Topological Uncertainty in Wireless Networks

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Abstract—This work provides an analysis of topological uncertainty in wireless networks. Here, we model a wireless network as a random geometric graph (RGG) and quantify topological uncertainty in terms of the Shannon entropy of the underlying graph model. Direct pairwise connections between nodes are probabilistic in general, and thus our analysis covers cases where channel randomness contributes to uncertainty in the network topology. We derive a simple bound on RGG entropy. We then consider a pairwise connection model based on small-scale Rayleigh fading and study the behavior of the entropy bound as the number of nodes n in the network grows large while the typical connection range r_0 increases or decreases. We present three key results that quantify the rate of growth or decay of r_0 , as a function of the number of nodes n , that must be obeyed in order for the entropy bound to converge to a positive limit. Although the contributions of this paper are theoretical, they have applications in systems such as ad hoc networks employing opportunistic routing and device-to-device (D2D) networks for future cellular communication (5G and beyond).

Index Terms—Random geometric graphs, network topology, routing, entropy, connectivity.

I. INTRODUCTION

The availability of topology information at a central entity or in a distributed nature is of great importance in many wireless networks, including mobile ad hoc networks and multihop cellular device-to-device (D2D) networks [1]. Such information provides a view of connectivity at a fundamental level, but it can also be exploited in practice to adjust physical layer modulation and coding choices as well as routing protocols [2], [3]. In many wireless networking contexts, the network topology is uncertain, perhaps due to fluctuations in the channel (fading) or owing to node mobility. In such cases, it is enormously beneficial to be able to quantify this uncertainty. Tasks such as network synchronization [4], evaluating route stability [5], and communicating topology information throughout the network in order to maintain connectivity [6] are fundamentally governed by the level of topological uncertainty in the network.

Topological uncertainty has been studied in the fields of biology, chemistry, sociology, and computer science for many years through the mathematical formalism of graph entropy [7], [8]. Mostly, research has been focused on networks and graphs whose distributions do not rely on an underlying spatial embedding. The canonical example is the Erdős-Rényi (ER) graph, in which each edge exists independently with probability p . Such models often lend themselves to applications involving, for example, database storage. Indeed, the structural entropy of ER graphs was recently investigated in [9]. In contrast to

these “non-spatial” network models, the topology of a wireless communication network inherently depends upon the geometric and physical nature of the space in which it is situated, a dependence that arises through propagation effects. Topological uncertainty in dynamic mobile ad hoc networks was investigated in [6], and [10] treated self-organization of networks using a graph entropy framework. Apart from these examples, topological uncertainty as it relates specifically to wireless communication networks has been left relatively unexplored.

In this paper, we analyze uncertainty in wireless networks through the formalism of random geometric graphs (RGGs). The network model that we employ accounts for random node positions and link states affected by small-scale fading. This is the first time such a model has been used to study topological uncertainty. As with previous works, we quantify uncertainty in terms of the Shannon entropy of a graph. We derive a simple bound on RGG entropy and study the behavior of this bound as the number of nodes in the network n grows large while the typical connection range r_0 increases or decreases. We consider both hard and soft (probabilistic) pairwise connection models, the soft models pertaining to the case where Rayleigh fading is present in the channel. We present three key results of this work as lemmas that quantify the rate of growth or decay of r_0 , as a function of the number of nodes n , that must be obeyed in order for the entropy bound to converge to a positive limit. Although the contributions of this paper are theoretical, we discuss practical implications of the results and make suggestions for further study.

The rest of the paper is organized as follows. In section II, we give details of the network and pairwise connection models considered in this work. We quantify topological uncertainty for wireless networks in section III by analyzing the entropy of RGGs. Section IV deals with specific questions surrounding the behavior of RGG entropy as the number of nodes in the network grows while the typical connection range for each node increases or decreases. Conclusions and areas for future study are given in section V.

Notation: Throughout this paper, we use the following notation. The expectation operation is denoted by $\mathbb{E}[\cdot]$. $|\mathcal{A}|$ is the size of the set $\mathcal{A}$. $\mathcal{K}_d$ signifies a convex, compact space in d dimensions, where $V(\mathcal{K}_d)$ is the volume of $\mathcal{K}_d$. For two dimensions $L(\mathcal{K}_2)$ and $A(\mathcal{K}_2)$ are the perimeter length and the area of $\mathcal{K}_2$; in three dimensions, $A(\mathcal{K}_3)$ is the surface area of $\mathcal{K}_3$. $\mathbb{I}(\cdot)$ is the indicator function, which is equal to one if the argument is true and zero otherwise. $\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt$ is

the gamma function. For nonnegative functions $f(n)$ and $g(n)$, the notation $f(n) = O(g(n))$ signifies that for some $k_1 > 0$, there exists an $n_1 > 0$ such that $f(n) \leq k_1 g(n)$ for all $n > n_1$; moreover, the notation $f(n) = \Omega(g(n))$ signifies that for some $k_2 > 0$, there exists an $n_2 > 0$ such that $f(n) \geq k_2 g(n)$ for all $n > n_2$. The asymptotic equivalence of two functions $f(n)$ and $g(n)$ is defined as the condition $\lim_{n \rightarrow \infty} f(n)/g(n) = 1$, and it is indicated symbolically as $f(n) \sim g(n)$.

II. NETWORK AND PAIRWISE CONNECTION MODELS

A. Network Model

Consider a set of n nodes (wireless devices) $\mathcal{V}$ embedded in $\mathcal{K}_d \subset \mathbb{R}^d$, which has volume $K_d = |\mathcal{K}_d|$. The locations of the nodes are denoted by $\mathbf{r}_1, \dots, \mathbf{r}_n$ and are independent and identically distributed (i.i.d.) in $\mathcal{K}_d$ according to a probability density $\lambda(\mathbf{r})$. In this work, we will consider a uniform distribution of nodes, i.e., $\lambda(\mathbf{r}) := \lambda = 1/K_d$. The triple $(\mathcal{V}, \mathcal{K}_d, \lambda)$ represents a binomial point process (BPP). In the following, we use the index i and the location $\mathbf{r}_i$ interchangeably to refer to node i .

We model a wireless network as an instantiation of an undirected RGG G . We denote the set of all possible graphs by $\mathcal{G}$, and each graph $g \in \mathcal{G}$ is described by the node set $\mathcal{V}$, the particular embedding mentioned above, and the resulting edge set $\mathcal{E}_g$, where each edge represents a directly established communication link between two nodes in the network. An edge (i, j) exists between nodes i and j , separated by a (Euclidean) distance $r_{i,j} = \|\mathbf{r}_i - \mathbf{r}_j\|$, with probability $p(r_{i,j}) := p_{i,j}$.

Notationally, we stress that G is a random variable, which can take values in $\mathcal{G}$. Consequently, one might say that $g = (\mathcal{V}, \mathcal{E}_g)$ exists *a posteriori*, but the *probability* that $G = g$ is prescribed by the probability measure p and the probability density λ .

B. Pairwise Connection Model

We call the probability $p_{i,j}$ the *pairwise connection function* for nodes i and j . Allowing edges to exist probabilistically will enable us to consider the effect that multipath fading has on network topology. Moreover, certain functions $p_{i,j}$ can be parameterized such that the conventional unit disk model (a.k.a. Gilbert disk) can be recovered as a special case. Indeed, if $p(r) = \mathbb{I}(r \leq r_0)$ for $r_0 > 0$ and $\mathcal{K}_d = [0, 1]^d$, then we have the canonical RGG formalism with unit disk connection function [11], [12]. Note that previous works have generally only considered unit disk pairwise connection functions [10] or completely different random graph formalisms [9].

A typical pairwise connection function, modeled on the outage probability of a Rayleigh fading channel in wireless networks, has the form [13]

$$p(r) = \exp(-(r/r_0)^\eta) \quad (1)$$

where r_0 defines the typical connection range and $\eta \geq 1$ determines how soft the connection function is, with larger η yielding a harder cutoff on the connectivity range. Physically, η represents the path loss exponent, and (1) signifies

the probability that the channel linking two nodes separated by a distance r can support a prescribed rate [13]. Letting $\eta \rightarrow \infty$, we recover the hard disk connection function. The typical connection range is usually defined by system variables including the transmission frequency, antenna gains, and the noise figure associated with the receiver amplifiers.

III. QUANTIFYING TOPOLOGICAL UNCERTAINTY WITH SHANNON ENTROPY

We quantify topological uncertainty in the context of wireless networks by studying the Shannon entropy of the underlying RGG model. The entropy of a graph G can be written as¹

$$H(G) = \mathbb{E}[-\log \mathbb{P}(G)] = - \sum_{g \in \mathcal{G}} \mathbb{P}(g) \log \mathbb{P}(g). \quad (2)$$

Note that the sum is over all graphs; the node configuration is marginalized in this definition. Explicitly accounting for this marginalization, we can write

$$\mathbb{P}(g) = \frac{1}{K_d^n} \int_{\mathcal{K}_d^n} \mathbb{P}(g|\mathbf{r}_1, \dots, \mathbf{r}_n) d\mathbf{r}_1 \cdots d\mathbf{r}_n. \quad (3)$$

Working with pair distances $r_{i,j}$ instead of node positions, we have that

$$\begin{aligned} \mathbb{P}(g|\mathbf{r}_1, \dots, \mathbf{r}_n) &= \mathbb{P}(g|r_{1,2}, \dots, r_{n-1,n}) \\ &= \prod_{(i,j) \in g} p(r_{i,j}) \prod_{(i,j) \notin g} (1 - p(r_{i,j})). \end{aligned} \quad (4)$$

Let us assume that the pair distances are distributed as $r_{1,2}, \dots, r_{n-1,n} \sim f(r_{1,2}, \dots, r_{n-1,n})$. Instead of averaging over the node positions as in (3), we may average over the pair distances

$$\begin{aligned} \mathbb{P}(g) &= \int_{[0,D]^n} f(r_{1,2}, \dots, r_{n-1,n}) \\ &\times \prod_{(i,j) \in g} p(r_{i,j}) \prod_{(i,j) \notin g} (1 - p(r_{i,j})) \prod_{i < j} dr_{i,j}. \end{aligned} \quad (5)$$

To make progress, we assume the pair distances are i.i.d., i.e., $f(r_{1,2}, \dots, r_{n-1,n}) \simeq \prod_{i < j} f(r_{i,j})$. Invoking this assumption, it is easy to see that

$$\mathbb{P}(g) \simeq \bar{p}^{|\mathcal{E}_g|} (1 - \bar{p})^{\binom{n}{2} - |\mathcal{E}_g|} \quad (6)$$

where $\mathcal{E}_g$ denotes the edge set for the graph g (and thus $|\mathcal{E}_g|$ is the number of edges in g),

$$\bar{p} := \int_0^D p(r) f(r) dr \quad (7)$$

is the average pairwise connection probability, and D is the diameter of the set $\mathcal{K}_d$, defined as

$$D = \sup_{\mathbf{r}_i, \mathbf{r}_j \in \mathcal{K}_d} \|\mathbf{r}_i - \mathbf{r}_j\|. \quad (8)$$

¹To aid exposition, all logarithms will have base e throughout this paper unless specified otherwise.

Substituting into (2), we have

$$\begin{aligned} H(G) &\simeq - \sum_{k=0}^{\binom{n}{2}} \binom{\binom{n}{2}}{k} \bar{p}^k (1-\bar{p})^{\binom{n}{2}-k} \log \left(\bar{p}^k (1-\bar{p})^{\binom{n}{2}-k} \right) \\ &= \binom{n}{2} H(\bar{p}) \end{aligned} \quad (9)$$

where

$$H(\bar{p}) = -\bar{p} \log \bar{p} - (1-\bar{p}) \log(1-\bar{p}) \quad (10)$$

is the binary entropy function.

The structure of (9) is somewhat predictable. Indeed, if $p_{i,j} = p$ for all $i < j$, G is an ER graph. The entropy of ER graphs is well known to be

$$H_{\text{ER}}(G) = \binom{n}{2} H(p). \quad (11)$$

So by assuming pair distances are independent in the RGG case, we obtain an ER-like result, but where the pairwise connection probability is averaged over the pair distance distribution.

The independence assumption that led to (6) deserves comment. In fact, it can easily be reasoned that pair distances are not necessarily independent when the network domain is finite. Consider a simple example of three nodes positioned on the interior of a circle. Suppose node i resides on the boundary, and nodes j and k are positioned such that $r_{i,j}$ and $r_{i,k}$ are both approximately D , the diameter of the circle. Clearly, nodes j and k must lie very close to each other, and so knowledge of $r_{i,j}$ and $r_{i,k}$ provides statistical information about $r_{j,k}$.

If we return to the original definition of the entropy of G provided in (2), we can make use of the fact that, for a given number of nodes n , the distribution of G is equivalent to the distribution of the edge set. Let $X_{i,j}$ denote a Bernoulli random variable that models the existence (or not) of edge (i,j) . It follows that we can write

$$H(G) = H(X_{1,2}, X_{1,3}, \dots, X_{n-1,n}) \leq \sum_{i < j} H(X_{i,j}) \quad (12)$$

where equality holds if all $\{X_{i,j}\}$ are independent. The random variable $X_{i,j}$ is physically related to nodes i and j , but since pair distance information is not included in (12), we can write

$$\mathbb{P}(X_{i,j}) = \int_0^D \mathbb{P}(X_{i,j}|r_{i,j}) f(r_{i,j}) dr_{i,j} = \bar{p} \quad (13)$$

for all $i < j$. It follows that

$$H(G) \leq \binom{n}{2} H(\bar{p}). \quad (14)$$

Thus, the approximation given in (9) is an upper bound on $H(G)$. It is important to note that although we have arrived at this bound by simple means via (12), this approach abstracts the pair distance considerations taken earlier, and as a result it is perhaps less physically intuitive. Indeed, without considering the effects of the embedding geometry, it is not immediately obvious that (12) does not hold with equality.

Fig. 1. Entropy of an RGG with $n = 5$ nodes and pair connection functions with typical range r_0 for a unit square. Solid lines: upper bound $h(n)$. Markers: numerical simulations for $H(G)$.

IV. EFFECT OF FADING CHANNELS ON UNCERTAINTY

We now study topological uncertainty in fading channels by analyzing the entropy approximation (bound) given in the previous section, which we denote by $h(n) = \binom{n}{2} H(\bar{p})$ to aid exposition. Our focus is on systems that experience diffuse multipath propagation. In other words, we investigate the case when the small-scale fading model is Rayleigh, and the pairwise connection function that will form the basis of this analysis is that given in (1).

Before proceeding with a detailed analysis, it is instructive to examine Fig. 1, which shows the graph entropy $H(G)$ and the bound $h(n)$ for a five-node network plotted against the typical connection range r_0 for path loss exponents $\eta = 2, 3, 4, \infty$. The bounding geometry is a unit square in this example. A few important things can be noted from the figure. First, the bound $h(n)$ is relatively accurate, particularly for very soft connection functions (e.g., $\eta = 2$) and for small/large r_0 . Second, the entropy decreases as $r_0 \rightarrow 0$ and $r_0 \rightarrow \infty$. Third, the maxima of $h(n)$ and $H(G)$ appear to coincide fairly well². Motivated by these observations, we will analyze the small/large typical connection range regimes and develop scaling laws that link r_0 and the number of nodes in the network n to the entropy bound $h(n)$.

A. Small Typical Connection Range

Our main results relate to the case where the typical connection range decreases as the number of nodes in the network grows large (see Fig. 2). Intuitively, one can recognize that it is possible for the entropy bound to tend to zero in such a case, which would signify that the network is completely disconnected³. Hence, it is of fundamental interest to ascertain

²This is an interesting property that warrants further study, but we refrain from addressing it in this paper.

³Of course, $h(n) = 0$ may also signify that the network is completely connected (a complete graph); but this is impossible in the scaling regime considered here ($r_0 \rightarrow 0$). We will consider $r_0 \rightarrow \infty$ in the next section.

Fig. 2. Illustration of a small typical connection range in $\mathcal{K}_2$. The dots represent nodes with typical connection range r_0 , indicated by the blue shading. Note that unless $\eta = \infty$, there is a nonzero probability of connection for each pair of nodes, regardless of proximity. The shaded region around the boundary has area roughly equal to $l_2 r_0$.

the conditions under which such an event would (or would not) occur. To make progress, we will need to evaluate $\bar{p}$, which requires knowledge of the pair distance density function $f(r)$.

1) *Pair Distance Distribution:* From the theory of stochastic geometry [14], it is known that in d dimensions, the *set covariance* of a convex set $\mathcal{K}_d$ is given by

$$c_{\mathcal{K}_d}(\mathbf{r}) = \int_{\mathbb{R}^d} \mathbb{I}(\mathbf{x} \in \mathcal{K}_d) \mathbb{I}(\mathbf{x} - \mathbf{r} \in \mathcal{K}_d) d\mathbf{x}, \quad \mathbf{r} \in \mathbb{R}^d. \quad (15)$$

The *isotropized set covariance* is given by

$$\bar{c}_{\mathcal{K}_d}(r) = \int_{S^{d-1}} c_{\mathcal{K}_d}(s\mathbf{u}) d\mathbf{u}, \quad r \geq 0 \quad (16)$$

where $\mathbf{u}$ is a vector denoting a point on the $(d-1)$ -dimensional unit sphere S^{d-1} embedded in $\mathbb{R}^d$. It is known that in two dimensions the isotropized set covariance for uniformly distributed points in $\mathcal{K}_2$ is well approximated by the first-order expression

$$\bar{c}_{\mathcal{K}_2}(r) \simeq A(\mathcal{K}_2) - \frac{L(\mathcal{K}_2)}{\pi} r. \quad (17)$$

In three dimensions, we have the analogous approximation

$$\bar{c}_{\mathcal{K}_3}(r) \simeq V(\mathcal{K}_3) - \frac{A(\mathcal{K}_3)}{4} r. \quad (18)$$

In general, the pair distance probability density function (PDF) is given by [14]

$$f(r) = \frac{2\pi^{d/2} r^{d-1} \bar{c}_{\mathcal{K}_d}(r)}{\Gamma(d/2) V(\mathcal{K}_d)^2}. \quad (19)$$

2) *Entropy Scaling Limits in Two Dimensions:* Consider the case where the nodes in $\mathcal{V}$ are (randomly) uniformly distributed in $\mathcal{K}_2$. For notational simplicity, let the area and perimeter length of $\mathcal{K}_2$ be denoted by $a_2 := A(\mathcal{K}_2)$ and $l_2 := L(\mathcal{K}_2)$, respectively. We focus on the case when $r_0 \ll \pi a_2 / l_2$, which suggests the typical connection range is small relative to the

size of the domain (see Fig. 2). Based on this assumption and the fact that the pairwise connection function decays exponentially in the argument (cf. (1)), we can invoke the results given in section IV-A1 to write the pair distance PDF as

$$f(r) \simeq \frac{2\pi r}{a_2} \left(1 - \frac{l_2 r}{\pi a_2}\right). \quad (20)$$

Again, by the exponential decay of the integrand of (7), we have that

$$\bar{p} \simeq \int_0^\infty p(r) f(r) dr \simeq \bar{p}_0 \left(1 - \frac{l_2 r_0}{\pi a_2} \frac{\Gamma(3/\eta)}{\Gamma(2/\eta)}\right) \quad (21)$$

where

$$\bar{p}_0 = \frac{2\pi r_0^2 \Gamma(2/\eta)}{\eta a_2} \quad (22)$$

is the leading-order average pairwise connection probability, which quantifies the average proportion of $\mathcal{K}_2$ that is covered by a node in the bulk of the domain. We observe this proportion clearly by considering large η , for which we can write

$$\bar{p}_0 = \frac{\pi r_0^2}{a_2} - \frac{2\pi \gamma r_0^2}{\eta a_2} + O\left(\frac{1}{\eta^2}\right) \quad (23)$$

where $\gamma \simeq 0.5772$ is the Euler-Mascheroni constant. Note that the first term on the right-hand side is exactly the proportional hard disk coverage area.

Using (21) and considering the case when $r_0 \simeq 0$, we can write the binary entropy as

$$H(\bar{p}) = u_2 r_0^2 \left(2 \log\left(\frac{1}{r_0}\right) + 1 - \log u_2\right) + O(r_0^3 \log r_0) \quad (24)$$

where $u_2 = 2\pi \Gamma(2/\eta) / (\eta a_2)$ can be interpreted as the fractional area of a *soft* unit disk. From this expression, we can develop the following lemma that gives minimal conditions on the typical connection range that must be satisfied in order to ensure the graph is not completely disconnected.

Lemma 1: As $n \rightarrow \infty$, the entropy of a graph in $\mathcal{K}_2$ can be bounded away from zero only if

$$r_0^2 \log\left(\frac{1}{r_0}\right) = \Omega\left(\frac{1}{n^2}\right). \quad (25)$$

The entropy bound will tend to a limit $h(n) \rightarrow \ell_h > 0$ as $n \rightarrow \infty$ if

$$\begin{aligned} r_0(n) &= \exp\left(\frac{1}{2} W_m\left(-\frac{2\ell_h}{u_2 n(n-1)}\right)\right) \\ &= \left(\frac{\ell_h}{u_2 n^2 \log n}\right)^{\frac{1}{2}} \left(1 + O\left(\frac{1}{\log n}\right)\right) \end{aligned} \quad (26)$$

where $W_m(x)$ is the lower branch ($-1/e \leq x < 0$ and $W_m \leq -1$) of the solution to $x = W \exp W$.

Proof: Eq. (25) follows directly from (24) and the fact that $h(n) \sim n^2 H(\bar{p})/2$ as $n \rightarrow \infty$. Eq. (26) can be obtained directly by equating $h(n) = \ell_h$ and solving for r_0 when only the leading order of (24) is retained. $\square$

This lemma provides interesting insight into the topological uncertainty of networks. By scaling the connection range r_0

like $O(1/n)$, for example, (25) is met and the entropy bound increases with n . But by reducing the connection range by only a factor of $1/\sqrt{\log n}$ more, the uncertainty in the network stabilizes (by (26)). We can easily relate this to a physical system by considering the transmit power of each node P_T . Employing a simplified path loss model [15], it is straightforward to show that $r_0 \propto P_T^{1/\eta}$. Hence, to achieve stability in uncertainty in the limit of large n , the transmit power must scale like

$$P_T \propto (n^2 \log n)^{-\frac{\eta}{2}} \quad (27)$$

From this expression, it is clear that the macroscopic path loss behavior, which is linked to the physical properties of the environment, governs transmit power scaling (through the exponent η). Since the path loss exponent is an empirical parameter, it is inexact for any practical deployment. Thus, we can surmise that, in practice, the limit ℓ_h can at best provide a good approximation for the uncertainty of the network topology. Quantifying the accuracy for finite n subject to perturbations in η is an interesting topic left for consideration in the future.

3) *Entropy Scaling Limits in Three Dimensions:* One can obtain a similar result to that stated in Lemma 1 for networks embedded in $\mathcal{K}_3$. We state the result here without providing a proof since the associated calculations are effectively the same as those given for the two-dimensional case above.

Lemma 2: As $n \rightarrow \infty$, the entropy of a graph in $\mathcal{K}_3$ can be bounded away from zero only if

$$r_0^3 \log \left(\frac{1}{r_0} \right) = \Omega \left(\frac{1}{n^2} \right). \quad (28)$$

The entropy bound will tend to a limit $h(n) \rightarrow \ell_h > 0$ as $n \rightarrow \infty$ if

$$\begin{aligned} r_0(n) &= \exp \left(\frac{1}{3} W_m \left(-\frac{2\ell_h}{u_3 n(n-1)} \right) \right) \\ &= \left(\frac{\ell_h}{u_3 n^2 \log n} \right)^{\frac{1}{3}} \left(1 + O \left(\frac{1}{\log n} \right) \right) \end{aligned} \quad (29)$$

where $u_3 = 4\pi\Gamma(3/\eta)/(\eta V(\mathcal{K}_3))$ is the fractional volume of a soft unit ball.

There are clear similarities between the two and three-dimensional cases, as brought out in Lemmas 1 and 2. The only differences arise in the parameters u_2 and u_3 , which have analogous physical meanings, and the exponents. With regard to the exponents, the additional dimension in the three-dimensional case results in an exponent of $1/3$ instead of $1/2$, which implies a slower scaling of the typical connection range relative to a two-dimensional network. In practice, this could translate to a larger aggregate transmit power to maintain connectivity or coverage. One can conjecture that a general result holds for d dimensions, and in fact it is possible to quantify such a result. However, this is of mathematical interest only, and since we focus on wireless networks in this paper, it is omitted here.

B. Large Typical Connection Range

So far, we have considered small typical connection ranges. However, some networks, such as confined D2D clusters in 5G networks, may exhibit large connection ranges relative to the bounding geometry. To study such cases, we return to the definition of the average pairwise connection function given in (7). Suppose the typical connection range is large, which we define more quantitatively by the relation $r_0 \gg D$. In this case, we can expand the exponential in the integrand of (7) to obtain

$$\bar{p} = 1 - \frac{\mu_\eta}{r_0^\eta} + \frac{\mu_{2\eta}}{2r_0^{2\eta}} + \dots \quad (30)$$

where $\mu_m = \mathbb{E}[r^m]$. Intuitively, as the connection range increases relative to the domain diameter D , one could imagine that the uncertainty, or disorder, in the network topology would recede. Indeed, using the expression above, we find that

$$H(\bar{p}) = \frac{\eta \mu_\eta \log r_0}{r_0^\eta} + O \left(\frac{1}{r_0^\eta} \right), \quad 1 < \eta < \infty \quad (31)$$

and thus $h(n) \rightarrow 0$ as $r_0 \rightarrow \infty$. Note that the only statistical parameter entering this equation is the η th moment of the pair distance μ_η , which is dependent upon the geometry of $\mathcal{K}_d$.

Interestingly, (31) shows that the uncertainty arising from the soft connection function decays exponentially with η in contrast to the power law decay in the connection range r_0 . For the hard disk model, $\bar{p} = 1$ when $r_0 > D$ and $H(G) = 0$ since G is almost surely a complete graph. But for the soft pairwise connection model, uncertainty remains. Clearly, the bound $h(n)$ grows like $O(n^2)$ for fixed connection range r_0 . This makes perfect sense; with each node we add to the network, we increase the number of fading links by one less than the total number of nodes, and uncertainty will increase precisely as a result of this addition coupled with the severe fading effects. This suggests that in order to ensure uncertainty remains constant, or indeed decreases, with an increase in the number of nodes, we must increase the typical connection range. Practically, this can be done by increasing the transmit power, reducing the rate of the transmission, or employing diversity methods, among many other possibilities. The rate at which the typical connection range must increase in order to ensure uncertainty does not diverge is captured in the following lemma, which we state without proof⁴.

Lemma 3: As $n \rightarrow \infty$, the entropy of a graph in $\mathcal{K}_d$ can be bounded away from ∞ only if

$$\frac{r_0^\eta}{\log r_0} = \Omega(n^2). \quad (32)$$

The entropy bound will tend to a limit $h(n) \rightarrow \ell_h > 0$ as $n \rightarrow \infty$ if

$$\begin{aligned} r_0(n) &= \exp \left(-\frac{1}{\eta} W_m \left(-\frac{2\ell_h}{\mu_\eta n(n-1)} \right) \right) \\ &= \left(\frac{\mu_\eta n^2 \log n}{\ell_h} \right)^{\frac{1}{\eta}} \left(1 + O \left(\frac{1}{\log n} \right) \right). \end{aligned} \quad (33)$$

⁴The proof is similar to that given for the previous two lemmas.

Again, we see that the physical propagation environment plays an important role in the required connection range scaling. It is interesting to note that, by using similar arguments to those discussed above to link the transmit power to the connection range, P_T must increase like $O(n^2 \log n)$ to achieve topological certainty as more nodes enter the network, which does not depend on the path loss exponent η . Thus, this scaling is robust to uncertain knowledge of the physical environment. Nevertheless, a (greater than) square-law power scaling could pose significant practical problems for sensor or data fusion networks that rely on a stable topological structure for information passing.

V. CONCLUSIONS

In this paper, we analyzed topological uncertainty in wireless networks using an RGG formalism with soft pairwise connection functions. We derived a simple bound on RGG entropy and showed that, for a pairwise connection function based on Rayleigh fading, the typical connection range must decay like $O(1/(n^2 \log n)^{1/d})$ for the bound to converge to a limit in $d = 2, 3$ dimensional networks. Furthermore, we analyzed the large typical connection range regime and proved that r_0 must increase like $O((n^2 \log n)^{1/\eta})$ to yield a similar limit. The framework considered in this paper, along with the scaling results presented, provide insight into how network organization alters in dynamic systems, for example in ad hoc or D2D scenarios where devices enter/leave the network. This perspective can be useful for designing opportunistic routing protocols and understanding the amount of control overhead needed to maintain network connectivity. Numerous open problems exist, including the study of non-Rayleigh pairwise connection models and their effect on topological uncertainty, maximum entropy conditions on the network, structural entropy of RGGs and the application to topology compression, and the effect of imprecise path loss exponents on topological uncertainty stability.

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