Supplementary Materials

To:

Scenario-Free Analysis of Financial Stability with Interacting Contagion Channels

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Note: References to equations, figures, and table numbers without the "S." prefix refer to the main manuscript.

Notation	Description
E	Equity
D	Debt
λ	Leverage
$\hat{\lambda}$	Critical leverage
ν	Largest eigenvalue
A	Shock Transmission Matrix
$\boldsymbol{x}$	Shock vector
$\boldsymbol{x}^l$	Liquidity shocks vector
$\boldsymbol{x}^v$	Valuation shocks vector
S	Short-term loan
L	Long-term loan
n_s	Total number of shares in security s in circulation
μ_s	Price-impact factor for security s
p_s	Price of security s
δ_i	Risk-adjustment factor for institution i
ϕ_l	Fraction of (institutions that are) liquidity sinks
ϕ_v	Fraction of (institutions that are) valuation sinks
F	Fraction of short-term lenders (institutions that provide short-term loans)
Λ	Fraction of leverage targeters (institutions that are leverage targeting)
C_s	Market capitalization of security s
N	Number of institutions
N^v	Number of leveraged institutions
N^w	Number of distinct securities
N^s	Number of blocks of shares of security s
N^d	Number of debts (loans)
N_i^s	Number of blocks of security s received by institution i
N_i^d	Number of loans received by institution i
N_{ij}^d	Number of loans from institution i to institution j

Table S.1: Notation

S.1 Validation Tests

In Figure S.1, we compare the overestimation of the critical leverage of the mean-field model to the overestimation in randomly generated financial systems. The overestimation is calculated as the percentage increase from the true critical leverage to the counterparty risk critical leverage (i.e. the critical leverage found when considering only pure counterparty risk contagion). The financial systems are generated using the algorithm outlined in section 3.1 and system parameters derived in section 3.2. The systems include $N = 100$ institutions, such that the fraction of leverage targeting institutions Λ can be increased in 100 increments. The figure shows that the overestimation becomes arbitrarily large as $\Lambda \to 1$ for any $\phi_l < 1$, and that the overestimation in the mean-field model closely approximates the overestimation in the randomly generated financial systems.

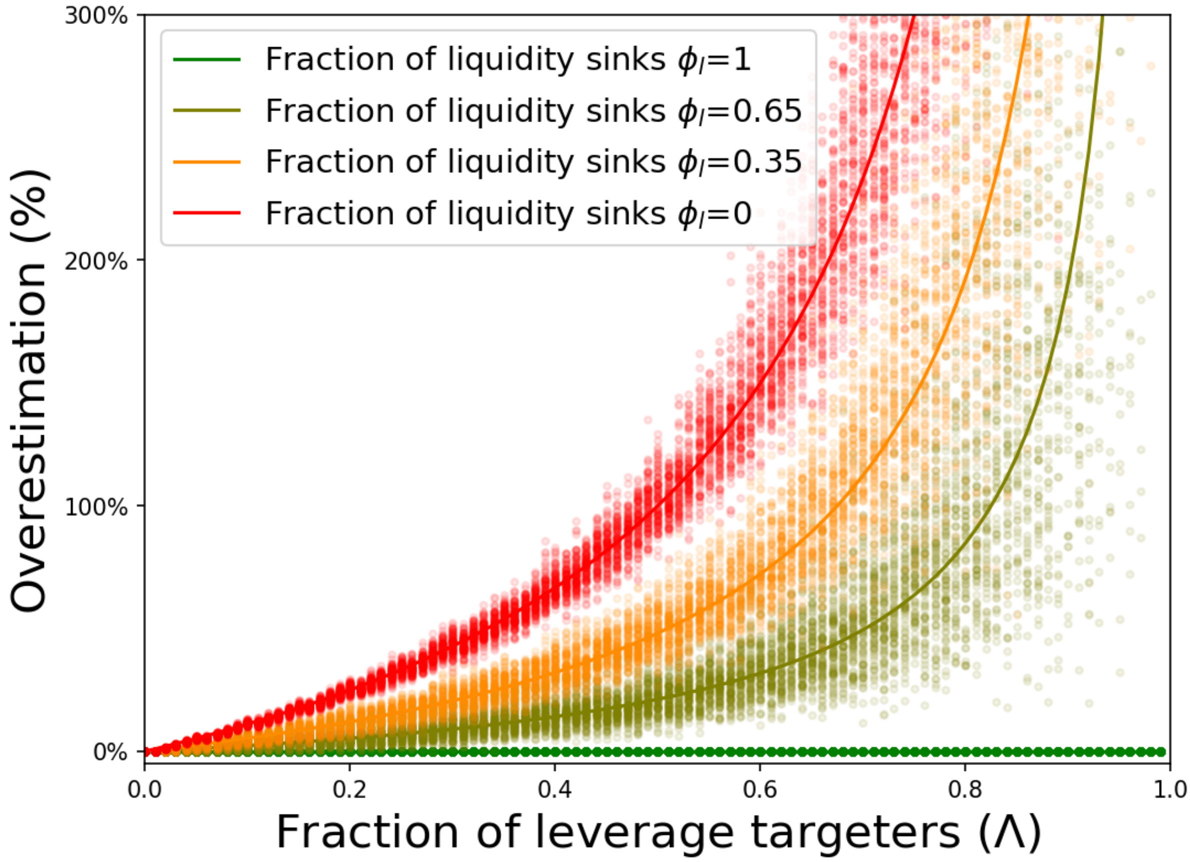

Figure S.1: **Overestimation of the Critical Leverage.** Comparison of the percentage increase from the true critical leverage to the counterparty risk critical leverage of randomly generated systems (dots) and the mean-field model (solid lines). Fixed parameters: $F = 0.5$, $\phi_v = 0.2$, $\mu_s = .1$, $\delta_i = .1$, $N = 100$, $N^d = 10$, $N^s = 100$, and $N^w = 10$.

Figure S.2 compares the mean-field critical leverage to the critical leverages of randomly generated financial systems. The financial systems are generated using the algorithm outlined in section 3.1, but with various modifications to the algorithm that generate additional heterogeneity. We use the system parameters derived in section 3.2 and set the number of loans and blocks of securities $N^d = N^s = 10$ to generate sparse financial systems. Each labeled column of the figure presents the critical leverages of systems that were generated with a single modification to the algorithm:

- For systems in the column “Pareto Weights”, the weight of each generated edge is drawn from a Pareto distribution with shape equal to two and scale equal to one, to create additional heterogeneity in the distribution of edge weights.
- For systems in the column “Pareto In-Degree”, the number of loans received by each institution is drawn from a Pareto distribution with shape equal to two and scale equal to one (rounded down and limited to 100 for computational efficiency), to create additional heterogeneity in the distribution of institutions’ in-degrees.
- For systems in the column “Pareto Out-Degree”, the number of loans made by each institution is drawn from a Pareto distribution with shape equal to two and scale

equal to one (rounded down and limited to 100 for computational efficiency), to create additional heterogeneity in the distribution of institutions' out-degrees.

- For systems in the column "In Core-Periphery", the set of non-sink institutions is divided into two halves; one half is excluded from receiving loans when each institution makes its N^d loans, to create a core of institutions that receive the majority of loans. Note that institutions designated as leveraged that do not receive any loans, are allocated a single random loan at the end of the algorithm (see section 3.1).

- For systems in the column "Out Core-Periphery", the set of non-sink institutions is divided into two halves; one half is excluded from making loans when each institution makes its N^d loans, to create a core of institutions that make the majority of loans. Institutions designated as short-term lenders that are excluded from making loans are allocated a single loan to a randomly chosen institution.

Comparison with figure 4 shows that the additional sources of heterogeneity increase the variation in critical leverages in figure S.2, in particular when institutions' in- or out degrees are drawn from a Pareto distribution.

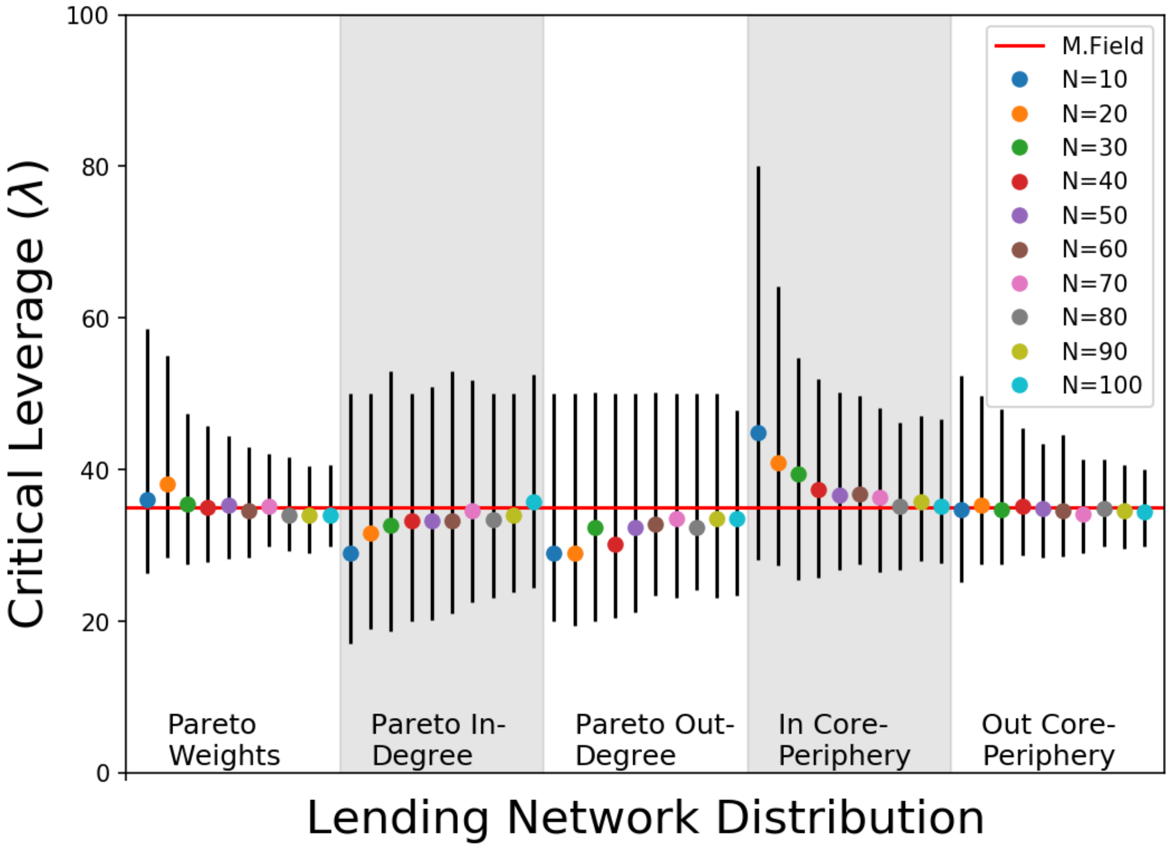

Figure S.2: **Generating Financial Systems with Additional Sources of Heterogeneity.** Comparison of the mean-field critical leverage (red line) to the critical leverages of randomly generated financial systems. For various modifications to the generation algorithm (which introduce additional heterogeneity), we generate 500 random systems and plot the median (colored dot) and the 15^{th} to 85^{th} percentile interval (black bars) of the distribution of critical leverages. Fixed parameters: $F = 0.5$, $\phi_v = 0.2$, $\mu_s = .1$, $\delta_i = .1$, $N = 100$, $N^d = 10$, $N^s = 10$, and $N^w = 10$.

S.2 Eigenvalue Time Dependence

We show that the shock transmission matrix' largest eigenvalue, when it is equal to one, is independent of the relative speeds at which the various channels act. When the largest eigenvalue $\nu = 1$, the corresponding (right) eigenvector $\boldsymbol{v}$ is invariant under multiplication by the shock transmission matrix,

$$\boldsymbol{v}_{t+1} = A\boldsymbol{v}_t = \nu\boldsymbol{v}_t = \boldsymbol{v}_t. \tag{S.1}$$

Hence, for any element v_k (corresponding to the network's k^{th} node) of the eigenvector $\boldsymbol{v}$, we have that

$$v_{k,t+1} = v_{k,t}. \tag{S.2}$$

From the matrix-vector product, we know that

$$v_{k,t+1} = \sum_{y \in \mathcal{A}} w_{yk} v_{y,t}, \tag{S.3}$$

where $\mathcal{A}$ denotes the network's set of nodes and $w_{yk} = A_{ky}$ denotes the weight of the edge from node y to node k. Hence,

$$v_{k,t} = \sum_{y \in \mathcal{A}} w_{yk} v_{y,t}. \tag{S.4}$$

Let us now take a specific pair of nodes i, k. We add a "dummy node" *between* nodes i and k, by replacing the edge from i to k by an edge with identical weight from j to k and adding an edge with weight equal to one from node i to node j. Using $\hat{w}_{xy}$ to denote edges in the new network, we have that $\hat{w}_{ij} = 1$, $\hat{w}_{ik} = 0$, $\hat{w}_{jk} = w_{ik}$, and $\hat{w}_{yx} = w_{yx}$ for any pair of nodes yx other than the pairs ij, ik and jk. Hence, any shock that was previously transmitted directly from i to k is now delayed by one iteration before arriving at node k, while the shock's magnitude is unaffected.

Compared to $\boldsymbol{v}$, the new network's eigenvector $\hat{\boldsymbol{v}}$ has an additional entry, $\hat{v}_j$. For all $y \neq j$ we set $\hat{v}_{y,t} = v_{y,t}$, and since $\hat{v}_{j,t+1} = \hat{w}_{ij}\hat{v}_{i,t} = \hat{v}_{i,t}$, we set $\hat{v}_{j,t} = v_{i,t}$. Using $\hat{\mathcal{A}}$ to denote the new network's set of nodes, we immediately see that

$$\hat{v}_{k,t+1} = \sum_{y \in \hat{\mathcal{A}}} \hat{w}_{yk} \hat{v}_{y,t} = \sum_{y \in \mathcal{A}} w_{yk} v_{y,t} = v_{k,t} = \hat{v}_{k,t}, \tag{S.5}$$

so the invariance in equation (S.2) is conserved (as well as the invariance of the entire eigenvector (S.1), as the rest of the network remains unchanged). Hence, when the largest eigenvalue is equal to one, we can add a dummy node that slows down shock transmission without affecting the invariance of the corresponding eigenvector under multiplication by the shock transmission matrix.

Note from Figure 2 that each institution is represented by both a liquidity and valuation node, such that liquidity and valuation shocks never travel over the same edges. In principle, we can add any number of dummy nodes to "tune" the relative speeds at which contagion channels operate with any desired granularity, without affecting the network's stability. Thus, when the largest eigenvalue is equal to one, the network's stability is independent of the relative speeds at which contagion channels operate.

S.3 Derivation of the Mean-Field Model

Here, we show that the shock transmission matrix reduces to a 2×2 matrix when passively leveraged institutions have the same risk-adjustment factor $\delta_i = \delta$, all leveraged institutions have the same leverage $\lambda_i = \lambda$, and $N^s/N, N^d/N, N \to \infty$. We denote the number of leveraged institutions as $N^v = (1 - \phi_v)N$ and the number of non-sink institutions as $N^l = (1 - \phi_l)N^v$.

We simplify the notation of the shock transmission matrix in three steps:

1. We reorder the nodes to move all empty columns (corresponding to liquidity sinks' absorption of liquidity shocks and valuation sinks' absorption of valuation shocks) to the right of the matrix. The resulting matrix is lower block triangular and its eigenvalues are given by those of the only non-zero diagonal block, i.e. the upper-left diagonal block. Hence, we can obtain a reduced matrix by removing liquidity sinks' liquidity nodes and valuation sinks' valuation nodes from the matrix without affecting the largest eigenvalue.

2. We reorder the nodes in the reduced matrix obtained in step 1. to move the empty rows that correspond to the transmission of liquidity shocks to valuation sinks to the bottom of the matrix. (Valuation sinks have no leverage target and no short-term debt so do not receive liquidity shocks.) The resulting matrix is upper block triangular and its eigenvalues are given by those of the only non-zero diagonal block, i.e. the upper-left diagonal block. Hence, we can completely remove valuation sinks from the matrix without affecting the largest eigenvalue.

3. For simplicity, we reorder the nodes to move the columns corresponding to liquidity sinks' shock transmission in response to valuation shocks to the right end of the matrix (i.e. liquidity sinks' valuation nodes are moved to the bottom of the shock vector).

We refer to resulting matrix as the *simplified* shock transmission matrix. The liquidity shock vector $\boldsymbol{x}^l$ is of length N^l and the valuation shock vector $\boldsymbol{x}^v$ is of length N^v. Hence, the dimensions of the simplified shock transmission matrix' funding contagion quadrant are $N^l \times N^l$, of the counterparty risk contagion quadrant $N^v \times N^v$, of the overlapping portfolio contagion quadrant $N^v \times N^l$, and of the leverage targeting contagion quadrant $N^l \times N^v$.

S.3.1 Simplified Shock Transmission Matrix

We first derive the simplified shock transmission matrix to which systems converge as $N^s/N, N^d/N, N \to \infty$, after which we discuss how to reduce the resulting shock transmission matrix to the 2×2 matrix.

- When $N^s/N \to \infty$ for all securities s, the market cap of each security s is distributed homogeneously over all institutions,

$$\lim_{N^s/N\to\infty} \frac{N_i^s}{N^s} = \mathbb{E}\left(\frac{N_i^s}{N^s}\right) = \frac{1}{N}, \tag{S.6}$$

where $\mathbb{E}(\dots)$ denotes the expectation. All non-sink institutions that do not provide short-term lending have security $\hat{s}$ at the top of their pecking order, which is the most liquid security among the N^w distinct securities. The price-impact factor of security $\hat{s}$ is denoted as μ (i.e. without any subscript).

- When $N^d/N, N \to \infty$, each leveraged institution's debt is distributed equally over all $N-1$ other institutions,

$$\lim_{N^d/N,N\to\infty} \frac{N_{ji}^d}{N_i^d} = \lim_{N\to\infty} \mathbb{E}\left(\frac{N_{ji}^d}{N_i^d}\right) = \lim_{N\to\infty} \frac{1}{N-1} = \frac{1}{N}. \tag{S.7}$$

The distribution of equities E_i that solves the balance sheet identity (13) and equations (S.6) and (S.7) is $E_i(1+\lambda_i) = E_j(1+\lambda_j)$ for any institutions i and j, where $\lambda_i = 0$ if institution i is a valuation sink and $\lambda_i = \lambda$ otherwise. That is, all leveraged institutions have the same equity and debt, and all valuation sinks have the same equity, which is equal to the sum of the equity and debt of any leveraged institution.

As equations (S.6) and (S.7) tell us how securities and total debt are distributed (and hence how overlapping portfolio contagion and counterparty risk contagion are distributed), let us now consider how short-term lending is distributed.

When $N^d/N, N \to \infty$, the fraction of short-term lender i's total short-term lending S_i provided to any leveraged institution j is equal to

$$\begin{aligned}\lim_{N^d/N,N\to\infty} \frac{S_{ij}}{S_i} &= \lim_{N^d/N,N\to\infty} \frac{\frac{N_{ij}^d}{N_j^d} D_j}{\sum_{k=1}^{N^v} \frac{N_{ik}^d}{N_k^d} D_k} = \lim_{N^d/N,N\to\infty} \frac{\frac{N_{ij}^d}{N_j^d}}{\sum_{k=1}^{N^v} \frac{N_{ik}^d}{N_k^d}} \\ &= \lim_{N^d/N,N\to\infty} \frac{N_{ij}^d}{\sum_{k=1}^{N^v} N_{ik}^d} = \lim_{N^d/N,N\to\infty} \frac{\frac{N^d}{N^v-1}}{N^d} = \lim_{N\to\infty} \frac{1}{N^v-1} = \frac{1}{N^v},\end{aligned} \tag{S.8}$$

where k runs over all leveraged institutions, and we have used that:

- When $N^s/N, N^d/N, N \to \infty$, all N^v leveraged institutions have the same debt D (as discussed above).

- $\lim_{N^d/N\to\infty} N_k^d = \mathbb{E}\left(N_k^d\right) = \frac{N^d N}{N^v}$ is the same for all leveraged institutions k (including institution j).

- The number of loans any leveraged institution i provides to another leveraged institution j is $\lim_{N^d/N\to\infty} N^d_{ij} = \mathbb{E}\left(N^d_{ij}\right) = \frac{N^d}{N^v-1}$, as institution i cannot lend to itself.

From equation (S.8) follows that the funding contagion transmission of a (non-sink) short-term lender, as given by the corresponding column in the simplified shock transmission matrix, is equal to

$$\left[\frac{1}{N^v},\ldots,\frac{1}{N^v},0,\frac{1}{N^v},\ldots,\frac{1}{N^v},0,\ldots,0\right]^T, \tag{S.9}$$

where for institution i, the i^{th} entry is zero (no lending to itself) and the last N^v terms are zero, which corresponds to the institution's non-transmission of overlapping portfolio contagion.

When $N\to\infty$, the institutions become a continuum and shock transmission to individual institutions vanishes. Hence, for $N\to\infty$, the funding contagion transmission vector reduces to[1]

$$\left[\frac{1}{N^v},\ldots,\frac{1}{N^v},0,\ldots,0\right]^T, \tag{S.10}$$

such that each (non-sink) short-term lender's funding contagion is distributed homogeneously over the continuum of leveraged institutions.

From equation (S.7), it follows that for each passively leveraged institution, the counterparty risk contagion transmission as given by the corresponding column of the simplified shock transmission matrix is equal to

$$\left[0,\ldots,0,\frac{\delta\lambda}{N},\ldots,\frac{\delta\lambda}{N},0,\frac{\delta\lambda}{N},\ldots,\frac{\delta\lambda}{N}\right]^T, \tag{S.11}$$

where for institution i, the i^{th} entry is zero (no debt to itself), and the first N^l entries of the vector are zero, which corresponds to the institution's non-transmission of leverage targeting contagion.

Similar to funding contagion, when $N\to\infty$, the counterparty risk contagion transmission vector reduces to

$$\left[0,\ldots,0,\frac{\delta\lambda}{N},\ldots,\frac{\delta\lambda}{N}\right]^T, \tag{S.12}$$

such that each passively leveraged institution's counterparty risk contagion is distributed homogeneously over the continuum of institutions.

[1]Formally, the difference between S.9 and S.10 vanishes in the limit $N\to\infty$; $\lim_{N\to\infty}\left|\left|\left[\frac{1}{N^v},\ldots,\frac{1}{N^v},0,\ldots,0\right]^T - \left[\frac{1}{N^v-1},\ldots,\frac{1}{N^v-1},0,\frac{1}{N^v-1},\ldots,\frac{1}{N^v-1},0,\ldots,0\right]^T\right|\right| = 0$.

Lastly, from equation (S.6), the overlapping portfolio contagion shock transmission vector for any non-sink institution that does not provide short-term lending is equal to

$$\left[0, \ldots, 0, \frac{\mu}{N}, \ldots, \frac{\mu}{N}\right]^T, \tag{S.13}$$

where the first N^l terms are zero, which corresponds to the institution's non-transmission of funding contagion. Hence, the overlapping portfolio contagion transmitted by any non-sink institution that does not provide short-term lending is distributed homogeneously over the continuum of institutions.

From equations (S.10), (S.12) and (S.13), we find that for $N^s/N, N^d/N, N \to \infty$, the simplified shock transmission matrix is given by

$$\left[\begin{array}{ccc||cccccccc}
I_1^f \frac{1}{N^v} & \cdots & I_{N^l}^f \frac{1}{N^v} & I_1^\lambda \lambda & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & 0 & I_2^\lambda \lambda & \ddots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \vdots \\
I_1^f \frac{1}{N^v} & \cdots & I_{N^l}^f \frac{1}{N^v} & 0 & \cdots & 0 & I_{N^l}^\lambda \lambda & 0 & \cdots & 0 \\
\hline\hline
\left(1 - I_1^f\right) \frac{\mu}{N} & \cdots & \left(1 - I_{N^l}^f\right) \frac{\mu}{N} & \left(1 - I_1^\lambda\right) \frac{\delta\lambda}{N} & \cdots & \cdots & \left(1 - I_{N^l}^\lambda\right) \frac{\delta\lambda}{N} & \cdots & \cdots & \left(1 - I_{N^v}^\lambda\right) \frac{\delta\lambda}{N} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\
\left(1 - I_1^f\right) \frac{\mu}{N} & \cdots & \left(1 - I_{N^l}^f\right) \frac{\mu}{N} & \left(1 - I_1^\lambda\right) \frac{\delta\lambda}{N} & \cdots & \cdots & \left(1 - I_{N^l}^\lambda\right) \frac{\delta\lambda}{N} & \cdots & \cdots & \left(1 - I_{N^v}^\lambda\right) \frac{\delta\lambda}{N}
\end{array}\right]. \tag{S.14}$$

where $I_i^f = 1$ if institution i is a short-term lender and $I_i^f = 0$ otherwise, and $I_i^\lambda = 1$ if institution i has a leverage target and $I_i^\lambda = 0$ otherwise.

S.3.2 Reduction to 2×2 Matrix

We now show that the simplified shock transmission matrix in equation (S.14) can be reduced to a 2×2 matrix with largest eigenvalue identical to that of the shock transmission matrix in equation (S.14). We do so by showing that system is uniquely determined by the dynamics of the aggregate liquidity and valuation shocks x_t^l and x_t^v.

Let $x_{t,i}^l$ denote the liquidity shock received by (non-sink) institution i at time t, such that $\boldsymbol{x}_t^l = \left[x_{t,1}^l, \ldots, x_{t,N^l}^l\right]^T$ and let $x_{t,j}^v$ be the valuation shock received by (leveraged) institution j at time t, such that $\boldsymbol{x}_t^v = \left[x_{t,1}^v, \ldots, x_{t,N^v}^v\right]^T$. Furthermore, let $x_t^l = \sum_{i=1}^{N^l} x_{t,i}^l$ be the aggregate liquidity shock received by all non-sink institutions at time t and $x_t^v = \sum_{i=1}^{N^v} x_{t,i}^v$ be the aggregate valuation shock received by all leveraged institutions at time t. Lastly, at time t, let the fraction of the aggregate liquidity shock x_t^l received by non-sink institutions with short-term lending be denoted as $\hat{F}_t = \sum_{i=1}^{N^l} I_i^f x_{t,i}^l / x_t^l$, such that $(1 - \hat{F}_t) = \sum_{i=1}^{N^l} \left(1 - I_i^f\right) x_{t,i}^l / x_t^l$, and let the fraction of the aggregate valuation shock x_t^v received by leverage targeting institutions be denoted as $\hat{\Lambda}_t = \sum_{i=1}^{N^v} I_i^\lambda x_{t,i}^v / x_t^v$, such that $(1 - \hat{\Lambda}_t) = \sum_{i=1}^{N^v} \left(1 - I_i^\lambda\right) x_{t,i}^v / x_t^v$.

We use the following properties throughout the derivation:

$$\frac{N^l}{N^v} = 1 - \phi_l, \tag{S.15}$$

$$\frac{\sum_{i=1}^{N^v} I_i^\lambda}{N^v} = \Lambda, \tag{S.16}$$

$$\lim_{N\to\infty} \frac{\sum_{i=1}^{N^l} I_i^f}{N^l} = F, \tag{S.17}$$

$$\lim_{N\to\infty} \frac{\sum_{i=1}^{N^l} I_i^f}{N^v} = \lim_{N\to\infty} \frac{(1-\phi_l)\sum_{i=1}^{N^l} I_i^f}{N^l} = (1-\phi_l)F, \tag{S.18}$$

$$\lim_{N\to\infty} \frac{\sum_{i=1}^{N^l} I_i^\lambda}{N^v} = \lim_{N\to\infty} \frac{(1-\phi_l)\sum_{i=1}^{N^l} I_i^\lambda}{N^l} = (1-\phi_l)\Lambda, \tag{S.19}$$

$$\lim_{N\to\infty} \sum_{i=1}^{N^l} I_i^f I_i^\lambda = \sum_i^{N^l} \mathbb{E}\left(I_i^f I_i^\lambda\right) = \sum_i^{N^l} \mathbb{E}\left(I_i^f\right) \mathbb{E}\left(I_i^\lambda\right) = F\Lambda \sum_i^{N^l} 1 = F\Lambda N^l. \tag{S.20}$$

The first identity is simply a restatement of the fact that we designate a fraction ϕ_l of leveraged institution as liquidity sinks. The second is a restatement of the definition that the fraction of leverage targeters Λ is equal to the fraction of leveraged institutions that have a leverage target. The third and fourth identities use that as $N \to \infty$, the fraction of institutions that provide short-term lending is equal to the fraction of non-sink institutions that provide short-term lending (because sinks and short-term lenders are designated independently). Similarly, the fifth identity uses that as $N \to \infty$, the fraction of non-sink institutions that have a leverage target is equal to Λ (because liquidity sinks and leverage strategies are designated independently). Lastly, the sixth identity gives the number of non-sink short-term lenders that are leverage targeting and follows from the fact that short-term lenders and leverage targeting institutions are designated independently.

Plugging the simplified shock transmission matrix in equation (S.14) into equation (2) yields

$$
\boldsymbol{x}_{t+1} = A\boldsymbol{x}_t = A \begin{bmatrix} \boldsymbol{x}_t^l \\ \boldsymbol{x}_t^v \end{bmatrix} = \begin{bmatrix} \frac{1}{N^v} \sum_{i=1}^{N^l} I_i^f x_{t,i}^l + \lambda I_1^\lambda x_{t,1}^v \\ \vdots \\ \frac{1}{N^v} \sum_{i=1}^{N^l} I_i^f x_{t,i}^l + \lambda I_{N^l}^\lambda x_{t,N^l}^v \\ \frac{\mu}{N} \sum_{i=1}^{N^l} \left(1 - I_i^f\right) x_{t,i}^l + \frac{\delta\lambda}{N} \sum_{j=1}^{N^v} \left(1 - I_j^\lambda\right) x_{t,j}^v \\ \vdots \\ \frac{\mu}{N} \sum_{i=1}^{N^l} \left(1 - I_i^f\right) x_{t,i}^l + \frac{\delta\lambda}{N} \sum_{j=1}^{N^v} \left(1 - I_j^\lambda\right) x_{t,j}^v \end{bmatrix} \tag{S.21}
$$

$$
= \begin{bmatrix} \frac{1}{N^v} \hat{F}_t x_t^l + \lambda I_1^\lambda x_{t,1}^v \\ \vdots \\ \frac{1}{N^v} \hat{F}_t x^l + \lambda I_{N^l}^\lambda x_{t,N^l}^v \\ \frac{\mu}{N}(1 - \hat{F}_t) x_t^l + \frac{\delta\lambda}{N}(1 - \hat{\Lambda}_t) x_t^v \\ \vdots \\ \frac{\mu}{N}(1 - \hat{F}_t) x_t^l + \frac{\delta\lambda}{N}(1 - \hat{\Lambda}_t) x_t^v \end{bmatrix}. \tag{S.22}
$$

Hence, at time $t+1$, we have for any non-sink institution i that

$$
x_{t+1,i}^l = \frac{\hat{F}_t}{N^v} x_t^l + \lambda I_i^\lambda x_{t,i}^v, \tag{S.23}
$$

and for any leveraged institution i that

$$
x_{t+1,i}^v = \frac{(1 - \hat{F}_t)\mu}{N} x_t^l + \frac{(1 - \hat{\Lambda}_t)\delta\lambda}{N} x_t^v. \tag{S.24}
$$

Equation (S.24) shows that $x_{t+1,i}^v$ is the same for any (leveraged) institution i, because the right-hand side of equation (S.24) does not depend on i. The aggregate valuation shock at time $t+1$ is given by

$$
x_{t+1}^v = \sum_{i=1}^{N^v} x_{t+1,i}^v = (1 - \hat{F}_t)(1 - \phi_v)\mu x_t^l + (1 - \hat{\Lambda}_t)(1 - \phi_v)\delta\lambda x_t^v, \tag{S.25}
$$

and hence we have for any leveraged institution i that

$$
x_{t+1,i}^v = \frac{x_{t+1}^v}{N^v}. \tag{S.26}
$$

From $x_{t+1,i}^v = x_{t+1}^v / N^v$ follows that

$$
\hat{\Lambda}_{t+1} = \frac{\sum_{i=1}^{N^v} I_i^\lambda x_{t+1,i}^v}{x_{t+1}^v} = \frac{\sum_{i=1}^{N^v} I_i^\lambda \frac{x_{t+1}^v}{N^v}}{x_{t+1}^v} = \frac{\sum_{i=1}^{N^v} I_i^\lambda}{N^v} = \Lambda. \tag{S.27}
$$

Furthermore, from $x_{t+1,i}^v = x_{t+1}^v / N^v$ also follows that, at time $t+2$,

$$x^l_{t+2,i} = \frac{\hat{F}_{t+1}}{N^v} x^l_{t+1} + \frac{\lambda I^\lambda_i}{N^v} x^v_{t+1}, \tag{S.28}$$

$$x^l_{t+2} = \hat{F}_{t+1}(1-\phi_l)x^l_{t+1} + \lambda x^v_{t+1} \sum_{i=1}^{N^l} \frac{I^\lambda_i}{N^v} = \hat{F}_{t+1}(1-\phi_l)x^l_{t+1} + \Lambda(1-\phi_l)\lambda x^v_{t+1}. \tag{S.29}$$

Using equations (S.28) and (S.29), we find that

$$\hat{F}_{t+2} = \frac{\sum_{i=1}^{N^l} I^f_i x^l_{t+2,i}}{x^l_{t+2}} = \frac{\sum_{i=1}^{N^l} I^f_i \left(\frac{\hat{F}_{t+1}}{N^v} x^l_{t+1} + \frac{\lambda I^\lambda_i}{N^v} x^v_{t+1}\right)}{\hat{F}_{t+1}(1-\phi_l)x^l_{t+1} + \Lambda(1-\phi_l)\lambda x^v_{t+1}} \tag{S.30}$$

$$= \frac{\hat{F}_{t+1} x^l_{t+1} \frac{\sum_{i=1}^{N^l} I^f_i}{N^v} + \lambda x^v_{t+1} \frac{\sum_{i=1}^{N^l} I^f_i I^\lambda_i}{N^v}}{\hat{F}_{t+1}(1-\phi_l)x^l_{t+1} + \Lambda(1-\phi_l)\lambda x^v_{t+1}} \tag{S.31}$$

$$= \frac{F\left(\hat{F}_{t+1}(1-\phi_l)x^l_{t+1} + \Lambda(1-\phi_l)\lambda x^v_{t+1}\right)}{\hat{F}_{t+1}(1-\phi_l)x^l_{t+1} + \Lambda(1-\phi_l)\lambda x^v_{t+1}} = F. \tag{S.32}$$

Equations (S.24) and (S.28) do not depend on individual shocks but only on the aggregate liquidity and valuation shocks. Therefore, we find that for $t > 1$ (where the initial exogenous shock occurs at $t = 0$)[2], the system's shock propagation is uniquely determined by the dynamics of x^l_t and x^v_t, which we find by plugging $\hat{F}_t = F$ and $\hat{\Lambda}_t = \Lambda$ into equations (S.25) and (S.29):

$$\begin{aligned} x^l_{t+1} &= F(1-\phi_l)x^l_t + \Lambda(1-\phi_l)\lambda x^v_t, \\ x^v_{t+1} &= (1-F)(1-\phi_v)\mu x^l_t + (1-\Lambda)(1-\phi_v)\delta\lambda. \end{aligned} \tag{S.33}$$

Equation (S.33) can be written in matrix form as

$$\begin{bmatrix} x^l_{t+1} \\ x^v_{t+1} \end{bmatrix} = \left[\begin{array}{c||c} F(1-\phi_l) & \Lambda(1-\phi_l)\lambda \\ \hline\hline (1-F)(1-\phi_v)\mu & (1-\Lambda)(1-\phi_v)\delta\lambda \end{array}\right] \begin{bmatrix} x^l_t \\ x^v_t \end{bmatrix}, \tag{S.34}$$

which gives the 2×2 matrix.

Let ν be the largest eigenvalue of the 2×2 matrix and $\boldsymbol{v}$ the corresponding eigenvector, such that

[2]This limitation is not an artifact of the derivation but an actual constraint: When $\boldsymbol{x}_{t=0}$ consists of a single valuation shock to a leverage targeting institution i, $\boldsymbol{x}_{t=1}$ consists of a single liquidity shock to the same institution i, so equation (S.34) does not hold for $t = 0$. Depending on whether or not institution i has made short-term loans, institution i transmits either a pure funding or pure overlapping portfolio contagion shock (so either $x^l_{t=2} = 0$ or $x^v_{t=2} = 0$) so equation (S.34) also does not hold for $t = 1$. Because the funding or overlapping portfolio contagion shock is distributed homogeneously over all institutions, equation (S.34) holds from $t = 2$ onward. However, this $\boldsymbol{x}_{t=0}$ is not an eigenvector of A, because $\boldsymbol{x}_{t=1}$ is orthogonal to $\boldsymbol{x}_{t=0}$. When $\boldsymbol{x}_{t=0}$ *is* an eigenvector of A, (S.34) holds for $t = 1$.

$$\hat{A}\boldsymbol{v} = \hat{A}\begin{bmatrix}\boldsymbol{v}^l \\ \boldsymbol{v}^v\end{bmatrix} = \nu \begin{bmatrix}\boldsymbol{v}^l \\ \boldsymbol{v}^v\end{bmatrix}. \tag{S.35}$$

When we rewrite equations (S.24) and (S.28) as a matrix-vector product and use that $\hat{F}_t = F$ and $\hat{\Lambda}_t = \Lambda$ for $t > 1$, we find that

$$\boldsymbol{x}_{t+1} = \begin{bmatrix}\frac{F(1+h)}{N^v} & \frac{\lambda I_i^\lambda}{N^v} \\ \vdots & \vdots \\ \frac{F(1+h)}{N^v} & \frac{\lambda I_i^\lambda}{N^v} \\ \frac{(1-F)\mu}{N} & \frac{(1-\Lambda)\delta\lambda}{N} \\ \vdots & \vdots \\ \frac{(1-F)\mu}{N} & \frac{(1-\Lambda)\delta\lambda}{N}\end{bmatrix} \begin{bmatrix}x_t^l \\ x_t^v\end{bmatrix}, \tag{S.36}$$

and because the matrix is constant, we see that $\boldsymbol{x}_t$ grows by ν when $\left[x_t^l, x_t^v\right]^T$ grows by ν. Therefore, ν is also the largest eigenvalue of the (full) shock transmission matrix A.