

STRATIFYING QUOTIENT STACKS AND MODULI STACKS

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ABSTRACT. Recent results in geometric invariant theory (GIT) for non-reductive linear algebraic group actions allow us to stratify quotient stacks of the form $[X/H]$, where X is a projective scheme and H is a linear algebraic group with internally graded unipotent radical acting linearly on X , in such a way that each stratum $[S/H]$ has a geometric quotient S/H . This leads to stratifications of moduli stacks (for example, sheaves over a projective scheme) such that each stratum has a coarse moduli space.

1. INTRODUCTION

Let $H = U \rtimes R$ be a linear algebraic group over an algebraically closed field of characteristic 0 with internally graded unipotent radical U ; that is, the Levi subgroup R of H has a central one-parameter subgroup (1-PS) $\lambda : \mathbb{G}_m \rightarrow R$ which acts on $\mathrm{Lie} U$ with all weights strictly positive. Of course any reductive group G has this form with both U and the central one-parameter subgroup λ being trivial. Suppose that H acts linearly on an irreducible projective scheme X with respect to an ample line bundle L over X . The aim of this paper is to describe a stratification of the quotient stack $[X/H]$ such that each stratum $[S/H]$ (where S is an H -invariant quasi-projective subscheme of X) has a geometric quotient S/H . When $H = R$ is reductive this stratification refines the Hesselink–Kempf–Kirwan–Ness (HKKN) stratification associated to the linear action on X (cf. [18, 22, 23, 27]). Potential applications of this construction include moduli stacks which can be filtered by quotient stacks with compatible linearisations; for example, it can be applied to moduli of sheaves of fixed Harder–Narasimhan type over a projective scheme [6], and moduli of unstable projective curves [21].

When H is reductive Mumford’s geometric invariant theory (GIT) [26] allows us to find open subschemes $X^s \subseteq X^{ss}$ of X , the stable and semistable loci, such that X^s has a geometric quotient X^s/H and X^{ss} has a good quotient

$$\phi : X^{ss} \rightarrow X//H = \mathrm{Proj} \left(\bigoplus_{m \geq 0} H^0(X, L^{\otimes m})^H \right).$$

Here $X^s = X^{s,H} = X^{s,H,\mathcal{L}}$ and $X^{ss} = X^{ss,H} = X^{ss,H,\mathcal{L}}$ depend on the choice of linearisation \mathcal{L} (that is, the ample line bundle L and the lift of the action of H to an action on L) and the GIT quotient $X//H = X//_{\mathcal{L}}H$ is a projective scheme with X^s/H as an open subscheme. Moreover when $x, y \in X^{ss}$ then $\phi(x) = \phi(y)$ if and only if the closures of the H -orbits of x and y meet in X^{ss} . The Hilbert–Mumford criteria allow us to determine the stable and semistable loci in a simple way without needing to understand the algebra of invariants $\bigoplus_{m \geq 0} H^0(X, L^{\otimes m})^H$.

The best situation occurs when $X^{ss} = X^s \neq \emptyset$; then $X//H = X^s/H$ is both a projective scheme and a geometric quotient of the open subscheme X^s of X . More generally if $X^s \neq \emptyset$

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then the projective completion $X//H$ of the geometric quotient X^s/H has a ‘partial desingularisation’ $\tilde{X}//H = \tilde{X}^{ss}/H$ where $\psi : \tilde{X}^{ss} \rightarrow X^{ss}$ is obtained as follows [24]. If $X^{ss} \neq X^s$ then there exists $x \in X^{ss}$ whose stabiliser in H is reductive of dimension strictly bigger than 0. To construct \tilde{X}^{ss} we first blow up X^{ss} along its closed subscheme where the stabiliser has maximal dimension in X^{ss} (such stabilisers are always reductive) or equivalently blow up X along the closure of this subscheme in X . We then remove the complement of the semistable locus for a small ample perturbation of the pullback linearisation. The maximal dimension of a stabiliser in this new semistable locus is strictly less than in X^{ss} . When $X^s \neq \emptyset$, repeating this process finitely many times leads to $\tilde{X}^{ss} = \tilde{X}^s \neq \emptyset$ and the partial desingularisation $\tilde{X}//H = \tilde{X}^s/H$.

When H is non-reductive, then the graded algebra $\bigoplus_{m \geq 0} H^0(X, L^{\otimes m})^H$ is not necessarily finitely generated and in general the attractive properties of Mumford’s GIT fail [2]. However when the unipotent radical U of $H = U \rtimes R$ is graded in the sense described above by a central 1-PS $\lambda : \mathbb{G}_m \rightarrow R$ of the Levi subgroup R , then after twisting the linearisation by an appropriate rational character, so that it becomes ‘graded’ itself in the sense of [5], some of the desirable properties of classical GIT still hold [3, 4]. More precisely, we first quotient by the linear action of the graded unipotent group $\hat{U} := U \rtimes \lambda(\mathbb{G}_m)$ using the results of [3, 4] described in §2.3, then we quotient by the residual action of the reductive group $H/\hat{U} \cong R/\lambda(\mathbb{G}_m)$. In the best case, when the U -action is free on a certain open subscheme X_{\min}^0 of X (cf. the condition $(*)$ in Definition 2.11 and Theorem 2.13), one can construct a geometric quotient of an open subscheme of ‘stable points’ for the \hat{U} -action such that the quotient is projective and this stable set has a Hilbert–Mumford type description. If the U -action has positive dimensional stabilisers generically, one can conclude the same results if we assume a weaker condition $(***)$ (cf. Theorem 2.17). Even when this weaker condition fails, one can perform an iterated sequence of blow-ups of X along H -invariant subschemes to obtain $\psi : \tilde{X} \rightarrow X$ such that \tilde{X} has an induced linear H -action satisfying $(***)$. Hence, there is a projective and geometric \hat{U} -quotient of an open subscheme of \tilde{X} that contains as an open subscheme a geometric \hat{U} -quotient of an open subscheme of X , as ψ is an isomorphism away from the exceptional divisor.

Now suppose that G is a reductive group acting linearly on a projective scheme X with respect to an ample line bundle L . Associated to this linear G -action and an invariant inner product on $\text{Lie } G$, there is a stratification (the ‘HKKN stratification’ cf. [18, 22, 23, 27])

$$X = \bigsqcup_{\beta \in \mathcal{B}} S_\beta$$

of X by locally closed subschemes S_β , indexed by a partially ordered finite set \mathcal{B} , such that

- (1) if $X^{ss} \neq \emptyset$, then \mathcal{B} has a minimal element 0 such that $S_0 = X^{ss}$,
- (2) for $\beta \in \mathcal{B}$, the closure of S_β is contained in $\bigcup_{\beta' \geq \beta} S_{\beta'}$, and
- (3) for $\beta \in \mathcal{B}$, we have $S_\beta \cong G \times^{P_\beta} Y_\beta^{ss}$, where $G \times^{P_\beta} Y_\beta^{ss}$ is the quotient of $G \times Y_\beta^{ss}$ by the diagonal action of a parabolic subgroup P_β of G acting on the right on G and on the left on a P_β -invariant locally closed subscheme Y_β^{ss} of X .

In fact, Y_β^{ss} is an open subscheme of a projective subscheme \bar{Y}_β of X that is determined by the action of a Levi subgroup L_β of P_β with respect to the restriction of the G -linearisation $L \rightarrow X$ to the P_β -action on \bar{Y}_β twisted by a rational character χ_β of P_β . The index β determines a central (rational) 1-PS $\lambda_\beta : \mathbb{G}_m \rightarrow L_\beta$ and χ_β is the corresponding rational character, where the choice of invariant inner product allows us to identify characters and co-characters of a fixed

maximal torus (cf. Remark 2.2). Furthermore P_β is the parabolic subgroup $P(\lambda_\beta)$ determined by the 1-PS λ_β , which grades the unipotent radical U_β of P_β . Thus by Property (3) above, to construct a G -quotient of (an open subset of) an unstable stratum S_β , we can study the linear P_β -action on \bar{Y}_β and apply the results described above for the action of $\hat{U} := U_\beta \rtimes \lambda_\beta(\mathbb{G}_m)$.

The G -action on the stratum S_β has a categorical quotient $Z_\beta // L_\beta$ induced by the morphism

$$p_\beta : Y_\beta^{ss} \rightarrow Z_\beta^{ss} = Z_\beta^{ss, L_\beta / \lambda_\beta(\mathbb{G}_m)} \quad y \mapsto p_\beta(y) := \lim_{t \rightarrow 0} \lambda_\beta(t)y,$$

where Z_β is the union of those connected components of the fixed point set for the action of $\lambda_\beta(\mathbb{G}_m)$ on X over which λ_β acts on the fibres of L with weight given by the restriction of β . However this categorical quotient is in general far from being a geometric quotient; it identifies y with $p_\beta(y)$, which lies in the orbit closure but typically not the orbit of y .

In this article we will show that, applying the blow-up sequence needed to construct a quotient by an action of a linear algebraic group with internally graded unipotent radical to the P_β -action on the projective subscheme \bar{Y}_β of X , we can refine the stratification $\{S_\beta | \beta \in \mathcal{B}\}$ to obtain a stratification of X such that each stratum is a G -invariant quasi-projective subscheme of X with a geometric quotient by the action of G . This refined stratification is a further refinement of the construction described in [25]. The quotient stack $[X/G]$ has an induced stratification $\{\Sigma_\gamma | \gamma \in \Gamma\}$ such that each stratum Σ_γ has the form

$$\Sigma_\gamma \cong [W_\gamma / H_\gamma]$$

where W_γ is a quasi-projective subscheme of X acted on by a linear algebraic subgroup H_γ of G with internally graded unipotent radical, and this action has a geometric quotient W_γ / H_γ . Moreover under appropriate hypotheses (involving condition $(***)$) for the actions of the subgroups H_γ , the geometric quotients W_γ / H_γ are themselves projective.

This will follow from the following theorem, which is proved in §3.

Theorem 1.1. *Let $H = U \rtimes R$ be a linear algebraic group with internally graded unipotent radical U acting on a projective scheme X over an algebraically closed field \mathbb{k} of characteristic 0 with respect to an ample linearisation and fix an invariant inner product on $\text{Lie } R$. Then X has a stratification $\{\mathcal{S}_\gamma | \gamma \in \Gamma\}$ induced by the linearisation \mathcal{L} and grading $\lambda : \mathbb{G}_m \rightarrow R$ for the action of H on X , such that the following properties hold.*

(i) *The index set Γ is finite and partially ordered such that for all $\gamma \in \Gamma$, we have*

$$\overline{\mathcal{S}_\gamma} \subseteq \mathcal{S}_\gamma \cup \bigcup_{\delta \in \Gamma, \delta > \gamma} \mathcal{S}_\delta.$$

- (ii) *Each \mathcal{S}_γ is an H -invariant quasi-projective subscheme of X with a geometric quotient \mathcal{S}_γ / H .*
- (iii) *If Y is an H -invariant projective subscheme of X then the stratification $\{\mathcal{S}_\gamma^Y | \gamma \in \Gamma^Y\}$ of Y induced by the restriction $\mathcal{L}|_Y$ of the linearisation \mathcal{L} to Y is (up to taking connected components) the restriction to Y of the stratification $\{\mathcal{S}_\gamma | \gamma \in \Gamma\}$ of X , so that there is a map of indexing sets $\phi_Y : \Gamma^Y \rightarrow \Gamma$ such that if $\gamma \in \Gamma^Y$ then \mathcal{S}_γ^Y is a connected component of $\mathcal{S}_{\phi_Y(\gamma)} \cap Y$;*

Moreover, if $H = G$ is reductive, then this stratification satisfies the following additional properties.

- (iv) *The stratification $\{\mathcal{S}_\gamma | \gamma \in \Gamma\}$ is a refinement of the HKKN stratification $\{S_\beta | \beta \in \mathcal{B}\}$ for the linearisation \mathcal{L} (cf. §2.1.1).*
- (v) *If $\beta \in \mathcal{B}$ (which we recall determines a 1-PS $\lambda_\beta : \mathbb{G}_m \rightarrow P_\beta \leq G$) satisfies*

$$(\dagger) \quad x \in Z_\beta^{ss} \quad \Rightarrow \quad \dim(\text{Stab}_G(x) / \lambda_\beta(\mathbb{G}_m)) = 0,$$

then the connected components of GZ_β^{ss} and $S_\beta \setminus GZ_\beta^{ss}$ (if these are nonempty) are strata in the refined stratification $\{\mathcal{S}_\gamma | \gamma \in \Gamma\}$.

As a consequence, we obtain a stratification of the quotient stack $[X/H]$ by locally closed substacks, each of which admits a coarse moduli space (cf. Corollary 3.5). Inspired by the reductive GIT notion of a good quotient, Alper introduces a notion of a good moduli space for a stack [1]. However, in general the strata appearing in this stratification of $[X/H]$ will not admit good moduli spaces, because a necessary condition for a stack to admit a good moduli space is that its closed points have reductive stabiliser groups (cf. [1, Proposition 12.14]). In general (even when $H = G$ is reductive) the points in the strata of $[X/H]$ will have non-reductive stabiliser groups.

If $H = G$ is reductive, then a stacky version of the HKKN stratification has been studied by Halpern-Leister, and by abstracting this concept he obtains a notion of a Θ -stratification [16]. Indeed, the linearisation of the G -action on X and the choice of invariant inner product is precisely the data required to construct a Θ -stratification of $[X/G]$, and this Θ -stratification is the stratification $\{[\mathcal{S}_\beta/G] : \beta \in \mathcal{B}\}$ obtained from the HKKN stratification of X . The stratification described above thus refines this Θ -stratification without depending on any additional data.

Since the construction of the refined stratification involves studying the blow-up procedures used in partial desingularisations of reductive GIT quotients [24] and for constructing geometric quotients by linear algebraic groups with internally graded unipotent radical [4], one can ask how this compares with the stack-theoretic blow-up constructions. The ideas in [24] have been generalised to stacks by Edidin and Rydh [15] to show that for a smooth Artin stack \mathfrak{X} admitting a stable good moduli space, there is a sequence of birational morphisms of smooth Artin stacks $\mathfrak{X}_n \rightarrow \cdots \mathfrak{X}_1 \rightarrow \mathfrak{X}$ such that the good moduli space of \mathfrak{X}_n is an algebraic space with only tame quotient singularities and is a partial desingularisation of the good moduli space of \mathfrak{X} . However, for H non-reductive, it is often the case that $[X/H]$ will not be a good moduli space, and so one cannot apply this result.

The picture provided by Theorem 1.1 has potential applications to moduli stacks which are filtered by quotient stacks, and to the construction of moduli spaces of ‘unstable’ objects (for example, moduli of sheaves over a projective scheme [6] or moduli of projective curves [21]). Suppose that \mathcal{M} is a moduli stack which can be expressed as an increasing union

$$\mathcal{M} = \bigcup_{n \geq 0} \mathcal{U}_n$$

of open substacks of the form

$$\mathcal{U}_n \cong [V_n/G_n]$$

where $[V_n/G_n]$ is the quotient stack associated to a linear action on a quasi-projective scheme V_n by a group G_n which is reductive (or more generally has internally graded unipotent radical). We can look for suitable ‘stability conditions’ on \mathcal{M} : linearisations $(\mathcal{L}_n)_{n \geq 0}$ for the actions of G_n on projective completions \overline{V}_n of V_n and invariant inner products on $\text{Lie } G_n$ which are compatible in the sense that the stratification induced by \mathcal{L}_n on $[V_n/G_n]$ restricts to the stratification induced by \mathcal{L}_m on $[V_m/G_m]$ when $n > m$.

This situation arises for sheaves over a projective scheme [20, 6], for example, and also for projective curves [21], and we obtain a stratification $\{\Sigma_\gamma | \gamma \in \Gamma\}$ of the stack \mathcal{M} such that each stratum Σ_γ is isomorphic to a quotient stack $[W_\gamma/H_\gamma]$, where W_γ is quasi-projective acted on by a linear algebraic group H_γ with internally graded unipotent radical, and there is a geometric

quotient W_γ/H_γ which is a coarse moduli space for Σ_γ . The geometric quotient W_γ/H_γ will be projective if semistability coincides with stability in an appropriate sense for the action of H_γ on a suitable projective completion of W_γ with respect to an induced linearisation.

The layout of this article is as follows. In §2, we will review classical and non-reductive GIT, describing how to construct quotients by actions of linear algebraic groups with internally graded unipotent radical. The heart of the paper is §3, in which we describe how to stratify a quotient stack $[X/H]$ into strata $\Sigma_\gamma = [W_\gamma/H_\gamma]$ where the action of H_γ on W_γ has a geometric quotient W_γ/H_γ . The argument is an inductive one, so the assumption on X and H is that H is a linear algebraic group with internally graded unipotent radical, and X is a projective scheme which has an amply linearised action of H . In §4, this construction is applied to stacks which are suitably filtered by quotient stacks, and §5 contains a brief discussion of examples including moduli of unstable curves and moduli of sheaves of given Harder–Narasimhan type over a fixed projective scheme.

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2. CLASSICAL AND NON-REDUCTIVE GIT

2.1. Classical GIT for reductive groups. In Mumford’s GIT [26], a linearisation of an action of a reductive group G on a projective scheme X over an algebraically closed field \mathbb{k} of characteristic 0 is given by a line bundle L (which we will always assume to be ample) on X and a lift of the action to L . Since G is reductive, the algebra of G -invariant sections $\widehat{\mathcal{O}}_L(X)^G$ is finitely generated as a graded algebra with associated projective scheme $X//G = \text{Proj}(\widehat{\mathcal{O}}_L(X)^G)$.

$$\begin{array}{ccc} (X, L) & \rightsquigarrow & \widehat{\mathcal{O}}_L(X) := \bigoplus_{k=0}^{\infty} H^0(X, L^{\otimes k}) \\ \downarrow & & \cup \\ X//G & \rightsquigarrow & \widehat{\mathcal{O}}_L(X)^G \quad \text{algebra of invariants.} \end{array}$$

The inclusion of $\widehat{\mathcal{O}}_L(X)^G$ in $\widehat{\mathcal{O}}_L(X)$ determines a rational map $X \dashrightarrow X//G$ which fits into a diagram

$$\begin{array}{ccc} X & \dashrightarrow & X//G \quad \text{projective} \\ \cup & & \parallel \\ \text{semistable } X^{ss} & \longrightarrow & X//G \\ \cup & & \cup \quad \text{open} \\ \text{stable } X^s & \longrightarrow & X^s/G \end{array}$$

where X^s (the stable locus) and X^{ss} (the semistable locus) are open subschemes of X , there is a geometric quotient X^s/G for the action of G on X^s , the GIT quotient $X//G$ is a good quotient for the action of G on X^{ss} via the G -invariant surjective morphism $\phi : X^{ss} \rightarrow X//G$, and

$$\phi(x) = \phi(y) \Leftrightarrow \overline{Gx} \cap \overline{Gy} \cap X^{ss} \neq \emptyset.$$

The semistable and stable loci X^{ss} and X^s of X are characterised by the following properties ([26, Chapter 2], [28]).

Proposition 2.1 (Hilbert–Mumford criteria for reductive groups). *Let T be a maximal torus of G .*

- (1) *A point $x \in X$ is semistable (respectively stable) for the G -action on X if and only if for every $g \in G$ the point gx is semistable (respectively stable) for the T -action.*

- (2) A point $x \in X \subset \mathbb{P}^n$ with homogeneous coordinates $[x_0 : \dots : x_n]$ is semistable (respectively stable) for a diagonal T -action on \mathbb{P}^n with weights $\alpha_0, \dots, \alpha_n$ if and only if

$$0 \in \text{Conv}\{\alpha_i : x_i \neq 0\}$$

(respectively 0 is contained in the interior of this convex hull).

2.1.1. *The HKKN stratification.* Associated to the linear action of G on X and an invariant inner product on the Lie algebra of G , there is a stratification (the ‘HKKN stratification’, which in the case $\mathbb{k} = \mathbb{C}$ is the Morse stratification for the norm-square of an associated moment map [18, 22, 23, 27]).

Remark 2.2. Let us clarify what is meant by this invariant inner product, whose associated norm we denote by $\| - \|$. If $\mathbb{k} = \mathbb{C}$, then G is the complexification of its maximal compact group K ; then the Lie algebra of K is a real vector space, and we choose an inner product on this Lie algebra that is invariant under the adjoint action of K . In fact, we will also assume that we fix a maximal compact torus $T_c \subset K$ such that the inner product is integral on the co-character lattice $X_*(T_c) \subset \text{Lie } K$. For an arbitrary algebraically closed field \mathbb{k} of characteristic zero, one can fix a maximal torus T of G and choose an inner product on the co-character space $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ that is invariant for the Weyl group of T and is integral on the co-character lattice (for example, see [19, §2]). Then this inner product gives an identification between characters and co-characters (i.e. 1-PSs) of T .

The HKKN stratification associated to the action of G on X with respect to L and the norm $\| - \|$ is a stratification

$$X = \bigsqcup_{\beta \in \mathcal{B}} S_{\beta}$$

of X by locally closed subschemes S_{β} , indexed by a partially ordered finite subset \mathcal{B} of rational elements in a positive Weyl chamber for the reductive group G , with the following properties.

- (1) If $0 \in \mathcal{B}$, then this is the minimal element and $S_0 = X^{ss}$.

Moreover, for each $\beta \in \mathcal{B}$, we additionally have the following properties:

- (2) the closure of S_{β} is contained in $\bigcup_{\beta' \geq \beta} S_{\beta'}$ where $\gamma \geq \beta$ if and only if $\gamma = \beta$ or $\|\gamma\| > \|\beta\|$;
(3) $S_{\beta} \cong G \times^{P_{\beta}} Y_{\beta}^{ss} := (G \times Y_{\beta}^{ss}) / P_{\beta}$ where this quotient is of the diagonal action of P_{β} on the right on G and on the left on Y_{β}^{ss} .

Here P_{β} is a parabolic subgroup of G which acts on a locally closed subscheme Y_{β}^{ss} of X .

More precisely, $\beta \in \mathcal{B}$ determines a (rational) 1-PS $\lambda_{\beta} : \mathbb{G}_m \rightarrow G$ and an associated parabolic subgroup $P_{\beta} = P(\lambda_{\beta}) = U_{\beta} \rtimes L_{\beta}$ with Levi subgroup $L_{\beta} = \text{Stab}_G(\beta)$ such that the conjugation action of $\lambda_{\beta}(\mathbb{G}_m)$ on $\text{Lie } U_{\beta}$ has strictly positive weights; thus P_{β} has internally graded unipotent radical. Let Z_{β} be the union of components in the fixed locus $X^{\lambda_{\beta}(\mathbb{G}_m)}$ on which this 1-PS acts on the fibres of L with weight given by the restriction of β , and let $Y_{\beta} \subset X$ be the subscheme of points $x \in X$ such that $\lim_{t \rightarrow 0} \lambda_{\beta}(t)y \in Z_{\beta}$; thus there is a retraction

$$p_{\beta} : Y_{\beta} \rightarrow Z_{\beta} \quad y \mapsto p_{\beta}(y) := \lim_{t \rightarrow 0} \lambda_{\beta}(t)y$$

which is equivariant with respect to the quotient homomorphism $q_{\beta} : P_{\beta} \rightarrow L_{\beta}$ obtained by identifying L_{β} with P_{β}/U_{β} . Let \mathcal{L}_{β} denote the restriction of the G -linearisation \mathcal{L} on X to the P_{β} -action on \overline{Y}_{β} twisted by the (rational) character χ_{β} of P_{β} corresponding to the 1-PS λ_{β} (via the norm $\| - \|$). We also let \mathcal{L}_{β} denote the restriction of this linearisation to the L_{β} -action on

Z_β . Then Y_β^{ss} (respectively Z_β^{ss}) is the semistable locus for the P_β -action on \bar{Y}_β (respectively the $L_\beta/\lambda_\beta(\mathbb{G}_m)$ -action on Z_β) linearised by the twisted linearisation \mathcal{L}_β ; furthermore, $Y_\beta^{ss} = p_\beta^{-1}(Z_\beta^{ss})$.

Finally, we make the following observation about quotienting the unstable strata (cf. [20]).

Remark 2.3. The G -action on S_β has a categorical quotient $Z_\beta//L_\beta$ induced by the map $p_\beta : Y_\beta^{ss} \rightarrow Z_\beta^{ss}$. In general, this quotient is far from being a geometric quotient (even after restriction to the pre-image of any nonempty open subscheme of $Z_\beta//L_\beta$), as $y \in Y_\beta^{ss}$ is identified with $p_\beta(y) \in \overline{Gy}$.

By (3), constructing a quotient of the G -action on a G -invariant open subset of S_β is equivalent to constructing a P_β -quotient of a P_β -invariant open subset of Y_β^{ss} (or its closure); the latter perspective will lead to a geometric quotient by using GIT for the non-reductive group P_β , whose unipotent radical U_β is internally graded by λ_β (cf. §2.3).

2.1.2. Partial desingularisations of reductive GIT quotients. The geometric quotient X^s/G has at most orbifold singularities when X is nonsingular, since the stabiliser subgroups of stable points are finite subgroups of G . If $X^{ss} \neq X^s \neq \emptyset$, the singularities of $X//G$ are typically more severe even when X is itself nonsingular, but $X//G$ has a ‘partial desingularisation’ $\tilde{X}//G$ which is also a projective completion of X^s/G and is itself a geometric quotient

$$\tilde{X}//G = \tilde{X}^{ss}/G$$

by G of an open subscheme $\tilde{X}^{ss} = \tilde{X}^s$ of a G -equivariant blow-up \tilde{X} of X [24]. Here \tilde{X}^{ss} is obtained from X^{ss} by successively blowing up along the subschemes of semistable points stabilised by reductive subgroups of G of maximal dimension and removing the complement of the semistable locus from the resulting blow-up.

For the construction of the partial desingularisation $\tilde{X}//G$ in [24], it is assumed that $X^s \neq \emptyset$. There exist semistable points of X which are not stable if and only if there exists a non-trivial connected reductive subgroup of G fixing a semistable point. Let $r > 0$ be the maximal dimension of a reductive subgroup of G fixing a point of X^{ss} and let $\mathcal{R}(r)$ be a set of representatives of conjugacy classes of all connected reductive subgroups R of dimension r in G such that

$$Z_R^{ss} := \{x \in X^{ss} : Rx = x\}$$

is non-empty. Then

$$Z_{\mathcal{R}(r)}^{ss} := \bigcup_{R \in \mathcal{R}(r)} GZ_R^{ss}$$

is a disjoint union of closed G -invariant subschemes of X^{ss} . The action of G on X^{ss} lifts to an action on the blow-up $X_{(1)}$ of X^{ss} along $Z_{\mathcal{R}(r)}^{ss}$, and this action can be linearised so that the complement of $X_{(1)}^{ss}$ in $X_{(1)}$ is the proper transform of the closed subscheme $\pi^{-1}(\pi(GZ_{\mathcal{R}(r)}^{ss}))$ of X^{ss} where $\pi : X^{ss} \rightarrow X//G$ is the quotient map (see [24, 7.17]). The G -linearisation on $X_{(1)}$ used here is (a tensor power of) the pullback of the ample line bundle L on X along $\psi_{(1)} : X_{(1)} \rightarrow X$ perturbed by a sufficiently small multiple of the exceptional divisor $E_{(1)}$; then, if the perturbation is sufficiently small, we have

$$\psi_{(1)}^{-1}(X^s) \subseteq X_{(1)}^s \subseteq X_{(1)}^{ss} \subseteq \psi_{(1)}^{-1}(X^{ss}) = X_{(1)},$$

and the stable and semistable loci $X_{(1)}^s$ and $X_{(1)}^{ss}$ will be independent of the choice of perturbation. Moreover, no point $x \in X_{(1)}^{ss}$ is fixed by a reductive subgroup of G of dimension at least r , and $x \in X_{(1)}^{ss}$ is fixed by a reductive subgroup R of dimension less than r in G if and only if it belongs to the proper transform of the closed subscheme Z_R^{ss} of X^{ss} .

Remark 2.4. In [24], X itself is blown up along the closure $\overline{Z_{\mathcal{R}(r)}^{ss}}$ of $Z_{\mathcal{R}(r)}^{ss}$ in X (or in a projective completion of X^{ss} with a G -equivariant morphism to X which is an isomorphism over X^{ss}). This gives a projective scheme $\overline{X}_{(1)}$ and blow-down map $\overline{\psi}_{(1)} : \overline{X}_{(1)} \rightarrow X$ restricting to $\psi_{(1)} : X_{(1)} \rightarrow X$ where $(\overline{\psi}_{(1)})^{-1}(X^{ss}) = X_{(1)}$. For a sufficiently small perturbation of the pullback to $\overline{X}_{(1)}$ of the linearisation on X , we have $(\overline{\psi}_{(1)})^{-1}(X^s) \subseteq \overline{X}_{(1)}^s \subseteq \overline{X}_{(1)}^{ss} \subseteq (\overline{\psi}_{(1)})^{-1}(X^{ss}) = X_{(1)}$, and moreover the restriction of the linearisation to $X_{(1)}$ is obtained from the pullback of L by perturbing by a sufficiently small multiple of the exceptional divisor $E_{(1)}$.

If $r > 1$, we can apply the same procedure to $X_{(1)}^{ss}$ to obtain $X_{(2)}^{ss}$ such that no point of $X_{(2)}^{ss}$ is fixed by a reductive subgroup of G of dimension at least $r - 1$. If $X^s \neq \emptyset$ then repeating this process at most r times gives us $\psi : \tilde{X}^{ss} \rightarrow X^{ss}$ such that ψ is an isomorphism over X^s and no positive-dimensional reductive subgroup of G fixes a point of \tilde{X}^{ss} . The partial desingularisation $\tilde{X} // G = \tilde{X}^{ss} / G$ can be obtained by blowing up $X // G$ along the proper transforms of $\pi(GZ_R^{ss}) \subset X // G$ in decreasing order of the dimension of R .

Remark 2.5. Suppose for simplicity that X is irreducible.

- a) If $X^s \neq \emptyset$, then this is the situation considered in [24] and the partial desingularisation construction is described above. If $X^{ss} = X^s$, then $\tilde{X} = X$.
- b) If $X^{ss} = \emptyset$, then there is an unstable stratum S_β with $\beta \neq 0$ in the HKKN stratification (cf. §2.1.1) which is a non-empty open subscheme of X , and thus when X is irreducible $X = \overline{S_\beta}$. Then constructing a quotient of a non-empty open subscheme of X reduces to non-reductive GIT for the action of the parabolic subgroup P_β on $\overline{Y_\beta}$ as described in §2.3 below, where a blow-up sequence may also need to be performed.
- c) If $X^s = \emptyset \neq X^{ss}$ then the partial desingularisation construction can be applied to X^{ss} , and there are different ways in which it can terminate.
 - (i) If $X^{ss} = GZ_R^{ss} \cong G \times^{N_R} Z_R^{ss}$ for a positive-dimensional connected reductive subgroup R of G with normaliser N_R in G , then N_R and N_R/R are also reductive, and

$$X // G \cong Z_R // N_R \cong Z_R // (N_R/R)$$

where Z_R is the closed subscheme of X which is the fixed point set for the action of R . Then we can apply induction on the dimension of G to study this case.

- (ii) If $GZ_R^{ss} \neq X^{ss}$ for each positive-dimensional connected reductive subgroup R of G , then we can perform the first blow-up in the partial desingularisation construction to obtain $\psi_{(1)} : X_{(1)} \rightarrow X^{ss}$ such that $X_{(1)}^{ss} \subseteq X_{(1)}$ and $X_{(1)}^s = \emptyset$ as above (as $X_{(1)}^s$ is open and $X_{(1)}^s \setminus E_{(1)} \cong X^s = \emptyset$, where $E_{(1)}$ is the exceptional divisor). If $X_{(1)}^{ss} = \emptyset$, then $X_{(1)}$ has a dense open stratum $S_{(1),\beta}$ for $\beta \neq 0$ as in Case *b*). If we have $X_{(1)}^{ss} = GZ_{(1),R}^{ss}$ for a positive-dimensional connected reductive subgroup R of G , where $Z_{(1),R}^{ss} = \{x \in X_{(1)}^{ss} : Rx = x\}$, then we proceed as in Case *i*) above. Otherwise we can repeat the process, until it terminates in one of these two ways.

2.2. GIT for non-reductive groups. Now suppose that X is a projective scheme over an algebraically closed field \mathbb{k} of characteristic 0 and let H be a linear algebraic group, with unipotent radical U , acting on X with respect to an ample linearisation L .

Definition 2.6. (Semistability for the unipotent group cf. [14, §4] and [14, 5.3.7]). For an invariant section $f \in I = \bigcup_{m>0} H^0(X, L^{\otimes m})^U$, let X_f be the U -invariant affine open subset of X on which f does not vanish, and let $\mathcal{O}(X_f)$ denote its coordinate ring.

- (1) The *semistable locus* for the U -action on X linearised by L is $X^{ss,U} = \bigcup_{f \in I^{\text{fg}}} X_f$ where

$$I^{\text{fg}} = \{f \in I \mid \mathcal{O}(X_f)^U \text{ is finitely generated}\}.$$

- (2) The *(locally trivial) stable locus* for the linearised U -action on X is $X^{\text{lt},U} = \bigcup_{f \in I^{\text{lt}}} X_f$ where

$$I^{\text{lt}} := \{f \in I^{\text{fg}} \mid q_U : X_f \rightarrow \text{Spec}(\mathcal{O}(X_f)^U) \text{ is a locally trivial geometric quotient}\}.$$

- (3) The *enveloped quotient* of $X^{ss,U}$ by the linear U -action is $q_U : X^{ss,U} \rightarrow q_U(X^{ss,U})$, where $q_U : X^{ss,U} \rightarrow \text{Proj}(\widehat{\mathcal{O}}_L(X)^U)$ is the natural morphism of schemes and $q_U(X^{ss,U})$ is a dense constructible subset of the *enveloping quotient*

$$X \wr U = \bigcup_{f \in I^{\text{fg}}} \text{Spec}(\mathcal{O}(X_f)^U).$$

Remark 2.7.

- (1) Even when $\widehat{\mathcal{O}}_L(X)^U$ is finitely generated, so that $X \wr U = \text{Proj}(\widehat{\mathcal{O}}_L(X)^U)$, the enveloped quotient $q_U(X^{ss,U})$ is not necessarily a subscheme of $X \wr U$ (for example, see [14, §6]).
- (2) The enveloping quotient $X \wr U$ has quasi-projective open subschemes ('inner enveloping quotients' $X //_{\circ} U$) that contain the enveloped quotient $q_U(X^{ss})$ and have ample line bundles which under the natural map $q_U : X^{ss} \rightarrow X \wr U$ pull back to positive tensor powers of L (see [2] for details).

The H -semistable locus $X^{ss} = X^{ss,H}$ and enveloped and (inner) enveloping quotients

$$q_H : X^{ss} \rightarrow q_H(X^{ss}) \subseteq X //_{\circ} H \subseteq X \wr H$$

for the linear action of H are defined as for the unipotent case in Definition 2.6 and Remark 2.7 (cf. [2]), but the definition of the stable locus $X^{\text{lt}} = X^{\text{lt},H}$ for the linear action of H combines (and extends) the definitions for unipotent and reductive groups.

Definition 2.8. (Stability for linear algebraic groups). Let H be a linear algebraic group acting linearly on X with respect to an ample line bundle L ; then the *(locally trivial) stable locus* is the open subscheme $X^{\text{lt}} = \bigcup_{f \in I^{\text{lt}}} X_f$ of X^{ss} , where $I^{\text{lt}} \subseteq \bigcup_{r>0} H^0(X, L^{\otimes r})^H$ is the subset of H -invariant sections f satisfying the following conditions:

- (1) the H -invariant open subscheme X_f is affine;
- (2) the H -action on X_f is closed with finite stabiliser groups; and
- (3) the restriction of the U -enveloping quotient map

$$q_U : X_f \rightarrow \text{Spec}((\widehat{\mathcal{O}}_L(X)^U)_{(f)})$$

is a locally trivial geometric quotient for the U -action on X_f .

2.3. GIT for linear algebraic groups with internally graded unipotent radicals. Now suppose that $H = U \rtimes R$ is a linear algebraic group with internally graded unipotent radical U in the following sense.

Definition 2.9. We say $H = U \rtimes R$ has *internally graded unipotent radical* U if there is a central 1-PS $\lambda : \mathbb{G}_m \rightarrow R$ whose conjugation action on the Lie algebra of U has strictly positive weights. We write $\widehat{U} = U \rtimes \lambda(\mathbb{G}_m) \leq H$ for the associated semi-direct product.

Suppose also that H acts linearly on X with respect to an ample line bundle L . It is shown in [3, 4] that the algebra of H -invariant sections is finitely generated provided:

- a) L is replaced with a suitable tensor power $L^{\otimes m}$, with $m \geq 1$ sufficiently divisible, and the linearisation of the action of H is twisted by a suitable (rational) character, and
- b) condition $(*)$ described below (also known as ‘semistability coincides with stability for the unipotent radical U ’) holds.

Moreover, in this situation the natural quotient morphism q_H from the semistable locus $X^{ss,H}$ to the enveloping quotient $X \mathbin{\mathbb{H}} H$ is surjective, and expresses the projective scheme $X // H = X //_{\circ} H = X \mathbin{\mathbb{H}} H$ as a good quotient of $X^{ss,H}$. Furthermore this locus $X^{ss,H}$ can be described using Hilbert–Mumford criteria. It is also shown in [4] that when condition $(*)$ is not satisfied, but is replaced with a slightly weaker condition, such as $(**)$ below, then there is a sequence of blow-ups of X along H -invariant subschemes (similar to that of [24] when H is reductive) resulting in a projective scheme \widehat{X} with an induced linear action of H satisfying condition $(*)$. In fact, these results can be generalised to allow actions where the U -action has positive dimensional stabilisers (cf. Theorems 2.17 and 2.18 below). Before giving a precise description of the condition $(*)$ and its variants, we define the notion of an adapted linearisation, which is also needed for the statement of the main results of [3, 4].

Let $\chi : H \rightarrow \mathbb{G}_m$ be a character of H ; the restriction to \widehat{U} of χ contains U in its kernel and can be identified naturally with an integer so that the integer 1 corresponds to the character of \widehat{U} which fits into the exact sequence $U \hookrightarrow \widehat{U} \rightarrow \lambda(\mathbb{G}_m)$. By replacing L with a sufficiently high power, we can without loss of generality assume that L is very ample. Let $\omega_{\min} := \omega_0 < \omega_1 < \dots < \omega_{\max}$ be the weights with which the 1-PS $\lambda : \mathbb{G}_m \rightarrow \widehat{U} \leq \widehat{H}$ acts on the fibres of the tautological line bundle $\mathcal{O}_{\mathbb{P}((H^0(X,L)^*)}(-1)$ over points of the fixed locus $\mathbb{P}((H^0(X,L)^*)^{\lambda(\mathbb{G}_m)})$. Without loss of generality we may assume that there exist at least two distinct such weights, as otherwise the U -action on X is trivial, in which case we can take a quotient by the action of the reductive group $R = H/U$.

Definition 2.10. For a character χ of H as above and a positive integer c , we say the rational character χ/c is *adapted* to the linear action of \widehat{U} if

$$(2.1) \quad \omega_{\min} := \omega_0 < \frac{\chi}{c} < \omega_1.$$

Furthermore, we say L is adapted to the \widehat{U} -action if $\omega_{\min} := \omega_0 < 0 < \omega_1$.

If the rational character χ/c is adapted to the linear action of \widehat{U} , then the H -linearisation $\mathcal{L}_\chi^{\otimes c}$ on X given by twisting the ample line bundle $L^{\otimes c}$ by the character χ (that is, so that the weights ω_j are replaced with $\omega_j c - \chi$) is adapted. Let $X_{\min+}^{s,\mathbb{G}_m} = X_{\min+}^{s,\lambda(\mathbb{G}_m)}$ denote the stable set in X for the linear action of \mathbb{G}_m via λ with respect to the adapted linearisation $\mathcal{L}_\chi^{\otimes c}$ and, for a maximal torus T of H containing $\lambda(\mathbb{G}_m)$, let $X_{\min+}^{s,T}$ denote the stable set in X for the linear action of T with

respect to the adapted linearisation $L_X^{\otimes c}$. By the theory of variation of (classical) GIT [13, 34], the stable set $X_{\min+}^{s, \lambda(\mathbb{G}_m)}$ is independent of the choice of adapted rational character χ/c . In fact, by the Hilbert–Mumford criterion, we have $X_{\min+}^{s, \lambda(\mathbb{G}_m)} = X_{\min}^0 \setminus Z_{\min}$, where if V_{\min} denotes the weight space of ω_{\min} in $V = H^0(X, L)^*$, then

$$Z_{\min} := X \cap \mathbb{P}(V_{\min}) = \left\{ x \in X^{\lambda(\mathbb{G}_m)} : \lambda(\mathbb{G}_m) \text{ acts on } L^*|_x \text{ with weight } \omega_{\min} \right\}$$

and

$$X_{\min}^0 := \{x \in X \mid \lim_{t \rightarrow 0, t \in \mathbb{G}_m} \lambda(t) \cdot x \in Z_{\min}\}.$$

Definition 2.11. (Conditions $(*)$ – $(***)$) generalising ‘semistability equals stability’ cf. [4]) With the above notation, we define the following conditions for the \widehat{U} -action on X .

$$(*) \quad \text{Stab}_U(z) = \{e\} \text{ for every } z \in Z_{\min}.$$

$$(**) \quad \text{Stab}_U(x) = \{e\} \text{ for generic } x \in X_{\min}^0.$$

Moreover, if $U \geq U^{(1)} \geq \dots \geq U^{(s)} \geq \{e\}$ denotes the derived series of U , we define condition

$$(***) \quad \text{for } 1 \leq j \leq s, \text{ there exists } d_j \in \mathbb{N} \text{ such that } \dim \text{Stab}_{U^{(j)}}(x) = d_j \text{ for all } x \in X_{\min}^0.$$

Note that condition $(*)$ holds if and only if we have $\text{Stab}_U(x) = \{e\}$ for all $x \in X_{\min}^0$. This condition is also referred to in [4] as the condition ‘semistability coincides with stability’ for the action of \widehat{U} (or, when the 1-PS $\lambda : \mathbb{G}_m \rightarrow R$ is fixed, for the linear action of U).

Definition 2.12. Let $T \leq R$ be a maximal torus containing $\lambda(\mathbb{G}_m)$. The *minimal stable set* for a linear H -action for which condition $(*)$ holds for the action of the graded unipotent group \widehat{U} is

$$X_{\min+}^{s, H} := \bigcap_{h \in H} h X_{\min+}^{s, T}.$$

We note that the minimal stable set for the \widehat{U} -action satisfies

$$X_{\min+}^{s, \widehat{U}} = \bigcap_{u \in U} u X_{\min+}^{s, \lambda(\mathbb{G}_m)} = X_{\min}^0 \setminus U Z_{\min}.$$

Theorem 2.13. (*\widehat{U} -Theorem when semistability coincides with stability for \widehat{U}*) [4] Suppose that the linearisation for the action of H on X is adapted as in Definition 2.10 and that the \widehat{U} -action on X satisfies condition $(*)$. Then the following statements hold.

- (i) The open subscheme $X_{\min+}^{s, \widehat{U}}$ of X has a projective geometric quotient $X//\widehat{U} = X_{\min+}^{s, \widehat{U}}/\widehat{U}$ by \widehat{U} .
- (ii) The open subscheme $X_{\min+}^{s, H}$ of X has a good quotient $X//H = (X//\widehat{U})//R$ by $H = U \rtimes R$, which is also projective.

Remark 2.14. In the proof of Theorem 2.13 (and its variants below), one replaces the adapted linearisation by a ‘well adapted’ linearisation (which can be achieved by twisting by a rational character); this is a slightly stronger notion. This strengthening does not alter $X_{\min+}^{s, \widehat{U}}$ or its quotient $X//\widehat{U}$, but it affects what can be said about induced ample line bundles on

$X//\widehat{U}$ and $X//H = (X//\widehat{U})/(R/\lambda(\mathbb{G}_m))$. The proofs in [3, 4] that the algebras of invariants $\bigoplus_{m \geq 0} H^0(X, L_{m\chi}^{\otimes cm})^{\widehat{U}}$ and

$$\bigoplus_{m \geq 0} H^0(X, L_{m\chi}^{\otimes cm})^H = \left(\bigoplus_{m \geq 0} H^0(X, L_{m\chi}^{\otimes cm})^{\widehat{U}} \right)^{(R/\lambda(\mathbb{G}_m))}$$

are finitely generated, and that the enveloping quotients $X//\widehat{U} = X_{\min}^{s, \widehat{U}}/\widehat{U}$ and $X//H$ are the associated projective schemes, require that the linearisation is twisted by a well adapted rational character χ/c . More precisely, it is shown in [3, 4] that, given a linear action of H on X with respect to an ample line bundle L , there exists $\epsilon > 0$ such that if χ/c is a rational character of \mathbb{G}_m (lifting to H) with c sufficiently divisible and $\chi : H \rightarrow \mathbb{G}_m$ a character of H such that

$$\omega_{\min} < \frac{\chi}{c} < \omega_{\min} + \epsilon,$$

then the algebras of invariants $\bigoplus_{m \geq 0} H^0(X, L_{m\chi}^{\otimes cm})^{\widehat{U}}$ and $\bigoplus_{m \geq 0} H^0(X, L_{m\chi}^{\otimes cm})^H$ are finitely generated, and the associated projective schemes $X//\widehat{U}$ and $X//H$ satisfy the conclusions of Theorem 2.13.

Theorem 2.13 describes the good case when semistability coincides with stability for the linear action of \widehat{U} . The following versions proved in [4] apply more generally.

Theorem 2.15. (\widehat{U} -Theorem giving projective completions) [4] Suppose that the linear action of \widehat{U} on X is adapted and satisfies condition (**). Then there exists a sequence of blow-ups along H -invariant projective subschemes, resulting in a projective scheme \widetilde{X} (with blow-down map $\widetilde{\psi} : \widetilde{X} \rightarrow X$) such that the conditions of Theorem 2.13 still hold for a suitable ample linearisation, and such that the quotient given by that theorem is a geometric quotient of an open subscheme $\widetilde{X}^{s, H}$ of \widetilde{X} .

Remark 2.16. This blow-up sequence is constructed in two stages: one first blows up by considering the stabiliser subgroups for the unipotent group U , and then one blows up by considering the stabiliser subgroups for the reductive group R .

In the first step, one performs a blow-up sequence to obtain $\widehat{\psi} : \widehat{X} \rightarrow X$ such that the \widehat{U} -action on \widehat{X} satisfies condition (*) with respect to a linearisation $\widehat{\mathcal{L}}$, which is an arbitrarily small perturbation of $\widehat{\psi}^*(\mathcal{L})$. The centres of the blow-ups used to obtain \widehat{X} from X are determined by the dimensions of the stabilisers in U of the limits $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ for $x \in X_{\min}^0$ (for details, see [4]). Then one can construct a projective and geometric quotient of the \widehat{U} -action on $\widehat{X}^{s, \widehat{U}}$ by Theorem 2.13 and, as $\widehat{\psi}$ is an isomorphism away from the exceptional divisor, one obtains a geometric quotient of a \widehat{U} -invariant open subset $X^{s, \widehat{U}}$ of X as an open subscheme of $\widehat{X} //_{\widehat{\mathcal{L}}} \widehat{U}$, where $X^{s, \widehat{U}}$ is the image under $\widehat{\psi}$ of the intersection of $\widehat{X}^{s, \widehat{U}}$ with the complement of the exceptional divisor in \widehat{X} . Another characterisation of this stable locus is as

$$X^{s, \widehat{U}} = \{x \in \psi(\widehat{X}^{s, \widehat{U}}) \mid \dim \text{Stab}_U(\lim_{t \rightarrow 0} \lambda(t) \cdot x) = 0\}.$$

If one is only interested in obtaining a good quotient for the H -action, then the second stage of the blow-up procedure is not needed: one can then take a reductive GIT quotient of the residual action on $\widehat{X} //_{\widehat{\mathcal{L}}} \widehat{U}$ of $H/\widehat{U} = R/\lambda(\mathbb{G}_m)$, and thus one obtains a good quotient of the H -action on an open subset of X as an open subscheme of $(\widehat{X} //_{\widehat{\mathcal{L}}} \widehat{U}) / (R/\lambda(\mathbb{G}_m))$. Moreover, this good quotient restricts to a geometric quotient on an open subscheme of stable points.

Finally, to go from \widehat{X} to the blow-up \widetilde{X} in Theorem 2.15, one performs an additional blow up sequence by considering the stabiliser groups for the action of the reductive group $R/\lambda(\mathbb{G}_m)$ as in the partial desingularisation procedure described in §2.1.2.

This gives us an H -invariant open subscheme $X^{s,H}$ of X with a geometric quotient by H which is an open subscheme of the projective scheme $\widetilde{X}/\widetilde{\mathcal{L}}H$ (and also of $\widehat{X}/\widehat{\mathcal{L}}H$). Here $X^{s,H}$ is the image under $\widehat{\psi}$ of the intersection of $\widehat{X}^{s,H}$ with the complement of the exceptional divisor in \widehat{X} .

Theorem 2.13 can be generalised by weakening the condition $(*)$ further to $(***)$ to allow for actions with positive dimensional stabiliser groups generically.

Theorem 2.17. (*\widehat{U} -Theorem with positive-dimensional stabilisers in U*) [4] Suppose that condition $(***)$ holds for an adapted linear \widehat{U} -action on X . Then the conclusions of Theorem 2.13 hold.

In fact, this theorem still holds if we replace the derived series in condition $(***)$ with any series $U \geq U^{(1)} \geq \dots \geq U^{(s)} \geq \{e\}$ which is normalised by H and whose successive quotients $U^{(j)}/U^{(j+1)}$ are abelian, provided that $(***)$ holds for this series.

Finally, there is a version of the theorem without requiring any hypothesis related to semistability coinciding with stability.

Theorem 2.18. (*\widehat{U} -Theorem with positive-dimensional stabilisers in U , giving projective completions*) [4] For a linear H -action on X with respect to an adapted ample linearisation \mathcal{L} , there is a sequence of blow-ups along H -invariant projective subschemes, resulting in a projective scheme \widetilde{X} such that condition $(***)$ holds for a suitable linearisation, and such that the H -quotient given by Theorem 2.13 is a geometric quotient of an open subscheme $\widetilde{X}^{s,H}$ of \widetilde{X} .

This theorem provides a non-reductive analogue of the partial desingularisation construction for reductive GIT described at the end of §2.1.

As before, this gives us an H -invariant open subscheme $X^{s,H}$ of X with a geometric quotient by H which is open in the projective scheme $X/\mathcal{L}H$; here $X^{s,H}$ is the image under ψ of the intersection of $\widetilde{X}^{s,H}$ with the complement of the exceptional divisor in \widetilde{X} .

Remark 2.19. As for Theorem 2.15 (cf. Remark 2.16), one first constructs a blow-up $\widehat{X} \rightarrow X$ by considering stabiliser subgroups for the unipotent action, then one constructs a further blow-up sequence $\widetilde{X} \rightarrow \widehat{X}$ by considering the stabiliser subgroups for the residual action of the reductive subgroup $R/\lambda(\mathbb{G}_m)$. The slight difference is that in the first step, the centres of the blow-ups used to obtain \widehat{X} from X are determined by the dimensions of the $U^{(j)}$ -stabilisers of the limit $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ and of x itself for $x \in X_{\min}^0$. We will see in the next section that by applying Theorem 2.18 to the closures of the subschemes where these dimensions take different values, and combining this with the partial desingularisation construction of [24] for reductive GIT quotients, X can be stratified so that each stratum is a locally closed H -invariant subscheme of X with a geometric quotient by the action of H .

3. STRATIFYING QUOTIENT STACKS

Let $H = U \rtimes R$ be a linear algebraic group with internally graded unipotent radical U (cf. Definition 2.9) acting on a projective scheme X over an algebraically closed field \mathbb{k} of characteristic 0 with respect to an ample linearisation \mathcal{L} . We fix an invariant inner product on the Lie algebra $\mathrm{Lie} R$ of the Levi factor, just as in the construction of the HKKN stratification (cf. §2.1.1).

The aim of this section is to prove Theorem 1.1 stated in §1. We will prove this result using a recursive argument involving the dimensions of X and of H and the number of irreducible components of X . The idea will be to start by defining a ‘minimum’ stratum, which will be a non-empty H -invariant open subscheme of X , and then proceed recursively.

We will assume that H is connected and that X is reduced.

Remark 3.1. These assumptions do not involve any significant loss of generality.

- (1) The HKKN stratification $\{S_\beta \mid \beta \in \mathcal{B}\}$ is usually indexed by a finite subset \mathcal{B} of a positive Weyl chamber. Then the strata are not necessarily connected even when X is irreducible, and it is often useful to refine the stratification so that the strata are the connected components of S_β , or are unions of some but not all of these connected components (cf. [23]). There is a similar ambiguity in the construction of the refined stratification defined in this section: at some points we take connected components, but this is not crucial to the definition. Indeed if we wish to allow the group H to be disconnected then we cannot assume that the strata are connected since they are required to be H -invariant. Then instead of taking connected components (which will be invariant under the component H_0 of the identity in H), we can take their H -sweeps, which will be disjoint unions of at most $|H/H_0|$ of these connected components.
- (2) If X is non-reduced, then we can define the stratification on X by using a positive power of L to define an H -equivariant embedding of X into a projective space \mathbb{P}^n , and then take the fibre product of X with the stratification on \mathbb{P}^n . Indeed, we will see that the stratification is functorial for equivariant closed immersions (this is essentially the third statement in Theorem 1.1). This follows as the reductive notions of GIT (semi)stability are functorial and since in the non-reductive case, we are assuming that H has internally graded unipotent radical. The stable loci when H has internally graded unipotent radical and adapted linearisation are also functorial, as they have Hilbert–Mumford style descriptions (see Definition 2.12). We note that in the more general non-reductive GIT set up described in §2.2 the notions of (semi)stability are not functorial, as taking H -invariants is not exact, and so there can be invariants which do not extend to the ambient space. The advantage of assuming that X is reduced is that the complement to the open stratum then has a canonical scheme structure; thus it is easier to recursively define the stratification.

Let us describe the recursive construction when H is connected and X is reduced. For each linear action on X of $H = U \rtimes R$ with internal grading $\lambda : \mathbb{G}_m \rightarrow R$ and linearisation \mathcal{L} with underlying ample line bundle L , we will first use recursion to define a *nonempty* H -invariant open subscheme $\mathcal{S}_0(X, H, \lambda, \mathcal{L})$ of X that admits a geometric quotient $\mathcal{S}_0(X, H, \lambda, \mathcal{L})/H$; this will be done by considering seven different cases. After defining the open stratum, we will define the stratification $\{\mathcal{S}_\gamma \mid \gamma \in \Gamma\}$ of X with strata $\mathcal{S}_\gamma = \mathcal{S}_\gamma(X, H, \lambda, \mathcal{L})$ and index set $\Gamma = \Gamma(X, H, \lambda, \mathcal{L})$ by letting X_1, \dots, X_k be the connected components of the projective subscheme of X equal to the complement of $\mathcal{S}_0(X, H, \lambda, \mathcal{L})$, letting $\mathcal{S}_{0,i}(X, H, \lambda, \mathcal{L})$ for $1 \leq i \leq m$ be the connected components of $\mathcal{S}_0(X, H, \lambda, \mathcal{L})$, and setting

$$(3.1) \quad \Gamma(X, H, \lambda, \mathcal{L}) := \{0\} \times \{1, \dots, m\} \cup \bigcup_{1 \leq j \leq k} \{X_j\} \times \Gamma(X_j, H, \lambda, \mathcal{L}|_{X_j}).$$

The strata indexed by $(0, i) \in \Gamma(X, H, \lambda, \mathcal{L})$ for $1 \leq i \leq m$ are then the connected components of the open subscheme $\mathcal{S}_0(X, H, \lambda, \mathcal{L})$ constructed using the case by case argument below, whereas

the stratum indexed by an element (X_j, γ) for $1 \leq j \leq k$ and $\gamma \in \Gamma(X_j, H, \lambda, \mathcal{L}|_{X_j})$ is

$$(3.2) \quad \mathcal{S}_{(X_j, \gamma)}(X, H, \lambda, \mathcal{L}) := \mathcal{S}_\gamma(X_j, H, \lambda, \mathcal{L}|_{X_j}),$$

where the strata $\mathcal{S}_\gamma(X_j, H, \lambda, \mathcal{L}|_{X_j})$ are constructed by induction. The partial order on Γ then naturally comes from the partial orders on each $\Gamma(X_j, H, \lambda, \mathcal{L}|_{X_j})$ with $(0, i) < (X_j, \gamma)$ for all $1 \leq i \leq m$ and $1 \leq j \leq k$ and $\gamma \in \Gamma(X_j, H, \lambda, \mathcal{L}|_{X_j})$.

Let us now describe how to define $\mathcal{S}_0(X, H, \lambda, \mathcal{L})$ in the seven different cases. Let

$$U = U^{(0)} \geq U^{(1)} = [U, U] \geq \dots \geq U^{(s)} \geq \{e\}$$

be the derived series of U . Let

$$Z_{\min}^{d_0} = \{x \in Z_{\min} \mid \dim(\text{Stab}_U(x)) = d_0\}$$

where d_0 is the minimal value of $\dim(\text{Stab}_U(x))$ for $x \in Z_{\min}$. Then $Z_{\min}^{d_0}$ is a nonempty H -invariant open subscheme of Z_{\min} , and $UZ_{\min}^{d_0}$ is a nonempty locally closed subscheme of X .

Case 1. First assume that the central one-parameter subgroup $\lambda : \mathbb{G}_m \rightarrow R$ of R has at least two distinct weights for the linear action of H on X ; equivalently Z_{\min} is a projective subscheme of X with $Z_{\min} \neq X$. Under this assumption we have two possibilities to consider.

Case 1(a). Suppose that $UZ_{\min}^{d_0}$ is open in X . Then we define

$$\mathcal{S}_0(X, H, \lambda, \mathcal{L}) = U\{x \in \mathcal{S}_0(Z_{\min}, R/\lambda(\mathbb{G}_m), \lambda_0, \mathcal{L}|_{Z_{\min}}) \mid \dim(\text{Stab}_{U_j}(x)) = d_j \text{ for } 1 \leq j \leq s\}$$

where λ_0 is the trivial one-parameter subgroup that grades the trivial unipotent radical of the reductive group $R/\lambda(\mathbb{G}_m)$ and $d_j := \min\{\dim(\text{Stab}_{U_j}(x)) : x \in \mathcal{S}_0(Z_{\min}, R/\lambda(\mathbb{G}_m), \lambda_0, \mathcal{L}|_{Z_{\min}})\}$.

By induction we can assume that $\mathcal{S}_0(Z_{\min}, R/\lambda(\mathbb{G}_m), \lambda_0, \mathcal{L}|_{Z_{\min}})$ is a nonempty R -invariant open subscheme of Z_{\min} with a geometric quotient by the action of $R/\lambda(\mathbb{G}_m)$ (or equivalently by the action of R , since the central one-parameter subgroup $\lambda(\mathbb{G}_m)$ of R acts trivially on Z_{\min}). Indeed, we can construct such an open subscheme as in Case 2 described below. Thus $\mathcal{S}_0(X, H, \lambda, \mathcal{L})$ is an H -invariant nonempty open subscheme of X , and by the proof of Theorem 2.18 (see [4, Remark 2.10]) it has a geometric quotient

$$\begin{aligned} \mathcal{S}_0(X, H, \lambda, \mathcal{L})/H &\cong \mathcal{S}_0(Z_{\min}, R/\lambda(\mathbb{G}_m), \lambda_0, \mathcal{L}|_{Z_{\min}})/R \\ &= \mathcal{S}_0(Z_{\min}, R/\lambda(\mathbb{G}_m), \lambda_0, \mathcal{L}|_{Z_{\min}})/(R/\lambda(\mathbb{G}_m)). \end{aligned}$$

Case 1(b). Suppose that $UZ_{\min}^{d_0}$ is not open in X . Recall that X_{\min}^0 is an H -invariant open subscheme of X and that the morphism $p : X_{\min}^0 \rightarrow Z_{\min}$ defined by

$$p(x) = \lim_{t \rightarrow 0} \lambda(t)x$$

satisfies $p(urx) = rp(x)$ for all $x \in X_{\min}^0$, $u \in U$ and $r \in R$ (cf. [4]). We set

$$\mathcal{S}_0(X, H, \lambda, \mathcal{L}) = \left\{ x \in X_{\min}^0 \setminus UZ_{\min} \mid \begin{array}{l} p(x) \in \mathcal{S}_0(Z_{\min}, R/\lambda(\mathbb{G}_m), \lambda_0, \mathcal{L}|_{Z_{\min}}) \text{ and for } 1 \leq j \leq s \\ \dim(\text{Stab}_{U_j}(x)) = \delta_j \text{ and } \dim(\text{Stab}_{U_j}(p(x))) = d_j \end{array} \right\}$$

where d_j and δ_j are defined inductively, in decreasing order of j , as follows. Assuming that d_i and δ_i are defined for $i > j$, we define

$$d_j := \min \left\{ \dim(\text{Stab}_{U_j}(z)) \mid \begin{array}{l} \exists x \in X_{\min}^0 \text{ with } z = p(x) \in \mathcal{S}_0(Z_{\min}, R/\lambda(\mathbb{G}_m), \lambda_0, \mathcal{L}|_{Z_{\min}}) \\ \text{and } \dim(\text{Stab}_{U_i}(z)) = d_i \text{ for all } i > j \end{array} \right\}$$

and

$$\delta_j := \min \left\{ \dim(\text{Stab}_{U_j}(x)) \mid x \in X_{\min}^0 \text{ such that } p(x) \in \mathcal{S}_0(Z_{\min}, R/\lambda(\mathbb{G}_m), \lambda_0, \mathcal{L}|_{Z_{\min}}) \right. \\ \left. \text{and } \dim(\text{Stab}_{U_i}(x)) = \delta_i \text{ for all } i > j \right\},$$

where for $j = s$, the conditions appearing in the second line of these definitions are empty.

Note that if X and Z_{\min} are irreducible then we can define d_j and δ_j much more simply as the generic values taken by $\dim(\text{Stab}_{U_j}(z))$ for $z \in Z_{\min}$ and by $\dim(\text{Stab}_{U_j}(x))$ for $x \in X_{\min}^0$, but in this approach to defining the stratification $\{\mathcal{S}_\gamma \mid \gamma \in \Gamma\}$ we need to allow X to be reducible.

Using the proof of Theorem 2.18 again (cf. [4, Remark 2.10]), we have that $\mathcal{S}_0(X, H, \lambda, \mathcal{L})$ is an H -invariant nonempty open subscheme of X with a geometric quotient $\mathcal{S}_0(X, H, \lambda, \mathcal{L})/\widehat{U}$, and by the inductive construction a geometric quotient

$$\mathcal{S}_0(X, H, \lambda, \mathcal{L})/H = (\mathcal{S}_0(X, H, \lambda, \mathcal{L})/\widehat{U})/(R/\lambda(\mathbb{G}_m)).$$

Case 2. Now assume that the central one-parameter subgroup $\lambda : \mathbb{G}_m \rightarrow R$ which grades U acts trivially on X , so the unipotent radical U of $H = U \rtimes R$ must also act trivially on X . Then without loss of generality we can assume that $H = R$ is reductive and that $\lambda = \lambda_0$ is trivial.

Case 2(a). Suppose that the stable locus $X^{s,R}$ for the linear action of R on X is nonempty. Then we let

$$\mathcal{S}_0(X, H, \lambda, \mathcal{L}) = X^{s,R}$$

and by classical GIT this has a geometric quotient $X^{s,R}/H = X^{s,R}/R$.

Case 2(b). Suppose that the semistable locus $X^{ss,R}$ for the linear action of R on X is empty. Then there is a stratum S_β from the HKKN stratification for X (associated to our invariant inner product on R) such that $\beta \neq 0$ and

$$S_\beta = HY_\beta^{ss} = RY_\beta^{ss} \cong R \times^{P_\beta} Y_\beta^{ss}$$

is nonempty and open in X (see §2.1.1 and Remark 2.5). Then we have the following two subcases to consider.

Case 2(b)i. Suppose that $\overline{Y_\beta^{ss}} \neq X$. Then by induction on $\dim X$ and the number of irreducible components of X ,

$$\mathcal{S}_0(X, H, \lambda, \mathcal{L}) = R(\mathcal{S}_0(\overline{Y_\beta^{ss}}, P_\beta, \lambda_\beta, \mathcal{L}|_{\overline{Y_\beta^{ss}}}) \cap Y_\beta^{ss}) \cong R \times^{P_\beta} (\mathcal{S}_0(\overline{Y_\beta^{ss}}, P_\beta, \lambda_\beta, \mathcal{L}|_{\overline{Y_\beta^{ss}}}) \cap Y_\beta^{ss})$$

is a nonempty R -invariant (and hence H -invariant) open subscheme of X and has a geometric quotient

$$\mathcal{S}_0(X, H, \lambda, \mathcal{L})/H = \mathcal{S}_0(X, H, \lambda, \mathcal{L})/R \cong (\mathcal{S}_0(\overline{Y_\beta^{ss}}, P_\beta, \lambda_\beta, \mathcal{L}|_{\overline{Y_\beta^{ss}}}) \cap Y_\beta^{ss})/P_\beta.$$

Case 2(b)ii. Suppose that $\overline{Y_\beta^{ss}} = X$. Then $P_\beta = R$ so β defines a rational character of R and corresponds to a nontrivial central one-parameter subgroup $\lambda_\beta : \mathbb{G}_m \rightarrow R$. If $Z_\beta \neq X$ then the one-parameter subgroup $\lambda_\beta(\mathbb{G}_m)$ acts nontrivially on X , and thus we can use Case 1 and induction to define

$$\mathcal{S}_0(X, H, \lambda, \mathcal{L}) = \mathcal{S}_0(X, R, \lambda_\beta, \mathcal{L})$$

so that it is a nonempty R -invariant (and hence H -invariant) open subscheme of X and has a geometric quotient

$$\mathcal{S}_0(X, H, \lambda, \mathcal{L})/H = \mathcal{S}_0(X, R, \lambda_\beta, \mathcal{L})/R.$$

If $Z_\beta = X$ then $\lambda_\beta(\mathbb{G}_m)$ acts trivially on X and we can use induction on the dimension of H to define

$$\mathcal{S}_0(X, H, \lambda, \mathcal{L}) = \mathcal{S}_0(X, R/\lambda_\beta(\mathbb{G}_m), \lambda_0, \mathcal{L})$$

which is a nonempty R -invariant (and hence H -invariant) open subscheme of X with geometric quotient $\mathcal{S}_0(X, H, \lambda, \mathcal{L})/H = \mathcal{S}_0(X, R/\lambda_\beta(\mathbb{G}_m), \lambda_0, \mathcal{L})/(R/\lambda_\beta(\mathbb{G}_m))$.

Case 2(c). Suppose now that $X^{s,R} = \emptyset \neq X^{ss,R}$. Recall from Remark 2.5 that the partial desingularisation construction [24] for the linear action of the reductive group R on X can be applied to $X^{ss,R}$, although, for simplicity, X was assumed to be irreducible in Remark 2.5, which is no longer the case here. This construction terminates with a birational projective morphism $\psi : \tilde{X} \rightarrow X$ and R -linearisation $\tilde{\mathcal{L}}$ for an ample line bundle \tilde{L} on \tilde{X} which is a positive integer multiple of $\psi^*L \otimes \mathcal{O}(-\epsilon E)$ where E is the exceptional divisor and $0 < \epsilon \ll 1$ is rational. Then we have the following two subcases to consider.

Case 2(c)i. Suppose that there is a positive-dimensional connected reductive subgroup R' of R such that

$$R\tilde{Z}_{R'}^{ss} \cong R \times^{N_{R'}} \tilde{Z}_{R'}^{ss}$$

is open and nonempty in $\tilde{X}^{ss,R}$. Here $N_{R'}$ is the normaliser of R' in R and

$$\tilde{Z}_{R'}^{ss} = \{x \in \tilde{X}^{ss,R} \mid R'x = x\} = \tilde{Z}_{R'} \cap X^{ss,R}$$

with $\tilde{Z}_{R'}$ the union of those connected components of the fixed point set $\tilde{X}^{R'}$ over which R' acts trivially on the fibres of $\tilde{\mathcal{L}}$. Moreover $N_{R'}$ and its quotient group $N_{R'}/R'$ are reductive, and $\tilde{Z}_{R'}^{ss}$ coincides with both the semistable locus and the stable locus for the induced action of $N_{R'}/R'$ on $\tilde{Z}_{R'}$.

In this situation $R\tilde{Z}_{R'}^{ss} \setminus E \cong R \times^{N_{R'}} (\tilde{Z}_{R'}^{ss} \setminus E)$ is a nonempty R -invariant (and hence H -invariant) open subscheme of $\tilde{X} \setminus E$ with a geometric quotient

$$(R\tilde{Z}_{R'}^{ss} \setminus E)/R \cong (\tilde{Z}_{R'}^{ss} \setminus E)/N_{R'}.$$

Since ψ restricts to an H -equivariant isomorphism from $\tilde{X} \setminus E$ to an open subscheme of X , it follows that if we set

$$\mathcal{S}_0(X, H, \lambda, \mathcal{L}) = \psi(R\tilde{Z}_{R'}^{ss} \setminus E)$$

then $\mathcal{S}_0(X, H, \lambda, \mathcal{L})$ is a nonempty R -invariant (and hence H -invariant) open subscheme of X with a geometric quotient

$$\mathcal{S}_0(X, H, \lambda, \mathcal{L})/H \cong (\tilde{Z}_{R'}^{ss} \setminus E)/N_{R'}.$$

Case 2(c)ii. By Remark 2.5, if we are not in the situation of 2(c)i), then the semistable locus $\tilde{X}^{ss,R}$ is empty, and as in Case 2(b) there is a nonempty open stratum

$$\tilde{S}_\beta \cong R \times^{P_\beta} \tilde{Y}_\beta^{ss}$$

with $\beta \neq 0$ in the HKKN stratification of \tilde{X} . Then $\tilde{S}_\beta \setminus E \cong R \times^{P_\beta} (\tilde{Y}_\beta^{ss} \setminus E)$ is nonempty and open in \tilde{X} .

If $\overline{\widetilde{Y}_\beta^{ss} \setminus E} \neq \widetilde{X}$ then as in Case 2(b)i) $\mathcal{S}_0(X, H, \lambda, \mathcal{L}) = \psi(R\mathcal{S}_0(\overline{\widetilde{Y}_\beta^{ss} \setminus E}, P_\beta, \lambda_\beta, \widetilde{\mathcal{L}}|_{\overline{\widetilde{Y}_\beta^{ss} \setminus E}}) \setminus E)$ is a nonempty R -invariant (and hence H -invariant) open subscheme of X with a geometric quotient

$$\mathcal{S}_0(X, H, \lambda, \mathcal{L})/H \cong (\mathcal{S}_0(\overline{\widetilde{Y}_\beta^{ss} \setminus E}, P_\beta, \lambda_\beta, \widetilde{\mathcal{L}}|_{\overline{\widetilde{Y}_\beta^{ss} \setminus E}}) \setminus E)/P_\beta.$$

Finally if $\overline{\widetilde{Y}_\beta^{ss} \setminus E} = \widetilde{X}$ then as in Case 2(b)ii) we can either reduce to Case 1 or use induction on the dimension of H .

This completes the recursive definition of $\mathcal{S}_0(X, H, \lambda, \mathcal{L})$; then we define the stratification $\{\mathcal{S}_\gamma | \gamma \in \Gamma\}$ and its indexing set Γ as in (3.1) and (3.2), and Theorem 1.1 follows from the construction. Of course, if the dimension of X or H is 0, we set $\mathcal{S}_0(X, H, \lambda, \mathcal{L}) = X$.

Example: Ordered points on the projective line. This is a familiar example (cf. [23, 28]). Let $H = \mathrm{SL}(2)$ so that U is trivial and $R = H$. Fix $n \geq 1$ and consider the diagonal action of H on $(\mathbb{P}^1)^n$. The linearisation is the n th tensor power of the standard representation on \mathbb{C}^2 via the Segre embedding, and the indexing set of the HKKN stratification is

$$\mathcal{B} = \{(2r - n) \mid n \geq r > n/2\} \cup \{0\}.$$

If $\beta = 2r - n$ with $r > n/2$ then a sequence lies in $Z_\beta = Z_\beta^{ss}$ if and only if it contains r entries equal to $[1 : 0]$ and $n - r$ entries equal to $[0 : 1]$, while $Y_\beta = Y_\beta^{ss}$ consists of sequences with precisely r entries equal to $[1 : 0]$, and finally the HKKN stratum S_β consists of sequences such that exactly r entries coincide. Thus Z_β, Y_β and S_β all have $\binom{n}{r}$ connected components. The semistable stratum corresponds to $\beta = 0$ and consists of sequences in which no point of \mathbb{P}^1 occurs strictly more than $n/2$ times.

In order to describe the refined stratification, note first that the U_β -stabilisers in Z_β when $\beta = 2r - n$ are trivial for $n/2 < r < n$. Therefore in the refined stratification, S_β decomposes as the disjoint union of $S_{2r-n}^{r, n-r} = \mathrm{SL}(2)Z_\beta$, consisting of sequences supported at two points of multiplicity r and $n - r$ respectively, and its complement $S_{2r-n}^{r, < n-r}$ in S_β ; when $r = n - 1$ this complement is empty. When $r = n$ the stabilisers for $U_\beta = U_n$ are not trivial, but they have the same dimension for all points in Z_n and the stratum S_n remains intact.

The semistable stratum S_0 behaves differently for even and odd n . For odd n , the $\mathrm{SL}(2)$ -semistable locus S_0 coincides with the stable locus and no blow-up is needed. However, for even n we need to split S_0 into smaller strata to get geometric quotients: the stable locus $S_0^{< n}$ where strictly fewer than $n/2$ points coincide, the (connected components of the) centre $S_0^{n/2, n/2}$ of the blow-up where half the entries of the sequence coincide at each of two points in \mathbb{P}^1 , and the (connected components of the) strata $S_0^{n/2, < n/2}$ where half the entries coincide but the other half do not.

Thus modulo taking connected components the refined stratification has the following form:

$$\begin{aligned} n \text{ odd: } (\mathbb{P}^1)^n &= S_0 \sqcup S_{n-2} \sqcup S_n \sqcup \bigsqcup_{n/2 < r < n-1} \underbrace{S_{2r-n}^{r, n-r} \sqcup S_{2r-n}^{r, < n-r}}_{S_{2r-n}}; \\ n \text{ even: } (\mathbb{P}^1)^n &= \underbrace{S_0^{< n} \sqcup S_0^{n/2, n/2} \sqcup S_0^{n/2, < n/2}}_{S_0} \sqcup S_{n-2} \sqcup S_n \sqcup \bigsqcup_{n/2 < r < n-1} \underbrace{S_{2r-n}^{r, n-r} \sqcup S_{2r-n}^{r, < n-r}}_{S_{2r-n}}. \end{aligned}$$

Remark 3.2. If we only require good quotients rather than geometric quotients of the strata S_γ , then we can simplify the algorithm for constructing the stratification $\{\mathcal{S}_\gamma | \gamma \in \Gamma\}$ somewhat, by combining the Cases 2(a) and (c) in the definition of $\mathcal{S}_0(X, H, \lambda, \mathcal{L})$ into the following single case.

Case 2(a'). Suppose that the semistable locus $X^{ss,R}$ for the linear action of R on X is nonempty. Then we let

$$\mathcal{S}_0(X, H, \lambda, \mathcal{L}) = X^{ss,R}$$

and by classical GIT this has a good quotient $X//R$.

We note that this stratification is not intrinsic to the stack $\mathfrak{X} := [X/H]$, as it depends on the presentation of \mathfrak{X} as a quotient stack $[X/G]$ and a choice of ample linearisation \mathcal{L} on X , as well as an invariant inner product on $\mathrm{Lie} R$. Since for reductive groups $H = R$, this stratification refines the HKKN stratification, which itself depends on a choice of such a linearisation and an inner product, this dependence is to be expected.

Remark 3.3. Note that when $H = R$ is reductive then condition (\dagger) in Theorem 1.1 holds for $\beta = 0$ if and only if $X^{ss} = X^s$. Moreover (\dagger) holds for $\beta \neq 0$ if and only if semistability coincides with stability for the induced linear action of $\mathrm{Stab}(\beta)/\lambda_\beta(\mathbb{G}_m)$ on Z_β and also condition $(*)$ holds for the action of the graded unipotent radical $\widehat{U}_\beta = U_\beta \rtimes \lambda_\beta(\mathbb{G}_m)$ of the parabolic subgroup $P_\beta \leq R$.

There is a notion of a coarse moduli space of an Artin stack \mathfrak{X} (over \mathbb{k}), which is a morphism $\pi : \mathfrak{X} \rightarrow Y$ to an algebraic space Y that is a categorical quotient (i.e. π is initial among all morphisms from \mathfrak{X} to an algebraic space) and such that π induces a bijection on point sets $[\mathfrak{X}(\mathbb{k})] \cong Y(\mathbb{k})$.

Remark 3.4. In general, if G is an algebraic group acting on a scheme X with a categorical quotient $q : X \rightarrow Y$, then $\pi : [X/G] \rightarrow Y$ is not necessarily a coarse moduli space, as π only induces a bijection on points sets if Y is an orbit space. However, if q is a geometric quotient, then π is a coarse moduli space for $\mathfrak{X} := [X/G]$. Indeed, since geometric quotients are orbit spaces, π clearly induces a bijection $[\mathfrak{X}(\mathbb{k})] = X(\mathbb{k})/G(\mathbb{k}) \cong Y(\mathbb{k})$. To prove it is also a categorical quotient, one can apply a result of Rydh [29, Theorem 3.16], which says that if q is universally open and strongly geometric (in the sense of [29, Definition 2.2]), then π is a categorical quotient. Since q is a geometric quotient it is surjective, universally open and the fibres of q are single orbits; furthermore, the natural morphism $j : G \times X \rightarrow X \times_Y X$ is surjective. In particular, j , q and q^{cons} (the induced map for the constructible topology) are all universally submersive; thus q is a strongly geometric quotient by [29, Lemma 2.3].

Alper [1] introduced a notion of a good moduli space for stacks, which is inspired by the notion of a good quotient in reductive GIT. However, this notion is suited to reductive groups and is less favourable for our set up where we have stacks with potentially non-reductive stabiliser groups. Indeed $BG = [\mathrm{Spec} \mathbb{k}/G] \rightarrow \mathrm{Spec} \mathbb{k}$ is a good moduli space if and only if G/\mathbb{k} is linearly reductive, as the stabiliser groups of the closed points of a stack admitting a good moduli space are all linearly reductive by [1, Proposition 12.14].

By Remark 3.4, we obtain the following corollary of Theorem 1.1.

Corollary 3.5. *Let $\mathfrak{X} = [X/H]$ be a quotient stack, where X and H are as in the statement of Theorem 1.1, and so we have a stratification $\{\mathcal{S}_\gamma | \gamma \in \Gamma\}$ of X into H -invariant quasi-projective subschemes which*

each admit geometric quotients S_γ/H . Then there is an induced stratification $\{\mathfrak{X}_\gamma := [S_\gamma/H] : \gamma \in \Gamma\}$ of \mathfrak{X} such that each stratum has a coarse moduli space $\mathfrak{X}_\gamma \rightarrow S_\gamma/H$.

We note that, in general, it is not the case that $[S_\gamma/H]$ is isomorphic to S_γ/H ; this only happens if additionally the action of H on S_γ is free.

4. STACKS FILTERED BY QUOTIENT STACKS

In §3, we constructed a stratification $\{S_\gamma | \gamma \in \Gamma\}$ of a projective scheme X acted on linearly by a linear algebraic group $H = U \rtimes R$ with internally graded unipotent radical U , such that each stratum is an H -invariant quasi-projective subscheme of X with a geometric quotient by the action of H . The quotient stack $[X/H]$ then has an induced stratification $\{\Sigma_\gamma | \gamma \in \Gamma\}$ such that each stratum Σ_γ has the form

$$\Sigma_\gamma := [S_\gamma/H] \cong [W_\gamma/H_\gamma]$$

where W_γ is a quasi-projective subscheme of X acted on by a linear algebraic subgroup H_γ of H with internally graded unipotent radical, and this action has a geometric quotient W_γ/H_γ . Moreover under appropriate hypotheses of ‘semistability coinciding with stability’ for the actions of the subgroups H_γ , the geometric quotients W_γ/H_γ are themselves projective.

We can apply this to Artin stacks which are filtered by quotient stacks admitting stratifications that are compatible in the following sense.

Definition 4.1. Let \mathcal{M} be an Artin stack.

- (1) We say \mathcal{M} is *filtered by quotient stacks* if \mathcal{M} can be expressed as an increasing union

$$\mathcal{M} = \bigcup_{n \geq 0} \mathcal{M}_n$$

of open substacks with presentations of the form

$$\mathcal{M}_n \cong [V_n/H_n]$$

where $[V_n/H_n]$ is the quotient stack associated to a linear action on a quasi-projective scheme V_n by a group $H_n = U_n \rtimes R_n$ with internally graded unipotent radical.

- (2) We say \mathcal{M} is *filtered by quotient stacks admitting compatible stratifications* if \mathcal{M} is filtered as above and moreover, for each $n \in \mathbb{N}$, there is an ample H_n -linearisation \mathcal{L}_n on a projective completion \overline{V}_n of V_n and a choice of invariant inner product on the Lie algebra of the Levi factor $R_n \leq H_n$ such that for $n' > n$, the pullback along $\mathcal{M}_n \hookrightarrow \mathcal{M}_{n'}$ of the stratification of $\mathcal{M}_{n'}$ (obtained by restricting the stratification on $[\overline{V}_{n'}/H_{n'}]$ defined by $\mathcal{L}_{n'}$ and this norm to $\mathcal{M}_{n'}$) coincides with the corresponding stratification on \mathcal{M}_n .
- (3) We additionally say that \mathcal{M} is *filtered by quotient stacks admitting compatible stratifications which are asymptotically stable*, if for each n and each stratum $\Sigma_{\gamma,n}$ of $\mathcal{M}_n \cong [V_n/H_n]$ there exists $N \geq n$ such that if $l > m \geq N$ and $\Sigma_{\gamma,l} \hookrightarrow \mathcal{M}_l$ and $\Sigma_{\gamma,m} \hookrightarrow \mathcal{M}_m$ are the strata whose restrictions to \mathcal{M}_n are $\Sigma_{\gamma,n}$, then $\Sigma_{\gamma,l}$ is the image of $\Sigma_{\gamma,m}$ under the inclusion $\mathcal{M}_m \hookrightarrow \mathcal{M}_l$.

Then we obtain the following corollary to Theorem 1.1.

Corollary 4.2. *Let \mathcal{M} be an Artin stack that is filtered by quotient stacks admitting compatible stratifications which are asymptotically stable; then there is a stratification $\{\Sigma_\gamma | \gamma \in \Gamma\}$ of the stack \mathcal{M} such that each stratum Σ_γ is isomorphic to a quotient stack $[W_\gamma/H_\gamma]$, where W_γ is quasi-projective and acted*

on by a linear algebraic group $H_\gamma = U_\gamma \rtimes R_\gamma$ with internally graded unipotent radical U_γ . Moreover, there is a geometric quotient W_γ/H_γ which is a coarse moduli space for Σ_γ . In particular, \mathcal{M} has a stratification by quotient stacks which all have coarse moduli spaces.

Remark 4.3. In fact, the same result holds if we consider stacks which admit an increasing open cover indexed by a partially ordered filtered set that contains a cofinal copy of \mathbb{N} with analogous properties.

We have the following criterion for these moduli spaces to be projective (cf. Remark 3.3).

Remark 4.4. In the situation of Corollary 4.2, for a stratum $\Sigma_\gamma \cong [W_\gamma/H_\gamma]$ where the unipotent radical U_γ is graded by $\lambda_\gamma : \mathbb{G}_m \rightarrow R_\gamma$, the coarse moduli space W_γ/H_γ will be projective if $(*)$ holds for the action of $U_\gamma \rtimes \lambda_\gamma(\mathbb{G}_m)$ on a suitable projective completion of W_γ with respect to an induced linearisation, and semistability coincides with stability for the induced action of $R_\gamma/\lambda_\gamma(\mathbb{G}_m)$ on the analogue of Z_{\min} in this projective completion.

5. APPLICATIONS

We can apply the previous section to the construction of moduli spaces of ‘unstable’ objects, including moduli of sheaves over a projective scheme [6], projective curves [21]) and hypersurfaces in a complete toric variety.

5.1. Moduli of hypersurfaces in a complete toric variety. Let Y be a complete simplicial toric variety. It was observed in [3, §4] (using the description of $\text{Aut}(Y)$ given in [11]) that the automorphism group $H = \text{Aut}(Y)$ of Y is a linear algebraic group with internally graded unipotent radical.

By considering suitable Hilbert schemes, each component of the moduli stack of hypersurfaces in Y can be expressed itself as a quotient stack $[V/H]$ where V is a projective scheme with a linear action of $H := \text{Aut}(Y)$ with an ample linearisation. Thus we can apply Theorem 1.1 directly to the moduli stack \mathcal{M} to obtain a stratification $\{\Sigma_\gamma | \gamma \in \Gamma\}$ of \mathcal{M} such that each stratum Σ_γ is isomorphic to a quotient stack $[W_\gamma/H_\gamma]$, where W_γ is a quasi-projective scheme acted on by a linear algebraic group H_γ with internally graded unipotent radical, and there is a geometric quotient W_γ/H_γ which is a coarse moduli space for Σ_γ .

A simple example is the automorphism group of the weighted projective plane $Y = \mathbb{P}(1, 1, 2) = (\mathbb{k}^3 \setminus \{0\})/\mathbb{G}_m$, for \mathbb{G}_m acting linearly on \mathbb{k}^3 with weights 1, 1, 2. This is given by

$$\text{Aut}(\mathbb{P}(1, 1, 2)) \cong U \rtimes R$$

where $R \cong \text{GL}(2)$ is reductive and $U \cong (\mathbb{G}_a)^3$ is unipotent, with elements $(\lambda, \mu, \nu) \in \mathbb{k}^3$ acting on $\mathbb{P}(1, 1, 2)$ via

$$[x, y, z] \mapsto [x, y, z + \lambda x^2 + \mu xy + \nu y^2].$$

The central one-parameter subgroup \mathbb{G}_m of $R \cong \text{GL}(2)$ acts on the Lie algebra of U with all positive weights, providing an internal grading of the unipotent radical U .

5.2. Moduli of sheaves over a projective scheme. Let $(Y, \mathcal{O}_Y(1))$ denote a polarised projective scheme over an algebraically closed field \mathbb{k} of characteristic 0. For a fixed Hilbert polynomial P , we let $\mathcal{M} := \mathcal{M}_{Y,P}$ denote the moduli stack of coherent sheaves over Y with Hilbert polynomial P with respect to $\mathcal{O}_Y(1)$. We can filter this stack by quotient stacks in several ways: for example, one can use Castelnuovo–Mumford regularity or the Harder–Narasimhan type. In both filtrations, we will filter by quotient stacks which are quotients by general linear groups

of open subschemes of Quot schemes, which appear in Simpson's GIT construction [33] of the moduli space of semistable sheaves on Y with Hilbert polynomial P . In this section, we will describe the two ways to filter \mathcal{M} and explain how to use a filtration involving Harder–Narasimhan types to apply the results in §4.

Let us recall Simpson's reductive GIT construction of moduli spaces of semistable sheaves. By the Le Potier estimates [33, Theorem 1.1], the family of semistable sheaves over Y with Hilbert polynomial P is bounded; hence, for $n \gg 0$, all such sheaves are n -regular in the sense of Castelnuovo–Mumford. For an n -regular sheaf \mathcal{E} , the evaluation map

$$H^0(\mathcal{E}(n)) \otimes \mathcal{O}_Y(-n) \rightarrow \mathcal{E}$$

is surjective and $\dim H^0(\mathcal{E}(n)) = P(\mathcal{E}, n)$, as all the higher cohomology groups of $\mathcal{E}(n)$ vanish; hence, any n -regular sheaf with Hilbert polynomial P determines a closed point in the Quot scheme

$$Q_n := \text{Quot}(\mathcal{O}_Y(-n)^{\oplus P(n)}, P)$$

parameterising quotient sheaves of $\mathcal{O}_Y(-n)^{\oplus P(n)}$ with Hilbert polynomial P . In fact, it determines a point in the open subscheme $Q_{n\text{-reg}}$ consisting of quotients $q : \mathcal{O}_Y(-n)^{\oplus P(n)} \rightarrow \mathcal{F}$ such that \mathcal{F} is n -regular and $H^0(q(n))$ is an isomorphism. If $Q_{n\text{-reg}}^{ss}$ denotes the open subset, where additionally the quotient sheaf is semistable, then Simpson constructs the moduli space of semistable sheaves as a quotient of the natural $\text{SL}_{P(n)}$ -action on the closure of $Q_{n\text{-reg}}^{ss}$. The action is linearised by choosing $m \gg n$ to construct an embedding of the Quot scheme into a Grassmannian and then composing with a Plücker embedding of the Grassmannian into projective space; we let $\mathcal{L}_{m,n}$ denote this linearisation.

Since n -regularity is an open condition, we obtain an increasing open filtration of \mathcal{M} by the substacks $\mathcal{M}_{n\text{-reg}}$ of n -regular sheaves; furthermore, there are isomorphisms $\mathcal{M}_{n\text{-reg}} \cong [Q_{n\text{-reg}} / \text{GL}_{P(n)}]$ for all n . Hence \mathcal{M} is filtered by quotient stacks using n -regularity.

Let us explain a second way to filter \mathcal{M} using Harder–Narasimhan (HN) types [17]. We recall that every coherent sheaf \mathcal{E} has a unique HN filtration $0 = \mathcal{E}^{(0)} \subsetneq \mathcal{E}^{(1)} \subsetneq \dots \subsetneq \mathcal{E}^{(r)} = \mathcal{E}$ (where here we use a Rudakov type notion of semistability to define HN filtrations for any coherent sheaf; for example, see [19, Definition 4.11]). The HN type of \mathcal{E} is $\tau(\mathcal{E}) := (P(\mathcal{E}_1), \dots, P(\mathcal{E}_r))$, where $\mathcal{E}_i := \mathcal{E}^{(i)} / \mathcal{E}^{(i-1)}$. The set of HN types for sheaves with Hilbert polynomial P is partially ordered, such that the semistable HN type given by the single polynomial (P) is minimal. We let \mathcal{M}^τ and $\mathcal{M}^{<\tau}$ denote the substacks of sheaves of HN type τ and of HN type less than τ respectively. By work of Shatz [32], the HN type is upper semi-continuous, and so $\mathcal{M}^{<\tau}$ is an open substack of \mathcal{M} . Furthermore, as there are only finitely many HN types less than a given HN type τ , these sheaves form a bounded family, and so $\mathcal{M}^{<\tau}$ is a quotient stack of the form $[Q_{n\text{-reg}}^{<\tau} / \text{GL}_{P(n)}]$ for $n \gg 0$ (depending on τ); here, $Q_{n\text{-reg}}^{<\tau}$ is the open subscheme of $Q_{n\text{-reg}}$ parameterising quotient sheaves with HN type less than τ .

Remark 5.1. The relationship between the HKKN stratifications and the HN stratifications on $Q_{n\text{-reg}}$ is studied in [19, 20], where it is shown that for a HN type τ and for $m \gg n \gg 1$ (depending on τ), there is an HKKN index $\beta_{n,m}(\tau)$ for the $\text{SL}_{P(n)}$ -action on Q_n with respect to $\mathcal{L}_{n,m}$ (and the Killing form on $\text{SL}_{P(n)}$) such that the following statements hold.

- (1) All sheaves with HN type τ are n -regular, and thus $\mathcal{M}_\tau \cong [Q_{n\text{-reg}}^\tau / \text{GL}_{P(n)}]$, where $Q_{n\text{-reg}}^\tau$ is the HN stratum in $Q_{n\text{-reg}}$ indexed by τ .
- (2) The HN stratum $Q_{n\text{-reg}}^\tau$ is contained as a closed subscheme in the HKKN stratum $S_{\beta_{n,m}(\tau)}$.

For the precise definition of $\beta_{n,m}(\tau)$, see [19, Definition 4.18]. For fixed n and m , the assignment $\tau \mapsto \beta_{n,m}(\tau)$ is not injective; however, for two HN types $\tau \neq \tau'$ and $m \gg n \gg 0$, we have $\beta_{n,m}(\tau) \neq \beta_{n,m}(\tau')$. In fact, for a finite number of HN types τ_1, \dots, τ_l , we can pick $m \gg n \gg 1$ so that all the indices $\beta_{n,m}(\tau_i)$ are distinct and both the properties (1) and (2) above hold for all of these HN types.

Furthermore, in [19, Theorem 1.1] an asymptotic HKKN stratification on \mathcal{M} is constructed using the HKKN stratifications on each $\mathcal{M}_{n-\text{reg}}$ and it is shown that this asymptotic HKKN stratification coincides with the HN stratification on \mathcal{M} . In fact, the HKKN stratifications on $\mathcal{M}_{n-\text{reg}}$ are not compatible stratifications which are asymptotically stable in the sense of Definition 4.1, and so this asymptotic construction of a HKKN stratification is slightly more involved (for details, see [19, §4]).

We claim that if we consider the filtration given by the open substacks $\mathcal{M}^{<\tau}$ and choose an appropriate presentation of these stacks, then the HKKN stratifications on each $\mathcal{M}^{<\tau}$ coincide with the HN stratifications, and thus in particular, these stratifications are compatible and asymptotically stable in the sense of Definition 4.1. For each HN type τ , as there are only finitely many HN types less than τ , we can pick $m \gg n \gg 1$ as in Remark 5.1, so that both statements in this remark hold for all these HN types and they all have different HKKN indices. Indeed, if we compare the stratifications on $Q_{n-\text{reg}}^{<\tau}$, then we have $Q_{n-\text{reg}}^{\tau'} \hookrightarrow S_{\beta_{n,m}(\tau')}$ for all $\tau' < \tau$, and as $Q_{n-\text{reg}}^{<\tau}$ is the disjoint union of the HN strata $Q_{n-\text{reg}}^{\tau'}$, we see that in fact the HN and the HKKN stratifications must agree.

Finally let us consider the refinement of the HKKN stratification on each $\mathcal{M}^{<\tau}$ provided by Theorem 1.1. Since the refinement will come from considering various stabiliser groups and stable loci, and these can be studied intrinsically for the sheaves themselves, we see that the stratifications on each $\mathcal{M}^{<\tau} \cong [Q_{n-\text{reg}}^{<\tau} / \text{GL}_{P(n)}]$ are compatible and asymptotically stable in the sense of Definition 4.1. Therefore, there is an induced stratification $\{\Sigma_\gamma | \gamma \in \Gamma\}$ of \mathcal{M} by quotient stacks that refines the HN stratification and such that each quotient stack has a presentation $\Sigma_\gamma \cong [W_\gamma / H_\gamma]$, where W_γ is a quasi-projective scheme acted on by a linear algebraic group H_γ with internally graded unipotent radical, and there is a geometric quotient W_γ / H_γ which is a coarse moduli space for Σ_γ . In fact, we can say what the open stratum is: provided that there exists a stable sheaf with Hilbert polynomial P , the open stratum is the moduli stack \mathcal{M}^s of stable sheaves. The precise construction and description of these coarse moduli spaces is described in [6].

5.3. Moduli of unstable curves. Another classical family of moduli spaces constructed as reductive GIT quotients is given by the moduli spaces of stable curves of fixed genus $g \geq 2$. Here we can use special linear group actions on Hilbert schemes of curves embedded in projective spaces, and exploit the Rosenlicht–Serre description of singular curves in terms of their normalisations together with additional data describing the singularities, to look for an associated stratification $\{\Sigma_\gamma | \gamma \in \Gamma\}$ of the moduli stack of projective curves, allowing us to construct geometric quotients of the unstable strata [21]. In this case of projective curves the linear actions can be set up so that (almost always) the condition that semistability coincides with stability is satisfied. Thus we expect the coarse moduli spaces of ‘unstable curves’ given by the geometric quotients W_γ / H_γ of the strata $\Sigma_\gamma \cong [W_\gamma / H_\gamma]$ to be projective.

REFERENCES

- [1] J. Alper, Good moduli spaces for Artin stacks, *Ann. Inst. Fourier (Grenoble)*, 63 (6), 2349–2402, 2013.

- [2] G. Bérczi, B. Doran, T. Hawes, F. Kirwan, Constructing quotients of algebraic varieties by linear algebraic group actions, arXiv:1512.02997, to appear in "Handbook of Group Actions" Vol 3, editors Lizhen Ji, Athanase Papadopoulos, Shing-Tung Yau.
- [3] G. Bérczi, B. Doran, T. Hawes, F. Kirwan, Geometric invariant theory for graded unipotent groups and applications, arXiv: 1601.00340 .
- [4] G. Bérczi, B. Doran, T. Hawes, F. Kirwan, Projective completions of graded unipotent quotients, arXiv:1607.04181.
- [5] G. Bérczi, B. Doran, F. Kirwan, Graded linearisations, arXiv:1703.05226, to appear in Modern Geometry a celebration of the work of Simon Donaldson, editors Vincente Munoz, Ivan Smith, Richard Thomas, AMS Proceedings of Symposia in Pure Mathematics.
- [6] G. Bérczi, V. Hoskins, J. Jackson, F. Kirwan, Quotients of unstable subvarieties and moduli spaces of sheaves of fixed Harder–Narasimhan type II, in preparation.
- [7] G. Bérczi, J. Jackson, F. Kirwan, Variation of non-reductive GIT, in preparation.
- [8] G. Bérczi, F. Kirwan, Graded unipotent groups and Grosshans theory, arXiv:1511.06983, to appear in *Forum of Mathematics, Sigma*.
- [9] L. Brambila-Paz, O. Mata, N. Nitsure, Moduli stacks and moduli schemes for rank 2 unstable bundles, arXiv:0911.2301.
- [10] M. Brion, On the geometry of algebraic groups and homogeneous spaces, *J. Algebra*, 329, 52–71, 2011.
- [11] D.A. Cox, The homogeneous coordinate ring of a toric variety, *J. Algebraic Geometry* 4 (1995) 17–50.
- [12] Jacques Dixmier. *Enveloping algebras*, volume 11 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1996. Revised reprint of the 1977 translation.
- [13] I. Dolgachev and Y. Hu, Variation of geometric invariant theory quotients, *Inst. Hautes Études Sci. Publ. Math.*, 87(1), 5–56, 1998 (with an appendix by N. Ressayre).
- [14] B. Doran and F. Kirwan, Towards non-reductive geometric invariant theory, *Pure Appl. Math. Q.* 3 (2007), 61–105.
- [15] D. Edidin and D. Rydh, Canonical reduction of stabilizers for Artin stacks with good moduli spaces, arXiv: 1710.03220.
- [16] D. Halpern-Leistner, Theta-stratifications, Theta-reductive stacks, and applications, to appear in the proceedings of the 2015 AMS summer institute in Salt Lake City.
- [17] G. Harder and M. S. Narasimhan. On the cohomology groups of moduli spaces of vector bundles on curves. *Math. Ann.*, 212 (3) 215–248, 1975.
- [18] W.H. Hesselink, Uniform instability in reductive groups, *J. Reine Angew. Math.* 304 (1978), 299–316.
- [19] V. Hoskins, Stratifications for moduli of sheaves and moduli of quiver representations, to appear in *Algebraic Geometry* (arXiv: 1407.4057).
- [20] V. Hoskins and F. Kirwan, Quotients of unstable subvarieties and moduli spaces of sheaves of fixed Harder–Narasimhan type, *Proc. London Math. Soc.*, 105(4), 852–890, 2012.
- [21] J. Jackson, Oxford DPhil thesis, in preparation.
- [22] G. Kempf, Instability in invariant theory, *Ann. of Math.* 108 (2), 299–316, 1978.
- [23] F. Kirwan, Cohomology of quotients in symplectic and algebraic geometry. Mathematical Notes, 31. Princeton University Press, Princeton, NJ, 1984.
- [24] F. Kirwan, Partial desingularisations of quotients of nonsingular varieties and their Betti numbers. *Ann. of Math.* (2) 122 (1985), no. 1, 41–85.
- [25] F. Kirwan, Refinements of the Morse stratification of the normsquare of the moment map, in 'The breadth of symplectic and Poisson geometry', *Progr. Math.*, 232, 327–362, 2005.
- [26] D. Mumford, J. Fogarty and F. Kirwan, Geometric invariant theory, 3rd edition, Springer, 1994.
- [27] L. Ness, A stratification of the null cone via the moment map, *Amer. J. Math.* 106, 1984, 1281–1329.
- [28] P.E. Newstead, Introduction to moduli problems and orbit spaces, Tata Institute Lecture Notes, Springer, 1978.
- [29] D. Rydh, Existence and properties of geometric quotients, *J. Algebraic Geom.* 22 (2013), no. 4, 629669.
- [30] The Stacks Project Authors. *Stacks Project*. <http://stacks.math.columbia.edu>, 2015.
- [31] J. -P. Serre, Espaces fibrés algébriques, Séminaire C. Chevalley; 2e année: 1958. Anneaux de Chow et applications séminaire Claude Chevalley, 1–37.
- [32] S. Shatz, The decomposition and specialization of algebraic families of vector bundles, *Compositio Math.* 35, 163–187, 1977.

- [33] C. Simpson, Moduli of representations of the fundamental group of a smooth projective variety, *Publications Mathématiques de l'IHÉS*, 79(1), 47–129, 1994.
- [34] M. Thaddeus, Geometric invariant theory and flips, *J. Amer. Math. Soc.*, 9(3), 691–723, 1996.

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