

On Young Modules of Defect 2 Blocks of Symmetric Group Algebras

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In this paper, we obtain the ordinary characters and module structures of the Young modules of defect 2 blocks of symmetric group algebras. We also study how these modules induce and restrict in a $[2 : 1]$ -pair. © 1999 Academic Press

INTRODUCTION

Defect 2 blocks of symmetric group algebras have been studied extensively by K. Erdmann, S. Martin, M. J. Richards, and J. C. Scopes. S. Martin [10] constructed the Ext-quiver of the principal block of $k\mathfrak{S}_{2p}$, where k is a field of odd characteristic p . He and K. Erdmann [4] later gave the relations for this Ext-quiver. J. C. Scopes [16] showed that these defect 2 blocks share many common properties.

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M. J. Richards [14] gave a combinatorial algorithm for calculating the entries of the decomposition matrix of any of these blocks. He also showed that their diagonal Cartan entries are bounded above by 6. We shall describe his algorithm for calculating the decomposition matrix in the next section.

Young modules are the indecomposable summands of the permutation modules of Young subgroups. Understanding these modules will help us in understanding Schur algebras, because one way of defining Schur algebras is as endomorphism rings of direct sums of Young modules.

In this paper, we restrict our attention to the Young modules of the defect 2 blocks of symmetric group algebras. We shall obtain their ordinary characters and their module structures in Section 2 (we mean “module structure” in the weak sense of describing the Loewy and socle layers). In Section 3, we shall see how these modules induce and restrict when two defect 2 blocks form a $[2 : 1]$ -pair.

The implications of these results for Schur algebras will be discussed in a subsequent paper.

1. PRELIMINARIES

In this section, we give a short account of the background theory. For a more detailed account, we refer the reader to [9] for general theory of group representations, [7] for representation theory of symmetric groups, and [13] for Schur algebras and their representation theory.

Denote by \mathfrak{S}_n the symmetric group on n letters. Let k be an algebraically closed field of prime characteristic p . Given a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ of n , we obtain the permutation module $M^\lambda = k \uparrow_{k \mathfrak{S}_\lambda}^{k \mathfrak{S}_n}$ by inducing the trivial module of the Young subgroup $\mathfrak{S}_\lambda \cong \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \dots$ to \mathfrak{S}_n . One of the most important submodules of M^λ is the Specht module S^λ , which is a p -modular reduction of an ordinary irreducible representation of \mathfrak{S}_n whose character is denoted by χ^λ . As λ runs through the partitions of n , the χ^λ give a complete list of irreducible characters of \mathfrak{S}_n . The Specht module S^λ has a simple, self-dual head D^λ if the partition λ is p -regular, and as λ runs through the p -regular partitions of n , the set of D^λ is a complete list of mutually non-isomorphic simple modules of $k \mathfrak{S}_n$.

The Young module Y^λ is defined to be the unique indecomposable direct summand of M^λ which contains S^λ as a submodule. It is the p -modular reduction of a unique (up to isomorphism) ordinary representation of \mathfrak{S}_n ; we may therefore define $\text{ch}(Y^\lambda)$ unambiguously as the character of any such ordinary representation. It turns out that the multiplicity of χ^λ as a constituent of $\text{ch}(Y^\lambda)$ is one, and any other constituent has the form χ^μ for some $\mu > \lambda$. Any Young module is self-dual, and as a consequence, if λ is

p -regular, then S^λ is simple if, and only if, $\text{ch}(Y^\lambda)$ is irreducible. Also, Y^λ is projective if, and only if, λ is p -restricted, and if so, $Y^\lambda \cong P(D^{\lambda'} \otimes \text{sgn})$, where sgn is the natural signature representation of \mathfrak{S}_n ; in particular, any indecomposable projective module of $k\mathfrak{S}_n$ is a Young module.

Two Specht modules S^λ and S^μ of $k\mathfrak{S}_n$ lie in the same block if, and only if, λ and μ have the same p -core (Nakayama's "Conjecture"). Hence a block of $k\mathfrak{S}_n$ is determined by a p -core partition τ of $n - wp$ (where w is a nonnegative integer, known as the weight of the block). An irreducible character, Specht module, simple module, or Young module of $k\mathfrak{S}_n$ lies in this block if, and only if, its associated partition of n has p -core τ ; by "a partition λ of B " we will mean any such partition.

The Branching Rule provides a Specht filtration for the restricted Specht module $S^\lambda \downarrow_{\mathfrak{S}_{n-1}}$ and the induced Specht module $S^\lambda \uparrow^{\mathfrak{S}_{n+1}}$. The Specht module S^μ is a factor in this filtration if, and only if, μ can be obtained from λ by removing or adding a node to its Young diagram.

G. D. James devised an intuitive way to understand manipulations of p -hooks which involves β -numbers and an abacus with p runners. We refer the reader to [8] for a detailed account.

Let B be a block of $k\mathfrak{S}_n$ having weight w . Unless B has an empty p -core, we can find a runner in an abacus display of the p -core of B having more beads, say b beads more, than that on its immediate left. Interchanging these two runners produces an abacus display of another p -core; let \tilde{B} be the block of $k\mathfrak{S}_{n-b}$ having this p -core. Then B and \tilde{B} are said to form a $[w : b]$ -pair. Scopes [15] showed that if $b \geq w$, then the blocks B and \tilde{B} are Morita equivalent and thus have equivalent module categories.

We will be discussing defect 2 blocks of symmetric algebras. These do not occur in even characteristic, and in odd characteristic they are exactly the blocks having weight 2. Their defect groups are isomorphic to $C_p \times C_p$. From now on, we will assume that our field k has odd characteristic p .

We remind the reader that the defect 2 blocks of symmetric group algebras are found to enjoy the following common properties, as shown by Scopes [16]:

1. The entries in the decomposition matrix are bounded above by 1.
2. The diagonal Cartan entries are bounded below by 3.
3. The non-diagonal Cartan entries are bounded above by 2.
4. The simple modules do not self-extend.
5. The Ext^1 -space between two simple modules is at most one-dimensional.
6. The principal indecomposable modules have a common Loewy length 5 and are stable (i.e., the Loewy series coincide with the socle series).

We now describe the combinatorial algorithm of M. J. Richards [14] for calculating the decomposition matrix of a defect 2 block B of $k\mathfrak{S}_n$. Given a partition λ of B , let $\partial\lambda$ be the absolute difference between the leg lengths of the two p -hooks removed from λ and its intermediate partition to obtain its p -core. If $\partial\lambda = 0$, we further say, following Richards, that λ is black if either λ has two p -hooks and the larger leg length is even, or λ has a p -hook and a $2p$ -hook and the leg length of the $2p$ -hook is congruent to 0 or 3 (mod 4); otherwise λ is white. Given a p -regular partition μ , define $\mu^{*'}$ to be the next smaller partition (with respect to the lexicographic ordering) having the same ∂ -value, and if this value is 0, it has the same color as well. We have the following theorem:

THEOREM 1.1 (Richards). *The partition $\mu^{*'}$ satisfies $D^{\mu^*} \cong D^\mu \otimes \text{sgn}$, where μ^* is the conjugate partition to $\mu^{*'}$. Moreover, we have*

$$[S^\lambda : D^\mu] = \begin{cases} 1, & \text{if } \lambda \in \{\mu, \mu^{*'}\}; \\ 1, & \text{if } \mu \triangleright \lambda \triangleright \mu^{*'} \text{ and } \partial\mu - \partial\lambda = \pm 1; \\ 0, & \text{otherwise.} \end{cases}$$

Recall the abacus notation for weight 1 and weight 2 partitions:

DEFINITION 1.2. Let τ be a p -core with r parts, and let $b \geq p + r$ be a fixed integer. Every weight 1 partition λ may be displayed on an abacus with b beads. The *abacus notation* for λ is defined as follows: if the bead of weight 1 in the abacus display of λ lies on runner i , denote λ by $[i]$.

DEFINITION 1.3. Let τ be a p -core with r parts, and let $b \geq 2p + r$ be a fixed integer. Every weight 2 partition λ may be displayed on an abacus with b beads. The *abacus notation* for λ is defined as follows: if the abacus display of λ has

1. one bead of weight 2 on runner i , denote λ by $\langle i \rangle$;
2. two beads of weight 1 on runners i and j , denote λ by $\langle i, j \rangle$.

2. NON-PROJECTIVE YOUNG MODULES

Let B be a block of $k\mathfrak{S}_n$ of defect 2, and let $\tau = (\tau_1, \tau_2, \dots)$ be the corresponding p -core partition of $n - 2p$. The characters of projective indecomposable modules of B are determined by Theorem 1.1. Furthermore, the work of Scopes leads to a description of the Loewy and socle layers of these modules (see [17] for details). So in this section we restrict our attention to non-projective Young modules.

We choose an abacus for the p -core τ such that the p th runner has at least as many beads as any other runner. Order the set of p bottommost beads (one for each runner) so that the corresponding β -numbers run from

largest to smallest, and for $0 \leq i \leq p-1$, let d_i be the runner on which the $(i+1)$ th bead in this order lies. Because of our particular choice of abacus, we have $d_0 = p$. Let h be the number of beads on the p th runner of the abacus, and let a be the number of runners on which there are h beads. Then d_0, \dots, d_{a-1} are the runners, listed in descending order, on which there are h beads. The statements in the following lemma are easily verified.

LEMMA 2.1. 1. We have $a = p$ if and only if B is the principal block of \mathfrak{S}_{2p} , and $a = 1$ if and only if $\tau_1 - \tau_2 = p - 1$.

2. The first $p + a$ partitions of B in the lexicographic order are

$$\langle d_0 \rangle > \dots > \langle d_{a-1} \rangle > \langle d_0, d_1 \rangle > \langle d_0, d_2 \rangle > \dots > \langle d_0, d_{p-1} \rangle,$$

and each of these partitions is always p -regular, except $\langle d_0, d_{p-1} \rangle$, which is p -regular if and only if $a = p$.

3. The following relations hold in the dominance order:

$$\begin{aligned} &\langle d_0 \rangle \triangleright \dots \triangleright \langle d_{a-1} \rangle; \\ &\langle d_0, d_1 \rangle \triangleright \dots \triangleright \langle d_0, d_{a-1} \rangle \triangleright \langle d_0, d_0 \rangle \triangleright \langle d_0, d_a \rangle \triangleright \dots \triangleright \langle d_0, d_{p-1} \rangle; \\ &\text{and } \langle d_i \rangle \triangleright \langle d_0, d_i \rangle, \quad \text{for } 0 < i < a. \end{aligned}$$

4.

$$\partial \lambda = \begin{cases} i, & \text{if } \lambda = \langle d_i \rangle, \quad 0 \leq i \leq a-1; \\ a-1, & \text{if } \lambda = \langle d_0, d_0 \rangle; \\ i-1, & \text{if } \lambda = \langle d_0, d_i \rangle, \quad 1 \leq i \leq a-1; \\ i, & \text{if } \lambda = \langle d_0, d_i \rangle, \quad a \leq i \leq p-1. \end{cases}$$

Furthermore, $\langle d_0 \rangle$ is black, $\langle d_0, d_0 \rangle$ is white if $a = 1$, and $\langle d_0, d_1 \rangle$ is white if $a > 1$.

5. The non- p -restricted partitions of B are

$$\langle d_0 \rangle > \dots > \langle d_{a-1} \rangle > \langle d_0, d_1 \rangle > \langle d_0, d_a \rangle > \dots > \langle d_0, d_{p-1} \rangle,$$

where we have $\langle d_0, d_0 \rangle$ instead of $\langle d_0, d_1 \rangle$ if $a = 1$.

THEOREM 2.2 (Characters of non-projective Young modules of B). Define

$$\lambda^{(*)} = (\tau_1 + p, \tau_2 + p, \tau_3, \dots),$$

a non- p -restricted partition of B . Label the remaining p non- p -restricted partitions of B according to lexicographic order:

$$\lambda^{(0)} > \lambda^{(1)} > \dots > \lambda^{(p-1)}.$$

If B is the principal block of \mathfrak{S}_{2p} then let $a = p$; otherwise let $a = \min\{i \mid \lambda_1^{(i)} - \tau_1 = p\}$. Also, let δ be the smallest partition in B which is strictly larger than $\lambda^{(a)}$ in the lexicographic order. Then

$$\begin{aligned} \text{ch}(Y^{\lambda^{(0)}}) &= \chi^{\lambda^{(0)}}. \\ \text{ch}(Y^{\lambda^{(*)}}) &= \begin{cases} \chi^{\lambda^{(*)}}, & \text{if } a = 1; \\ \chi^{\lambda^{(*)}} + \chi^{\lambda^{(1)}}, & \text{if } a > 1. \end{cases} \end{aligned}$$

For $1 \leq i \leq p-1$,

$$\text{ch}(Y^{\lambda^{(i)}}) = \begin{cases} \chi^{\lambda^{(i)}} + \chi^{\lambda^{(i-1)}}, & \text{if } i \neq a; \\ \chi^{\lambda^{(a)}} + \chi^{\lambda^{(a-1)}} + \chi^\delta, & \text{if } i = a. \end{cases}$$

Remark. We have avoided the use of abacus notation in the statement of the theorem. The partitions appearing there have the following abacus notations:

$$\begin{aligned} \lambda^{(i)} &= \begin{cases} \langle d_i \rangle, & \text{if } 0 \leq i < a; \\ \langle d_0, d_i \rangle, & \text{if } a \leq i < p. \end{cases} \\ \lambda^{(*)} &= \begin{cases} \langle d_0, d_0 \rangle, & \text{if } a = 1; \\ \langle d_0, d_1 \rangle, & \text{if } a > 1. \end{cases} \\ \delta &= \langle d_0, d_0 \rangle. \end{aligned}$$

In particular note that the number a defined in the statement of the theorem agrees with the a appearing earlier.

In the proof we will not use any prior knowledge of the block B (e.g., decomposition numbers).

Proof. We first consider the Young modules $Y^{\lambda^{(0)}}$ and $Y^{\lambda^{(*)}}$. We apply the Green correspondence in the context of Young modules [5, 13]. We can remove two horizontal p -hooks from the first row of $\lambda^{(0)}$ and two horizontal p -hooks from $\lambda^{(*)}$, one each from the first and second rows. Thus the Young subgroup $\mathfrak{S}_{(p^2, 1^{n-2p})}$ is a Young vertex for both $Y^{\lambda^{(0)}}$ and $Y^{\lambda^{(*)}}$, and their Green correspondents in the normalizer $(\mathfrak{S}_p \wr \mathfrak{S}_2) \times \mathfrak{S}_{n-2p}$ of this subgroup are $Y^{(2)} \otimes Y^\tau$ and $Y^{(1^2)} \otimes Y^\tau$, respectively, where $Y^{(2)}$ and $Y^{(1^2)}$ are regarded as modules for $\mathfrak{S}_p \wr \mathfrak{S}_2$ via the natural homomorphism from $\mathfrak{S}_p \wr \mathfrak{S}_2$ to \mathfrak{S}_{2p} . Notice that we have $\text{ch}(Y^{(2)}) = \chi^{(2)}$, $\text{ch}(Y^{(1^2)}) = \chi^{(1^2)}$, and $\text{ch}(Y^\tau) = \chi^\tau$, because these partitions are p -cores. So the sum of the characters of the Green correspondents of $Y^{\lambda^{(0)}}$ and $Y^{\lambda^{(*)}}$ is equal to $(\chi^{(p)} \otimes \chi^{(p)} \otimes \chi^\tau) \uparrow_{\mathfrak{S}_p \times \mathfrak{S}_p \times \mathfrak{S}_{n-2p}}^{(\mathfrak{S}_p \wr \mathfrak{S}_2) \times \mathfrak{S}_{n-2p}}.$

By a theorem of Marichal and Puig (see [1, Theorem 2.2]), the block of the subgroup $\mathfrak{S}_{2p} \times \mathfrak{S}_{n-2p}$ which corresponds to B under Brauer's first main theorem of block theory is the product of the principal block of \mathfrak{S}_{2p} and the block of defect 0 of \mathfrak{S}_{n-2p} corresponding to τ . By Brauer's second main theorem, the Green correspondents of $Y^{\lambda^{(0)}}$ and $Y^{\lambda^{(*)}}$ in $\mathfrak{S}_{2p} \times \mathfrak{S}_{n-2p}$ lie in this block. Using the Littlewood–Richardson Rule we see that the truncation of $(\chi^{(p)} \otimes \chi^{(p)}) \uparrow_{\mathfrak{S}_p \times \mathfrak{S}_p}^{\mathfrak{S}_{2p}}$ to the principal block of \mathfrak{S}_{2p} is $\chi^{(2p)} + \chi^{(2p-1, 1)} + \chi^{(p^2)}$. Hence the sum of the characters of the Green correspondents of $Y^{\lambda^{(0)}}$ and $Y^{\lambda^{(*)}}$ in $\mathfrak{S}_{2p} \times \mathfrak{S}_{n-2p}$ is $\Phi = \Phi_1 + \Phi_2 + \Phi_3$, where $\Phi_1 = \chi^{(2p)} \otimes \chi^\tau$, $\Phi_2 = \chi^{(2p-1, 1)} \otimes \chi^\tau$, and $\Phi_3 = \chi^{(p^2)} \otimes \chi^\tau$. By the Green correspondence, the sum of $\text{ch}(Y^{\lambda^{(0)}})$ and $\text{ch}(Y^{\lambda^{(*)}})$ is a subcharacter of the $\Phi \uparrow^B$.

Actually it was easy to see from the very start that the character of $Y^{\lambda^{(0)}}$ is $\chi^{\lambda^{(0)}}$ because $\lambda^{(0)}$ is the largest partition of B . Using the Littlewood–Richardson rule we see that $\Phi_1 \uparrow^B$ is the sum of $\chi^{\lambda^{(0)}}$ and characters of the form χ^μ , where $\mu \leq \lambda^{(*)}$; $\Phi_2 \uparrow^B$ is the sum of $\chi^{(2p-1+\tau_2, \tau_1+1, \tau_3, \dots)}$ and characters of the form χ^μ , where $\mu \leq \lambda^{(*)}$; and all constituents of $\Phi_3 \uparrow^B$ are of the form χ^μ , where $\mu \leq \lambda^{(*)}$.

If $a = 1$, then $\tau_1 - \tau_2 = p - 1$ and $(2p - 1 + \tau_2, \tau_1 + 1, \tau_3, \dots) = \lambda^{(*)}$; thus we may conclude by our analysis that $\text{ch}(Y^{\lambda^{(*)}}) = \chi^{\lambda^{(*)}}$. If on the other hand $a > 1$, then $\tau_1 - \tau_2 < p - 1$ and $(2p - 1 + \tau_2, \tau_1 + 1, \tau_3, \dots)$ is a non- p -restricted partition strictly larger than $\lambda^{(*)}$, and it is the largest non- p -restricted partition of B other than $\lambda^{(0)}$, so it must be $\lambda^{(1)}$. So in this case we have either $\text{ch}(Y^{\lambda^{(*)}}) = \chi^{\lambda^{(*)}}$ or $\text{ch}(Y^{\lambda^{(*)}}) = \chi^{\lambda^{(*)}} + \chi^{\lambda^{(1)}}$. If the former were true, then the Specht module $S^{\lambda^{(*)}}$ would be simple, but by a theorem of James (Theorem 2.10 of [6]) this is false, as $\lambda^{(*)}$ is p -regular and the p -power diagram of $\lambda^{(*)}$ contains two distinct entries in one of its columns: the hook length of the $(\tau_1 + 2)$ th node in the first row of $\lambda^{(*)}$ is p , while the hook length of the node directly below it is less than p .

We now turn to the Young modules $Y^{\lambda^{(i)}}$. First note that $\lambda_1^{(i)} - \lambda_2^{(i)} \geq p$. Thus we may define partitions

$$\nu^{(i)} = (\lambda_1^{(i)} - p, \lambda_2^{(i)}, \dots)$$

and see easily that

$$\nu^{(0)} > \nu^{(1)} > \dots > \nu^{(p-1)}.$$

Each partition $\nu^{(i)}$ has no horizontal p -hook, so the Young subgroup $\mathfrak{S}_{(p, 1^{n-p})}$ is a Young vertex of $Y^{\lambda^{(i)}}$, and the Green correspondent of $Y^{\lambda^{(i)}}$ in the normalizer $\mathfrak{S}_p \times \mathfrak{S}_{n-p}$ is $S^{(p)} \otimes Y^{\nu^{(i)}}$.

The partitions $\nu^{(i)}$ are in a block of cyclic defect, where the characters of Young modules are well known:

$$\text{ch}(Y^{\nu^{(i)}}) = \chi^{\nu^{(i)}} + \chi^{\nu^{(i-1)}}, \quad 1 \leq i \leq p-1.$$

By the Green correspondence, then, for $1 \leq i \leq p-1$, the induction of the character

$$\Psi_i = \chi^{(p)} \otimes \chi^{\nu^{(i)}} + \chi^{(p)} \otimes \chi^{\nu^{(i-1)}}$$

from $\mathfrak{S}_p \times \mathfrak{S}_{n-p}$ to B is the sum of $\text{ch}(Y^{\lambda^{(i)}})$ and the character of some projective module. To calculate this induced character we will use abacus notation.

The abacus notation for the partition $\nu^{(i)}$ is $[d_i]$. As a preliminary step we now calculate the induction of the character

$$\chi^{(p)} \otimes \chi^{[d_i]}$$

from $\mathfrak{S}_p \times \mathfrak{S}_{n-p}$ to B , using the Littlewood–Richardson rule, translated appropriately to the abacus. Starting from the abacus display for $[d_i]$, we consider all possible ways of making a series of p horizontal bead moves (each such move corresponds to adding an indent node) so that throughout the process a bead never occupies a previously occupied place (this condition corresponds to having to add nodes in distinct columns to the Young diagram of $[d_i]$) and so that the final abacus display has in each runner the same number of beads as the original display (this so that we end up with the display of a partition in the block B). The resulting displays are those partitions μ belonging to B such that χ^μ is a constituent of the induced character (the multiplicity is always just one). Note that a “successful” series of p bead moves involves moving a bead from each runner to the next one at some point.

Let us start with the case $i \geq a$. In the abacus display for $[d_i]$, there are h beads in the p th runner and no beads in any runner in rows greater than h . So during a successful series of p bead moves the only bead which can at some point move into the p th runner is the bottommost bead in this runner; it must move p times, ending up on the runner it started on, but one place lower. So in this case, we get only the partition $\langle d_0, d_i \rangle$.

Now consider the case $i < a$. In the abacus display for $[d_i]$, there are no beads in rows greater than $h+1$, and the only bead in row $h+1$ is on runner d_i . Furthermore the beads in row h occur on the runners $d_0, \dots, \hat{d}_i, \dots, d_{a-1}$.

If $i = a-1$, then the bottommost bead on runner d_{a-1} must be moved to runner p , at least, while a bead moving into runner d_{a-1} must come from runner p (having originated there on row $h+1$ or on row $h-1$). So here we obtain the two partitions $\langle d_{a-1} \rangle$ and $\langle d_0, d_0 \rangle$.

If $i < a-1$, the bottommost bead in runner d_i must be moved at least to runner p . Next, note that a bead moving into runner d_i must come from runner p (originating on row $h+1$) or from runner d_{i+1} (on row h). In the latter case we finish by moving either of the two bottommost beads on

runner p (on rows h and $h+1$) to runner d_{i+1} . We get the following three partitions as a result: $\langle d_i \rangle$, $\langle d_{i+1} \rangle$, and $\langle d_0, d_{i+1} \rangle$.

Now we can proceed with the calculation of the character of $Y^{\lambda^{(i)}}$ for $1 \leq i \leq p-1$. Suppose first that $i < a-1$. Then inducing

$$\Psi_i = \chi^{(p)} \otimes \chi^{[d_i]} + \chi^{(p)} \otimes \chi^{[d_{i-1}]}$$

from $\mathfrak{S}_p \times \mathfrak{S}_{n-p}$ to B we obtain

$$(\chi^{\langle d_i \rangle} + \chi^{\langle d_{i+1} \rangle} + \chi^{\langle d_0, d_{i+1} \rangle}) + (\chi^{\langle d_{i-1} \rangle} + \chi^{\langle d_i \rangle} + \chi^{\langle d_0, d_i \rangle}).$$

The character of $Y^{\lambda^{(i)}} = Y^{\langle d_i \rangle}$ is a subcharacter of this induced character, and the only constituent χ^μ of this induced character for which $\mu > \langle d_i \rangle$ is $\chi^{\langle d_{i-1} \rangle}$. So there are only two possibilities: either $\text{ch}(Y^{\langle d_i \rangle}) = \chi^{\langle d_i \rangle}$ or $\text{ch}(Y^{\langle d_i \rangle}) = \chi^{\langle d_i \rangle} + \chi^{\langle d_{i-1} \rangle}$. The first would imply that $Y^{\langle d_i \rangle}$ is a simple module with cyclic vertex lying in a block with non-cyclic defect groups, which is impossible by a theorem of Erdmann [3]. Translating back from abacus notation, we see that $\text{ch}(Y^{\lambda^{(i)}})$ is as stated.

Next we consider $i = a-1$. Inducing Ψ_{a-1} to B we get

$$(\chi^{\langle d_{a-1} \rangle} + \chi^{\langle d_0, d_0 \rangle}) + (\chi^{\langle d_{a-2} \rangle} + \chi^{\langle d_{a-1} \rangle} + \chi^{\langle d_0, d_{a-1} \rangle}).$$

The character of $Y^{\lambda^{(a-1)}} = Y^{\langle d_{a-1} \rangle}$ is a subcharacter of this induced character, and the only constituent χ^μ of this induced character for which $\mu > \langle d_{a-1} \rangle$ is $\chi^{\langle d_{a-2} \rangle}$. So there are only two possibilities: either $\text{ch}(Y^{\langle d_{a-1} \rangle}) = \chi^{\langle d_{a-1} \rangle}$ or $\text{ch}(Y^{\langle d_{a-1} \rangle}) = \chi^{\langle d_{a-1} \rangle} + \chi^{\langle d_{a-2} \rangle}$. The former is precluded by the same argument used above, so we see that $\text{ch}(Y^{\lambda^{(a-1)}})$ is as stated.

Next we look at the case $i = a$. The character $\Psi_a \uparrow^B = \chi^{\langle d_0, d_a \rangle} + (\chi^{\langle d_{a-1} \rangle} + \chi^{\langle d_0, d_0 \rangle})$ is the sum of the character of $Y^{\lambda^{(a)}} = Y^{\langle d_0, d_a \rangle}$ and the character of some projective B -module. The character of any non-zero projective module in a block of positive defect cannot be irreducible, and the same is true for the character of $Y^{\langle d_0, d_a \rangle}$ by the argument given above. Thus the projective module here must be zero. We conclude, translating back from abacus notation, that $\text{ch}(Y^{\lambda^{(a)}})$ is as stated.

Finally we come to the case $i > a$. Here $\Psi_i \uparrow^B = \chi^{\langle d_0, d_i \rangle} + \chi^{\langle d_0, d_{i-1} \rangle}$, so arguing as before we see that the character of $Y^{\lambda^{(i)}}$ is exactly this induced character, and translating notation, that $\text{ch}(Y^{\lambda^{(i)}})$ is as stated. ■

We now turn to giving Loewy layers of non-projective Young modules, first noting the following lemma:

LEMMA 2.3. *Let λ be a non- p -restricted partition of B . Then Y^λ either is simple or has Loewy length 3; in the latter case, its first Loewy layer is isomorphic to its third Loewy layer.*

Proof. This follows from the fact that the simple modules of B do not self-extend, the Young modules are self-dual and indecomposable, and the principal indecomposable modules of B have common Loewy length 5. ■

THEOREM 2.4 (Module structures of non-projective Young modules of B). *Retain the notation of Theorem 2.2 and in addition define one additional partition: if $2 < a < p$ let ε be the smallest partition of B which is larger than δ . In the following module structures, we write λ in place of D^λ to avoid clutter. Also, $\lambda^{(i)}$ is to be treated as zero whenever it is undefined (or when the partition is p -singular). If $a = 1$ then*

$$Y^{\lambda^{(0)}} = \lambda^{(0)}; \quad Y^{\lambda^{(*)}} = \lambda^{(*)}; \quad Y^{\lambda^{(1)}} = \begin{array}{cc} \lambda^{(0)} & \lambda^{(*)} \\ & \lambda^{(1)} \\ \lambda^{(0)} & \lambda^{(*)} \end{array};$$

$$Y^{\lambda^{(2)}} = \begin{array}{ccc} & \lambda^{(1)} & \\ \lambda^{(0)} & \lambda^{(*)} & \lambda^{(2)} \\ & \lambda^{(1)} & \end{array};$$

$$Y^{\lambda^{(i)}} = \begin{array}{cc} & \lambda^{(i-1)} \\ \lambda^{(i-2)} & \lambda^{(i)} \\ & \lambda^{(i-1)} \end{array}, \quad \text{if } 2 < i < p-1;$$

and

$$Y^{\lambda^{(p-1)}} = \begin{array}{c} \lambda^{(p-2)} \\ \lambda^{(p-3)} \\ \lambda^{(p-2)} \end{array}, \quad \text{if } p > 3.$$

If $a > 1$ then

$$Y^{\lambda^{(0)}} = \lambda^{(0)}; \quad Y^{\lambda^{(*)}} = \begin{array}{cc} & \lambda^{(1)} \\ \lambda^{(0)} & \lambda^{(*)} \\ & \lambda^{(1)} \end{array}; \quad Y^{\lambda^{(1)}} = \begin{array}{c} \lambda^{(0)} \\ \lambda^{(1)} \\ \lambda^{(0)} \end{array};$$

$$Y^{\lambda^{(i)}} = \begin{array}{cc} & \lambda^{(i-1)} \\ \lambda^{(i-2)} & \lambda^{(i)} \\ & \lambda^{(i-1)} \end{array},$$

if $2 \leq i \leq a-1$ or $a+2 \leq i \leq p-2$;

$$Y^{\lambda^{(a)}} = \begin{array}{ccc} & \lambda^{(a-1)} & \delta \\ \lambda^{(a-2)} & \varepsilon & \lambda^{(a)} \\ & \lambda^{(a-1)} & \delta \end{array},$$

where ε is replaced by $\lambda^{(a-2)}$ and $\lambda^{(*)}$ if $a = 2$;

$$Y^{\lambda^{(a+1)}} = \begin{array}{cc} & \lambda^{(a)} \\ \delta & \lambda^{(a+1)} \\ & \lambda^{(a)} \end{array}, \quad \text{if } a \leq p-2;$$

and

$$Y^{\lambda^{(p-1)}} = \begin{array}{c} \lambda^{(p-2)} \\ \lambda^{(p-3)} \\ \lambda^{(p-2)} \end{array}, \quad \text{if } a \leq p-3.$$

Proof. We first calculate the first few rows of the decomposition matrix of B , using Theorem 1.1 and Lemma 2.1.

If $a = 1$, the first $p + 1$ rows of the decomposition matrix of B are

$\partial\lambda$	abacus	λ	$\lambda^{(0)}$	$\lambda^{(*)}$	$\lambda^{(1)}$	$\lambda^{(2)}$	\dots	$\lambda^{(p-2)}$
0(black)	$\langle d_0 \rangle$	$\lambda^{(0)}$	1					
0(white)	$\langle d_0, d_0 \rangle$	$\lambda^{(*)}$		1				
1	$\langle d_0, d_1 \rangle$	$\lambda^{(1)}$	1	1				
2	$\langle d_0, d_2 \rangle$	$\lambda^{(2)}$			1	1		
\vdots	\vdots	\vdots				\ddots	\ddots	
$p-2$	$\langle d_0, d_{p-2} \rangle$	$\lambda^{(p-2)}$					1	1
$p-1$	$\langle d_0, d_{p-1} \rangle$	$\lambda^{(p-1)}$						1

If $a > 1$, then the first $p + a$ rows of the decomposition matrix of B are

$\partial\lambda$	abacus	λ	$\lambda^{(0)}$	$\lambda^{(1)}$	$\lambda^{(2)}$	\dots	$\lambda^{(a-1)}$	$\lambda^{(*)}$	ε	δ	$\lambda^{(a)}$	\dots	$\lambda^{(p-2)}$
0(black)	$\langle d_0 \rangle$	$\lambda^{(0)}$	1										
1	$\langle d_1 \rangle$	$\lambda^{(1)}$	1	1									
2	$\langle d_2 \rangle$	$\lambda^{(2)}$		1	1								
\vdots	\vdots	\vdots			\ddots	\ddots							
$a-1$	$\langle d_{a-1} \rangle$	$\lambda^{(a-1)}$					1	1					
0(white)	$\langle d_0, d_1 \rangle$	$\lambda^{(*)}$		1				1					
1	$\langle d_0, d_2 \rangle$		1	1	1			1	1				
\vdots	\vdots				\ddots	\ddots			\ddots	\ddots			
$a-2$	$\langle d_0, d_{a-1} \rangle$	ε					1	1		1	1		
$a-1$	$\langle d_0, d_0 \rangle$	δ	1_x					1	1_x		1_y	1	
a	$\langle d_0, d_a \rangle$	$\lambda^{(a)}$									1	1	
\vdots	\vdots	\vdots									\ddots	\ddots	
$p-2$	$\langle d_0, d_{p-2} \rangle$	$\lambda^{(p-2)}$										1	1
$p-1$	$\langle d_0, d_{p-1} \rangle$	$\lambda^{(p-1)}$											1

where the entries 1_x occur if and only if $a = 2$, and the entry 1_y occurs if and only if $a > 2$. We actually do not need to know the rows $\langle d_0, d_2 \rangle, \dots, \langle d_0, d_{a-1} \rangle$; they are included for the sake of completeness.

Theorem 2.2 along with the decomposition numbers given here determine the composition factors of the non-projective Young modules of B . As each of these modules has only a small number of composition factors, their module structures are determined by Lemma 2.3 and are indeed as stated. ■

3. CORRESPONDENCE OF YOUNG MODULES IN $[2 : 1]$ -PAIRS

Let B be a defect 2 block of $k\mathfrak{S}_n$, where k is a field of odd characteristic p , whose p -core has an abacus display having one bead more on the i th runner than on the $(i-1)$ th runner. Let \tilde{B} be the defect 2 block of $k\mathfrak{S}_{n-1}$, whose p -core has an abacus display which is obtained from that of B by

interchanging the i th and $(i - 1)$ th runners. Then B and \tilde{B} form a $[2 : 1]$ -pair.

In this section, we investigate how the Young modules of B restrict to \tilde{B} and how the Young modules of \tilde{B} induce to B . With the structures of the Young modules known in the last section, one may proceed to do this by looking at the module structures. However, we prefer a “conceptual” approach and shall make as little use as possible of the results shown in the last section.

We recall the following terminology:

DEFINITION 3.1. With respect to the $[2 : 1]$ -pair B and \tilde{B} ,

1. A partition of B is *exceptional* if it has two or more beads on the i th runner of its abacus display which may be moved to their respective unoccupied preceding positions on the $(i - 1)$ th runner. Otherwise, it is *non-exceptional*.

2. A Specht module S^λ of B is *exceptional* if λ is exceptional. Otherwise, it is *non-exceptional*.

3. A simple module D^λ of B is *exceptional* if $D^\lambda \downarrow_{\tilde{B}}$ is not simple. Otherwise, it is *non-exceptional*.

4. A partition of \tilde{B} is *exceptional* if it has two or more beads on the $(i - 1)$ th runner of its abacus display which may be moved to their respective unoccupied succeeding positions on the i th runner. Otherwise, it is *non-exceptional*.

5. A Specht module $S^{\tilde{\lambda}}$ of B is *exceptional* if $\tilde{\lambda}$ is exceptional. Otherwise, it is *non-exceptional*.

6. A simple module $D^{\tilde{\lambda}}$ of B is *exceptional* if $D^{\tilde{\lambda}} \uparrow^B$ is not simple. Otherwise, it is *non-exceptional*.

Scopes [16] studied $[2 : 1]$ -pairs extensively. She found that there are three exceptional partitions of B and three exceptional partitions of \tilde{B} , denoted as $\alpha = \langle i, i \rangle$, $\beta = \langle i, i - 1 \rangle$, and $\gamma = \langle i - 1 \rangle$, and $\tilde{\alpha} = \langle i \rangle$, $\tilde{\beta} = \langle i, i - 1 \rangle$, and $\tilde{\gamma} = \langle i - 1, i - 1 \rangle$, respectively. Also, D^α and $D^{\tilde{\alpha}}$ are the only exceptional simple modules of B and \tilde{B} , respectively. The simple module D^α (resp. $D^{\tilde{\alpha}}$) is a composition factor of S^α , S^β , and S^γ (resp. $S^{\tilde{\alpha}}$, $S^{\tilde{\beta}}$, and $S^{\tilde{\gamma}}$) and occurs in no other Specht module. Both $D^\alpha \downarrow_{\tilde{B}}$ and $D^{\tilde{\alpha}} \uparrow^B$ have Loewy length 3 and simple heads isomorphic to $D^{\tilde{\alpha}}$ and D^α , respectively.

The exceptional Specht modules have the following relations:

$$\begin{aligned} S^\alpha \downarrow_{\tilde{B}} &\sim S^{\tilde{\alpha}} \oplus S^{\tilde{\beta}}; & S^{\tilde{\alpha}} \uparrow^B &\sim S^\alpha \oplus S^\beta; \\ S^\beta \downarrow_{\tilde{B}} &\sim S^{\tilde{\alpha}} \oplus S^{\tilde{\gamma}}; & S^{\tilde{\beta}} \uparrow^B &\sim S^\alpha \oplus S^\gamma; \\ S^\gamma \downarrow_{\tilde{B}} &\sim S^{\tilde{\beta}} \oplus S^{\tilde{\gamma}}; & S^{\tilde{\gamma}} \uparrow^B &\sim S^\beta \oplus S^\gamma. \end{aligned}$$

Also, β (resp. γ) is p -regular if, and only if, $\tilde{\gamma}$ (resp. $\tilde{\beta}$) is p -regular, and if so, $D^\beta \downarrow_{\tilde{B}} \cong D^{\tilde{\gamma}}$ and $D^{\tilde{\gamma}} \uparrow^B \cong D^\beta$ (resp. $D^\gamma \downarrow_{\tilde{B}} \cong D^{\tilde{\beta}}$ and $D^{\tilde{\beta}} \uparrow^B \cong D^\gamma$).

For each non-exceptional partition λ of B , we can find a unique bead on the i th runner of its abacus display which may be moved to its (unoccupied) preceding position. Moving this bead produces a non-exceptional partition, $f(\lambda)$ say, of \tilde{B} . This function f is a bijection between the non-exceptional partitions of the $[2 : 1]$ -pair and has the following properties:

1. λ is p -regular (resp. p -restricted) if, and only if, $f(\lambda)$ is p -regular (resp. p -restricted).
2. $\lambda > \mu$ if, and only if, $f(\lambda) > f(\mu)$.
3. $\lambda > \alpha$ (resp. $\mu < \alpha$) if, and only if, $f(\lambda) > \tilde{\alpha}$ (resp. $f(\mu) < \tilde{\alpha}$).
4. $\lambda > \gamma$ (resp. $\mu < \gamma$) if, and only if, $f(\lambda) > \tilde{\gamma}$ (resp. $f(\mu) < \tilde{\gamma}$).
5. If λ is p -regular, then $D^\lambda \downarrow_{\tilde{B}} \cong D^{f(\lambda)}$ and $D^{f(\lambda)} \uparrow^B \cong D^\lambda$.

Let D^λ be a non-exceptional simple module of B with $D^\lambda \downarrow_{\tilde{B}} \cong D^{\tilde{\lambda}}$. Scopes [16] showed that if D^λ (resp. $D^{\tilde{\lambda}}$) occurs twice in $P(D^\alpha)$ (resp. $P(D^{\tilde{\alpha}})$), then it occurs in the second and fourth Loewy layers, while if it occurs once, then it occurs in the third Loewy layer. Moreover, $[P(D^\alpha) : D^\lambda] = 2$ if and only if $[P(D^{\tilde{\alpha}}) : D^{\tilde{\lambda}}] = 1$, while $[P(D^\alpha) : D^\lambda] = 1$ if and only if $[P(D^{\tilde{\alpha}}) : D^{\tilde{\lambda}}] = 2$.

We first show that the Ext-quivers of all defect 2 blocks are bipartite. This result appeared in the doctoral thesis of the second author [17].

THEOREM 3.2. *The Ext-quivers of all defect 2 blocks are bipartite.*

Proof. It can be checked readily that the Ext-quiver of the principal block of $k\mathfrak{S}_{2p}$, constructed by S. Martin [10], is bipartite. Thus it suffices to show that if B and \tilde{B} form a $[2 : 1]$ -pair, then B will inherit this property from \tilde{B} . But if $\tilde{\Lambda}$ (with $D^{\tilde{\alpha}} \in \tilde{\Lambda}$) and its complement form a partition of the simple modules of \tilde{B} displaying the bipartite nature of its Ext-quiver, then

$$\Lambda = \{D^\lambda \mid D^\lambda \text{ non-exceptional and } D^\lambda \downarrow_{\tilde{B}} \in \tilde{\Lambda}\}$$

and its complement form a partition of the simple modules of B displaying the bipartite nature of its Ext-quiver. ■

Using what we know about defect 2 blocks, we have

THEOREM 3.3. *Suppose λ is a p -restricted partition of B . Let $\tilde{\lambda}$ be defined by $D^{\tilde{\lambda}'} \cong \text{soc}(D^{\lambda'} \downarrow_{\tilde{B}})$. Then $Y^\lambda \downarrow_{\tilde{B}}$ is isomorphic to a direct sum of $Y^{\tilde{\lambda}}$ and possibly $Y^{\tilde{\gamma}}$. Moreover, $Y^{\tilde{\gamma}}$ is a summand of the restricted module if, and only if, D^α occurs twice or more in Y^λ .*

Proof. Since λ is p -restricted, Y^λ is isomorphic to the projective cover of $D^{\lambda'} \otimes \text{sgn}$. Thus,

$$\begin{aligned} Y^\lambda \downarrow_{\tilde{B}} &\cong P(D^{\lambda'} \otimes \text{sgn}) \downarrow_{\tilde{B}} \\ &\cong P(\text{soc}((D^{\lambda'} \otimes \text{sgn}) \downarrow_{\tilde{B}})) (\oplus P(D^{\tilde{\alpha}})) \\ &\cong Y^{\tilde{\lambda}} (\oplus Y^{\tilde{\gamma}}) \end{aligned}$$

with the extra summand of $Y^{\tilde{\gamma}}$ if, and only if, D^α occurs twice or more in Y^λ , and where $\tilde{\lambda}$ is such that $D^{\tilde{\lambda}} \cong \text{soc}((D^{\lambda'} \otimes \text{sgn}) \downarrow_{\tilde{B}}) \otimes \text{sgn} \cong \text{soc}(D^{\lambda'} \downarrow_{\tilde{B}})$. ■

Similarly, we have

THEOREM 3.4. *Suppose $\tilde{\lambda}$ is a p -restricted partition of \tilde{B} . Let λ be defined by $D^{\lambda'} \cong \text{soc}(D^{\tilde{\lambda}'} \uparrow^B)$. Then $Y^{\tilde{\lambda}} \uparrow^B$ is isomorphic to a direct sum of Y^λ and possibly Y^γ . Moreover, Y^γ is a summand of the restricted module if, and only if, $D^{\tilde{\alpha}}$ occurs twice or more in $Y^{\tilde{\lambda}}$.*

Thus, it remains to study the Young modules of B and \tilde{B} , the associated partitions of which are not p -restricted.

PROPOSITION 3.5. *Let λ be a non- p -restricted partition of B . Then $Y^\lambda \downarrow_{\tilde{B}}$ has a well-defined non-projective summand, $Y^{\tilde{\mu}}$ say. Moreover, Y^λ is the non-projective summand of $Y^{\tilde{\mu}} \uparrow^B$.*

Thus, there is a natural one-to-one correspondence between the non-projective Young modules of B and those of \tilde{B} .

Proof. J. Rickard has shown that B and \tilde{B} are derived equivalent and therefore stably equivalent. A proof of this (see [2]) actually shows that a stable equivalence is induced by restriction and induction. Since these functors preserve projectives and direct sums, we see from standard results of stable equivalence that $Y^\lambda \downarrow_{\tilde{B}}$ has a well-defined non-projective summand. Also, as restriction and induction both send Young modules to direct sums of Young modules, this summand must be of the form $Y^{\tilde{\mu}}$ for some non- p -restricted partition $\tilde{\mu}$ of \tilde{B} . It is also clear from the stable equivalence that Y^λ is the non-projective summand of $Y^{\tilde{\mu}} \uparrow^B$. ■

LEMMA 3.6. *Suppose α is not p -restricted. Then Y^α corresponds to $Y^{\tilde{\beta}}$. Similarly, if β is not p -restricted, then Y^β corresponds to $Y^{\tilde{\alpha}}$.*

Proof. The constituents χ^λ of $\text{ch}(Y^\alpha)$ satisfy $\lambda \geq \alpha$. This implies that the constituents $\chi^{\tilde{\lambda}}$ of the character of $Y^\alpha \downarrow_{\tilde{B}}$ satisfy $\tilde{\lambda} \geq \tilde{\beta}$. Thus $Y^{\tilde{\beta}}$ is a summand of $Y^\alpha \downarrow_{\tilde{B}}$. Since $\tilde{\beta}$ is not p -restricted, the first assertion follows from Proposition 3.5.

The second assertion uses a similar argument applied to $Y^{\tilde{\alpha}}$. ■

PROPOSITION 3.7. *Suppose λ is a non-exceptional, non- p -restricted partition of B . Then Y^λ corresponds to $Y^{f(\lambda)}$.*

Proof. Let ρ be the least partition with respect to the lexicographic ordering in $\{\mu \mid \chi^\mu \text{ is a constituent of the character of } Y^{f(\lambda)} \uparrow^B\}$. Then Y^ρ is a summand of $Y^{f(\lambda)} \uparrow^B$ and $\rho \leq \lambda$.

If ρ is a non-exceptional partition of B , then $\chi^{f(\rho)}$ is a constituent of $\text{ch}(Y^\lambda)$. Thus, $\tilde{\rho} \geq \tilde{\lambda}$, and so $\rho \geq \lambda$. This shows that $\rho = \lambda$ and hence $f(\rho) = f(\lambda)$. Thus Y^λ is a summand of $Y^{f(\lambda)} \uparrow^B$. Since λ is not p -restricted, we see that Y^λ corresponds to $Y^{f(\lambda)}$ by Proposition 3.5.

If ρ is an exceptional partition of B , then ρ cannot be α , since there is no Specht module of \tilde{B} which when induced to B gives a Specht filtration with α being the least partition among the partitions associated with the Specht factors. If $\rho = \beta$, then $\chi^{\tilde{\alpha}}$ must be a constituent of $\text{ch}(Y^{f(\lambda)})$, and so $f(\lambda) < \tilde{\alpha}$. This then implies that $\lambda < \alpha$ and thus $\lambda < \beta$, since β is the next partition of B with respect to the lexicographic ordering after α , giving us a contradiction. Hence, $\rho = \gamma$, and $Y^{f(\lambda)} \uparrow^B = U_1 \oplus Y^\gamma$ for some U_1 . Note that χ^λ is a constituent of the character of U_1 since χ^λ is not a constituent of $\text{ch}(Y^\gamma)$. Let ρ_1 be the least partition in $\{\mu \mid \chi^\mu \text{ is a constituent of the character of } U_1\}$. Applying the arguments used above shows that $\rho_1 = \lambda$ or γ . Thus, by repeating these arguments if necessary, we will arrive at the conclusion that Y^λ corresponds to $Y^{f(\lambda)}$. ■

We use the fact that a Specht constituent cannot occur more than once in the character of any Young module for the proposition below.

THEOREM 3.8. *Suppose λ is a non- p -restricted partition of B . Suppose further that Y^λ corresponds to $Y^{\tilde{\lambda}}$ of \tilde{B} in the sense of Proposition 3.5. Then $Y^\lambda \downarrow_{\tilde{B}}$ is isomorphic to a direct sum of $Y^{\tilde{\lambda}}$ and possibly $Y^{\tilde{\gamma}}$. Moreover, $Y^{\tilde{\gamma}}$ is a summand of the restricted module if, and only if, D^α occurs twice in Y^λ .*

Similarly, $Y^{\tilde{\lambda}} \uparrow^B$ is isomorphic to a direct sum of Y^λ and possibly Y^γ , with the latter occurring if, and only if, $D^{\tilde{\alpha}}$ occurs twice in $Y^{\tilde{\lambda}}$.

Proof. We know that $Y^\lambda \downarrow_{\tilde{B}}$ is isomorphic to a direct sum of $Y^{\tilde{\lambda}}$ and possibly some projective modules. We only need to consider the case where Y^λ has Loewy length 3, the case where Y^λ is a simple module being trivial. We know that D^α occurs not more than three times in Y^λ , and $D^{\tilde{\alpha}}$ occurs not more than three times in $Y^{\tilde{\lambda}}$. In fact, D^α cannot occur three times in Y^λ if Y^λ has Loewy length 3, otherwise D^α must occur exactly once in each of the three Loewy layers since Y^λ is self-dual. But this is impossible in view of the fact that Y^λ is indecomposable and the Ext-quiver of B is bipartite. Thus, $[Y^\lambda : D^\alpha] \leq 2$. If $[Y^\lambda : D^\alpha] = 0$, then $Y^\lambda \downarrow_{\tilde{B}}$ is an indecomposable module and thus is isomorphic to $Y^{\tilde{\lambda}}$. If $[Y^\lambda : D^\alpha] = 1$, then D^α occurs in the second Loewy layer of Y^λ , and this further shows that $Y^\lambda \downarrow_{\tilde{B}}$ has

Loewy length 3. Thus $Y^\lambda \downarrow_{\tilde{B}}$ does not have any projective summand and is isomorphic to $Y^{\tilde{\lambda}}$. If $[Y^\lambda : D^\alpha] = 2$, then D^α must occur once in the first Loewy layer and once in the third Loewy layer (it cannot occur twice in the second Loewy layer; otherwise $Y^\lambda \downarrow_{\tilde{B}}$ would be an indecomposable Loewy length 3 module and have four copies of D^α , which is impossible). Let $U \cong Y^\lambda / D^\alpha$ have the following module structure:

$$\begin{array}{c} X_1 \oplus D^\alpha \\ X_2 \quad , \\ X_1 \end{array}$$

where X_1 and X_2 are semi simple. Then $U \downarrow_{\tilde{B}}$ has Loewy length at most 4, with the fourth Loewy layer consisting of simple modules extending $D^{\tilde{\alpha}}$ (these are summands of X_1 by the bipartite nature of the Ext-quiver of \tilde{B}). Now $Y^\lambda \downarrow_{\tilde{B}}$ is an extension of $U \downarrow_{\tilde{B}}$ by $D^\alpha \downarrow_{\tilde{B}}$. Hence only the socle of $D^\alpha \downarrow_{\tilde{B}}$ can occur in the fifth Loewy layer of $Y^\lambda \downarrow_{\tilde{B}}$, and so the only projective Young module capable of being a summand of $Y^\lambda \downarrow_{\tilde{B}}$ is $Y^{\tilde{\gamma}}$. But since $Y^\lambda \downarrow_{\tilde{B}}$ has four copies of $D^{\tilde{\alpha}}$, it must have a projective summand. Therefore, it must be isomorphic to $Y^{\tilde{\lambda}} \oplus Y^{\tilde{\gamma}}$. ■

For the remainder of this section, we shall investigate how the non- p -restricted partitions of B and \tilde{B} correspond under the notation introduced in Section 2.

Let τ and $\tilde{\tau}$ be the p -cores of B and \tilde{B} , respectively. Denote the partitions $(\tau_1 + p, \tau_2 + p, \tau_3, \dots)$ and $(\tilde{\tau}_1 + p, \tilde{\tau}_2 + p, \tilde{\tau}_3, \dots)$ by $\lambda^{(*)}$ and $\tilde{\lambda}^{(*)}$, respectively. Label the remaining non- p -restricted partitions of B and \tilde{B} by $\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(p-1)}$ and $\tilde{\lambda}^{(0)}, \tilde{\lambda}^{(1)}, \dots, \tilde{\lambda}^{(p-1)}$, respectively, such that $\lambda^{(0)} > \lambda^{(1)} > \dots > \lambda^{(p-1)}$ and $\tilde{\lambda}^{(0)} > \tilde{\lambda}^{(1)} > \dots > \tilde{\lambda}^{(p-1)}$. As before, we have, in abacus notation,

$$\lambda^{(j)} = \begin{cases} \langle d_j \rangle, & \text{if } 0 \leq j < a; \\ \langle d_0, d_j \rangle, & \text{if } a \leq j < p. \end{cases}$$

$$\lambda^{(*)} = \begin{cases} \langle d_0, d_0 \rangle, & \text{if } a = 1; \\ \langle d_0, d_1 \rangle, & \text{if } a > 1. \end{cases}$$

Remark. Since $a = p$ occurs only when the block concerned is the principal block of $k\mathfrak{S}_{2p}$, we see that we must have $a < p$ for B .

Similarly, we define \tilde{d}_j ($0 \leq j < p$) and \tilde{a} so that, in abacus notation, we have

$$\tilde{\lambda}^{(j)} = \begin{cases} \langle \tilde{d}_j \rangle, & \text{if } 0 \leq j < \tilde{a}; \\ \langle \tilde{d}_0, \tilde{d}_j \rangle, & \text{if } \tilde{a} \leq j < p. \end{cases}$$

$$\tilde{\lambda}^{(*)} = \begin{cases} \langle \tilde{d}_0, \tilde{d}_0 \rangle, & \text{if } \tilde{a} = 1; \\ \langle \tilde{d}_0, \tilde{d}_1 \rangle, & \text{if } \tilde{a} > 1. \end{cases}$$

Since the abacus display of the p -core of \tilde{B} is obtained from that of B by interchanging two consecutive runners, we see that

$$\tilde{a} - a = \begin{cases} 1, & \text{if } i = d_0; \\ 0, & \text{otherwise.} \end{cases}$$

PROPOSITION 3.9. *The Young module $Y^{\lambda^{(*)}}$ corresponds to $Y^{\tilde{\lambda}^{(*)}}$.*

Proof. If $a = 1$, then $\lambda^{(*)} = \langle d_0, d_0 \rangle$. If further $i = d_0$, then $\lambda^{(*)} = \alpha$. We have seen that $Y^{\lambda^{(*)}}$ would then correspond to $Y^{\tilde{\beta}} = Y^{\langle d_0-1, d_0 \rangle}$. But in this case, $\langle d_0 - 1, d_0 \rangle$ is precisely $\langle \tilde{d}_0, \tilde{d}_1 \rangle = \tilde{\lambda}^{(*)}$. If, on the other hand, $i \neq d_0$, then $Y^{\lambda^{(*)}}$ would correspond to $Y^{\langle d_0, d_0 \rangle}$ of \tilde{B} . But $\langle d_0, d_0 \rangle$ is precisely $\tilde{\lambda}^{(*)}$.

If $a > 1$, then $\lambda^{(*)} = \langle d_0, d_1 \rangle$. A little thought shows that, regardless of what the value of i is, $Y^{\langle d_0, d_1 \rangle}$ of B corresponds to $Y^{\langle \tilde{d}_0, \tilde{d}_1 \rangle} = Y^{\tilde{\lambda}^{(*)}}$ of \tilde{B} . ■

PROPOSITION 3.10. *For $0 \leq j < p$, the Young module $Y^{\lambda^{(j)}}$ corresponds to $Y^{\tilde{\lambda}^{(j)}}$.*

Proof. Suppose that the partitions concerned are all non-exceptional. By Proposition 3.5, $Y^{\lambda^{(j)}}$ corresponds to $Y^{f(\lambda^{(j)})}$. But f preserves lexicographic ordering, so that $f(\lambda^{(0)}) > f(\lambda^{(1)}) > \dots > f(\lambda^{(p-1)})$. Since none of these partitions equals $\tilde{\lambda}^{(*)}$ by the previous proposition, we conclude that $f(\lambda^{(j)}) = \tilde{\lambda}^{(j)}$ for all j .

The only possible exceptional partition of the form $\lambda^{(j)}$ is $\lambda^{(a)}$, and in that case, $\lambda^{(a)} = \beta$ and $i = d_0 = d_a + 1$. Moreover,

$$\tilde{d}_j = \begin{cases} i - 1 (= d_0 - 1 = d_a), & \text{if } j = 0; \\ i (= d_0 = d_a + 1), & \text{if } j = a (= \tilde{a} - 1); \\ d_j, & \text{otherwise.} \end{cases}$$

Now, Y^β corresponds to $Y^{\tilde{\alpha}} = Y^{\langle d_0 \rangle}$. But $\langle d_0 \rangle$ of \tilde{B} is precisely $\tilde{\lambda}^{(\tilde{a}-1)} = \tilde{\lambda}^{(a)}$. It is also not difficult to check that $f(\lambda^{(j)}) = \tilde{\lambda}^{(j)}$ for $j \neq a$. ■

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