Topics in Ergodic Control and Backward Stochastic Differential Equations

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To Louise the Owl, Mr. Wibbles
and other friends

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Abstract

The core of this thesis focuses on a number of different aspects of ergodic stochastic control in connection with backward stochastic differential equations (BSDEs for short). Chapter 1 serves as an introduction to the problem formulation in various contexts and states a number of results we will be using in the sequel. Chapter 2 deals with the so called weak formulation, where the control is represented as a change of measure. The optimal value and feedback control are obtained using a relatively recent object called ergodic BSDEs. In order to achieve this we establish the existence and uniqueness of solutions to these equations along the way.

Chapter 3 is concerned with non zero-sum games where multiple players control the drift of a process, and their payoffs depend on its ergodic behaviour. We establish their connection with systems of Ergodic BSDEs, and prove the existence of a Nash equilibrium under general conditions.

In Chapter 4 we show a novel duality between the existence of a solution to an infinite horizon adjoint BSDE and strong dissipativity of the forward process. Thus the link between ergodicity of the controlled process and the infinite horizon stochastic maximum principle is established. Finally, in Chapter 5 we provide conclusions, conjectures and directions for future research.

Contents

Chapter 1

Introduction

The aim of this chapter is to give a brief introduction to the area of stochastic optimal control, the theory of backward stochastic differential equations (BSDEs) and to provide a connection between the two. The first section describes a general formulation of a stochastic optimisation problem and then moves on to the ergodic cost case. The second attempts to elucidate the logic behind ergodic BSDEs and to explain why they prove useful for long run average (another term for ergodic which we will use in the sequel) optimisation. The last section serves effectively as a collection of abstracts to the remaining chapters, outlining the main results of each.

1.1 Stochastic optimal control

This section explains the essence of the area of stochastic optimal control, as well as describing additional challenges that ergodic cost functionals bring.

1.1.1 Classical control

A typical stochastic control problem can be formulated the following way. We are given a system, parametrised by time, that evolves in some (metric) space. In order to capture the uncertainty of its evolution, we introduce a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and represent the dynamics as a stochastic differential equation on the state, i.e. through a given mapping $t \mapsto X_t(\omega)$. At each point in time the dynamics are influenced by a control u, taking values in a space, which may encode certain problem specific constraints. A control should take into account all the information available until the current time (but no more); in other words, the process $\{u_t\}_{t\geq 0}$ needs to be predictable. Controls satisfying these requirements are called admissible. The final piece that is required is the cost functional, i.e. a criterion which would allow us to compare the performance of different controls. Typical criteria used in stochastic optimal control are

$$J(x_0, u) = \mathbb{E}\Big[\int_0^T L(t, X_t^u, u_t)dt + g(X_T^u)\Big] \tag{1.1}$$

for finite horizon optimisation, and

$$\bar{J}(x_0, u) = \mathbb{E}\Big[\int_0^\infty e^{-\alpha t} L(t, X_t^u, u_t)dt\Big]$$

for the infinite horizon case. The function L is called a running cost function, g is a terminal reward and $\alpha > 0$ is a discount factor. The objective is to minimise/maximise the cost functional over the space of all admissible controls. Below is a famous example of a control problem that comes from finance.

Example 1.1.1. ***Merton's optimal investment-consumption problem*** *Consider a financial market with two assets: a stock and a bond. The dynamics of the stock price is modelled by the standard geometric Brownian motion, i.e.*

$$dS(t) = S(t)[\mu dt + \sigma dW_t],$$

where μ, σ are given constants, $\sigma \neq 0$ and W is a standard one-dimensional Brownian motion. We further assume that the bond yields a constant interest rate r which is reinvested, i.e.

$$dB(t) = rB(t)dt.$$

For each time $t \geq 0$, we denote the consumption rate by $c(t)$ and proportion of wealth invested in the stock by $\pi(t)$. The wealth dynamics are given by

$$dZ(t) = Z(t)[(r + \pi(t))(\mu - r)dt + \pi(t)dW_t] - c(t)dt.$$

Here, the state process is $Z = Z^{\pi,c}$ and the admissible controls are $\pi(t) \in [0, 1], c(t) \geq 0$ for each $t \geq 0$. We assume it is possible to transfer funds between the stock holdings and the bond holdings instantaneously and without a loss (along with the usual assumptions of linear pricing and infinite liquidity), so it is not necessary to keep track of holdings in each asset separately. The final restriction for this problem is that $Z(t) \geq 0$ for all $t \geq 0$. In other words, the set of admissible controls $\mathcal{A}$ is given by

$$\mathcal{A} := \{(\pi(\cdot), c(\cdot))|\text{bounded, adapted processes such that } Z^{\pi,c} \geq 0 \text{ a.s.}\}$$

The objective functional is the expected discounted utility of consumption:

$$J = \mathbb{E}\left[\int_0^\infty e^{-\alpha t} U(c(t))dt\right],$$

where $U : \mathbb{R}_+ \to \mathbb{R}$ is the utility function. For the explicit solution to this problem for certain classes of utility functions, see Soner [45].

Remark 1.1.1. *The main technique used to solve control problems with additive objective functionals (i.e. of the form (1.1)) is the* dynamic programming principle. *To understand what it is, assume we want to find optimal control on the interval $[t, T]$. Denote by $v(s, x)$, where $s \geq t$, the optimal value on the interval $[s, T]$ where at time s the controlled process $X_s^u = x$. Then for any stopping time $\tau \geq t$*

$$\begin{aligned} v(t, x) &= \inf_u \mathbb{E}\left[\int_t^T L(s, X_s^{u,t,x}, u_s)ds + g(X_T^{u,t,x})\big|\mathcal{F}_t\right] \\ &= \inf_u \mathbb{E}\left[\int_t^\tau L(s, X_s^{u,t,x}, u_s)ds + v(\tau, X_s^{u,t,x})\big|\mathcal{F}_t\right]. \end{aligned}$$

Intuitively, the statement above says that if something is optimal at a later time, it remains optimal now (for details and rigorous proofs the reader is referred to Fleming and Soner [15]). In the sequel we will mostly be dealing with non-additive functionals, where this naive approach does not work, but dynamic programming still remains a tool of crucial importance.

Of primary interest to the current work is the so-called "weak" formulation, namely the case when the control acts only on the drift and the compensator of the jump measure. We will see in the sequel that, in this case, it is possible to express the value function for our control problem through a BSDE. Using the so-called comparison theorem for BSDEs, the value function for the optimal control can then be found by minimising/maximising the driver of the BSDE, which plays the role of the Hamiltonian. For a detailed account of this case the reader is referred to Chapter 21 in Cohen and Elliott, [10]. For general theory see the books of Krylov [30], Yong and Zhou [47], Fleming and Soner [15] and Pham [37].

1.1.2 Ergodic control

The area of ergodic optimal control is concerned with the problem of minimising a time averaged penalty (cost) over infinite time horizon. It is applicable in situations where the controlled system settles to a limiting behaviour relatively quickly and one is essentially comparing various possible (control dependent) equilibrium behaviours.

In mathematical terms, given a separable metric space $\mathcal{U}$, the value functional takes the form

$$J(x,u) = \limsup_{T\to\infty} T^{-1}\mathbb{E}\bigg[\int_0^T L(X_t^u, u_t)dt\bigg], \tag{1.2}$$

where the process $\{X_t^u\}_{t\geq 0}$ represents the state process under the controlled forward dynamics, u is an $\mathcal{F}_t$-predictable $\mathcal{U}$-valued process, and L is a bounded measurable cost function.

One can see from (1.2) that the ergodic cost functional differs from the standard "integral" cost in one crucial way: it is unaffected by what happens on any finite time interval. An immediate consequence of this observation is that a completely new set of analytical tools is required to study this problem, since one can no longer split the time horizon into an initial segment and the rest in order to apply the dynamic programming heuristic.

Nevertheless, dynamic programming is not entirely useless in the case of long run average cost. The main technique found in the literature for this situation is ergodic Hamilton–Jacobi–Bellman equations (HJB for short) – an object obtained by sending the discount rate to zero in the infinite horizon discounted cost control problem (the so-called vanishing discount approach). It allows one to characterise optimal policies under certain technical hypotheses. Another approach, which is an area of active research at the moment, is to represent control as an infinite dimensional convex program. For an exhaustive survey of theoretical foundations behind these methods, as well as the numerical schemes they facilitate, see Arapostathis, Borkar and Ghosh [3].

In the present work we focus instead on the probabilistic approach to the problem of ergodic optimal control, based on theory of ergodic backward stochastic differential equations. The next section provides a concise exposition of the results required to understand what these objects are and how it is connected to the original problem.

1.2 Backward Stochastic Differential Equations

1.2.1 Classical BSDEs: a brief introduction

The theory of Backward Stochastic Differential Equations (henceforth BSDEs) has always been closely related to stochastic optimal control. The simplest (linear) case was originally studied by Bismut, who introduced it as the equation for the adjoint process in the stochastic version of Pontryagin maximum principle in 1973 (see [5]). The notion was generalised by Pardoux and Peng in 1990 (see [34]), who proved that non-linear BSDEs were well posed. Since then BSDEs have become a tool of crucial importance in the study of various aspects of decision making under uncertainty. A comprehensive survey on various applications of this theory, with an emphasis on mathematical finance, can be found in El Karoui, Peng and Quenez [27].

The general idea behind a BSDE is to specify a process through its dynamics and its terminal value at a fixed, deterministic time $T > 0$. For ordinary differential equations, under the usual Lipschitz assumption, there is no fundamental difference between the initial value problem and the terminal value problem (the equivalence is established by applying the time reversal $t \mapsto T - t$). The difficulty in the stochastic world stems from the fact that the terminal value is allowed to be a random variable, but we look for a solution which is adapted to a given filtration. The following classic example (see p. 467 in [10]) illuminates this problem and suggests intuition as to what the correct way of formulating the terminal value problem for SDEs should be.

Example 1.2.1. *Let $\mathcal{F}^W$ be the filtration generated by a Brownian motion W, and suppose $\phi : \mathbb{R} \to \mathbb{R}$ is a Borel function. Consider the equation on $[0, T]$ given by*

$$dY_t = 0, \qquad Y_T = \phi(W_T).$$

The only possible solution, namely $Y_t = \phi(W_T)$ for all $t \in [0, T]$ is clearly not adapted for general functions ϕ.

This suggests that one solution process $\{Y_t\}_{t\in[0,T]}$ is not enough. Instead, we remember that by the martingale representation theorem, if $\mathbb{E}\big[|\phi(W_T)|\big] < \infty$ there exists a predictable process $\{Z_t\}_{t\in[0,T]}$ such that

$$\phi(W_T) = \mathbb{E}\big[\phi(W_T)\big] + (Z \bullet W)_T,$$

and thus we obtain the existence of a pair of processes (Y, Z) satisfying the equation

$$dY_t = Z_t dW_t, \qquad Y_T = \phi(W_T)$$

by setting

$$Y_t := \mathbb{E}\big[\phi(W_T)|\mathcal{F}_t\big] = \mathbb{E}\big[\phi(W_T)\big] + (Z \bullet W)_t.$$

Conceptually, the Z component corrects the possible "non adaptedness" by allowing us to control the randomness of the process Y in such a way as to guarantee that it hits the stochastic terminal value. This leads us to the following definition:

Definition 1.2.1. *Let $m < \infty$. Consider a function*

$$\phi : \Omega \times [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m \times N} \to \mathbb{R}^m$$

such that $\phi(\omega, t, y, z)$ is progressively measurable in (ω, t) and Borel measurable in (y, z). For such a function and an $\mathcal{F}_T$-measurable $\mathbb{R}^m$-valued random variable ξ, a Backward Stochastic Differential Equation is the equation

$$dY_t = -\phi(\omega, t, Y_t, Z_t)dt + Z_t dW_t, \qquad Y_T = \xi. \tag{1.3}$$

The function ϕ is called the driver of a BSDE. A solution to a BSDE is a pair of processes (Y, Z) satisfying the above equation, such that Y is $\mathbb{R}^m$-valued, càdlàg and adapted and Z is $\mathbb{R}^{m \times N}$-valued and predictable.

Given the time horizon T, we denote by $\mathcal{S}^2(0, T)$ the set of $\mathbb{R}^m$-valued progressively measurable processes Y such that

$$\mathbb{E}\big[\sup_{t \in [0,T]} |Y_t|^2\big] < \infty,$$

and by $\mathcal{L}^2_T(\langle W \rangle)$ the set of $\mathbb{R}^{m \times N}$-valued predictable processes Z such that

$$\mathbb{E}\big[\int_0^T |Z_t|^2 dt\big] < \infty.$$

We are now ready to state a fundamental result about BSDEs (for a proof see, e.g. Theorem 19.1.7 in Cohen and Elliott [10]):

Theorem 1.2.1. *Suppose the following conditions are satisfied:*

- *The driver satisfies the following integrability condition:*

$$\mathbb{E}\left[\int_0^T \phi(\omega, t, 0, 0)^2 dt\right] < \infty$$

- *The terminal value is square integrable, i.e. $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R}^m)$.*

- *The driver ϕ is Lipschitz in (y, z). In other words there exists a constant $K > 0$ such that*

$$\|\phi(\omega, t, y, z) - \phi(\omega, t, y', z')\|^2 \le K(\|y - y'\|^2 + \|z - z'\|^2)$$

holds for all $y, y' \in \mathbb{R}^m$, $z, z' \in \mathbb{R}^{m \times N}$ $dt \times d\mathbb{P} - a.e.$

Then there exists a unique solution $(Y, Z) \in \mathcal{S}^2(0, T) \times \mathcal{L}^2_T(\langle W \rangle)$ to the BSDE (1.3).

Remark 1.2.1. *For the sake of parsimony, in the introduction we only deal with diffusion driven BSDEs and corresponding control problems. In the sequel, most importantly in Chapter 2, we will state and use certain results about jump processes as well. For a self contained theoretic account of a general case the interested reader is referred to Cohen and Elliott, [10].*

1.2.2 BSDEs as adjoint equations

As we pointed out earlier, from the probabilistic perspective there exist two main formulations of control problems. The first one, called "weak", describes the situation where only drift of the forward process is controlled, allowing one to treat each control (using Girsanov's theorem) as a change of measure. The second one pertains to the more general scenario where volatility is affected as well. The former will be studied in great detail in the subsequent sections of the introduction as well as the main text of the thesis. For the remainder of this section we therefore turn our attention to the latter by stating a version of the stochastic maximum principle. Let the controlled dynamics of $\{X_t^u\}_{t\geq 0}$ be governed by the Itô SDE:

$$dX_t^u = b(t, X_t^u, u_t)dt + \sigma(t, X_t^u, u_t)dW_t, \quad X_0^u = x_0 \in \mathbb{R}^n, \tag{1.4}$$

where W denotes a standard Brownian motion under the measure $\mathbb{P}$. We consider the problem of minimising the functional

$$J(x_0, u) = \mathbb{E}\left[\int_0^T L(t, X_t^u, u_t)dt + g(X_T^u)\right]. \tag{1.5}$$

Definition 1.2.2. *The function $H : [0, T] \times \mathbb{R}^n \times \mathcal{U} \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}$ defined by*

$$H(t, x, u, y, z) := b'(t, x, u)y + \mathrm{Tr}(\sigma'(t, x, u)z) + L(t, x, u), \tag{1.6}$$

is called "generalised Hamiltonian" and it is assumed that H is differentiable in x. Furthermore, the BSDE

$$dy(t) = -\nabla_x H\big(t, X_t^u, u_t, y(t), z(t)\big)dt + z(t)dW_t, \quad y(T) = \nabla_x g(X_T), \tag{1.7}$$

is called "first order adjoint equation" of the problem (1.5).

The following condition for the necessary stochastic maximum principle can then be established (see Theorem 3 in [35]):

Theorem 1.2.2. *Suppose $\{u_t\}_{t\geq 0}$ is the optimal control process and $\{X_t^u\}_{t\geq 0}$ are the corresponding controlled dynamics for the problem (1.5). Then there exists a solution (y, z) to the BSDE (1.7).*

Remark 1.2.2. *Theorem 1.2.2 illustrates the importance of BSDEs, stating that the existence of solutions to certain equations is necessary for the control to be optimal. We will return to this in the ergodic framework in Chapter 4.*

1.2.3 BSDEs in infinite horizon

So far we have only discussed BSDEs with finite deterministic terminal time. Given that our ultimate goal is to deal with the ergodic cost functional, some theory in infinite horizon is clearly required. In this section we make a important step by considering the following equation:

$$Y_t = Y_T + \int_t^T [\phi(s, Y_s, Z_s) - \alpha Y_s]ds - \int_t^T Z_s dW_s, \tag{1.8}$$

where $\alpha > 0$ and $0 \leq t \leq T < 0$. This type of BSDE was initially introduced by Briand and Hu in [6] and then generalised by Royer in [43] and Hu and Tessitore in [24]. The main result is as follows (see Chapter 2 for the proof of a more general version):

Theorem 1.2.3. *Suppose the following conditions on the driver ϕ hold:*

- *ϕ is uniformly Lipschitz in z with Lipschitz constant K.*
- *ϕ is continuous w.r.t y and there exists a continuous and increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

  $$|\phi(t, y, z)| \leq \psi(|y|) + |\phi(t, 0, z)|$$

  holds for all $t \geq 0, y \in \mathbb{R}, z \in H$.
- *ϕ is monotone w.r.t y, that is for all $t \geq 0, y, y' \in \mathbb{R}, z \in H$*

  $$(y - y')(\phi(t, y, z) - \phi(t, y', z)) \leq 0 \quad \mathbb{P} - a.s.$$

- *There exists a constant $M > 0$ such that for all $t \geq 0$, $|\phi(t, 0, 0)| < M$ $\mathbb{P} - a.s.$*

Then there exits a solution pair (Y, Z) to the BSDE (1.8) such that $\{Y_t\}_{t\geq 0}$ is a continuous adapted process bounded by M/α. Moreover, the solution is unique in this class.

The reason the equation (1.8) is of interest to us is that it allows us to deal with infinite horizon optimisation. In order to see how, consider the problem of minimising

$$J(x_0, u) = \mathbb{E}\left[\int_0^\infty e^{-\alpha t} g(X_t^u, u_t) dt\right]$$

over the space of predictable processes $u : \Omega \times \mathbb{R}_+ \to \mathcal{U}$, where $\{X_t^u\}_{t\geq 0}$ represents the forward process associated with control u, with dynamics given by

$$dX_t^u = (AX_t^u + F(X_t^u) + G(X_t^u)R(u_t))dt + G(X_t^u)dW_t, \quad X_o^u = x_0. \tag{1.9}$$

In other words the control u affects the drift of X. We define the Hamiltonian as

$$\phi(x, z) := \inf_{u \in \mathcal{U}} \{g(x, u) + zR(u)\}. \tag{1.10}$$

Then one can find an optimal pair $(\bar{u}, \bar{X})$ by solving the BSDE (1.8) with ϕ as the driver. The proof is a combination of Itô's formula and a measurable selection theorem, and we will see a very similar one in the next section. For a detailed account of the case above see Hu and Tessitore [24].

1.2.4 From BSDEs to Ergodic BSDEs

Over the course of the previous sections we showed that BSDEs provide an extremely useful and flexible probabilistic approach to dealing with both finite and infinite horizon linear stochastic optimal control problems. Therefore it is reasonable to expect that there is a BSDE–based framework that would prove natural for understanding the case of ergodic cost. One such framework that emerged relatively recently is based on Ergodic BSDEs, an extension of BSDEs which takes the form

$$Y_t = Y_T + \int_t^T [\phi(X_u^x, Z_u) - \lambda] du - \int_t^T Z_u dW_u, \tag{1.11}$$

where $\{X_t^x\}_{t \geq 0}$ is a forward process starting at x taking values in a separable Hilbert space H. The constant $\lambda \in \mathbb{R}$ is a part of the solution. The equations of this type are fairly recent and were first introduced in 2009 by Furhman, Hu and Tessitore, [16]. Since then they have been also studied in the context of Markov chains by Cohen and Hu, [12] and in discrete time case by Allan and Cohen, [2].

Remark 1.2.3. *It is worth pointing out that an EBSDE is not strictly speaking a BSDE since there is no terminal value. However, given that a solution is obtained as a limit of solutions to infinite horizon BSDEs of the form (1.8) as $\alpha \to 0$, the name is appropriate. Another way of thinking about it is the following: in a standard forward-backward system one "runs" the forward component up to maturity, and then solves the backward component in reverse. In the ergodic case one does the same, but, instead of the value of the forward process at a certain deterministic finite time, one obtains instead the invariant law by letting the forward process run for long enough. This information is captured in the additional component λ of the solution.*

We leave the details of construction of solutions to the equation (1.11) to Chapter 2 (we will deal with a more general version that includes jumps and time inhomogeneity of the forward dynamics). For now we just state that there indeed exists a solution triple (Y, Z, λ) such that $\{Y_t\}_{t\geq 0}$ is polynomially growing in the forward process. In other words

$$|Y_t| < c_x(1 + \|X_t^x\|^2). \tag{1.12}$$

Using this bound we provide below a heuristic argument that allows one to gain a better understanding of the term λ. To this end consider the finite horizon equation

$$Y_t^{T,x} = \int_t^T \phi(X_u^x, Z_u^{T,x})du - \int_t^T Z_u^{T,x} dW_u, \tag{1.13}$$

with zero terminal value. We know that under certain conditions on the driver ϕ (most importantly, ϕ is Lipschitz in z), for each $T > 0$ there exists a unique solution $(Y, Z) \in \mathcal{S}^2 \times \mathcal{L}^2(W)$ to (1.13). Then

$$Y_0^{T,x} - Y_0 = -Y_T + \int_0^T [\phi(X_u^x, Z_u^{T,x}) - \phi(X_u^x, Z_u)]du + \lambda T - \int_t^T [Z_u^{T,x} - Z_u]dW_u.$$

Given that ϕ is Lipschitz in z, we can change measure to remove the drift. Then, taking expectations with respect to the new measure $\mathbb{Q}$, we obtain

$$T\lambda = Y_0^{T,x} - Y_0 + \mathbb{E}^Q[Y_T],$$

and, taking (1.12) into account and assuming that X^x has finite second moment, we see that

$$\frac{Y_0^{T,x}}{T} \to \lambda$$

uniformly on all bounded sets in H. For a rigorous proof of this and some other large time behaviour results see Hu, Madec and Richou [22].

We are finally ready to return to the original problem of ergodic optimal control. Consider the framework of the previous section where the control u affects the drift of the forward process X^u. We assume that under some measure $\mathbb{P}^{u,T} \sim \mathbb{P}$ the dynamics of the controlled process X^u on $[0, T]$ are given by (1.9). Our goal is to minimise the functional

$$J(x_0, u) = \limsup_{T\to\infty} T^{-1}\mathbb{E}^{u,T}\left[\int_0^T L(X_t, u_t)dt\right].$$

Let (Y, Z, λ) be a solution to the EBSDE (1.11) with the driver defined as in (1.10), such that the bound (1.12) holds. Then

$$\begin{aligned}
-dY_t &= [\phi(X_t^u, Z_t) - \lambda]dt - Z_t dW_t \\
&= [\phi(X_t^u, Z_t) - \lambda]dt - Z_t dW_t^u - Z_t R(u_t)dt,
\end{aligned}$$

and thus

$$\lambda = \frac{1}{T}\mathbb{E}^{u,T}[Y_T - Y_0] + \mathbb{E}^{u,T}\frac{1}{T}\int_0^T [\phi(X_t^u, Z_t) - Z_t R(u_t) - L(X_t^u, Z_t)]dt$$
$$+ \frac{1}{T}\mathbb{E}^{u,T}\int_0^T L(X_t^u, Z_t)dt.$$

By definition of the Hamiltonian, we deduce that

$$\frac{1}{T}\mathbb{E}^{u,T}\int_0^T L(X_t^u, Z_t)dt \geq \frac{1}{T}\mathbb{E}^{u,T}[Y_0 - Y_T] + \lambda.$$

In the sequel we will show that there exists a constant $C > 0$, independent of T, such that

$$\mathbb{E}^{u,T}\|X_T^u\|^2 \leq C(1 + \|x_0\|^2),$$

where, as always, $x_0 = X_0^u$. Then, using the bound (1.12), we immediately see that

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}^{u,T}[Y_0 - Y_T] = 0,$$

implying that

$$J(x_0, u) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}^{u,T}\int_0^T L(X_t^u, Z_t)dt \geq \lambda.$$

It is also clear that we have equality if the infimum in the Hamiltonian is attained. Moreover, even if the infimum in (1.10) is not attained, there exists a control $\{\tilde{u}_t\}_{t\geq 0}$, such that $J(x_0, \tilde{u}) = \lambda$. For details about this last statement, see Allan and Cohen, [2].

1.3 Structure of the thesis

The aim of this section is to provide a short summary of results contained in each chapter of the sequel.

1.3.1 EBSDEs with jumps and time dependence

In this chapter we extend the existing theory in two natural ways. The first generalisation is to add jumps to the diffusion setting of Furhman et al. in [14]. In other words, our aim is to be able to use an EBSDE–based approach for ergodic optimal control problems in the case where stochastic dynamics are given with reference to a Lévy process. Optimal control of jump diffusions has been of great interest recently, primarily due to its possible application to network control problems and hybrid stochastic systems. From the standpoint of finance it allows us to factor shocks into the model. The second extension is to incorporate time-dependency. This will allow us to consider dynamics with seasonal components, such as business cycles.

In more technical terms, we look at the framework of real-valued forward-backward stochastic differential equations with jumps, as introduced by Barles in [4] and later extended by Royer in [44]. In this setting the filtration is generated by a Wiener process on a separable Hilbert space H together with a Poisson random measure.

The main result of the chapter is a (constructive) proof of existence of a Markovian solution to an ergodic BSDE

$$Y_t = Y_T + \int_t^T [f(X_u, Z_u, U_u) - \lambda] du - \int_t^T Z_u^* dW_u - \int_t^T \int_{H \setminus \{0\}} U_s(x) \tilde{N}(ds, dx),$$

where the forward dynamics are given by an H-valued process X, governed by the non-autonomous mild Itô SDE:

$$dX_t = A(t)X_t dt + F_t(X_t)dt + G(t)dL_t, \quad X_\tau = x.$$

Here L is an H-valued square-integrable Lévy martingale, $\{A(t)\}_{t\geq 0}$ generates an exponentially bounded evolution family, all coefficients are T-periodic for some $T > 0$ and satisfy a number of technical assumptions (see Assumption 2 in Chapter 2). We note that existence and uniqueness of solutions to equations of this form were studied by Knäble in [29] and by Peszat and Zabczyk in [36].

We also prove that the Markovian solution we construct is unique (up to a shift) in the class of polynomially growing functions. We conclude by showing applications to classical ergodic stochastic control, as well as to a problem related to power plant evaluation.

1.3.2 Ergodic stochastic differential games

The main goal of this chapter is to prove the existence of an equilibrium in the case where we have two drift controlling players who are ergodic, i.e. they choose their strategies in infinite horizon and optimise the long run average. Similarly to Chapter 2, in present work we are concerned with the "weak" game formulation (however, in this chapter we deal with the noise driven by a finite dimensional diffusion, so there are no jumps or time inhomogeneity). In other words, given an arbitrary pair of admissible controls (u, v) (i.e. predictable stochastic processes $(\{u_t\}_{t\geq 0}, \{v_t\}_{t\geq 0})$), we define the Girsanov density

$$\rho_T^{u,v} := \exp\left(\int_0^T R(u_t, v_t) dW_t - \frac{1}{2} \int_0^T \|R(u_t, v_t)\|^2 ds \right)$$

for some function $R : \mathcal{U}_1 \times \mathcal{U}_2 \to \mathbb{R}^N$, satisfying $\|R(\cdot,\cdot)\| \le \bar{C}$ for some $\bar{C} > 0$. This generates the probability $\mathbb{P}^{u,v,T} := \rho_T^{u,v}\mathbb{P}$. For each agent $(i = 1, 2)$ we define ergodic payoffs

$$J^i(x_0, u, v) = \limsup_{T\to\infty} T^{-1}\mathbb{E}^{u,v,T}\left[\int_0^T L_i(X_t, u_t, v_t)dt\right].$$

The goal is to find a 'Nash' equilibrium, that is, an admissible control (u^*, v^*), such that

$$J^1(x_0, u^*, v^*) \le J^1(x_0, u, v^*) \text{ and } J^2(x_0, u^*, v^*) \le J^2(x_0, u^*, v)$$

holds for all admissible (u, v). In a standard fashion (see, e.g. [21]), we impose the so-called generalised Isaac's condition on the Hamiltonian, in order to ensure the attainability of the infimum simultaneously for both players. We then prove that the existence of a saddle point for the game follows from the existence of a solution to a system of Ergodic BSDEs with continuous coefficients. Using a modified version of Picard iteration (not dissimilar in nature to the fixed point construction used by Hu and Tang in [23]), along with an array of estimates for solutions of EBSDEs, established in [14] and [11], we prove that such a system does admit a solution. We then extend these results to the case of finite or countably many players, as well as two asymmetric agents.

As a bonus we obtain the existence of a solution to a class of one-dimensional EBSDEs with continuous drivers of linear growth. This is an important class, since it is known that for drivers with quadratic growth the existence of solutions requires much heavier assumptions (for details in a PDE framework see Robertson and Xing [42]).

1.3.3 Classical adjoints for ergodic stochastic control

In this chapter we consider ergodic optimal control of a diffusion process $\{X_t^u\}_{t\ge0}$, taking values in $\mathbb{R}^n$, where both drift and volatility are controlled. In other words, we are trying to minimise the value functional

$$J(x, u) = \limsup_{T\to\infty} T^{-1}\mathbb{E}\left[\int_0^T L(X_t^u, u_t)dt\right],$$

where the dynamics of $(X_t^u)_{t\ge0}$ are governed by

$$dX_t = b(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dW_t, \quad X_0 = x_0.$$

We observe that the classical finite horizon Hamiltonian (see, e.g. [35]) defined by

$$H(t, x, u, y, z) := b'(t, x, u)y + \mathrm{Tr}(\sigma'(t, x, u)z) + L(t, x, u)$$

is appropriate for this case as well. We show that the corresponding adjoint BSDE becomes an infinite horizon one, with a 'discount' term $\nabla_x b(t, X_t^u, u_t)Y(t)$. We then establish the equivalence between the conditions required for the adjoint equation to admit a unique solution and strong dissipativity of the forward process $(X_t^u)_{t\geq 0}$ under all admissible controls. We show that in our framework the latter implies ergodicity. In other words, we prove that the 'strong' control formulation via adjoints is only viable for systems that regularise quickly, and in some sense the converse also holds.

Chapter 2

Ergodic BSDEs with jumps and time dependence

2.1 Introduction

In this Chapter we look at ergodic BSDEs in the case where the forward dynamics are given by the solution to a non-autonomous (time-periodic coefficients) Ornstein–Uhlenbeck-like SDE with Lévy noise, taking values in a separable Hilbert space. We establish the existence of a unique bounded solution to an infinite horizon discounted BSDE. We then use the vanishing discount approach, together with coupling techniques, to obtain a Markovian solution to the EBSDE. We also prove uniqueness under certain growth conditions. Applications are then given, in particular to risk-averse ergodic optimal control and power plant evaluation under uncertainty. The Chapter is organised as follows: in Section 2 we introduce the necessary notation and discuss the preliminaries; Section 3 is devoted to the results concerning solutions to the forward SDE; in Section 4 EBSDEs are introduced and main results are proven. Section 5 contains several examples of the application of EBSDEs to optimal ergodic control.

2.2 Notation and general assumptions

For the rest of this chapter, let H be a separable real Hilbert space with scalar product $\langle \cdot, \cdot \rangle_H$ and norm $\| \cdot \|_H$. To simplify notation we will denote them respectively $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. Since we shall be working with general separable Hilbert spaces, we will require a number of extensions of classical results. The main purpose of this section is to state them. We start with a definition of Q-Wiener and Lévy processes on a general Hilbert space H:

Definition 2.2.1. *A stochastic process $L = (L(t), t \geq 0)$ taking values in H is called a Lévy process if $L(0) = 0$, the process L is stochastically continuous, and it has stationary, independent increments, in the sense that the law $\mathcal{L}(L(t) - L(s))$ depends only on the difference $t - s$. By stochastic continuity we mean that for every $\epsilon > 0$ and $t \geq 0$, $\lim_{s \to t} \mathbb{P}(|L(s) - L(t)| > \epsilon) = 0$.*

Remark 2.2.1. *A useful way of thinking about the Lévy process taking values in a Hilbert space is through the series expansion, i.e. assuming that $\{e_n\}_{n\geq 1}$ is an orthonormal basis of H, we have*

$$L(t) = \sum_{n\geq 1} \langle L(t), e_n \rangle e_n = \sum_{n\geq 1} L_n(t) e_n,$$

where L_n are real-valued càdlag Lévy processes.

Definition 2.2.2. *An H-valued stochastic process $\{W_t, t \geq 0\}$ is called a Q-Wiener process if*

- *$W_0 = 0$,*
- *W has continuous trajectories,*
- *W has independent increments,*
- *the law of $W_t - W_s$ is Gaussian with mean zero and covariance $(t - s)Q$, for all $0 \leq s \leq t$ in the sense that for any $h \in H$ and $0 \leq s \leq t$, the real-valued random variable $\langle h, W_t - W_s \rangle_H$ is Gaussian with mean zero and variance $(t-s)\langle Qh, h\rangle_H$.*

For a given process $\{L_t\}_{t\geq 0}$ and a set $A \in H$ we denote by $N_t(A) = N(t, A)$ the (random) number of 'jumps of size A' up to time t, that is $N(t, A) := card\{s \in [0, t] | \Delta L_s \in A\}$. Denoting $\mathcal{B}(H)$ the Borel σ-algebra, we say that $A \in \mathcal{B}(H)$ is bounded below if $0 \notin \bar{A}$, where $\bar{A}$ denotes the closure of A. Proof of the following result can be found, for example, in [1]:

Proposition 2.2.1. *If A is bounded below, then $N(\cdot, A) = \{N(t, A), t \geq 0\}$ is a Poisson process with intensity $\nu(A) = \mathbb{E}[N(1, A)]$.*

We remark that since we assume H to be separable, it is also Polish, and therefore the space $B = H \backslash \{0\}$ endowed with its Borel σ-field $\mathcal{B}$ is a Blackwell space. We need this since stochastic integration with respect to Poisson measures is well defined on Blackwell spaces. Following [41] we adopt the definition of the Itô stochastic integral with respect to the compensated random measure $\tilde{N}(dt, dx) := N(dt, dx) - \nu(dx)dt$

as an isometry, which extends the classical isometry on simple predictable processes. That is if we define the set of progressively measurable processes

$$\mathcal{L}^2(\tilde{N}) = \left\{ \mathcal{P} \otimes \mathcal{B} - \text{measurable processes } \sigma : \mathbb{E}\left[\int_0^t \int_B \|\sigma(s,x)\|^2 \nu(dx)ds \right] < \infty \right\},$$

then, for every $\sigma \in \mathcal{L}^2(\tilde{N})$, we have

$$\mathbb{E}\left[\left\| \int_0^t \int_B \sigma(s,x)\tilde{N}(ds,dx) \right\|^2 \right] = \mathbb{E}\left[\int_0^t \int_B \|\sigma(s,x)\|^2 \nu(dx)ds \right].$$

As we shall see below, any Lévy process that is also a martingale can be represented as a sum of a Wiener process and a compensated Poisson process. Therefore, combining the above with the standard integration theory for Brownian motion, we have a well defined stochastic integrand.

Remark 2.2.2. *It is well known that in finite dimensional spaces any Lévy process has a càdlàg modification. However, in general this property fails in Banach spaces. But since we work with Lévy martingales, the processes we consider can be assumed to satisfy this property (see, e.g. [36]).*

The following version of the celebrated Lévy–Itô decomposition for an H-valued Lévy process can be found, for example, in [29]:

Theorem 2.2.1. ***(Itô–Lévy Decomposition)*** *If L is an H-valued Lévy process, there is a drift vector $b \in H$, a Q-Wiener process W on H and a random measure N, such that $\mathcal{W}$ is independent of $N_t(A)$ for any A that is bounded below, and we have*

$$L_t = bt + W(t) + \int_{||x||<1} x\tilde{N}(t,dx) + \int_{||x||\geq 1} xN_t(dx).$$

Remark 2.2.3. *For the rest of this chapter we will only be interested in the case of Lévy martingales, and therefore the decomposition above takes the following form*

$$L_t = W(t) + \int_0^t \int_B x\tilde{N}(dt,dx).$$

where $\tilde{N}(dt,dx)$ is the compensated Poisson random measure.

Assumption 2.2.1. *Since we will mainly be dealing with square-integrable Lévy martingales we will require the following condition to hold:*

$$\int_B \|x\|^2 \nu(dx) < \infty.$$

Given the fact that our Lévy process is square integrable, this assumption says that there are not too many big jumps. It is not necessary in order to introduce stochastic integration with respect to Lévy processes in a separable Hilbert space, but it will prove crucial for the coupling argument in Section 3.3.

Throughout the sequel we will be repeatedly using methods involving measure changes. To that end, we need a version of the Girsanov theorem. The following is a reformulation of Theorem 15.3.10 in [10]:

Theorem 2.2.2. *Suppose we have uniformly bounded functions $\beta : \Omega \times [0,T] \to H$ and $\gamma : B \times \Omega \times [0,T] \to \mathbb{R}^+$, such that $\gamma(\cdot,\omega,t) - 1 \in L^2(\nu(dx))$ for all $(\omega,t) \in \Omega \times [0,T]$. We define*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}\bigg(\int_{[0,\cdot]} \beta(\omega,t) dW_t + \int_{[0,\cdot]} \int_B (\gamma(x,\omega,t) - 1)\tilde{N}(dx,dt) \bigg)_T,$$

where $\mathcal{E}$ denotes the Doléans-Dade exponential. Then $\Lambda_t := \frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{F}_t}$ is a positive square integrable martingale, and under $\mathbb{Q}$

$$W^{\mathbb{Q}} := W - \int_{[0,\cdot]} \beta(\omega,t)dt,$$

is a Wiener process, where the integral is understood as a series (see Remark 2.2.1). The compensator of N under $\mathbb{Q}$ is given by

$$\nu^{\mathbb{Q}}(dx,dt) := \gamma(x,\omega,t)\nu(dx)dt.$$

Remark 2.2.4. *In general, the assumption that β is uniformly bounded is stronger than is necessary. However, it suffices for the purposes of this work, since it allows us to eliminate bounded drifts by changing measure.*

2.3 The forward SDE

In this section we study the properties of the 'forward' process, henceforth denoted $\{X\}_{t\geq\tau}$, for some $\tau \geq 0$. Its role can be understood intuitively as a source of stochasticity in the driver of a BSDE. We first solve the dynamics of X in the forward way, then plug the obtained values into the BSDE while running it backwards. In our case, we assume that X is a solution to an Ornstein–Uhlenbeck type equation driven by Lévy noise on a separable Hilbert space H. We also assume that the coefficients are time periodic. This constitutes a natural way to extend the present theory and is of interest in various applications (see Section 2.5).

2.3.1 Context

We start this section by looking at a family $\{A_t\}_{t\geq 0}$ of linear operators on H with common domain $D(A)$ dense in H, assuming that $A : \mathbb{R}^+ \times D(A) \to H$ generates an exponentially bounded evolution family according to the following definition (see [29]):

Definition 2.3.1. *An exponential bounded evolution family on H is a two-parameter family $\{U(t,s)\}_{t\geq s}$ of bounded linear operators on H such that*

- *$U(s,s) = I$ and $U(t,s)U(s,r) = U(t,r)$ for $r \leq s \leq t$,*
- *for each $x \in H$, the map $(t,s) \to U(t,s)x$ is continuous on $\{(s,t) : 0 \leq s \leq t < \infty\}$, and*
- *there exists $M > 0$ and $\mu > 0$ such that $||U(t,s)||_{op} \leq Me^{-\mu(t-s)}$ for $s \leq t$.*

Remark 2.3.1. *By 'generates' we mean that for $0 \leq s \leq t$ we have*

$$\frac{d}{dt}U(t,s)x = A(t)U(t,s)x$$

for all $x \in H$.

Remark 2.3.2. *One way of thinking about exponential bounded evolution families is as a time-dependent infinite dimensional modification of the familiar case where A is a real $d \times d$ matrix, the eigenvalues of which have non-positive real parts. Then U takes the form e^{tA}, and all conditions are satisfied.*

We now consider the H-valued process X given by the following non-autonomous mild Itô SDE

$$X(t,\tau,x) = U(t,\tau)x + \int_\tau^t U(t,s)F_s(X(s,\tau,x))ds + \int_\tau^t U(t,s)G(s)dL(s), \tag{2.1}$$

which is a mild version of the following Cauchy problem,

$$dX_t = A(t)X_t dt + F_t(X_t)dt + G(t)dL_t, \quad X_\tau = x, \quad t \geq \tau. \tag{2.2}$$

Conditions for existence and uniqueness of the solution will be formulated in Theorem 2.3.1. For the rest of the chapter we assume the following

Assumption 2.3.1. *(i) The family A_t generates an exponentially bounded evolution family. Their adjoints $A^*(t)$ also have a common domain, which is dense in H.*

(ii) $F : \mathbb{R}^+ \times H \to H$ *is a uniformly bounded family of measurable maps with common domain* $D(F)$*, which is dense in* H*.*

(iii) $(\Omega, \mathcal{F}, \mathbb{P})$ *is a complete probability space, and the pair* $(W, \tilde{N})$ *that comes from the Itô–Lévy decomposition of* L *has the predictable representation property* [1] *in the filtration* $\{\mathcal{F}_t\}_{t\geq 0}$*.*

(iv) $\{G_t\}_{t\geq 0}$ *is a uniformly bounded family of linear operators in* $L(H, H)$ *with common domain* $D(G)$ *dense in* H *and with bounded inverses.*

(v) The linear operator $U(t, \cdot)G(\cdot)$ *is uniformly bounded in the Hilbert–Schmidt norm* $\|\cdot\|_t$*, defined by*

$$\|S\|_t := \left[\mathbb{E}\left(\int_0^t \mathrm{Tr}(S_u Q S_u^*) du\right)\right]^{\frac{1}{2}},$$

where Q *is the covariance operator of the Wiener part of* L*.*

(vi) Coefficients $A(t)$*,* $F(t, \cdot)$ *and* $G(t)$ *are* T^**– periodic for some* $T^* \geq 0$*, that is* $A(t + T^*) = A(t)$*, and similarly for* F *and* G*.*

Remark 2.3.3. *The norm* $\|\cdot\|_t$ *defined above allows for the following isometry:*

$$\mathbb{E}\left(\left\|\int_0^t S_t dW_t\right\|^2\right) = \|S\|_t^2,$$

where $\mathcal{W}$ *is a* Q*-Wiener process and* Q *is a trace class operator.*

For the situation with autonomous coefficients, namely when $A_t = A$, $G(t) = G$ and $F(t, \cdot) = F(\cdot)\ \forall t \geq 0$, the following theorem is a direct corollary of Theorem 9.29 in [36]:

Theorem 2.3.1. *Suppose that* A*,* F*,* G *are time homogenous and assume that*

(i) F *and* G *satisfy Assumption 2.3.1,*

(ii) F *is Lipschitz-continuous.*

Then, for all $\tau \geq 0$*, and any* $\mathcal{F}_\tau$*-measurable square integrable random variable* $\bar{X}_\tau$ *in* H*, the equation*

$$dX_t = (AX_t + F(X_t))dt + GdL_t, \quad X(\tau) = \bar{X}_\tau \tag{2.3}$$

[1]That is, every $\mathcal{F}_t$-local martingale can be represented as a sum of integrals with respect to W and $\tilde{N}$ with predictable integrands.

has a unique (up to modification) mild solution (i.e. the integral form) with a càdlàg version. Moreover, $\forall 0 \leq \tau \leq T < \infty$, there exists $C < \infty$ such that, for all $x, y \in H$,

$$\sup_{t\in[\tau,T]} \mathbb{E}\|X(t,\tau,x) - X(t,\tau,y)\|^2 \leq C\|x-y\|^2. \tag{2.4}$$

Remark 2.3.4. *Suppose now that F is bounded and measurable, and can be approximated as a uniform limit of Lipschitz functions. Then one could adapt the proof of Theorem 10.14 in [36] to show that there exists a unique càdlàg mild solution to the equation (2.3). In other words, there exists an adapted H-valued càdlàg process $\{X_t\}_{t\geq\tau}$, such that the equation*

$$X_t = e^{(t-\tau)A}x + \int_\tau^t e^{(s-\tau)A}F(X_s)ds + \int_\tau^t e^{(t-\tau)A}Gd\tilde{L}(s),$$

is satisfied $\mathbb{P}-a.s.$ Moreover, the estimate (2.4) still holds.

Remark 2.3.5. *Theorem 2.3.1 can be extended to the non-autonomous case in a straightforward manner. The linear case has been treated in [29]. For semilinear equations of the form of (2.1), one can prove existence by the standard fixed-point argument, and uniqueness by Grönwall's lemma. Since this is not the primary interest of this work, we omit the proof.*

To simplify notation, we write $U_t^s = U(t,s)$ and $U_t = U(t,0)$. Making sure that the stochastic convolutions in (2.1) exist in the sense of Böchner integral, the following result can be found, for example, in [29]:

Theorem 2.3.2. *If U is an exponentially bounded family and G satisfies Assumption 2.3.1, then the stochastic convolution $X_{U,G} := \int_\tau^t U(t,r)G(r)dL(r)$ exists in the following sense:*

$$\int_\tau^t U(t,r)G(r)dL(r) = \int_\tau^t U(t,r)G(r)dW(r) + \int_\tau^t \int_B U(t,r)G(r)x\tilde{N}(dr,dx),$$

where the summands exist as martingale integrals as discussed above.

Definition 2.3.2. *Whenever $f : H \to \mathbb{R}$ is measurable and bounded, we call*

$$P(s,t)[f](x) := \mathbb{E}\big[f(X(t,s,x))\big]$$

the two-parameter transition semigroup associated with the solution X of (2.1). To simplify notation, in the sequel we will be particularly interested in the case $s = 0$, and we write $X_t^x := X(t,0,x)$ and $\mathcal{P}_t[f](x) = \mathbb{E}[f(X_t^x)]$. However, all our results, including the coupling estimate, can be easily extended to the more general $P(s,t)[f](\cdot)$ case.

2.3.2 Coupling estimate

The goal of this subsection is to obtain the exponential convergence of laws corresponding to two solutions of (2.1) with different initial conditions. We need this convergence to be uniform in the class of processes with bounded nonlinear part. In other words, our aim is to prove the following theorem:

Theorem 2.3.3. *Let $F : \mathbb{R}^+ \times H \to H$ be any Lipschitz function and $\{A_t\}_{t\geq 0}$ be fixed and generate an exponentially bounded evolution family. Then there exist constants $C > 0$ and $\rho > 0$ such that, for any bounded continuous function $\psi : H \to \mathbb{R}$,*

$$|P(\tau,t)[\psi](x) - P(\tau,t)[\psi](y)| \leq C(1 + ||x||^2 + ||y||^2)e^{-\rho(t-\tau)} \sup_{u\in H} ||\psi(u)|| \qquad (2.5)$$

where our constants C and ρ depend only on $\sup_{u\in H} F(u)$ and on the constant μ of the evolution family $\{A_t\}_{t\geq 0}$.

This estimate will be crucial in the sequel when we show the existence of a solution to an EBSDE. In our proof, we follow the derivation of Theorem 2.4 in [14] and Theorem 2.8 in [40]. We require a number of results from the theory of coupling. A survey can be found in [32]. The rest of the section is organised as follows: we begin by stating the necessary facts from the theory of coupling (see [32] or more details). Having obtained the necessary machinery (most importantly Lemmas 2.3.2 and 2.3.3) we then prove Theorem 2.3.3.

Definition 2.3.3. *Given two probability measures μ_X and μ_Y on measurable spaces R_X and R_Y, a coupling is a random variable (Z_X, Z_Y) taking values in the product space $R_X \times R_Y$, whose joint distribution has marginal distributions μ_X and μ_Y respectively.*

Definition 2.3.4. *Two processes X and Y are said to admit a successful coupling on $[T_1, T_2]$ when there exists $t \in [T_1, T_2]$ such that $X_t = Y_t$.*

Lemma 2.3.1. *(Theorem 5.2 in [32]) For any two probability measures (μ_1, μ_2) on a measurable space $(E, \mathcal{E})$ there exists a coupling (Z, Z') such that*

- $\|\mu_1 - \mu_2\|_{TV} = 2\mathbb{P}(Z \neq Z')$,
- *Z and Z' are independent conditional on $\{Z \neq Z'\}$, provided that the latter event has positive probability,*

- $\mathbb{P}(Z = Z', Z \in A) = (\mu_1 \wedge \mu_2)(A)$ [2],

where

$$\|\mu_1 - \mu_2\|_{TV} = \sup_{\Gamma \in \mathcal{E}} |\mu_1(\Gamma) - \mu_2(\Gamma)|$$

is the standard total variation norm for measures on $(E, \mathcal{E})$.

Remark 2.3.6. *The lemma above shows a slightly different way of thinking about couplings. Given the marginal laws, we 'manually' construct random variables following them. In the process of this construction our goal is to tweak these variables in such a way as to maximise the probability of them meeting. In that case, by a coupling we mean the pair (X, X') of random variables constructed.*

In the sequel we will require the following auxiliary lemma, where (in principle) we couple the terminal values of solutions to (2.1) by constructing a *bridge*.

Lemma 2.3.2. *Fix $\tilde{T} > 0$ and consider $X^{x,k}$ and $X^{y,k}$ the mild solutions (with $\tau = k\tilde{T}$) of (2.1) for $k \geq 0$ with initial conditions $x \in B_R(0)$ and $y \in B_R(0)$, $x \neq y$. We denote by μ_x^k and μ_y^k the respective laws of $X^{k,x}_{(k+1)\tilde{T}}$ and $X^{k,y}_{(k+1)\tilde{T}}$. Set*

$$Y_t^k = X_t^y + \frac{(k+1)\tilde{T} - t}{(k+1)\tilde{T}} U_t^\tau (x - y)$$

and observe that the law of $Y^k_{(k+1)\tilde{T}}$ is μ_y^k [3]. Then, for every $p > 1$, there exists C such that

$$\int_H \left(\frac{d\mu_y^k}{d\mu_x^k}(u)\right)^p d\mu_x^k(u) \leq C. \tag{2.6}$$

Proof: Given that X^y satisfies (2.2), we immediately notice that

$$dY_t = (A(t)Y_t + F_t(Y_t))dt + G(t)dL(t) - \left(\frac{1}{(k+1)\tilde{T}} U_t^\tau (x-y) + F_t(Y_t) - F_t(X_t^y)\right)dt.$$

We now define

$$b^*(s) = \left(\frac{1}{(k+1)\tilde{T}} U_t^\tau (x-y) + F_t(Y_t) - F_t(X_t^y)\right) G^{-1}(t).$$

[2] By $\mu_1 \wedge \mu_2$ we denote the greatest minorization of μ_1 and μ_2, namely the subprobability with the density $g_1 \wedge g_2$ with respect to $\lambda = \mu_1 + \mu_2$, where $g_1 = d\mu_1/d\lambda$ and $g_2 = d\mu_2/d\lambda$.

[3] This is simply the consequence of the fact that $Y^k_{(k+1)\tilde{T}} = X^y_{(k+1)\tilde{T}}$.

By assumption, G is an invertible operator and there exists $C_1 > 0$ such that $\|G^{-1}\|_{op} \leq C_1$. Given our assumptions, it is clear that

$$\|b^*(s)\| \leq 2C_1\bigg(\|F\|_\infty + \frac{MR}{\tilde{T}}\bigg),$$

where M comes from the definition of U. We define

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}\bigg(\int_{k\tilde{T}}^{(k+1)\tilde{T}} b^*(t)dW_t\bigg).$$

Since $b(\cdot)$ is uniformly bounded, by Theorem 2.2.2, the process $\Lambda_s := \frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{F}_s}$, defined on $[k\tilde{T}, (k+1)\tilde{T}]$, is a positive square integrable martingale and $\mathbb{Q} \sim \mathbb{P}$. Moreover, under $\mathbb{Q}$, $\tilde{L}_t = L_t - \int_0^t b^*(s)ds$ is a Lévy process with the same triplet (in the sense of Lévy–Khintchine representation) as L under $\mathbb{P}$.

It is clear that $Y_{(k+1)\tilde{T}}$ has the law μ_x^k under $\mathbb{Q}$ and μ_y^k under $\mathbb{P}$. We notice that

$$\int_H \bigg(\frac{d\mu_y^k}{d\mu_x^k}(u)\bigg)^p d\mu_x^k(u) \leq \mathbb{E}\big[\Lambda^p_{(k+1)\tilde{T}}\big],$$

and the claim follows using that, for every $p > 1$, the process $\mathcal{E}(\int_{k\tilde{T}}^{\cdot} pb^*(s)dW_s)$ is a true martingale.

$\square$

Remark 2.3.7. *Since the coefficients in (2.1) depend on time, the laws of solutions with the same initial conditions on various time segments of length $\tilde{T}$ are different. However, since the bound on $b^*(\cdot)$ holds uniformly in time, the bound (2.6) does as well.*

The following lemma can be found in [32]:

Lemma 2.3.3. *Let μ_1 and μ_2 be two equivalent probability measures on some space E. If there exist constants $C > 0$ and $p > 1$ such that*

$$\int_E \bigg[\frac{d\mu_1}{d\mu_2}(x)\bigg]^{p+1} d\mu_2(x) < C,$$

then

$$\int_E \bigg(1 \wedge \frac{d\mu_1}{d\mu_2}(x)\bigg) d\mu_2(x) \geq \bigg[1 - \frac{1}{p}\bigg]\bigg(\frac{1}{pC}\bigg)^{\frac{1}{p-1}}$$

and hence in the notation of Lemma 2.3.1 we get

$$\mathbb{P}(Z = Z') = \big(\mu_1 \wedge \mu_2\big)(E) \geq \bigg[1 - \frac{1}{p}\bigg]\bigg(\frac{1}{pC}\bigg)^{\frac{1}{p-1}}.$$

Proof: (**Of Theorem 2.3.3**) We concentrate on the case of $\tau = 0$ for notational simplicity. As will be clear from the proof, the result can be easily extended to the general two-parameter semigroup. We fix the initial conditions $x, y \in H$. For any two processes X^y and X^x with laws corresponding to the solutions of (2.1) with initial conditions y and x respectively, we denote their respective laws μ_x and μ_y. We now consider the coupled process

$$Y_t = \begin{cases} X_t^y, & t < T, \\ X_t^x, & t \geq T. \end{cases} \tag{2.7}$$

where

$$T = \inf\{s : X_s^x = X_s^y\}$$

is the first meeting time of X^x and X^y. We notice that Y and X^y have the same law. Now, for any bounded $\phi : H \to \mathbb{R}$,

$$\begin{aligned} |\mathcal{P}_t[\phi](x) - \mathcal{P}_t[\phi](y)| &= |\mathbb{E}\phi(X_t^x) - \mathbb{E}\phi(Y_t)| \\ &= |\mathbb{E}\big([\phi(X_t^x) - \phi(Y_t)]\mathbf{1}_{\{T>t\}}\big)| \leq 2 \sup_{x \in H} |\phi(x)| \mathbb{P}(T > t) \end{aligned} \tag{2.8}$$

and by Markov's inequality, for any $\rho > 0$

$$\mathbb{P}(T > t) \leq \mathbb{E}\big[e^{\rho T}\big] e^{-\rho t}.$$

Therefore, in order to arrive at our result, we will construct X^x and X^y, and then prove that there exist constants $\tilde{C} > 0$ and $\rho > 0$, such that

$$\mathbb{E}\big[e^{\rho T}\big] \leq \tilde{C}(1 + \|x\|^2 + \|y\|^2). \tag{2.9}$$

Remark 2.3.8. *One important thing to understand is what we mean by "construct". Since we are trying to prove the convergence of laws, we do not have to work with the original solutions to our forward equation, but can instead patch together the pieces constructed on various time intervals. On each such interval* $[k\tilde{T}, (k+1)\tilde{T}]$ *we let* $X_s^x = X(s, k\tilde{T}, X_{k\tilde{T}}^x)$ *with the Lévy process* L *in (2.1) replaced by* $\tilde{L}$*, and* $X_s^y = X(s, k\tilde{T}, X_{k\tilde{T}}^y)$ *with* L *replaced by* $\bar{L}$*, where* $\tilde{L}$ *and* $\bar{L}$ *are Lévy processes with the same law as* L*. Since the law of the solution does not depend on the choice of the noise,* X^x *and* X^y *have the same laws on* $[k\tilde{T}, (k+1)\tilde{T}]$ *as the original solutions.*

We proceed the following way:

- **(Step 1)** We start by showing that we can choose a time step $\tilde{T} > 0$ and a radius $R > 0$, such that, if we observe two independent solution processes X^x and X^y only at times $\{n\tilde{T}\}_{n\in\mathbb{N}}$, there is an *exponential bound* on the waiting time for both $X^y_{n\tilde{T}}$ and $X^x_{n\tilde{T}}$ to enter $B_R(0)$. The independence here is understood in the the sense that we take two independent copies ($\tilde{L}$ and $\bar{L}$) of the Lévy process L, as in Remark 2.3.8.

- **(Step 2)** Once $X^x_{k\tilde{T}}$ and $X^y_{k\tilde{T}}$ are in $B_R(0)$ for some $k \geq 0$, we lift the independence assumption and construct two solutions $X^{X^x_{k\tilde{T}},k\tilde{T}}$ and $X^{X^x_{k\tilde{T}},k\tilde{T}}$ to (2.1) on $[k\tilde{T},(k+1)\tilde{T}]$ with initial conditions $X^x_{k\tilde{T}}$ and $X^y_{k\tilde{T}}$ respectively. We then infer that, for the constructed solutions, the probability of them meeting on $[k\tilde{T},(k+1)\tilde{T}]$ is bounded from below uniformly in k.

- **(Step 3)** We then iterate these arguments to show that the probability that the two processes we are constructing have not met decays exponentially in time.

Step 1: We begin the formal derivation by showing that there exist positive constants μ, c and D such that

$$\mathbb{E}\|X^x_t\|^2 \leq D(\|x\|^2 e^{-2\mu t} + c).$$

In order to proceed, we define

$$\begin{aligned} V_t &= U(t,0)x + \int_0^t U(t,s)F(X_s)ds, \\ Z_t &= \int_0^t U(t,r)G(r)dW(r), \end{aligned}$$

and

$$Q_t = \int_0^t \int_B U(t,r)G(r)x\tilde{N}(dr,dx).$$

We notice that

$$\begin{aligned} \|V_t\| &\leq \|U_t x\| + \left\| \int_0^t U_t^s F(X_s)ds \right\| \leq \|U_t\|_{op}\|x\| + \bar{F}\int_0^t \|U_t^s\|_{op}ds \\ &\leq Me^{-\mu t}\|x\| + \bar{F}\int_0^t e^{-\mu(t-s)}ds \leq Me^{-\mu t}\|x\| + \frac{\bar{F}}{\mu}. \end{aligned}$$

and thus, by using the inequality $(a+b)^2 \leq 2(a^2+b^2)$,

$$\|V_t\|^2 \leq 2\big(M^2 e^{-2\mu t}\|x\|^2 + \frac{\bar{F}^2}{\mu^2}\big).$$

Using isometries and independence of W and $\tilde{N}$, we also see that

$$\begin{aligned}\mathbb{E}\|Z_t+Q_t\|^2 &= \mathbb{E}\langle Z_t+Q_t, Z_t+Q_t\rangle_H = \mathbb{E}\|Z_t\|^2+\mathbb{E}\|Q_t\|^2\\ &= \|U(t,\cdot)G\|_t^2+\int_0^t\int_B \|U_t^sG(s)x\|^2\nu(ds)ds\\ &\leq C+\|G\|_{op}\int_0^t\int_B\|U_t^s\|_{op}^2\|x\|^2\nu(dx)ds\\ &\leq C+\|G\|_{op}M^2\tilde{D}\leq \tilde{C},\end{aligned}$$

for some constant $\tilde{C}$, where $\tilde{D} = \int_B \|x\|^2\nu(dx)$, M comes from the definition of U, and $\|\cdot\|_t$ is defined as in Assumption 2.3.1. We can now use the fact that $(a+b)^2 \leq 2(a^2+b^2)$ again to get

$$\mathbb{E}\|X_t^x\|^2 \leq D(\|x\|^2e^{-2\mu t}+c) \tag{2.10}$$

for some constants D and c. We remark that all the bounds above hold uniformly in time, that is, even though we do not have the Markov property, we still obtain that for any two solutions X^x and X^y of (2.1),

$$\mathbb{E}\big(\|X^x_{(k+1)\tilde{T}}\|^2+\|X^y_{(k+1)\tilde{T}}\|^2\big|\mathcal{F}_{k\tilde{T}}\big) \leq De^{-2\mu\tilde{T}}(\|X^x_{k\tilde{T}}\|^2+\|X^y_{k\tilde{T}}\|^2)+2Dc, \quad k\geq 0 \tag{2.11}$$

for any fixed $\tilde{T}$. We now define, for fixed $R>0$

$$A_k=\{\|X^x_{k\tilde{T}}\|^2+\|X^y_{k\tilde{T}}\|^2>R\},\quad B_k=\bigcap_{j=0}^k A_j.$$

And by Chebyshev's inequality and (2.11) we obtain

$$\mathbb{P}(A_{k+1}\big|\mathcal{F}_{k\tilde{T}}) \leq \frac{De^{-2\mu\tilde{T}}}{R}(\|X^x_{k\tilde{T}}\|^2+\|X^y_{k\tilde{T}}\|^2)+2\frac{Dc}{R}. \tag{2.12}$$

We now define the matrix

$$C=\begin{pmatrix} De^{-2\mu\tilde{T}} & 2Dc\\ \frac{D}{R}e^{-2\mu\tilde{T}} & \frac{2Dc}{R}\end{pmatrix}.$$

After multiplying (2.11) and (2.12) by $\mathbf{1}_{B_k}$, taking an expectation and noticing that $\mathbf{1}_{B_{k+1}}\leq \mathbf{1}_{B_k}$ we have

$$\begin{pmatrix}\mathbb{E}\big(\|X^x_{(k+1)\tilde{T}}\|^2+\|X^y_{(k+1)\tilde{T}}\|^2\big)\mathbf{1}_{B_k+1}\\ \mathbb{P}(B_{k+1})\end{pmatrix}\leq C\begin{pmatrix}\mathbb{E}\big(\|X^x_{k\tilde{T}}\|^2+\|X^y_{(k+1)\tilde{T}}\|^2\big)\mathbf{1}_{B_k}\\ \mathbb{P}(B_k)\end{pmatrix}$$

the inequality being componentwise. Thus, iterating this procedure we arrive at

$$\begin{pmatrix}\mathbb{E}\big(\|X^x_{k\tilde{T}}\|^2+\|X^y_{k\tilde{T}}\|^2\big)\mathbf{1}_{B}\\ \mathbb{P}(B_k)\end{pmatrix}\leq C^k\begin{pmatrix}\|x\|^2+\|y\|^2\\ 1\end{pmatrix},$$

and premultiplying by the row vector $(0,1)$ on both sides we see that

$$\mathbb{P}(B_k) \leq (0,1)C^k \begin{pmatrix} \|x\|^2 + \|y\|^2 \\ 1 \end{pmatrix}.$$

The above discussion is true for any choice of $\tilde{T}$ and R, but now we want to obtain an exponential bound. The set of eigenvalues of C is $\{0, \frac{2Dc}{R} + De^{-2\mu\tilde{T}}\}$, and we need them both to be smaller than one. Therefore, we choose $R = 8Dc$ and $\tilde{T}$ such that $e^{-2\mu\tilde{T}} \leq \frac{1}{4D}$, so that $\frac{Dc}{R} + De^{-2\mu\tilde{T}} \leq \frac{1}{2}$. Given the fact that the corresponding eigenvectors constitute a basis in $\mathbb{R}^2$, the vector $(0,1)$ can be represented in the eigenvector basis, and therefore there exists a constant $\bar{C} > 0$ such that

$$\mathbb{P}(B_k) \leq \bar{C}\left(\frac{1}{2}\right)^k (1 + \|x\|^2 + \|y\|^2).$$

We now define the first hitting time of $B_R(0)$ on our discretised timeline as

$$\tau = \inf\{k\tilde{T} : \|X^x_{k\tilde{T}}\|^2 + \|X^y_{(k+1)\tilde{T}}\|^2 \leq R, k \in \mathbb{N}\},$$

and then

$$\mathbb{P}(\tau \geq k\tilde{T}) \leq \mathbb{P}(B_k) \leq \bar{C}\left(\frac{1}{2}\right)^k (1 + \|x\|^2 + \|y\|^2). \tag{2.13}$$

Take a constant $\tilde{\beta}$ such that $\tilde{\beta}\tilde{T} < \ln 2$. Then

$$\mathbb{E}\left(e^{\tilde{\beta}\tau}\right) = \sum_{k=0}^{\infty} e^{\tilde{\beta}k\tilde{T}}\mathbb{P}(\tau = k\tilde{T}) \leq \sum_{k=0}^{\infty} e^{\tilde{\beta}k\tilde{T}}\mathbb{P}(\tau \geq k\tilde{T}) \leq \frac{\bar{C}}{1 - \frac{e^{\tilde{\beta}\tilde{T}}}{2}}(1 + \|x\|^2 + \|y\|^2)$$

and therefore there exists a constant C_2 such that, for every $\gamma \leq \tilde{\beta}$,

$$\mathbb{E}\left(e^{\gamma\tau}\right) \leq C_2(1 + \|x\|^2 + \|y\|^2). \tag{2.14}$$

The first step of the proof is now concluded.

Step 2: We use the notation introduced in the proof of Lemma 2.3.2.

- By Lemma 2.3.1, on the interval $[k\tilde{T}, (k+1)\tilde{T}]$, there exists a pair of processes $(\tilde{X}^{x,k\tilde{T}}, \tilde{Y}^{k\tilde{T}})$ with terminal time laws μ^k_x and μ^k_y respectively, such that

$$\mathbb{P}(\tilde{X}^{x,k\tilde{T}}_{(k+1)\tilde{T}} = \tilde{Y}^{k\tilde{T}}_{(k+1)\tilde{T}}) = \frac{1}{2}\|\mu^k_x - \mu^k_y\|_{TV}.$$

- We remember that we are in the case where $x, y \in B_R(0)$. Taking $p = 3$ in Lemma 2.3.2, and applying Lemma 2.3.3, we know that there exists a constant C, such that

$$\|\mu^k_x - \tilde{\mu}^k\|_{TV} \geq \frac{1}{2C},$$

and therefore

$$\mathbb{P}(\tilde{X}^{x,k\tilde{T}}_{(k+1)\tilde{T}} = \tilde{Y}^k_{(k+1)\tilde{T}}) \geq \frac{1}{4C}.$$

- We immediately see that the pair of processes defined by

$$\left(\tilde{X}_t^{x,k\tilde{T}}, \tilde{X}_t^{y,k\tilde{T}} = \tilde{Y}_t^{k\tilde{T}} - \frac{\tilde{T}-t}{\tilde{T}} U_t^{k\tilde{T}}(x-y)\right)_{t\in[k\tilde{T},(k+1)\tilde{T}]},$$

are successfully coupled with probability bounded from below, since

$$\begin{aligned} &\mathbb{P}(\tilde{X}_s^{x,k} = \tilde{X}_s^{y,k} \text{ for some } s \in [k\tilde{T},(k+1)\tilde{T}]) \\ &\geq \mathbb{P}(\tilde{X}_{(k+1)\tilde{T}}^{x,k} = \tilde{Y}_{(k+1)\tilde{T}}^{k}) \geq \frac{1}{4C}. \end{aligned} \tag{2.15}$$

Thus the second step is complete.

Step 3: We are now ready to construct the processes X^x, X^y we used in (2.7) on individual time intervals of duration $\tilde{T}$ and then patch them all together. Assume that we have constructed X^y and X^x on $[0,k\tilde{T}]$. We now proceed in the following way:

- If $X_{k\tilde{T}}^y$ and $X_{k\tilde{T}}^x$ are in $B_R(0)$, then on $[k\tilde{T},(k+1)\tilde{T}]$ we set $X_t^x = \tilde{X}_t^{X_{k\tilde{T}}^x,k\tilde{T}}$ and $X_t^y = \tilde{X}_t^{X_{k\tilde{T}}^y,k\tilde{T}}$ where $\tilde{X}$ is the maximal coupling constructed in Step 2.
- If at least one process does not finish in the ball, then on the next timestep we set $X_t^x = \bar{X}_t^{X_{k\tilde{T}}^x}$ and $X_t^y = \bar{X}_t^{X_{k\tilde{T}}^y}$, where $\{\bar{X}_t^x\}_{t\geq k\tilde{T}}$ and $\{\bar{X}_t^y\}_{t\geq k\tilde{T}}$ are two independent solutions to (2.1) with $\tau = k\tilde{T}$ and initial conditions $X_{k\tilde{T}}^x$ and $X_{k\tilde{T}}^y$ respectively.

We have thus constructed X^x and X^y on the entire time line. We now proceed to prove an exponential bound on their first meeting time. For that, we define a family $\{z_k\}_{\{k\in\mathbb{N}\}}$ as follows: $z_0 = 0$ and $z_{n+1} = \inf\{k > z_n : k \in \mathbb{N}, X_{k\tilde{T}}^x, X_{k\tilde{T}}^y \in B_R(0)\}$. By (2.14) we get

$$\mathbb{E}\big[e^{\gamma z_1 \tilde{T}}\big] \leq C_2(1 + \|x\|^2 + \|y\|^2)$$

and thus

$$\mathbb{E}\big[e^{\gamma(z_{n+1}-z_n)\tilde{T}} \big| \mathcal{F}_{z_n\tilde{T}}\big] \leq C_1(1 + \|X_{z_n\tilde{T}}^x\|^2 + \|X_{z_n\tilde{T}}^y\|^2).$$

Since $e^{-\gamma z_n \tilde{T}}$ is $\mathcal{F}_{z_n\tilde{T}}$-measurable and $\|X_{z_n\tilde{T}}^{x,y}\| \leq R$, we get

$$\mathbb{E}\big[e^{\gamma z_{n+1}\tilde{T}}\big] \leq C_1^{n+1} C_2^n (1 + \|x\|^2 + \|y\|^2),$$

where $C_1 = 1 + 2R^2$. We now set

$$\bar{k} = \inf\{k : X_{z_k\tilde{T}}^x = X_{z_k\tilde{T}}^y\}.$$

Since $X^{x,y}_{z_k\tilde{T}} \in B_R(0)$ for every $k > 0$, we have, from (2.15)

$$\mathbb{P}(\bar{k} > k+1 | \bar{k} > k) \leq \left(1 - \frac{1}{4C}\right).$$

As $\mathbb{P}(\bar{k} > k+1) = \mathbb{P}(\bar{k} > k+1|\bar{k} > k)\mathbb{P}(\bar{k} > k)$ we conclude that

$$\mathbb{P}(\bar{k} > k) \leq \left(1 - \frac{1}{4C}\right)^k.$$

We now choose $0 < \alpha < \gamma$ such that

$$\left(1 - \frac{1}{4C}\right)^{1-\alpha/\gamma} C_1^{\alpha/\gamma} C_2^{\alpha/\gamma} < 1,$$

and then, using Hölder's inequality, we see that

$$\begin{aligned}
\mathbb{E}(e^{\alpha z_{\bar{k}}\tilde{T}}) &= \mathbb{E}\big(\mathbb{E}(e^{\alpha z_{\bar{k}}\tilde{T}}|\bar{k})\big) \leq \sum_{k\geq 0} \mathbb{E}\big[e^{\alpha z_k \tilde{T}} \mathbf{1}_{\bar{k}=k}\big] \\
&\leq \sum_{k\geq 0} (\mathbb{P}(\bar{k} = k))^{1-\alpha/\gamma} (\mathbb{E}e^{\alpha z_k \tilde{T}})^{\alpha/\gamma} \\
&\leq \sum_{k\geq 0} (\mathbb{P}(\bar{k} > k-1))^{1-\alpha/\gamma} (\mathbb{E}e^{\alpha z_k \tilde{T}})^{\alpha/\gamma} \\
&\leq \sum_{k\geq 0} \left(1 - \frac{1}{4C}\right)^{(k-1)(1-\alpha/\gamma)} (C_1^k C_2^{k-1}(1 + \|x\|^2 + \|y\|^2))^{\alpha/\gamma} \\
&\leq C_3(1 + \|x\|^2 + \|y\|^2)
\end{aligned}$$

for some constant C_3. For each $\rho \leq \alpha$ we get

$$\mathbb{E}\big[e^{\rho T}\big] \leq \mathbb{E}\big[e^{\rho(z_{\bar{k}}+1)\tilde{T}}\big] \leq \tilde{C}(1 + \|x\|^2 + \|y\|^2)$$

where $\tilde{C} = C_3 e^{\rho\tilde{T}}$. □

Lemma 2.3.4. *The estimate (2.5) can be extended to the case where F is bounded and measurable, and there exists a uniformly bounded sequence of Lipschitz (in the second argument) functions $\{F_n\}_{n\geq 1}$ such that*

$$\lim_n F_n(t,x) = F(t,x), \quad \forall x \in H, t \geq 0.$$

Proof: The proof uses standard Girsanov arguments and is identical to Corollary 2.5 in [14].

□

Remark 2.3.9. *The reason Lemma 2.3.4 is necessary is because in order to construct a solution to the EBSDE in the sequel, we will first have to change measure. From Girsanov's theorem, we know that under the new measure the forward process will have additional bounded drift. We therefore need to ensure that the estimate (2.5) still holds.*

2.3.3 Recurrence

This section is devoted to proving that under certain assumptions the forward process (2.1) eventually enters any open ball in H with probability one. We establish this for the case of time periodic coefficients and Lévy noise with non-trivial diffusion component. This is a natural extension of existing theory and interesting in its own right. In the sequel we will need a slightly weaker property, namely the eventual return to any open ball around zero, in order to prove the uniqueness of the Markovian solution to an EBSDE. We start by formulating an additional assumption:

Assumption 2.3.2. *For notational simplicity suppose that in (2.1) $\tau = 0$. Then we assume that the process $Z_A(t)$ defined by*

$$Z_A(t) = \int_0^t U_s G(s) dL_s.$$

spans the entire space H, that is, $\mathbb{P}(Z_A(t) \in V) > 0$ for all $t > 0$ and any open non-empty $V \in H$.

Remark 2.3.10. *This assumption may seem overly restrictive, as one can think of many Lévy processes that do not span the entire space. For example, the case when one-dimensional components $\{L_n(t)\}$ are supported on the integers. Even in a more general case, one could think of a Lévy process $L(t)$ supported on a subspace. However, since we focus our attention on the case where $G(s)$ is invertible for every $s \geq 0$, and L has a non-trivial diffusion component, the assumption is reasonable.*

Lemma 2.3.5. *If the process $Z_A(t)$ satisfies Assumption 2.3.2 for all $t > 0$, then process X_t^x satisfying (2.1) is irreducible. In other words*

$$\mathbb{P}(X_t^x \in B_\epsilon(z)) > 0$$

for any $t > 0$, $z \in H$, $\epsilon > 0$.

Remark 2.3.11. *Here, and in the sequel, we denote by $B_R(x)$ the open ball of radius R around some $x \in H$.*

Proof: We follow the proof of Proposition 3.3 in [40]. We fix $T > 0$, $y \in H$, $\epsilon > 0$. For the rest of the proof we also denote $X_t = X_t^x$. Then

$$X_{t+a} = U_{t+a}^t X_t + \int_t^{t+a} U_{t+a}^s F_s(X_s) ds + \int_t^{t+a} U_{t+a}^s G(s) dL_s.$$

Let z be any element in the support of the distribution of the random variable $U_{t+a}^s X_t$. Then, by definition, the event

$$B = \{|U_{t+a}^s X_t - z| < \epsilon/3\}$$

is of positive probability. Since $\|F\|_\infty = \sup_{t\geq 0, x\in H} F_t(x) < \infty$, using the definition of U, we have

$$\begin{aligned}
\left| \int_t^{t+a} U_{t+a}^s F_s(X_s) ds \right| &\leq \int_t^{t+a} \|U_{t+a}^s\|_{op} \|F\|_\infty ds \\
&\leq M\|F\|_\infty \int_t^{t+a} e^{-\mu(t+a-s)} ds \\
&\leq ca,
\end{aligned}$$

for some $c > 0$. We then write

$$X_{t+a} - y = (U_{t+a}^t X_t - z) + \int_t^{t+a} U_{t+a}^s F_s(X_s) ds + \left(\int_t^{t+a} U_{t+a}^s G(s) dL_s - y + z \right). \quad (2.16)$$

The event

$$C = \left\{ \left| \int_t^{t+a} U_{t+a}^s G(s) dL_s - y + z \right| < \epsilon/3 \right\}$$

is of positive probability by Assumption 2.3.2. Since X_t and the increments of L on $[t, t+a]$ are independent, so are the events B and C. Therefore $B \cap C$ has positive probability. Given (2.16), we have shown that

$$|X_{t+a} - y| \leq \epsilon/3 + ca + \epsilon/3$$

with positive probability on $B \cap C$. We now choose a so that $ca < \epsilon/3$ and $T - a \geq 0$. Setting $t = T - a$, we obtain

$$\mathbb{P}(|X_T - y| \leq \epsilon) \geq \mathbb{P}(B \cap C) > 0,$$

which is the result.

□

In order to proceed, we need a few results concerning the invariant measure for the solution to the equation (2.1). We begin by considering the linear problem

$$dX_t = A(t) X_t dt + G(t) dL_t, \quad X_\tau = x,$$

which can be reduced to the autonomous case by the standard technique of enlarging the state space, i.e. by considering the evolution of the vector $(X, y) \in H \times \mathbb{R}_+$ given by

$$\begin{cases} dX_t = A(y(t))X_t dt + G(y(t))dL_t & X(0) = x \\ dy(t) = dt & y(0) = \tau \end{cases}$$

Following [29] we define a one-parameter semigroup as

$$P_s u(t, x) := P(t, t + s)u(t + s, \cdot)(x)$$

meaning that we apply the two-parameter semigroup to u as a function of x only. It is clear from the definition that P_τ is a Markovian semigroup, which gives us the opportunity to use the powerful existing theory. In order to establish existence and uniqueness of the invariant measure, we need to define the corresponding "periodic" L^2-space on which the semigroup is a contraction. We denote

$$L^2_*(\mu) := \Big\{ f : \mathbb{R} \times H \to \mathbb{R} \text{ measurable } : f(t + T^*, x) = f(t, x) \quad \mu - a.e. \\ \text{and } \int_{[0,T^*]\times H} |f(y)|^2 \mu(dy) < \infty \Big\}.$$

for some measure μ. It is clear that L^2_* is a Hilbert space. The following result was established in [29].

Proposition 2.3.1. *There exists a unique invariant measure for the semigroup P. In other words for every bounded measurable function u such that $u(t + T^*, x) = u(t, x)$ for each $t > 0$ and $x \in H$ we have:*

$$\int_{[0,T^*]\times H} P_s u(t, x)\mu(dt, dx) = \int_{[0,T^*]\times H} u(t, x)\mu(dt, dx).$$

Furthermore, on $L^2_*(\mu)$ the semigroup P_s is a contraction.

We deduce that there also exists a unique invariant measure μ corresponding to the original semilinear problem

$$dX_t = A(t)X_t dt + F_t(X_t)dt + G(t)dL_t, \quad X_\tau = x.$$

This can easily be shown by a change of measure to reduce to the linear case. We leave details to the reader. We are now ready to prove the main result of this section, namely the recurrence of the forward process $\{X_t\}_{t\geq\tau}$. We present two proofs: one is applicable for the case of $\dim H < \infty$, and is elementary, in the sense that it does not rely on the existence of the invariant measure. The second one deals with the case of $\dim H = \infty$.

Theorem 2.3.4. *For any $x_0, x \in H$, $s \geq 0$ and for any fixed $\epsilon > 0$, we define $\tau := \inf\{t \geq s : X_t^x \in B_\epsilon(x_0)\}$. Then $\mathbb{P}(\tau > T) \to 0$ as $T \to \infty$.*

Proof: (Intuition, $\dim H < \infty$**)** From Step 1 of the proof of Theorem 2.3.3 we know that we can find a radius $R > 0$, such that the probability that our process returns to the ball $\bar{B}_R(0)$ is not trivial. We then discretise time with a step $\tilde{T}$. We know that the discretised process will return to $\bar{B}_R(0)$ infinitely often, and by Lemma 2.3.5 the probability of the jump from $\bar{B}_R(0)$ to any open ball is bounded from below. We then invoke a Borel–Cantelli type argument to demonstrate the claim.
(Formal proof, $\dim H < \infty$**)** We start by introducing a family of events $\{E_n\}_{n \geq 1}$ as

$$E_n = \{\text{there exists } k = 1 \dots n : X_{k\tilde{T}}^x \in B_\epsilon(x_0)\},$$

and immediately notice that

$$\mathbb{P}(E_n \big| \bar{E}_{n-1}) = \mathbb{P}(\{X(n\tilde{T}, (n-1)\tilde{T}, X_{(n-1)\tilde{T}}^x) \in B_\epsilon(x_0)\}),$$

where $\bar{E}$ denotes the complement of E and $X(t, s, x)$ is the value at time t of the solution to (2.1) starting at time $\tau = s$ with $X_\tau = x$. Therefore,

$$\begin{aligned}
\mathbb{P}(E_n | \bar{E}_{n-1}) &= \mathbb{P}\big(X(n\tilde{T}, (n-1)\tilde{T}, X_{(n-1)\tilde{T}}^x) \in B_\epsilon(x_0)\big) \\
&\geq \mathbb{P}\big(X_{(n-1)\tilde{T}}^x \in B_R(0), X(n\tilde{T}, (n-1)\tilde{T}, X_{(n-1)\tilde{T}}^x) \in B_\epsilon(x_0)\big) \\
&= \mathbb{P}\big(X_{(n-1)\tilde{T}}^x \in B_R(0)) \mathbb{P}(X(n\tilde{T}, (n-1)\tilde{T}, X_{(n-1)\tilde{T}}^x) \in B_\epsilon(x_0)\big).
\end{aligned}$$

Since coefficients in (2.1) are T^*-periodic and $\bar{B}_R(0)$ is compact (and therefore $[0, T^*] \times \bar{B}_R(0)$ is compact), and given the stability of solutions to (2.1) with respect to the initial value (as stated in (2.4)), there exists $\delta > 0$ such that

$$\mathbb{P}\left(X(n\tilde{T}, (n-1)\tilde{T}, X_{n\tilde{T}}^x) \in B_\epsilon(x_0) \middle| X_{n\tilde{T}}^x \in \bar{B}_R(0)\right) > \delta.$$

Hence

$$\sum_{n \geq 1} \mathbb{P}(E_n | \bar{E}_{n-1}) \geq \delta \sum_{n \geq 1} \mathbb{P}(X_{n\tilde{T}}^x \in \bar{B}_R(0)).$$

In "Step 1" of the the proof of Theorem 2.3.3 we showed that

$$\mathbb{E}\|X_t^x\|^2 \leq L(\|x\|^2 e^{-2\mu t} + c),$$

and therefore by Markov's inequality we have

$$\mathbb{P}(\|X_t^x\|^2 > R) \leq \frac{L}{R}(\|x\|^2 e^{-2\mu t} + c).$$

It is clear that we can choose R so that

$$1 - \mathbb{P}(\|X^x_{k\tilde{T}}\|^2 > R) \geq 1/k$$

for all $k \geq 1$. Therefore

$$\sum_{n \geq 1} \mathbb{P}(E_n | \bar{E}_{n-1}) \geq \delta \sum_n 1/n = \infty,$$

and thus by the counterpart of the Borel–Cantelli Lemma (see [7]), we conclude that

$$\mathbb{P}(\tau < \infty) = \mathbb{P}(\cup_n E_n) = 1,$$

concluding the proof. $\square$

Proof: (dim $H = \infty$) Let the function $\psi : H \to \mathbb{R}$ be bounded and continuous. From Theorem 2.5 we know that for any $x, y \in H$ and $0 \leq t \leq t'$ we have

$$\begin{aligned} |P_s[\psi](t,x) - P_s[\psi](t',y)| &= |P_s[\psi](t', X^{t,x}_{t'}) - P_s[\psi](t',y)| \\ &\leq C(1 + ||X^{t,x}_{t'}||^2 + ||y||^2) e^{-\rho(s-t')} \sup_{u \in H} ||\psi(u)||. \end{aligned} \tag{2.17}$$

Using (2.17) and the fact that there exists a unique invariant measure μ for the semigroup P_t, one can show (see, e.g. [38]) that μ is exponentially mixing. In other words,

$$P_s[\psi](t,x) \to \mu(\psi) = \int_{[0,T^*] \times H} \psi(t,x) \mu(dt, dx).$$

We now set $\psi(t,x) = \mathbf{1}_{x \in A}$ for some open set $A \subset H$. Then $P_s[\psi](t,x) = \mathbb{P}(X^{t,x}_s \in A)$. By Theorem 2.3.5 we know that for all $0 \leq t \leq s$, $x \in H$ we have $\mathbb{P}(X^{t,x}_s \in A) > 0$. Therefore

$$\mu([0,T^*] \times A) = \int_{[0,T^*] \times H} \mathbb{P}(X^{t,x}_s \in A) \mu(dt, dx) = \delta_A > 0$$

for some constant δ_A. Setting $t = 0$ and $A = B_\epsilon(x_0)$ we have

$$\liminf_{t \to \infty} \mathbb{P}(X^x_t \in B_\epsilon(x_0)) = \mu([0,T^*] \times B_\epsilon(x_0)) = \delta_\epsilon > 0$$

and thus, by Proposition 3.4.5 in [39], the claim follows.

$\square$

2.4 Backwards SDEs

We now move from the 'forward' process X to consider the 'backwards' part of our problem. This section is organised as follows: we start by introducing the class of discounted BSDEs in infinite horizon and proving that they admit a bounded solution. Then we use the coupling estimate obtained in the previous section to prove existence of a solution to our EBSDE. The next subsection is devoted to the uniqueness of the Markovian solution. We conclude by providing an alternative representation for the solution. Similarly to [44] we impose certain assumptions on the driver of our BSDE.

Definition 2.4.1. *Henceforth we assume that the driver of a BSDE with jumps is a measurable function* $f : \Omega \times \mathbb{R}^+ \times \mathbb{R} \times H \times \mathcal{L}^2(B, \mathcal{B}, \nu) \to \mathbb{R}$*, where* ν *denotes the compensator of the Poisson random measure.*

Assumption 2.4.1. *For all* T *we have the following conditions on our driver* $f(\omega, t, y, z, u)$*:*

- f *is predictable in* (ω, t)*.*
- f *is continuous w.r.t* y *and there exists an* $\mathbb{R}^+$*-valued process* $(\phi_t)_{0 \le t \le T}$ *such that* $\mathbb{E}\left(\int_0^T \phi_s^2 ds\right) < \infty$ *and*

$$|f(\omega, t, y, z, u)| \le \phi_t + K\left(|y| + ||z|| + \int_B |u(v)|^2 \nu(dv)\right)^{1/2}$$

- f *is "monotonic" w.r.t* y*, that is* $\exists \alpha \in \mathbb{R}$ *such that* $\forall t \ge 0, \forall y, y' \in \mathbb{R}, \forall z \in H, \forall u \in \mathcal{L}^2(B, \mathcal{B}, \nu)$

$$(y - y')(f(\omega, t, y, z, u) - f(\omega, t, y', z, u)) \le \alpha |y - y'|^2 \quad \mathbb{P} - a.s.$$

- f *is Lipschitz w.r.t.* z *and* u*. In particular* $\exists K \ge 0 : \forall t \in [0, T], \forall y \in \mathbb{R}, \forall z, z' \in H, \forall u, u' \in \mathcal{L}^2(B, \mathcal{B}, \nu)$

$$|f(\omega, t, y, z, u) - f(\omega, t, y, z', u')| \le K||z - z'|| + K\left(\int_B |u(v) - u'(v)|^2 \nu(dv)\right)^{1/2}$$

In order to have a comparison theorem, we make the following further assumption.

Assumption 2.4.2. *There exists* $-1 < C_1 \le 0$ *and* $C_2 \ge 0$ *such that*

$$\forall x \in H, \quad \forall z \in H^*, \quad \forall u, u' \in \mathcal{L}^2(B, \mathcal{B}, \nu, \mathbb{R})$$

we have

$$f(\omega, t, y, z, u) - f(\omega, t, y, z, u') \le \int_B (u(v) - u'(v)) \gamma_t^{\omega, z, u, u'}(v) \nu(dv),$$

where $\gamma^{\omega, t, z, u, u'} : \Omega \times B \to \mathbb{R}$ *is measurable (predictable* $\times$ *Borel) in all arguments and satisfies*

$$C_1(1 \wedge ||x||) \le \gamma^{\omega, t, z, u, u'}(x) \le C_2(1 \wedge ||x||)$$

for all $v \in B$*.*

The following existence theorem for finite horizon BSDEs with jumps can be found in [44]. In that paper the case of finite-dimensional Brownian motion is considered. The extension to the infinite dimensional case where W is a Q-Wiener processes is immediate (for details see [10]).

Theorem 2.4.1. *Under Assumption 2.4.1, there exists a unique solution* $(Y, Z, U) \in (\mathcal{S}^2 \times \mathcal{L}^2(W) \times \mathcal{L}^2(\nu))$, *for any terminal condition* $\eta \in \mathcal{L}^2(\mathcal{F}_T)$, *to the equation*

$$Y_t = \eta + \int_t^T f(\omega, u, Y_u, Z_u, U_u)du - \int_t^T Z_u^* dW_u - \int_t^T \int_B U_s(x)\tilde{N}(ds, dx).$$

Lemma 2.4.1. *For every* Y, Z, U, U' *under Assumption 2.4.2 there exists a process* $\gamma_t = \gamma^{\omega, Y_t, Z_t, U_t, U'_t}$ *such that*

$$f(\omega, t, Y_t, Z_t, U_t) - f(\omega, t, Y_t, Z_t, U'_t) = \int_B (U(v) - U'(v))\gamma_t(v)\nu(dv) \tag{2.18}$$

Proof: We first notice that $\forall t > 0, \forall \omega \in \Omega, \forall z \in H^*, \forall u, u' \in \mathcal{L}^2(B, \mathcal{B}, \nu, \mathbb{R})$ there exist $\gamma_{1,t}^{\omega,z,u,u'}(v)$ and $\gamma_{2,t}^{\omega,z,u,u'}(v)$, satisfying $C_1(1 \wedge ||v||) \leq \gamma_{i,t}(v) \leq C_2(1 \wedge ||v||)$ for $i = 1 \ldots 2$, such that

$$f(\omega, t, y, z, u) - f(\omega, t, y, z, u') \leq \int_B (u(v) - u'(v))\gamma_{1_t}^{\omega,y,z,u,u'}(v)\nu(dv)$$

and

$$f(\omega, t, y, z, u) - f(\omega, t, y, z, u') \geq \int_B (u(v) - u'(v))\gamma_{2,t}^{\omega,y,z,u,u'}(v)\nu(dv).$$

Then there exists $\alpha_t = \alpha(t, \omega, y, z, u, u')$ such that

$$\begin{aligned} f(\omega, t, y, z, u) &- f(\omega, t, y, z, u') \\ &= \int_B (u(v) - u'(v))(\alpha_t \gamma_{1,t}^{\omega,z,y,u,u'}(v) + (1 - \alpha_t)\gamma_{2,t}^{x,z,y,u,u'}(v))\nu(dv) \end{aligned}$$

and we immediately see that

$$\alpha_t = \frac{f(\omega, t, y, z, u) - f(\omega, t, y, z, u') - \int_B (u(v) - u'(v))\gamma_{2,t}^{\omega,z,y,u,u'}(v)\nu(dv)}{\int_B (u(v) - u'(v))(\gamma_{1,t}^{\omega,y,z,u,u'}(v) - \gamma_{2,t}^{\omega,y,z,u,u'}(v))\nu(dv)} \in [0, 1],$$

noticing that if the denominator is zero then $\alpha_t = 1$ satisfies the claim. Now for each $s \in [0, t]$ and $v \in B$ we can explicitly define

$$\gamma_t(v) = \alpha(t, \omega)\gamma_{1_t}^{\omega, Y_t, Z_t, U_t, U'_t}(v) + (1 - \alpha(t, \omega))\gamma_{2,t}^{\omega, Y_t, Z_t, U_t, U'_t}(v)$$

where $\alpha(t, \omega) = \alpha(t, \omega, Y_t, Z_t, U_t, U'_t)$, and it is clear that γ_t satisfies (2.18).

□

2.4.1 Infinite horizon BSDEs

In this section we show that there exists a unique bounded solution to the infinite-horizon BSDE with discounting, that is the equation

$$Y_T = Y_t - \int_t^T (-\alpha Y_u + f(\omega, u, Z_u, U_u))du + \int_t^T Z_u dW_u + \int_t^T \int_B U_s(x)\tilde{N}(ds, dx), \tag{2.19}$$

which will prove crucial to the study of Ergodic BSDEs in the next section. In order to proceed we will require Tanaka's formula for general semimartingales. The following version can be found, for example, in [10]. Here we use the convention that $\text{sign}(x) = x/|x|$ for $x \neq 0$ and $\text{sign}(0) = 0$.

Lemma 2.4.2. ***(Tanaka's formula)*** *Let X be a semimartingale and $a \in \mathbb{R}$. Then there exists a continuous increasing local time process L^a and a pure jump process $L^{X,a}$, with $L^a(0) = 0$ (unique $\mathbb{P}-$a.s.), such that X allows the following representation:*

$$d|X_t - a| = \text{sign}(X_{t-} - a)dX_t + dL_t^a + \Delta L_t^{X,a},$$

where

$$\Delta L_t^{X,a} = |X_t - a| - |X_{t-} - a| - \text{sign}(X_{t-} - a)\Delta X_t$$

is a 'local-time' jump process.

Remark 2.4.1. *If we consider the above process ΔL^X, we notice that*

$$\begin{aligned}
\Delta L_t^{X,a} &= |X_t - a| - |X_{t-} - a| - \text{sign}(X_{t-} - a)\Delta(X_t - a) \\
&= |X_t - a| - \text{sign}(X_{t-} - a)(X_t - a) \\
&= |X_t - a|(1 - \text{sign}((X_{t-} - a)(X_t - a))) \\
&\geq 0.
\end{aligned}$$

Theorem 2.4.2. *Let $\alpha > 0$ and $f : \Omega \times \mathbb{R}^+ \times H^* \times \mathbb{R} \to \mathbb{R}$ be such that*

- *f satisfies Assumptions 2.4.1 and 2.4.2*
- *$|f(w, t, 0, 0)|$ is uniformly bounded by $C \in \mathbb{R}$*

Then there exists an adapted solution (Y, Z, U), with Y càdlàg and $Z \in \mathcal{L}^2(W)$, $U \in \mathcal{L}^2(\tilde{N})$to the infinite horizon equation (2.19) for all $0 \leq t \leq T < \infty$, satisfying $|Y_t| \leq C/\alpha$, and this solution is unique among bounded adapted solutions.

Furthermore, if (Y^T, Z^T, U^T) *denotes the (unique) adapted square integrable solution to*

$$Y_t^T = \int_t^T (-\alpha Y_u^T + f(\omega, u, Z_u^T, U_u^T))du - \int_t^T (Z_u^T)dW_u - \int_t^T \int_B U_s(x)\tilde{N}(ds, dx) \quad (2.20)$$

then $\lim_{T\to\infty} Y_t^T = Y_t$ *a.s., uniformly on compact sets in* t.

Proof: We start by proving that if a bounded solution exists, it is unique. Suppose we have two bounded solutions (Y, Z, U) and (Y', Z', U') to (2.19). We denote $\delta Y := Y - Y'$, $\delta Z := Z - Z'$ and $\delta U = U - U'$. We also denote

$$\alpha_s := \begin{cases} \frac{f(\omega,s,Z_s,U_s)-f(\omega,s,Z'_s,U_s)}{||\delta Z||^2}\delta Z & \text{if } Z \neq Z', \\ 0 & \text{otherwise.} \end{cases}$$

Now define $M_t = \int_0^t \alpha_s dW_s + \int_0^t \int_B \gamma_s(x)\tilde{N}(ds, dx)$, where $\gamma = \gamma^{\omega,t,Z',U,U'}$ is defined as in Lemma 2.4.1. Then we can use Theorem 2.2.2 to show that there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that under $\mathbb{Q}$ the process

$$K_t = \int_t^T (f(\omega, u, Z_u, U_u) - f(\omega, u, Z'_u, U'_u))du + \int_t^T \delta Z_u^* dW_u + \int_t^T \int_B \delta U_s(x)\tilde{N}(ds, dx)$$

is a martingale. We now apply Tanaka's formula and Remark 2.4.1 to see that for all $s \leq t \leq T$ we have

$$E_{\mathbb{Q}}[e^{-\alpha t}|\delta Y_t| - e^{-\alpha s}|\delta Y_s|\,|\mathcal{F}_s] \geq 0,$$

and hence

$$|\delta Y_s| \leq e^{-\alpha(t-s)} E_{\mathbb{Q}}[|\delta Y_t|\,|\mathcal{F}_s] \leq e^{-\alpha(t-s)}C,$$

for C a bound on $|\delta Y_t|$. This bound is independent of T and collapses as $t \to \infty$. Hence $|\delta Y_s| = 0$, from which we see $Y_s = Y'_s$ a.s. for every s, and hence $Y = Y'$ up to indistinguishability as Y and Y' are càdlàg.

We now show that a bounded solution exists. We first notice that there indeed exists a unique solution to the T-horizon BSDE (2.20). In order to see this, by a standard comparison argument it suffices to check that our new driver, namely $F(\omega, t, y, z, u) := -\alpha y + f(\omega, t, z, u)$ satisfies Assumption 2.4.1, provided that f does. This is clear given that our additional term does not depend on (z, u), is continuous and monotonic. We denote the solution as (Y^T, Z^T, U^T). We now prove that Y^T is bounded. Similarly to above, we define

$$\beta_s := \begin{cases} \frac{f(\omega,s,Z_s^T,U_s^T)-f(\omega,s,0,U_s^T)}{||Z^T||^2} Z_s^T & \text{if } Z_s^T \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

and denote $\tilde{M}_t = \int_0^t \beta_s dW_s + \int_0^t \int_B \tilde{\gamma}_s(x)\tilde{N}(ds,dx)$, where $\tilde{\gamma} = \gamma^{\omega,0,U,0}$ is defined as in Lemma 2.4.1. Then, as above, there exists a probability measure $\tilde{\mathbb{Q}} \sim \mathbb{P}$, under which the process

$$\tilde{K}_t = \int_t^T (f(\omega,u,Z_u,U_u) - f(\omega,u,0,0))du + \int_t^T Z_u^* dW_u + \int_t^T \int_B U_s(x)\tilde{N}(ds,dx)$$

is a martingale, and therefore applying Tanaka's formula and Itô's formula to $e^{-\alpha t}|Y_t^T|$ we see that

$$|Y_t^T| \le e^{\alpha t} E_{\tilde{\mathbb{Q}}}\Big[\int_t^T e^{-\alpha u}|f(\omega,u,0,0)|du\Big|\mathcal{F}_t\Big] \le C/\alpha \tag{2.21}$$

where C is the bound on $|f(\omega,t,0,0)|$. Thus Y^T is uniformly bounded. We now show that Y^T forms a Cauchy sequence in T uniformly on compacts in t. For every $T' \ge T$ we define

$$\rho_s := \begin{cases} \frac{f(\omega,s,Z_s^T,U_s^T)-f(\omega,s,Z^{T'},U_s^T)}{||Z^T-Z^{T'}||^2}(Z_s^T - Z_s^{T'}) & \text{if } Z_s^T \ne Z_s^{T'}, \\ 0 & \text{otherwise,} \end{cases}$$

and denote $\bar{M}_t = \int_0^t \beta_s dW_s + \int_0^t \int_B \bar{\gamma}_s(x)\tilde{N}(ds,dx)$, where $\bar{\gamma} = \gamma^{\omega,Z^{T'},U^T,U^{T'}}$ is defined as in Lemma 2.4.1. As above, applying Tanaka's formula and inequality (2.21), we observe

$$|Y_t^T - Y_t^{T'}| \le e^{-\alpha(T-t)} E_{\bar{\mathbb{Q}}}[|Y_t^{T'} - Y_t^T|\,|\mathcal{F}_t] \le 2Ce^{-\alpha(T-t)}/\alpha, \tag{2.22}$$

where $\bar{\mathbb{Q}}$ is defined in a similar way to $\mathbb{Q}$. Hence we see that Y_t^T is a Cauchy sequence in T, therefore the limit exists, and we denote it Y_t. The bound established in inequality (2.21) also holds for Y_t, and convergence uniformly on any compact interval is clear from (2.21). We now show that Z_t^T and U_t^T are Cauchy sequences. We denote $\tilde{Z}_t = Z_t^T - Z_t^{T'}$ and $\tilde{U}_t = U_t^T - U_t^{T'}$. Apply Itô's formula to $\big(\tilde{Y}_t\big)^2$, where $\tilde{Y}_t := Y_t^T - Y_t^{T'}$. Then, after standard calculations under $\bar{\mathbb{Q}}$, we see that, for each $t < T$,

$$\tilde{Y}_t^2 = \tilde{Y}_0^2 + \mathbb{E}_{\bar{\mathbb{Q}}}\left(\int_0^t \int_B |\tilde{U}_s(v)|^2 \nu(dv)ds\right) + \mathbb{E}_{\bar{\mathbb{Q}}}\bigg(\int_0^t ||\tilde{Z}_s||^2 ds\bigg) + 2\alpha \mathbb{E}_{\bar{\mathbb{Q}}}\bigg(\int_0^t \tilde{Y}_s^2 ds\bigg).$$

Given (2.22) our claim follows. Therefore the limit as $T \to \infty$ exists for sequences $\{Z_t^T\}$ and $\{U_t^T\}$. Taking Z and U as their respective limits, we have our desired solution (Y, Z, U).

□

Assumption 2.4.3. ***(Markovian structure)*** *In the sequel we will assume that the driver f is Markovian, that is*

$$f(\omega,t,Z_t,U_t) = \bar{f}(X_t(\omega),Z_t,U_t)$$

for some measurable $\bar{f}$. For convenience we simply write f for $\bar{f}$.

Corollary 2.4.1. *Let* $(Y^{\alpha,x,s}, Z^{\alpha,x,s}, U^{\alpha,x,s})$ *be the unique bounded solution to the discounted BSDE*

$$\begin{aligned} Y_T^{\alpha,x,s} = Y_t^{\alpha,x,s} - \int_t^T (-\alpha Y_u^{\alpha,x,s} + f(X(t,s,x), Z_u^{\alpha,x,s}, U_u^{\alpha,x,s}))du \\ + \int_t^T (Z_u^{\alpha,x,s})^* dW_u + \int_t^T \int_B U_s^{\alpha,x,s}(x)\tilde{N}(ds,dx), \end{aligned} \tag{2.23}$$

on $[s,T]$ *for some* $s \geq 0$*. We define a function* v^α *by* $v^\alpha(s,x) = Y_s^{\alpha,x,s}$*. Then, provided* v^α *is measurable,* $Y_t^{\alpha,x,s} = v^\alpha(t, X(t,s,x))$ *is a solution, and by uniqueness we also get that* $v^\alpha(s,x)$ *is bounded. It is also not hard to see that processes* Z *and* U *are Markovian, in other words the solution triplet* (Y_t, Z_t, U_t) *can be represented as*

$$(v^\alpha(t,X_t), \xi^\alpha(t,X_t), \psi^\alpha(t,X_t))$$

for some deterministic $v^\alpha, \xi^\alpha, \psi^\alpha$*. For details on Markovian representations see, e.g. [10].*

In what follows we will repeatedly use changes of measure to eliminate various parts of the driver in our BSDE. In view of Lemma 2.3.4, in order to use the result of Theorem 2.3.3, we need to ensure that under the new measure, the nonlinearity of the drift of the process $\{X_t\}_{t\geq 0}$ can be approximated as a uniform limit of Lipschitz functions. Hence we require the following assumption:

Assumption 2.4.4. *With the notation of Corollary 2.4.1, the function* $\vartheta^\alpha : \mathbb{R}^+ \times H \to \mathbb{R}$*, defined by*

$$\vartheta^\alpha(t,x) := f(x, 0, \psi^\alpha(t,x)) - f(x,0,0)$$

can be represented pointwise as a limit of a uniformly bounded family of Lipschitz (in x*) functions for all* $\alpha > 0$*.*

2.4.2 Ergodic BSDEs

Now we use the same technique we employed in Theorem 2.4.2 to obtain a solution for the Ergodic BSDE

$$Y_t = Y_T + \int_t^T [f(X_u^x, Z_u, U_u) - \lambda]du - \int_t^T Z_u^* dW_u - \int_t^T \int_B U_s(x)\tilde{N}(ds,dx), \tag{2.24}$$

where $0 \leq t \leq T < \infty$, and $f : H \times H^* \times \mathbb{R} \to \mathbb{R}$ is a given function, Y is a real-valued càdlàg stochastic process, Z is a predictable process in H^*. We change measure in such a way to get rid of the drift term, then take expectations, and then send T to infinity. In our case, the generator depends on ω through the forward process $X(t,s,x)$, and we define measure $\mathbb{Q}^{x,\alpha,T}$ to be such that the process

$$\tilde{K}_t = \int_t^T (f(X(u,s,x), Z^{\alpha_u,x,s}, U^{\alpha_u,x,s}) - f(X(u,s,x),0,0))du \\ + \int_t^T (Z_u^\alpha)^* dW_u + \int_t^T \int_B U_s^\alpha(x)\tilde{N}(ds,dx)$$

is a $\mathbb{Q}^{x,\alpha,T}$-martingale on $[s,T]$. Then, under $\mathbb{Q}^{x,\alpha,T}$, we have

$$e^{-\alpha s} v^\alpha(s,x) = E_{\mathbb{Q}^{x,\alpha,T}}\Big[e^{-\alpha T} v^\alpha(T, X(T,s,x)) + \int_{]s,T]} e^{-\alpha u} f(X(u,s,x),0,0)du\Big].$$

As $|v^\alpha(t, X(t,s,x))| \le C/\alpha$ for all $0 \le t \le T$, letting $T \to \infty$ we obtain

$$e^{-\alpha s} v^\alpha(s,x) = \lim_{T\to\infty} E_{\mathbb{Q}^{x,\alpha,T}}\Big[\int_{]s,T]} e^{-\alpha u} f(X(u,s,x),0,0)du\Big].$$

In order to proceed we notice that under the new measure our forward SDE takes the form

$$\begin{cases} dX_t = A(t)X_t dt + F^{\mathbb{Q}}(t, X_t)dt + G(t)dL_t \\ X_s = x, \end{cases} \tag{2.25}$$

where $F^{\mathbb{Q}}(\cdot,\cdot)$ is the nonlinearity under measure $\mathbb{Q}$ that includes new drift terms.

Lemma 2.4.3. *The map $F^{\mathbb{Q}}(t,x)$ is bounded and can be represented as a pointwise limit of a uniformly bounded family of Lipschitz functions.*

Proof: We know explicitly the structure of $F^{\mathbb{Q}}$. Define

$$\rho_s(x) := \begin{cases} \frac{f(x,\xi^\alpha(s,x),\psi^\alpha(s,x)) - f(x,0,\psi^\alpha(s,x))}{||\xi^\alpha(s,x)||^2}(\xi^\alpha(s,x)) & \text{if } \xi^\alpha(s,x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

where $Z_s = \xi^\alpha(s, X_s)$ and $U_s = \psi(s, X_s)$ as in Corollary 2.4.1. Using Lemma 2.4.1, define $\{\gamma_u\}_{u\ge s}$ to be such that

$$f(X(t,s,x),0,U_t) - f(X_t^x,0,0) = \int_B U(v)\gamma_t(X(t,s,x),v)\nu(dv) \tag{2.26}$$

for all $t \ge 0$. Then $\bar{F}(t, X(t,s,x)) := F^{\mathbb{Q}}(t, X(t,s,x)) - F_t(X(t,s,x))$ can be written as

$$\bar{F}(t,X(t,s,x)) = G(t)\rho_t(X(t,s,x)) + \int_B \gamma_t(X(t,s,x),r)\big[G(t)r\big]\nu(dr).$$

The first argument is bounded due to the fact that f is Lipschitz in Z. By arguments identical to Lemma 3.4 in [14], one can also show that it is a pointwise limit of

Lipschitz functions. The second term depends on $X(t,s,x)$ only though the process γ_t. By (2.26) and Assumption 2.4.4, we conclude the result.

$\square$

Lemma 2.4.4. *For v^α defined as in Corollary 2.4.1, and for an arbitrary $x_0 \in H$, there exist bounds C' and C such that*

$$|v^\alpha(s,x) - v^\alpha(s,x_0)| < C'(1 + \|x\|^2 + \|x_0\|^2) \quad \text{and} \quad \alpha|v^\alpha(s,x)| < C$$

uniformly in x, s and α.

Proof: With $\mathbb{Q}^{x,\alpha,T}$ as above, we denote $\mathcal{P}^\alpha(t,s)[f](x) = \mathbb{E}^{\mathbb{Q}^{x,\alpha,T}}[f(X(t,s,x))]$, where $X(t,s,x)$ is the mild solution to (2.25). Then we obtain

$$\begin{aligned}
&|v^\alpha(s,x) - v^\alpha(t,x_0)| \\
&\quad \le e^{\alpha s}\Big| \lim_{T\to\infty} E^{\mathbb{Q}^{x,\alpha,T}}\Big[\int_{]s,T]} e^{-\alpha u} f(X(u,s,x),0,0)du\Big] \\
&\qquad - \lim_{T\to\infty} E^{\mathbb{Q}^{x_0,\alpha,T}}\Big[\int_{]s,T]} e^{-\alpha(u-s)} f(X(u,s,x_0),0,0)du\Big]\Big| + C''|s-t| \\
&\quad \le \int_s^\infty e^{-\alpha(u-s)}\big|\mathcal{P}^\alpha(u,s)[f(\cdot,0,0)](x) - \mathcal{P}^\alpha(u,s)[f(\cdot,0,0)](x_0)\big|du \\
&\qquad + C''|s-t| \\
&\quad \le C'(1 + \|x\|^2 + \|x_0\|^2) + C''|s-t|,
\end{aligned}$$

for every $0 \le t \le s$, where C' and C'' are independent of α. For the last step we use the result of Theorem 2.3.3. The inequality $\alpha|v^\alpha(s,x)| < C$ follows from Theorem 2.4.2.

$\square$

Remark 2.4.2. *We notice that, due to the periodic structure, $v^\alpha(t+T^*,x) = v^\alpha(t,x)$, and hence, for fixed $s \ge 0$ and $x_0 \in H$,*

$$\begin{aligned}
\|v^\alpha(t,x) - v^\alpha(s,x_0)\| &\le \|v^\alpha(t,x) - v^\alpha(t,x_0)\| + \|v^\alpha(t,x_0) - v^\alpha(0,x_0)\| \\
&\le C'(1 + \|x\|^2) + \sup_{u\in[s,s+T^*]} \|v^\alpha(u,x_0) - v^\alpha(s,x_0)\| \\
&\le C'(1 + \|x\|^2) + C''T^*.
\end{aligned}$$

Given that v^α is uniformly Lipschitz in time, we have

$$\|v^\alpha(t,x) - v^\alpha(s,x_0)\| \le C'(1 + \|x\|^2)$$

for some new constant C'.

Lemma 2.4.5. *There exists a bound $\bar{C}$, such that*

$$\|\nabla_x v^\alpha(t,x)\| \le \bar{C}(1+\|x\|^2).$$

holds uniformly in x, t and α.

Proof: The result can be obtained by following exactly the same reasoning as in the proof of Theorem 4.2 in [17].

□

Theorem 2.4.3. *There exists a sequence $\alpha_n \to 0$, a bounded deterministic function $v : \mathbb{R}^+ \times H \to \mathbb{R}$ and a constant $\lambda \in \mathbb{R}$, such that*

$$(v^{\alpha_n}(s,x) - v^{\alpha_n}(s,x_0)) \to v(s,x) \quad \text{ and } \quad \alpha_n v^{\alpha_n}(s,x) \to \lambda$$

for all $s \ge 0$, $x \in H$.

Proof: Since H is a separable space, there exists a dense subset $V \subset \mathbb{R}_+ \times H$. On V we can use a diagonal procedure to construct a sequence $\alpha_n \searrow 0$ such that

$$(v^{\alpha_n}(s,x) - v^{\alpha_n}(s_0,x_0)) \to v(s,x) \quad \text{ and } \quad \alpha_n v^{\alpha_n}(s_0,x_0) \to \lambda$$

for some function $v : V \to \mathbb{R}$ and a real number λ. By Lemmas 2.4.4 and 2.4.5 we know that the functions v^α are locally Lipschitz in both time and space. We can therefore extend v by continuity to the whole $\mathbb{R}^+ \times H$, proving that

$$v^{\alpha_n}(s,x) - v^{\alpha_n}(s,x_0) \to v(s,x)$$

for all $x \in H$ and $s \ge 0$. We notice that, for $t \ge s$,

$$\begin{aligned}
\alpha_n v^{\alpha_n}(t,x) &= \alpha_n(v^{\alpha_n}(s_0,x_0)) + \alpha_n(v^{\alpha_n}(t,x) - v^{\alpha_n}(s_0,x_0)) \\
&= \alpha_n v^{\alpha_n}(s_0,x_0) + \alpha_n(v^{\alpha_n}(t,x) - v^{\alpha_n}(t,X(t,s,x_0)) \\
&\quad + \alpha_n \int_{]s,t]} e^{-\alpha_n u} f(X(u,s,x_0),0,0)du \\
&\to \lambda
\end{aligned}$$

since

$$\begin{aligned}
&\left|\alpha_n(v^{\alpha_n}(t,x) - v^{\alpha_n}(t,X(t,s,x_0)) + \alpha_n \int_{]s,t]} e^{-\alpha_n u} f(X(u,s,x_0),0,0)du\right| \\
&\qquad \le \alpha_n C''|t-s_0| + \alpha_n C'(1+\|X(t,s_0,x_0)\|^2 + \|x_0\|^2) \\
&\qquad \to 0.
\end{aligned}$$

We have thereby proven that λ is indeed a constant independent of time.

□

Theorem 2.4.4. *Let v and λ be constructed as above. We also set $x_0 = 0 \in H$ and $s_0 = 0 \in \mathbb{R}$ for the sake of simplicity. Then, if we define*

$$Y_t^x = v(t, X_t^x),$$

there exist processes Z^x and U^x such that the quadruple (Y^x, Z^x, U^x, λ) solves the EBSDE

$$Y_t^x = Y_T^x + \int_t^T [f(X_u^x, Z_u^x, U_u^x) - \lambda] du - \int_t^T (Z_u^x)^* dW_u - \int_t^T \int_B U_s^x(x) \tilde{N}(ds, dx) \quad (2.27)$$

for $0 \le t \le T < \infty$. Moreover, if there exists any other solution (Y', Z', U', λ') that satisfies

$$|Y_t'| < c_x(1 + \|X_t^x\|^2), \quad (2.28)$$

for some constant c that may depend on x, then $\lambda = \lambda'$.

Proof: We look at the discounted BSDE

$$\begin{aligned} Y_T^{\alpha,x} = Y_t^{\alpha,x} - \int_t^T (-\alpha Y_u^{\alpha,x} - \alpha v^\alpha(0,0) + f(X_t^x, Z_u^{\alpha,x}, U_u^{\alpha,x}))du \\ + \int_t^T (Z_u^{\alpha,x})^* dW_u + \int_t^T \int_B U_s^{\alpha,x}(x) \tilde{N}(ds, dx). \end{aligned}$$

It is clear that the unique bounded solution is $Y_t^{\alpha,x} = v^\alpha(t, X_t^x) - v^\alpha(0,0)$. We remember that $|v^\alpha(s, X_s^x) - v^\alpha(0,0)| \le (1 + \|X_t^x\|^2)$. We conclude, by the dominated convergence theorem, that

$$\mathbb{E} \int_0^T |Y_t^{\alpha,x} - Y_t^{\alpha_m,x}|^2 dt \to 0 \quad \text{and} \quad \mathbb{E}|Y_T^{\alpha,x} - Y_T^{\alpha_m,x}|^2 \to 0$$

as $n \to \infty$.

We now prove that the sequences $Z^{\alpha,x}$ and $U^{\alpha,x}$ are also Cauchy. Denote $\bar{Y} = Y^{\alpha_n,x} - Y^{\alpha_m,x}$, $\bar{Z} = Z^{\alpha_n,x} - Z^{\alpha_m,x}$, $\bar{U} = U^{\alpha_n,x} - U^{\alpha_m,x}$. We then have

$$\bar{Y}_T = \bar{Y}_t - \int_t^T \big(-\alpha \bar{Y}_u + \bar{f}(u) \big) du + \int_t^T (\bar{Z}_u)^* dW_u + \int_t^T \int_B \bar{U}_s(x) \tilde{N}(ds, dx),$$

where $\bar{f}(u) = f(X_u^x, Z_u^{\alpha_n,x}, U_u^{\alpha_n,x}) - f(X_u^x, Z_u^{\alpha_m,x}, U_u^{\alpha_m,x})$. By standard arguments, we know that, for any $\beta \ge 4K + 1/2$ (where K is the Lipschitz constant of f), and $\beta > \max(\alpha_n, \alpha_m)$, we have

$$\begin{aligned} e^{\beta t} \mathbb{E} \|\bar{Y}_t\|^2 + \frac{1}{2} \int_t^T e^{\beta s} \mathbb{E} \Big(\|\bar{Z}_t\|^2 + \int_B \|\bar{U}_s(v)\|^2 \nu(dv) \Big) dt \\ \le \mathbb{E} \Big[\|\bar{Y}_T\|^2 + \frac{4}{2\beta - 1} \int_t^T e^{\beta s} \|\delta f_s\|^2 ds \Big], \end{aligned}$$

where

$$\delta f_s = (\alpha_n - \alpha_m) Y_s^{\alpha_m, x}.$$

By Theorem 2.4.3 and using the bound on $\mathbb{E}[\|X_t^x\|^2]$ obtained in Step 1 of Theorem 2.3.3, we know that there exists $C = C(x)$ such that $\mathbb{E}[\|Y_s^{\alpha_m,x}\|^2] \leq C$, and thus

$$\mathbb{E}\left[\frac{4}{2\beta - 1} \int_t^T e^{\beta s} \|\delta f_s\|^2 ds\right] \leq \frac{4CT}{2\beta - 1}(\alpha_n - \alpha_m)^2,$$

and hence we immediately see that sequences $\{Z^{\alpha_n,x}\}_{n\geq 1}$ and $\{U^{\alpha_n,x}\}_{n\geq 1}$ are Cauchy. Denoting Z^x and U^x their corresponding limits, we get the first part of the result. In order to prove uniqueness, suppose there exists another solution (Y', Z', U', λ') with polynomial growth. Let $\tilde{Y} = Y^x - Y'$, $\tilde{Z} = Z^x - Z'$, $\tilde{U} = U^x - U'$ and $\tilde{\lambda} = \lambda - \lambda'$. Then

$$\begin{aligned} \tilde{Y}_t = \tilde{Y}_T + \int_{]t,T]} [f(X_u^x, Z_u^x, U_u^x) - f(X_u^x, Z_u', U_u') - \tilde{\lambda}] du \\ - \int_{]t,T]} \tilde{Z}_u^* dW_u - \int_{]t,T]} \int_B \tilde{U}_s(r) \tilde{N}(ds, dr) \end{aligned}$$

By the standard Girsanov's argument there exists a probability measure $\mathbb{Q}^T \sim \mathbb{P}$ such that under $\mathbb{Q}^T$ the process

$$\begin{aligned} K_t = \int_t^T (f(X_u^x, Z_u^x, U_u^x) - f(X_u^x, Z_u', U_u')) du \\ + \int_t^T \delta \tilde{Z}_u^* dW_u + \int_t^T \int_B \tilde{U}_s(r) \tilde{N}(ds, dr) \end{aligned}$$

is a martingale on $[0, T]$. Then we see that

$$\tilde{\lambda} = T^{-1} \mathbb{E}^{\mathbb{Q}^T} [\tilde{Y}_T - \tilde{Y}_0].$$

Given the growth condition (2.28) and the estimate (2.10), by sending $T \to \infty$ we obtain $\tilde{\lambda} = 0$ and thus the uniqueness of λ is proven.

□

We are now ready to prove the main uniqueness result for Markovian solutions to our EBSDE, where by "Markovian" we mean that, if Y is a solution, then there exists a continuous deterministic function v, such that $Y_t = v(t, X_t^x)$ for all $t > 0$. In the proof we will use the fact that the coefficients in the forward process are time dependant but T^*-periodic for some $T^* > 0$. Recalling the construction of the solution in Theorem 2.4.3 we immediately see that it is T^*-periodic in the first argument. Therefore, it is sensible to establish uniqueness in the class of Markovian solutions for which

$$v(t, x) = v(t + T^*, x) \quad \forall t > 0, x \in H. \tag{2.29}$$

Theorem 2.4.5. *Let (Y, Z, U, λ) and (Y', Z', U', λ') be two Markovian solutions to the EBSDE (2.27). If Y, Y' satisfy the growth condition (2.28), v, v' satisfy (2.29) and $v'(0,0) = v(0,0)$, then $v = v'$ a.e.*

Proof: From Theorem (2.4.4) we know that $\lambda = \lambda'$. We now show that in this case $Y = Y'$. Denoting $\tilde{Y} = Y^x - Y'$, $\tilde{Z} = Z^x - Z'$, $\tilde{U} = U^x - U'$ and defining $\mathbb{Q}^T$ as in the proof of Theorem 2.27, we immediately have for all $t < T$

$$\tilde{Y}_t = \mathbb{E}^{\mathbb{Q}^T}[\tilde{Y}_T | \mathcal{F}_t]$$

for all T. Given the Markovian representation of our solutions we can rewrite the above as

$$\tilde{v}(t,x) = \mathbb{E}^{\mathbb{Q}^T}[\tilde{v}(T, X_T^{t,x}) | \mathcal{F}_t], \tag{2.30}$$

where $\tilde{v}(t,x) := v(t,x) - v'(t,x)$. Now, since (2.30) holds for any T, we obtain

$$\tilde{v}(t,x) = \mathbb{E}^{\mathbb{Q}^{kT^*}}[\tilde{v}(kT^*, X_{kT^*}^{t,x}) | \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}^{kT^*}}[\tilde{v}(0, X_{kT^*}^{t,x}) | \mathcal{F}_t],$$

for all k such that $kT^* \geq t$. The next ingredient we will require is following estimate which can be shown with a technique identical to the one used to obtain (2.10):

$$\mathbb{E}^{\mathbb{Q}^T}\left[\|X_t^x\|^4\right] < c(1 + \|x\|^4), \quad t \in [0,T]$$

where c is independent of T. We now notice that, for any $\epsilon > 0$, there exists $\delta > 0$ such that $|\tilde{v}(0,x)| \leq \epsilon$ if $\|x\| < \delta$, due to the fact that $\tilde{v}$ is locally Lipschitz $\tilde{v}(0,0) = 0$. Set $\tau = \inf\{kT^* : \|X_{kT^*}^{t,x}\| < \epsilon, k \in \mathbb{N}\}$. We then see that

$$\begin{aligned}
|\tilde{v}(t,x)| &= \left|\mathbb{E}^{\mathbb{Q}^{kT^*}}[\tilde{v}(0, X_{kT^* \wedge \tau}^{t,x}) | \mathcal{F}_t]\right| \\
&\leq \mathbb{E}^{\mathbb{Q}^{kT^*}}\left[|\tilde{v}(0, X_\tau^{t,x})| \mathbf{1}_{\{\tau < kT^*\}}\right] + \mathbb{E}^{\mathbb{Q}^{kT^*}}\left[|\tilde{v}(0, X_{kT^*}^{t,x})| \mathbf{1}_{\{\tau \geq kT^*\}}\right] \\
&\leq \epsilon + (\mathbb{Q}^{kT^*}(\tau > kT^*))^{\frac{1}{2}} \left(\mathbb{E}^{\mathbb{Q}^{kT^*}} |\tilde{Y}_{kT^*}|^2\right)^{\frac{1}{2}} \\
&\leq \epsilon + C(\mathbb{Q}^{kT^*}(\tau > kT^*))^{\frac{1}{2}} \left(\mathbb{E}^{\mathbb{Q}^{kT^*}}\left[1 + |X_{kT^*}^{t,x}|^4\right]\right)^{\frac{1}{2}} \\
&\to \epsilon.
\end{aligned}$$

The last step of the derivation above follows from the fact that

$$\mathbb{Q}^{kT^*}(\tau > kT^*) \to 0 \text{ as } k \to \infty.$$

In order to see this, we look at the discretised process $\{X_{kT^*}^{t,x}\}_{k \in \mathbb{N}}$. We immediately see that it is irreducible. We therefore can prove the desired recurrence by following the proof of Theorem 2.3.4 with time step chosen as the first multiple of T^* larger than $\tilde{T}$.

□

Remark 2.4.3. *We briefly outline the connection between solutions to finite horizon BSDEs and their ergodic counterpart. The following discussion is inspired by Theorem 4.1 in [22]. Fix $T > 0$ and denote $(Y^{T,x}, Z^{T,x}, U^{T,x})$ the (unique) solution to the following BSDE*

$$Y_t^{T,x} = \int_t^T f(X_u^x, Z_u^{T,x}, U_u^{T,x})du - \int_t^T (Z_u^{T,x})^* dW_u - \int_t^T \int_B U_s^{T,x}(x)\tilde{N}(ds, dx).$$

Let $v(\cdot, \cdot)$ be the Y-component of the Markovian solution to EBSDE (2.24) with ergodic component λ. Then we immediately see that

$$Y_0^{T,x} - v(0, x) = -\mathbb{E}v(T, X_T^x) + \lambda T.$$

By polynomial growth we know that

$$\|v(0, x)\| \leq C'(1 + \|x\|^2), \quad \|v(T, X_T^x)\| \leq C'(1 + \|X_T^x\|^2).$$

By Theorem 2.3.1 we know that

$$\mathbb{E}\|X_T^x\|^2 \leq C''(1 + \|x\|^2)$$

for some constant $C'' > 0$ independent of time. Therefore we obtain the following limiting behaviour:

$$\frac{Y_0^{T,x}}{T} \to \lambda. \tag{2.31}$$

In the context of control (see Section 2.5 for details) (2.31) shows that, as one goes from finite horizon optimisation to the ergodic one, even though optimal policies do not have to converge, optimal (average) values do.

2.4.3 Alternative representation for λ

In this section we show the representation of λ as an integral with respect to a certain invariant measure. We established in Section 3.3 that there exists a unique invariant measure μ corresponding to the semilinear problem

$$dX_t = A(t)X_t dt + F_t(X_t)dt + G(t)dL_t, \quad X_\tau = x.$$

In particular, the following holds:

$$\int_{[0,T^*]\times H} P_s u(t, x)\mu(dt, dx) = \int_{[0,T^*]\times H} u(t, x)\mu(dt, dx),$$

where P_s is the corresponding semigroup. We recall that the Markovian solution to the EBSDE (2.27) constructed in Theorem 2.4.4 is T^*-periodic, that is the quadruple (Y, Z, U, λ) has a representation (v, ξ, ψ, λ), where v, ξ and ψ are T^*-periodic in time.

Theorem 2.4.6. *The value λ in the EBSDE solution (v, ξ, ψ, λ) satisfies*

$$\lambda = \int_{[0,T^*]\times H} f(x, \xi(t,x), \psi(t,x))\mu(dt, dx),$$

where μ is the unique invariant measure.

Proof: The invariance of μ implies that for any fixed times T and $s \leq T$, and any bounded measurable function u such that $u(t + T^*, x) = u(t, x)$ we have

$$\int_{[0,T^*]\times H} \mathbb{E}u(T, X_T^{s,x})\mu(dt, dx) = \int_{[0,T^*]\times H} u(t, x)\mu(dt, dx).$$

We write

$$v(t,x) = \mathbb{E}^{\mathbb{P}_{x,t}}\left[v(T, X_T^{s,x}) + \int_t^T (f(X_t^{s,x}, \xi(s, X_t^{s,x}), \psi(s, X_t^{s,x})) - \lambda)ds\right],$$

where the subscript (x, t) indicates that the forward equation was started at time t with the value x. Then by the invariance property, integrating both sides with respect to μ, we obtain the result.

□

Remark 2.4.4. *The representation above gives us an intuitive idea of how to interpret λ. If one thinks about the driver f as a cost function of the optimally controlled dynamical system for the law of X, then λ is the cost of one cycle.*

2.5 Applications

2.5.1 Classical Ergodic Control

In this section we show how general ergodic control problems can be seen in the framework of EBSDEs for the case of controlled drift. Denote by $L : H \times \mathcal{U} \to \mathbb{R}$ a bounded measurable cost function such that

$$|L(x, u) - L(x', u)| \leq C\|x - x'\|,$$

for some $C > 0$. We consider the problem of minimising

$$J(x_0, u) = \limsup_{T\to\infty} T^{-1}\mathbb{E}^{u,T}\left[\int_0^T L(X_t, u_t)dt\right],$$

over the space $\mathcal{U}$ of controls, a separable metric space in which $u_t(\omega)$ takes values. We further assume that under $\mathbb{P}^{u,T} \sim \mathbb{P}$ the dynamics of the controlled process X on $[0, T]$ are given by

$$dX_t = (A(t)X_t + F_t(X_t))dt + G(t)\left(R(u_t)dt + \int_B \gamma(u(t), y)\nu(dy)\right)dt + G(t)dL_t,$$

with $X_0 = x_0$. We further assume that $\|R(u)\| \leq C'$ and $\gamma(u(t), y)$ is a measurable function such that there exist a constant $0 \leq C < 1$ such that for every $u \in \mathcal{U}$

$$-C(1 \wedge ||\xi||) \leq \gamma(u, \xi) \leq C(1 \wedge ||\xi||)$$

for all $\xi \in B$. We define the Hamiltonian

$$f(x, z, r) = \inf_{u \in \mathcal{U}} \left\{ L(x, u) + zR(u) + \int_B \gamma(u, \xi) r(\xi) \nu(d\xi) \right\}, \tag{2.32}$$

where $x \in H$, $z \in H$ and $r : B \to \mathbb{R}$. Immediately we notice that $f(x, 0, 0)$ is bounded. It is also easy to check that f satisfies Assumptions 2.4.1 and 2.4.2. Therefore, the EBSDE with driver $f(x, z, r)$ admits a unique (in the class of processes with polynomial growth) Markovian solution (Y, Z, U, λ). If the infimum in (2.32) is attained, then, by a well known result (see [10]), there exists (assuming the continuum hypothesis) a Borel-measurable function $\kappa : H \times H^* \times \mathcal{L}^2(B, \mathcal{B}, \nu, \mathbb{R}) \to \mathcal{U}$ such that

$$f(x, z, r) = L(x, \kappa(x, z, r)) + zR(\kappa(x, z, r)) + \int_B \gamma(\kappa(x, z, r), \xi) r(\xi) \nu(d\xi).$$

Theorem 2.5.1. *Let the quadruple (Y, Z, U, λ) be the unique Markovian solution satisfying $|Y_t| \leq c(1 + \|X_t\|^2)$ for all $t \geq 0$ and some $c > 0$. Then the following hold:*

(i) For an arbitrary control $u \in \mathcal{U}$ we have $J(x_0, u) = \lambda$ if

$$f(X_t, Z_t, U_t) = L(X_t, u(t)) + Z_t R(u(t)) + \int_B \gamma(u(t), \xi) r(\xi) \nu(d\xi) \quad d\mathbb{P} \times dt - a.e.$$

(ii) If the infimum is attained in (2.32), then the control $\bar{u}(t) = \kappa(X_t, Z_t, U_t)$ verifies $J(x_0, \bar{u}) = \lambda$.

(iii) Even if the infimum in (2.32) is not attained, there exists a control $\{\tilde{u}_t\}_{t \geq 0}$, such that $J(x_0, \tilde{u}) = \lambda$.

Proof: Identical to the proofs of Theorem 8 in [12] and Theorem 5.1 in [2]. □

2.5.2 Power plant evaluation

In this section we present a model for power plant evaluation using Ergodic BSDEs. We show how due to the properties of gas and electricity the problem falls very naturally into the theoretical framework we have developed. We begin by defining a mathematical model of a power plant.

Definition 2.5.1. *We denote by $\{E(t)\}_{t\geq 0}$ and $\{G(t)\}_{t\geq 0}$ the electricity and gas price processes respectively. We assume that a power plant allows its owner to convert gas into electricity instantaneously, generating profit if $E(t) - cG(t) > 0$, where c is some conversion constant. The quantity $X(t) := E(t) - cG(t)$ is called the spark spread.*

In existing literature (for an overview see, for example, [8]) the value of a power plant is approximated as a sum of spread options on spot power with different maturities, namely european options with payoffs $X^+_{T_j}$, where $\{T_j, j \in J\}$ represent the future hours of production over the plant's lifetime. In other words

$$VP_t = \sum_{j\in J} \exp(-r(T_j - t))\mathbb{E}^{\mathbb{Q}}\Big((X_{T_j})^+ \Big| \mathcal{F}_t \Big).$$

A flaw of this approach is that it relies heavily on the current state of the world, characterised by the short term dynamics of the electricity and gas prices. However, it is clear that one might want to evaluate the power plant before investing into its construction, and by the time the plant begins operation all the short term parameters will have changed. In the rest of the section we provide an alternative method for evaluation, assuming only that the price processes follow ergodic behaviour. In terms of the problem in question, this means that the present state is not important for the calculation of the long term (ergodic) average.

We develop a slightly simplified model, where we do not give the dynamics of electricity and gas prices separately, but instead assume that the evolution of the spark spread X is governed by the following equation:

$$dX_t = \theta_t(\kappa_t - X_t)dt + G(t)\big[dW_t - \int_B x\tilde{N}(dt, dx)\big], \quad X_\tau = x, \tag{2.33}$$

where $B = \mathbb{R}\backslash\{0\}$, $\{\theta_t\}_{t\geq 0}$ is a positive process that describes the rate of mean reversion, $\{\kappa_t\}_{t\geq 0}$ is a non-negative process of the mean and $\tilde{N}$ is a compensated Poisson random measure on $\mathbb{R}_+ \times B$ with the compensator $\eta(dt, dx) = \nu(dx)dt$. We also assume that all the processes are periodic in time with period $T^* =$ one year . The goal is to find the average yearly profit of the plant, namely

$$\lambda = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\int_t^T (X_s)^+ ds,$$

where $(x)^+ := \max(x, 0)$. It is important to notice that, in reality, the difficulty in finding λ comes from the fact that the vector of parameters (θ, κ, ν) is not known exactly. Therefore, we face the risk averse problem of determining the worst-case average under a range of plausible parameters, namely

$$\lambda = \inf_{u \in \mathcal{U}} \lim_{T \to \infty} \frac{1}{T} \mathbb{E}^u \int_t^T (X_s^u)^+ ds,$$

where $\mathcal{U}$ denotes a space of possible values for $u = (\theta, \kappa, \nu)$, and under $\mathbb{P}^u \sim \mathbb{P}$ the dynamics of X are given by

$$\begin{aligned} dX_t = \theta_t(\kappa_t - X_t)dt + R(X_t, u(t))dt + \int_B \gamma(u(t), y)\nu(dy)dt \\ + G(t)\big[dW_t - \int_B x\tilde{N}(dt, dx)\big]. \end{aligned}$$

These parameters control the rate of mean reversion through R and the rate of spikes through γ. In order to make the model more realistic, without loss of clarity one can also consider the problem of minimising a generalised functional

$$\lambda = \inf_{u \in \mathcal{U}} \lim_{T \to \infty} \frac{1}{T} \mathbb{E}^u \int_t^T L(X_s, u(s)) ds,$$

where $L(x, u)$ incorporates a penalty corresponding to the perceived likelihood of the parameters being realised. Following exactly the same logic as in the derivation of (2.32), we define the Hamiltonian

$$f(x, z, r) = \inf_{u \in \mathcal{U}} \left\{ L(x, u) + zR(x, u) + \int_B \gamma(u, \xi) r(\xi) \nu(d\xi) \right\},$$

and proceed to solve the EBSDE with the driver f.

Remark 2.5.1. *It is clear that once λ is known, the risk-averse discounted expected revenue of the power plant with estimated lifetime of N years can be calculated by*

$$v(N) = \lambda \int_0^N e^{-r(t)} dt,$$

where $r(t)$ is a (deterministic) discount rate.

Remark 2.5.2. *As we mentioned at the beginning of this section, imposing the Ornstein–Uhlenbeck dynamics on the spark spread is restrictive. Ideally one would like to model electricity and gas processes separately. If we assume that the marginal price processes follow sums of OU processes (as in [33], where the authors focus mainly on the copula-based approach) we end up with a two-dimensional problem, where the ergodicity required for the existence of a solution to EBSDE is obtained through the fact that the sum of ergodic processes is itself ergodic. The reason we present a simplified version is that it naturally demonstrates the theoretical framework we developed throughout this chapter, and gives a clear illustration of how EBSDEs can be applied to this class of problems.*

Chapter 3

Nash Equilibria for non Zero-Sum Ergodic Stochastic Differential Games

3.1 Introduction

The area of non-zero sum stochastic differential games has been a subject of intensive research over the past several decades. It deals with the situation where multiple players are each trying to maximise their payoff, but rewards of all players do not sum up to any constant. The basic aim is to find a Nash equilibrium point – a set of strategies for all players, such that none of them would benefit from unilaterally deviating. The study of the connection between this problem and backward stochastic differential equations (BSDEs) was pioneered by Hamadéne, Lepeltier and Peng in [26], building up on their previous work on zero-sum games in [20]. Since then it has been discussed in multiple contexts, e.g. games of control and stopping by Karatzas and Li in [13] and risk-sensitive control by El Karoui and Hamadéne in [25]. The main goal of the chapter is to prove the existence of an equilibrium in the case where we have two drift controlling players who are ergodic, i.e. they choose their strategies in infinite horizon and optimise the long run average. In a standard fashion (see, e.g. [21]), we impose the so-called generalised Isaac's condition on the Hamiltonian, in order to ensure the attainability of the infimum simultaneously for both players. We then prove that the existence of a saddle point for the game follows from the existence of a solution to a system of Ergodic BSDEs with continuous coefficients. Using a modified version of Picard iteration (not dissimilar in nature to the fixed point construction used by Hu and Tang in [23]), along with an array of estimates for solutions of EBSDEs, established in [14] and [11], we prove that such a system does admit a solution. We then extend these results to the case of finite or countably many players. As a byproduct of developing the machinery required to deal with ergodic games, we obtain the existence of a solution to a class of EBSDEs with continuous drivers of linear growth. The concluding section is dedicated to asymmetric games, where one player is purely ergodic, and the other also optimises in infinite horizon, but values the present more than the future (i.e. uses discounting). We show that the value of such a game exists, and demonstrate convergence to the ergodic system as we send the discount rate to zero.

3.2 Setup

We begin by describing the uncontrolled forward process $\{X_t\}_{t\geq 0}$, the drift of which will be affected by the players in the sequel. Let the dynamics of $\{X_t\}_{t\geq 0}$ be governed by the Itô SDE:

$$dX_t = (AX_t + F(X_t))dt + \sigma(X_t)dW_t, \quad X_0 = x_0, \tag{3.1}$$

where W denotes a standard Brownian motion under the measure $\mathbb{P}$. We need a few preliminary assumptions.

Assumption 3.2.1. *Conditions to guarantee existence of a strong solution to (3.1):*

- *The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, $\{\mathcal{F}_t\}_{t\geq 0}$ is an augmented filtration satisfying the usual conditions, and $\{W_t\}_{t\geq 0}$ is a Brownian motion.*

- *The process $\{W_t\}_{t\geq 0}$ has predictable representation property in the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.*

- *The operator A is dissipative and generates a stable semigroup $\{e^{tA}\}_{t\geq 0}$. In other words, there exist constants $\mu > 0$, $M > 0$ such that*

$$\langle Ax, x\rangle \leq -\mu\|x\|^2 \text{ and } \|e^{tA}\| \leq Me^{-\mu t}$$

hold for all $x \in \mathbb{R}^n$ and all $t \geq 0$. Here and for the rest of the chapter $\|\cdot\|$ denotes the Euclidean norm (or Frobenius norm for a matrix).

- *The map F is uniformly Lipschitz and bounded.*

- *There exist constants $\underline{\sigma}$ and $\overline{\sigma}$ such that*

$$\underline{\sigma} \leq \|\sigma(x)\| + \|\sigma^{-1}(x)\| \leq \overline{\sigma}$$

holds for all $x \in \mathbb{R}^n$, $t \geq 0$.

The following result is a straightforward application of Grönwall's lemma and the dissipativity condition:

Theorem 3.2.1. *If Assumption 3.2.1 is satisfied, then there exists a unique strong solution to the equation (3.1). Moreover,*

$$\mathbb{E}\big(|X_t|^2\big) < C(1 + \mathbb{E}|x_0|^2),$$

where the constant C depends on M, μ, but is independent of t.

Proof: Fix a constant $k > 0$. Applying Itô's formula to $e^{kt}\|X_t\|^2$ and taking expectations, we obtain

$$e^{kt}\mathbb{E}\|X_t\|^2 = \mathbb{E}\|x_0\|^2 + \int_0^t ke^{ks}\|X_s\|^2 ds + \mathbb{E}\int_0^t e^{ks}G(s)ds,$$

where

$$\begin{aligned} G(s) &= 2\langle AX_t + F(X_t), X_s\rangle + \|\sigma(X_s)\|^2 \\ &\leq -2\mu\|X_s\|^2 + \langle F(X_s), X_s\rangle + \|\sigma(X_s)\|^2. \end{aligned}$$

Taking into account Assumption 3.2.1 and applying Cauchy–Schwarz, we see that for any $\epsilon > 0$, there exists C^ϵ, such that

$$G(s) \leq -2(\mu - \epsilon)\|X_s\|^2 + C^\epsilon,$$

for $0 \leq s \leq t$. Then, choosing $\epsilon < \mu$ and $k < 2(\mu - \epsilon)$, we arrive at

$$e^{kt}\mathbb{E}\|X_t\|^2 \leq \mathbb{E}\|x_0\|^2 + \frac{C^\epsilon}{k}e^{kt},$$

concluding the proof.

□

Remark 3.2.1. *Define the new process $\bar{X}_t := \|X_t\|^2$. As a straightforward application of Itô's lemma it follows that*

$$d[\bar{X}]_t = \langle\sigma(X_t)X_t, \sigma(X_t)X_t\rangle dt \leq \bar{\sigma}^2\bar{X}dt.$$

Therefore by exactly the same reasoning as in the proof of Theorem 3.2.1, one can show that

$$\mathbb{E}\|X_t\|^4 \leq C(1 + \mathbb{E}\|x_0\|^4), \tag{3.2}$$

for some constant C independent of time.

Remark 3.2.2. *If $F(\cdot)$, the nonlinear part of the drift, is a bounded measurable map, there still exists a solution to (3.1) in the weak sense. In other words, there exists a measure $\tilde{\mathbb{P}} \ll \mathbb{P}$ and an $\{\mathcal{F}_t\}_{t\geq 0}$-Brownian motion $\tilde{W}$ such that*

$$X_t = e^{tA}x_0 + \int_0^t e^{sA}F(X_s)ds + \int_0^t e^{sA}\sigma(X_s)d\tilde{W}(s)$$

holds, and such solution is unique in law. Moreover, since this solution can be obtained through a measure change from a strong solution to the equation without drift, $\{\tilde{W}_t\}_{t\geq 0}$ has predictable representation property in the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. For details see, e.g. Chapters 15 and 18 in [10].

Let $\mathcal{U}_1 \times \mathcal{U}_2$ be a separable metric space of controls in which $(u_t(\omega), v_t(\omega))$ takes values. Denote by $L_i : \mathbb{R}^N \times \mathcal{U}_1 \times \mathcal{U}_2 \to \mathbb{R}$, $i = 1, 2$, bounded measurable cost functions, such that

$$|L_i(x, u, v) - L_i(x', u, v)| \leq C\|x - x'\|,$$

for some $C > 0$. Given an arbitrary pair of admissible controls (u, v) (i.e. predictable stochastic processes $(\{u_t\}_{t\geq 0}, \{v_t\}_{t\geq 0})$), we define the Girsanov density

$$\rho_T^{u,v} := \exp\left(\int_0^T R(u_t, v_t)dW_t - \frac{1}{2}\int_0^T \|R(u_t, v_t)\|^2 ds\right)$$

for some function $R : \mathcal{U}_1 \times \mathcal{U}_2 \to \mathbb{R}^N$, satisfying $\|R(\cdot, \cdot)\| \leq \bar{C}$ for some $\bar{C} > 0$. We define the probability $\mathbb{P}^{u,v,T} := \rho_T^{u,v}\mathbb{P}$. For each agent $(i = 1, 2)$ we define ergodic payoffs

$$J^i(x_0, u, v) = \limsup_{T\to\infty} T^{-1}\mathbb{E}^{u,v,T}\left[\int_0^T L_i(X_t, u_t, v_t)dt\right]. \tag{3.3}$$

The goal is to find an admissible control (u^*, v^*), such that

$$J^1(x_0, u^*, v^*) \leq J^1(x_0, u, v^*) \text{ and } J^2(x_0, u^*, v^*) \leq J^2(x_0, u^*, v)$$

holds for all admissible (u, v).

In order to use dynamic principles to determine optimal u and v, we define the Hamiltonian functions $\{H_i\}_{i=1,2}$, as

$$H_i(x, z_i, u, v) = z_i R(u, v) + L_i(x, u, v) \tag{3.4}$$

for $(u, v) \in \mathcal{U}_1 \times \mathcal{U}_2$. We notice that for a fixed pair of controls (u, v), for each player $(i = 1, 2)$, the payoff (3.3) is an ergodic average. Consider the following ergodic backward stochastic differential equation (EBSDE), in which the Hamiltonian plays the role of the driver term:

$$Y_t^i = Y_T^i + \int_t^T [H_i(X_s, Z_s^i, u_s, v_s) - \lambda^i]ds - \int_t^T Z_s^i dW_s. \tag{3.5}$$

Suppose there exists a solution triplet (Y^i, Z^i, λ^i) to (3.5), and there also exists a constant $C = C(x_0) > 0$ independent of T, such that $\mathbb{E}^{u,v,T}\|Y_t\| < C$ for all $0 \le t \le T$. Then, by changing measure and dividing by T, we immediately see that

$$J^i(x_0, u, v) = \lambda^i.$$

We refer to (3.5) as the BSDE corresponding to the value function (3.3). In order to proceed we need the following assumptions:

Assumption 3.2.2. *Part (i) below is called the generalised Isaac's condition. Part (ii) is a technical condition, and the motivation behind it will be discussed later.*

(i) There exist measurable maps u_1^*, u_2^*, defined on $\mathbb{R}^{3N}$, with values in $\mathcal{U}_1$ and $\mathcal{U}_2$ respectively, such that

$$H_1(x, z_1, u_1^*(x, z_1, z_2), u_2^*(x, z_1, z_2)) \le H_1(x, z_1, u, u_2^*(x, z_1, z_2))$$

and

$$H_2(x, z_1, u_1^*(x, z_1, z_2), u_2^*(x, z_1, z_2)) \le H_2(x, z_1, u_1^*(x, z_1, z_2), v)$$

holds for all $(x, z_1, z_2, u, v) \in \mathbb{R}^{3N} \times \mathcal{U}_1 \times \mathcal{U}_2$.

(ii) The mapping $(z_1, z_2) \in \mathbb{R}^{2N} \to (H_1^*, H_2^*)(x, z_1, z_2)$ is continuous for any fixed x.

One way to think about the condition above is as the natural extension of the standard assumptions used in the zero-sum framework to the non zero-sum case. Indeed, consider the case of two players, one maximising and the other minimising a functional

$$J(x_0, u, v) = \limsup_{T\to\infty} T^{-1}\mathbb{E}^{u,v,T}\left[\int_0^T L(X_t, u_t, v_t)dt\right],$$

by choosing controls $(u_t, v_t) \in \mathcal{U}_1 \times \mathcal{U}_2$ for $t \ge 0$. Define the Hamiltonian as $H(x, z, u, v) = zR(u, v) + L(x, u, v)$. Then, using a measurable selection argument, one could show that the standard Isaac's condition

$$\max_{v\in\mathcal{U}_2} \min_{u\in\mathcal{U}_1} H(x, z, u, v) = \min_{u\in\mathcal{U}_1} \max_{v\in\mathcal{U}_2} H(x, z, u, v), \quad \forall x, z \in \mathbb{R}^N$$

is equivalent to the existence of Borel-measurable functions $u^* : \mathbb{R}^{2N} \to \mathcal{U}_1$ and $v^* : \mathbb{R}^{2N} \to \mathcal{U}_2$, such that

$$H(x,z,u^*(x,z),v^*(x,z)) \le H(x,z,u,v^*(x,z)) \text{ for all } u \in \mathcal{U}_1,$$
$$H(x,z,u^*(x,z),v^*(x,z)) \ge H(x,z,u^*(x,z),v) \text{ for all } v \in \mathcal{U}_2.$$

For details, see [20]. Given this connection, one can see that (i) in the Assumption 3.2.2 is a reasonable generalisation of the Isaac's condition.

Remark 3.2.3. *We notice that, for fixed u, v, the function H_i is Lipschitz in z_i. However, Assumption 3.2.2 only gives us the existence of a so called "closed loop" control, where the optimal processes u^*, v^* depend on (z_1, z_2), and only continuity is guaranteed. Therefore the standard method of proving the existence of a solution to the corresponding Ergodic BSDE breaks down.*

Remark 3.2.4. *In the game we consider, both players control the drift. In [21] the authors considered the case where the resulting $R(u, v)$ can be unbounded, but is of linear growth with respect to the underlying process $\{X_t\}_{t\ge0}$. However, since we are in the ergodic framework, we restrict our attention to the bounded case.*

Remark 3.2.5. *One may also notice the similarity between the generalised Isaac's condition and the definition of a saddle point. Indeed, one could think of part (i) of Assumption 3.2.2 as ensuring the existence of Nash equilibrium controls on the infinitesimal scale.*

3.3 Ergodic BSDEs with continuous coefficients and stochastic games

In this section we prove that, under our assumptions, the game has value, that is there exists a Nash equilibrium as defined in the preceding section. We begin by considering two players of ergodic type, then proceed to generalise to the case of $n \in \mathbb{N} \cup \{+\infty\}$ players. Finally, we show a connection with PDEs.

3.3.1 Games with two players of ergodic type

In order to establish the existence of a Nash equilibrium, the following two-step programme needs to be carried out:

- Assuming there exists a solution to the EBSDE corresponding to control pair (u^*, v^*) as defined in Assumption 3.2.2, establish its optimality.

- Prove the existence of a unique solution to the EBSDE with continuous coefficients, guaranteeing this assumption holds.

In order to use the results of [14] to establish the existence of a solution to a one-dimensional ergodic BSDE, the following set of assumptions on the driver needs to be made:

Assumption 3.3.1. *The driver $f : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a measurable map. Moreover, there exist constants l, κ such that*

$$\big|f(x,0)\big| \leq l, \quad \big|f(x,z) - f(x,\bar{z})\big| \leq \kappa \|z - \bar{z}\|$$

holds for all $x, z, \bar{z} \in \mathbb{R}^N$.

We begin by stating certain crucial results from the theory of Ergodic BSDEs. The following theorem establishes existence and uniqueness of solutions. We omit the proof, as it can be easily constructed using the arguments in [14].

Theorem 3.3.1. *Let the driver f satisfy Assumption 3.3.1. Then there exists a solution (Y, Z, λ), where $\{Y_t\}_{t\geq 0}$ is adapted, $\{Z_t\}_{t\geq 0}$ is predictable, and $\lambda \in \mathbb{R}$ is a constant, to the EBSDE*

$$Y_t = Y_T + \int_t^T [f(X_s, Z_s) - \lambda] ds - \int_t^T Z_s dW_s,$$

such that $\|Y_t\| \leq C(1+\|X_t\|^2)$ for some constant $C > 0$. There also exists a deterministic function $v : \mathbb{R}^N \to \mathbb{R}$, such that $Y_t = v(X_t)$. Moreover, the solution is unique in the class of Markovian solutions, for which the Y component is of polynomial growth with respect to X.

We now show that if both players can find a Markovian solution of polynomial growth to their respective EBSDEs with optimal pair of controls (u^*, v^*), then by so doing they find a Nash equilibrium.

Theorem 3.3.2. *Assume that there exist two triplets (Y^i, Z^i, λ^i), $i = 1, 2$, such that*

$$Y_t^i = Y_T^i + \int_t^T [H_i(X_s, Z_s^i, u^*(Z_s^1, Z_s^2), v^*(Z_s^1, Z_s^2)) - \lambda^i] ds - \int_t^T Z_s^i dW_s, \qquad (3.6)$$

holds for all $0 \leq t \leq T < 0$, where the pair (u^*, v^*) is as in Assumption 3.2.2. Moreover, assume there exist two deterministic functions with polynomial growth $y_i(x)$, $i = 1, 2$, such that $Y_t^i = y_i(X_t)$ holds $\mathbb{P}$-a.s. for all $t \geq 0$. Then the control $(u^*(X_s, Z_s^1, Z_s^2), v^*(X_s, Z_s^1, Z_s^2))$ is a Nash equilibrium.

Proof: We focus on the case $i = 1$. For $t \geq 0$ write $u^*_t = u^*(Z^1_t, Z^2_t)$ and $v^*_t = v^*(Z^1_t, Z^2_t)$. The control (u^*, v^*) is clearly admissible. For an arbitrary admissible control u, the pair (u, v^*) generates a measure $\mathbb{P}^{u,v^*}$, under which $dW^{u,v^*}_t = dW - R(u_t, v^*_t)dt$ is a Brownian motion. Then, since (Y^1, Z^1, λ^1) solves the EBSDE,

$$\begin{aligned}\lambda^1 T = Y^1_T - Y^1_0 + \int_0^T [H_i(X_s, Z^1_s, u^*_s, v^*_s) - Z^1_s R(u_t, v^*_s) - L^1(X_s, u_s, v^*_s)]ds \\ + \int_0^T Z^i_s dW^{u,v^*}_s + \int_0^T L^1(X_s, u_s, v^*_s)ds.\end{aligned}$$

Taking expectations with respect to $\mathbb{P}^{u,v^*}$, and remembering that λ^1 is a constant, we arrive at

$$\lambda^1 \leq \frac{1}{T}\mathbb{E}^{u,v^*}[Y^1_T - Y^1_0] + \frac{1}{T}\mathbb{E}^{u,v^*}\left[\int_0^T L^1(X_s, u_s, v^*_s)ds\right],$$

since

$$H_i(X_s, Z^1_s, u^*_s, v^*_s) - Z^1_s R(u_t, v^*_s) - L^1(X_s, u_s, v^*_s) \leq 0$$

by the definition of (u^*, v^*). We recall that Y^1 is of polynomial growth in X, and the latter has all moments under $\mathbb{P}^{u,v^*}$ uniformly in t (for details see, e.g. [14]). Thus, taking lim sup on both sides, we obtain

$$\lambda^1 \leq \limsup_{T\to\infty} \mathbb{E}^{u,v^*}\left[\int_0^T L^1(X_s, u_s, v^*_s)ds\right] = J^1(u, v^*).$$

For $u = u^*$ the inequality everywhere becomes equality by the definition of the pair (u^*, v^*) in Assumption 3.2.2, finishing the proof.

□

Remark 3.3.1. *The result of Theorem 3.3.2 shows that even in the case where admissible controls are allowed to be non-Markov, the values λ^i, $i = 1, 2$ obtained as solutions to the system (3.6) are still optimal. In other words, using the generalised Isaac's condition, we can show the existence of "closed loop" or "feedback" controls that are optimal in the larger class of "open loop" or "predictable" controls.*

We now proceed to the second step of our programme. We prove that there exists a solution to the system of ergodic BSDEs (3.6). The following auxiliary result is a weak version of a comparison theorem for the ergodic values λ, which holds in multiple dimensions.

Lemma 3.3.1. *Let f^i, $i = 1, 2$ be two drivers, such that for any (x, z_1, z_2)*

$$|f^i(x, z_1, z_2)| \leq C\|z_i\| + \bar{C}$$

for some constants $C, \bar{C} > 0$. Suppose that there exist Markov solutions (Y^i, Z^i, λ^i), $i = 1, 2$ to the corresponding EBSDEs. Let λ be the ergodic part of the solution (Y, Z, λ) to the EBSDE

$$-dY_t = [(C\|Z_t\| + \bar{C}) - \lambda]ds - Z_t dW_t.$$

Then $\lambda^i \le \lambda$ for $i = 1, 2$.

Proof: Define a measure $\mathbb{Q}^T$ as follows:

$$\frac{d\mathbb{Q}^T}{d\mathbb{P}} := \mathcal{E}\bigg(\int_0^T \frac{f(Z_t) - f(Z_t^i)}{\|Z_t - Z_t^i\|^2}(Z_t - Z_t^i)dW_t\bigg),$$

where $f(z) = C\|z\| + \bar{C}$. Then under $\mathbb{Q}^T$, the process $\{X_t\}_{t\ge 0}$ is ergodic with some invariant measure μ. Since we only consider Markovian solutions, we know that $Y_t^i = v^i(X_t)$, $Y_t = v(X_t)$ and $Z_t^i = \xi^i(X_t)$ (for details on Markov representation, see e.g. [16]). Then, integrating the difference $v^i(x) - v(x)$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} (v^i(x) - v(x))\mu(dx) \\
&= \int_{\mathbb{R}^N} \bigg[\mathbb{E}^{Q^T}(v^i(X_T) - v(X_T)) + \int_0^T (f^i - f)(X_t, Z_t^1, Z_t^2)dt - T(\lambda^i - \lambda)\bigg]\mu(dx) \\
&= \int_{\mathbb{R}^N} (v^i(x) - v(x))\mu(dx) - T(\lambda^i - \lambda) + \int_0^T \int_{\mathbb{R}^N} (f^i - f)(x, \xi^i(x), \xi^2(x))\mu(dx)dt,
\end{aligned}$$

from which we immediately see that

$$\lambda^i - \lambda = \int_{\mathbb{R}^N} (f^i(x, \xi^1(x), \xi^2(x)) - f(\xi^i(x)))\mu(dx) \le 0,$$

finishing the proof.

□

There are two apparent ways to attack the problem of proving the existence of a solution to (3.6):

(i) For any $t \ge 0$, equation (3.6) gives us a map $Y_{t+1}^i \to Y_t^i$ for $i \in 1, 2$. If we integrate with respect to the ergodic measure π for the forward process $\{X_t\}_{t\ge 0}$, we have a new map. Now we know that the solution to (3.6) is its fixed point. To proceed, we would need to prove certain compactness and continuity properties for this map.

(ii) We could attempt construction of a solution iteratively using a form of Picard iteration.

In this work we focus on plan (ii). The main result of this section is as follows:

Theorem 3.3.3. *Suppose Assumptions 3.2.1 and 3.2.2 are satisfied. Then there exist solution triplets (Y^i, Z^i, λ^i), $i = 1, 2$, where Y^i is adapted, Z^i predictable, and λ^i a constant, such that*

$$Y_t^i = Y_T^i + \int_t^T [H_i(X_s, Z_s^i, u^*(Z_s^1, Z_s^2), v^*(Z_s^1, Z_s^2)) - \lambda^i]ds - \int_t^T Z_s^i dW_s,$$

holds for all $0 \leq t \leq T < 0$. Moreover, there exist deterministic functions with polynomial growth $y^1(x), y^2(x)$, such that $Y_t^i = y^i(X_t)$, $i = 1, 2$ holds $\mathbb{P}$ - a.s. for all $t \geq 0$.

Proof: We begin by setting $Z^{i,0} \equiv 0$ for $i = 1, 2$. On each step, given a set of Markovian solutions $\{(Y^{i,n-1}, Z^{i,n-1}, \lambda^i)\}_{i=1,2}$ obtained previously, we define the next iteration as a solution to the EBSDE

$$dY_t^{i,n} = -[f^{i,n}(X_t, Z_t^{i,n}) - \lambda^{i,n}]dt - \int_t^T Z_t^{i,n} dW_t, \quad i = 1, 2 \tag{3.7}$$

where

$$f^{i,n}(X_s, Z^{i,n}) = H_i(X_s, Z_s^{i,n}, u^*(Z_s^{1,n-1}, Z_s^{2,n-1}), v^*(Z_s^{1,n-1}, Z_s^{2,n-1})).$$

This notation makes sense, since $\{Z^{i,n-1}\}_{i=1,2}$ are Markovian. In other words, there exist deterministic functions $\{\xi^{i,n-1}\}_{i=1,2}$, such that $Z_t^{i,n-1} = \xi^{i,n-1}(X_t)$. We note that for all $n \geq 1$, the drivers $\{f^{i,n}\}_{i=1,2}$ are uniformly Lipschitz in the z component, since

$$\left|f^{i,n}(x, z) - f^{i,n}(x, z')\right| = \left|R((u^*, v^*)(\xi^{1,n-1}, \xi^{2,n-1}))\right| \|z - z'\| \leq C\|z - z'\|.$$

We therefore know that there exists a solution $\{(Y^{i,n}, Z^{i,n}, \lambda^{i,n})\}_{i=1,2}$ to (3.7) [1].

It remains to show that there exists a convergent subsequence with limit $\{(Y^i, Z^i, \lambda^i)\}_{i=1,2}$. By Lemma 3.3.1, there exists a constant $\lambda > 0$, such that

$$\|\lambda^{i,n}\| \leq \lambda, \quad i = 1, 2$$

holds for all $n \geq 1$. We can therefore find a subsequence $\{n_k\}_{k \in \mathbb{N}}$, such that $\lambda^{i,n_k} \to \lambda^i$, $i = 1, 2$ for some $\{\lambda^i\}_{i=1,2}$. One can also show (see Theorem 8 in [11]), that for

[1] It is important to note that the existence results in [11] and [14] assume that the volatility of the forward dynamics is constant. However, since in the present chapter we are dealing with finite-dimensional case, the proof of the ergodicity of the forward process (and thus existence of a solution to the EBSDE) can be adapted to incorporate state-dependence. For details see Section 4.3.3.

all $n \geq 1$, there exist deterministic functions $\{v^{i,n}\}_{i=1,2}$, such that $Y_t^{i,n} = v^{i,n}(X_t)$, $Z_t^{i,n} = \nabla v^{i,n}(X_t)\sigma(X_t)$ and

$$\|Y_t^{i,n}\| = \|v^{i,n}(X_t)\| \leq C(1 + \|X_t\|^2) \tag{3.8}$$

holds for some constant $C > 0$ independent of n (*As shown in [11], this constant depends on* $\sup_{x\in\mathbb{R}^n} F(x)$ *and on the Lipschitz constant of* $f^{i,n}(x,z)$ *in the* z *component. The fact that the latter is uniform in* n *follows from the definition of the Hamiltonian.*). Moreover, we have the following gradient estimate:

$$\|\nabla v^{i,n}(x)\| \leq C(1 + \|x\|^2) \tag{3.9}$$

for $i = 1, 2, n \in \mathbb{N}$. Since $\mathbb{R}^N$ is separable, there exists a dense countable subset $B \subset \mathbb{R}^n$. We can therefore extract a further subsequence $\{n_{k_l}\}$, such that

$$v^{i,n_{k_l}}(x) \to v^i(x), \quad i = 1, 2$$

for all $x \in B$, and some $v^i : B \to \mathbb{R}$. Given (3.9), we know that the sequence $\{v^{i,n}\}_{n\in\mathbb{N}}$ is locally Lipschitz uniformly in n, and thus we can extend $\{v^i\}_{i=1,2}$ to the whole of $\mathbb{R}^N$ by continuity. *NB: For the sequel we use* n *instead of* n_{k_l} *for notational simplicity, assuming we work with this subsequence.*

We now prove that the sequence $\{Z^{i,n}\}_{n\in\mathbb{N}}$ is Cauchy. It is known (see, e.g. [14]) that, since $Z^{i,n}$ is Markov, it has the following representation:

$$Z_t^{i,n} = \nabla v^{i,n}(X_t)\sigma(X_t), \qquad \text{for all } t \geq 0.$$

Hence, by (3.9), we see that

$$\mathbb{E}\|Z_t^{i,n}\| \leq \bar{C}\mathbb{E}(1 + \|X_t\|^2) \leq \tilde{C}(1 + \|x_0\|^2), \tag{3.10}$$

for some constants $\bar{C}, \tilde{C}$ that are uniform in t, n. Now, set $Y_t^{i,n} = v^{i,n}(X_t)$ for $i = 1, 2$ and $n \in \mathbb{N}$. Denote $\bar{Y} = Y^{i,n} - Y^{i,m}$, $\bar{Z} = Z^{i,n} - Z^{i,m}$, $\bar{\lambda}^i = \lambda^{i,n} - \lambda^{i,m}$ for $i = 1, 2$. For all $T \geq 0$, applying Itô's lemma to $\bar{Y}^2$ and taking expectations, we obtain

$$\bar{Y}_0^2 = \mathbb{E}\big[\bar{Y}_T^2\big] - 2\mathbb{E}\left[\int_0^T \bar{Y}_t \bar{f}(t)dt\right] + \bar{\lambda}\mathbb{E}\left[\int_0^T \bar{Y}_t dt\right] + \mathbb{E}\left[\int_0^T \|\bar{Z}_t\|^2 dt\right], \tag{3.11}$$

where $\bar{f}(t) = f^{i,n}(X_t, Z^{i,n}) - f^{i,m}(X_t, Z^{i,m})$ in the notation above. Using the definition of $\bar{f}(t)$, estimate (3.9) and Theorem 3.2.1, we see that

$$\mathbb{E}\|\bar{f}(t)\|^2 \leq \bar{c}\mathbb{E}(1 + \|X_t\|^4) \leq c(1 + \|x_0\|^4),$$

where c and $\bar{c}$ are some constants that do not depend on n, m. Thus, applying Cauchy–Schwartz and the dominated convergence theorem, we conclude that

$$\mathbb{E}\bigg[\int_0^T \bar{Y}_s \bar{f}(t) dt\bigg] \to 0,$$

as $n, m \to \infty$. The other terms in (3.11) tend to zero by construction, and thus $\{Z^{i,n}\}_{n\geq 0}$ is convergent in $\mathcal{L}^2_T(W)$ for all $T \geq 0$, where

$$\mathcal{L}^2_T(W) := \bigg\{\text{predictable processes } \theta : \mathbb{E}\bigg[\int_0^T \|\theta_t\|^2 dt\bigg] < \infty\bigg\}.$$

In other words, there exist limits Z^1, Z^2, such that

$$\mathbb{E}\bigg[\int_0^T \|Z^{i,n}_t - Z^i_t\|^2 dt\bigg] \to 0, \quad i = 1, 2$$

for all $T \geq 0$.

It remains to show the driver term is also convergent in the appropriate topology. For all $T \geq 0$ we have

$$\begin{aligned}
&\mathbb{E}\int_0^T \big|f^{i,n}(X_s, Z^{i,n}_s) - H_i(X_s, Z^i_s, u^*(Z^1_s, Z^2_s), v^*(Z^1_s, Z^2_s))\big| ds \\
&\leq \mathbb{E}\int_0^T \big|f^{i,n}(X_s, Z^{i,n}_s) - f^{i,n}(X_s, Z^{i,n-1}_s)\big| ds \\
&\quad + \mathbb{E}\int_0^T \big|f^{i,n}(X_s, Z^{i,n-1}_s) - H_i(X_s, Z^i_s, u^*(Z^1_s, Z^2_s), v^*(Z^1_s, Z^2_s))\big| ds.
\end{aligned} \tag{3.12}$$

By definition of the Hamiltonian,

$$\big|f^{i,n}(X_s, Z^{i,n}_s) - f^{i,n}(X_s, Z^{i,n-1}_s)\big| \leq C\big|Z^{i,n}_s - Z^{i,n-1}_s\big|,$$

showing the convergence of the first term in (3.12) to zero. We recall that convergence in L^2 implies convergence in measure, which in turn implies convergence *a.e.* for a subsequence. Therefore there exists a subsequence $\{n_k\}_{k\in\mathbb{N}}$, such that

$$Z^{i,n_k} \to Z^i \quad \mathbb{P} \otimes dt - a.e., \quad i = 1, 2.$$

Recall that

$$f^{i,n}(X_s, Z^{i,n-1}_s) = H_i(X_s, Z^{i,n-1}_s, u^*(Z^{1,n-1}_s, Z^{2,n-1}_s), v^*(Z^{1,n-1}_s, Z^{2,n-1}_s))$$

and, given the continuity assumption on $f^{i,n}(X_s, Z^{i,n-1}_s)$ in $(Z^{1,n-1}_s, Z^{2,n-1}_s)$, we conclude that

$$f^{i,n_k}(X_s, Z^{i,n_{k-1}}_s) \to H_i(X_s, Z^i_s, u^*(Z^1_s, Z^2_s), v^*(Z^1_s, Z^2_s)) \quad \mathbb{P} \otimes dt - a.e.$$

and convergence of the second term in (3.12) follows by the dominated convergence theorem. We have shown that

$$\lim_{n\to\infty} f^{i,n_k}(X_s, Z_s^{i,n_k}) = H_i(X_t, Z_s^i, u^*(Z_s^1, Z_s^2), v^*(Z_s^1, Z_s^2)) \quad \text{in } \mathcal{L}_T^2(\mathbb{R}^N)$$

for all $T \geq 0$. This concludes the proof.

$\square$

Remark 3.3.2. *We can now see why part (ii) in Assumption 3.2.2 was necessary. In order to use the BSDE approach, we had to prove that a solution to a certain system of equations exists. We constructed such solution as a limit, and then had to show that the driver terms converged as well. Therefore, continuity of the Hamiltonian (which played the role of the driver) in* (z_1, z_2) *was naturally required.*

Remark 3.3.3. *One interesting question is the connection between solutions to finite horizon games (i.e. systems of BSDEs with continuous coefficients) and their ergodic counterpart. For a detailed account of the one dimensional Lipschitz case, see Hu, Madec and Richou [22]. In the present context, given that at the optimum the drivers are only continuous, a measure change technique is not available, so the techniques used in [22] cannot be easily transferred.*

One simple line of attack would be to fix one player's strategy at the ergodic optimum and focus on the long-run behaviour of the other player. This reduces the problem to the known case, but raises more questions than it answers, since the control is no longer closed loop. Thus, even though convergence of the solution is guaranteed (as in [22]), the interpretation of this system of equations as a stochastic game is lost.

Considering the system of BSDEs with continuous coefficients directly is a challenging task. In particular, as there are no uniqueness results in this context, and measure change techniques are not available, it proves difficult to demonstrate convergence of solutions without significant further exploration of this theory. We leave these questions for future research.

3.3.2 Quick detour into one dimensional EBSDEs

In this section we leave the world of games and control and apply the machinery established above to the case of one dimensional ergodic BSDE with a continuous driver of linear growth (in the z component). We begin by proving an auxiliary lemma that establishes a useful representation of this class of functions.

Lemma 3.3.2. *Suppose we are given a function $f : \mathbb{R}^d \to \mathbb{R}$, that is continuous and of linear growth, that is, there exists a constant $\kappa > 0$, such that $|f(x)| \leq \kappa(1 + \|x\|)$ for all $x \in \mathbb{R}^N$. Then there exist two uniformly bounded functions $\phi(\cdot), \psi(\cdot)$, such that*

$$f(x) = \phi(x)x + \psi(x),$$

holds for all $x \in \mathbb{R}^d$.

Proof: We first notice that given that f is of linear growth, there exists a constant $C > 0$ such that $\|f(x)\| \leq C\|x\|$ for all $x : \|x\| \geq 1$. Define

$$\phi(x) := \mathbf{1}_{\|x\|\geq 1}\frac{f(x)}{\|x\|^2}x, \quad \psi(x) := f(0) + \mathbf{1}_{\|x\|<1}f(x).$$

The claim follows immediately.

□

Remark 3.3.4. *Notice that we have no continuity assumption on the individual components ϕ, ψ. One can clearly see that this decomposition is inspired by the structure of the Hamiltonian in the games framework.*

Taking into account the result of Lemma 3.3.2, we are therefore interested in the equation

$$Y_t = Y_T + \int_t^T [f(X_s, Z_s) - \lambda]ds - \int_t^T Z_s dW_s, \tag{3.13}$$

where one can find functions $\phi(\cdot), \psi(\cdot)$, such that $f(x, z) = \phi(x, z)z + \psi(x, z)$, and there exist constants $c, C > 0$, such that

$$\|\phi(x, z)\| \leq c, \quad \|\psi(x, z)\| \leq C, \tag{3.14}$$

holds for all $x, z \in \mathbb{R}^N$. In general, the existence of a solution to (3.14) does not follow from the standard theorems about EBSDEs, for the driver $f(\cdot, \cdot)$ may not be Lipschitz. However, the technique developed for the game setup still gives the following result:

Theorem 3.3.4. *Suppose the driver $f : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a measurable map, such that $f(x, \cdot)$ is continuous for all $x \in \mathbb{R}^N$ and there exist a constant $\kappa > 0$, such that*

$$\big|f(x, z)\big| \leq \kappa(1 + \|z\|).$$

Then there exists a solution (Y, Z, λ) to the EBSDE (3.13). Moreover, there exists a deterministic locally Lipschitz function $v : \mathbb{R}^N \to \mathbb{R}$, such that $Y_t = v(X_t)$ for all $t \geq 0$.

Proof: We use the iteration method developed in the previous section in the games context. Set $Z_s^0 \equiv 0$, and let (Y^n, Z^n, λ^n) be a solution to

$$Y_t^n = Y_T^n + \int_t^T [\phi(X_s, Z_s^{n-1})Z_s^n + \psi(X_s, Z_s^{n-1}) - \lambda^n]ds - \int_t^T Z_s^n dW_s, \qquad (3.15)$$

where the process $\{Z_s^{n-1}\}_{s\geq 0}$ is a known Markovian solution we obtained on the previous iteration. In other words, $Z_s^{n-1} = \xi^{n-1}(X_s)$ for some deterministic function $\xi^{n-1} : \mathbb{R}^N \to \mathbb{R}^N$. In the notation of Lemma 3.3.2, set

$$f^n(x, z) = \phi(x, \xi^{n-1}(x))z + \psi(x, \xi^{n-1}(x)).$$

We notice that there exist constants $c, C > 0$, such that $\|f^n(x, 0)\| \leq C$, and $\|f^n(x, z) - f^n(x, z')\| \leq c\|z - z'\|$, so by Theorem 3.3.1 there exists a solution (Y^n, Z^n, λ^n) to (3.15). There also exist deterministic functions $y^n : \mathbb{R}^N \to \mathbb{R}, \xi^n : \mathbb{R}^N \to \mathbb{R}^N$, such that $Y_s^n = v^n(X_s), Z_s^n = \xi^n(X_s)$. Moreover, estimates (3.8) and (3.9) hold uniformly in n. We can therefore use the diagonalisation procedure on some dense countable subset $B \subset \mathbb{R}^N$, in order to obtain the function $v : B \to \mathbb{R}$, and $\lambda \in \mathbb{R}$, such that

$$v(x) = \lim_{k\to\infty} v^{n_k}(x) \quad \text{for all } x \in B, \quad \text{and } \lambda = \lim_{k\to\infty} \lambda^{n_k}$$

for some subsequence $\{n_k\}_{k\geq 0}$. We can then extend v to the entirety of $\mathbb{R}^N$ by continuity. We then proceed to show convergence of $\{Z^{n_k}\}_{k\geq 0}$ in $\mathcal{L}_T^2(W)$ for all $T \geq 0$ in exactly the same way as in the proof of Theorem 3.3.3. We finally extract a further subsequence that converges $\mathbb{P} \otimes dt - a.e.$, and use the fact that

$$\begin{aligned}
&\|f^n(X_s, Z_s^n) - f(X_s, Z_s)\| \\
&\qquad \leq \|f^n(X_s, Z_s^n) - f^n(X_s, Z_s^{n-1})\| + \|f^n(X_s, Z_s^{n-1}) - f(X_s, Z_s)\| \\
&\qquad \leq c\|Z_s^n - Z_s^{n-1}\| + \|f(X_s, Z_s^{n-1}) - f(X_s, Z_s)\|.
\end{aligned}$$

Using the dominated convergence theorem along with the continuity of f, we conclude the convergence of the driver term.

□

Remark 3.3.5. *We notice that the proof above extends naturally to the case of multiple dimensional EBSDEs with an appropriate diagonal boundedness structure. In other words, consider the system*

$$Y_t^i = Y_T^i + \int_t^T [f^i(X_s, \mathbf{Z}_s) - \lambda]ds - \int_t^T Z_s^i dW_s, \quad i = 1, \ldots, n,$$

where $\mathbf{Z}$ *denotes the vector of* Z*-components. Suppose for all* $i = 1, \dots, n$ *and for all* $x \in \mathbb{R}^N$*, the map* $\mathbf{z} \to f^i(x, \mathbf{z})$ *is continuous, and* $f^i(x, \mathbf{z})$ *is of linear growth in* z^i*. Then the system admits a solution.*

Remark 3.3.6. *Similar to the setting of finite horizon BSDEs with continuous coefficients considered by Lepeltier and San Martin (see [31]), we do not obtain any uniqueness result. Of course one could only hope for the uniqueness of the ergodic values, since* $Y + c$ *for any constant* c *also solves the same equation. The difficulty in applying the standard technique to establish uniqueness for* λ*'s is that this uses a change of measure to remove a drift term of the form* $[f(X_s, Z_s) - f(X_s, Z'_s)]ds$*. This is not possible in the present framework due to the fact that* $f(x, \cdot)$ *is not Lipschitz.*

3.3.3 Connection to PDEs

In this section we briefly show the relation between Theorem 3.3.3 and the existence of a solution to the following system of elliptic ergodic HJB equations:

$$\begin{cases} \mathcal{L}y^1(x) + H_1(x, \nabla y^1(x)\sigma(x), u^*(x), v^*(x)) = \lambda^1, \\ \mathcal{L}y^2(x) + H_2(x, \nabla y^2(x)\sigma(x), u^*(x), v^*(x)) = \lambda^2, \end{cases} \tag{3.16}$$

where

$$u^*(x) = u^*(\nabla y^1(x)\sigma(x), \nabla y^2(x)\sigma(x)), \quad v^*(x) = v^*(\nabla y^1(x)\sigma(x), \nabla y^2(x)\sigma(x)),$$

the functions $y^i(\cdot)$, $i = 1, 2$ are as in Theorem 3.3.3, and $\mathcal{L}$ is a linear operator defined as

$$\mathcal{L}v(x) = \frac{1}{2}Tr\Big(\sigma(x)\sigma^*(x)\nabla^2 v(x)\Big) + \langle Ax + F(x), \nabla v(x)\rangle.$$

We notice that if we define the transition semigroup $\{P_t\}_{t\geq 0}$ associated with the process $\{X_t\}_{t\geq 0}$ by

$$P_t[\phi](x) = \mathbb{E}[\phi(X^t)],$$

for all measurable functions $\phi : \mathbb{R}^N \to \mathbb{R}$ of polynomial growth, then $\mathcal{L}$ is the generator of $\{P_t\}_{t\geq 0}$. Similarly to [14], we adopt the following definition:

Definition 3.3.1. *Two pairs* $\{(y^i, \lambda^i)\}_{i=1,2}$ *(*$y^i : \mathbb{R}^N \to \mathbb{R}$*,* $\lambda^i \in \mathbb{R}$*) are called a mild solution to (3.16), if*

(i) The functions $\{y^i(x)\}_{i=1,2}$ *are in* C^1 *and of polynomial growth.*

(ii) The gradient functions $\{\nabla y^i(x)\}_{i=1,2}$ *are of polynomial growth.*

(iii) For all $0 \leq t \leq T$, for all $x \in \mathbb{R}^N$, and for $i = 1, 2$,

$$y^i(x) = P_{T-t}[y^i](x) + \int_t^T \big(P_{T-s}\big[H_i(x, \nabla y^i(\cdot)\sigma(\cdot), u^*(\cdot), v^*(\cdot))\big](x) - \lambda^i\big)ds,$$

where u^*, v^* are as above.

We therefore automatically have the following result:

Theorem 3.3.5. *Suppose conditions in Theorem 3.3.3 are satisfied. Then $\{(y^i, \lambda^i)\}_{i=1,2}$ is a unique mild solution to the system (3.16). Conversely, if $\{(\tilde{y}^i, \tilde{\lambda}^i)\}_{i=1,2}$ is a mild solution to* (3.16)*, then, setting $Y_t^i = \tilde{y}^i(X_t)$, $i = 1, 2$, we obtain a solution to the system of EBSDEs (3.6).*

3.3.4 Games with $n > 2$ players

In this section we consider a game between $n \in \mathbb{N} \cup \{\infty\}$ ergodic players. As in the case of two players, we show that the existence of a solution to a certain system of ergodic BSDEs guarantees the existence of a saddle point (provided a version of generalised Isaac's condition is assumed). We then proceed to prove the former. We show that the scenario of finitely many players is a relatively straightforward extension of Theorem 3.3.3, but extra care is required for the countable case.

Let $\mathbf{u} = (u_1, \dots, u_n)$ be the vector of controls, such that $\mathbf{u}(\omega)$ is a predictable process with values in a separable metric space $\mathcal{U} = \mathcal{U}_1 \times \cdots \times \mathcal{U}_n$. Denote $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{R}^{n \times N}$. We define the Hamiltonian functions $\{H_i\}_{i=1,\dots,n}$, as

$$H_i(x, z_i, \mathbf{u}) = z_i R(\mathbf{u}) + L_i(x, \mathbf{u}),$$

where, similar to the two player case, $L_i : \mathbb{R}^N \times \mathcal{U} \to \mathbb{R}$, $i = 1 \dots n$ are measurable cost functions Lipschitz in x, and the function $R : \mathcal{U} \to \mathbb{R}^N$ corresponds to the control of the drift by players through the change of measure $\mathbb{P}^{u,v,T} := \rho_T^{u,v}\mathbb{P}$, where

$$\rho_T^{\mathbf{u}} := \exp\left(\int_0^T R(\mathbf{u}_t)dW_t - \frac{1}{2}\int_0^T \|R(\mathbf{u}_t)\|^2 ds\right).$$

The payoffs are defined as

$$J^i(x_0, \mathbf{u}) = \limsup\nolimits_{T\to\infty} T^{-1}\mathbb{E}^{\mathbf{u},T}\left[\int_0^T L_i(X_t, \mathbf{u}_t)dt\right],$$

for $i = 1 \dots n$, and the goal is to find admissible vector of controls $\mathbf{u}^*$, such that for all $i = 1 \dots n$,

$$J^i(x_0, u_i^*, \mathbf{u}_{-i}) \geq J^i(\mathbf{u}^*),$$

holds for all admissible controls $\mathbf{u}_{-i}$, where $\mathbf{u}_{-i}$ denotes the vector $\mathbf{u}$ without its i-th component. The generalised Isaac's conditions become

Assumption 3.3.2. *Suppose $n < \infty$.*

(i) There exists a measurable map $\mathbf{u}^* : \mathbb{R}^{(n+1)N} \to \mathcal{U}$, such that for each $i = 1 \dots n$

$$H_i^*(x, \mathbf{z}) := H_i(x, z_i, \mathbf{u}^*(x, \mathbf{z})) \leq H_i(x, z_i, u_i, \mathbf{u}_{-i}^*(x, \mathbf{z}))$$

holds for all $(x, \mathbf{z}, u_i) \in \mathbb{R}^{(n+1)N} \times \mathcal{U}_i$.

(ii) The mapping $\mathbf{z} \in \mathbb{R}^{n \times N} \to \mathbf{H}^*(x, \mathbf{z})$ is continuous for any fixed x. Here $\mathbf{H}^*$ denotes the vector of H_i^*.

As an immediate extension of the two player case we get the following result, the proof of which is identical to that of Theorem 3.3.2 with certain amount of notational care:

Theorem 3.3.6. *Given Assumption 3.3.2, suppose that there exist $n \geq 2$ triplets (Y^i, Z^i, λ^i), $i = 1 \dots n$, such that*

$$Y_t^i = Y_T^i + \int_t^T [H_i(X_s, Z_s^i, \mathbf{u}^*(\mathbf{Z}_s)) - \lambda^i] ds - \int_t^T Z_s^i dW_s, \tag{3.17}$$

holds for all $0 \leq t \leq T < 0$. Moreover, assume there exist n deterministic functions with polynomial growth $y_i(x)$, $i = 1 \dots n$, such that $Y_t^i = y_i(X_t)$ holds $\mathbb{P}$ - a.s. for all $t \geq 0$. Then the control $\mathbf{u}^*(X_s, \mathbf{Z}_s)$ is a Nash equilibrium.

For the case of finitely many players, we can show that the system (3.17) admits a solution using exactly the same Picard iteration as in the proof of Theorem 3.3.3. In other words, we set $\mathbf{Z}^0 \equiv 0$, and given a set of Markovian solutions $\{(Y^{i,n-1}, Z^{i,n-1}, \lambda^i)\}_{i=1,\dots,n}$ obtained previously, we define the next iteration as a solution to

$$dY_t^{i,n} = -[f^{i,n}(X_t, Z_t^{i,n}) - \lambda^{i,n}]dt - \int_t^T Z_t^{i,n} dW_t, \quad i = 1, \dots, n,$$

where

$$f^{i,n}(X_s, Z^{i,n}) = H_i(X_s, Z_s^{i,n}, \mathbf{u}^*(\mathbf{Z}_s^{n-1}).$$

The rest of the proof is identical.

3.3.4.1 Infinite player games

For a countable number of players additional care is needed. First, condition (ii) in Assumption 3.3.2 needs to be reformulated. This is due to the fact that in infinite dimensions not all norms are equivalent, and therefore in order to have a notion of continuity, we need to specify the topology on infinite dimensional vectors $\mathbf{z}$. Let θ^i denote the i-th component of the vector θ, and define

$$\mathcal{L}^2_{\epsilon,loc} := \left\{ \text{predictable processes } \theta : \mathbb{E}\left[\sum_{i=1}^{\infty} \frac{1}{i^{1+\epsilon}} \int_0^T \|\theta^i_t\|^2 dt \right] < \infty, \text{ for all } T \right\}.$$

Given the fact that constants in the estimate (3.10) do not depend on time or the number of iteration, we know that there exists a constant $C > 0$ such that

$$\mathbb{E}\left[\int_0^T \|Z^{i,n}_t\| \right] < CT, \qquad \text{for all } T > 0,$$

holds uniformly in i, n. Hence $\mathbf{Z}^n \in \mathcal{L}^2_{\epsilon,loc}$ for all $n \geq 0$. It is clear that componentwise convergence of $\mathbf{Z}^n$ to $\mathbf{Z}$ in $\mathcal{L}^2_T(W)$ for all $T > 0$ guarantees convergence of $\mathbf{Z}^n$ to $\mathbf{Z}$ in $\mathcal{L}^2_{\epsilon,loc}$. We also notice that, in the proof of Theorem 3.3.3 we use continuity of the vector of Hamiltonians $\mathbf{H}^*$ for each component separately. Therefore, the infinite dimensional analogue of (i) in Assumption 3.3.2 remains exactly the same, while (ii) becomes

(ii) The mapping $\mathbf{z} \in \mathcal{L}^2_{\epsilon,loc} \to H^*_i(x, \mathbf{z})$ is continuous (for $\mathbb{P} \times dt$ almost all (ω, t)) for each $i \in \mathbb{N}$.

Secondly, the construction of a convergent subsequence for the vector of ergodic values λ^k needs modification. This is due to the fact that infinite dimensional cubes are not compact. However, the result of Lemma 3.3.1 clearly holds, ensuring that each component $\lambda^{i,k}$ lies in a compact (uniformly in k), and thus we can use a diagonalisation procedure to construct a limit λ. Using **(ii)** instead of (ii), the rest of the proof remains the same.

Remark 3.3.7. *Even though the topology $\mathcal{L}^2_{\epsilon,loc}$ looks artificial, it has an intuitive interpretation. As pointed out in Remark 3.2.5, the generalised Isaac's condition is an equivalent of Nash equilibrium on the infinitesimal scale. Hence the Hamiltonians $\{H_i\}_{i\geq 1}$ in some sense represent "local values" for each player. We also note that the vector $\{\mathbf{Z}_t\}_{t\geq 0}$ can be seen as a vector of strategies, since it determines the values of optimal controls $\{\mathbf{u}^*_t\}_{t\geq 0}$.*

Therefore continuity of $\{H_i\}_{i\geq 1}$ in $\mathbf{z} \in \mathcal{L}^2_{\epsilon,loc}$ means that, when finitely many "neighbours" (the term can be made rigorous if we put a graph structure on the space of agents) of player $i \in \mathbb{N} \cup \{\infty\}$ perturb their strategies only slightly, the change in his local value will be insignificant as well, even if distant agents make substantial changes.

Remark 3.3.8. *Further generalisations may include the case of infinite dimensional forward process, the presence of Lévy noise, as well as the introduction of time periodicity. Given that the corresponding existence and uniqueness results were established in [11] (for the case where X takes values in a separable Hilbert spaces), we expect that all the results of the present work should still hold true.*

3.4 Stochastic games with asymmetric agents

In this section we consider a game between two players of different type. The main goal is to understand whether there is a fundamental difference between an ergodic player (as defined above), and one who discounts the future as the discount rate becomes negligible. We define the payoff structure as follows:

$$
\begin{aligned}
J^1(x_0, u, v) &= \limsup_{T\to\infty} T^{-1}\mathbb{E}^{u,v,T}\left[\int_0^T L_1(X_t, u_t, v_t)dt\right],\\
J^2(x_0, u, v) &= \mathbb{E}^{u,v,T}\left[\int_0^\infty e^{-\alpha t} L_2(X_t, u_t, v_t)dt\right],
\end{aligned}
$$

where α is some positive constant. In the sequel we will be calling the player of the first type 'ergodic', and of the second 'discounting'. We begin by proving an analogue of Theorem 3.3.2 for this case:

Theorem 3.4.1. *Let the Hamiltonians $\{H_i\}_{i=1,2}$ be defined as in (3.4) and suppose Assumption 3.2.2 holds. Suppose further that there exist a triplet (Y, Z, λ), and a pair $(\bar{Y}, \bar{Z})$ such that*

$$
\begin{cases}
Y_t = Y_T + \int_t^T [H_1(X_s, Z_s, u^*(Z_s, \bar{Z}_s), v^*(Z_s, \bar{Z}_s)) - \lambda]ds - \int_t^T Z_s dW_s,\\
\bar{Y}_t = \bar{Y}_T + \int_t^T [H_2(X_s, \bar{Z}_s, u^*(Z_s, \bar{Z}_s), v^*(Z_s, \bar{Z}_s)) - \alpha \bar{Y}_s]ds - \int_t^T \bar{Z}_s dW_s,
\end{cases}
\tag{3.18}
$$

holds for all $0 \leq t \leq T < 0$. Moreover, assume there exist two deterministic functions with polynomial growth $y(x), \tilde{y}(x)$, such that $Y_t = y(X_t)$ and $\bar{Y}_t = \tilde{y}(X_t)$ holds $\mathbb{P}$-a.s. for all $t \geq 0$. Then the control $(u^*(X_s, Z_s, \bar{Z}_s), v^*(X_s, Z_s, \bar{Z}_s))$ is a Nash equilibrium.

Proof: We first notice that, since the Hamiltonian structure remains the same, the proof for the ergodic player is identical to that in Theorem 3.3.2. We therefore focus on the discounting one. For an arbitrary admissible control v, the pair (u^*, v) generates a measure $\mathbb{P}^{u^*,v}$ under which $dW_t^{u^*,v} = dW - R(u_t^*, v_t)dt$ is a Brownian motion. Then

$$d(e^{-\alpha t}\bar{Y}_t) = -e^{-\alpha t}\big([H_2(X_t, \bar{Z}_t, u_t^*, v_t^*) + \bar{Z}_t R(u_t^*, v_t)]dt + \bar{Z}_t dW_t^{u,v^*}\big),$$

and thus, integrating and taking expectations with respect to $\mathbb{P}^{u^*,v}$, we obtain

$$\begin{aligned}\bar{Y}_0 = \mathbb{E}^{u^*,v}\bigg[\int_0^T \big(H_2(X_t, \bar{Z}_t, u_t^*, v_t^*) - \bar{Z}_t R(u_t^*, v_t) - L_2(X_t, u_t^*, v_t)\big)dt\bigg] \\ + e^{-\alpha T}\mathbb{E}^{u^*,v}\big[\bar{Y}_T\big] + \mathbb{E}^{u^*,v}\bigg[\int_0^T L_2(X_t, u_t^*, v_t)dt\bigg].\end{aligned}$$

Taking the lim sup on both sides, we obtain

$$\bar{Y}_0 \leq \mathbb{E}^{u^*,v}\bigg[\int_0^\infty L_2(X_t, u_t^*, v_t)dt\bigg] = J^2(x_0, u^*, v),$$

and the equality holds for $v = v^*$.

$\square$

Theorem 3.4.2. *The system (3.18) admits a Markovian solution in the sense of Theorem 3.4.1.*

Sketch of the proof: Since the rigorous proof is almost identical to that of Theorem 3.3.3, we only outline the minor differences. Our first aim is to construct a solution to the system (3.18) using Picard iteration (3.7). Notice that for a fixed $\alpha > 0$, the uniform estimates (3.8) and (3.9) hold for the ergodic BSDE as before, while for the discounted one, (3.8) is replaced by

$$\|v^{2,n}(x)\| \leq C/\alpha, \quad \|v^{2,n}(x) - v^{2,n}(0)\| \leq C(1 + \|x\|^2),$$

where the constant $C > 0$ is independent of n and α (for details see Theorem 8 in [11]). This allows us to follow the rest of the proof of Theorem 3.3.3 with only slight modifications.

$\square$

We are now interested in what happens when we send the discount rate α to zero. The idea is to extract a convergent subsequence from a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ and then study the limiting object. We begin by showing the following result:

Theorem 3.4.3. *Let $y^\alpha(x), \tilde{y}^\alpha(x)$ be the Y components of the Markovian solution to (3.18), and λ^α be the ergodic value of the first player. Then there exists a sequence $\alpha_n \downarrow 0$ and two functions of polynomial growth $y(\cdot)$ and $\tilde{y}(\cdot)$, such that*

$$y^{\alpha_n}(x) \to y(x), \quad \lambda^{\alpha_n} \to \lambda, \quad \big(\tilde{y}^{\alpha_n}(x) - \tilde{y}^{\alpha_n}(0)\big) \to \tilde{y}(x), \quad \alpha_n \tilde{y}^{\alpha_n}(0) \to \tilde{\lambda},$$

holds for all $x \in \mathbb{R}^N$.

Proof: We know that, by construction, the functions $y^{\alpha_n}(\cdot)$ and $\tilde{y}^{\alpha_n}(\cdot)$ are locally Lipschitz, uniformly in n. Also,

$$\|y^{\alpha_n}(x)\| \leq C(1 + \|x\|^2), \quad \|\tilde{y}^{\alpha_n}(x) - \tilde{y}^{\alpha_n}(0)\| \leq C(1 + \|x\|^2)$$

holds for all $x \in \mathbb{R}^N$. By Lemma 3.3.1 we also know that $\{\lambda^{\alpha_n}\}_{n\geq 1}$ lie in a compact. We can therefore use the diagonalisation procedure to construct the limit on a dense subset of $\mathbb{R}^N$ (e.g. $\mathbb{Q}^N$), and then extend by continuity to the entire space.

$\square$

Using arguments identical to the those in the proof of Theorem 3.3.3, we conclude that as the discount rate of the second player goes to zero, we can extract a subsequence such that

$$J^{1,\alpha_n}(x_0, u^*, v^*) = \lambda^{\alpha_n} \to \lambda, \quad \alpha_n J^{2,\alpha}(x_0, u^*, v^*) = \alpha_n \tilde{y}^{\alpha_n}(x_0) \to \tilde{\lambda},$$

and we can find $(\lambda, \tilde{\lambda})$ by solving the following system:

$$\begin{cases} Y_t = Y_T + \int_t^T [H_1(X_s, Z_s, u^*(Z_s, \bar{Z}_s), v^*(Z_s, \bar{Z}_s)) - \lambda] ds - \int_t^T Z_s dW_s, \\ \bar{Y}_t = \bar{Y}_T + \int_t^T [H_2(X_s, \bar{Z}_s, u^*(Z_s, \bar{Z}_s), v^*(Z_s, \bar{Z}_s)) - \tilde{\lambda}] ds - \int_t^T \bar{Z}_s dW_s. \end{cases}$$

This is in line with our intuition. Indeed, we know that for the case of one player (i.e. a standard optimal control problem), the EBSDE is obtained a a limit of discounted infinite horizon BSDEs.

Chapter 4

Classical Adjoints for Ergodic Stochastic Control

In this chapter we consider ergodic optimal control of a diffusion process $\{X_t^u\}_{t\geq 0}$, taking values in $\mathbb{R}^n$, where both drift and volatility are controlled. We establish a novel strong duality between the existence of a unique solution to the infinite horizon adjoint BSDE and strong dissipativity of X^u. We then proceed to show that the latter implies irreducibility, the strong Feller property and exponential ergodicity. We conclude by discussing the connection with ergodic BSDEs.

4.1 Setup

In this section we introduce the preliminary concepts and definitions we will be using repeatedly in the sequel. We also state the principle problem of interest and discuss its connections to the existing theory.

4.1.1 The forward dynamics

We begin by introducing the forward process $\{X_t^u\}_{t\geq 0}$. Let the controlled dynamics of $\{X_t^u\}_{t\geq 0}$ be governed by the Itô SDE:

$$dX_t^u = b(t, X_t^u, u_t)dt + \sigma(t, X_t^u, u_t)dW_t, \quad X_0 = x_0 \in \mathbb{R}^n, \tag{4.1}$$

where W denotes a standard Brownian motion under the measure $\mathbb{P}$.

Definition 4.1.1. *Let $\mathcal{U}$ be a separable metric space. We call a stochastic process $\{u_t\}_{t\geq 0}$ admissible if the following conditions are satisfied:*

(i) It takes values in $\mathcal{U}$ and is predictable in the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

(ii) It is Markov, i.e. $u_t = \alpha_t(X_t)$ *for some Borel measurable function* α_t.

(iii) There exists a strong, Markovian solution to the equation

$$dX_t = b(t, X_t, \alpha_t(X_t))dt + \sigma(t, X_t, \alpha_t(X_t))dW_t, \tag{4.2}$$

which is nothing else but equation (4.1).

Remark 4.1.1. *This class of controls may seems restrictive, but in fact for any* $\epsilon > 0$ *one can find a Markov control that is* ϵ*-optimal. For details see Chapter 5 in [30]. For notational simplicity in the sequel we use* $b^u(t, X_t)$ *instead of* $b(t, X_t, \alpha_t(X_t))$ *and* $\sigma^u(t, x)$ *for* $\sigma(t, x, \alpha(t, x))$.

Assumption 4.1.1. *Let a stochastic process* $\{u_t\}_{t\geq 0}$ *satisfy (i) and (ii) in Definition (4.1.1). In order to ensure (iii), the following conditions need to be satisfied:*

- *The triple* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a complete probability space,* $(\mathcal{F}_t)_{t\geq 0}$ *is an augmented filtration satisfying the usual conditions, and* $(W_t)_{t\geq 0}$ *is a Brownian motion.*
- *The coefficients* $b^u(t, x)$ *and* $\sigma^u(t, x)$ *are measurable in* t*, locally Lipschitz in* x*, that is, for every* T, N*, there exist a constant* K *depending only on* T *and* N*, such that*

  $$|b^u(t, x) - b^u(t, y)| + |\sigma^u(t, x) - \sigma^u(t, y)| < K|x - y|$$

  holds for all $|x|, |y| \leq N$ *and all* $t \leq T$.
- *The linear growth condition holds. In other words, for all* $u \in \mathcal{U}, x \in \mathbb{R}^n, t \geq 0$,

  $$|b^u(t, x)| + |\sigma^u(t, x)| \leq \bar{K}(1 + |x|),$$

  where $\bar{K}$ *is a constant.*
- $x_0 \in \mathbb{R}^n$ *is a constant.*

The first part of the following result is standard. The second is an easy application of Grönwall's lemma (for details see, e.g. Theorem 6.30 in [28]):

Theorem 4.1.1. *If Assumption 4.1.1 is satisfied then, for any fixed control process* $\{u_t\}_{t\geq 0}$*, there exists a unique strong solution to the equation (4.1). Moreover,*

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |X_t^u|^2\right) < C(1 + \mathbb{E}|x_0|^2),$$

where the constant C *depends only on* K *and* T.

For the rest of the chapter we also make the following assumptions:

Assumption 4.1.2. *The following conditions hold for all admissible controls u:*

(i) Functions $b^u(t,\cdot)$, $\sigma^u(t,\cdot)$ and $L(t,\cdot,u_t)$ are twice continuously differentiable for all $t \in \mathbb{R}^+$.

(ii) There exist a constant $\overline{\sigma} > 0$, such that

$$\|(\sigma^u)^{-1}(t,x)\|_{op} \leq \overline{\sigma}$$

holds for all $x \in \mathbb{R}^n, t \geq 0$.

(iii) There exists a constant $\tilde{C} > 0$, such that

$$\|b^u(t,0)\| \leq \tilde{C}$$

holds for all $t \geq 0$.

Remark 4.1.2. *Note that Lipschitz continuity of the coefficients is stronger than what is generally required for the purposes of control. However, in the sequel we will need certain estimates for the stability of the forward process in its initial value, and stronger regularity assumptions will come into play.*

The following definition will prove crucial in the sequel, since it is very closely related to ergodicity of the solution process to (4.1).

Definition 4.1.2. *Let the dynamics of $\{X^u_t\}_{t\geq 0}$ be governed by (4.1). We say that X^u has a strongly dissipative drift (or X^u is strongly dissipative), if there exists a constant $\mu > 0$ such that*

$$\langle b^u(t,x) - b^u(t,u), x - y\rangle \leq -\mu\|x-y\|^2$$

holds for all $t \geq 0$.

4.1.2 The problem

We consider the ergodic control problem, namely that of minimising (over the space of admissible controls) the functional

$$J(x_0,u) = \limsup_{T\to\infty} T^{-1}\mathbb{E}\left[\int_0^T L(t,X^u_t,u_t)dt\right], \tag{4.3}$$

where the dynamics of the controlled process $\{X_t^u\}_{t\geq 0}$ are given by (4.1), and the coefficients $b^u(\cdot)$ and $\sigma^u(\cdot)$ satisfy Assumptions 4.1.1 and 4.1.2 for all admissible controls $\{u_t\}_{t\geq 0}$. The finite horizon case, namely the problem of minimising

$$J(x_0, u) = \mathbb{E}\bigg[\int_0^T L(t, X_t^u, u_t)dt + g(X_T^u)\bigg], \tag{4.4}$$

is well understood (see, e.g. the seminal paper by Peng [35]). The adjoint equation is the BSDE

$$dY(t) = -\nabla_x H\Big(t, X_t^u, u_t, Y(t), Z(t)\Big)dt + Z(t)dW_t, \quad Y(T) = \nabla_x g(X_T), \tag{4.5}$$

where $H : [0,T] \times \mathbb{R}^n \times \mathcal{U} \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}$ is the Hamiltonian, defined by

$$H(t, x, u, y, z) := b'(t, x, u)y + \mathrm{Tr}(\sigma'(t, x, u)z) + L(t, x, u). \tag{4.6}$$

and it is assumed that H is differentiable in x (*NB: here and for the rest of the chapter $'$ denotes the transpose*). The following condition for the necessary stochastic maximum principle can then be established (see Theorem 3 in [35]):

Theorem 4.1.2. *Suppose $\{u_t\}_{t\geq 0}$ is the optimal control process and $\{X_t^u\}_{t\geq 0}$ are the corresponding controlled dynamics for the problem (4.4). Then there exists a solution to the BSDE (4.5) on finite horizons.*

Remark 4.1.3. *We note that, in [35], since no convexity on the Hamiltonian is assumed, an additional second order adjoint equation appears. However in the present work we will focus on the first order BSDE.*

One can also show the connection between the adjoint process $\{Y(t)\}_{t\geq 0}$ and the dynamic programming principle. The following result can be found, for example, in Chapter 6 of [37]:

Lemma 4.1.1. *Let $\{X_s^{u,t,x}\}$ denote the solution to (4.1) with the initial condition $X_t^{u,t,x} = x$. Define the value function $v(\cdot,\cdot)$ as follows:*

$$v(t, x) := \inf_u \mathbb{E}\bigg[\int_t^T L(s, X_s^{u,t,x}, u_s)ds + g(X_T^{u,t,x})\bigg].$$

Then

$$\bar{Y}(t) = \nabla_x v(t, \bar{X}_t),$$

where $\bar{Y}$ is the first component of the solution to the adjoint BSDE with optimal control process $\{\bar{u}_t\}_{t\in[0,T]}$ and the corresponding controlled forward process $\{\bar{X}\}_{t\geq 0}$.

Remark 4.1.4. *From Lemma 4.1.1, we see that the process $\bar{Y}$ can be thought of as the optimal marginal value of a change in X. This will be an intuitive starting point when developing an infinite horizon analogue for the adjoint BSDE in the next section.*

4.2 Stochastic maximum principle approach to the problem of ergodic optimal control

The aim of this section is to extend the finite horizon framework to the ergodic case, showing an analogue of the adjoint equation and establishing existence and uniqueness of its solutions. For the rest of the chapter, we make the following assumption:

Assumption 4.2.1. *The space of controls $\mathcal{U}$ is a locally convex real topological vector space. (For the sake of simplicity we consider $\mathcal{U} = \mathbb{R}^k$, but the subsequent analysis does not change provided that the Gâteaux derivative is well defined.)*

4.2.1 Adjoint BSDE in infinite horizon

In this subsection we explain (*in a heuristic way*) why the adjoint equation to the ergodic control problem becomes an infinite horizon BSDE. For the remainder of this subsection, we denote the optimal control pair as $(\bar{u}, \bar{X})$. We first state (without a proof) a proposition that can be easily verified:

Proposition 4.2.1. *The following statements about infinite horizon optimisation are equivalent:*

- *The control process $\{\bar{u}_t\}_{t\geq 0}$ minimises the functional $J(x_0, \cdot)$ as defined in (4.3).*
- *The control process $\{\bar{u}_t\}_{t\geq 0}$ formally minimises the (non averaged) functional defined by*

$$\bar{J}(x_0, \cdot) = \mathbb{E}\Big[\int_0^\infty L(t, X_t^{\cdot}, \cdot)dt\Big] \tag{4.7}$$

  in the sense that (since the right hand side in (4.7) may diverge) we have

$$\limsup_{T\to\infty} \frac{\mathbb{E}\big[\int_0^T L(t, X_t^{\bar{u}}, \bar{u}_t)dt\big]}{\mathbb{E}\big[\int_0^T L(t, X_t^{u}, u_t)dt\big]} \leq 1$$

  for all admissible controls u.

The problem of minimising (4.7) is in some sense simpler than the ergodic one. This is because, considered formally, optimality in (4.7) implies optimality over every finite horizon by the dynamic programming principle, whereas the ergodic problem only considers optimality in the limit (and so the dynamic programming principle is not necessary for optimality, see Theorem 5.1 in [2] for an example of this phenomenon). Similarly to Lemma 4.1.1, suppose we can define

$$v(t,x) := \inf_u \mathbb{E}\left[\int_t^\infty L(s, X_s^{u,t,x}, u_s)ds\right].$$

We can directly employ the dynamic programming principle to see that, for any $T \geq 0$, we have a finite horizon problem of finding

$$\inf_u \bar{J}(x_0, u) = \inf_u \mathbb{E}\left[\int_0^T L(t, X_t^u, u_t)dt\right] + v(T, X_T^u). \tag{4.8}$$

Then, since $\{\bar{u}\}_{t\geq 0}$ minimises $\bar{J}(x_0, \cdot)$, it is also optimal for (4.8) for every $T > 0$. Therefore, by Theorem 4.1.2, we infer that the equation

$$\bar{Y}(t) = \bar{Y}(T) + \int_t^T \nabla_x H(s, \bar{X}_s, \bar{u}_s, \bar{Y}(s), \bar{Z}(s))ds - \int_t^T \bar{Z}(s)dW_s \tag{4.9}$$

should hold for all $0 \leq t \leq T < \infty$. This suggests strongly that (4.9) is a good candidate for the adjoint equation in an ergodic maximum principle.

Since, unlike in the finite horizon case, there is no terminal condition, we need to say something about the limiting (as $T \to \infty$) behaviour of the 'optimal' solution to (4.9) (solution with the optimal control pair $(\bar{u}, \bar{X})$). As pointed out in Remark 4.1.4, the process $\bar{Y}$ can be thought of as the marginal value of a change in X. Therefore the condition $\bar{Y}(T) \to 0$ as $T \to \infty$ (which would be the direct analogue of the finite horizon case) does not seem to be the right one. In the next section we will discuss this in more detail.

4.2.2 Stochastic Maximum Principle

In the previous section we have explained why (4.9) is a reasonable candidate for the adjoint equation associated with the problem of ergodic control. One can therefore expect that the existence of a solution to (4.9) is necessary for optimality, given dynamic programming should hold. The goal of this section is to show that, under certain additional assumptions, it is also sufficient. The main result is the following theorem (similar in spirit to the result in [19]), which is a version of the stochastic maximum principle:

Theorem 4.2.1. *Suppose the control process $\{\bar{u}_t\}_{t\geq 0}$ is admissible, and denote the associated forward process $\{\bar{X}_t\}_{t\geq 0}$. Suppose that the corresponding infinite horizon adjoint BSDE (4.9) has solution $(\bar{Y}, \bar{Z})$. Suppose further that, for any admissible control process $\{u_t\}_{t\geq 0}$,*

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\Big[\bar{Y}_T(X^u_T - \bar{X}_T)\Big] = 0. \tag{4.10}$$

Moreover, assume that the Hamiltonian H as defined in (4.6) is convex [1] in (x, u), differentiable with respect to x, u, and

$$H(t, \bar{X}_t, \bar{u}_t, \bar{Y}(t), \bar{Z}(t)) = \min_{u\in\mathcal{U}} H(t, \bar{X}_t, u, \bar{Y}(t), \bar{Z}(t)). \tag{4.11}$$

Then the control $\{\bar{u}_t\}_{t\geq 0}$ is optimal.

Remark 4.2.1. *All conditions of this theorem are intuitive apart from (4.10). However, it also has an economic interpretation. It tells us that if the the optimal marginal value of a change in X is positive for some large time T, then $X^u_T - \bar{X}_T$, the difference in the state between an arbitrary controlled process and the optimal one, is on average sublinear in time.*

Proof: In order to prove optimality of $\bar{u}$, we need to show that, for all admissible controls $\{u_t\}_{t\geq 0}$,

$$J(x_0, \bar{u}) - J(x_0, u) \leq 0.$$

Since $\limsup$ is subadditive, we have, for any functions $g, f : \mathbb{R}^+ \to \mathbb{R}$,

$$\limsup_{T\to\infty} g(T) - \limsup_{T\to\infty} f(T) \leq \limsup_{T\to\infty} (g - f)(T),$$

and therefore it is enough to show that

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\bigg[\int_0^T \big(L(t, \bar{X}_t, \bar{u}_t) - L(t, X^u_t, u_t)\big)dt\bigg] \leq 0. \tag{4.12}$$

By definition of the Hamiltonian in (4.6), we can write

$$\begin{aligned} L(t, \bar{X}_t, \bar{u}_t) - L(t, X^u_t, u_t) &= H(t, \bar{X}_t, \bar{u}_t, \bar{Y}(t), \bar{Z}(t)) - H(t, X^u_t, u_t, \bar{Y}(t), \bar{Z}(t)) \\ &\quad + \mathrm{Tr}\Big((\sigma'(t, X^u_t, u_t) - \sigma'(t, \bar{X}_t, \bar{u}_t))\bar{Z}(t)\Big) \\ &\quad + \Big(b'(t, X^u_t, u_t) - b'(t, \bar{X}_t, \bar{u}_t)\Big)\bar{Y}(t). \end{aligned}$$

[1] Note that convexity is quite restrictive, since it implies that the coefficients b and σ are affine in (x, u). However, this is a standard assumption used in the literature when discussing sufficient stochastic maximum principles.

By convexity of $H(t,\cdot,\cdot,y,z)$ for all $(t,y,z)\in\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^n$, we have

$$\begin{aligned}H(t,\tilde{x},\tilde{u},y,z) &- H(t,x,u,y,z)\\ &\leq \nabla_x H(t,\tilde{x},\tilde{u},y,z)'(\tilde{x}-x)+\nabla_u H(t,\tilde{x},\tilde{u},y,z)'(\tilde{u}-u).\end{aligned}\tag{4.13}$$

Given (4.11), we know that

$$\nabla_u H(t,\tilde{x},\tilde{u},y,z)'(\tilde{u}-u)\leq 0,$$

and thus (4.13) becomes

$$H(t,\tilde{x},\tilde{u},y,z)-H(t,x,u,y,z)\leq -\nabla_x H(t,\tilde{x},\tilde{u},y,z)'(x-\tilde{x}).$$

We recall that the dynamics of $\{\bar{Y}_t\}_{t\geq 0}$ are given by

$$d\bar{Y}_t=-\nabla_x H(t,\bar{X}_t,\bar{u},\bar{Y}_t,\bar{Z}_t)dt+\bar{Z}_t dW_t.$$

Therefore, applying Itô's formula to $\bar{Y}_T(X_T^u-\bar{X}_T)$ (note that $\bar{X}_0=X_0^u$) and substituting the result into (4.10), we arrive at

$$\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\int_0^T\big(L(t,\bar{X}_t,\bar{u}_t)-L(t,X_t^u,u_t)\big)dt\right]\leq\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\Big[\bar{Y}_T(X_T-\bar{X}_T)\Big],$$

concluding the proof.

□

4.2.3 Illustration in one dimension

In Theorem 4.2.1 we have seen the close relationship between optimality of a control u and the existence of solutions to the adjoint BSDE (4.9). We now wish to ask, for a given control, can we see that this BSDE admits a solution? Given we are working over infinite horizons, this is a somewhat delicate question.

In this subsection we consider the simplest case, where the process $\{X_t^u\}_{t\geq 0}$ is one-dimensional, in order to motivate the subsequent discussion. In order to separate Y in the driver, we write

$$\begin{aligned}f(s,x,u_s,z) :&= H_x(s,x,u_s,y,z)-b_x^u(s,x)y\\ &= \nabla_x\big(\operatorname{Tr}([\sigma^u]'(t,x)z)+L(t,x,u)\big).\end{aligned}$$

The first thing we notice is that the adjoint BSDE

$$Y(t)=Y(T)+\int_t^T[f(s,X_s^u,u_s,Z(s))-b_x^u(s,X_s^u)Y(s)]ds-\int_t^T Z(s)dW_s\tag{4.14}$$

looks very similar to a 'discounted' one, where the role of the discount factor is played by the coefficient $b_x(s, X_s^u, u_s)$. It is known in the theory of infinite horizon discounted BSDEs that, in order to guarantee the existence of a solution to (4.14), it is reasonable to assume (see, e.g. [43]) that there exists a constant $k > 0$, such that

$$b_x^u(s, x) < -k \tag{4.15}$$

for all $s \geq 0$, $x \in \mathbb{R}^n$. The reason is the following: if we allow the 'discount' to be positive, then there is little hope for a solution in general. To illustrate why, consider the following equation:

$$dY_t = -rY_t dt + Z_t dW_t.$$

We immediately see that

$$Y_0 = \mathbb{E}\big[e^{rT} Y_T\big], \text{ for all } T \geq t.$$

Therefore, if Y_T does not vanish, there is no bounded solution, since e^{rT} explodes as $T \to \infty$. The case where b_x is nonpositive is still an open question (it is conjectured that the solution exists if the discount is negative on a set of positive ergodic measure). The following lemma is trivial, but we state it since its generalisation will play an important role in the sequel.

Lemma 4.2.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be of class C^1 and let $k > 0$ be a constant. Then the following conditions are equivalent:*

(i) $g_x(x) \leq -k$ *for all* $x \in \mathbb{R}$.

(ii) $(g(x) - g(y))(x - y) \leq -k(x - y)^2$.

Proof: (i)→(ii): By the mean value theorem, we have, for some $\bar{x} \in (x, y)$,

$$(g(x) - g(y))(x - y) = g_x(\bar{x})(x - y)^2 \leq -k(x - y)^2.$$

(ii)→(i): Dividing both sides by $(x - y)^2$, we obtain

$$\frac{g(x) - g(y)}{x - y} \leq -k.$$

Setting $x = y + h$, and passing to the limit, the result follows.

□

Taking $g = b$ we notice that the condition (ii) is nothing else but the strong dissipativity (in the sense of Definition 4.1.2) of the drift of the process $\{X^u\}_{t \geq 0}$. In the sequel we will establish (see Theorem 4.3.3) that the latter implies ergodicity. Therefore Lemma 4.2.1 shows the equivalence between the condition required for the existence of a solution to the adjoint BSDE (4.14) in infinite horizon for some control process $\{u_t\}_{t\geq 0}$, and the one ensuring ergodicity of the forward process. This conclusion is relatively intuitive: in order for an admissible control process $\{u_t\}_{t\geq 0}$ to be potentially optimal, we should know that the corresponding controlled process is ergodic! What is surprising is that the converse is also true – given the dissipativity of the controlled dynamics, one can hope for classical solutions to the infinite horizon adjoint BSDE.

The question becomes: what happens in the multidimensional setting? Is it possible to generalise this connection? The rest of the chapter shows that the answer is yes.

4.2.4 General case

In the general multiple dimensional case, the notion of 'discounting' becomes nontrivial. The following definition provides a set of conditions (inspired by [43]) required to ensure existence and uniqueness of a bounded solution to the multidimensional infinite horizon adjoint BSDE (4.18):

Definition 4.2.1. *We call an admissible control process $\{u_t\}_{t\geq 0}$ "potentially optimal" if the following conditions are satisfied:*

- *There exists a constant $k > 0$, such that*

$$\langle \nabla_x b^u(t,x)y, y\rangle \leq -k\|y\|^2 \tag{4.16}$$

  holds for all $t \geq 0$, $x, y \in \mathbb{R}^n$. Moreover

$$\limsup_{\|x\|\to\infty} \frac{\|\sigma(t,x,u_t)\|^2}{\|x\|^2} = \bar{k} < 2k \tag{4.17}$$

  holds for all $t \geq 0$.

- *The driver $\nabla_x H(t,x,u_t,y,z)$ is uniformly Lipschitz in z. Moreover, there exists a constant $C > 0$ such that*

$$\|\nabla_x L(t,x,u_t)\| \leq C$$

  holds for all $(t,x,u) \in \mathbb{R}^+ \times \mathbb{R}^n$.

Remark 4.2.2. *While it is clear that (4.16) is just the multidimensional analogue of (4.14), condition (4.17) is purely technical and therefore may seem a little artificial. In order to gain some intuition about what it actually means, consider the standard case of a one-dimensional wealth process*

$$dV_t = rV_t dt + \pi_t \sigma V_t dW_t - c_t dt,$$

where $\{\pi_t\}_{t\geq 0}$ represents a fraction of wealth invested in a risky asset with volatility σ, and $\{c_t\}_{t\geq 0}$ is the consumption process. Then condition (4.17) states that for an (potentially) optimal policy $\pi^* = \pi^*(t,x)$,

$$\limsup_{\|x\|\to\infty} \pi^*(t,x)\sigma < 2k$$

for some $k > 0$. This is equivalent to saying that as one becomes more and more rich, it is optimal to risk only sufficiently small fractions of ones's wealth.

We are now ready to prove the existence and uniqueness of a bounded solution to the infinite horizon adjoint BSDE

$$Y(t) = Y(T) + \int_t^T \nabla_x H(s, X_s^u, u_s, Y(s), Z(s))ds - \int_t^T Z(s)dW_s. \tag{4.18}$$

Theorem 4.2.2. *For any "potentially optimal" process $\{u_t\}_{t\geq 0}$ there exists an adapted solution (Y, Z), with Y càdlàg and $Z \in \mathcal{L}^2(W)$ to the infinite horizon equation (4.18) for all $0 \leq t \leq T < \infty$, satisfying $\|Y(t)\| \leq C'$ for some $C' \in \mathbb{R}$, and this solution is unique among bounded adapted solutions.*

Furthermore, if (Y^T, Z^T) denotes the (unique) adapted square integrable solution to

$$Y^T(t) = \int_t^T \nabla_x H(t, X_s^u, u_s, Y^T(s), Z^T(s))ds - \int_t^T Z^T(s)dW_s, \tag{4.19}$$

then $\lim_{T\to\infty} Y^T(t) = Y(t)$ a.s., uniformly on compact sets in t.

Proof: We start by proving that if a bounded solution exists, it is unique. Suppose we have two bounded solutions (Y, Z) and (Y', Z') to (4.18). We denote $\delta Y := Y - Y'$, $\delta Z := Z - Z'$. Applying Itô's lemma to $\|\delta Y\|^2$, we obtain

$$d\|\delta Y(t)\|^2 = 2\langle \delta Y(t), d(\delta Y(t))\rangle + \|\delta Z(t)\|^2 dt.$$

We recall that the Hamiltonian is given by

$$H(t, x, u_t, y, z) := [b^u]'(t,x,u)y + \mathrm{Tr}([\sigma^u]'(t,x)z) + L(t, x, u_t).$$

We define an auxiliary function

$$
\begin{aligned}
f(s,x,u_s,z) &:= \nabla_x H(s,x,u,y,z) - \nabla_x b^u(s,x)y \\
&= \nabla_x L(s,x,u) + \nabla_x \operatorname{Tr}\left(\sigma^u(s,x)z\right)
\end{aligned}
$$

to represent the "undiscounted" part of the driver term in (4.18). We emphasise that f *does not depend* on y. Then, by Girsanov's theorem, one can show that there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that the process

$$
M_t := \int_t^T \delta Z(s)dW_s + \int_t^T (f(s,X_s^u,u_s,Z(s)) - f(s,X_s^u,u_s,Z'(s)))ds
$$

is a $\mathbb{Q}$-martingale [2]. Thus, for all $0 \le s \le t \le T$, we see that

$$
\begin{aligned}
\mathbb{E}_s^Q \|\delta Y(t)\|^2 = \|\delta Y(s)\|^2 &+ \mathbb{E}_s^Q \int_s^t \|\delta Z(l)\|^2 dl \\
&- 2\mathbb{E}_s^Q \int_s^t \langle \nabla_x b^u(l,X_l^u)\delta Y(l), \delta Y(l)\rangle dl
\end{aligned}
$$

where $\mathbb{E}_s^Q$ denotes the conditional expectation under $\mathbb{Q}$ given $\mathcal{F}_s$. Given that the process $\{u_t\}_{t\ge 0}$ is *"potentially optimal"*, we know that

$$
-2\langle \nabla_x b^u(l,X_l^u)\delta Y(l), \delta Y(l)\rangle \ge 2k\delta\|Y(l)\|^2,
$$

and hence we immediately see that

$$
\frac{d}{dt}\mathbb{E}_s^Q \|\delta Y(t)\|^2 \ge 2k\mathbb{E}_s^Q \|\delta Y(t)\|^2,
$$

[2] In general, applying Girsanov's theorem in the case where M is multidimensional may not be possible. However, we notice that the driver term f has a very special structure, i.e.

$$
f(s,X_s^u,u_s,Z(s)) - f(s,X_s^u,u_s,Z'(s)) = \nabla_x \operatorname{Tr}\left(\sigma^u(s,X_s^u)\delta Z(s)\right).
$$

Denote the vector above as a. Then, for any $i = 1,\dots,n$

$$
|a_i| = \sum_{j,k} [\delta Z(s)]_{jk} \frac{\partial \sigma^u(s,X_s^u)}{\partial x_i}.
$$

Taking into account the fact that σ^u is uniformly Lipschitz on $[0,T]$, there exists a constant $c > 0$ such that

$$
\|a\| < c\|\delta Z(s)\|_F, \qquad (\star)
$$

where $\|\cdot\|_F$ is the Frobenius norm. Set $\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(\int_{[0,\cdot]} \beta_t dW_t)$, where the vector β is defined as follows:

$$
\beta_s := \begin{cases} \frac{\nabla_x \operatorname{Tr}\left(\sigma^u(s,X_s^u)\delta Z(s)\right)}{||\delta Z(s)||^2}\delta Z(s) & \text{if } \delta Z(s) \neq 0, \\ 0 & \text{otherwise.} \end{cases}
$$

Taking into account $(\star)$ and the fact that all the norms in finite dimensions are equivalent, we see that our change of measure is well defined.

leading to

$$\mathbb{E}_s^Q \|\delta Y(t)\|^2 \geq \|\delta Y(s)\|^2 e^{2k(t-s)},$$

and hence

$$\|\delta Y(s)\|^2 \leq \mathbb{E}_s^Q \|\delta Y(t)\|^2 e^{-2k(t-s)} \leq 2C' e^{-2k(t-s)}, \tag{4.20}$$

where $\bar{C}$ is the bound on $\|Y\|$. The right hand side in (4.20) is independent of T and collapses as $t \to \infty$. Hence $|\delta Y_s| = 0$, from which we see $Y_s = Y_s'$ a.s. for every s, and hence $Y = Y'$ up to indistinguishability as Y and Y' are càdlàg.

We now show that a bounded solution exists. We first notice that there indeed exists a unique solution to the T-horizon BSDE (4.19) (see, for example, [43]). In order to prove that Y^T is bounded, we proceed similarly to above. Applying Itô's lemma to $\|Y^T\|^2$, we obtain

$$d\|Y^T(t)\|^2 = 2\langle Y^T(t), dY^T(t)\rangle + \|Z^T(t)\|^2 dt.$$

As before, we know that there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that the process

$$M_t := \int_t^T Z^T(s) dW_s + \int_t^T (f(s, X_s^u, u_s, Z^T(s)) - f(s, X_s^u, u_s, 0)) ds$$

is a martingale. Thus, taking conditional expectations under $\mathbb{Q}$, for all $0 \leq s \leq t \leq T$, we see that

$$\begin{aligned} \frac{d}{dt}\mathbb{E}_s^Q \|Y_t^T\|^2 &\geq -2\mathbb{E}\langle K_t^{Y^T,0}, Y_t^T\rangle + 2k\mathbb{E}_s^Q \|\delta Y_t\|^2 \\ &\geq -2\mathbb{E}C\|Y_t^T\| + 2k\mathbb{E}_s^Q \|\delta Y_t\|^2 \end{aligned}$$

as a simple application of Cauchy–Schwarz, where C is the uniform bound on $\|f(s, X_s^u, u_s, 0)\| = \|\nabla_x L(t, X_t^u, u_t)\|$ from Definition 4.2.1. It is clear that, for any $\epsilon > 0$, there exists a constant $C^\epsilon > 0$ such that $Cx \leq \epsilon x^2 + C^\epsilon$ for all $x \geq 0$. Using this fact, we arrive at

$$\frac{d}{dt}\left[\mathbb{E}_s^Q \|Y_t^T\|^2 - \frac{C^\epsilon}{k-\epsilon}\right] \geq 2(k-\epsilon)\left[\mathbb{E}_s^Q \|Y_t^T\|^2 - \frac{C^\epsilon}{k-\epsilon}\right].$$

By Grönwall's lemma we see that

$$\mathbb{E}_s^Q \|Y_t^T\|^2 - \frac{C^\epsilon}{k-\epsilon} \geq e^{2(k-\epsilon)(t-s)}\left[\|Y_s^T\|^2 - \frac{C^\epsilon}{k-\epsilon}\right].$$

Choosing $\epsilon < k$, and taking $t = T$, we conclude that

$$\|Y_s^T\|^2 \leq -e^{-2(k-\epsilon)(T-s)}\frac{C^\epsilon}{k-\epsilon} + \frac{C^\epsilon}{k-\epsilon} \leq \frac{C^\epsilon}{k-\epsilon},$$

and thus Y^T is uniformly bounded. Denoting $\delta Y := Y^T - Y^{T'}$, $\delta Z := Z^T - Z^{T'}$, we proceed exactly as we did to prove uniqueness, to arrive at

$$\|\delta Y_s\|^2 \leq \mathbb{E}_s^Q \|\delta Y_t\|^2 e^{-2k(t-s)} \leq \frac{2C^\epsilon}{k-\epsilon} e^{-2k(t-s)}, \tag{4.21}$$

where $\mathbb{Q} \sim \mathbb{P}$ is the probability measure such that the process

$$\begin{aligned} M_t := &\int_t^T \langle \delta Y(s), \delta Z_s \rangle dW_s \\ &+ \int_t^T \langle \delta Y(s), (f(s, X_s^u, u_s, Z^T(s)) - f(s, X_s^u, u_s, Z^{T'}(s))) \rangle ds \end{aligned}$$

is a martingale for $0 \leq t \leq T \leq T'$. From (4.21) we see that Y_t^T is a Cauchy sequence in T. Therefore the limit exists, and we denote it Y_t. The uniform bound also holds for Y_t, and convergence uniformly on compacts is clear from (4.21). In order to see that Z^T is Cauchy as well, we apply Itô's formula to $\|\delta Y\|^2$, and take expectations under $\mathbb{Q}$. Then

$$\mathbb{E}^Q \int_0^t \|\delta Z(s)\|^2 ds \leq -2k\mathbb{E}^Q \int_0^t \|\delta Y(s)\|^2 + \mathbb{E}\|\delta Y(t)\|^2 - \|\delta Y(0)\|^2,$$

and, given (4.21), the claim follows. Therefore, the limit as $T \to \infty$ exists for the sequence $\{Z_t^T\}$. Taking Z as its limit we obtain our desired solution (Y, Z).

□

Having proven the existence of a unique solution to the adjoint BSDE on an infinite horizon, we aim to establish a relation to ergodicity of $\{X_t^u\}_{t\geq 0}$ under all controls u. We begin with the following lemma:

Lemma 4.2.2. *Suppose for all $t \in \mathbb{R}^+$, the function $b^u(t, \cdot) \in C^1$. Then the following conditions are equivalent:*

(i) The function $b^u(t, \cdot)$ is strongly dissipative, that is

$$\langle b^u(t,x) - b^u(t,y), x - y \rangle \leq -k\|x - y\|^2$$

holds for all $t \geq 0$, $x, y \in \mathbb{R}^n$.

(ii) The gradient matrix $\nabla_x b^u(t, \cdot)$ is negative definite, that is

$$\langle \nabla_x b^u(t,x) y, y \rangle \leq -k\|y\|^2$$

holds for all $t \geq 0$, $x, y \in \mathbb{R}^n$.

Proof: **(i) $\to$ (ii):** Given any $\alpha \in [0,1]$, let $x(\alpha) := \alpha x + (1-\alpha)y$. Define the function $g : [0,1] \to \mathbb{R}$ by

$$g(\alpha) = \langle b^u(t, x(\alpha)) - b^u(t,u), x - y \rangle - k(1-\alpha)\|x-y\|^2,$$

where k is the same as in Assumption 4.2.1. We notice that $g(0) = -k\|x-y\|^2$, and $g(1) = \langle b^u(t,x) - b^u(t,y), x-y\rangle$. We see also that

$$g_\alpha(\alpha) = \langle \nabla_x b^u(t, x(\alpha))(x-y), x-y \rangle + k\|x-y\|^2,$$

and therefore $g_\alpha(\alpha) \leq 0$ for all $t \in [0,1]$. Therefore $g(1) \leq g(0)$, finishing the proof.
(ii) $\to$ (i): By assumption, we know that, for all $y \in \mathbb{R}^n$ and for all $\epsilon > 0$,

$$\langle b^u(t, x+\epsilon y) - b^u(t,u), \epsilon y \rangle \leq -k\epsilon^2 \|y\|^2.$$

Dividing both sides by ϵ^2, and taking the limit, we arrive at

$$\left\langle \lim_{\epsilon \to 0} \frac{b^u(t, x+\epsilon y) - b^u(t,y)}{\epsilon}, y \right\rangle \leq -k\|y\|^2,$$

and the result follows, given that the function b is continuously differentiable.

$\square$

We have thus proven that the conditions proposed for the adjoint BSDE to admit a solution are equivalent to strong dissipativity of the forward process. In the next section we will show that this in turn implies ergodicity, in the sense that the laws of two solutions to the equation (4.1) with different initial values get exponentially close in time. This will allow us to establish the connection to the representation via ergodic BSDEs.

Remark 4.2.3. *One could wish to establish connection between the solvability of the adjoint BSDE in infinite horizon and weak dissipativity of the forward process. The reason being that it is the largest class for which we can expect to generally prove ergodicity. However, even in one dimension, the condition*

$$\langle \nabla_x b^u(t,x)y, y \rangle \leq -k\|y\|^2$$

cannot be relaxed without losing the existence of solutions to the adjoint equation on infinite horizons. By Lemma 4.2.2 under this condition the drift b is strongly dissipative.

4.3 Structural properties of the forward process

The main goal of this section is to show that, if Assumptions 4.1.1 and 4.2.1 are satisfied, then, for each *"potentially optimal"* $\{u_t\}_{t\geq 0}$, the process X^u is strong Feller and irreducible. We then proceed to prove that the laws of two processes satisfying (4.1) and started at different points get exponentially close as $t \to \infty$.

Remark 4.3.1. *In the sequel we omit u from the dynamics of X for notational simplicity.*

We start with an auxiliary lemma that will be crucial in showing the exponential convergence of laws.

Lemma 4.3.1. *Let $\{X_t^x\}_{t\geq 0}$ be a solution to (4.1) with $X_0^x = x$. Suppose also that Assumption 4.2.1 is satisfied. Then there exist constants $c \geq 0$ and $\mu > 0$ such that*

$$\mathbb{E}\|X_t^x\|^2 \leq \|x\|^2 e^{-\mu t} + c$$

for all $t \geq 0$.

Proof: Let $\mu > 0$. Applying Itô's formula to $e^{\mu t}\|X_t^x\|^2$ and taking expectations, we obtain

$$\mathbb{E}e^{\mu t}\|X_t^x\|^2 = \|x\|^2 + \int_0^t \mu e^{\mu s}\|X_s\|^2 ds + \mathbb{E}\int_0^t G(s)ds,$$

where

$$\begin{aligned} G(s) &= 2\langle b(s, X_s^x), X_s^x\rangle + \|\sigma(s, X_s^x)\|^2 \\ &\leq -2k\|X_s^x\|^2 + \langle b(s,0)X_s^x\rangle + \|\sigma(s, X_s^x)\|^2. \end{aligned}$$

Taking (4.17) into account and applying Cauchy–Schwarz, we see that for any $\epsilon > 0$, there exists C^ϵ, such that

$$G(s) \leq -2\mu^\epsilon\|X_s^x\|^2 + C^\epsilon,$$

where $\mu^\epsilon := k - \epsilon - \bar{k}/2$ and $0 \leq s \leq t$. Then we can pick any $\epsilon < k - \bar{k}/2$, and $\mu < 2\mu^\epsilon$ to arrive at

$$e^{\mu t}\mathbb{E}\|X_t^x\|^2 \leq \|x\|^2 + \frac{C^\epsilon}{\mu}e^{\mu t}.$$

We conclude the proof by setting $c = \frac{C^\epsilon}{\mu}$.

□

4.3.1 Strong Feller Property

We begin by defining a number of objects we will be repeatedly using in the sequel.

Definition 4.3.1. *Here and for the rest of the chapter, we use the standard notation*

$$P_t\psi(x) := \mathbb{E}\big[\psi(X_t^x)\big]$$

for the transition operator (or the semigroup associated with the process $\{X_t^x\}_{t\geq 0}$*), and* $\|\psi\|_0 = \sup_{u\in\mathbb{R}^n} \|\psi(u)\|$*. We also write*

$$P_t(x, A) = P_t[\mathbf{1}_A](x), \quad A \in \mathbb{R}^n$$

for the law of the process $\{X_t^x\}_{t\geq 0}$*.*

We first state the main result of this section, the proof of which will require two auxiliary lemmas:

Theorem 4.3.1. *Suppose Assumptions 4.1.1 holds and the control process u is 'potentially optimal'. Then for all* $\psi \in \mathcal{B}_b(\mathbb{R}^n)$ *and* $t > 0$*,*

$$|P_t\psi(x) - P_t\psi(y)| \leq C_t\|x - y\|\|\psi(u)\|_0 \tag{4.22}$$

holds for some constant C_t*.*

Lemma 4.3.2. *Let* $c > 0$ *and* $t > 0$ *be fixed. Then the following conditions are equivalent:*

1. *For all* $\psi \in C_b^2(\mathbb{R}^n)$ *and for all* $x, y \in \mathbb{R}^n$ *we have*

$$|P_t\psi(x) - P_t\psi(y)| \leq c\|x - y\|\|\psi(u)\|_0.$$

2. *For all* $\psi \in B_b(\mathbb{R}^n)$ *and for all* $x, y \in \mathbb{R}^n$ *we have*

$$|P_t\psi(x) - P_t\psi(y)| \leq c\|x - y\|\|\psi(u)\|_0.$$

3. *For all* $x, y \in \mathbb{R}^n$

$$|P_t(x, \cdot) - P_t(y, \cdot)|_{TV} \leq c\|x - y\|.$$

Proof: See Lemma 9.36 in [39].

□

In addition to the stochastic differential equation (4.1), we now introduce a corresponding velocity process (a directional derivative with respect to the initial value) $t \to V^h(t), t \geq 0$, defined by

$$V^h(t) := \langle D_x X_t^x, h \rangle$$

for $h \in \mathbb{R}^n$ and $t \geq 0$. We notice that this is nothing but a directional derivative of the stochastic flow $x \mapsto X^x$ defined by the SDE (4.1). We note that under our regularity assumptions on the coefficients b and σ this is a meaningful object (see Theorem 9.6 in [39] for details). We now prove a crucial estimate on the norm of $V^h(t)$.

Lemma 4.3.3. *Suppose the control process u is 'potentially optimal'. Then, for all $t \geq 0$, there exists a constant c_t, such that*

$$\mathbb{E}\big[\|V^h(t)\|^2\big] \leq c_t \|h\|^2,$$

for all $h \in \mathbb{R}^n$.

Proof: By definition, the process X^x satisfies the integral version of (4.1), that is

$$X_t^x = x + \int_0^t b(s, X_s^x)ds + \int_0^t \sigma(s, X_s^x)dW_s.$$

taking directional derivatives of both sides, we obtain

$$\langle D_x X_t^x, h \rangle = h + \int_0^t \nabla_x b(s, X_s^x)\langle D_x X_s^x, h \rangle ds + \int_0^t \nabla_x \sigma(s, X_s^x)\langle D_x X_s^x, h \rangle dW_s,$$

which can be immediately rewritten as

$$V^h(t) = h + \int_0^t \nabla_x b(s, X_s^x)V^h(s)ds + \int_0^t \nabla_x \sigma(s, X_s^x)V^h(s)dW_s.$$

By Assumption 4.1.1 the function $\sigma(t, \cdot)$ is locally Lipschitz. Combining this with condition (4.17) we see that there exists a constant $l > 0$ such that

$$\|\nabla_x \sigma(s, X_s^x)h\|^2 \leq l\|h\|^2.$$

Applying Itô's lemma to $\|V^h(t)\|^2$, taking expectations and using (4.16), we arrive at

$$\mathbb{E}\big[\|V^h(t)\|^2\big] \leq \|h\|^2 + (l-k)\int_0^t \mathbb{E}\big[\|V^h(s)\|^2\big]ds,$$

and the conclusion follows by Grönwall's lemma with $c_t = e^{(l-k)t}$.

□

In order to proceed, we need a version of the Bismut–Elworthy formula:

Lemma 4.3.4. *In the notation above, the following holds:*

$$\mathbb{E}\left[\psi(X_t^x)\int_0^t \sigma^{-1}(s,X_s^x)V^h(s)dW_s\right] = t\langle h, D_x P_t\psi(x)\rangle, \tag{4.23}$$

where $\psi \in C_b^2(\mathbb{R}^n)$.

Proof: See, for example, [46].

□

Proof: (Theorem 4.3.1). Using formula (4.23), we deduce

$$\|\langle h, D_x P_t\psi(x)\rangle\|^2 \le \frac{1}{t^2}\|\psi(u)\|_0^2 \mathbb{E}\left[\int_0^t \|\sigma^{-1}(s,X_s^x)V^h(s)\|^2 ds\right],$$

and, using Lemma 4.3.3, we obtain the following estimate:

$$\|\langle h, D_x P_t\psi(x)\rangle\|^2 \le \frac{\overline{\sigma}^2}{t^2}\|\psi(u)\|_0^2\|h\|^2\int_0^t c_s ds,$$

and therefore

$$\|D_x P_t\psi(x)\| \le C_t\|\psi(u)\|_0,$$

where

$$C_t := \frac{\overline{\sigma}}{t}\left(\int_0^t c_s ds\right)^{1/2}.$$

Thus

$$|P_t\psi(x) - P_t\psi(y)| \le C_t\|x-y\|\|\psi(u)\|_0$$

holds for any $x, y \in \mathbb{R}^n$, $t \ge 0$ and $\psi \in C_b^2(\mathbb{R}^n)$. The conclusion follows immediately by Lemma 4.3.2.

□

Remark 4.3.2. *It is straightforward to see that all the estimates established above would still hold if we replaced* $P_t\psi(x)$ *with* $P(\tau,t)[f](x) = \mathbb{E}\psi(X_t^{x,\tau})$*, where* $X^{x,\tau}$ *is the solution to*

$$dX_t^{x,\tau} = b(t, X_t^{x,\tau})dt + \sigma(t, X_t^{x,\tau})dW_t, \quad X_\tau = x. \tag{4.24}$$

In this case all time dependent constants C_t *will become* $C_{t-\tau}$.

4.3.2 Irreducibility

The goal of this section is to prove the irreducibility of the forward process, i.e. that it at any time it can be in any open ball with positive probability. The main result is as follows:

Theorem 4.3.2. *Let $\{X_t^x\}_{t\geq 0}$ be the solution to (4.1), and let $B_r(x)$ denote an open ball of radius r around $x \in \mathbb{R}^n$. Then*

$$\mathbb{P}(X_t^x \in B_r(z)) > 0$$

for any $t > 0$, $z \in \mathbb{R}^n$, $r > 0$.

Proof: We first notice that irreducibility for X^x is equivalent to irreducibility of the (strong) solution to

$$dY_t = \Big(b(t, Y_t) + \phi(t)\Big)dt + \sigma(t, Y_t)dW_t, \quad Y_0 = x,$$

where $\phi(\cdot, \cdot)$ is a bounded function. In order to see this, we recall that $\sigma^{-1}(t, x)$ is bounded and therefore, for any $T > 0$, we can find a probability measure $\mathbb{Q}^T \sim \mathbb{P}$, such that under $\mathbb{Q}^T$, the process $dW_t^Q = dW_t + \phi(t)dt$ defines a Brownian motion for all $t \in [0, T]$. Since these measures are equivalent, so are the laws of X^x and Y on $[0, T]$. For details see Lemma 7.3.2 in [38]. The rest of the proof is inspired by Lemma 7.3.3 in [38]. We fix $t > 0$, $x, z \in \mathbb{R}^n$, $r > 0$. We will now construct a bounded function ϕ, such that the corresponding solution satisfies

$$\mathbb{P}(\|Y_t - z\| < r) > 0.$$

By the reasoning above, this is enough to prove the claim. Let $t_1 \in [0, t)$ be a moment of time to be chosen later. We then define ϕ in the following way:

$$\phi(s) := \begin{cases} 0, & s \in [0, t_1), \\ \xi(s, Y_{t_1}), & s \in [t_1, t], \end{cases}$$

where

$$\xi(s, u) = \begin{cases} \frac{y - b(s,0) - u}{t - t_1}, & \|u\| < R, \\ 0, & \text{otherwise}, \end{cases}$$

where $y, R > 0$ are constants to be chosen later. We start by picking any y such that $\|y - a\| < r/3$. Then, if $\|Y_{t_1}\| < R$, we have

$$Y_t = y + \int_{t_1}^{t} [b(s, Y_s) - b(s, 0)]ds + \int_{t_1}^{t} \sigma(s, Y_s)dW_s =: y + I,$$

and thus

$$\mathbb{P}\Big(\|Y_t - z\| < r\Big) \geq \mathbb{P}\Big(\|Y_{t_1}\| < R, \|I\| \leq 2r/3\Big) \geq \mathbb{P}\Big(\|Y_{t_1}\| < R\Big) - \mathbb{P}\Big(\|I\| > 2r/3\Big),$$

and it remains to pick t_1 such that

$$\mathbb{P}(\|I\| > 2r/3) \leq 1/4,$$

and $R > 0$ such that

$$\mathbb{P}(\|Y_{t_1}\| < R) \geq 3/4.$$

We notice that the estimate in Lemma 4.3.1 is true for the process Y as well. Therefore, taking into account that the function $b(s, \cdot)$ is Lipschitz, and applying Markov's inequality, we have

$$\mathbb{P}(\|I\| > 2r/3) \leq \frac{3}{2r}\mathbb{E}\big[\|I\|\big] \leq \frac{3\Big((t-t_1)K\big[\|x\| + c\big] + \sqrt{t-t_1}\bar{\sigma}\Big)}{2r},$$

and hence the choice of t_1 is always possible. The choice of R is a direct application of Lemma 4.3.1.

□

4.3.3 Main result

We are now ready to prove the exponential convergence of laws for two solutions of the SDE (4.1) with different initial values. This result makes it possible to talk about Ergodic BSDEs in the present framework, and gives us hope to relate two existing approaches to Ergodic Optimal Control.

Theorem 4.3.3. *Suppose Assumptions 4.1.1 and 4.2.1 hold. Then there exist constants $C > 0$ and $\rho > 0$ such that, for any bounded continuous function $\psi : \mathbb{R}^n \to \mathbb{R}$,*

$$|P(\tau, t)[\psi](x) - P(\tau, t)[\psi](y)| \leq C(1 + ||x||^2 + ||y||^2)e^{-\rho(t-\tau)}\|\psi(u)\|_0, \tag{4.25}$$

where $P(\tau, t)[f](x)$ is defined as in Remark 4.3.2.

Proof: We will not provide a detailed proof, since most of it is would closely follow the proof of Theorem 2.3.3 in Chapter 2, but we will outline a strategy. The main idea is to construct a discrete time Markov chain $(\bar{X}_{k\tilde{T}}, \bar{Y}_{k\tilde{T}})$ with a time step $\tilde{T}$ on the space $\mathbb{R}^{2n}$, whose transition operator is determined by the law of the processes $\{X_t^x\}_{t\geq 0}$ and $\{X_t^y\}_{t\geq 0}$. We then prove the necessary convergence of laws for this chain.

- **(Step 1):** The first step is almost identical to the corresponding one in the proof of Theorem 2.3.3, except for a new proof of Lemma 4.3.1. We start by showing that we can choose a time step $\tilde{T} > 0$ and a radius $R > 0$, such that, if we observe two independent solution processes X^x and X^y only at times $\{n\tilde{T}\}_{n\in\mathbb{N}}$, there is an *exponential bound* on the waiting time for both $X^y_{n\tilde{T}}$ and $X^x_{n\tilde{T}}$ to enter $B_R(0)$. The independence here is understood in the the sense that we take two independent copies ($\tilde{W}$ and $\bar{W}$) of the Brownian motion W. We set
$$\bar{X}_{k\tilde{T}} = X^x_{k\tilde{T}}, \quad \bar{Y}_{k\tilde{T}} = X^y_{k\tilde{T}},$$
for $k \leq \tau := \inf\{n : X^x_{n\tilde{T}} \in B_R(0), X^y_{n\tilde{T}} \in B_R(0)\}$.

- **(Step 2):** We will explain this part in a little more detail, since the method here is different from that in Chapter 2. Once $X^x_{k\tilde{T}}$ and $X^y_{k\tilde{T}}$ are in $B_R(0)$ for some $k \geq 0$, we lift the independence assumption and construct two solutions $X^{X^x_{k\tilde{T}},k\tilde{T}}$ and $X^{X^x_{k\tilde{T}},k\tilde{T}}$ to (4.24) on $[k\tilde{T},(k+1)\tilde{T}]$ with initial conditions $X^x_{k\tilde{T}}$ and $X^y_{k\tilde{T}}$ respectively. By irreducibility of solutions to (4.24) and compactness of $B_R(0)$, for any fixed $\epsilon > 0$, there exists a constant $\delta_\epsilon > 0$ such that
$$\mathbb{P}\left(X^{X^x_{k\tilde{T}},k\tilde{T}}_{(k+0.5)\tilde{T}} \in B_\epsilon(0), X^{X^y_{k\tilde{T}},k\tilde{T}}_{(k+0.5)\tilde{T}} \in B_\epsilon(0)\right) > \delta_\epsilon.$$
We proceed as follows: we know (see Theorem 5.2 in [32]) that there exists a coupling (Z, Z') such that
$$\mathbb{P}(Z \neq Z') = \left\|P_{(k+1)\tilde{T}}(X^x_{(k+0.5)\tilde{T}},\cdot) - P_{(k+1)\tilde{T}}(X^y_{(k+0.5)\tilde{T}},\cdot)\right\|_{TV},$$
where $P_{(k+1)\tilde{T}}(x,\cdot)$ is the law of the solution to (4.24) with $X_{(k+0.5)\tilde{T}} = x$, and Z, Z' are independent conditioned on the event $\{Z \neq Z'\}$. By Theorem 4.3.1 and Lemma 4.3.2, there exists $\epsilon > 0$ such that, for $x, y \in B_\epsilon(0)$,
$$\left\|P_{(k+1)\tilde{T}}(X^x_{(k+0.5)\tilde{T}},\cdot) - P_{(k+1)\tilde{T}}(X^y_{(k+0.5)\tilde{T}},\cdot)\right\|_{TV} \leq \frac{1}{2},$$
and therefore
$$\mathbb{P}(Z \neq Z') \geq \frac{1}{2}.$$
Now we set
$$\bar{X}_{(k+1)\tilde{T}} = \begin{cases} Z, & (X^x_{(k+0.5)\tilde{T}}, X^y_{(k+0.5)\tilde{T}}) \in B_\epsilon(0) \times B_\epsilon(0), \\ X^x_{(k+1)\tilde{T}}, & \text{otherwise.} \end{cases}$$

Define $\bar{Y}_{(k+1)\tilde{T}}$ similarly. We then infer that, for the constructed solutions,

$$\mathbb{P}(\bar{X}_{(k+1)\tilde{T}} = \bar{Y}_{(k+1)\tilde{T}}) \geq \frac{\delta_\epsilon}{2},$$

which is a bound from below uniformly in k.

- **(Step 3):** We then iterate these arguments to show that the probability that the two processes we are constructing have not met decays exponentially in time.

□

4.4 Connection to Ergodic BSDEs

In this section we explore an alternative way of looking at optimal ergodic control problems. Recall that we aim to minimise

$$J(x_0, u) = \limsup_{T\to\infty} T^{-1}\mathbb{E}\left[\int_0^T L(t, X_t^u, u_t)dt\right], \tag{4.26}$$

over the space $\mathcal{U}$ of controls, where L is uniformly bounded by some constant $C > 0$. If we assume that we can attack this problem though the adjoint BSDE approach, by Lemma 4.2.2 the process X_t^u is ergodic under all *"potentially optimal"* control processes $\{u_t\}$. In order to ensure the existence of an ergodic measure, we impose the following conditions:

Assumption 4.4.1. *There exists a positive period T^*, such that*

(i) For all admissible control processes $\{u_t\}_{t\geq 0}$, $u_{t+T^*} = u_t$.

(ii) The coefficients b and σ in the dynamics of X are T^*-periodic.

(iii) The cost function L is T^*-periodic.

We know (see, e.g. [29]), that there exists a unique invariant measure μ for the semigroup associated with the process X^u, such that there exists ergodic average λ^u, satisfying

$$\bar{\lambda^u} = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T L(t, X_t^u, u_t)dt = \int_{[0,T^*]\times\mathbb{R}^n} L(t,x)\mu(dt,dx).$$

As a special case of a much more general result (see, e.g. Theorem 2.4.4) one could also prove the following:

Theorem 4.4.1. *For any fixed control process $\{u_t\}_{t\geq 0}$, there exists a solution triple (Y^u, Z^u, λ^u) to the equation*

$$Y_t^u = Y_T^u + \int_t^T [L(s, X_t^u, u_s) - \lambda^u]du - \int_t^T Z_s^u dW_s. \tag{4.27}$$

There also exists a deterministic function $v^u(t,x)$, such that $Y_t^u = v^u(t, X_t^u)$, v^u is locally Lipschitz in space and globally in time, and

$$\|v^u(t,x)\| \leq C^u(1 + \|x\|^2)$$

for some constant $C^u > 0$.

Remark 4.4.1. *Note that in the present context, the driver function L is time dependent. This fact poses no problem to the construction of a solution to (4.27) via vanishing discount approach, since the existence of a solution to the discounted infinite horizon BSDE can be established in the most general case (see, e.g. Theorem 2.4.3). One could also still show that if there is another Markovian solution $(\tilde{v}^u, \tilde{Z}^u, \tilde{\lambda}^u)$, such that $\tilde{v}^u$ is of polynomial growth in x, then $\lambda^u = \tilde{\lambda}^u$.*

We now demonstrate that for a fixed control process u, λ^u is the corresponding ergodic average. In other words,

$$\lambda^u = \bar{\lambda}^u.$$

In order to see this, denote $(Y^{u,T}, Z^{u,T})$ to be a unique solution to the T-horizon BSDE

$$Y_t^{u,T} = \int_t^T L(s, X_t^u, u_s)du - \int_t^T Z_s^{u,T} dW_s.$$

It is clear that

$$Y_0^{u,T} = \mathbb{E}\int_0^T L(t, X_t^u, u_t)dt.$$

Comparing the dynamics of Y^u and $Y^{u,T}$ and taking expectations, we immediately see that

$$Y_0^{u,T} - v^u(0, x_0) = -\mathbb{E}v^u(T, X_T^u) + \lambda T.$$

By polynomial growth we know that

$$\|v^u(0, x_0)\| \leq C^u(1 + \|x_0\|^2), \quad \|v^u(T, X_T^u)\| \leq C^u(1 + \|X_T^u\|^2).$$

Recall (see Theorem 2.3.3) that

$$\mathbb{E}\|X_T^u\|^2 \leq \bar{C}^u(1 + \|x_0\|^2)$$

for some constant $\bar{C}^u > 0$ independent of time. Therefore

$$J(x_0, u) = \lim \sup_{T \to \infty} \frac{Y^{u,T}}{T} = \lambda^u.$$

Then the optimal control process $\{u_t^*\}$ satisfies

$$\lambda^{u^*} = \inf_u \lambda^u.$$

We note that since we control both drift and volatility of the forward process, it can not be represented as measure change, and therefore we are not able to arrive at the optimal value by solving just one ergodic BSDE with infimum of some Hamiltonian in the driver.

4.5 Conclusion

We have established a duality between the conditions implying the ergodicity of the forward process and the conditions required for the stochastic maximum principle approach to the problem of optimisation to make sense. The result confirms our intuition: one would expect that in order to have any hope for solving an ergodic control problem via probabilistic techniques, the ergodicity of the controlled process needs to be assumed. The converse is also true, in a sense that if the forward process is strongly dissipative under some set of controls, then we can deal with the problem via the adjoint equation technique. We have shown that this is indeed the case. One direction for future work would be to find a different approach that would allow us to attack the case of weakly dissipative drifts within the strong formulation.

Chapter 5

Conclusions

5.1 Future research

In this section we briefly describe two natural extensions of the previous work.

5.1.1 Mean field games

The first one is concerned with stochastic ergodic games with infinitely (countably) many players. As pointed out in the corresponding section of Chapter 3, the main difficulty with extending our results directly is the fact that in order to talk about continuity in infinite dimensions, we need to specify the topology. We recall that the infinite dimensional vector $\mathbf{z}$ represents the set of strategies for all players, and H_i corresponds to the hamiltonian of the i-th agent. We assume that the map

$$\mathbf{z} \in L^2_{\epsilon,loc} \to H_i(x, \mathbf{z})$$

is continuous for all $i \in \mathbb{N}$. This makes sense since we can show, while constructing a Markovian solution to the infinite dimensional system of EBSDEs, that $\mathbf{z}^n \to \mathbf{z}$ in $L^2_{\epsilon,loc}$. However, this topology is a little weaker than what is required to deal with mean field games. To be able to consider objects like $\lim_{n\to\infty} \frac{1}{n} \sum_1^n \|z_t^i\|$, which arise naturally in the mean field structure, we want to be able to show that, for any fixed $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^N$, our approximating sequence

$$\mathbf{z}^n \to \mathbf{z} \quad \text{weakly in } l^\infty.$$

Mathematically, the difficulty is that in order to attack this problem one would need to deal with all agents simultaneously, which potentially leads to more complexity than the cases we have considered.

5.1.2 Turnpikes and robust Kelly

The second running project deals with the connection between the Kelly betting criterion, the Turnpike theorems, and the ergodic optimal control. For example, if we think about the portfolio allocation/consumption problem. We have established that ergodicity of the forward process under the control $\{c_t\}_{t\geq 0}$ is necessary for the latter to be optimal. We therefore need to consume enough to make the wealth process recurrent. We also cannot consume too much, for otherwise we will collapse to zero too quickly. Therefore, it is conjectured that the ergodic average of our consumption should correspond to the optimal growth rate of our wealth.

A connection between ergodic HJB equations and turnpike theorems has been established by Guasoni, Robertson, Xing and Kardaras in [18]. Their setup has the restriction that returns may only be related through the behaviour of a one dimensional state variable, which they assume to be ergodic. We expect that a fully probabilistic approach will allow for this to be relaxed. We also wish to consider robust versions of this, where we account for possible misspecifications of the model over infinite horizon. Further financial applications connected with entropic risk are considered by Chong, Hu, Liang and Zariphopoulou in [9]

5.2 Final remarks

Three papers have been written using the material in this thesis. All of them deal with various aspects of ergodic optimal control and associated BSDEs. One extends the existing theory of ergodic BSDEs to the case of time-periodic structures with jumps. The second one established the existence of a Nash equilibrium in the context of ergodic games with multiple players. The third one ties the ergodicity of the controlled process in the strong formulation with viability of the stochastic maximum principle. The last two can potentially be extended and improved to incorporate some interesting cases and show hidden connections between different parts of long term optimisation.

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