemma 1.11. *There are subsets of $P^{(\lambda)}$ and P which are dense in $P^{(\lambda)}$ and P respectively but are not isomorphic to $P^{(\lambda)}$ and P respectively.*

Proof. By Lemma 1.3 $P^{(\lambda)}$ is dense in P but is not isomorphic to P. The set $P^{(\lambda)}-\{\omega_\lambda\}$ is dense in $P^{(\lambda)}$ (where ω_λ signifies the word in $P^{(\lambda)}$ with this as its only element). However it contains an ω_λ-sequence with no supremum, namely $(\mu\omega_\lambda)_{\mu<\omega_\lambda}$, so it cannot have the same order type as $P^{(\lambda)}$.

Theorem 2.12. *Let N^X, N^Y be convex subsets of N, both with cofinality δ_κ and coinitiality ω^*. Let X, Y be subsets of N^X, N^Y, with X and $X' := N^X - X$ dense in N^X and Y and $Y' := N^Y - Y$ dense in N^Y. Then there exists an order isomorphism $f : N^X \to N^Y$ such that the restriction to X is an order isomorphism between X and Y.*

Corollary. *If X and $X' := N - X$ are both dense in N then $X \simeq X'$.*

This follows since we can define $Y := X'$ and apply the theorem.

The proof of this theorem is rather long and uses some technical lemmas, which will be given in the next chapter. Here one interesting consequence of it will be shown.

Remember the set R was defined in the following way.

$$R := \{x \in N \mid \text{length}(x) \equiv 1 \bmod 2\}.$$

It is obvious that this set is dense in N with dense complement in N.

Lemma 1.12. *Every subset which is dense in R is isomorphic to R.*

Proof. Let X be a dense subset of R. Then $N - R \subseteq N - X = X'$ so X' is dense in N. Obviously X is also a dense subset of N. Since X and X' are both dense in N the corollary above shows $X \simeq R$.

Now the automorphism groups of M, N, P, $P^{(\lambda)}$, Q and R will be examined, with the aim of proving that they are all order 2-transitive. In the cases of M and P we have to add the condition that $\text{cof}(x_i) = \text{cof}\,(y_i)$ to be able to find an element of $\text{Aut}(M)$ or $\text{Aut}(P)$ mapping the pair (x_1, x_2) to (y_1, y_2).

Lemma 1.13. *Suppose* $(u_1, \dots, u_n), (v_1, \dots, v_n) \in M^n$ *for some* $n < \omega$ *and* $u_i < u_{i+1}$ *and* $v_i < v_{i+1}$ *for* $i = 1, \dots, n-1$. *If also* $\mathrm{cof}(u_i) = \mathrm{cof}(v_i)$ *for* $i = 1, \dots, n$ *then there exists* $g \in Aut(M)$ *such that* $g(u_i) = v_i$ *for* $i = 1, \dots, n$.

Proof. It is enough to show that if $u_1, u_2, v_1, v_2 \in M$ with $u_1 < u_2$ and $v_1 < v_2$ and $\mathrm{cof}(u_2) = \mathrm{cof}(v_2)$ then $(u_1, u_2) \simeq (v_1, v_2)$ and $(-\infty, u_2) \simeq (-\infty, v_2)$ and $(u_2, \infty) \simeq (v_2, \infty)$. The proof will be in four parts and all intervals will be intervals in M.

(1) Assume first of all that the last elements of u_2 and v_2 are both ω_λ. Then

$$(w\alpha_1\alpha_2 \dots \alpha_n, w\beta_1\beta_2 \dots \beta_{m-1}\omega_\lambda) \simeq (1, \omega_\lambda).$$

That is, any interval (u, v), where $\mathrm{cis}(u, v) = w$ and $\mathrm{length}(x) > \mathrm{length}(w)$, is isomorphic to the interval bounded by the one element word 1 and the one element word ω_λ.

Define an ascending sequence of ordinals $\Lambda_1, \dots, \Lambda_m$ by

$$\Lambda_1 := \beta_1 - (\alpha_1 + 1)$$
$$\Lambda_i := \Lambda_{i-1} + \beta_i \qquad \text{for } i = 1, \dots, m-1.$$

Now we can define

$$U_{n-i} := \{\alpha_1\alpha_2 \dots \alpha_{n-i}(\alpha_{n-i+1} + \mu) \mid 0 < \mu < \omega_\kappa\} \quad \text{for } i = 1, \dots, n-1,$$
$$V_1 := \{\mu \mid \alpha_1 + 1 < \mu < \beta_1\},$$
$$V_i := \{\beta_1\beta_2 \dots \beta_{i-1}\mu \mid 0 < \mu < \beta_i\} \qquad \text{for } i = 2, \dots, m-1,$$
$$V_m := \{\beta_1\beta_2 \dots \beta_{m-1}0\mu \mid 0 < \mu < \omega_\kappa\},$$
$$V^* := \{\beta_1\beta_2 \dots \beta_{m-1}\mu \mid 0 < \mu < \omega_\lambda\},$$

and also

$$X_{n-i} := \{(i+1)\mu \mid 0 < \mu < \omega_\kappa\} \qquad \text{for } i = 1, \dots, n-1,$$
$$Y_1 := \{(n+1)\mu \mid 0 < \mu < \Lambda_1\}$$
$$Y_i := \{(n+1)\mu \mid \Lambda_{i-1} < \mu < \Lambda_i\} \quad \text{for } i = 2, \dots, m-1,$$
$$Y_m := \{(n+1)\mu \mid \Lambda_{m-1} < \mu < \omega_\kappa\},$$
$$Y^* := \{\mu \mid (n+1) < \mu < \omega_\lambda\}.$$

Then we have isomorphisms

$$\begin{aligned} &g_i : X_{n-i} \to U_{n-i} \qquad \text{for } i = 1, \dots, n-1, \\ &f_i : X_i \to U_i \qquad \text{for } i = 1, \dots, m, \\ &f^* : Y^* \to V^*. \end{aligned}$$

Notice that $i < j$ and $x \in \mathrm{Im}(g_i)$, $y \in \mathrm{Im}(g_j)$ implies $x < y$ and similarly for $x \in \mathrm{Im}(f_i)$ and $y \in \mathrm{Im}(f_j)$. Also $x \in \mathrm{Im}(g_i)$ or $\mathrm{Im}(f_i)$ and $z \in \mathrm{Im}(f^*)$ implies $x < z$.

Now suppose $x = \gamma_1\gamma_2 \dots \gamma_k \in (1, \omega_\lambda)$. If $\gamma_1 \leqslant n$ then either $\gamma_1 = 1$ and $k > 1$ or for some $i = 1, \dots, n-1$ we have $\gamma_1 = i+1$ and $k = 1$ or $\gamma_1 = i+1$ and $\gamma_2 = 0$ and $k > 2$ or $\gamma_1\gamma_2 \in X_{n-i}$. These are all the possibilities for x so if we define

$$f(x) := \begin{cases} w\alpha_1\alpha_2 \dots \alpha_n\gamma_2 \dots \gamma_k & \gamma_1 = 1,\ k > 1 \\ w\alpha_1\alpha_2 \dots \alpha_{n-i}(\alpha_{n-i+1} + 1) & \gamma_1 = i+1,\ k = 1 \\ w\alpha_1\alpha_2 \dots \alpha_{n-i}(\alpha_{n-i+1} + 1)0\gamma_3 \dots \gamma_k & \gamma_1\gamma_2 = (i+1)0,\ k > 2 \\ w\alpha_1\alpha_2 \dots \alpha_{n-i} g_i(\gamma_1\gamma_2)\gamma_3 \dots \gamma_k & \gamma_1\gamma_2 \in X_{n-i} \end{cases}$$

then f is defined on all $x \in (1, \omega_\lambda)$ with $\mathrm{first}(x) \leqslant n$. Now suppose $x = \gamma_1\gamma_2 \dots \gamma_k \in (1, \omega_\lambda)$ and $\gamma_1 = n+1$. Then $k = 1$ or $\gamma_2 = 0$ and $k > 2$ or for some $i = 1, \dots, m$ we have $\gamma_1\gamma_2 \in Y_i$ or $i < m$ and $\gamma_2 = \Lambda_i$ and $k = 2$ or $\gamma_2\gamma_3 = \Lambda_i 0$ and $k > 3$. Thus (remembering $\gamma_1 = n+1$) we can define

$$f(x) := \begin{cases} w(\alpha_1 + 1) & k = 1 \\ w(\alpha_1 + 1)0\gamma_3 \dots \gamma_k & \gamma_2 = 0,\ k > 3 \\ wf_1(\gamma_1\gamma_2)\gamma_3 \dots \gamma_k & \gamma_1\gamma_2 \in Y_1 \\ w\beta_1\beta_2 \dots \beta_{i-1} f_i(\gamma_1\gamma_2)\gamma_3 \dots \gamma_k & \gamma_1\gamma_2 \in Y_i,\ 1 < i \leqslant m \\ w\beta_1\beta_2 \dots \beta_i & \gamma_2 = \Lambda_i,\ k = 2 \\ w\beta_1\beta_2 \dots \beta_i 0\gamma_3 \dots \gamma_k & \gamma_2\gamma_3 = \Lambda_i 0,\ k > 3 \end{cases}$$

and f is defined on all elements x of $(1, \omega_\lambda)$ such that $\mathrm{first}(x) = n+1$. Finally if $\mathrm{first}(x) > n+1$ then $\mathrm{first}(x) = \gamma_1 \in Y^*$ so defining

$$f(x) := w\beta_1\beta_2 \dots \beta_{m-1} f^*(\gamma_1)\gamma_2 \dots \gamma_k$$

defines f on all elements remaining in $(1, \omega_\lambda)$.

This mapping is onto (u, v) since $x \in (u, v)$ implies $x = w\gamma_1\gamma_2 \ldots \gamma_k$ and one of the following is true. If $\gamma_1 = \alpha_1$ then we must have $\gamma_1\gamma_2 \cdots \gamma_n = \alpha_1\alpha_2 \ldots \alpha_n$ and $k > n$ or for some $i = 1, \ldots, n-1$ that $\gamma_1\gamma_2 \cdots \gamma_{n-i} \in U_{n-i}$ or $\gamma_1\gamma_2 \ldots \gamma_{n-i} = \alpha_1\alpha_2 \ldots \alpha_{n-i-1}(\alpha_{n-i} + 1)$ and $k = n - i$ or $\gamma_1\gamma_2 \cdots \gamma_{n-i-1}\gamma_{n-i}\gamma_{n-i+1} = \alpha_1\alpha_2 \ldots \alpha_{n-i-1}(\alpha_{n-i}+1)0$ and $k > n-i+1$. All such elements are in $\mathrm{Im}(f)$.

If $\alpha_1 < \gamma_1 < \beta_1$ then $\gamma_1 = \alpha_1 + 1$ and $k = 1$ or $\gamma_1\gamma_2 = (\alpha_1 + 1)0$ and $k > 2$ or $\gamma_1 \in V_1$. Thus $x \in \mathrm{Im}(f)$.

If $\gamma_1 = \beta_1$ then for some $i = 2, \ldots, m$ we have $\gamma_1\gamma_2 \ldots \gamma_i \in V_i$ or $\gamma_1\gamma_2 \ldots \gamma_i = \beta_1\beta_2 \ldots \beta_i$ and $k = i$ or $\gamma_1\gamma_2 \ldots \gamma_{i+1} = \beta_1\beta_2 \ldots \beta_i 0$ and $k > i + 1$ or $\gamma_1\gamma_2 \ldots \gamma_m \in V^*$. Again all such elements are in $\mathrm{Im}(f)$.

It only remains to prove that f is order-preserving. Suppose then that $x, y \in (1, \omega_\lambda)$ with $x < y$. We must have $x = z\gamma_1\gamma_2 \ldots \gamma_k$ and $y = z\delta_1\delta_2 \ldots \delta_l$ where $\gamma_1 < \delta_1$ (and z may be the empty word). Suppose z is the empty word and γ_1, δ_1 are both in X_{n-i} for some i. Then

$$\begin{aligned} f(x) &= w\alpha_1\alpha_2 \ldots \alpha_{n-i} g_i(\gamma_1)\gamma_2 \ldots \gamma_k \\ &< w\alpha_1\alpha_2 \ldots \alpha_{n-i} g_i(\delta_1)\delta_2 \ldots \delta_k = f(y) \end{aligned}$$

since g_i is order-preserving. The cases when $z = 1$ or $z = (n+1)$ or z is the empty word and $\gamma_1\gamma_2$, $\delta_1\delta_2$ are both in Y_i or γ_1, δ_1 are both in Y^* are similar. If $x = (i+1)$ then $f(x) = w\alpha_1\alpha_2 \ldots \alpha_{n-i-1}(\alpha_{n-i} + 1) < f(y)$ if $y \in X_{n-i+j}$ for some $j \geqslant 1$ and similarly when x or y is $(i+1)0\gamma_3 \ldots \gamma_k$ or $(n+1)\Lambda_i$ or $(n+1)\Lambda_i 0\gamma_4 \ldots \gamma_k$.

If $\mathrm{length}(z) \leqslant 1$ and γ_1, δ_1 are not both in one of X_i, Y_i or Y^* then f is order-preserving by the remarks about $\mathrm{Im}(g_i)$, $\mathrm{Im}(f_i)$ and $\mathrm{Im}(f^*)$. If $\mathrm{length}(z) > 1$ then

$$f(x) = f(z)\gamma_1\gamma_2 \ldots \gamma_k < f(z)\delta_1\delta_2 \ldots \delta_l = f(y).$$

Thus f is an order-isomorphism between $(1, \omega_\lambda)$ and (u, v).

(2) Now it will be shown that the above result holds, whatever the last element of u_2 and v_2 is, as long as $\mathrm{cof}(u_2) = \mathrm{cof}(v_2)$. Suppose then that

$$u_1 = x\alpha_{1,1}\dots\alpha_{1,n_1}, \qquad u_2 = x\alpha_{2,1}\dots\alpha_{2,n_2}, \qquad \text{with } \alpha_{1,1} < \alpha_{2,1},$$
$$y_1 = v\beta_{1,1}\dots\beta_{1,m_1}, \qquad v_2 = y\beta_{2,1}\dots\beta_{2,m_2}, \qquad \text{with } \beta_{1,1} < \beta_{2,1}.$$

This implies $\mathrm{cis}(u_1, u_2) = x$ and $\mathrm{cis}(v_1, v_2) = y$. We wish to prove that

$$(x\alpha_{1,1}\dots\alpha_{1,n_1}, x\alpha_{2,1}\dots\alpha_{2,n_2}) \simeq (y\beta_{1,1}\dots\beta_{1,m_1}, y\beta_{2,1}\dots\beta_{2,m_2}).$$

Suppose that $\mathrm{cof}(\alpha_{2,n_2}) = \mathrm{cof}(\beta_{2,m_2}) = \omega_\iota \leqslant \omega_\kappa$. Then we can find ω_ι-sequences of ordinals $(\gamma_\mu)_{1\leqslant\mu<\omega_\iota}$ and $(\eta_\mu)_{1\leqslant\mu<\omega_\iota}$ such that

$$\alpha_{2,n_2} = \sup(\gamma_\mu)_{1\leqslant\mu<\omega_\iota}$$
$$\beta_{2,m_2} = \sup(\eta_\mu)_{1\leqslant\mu<\omega_\iota}.$$

Then for $1 \leqslant \mu < \omega_\iota$ define

$$x_\mu := x\alpha_{1,1}\dots\alpha_{1,n_1-1}\gamma_\mu\omega_\lambda,$$
$$y_\mu := y\beta_1\dots\beta_{m-1}\eta_\mu\omega_\lambda$$

and

$$I_0 := (u_1, x_1), \qquad I_\mu := (x_\mu, x_{\mu+1}),$$
$$J_0 := (v_1, y_1), \qquad J_\mu := (y_\mu, y_{\mu+1}).$$

Notice that

$$(x_\mu)_{\mu<\omega_\iota} \simeq \omega_\iota \simeq (y_\mu)_{\mu<\omega_\iota}$$

and $I_\mu \simeq J_\mu$ for all $\mu < \omega_\iota$ by case (1). Thus

$$\begin{aligned}(x\alpha_{1,1}\dots\alpha_{1,n_1}, x\alpha_{2,1}\dots\alpha_{2,n_2}) &= \left(\bigcup_{\mu<\omega_\iota} I_\mu\right) \cup \omega_\iota \\ &\simeq \left(\bigcup_{\mu<\omega_\iota} J_\mu\right) \cup \omega_\iota \\ &= (y\beta_{1,1}\dots\beta_{1,m_1}, y\beta_{2,1}\dots\beta_{2,m_2}).\end{aligned}$$

(3) Now we will see that an interval $U = (w, w\beta_1\beta_2\dots\beta_m)$ where the first word is a proper initial segment of the second word, is isomorphic to some convenient interval where the first word is not an initial segment of the second word. What was proved above then shows U is isomorphic to every interval of this form. The interval we will use is $((0)(1), \beta_m)$, that is the interval bounded by the two element word $(0)(1)$ and the one element word β_m $(= \text{last}(w\beta_1\beta_2\dots\beta_m))$. Define $V_2, \dots, V_m$ and $V^\star$ as in (1). Define $\Lambda_1 := \beta_1$ and $\Lambda_{i+1} := \Lambda_i + \beta_{i+1}$ for $i = 1, \dots, m-2$ and

$$
\begin{aligned}
V_1 &:= \{\mu \mid 1 < \mu < \beta_1\},\\
Y_1 &:= \{(0)(1)\mu \mid 1 < \mu < \beta_1\},\\
Y_i &:= \{(0)(1)\mu \mid \Lambda_{i-1} < \mu < \Lambda_i\} \qquad \text{for } i = 2, \dots, m-1,\\
Y_m &:= \{(0)(1)\mu \mid \Lambda_{m-1} < \mu < \omega_\kappa\},\\
Y^\star &:= \{\mu \mid 0 < \mu < \beta_m\}.
\end{aligned}
$$

Then, as above, we have order isomorphisms

$$
\begin{aligned}
&h_i : Y_i \to U_i \quad \text{for } i = 1, \dots, m,\\
&h^\star : Y^\star \to V^\star.
\end{aligned}
$$

Notice that again $i < j$ and $x \in \text{Im}(h_i)$, $y \in \text{Im}(h_j)$ implies $x < y$ and $x \in \text{Im}(h_i)$ and $z \in \text{Im}(h^\star)$ implies $x < z$.

Now suppose $x = \gamma_1\gamma_2\dots\gamma_k \in ((0)(1), \beta_n)$ and $\text{first}(x) = 0$. Then $\gamma_1\gamma_2\gamma_3 = (0)(1)(0)$ and $k > 3$ or $\gamma_1\gamma_2\gamma_3 = (0)(1)(1)$ and $k = 3$ or $\gamma_1\gamma_2\dots\gamma_4 = (0)(1)(1)(0)$ and $k > 4$ or for some $i = 2, \dots, m$ we have $\gamma_1\gamma_2\gamma_3 \in Y_i$ or $\gamma_1\gamma_2\gamma_3 = (0)(1)\Lambda_{i-1}$ and $k = 3$ or $\gamma_1\gamma_2\dots\gamma_4 = (0)(1)\Lambda_{i-1}0$ and $k > 4$. Thus if we define

$$
h(x) := \begin{cases}
w0\gamma_4\dots\gamma_k & \gamma_2\gamma_3 = (1)(0),\ k > 3\\
w1 & \gamma_2\gamma_3 = (1)(1),\ k = 3\\
w(1)(0)\gamma_5\dots\gamma_k & \gamma_2\gamma_3\gamma_4 = (1)(1)(0),\ k > 4\\
w\beta_1\beta_2\dots\beta_{i-1}h_i(\gamma_1\gamma_2\gamma_3)\gamma_4\dots\gamma_k & \gamma_1\gamma_2\gamma_3 \in Y_i\\
w\beta_1\beta_2\dots\beta_i & \gamma_3 = \Lambda_i,\ k = 3\\
w\beta_1\beta_2\dots\beta_i0\gamma_5\dots\gamma_k & \gamma_3\gamma_4 = \Lambda_i0,\ k > 4
\end{cases}
$$

then h is defined on all elements x of $((0)(1), \beta_n)$ with first$(x) = 0$. If first$(x) > 0$ then first$(x) = \gamma_1 \in Y^*$ so defining

$$h(x) := w\beta_1\beta_2 \ldots \beta_{m-1}h^*(\gamma_1)\gamma_2 \ldots \gamma_k$$

defines h on all elements remaining in $((0)(1), \beta_n)$.

This mapping is onto (u, v) since $x \in (w, w\beta_1\beta_2 \ldots \beta_n)$ implies $x = w\gamma_1\gamma_2 \ldots \gamma_k$ and one of the following is true. If $\gamma_1 < \beta_1$ then $\gamma_1 = 0$ and $k > 1$ or $\gamma_1 = 1$ and $k = 1$ or $\gamma_1 \in V_1$. Thus $x \in \mathrm{Im}(h)$.

If $\gamma_1 = \beta_1$ then for some $i = 2, \ldots, m$ we have $\gamma_1\gamma_2 \ldots \gamma_i \in V_i$ or $\gamma_1\gamma_2 \ldots \gamma_i = \beta_1\beta_2 \ldots \beta_i$ and $k = i$ or $\gamma_1\gamma_2 \ldots \gamma_{i+1} = \beta_1\beta_2 \ldots \beta_i 0$ and $k > i + 1$ or $\gamma_1\gamma_2 \ldots \gamma_m \in V^*$ and again $x \in \mathrm{Im}(h)$.

The proof that h is order-preserving is similar to that in part (1).

(4) For any two words u and v in M we can define an isomorphism between (u, ∞) and (v, ∞) by finding cofinal sequences $(u_\mu)_{\mu<\omega_\kappa}$ and $(v_\mu)_{\mu<\omega_\kappa}$ in the intervals, such that $u = u_0$ and $v = v_0$ and cof$(u_\mu) =$ cof(v_μ) for all $\mu > 0$. Then, by (2) and (3) we can map (u_0, u_1) to (v_0, v_1) and $[u_\mu, u_{\mu+1})$ to $[v_\mu, v_{\mu+1})$ for all $\mu > 0$, and the union of these mappings will be an isomorphism between the two intervals.

In exactly the same way, if cof$(u) =$ cof$(v) = \omega_\iota$ then we can find sets of words $(u_\mu)_{\omega^* \prec \mu < \omega_\iota}$ and $(v_\mu)_{\omega^* \prec \mu < \omega_\iota}$ which are coinitial in M, have suprema u and v respectively, and whose elements have the same cofinalities as each other. Then the union of the isomorphisms between these intervals is an isomorphism between $(-\infty, u)$ and $(-\infty, v)$.

Lemma 1.14. *$P^{(\lambda)}$ and N are both order 2-transitive.*

Proof. For any isomorphism f it is true that cof$(x) =$ cof$(f(x))$. Therefore if $f \in \mathrm{Aut}(M)$ the restriction of f to $P^{(\lambda)}$ is in $\mathrm{Aut}(P^{(\lambda)})$. By Lemma 1.13 $\{f|_{P^{(\lambda)}} \mid f \in \mathrm{Aut}(M)\}$ is an order 2-transitive subgroup of $\mathrm{Aut}(P^{(\lambda)})$. It follows immediately that $\mathrm{Aut}(P^{(\lambda)})$ is 2-transitive.

Similarly $\{f|_N \mid f \in \mathrm{Aut}(M)\}$ is an order 2-transitive subgroup of $\mathrm{Aut}(N)$.

Lemma 1.15. *Suppose $(x_1, \dots, x_n), (y_1, \dots, y_n) \in P^n$ for some $n < \omega$ and $x_i < x_{i+1}$ and $y_i < y_{i+1}$ for $i = 1, \dots, n-1$. If also $\mathrm{cof}(x_i) = \mathrm{cof}(y_i)$ for $i = 1, \dots, n$ then there exists $g \in Aut(P)$ such that $g(x_1, \dots, x_n) = (y_1, \dots, y_n)$.*

Proof. As above the elements of $\mathrm{Aut}(M)$ form a subgroup of $\mathrm{Aut}(P)$ which is transitive on pairs $(x_1, x_2), (y_1, y_2)$ in P^2 with $\mathrm{cof}(x_i) = \mathrm{cof}(y_i)$ for $i = 1, 2$ and $x_1 < x_2$ and $y_1 < y_2$.

Lemma 1.16. *The automorphism group of R acts order 2-transitively on R.*

Proof. By Theorem 2.12, we know that $(x_1, x_2)_R \simeq (y_1, y_2)_R$ for all x_1, x_2, y_1, $y_2 \in N$ with $x_1 < x_2$ and $y_1 < y_2$, and that

$$(-\infty, x_1)_R \simeq (-\infty, y_1)_R,$$
$$(x_1, \infty)_R \simeq (y_1, \infty)_R.$$

Hence the automorphism group of this set is 2-transitive. Notice that we can use Theorem 2.12 in exactly the same way to prove the lemma for the automorphism group of N.

Lemma 1.17. $[\mathrm{Aut}(N){:}\mathrm{Aut}(M)] = 2^{\aleph_\kappa}$.

Proof. Let $Y := \{xi \mid x \in N_1,\ i \in \omega\}$. Suppose now that $X \subset N_1$ and $X \neq \emptyset$. Define $f_X : N \longrightarrow N$ by specifying that for $i \in \omega$

$$f_X(xi) := \begin{cases} x\,\underbrace{1\dots1}_{i+1 \text{ ones}} & \text{if } x \in X \\ xi & \text{if } x \notin X \end{cases}$$

and for all $y \in Y$ defining f_X to be the isomorphism between $(\leftarrow, y)_Y$ and $(\leftarrow, f_X(y))_{f_X(Y)}$ which exists by Lemma 1.14.

This gives an automorphism of N. Obviously two different subsets of N_1 will give me two different isomorphisms, and none of these could be the restriction of a mapping in $\mathrm{Aut}(M)$, since they are sending sequences with a least upper bound in M to sequences with no least upper bound. Since $N_1 \simeq \omega_\kappa$ the result follows.

Lemma 1.18. *If* $\omega_\lambda > \omega$ *then* $[\mathrm{Aut}(P^{(\lambda)}):\mathrm{Aut}(M)] = 2^{\aleph_\kappa}$.

Proof. This proof is exactly the same as the preceding one.
Let $Y := \{xi\omega_\lambda \mid x \in P_1^{(\lambda)},\ i < \omega\}$. Suppose $X \subset P_1^{(\lambda)}$ and $X \neq \emptyset$. Define $f_X : Y \longrightarrow P^{(\lambda)}$ by specifying that, for $i \in \omega$

$$f_X(xi\omega_\lambda) := \begin{cases} x \underbrace{\omega_\lambda \dots \omega_\lambda}_{(i+1) \text{ many } \omega_\lambda} & \text{if } x \in X \\ xi\omega_\lambda & \text{if } x \notin X. \end{cases}$$

Notice that, if $i \in \omega$ and $y = xi\omega_\lambda \in Y$ then $x(i-1)\omega_\lambda$ is the predecessor of y in Y and $(\leftarrow, y)_Y = (x(i-1)\omega_\lambda, y)$ if $i > 0$ and $(\leftarrow, f_X(y))_{f_X(Y)}$ is the interval $(x(i-1)\omega_\lambda, y)$ if $x \notin X$ and the interval

$$(x \underbrace{\omega_\lambda \dots \omega_\lambda}_{i \text{ many } \omega_\lambda}, x \underbrace{\omega_\lambda \dots \omega_\lambda}_{(i+1) \text{ many } \omega_\lambda})$$

if $x \in X$. If $i = 0$ then

$$(\leftarrow, y)_Y = (x_0 0\omega_\lambda, x_0 1\omega_\lambda) \cup \left(\bigcup_{0<i<\omega} [x_0 i\omega_\lambda, x_0(i+1)\omega_\lambda) \right)$$

if x has a predecessor $x_0 \in {P_1}^{(\lambda)}$ and in this case $(\leftarrow, f_X(y))_{f_X(Y)}$ is the same if $x \notin X$ and

$$(\leftarrow, f_X(y))_{f_X(Y)} = (x_0\omega_\lambda, x_0\omega_\lambda\omega_\lambda) \cup \left(\bigcup_{0<i<\omega} [x_0 \underbrace{\omega_\lambda \dots \omega_\lambda}_{(i+1) \text{ many } \omega_\lambda}, x_0 \underbrace{\omega_\lambda \dots \omega_\lambda}_{(i+2) \text{ many } \omega_\lambda}) \right)$$

Thus in both these cases Lemma 1.14 shows there is an isomorphism $g_{X,y}$ mapping $(\leftarrow, y)_Y$ onto $(\leftarrow, f_X(y))_{f_X(Y)}$. The case when x has no predecessor in $P_1^{(\lambda)}$ is similar. Thus the union of the map f_X and the maps $g_{X,y}$ is an automorphism h_X of $P^{(\lambda)}$.

This prescription gives $2^{\aleph_\kappa}$ different automorphisms h_X of $P^{(\lambda)}$ none of which could be the restriction of a mapping in $\mathrm{Aut}(M)$, since again they are mapping an ω-sequence which has a least upper bound in M onto one that does not.

Lemma 1.19. *Suppose ω_κ is regular. Then every element of $\mathrm{Aut}(R)$ extends to a unique element of $\mathrm{Aut}(N)$, but if H is the subgroup of $\mathrm{Aut}(N)$ formed in this way then $[\mathrm{Aut}(N) : H] = 2^{\aleph_\kappa}$.*

Proof. Suppose $x \in N - R$ and $f \in \mathrm{Aut}(R)$. By the proof of Lemma 1.5 there is an ω_κ-sequence $(x_\mu)_{\mu<\omega_\kappa}$ contained in R which has supremum x. Then $(f(x_\mu))_{\mu<\omega_\kappa}$ is a bounded ω_κ-sequence in N which must have a supremum y_x by Lemma 1.5. If we define

$$f_1(x) := \begin{cases} y_x & \text{if } x \in N - R \\ f(x) & \text{if } x \in R \end{cases}$$

then f_1 is an automorphism of N that extends f. This is obviously the only way an extension of f which is order-preserving can be defined.

On the other hand, suppose X is any non-empty subset of R_1. Define $Y := \{x0\nu \mid x \in R_1, \nu \text{ a successor in } \omega_\kappa\}$. Define an automorphism f_X in the following way, letting ν range over all the successors in ω_κ.

$$f_X(x0\nu) := \begin{cases} x\nu(1) & \text{if } x \in X \\ x0\nu & \text{if } x \notin X. \end{cases}$$

For all $y \in Y$ define f_X to be the isomorphism between $(\leftarrow, y)_Y$ and the interval $(\leftarrow, f_X(y))_{f_X(Y)}$, which exists by Theorem 2.12. This gives an automorphism of N. Obviously two different subsets of R_1 will give me two different isomorphisms, and none of these could be the extension of a mapping in $\mathrm{Aut}(R)$, since they are sending ω_κ-sequences with a least upper bound in R to ω_κ-sequences with no least upper bound.

Lemma 1.20. *The automorphism group of Q acts order 2-transitively on Q.*

Proof. In this proof ordinal multiplication <u>and</u> the formation of words in Q will be denoted by the juxtaposition of elements. A one element word $\gamma_1 s_1$ where $\gamma_1 \in \omega$, for example 11, will usually be written $(\gamma_1)(s_1)$ (so it would be

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Lemma 1.20. *The automorphism group of Q acts order 2-transitively on Q.*

Proof. In this proof ordinal multiplication <u>and</u> the formation of words in Q will be denoted by the juxtaposition of elements. A one element word $\gamma_1 s_1$ where $\gamma_1 \in \omega$, for example 11, will usually be written $(\gamma_1)(s_1)$ (so it would be

(1)(1) in this case) to avoid confusion. Let $u, v \in Q$ with $u = \alpha_1 q_1 \ldots \alpha_n q_n, v = \beta_1 r_1 \ldots \beta_m r_m$.
An isomorphism will be given to show that

$$(-\infty, v] \simeq (-\infty, (0)(1)].$$

Let us define

$$\begin{aligned} x_0 &:= (0)(0), \\ x_{\mu+1} &:= (0)(0)\mu(0) \qquad \text{if } 0 \leqslant \mu < \omega, \\ x_\mu &:= (0)(0)\mu(0) \qquad \text{if } \omega \leqslant \mu < \omega_\kappa \end{aligned}$$

and $X := \{x_\mu \mid \mu < \omega_\kappa\}$. Then

$$\begin{aligned} & X_0 := (\leftarrow, x_0]_X \cap Q_1 \simeq \mathbb{Q} + 1, \\ \text{for all } \mu > 0 \qquad & X_\mu := (\leftarrow, x_\mu]_X \cap Q_2 \simeq \mathbb{Q} + 1, \\ \text{and} \qquad & X^* := (\leftarrow, (0)(1)]_X \cap Q_1 \simeq \mathbb{Q} + 1. \end{aligned}$$

Notice also that, for any $\Lambda < \omega_\kappa$, we have

$$X^\Lambda := \bigcup_{\Lambda < \mu < \omega_\kappa} (\leftarrow, x_\mu]_X \cap Q_2 \simeq \omega_\kappa \times \mathbb{Q}.$$

The isomorphism between $(-\infty, (0)(1)]$ and $(-\infty, v]$ will be defined by finding a set $V = \{v_\mu \mid \mu < \Lambda\}$ of order type Λ where $\Lambda < \omega_\kappa$, and integers $(n_\mu)_{\mu<\omega_\kappa}$ such that $(\leftarrow, v_\mu)_V \cap Q_{n_\mu}$ has order type $\mathbb{Q}$ for all $\mu < \Lambda$, and a set V^Λ of order type $\omega_\kappa \times \mathbb{Q}$. These sets will be such that every element of Q which is smaller than v has an initial segment in $(\leftarrow, v_\mu)_V \cap Q_{n_\mu}$ for some μ or in V^Λ or in $(\leftarrow, v)_{V^\Lambda}$ (we will see later that the sets above satisfy this condition for elements of $(-\infty, (0)(1)]$).

Define ordinals $\Lambda_1, \Lambda_2, \ldots, \Lambda_m$ inductively by

$$\begin{aligned} \Lambda_1 &:= \beta_1, \\ \Lambda_{i+1} &:= \begin{cases} \Lambda_i + \beta_{i+1} + 1 & \text{if } \beta_{i+1} < \omega, \\ \Lambda_i + \beta_{i+1} & \text{if } \beta_i \geqslant \omega. \end{cases} \end{aligned}$$

Now we can use these ordinals to define the set V of order type ω_κ that we require. Define, $v_\mu := \mu r_1$ for all $\mu \leqslant \beta_1$. Notice that

$$\begin{aligned}(-, v_\mu]_{\{v_\mu \ \mid\ \mu < \beta_1\}} \cap Q_1 &= \{v_{\mu-1} q \mid q \in \mathbb{Q},\ q > r_{\mu-1}\} \\ &\quad \cup \{v_\mu q \mid q \in \mathbb{Q},\ q \leqslant r_\mu\} \\ &\simeq \mathbb{Q} + 1\end{aligned}$$

if μ is a successor and

$$\begin{aligned}(\leftarrow, v_\mu]_{\{v_\mu \ \mid\ \mu < \beta_1\}} \cap Q_1 \cup \{v_\mu q \mid q \in \mathbb{Q},\ q \leqslant r_\mu\} \\ \simeq \mathbb{Q} + 1\end{aligned}$$

if μ is not a successor, for all $\mu \leqslant \beta_1$. Let $n_\mu := 1$ for all $\mu \leqslant \beta_1$. For $i = 1, \ldots, m-2$ define

$$\begin{aligned}v_{\Lambda_i} &:= \beta_1 r_1 \ldots \beta_i r_i, & \\ v_{\Lambda_i + \mu + 1} &:= \beta_1 r_1 \ldots \beta_i r_i \mu r_{i+1} & \text{if } 0 \leqslant \mu < \omega, \\ v_{\Lambda_i + \mu} &:= \beta_1 r_1 \ldots \beta_i r_i \mu r_{i+1} & \text{if } \omega \leqslant \mu < \beta_{i+1}, \\ v_{\Lambda_m + \mu + 1} &:= \beta_1 r_1 \ldots \beta_{m-1} r_{m-1} \mu (r_m - 1) & \text{if } 0 \leqslant \mu < \omega \\ v_{\Lambda_m + \mu} &:= \beta_1 r_1 \ldots \beta_{m-1} r_{m-1} \mu (r_m - 1) & \text{if } \omega \leqslant \mu \leqslant \beta_m.\end{aligned}$$

Let $V := \{v_\mu \mid \mu \leqslant \Lambda_m\}$ and for $i = 1, \ldots, m-1$ let $n_\mu := i+1$ if $\Lambda_i < \mu \leqslant \Lambda_{i+1}$ to give

$$V_\mu := (\leftarrow, v_\mu]_V \cap Q_{n_\mu} \simeq \mathbb{Q} + 1.$$

for all $\mu \leqslant \Lambda_m$. Now we can define

$$\begin{aligned}V^{\Lambda_m} &:= \{x \in Q \mid x = \beta_1 r_1 \ldots \beta_{m-1} r_{m-1} \beta_m (r_m - 1) \gamma s,\ \gamma s \in \omega_\kappa \times \mathbb{Q}\} \\ &\subset Q_{m+1} \\ &\simeq \omega_\kappa \times \mathbb{Q}.\end{aligned}$$

Notice that

$$\begin{aligned}V^* &:= (\leftarrow, v]_{V^{\Lambda_m}} \cap Q_m \\ &= \{\beta_1 r_1 \ldots \beta_{m-1} r_{m-1} \beta_m q \mid q \in \mathbb{Q},\ r_m - 1 < q \leqslant r_m\} \\ &\simeq \mathbb{Q} + 1.\end{aligned}$$

Thus we have an isomorphism

$$f^*: X^* \to V^*.$$

We also have an order isomorphism

$$f_\mu: X_\mu \to V_\mu$$

for each $\mu \leqslant \Lambda_m$. Finally we have an isomorphism

$$f^{\Lambda_m}: X^{\Lambda_m} \to V^{\Lambda_m} \qquad \text{where } X^{\Lambda_m} = \bigcup_{\Lambda_m < \mu < \omega_\kappa} (\leftarrow, x_\mu]_X \cap Q_2$$

Notice that $\mu < \nu$ and $x \in \mathrm{Im}(f_\mu)$, $y \in \mathrm{Im}(f_\nu)$ implies that $x < y$. Also $w \in \mathrm{Im}(f^{\Lambda_m})$, $z \in \mathrm{Im}(f^*)$ implies $x, y < w < z$.

Now we can define an isomorphism between the two intervals in Q as follows. Suppose $x \in Q$ and $x \leqslant (0)(1)$. We must have $x = 0s_1\gamma_2 s_2 \ldots \gamma_k s_k$. Moreover either

(1) $s_1 < 0$ so $0s_1 \in X_0$;

(2) $s_1 = 0$ and $\gamma_2 \leqslant \Lambda_m$ so $00\gamma_2 s_2 \in X_{\gamma_2}$ if $s_2 < 0$ or $00\gamma_2 s_2 \in X_{\gamma_2+1}$ if $s_2 \geqslant 0$;

(3) $s_1 = 0$ and $\gamma_2 > \Lambda_m$ so $00\gamma_2 s_2 \in X^{\Lambda_m}$;

(4) $0 < s_1 \leqslant 1$ so $0s_1$ in X^*.

Thus if we define f on all elements of Q with initial segments in X_μ for some $\mu \leqslant \Lambda_m$ or in X^{Λ_m} or in X^* then f will be defined on all of $(-\infty, (0)(1)]$. So (with $x = 0\gamma_2 s_2 \ldots \gamma_k s_k$) define

$$f(x) := \begin{cases} f_0(0s_1)\gamma_2 s_2 \ldots \gamma_k s_k & \text{if } 0s_1 \in X_0, \\ f_{\gamma_2}(00\gamma_2 s_2)\gamma_3 s_3 \ldots \gamma_k s_k & \text{if } 00\gamma_2 s_2 \in X_{\gamma_2} \\ f_{\gamma_2+1}(00\gamma_2 s_2)\gamma_3 s_3 \ldots \gamma_k s_k & \text{if } 00\gamma_2 s_2 \in X_{\gamma_2+1} \\ f^{\Lambda_m}(00\gamma_2 s_2)\gamma_3 s_3 \ldots \gamma_k s_k & \text{if } \gamma_2 s_2 \in X^{\Lambda_m} \\ f^*(0s_1)\gamma_2 s_2 \ldots \gamma_k s_k & \text{if } \gamma_1 s_1 \in X^*. \end{cases}$$

By the comment above the definition it is defined on the whole of $(-\infty, (0)(1)]$. Suppose $x \in (-\infty, v]$ and $x = \delta_1 t_1 \ldots \delta_l t_l$. Then one of the following holds;

(1) $\delta_1 t_1 < \beta_1 r_1$ so $\delta_1 s_1 \in V_0$ or V_{δ_1} or V_{δ_1+1};

(2) for some i with $1 \leqslant i < m-2$ we have $\delta_1 t_1 \ldots \delta_i t_i = \beta_1 r_1 \ldots \beta_i r_i$ and $\delta_{i+1} t_{i+1} < \beta_{i+1} r_{i+1}$ so $\delta_1 t_1 \ldots \delta_{i+1} t_{i+1} \in V_{\Lambda_i + \delta_{i+1}}$ if $t_{i+1} \leqslant r_{i+1}$ or $\delta_1 t_1 \ldots \delta_{i+1} t_{i+1} \in V_{\Lambda_i + \delta_{i+1} + 1}$ if $t_{i+1} > r_{i+1}$;

(3) $\delta_1 t_1 \ldots \delta_{m-1} t_{m-1} = \beta_1 r_1 \ldots \beta_{m-1} r_{m-1}$ and $\delta_m t_m < \beta_m (r_m - 1)$ which implies $\delta_1 t_1 \ldots \delta_m t_m \in V_{\Lambda_{m-1} + \delta_m}$ if $t_m \leqslant (r_m - 1)$ or $\delta_1 t_1 \ldots \delta_m t_m \in V_{\Lambda_{m-1} + \delta_m + 1}$ if $t_m > (r_m - 1)$;

(4) $\delta_1 t_1 \ldots \delta_m t_m = \beta_1 r_1 \ldots \beta_{m-1} r_{m-1} \beta_m (r_m - 1)$ so $\delta_1 t_1 \ldots \delta_{m+1} t_{m+1} \in V^*$;

(5) $\delta_1 t_1 \ldots \delta_{m-1} t_{m-1} = \beta_1 r_1 \ldots \beta_{m-1} r_{m-1}$ and $r_m - 1 < t_m \leqslant r_m$ which implies $\delta_1 t_1 \ldots \delta_m t_m \in V^{\Lambda_m}$.

Since we have mappings onto all the elements in these sets, we have a mapping onto every word in $(-\infty, v]$.

It only remains to show that f is order preserving. Suppose then that $x, y \in (-\infty, (0)(1))$ with $x < y$. Then we must have some (possibly empty) word w such that $x = w\gamma_1 s_1 \ldots \gamma_k s_k$ and $y = w\delta_1 t_1 \ldots \delta_l t_l$. We then have the following possibilities for x and y.

(1) Suppose w is the empty word or $(0)(0)$ and $w\gamma_1 s_1$ and $w\delta_1 t_1$ are both in X_μ for some μ. Then

$$f(x) = f_\mu(w\gamma_1 s_1)\gamma_2 s_2 \ldots \gamma_k s_k < f_\mu(w\delta_1 t_1)\delta_2 t_2 \ldots \delta_l t_l$$

since f_μ is an order isomorphism.

(2) Suppose w is the empty word or $(0)(0)$ and $w\gamma_1 s_1$ and $w\delta_1 t_1$ are both in X^{Λ_m} or $w\gamma_1 s_1$ and $w\delta_1 t_1$ are both in X^*. As in (1) we then have $f(x) < f(y)$ because f^{Λ_m} and f^* were order isomorphisms.

(3) Suppose w is the empty word or $(0)(0)$ and $w\gamma_1 s_1 \in X_\mu$. If $w\delta_1 t_1 \in X_\nu$ for $\nu > \mu$ or X^{Λ_m} or X^* then by the comment above about $\mathrm{Im}(f_\mu)$ and $\mathrm{Im}(f^{\Lambda_m})$ and $\mathrm{Im}(f^*)$ we have $f(x) < f(y)$. The same thing is true if $w\gamma_1 s_1 \in X^{\Lambda_m}$ and $w\delta_1 t_1 \in X^*$.

(4) Suppose w is not an initial segment of $(0)(0)$. Then

$$f(x) = f(w)\gamma_1 s_1 \ldots \gamma_k s_k < f(w)\delta_1 t_1 \ldots \delta_l t_l = f(y).$$

Thus f is an order isomorphism and any two intervals of Q with a least upper bound and no lower bound are isomorphic.

Pick an element $\delta_1 t_1$ in $\omega_\kappa \times \mathbb{Q}$ which is smaller than $\beta_1 r_1$.

Assume firstly that either $m > 1$ or that $m = 1$ and $\delta_1 t_1 < \beta_1(r_1 - 1)$. Then since we have $v_\mu = \mu r_1$ for all $\mu < \beta_1$ we must have (now considering $\delta_1 t_1$ as a one element word of Q) that $(\delta_1 t_1, v_{\nu_0}) \cap Q_1 \simeq \mathbb{Q}$ for some $\nu_0 \leqslant \Lambda_1$. Consider the sequence $(v_\mu)_{\nu_0 \leqslant \mu \leqslant \Lambda_m}$. We can re-index this to get $(v'_\mu)_{\mu \leqslant \Lambda}$ for some $\Lambda \leqslant \Lambda_m$ and we can re-index the sequence $(n_\mu)_{\nu_0 \leqslant \mu \leqslant \Lambda_m}$ in the same way to get $(n'_\mu)_{\mu \leqslant \Lambda}$ such that if $Z := \{v'_\mu \mid 0 < \mu < \omega_\kappa\}$ we have that

$$Z_\mu := (\leftarrow, v'_\mu]_Z \cap Q_{n'_\mu} \simeq \mathbb{Q} + 1 \qquad \text{for all } \mu \text{ with } 0 \leqslant \Lambda.$$

Remember

$$(\delta_1 t_1, v_{\nu_0}] \cap Q_1 = (\delta_1 t_1, v'_0] \cap Q_1 \simeq \mathbb{Q} + 1. \tag{a}$$

Thus we can find an isomorphism

$$g_0 : X_0 \to \{\gamma s \mid \gamma s \leqslant v'_0,\ \delta_1 < \gamma \text{ or } \gamma = \delta_1 \text{ and } s > t_1\}$$

since the set on the right is the set (a) above. Also we have isomorphisms g_μ between X_μ and Z_μ for all μ with $0 < \mu \leqslant \Lambda$. Again

$$X^\Lambda := \bigcup_{\Lambda < \mu < \omega_\kappa} (\leftarrow, x_\mu]_X \simeq \omega_\kappa \times \mathbb{Q}$$

and so we have $g^\Lambda : X^\Lambda \to V^{\Lambda_m}$. We can use the g_μ and g^Λ and f^* to define an isomorphism

$$g : \{x \in Q \mid x \leqslant v \text{ and } \delta_1 q \leqslant x \text{ for all } q \in \mathbb{Q} \text{ with } t_1 < q\} \to (-\infty, (0)(1))$$

in exactly the same way as above (since every element in this set has an initial segment in exactly one of Z_μ for some μ with $0 < \mu \leqslant \Lambda$ or in V^* or in $(\delta_1 t_1, v'0] \cap Q_1$).

If $m = 1$ and $\beta_1(r_1 - 1) < \delta_1 t_1 < \beta_1 r_1$ then we can do the same thing by defining

$$
\begin{aligned}
& v_0' := \beta_1(\tfrac{r_1+t_1}{2}) \\
\text{and} \qquad & V^0 := \{x \in Q \mid x = \beta_1(\tfrac{r_1+t_1}{2})\gamma s \text{ for some } \gamma s \in \omega_\kappa \times \mathbb{Q}\} \\
\text{and} \qquad & V^* := \{x = \beta_1 s \mid s \in \mathbb{Q},\ \tfrac{r_1+t_1}{2} < s \leqslant r_1\}
\end{aligned}
$$

and then finding an isomorphism between X_0 and $(\leftarrow, v_0']_{\{\delta_1 t_1\}} \cap Q_1$, an isomorphism between X^0 and V^0 and an isomorphism between X^* and V^* and then proceeding as above. Thus

$$\{x \in Q \mid x \leqslant v \text{ and } \delta_1 q \leqslant x \text{ for all } q \in \mathbb{Q} \text{ with } t_1 < q\} \simeq (-\infty, (0)(1)]$$

for all $\delta_1 t_1 < \beta_1 r_1$.

Now we shall see that if $u = \alpha_1 q_1 \ldots \alpha_n q_n \in Q$ then

$$(u, \infty) \simeq ((0)(0), \infty).$$

Notice that for $i = 0, 1, \ldots, n-2$

$$
\begin{aligned}
& X_i := ((i)(0), (i+1)(0)] \cap Q_1 \simeq \mathbb{Q} + 1 \\
\text{and} \qquad & Y_i := \{(i+1)(0)\gamma s \mid \gamma s \in \omega_\kappa \times \mathbb{Q}\} \simeq \omega_\kappa \times \mathbb{Q} \\
\text{and} \qquad & Y_{n-1} := \{(n-1)s \mid s \in \mathbb{Q},\ s > 0\} \\
& \qquad \cup \{\gamma s \mid \gamma s \in \omega_\kappa \times \mathbb{Q},\ \gamma \geqslant n\} \simeq \omega_\kappa \times \mathbb{Q}.
\end{aligned}
$$

Similarly for $i = 0, \ldots, n-2$ we have

$$
\begin{aligned}
U_i & := \{\alpha_1 q_1 \ldots \alpha_{n-i-1} q_{n-i-1} \alpha_{n-i} s \mid s \in \mathbb{Q},\ q_{n-i} < s \leqslant q_{n-i} + 1, \} \\
& \simeq \mathbb{Q} + 1 \\
\text{and} V_i & := \{\alpha_1 q_1 \ldots \alpha_{n-i-1} q_{n-i-1} \alpha_{n-i} s \mid s \in \mathbb{Q}, s > q_{n-i} + 1\} \\
& \cup \{\alpha_1 q_1 \ldots \alpha_{n-i-1} q_{n-i-1} \gamma s \mid \gamma \in \omega_\kappa,\ \gamma > \alpha_{n-i},\ s \in \mathbb{Q}\} \\
& \simeq \omega_\kappa \times \mathbb{Q} \\
\text{and } V_{n-1} & := \{\alpha_1 s \mid s \in \mathbb{Q},\ s > q_1\} \cup \{\gamma s \mid \gamma > \alpha_1,\ q \in \mathbb{Q}\} \\
& \simeq \omega_\kappa \times \mathbb{Q}.
\end{aligned}
$$

We can then find isomorphisms $g_i : X_i \to U_i$ for $i = 0, 1, \ldots, n-1$ and $h_i : Y_i \to V_i$ for $i = 0, 1, \ldots, n-1$. Notice that $x \in \mathrm{Im}(g_i)$ and $y \in \mathrm{Im}(h_i)$ implies $x < y$. Similarly $x \in \mathrm{Im}(g_i) \cup \mathrm{Im}(h_i)$ and $y \in \mathrm{Im}(g_j) \cup \mathrm{Im}(h_j)$ for some $i < j$ implies $x < y$.

We can use these isomorphisms to define an order isomorphism between the intervals $((0)(0), \infty)$ and (u, ∞) as follows.
Suppose $x \in ((0)(0), \infty)$ and that $x = \gamma_1 s_1 \ldots \gamma_k s_k$. Then if $\gamma_1 s_1 > (n-1)0$ then we have $\gamma_1 s_1 \in Y_{n-1}$. If not and $\gamma_1 = n-1$ and $s_1 = 0$ then if $k = 1$ we have $\gamma_1 s_1 \in X_{n-2}$ and if $k = 2$ we have $x \in Y_{n-2}$. Assume then that $\gamma_1 = i$ for some $i \in \{0, \ldots, n-2\}$. If $s_1 = 0$ and $\mathrm{length}(x) > 1$ then $\gamma_1 s_1 \gamma_2 s_2 \in Y_i$ and if $s_1 < 0$ or $s = 0$ and $\mathrm{length}(x) = 1$ then $\gamma_1 s_1 \in X_i$.

As in the case of intervals unbounded below this implies that if we define a mapping on all elements with first element in these sets it will be defined on all of $((0)(0), \infty)$. So, taking $x = \gamma_1 s_1 \ldots \gamma_k s_k$ define

$$g(x) := \begin{cases} g_i(\gamma_1 s_1)\gamma_2 s_2 \ldots \gamma_k s_k & \text{if } \gamma_1 s_1 \in X_i - \{(j+1)0 \mid 0 \leqslant j < n\} \\ g_i(\gamma_1 s_1) & \text{if } \gamma_1 s_1 = (i+1)0,\ 0 \leqslant i < n \\ h_i(\gamma_1 s_1)\gamma_2 s_2 \ldots \gamma_k s_k & \text{if } \gamma_1 s_1 \gamma_2 s_2 \in Y_i. \end{cases}$$

By the comment before the definition g is defined on every element of $((0)(0), \infty)$. Suppose $y \in (u, \infty)$ with $y = \delta_1 t_1 \ldots \delta_l t_l$. Then if $\delta_1 t_1 > \alpha_1 q_1$ we have $\delta_1 t_1 \in V_m$. If not, then we must have that for some $i \in \{0, \ldots, n-1\}$ we have $\delta_1 t_1 \ldots \delta_{n-i} t_{n-i} = \gamma_1 s_1 \ldots \gamma_{n-i} s_{n-i}$ and $\delta_{n-i+1} = \alpha_{n-i+1}$ and $q_{n-i+1} < t_{n-i+1} \leqslant q_{n-i+1} + 1$ which implies that $\delta_1 t_1 \ldots \delta_{n-i} t_{n-i} \delta_{n-i+1} t_{n-i+1} \in U_i$ or $\delta_1 t_1 \ldots \delta_{n-i} t_{n-i} = \gamma_1 s_1 \ldots \gamma_{n-i} s_{n-i}$ and $\delta_{n-i+1} = \alpha_{n-i+1}$ and $q_{n-i+1} + 1 < t_{n-i+1}$ or $\delta_{n-i+1} > \alpha_{n-i+1}$ so $\delta_1 t_1 \ldots \delta_{n-i} t_{n-i} \delta_{n-i+1} t_{n-i+1} \in V_i$.

Since we have isomorphisms onto all of the U_i and V_i we know g is onto (u, ∞).

Now we need to show that g is order preserving. Suppose then that $x, y \in ((0)(0), \infty)$ with $x < y$. Then we must have some (possibly empty) word w

such that $x = w\gamma_1 s_1 \dots \gamma_k s_k$ and $y = w\delta_1 t_1 \dots \delta_l t_l$. We then have the following possibilities for x and y.

(1) Suppose w is the empty word and both $\gamma_1 s_1$ and $\delta_1 t_1$ are in X_i for some $i \leqslant n-1$. Then

$$g(x) = g_i(\gamma_1 s_1)\gamma_2 s_2 \dots \gamma_k s_k < g_i(\delta_1 t_1)\delta_2 t_2 \dots \delta_l t_l = g(y).$$

since g_i is an order isomorphism.

(2) Suppose w is $(i+1)(0)$ for some $i < n$ so both $w\gamma_1 s_1$ and $w\delta_1 t_1$ are in Y_i for some $i \leqslant n$. As in (1) we then have $g(x) < g(y)$ because h_i is an order isomorphism.

(3) Suppose w is the empty word and $w\gamma_1 s_1 \in X_i \cup Y_i$. If $w\delta_1 t_1 \in X_j \cup Y_j$ for $j > i$ then by the comment above about $\mathrm{Im}(g_i) \cup \mathrm{Im}(h_i)$ and $\mathrm{Im}(g_j) \cup \mathrm{Im}(h_j)$ we have $g(x) < g(y)$.

(4) Suppose w is not the empty word or $(i+1)(0)$ for some $i < n$. Then

$$g(x) = g(w)\gamma_1 s_1 \dots \gamma_k s_k < g(w)\delta_1 t_1 \dots \delta_l t_l = g(y).$$

Thus g is an order isomorphism and any two intervals (u_1, ∞) and (u_2, ∞) are isomorphic to $((0)(0), \infty)$ and thus to each other.

Suppose $\delta_1 t_1 \in \omega_\kappa \times \mathbb{Q}$ and $\delta_1 t_1 > \alpha_1 q_1$. Then

$$\begin{aligned} U_{n-1} := \{\gamma s \mid \alpha_1 q_1 < \gamma s < \delta_1 t_1\} & \qquad \text{(a)} \\ \cup \{x \in Q \mid x > u,\ x < \delta_1 q \text{ for all } q \in \mathbb{Q} \text{ with } t_1 < q\} \cap Q_2 & \qquad \text{(b)} \\ \simeq \omega_\kappa \times \mathbb{Q}. & \end{aligned}$$

To see this, consider the sequence $(x_\mu)_{\mu < \omega_\kappa}$ where x_μ is defined by $x_\mu := \delta_1 t_1 \mu 0$ for all $\mu < \omega_\kappa$. This is cofinal and contained in U_n. Moreover we have that $(x_\mu, x_{\mu+1}] \cap Q_2 \simeq \mathbb{Q} + 1$ and

$$\{x \in Q \mid x > u,\ x < \delta_1 q \text{ for all } q \in \mathbb{Q} \text{ with } t_1 < q\} \cap Q_2 = \bigcup_{\mu < \omega_\kappa} ((x_\mu, x_{\mu+1}] \cap Q_2)$$

so every element of this set has an initial segment in (a) or (b) above. Now we can find g_n an order isomorphism between U_{n-1} and Y_{n-1}. If we replace h_n with g_n in the definition of g above then we have

$$\{x \in Q \mid x > u,\ x < \delta_1 q \text{ for all } q \in \mathbb{Q} \text{ with } t_1 < q\} \simeq ((0)(0), \infty).$$

Suppose now that $u, v \in Q$ with $u < v$. Then we must have

$$u = w\alpha_1 q_1 \ldots \alpha_n q_n \qquad \text{and} \qquad v = w\beta_1 r_1 \ldots \beta_m r_m$$

with $\alpha_1 q_1 < \beta_1 r_1$ for some $w \in Q \cup \{$ empty word $\}$. Pick $\delta_1 t_1 \in \omega_\kappa \times \mathbb{Q}$ with $\alpha_1 q_1 < \delta_1 t_1 < \beta_1 r_1$. Define $u_1 = \alpha_1 q_1 \ldots \alpha_n q_n$ and $v_1 = \beta_1 r_1 \ldots \beta_m r_m$. Then

$$\begin{aligned}(u_1, v_1) &= \{x \in Q \mid x > u_1, x < \delta_1 q \text{ for all } q \in \mathbb{Q} \text{ with } t_1 < q\} \\ &\quad \cup \{x \in Q \mid x < v_1, x \geqslant \delta_1 q \text{ for all } q \in \mathbb{Q} \text{ with } t_1 < q\} \\ &\simeq ((0)(0), \infty) \cup (-\infty, (0)(1))\end{aligned}$$

by the work we have already done. Obviously

$$(u, v) \simeq (u_1, v_1)$$

so

$$(u, v) \simeq ((0)(0), \infty) \cup (-\infty, (0)(1))$$

for any $u, v \in Q$ with $u < v$ and Q is order 2-transitive.

so every element of this set has an initial segment in (a) or (b) above. Now we can find g_n an order isomorphism between U_{n-1} and Y_{n-1}. If we replace h_n with g_n in the definition of g above then we have

$$\{x \in Q \mid x > u,\ x < \delta_1 q \text{ for all } q \in \mathbb{Q} \text{ with } t_1 < q\} \simeq ((0)(0), \infty).$$

Suppose now that $u, v \in Q$ with $u < v$. Then we must have

$$u = w\alpha_1 q_1 \ldots \alpha_n q_n \qquad \text{and} \qquad v = w\beta_1 r_1 \ldots \beta_m r_m$$

with $\alpha_1 q_1 < \beta_1 r_1$ for some $w \in Q \cup \{\text{ empty word }\}$. Pick $\delta_1 t_1 \in \omega_\kappa \times \mathbb{Q}$ with $\alpha_1 q_1 < \delta_1 t_1 < \beta_1 r_1$. Define $u_1 = \alpha_1 q_1 \ldots \alpha_n q_n$ and $v_1 = \beta_1 r_1 \ldots \beta_m r_m$. Then

$$\begin{aligned}(u_1, v_1) &= \{x \in Q \mid x > u_1, x < \delta_1 q \text{ for all } q \in \mathbb{Q} \text{ with } t_1 < q\} \\ &\quad \cup \{x \in Q \mid x < v_1, x \geqslant \delta_1 q \text{ for all } q \in \mathbb{Q} \text{ with } t_1 < q\} \\ &\simeq ((0)(0), \infty) \cup (-\infty, (0)(1))\end{aligned}$$

by the work we have already done. Obviously

$$(u, v) \simeq (u_1, v_1)$$

so

$$(u, v) \simeq ((0)(0), \infty) \cup (-\infty, (0)(1))$$

for any $u, v \in Q$ with $u < v$ and Q is order 2-transitive.

Chapter Two

Universals and Order Types Determined by Denseness Conditions

In the first chapter some fairly elementary facts about the order-types of L, M, N, P, $P^{(\lambda)}$, Q and R were demonstrated, including that these sets are non-isomorphic, except for L and N and possibly R and N. Now we shall see that all of these sets are $\equiv$-equivalent, that is they embed and can be embedded into each other, and that there are, in fact, $2^{\aleph_\kappa}$ $\equiv$-equivalent, non-isomorphic sets.

This chapter also includes proofs that denseness in Q determines order type, and when ω_κ is singular denseness in N determines order type and that, in all cases, denseness in N together with denseness of the complement in N determines order type.

To show that all the sets mentioned are $\equiv$-equivalent it will be proved that they are all $\eta_{\omega_1,\omega_{\kappa+1}}$-universals. Since $L \simeq N$ and we have already seen that $N, P, P^{(\lambda)}$ and R are dense in M and obviously M is dense in itself, it is enough to show that denseness in M implies $\eta_{\omega_1,\omega_{\kappa+1}}$-universality, except of course in the case of Q which will be dealt with separately. Firstly notice the following facts about words of a particular length in the sets given.

Lemma 2.1. *For any $n < \omega$ we have*

$$M_n \simeq N_n \simeq P_n \simeq P_n^{(\lambda)} \simeq \omega_\kappa^n,$$

$$R_n \simeq \omega_\kappa^n \text{ if } n \equiv 1 \text{ mod } 2 \text{ and } R_n = \emptyset \text{ if } n \equiv 0 \text{ mod } 2,$$

$$Q_n \simeq (\omega_\kappa \times \mathbb{Q})^n.$$

Proof. If $n \equiv 0$ mod 2 then the assertion of the lemma about R_n is just the definition of R. In each of the other cases an inductive argument shows the lemma to be true. Let X be any of the sets above except R. Let S_i be the set of initial segments (not necessarily proper) of length i of words in X. We wish to prove $S_i \simeq (\omega_\kappa \times \mathbb{Q})^i$ if S is Q and $S_i \simeq \omega_\kappa^i$ otherwise. The assertion is obviously

true for S_1, so we are trying to show that $S_n \simeq (S_1)^n$, for all $n < \omega$. Suppose it is true when $j = n$. Then

$$\begin{aligned} S_{n+1} &= \sum_{x \in S_n} Y_x \qquad Y_x \simeq S_1 \text{ for all } x \in S_n \\ &\simeq (S_1)^n \times S_1 \\ &\simeq (S_1)^{n+1} \end{aligned}$$

which proves the assertion. Now if X is Q then $Q_n = S_n$ so the lemma is true. Otherwise

$$\begin{aligned} S_n = &\{\alpha_1\alpha_2 \ldots \alpha_n \mid n < \omega,\ \alpha_i \in \omega_\kappa,\ \alpha_n \text{ a successor }\} \\ &\cup \{\alpha_1\alpha_2 \ldots \alpha_n \mid n < \omega,\ \alpha_i \in \omega_\kappa,\ \alpha_n \text{ a limit }\} \\ &\cup \{\alpha_1\alpha_2 \ldots \alpha_n \mid n < \omega,\ \alpha_i \in \omega_\kappa,\ \alpha_n = 0\}. \end{aligned}$$

Since both the first two setsare moieties of $S_n$with order type ω_κ^n and they are also N_n and P_n the result follows for N, P and M. It is then immediate for R. Similarly

$$\begin{aligned} P_n &= \bigcup_{\substack{\omega_\lambda < \omega_\kappa \\ \omega_\lambda \text{ regular}}} \{\alpha_1\alpha_2 \ldots \alpha_n \mid \alpha_n \text{ an } \omega_\lambda\text{-limit }\} \\ &= \bigcup_{\substack{\omega_\lambda < \omega_\kappa \\ \omega_\lambda \text{ regular}}} P_n^{(\lambda)} \end{aligned}$$

and these are moieties of P_n, each with order type ω_κ^n.

Lemma 2.2. *Every interval of M embeds every ordinal $< \omega_{\kappa+1}$ and every countable reverse ordinal.*

Proof. Assume that γ is an ordinal which is less than $\omega_{\kappa+1}$ and that all intervals of M embed η for all ordinals $\eta < \gamma$. Suppose $u,\ v \in M$ with $u < v$, and $u = \alpha_1\alpha_2 \ldots \alpha_n$ and $v = \beta_1\beta_2 \ldots \beta_m$.
If $n > m$ we have

$$\omega_\kappa \simeq \{\alpha_1\alpha_2 \ldots \alpha_n\gamma \mid 0 < \gamma < \omega_\kappa\} \subset (u, v)$$

and if $n \leqslant m$ we have

$$\omega_\kappa \simeq \{\beta_1\beta_2\ldots\beta_{m-1}(\beta_m - 1)\gamma \mid 0 < \gamma < \omega_\kappa\} \subset (u,v)$$

so (u,v) embeds ω_κ. Hence it embeds cof(γ) so we may assume that cof$(\gamma) < \gamma$. Then we can write γ as the sum of cof(γ) smaller ordinals and by embedding them in the appropriate intervals of (u,v) we can construct an embedding of γ into (u,v).

This also shows that M is dense, and it therefore embeds every countable reverse ordinal.

Lemma 2.3. *If X is a subset of M which is dense in M and $\alpha_1\alpha_2\ldots\alpha_n \in M$ then there is a word in X which has $\alpha_1\alpha_2\ldots\alpha_n$ as an initial segment. Therefore every interval of X embeds every countable reverse ordinal and every ordinal less than $\omega_{\kappa+1}$.*

Proof. Let $\alpha_1\alpha_2\ldots\alpha_n \in M$. Then $\alpha_1\alpha_2\ldots\alpha_{n-1}(\alpha_n+1)$ is also in M. Any element of X between $\alpha_1\alpha_2\ldots\alpha_n$ and $\alpha_1\alpha_2\ldots\alpha_{n-1}(\alpha_n+1)$ must have $\alpha_1\alpha_2\ldots\alpha_n$ as an initial segment.

Let γ be a countable reverse ordinal or an ordinal less than $\omega_{\kappa+1}$ and let g be an embedding of γ into M. If $\alpha \in \gamma$ then let x_α be a word in X which has $g(\alpha)$ as an initial segment. The set $\{x_\alpha \mid \alpha \in \gamma\}$ has order type γ.

Lemma 2.4. *If X is a subset which is dense in M then* ordertype(X) *is an $\eta_{\omega_1,\omega_{\kappa+1}}$-universal – in particular the ordertype of M is an $\eta_{\omega_1,\omega_{\kappa+1}}$-universal.*

Proof. Since $M_n \simeq \omega_\kappa^n$ by Lemma 2.1 and $M = \bigcup_{n<\omega} M_n$ we know that M is a countable union of scattered sets. Obviously this also shows that $|M| = \aleph_\kappa$. Since $\omega_1^* \nleq M_n$, for any n and since M is the union of only countably many sets M_n we know $\omega_1^* \nleq M$. The same facts must be true for any subset X of M. If X is also dense in M then Lemma 2.3 shows that all intervals of X embed α^* and β for all $\alpha < \omega_1$ and $\beta < \omega_{\kappa+1}$, whence ordertype$(X)$ is an $\eta_{\omega_1,\omega_{\kappa+1}}$-universal.

Corollary. *The order types of N, R, P and $P^{(\lambda)}$ for all $\omega_\lambda < \omega_\kappa$, are all $\eta_{\omega_1,\omega_{\kappa+1}}$-universals.*

Proof. By Lemma 1.3 these sets are all dense in M so the result follows.

Lemma 2.5. *The order type of Q is an $\eta_{\omega_1,\omega_\kappa+1}$-universal.*

Proof. By Lemma 2.1 we know that Q is a countable union of scattered sets. Indeed the words of any particular length are a countable union of scattered sets and the whole set is the (countable) union of the subsets each consisting of the words of a particular length. It also follows immediately from Lemma 2.1 that $\omega_1^* \not\leqslant Q_j$ for any $j < \omega$ and hence $\omega_1^* \not\leqslant Q$ and that Q has cardinality $\aleph_\kappa$.

Assume that β is an ordinal of cardinality ($\leqslant$) $\aleph_\kappa$ and that all intervals of Q embed γ for all ordinals $\gamma < \beta$. Suppose $u, v \in Q$ with $u < v$. If $n = \max(\text{length}(u), \text{length}(v))$ then $u, v \in Q_1 \cup \ldots \cup Q_n$. We can pick $x \in (u, v) \cap Q_n$ and then

$$\omega_\kappa \times \mathbb{Q} \simeq \{x\alpha q \mid \alpha q \in \omega_\kappa \times \mathbb{Q}\} \subset (u, v).$$

Then since $\text{cof}(\beta) \leqslant \omega_\kappa$ and (u, v) embeds ω_κ it embeds $\text{cof}(\beta)$ so we may assume that $\text{cof}(\beta) < \beta$. Then we can write β as the sum of $\text{cof}(\beta)$ smaller ordinals and by embedding them in the appropriate intervals of (u, v) we can construct an embedding of β into (u, v).

Corollary (to Lemmas 2.4 and 2.5). *If X is any of $M, N, P, P^{(\lambda)}, R$ or Q and $x \in X$ then the coinitiality of x in X is ω^*.*

Proof. All these sets have order types which are $\eta_{\omega_1,\omega_\kappa+1}$-universals whence the set are dense and have no uncountable descending chains. The corollary follows immediately.

Lemma 2.6. *If X and Y are any two of M, N, P, Q, R and $P^{(\lambda)}$, for any ω_λ then* $\text{ordertype}(X) \equiv \text{ordertype}(Y)$.

Proof. This is implied by the corollary to Theorem 0.6, since by Lemma 2.5 and the corollary to Lemma 2.4 the order types of all these sets are $\eta_{\omega_1,\omega_\kappa+1}$-universals.

It is self-evident that we can form a new $\eta_{\omega_1,\omega_\kappa+1}$-universal with cofinality ω_λ by forming the product $\omega_\lambda \times X$, for any regular ω_λ with $\omega \leqslant \omega_\lambda < \omega_\kappa$ where X is any of $M, N, P, P^{(\lambda)}, Q$ or R.

Lemma 2.7. *There are $2^{\aleph_\kappa}$ non-isomorphic subsets of M which are $\equiv$-equivalent to $\eta_{\omega_1,\omega_{\kappa+1}}$. Hence there are $2^{\aleph_\kappa}$ different order types which are $(\mathcal{U}, \eta_{\omega_1,\omega_{\kappa+1}})$-universals, with $\mathcal{U} = \{0,1\}$.*

Proof. Let $W := \{\alpha \in \omega_\kappa \mid \alpha$ has a finite tail which is odd $\}$. Note that ordertype$(W) = \omega_\kappa$. Let X be a subset of W. Now define

$$M_X := \{\alpha_1\alpha_2\ldots\alpha_n \mid n < \omega,\ \alpha_1 \in W,\ \alpha_n \neq 0,\ \alpha_n \text{ a successor if } \alpha_1 \in X\} \cup \{x = \alpha + 1 \mid \alpha \in W\}.$$

Notice that if $\alpha \in W$ then $(\alpha+1, \alpha+2)_{M_X} = \emptyset$ for all X.

Suppose that $X, Y \subseteq W$ and that $f : M_X \to M_Y$ is an isomorphism. We can write

$$\text{ordertype}(M_X) = \sum_{\alpha \in W} 1 + \phi_\alpha + 1$$
$$\text{ordertype}(M_Y) = \sum_{\alpha \in W} 1 + \psi_\alpha + 1$$

where ϕ_α = ordertype(M) if $\alpha \notin X$ and ϕ_α = ordertype(N) if $\alpha \in X$ and similarly for ψ_α. It is then easy to see that each one element word $x = (\alpha + 1)$ where $\alpha \in W$ must be mapped to itself by f because these words are the only elements of M_X or M_Y which have successors and they form a well-ordered subset of M_X or M_Y. Then f must be an isomorphism between sets of order type $1 + \phi_\alpha$ and $1 + \psi_\alpha$ for all $\alpha \in W$. Since $N \not\preccurlyeq M$ this implies $\alpha \in X$ if and only if $\alpha \in Y$ so $X \simeq Y$.

This means there are at least $2^{\aleph_\kappa}$ subsets M_X of M which are all pair-wise non-isomorphic. Obviously then ordertype$(M_X) \preccurlyeq$ ordertype(M) and since (in the notation above) ϕ_α = ordertype(M) if $\alpha \notin X$ we have ordertype$(M) \preccurlyeq$ ordertype(M_X). By the second corollary to Lemma 0.6 M_X is an $(\mathcal{U}, \eta_{\omega_1,\omega_{\kappa+1}})$-universal with $\mathcal{U} = \{0,1\}$. This completes the proof of the lemma.

Now we will see that, if ω_κ is regular, then a subset which is dense in N, with dense complement, has its order type completely determined. If ω_κ is singular then denseness in N alone is enough to determine order type.

Before a proof that denseness with dense complement determines the order type of a subset of N (in fact it determines the order type of a subset of any interval of N with cofinality δ_κ) some more technical lemmas are needed. To prove the assertion an automorphism of N which restricts to an isomorphism between two subsets X and Y satisfying the denseness conditions will be defined, using induction on the length of words. This is facilitated by the fact that $N_1 \cup \ldots \cup N_n$ is well-ordered for any $n < \omega$. The isomorphism will be defined at each stage on a subset of X containing $X \cap (N_1 \cup \ldots \cup N_n)$ and a subset of Y containing $Y \cap (N_1 \cup \ldots \cup N_n)$ and similar sets containing all the words of length $\leqslant n$ in the complements of X or Y. To ensure that the isomorphism constructed can be extended to an order-isomorphism on longer words it is necessary to ensure that sequences in X *with* a supremum in X are being mapped onto the same thing in Y, and similarly for sequences in X without a supremum in X. The first chapter showed that all the elements of X or Y have cofinality $\delta_\kappa = \text{cof}(\omega_\kappa)$ so the only sequences which may cause a problem are δ_κ-sequences.

The first of the following lemmas shows that, in N, a sequence with supremum x essentially consists of elements which are longer than x (so if the mapping is defined on all elements shorter than n then the supremum of a sequence is not going to be added in at a later stage). Obviously, however, the set X could contain all the elements of length say 2, while Y has none of these. So at the second stage of the induction elements of length 2 in X may be mapped to much longer elements of Y. Thus the point of Lemmas 2.9, 2.10 and 2.11 is to show that we can find subsets of Y with the correct order type and whose δ_κ-sequences have a least upper bound in Y or not depending on whether the sequences which are mapping to them have one in X. In Lemmas and Theorems 2.8 to 2.15 an interval (x, y) will always be an interval in N.

Lemma 2.8. *If* $x = \sup_N(x_\mu)_{\mu<\delta_\kappa}$ *then there exists* $\mu < \delta_\kappa$ *such that* $\text{length}(x) < \text{length}(x_\mu)$.

Proof. Suppose that if $x = \beta_1\beta_2 \ldots \beta_m$ and let $y := \beta_1\beta_2 \ldots \beta_{m-1}(\beta_m - 1)$. Note that $(\beta_m - 1)$ exists since $x \in N$ but that y need not be in N. Nevertheless, since

$x = \sup(x_\mu)$ we must have some μ_0 such that $y < x_\mu < x$ in the lexicographic order, for all $\mu \geqslant \mu_0$. But this implies that

$$x_\mu = \beta_1\beta_2 \dots \beta_{m-1}(\beta_m - 1)\gamma_1\gamma_2 \dots \gamma_{l_\mu}$$

for some $l_\mu \geqslant 1$. Thus $\text{length}(x) < \text{length}(x_\mu)$ for all $\mu \geqslant \mu_0$.

Consider the set M and let X be a convex subset of M. Let m be the smallest integer such that $M_m \cap X$ is not empty. Suppose $x = \alpha_1\alpha_2 \dots \alpha_m \in X$. Then all the elements of X must have $\alpha_1\alpha_2 \dots \alpha_{m-1}$ as an initial segment. Otherwise if we have say $\alpha_1\alpha_2 \dots \alpha_{m-2}\beta$ with $\beta > \alpha_{m-1}$ occurring as an initial segment of something in X, then the convexity of X implies $\alpha_1\alpha_2 \dots \alpha_{m-2}\beta \in X$, contradicting the fact that m is the length of the shortest element in X. Obviously then $\text{ordertype}(X \cap M_m) \leqslant \omega_\kappa$. In N all this is still true but it needs more proof since β could be a limit ordinal, in which case $\alpha_1\alpha_2 \dots \alpha_{m-2}\beta \notin N$.

Lemma 2.9. *Remember that δ_κ is the cofinality of ω_κ. Let X be a non-empty convex subset of N. Let m be the smallest integer such that $X \cap N_m \neq \emptyset$. Then*

(1) *no δ_κ-sequence in $X \cap N_m$ has a least upper bound in X;*

(2) ordertype$(X \cap N_m) \leqslant \omega_\kappa$ *and if we have equality then $X \cap N_m$ is cofinal in X.*

Proof.

(1) Let $y \in N$. By Lemma 2.8 if y is the supremum of a δ_κ-sequence in $X \cap N_m$ then $\text{length}(y) < m$ so $y \notin X$.

(2) Let $x = \beta_1\beta_2 \dots \beta_m \in X \cap N_m$. Define

$$S := \{\beta_1\beta_2 \dots \beta_{m-1}\lambda \mid \lambda \in \omega_\kappa,\ \lambda \text{ a successor ordinal}\} \simeq \omega_\kappa.$$

Then we wish to prove that $X \cap N_m \subseteq S$.

Suppose that $v \in X \cap N_m$, that $v < x$ and that for some $\alpha_l < \beta_l$ we have $v = \beta_1\beta_2 \dots \beta_{l-1}\alpha_l \dots \alpha_m$. If $v < y$ for all $y \in S$ then we must have $l < m$. We can then define $u := \beta_1\beta_2 \dots \beta_{l-1}(\alpha_l + 1)$ and $v < u < x$ so $u \in X$ and $\text{length}(u) < m$ which is a contradiction. Hence there are no elements of $X \cap N_m$ which are smaller than all the elements of S.

Let $y := \beta_1\beta_2 \ldots \beta_{m-2}(\beta_{m-1}+1)$ and suppose there exists $z \in X$ with $z > v$ for all $v \in S$. Since $y = \sup_N(S)$ it follows that $x < y \leqslant z$ and so $y \in X$. This is a contradiction since length$(y) < m$. Thus S is cofinal in X. This, together with the paragraph above shows there are no elements of $X \cap N_m$ which are greater than or smaller than all the elements of S and so $S \subset N_m$ implies $X \cap N_m \subseteq S$. Obviously ordertype$(X \cap N_m) \leqslant$ ordertype$(S) = \omega_\kappa$ and if we have equality $X \cap N_m$ is cofinal in S and hence also in X.

Lemma 2.10. *Let X be a subset which is dense in N. Let $x, y \in X$ with $x < y$ and let β be any ordinal $< \omega_\kappa$. Then there exists $B \subset (x,y) \cap X$ such that* ordertype$(B) = \beta$ *and*

(1) *all δ_κ-sequences contained in B have no supremum in N.*

Proof. Suppose that $\text{cis}(x,y) = \alpha_1\alpha_2 \ldots \alpha_m$ so $x = \alpha_1\alpha_2 \ldots \alpha_m\alpha_{m+1} \ldots \alpha_n$ and $y = \alpha_1\alpha_2 \ldots \alpha_m\beta_{m+1} \ldots \beta_m$ with $\alpha_{m+1} < \beta_{m+1}$ or length$(x) = m$. Since X is dense and $X \cap N_m$ is well-ordered for any $m < \omega$, there can be no bound on the length of elements in any interval of X so let $z \in (x,y)_X$ with length$(z) = p > n, m$. So $z = \alpha_1\alpha_2 \ldots \alpha_m\gamma_{m+1} \ldots \gamma_p$. Then there exists j with $m < j \leqslant p$ such that $\alpha_i = \gamma_i$ for $i = m+1, \ldots, j-1$ and $\alpha_j < \gamma_j$ or $j < p$ and length$(x) = j$. Also there exists $k \leqslant m$ such that $\gamma_i = \beta_i$ for $i = m+1, \ldots, k-1$ and $\gamma_k < \beta_k$. Then if we define $C_0 := \{z\lambda \mid \lambda \text{ a successor in } \omega_\kappa\}$ we have that

$$C_0 \simeq \{\delta_{p+1} \mid \delta_1\delta_2 \ldots \delta_{p+1} \in C_0\} \simeq \omega_\kappa.$$

By Lemma 2.3 for every $v \in C_0$ there exists $v' \in X$ which has v as an initial segment. Define $C := \{v' \mid v \in C_0\}$. Then

$$C \simeq \{\delta_{p+1} \mid \delta_1\delta_2 \ldots \delta_r \in C\} \simeq \omega_\kappa$$

and moreover every element of C is determined by its $(p+1)$th letter. Also if $u \in C$ then $u = \alpha_1\alpha_2 \ldots \alpha_m\gamma_{m+1} \cdots \gamma_p\eta_{p+1}\eta_{p+2} \ldots \eta_r$. We already know there exists j with $m < j \leqslant p$ such that $\alpha_i = \gamma_i$ for $i = m+1, \ldots, j-1$ and $\alpha_j < \gamma_j$ or $j < p$ and length$(x) = j$ and also there exists $k \leqslant m$ such that $\gamma_i = \beta_i$ for

$i = m+1, \ldots, k-1$ and $\gamma_k < \beta_k$. Hence $x < u < y$ so $u \in (x,y)$ and since u was an arbitrary element of C we have $C \subset (x,y)$.

Now if $\beta < \omega_\kappa$ then we can find an initial segment B of C of order type β. Notice that if ω_κ is regular so $\delta_\kappa = \omega_\kappa$ then B has no δ_κ-sequences contained in it and (1) is vacuously true. Assume then that $\delta_\kappa < \omega_\kappa$. Let $(x_\mu)_{\mu<\delta_\kappa}$ be a δ_κ-sequence in B. Then

$$\{\alpha_{p+1} \mid \alpha_{p+1} \text{ is the } (p+1)\text{th letter of } x_\mu \text{ for some } \mu < \delta_\kappa\} \simeq \delta_\kappa$$

so the supremum of this set of ordinals is a limit ordinal, say γ. Suppose $v = \delta_1\delta_2\ldots\delta_s$ is any upper bound of the sequence in N. Then, since δ_s must be a successor, we know $\delta_s \neq \gamma$ so $\delta_1\delta_2\ldots\delta_{s-1}(\delta_s - 1)1$ is a lower upper bound for the sequence. This shows that B satisfies (1).

Lemma 2.11. *Let X be any subset of N which has cofinality δ_κ and is dense in its convex closure in N. Let U_0 be a subset of X which has order type $\leqslant \omega_\kappa$, with U_0 cofinal in X if equality holds. Suppose U_0 satisfies the condition*

(1) *all δ_κ-sequences contained in it have no supremum in X.*

Then there exists a cofinal subset U of X, with order type ω_κ and with $U_0 \subseteq U$, which also satisfies (1).

Proof. Suppose first that ω_κ is a regular initial ordinal. Then choose U to be any cofinal set in X which contains U_0 and has order type ω_κ, (as guaranteed by Lemma 2.3). Since $\delta_\kappa = \omega_\kappa$ we know U must satisfy (1) as there are no δ_κ-sequences in U which are bounded in X and hence no δ_κ-sequences with a least upper bound in X.

Suppose then that ω_κ is singular so $\delta_\kappa < \omega_\kappa$. We can write ω_κ as a δ_κ-sum of ordinals $\omega_\kappa = \sum_{\mu<\delta_\kappa} \gamma_\mu$ where $\gamma_\mu < \omega_\kappa$ for all μ, and use this to find U.

If ordertype$(U_0) = \omega_\kappa$ then we simply take $U := U_0$. If ordertype$(U_0) < \omega_\kappa$ then let U_0' be any cofinal subset of X with ordertype$(U_0') = \delta_\kappa$. We can write $U_0' = (u_\nu)_{\nu<\delta_\kappa}$. Since ordertype$(U_0) < \omega_\kappa$ there exists $\nu_0 < \delta_\kappa$ such that $\gamma_\nu \not\leq$ ordertype(U_0) for all $\nu > \nu_0$ so certainly ordertype$((u_\nu, u_{\nu+1}) \cap U_0) < \gamma_\nu$, for all

$\nu > \nu_0$. Hence Lemma 2.10 (together with its proof) shows we can find a set U_ν with order type γ_ν such that

$$(u_\nu, u_{\nu+1}) \cap U_0 \subset U_\nu \subset (u_\nu, u_{\nu+1}) \cap X$$

and such that no δ_κ-sequences in U_v have a supremum in N, and therefore there are definitely no δ_κ-sequences in U_v with a supremum in X so U_v satisfies (1). Now define

$$U := U_0 \cup \left(\bigcup_{\nu_0 < \nu < \delta_\kappa} U_\nu \right)$$

Then any bounded δ_κ-sequence in U has a terminal segment in U_ν for some ν or is contained in U_0, and so has no supremum in X. Any δ_κ-sequence cofinal in U is cofinal in X and therefore has no supremum in X. Thus U satisfies (1). Moreover $U_0 \subset U$ and ordertype$(U) = \omega_\kappa$ so the lemma is true.

Remember that for $X \subseteq Y$ and an element x of Y we defined

$$(\leftarrow, x)_X := \{v \in Y \mid v < x \text{ and if } u \in X \text{ with } u < x \text{ then } u < v\}.$$

Remember also that (if Y is dense) this definition implies that $(\leftarrow, x)_X = \emptyset$ if and only if we can find a sequence in X which has supremum x when the sequence is considered as a sequence in Y. Since δ_κ was defined to be the cofinality of ω_κ, in the case of N we would have to have a δ_κ-sequence in $X \subset N$, with supremum x in N if we are to have $(\leftarrow, x)_X = \emptyset$.

Note 1. If X is a convex subset of N with cofinality δ_κ and U, V are subsets of X which are cofinal in X, both with order type ω_κ, then $U \cup V \simeq \omega_\kappa$.

Note 2. If $U \subseteq N_i$ for some $i < \omega$ then $(\leftarrow, u)_U \neq \emptyset$ for all $u \in U$. Indeed, if $u = \alpha_1\alpha_2 \ldots \alpha_n$ then $\alpha_1\alpha_2 \ldots \alpha_{n-1}(\alpha_n - 1)1 \in (\leftarrow, u)_U$.

Note 3. Suppose $U \subset N^U$, where N^U is a convex subset of N with coinitiality ω. If all δ_κ-sequences in U have no suprema then $(\leftarrow, u)_U \cap N^U \neq \emptyset$ for all $u \in U$. Otherwise, of course, we would have to have u the supremum of a δ_κ-sequence contained in U or u the least element of N^U.

Theorem 2.12. *Let N^X, N^Y be convex subsets of N, both with cofinality δ_κ and coinitiality ω^*. Let X, Y be subsets of N^X, N^Y, with X and $X' := N^X - X$ dense in N^X and Y and $Y' := N^Y - Y$ dense in N^Y. Then there exists an order isomorphism $f : N^X \to N^Y$ such that the restriction to X is an order isomorphism between X and Y.*

Proof. It is enough to prove that for all $m \geqslant 1$ there are cofinal sets $X_m \subset N^X$ and $Y_m \subset N^Y$ with order type ω_κ^m, and an order isomorphism $f_m : X_m \to Y_m$ such that

(1) $(N_1 \cup \ldots \cup N_m) \cap N^X \subseteq X_m$ and $(N_1 \cup \ldots \cup N_m) \cap N^Y \subseteq Y_m$;

(2) if $1 \leqslant i < m$ then $X_i \subset X_m$, $Y_i \subset Y_m$ and $f_i = f_m|_{X_i}$;

(3) if $u \in X \cap X_m$ then $f_m(u) \in Y \cap Y_m$ and if $u \in X' \cap X_m$ then $f_m(u) \in Y' \cap Y_m$;

(4) for all δ_κ-sequences $(v_\chi)_{\chi<\delta_\kappa}$ contained in X_m such that there exists $w = \sup_{N^X}(v_\chi)_{\chi<\delta_\kappa}$ we have that $w \in X_m$ and for all δ_κ-sequences $(v_\chi)_{\chi<\delta_\kappa}$ contained in Y_m such that there exists $w = \sup_{N^Y}(v_\chi)_{\chi<\delta_\kappa}$ we have that $w \in Y_m$;

(5) for all $x \in X_m$ we have $(\leftarrow, x)_{X_m} \cap N^X = \emptyset$ if and only if $(\leftarrow, f_m(x))_{Y_m} \cap N^Y = \emptyset$.

Then, since each f_m is an extension of the others by (2), the map $\bigcup_{m\geqslant 1} f_m$ will be an isomorphism between N^X and N^Y which restricts by (3) to an isomorphism between X and Y.

If $m = 1$ let j be the smallest integer such that ordertype$(N^X \cap N_j) \neq \emptyset$, and let l be the smallest integer such that ordertype$(N^Y \cap N_l) \neq \emptyset$. Lemma 2.9 shows ordertype$(X \cap N_j) \leqslant \omega_\kappa$ and ordertype$(Y \cap N_l) \leqslant \omega_\kappa$ and that these sets are cofinal in N^X and N^Y respectively if we have equality. It also shows that no δ_κ-sequences in $N^X \cap N_j$ have a supremum in N^X and no δ_κ-sequences in $N^Y \cap N_l$ have a supremum in N^Y. Hence Lemma 2.11 shows that we can find cofinal subsets U of X and V of Y, of order type ω_κ with $X \cap N_j \subseteq U$ and $Y \cap N_l \subseteq V$ where all δ_κ-sequences have no least upper bound in N^X or N^Y respectively. Note 3 shows $(\leftarrow, u)_U \neq \emptyset$ and $(\leftarrow, v)_V \neq \emptyset$ for all $u \in U$, $v \in V$.

Since U and V have the same order type let g be the order isomorphism between them. Now let $u \in U$ and consider the order type of

$$(\leftarrow, u)_U \cap X' \cap N_j \qquad \text{and} \qquad (\leftarrow, g(u))_V \cap Y' \cap N_l. \tag{$\dagger$}$$

They must both be less than ω_κ since they are bounded subsets of $N^X \cap N_j$ and $N^Y \cap N_l$ respectively, which both have order type $\leqslant \omega_\kappa$. If

$$(\leftarrow, u)_U \cap X' \cap N_j \simeq (\leftarrow, g(u))_V \cap Y' \cap N_l$$

let g_u be the order isomorphism between the two sets and define

$$U_u := (\leftarrow, u)_U \cap X' \cap N_j.$$

If the two sets in ($\dagger$) are not isomorphic assume that we have

$$\text{ordertype}((\leftarrow, u)_U \cap X' \cap N_j) < \text{ordertype}((\leftarrow, g(u))_V \cap Y' \cap N_l).$$

We know $(\leftarrow, u)_U \cap N^X \neq \emptyset$. The denseness of X' in N^X then implies that $(\leftarrow, u)_U \cap X' \neq \emptyset$. By Lemma 2.9 no δ_κ-sequences in $X' \cap N_j$ have a supremum in N^X, and since $(\leftarrow, u)_U \cap X' \cap N_j$ is bounded in N^X no δ_κ-sequences in it have a supremum in N (for $y = \sup_N (x_\gamma)_{\gamma < \delta_\kappa}$ and $y \notin N^X$ implies (x_γ) is cofinal in N^X). Similarly there are no δ_κ-sequences in $(\leftarrow, g(u))_V \cap Y' \cap N_l$ with a supremum in N. Therefore by Lemma 2.10 we can find U_u with

$$(\leftarrow, u)_U \cap X' \cap N_j \subset U_u \subset (\leftarrow, u)_U \cap X'$$

where U_u has the same order type as $(\leftarrow, g(u))_V \cap Y' \cap N_l$ and no δ_κ-sequences in U_u have a supremum in N. Let g_u be the isomorphism between U_u and $(\leftarrow, g(u))_V \cap Y' \cap N_l$.

A similar argument shows that if

$$\text{ordertype}((\leftarrow, u)_U \cap X' \cap N_j) > \text{ordertype}((\leftarrow, g(u))_V \cap Y' \cap N_l)$$

we can define $U_u := (\leftarrow, u)_U \cap X' \cap N_j$ and find $g_u : U_u \rightarrow (\leftarrow, g(u))_V \cap Y'$ such that $(\leftarrow, g(u))_V \cap Y' \cap N_l \subseteq g_u(U_u)$, and no δ_κ-sequences in $g_u(U_u)$ have a supremum in N. Now define

$$X_1 := U \cup \left(\bigcup_{u \in U} U_u \right)$$

$$Y_1 := V \cup \left(\bigcup_{u \in U} g_u(U_u) \right)$$

$$f_1(v) := \begin{cases} g(v) \text{ if } v \in U \\ g_u(v) \text{ if } v \in U_u \text{ for some } u \in U. \end{cases}$$

Then by Note 1, X_1 has order type ω_κ and by construction f_1 is an isomorphism (so Y_1 also has order type ω_κ). We constructed U to contain all of $X \cap N_j$. Since X and Y are dense in N^X and N^Y respectively they are cofinal in the latter sets whence U and V are also. If $v \in X' \cap N_j$ then the cofinality of U in N^X ensures that there is an element of U which is greater than v. If x is the least such element of U then $v \in (\leftarrow, x)_U$. Then

$$v \in (\leftarrow, x)_U \cap X' \cap N_j \subseteq U_x$$

so the fact that v was an arbitrary element of $X' \cap N_j$ shows that

$$X' \cap N_j \subseteq \bigcup_{x \in U} U_x.$$

Similarly we constructed V to contain $Y \cap N_l$ and by the same type of argument as above

$$Y' \cap N_l \subseteq \bigcup_{x \in U} g_x(U_x).$$

Since $N^X \cap N_j = (X \cap N_j) \cup (X' \cap N_j)$ and $N^Y \cap N_l = (Y \cap N_l) \cup (Y' \cap N_l)$ we have that $N^X \cap N_j \subseteq X_1$ and $N^Y \cap N_l \subseteq Y_1$. Obviously $j, l \geqslant 1$ whence $N^X \cap N_1 \subseteq X_1$ and $N^Y \cap N_1 \subseteq Y_1$ so part (1) of the inductive hypothesis is satisfied. Part (2) is not applicable when $m = 1$. Since g is an isomorphism between a subset of X and a subset of Y and the map $\bigcup_{u \in U} g_u$ is an isomorphism between a subset of X' and a subset of Y' we know that $f_1 : X_1 \rightarrow Y_1$ satisfies part (3) of the inductive hypothesis. Part (4) is true since, by construction, no δ_κ-sequence in X_1 has a supremum in N^X and no δ_κ-sequence in Y_1 has a supremum in N^Y. Note 3 shows that this implies $(\leftarrow, u)_{X_1} \cap N^X \neq \emptyset$ for all $u \in X_1$ and $(\leftarrow, v)_{Y_1} \cap N^Y \neq \emptyset$ for all $v \in Y_1$ so part (5) of the inductive hypothesis holds.

Suppose the inductive hypothesis holds when $m = n$. So X_n has order type ω_κ^n and we know that $Y_n = f_n(X_n)$ and $(N_1 \cup \ldots \cup N_n) \cap N^X \subseteq X_n$ and $(N_1 \cup \ldots \cup N_n) \cap N^Y \subseteq Y_n$. Suppose $x \in X_n$. Assume that $(\leftarrow, x)_{X_n} \neq \emptyset$. Then part (5) of the inductive hypothesis shows that $(\leftarrow, f_n(x))_{Y_n} \neq \emptyset$.

Then since $\mathrm{cof}_N(y) = \delta_\kappa$ for all $y \in N$ by Lemma 1.4 we know $(\leftarrow, x)_{X_n}$ and $(\leftarrow, f_n(x))_{Y_n}$ are both convex subsets of N with cofinality δ_κ and coinitiality ω^*. We are assuming that all words of length $\leqslant n$ are contained in X_n and Y_n. So when we define j_x, l_x to be the least integers such that $(\leftarrow, x)_{X_n} \cap N_{j_x} \neq \emptyset$ and $(\leftarrow, f_n(x))_{Y_n} \cap N_{l_x} \neq \emptyset$ we must then have $j_x, l_x \geqslant n+1$. The case $m = 1$ shows that we can find sets U_x and V_x of order type ω_κ which are cofinal in $(\leftarrow, x)_{X_n}$ and $(\leftarrow, f_n(x))_{Y_n}$ and an isomorphism $f_x : U_x \to V_x$, such that

(1)* $(\leftarrow, x)_{X_n} \cap N_{j_x} \subseteq U_x \subset (\leftarrow, x)_{X_n}$ and $(\leftarrow, f_n(x))_{Y_n} \cap N_{l_x} \subseteq V_x \subset (\leftarrow, f_n(x))_{Y_n}$;

(3)* $y \in X \cap U_x$ if and only if $f_x(y) \in Y \cap V_x$;

(4)* no δ_κ-sequences in $(\leftarrow, x)_{X_n}$ have a supremum in $(\leftarrow, x)_{X_n}$ and no δ_κ-sequences in $(\leftarrow, f_n(x))_{Y_n}$ have a supremum in $(\leftarrow, f_n(x))_{Y_n}$;

(5)* $(\leftarrow, u)_{U_x} \neq \emptyset$ and $(\leftarrow, f_x(u))_{V_x} \neq \emptyset$ for all $u \in U_x$.

Therefore, we can define

$$X_{n+1} := X_n \cup \left(\bigcup_{x \in X_n} U_x \right)$$

$$Y_{n+1} := Y_n \cup \left(\bigcup_{x \in X_n} V_x \right)$$

$$f_{n+1}(w) := \begin{cases} f_n(w) \text{ if } w \in X_n \\ f_x(w) \text{ if } w \in U_x \text{ for some } x \in X_n. \end{cases}$$

For each $x \in X$ we have $U_x \simeq \omega_\kappa$. Since X_n had order type ω_κ^n and the sets above are between the elements of X_n the order type of X_{n+1} is ω_κ^{n+1}. Since

$X_1 \subseteq X_i$ and $Y_1 \subseteq Y_i$ for all $i \leqslant n+1$ and X_1 and Y_1 are cofinal in N^X and N^Y respectively we know that X_i and Y_i are also cofinal for all $i \leqslant n+1$. By construction f_{n+1} is an isomorphism (so Y_{n+1} also has order type ω_κ^{n+1}).

Since

$$(\leftarrow, x)_{X_n} \cap N_{j_x} \subseteq U_x \qquad \text{and} \qquad (\leftarrow, f_n(x))_{Y_n} \cap N_{l_x} \subseteq V_x$$

for all $x \in X_n$, we know

$$\left(\bigcup_{x \in X_n} (N_{j_x} \cap N^X)\right) - X_n \subseteq \bigcup_{x \in X_n} (\leftarrow, x)_{X_n} \cap N_{j_x} \subseteq \bigcup_{x \in X_n} U_x$$

and

$$\left(\bigcup_{x \in X_n} (N_{l_x} \cap N^Y)\right) - Y_n \subseteq \bigcup_{x \in X_n} (\leftarrow, f_n(x))_{Y_n} \cap N_{l_x} \subseteq \bigcup_{x \in X_n} V_x.$$

Since $j_x, l_x \geqslant n+1$ and X_n, Y_n are cofinal in N^X, N^Y this shows that part (1) of the inductive hypothesis is satisfied. Part (2) of the inductive hypothesis holds since X_{n+1} is defined as a superset of X_n and f_{n+1} is defined as f_n on X_n. Part (3) holds since f_n satisfies (3) and f_x satisfies (3)* for all $x \in X_n$.

We are assuming that the fourth part of the inductive hypothesis holds on X_n and Y_n. Therefore the only δ_κ-sequences we have to consider are those with a terminal segment in U_x or V_x for some $x \in X_n$. The fact that U_x and V_x satisfy (4)* show that any δ_κ-sequences bounded in them have no supremum in N and therefore satisfy (4). Since we picked U_x and V_x to be cofinal in $(\leftarrow, x)_{X_n}$ and $(\leftarrow, f_n(x))_{Y_n}$ respectively, any cofinal sequences have x or $f_n(x)$ as their supremum. Obviously these elements are in X_{n+1} or Y_{n+1} so all δ_κ-sequences cofinal in them also satisfy (4).

Part (5) holds since U_x and V_x are cofinal in $(\leftarrow, x)_{X_{n+1}}$ and $(\leftarrow, f_{n+1}(x))_{Y_{n+1}}$ and so $(\leftarrow, x)_{X_{n+1}} = \emptyset$ and $(\leftarrow, f_{n+1}(x))_{Y_{n+1}} = \emptyset$ for all $x \in X_n$ and by construction $(\leftarrow, x)_{X_{n+1}} \neq \emptyset$ and $(\leftarrow, f_{n+1}(x))_{Y_{n+1}} \neq \emptyset$ for all $x \in X_{n+1} - X_n$. This completes the proof of the theorem.

In fact a stronger theorem is true if ω_κ is singular, for in this case denseness in N alone is enough to determine the order type of a subset. Obviously Theorem

2.12 is a special case of this result, when ω_κ is singular. Thus it is only necessary to prove Theorem 2.12 in the regular case. However, it is interesting to see that the same techniques can be used in the singular case, even though extra arguments are needed at some points. Lemmas 2.10 and 2.11, whose proofs are much simpler in the regular case, are also needed in the proof of Theorem 2.15.

Assume then that ω_κ is singular. If X is a dense subset of N then for each $m < \omega$ we wish to construct a mapping between $(\leftarrow, x)_{N_m} \cap N_{m+1}$ and the set $(\leftarrow, x)_{X_m} \cap X \cap N_{m+1}$, for some subset X_m of X. Lemma 2.9 shows that the first of these sets has order type ω_κ and is cofinal in $(\leftarrow, x)_{N_m}$. The next lemma shows what conditions are needed on X_m to ensure that $(\leftarrow, x)_{X_m} \cap N_{m+1}$ does not have order type larger than ω_κ and that an ω_κ-sequence in this set is cofinal in $(\leftarrow, x)_{X_m}$. In what follows X' is defined to be $N - X$ for any subset X of N.

Lemma 2.13. *Let m be a positive integer. Let $X \subset N$ and let $X_m \subset X$ satisfying*

(1) *$x \in N_1 \cup \ldots \cup N_{m-1}$ implies there exists a δ_κ-sequence $(x_\mu)_{\mu<\delta_\kappa}$ contained in X_m with $x = \sup_N(x_\mu)$;*

(2) *$x \in X \cap N_m$ implies $x \in X_m$;*

(3) *$x \in X' \cap N_m$ implies there exists a δ_κ-sequence $(x_\mu)_{\mu<\delta_\kappa}$ contained in X_m with $x = \sup_N(x_\mu)$.*

Then for any $y \in X_m$ we have that $\text{ordertype}((\leftarrow, y)_{X_m} \cap N_{m+1}) \leqslant \omega_\kappa$ *and $(\leftarrow, y)_{X_m} \cap N_{m+1}$ is cofinal in $(\leftarrow, y)_{X_m}$ if we have equality.*

Proof. Let $y \in X_m$. By (1) if $\text{length}(y) < m$ then $(\leftarrow, y)_{X_m} = \emptyset$. So assume $\text{length}(y) \geqslant m$, so $y = \beta_1\beta_2\ldots\beta_n$. If $n = m$ define

$$S := \{\beta_1\beta_2\ldots\beta_{m-1}(\beta_m - 1)\lambda \mid \lambda \text{ a successor in } \omega_\kappa\} \simeq \omega_\kappa.$$

Then we will prove that we have $(\leftarrow, y)_{X_m} \cap N_{m+1} \subseteq S$.

Suppose, seeking a contradiction, that there exists $v \in (\leftarrow, y)_{X_m} \cap N_{m+1}$ with v smaller than all the elements of S so $v = \beta_1\beta_2\ldots\beta_{l-1}\alpha_l\ldots\alpha_{m+1}$, where $l < m$ and $\alpha_l < \beta_l$ or $l = m$ and $\alpha_m < \beta_m - 1$. Define $u := \beta_1\beta_2\ldots\beta_{l-1}(\alpha_l + 1)$. If $u \in X_m$ then $v < u < y$ implies $v \notin (\leftarrow, y)_{X_m}$. If $u \notin X_m$ then since length$(u) \leqslant m$ we can find a δ_κ-sequence $(x_\mu)_{\mu<\delta_\kappa}$ which is contained in X_m with $u = \sup_N(x_\mu)$. But then there exists $\nu < \delta_\kappa$ such that $v < x_\nu < u < y$ and again $v \notin (\leftarrow, y)_{X_m}$. This contradiction shows that there are no elements of $(\leftarrow, y)_{X_m} \cap N_{m+1}$ which are smaller than all the elements of S.

If $n > m$ then we can define

$$S := \{\beta_1\beta_2 \ldots \beta_m\lambda \mid \lambda \text{ a successor in } \omega_\kappa,\ \lambda \leqslant \beta_{m+1}\} \simeq \beta_{m+1} + 1.$$

If there exists $v \in (\leftarrow, y)_{X_m} \cap N_{m+1}$ which is smaller than all the elements of S then $v = \beta_1\beta_2 \ldots \beta_{l-1}\alpha_l \ldots \alpha_{m+1}$, where $\alpha_l < \beta_l$ and $l \leqslant m$. Thus we can define $u := \beta_1\beta_2 \ldots \beta_{l-1}(\alpha_l + 1)$ and $v < u < y$ and length$(u) \leqslant m$. As above we can use u to show $v \notin (\leftarrow, y)_{X_m}$. This contradiction shows there are no elements of $(\leftarrow, y)_{X_m}$ smaller than all the elements of S. Obviously there are no elements of $(\leftarrow, y)_{X_m} \cap N_{m+1}$ which are greater than all the elements of S in either case. Thus $S \subset N_{m+1}$ implies $(\leftarrow, y)_{X_m} \cap N_{m+1} \subseteq S$. This implies that

$$\text{ordertype}((\leftarrow, y)_{X_m} \cap N_{m+1}) \leqslant \text{ordertype}(S) \leqslant \omega_\kappa.$$

If we have

$$\text{ordertype}((\leftarrow, y)_{X_m} \cap N_{m+1}) = \omega_\kappa$$

then $m = n$, so S is cofinal in $(\leftarrow, y)_{X_m}$. In this case $(\leftarrow, y)_{X_m} \cap N_{m+1}$ is cofinal in S and hence also in $(\leftarrow, y)_{X_m}$.

Notice also that if $(v_\mu)_{\mu<\delta_\kappa}$ is contained in $(\leftarrow, y)_{X_m} \cap N_{m+1}$ and is not cofinal then if we define $\alpha_\mu :=$ the $(m+1)$th letter of v_μ then $(\alpha_\mu)_{\mu<\delta_\kappa}$ is a δ_κ-sequence of ordinals, with a least upper bound γ (a limit ordinal). If $u = \delta_1\delta_2 \ldots \delta_r$ is an upper bound for $(v_\mu)_{\mu<\delta_\kappa}$ then $\delta_r \neq \gamma$ since δ_r must be a successor. Hence $\delta_1\delta_2 \ldots \delta_{r-1}(\delta_r - 1)1$ is a smaller upper bound. So no bounded δ_κ-sequence in $(\leftarrow, y)_{X_m} \cap N_{m+1}$ has a least upper bound.

The conditions on X_m in the last lemma were chosen to facilitate the proof. The next lemma shows that these conditions can be formulated differently so that they are more convenient to use when proving the theorem.

Lemma 2.14. *Let X be a subset of N and X_m a subset of X. Suppose $(N_1 \cup \ldots \cup N_m) \cap X \subseteq X_m$ and for every element $u \in X' \cap N_i$ for $i \leqslant m$ there is a δ_κ-sequence in X_m with supremum u in N. Then $x \in N_1 \cup \ldots \cup N_{m-1}$ implies there is a δ_κ-sequence in X_m with supremum x.*

Proof. If $x \in (N_1 \cup \ldots \cup N_{m-1}) \cap X'$ then the assertion is one of the conditions stated. Suppose then that $x = \alpha_1\alpha_2 \ldots \alpha_n \in (N_1 \cup \ldots \cup N_{m-1}) \cap X$. Let $(\mu)_{\mu<\delta_\kappa}$ be a cofinal sequence of successors in ω_κ. For each $\mu < \omega_\kappa$ define

$$x_\mu := \alpha_1\alpha_2 \ldots \alpha_{n-1}(\alpha_n - 1)\mu.$$

Notice that length$(x_\mu) = n + 1 \leqslant m$ for all μ. Now if there is a cofinal subsequence of $(x_\mu)_{\mu<\delta_\kappa}$, say $(x_{\mu'})_{\mu'<\delta_\kappa}$ with $x_{\mu'} \in (N_1 \cup \ldots \cup N_m) \cap X$ then since $(N_1 \cup \ldots \cup N_m) \cap X \subseteq X_m$, again the assertion of the lemma is true. Otherwise, there exists $\mu_0 < \delta_\kappa$ with $x_\mu \in (N_1 \cup \ldots \cup N_m) \cap X'$ for all $\mu > \mu_0$. Then, for all $\mu > \mu_0$ we can find $(y^\mu_\nu)_{\nu<\delta_\kappa}$ contained in X_m with $\sup_N(y^\mu_\nu) = x_\mu$. But then inductively, for each $\mu > \mu_0$ we can pick ν_μ such that $y^\mu_{\nu_\mu} > y^\chi_{\nu_\chi}$ for all $\chi < \mu$. Then $(y^\mu_{\nu_\mu})_{\mu<\delta_\kappa}$ is contained in X_m and has supremum x in N. Thus the lemma is true.

Theorem 2.15. *Suppose ω_κ is singular. Let X be a subset of N, which is dense in N. Then $X \simeq N$.*

Proof. It is enough to prove that for all $m \geqslant 1$ there are cofinal sets $X_m \subset X$ and $Y_m \subset N$ with order type ω_κ^m, and an order isomorphism $f_m : X_m \to Y_m$ such that

(1) $(N_1 \cup \ldots \cup N_m) \cap X \subseteq X_m$ and $N_1 \cup \ldots \cup N_m \subseteq Y_m$;

(2) if $1 \leqslant i < m$ then $X_i \subset X_m$, $Y_i \subset Y_m$ and $f_i = f_m|_{X_i}$;

(3) if $u \in X' \cap N_m$ then there is a δ_κ-sequence in X_m with supremum u in N;

(4) for all δ_κ-sequences $(v_\chi)_{\chi<\delta_\kappa}$ contained in X_m such that there exists $w = \sup_X(v_\chi)_{\chi<\delta_\kappa}$ we have that $w \in X_m$ and for all δ_κ-sequences $(v_\chi)_{\chi<\delta_\kappa}$ contained in Y_m such that there exists $w = \sup_N(v_\chi)_{\chi<\delta_\kappa}$ we have that $w \in Y_m$;

(5) for all $x \in X_m$ we have $(\leftarrow, x)_{X_m} = \emptyset$ if and only if $(\leftarrow, f_m(x))_{Y_m} = \emptyset$.

Then, since by (2) each f_m is an extension of the others, the map $\bigcup_{m\geqslant 1} f_m$ will be an isomorphism between X and N.

Let $(\eta_\nu)_{\nu<\delta_\kappa}$ be a sequence of ordinals of order type δ_κ which forms a proper initial segment of ω_κ and all of whose elements are successor ordinals. Let $(\sigma_\nu)_{\nu<\delta_\kappa}$ be a cofinal subset of ω_κ, of order type δ_κ, again all of whose elements are successor ordinals.

To show that there exist subsets X_1 and Y_1 satisfying the inductive hypothesis consider ordertype($X \cap N_1$) and ordertype(N_1). Obviously ordertype$(X \cap N_1) \leqslant \omega_\kappa$, if we have equality it is cofinal in X, and by Lemma 2.8 no δ_κ-sequences contained in this set have a least upper bound in N. Then by Lemma 2.11, we can find U a cofinal subset of X, with order type ω_κ, which contains $X \cap N_1$ and has no δ_κ-sequences with a supremum in N. Let f be the isomorphism from U onto N_1. Now let $u \in U$ so $f(u) = \gamma$ for some one element word consisting of the successor ordinal γ. Consider $(\leftarrow, u)_U \cap X' \cap N_1$. This must have order type $< \omega_\kappa$. If $y \in (\leftarrow, u)_U \cap X' \cap N_1$ then y is a one element word α, where α is some successor ordinal. For each two element word $(\alpha - 1)\sigma_\nu$ by Lemma 1.3 there is a word in X with initial segment $(\alpha - 1)\sigma_\nu$. For each ν pick one such word y_ν. Now define

$$U_u := \{y_\nu \mid y \in (\leftarrow, u)_U \cap X' \cap N_1,\ \nu < \delta_\kappa\}$$

$$f_u(y_\nu) := (\gamma - 1)\alpha\eta_\nu.$$

Notice that by the proof of Lemma 2.3 we have that (y_ν) is a δ_κ-sequence in X with supremum y in N, and (since $y \notin X$) no supremum in X, and that again by the proof of Lemma 1.3 that $(f_u(y_\nu))$ is a δ_κ-sequence in N with no supremum in N. Also $y_\nu \in (\leftarrow, u)_{N_1 \cap X}$ and $f_u(y_\nu) \in (\leftarrow, f(u))_{N_1}$ and we have $y_\nu < z_\mu$ and $f_u(y_\nu) < f_u(z_\mu)$ if $y < z$. Now define

$$X_1 := U \cup \left(\bigcup_{u \in U} U_u\right)$$

$$Y_1 := N_1 \cup \left(\bigcup_{u \in U} f_u(U_u)\right)$$

$$f_1(v) := \begin{cases} f(v) \text{ if } v \in U \\ f_u(v) \text{ if } v \in U_u \text{ for some } u \in U. \end{cases}$$

Then since $\delta_\kappa < \omega_\kappa$ by Note 1, $X_1 \simeq Y_1 \simeq \omega_\kappa$ and both were picked to be cofinal in N. By construction X_1 and Y_1 contain all the words of length 1 so they satisfy part (1) of the inductive hypothesis, and part (2) is not applicable when $m = 1$. For each $y \in N_1 \cap X'$ we have $y = \sup(y_\nu)_{\nu<\delta_\kappa}$ and $\{y_\nu \mid \nu < \delta_\kappa\} \subset X_1$ so X_1 satisfies part (3) of the inductive hypotheses. No δ_κ-sequences in N_1 or U have a supremum in N and the elements of U_u and $f_u(U_u)$ were chosen so that this is also true of them so part (4) holds. This implies, by Note 3, that $(\leftarrow, x)_{X_1} \neq \emptyset$ and $(\leftarrow, f_1(x))_{Y_1} \neq \emptyset$ for all $x \in X_1$, whence part (5) holds. So we can find X_1, Y_1 and f_1 which satisfy the inductive hypotheses.

Suppose the inductive hypotheses hold when $m = n$. So X_n has order type ω_κ^n and we know that $Y_n = f_n(X_n)$ and $(N_1 \cup \ldots \cup N_n) \cap X \subset X_n$ and $N_1 \cup \ldots \cup N_n \subset Y_n$. Suppose $x \in X_n$. Assume that $(\leftarrow, x)_{X_n} \neq \emptyset$. Then part (5) of the inductive hypothesis shows that $(\leftarrow, f_n(x))_{Y_n} \neq \emptyset$.

Lemmas 2.13 and 2.14 show that the inductive hypothesis implies

$$\text{ordertype}((\leftarrow, x)_{X_n} \cap N_{n+1}) \leqslant \omega_\kappa \tag{a}$$

$$\text{and ordertype}((\leftarrow, f_n(x))_{Y_n}) \cap N_{n+1}) \leqslant \omega_\kappa \tag{b}$$

and that we have equality in (a) only if length$(x) = n$ and equality in (b) only if length$(f_n(x)) = n$ and then the set on the left is cofinal in each case. Also Lemma 2.8, together with the inductive hypothesis, shows there are no δ_κ-sequences in $(\leftarrow, x)_{X_n} \cap N_{n+1}$ with a supremum in $(\leftarrow, x)_{X_n}$ and similarly for $(\leftarrow, f_n(x))_{Y_n} \cap N_{n+1}$. So Lemma 2.11 shows that we can find cofinal subsets U_x of $(\leftarrow, x)_{X_n} \cap X$ and V_x of $(\leftarrow, f_n(x))_{Y_n}$, of order type ω_κ with

$$X \cap N_{n+1} \cap (\leftarrow, x)_{X_n} \subseteq U_x$$

and

$$N_{n+1} \cap (\leftarrow, f_n(x))_{Y_n} \subseteq V_x.$$

Furthermore it shows that we can pick U_x and V_x so that no δ_κ-sequences in them have a supremum in $(\leftarrow, x)_{X_n}$ and $(\leftarrow, f_n(x))_{Y_n}$ respectively. Let g_x be the order isomorphism between U_x and V_x. Now let $u \in U_x$. Suppose that we have ordertype$((\leftarrow, u)_{U_x} \cap N_{n+1} \cap X') = \gamma$. This must be less than ω_κ since we are considering a bounded subset of $(\leftarrow, x)_{X_n} \cap N_{n+1}$ which has order type at most ω_κ. Then $(\leftarrow, u)_{U_x} \cap N_{n+1} \cap X' = \{y_\mu \mid \mu < \gamma\}$ where $y_\mu = \alpha_1^\mu \alpha_2^\mu \ldots \alpha_{n+1}^\mu$, with α_{n+1} some successor ordinal. If we define

$$v_\nu^\mu := \alpha_1^\mu \alpha_2^\mu \ldots \alpha_n^\mu (\alpha_{n+1}^\mu - 1)\sigma_\nu \qquad \text{for all } \nu < \delta_\kappa$$

then $y_\mu = \sup_N(v_\nu^\mu)$ so there exists $\nu_0 < \delta_\kappa$ such that $v_{\nu_0}^0 \in (\leftarrow, u)_{U_x}$ (notice that $\mu > 0$ implies $v_\nu^\mu > y_0$ so $v_\nu^\mu \in (\leftarrow, u)_{U_x}$). By Lemma 1.3 we can pick a word y_ν^μ in X with initial segment v_ν^μ for each $\mu > 0$. Pick each y_ν^0 to have initial segment $v_{\nu_0+\nu}^0$. Then $y_\mu = \sup(y_\nu^\mu)_{\nu<\delta_\kappa}$ for all μ and $y_\nu^\mu \in (\leftarrow, x)_{X_n}$ and $\mu_1 < \mu_2$ implies $y_\nu^{\mu_1} < y_{\nu'}^{\mu_2}$ for all $\nu, \nu' \in \delta_\kappa$.

Now by Lemma 2.10 we can pick $C \subset (\leftarrow, g_x(u))_{V_x}$ such that C has no δ_κ-sequences with a supremum in N and C has order type γ so $C = \{z_\mu \mid \mu < \gamma\}$. If C has a greatest element $z_{\gamma-1}$ pick $z_\gamma \in (z_{\gamma-1}, f_n(x))$. If, for each μ

$$z_\mu = \beta_1^\mu \beta_2^\mu \ldots \beta_{j_\mu}^\mu$$

then define, for $\mu < \gamma$

$$z_\nu^\mu := \begin{cases} \beta_1^\mu \beta_2^\mu \ldots \beta_{j_\mu}^\mu \eta_\nu & \text{if } j_\mu > j_{\mu+1} \\ \beta_1^{\mu+1} \beta_2^{\mu+1} \ldots \beta_{j_{\mu+1}-1}^{\mu+1} (\beta_{j_{\mu+1}} - 1)\eta_\nu & \text{if } j_\mu \leqslant j_{\mu+1}. \end{cases}$$

Notice that $(z_\nu^\mu)_{\nu<\delta_\kappa}$ has no supremum for any μ and that $z_\nu^\mu \in (z_\mu, z_{\mu+1})$ so $z_\mu^\nu \in (\leftarrow, f_n(x))_{Y_n}$ for all μ and $\mu_1 < \mu_2$ implies $z_\nu^{\mu_1} < z_{\nu'}^{\mu_2}$ for all ν, ν'. Now define

$$U_{xu} := \{y_\nu^\mu \mid y_\mu \in (\leftarrow, u)_{U_u} \cap N_{m+1} \cap X',\ \nu < \delta_\kappa\}$$
$$g_{xu}(y_\nu^\mu) := (z_\nu^\mu).$$

Since $\mu_1 < \mu_2$ implies $y_\nu^{\mu_1} < y_{\nu'}^{\mu_2}$ and $z_\nu^{\mu_1} < z_{\nu'}^{\mu_2}$ the union of the maps $g_{x\mu}$ is an order-preserving isomorphism between a subset of X and a subset of N.

Therefore, we can define

$$X_{n+1} := X_n \cup \left(\bigcup_{x \in X_n} U_x \right) \cup \left(\bigcup_{\substack{x \in X_n, \\ u \in U_x}} U_{xu} \right)$$

$$Y_{n+1} := Y_n \cup \left(\bigcup_{x \in X_n} g_x(U_x) \right) \cup \left(\bigcup_{\substack{x \in X_n, \\ u \in U_x}} g_{xu}(U_{xu}) \right)$$

$$f_{n+1}(w) := \begin{cases} f_n(w) \text{ if } w \in X_n \\ g_x(w) \text{ if } w \in U_x \text{ for some } x \in X_n \\ g_{xu}(w) \text{ if } w \in U_{xu} \text{ for some } x \in X_n, u \in U_x. \end{cases}$$

By Note 1 since $\delta_\kappa < \omega_\kappa$ if $x \in X$ then

$$U_x \cup \left(\bigcup_{u \in U_x} U_{xu} \right) \simeq \omega_\kappa. \tag{†}$$

So since X_n had order type ω_κ^n and the sets (†) above are between the elements of X_n the order type of X_{n+1} is ω_κ^{n+1}. The facts that $X_1 \subseteq X_i$ and $Y_1 \subseteq Y_i$ for all $i \leqslant n+1$ and that X_1 and Y_1 were both cofinal in N show that all these sets are also cofinal. By construction f_{n+1} is an isomorphism (so Y_{n+1} also has order type ω_κ^{n+1}).

To show part (1) of the inductive hypothesis holds let $v \in N_{n+1}$. Since X_n was cofinal in N there is $x \in X_n$ with $v \leqslant x$. Let w be the least element in X_n with $v \leqslant w$, (whose existence is ensured by the well-orderedness of X_n). If $v = w$ then $v \in X_n$ and if not then $v \in (\leftarrow, w)_{X_n}$.

This shows that if $v \in N_{n+1} \cap X$ then $v \in X_n$ or $v \in (\leftarrow, w)_{X_n} \cap N_{n+1} \cap X \subseteq U_w$. Since v was an arbitrary element of $N_{n+1} \cap X$ it follows that

$$N_{n+1} \cap X \subseteq X_n \cup \left(\bigcup_{w \in X_n} U_w \right) \subseteq X_{n+1}.$$

A similar argument shows that $N_{n+1} \subseteq Y_{n+1}$ so part (1) of the inductive hypothesis is satisfied.

Part (2) of the inductive hypothesis holds since X_{n+1} is defined as a superset of X_n and f_{n+1} is defined as f_n on X_n. Now suppose that $v \in N_{n+1} \cap X'$. By the paragraph showing (1) holds when $m = n+1$ there exists $w \in X_n$ with $v \in (\leftarrow, w)_{X_n}$. Then the set U_w has been defined. Again since U_w was defined to be cofinal in $(\leftarrow, w)_{X_n}$ we can find an element of U_w which is greater than v and if u is the least such element then

$$v \in (\leftarrow, u)_{U_w} \cap X'.$$

Then the set U_{wu} was defined to contain a sequence $(v_\nu)_{\nu<\omega_\kappa}$ with supremum v in N. Thus part (3) of the inductive hypothesis holds.

We are assuming that the fourth part of the inductive hypothesis holds on X_n and Y_n. Therefore the only δ_κ-sequences we have to consider are those with a terminal segment in U_x or V_x for some $x \in X_n$. Any bounded sequences in U_x and V_x have no supremum in N, so they satisfy (4). Since we picked U_x and V_x to be cofinal in $(\leftarrow, x)_{X_n}$ and $(\leftarrow, f_n(x))_{Y_n}$ respectively, any cofinal sequences have x or $f_n(x)$ as their supremum. Obviously these elements are in X_{n+1} or Y_{n+1} so all cofinal δ_κ-sequences also satisfy (4). No δ_κ-sequences in U_{xu} or $g_{xu}(U_{xu})$ have a supremum in X and N respectively – this follows for U_{xu} because the same thing is true about $(\leftarrow, x)_{X_n} \cap N_{n+1} \cap X'$ and each sequence (y^ν_μ) and for $g_{xu}(U_{xu})$ because it is true for C and for the (z^μ_ν).

Part (5) holds since U_x and V_x are cofinal in $(\leftarrow, x)_{X_{n+1}}$ and $(\leftarrow, f_{n+1}(x))_{Y_{n+1}}$ and so $(\leftarrow, x)_{X_{n+1}} = \emptyset$ and $(\leftarrow, f_{n+1}(x))_{Y_{n+1}} = \emptyset$ for all $x \in X_n$ and by construction $(\leftarrow, x)_{X_{n+1}} \neq \emptyset$ and $(\leftarrow, f_{n+1}(x))_{Y_{n+1}} \neq \emptyset$ for all $x \in X_{n+1} - X_n$. This completes the proof of the theorem.

I had hoped to show here that Q is isomorphic to any subset dense in it. However, although this may be true (I have been unable to find a counter-example) I have not been able to perform the inductive construction needed. I include for interest some lemmas showing the structure of a dense subset of Q, and the properties it shares with Q.

If X is a set which is dense in Q and we are trying to construct an order isomorphism f between X and Q, defining f at each stage between a subset of X containing $X \cap (Q_1 \cup \ldots \cup Q_n)$ and $Q_1 \cup \ldots \cup Q_n$ then we must ensure that both descending and ascending countable sequences with suprema and infima in Q are mapped to the correct kind of sequences with suprema and infima in X, and similarly for sequences without suprema and infima. For the rest of this chapter (x, y) is an interval in Q.

The first lemma shows that a sequence with infimum x essentially consists of elements longer than x (in fact from some point onwards all the elements in the sequence must have x as an initial segment) and the second lemma that an ascending sequence with x as supremum, after some point consists of elements of length at least that of x. Remember that $(x_i)_{i\in\omega^*}$ is a strictly descending sequence indexed by ω, so if $i_0 \in \omega$ then $\{x_j \mid j > i_0\} \simeq \omega^*$.

Lemma 2.16. *Let X be a dense subset of Q. Suppose $x = \alpha_1 q_1 \ldots \alpha_n q_n \in X$ and $x = \inf_X (x_i)_{i\in\omega^*}$. Suppose that $x_i = \beta_1^i r_1^i \ldots \beta_{m_i}^i r_{m_i}^i$. Then there exists $i_0 \in \omega^*$ such that for all $j > i_0$*

$$\beta_1^j r_1^j \ldots \beta_n^j r_n^j = \alpha_1 q_1 \ldots \alpha_n q_n$$

and $\beta_{n+1}^j = 0$ and $(r_{n+1}^j)_{j>i_0}$ is coinitial in $\mathbb{Q}$.

Proof. Let $y := \alpha_1 q_1 \ldots \alpha_n q_n(0)(0)$. Then for some $i_0 \in \omega^*$ we must have $x_j \in (x, y) \cap X$ for all $j > i_0$. This implies $\alpha_1 q_1 \ldots \alpha_n q_n(0)$ is an initial segment of x_j for all $j > i_0$.

Suppose that $(r_{n+1}^j)_{j>i_0}$ has a lower bound, say s in $\mathbb{Q}$. Then it follows that v, an element of X with initial segment $\alpha_1 q_1 \ldots \alpha_n q_n(0)(s-1)$ is strictly smaller than all the elements of $(x_i)_{i\in\omega^*}$ which is a contradiction, since $x < v$. Thus $(r_{n+1}^j)_{j>i_0}$ is coinitial in $\mathbb{Q}$.

Lemma 2.17. *Let X be a dense subset of Q. Suppose $x = \beta_1 r_1 \ldots \beta_m r_m = \sup_X (x_i)_{i<\omega}$, where $x_i = \alpha_1^i q_1^i \ldots \alpha_{n_i}^i q_{n_i}^i$. Then there exists $i_0 < \omega$ such that for all $i > i_0$*

$$\alpha_1^i q_1^i \ldots \alpha_{m-1}^i q_{m-1}^i \alpha_m^i = \beta_1 r_1 \ldots \beta_{m-1} r_{m-1} \beta_m$$

and $\sup_{\mathbb{Q}} (q_m^i)_{i>i_0} = r_m$.

Proof. If y is defined by

$$y := \beta_1 r_1 \ldots \beta_{m-1} r_{m-1} \beta_m (r_m - 1)$$

then there exists i_0 such that for all $i > i_0$ we have $x_i \in (y, x)$. This proves the first part of the lemma. For the second part obviously we have $q^i_m < r_m$ for all i. If there is $s \in \mathbb{Q}$ with $q^i_m < s < r_m$ for all i then any element v of X with initial segment $\beta_1 r_1 \ldots \beta_{m-1} r_{m-1} \beta_m s$ will be an upper bound for the elements x_i, which is a contradiction since $v < x$.

For any integer $n > 0$ and any $x \in Q_n$, the interval $(x, \rightarrow)_{Q_n}$ has cofinality δ_κ. In fact

$$(x, \rightarrow)_{Q_n} \cap Q_{n+1} = \{x\alpha q \mid \alpha q \in \omega_\kappa \times \mathbb{Q}\}$$

and this set is cofinal in $(x, \rightarrow)_{Q_n}$. Suppose then that f_n is an isomorphism between some subset X_n of X and Q_n. If we are to find a subset X_{n+1} of X containing all the words in X of length $n+1$ and such that f_n can be extended to an isomorphism $f_{n+1}: X_{n+1} \to Q_1 \cup \ldots \cup Q_{n+1}$ then we want that

$$\text{ordertype}((x, \rightarrow)_{X_n} \cap X \cap Q_{n+1}) \preccurlyeq \omega_\kappa \times \mathbb{Q}$$

for all $x \in X_n$. The next lemma shows the conditions that X_n must satisfy for this to hold.

Lemma 2.18. *Let n be a positive integer. Let X be a subset of Q and let Y be a subset of X satisfying*

(1) $x \in X \cap Q_{n-1}$ *implies* $x \in Y$;

(2) $x \in X' \cap Q_{n-1}$ *implies there exists an ω^*-sequence $(y_\mu)_{\mu \in \omega^*}$ contained in Y with $x = \inf_Q(y_\mu)$;*

(3) $x \in Q_1 \cup \ldots \cup Q_{n-2}$ *implies there exists an ω^*-sequence $(y_\mu)_{\mu \in \omega^*}$ contained in Y with $x = \inf_Q(y_\mu)$.*

Then for any $y \in Y$ we have that $\text{ordertype}((y, \rightarrow)_Y \cap Q_n) \preccurlyeq \omega_\kappa \times \mathbb{Q}$ *and that $(y, \rightarrow)_Y \cap Q_n$ is cofinal in $(y, \rightarrow)_Y$ if $\omega_\kappa \preccurlyeq$* $\text{ordertype}((y, \rightarrow)_Y \cap Q_n)$.

Proof. Let $y \in Y$. By assumption (3), if $\text{length}(y) < n-1$ then $(y, \rightarrow)_Y = \emptyset$. So assume $\text{length}(y) \geqslant n-1$, so $y = \beta_1 r_1 \ldots \beta_m r_m$ with $m \geqslant n-1$. Define

$$S := \{\beta_1 r_1 \ldots \beta_{n-1} r_{n-1} \beta r \mid \beta \in \omega_\kappa,\ r \in \mathbb{Q}\} \simeq \omega_\kappa \times \mathbb{Q}.$$

Then we need to prove that $(y, \rightarrow)_Y \cap Q_n \subseteq S$.

Suppose that there exists $v \in (y, \rightarrow)_Y$ which is larger than all the elements of S. Then $v = \beta_1 r_1 \ldots \beta_{l-1} r_{l-1} \alpha_l q_l \ldots \alpha_j q_j$, where $\alpha_l q_l > \beta_l r_l$ and $l \leqslant n-1$. Let $\gamma s \in \omega_\kappa \times \mathbb{Q}$ with $\beta_l r_l < \gamma s < \alpha_l q_l$. Define $u := \beta_1 r_1 \ldots \beta_{l-1} r_{l-1} \gamma s$. If $u \in X \cap \boldsymbol{Q}_{n-1}$ then $u \in Y$ and $y < u < v$ implies $v \notin (y, \rightarrow)_Y$. If $u \in X' \cap Q_{n-1}$ or $\text{length}(x) < n-1$ then we can find an ω^*-sequence $(x_i)_{i \in \omega^*}$ which is contained in Y with $u = \inf_Q(x_i)$. But then there exists $j \in \omega^*$ such that $y < u < x_j < v$ and again $v \notin (y, \rightarrow)_Y$. This contradiction shows that there are no elements of $(y, \rightarrow)_Y$ which are greater than all the elements of S, so S is cofinal in $(y, \rightarrow)_Y$.

Obviously there are no elements of $(y, \rightarrow)_Y \cap Q_n$ which are smaller than all the elements of S and so $S \subset Q_n$ implies $(y, \rightarrow)_Y \cap Q_n \subseteq S$. Then

$$\text{ordertype}((y, \rightarrow)_Y \cap Q_n) \preccurlyeq \text{ ordertype}(S) \preccurlyeq \omega_\kappa \times \mathbb{Q}.$$

This shows that the first part of the assertion of Lemma 2.18 holds.

To prove the second part assume we have

$$\omega_\kappa \preccurlyeq \text{ ordertype}((y, \rightarrow)_Y \cap Q_n).$$

Then since $\omega_\kappa \not\preccurlyeq \gamma$ where γ is the order type of any proper initial segment of $\omega_\kappa \times \mathbb{Q}$ we know that $(y, \rightarrow)_Y \cap Q_n$ is cofinal in S and hence also in $(y, \rightarrow)_Y$. Hence the second part of the assertion of Lemma 2.18 is true.

Suppose ω_κ is regular and (X^1, X^2) is a Dedekind cut of a dense subset X of Q, with X^1 having cofinality $\delta_\kappa (= \omega_\kappa)$. Then we can find a cofinal set $A = \{x_\mu \mid \mu < \omega_\kappa\}$ of X^1 of order type ω_κ such that for some integer m the mth element of x_μ is $\mu 0$ for each μ. If ω_κ is singular then some Dedekind cuts of X have a cofinal sequence of this type. These sequences are used to divide an interval $(x, \rightarrow)_{X_n}$ into a cofinal ω_κ-sequence of intervals $(\leftarrow, x_\mu)_A$ where $(\leftarrow, x_\mu)_A \cap Q_m \simeq \mathbb{Q}$. This then facilitates the definition of a isomorphism between $(x, \rightarrow)_{X_n}$ and $(y, \rightarrow)_{Q_n}$ where y is any element of Q_n for the following reason. When we define y_α by $y_\alpha := y\alpha 0$ and $Y := \{y_\alpha \mid \alpha < \omega_\kappa\}$ we have

$$(y, \rightarrow)_{Q_n} = \bigcup_{\alpha < \omega_\kappa} (\leftarrow, y_\alpha)_Y \qquad \text{and} \qquad (\leftarrow, y_\alpha)_Y \cap Q_{n+1} \simeq \mathbb{Q}.$$

Lemma 2.19. *Suppose X is dense in Q and (X^1, X^2) is a Dedekind cut of X, where X^1 has cofinality δ_κ. Let $(x_\mu)_{\mu<\delta_\kappa}$ be a cofinal sequence in X^1 and $(y_i)_{i\in\omega^*}$ be a coinitial sequence in X^2. Then there are two possibilities for (x_μ) and (y_i).*

(1) *There exists $\mu_0 < \delta_\kappa$, some $i_0 \in \omega^*$ and some word $\alpha_1 q_1 \dots \alpha_n q_n$ such that if $x_\mu = \alpha_1^\mu q_1^\mu \dots \alpha_{m_\mu}^\mu q_{m_\mu}^\mu$ then for all $\mu > \mu_0$*

$$\alpha_1^\mu q_1^\mu \dots \alpha_n^\mu q_n^\mu = \alpha_1 q_1 \dots \alpha_n q_n$$

and the sequence of ordinals $(\alpha_{n+1}^\mu)_{\mu<\delta_\kappa}$ is cofinal in ω_κ. Also if $y_i = \beta_1^i r_1^i \dots \beta_{m_i}^i r_{m_i}^i$ then for all i with $i_0 < i$

$$\beta_1^i r_1^i \dots \beta_{n-1}^i r_{n-1}^i \beta_n^i = \alpha_1 q_1 \dots \alpha_{n-1} q_{n-1} \alpha_n$$

and $\inf_{\mathbb{Q}}(r_n^i)_{i\in\omega^*} = q_n$.

(2) *The above statement does not hold (in which case $\omega_\kappa = \omega$ or $\delta_\kappa < \omega_\kappa$).*

Notice that in case (1), if $\delta_\kappa < \omega_\kappa$ we can find an element z_α of X which has $\alpha_1 q_1 \dots \alpha_n q_n \alpha 0$ as an initial segment, for all $\alpha < \omega_\kappa$ with $\alpha \notin (\alpha_{n+1}^\mu)_{\mu<\delta_\kappa}$. Then

$$\{x_\mu \mid \mu < \delta_\kappa\} \cup \{z_\alpha \mid \alpha < \omega_\kappa,\ \alpha \notin (\alpha_{n+1}^\mu)_{\mu<\delta_\kappa}\}$$

is an ω_κ-sequence which is cofinal in X^1.

Proof. Assume first of all that ω_κ is regular and uncountable, so $\delta_\kappa = \omega_\kappa > \omega$. Let

$$S_i := \{z \mid z \text{ is an initial segment of length } i \text{ of some } x_\mu\}.$$

Since $(x_\mu)_{\mu<\omega_\kappa} \subset \bigcup_{i<\omega} S_i$ and ω_κ is uncountable and regular there exists a greatest integer n such that $\omega_\kappa \not\preccurlyeq \text{ordertype}(S_n)$. Since (x_μ) is bounded in Q we must have $n \geqslant 1$. Since ω_κ is regular we must have an element $\alpha_1 q_1 \dots \alpha_n q_n \in S_n$ which is an initial segment to a subset A of S_{n+1} of order type ω_κ. If (x_μ) is not to have order type greater than ω_κ we must then have some $\mu_0 < \omega_\kappa$ with $\alpha_1 q_1 \dots \alpha_n q_n$ as an initial segment of x_μ for all $\mu > \mu_0$ and the first part of (1) follows since $\text{ordertype}(A) = \omega_\kappa$. It is obvious that there exists i_0 such that y_i has $\alpha_1 q_1 \dots \alpha_{n-1} q_{n-1} \alpha_n$ as an initial segment. If for some $r \in \mathbb{Q}$ we have

$q_n < r < q_n^i$ for all i then $\alpha_1 q_1 \dots \alpha_{n-1} q_{n-1} \alpha_n r$ is smaller than all the y_i and larger than all the x_μ, which is a contradiction.

Now if $\delta_\kappa < \omega_\kappa$ or $\omega_\kappa = \omega$ and $\alpha_1 q_1 \dots \alpha_n q_n$ is an arbitrary element of Q we can pick an element x_α of X with initial segment $\alpha_1 q_1 \dots \alpha_n q_n \alpha$ for all $\alpha < \omega_\kappa$. Then $(x_\alpha)_{\alpha<\omega_\kappa}$ is an ω_κ-sequence in X and we can use it to define a Dedekind cut (X^1, X^2) of X in the following way. Let $X^2 := \{x \in X \mid x > x_\alpha$ for all $\alpha < \omega_\kappa\}$. Let $X^1 := X - X^2$. If $\delta_\kappa < \omega_\kappa$ and $(\alpha_\mu)_{\mu<\delta_\kappa}$ is a cofinal subsequence in ω_κ then $(x_{\alpha_\mu})_{\mu<\delta_\kappa}$ will define the same Dedekind cut of X as $(x_\alpha)_{\alpha<\omega_\kappa}$. Thus (1) can also hold in the singular case.

If $y \in Y$ for some $Y \subset X$ and $(y, \rightarrow)_Y$ has cofinality δ_κ call $(y, \rightarrow)_Y$ *good* if the Dedekind cut defined in the way described in the preceeding paragraph is of the type labelled (1) in Lemma 2.19. Similarly if (X^1, X^2) determined by a coinitial sequence in $(\leftarrow, y)_Y$ is of type (1) then $(\leftarrow, y)_Y$ is good. I may also call Dedekind cuts of type (1) good.

From now on we shall define Q_0 to be the empty set. Notice that, for any $n < \omega$, if $\alpha_1 q_1 \dots \alpha_{n-1} q_{n-1} \in Q \cup \{$ empty word $\}$ and we define $x_\alpha := \alpha_1 q_1 \dots \alpha_{n-1} q_{n-1} \alpha 0$ for all $\alpha < \omega_\kappa$ and define

$$A := \{x_\alpha \mid \alpha < \omega_\kappa\} \cup Q_{n-1}$$

then

$$(\leftarrow, x_\alpha)_A \cap Q_n = \{\alpha_1 q_1 \dots \alpha_{n-1} q_{n-1} \alpha q \mid q \in \mathbb{Q},\ q < 0\} \simeq \mathbb{Q}$$

if α is not a successor and

$$\begin{aligned}(\leftarrow, x_\alpha)_A \cap Q_n = &\{\alpha_1 q_1 \dots \alpha_{n-1} q_{n-1} \alpha q \mid q \in \mathbb{Q},\ q < 0\} \\ &\cup \{\alpha_1 q_1 \dots \alpha_{n-1} q_{n-1} (\alpha - 1) q \mid q \in \mathbb{Q},\ q > 0\} \\ &\simeq \mathbb{Q}\end{aligned}$$

if α is a successor. In each of these cases, $(\leftarrow, x_\alpha)_A \cap Q_n$ is cofinal in $(\leftarrow, x_\alpha)_A$. If and only if α is not a successor we have that $(\leftarrow, x_\alpha)_A \cap Q_n$ is coinitial in $(\leftarrow, x_\alpha)_A$. Also Lemma 2.17 shows that $(\leftarrow, x_\alpha)_A \cap Q_n$ is closed upwards in $(\leftarrow, x_\alpha)_A$ and that every element of $(\leftarrow, x_\alpha)_A \cap Q_n$ is the supremum of an ascending sequence in the set $(\leftarrow, x_\alpha)_A \cap Q_n$. Lemma 2.16 shows that no descending sequences in $(\leftarrow, x_\alpha)_A \cap Q_n$ have infima in $(\leftarrow, x_\alpha)_A$. Finally, it is also true that the interval $(x, \rightarrow)_{Q_n}$ has cofinality δ_κ for all $x \in Q_n$. These facts provide the motivation for the next two lemmas.

Lemma 2.20. *Let X be a dense subset of Q and Y a subset of X satisfying the conditions of Lemma 2.18, with $y \in Y$. Suppose $(y, \rightarrow)_Y$ has cofinality δ_κ and is good. Then there exists a sequence $A = (x_\mu)_{\mu<\omega_\kappa} \subset (y, \rightarrow)_Y \cap X$ which is cofinal in $(y, \rightarrow)_Y$. Moreover for all μ there exists $n_\mu \geqslant n$ such that $(\leftarrow, x_\mu)_A \cap Q_{n_\mu} \simeq \mathbb{Q}$. Also if μ is a successor then $(\leftarrow, x_\mu)_A \cap Q_{n_\mu}$ is not coinitial in $(\leftarrow, x_\mu)_A$ and $(\leftarrow, x_\mu)_A$ is good*

Warning: If $n_\nu > n$ for any ν then there are a finite number of elements x_μ in A such that μ is a successor and $(\leftarrow, x_\mu)_Y \cap Q_{n_\mu}$ is coinitial in $(\leftarrow, x_\mu)_Y$. Then $x_\mu = \beta_1 r_1 \ldots \beta_{n_\mu} r_{n_\mu}$ and $\{\beta_1 r_1 \ldots \beta_{n_\mu -1} r_{n_\mu -1} \beta_{n_\mu} q \mid q \in \mathbb{Q},\ -1 < q < 0\}$ fulfills the requirements on $(\leftarrow, x_\mu)_Y \cap Q_{n_\mu}$. The reader must make this substitution.

Proof. Notice that $n_\nu > n$ implies $(\leftarrow, x_\nu)_A \cap Q_n = \emptyset$.

By Lemma 2.18 ordertype$((y, \rightarrow)_Y \cap Q_n) \preccurlyeq \omega_\kappa \times \mathbb{Q}$. If we have that $\omega_\kappa \preccurlyeq$ ordertype$((y, \rightarrow)_Y \cap Q_n)$ then, again by Lemma 2.18, the set $(y, \rightarrow)_Y \cap Q_n$ is cofinal in $(y, \rightarrow)_Y$ and if $y = \alpha_1 q_1 \ldots \alpha_m q_m$ it is of the form

$$\{\alpha_1 q_1 \ldots \alpha_{n-1} q_{n-1} \mu q \mid \mu q \in \omega_\kappa \times \mathbb{Q},\ (\mu q > \alpha_n q_n \text{ if } m > n-1)\ \}.$$

In this case Lemma 2.3 shows we can pick an element x_μ in X with initial segment $\alpha_1 q_1 \ldots \alpha_{n-1} q_{n-1} \mu 0$ for all $\mu < \omega_\kappa$ (with $\mu > \alpha_n$ if $m \geqslant n$). Then if we define $A := \{x_{\alpha_n + 1 + \mu} \mid \mu \in \omega_\kappa\}$ the set A is cofinal in $(y, \rightarrow)_Y$. Also

$$\begin{aligned}(\leftarrow, x_\mu)_A \cap Q_n &\subseteq \{\alpha_1 q_1 \ldots \alpha_{n-1} q_{n-1} \mu q \mid q \in \mathbb{Q}, q < 0\} \\ &\simeq \mathbb{Q}\end{aligned}$$

for all ordinals μ which are not successors and

$$\begin{aligned}(\leftarrow, x_\mu)_A \cap Q_n &\subseteq \{\alpha_1 q_1 \dots \alpha_{n-1} q_{n-1} \mu q \mid q \in \mathbb{Q}, q < 0\} \\ &\cup \{\alpha_1 q_1 \dots \alpha_{n-1} q_{n-1} (\mu - 1) q \mid q \in \mathbb{Q},\ 0 < q\} \\ &\simeq \mathbb{Q}\end{aligned}$$

for all successor ordinals μ, which implies $(\leftarrow, x_\mu)_A$ is good, if μ is a successor. By Lemma 2.16 $(\leftarrow, x_\mu)_A \cap Q_n$ is not coinitial in $(\leftarrow, x_\mu)_A$ if μ is a successor since in this case $(\leftarrow, x_\mu)_A = (x_{\mu-1}, x_\mu)$ and $x_{\mu-1}$ has length $\geqslant n$.

Suppose $\omega_\kappa \not\prec$ ordertype$((y, \rightarrow)_Y \cap Q_n)$. Since $(y, \rightarrow)_Y$ is good we know from Lemma 2.19 that we can find a sequence $B = (z_\mu)_{\mu<\omega_\kappa}$ in X with some word $\beta_1 r_1 \dots \beta_{m-1} r_{m-1} \in Q$ which is an initial segment of z_μ for all μ and the mth ordinal element of z_μ being μ and the mth rational element of z_μ being 0, for each $\mu < \omega_\kappa$. Then

$$\begin{aligned}(\leftarrow, z_\mu)_B \cap Q_m &= \{\beta_1 r_1 \dots \beta_{m-1} r_{m-1} \mu q \mid q \in \mathbb{Q},\ 0 < q\} \\ &\simeq \mathbb{Q}\end{aligned}$$

if μ is a limit and

$$\begin{aligned}(\leftarrow, z_\mu)_B \cap Q_m &= \{\beta_1 r_1 \dots \beta_{m-1} r_{m-1} (\mu - 1) q \mid q \in \mathbb{Q},\ 0 < q\} \\ &\cup \{\beta_1 r_1 \dots \beta_{m-1} r_{m-1} \mu q \mid q \in \mathbb{Q},\ q < 0\} \\ &\simeq \mathbb{Q}\end{aligned}$$

if μ is a successor, which implies $(\leftarrow, z_\mu)_B$ is good. If $\alpha_1 q_1 \dots \alpha_{n-1} q_{n-1}$ is not an initial segment of $\beta_1 r_1 \dots \beta_{m-1} r_{m-1}$ then we would have (by the same reasoning as in Lemma 2.18) that

$$\omega_\kappa \times \mathbb{Q} \prec \text{ ordertype}((y, \rightarrow)_Y \cap Q_n)$$

contradicting Lemma 2.18. Define ordinals $\Lambda_n, \Lambda_{n+1}, \dots, \Lambda_{m-1}$ inductively by

$$\Lambda_n := \beta_n,$$

$$\Lambda_{i+1} := \Lambda_i + \beta_{i+1}.$$

Now we can use these ordinals to define a sequence as follows. If $m > n+1$, then for each $i \in \{0, \ldots, m-n-2\}$ let $v_{\Lambda_{n+i}+\mu}$ be an element of X such that $v_{\Lambda_{n+i}+\mu}$ has initial segment $\beta_1 r_1 \ldots \beta_{n+i} r_{n+i} \mu 0$, for μ such that $0 < \mu < \beta_{n+i+1}$. If $\mu = \Lambda_{n+i}$ let v_μ be an element of X with initial segment $\beta_1 r_1 \ldots \beta_{n+i} r_{n+i}$. Notice that

$$(\leftarrow, v_\mu)_{\{v_\nu \mid \nu < \Lambda_{m-1}\}} \cap Q_{n+i+1} = \{\beta_1 r_1 \ldots \beta_{n+i} r_{n+i} \mu q \mid q \in \mathbb{Q},\ q < 0\} \\ \simeq \mathbb{Q}$$

if $\Lambda_{n+i} < \mu \leqslant \Lambda_{n+i+1}$ and μ is not a successor and

$$\begin{aligned}(\leftarrow, v_\mu)_{\{v_\mu \mid \mu < \Lambda_{m-1}\}} \cap Q_{n+i+1} &= \{\beta_1 r_1 \ldots \beta_{n+i} r_{n+i} (\mu - 1) q \mid q \in \mathbb{Q},\ q > 0\} \\ &\cup \{\beta_1 r_1 \ldots \beta_{n+i} r_{n+i} \mu q \mid q \in \mathbb{Q},\ q < 0\} \\ &\simeq \mathbb{Q}\end{aligned}$$

if $\Lambda_{n+i} < \mu \leqslant \Lambda_{n+i+1}$ and μ is a successor and $\mu \neq \Lambda_{n+i} + 1$, which implies $(\leftarrow, v_\mu)_A$ is good, if μ is a successor and $\mu \neq \Lambda_{n+i} + 1$. For $i = 0, \ldots, m-2$ let $n_\mu := n + i + 1$ for all μ such that $\Lambda_{n+i} < \mu \leqslant \Lambda_{n+i+1}$. Let $n_\mu := n$ if $\mu \leqslant \beta_n$ and let $n_\mu := m$ if $\mu > \Lambda_{m-1}$. Now define

$$\begin{aligned} y_\mu &:= \begin{cases} x_\mu & \text{if } \mu < \beta_n \\ v_\mu & \text{if } \beta_n \leqslant \mu < \Lambda_{m-1} \end{cases} \\ y_{\Lambda_{m-1}+\mu} &:= z_\mu \qquad \text{if } 0 \leqslant \mu < \omega_\kappa. \end{aligned}$$

If $m = n + 1$ define

$$\begin{aligned} y_\mu &:= \begin{cases} x_\mu & \text{if } \mu < \beta_n \\ v_{\beta_n} & \text{if } \mu = \beta_n \end{cases} \\ y_{\Lambda_n+\mu} &:= z_\mu \qquad \text{if } 0 < \mu < \omega_\kappa. \end{aligned}$$

and $n_\mu := n$ if $\mu \leqslant \beta_n$ and $n_\mu := m$ if $\mu > \beta_n$. Then the sequence $C = (y_\nu)_{\nu < \omega_\kappa}$ is cofinal in $(y, \rightarrow)_Y$ and by the arguments above $(\leftarrow, y_\nu)_C \cap Q_{n_\nu} \simeq \mathbb{Q}$. Suppose that ν is a successor and $\nu \neq \Lambda_{n+i} + 1$ for some i. Then $(\leftarrow, y_\nu)_C \cap Q_{n_\nu}$ is not coinitial in $(\leftarrow, y_\nu)_C$ by Lemma 2.16 since then $(\leftarrow, y_\nu)_C = (y_{\nu-1}, y_\nu)$ and $y_{\nu-1}$ has length $\geqslant n_\nu$. If $\nu = \Lambda_{n+i} + 1$ then if $y_\nu = \beta_1 r_1 \ldots \beta_{n_\nu - 1} r_{n_\nu - 1}(1)(0)$ we have

$$(\leftarrow, y_\nu)_C \cap Q_{n_\nu} = \{\beta_1 r_1 \ldots \beta_{n_\nu-1} r_{n_\nu-1} 0q \mid q \in \mathbb{Q}\}$$
$$\cup \{\beta_1 r_1 \ldots \beta_{n_\nu-1} r_{n_\nu-1} 1q \mid q \in \mathbb{Q},\ q < 0\}$$

and this is coinitial in $(\leftarrow, y_\nu)_C$. However

$$\{\beta_1 r_1 \ldots \beta_{n_\nu-1} r_{n_\nu-1} 1q \mid q \in \mathbb{Q},\ -1 < q < 0\} \simeq \mathbb{Q}$$

and is not coinitial in $(\leftarrow, y_\nu)_C$ and a Dedekind cut determined by a coinitial sequence in this set is good. Thus the lemma is true.

Notice that this lemma implies $(y, \rightarrow)_Y \cap Q_n \subseteq \bigcup_{\mu < \omega_\kappa} (\leftarrow, y_\mu]_C$.

Lemma 2.21. *Suppose that X is dense in $\mathbb{Q}$ and that A is a subset of X. Suppose also that $u \in A$ and $(\leftarrow, u)_A \cap Q_n \simeq \mathbb{Q}$ and* length$(x) \geqslant n$ *for all $x \in (\leftarrow, u)_A$. Then there is a subset V of $X \cap (\leftarrow, u)_A$ such that*

(1) $(\leftarrow, u)_A \cap Q_n \cap X \subseteq V$;

(2) *for all $x \in (\leftarrow, u)_A \cap Q_n \cap X'$ there is an ω^*-sequence $(x_i)_{i \in \omega^*}$ in V with $x = \inf_X(x_i)$;*

(3) *V is cofinal in $(\leftarrow, u)_A$ and if $(\leftarrow, u)_A = (v, u)$ we can pick V to be coinitial in (v, u) if we wish;*

(4) *for all $x \in V$ there is an ω-sequence $(x_i)_{i \in \omega}$ in V with $x = \sup_X(x_i)$;*

(5) *$V \simeq \mathbb{Q}$;*

(6) *V is closed upwards in $(\leftarrow, u)_A$;*

(7) *no ω^*-sequence in V has an infimum in X;*

(8) *if $x \in V$ and $x = \inf_V(x_i)_{i \in \omega^*}$ then the interval $(x, \rightarrow)_V$ has cofinality δ_κ and $(x, \rightarrow)_V$ is good.*

Proof. First notice that, since $V \simeq \mathbb{Q}$ for all $x \in V$ we must have some sequence (x_i) in V such that $x = \inf_V(x_i)$. The extra condition that x is the infimum of an ω^*-sequence is added to facilitate the inductive arguments used in the proof.

coinitial in $(\leftarrow, y_\nu)_C$ by Lemma 2.16 since then $(\leftarrow, y_\nu)_C = (y_{\nu-1}, y_\nu)$ and $y_{\nu-1}$ has length $\geqslant n_\nu$. If $\nu = \Lambda_{n+i} + 1$ then if $y_\nu = \beta_1 r_1 \ldots \beta_{n_\nu - 1} r_{n_\nu - 1}(1)(0)$ we have

$$(\leftarrow, y_\nu)_C \cap Q_{n_\nu} = \{\beta_1 r_1 \ldots \beta_{n_\nu-1} r_{n_\nu-1} 0q \mid q \in \mathbb{Q}\}$$
$$\cup \{\beta_1 r_1 \ldots \beta_{n_\nu-1} r_{n_\nu-1} 1q \mid q \in \mathbb{Q},\ q < 0\}$$

and this is coinitial in $(\leftarrow, y_\nu)_C$. However

$$\{\beta_1 r_1 \ldots \beta_{n_\nu-1} r_{n_\nu-1} 1q \mid q \in \mathbb{Q},\ -1 < q < 0\} \simeq \mathbb{Q}$$

and is not coinitial in $(\leftarrow, y_\nu)_C$ and a Dedekind cut determined by a coinitial sequence in this set is good. Thus the lemma is true.

Notice that this lemma implies $(y, \rightarrow)_Y \cap Q_n \subseteq \bigcup_{\mu < \omega_\kappa} (\leftarrow, y_\mu]_C$.

Lemma 2.21. *Suppose that X is dense in $\mathbb{Q}$ and that A is a subset of X. Suppose also that $u \in A$ and $(\leftarrow, u)_A \cap Q_n \simeq \mathbb{Q}$ and* length$(x) \geqslant n$ *for all $x \in (\leftarrow, u)_A$. Then there is a subset V of $X \cap (\leftarrow, u)_A$ such that*

(1) $(\leftarrow, u)_A \cap Q_n \cap X \subseteq V$;

(2) *for all $x \in (\leftarrow, u)_A \cap Q_n \cap X'$ there is an ω^*-sequence $(x_i)_{i \in \omega^*}$ in V with $x = \inf_X (x_i)$;*

(3) *V is cofinal in $(\leftarrow, u)_A$ and if $(\leftarrow, u)_A = (v, u)$ we can pick V to be coinitial in (v, u) if we wish;*

(4) *for all $x \in V$ there is an ω-sequence $(x_i)_{i \in \omega}$ in V with $x = \sup_X (x_i)$;*

(5) $V \simeq \mathbb{Q}$;

(6) *V is closed upwards in $(\leftarrow, u)_A$;*

(7) *no ω^*-sequence in V has an infimum in X;*

(8) *if $x \in V$ and $x = \inf_V (x_i)_{i \in \omega^*}$ then the interval $(x, \rightarrow)_V$ has cofinality δ_κ and $(x, \rightarrow)_V$ is good.*

Proof. First notice that, since $V \simeq \mathbb{Q}$ for all $x \in V$ we must have some sequence (x_i) in V such that $x = \inf_V (x_i)$. The extra condition that x is the infimum of an ω^*-sequence is added to facilitate the inductive arguments used in the proof.

If $(\leftarrow, u)_A = (v, u)$ and $v = \beta_1 r_1 \ldots \beta_p r_p$ define

$$Z := \{\beta_1 r_1 \ldots \beta_p r_p 0q \mid q \in \mathbb{Q}\}$$

(we may have $p + 1 = n$). For each $z \in Z \cap X'$ pick $(z_i)_{i \in \omega^*}$ in X such that $z = \inf(z_i)$ and z_i has initial segment $z0$ for all i. Define

$$Z^* := \{z_i \mid z \in Z \cap X',\ i \in \omega^*\}.$$

If $(\leftarrow, u)_A \cap Q_n$ is cofinal in $(\leftarrow, u)_A$ define $Y := \emptyset$ and $Y^* = \emptyset$. If not put $u = \gamma_1 s_1 \ldots \gamma_k s_k$ and define

$$Y := \{\gamma_1 s_1 \ldots \gamma_{k-1} s_{k-1} \gamma_k q \mid q \in \mathbb{Q},\ s_k - 1 < q < s_k\}.$$

For each $y \in Y \cap X'$ pick $(y_i)_{i \in \omega^*}$ in X such that $y = \inf(y_i)$ and y_i has initial segment $y0$ for all i. Define

$$Y^* := \{y_i \mid y \in Y \cap X',\ i \in \omega^*\}.$$

For each x in the set $(\leftarrow, u)_A \cap Q_n \cap X'$ pick an ω^*-sequence $(x_i)_{i \in \omega^*}$ in X with $x = \inf_Q(x_i)$ and $x0$ an initial segment of x_i for all i. Let U be defined by

$$\begin{aligned} U :=\ & ((\leftarrow, u)_A \cap Q_n \cap X) \cup (Y \cap X) \cup Y^* \\ & \cup \{x_i \mid x \in (\leftarrow, u)_A \cap Q_n \cap X',\ i \in \omega^*\} \end{aligned}$$

if we do not care if V is not coinitial. Otherwise define

$$\begin{aligned} U :=\ & ((\leftarrow, u)_A \cap Q_n \cap X) \cup (Y \cap X) \cup Y^* \\ & \cup \{x_i \mid x \in (\leftarrow, u)_A \cap Q_n \cap X',\ i \in \omega^*\} \\ & \cup Z \cup Z^*. \end{aligned}$$

Notice that any superset of U satisfies (1), (2) and (3).
Let $U^0 := U$.

For $m \geqslant 0$ assume U^m is defined and define U^{m+1} inductively in the following way. For each $y = \alpha_1 q_1 \dots \alpha_k q_k \in U^m$ with $(\leftarrow, y)_{U^0 \cup \dots \cup U^m} \neq \emptyset$ pick an ω-sequence $(y_i)_{i \in \omega}$ in $(\leftarrow, y)_{U^0 \cup \dots \cup U^m} \cap X$ with $y = \sup_X(y_i)$, such that for each $i \in \omega$

$$y_i = \alpha_1 q_1 \dots \alpha_{k-1} q_{k-1} \alpha_k q_k^i \alpha_{k+1}^i q_{k+1}^i \dots \alpha_r^i q_r^i.$$

with $r > m, k$ and $\sup_{\mathbb{Q}}(q_k^i) = q_k$ (Lemma 2.3 shows we can do this).
Then let

$$U^{m+1} := \{y_i \mid y \in U^m,\ i \in \omega\}.$$

Now define V by

$$V := \bigcup_{m < \omega} U^m.$$

Also notice

$(*)(*)$ if the Dedekind cut determined by a coinitial sequence in $(\leftarrow, u)_A \cap Q_n$ is good and we do not add Z and Z^* then the Dedekind cut determined by a coinitial sequence in V is good.

We claim that V satisfies (1) - (8). Since $U \subseteq V$ we know V satisfies (1) - (3). The way U^{m+1} was defined from U^m ensures that V satisfies (4).

Since every element of V has cofinality ω we know that V is dense. V is cofinal in $(\leftarrow, u)_A$ so it has no greatest element. We know U^0 has no least element and so if for some m we have u a least element of $U^0 \cup \dots \cup U^m$ we know u cannot be the least element of U^0. This implies

$$(\leftarrow, u)_{U^0 \cup \dots \cup U^m} \neq \emptyset$$

and so U^{m+1} contains an element smaller than u. Thus u is not the least element of V and since this is true for any element of V we know V has no least element. U^0 is countable and by induction U^m countable implies that U^{m+1} is countable. Thus $V = \bigcup_{m<\omega} U^m$ is countable. Hence, by Cantor's Theorem, $V \simeq \mathbb{Q}$.

The rest of this proof (and the real content of this lemma) is concerned with showing that V is closed upwards in $(\leftarrow, u)_A$, that ω^*-sequences in V have no infima in X and $(x, \rightarrow)_V$ has cofinality δ_κ and is good for all $x \in V$. These things will be proved in two parts, the first dealing with sequences wholly contained in $U^0 \cup \ldots \cup U^m$ for some $m < \omega$ and the second with sequences which are not.

Notice that if $(x_i)_{i\in\omega^*}$ is any descending sequence and $y_i \in (x_{i-1}, x_i)$ for all i then (y_i) has an infimum if and only if (x_i) does. Similarly if $(x_i)_{i\in\omega}$ is an ascending sequence and $y_i \in (x_i, x_{i+1})$ for all i then $\sup(x_i) = \sup(y_i)$.

Case 1. For all sequences wholly contained in $U^0 \cup \ldots \cup U^m$

(6) If $x \in (\leftarrow, u)_X \cap X$ and $x = \sup_X(x_i)$ for some $(x_i) \subset U^0 \cup \ldots \cup U^m$ then $x \in U^0 \cup \ldots \cup U^m$.

(7) There are no descending sequences in $U^0 \cup \ldots \cup U^m$ with infima in the set $(\leftarrow, u)_A \cap X$.

(8) If $x \in U^0 \cup \ldots \cup U^m$ and $x = \inf_{U^0\cup\ldots\cup U^m}(x_i)_{i\in\omega^*}$ then $(x, \rightarrow)_{U^0\cup\ldots\cup U^m}$ has cofinality δ_κ and is good.

The proof is by induction on m.

(6) Any ω-sequence in with a terminal segment of elements each with an initial segment in $(\leftarrow, u)_A \cap Q_n$ must have its supremum x in this set by Lemma 2.17 (or $x = u$). Thus $x \notin (\leftarrow, u)_A \cap X$ or $x \in U^0$. The proof for sequences whose elements have initial segments in Z or Y is similar.

(7) It is obvious that there are no ω^*-sequences in $Y \cap X$ or $Z \cap X$ with infima in $(\leftarrow, u)_A$. By Lemma 2.16 there are no ω^*-sequences in $(\leftarrow, u)_A \cap Q_n \cap X$ with infima in $(\leftarrow, u)_A$ since $\text{length}(x) \geqslant n - 1$ for all $x \in (\leftarrow, u)_A$ and similarly for $Z \cap X$ and $Y \cap X$. The sequences $(y_i)_{i\in\omega^*}$ were picked to have no infimum in X. Suppose then that for some $(z_i)_{i\in\omega^*}$ we have $z_i \in (y^i_j)_{j\in\omega^*}$ for each i. Then z_i has initial segment y_i0 for some $y_i \in Q_n \cap X'$ and the assertion follows since $(\leftarrow, u)_A$ has no elements of length $n - 1$. The proof for Y^* and Z^* is similar.

(8) Suppose that $x \in U^0$. If $x \in (y_i)_{i\in\omega^*}$ for some descending sequence (y_i) with infimum in $X' \cap Q_n$ or some descending sequence in Y^* or Z^* then unless $x = y_0$ we know x has a predecessor in U^0 and x is not the infimum of a descending sequence in U^0.

Now assume x is the infimum in U^0 of an ω^*-sequence in U^0 and that $x = \alpha_1 q_1 \ldots \alpha_k q_k$ for some $k \geqslant n$ and $x \notin Z \cap X$ or Z^* or $Y \cap X$ or Y^*.

Consider the sequence $(x_i)_{i\in\omega^*}$ defined by $x_i := \alpha_1 q_1 \ldots \alpha_{n-1} q_{n-1} \alpha_n (q_n + \frac{1}{i})$. Notice that any lower bounds for this sequence which are greater than x must have $\alpha_1 q_1 \ldots \alpha_n q_n 0$ as an initial segment. However if $x \in (\leftarrow, u)_A \cap Q_n$ then x is the only element of U^0 with $\alpha_1 q_1 \ldots \alpha_n q_n$ as an initial segment even and if $x = y_0$ as in the paragraph above then x is the greatest element in U^0 with $\alpha_1 q_1 \ldots \alpha_n q_n 0$ as an initial segment.

Thus if there exists a subsequence $(x'_i)_{i\in\omega^*}$ of order type ω^* such that $x'_i \in (\leftarrow, u)_{U^0}$ for all i then $(x, \rightarrow)_{U^0}$ is bounded above by (x'_i) and (8) is true. If there exists j such that $x_i \in (\leftarrow, u)_{U^0} \cap X'$ for all $i > j$ then we can pick $y_i \in (x_{i-1}, x_i) \cap U^0$ for each $i > j$ to get a sequence which defines the same Dedekind cut as (x_i). In either case the sequence (z_μ) picked so that z_μ has initial segment $x(\mu)(0)$ is cofinal in the interval and the Dedekind cut (z_μ) determines is of the type labelled (1) in 2.19 so the interval is good.

The argument for elements x of $Z \cap X$ or Z^* is just the same except we need to consider the $(p+1)$th element of z. Similarly for $x \in Y \cap X$ or Y^* and its kth element.

Thus U^0 satisfies all three parts of the inductive hypothesis.
Suppose $m \geqslant 0$.

(6) Let $(x_i)_{i\in\omega} \subset U^{m+1}$. If there exists $i_0 \in \omega$ such that $(x_i)_{i_0<i}$ is contained in $(\leftarrow, x)_{U^0\cup\ldots\cup U^m}$ for some $x \in U^m$ then $x = \sup(x_i)$ and the proof is complete. If not then we have a subsequence $(x'_i)_{i\in\omega}$ of (x_i) such that for each i we have that $x_i \in (\leftarrow, y_i)_{U^0\cup\ldots\cup U^m}$ for some $y_i \in U^m$. Then $\sup(x_i) = \sup(y_i)$ which is in $U^0 \cup \ldots \cup U^m$ by the inductive hypothesis.

(7) Assume there are no sequences in $U^0\cup\ldots\cup U^m$ with infima in $(\leftarrow, u)_A \cap X$. Let $(x_i)_{i\in\omega^*}$ be a descending sequence in U^{m+1}. If there exists i_0 such that, for all $i > i_0$

$$x_i \in (\leftarrow, y)_{U^0\cup\ldots\cup U^m}$$

for some $y \in U^0 \cup \ldots \cup U^m$ then $\omega^* \preccurlyeq \omega$ which is not true. Thus we must have a subsequence $(x'_i)_{i \in \omega^*}$ of (x_i) with

$$x'_i \in (\leftarrow, y_i)_{U^0 \cup \ldots \cup U^m}$$

for distinct elements $y_i \in U^0 \cup \ldots \cup U^m$. Then $(y_i)_{i \in \omega^*}$ is an ω^*-sequence with no infimum in $(\leftarrow, u)_A \cap X$ by the inductive hypothesis whence (x'_i) and thus (x_i) have no infimum in $(\leftarrow, u)_A \cap X$.

(8) Suppose $x \in U^0 \cup \ldots \cup U^{m+1}$ and $x = \inf_{U^0 \cup \ldots \cup U^{m+1}}(x_i)_{i \in \omega^*}$ for $(x_i) \subset U^{m+1}$. As in the paragraph above we must have a subsequence $(x'_i)_{i \in \omega^*}$ with $x'_i \in (y_{i-1}, y_i)$ for some sequence $(y_i)_{i \in \omega^*} \subset U^0 \cup \ldots \cup U^m$. The Dedekind cut determined by $(y_i)_{i \in \omega^*}$ must have cofinality δ_κ and be good by the inductive hypothesis and then the fact that $(x_i)_{i \in \omega^*}$ determines the same Dedekind cut shows the third part of the inductive hypothesis holds.

Thus (6), (7) and (8) are all true for all sequences in $U^0 \cup \ldots \cup U^m$ for any $m < \omega$.

Case 2. Here we are concerned with sequences in V but not contained in the set $U^0 \cup \ldots \cup U^m$ for any integer m. If such an ascending sequence (x_i) contains a cofinal sequence (x'_i) which is contained in $U^0 \cup \ldots \cup U^m$ for some m then (x'_i) has a supremum in V by Case 1. Similarly, if (y_i) is a descending sequence with a coinitial sequence (y'_i) in $U^0 \cup \ldots \cup U^m$ then by Case (1) there is no infimum in X for (y'_i) and if y is the infimum of this sequence in $U^0 \cup \ldots \cup U^m$ then $(y, \rightarrow)_{U^0 \cup \ldots \cup U^m}$ is good. Thus it is enough to prove the following.

(6) If $x \in (\leftarrow, u)_A \cap X$ and $x = \sup_X(x_i)_{i \in \omega}$ for some $(x_i) \subset V$ with $x_i \in U^i$ for all $i < \omega$ then $x \in V$.

(7) If $(x_i)_{i \in \omega^*}$ is a sequence with $x_i \in U^i$ for all i then (x_i) has no infimum in X.

(8) If $(x_i)_{i \in \omega^*}$ is a sequence with $x_i \in U^i$ for all i and for some $x \in V$ we have $x = \inf_V(x_i)_{i \in \omega^*}$ then $(x, \rightarrow)_V$ has cofinality δ_κ and is good.

For each of the three parts essentially the same technique will be used in the proof. We will show that for any sequence contained in V and any integer m we can find l (dependant on m) such that all the elements of the sequence share an initial segment of length m with something in $U^0 \cup \ldots \cup U^l$. Thus any statements about sequences in V can be reduced to statements about sequences in $U^0 \cup \ldots \cup U^l$ for some $l < \omega$, which are then covered by Case 1.

(6) Suppose that $(x_i)_{i\in\omega} \subseteq V$ and that $x_i \in U^i$ for each $i \in \omega$. Let $l < \omega$. Then if we consider x_j for any $j \geqslant l+2$ we can find $y_j \in U^{l+1}$ with

$$y_j = \beta_1^j r_1^j \ldots \beta_{l+1}^j r_{l+1}^j \ldots \beta_k^j r_k^j$$

such that $\beta_1^j r_1^j \ldots \beta_{l+1}^j r_{l+1}^j$ is an initial segment of both y_j and x_j. But then if all the elements x_i for $i \geqslant l+2$ have $\alpha_1 q_1 \ldots \alpha_l q_l \alpha_{l+1}$ as an initial segment and their $(l+1)$th rational elements form an ascending sequence with supremum q in $\mathbb{Q}$ then the same is true of all the elements y_i. In this case $\sup(y_j) = \sup(x_i)$. By Case 1 we then know $\sup(x_i) \in V$. Since this is true for all $l < \omega$ and Lemma 2.17 shows that we must have some such initial segment common to all the x_i after some point, if the sequence is to have an supremum we know the sequence (x_i) has its supremum in V, if it has one in X.

(7) Suppose that $(x_i)_{i\in\omega^*} \subseteq V$ and that $x_i \in U^i$ for each $i \in \omega^*$. Let $l < \omega$. Then if we consider x_j for any $j \geqslant l+2$ we can find $y_j \in U^{l+1}$ with

$$y_j = \beta_1^j r_1^j \ldots \beta_{l+1}^j r_{l+1}^j \ldots \beta_m r_m$$

such that $\beta_1^j r_1^j \ldots \beta_{l+1}^j r_{l+1}^j$ is an initial segment of both y_j and x_j. But then if all the elements x_i for $i \geqslant l+2$ have $\alpha_1 q_1 \ldots \alpha_l q_l 0$ as an initial segment and their $(l+1)$th rational elements form a coinitial sequence in $\mathbb{Q}$ then the same is true of all the elements y_i. In this case if $v = \alpha_1 q_1 \ldots \alpha_l q_l \in (\leftarrow, u)_A \cap X$ then $v = \inf_X(y_i)$ which Case 1 shows is impossible. Thus $v \notin (\leftarrow, u)_A \cap X$. Since this is true for all $l < \omega$ and Lemma 2.16 shows that we must have some such initial segment common to all the x_i after some point, if the sequence is to have an infimum we know the sequence (x_i) can have no infimum in $(\leftarrow, u)_A \cap X$.

(8) Suppose $x = \inf_V(x_i)_{i\in\omega^*}$ where (x_i) is the descending sequence in the preceding paragraph. If $x \notin U^0 \cup \ldots \cup U^{l+1}$ then we must have $z \in U^0 \cup \ldots \cup U^m$ for some $m \geqslant l+1$ with $z > x$ and $z < y_i$ for all i which is a contradiction since $x = \inf_V(y_i)$. Thus $x \in U^0 \cup \ldots \cup U^l$ and so $x = \inf_{U^0\cup\ldots\cup U^l}(y_i)$. Then the assertion about $(x, \rightarrow)_V$ holds, by Case 1.

Ideally, we would now perform an inductive construction to show that a dense subset X of Q is isomorphic to Q. At the first stage we would need to find a cofinal ω_κ-sequence $A = (x_\mu)_{\mu<\omega_\kappa}$ in X, where $(\leftarrow, x_\mu)_A \cap Q_1 \simeq \mathbb{Q}$ for all μ. Lemma 2.21 to find V_μ satisfying (1) – (8) above. We would then map V_μ to

$$U_\mu := (\leftarrow, \mu 0)_{\{\nu 0 \mid \nu\in\omega_\kappa\}} \cap Q_1.$$

This last set has order type $\mathbb{Q}$. However V_μ lacks an important property that U_μ has. If a *gap* in X is a Dedekind cut (X^1, X^2) where X^1 has no greatest, and X^2 no least element then every gap in U_μ defines a gap in Q. Indeed the only gaps in U_μ (when μ is a successor) are

$$U^1_\mu = \{(\mu-1)q \mid q \in \mathbb{Q},\ 0 < q < r,\ r \in \mathbb{R}-\mathbb{Q}\}$$
$$\text{and } U^2_\mu = \{(\mu-1)q \mid q \in \mathbb{Q},\ r < q,\} \cup \{\mu q \mid q \in \mathbb{Q},\ q < 0\}$$

or

$$U^1_\mu = \{(\mu-1)q \mid q \in \mathbb{Q},\ q > 0\} \quad \text{and} \quad U^2_\mu = \{\mu q \mid q \in \mathbb{Q},\ q < 0\}.$$

The same is not true of V_μ and X. For example, a set which is cofinal in $(\leftarrow, x_\mu)_A \cap Q_1$ and a set coinitial in $(Y \cap X) \cup Y^\star$ will define a gap of V_μ, but only an interval in X, if length$(x_\mu) > 1$.

Proof. If α is an ordinal and $\beta \sim \alpha$ then, by Lemma 0.3 we know β is an ordinal and $\beta < \omega \times \alpha$. Hence every PWOP equivalence class of ordinals of cardinality $\aleph_\kappa$ has size at most $\aleph_0.\aleph_\kappa = \aleph_\kappa$. There are $\aleph_{\kappa+1}$ ordinals of cardinality $\aleph_\kappa$ and so we can find $\aleph_{\kappa+1}$ non-PWOP equivalent ordinals. By the corollary to Lemma 2.4 each of these is isomorphic to a subset of R. Therefore R has at least $\aleph_{\kappa+1}$ non-PWOP isomorphic subsets.

Lemma 3.3. *If $\xi \preccurlyeq \zeta$ then $\zeta \sim \zeta\dot{\cup}\xi$.*

Proof. Define

$$R_0 := \{x \in N \mid \text{ length}(x) \equiv 1 \bmod 4\}$$

Then R_0 and $R - R_0$ are dense in R so by Lemma 1.12

$$\text{ordertype}(R_0) = \text{ordertype}(R - R_0) = \zeta.$$

Then if $\xi \preccurlyeq \zeta$ we can find $X_0 \subseteq R_0$ where ordertype$(X_0) = \xi$. If $R_1 := R - X_0$ then $R - R_0 \subseteq R_1$ which implies R_1 is dense in R so ordertype$(R_1) = \zeta$, again by Lemma 1.12. Then the decomposition

$$R = X_0 \dot{\cup} R_1$$

shows that

$$\zeta \sim \xi\dot{\cup}\zeta.$$

Lemma 3.4. *If $\xi \equiv \zeta$ then $\xi \sim \zeta$. Hence if $\phi \preccurlyeq \xi$ then $\xi \sim \xi\dot{\cup}\phi$.*

Proof. Since $\zeta \preccurlyeq \xi$ it must be true that $\xi = \zeta\dot{\cup}\rho$ for some order type ρ. Then $\xi \preccurlyeq \zeta$ implies that $\rho \preccurlyeq \zeta$ (and $\phi \preccurlyeq \zeta$ if $\phi \preccurlyeq \xi$) so by Lemma 3.3 it follows that $\zeta \sim \zeta\dot{\cup}\rho$. So $\zeta \sim \xi$. If $\phi \preccurlyeq \xi$ then $\phi \preccurlyeq \zeta$ so

$$\begin{aligned} \xi &\sim \zeta \\ &\sim \zeta\dot{\cup}\phi \\ &\sim \xi\dot{\cup}\phi \end{aligned}$$

and the lemma is true.

This important lemma is needed in the first stage of the induction performed on Laver's classification of all order types which are the union of a countable number of scattered types. Of course, in the singular case, since ordertype(N) is also an $\eta_{\omega_1,\omega_{\kappa+1}}$-universal and is isomorphic to any dense subset of itself we could have used ordertype(N) in place of ζ to prove Lemma 3.3.

To prove Theorem 3.1 I need to show that G has no more than $\aleph_{\kappa+1}$ orbits on $\mathcal{P}(R)$. All order types of sets which can be written as the union of countably many scattered type and which do not embed w_1^* or $\omega_{\kappa+1}$ can occur as the order types of subsets of R. Remember that $\mathcal{K} := \bigcup_{\beta \in \mathrm{On}} \mathcal{K}_\beta$, where

$$\mathcal{K}_0 := \{0, 1\};$$

$$\mathcal{K}_\beta := \{\phi \mid \phi \text{ is a } (\mathcal{U}, \omega^*)\text{-unbounded sum or a } (\mathcal{U}, \omega_\lambda)\text{-unbounded sum or a } (\mathcal{U}, \eta_{\omega_1,\omega_{\lambda+1}})\text{-universal, for some } \mathcal{U} \subseteq \bigcup_{\delta<\beta} \mathcal{K}_\delta, \text{ some initial ordinal } \omega_\lambda\}$$

and that, by Theorem 0.10, every order type which is a countable union of scattered sets and does not embed ω^* is a finite sum of elements of $\mathcal{K}$. Theorem 0.11 shows that there are only $\aleph_{\kappa+1}$ non-$\equiv$-equivalent order types in $\mathcal{K}$. Thus it is enough to show that $\equiv$-equivalence implies $\sim$-equivalence, in $\mathcal{K}$. To do this we will perform transfinite induction on $\mathrm{ht}(\phi)$ defined by

$$\mathrm{ht}(\phi) := \min\{\beta \in \mathrm{On} \mid \phi \in \mathcal{K}_\beta\}.$$

Lemma 3.5. *Suppose that $\phi \in \mathcal{K} - \{1\}$. If ψ is any order type with $\psi \preccurlyeq \phi$ then $\phi \sim \phi \dot{\cup} \psi$.*

Proof. Suppose, first of all, that ϕ is an unbounded sum $\sum_{\alpha<\omega_\lambda} \phi_\alpha$ where ω_λ is an initial ordinal and $\mathrm{ht}(\phi_\alpha) < \mathrm{ht}(\phi)$ for all $\alpha < \omega_\lambda$. This part of the proof follows almost exactly the proof of Lemma 3 in [**5**]. As inductive hypothesis we may suppose that for each α the assumption of the lemma holds if ϕ_α is infinite or 0. Let ψ be an order type $\preccurlyeq \phi$. If all the summands are finite (that is 0 or

1) then either $\phi = 0$ and $\phi \sim \phi \dot{\cup} \psi$ or, since $\phi \neq 1$, we know $\aleph_\lambda$ many of the summands are 1, so $\phi = \omega_\lambda$ and certainly $\phi \sim \phi \dot{\cup} \psi$.

Suppose then that at least one of the types ϕ_α is infinite. From a particular embedding of a set of order type ψ into one of type ϕ we get a decomposition $\psi = \sum_{\alpha<\omega_\lambda} \psi_\alpha$ where $\psi_\alpha \preccurlyeq \phi_\alpha$ for all α. We define a function $f : \omega_\lambda \to \omega_\lambda$ recursively as follows. Assuming that $f(\alpha)$ is already defined for all $\alpha < \beta$, we define $f(\beta)$ by the following rules:
if ψ_β is finite then

$$f(\beta) := \min\{\gamma \mid f(\alpha) < \gamma \text{ for all } \alpha < \beta \text{ and } \phi_\gamma \text{ is infinite }\}$$

if ψ_β is infinite then

$$f(\beta) := \min\{\gamma \mid f(\alpha) < \gamma \text{ for all } \alpha < \beta \text{ and } \psi_\beta \preccurlyeq \phi_\gamma\}.$$

The existence of such numbers $\gamma < \omega_\lambda$ is guaranteed by the definition of unbounded sums. The point of the definition is that the function f is strictly increasing, $\phi_{f(\beta)}$ is infinite and $\psi_\beta \preccurlyeq \phi_{f(\beta)}$ for all β. By our inductive hypothesis we have $\phi_{f(\beta)} \sim \psi_\beta \dot{\cup} \phi_{f(\beta)}$ for all $\beta < \omega_\lambda$. If $\alpha < \omega_\lambda$ and X_α is a linearly ordered set of type ϕ_α then we have partitions $X_\alpha = X_{\alpha 0} \dot{\cup} X_{\alpha 1}$ where

$$\text{if } \alpha \notin \text{im}(f) \text{ then } X_{\alpha 0} = \emptyset \text{ and } X_{\alpha 1} = X_\alpha;$$

$$\text{if } \alpha = f(\gamma) \text{ then } X_{\alpha 0} \text{ has type } \psi_\gamma \text{ and } X_{\alpha 1} \simeq X_\alpha.$$

Then, if $X := \sum_{\alpha\in\omega_\lambda} X_\alpha$ we have $X = \sum_{\alpha\in\omega_\lambda} X_{\alpha 0} \dot{\cup} \sum_{\alpha\in\omega_\lambda} X_{\alpha 1}$ and so $\phi \sim \psi \dot{\cup} \phi$. A very similar argument deals with the case when ϕ is a $(\mathcal{U}, \omega^*)$-unbounded sum.

Now suppose that ϕ is a universal, so $\phi = \sum_{\alpha\in Y} \phi_\alpha$ with $\text{ht}(\phi_\alpha) < \text{ht}(\phi)$, where Y is some set whose order type is an $\eta_{\omega_1,\omega_{\lambda+1}}$-universal.

Suppose that $\mathcal{U} \subset \{0,1\}$. Since $\phi \neq 1, \phi = 0$, which is covered above or by the second corollary to Theorem 0.6 we know $\phi \equiv \zeta$. Then by Lemma 3.4, $\phi \sim \psi \dot{\cup} \phi$ for any $\psi \preccurlyeq \phi$.

So suppose that ϕ_γ is infinite for some $\gamma \in Y$. Then by the definition of universal

$$\sum_{\alpha \in Y} \phi'_\alpha \preccurlyeq \phi$$

where $\phi'_\alpha = \phi_\gamma$ for all $\alpha \in Y$. So we must have $Y_0 \subseteq Y$, with ordertype$(Y_0) \equiv$ ordertype(Y) and ϕ_γ infinite for all $\gamma \in Y_0$ (and finite for all $\gamma \in Y - Y_0$). Let $h : Y - Y_0 \to Y_0$ be an injective, order preserving function (there must be one since ordertype$(Y) \preccurlyeq$ ordertype(Y_0)). Now if X is a set of order type ϕ and X_0, X_1 are sets of order type $\sum_{\alpha \in Y_0} \phi_\alpha$, and $\sum_{\alpha \in Y - Y_0} \phi_\alpha$ respectively then $X = X_0 \dot{\cup} X_1$ and so

$$\begin{aligned}\phi &\sim \sum_{\alpha \in Y_0} \phi_\alpha \dot{\cup} \sum_{\alpha \in Y - Y_0} \phi_\alpha \\ &\sim \sum_{\alpha \in Y_0} \chi_\alpha\end{aligned}$$

where $\chi_\alpha := \phi_\alpha$ if $\alpha \notin \mathrm{Im}(h)$, and $\chi_\alpha := \phi_\gamma \dot{\cup} \phi_\alpha$ if $\alpha = h(\gamma)$.
But $\chi_\alpha = \phi_\gamma \dot{\cup} \phi_\alpha \sim \phi_\alpha$ by the inductive hypothesis, (since ϕ_γ is finite and ϕ_α is infinite, automatically we have $\phi_\gamma \preccurlyeq \phi_\alpha$). Then $\phi \sim \sum_{\alpha \in Y_0} \phi_\alpha$ where every ϕ_α is infinite. Then if $\psi \preccurlyeq \phi$, again from a particular embedding of a set of order type ψ into one of type ϕ we get a decomposition $\psi = \sum_{\alpha \in Y_0} \psi_\alpha$ where $\psi_\alpha \preccurlyeq \phi_\alpha$ for all α. But then

$$\begin{aligned}\phi &= \sum_{\alpha \in Y_0} \phi_\alpha \\ &\sim \sum_{\alpha \in Y_0} \phi_\alpha \dot{\cup} \psi_\alpha \\ &\sim \sum_{\alpha \in Y_0} \phi_\alpha \dot{\cup} \sum_{\alpha \in Y_0} \psi_\alpha \\ &\sim \phi \dot{\cup} \psi\end{aligned}$$

(again using the inductive hypothesis, which is now possible since $\phi_\alpha \neq 1$).

Theorem 3.1. *The group of* PWOP *automorphisms of R has $\aleph_{\kappa+1}$ orbits on $\mathcal{P}(R)$.*

Proof. If X_1, X_2 are subsets of R then there exists a PWOP permutation of R mapping one to the other if and only if the ordered pairs $(X_1, R - X_1), (X_2, R - X_2)$

are PWOP equivalent. Therefore the number of orbits of the group G is equal to

are PWOP equivalent. Therefore the number of orbits of the group G is equal to the number of non-PWOP isomorphic subsets of R, and by Lemma 3.2 we have at least $\aleph_{\kappa+1}$ of these. If $X \subseteq R$ then Theorem 0.10 shows that if ξ is the order type of X then ξ is a finite sum of types from $\mathcal{K}$. Theorem 0.11 says there are are $\aleph_{\kappa+1}$ $\equiv$-equivalence classes of order types of cardinality $\aleph_\kappa$ in $\mathcal{K}$ so if we can prove that $\equiv$-equivalence implies $\sim$-equivalence then the theorem is true. However this is an immediate corollary of the last lemma. For if $\psi \equiv \phi$ then $\psi \preccurlyeq \phi$ so $\phi \sim \phi\dot{\cup}\psi$ but also $\phi \preccurlyeq \psi$ so $\psi \sim \psi\dot{\cup}\phi$ which means $\psi \sim \phi$.

Theorem 3.6. *The group of* PWOP *automorphisms of X has $\aleph_{\kappa+1}$ orbits on $\mathcal{P}(X)$ for any set X with order type an $(\mathcal{U}, \eta_{\omega_1,\omega_{\kappa+1}})$-universal, with $\mathcal{U} = \{0,1\}$.*

Proof. If $\phi =$ ordertype(X) then $\phi \equiv \zeta$ by the second corollary to Lemma 0.7 so if Y is any subset of X we have ordertype$(Y) \preccurlyeq \zeta$. Thus $\psi \preccurlyeq \phi$ implies and is implied by $\psi \preccurlyeq \zeta$. Therefore the number of non-$\sim$-equivalent order types embeddable into ζ and ϕ is the same, namely $\aleph_{\kappa+1}$ whence there are $\aleph_{\kappa+1}$ subsets of X which are not PWOP isomorphic.

This completes the proof of the theorem, and the section of this chapter concerning PWOP automorphisms. We will now see how the set M together with the theorems of [7] show that being a countable union of scattered sets with no uncountable descending chains is equivalent to being a countable union of well-ordered sets. First we will see how the work in this thesis alone gives a similar fact for well-ordered sets.

Theorem 3.7. *Suppose α is an ordinal of cardinality $\aleph_\kappa$. If W is a set of order type α then W is a countable union of sets $(W_i)_{i<\omega}$ where for each i the order type of W_i is $\leqslant \omega_\kappa^i$.*

Proof. Since Lemma 2.1 shows

$$M = \bigcup_{i<\omega} M_i$$

are PWOP equivalent. Therefore the number of orbits of the group G is equal to the number of non-PWOP isomorphic subsets of R, and by Lemma 3.2 we have at least $\aleph_{\kappa+1}$ of these. If $X \subseteq R$ then Theorem 0.10 shows that if ξ is the order type of X then ξ is a finite sum of types from $\mathcal{K}$. Theorem 0.11 says there are are $\aleph_{\kappa+1}$ $\equiv$-equivalence classes of order types of cardinality $\aleph_\kappa$ in $\mathcal{K}$ so if we can prove that $\equiv$-equivalence implies $\sim$-equivalence then the theorem is true. However this is an immediate corollary of the last lemma. For if $\psi \equiv \phi$ then $\psi \preccurlyeq \phi$ so $\phi \sim \phi\dot{\cup}\psi$ but also $\phi \preccurlyeq \psi$ so $\psi \sim \psi\dot{\cup}\phi$ which means $\psi \sim \phi$.

Theorem 3.6. *The group of* PWOP *automorphisms of X has $\aleph_{\kappa+1}$ orbits on $\mathcal{P}(X)$ for any set X with order type an $(\mathcal{U}, \eta_{\omega_1,\omega_{\kappa+1}})$-universal, with $\mathcal{U} = \{0,1\}$.*

Proof. If $\phi =$ ordertype(X) then $\phi \equiv \zeta$ by the second corollary to Lemma 0.7 so if Y is any subset of X we have ordertype$(Y) \preccurlyeq \zeta$. Thus $\psi \preccurlyeq \phi$ implies and is implied by $\psi \preccurlyeq \zeta$. Therefore the number of non-$\sim$-equivalent order types embeddable into ζ and ϕ is the same, namely $\aleph_{\kappa+1}$ whence there are $\aleph_{\kappa+1}$ subsets of X which are not PWOP isomorphic.

This completes the proof of the theorem, and the section of this chapter concerning PWOP automorphisms. We will now see how the set M together with the theorems of [7] show that being a countable union of scattered sets with no uncountable descending chains is equivalent to being a countable union of well-ordered sets. First we will see how the work in this thesis alone gives a similar fact for well-ordered sets.

Theorem 3.7. *Suppose α is an ordinal of cardinality $\aleph_\kappa$. If W is a set of order type α then W is a countable union of sets $(W_i)_{i<\omega}$ where for each i the order type of W_i is $\leqslant \omega_\kappa^i$.*

Proof. Since Lemma 2.1 shows

$$M = \bigcup_{i<\omega} M_i$$

where the order type of M_i is ω_κ^i and since Lemma 2.2 shows $\alpha \leqslant \text{ordertype}(M)$ the result follows.

Theorem 3.8(Lauchli [6]). *Conversely, this is the best result possible in the following sense. Suppose α is an ordinal, of cardinality $\aleph_\kappa$, with $\aleph_\kappa$ regular and greater than $\aleph_0$, and that $\alpha \geqslant \omega_\kappa^\omega$. If we write a set of order type α as a countable union of well-ordered sets, $\bigcup_{i<\omega} A_i$ where the order type of A_i is α_i, then $\sup_{i<\omega} \alpha_i \geqslant \omega_\kappa^\omega$.*

Proof. By Lemma 0.4, if $\omega_\kappa^i \leqslant \alpha$ where $i < \omega$ and $\omega_\kappa > \omega$ and we write a set of order type α as a countable union of sets then ω_κ^i is embeddable in at least one of the sets. Since this is true for every $i < \omega$ if $\alpha \geqslant \omega_\kappa^\omega$ the result follows.

Theorem 3.7 above is surprising, but using Lemma 2.4 and a theorem from [7], we can get an even stronger result.

Theorem 3.9. *If U is a linearly ordered set of cardinality $\aleph_\kappa$ which is a countable union of scattered sets then we can write*

$$U = \bigcup_{i<\omega} U_i$$

where ordertype$(U_i) \leqslant \omega_\kappa^i$.

Proof. By Theorem 0.6 we know that $\text{ordertype}(U) \preccurlyeq \phi$ if ϕ is any $\eta_{\omega_1,\omega_{\kappa+1}}$-universal. By Lemma 2.4 we know ordertype(M) is an $\eta_{\omega_1,\omega_{\kappa+1}}$-universal. Lemma 2.1 shows that M has such a decomposition, whence U does also.

Theorem 3.10. *Any scattered set U which does not embed ω_1^* or $\omega_{\kappa+1}$ and such that $\omega_\kappa^\omega \preccurlyeq \text{ordertype}(U)$ can be written as a countable union of well-ordered sets U_i where ordertype$(U_i) = \omega_\kappa^i$.*

Proof. This is equivalent to the theorem above. Obviously a countable union of countable unions of well-ordered sets is still a countable union of well-ordered sets.

Conversely any scattered set is of course a countable union of scattered sets so if U is scattered then

$$U = \bigcup_{i<\omega} U_i$$

where ordertype(U_i) $\leqslant \omega_\kappa^i$. Obviously for all $i < \omega$

$$\omega_\kappa^i \leqslant \omega_\kappa^\omega \leqslant \text{ ordertype}(U)$$

so there is no bound on $i < \omega$ such that $\omega_\kappa^i \leqslant$ ordertype(U). This implies that for each $i < \omega$ we can find U_i' with order type ω_κ^i such that $U_i - (\bigcup_{j<i} U_j') \subseteq U_i'$.

Theorem 3.6 is a special case of either 3.9 or 3.10 since well-ordered sets are scattered and have no descending chains that are even countable. If X is a countable union of scattered sets and X has no uncountable descending chains then Theorem 3.9 gives us the number of orbits of the group of $\aleph_\kappa$-PWOP automorphisms of X on $\mathcal{P}(X)$ as the following two lemmas show.

Lemma 3.11. *Suppose X is a countable union of scattered sets, that $|X| = \aleph_\kappa$ and that $\omega_\kappa^\omega \leqslant$ ordertype(X) and $w_1^* \not\leqslant$ ordertype(X). Then ordertype(X) $\sim_0$ ω_κ^ω.*

Proof. By Theorem 3.10

$$X = \dot{\bigcup_{i<\omega}} X_i$$

where ordertype(X_i) $= \omega_\kappa^i$. By definition, if W is a set of order type ω_κ^ω then

$$W = \sum_{i<\omega} W_i$$

where ordertype(W_i) $= \omega_\kappa^i$. Hence ordertype(X) $\sim_0$ ordertype(W).

Corollary. *All ordinals between ω_κ^ω and $\omega_{\kappa+1}$ are $\aleph_0$-PWOP-isomorphic to each other.*

Lemma 3.12. *Suppose that for $i = 1, \ldots, j$ we have that α_i is an initial ordinal with $\omega_\kappa < \alpha_1 < \alpha_2 < \ldots < \alpha_j$ and $n_1, \ldots, n_j$ are non-zero integers. Then*

$$\omega_\kappa^\omega \times \alpha_1^{n_1} \times \ldots \times \alpha_j^{n_j} \leqslant \gamma < \omega_{\kappa+1} \times \alpha_1^{n_1} \times \ldots \times \alpha_j^{n_j}$$

implies

$$\omega_\kappa^\omega \times \alpha_1^{n_1} \times \ldots \times \alpha_j^{n_j} \sim_0 \gamma.$$

Proof. First notice that, if α is infinite and $\alpha, \beta < \gamma$ then $\beta + (\alpha \times \gamma) \simeq \alpha \times \gamma$. This is true because $\alpha \simeq 1 + \alpha$. Thus

$$\begin{aligned} \alpha \times \gamma &\preccurlyeq \beta + \alpha \times \gamma \\ &\preccurlyeq \gamma + \alpha \times \gamma \\ &\simeq \alpha \times \gamma. \end{aligned}$$

Therefore, since these are well-ordered sets $\beta + (\alpha \times \gamma) \simeq \alpha \times \gamma$. Suppose

$$\omega_\kappa^\omega \times \alpha_1^{n_1} \times \ldots \times \alpha_j^{n_j} < \gamma < \omega_{\kappa+1} \times \alpha_1^{n_1} \times \ldots \times \alpha_j^{n_j}.$$

Then $\gamma = \alpha \times \alpha_1^{n_1} \times \ldots \times \alpha_j^{n_j} + \beta$ where $\omega_\kappa^\omega \leqslant \alpha < \omega_{\kappa+1}$ and $\beta < \alpha_1^{n_1} \times \ldots \times \alpha_j^{n_j}$. Then

$$\begin{aligned} \gamma &= \alpha \times \alpha_1^{n_1} \times \ldots \times \alpha_j^{n_j} + \beta \\ &\sim \beta + \alpha \times \alpha_1^{n_1} \times \ldots \times \alpha_j^{n_j} \\ &\simeq \alpha \times \alpha_1^{n_1} \times \ldots \times \alpha_j^{n_j}. \end{aligned}$$

The corollary to Lemma 3.11 shows that $\alpha \sim_0 \omega_\kappa^\omega$ so let $f : A \to W$ be an $\aleph_0$-PWOP isomorphism, where A and W are sets of order type α and ω_κ^ω respectively. Then let B be a well-ordered set of type $\alpha_1^{n_1} \times \ldots \times \alpha_j^{n_j}$ and order $A \times B$ and $W \times B$ lexicographically, creating sets of order type $\alpha \times \alpha_1^{n_1} \times \ldots \times \alpha_j^{n_j}$ and $\omega_\kappa^\omega \times \alpha_1^{n_1} \times \ldots \times \alpha_j^{n_j}$ respectively. The natural extension of f witnesses $A \times B \sim_0 W \times B$, whence

$$\alpha \times \alpha_1^{n_1} \times \ldots \times \alpha_j^{n_j} \sim_0 \omega_\kappa^\omega \times \alpha_1^{n_1} \times \ldots \times \alpha_j^{n_j}.$$

Since we showed above that

$$\gamma \sim \alpha \times \alpha_1^{n_1} \times \ldots \times \alpha_j^{n_j}$$

we have that

$$\gamma \sim_0 \omega_\kappa^\omega \times \alpha_1^{n_1} \times \ldots \times \alpha_j^{n_j}.$$

We can use the lemma just proved to see what $\sim_\lambda$-equivalence classes of ordinals look like, for any cardinal $\aleph_\lambda$. Firstly we will prove that $\sim_\lambda$-equivalence classes of ordinals are convex. The lemma above tells us about one set of $\sim_\lambda$-equivalence classes (because, as mentioned at the start of the chapter, $\sim_0$-equivalence implies $\sim_\lambda$-equivalence). Bearing in mind the convexity lemma the two lemmas following it show which of the ordinals which are not covered by Lemma 3.12 are $\sim_\lambda$-equivalent.

Lemma 3.13. *Any $\sim_\lambda$-equivalence class of ordinals is convex.*

Proof. Suppose α, β and γ are ordinals with $\alpha \sim_\lambda \beta$ and $\alpha < \gamma < \beta$. We can assume without loss of generality that all three are limit ordinals since if α is any limit ordinal and n is a finite order type then $\alpha \simeq n + \alpha \sim \alpha + n$. Then, since α is a limit ordinal we know that

$$\alpha \sim \alpha \mathbin{\dot\cup} \alpha$$

and similarly for β and γ.

Let A and B be sets of order type α and β and C a subset of B of order type γ. Notice that there exists $C_1 \subset A$ with $C_1 \sim_\lambda C$ and $A - C_1 \sim_\lambda B - C$. Indeed if $A = \dot{\bigcup}_{\mu<\omega_\lambda} A_\mu$ and $f : A \to B$ witness $A \sim_\lambda B$ then $B = \dot{\bigcup}_{\mu<\omega_\lambda} f(A_\mu)$. Let $C_\mu := f(A_\mu) \cap C$. Then

$$\begin{aligned} C_1 &:= \left(\dot{\bigcup_{\mu<\omega_\lambda}} f^{-1}(C_\mu) \right) \sim_\lambda C \\ B - C &= \dot{\bigcup_{\mu<\omega_\lambda}} (f(A_\mu) - C_\mu) \\ &\sim_\lambda \dot{\bigcup_{\mu<\omega_\lambda}} (A_\mu - f^{-1}(C_\mu)) \\ &= A - C_1. \end{aligned}$$

Then

$$\begin{aligned} C &\sim_\lambda (C - A) \mathbin{\dot\cup} A \mathbin{\dot\cup} (A - C_1) \mathbin{\dot\cup} C_1 \\ &\sim (B - C) \mathbin{\dot\cup} (C - A) \mathbin{\dot\cup} C \mathbin{\dot\cup} A \\ &\sim B \end{aligned}$$

Hence $C \sim_\lambda B$ and so $\gamma \sim_\lambda \beta$.

Notice that, if $|\alpha| \leqslant \aleph_\lambda$ then $\alpha \times \beta \sim_\lambda \beta \times \alpha$ for any ordinal β.

Lemma 3.14. *Suppose $\aleph_\lambda$ is an infinite cardinal and for $i = 1, \dots, j$ we have that α_i is an initial ordinal, that n_1 is an integer (possibly 0), that n_i is a non-zero integer if $i > 1$ and $\omega_{\lambda+1} = \alpha_1 < \alpha_2 < \dots < \alpha_j$. Then*

$$\alpha_1^{n_1} \times \dots \times \alpha_j^{n_j} \sim_\lambda \beta$$

if and only if $\alpha_1^{n_1} \times \dots \times \alpha_j^{n_j} \leqslant \beta < \omega_{\lambda+1} \times \alpha_1^{n_1} \times \dots \times \alpha_j^{n_j}$.

Proof. Suppose (seeking a contradiction) that in the situation defined above we have

$$\alpha_1^{n_1} \times \dots \times \alpha_j^{n_j} \sim_\lambda \omega_{\lambda+1} \times \alpha_1^{n_1} \times \dots \times \alpha_j^{n_j}. \tag{†}$$

Let $\beta := \alpha_2^{n_2} \times \dots \times \alpha_j^{n_j}$. Then (†) implies that we can write sets A and B of order type $\omega_{\lambda+1}^{n_1} \times \beta$ and $\omega_{\lambda+1}^{n_1+1} \times \beta$ respectively as

$$A = \dot{\bigcup_{\mu<\omega_\lambda}} A_\mu \qquad \text{and} \qquad B = \dot{\bigcup_{\mu<\omega_\lambda}} B_\mu$$

where $A_\mu \simeq B_\mu$. We know that $\aleph_{\lambda+1}$ is a regular cardinal, and obviously $\omega_\lambda < \omega_{\lambda+1}$. Thus by Lemma 0.4 we have $\omega_{\lambda+1}^{n_1+1} \times \beta \leqslant \text{ordertype}(B_\mu)$ for some μ, whence $\omega_{\lambda+1}^{n_1+1} \times \beta \leqslant \text{ordertype}(A_\mu)$ and so $\omega_{\lambda+1}^{n_1+1} \times \beta \leqslant \text{ordertype}(A) = \omega_{\lambda+1}^{n_1} \times \beta$. This is absurd so we cannot have $\alpha_1^{n_1} \times \dots \times \alpha_j^{n_j} \sim_\lambda \omega_{\lambda+1} \times \alpha_1^{n_1} \times \dots \times \alpha_j^{n_j}$.

Now suppose that

$$\alpha_1^{n_1} \times \dots \times \alpha_j^{n_j} \leqslant \beta < \omega_{\lambda+1} \times \alpha_1^{n_1} \times \dots \times \alpha_j^{n_j}.$$

Then $\beta = \alpha \times \alpha_1^{n_1} \times \dots \times \alpha_j^{n_j} + \gamma$ where $\alpha < \omega_{\lambda+1}$ so $|\alpha| \leqslant \aleph_\lambda$, and $\gamma < \alpha_1^{n_1} \times \dots \times \alpha_j^{n_j}$. Then

$$\begin{aligned} \beta &= \alpha \times \alpha_1^{n_1} \times \dots \times \alpha_j^{n_j} + \gamma \\ &\sim \gamma + \alpha \times \alpha_1^{n_1} \times \dots \times \alpha_j^{n_j} \\ &\simeq \alpha \times \alpha_1^{n_1} \times \dots \times \alpha_j^{n_j} \\ &\sim_\lambda \alpha_1^{n_1} \times \dots \times \alpha_j^{n_j} \times \alpha \\ &\simeq \alpha_1^{n_1} \times \dots \times \alpha_j^{n_j}. \end{aligned} \tag{1}$$

(The statement (1) is true because of the comment at the beginning of the proof of Lemma 3.12.)

Theorem 3.15. *Suppose $\aleph_\lambda$ is any infinite cardinal and $\aleph_\lambda < \aleph_\kappa$. Then there are countably many $\sim_\lambda$-equivalence classes of ordinals of cardinality $\leqslant \aleph_\kappa$ if $\kappa < \omega_1$ and $|\kappa|$ many if $\kappa \geqslant \omega_1$.*

Proof. The preceding four lemmas show that the $\sim_\lambda$-equivalence classes of ordinals of cardinality $\leqslant \aleph_\kappa$ are of the form

$$\{\gamma \in \mathrm{On} \mid |\gamma| = \aleph_\iota \text{ for some cardinal } \aleph_\iota \leqslant \aleph_\lambda\}$$

or

$$\{\gamma \in \mathrm{On} \mid \alpha_1^{n_1} \times \ldots \times \alpha_j^{n_j} \leqslant \gamma < \omega_{\lambda+1} \times \alpha_1^{n_1} \times \ldots \times \alpha_j^{n_j}, \text{ for } i = 1, \ldots, j$$
$$0 \leqslant n_i < \omega,\ \alpha_i \text{ an initial ordinal with } \omega_{\lambda+1} \leqslant \alpha_1 < \ldots < \alpha_j \leqslant \omega_\kappa\}$$

or

$$\{\gamma \in \mathrm{On} \mid \omega_{\lambda+1}^{\omega} \times \alpha_1^{n_1} \times \ldots \times \alpha_j^{n_j} \leqslant \gamma < \omega_{\lambda+2} \times \alpha_1^{n_1} \times \ldots \times \alpha_j^{n_j}, \text{ for } i = 1, \ldots, j$$
$$0 < n_i < \omega,\ \alpha_i \text{ an initial ordinal with } \omega_{\lambda+1} < \alpha_1 < \ldots < \alpha_j \leqslant \omega_\kappa\}.$$

Suppose $\kappa < \omega_1$. Then the first two sets are countable and the third set has size $\aleph_0$. Hence the number of $\sim_\lambda$-equivalence classes is $\aleph_0$. If $\kappa \geqslant \omega_1$ then the first two sets have size $|\kappa|$ and again the third one has size $\aleph_0$ so the lemma holds.

Theorem 3.16. *The group of $\aleph_0$-PWOP automorphisms of R has $\aleph_0$ orbits on $\mathcal{P}(R)$ if $\kappa < \omega_1$ and $|\kappa|$ if $\kappa \geqslant \omega_1$.*

Proof. This follows from Theorem 3.15 together with its proof, since by Lemma 3.11 we know that ordertype$(R) \sim_0 \omega_\kappa^\omega$. Then, if A is a (proper) subset of a set W of order type ω_κ^ω it must be true that A has order type α for some ordinal $\alpha < \omega_\kappa^\omega$. Thus the number of orbits of the $\aleph_0$-PWOP automorphisms on $\mathcal{P}(W)$ is the same as the number of non $\aleph_0$-PWOP- isomorphic ordinals.

Conclusion

The question of whether there exist sets of cardinality $\aleph_\kappa$ with permutation groups acting on them, with $\aleph_\lambda$ orbits on the power set, has been fully answered in this thesis, in the case when $\aleph_\lambda = \aleph_{\kappa+1}$. However it is still open for $\aleph_\lambda$ bigger than $\aleph_{\kappa+1}$ if $\aleph_\kappa$ is larger than $\aleph_0$. Since answering this question, in the case when $\aleph_\kappa$ is countable requires Martin's Axiom (see [1]), it is probable that solving the problem when $\aleph_\lambda$ is larger than $\aleph_{\kappa+1}$ will require more than the normal axioms of Zermelo-Fraenkel set theory with the axiom of choice.

We also do not know, at present, whether the set Q, consisting of all finite words in $\omega_\kappa \times \mathbb{Q}$, with a lexicographic ordering, is order-isomorphic to all its dense subsets, but this is a question that should be amenable to the techniques already used in this work.

References

[1] U. Avraham and S. Shelah, Martin's axiom does not imply that every two $\aleph_1$-dense sets of reals are isomorphic, *Israel J. Math.* **38** (1981), 161–176.

[2] J. Baumgartner, All $\aleph_1$-dense sets of reals can be isomorphic, *Fund. Math.* **79** (1973), 101–106

[3] R. Fraïssé, *Theory of Relations,* Amsterdam: North-Holland, 1986.

[4] F. Hausdorff, Grundzüge einer Theorie der geordneten Mengen, *Math. Ann.* **65** (1908), 435–505.

[5] Läuchli, H., Neumann, Peter. M., On Linearly Ordered Sets and Permutation Groups of Countable Degree, *Arch. Math. Logic* **27** (1988), 189–192.

[6] Läuchli, H., Private communication to Peter Neumann, 1989.

[7] Laver, R., On Fraissé's Order Type Conjecture, *Ann. Math* **93** (1971), 89–111.

[8] E. C. Milner, R. Rado, The Pigeon-hole Principle for Ordinal Numbers, *Proc. London Math. Soc.(3)* **15** (1965), 750–68.

[9] C. St. J. Nash-Williams, On well-quasi-ordering infinite trees, *Proc. Cambridge Phil. Soc.,* **61** (1965), 697–720.

[10] Ivan Rival (ed.), *Ordered Sets,* Holland: D. Reidel Publishing Company, 1982.

[11] Joseph G. Rosenstein, *Linear Orderings,* New York: Academic Press, 1982.

[12] Shelah, S., Thomas, S. Implausible subgroups of infinite symmetric groups, *Bul. L.M.S.,* **20** (1988) 313–318.

[13] W. Sierpiński, *Cardinal and Ordinal Numbers,* Poland: Polish Scientific Publishers, 1965.

[14] W. Sierpínski, Sur les types d'ordre des ensembles lineaires, *Fund. Math.* **37** (1950), 253–264.

[15] Gerald Stoller, Example of a subgroup of S_∞ which has a set transitive property. *Bull. Amer. Math. Soc.,* **69** (1963), 220–221.