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LAYOFFS AND QUITS IN REPEATED GAMES

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Layoffs and Quits in Repeated Games

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Abstract

This paper studies games in which the players are not locked into their relationship for a fixed number of periods. We consider two-player games where player 1 can decide to let the opponent continue in the game or replace it with a new player. We also allow the possibility of player 2 quitting the game. When only layoffs can occur, cooperation takes place in finite horizons due to the threat that termination of the relationship imposes on player 2. However, quits limit that cooperation to those cases where the outside option for player 2 is small (lower than some Nash equilibrium of the stage game).

JEL Classification: C70, C72

Keywords: repeated games, Folk Theorem, layoffs, quit

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1 Introduction

It is well known in game theory that there is a big gap between repeated interactions that last for a finite number of periods, and those that last forever. In infinitely repeated games, any feasible payoff can give rise to a subgame perfect equilibrium (SPE thereafter), provided each player is sufficiently patient, and is guaranteed a minimum level of utility. This result is known in the literature as the folk theorem.[1] However, the same conclusions do not translate to finitely repeated games. Indeed, when the stage game has a unique Nash equilibrium (NE thereafter), any finite repetition of the stage game will yield a unique SPE: the repetition of the unique NE.

In the theory of repeated games it is implicitly assumed that players cannot walk away from the game, and start playing with a different opponent. In other words, the players are locked into the relationship. It turns out that this assumption is at the heart of the contrasting results from finitely and infinitely repeated games. We argue that the ability to change partners plays an important role in sustaining cooperation (specially in finite interactions).

In this paper we look at two-player games in which we allow one of them (that we call principal) to decide whether to let the opponent (or agent) continue in the game, or replace it with a new player.[2] This is the case in buyer-seller transactions, for instance. A buyer can always decide to stop purchasing from the current supplier, and switch to a different one. Furthermore, it is even possible to sign an exclusive agreement by which the seller commits to transact only with the buyer, unless the buyer is willing to break the agreement in the future. Effectively, this gives the buyer the power to dismiss the seller, but the seller cannot quit. Another example would be the employment relationship. Typically, employment contracts are at will. This enables the firm to dispose of the services of a worker at any time. But also, the worker can quit at any point without any contractual

[1] See Fudenberg and Maskin (1986).

[2] To avoid confusion, we will refer to the principal as 'he' and the agent as 'she' from now on.

obligation. This endogenous duration also arises naturally in other business transactions, and social interactions.

In accordance with the two examples above, we analyze two different frameworks. First, we look at repeated games in which the principal is allowed to dismiss the agent, but the agent is not allowed to quit. And secondly, we introduce the possibility of the agent leaving the game. In this environment, we derive a limiting folk theorem that expands cooperation results obtained in previous work. Cooperation is possible even when players are short-lived, and the game takes place for a finite number of periods. This result comes from the threat that now termination of the relationship imposes on the opponent.

Throughout the paper we assume that the value of the outside option of the agent is exogenous.[3] We then look at the possibilities of cooperation as a function of that value. When the agent is not allowed to quit, we show the new framework displays a more continuous behavior as a function of the time horizon. The results show that the longer the players live, the closer the set of sub-game perfect equilibria is to that of the game with an infinite horizon.

For finitely repeated games with dismissal, it is possible to sustain any feasible and individually rational payoff arbitrarily closely as a SPE. Furthermore, this is independent of the value of the outside option. When the agent faces poor prospects on the outside, the threat of dismissal is enough to guarantee cooperation when the horizon of the game and the discount factor are large enough. The existence of dismissals introduces the possibility of credibly punishing the agent in the final periods of the game.[4] Furthermore, being able to punish the agent allows for the construction of a punishment-reward phase for the principal in the earlier periods (if the agent fails to carry out the punishment or reward,

[3]This value might be thought of as arising from general equilibrium considerations in a market where several principals and agents get together to play a game, as in MacLeod and Malcomson (1989). We do not pursue this approach here.

[4]This is similar to the case where there are multiple equilibria, which can be used to punish and reward players towards the end of the game. See Benoit and Krishna (1985) for more details. However, in this model only the agent can be punished and rewarded this way.

she can then be punished through a dismissal). Hence, despite the agent being the only one facing the threat of dismissal, both players can be credibly punished. Cooperation can then be sustained in the early periods.

When the outside option is large, dismissals are desirable. The agent can then be punished by forcing her to stay in the game until the end. Cooperative behavior is rewarded with a dismissal towards the end of the game. Of course, this can only be sustained as long as the agent is not allowed to leave the game.

When the agent is allowed to quit, the possibilities for cooperation weaken. We show that when the outside option is low enough (lower than the payoff from a NE of the stage game) it is possible to construct an SPE in which it is never credible for her to quit. Cooperation attains for this case just as before. In contrast, when the outside option is above the payoff of any NE of the stage game no cooperation takes place in finitely repeated games. The outcome in the last period must be a NE. But no agent would ever agree to play since the outside option is larger. By backward induction, she does not want to play at any stage of the game. Hence, the game would not take place at all.

The results in the paper are presented under the assumption that there is a large pool of potential new players among which the principal can choose after a dismissal. However, it is enough to have a single potential player. A dismissal would then be analogous to switching opponents between the two of them. This does not significantly alter the strategic considerations of the game. But a new interpretation of the results above arises. They say that when we allow for third parties, we can sustain more cooperation by fostering competition between the two opponents, thereby opening up the possibility for credible punishments.[5] Furthermore, we can think of these different models as arising from ex-ante contracting. Principal and agent might write an exclusive contract by which both

[5] Other papers have made use of third parties in a similar fashion. Holmstrom (1982), for instance, argues that third parties in teams make free disposal contracts credible. And these contracts can achieve the first best.

parties agree to play for a certain number of periods with each other. This gives rise to the standard repeated game framework. But other types of contracts are conceivable. They might bring a third party, and agree (the three of them) on an exclusive contract that gives the principal discretion to choose between the two opponents each period. This would correspond to the game with dismissals. And finally, the contract might be at will if neither party can be forced to stay in the game. Here we have both layoffs and quits. The choice among these initial contracts can have a big impact on future cooperation prospects.

We are not the first to look at the problem of achieving cooperation in finitely repeated games. Several solutions have been proposed in the literature. We will describe here some of these attempts, without pretending to offer an exhaustive list. Benoit and Krishna (1985) point out that cooperation can be obtained when the stage game has multiple equilibria. They show that, for such games, any SPE payoff of the infinitely repeated game can be approximated arbitrarily closely by the outcome of an SPE of the finitely repeated game, as the number of repetitions goes to infinity. Smith (1994) generalizes their results by developing the notion of recursively distinct Nash payoffs (RDNP), which he shows to be necessary and sufficient for the folk theorem to obtain.[6] Our results suggest that any game (even those with a unique NE) can behave as a game with RDNP when dismissals take place.[7] Furthermore, when quits are possible, even games with multiple equilibria might fail to display cooperation behavior when outside options are large. Kreps, Milgrom, Roberts and Wilson (1982) stress the role of reputational effects in sustaining cooperation when there is imperfect information about the type of the player. Finally, Hirshleifer and Rasmusen (1988) analyze a game with ostracism, in which players can decide to expel those who defect. Ostracism, then, reduces the size of the group but can help sustain cooperation.

The paper proceeds as follows. Section 2 develops the basic intuitions contained in the

[6] The existence of this paper was pointed out to me after the completion of this work. Some of the insights presented here are generalized in Smith (1994).

[7] Essentially, games with dismissals satisfy the recursively distinct Nash payoffs condition in Smith (1994).

paper with a simple example based on the prisoner's dilemma. Section 3 introduces the models and derives some initial results for general two-player games. Section 4 contains the folk theorems for infinitely and finitely repeated games with dismissals. Section 5 covers the results when we introduce quits. Section 6 discusses the interpretation of the models. The conclusion is in section 7. Finally, some of the proofs are left to the appendix.

2 The Prisoner's Dilemma

In this section, we start with an exposition of the basic intuitions of the paper with an example based on the well known prisoner's dilemma. Lets consider the symmetric stage game, where the principal is the row player and the agent the column player:

	C	D
C	5,5	1,6
D	6,1	2,2

It is well known that the only Nash Equilibrium (NE) of the stage game is for both players to defect: (D,D). From a backward induction argument, defection is also the only SPE when the stage game is repeated a finite number of times. Indeed, it is the only NE of the finitely repeated game.[8] Only repetition for an infinite amount of periods can help sustain cooperation (for discount factors close to 1).

Lets allow the principal to decide whether to keep playing with the opponent (the agent), or replace it with a new player. The agent, however, cannot quit the game for now. Suppose that the per-period outside option for the agent is $\underline{u} = 0$. And consider the case where both players live for $T = 3$ periods, at which point the game must end. For simplicity, assume no discounting. Then, we can sustain cooperation in the first period by both players, even though both have short horizons, and only one faces the threat of

[8] This is because in the prisoners' dilemma the outcome of the unique NE turns out to give each player their minimax payoff, and hence no punishment is possible.

dismissal. The following strategy profile achieves this: play (C,C), (D,C) and (D,D) in each of the three periods, respectively, with no dismissal unless a deviation occurs. If player 2 deviates, she gets fired, and the play is restored on the equilibrium path with a new opponent as if no deviation occurred. If player 1 deviates at any point, then (D,D) is played thereafter ("unfair" dismissals can also be considered deviations). By threatening the agent with dismissal, we can reward the principal in the second period for cooperative behavior in the first. Hence, both players cooperate at $t = 1$.

Notice first that a NE is being played when punishing the principal for a deviation. Hence, no further deviation would be profitable here. Lets consider now the incentives of the principal. By defecting in the first period he gets 10 (=6+2+2), which is strictly worse than what the original strategy yields: 13 (=5+6+2). A dismissal would do no better. And in the last two periods, he is already maximizing. Also, when punishing the agent, the principal is indifferent between dismissing her or not. The agent, on the other hand, can get 6 (=6+0+0) by deviating at $t = 1$ and getting dismissed. This falls short of the payoff from conforming to the original strategy, 8 (=5+1+2). Similarly, defecting in the second period yields 2 (=2+0), instead of 3 (=1+2) otherwise. And in the final period she is maximizing.

Consider now the case in which the outside option increases to $\underline{u} = 4.5$. Since quits are not allowed, we can still sustain cooperation. Consider the strategy profile: play (C,C), (D,C) and (D,D) in each of the three periods, respectively, and dismiss the opponent only after the second period, unless a deviation occurs. After a deviation (by either player), choose (D,D) with no dismissals thereafter. Just as before, in case someone deviates a NE is played. Hence, no profitable deviation exists in this subgame. By the same calculation above, the principal does not want to deviate at $t = 1$. And he is maximizing thereafter. When the agent deviates in the first period, she loses the opportunity to get dismissed and gets 10 (=6+2+2), instead of 10.5 (=5+1+4.5). Deviating in the second period performs equally bad: 4 (=2+2), instead of 5.5 (=1+4.5). And in the last period she is exercising the outside option.

Finally, suppose the agent is allowed to quit the game at any point in time. When $\underline{u} = 0$, it is straightforward to check that, with the strategies above, the agent always obtains more than zero in any subgame. As a result, quitting cannot be credible. The same strategy profile constitutes a SPE in this case, too. However, when $\underline{u} = 4.5$ the strategies above no longer sustain cooperation. Both after the first and second period, the agent would rather quit and obtain 4.5 per period than 1 and 2 respectively. Indeed, since they must play a NE in the last period, the agent always quits before then. By a backward induction argument, a NE would take place in all previous periods, and as a result, she would decide not to participate in the game ex-ante.

The rest of the paper is concerned with extending the previous intuition to general two-player games. The results follow from strategies that have the following structure. There is a terminal stage, at which cooperation does not take place (since the horizon is too short). They correspond to the last two periods in the game above. The key is that the required length of this terminal stage is independent of the number of periods of the game. Finally, in the initial stage (corresponding to the rest of the periods prior to the terminal stage) cooperation takes place. Hence, as the horizon goes to infinity, cooperation takes place almost in every period.

3 Introduction to the Models

Consider the two-player stage game $g : \mathcal{A} = \mathcal{A}_P \times \mathcal{A}_A \longrightarrow \Re^2$, where $\mathcal{A}_i$ is the action space of player $i \in \{P, A\}$ (this stands for principal and agent, respectively). We will denote the minimax payoffs of the stage game by $\underline{v}_P$ and $\underline{v}_A$, respectively, and the corresponding strategy profiles by m^P and m^A. Let the set of feasible payoffs be $\mathcal{F}$. Furthermore, let $\overline{v}_i = \max_a g_i(a)$ and $\underline{\underline{v}}_i = \min_a g_i(a)$ be the maximum and minimum payoffs attainable for each player, respectively. Throughout, we will assume that mixed strategies are observable, as in Fudenberg and Maskin (1986). Hence, we can also think of the action space as the space of mixed strategies.

The stage game will be repeated for T periods (possibly $T = \infty$). For notational convenience, denote by t the period we are in ($t = 1, ..., T$). $\delta \in (0, 1)$ is the discount factor used by both players.

Next we incorporate the possibility of dismissals and quits. First, we will allow the principal to have an extra move after the play of the stage game at each period $t = 1, ..., T - 1$. He will be able to decide whether he wants to keep playing with the same agent or "fire" her and restore the game with a new agent. Formally, this adds a move $f_t \in \{0, 1\}$ after each play of g. $f_t = 0$ means that the principal keeps the opponent, and he will set $f_t = 1$ whenever he decides to replace the agent.[9]

We also introduce the possibility of the agent quitting the game if the principal allowed her to continue. $q_t \in \{0, 1\}$ will denote such decision for $t = 0, ...T - 1$. $q_t = 0$ means she decides to continue until the next period, and $q_t = 1$ means she quits. Notice we allow this decision at $t = 0$. When $q_0 = 1$ the agent does not want to play the game at all. To accommodate this framework, there ought to be a supply of potential new players. We assume there is a pool of agents, denoted by $i \in \{1, 2, 3, ...\}$. If we let i_t be the identity of the agent called to play at period t, then the following recursive procedure defines which of the players faces the principal at any point in time:

$$\begin{aligned} \text{if } f_t &= 0 \text{ and } q_t = 0 \text{ make } i_{t+1} = i_t \\ \text{if } f_t &= 1 \text{ or } q_t = 1 \text{ make } i_{t+1} = i_t + 1 \end{aligned}$$

Notice that the firing and quitting decisions are irreversible here. We will denote the resulting repeated game with layoffs and quits by $G^Q(\delta, T)$.[10]

[9] We could allow for stochastic dismissals (or equivalently, to randomize between dismissing and not doing so). However, it turns out this is unnecessary.

[10] Notice that the game does not change when a new player comes in after a dismissal or a quit. This implicitly assumes the principal is indifferent between the current opponent and any potential candidate, as long as they take the same actions.

When studying the game $G^Q(\delta, T)$, we will denote the actions taken at period t by $a^t = \left(a_P^t, a_A^t\right)$. The history up to period t is $h^t = \left(a^1, a^2, ..., a^{t-1}; f_1, f_2, ..., f_{t-1}; q_0, q_1, ..., q_{t-1}\right)$. The space of all possible histories up to period t is $H^t = \mathcal{A}^{t-1} \times \{0,1\}^{t-1} \times \{0,1\}^t$. At period t, a behavior strategy is a function $s_i^t : H^t \longrightarrow \mathcal{A}_i$. We will denote them $s_i = \left(s_i^1, s_i^2, ..., s_i^T\right)$, and $s = (s_P, s_A)$. Layoff and quit strategies are $f_t : H^t \times \mathcal{A} \longrightarrow \{0,1\}$ and $q_t : H^t \times \mathcal{A} \times \{0,1\} \longrightarrow \{0,1\}$, respectively.[11]

Note here that there is no explicit mention of the particular agent playing at each period. This is not necessary since the previous recursive formulas can be used to exactly pin down the identity of such player by simply knowing the time period and the past play, i.e., the past dismissal policy. Realize also that implicit in this formulation is the assumption that any new agent entering the game (when appropriate) will observe all the past history of play. We will keep this assumption throughout the paper.[12]

Finally, for the game where replacement of the opponent is possible, his outside option becomes a key element of the analysis. Lets denote by $\underline{u}$ the constant per-period outside option of the agent, which she receives after $f_t = 1$ or $q_t = 1$. The payoffs in game G^Q will then be given by

$$\begin{aligned} u_P &= \sum_{t=1}^{T} \delta^{t-1} g_P\left(a^t\right) \\ u_{A,i} &= \sum_{\substack{t=1 \\ i_t = i}}^{T} \delta^{t-1} g_A\left(a^t\right) + \sum_{\substack{t=1 \\ i_t \neq i}}^{T} \delta^{t-1} \underline{u} \end{aligned}$$

After having defined the game with dismissals and quits, it is easy to consider standard repeated games, and games with only layoffs, as all these games bare a close relationship with each other. Indeed, a game with layoffs, denoted $G^L(\delta, T)$, is equivalent to G^Q when we restrict q_t to equal 0 for all t. Similarly, a standard repeated game, $G(\delta, T)$, is equivalent

[11] Notice the sequential nature of the decisions: q_t is chosen after f_t. This is not crucial for our results.

[12] See the discussion section for a comment on how restrictive this assumption is.

to G^L when f_t is forced to be 0 for all t.

After introducing all the notation, we can begin exploring the properties of both models. It will be useful to emphasize here some preliminary observations about the relation between standard repeated games and games with layoffs and quits. In particular, we look at the shape of the set of individually rational payoffs for G^L and G^Q, and the relation between the equilibria of these games and those of games without layoffs or quits.

The first result defines a lower bound to the set of payoffs that can be sustained as an equilibrium in games with layoffs, and layoffs and quits:

Lemma 1. *In any NE of the game $G^L(\delta, T)$, possibly $T = \infty$, with payoffs (v_P, v_A) we must have $v_P \geq \underline{v}_P$, and $v_A \geq \min\{\underline{u}, \underline{v}_A\}$. In any NE of the game $G^Q(\delta, T)$, possibly $T = \infty$, we must have $v_P \geq \underline{v}_P$, and $v_A \geq \underline{u}$.*

Proof. It is clear by usual arguments that $v_P \geq \underline{v}_P$ in both games. Suppose now the agent plays the one-period best response to the principal's strategy, as long as she stays in the game. This gives her a payoff of at least $\underline{v}_A$, in case she is active in the game, and $\underline{u}$ if she was fired previously. Hence, in G^L she will get a payoff $v_A \geq \min\{\underline{u}, \underline{v}_A\}$. In game G^Q, the agent can guarantee herself at least $\underline{u}$ by quitting. Hence, $v_A \geq \underline{u}$ there. ■

As usual, any equilibrium should guarantee the principal at least his minimax payoff, since that is the worst possible punishment. For the agent, instead, there are two possible punishments. She can either get minimaxed or dismissed. When only layoffs are possible, the worst possible punishment is the minimum of those two payoffs, since she cannot quit and exercise the outside option in case it is higher than the minimax payoff. When the agent is allowed to quit, her worst possible punishment is dismissal, as the agent can always quit if a worse punishment is intended.

Not only the individually rational sets of G and G^L are similar. But there is also a close connection between equilibrium strategies for the standard repeated game and the modified game, as the next proposition makes clear. It states that any strategy that constitutes an equilibrium for $G(\delta, T)$ is also an equilibrium of $G^L(\delta, T)$, when appropriately modified.

Lemma 2. *If a strategy profile s is a NE (SPE) of the repeated game $G(\delta, T)$, possibly $T = \infty$, it is also a NE (SPE) of the modified repeated game $G^L(\delta, T)$, suitably modified.*

Proof. See the Appendix. ■

From an equilibrium (either Nash or SPE) strategy profile of a repeated game it is possible to construct an equilibrium for the corresponding game with dismissals. In it, all possible agents follow the same exact original strategy. Since they will all behave identically, the principal will never find it optimal to dismiss the opponent (in and off the equilibrium path). Indeed, he will be indifferent between $f_t = 0$ and 1. In turn, agents find it optimal to follow the original strategy, since the principal never fires the opponent. The additional strategic move, then, does not reduce the set of equilibrium outcomes.

The following corollary translates the statement about equilibrium strategies into one about equilibrium payoffs:

Corollary 1. *Suppose that v can be sustained as a NE (SPE) in $G(\delta, T)$, possibly $T = \infty$. Then, v can be sustained as a NE (SPE) in $G^L(\delta, T)$.*

In contrast, an analogous result to lemma 2 will not hold when quits are allowed. Indeed, as we will see shortly, by diminishing the ability to punish, quits might reduce the set of equilibria quite substantially when the outside option of the agent is large.

4 Layoffs

We now turn to the analysis of the case where player 1 can dismiss the opponent. However, we do not allow player 2 to quit the game voluntarily, unless she is fired. We will relax this assumption in the next section.

4.1 Infinitely Repeated Games with Dismissal

We begin by looking at infinitely repeated games with layoffs. The results in this section will only be an extension of the folk theorem for the modified framework. However, they will be a useful benchmark for the finite horizon case later on.

We saw in the previous analysis that any payoff that arises from a SPE of the standard game is also sustainable as a SPE when dismissals are allowed. From the Folk Theorem in Fudenberg-Maskin (1986), then, the following result is immediate.

Corollary 2. *Suppose that $v = (v_P, v_A)$ is feasible and such that $v_P > \underline{v}_P$, and $v_A > \underline{v}_A$. Then, $\exists \underline{\delta}$ such that v can be supported as a SPE of the game $G^L(\delta, \infty)$ for any $\delta \in (\underline{\delta}, 1)$.*

In light of this corollary, we can conclude that the modified game $G^L(\delta, \infty)$, has at least as many payoffs that can be supported as a SPE than the ordinary repeated game. But there could be room for supporting a wider range of payoffs. What the next proposition shows, is that indeed this is the case. Any payoff that is feasible and individually rational (as stated in lemma 1) is supported by a SPE.

Proposition 1. *Suppose that $v = (v_P, v_A)$ is feasible and such that $v_P > \underline{v}_P$, and $v_A > \min\{\underline{u}, \underline{v}_A\}$. Then, $\exists \underline{\delta} < 1$ such that v can be supported as a SPE of the game $G^L(\delta, \infty)$ for any $\delta \in (\underline{\delta}, 1)$.*

Proof. Let the action profile a be such that $v = g(a)$. Realize that the proposition follows directly from the previous corollary for $v_A > \underline{v}_A$. Hence, it only remains to be shown that any feasible $v = (v_P, v_A)$ such that $v_P > \underline{v}_P$, and $v_A > \underline{u}$ can be supported as a SPE. Consider the following strategy profile:

- Phase I:

  Play action profile a, and make $f_t = 0$. Play remains in Phase I as long as there is no individual deviation from a, and $f_t = 0$. Move to Phase II$_i$ if player i deviates unilaterally (also if the principal fires the short-run player).

- Phase II_A:

  If the agent deviates in period t unilaterally, then the principal chooses $f_t = 1$, and play is resumed in Phase I with a new opponent in period $t+1$. If the principal deviates (i.e., $f_t = 0$), move to Phase II_P.

- Phase II_P:

  Play m^P for N periods, and make $f_t = 0$, as long as there is no individual deviation, and then go back to Phase I. Move to Phase II_A if the agent deviates unilaterally. If the principal deviates in any way, restart the phase.

Choose N large enough so that $\overline{v}_P + N\underline{v}_P < (N+1)\, v_P$. Then, make $\underline{\delta}$ close enough to 1, so that we have $\overline{v}_P + \sum_{s=1}^{N} \underline{\delta}^s \underline{v}_P < \sum_{s=0}^{N} \underline{\delta}^s v_P$ and $(1-\underline{\delta})\,\overline{v}_A + \underline{\delta}\underline{u} < (1-\underline{\delta}) \sum_{s=0}^{N-1} \underline{\delta}^s \underline{\underline{v}}_A + \underline{\delta}^N v_A$. The first condition on $\underline{\delta}$ will be satisfied for some $\underline{\delta} < 1$, since in the limit the inequality holds (by the condition on N). The second condition will also be satisfied for some $\underline{\delta} < 1$, since in the limit, we obtain $\underline{u} < v_A$.

The previous strategy profile will then constitutes a SPE of the modified game for any $\delta \in (\underline{\delta}, 1)$. The agent does not want to deviate from Phase I, since otherwise, she receives her outside option thereafter, which is strictly worse (the second condition on $\underline{\delta}$ ensures this is not profitable). For the same reason, she does not deviate from Phase II_P. Similarly, the principal does not deviate from Phase I as long as he gets minimaxed for enough periods (the conditions on N and $\underline{\delta}$ guarantee this). He can certainly not gain by deviating in Phase II_P, since he is best-responding. Finally, in Phase II_A he prefers to fire the opponent, because this brings him back to Phase I, rather than getting minimaxed. ■

4.2 Finitely Repeated Games with Dismissal

We now turn to exploring the effects of allowing for dismissals in finitely repeated games. We will see that the standard backward induction argument that prevents cooperations in games such as the prisoner's dilemma will yield rather different predictions here. Indeed, we obtain that any feasible and individually rational payoff can be approximated arbitrarily closely with the payoff from a SPE as the horizon of the game grows arbitrarily large. This represents a convergence result that provides the continuity between finite and infinite horizon games that is lacking in the standard framework.

We start first by making a remark about the strategies in these games. The intuition behind this remark was the essence of the proof of lemma 2.

Remark 1. *In the finitely repeated game the life span of any agent equals the length of the game, at every point in time t. Consequently, both the incumbent agent, and any new agent that might be brought into the game face the same continuation game, and hence, share the same strategic considerations. This means that, at any point, one period deviations in the dismissal policy of the principal have no effect on the feasibility and optimality of the future strategies of the agent. In other words, if s_A^s for $s > t$ constitute a best response to s_P^s for $s > t$, then this will be independent of whether $f_t = 0$ or 1 (i.e., whether the same agent that played in period t continued through $t+1$, or was replaced by a new one). Moreover, if both the incumbent agent and a potential new player follow the same course of action, s_A^s for $s > t$, then the principal must be indifferent between $f_t = 0$ and 1. This feature of the game makes the construction of appropriate strategies easier, and reduces the number of possible deviations that need to be checked below.*

We are now ready to provide the analog to proposition 1 for finitely repeated games. For expositional purposes, however, we only present here part of the proof, corresponding to the strategies that sustain payoffs above the outside option of the agent. These strategies will prove useful for analyzing games with layoffs and quits later. The strategies that sustain payoffs above the minimax payoff of the agent are left for the Appendix.

Proposition 2. *Suppose that $v = (v_P, v_A) \in \mathcal{F}$ is such that $v_P > \underline{v}_P$, and $v_A > \min\{\underline{u}, \underline{v}_A\}$. Further assume that $\underline{u} \neq v_A^*$ for some NE payoff v^*. Then, $\forall \epsilon > 0, \exists \underline{T}, \underline{\delta}$ such that $\forall T > \underline{T}$ and $\delta \in (\underline{\delta}, 1)$, there exists a SPE of $G^L(\delta, T)$ within ϵ of v.*

Proof. Let a^* be a NE of the stage game with payoffs v^*, and (a_t, f_t) represent the action profile and firing decision taken at time t. The strategies below can sustain any $v = (v_P, v_A) \in \mathcal{F}$ such that $v_P > \underline{v}_P$ and $v_A > \underline{u}$ as a SPE for the case that $v_A^* > \underline{u}$:

- Phase I:

  Play $(a, 0)$ for $t = 1, ..., T - K - R$, then $\left(\overline{a}^P, 0\right)$ for $t = T - K - R + 1, ..., T - K$, and finally $(a^*, 0)$ for $t = T - K + 1, ..., T$. After a unilateral deviation of the principal at $t > T - K - R$, or the agent at $t > T - K$, do nothing. Otherwise, after a unilateral deviation by player i at period t, go to Phase II$_i$.

- Phase II$_A$:

  The principal chooses $f_t = 1$, and then start period $t + 1$ in Phase I with a new opponent. If he chooses $f_t = 0$ instead, do nothing (i.e., go back to Phase I as if he fired the agent).

- Phase II$_P$:

  $\left(m^P, 0\right)$ for N periods, and then go back to Phase I. If the principal deviates in any way, continue in this phase as if no deviation occurred. If the agent deviates, then go to Phase II$_A$.

Choose N large enough so that the following inequality is satisfied: $\overline{v}_P + N\underline{v}_P < (N+1)\, v_P$. Similarly, let K be such that $\overline{v}_A + (N - 1 + R + K)\, \underline{u} < (N + R)\, \underline{\underline{v}}_A + K \min\{v_A, v_A^*\}$. Let $R = N$. Finally, make $\underline{T}$ large enough so that $\frac{(\underline{T}-R-K)}{\underline{T}} v_i + \frac{R}{\underline{T}} g_i\left(\overline{a}^P\right) + \frac{K}{\underline{T}} v_i^* \in (v_i - \epsilon, v_i + \epsilon)$, for $i \in \{A, P\}$, and $\underline{\delta} < 1$ such that all the previous inequalities are still satisfied after introducing the appropriate discounting of payoffs.

Now we check that no player can gain by deviating from the previous strategy profile. The agent cannot gain by deviating in Phase I for $t = T - K + 1, ..., T$, since a NE is played. For $t = T - K - R + 1, ..., T - K$ no deviation is profitable, since $\overline{v}_A + \sum_1^{S+K-1} \delta^s \underline{u} < \sum_0^S \delta^s g_A\left(\overline{a}^P\right) + \sum_S^{S+K-1} \delta^s v_A^*$, for $S \leq R - 1$, by the previous paragraph. And similarly for $t = 1, ..., T - K - R$. In Phase II_A she takes no action. Finally, we can only move to Phase II_P for $t = 1, ..., T-K-R$. But the definition of K ensures that even if the principal deviates at $t = T - K - R$, it is worthwhile for the agent to carry on the punishment. For sure, also, that earlier deviations will be punished as well.

The principal does not have scope for deviation in Phases II_A and II_P, since in the first one only the firing decision takes place (and we already argued in the previous remark that he is indifferent about the dismissal policy). In the second, besides the firing decision, the principal is optimizing. In Phase I, optimal behavior is followed for $t > T - K - R$. Moreover, the conditions on N together with the low discounting, ensure no deviation is profitable at earlier stages.

This completes the proof for this case. The other two remaining cases sustain a payoff $v = (v_P, v_A) \in \mathcal{F}$ such that $v_P > \underline{v}_P$ and $v_A > \underline{v}_A$ as a SPE when $v_A^* > \underline{u}$ and $v_A^* < \underline{u}$. They are dealt with in the appendix. ■

For the case in which $v_A > \underline{u}$ and $v_A^* > \underline{u}$ the equilibrium path requires three stages. In the final periods, no cooperation can be achieved due to the short horizon left. Hence, a NE play is required in the final K periods, so that nobody wants to deviate. Furthermore, since $v_A^* > \underline{u}$ we can punish the agent as strongly as we want by increasing K. By dismissing her before the final stage of the game she looses approximately $K(v_A^* - \underline{u})$ (for large discount factors). Prior to this, the agent rewards the principal with his maximum payoff for R periods. This way, we can punish the principal for a deviation prior to these R periods by minimaxing him. We sustain this (both the punishment and reward for the principal) with the threat of dismissal, since the agent does not want to jeopardize the large rents of the final stage.

Then, we can sustain any $v = (v_P, v_A) >> (\underline{v}_P, \underline{u})$ during the initial periods (and for most of the game length, when T is large). We achieve this with the threat of punishment. Such punishments consist of minimaxing the principal and dismissing the agent, respectively. When she gets dismissed, she receives the outside option thereafter, missing the opportunity to obtain the future v_A and v_A^* (remember in this case $v_A^* > \underline{u}$). This punishment can be made as large as we want by increasing the length of the final periods (in which the NE is played, and nobody has an incentive to deviate). In particular, it can be made large enough so that she prefers to minimax the principal after a deviation, than facing the dismissal. Hence, we can sustain the punishment for him as well.

Sustaining a payoff $v = (v_P, v_A) >> (\underline{v}_P, \underline{v}_A)$ is very similar when $v_A^* > \underline{u}$. Just as before, the agent is rewarded in the final K periods with v_A^* if she did not deviate before, and punished with dismissal before those K periods otherwise. The previous R periods are also as above. The only difference is the punishment for the agent during the initial periods. In this case, she must be minimaxed for some periods (rather than dismissed). But the same intuition follows. Dismissals make it possible to punishing the agent towards the end of the game. This in turn sustains the punishment of the principal, and cooperation arises early on.

Finally, the case in which $v = (v_P, v_A) >> (\underline{v}_P, \underline{v}_A)$ and $v_A^* < \underline{u}$ reverses the roles of the dismissal and the NE. Now facing dismissal is more attractive than remaining in the game under the NE play. However, the inability of the agent to quit makes her unable to exercise her outside option. And this allows the principal to keep the system of punishment-reward. Punishment after a deviation now consists on the play of the NE, whereas cooperation is rewarded in the final periods with the possibility to leave and exercise the outside option (she is dismissed). The game will be finished with a new opponent.

Furthermore, the length of the final stages $(R + K)$ required to get cooperation is independent of the length of the game. Hence, we can sustain a payoff arbitrarily close to any SPE of the infinitely repeated game, as the number of periods increases. And this result is independent of the value of the outside option of the agent, so long as she cannot quit the game. In this case, we can keep the system of rewards and punishments even when her outside option is large. Next we discuss the case in which the agent cannot be forced to stay in the game against her will.

5 Quits

In this section we expand the previous framework to allow the agent to quit the game. Before, she could obtain a payoff below her outside option both on the equilibrium path or after a deviation. The former will not happen if she can decide not to take part in the game in the first place. The latter would make her quit if she could. To rule this out, we will expand the strategy space of the agent to allow for quits. Formally, she can choose $q_t \in \{0, 1\}$ before the play of period $t + 1$. As a reminder, $q_t = 0$ means she continues through next period, and $q_t = 1$ when she quits.

5.1 Infinitely Repeated Games with Dismissals and Quits

As in the previous section, we begin with the analysis of the infinitely repeated game, as a benchmark for the finite case. With an infinite horizon, a folk theorem still obtains when the agent is allowed to quit:

Proposition 3. *Suppose that $v = (v_P, v_A)$ is feasible and such that $v_P > \underline{v}_P$, and $v_A > \underline{u}$. Then, $\exists \underline{\delta}$ such that v can be supported as a SPE of the game $G^Q(\delta, \infty)$ for any $\delta \in (\underline{\delta}, 1)$.*

Proof. Consider a feasible payoff vector $v = (v_P, v_A)$ such that $v_P > \underline{v}_P$, and $v_A > \underline{u}$. It is immediate that such payoff can be approximated arbitrarily closely with the strategies in the proof of proposition 1. Furthermore, to check the same strategies are still a SPE of the game with quits, it only remains to be shown that the agent never has an incentive to quit. This is true, however, since in every subgame her payoff is at least $\underline{u}$. This completes the proof. ■

When the agent is allowed to quit, she will not accept anything less than her outside option in any equilibrium. This affects the set of payments that can be sustained as a SPE. But quits do not affect the ability to cooperate in infinitely repeated games. With an infinite horizon, the prospects of cooperation are large at any point in the game. Hence, the threat of dismissal is enough to guarantee cooperation.

However, we will see next that quits can alter the results quite dramatically for finitely repeated games. When the outside option is large, quits limit the ability to punish the agent in the final periods. And by backward induction everything unravels.

5.2 Finitely Repeated Games with Dismissals and Quits

We now turn to the finite horizon framework where we introduce the possibility of the agent quitting the game. Here the size of the outside option will play an important role on the ability to sustain cooperation, unlike in all the previous cases. When $\underline{u}$ is low enough (to be made precise shortly), dismissal still imposes a credible punishment, while a quit will never be subgame perfect. All the work done for the model without quits will also apply here. On the other hand, a high $\underline{u}$ will prevent all possible punishments from being implemented, and a negative result with respect to the sustainability of cooperation will arise.

We begin with the optimistic message of the next partial cooperation result.

Proposition 4. *Suppose that $v = (v_P, v_A) \in \mathcal{F}$ is such that $v_P > \underline{v}_P$, and $v_A > \underline{u}$. Moreover, assume there exists a NE of the stage game v^* such that $v_A^* > \underline{u}$. Then, $\forall \epsilon > 0, \exists \underline{T}, \underline{\delta}$ such that $\forall T > \underline{T}$ and $\delta \in (\underline{\delta}, 1)$, there exists a SPE of $G^Q(\delta, T)$ within ϵ of v.*

Proof. It follows from the proof of proposition 2. Here the only additional deviation that could take place would be a quit. However, it is easy to see that when $v_A^* > \underline{u}$, the strategies specified in the previous section for this case would yield a payoff larger than $\underline{u}$ to the agent at all times. Hence, quitting cannot be optimal. ■

Cooperation can be sustained as long as the outside option is low enough, namely, lower than a NE payoff. In this case, the threat of quitting would never be credible, and could not be part of any SPE. The strategies used in the previous section, with no quits, can be readily applied to this case as well. On the other hand, when the outside option is larger than the payoff from any NE the same result would not hold. In the last period, a NE would have to be played. However, any agent would rather exercise the outside option at that point. As a result, the game does not take place in the last period. The second to last period becomes the last one, now. And just as in the prisoner's dilemma, a backward induction argument extends the same outcome to the rest of the periods. This is summarized in the next proposition:

Proposition 5. *Suppose that* $v_A^* \leq \underline{u}$ *for any NE of the stage game* v^**. Then, for any finite* T*, the game* $G^Q(\delta, T)$ *will never be played (i.e.* $q_0 = 1$*).*

When the agent's outside option is large, all the strategic benefits of introducing dismissals disappears. This suggest that those games where players cannot reap the benefits of cooperation might not be played at all. Either the outside option is such that the threat of dismissal can sustain cooperation (at least for long horizons and large discount factors), or the game will not take place. This result is in sharp contrast to the case where only layoffs could take place. There, independently of the size of the outside option, a system of punishment-rewards could always be constructed. For low outside options, the reward was continuation in the game, and dismissal was used as a means to inflict punishment. When the outside option was high, the reverse occurred: those who had to be rewarded were allowed to leave the game through a dismissal, whereas punishment required continuation in the game. It is clear, however, that this last case could not be enforced in equilibrium when the agent is able to quit the game. Any hope for cooperative behavior there vanishes.

5.3 Cooperation with Quits in Finite Interactions

The negative result obtained when quits are possible arises from the assumption that the outside option is independent of the history of the game. Dropping this assumption can restore cooperation in those games. In particular, suppose that $\underline{u}_{t+1} = \underline{u}(f_t)$, and that $\underline{u}(0) > \underline{u}(1)$. The outside option, then, is larger when the agent quits than when she is dismissed by the principal.[13] It is easy to see how this small change allows us to restore the ability to punish the agent. After a deviation, she will get dismissed. She will obtain a payoff of $\underline{u}(1)$. Those who do not deviate will be allowed to quit towards the end of the game and obtain $\underline{u}(0)$ for the final K periods. The difference in outside options, $\underline{u}(0) - \underline{u}(1) > 0$, will deter any deviation for K large enough. (In any case, the final K periods of the game never take place, here.) Furthermore, this makes it possible to punish and reward the principal in the earlier periods. Cooperation then arises again. A strategy profile can then be constructed to sustain cooperation during the early part of the game, as the following result states:

Proposition 6. *Let the agent's outside option depend on f_t, and be such that $\underline{u}(0) > \underline{u}(1)$. Suppose that $v = (v_P, v_A) \in \mathcal{F}$ is such that $v_P > \underline{v}_P$, and $v_A > \underline{u}(0)$. Then, $\forall \epsilon > 0, \exists \underline{T}, \underline{\delta}$ such that $\forall T > \underline{T}$ and $\delta \in (\underline{\delta}, 1)$, there exists a SPE of $G^Q(\delta, T)$ within ϵ of v.*

Proof. See the Appendix. ■

Allowing the outside activity to depend on whether the agent quits or gets dismissed restores the full cooperation results.

6 Interpretation of the Models

In this section we elaborate on the interpretation of the models. We discuss the need for the assumption of a large supply of potential players for the results to hold. We also point out the relation between the models and the choice of ex-ante contracts to govern the relationship. Finally, we consider the implications of the results for the literature on relational contracts.

[13] Alternatively, the outside option might depend on a costless report $m \in \{0, 1\}$ sent by the principal to the outside market (a letter of reference or recommendation, for instance), the report being favorable or unfavorable. Since m is costless, it is immediate to obtain that the principal writing an unfavorable report only after a dismissal (but not after a quit) might be an equilibrium.

6.1 On the Number of Players

Throughout the paper, we have assumed there is an infinite supply of potential agents.[14] This, however, does not have to be the case. Indeed, all the results are still valid when there are only two such players. The game would start with the first opponent. When punishing her requires dismissal, the second opponent would be brought in. Punishing this second agent with dismissal could be achieved by bringing back the initial opponent. Moreover, since deviations do not occur in equilibrium, it does not affect the incentives of the first player to deviate in the first place. Switching opponents between the two agents would achieve the exact same strategic incentives than making use of an infinite supply of them. Hence, we can sustain the same results with the threat of switching between the two players, forcing them to compete for cooperation.

Indeed, we can further generalize the framework by allowing the principal to randomize between the two opponents at the beginning of each period. The agent that gets chosen plays the game, and the other one obtains the reservation utility $\underline{u}$. But both agents are bound to play this game for T periods (just as with the standard repeated game). In this case, the strategic interactions that arise from the game $G^L(\delta, T)$ with layoffs would be contained in this expanded framework. All the SPE derived before would still be equilibria here. However, there would be other (more equitable) ways of playing the game. Without getting into the formal details, the principal can, for instance, start by randomizing between the two agents equally in each period. After the deviation of one of them, we would switch to a regime where the other plays with probability one (essentially firing the deviator). Such an equilibrium, however, might require larger discount factors since each agent plays only half the time while the punishments are the same as before.[15]

With this alternative formulation, other examples come to mind besides those in the

[14] At least, as many agent as the number of periods the game lasts (in order to be able to threaten with dismissal at any point).

[15] This will be the case when the payoff to sustain is larger than the outside option for the agents.

introduction. Consider the internal organization of a firm where the principal assigns two agents to two different jobs. The first one requires cooperation from both principal and agent, while the second one offers a constant reward. Using the task assignment instrumentally (forcing the agents to compete for the good job) can then induce cooperation.

The assumption of two opponents competing against each other to play with the principal would seem more natural if in order to play the game some initial investments are required.[16] When these investments exist, it is unrealistic to think that any potential player would start the game at any point in time without any loss of utility by either part. (The incentives to invest in the relationship would certainly be stronger at $t = 0$ than at $t = T - 1$.) If we assume two agents initially agree to play the game and make such investments, the assumption of costless replacement of the opponent is less troublesome. The additional player would require further initial investments, but this cost might be outweighed by the gains from future cooperative behavior that can be achieved.

Notice also that when we think of the games as being played by a third party, rather than a sequence of potential agents, the assumption of all parties observing the past history does not appear to be so extreme. Since the second opponent might be called upon to play, she will have incentives to monitor the evolution of the game.

6.2 Ex-ante Contracts

In light of the previous discussion, there is another way of thinking about the results presented here, and the difference between them and previous work. In the repeated game framework, the players are locked in the relationship for the entire length of the game. We can think of this as arising from an ex-ante exclusivity contract, by which both parties agree to transact only with each other for the specified number of periods.

However, this might not necessarily be the only possible contract available to them.

[16] Specially when those investments are specific to the game. This, in turn, could justify the existence of rents to be made by the short-run players, above their outside option.

In a buyer-supplier relationship, for instance, we can envision a different contract being signed. The two parties could agree to transact for T periods, but giving the principal the option to substitute the services of the agent by those of a third party. This third party should also sign a contract giving away the right to use her services to the principal. This would give rise to the game G^L.

Interestingly, when the outside option of the agent is below a NE, we know the third player would never take place in the game on the equilibrium path. Hence, the option to use her services will never be exercised by the principal. Nevertheless, such option might be very valuable, since it allows the principal to sustain a constant threat to the opponent, and hence cooperation. Conversely, when the outside option of the agent is above the NE, the third player will be called upon to play on the equilibrium path (when no quits are allowed). She would have to be compensated ex-ante to forego $\underline{u}$ in the final periods and accept such a contract. However, the prospects of cooperation that arise from it, might make it worthwhile paying such cost (specially for long horizons).

In case we cannot contractually force the agent to stick to the terms of the contract ex-post, quits might be possible. This might correspond, for instance, to employment contracts, which are typically at will. Here we would be playing game G^Q.

6.3 Relational Contracts for Finite Relationships

The results obtained here can be applied to the relational contracting framework. Here, the principal's action consists in the offer of a bonus, b, to be paid after the agent's choice of effort, e. The principal's payoff is $e-b$, and the agent's is $b-c(e)$, where $c(e)$ represents the cost of effort. When b and e are binary choices the game resembles the prisoner's dilemma (with the only difference being the sequential nature of this game).

The above results suggests that allowing the principal to fire the agent can help sustain cooperation independently of the value of the outside option (so long as it is different from zero). However, when the agent can quit, a relational contract inducing positive effort (and bonus) can only occur when the outside option is low (negative, indeed). In other words, the principal must pay an efficiency wage if any bonus is to be self-enforced early on. This is in contrast to previous results for infinitely repeated games, where an efficiency wage and a bonus are substitute mechanisms to enforce effort (see Levin (2003)). When the interactions take place for a finite number of periods, having a larger efficiency wage makes the self-enforceable bonuses easier to sustain.

7 Conclusion

In this paper we study two-player repeated games in which player one (called principal) is allowed to dismiss the opponent (or agent), and the opponent can quit the game at any point in time. Unlike standard repeated games where all players are bound to remain in the game, here cooperation can arise in finite interactions. When only dismissals can take place, and the agent cannot quit, any feasible and individually rational payoff can be approximated arbitrarily close when the game takes place for a long horizon and the discount factor is large. Therefore, the outcomes of the finite case resemble those of the infinitely repeated game, when the number of periods is large.

In contrast, when the agent is allowed to quit cooperation may fail. When her outside option is large (above all NE payoffs of the stage game), she will not be willing to play in the last period. A backward induction argument extends this to the rest of the periods. As a result, the agent will not agree to begin the first period, and the game will never take place. However, when the outside option lies below some NE, it is possible to construct an equilibrium where quitting is never subgame perfect.

Enriching the repeated game framework allowing for layoffs and quits provides a new set of predictions. It brings us a better understanding of the role that outside options play in repeated interactions, as well as the workings of endogenous continuation decisions. Cooperation arises naturally here, even in games with a finite horizon, without relying on other explanations proposed in the literature, such as incomplete information.

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8 Appendix: Proofs

Proof of Lemma 2. Suppose that $s = (s_P, s_A)$ is a NE (SPE) of $G(\delta, T)$. Now consider the strategy $\widetilde{s} = (\widetilde{s}_P, \widetilde{s}_A)$ for the game $G^L(\delta, T)$, with

$$\begin{aligned} \widetilde{s}_i^t : \quad & \widetilde{H}^t \longrightarrow \mathcal{A}_i \\ & \widetilde{h}^t \longmapsto s_i(h^t) \end{aligned}$$

where $\widetilde{h}^t = \left(a^1, a^2, ..., a^{t-1}; f_1, f_2, ..., f_{t-1}\right)$, and $h^t = \left(a^1, a^2, ..., a^{t-1}\right)$ contain the same actions.

It is easy to see that the following strategy constitutes a NE (SPE) of the modified game: $f_t = 0$ for all t, and $\widetilde{s}_i^t = s_i^t$, for all i that might be called to play at period t. First, notice that since no agent is being replaced before dying, we will follow exactly the same equilibrium path as in the original game. This yields the same payoffs.

Since $\widetilde{s}_i^t$ is independent of $f_1, f_2, ..., f_{t-1}$, and the same for all i, the principal is indifferent between $f_t = 0$, and $f_t = 1$. Also, since $f_t = 0$, the agents that might be called to play (even if someone was fired previously) will be better-off following this strategy, since it was optimal in the original game. This follows from the assumption that both the principal and the agents have the same horizon (have the same T). Finally, provided that nobody is fired, $\widetilde{s}_i^t$ are optimal, since they are a NE (SPE) of the original game. ■

Proof of Proposition 2 (Cont.). It only remains to be seen that the proposition is true when $v_P > \underline{v}_P$ and $v_A > \underline{v}_A$. Lets consider action profiles a^i that deliver payoffs $v_j^i = g_j\left(a^i\right) > \underline{v}_i$ such that $\underline{v}_i < v_j^i < v_i$ for all i and j, and $v_i^j > v_i^i$ for all $i \neq j$.[17]

We begin by considering the case where $\underline{u} > v_A^*$. The following strategies constitute a SPE:

[17] The existence of such action profiles is shown in Abreu et al. (1994).

- Phase I:

  Play $(a, 0)$ for $t \leq T-K-R$, then $\left(\overline{a}^P, 0\right)$ for $t = T-K-R+1, ..., T-K-1$, $\left(\overline{a}^P, 1\right)$ at $t = T-K$, and finally $(a^*, 1)$ for $t = T-K+1, ..., T$.

  After a deviation of the principal at $t > T-K-R$, or the agent at $t > T-K$, do nothing. Otherwise, after a deviation of the principal at time $t \leq T-K-R$ go to Phase II^{a}_P. If the agent deviates at $t \leq T-K-R-N-M$ go to Phase II^{a}_A, but if she does so for $t = T-K-R-N-M+1, ..., T-K$ then go to Phase III.

- Phase II^{a}_i:

  $\left(m^i, 0\right)$ for N periods. After this go to Phase II^{b}_i, unless $i = P$ and $t \geq T-K-R-M$, in which case, go back to Phase I instead.

  If player i deviates in any way, continue in this phase as if no deviation occurred. If the opponent, j, deviates in any way, then go to Phase II^{a}_j, unless $j = A$ and $t > T-K-R-N-M$, in which case go to Phase III.

- Phase II^{b}_i:

  $\left(a^i, 0\right)$ for M periods, and then go back to Phase I.

  If the agent deviates from this phase, go to Phase II^{a}_A if $t \leq T-K-R-N-M$, or to Phase III otherwise. If the principal deviates, then go to Phase II^{a}_P.

- Phase III:

  $(a^*, 0)$ thereafter (do not respond to any deviation).

Choose N large enough so that the following inequalities are satisfied: $\overline{v}_i + N\underline{v}_i < (N+1)\, v^i_i$ for $i \in \{A, P\}$. Then, set $R = N$, and M such that $\overline{v}_i + N\underline{v}_i + Mv^i_i < N\underline{\underline{v}}_i + Mv^j_i + v_i$ for $i \in \{A, P\}$. Let K be large enough to satisfy $\overline{v}_A + (N+M+R+K-1)\, v^*_A < (N+M+R)\,\underline{\underline{v}}_i + K\underline{u}$. Finally, make $\underline{T}$ large enough so that $\frac{(\underline{T}-R-K)}{\underline{T}} v_P + \frac{R}{\underline{T}} g_P\left(\overline{a}^P\right) + \frac{K}{\underline{T}} v^*_P \in (v_P - \epsilon, v_P + \epsilon)$ and $\frac{(\underline{T}-R-K)}{\underline{T}} v_A + \frac{R}{\underline{T}} g_A\left(\overline{a}^P\right) + \frac{K}{\underline{T}}\underline{u} \in (v_A - \epsilon, v_A + \epsilon)$, and $\underline{\delta} < 1$

such that all the previous inequalities are still satisfied after introducing the appropriate discounting of payoffs.

Now we check that no player can gain by deviating from the previous strategy profile. The agent cannot gain by deviating in Phase I for $t = T - K + 1, ..., T$, since a NE is played. For $t = T - K - R - N - M + 1, ..., T - K$ no deviation is profitable, since $\overline{v}_A + \sum_{s=1}^{S+K} \delta^s v_A^* < \sum_{s=0}^{S} \delta^s v_A + \sum_{s=S+1}^{S+R} \delta^s g_A\left(\overline{a}^P\right) + \sum_{s=S+R+1}^{S+K+R} \delta^s \underline{u}$, for $S < N + M$, and $\overline{v}_A + \sum_{s=1}^{S+K} \delta^s v_A^* < \sum_{s=0}^{S} \delta^s g_A\left(\overline{a}^P\right) + \sum_{s=S+1}^{S+K} \delta^s \underline{u}$, for $S < R$, by the conditions on K and δ. And similarly the conditions on N and δ guarantee there is no profitable deviation at any $t = 1, ..., T - K - R - N - M$, since $\overline{v}_A + \sum_{s=1}^{N} \delta^s \underline{v}_A + \sum_{s=N+1}^{N+M} \delta^s v_A^A < \sum_{s=0}^{N+M} \delta^s v_A$. Next, notice we can only start Phase II_A^{a} at $t \leq T - K - R - N - M$, and hence, Phase II_A^{b} must start at $t \leq T - K - R - M$. Also, we can only move to Phase II_P^{a} for $t \leq T - K - R$, and to Phase II_P^{b} for $t \leq T - K - R - M$. In Phase II_A^{a} the agent is best-responding. And in Phase II_P^{a} no profitable deviation exists. If such deviation occurs at $t \leq T - K - R - N - M$, then $\overline{v}_A + \sum_{s=1}^{N} \delta^s \underline{v}_A + \sum_{s=N+1}^{N+M} \delta^s v_A^A < \sum_{s=0}^{S} \delta^s g_A\left(m^P\right) + \sum_{s=S+1}^{S+M} \delta^s v_A^P + \sum_{s=S+M+1}^{N+M} \delta^s v_A$ for $S < N$, from the definition of M. If the deviation occurs at $t > T - K - R - N - M$, then the condition on K implies that $\overline{v}_A + \sum_{s=1}^{S+K} \delta^s v_A^* < \sum_{s=0}^{S} \delta^s \underline{\underline{v}}_A + \sum_{s=S+1}^{S+K} \delta^s \underline{u}$, for $S < R + N + M$. Similarly, the conditions on N, K and δ also guarantee no deviation in Phases II_A^{b} and II_P^{b} are profitable. And in Phase III a NE is played.

The principal does not have scope for deviation in Phases II_P^{a} and III since he is best-responding. In Phase I, optimal behavior is followed for $t > T - K - R$. Moreover, the conditions on N together with the low discounting, ensure no deviation is profitable at earlier stages. Finally, the conditions on M, R and δ ensure no deviation is profitable in Phases II_A^{a}, II_A^{b} and II_P^{b} (the checks are similar to the ones for the agent).

The final case we need to consider has $\underline{u} < v_A^*$. Here, the previous strategies also work with a minor modification. Now, set $\left(\overline{a}^P, 0\right)$ at $t = T - K$ and $(a^*, 0)$ for the final K periods of Phase I, and $(a^*, 1)$ during Phase III, instead. Now, cooperation in the late periods is rewarded with continuation in the game, while a late deviation is punished with dismissal. The conditions on K, R, N, M, T and δ are analogous (reversing the roles of v_A^* and $\underline{u}$). This completes all the possible cases, and hence the proof. ■

Proof of Proposition 6. Let (a_t, f_t, q_t) represent the action profile, firing and quitting decisions taken at time t. Consider now the following strategy profile:

- Phase I:

  Play $(a, 0, 0)$ for $t = 1, ..., T-K-R$, then $\left(\overline{a}^P, 0, 0\right)$ for $t = T-K-R+1, ..., T-K-1$, $\left(\overline{a}^P, 0, 1\right)$ at $t = T-K$, and finally $(a^*, 0, 1)$ for $t = T-K+1, ..., T$

  After a unilateral deviation of the principal at $t = T-K-R+1, ..., T$ do nothing. Otherwise, after a unilateral deviation by player i at period t, go to Phase II_i.

- Phase II_A:

  The principal chooses $f_t = 1$, and then start period $t+1$ in Phase I with a new opponent. If the principal deviates and chooses $f_t = 0$ instead, do nothing (i.e., go back to Phase I as if he fired the agent).

- Phase II_P:

  $\left(m^P, 0, 0\right)$ for N_P periods, and then go back to Phase I.

  If the principal deviates in any way, continue in this phase as if no deviation occurred. If the agent deviates, then go to Phase II_A.

If no deviation has occurred, the agent is allowed to quit before the final K periods. Otherwise, she is dismissed. (In any case, the final K periods of the game never take place, here.) The difference in outside options, $\underline{u}(0) - \underline{u}(1) > 0$, will deter any deviation for K large enough. Checking this strategy profile is a SPE is analogous to the proof of Proposition 2. The conditions on the parameters are also very similar. ■