

Special Lagrangian submanifolds with isolated conical singularities. I. Regularity

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1 Introduction

Special Lagrangian m -folds (SL m -folds) are a distinguished class of real m -dimensional minimal submanifolds which may be defined in \mathbb{C}^m , or in *Calabi–Yau m -folds*, or more generally in *almost Calabi–Yau m -folds* (compact Kähler m -folds with trivial canonical bundle).

This is the first in a series of five papers [11, 12, 13, 14] studying SL m -folds with *isolated conical singularities*. That is, we consider an SL m -fold X in M with singularities at x_1, \dots, x_n in M , such that for some SL cones C_i in $T_{x_i}M \cong \mathbb{C}^m$ with $C_i \setminus \{0\}$ nonsingular, X approaches C_i near x_i in an asymptotic C^1 sense. Readers are advised to begin with the final paper [14], which surveys the series, and applies the results to prove some conjectures.

Having a good understanding of the singularities of special Lagrangian submanifolds will be essential in clarifying the Strominger–Yau–Zaslow conjecture on the Mirror Symmetry of Calabi–Yau 3-folds [25], and also in resolving conjectures made by the author [9] on defining new invariants of Calabi–Yau 3-folds by counting special Lagrangian homology 3-spheres with weights. The series of papers aims to develop such an understanding for simple kinds of singularities of SL m -folds.

This first paper lays the foundations for [11, 12, 13, 14], setting up definitions and notation, and proving some auxiliary results in symplectic geometry and asymptotic analysis that will be needed in [11, 12, 13]. However, we also prove results of independent interest on the *regularity* of SL m -folds with conical singularities, and also of *Asymptotically Conical* SL m -folds in \mathbb{C}^m .

We initially define SL m -folds X with conical singularities x in Definition 3.6 below, such that X approaches the cone C near x like $O(r^{\mu-1})$ in a C^1 sense for some *rate* $\mu \in (2, 3)$, where r is the distance to x in M . In §5 we use elliptic regularity to prove an $O(r^{\mu-1-k})$ asymptotic estimate on the k^{th} derivative of the difference between X and C near x , for all $k \geq 0$.

We also show that the rate $\mu \in (2, 3)$ can be improved, up to a limit depending on the eigenvalues of the Laplacian on $\Sigma = C \cap \mathcal{S}^{2m-1}$. These results in effect *strengthen* the definition of conical singularities of SL m -folds, showing that it is equivalent to a rather stronger definition.

Section 6 relates special Lagrangian geometry to *Geometric Measure Theory*. Our main result here is that a special Lagrangian integral current whose tangent cones are ‘Jacobi integrable’ and of multiplicity one is actually an SL m -fold with conical singularities. Thus we *weaken* the definition of conical singularities of SL m -folds.

In [11] we will study the *deformation theory* of compact SL m -folds X with conical singularities in an almost Calabi–Yau m -fold M . We will show that the moduli space \mathcal{M}_X of deformations of X as an SL m -fold with conical singularities in M is locally homeomorphic to the zeroes of a smooth map $\Phi : \mathcal{I}_{X'} \rightarrow \mathcal{O}_{X'}$ between finite-dimensional vector spaces, and if the *obstruction space* $\mathcal{O}_{X'}$ is zero then \mathcal{M}_X is a smooth manifold.

Then [12, 13] will consider *desingularizations* of a compact SL m -fold X with conical singularities x_1, \dots, x_n with cones C_1, \dots, C_n in an almost Calabi–Yau m -fold M . We will take nonsingular Asymptotically Conical SL m -folds L_1, \dots, L_n in \mathbb{C}^m asymptotic to C_1, \dots, C_n at infinity, and glue them in to X at x_1, \dots, x_n to get a smooth family of compact, *nonsingular* SL m -folds \tilde{N} in M which converge to X .

We begin in §2 by defining *Riemannian manifolds with conical singularities*, and developing the theory of *weighted Sobolev spaces* upon them, and the *Fredholm properties* of the Laplacian on these spaces, adapting results of Lockhart and McOwen [16, 17]. We give a detailed treatment, in the hope that §2 will be a useful reference for further work on manifolds with conical singularities.

Almost Calabi–Yau manifolds and special Lagrangian geometry are introduced in §3, and SL m -folds with conical singularities defined in §3.3. Then §4 proves *Lagrangian Neighbourhood Theorems* for SL m -folds X with conical singularities in almost Calabi–Yau m -folds M . Essentially these are special coordinate systems on M near X , in which the symplectic form ω on M has a canonical form, and which satisfy asymptotic conditions near the singular points x_1, \dots, x_n of X . These theorems will be important tools in [11, 12, 13].

In §5 we prove regularity results for the convergence of X to its cone C_i near a singular point x_i , with all derivatives. Section 6 introduces Geometric Measure Theory, recalls results on *tangent cones* due to Adams and Simon, and shows that under some conditions on its tangent cones, a special Lagrangian integral current is an SL m -fold with conical singularities, in the sense of §3.3.

We finish in §7 by extending many of the results of §4–§5 to *Asymptotically Conical* SL m -folds in \mathbb{C}^m , which are asymptotic to an SL cone C in \mathbb{C}^m at some rate λ . These results will be needed in [12, 13].

Throughout we shall for simplicity take all submanifolds to be *embedded*. Nearly all of our results generalize immediately to *immersed* submanifolds, with only cosmetic changes. However, this does *not* apply to the Geometric Measure Theory material in §6, where the tangent cones really do have to be embedded rather than immersed.

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2 Manifolds with conical singularities

We shall study a class of singular Riemannian manifolds with isolated singularities modelled on cones.

Definition 2.1 Let (X, d) be a metric space and x_1, \dots, x_n be distinct points in X , and define $X' = X \setminus \{x_1, \dots, x_n\}$. We call X a *Riemannian m -manifold with conical singularities* x_1, \dots, x_n if the following conditions hold:

- (a) X' has the structure of a smooth, connected m -manifold with a Riemannian metric g inducing the metric d on X' .
- (b) We are given $\epsilon \in (0, 1)$ small such that $d(x_i, x_j) > 2\epsilon$ for $1 \leq i < j \leq n$ and a compact, nonsingular Riemannian $(m-1)$ -manifold (Σ_i, g_{Σ_i}) for $i = 1, \dots, n$. Write points in $\Sigma_i \times (0, \epsilon)$ as (σ, r) . Define the *cone metric* h_i on $\Sigma_i \times (0, \epsilon)$ to be $h_i = r^2 g_{\Sigma_i} + dr^2$.
- (c) For $i = 1, \dots, n$ there exist $\nu_i > 0$ and a diffeomorphism $\phi_i : \Sigma_i \times (0, \epsilon) \rightarrow S_i = \{y \in X : 0 < d(x_i, y) < \epsilon\} \subset X'$ such that

$$|\nabla^k(\phi_i^*(g) - h_i)| = O(r^{\nu_i - k}) \quad \text{as } r \rightarrow 0, \text{ for all } k \geq 0. \quad (1)$$

Here the Levi-Civita connection ∇ and $|\cdot|$ are computed using h_i .

Let C_{Σ_i} be the *Riemannian cone* on (Σ_i, g_{Σ_i}) , to be defined in Definition 2.2. We call C_{Σ_i} the *cone* and ν_i the *rate* of the singular point x_i .

Usually we will also assume that X is *compact*. Equation (1) implies that near x_i the metric g and its derivatives are asymptotic to the cone metric h_i on $\Sigma_i \times (0, \epsilon)$. For applications it generally suffices for (1) to hold when $k \leq l$ for some l . However, we will show in §5 that for the singular SL m -folds we are interested in (1) holds for all $k \geq 0$ automatically, so we may as well assume it.

Various authors have studied analysis of elliptic operators on classes of spaces including manifolds with conical singularities. We shall quote parts of their work, adapting it for our purposes where necessary. We treat the subject at some length in the hope that this will be a useful reference for future work on manifolds with conical singularities.

We start in §2.1 by discussing Riemannian cones and harmonic functions on them. Section 2.2 defines Banach spaces of functions on X' using weights, and §2.3 gives elliptic regularity results for the Laplacian on these spaces. Finally, §2.4 and §2.5 discuss homology, cohomology and Hodge theory on X' and X .

2.1 Riemannian cones and harmonic functions

Riemannian cones are a class of singular Riemannian manifolds.

Definition 2.2 Let (Σ, g_Σ) be a compact Riemannian $(m-1)$ -manifold, not necessarily connected. Define the *cone* C_Σ on Σ to be $\{0\} \cup C'_\Sigma$ where $C'_\Sigma = \Sigma \times (0, \infty)$. Write points in C'_Σ as (σ, r) . Define a Riemannian metric, the *cone metric* g on C'_Σ by $g = dr^2 + r^2 g_\Sigma$.

Define a metric d on C_Σ to be that induced by g on the connected components of C'_Σ , together with $d(0, (\sigma, r)) = r$ for $(\sigma, r) \in C'_\Sigma$ and $d((\sigma, r), (\sigma', r')) = r + r'$ for σ, σ' in different connected components of Σ and $r, r' > 0$. Then (C_Σ, d) is a metric space, called the *Riemannian cone on Σ* . It is a singular Riemannian manifold, with an isolated singularity at the *vertex* 0. Often we will take d as given and refer to C_Σ as a Riemannian cone.

For $t > 0$, define the *dilation* $t : C_\Sigma \rightarrow C_\Sigma$ by $t0 = 0$ and $t(\sigma, r) = (\sigma, tr)$. Then $t^*(d) = td$ and $t^*(g) = t^2g$. For $\alpha \in \mathbb{R}$, we say that a function $u : C'_\Sigma \rightarrow \mathbb{R}$ is *homogeneous of order α* if $u \circ t \equiv t^\alpha u$ for all $t > 0$. Equivalently, u is homogeneous of order α if $u(\sigma, r) \equiv r^\alpha v(\sigma)$ for some function $v : \Sigma \rightarrow \mathbb{R}$.

Clearly, a Riemannian cone (C_Σ, d) is an example of a manifold with conical singularities. Here is an elementary lemma on *harmonic functions* on cones.

Lemma 2.3 *In the situation of Definition 2.2, let $u(\sigma, r) \equiv r^\alpha v(\sigma)$ be a homogeneous function of order α on $C'_\Sigma = \Sigma \times (0, \infty)$, for $v \in C^2(\Sigma)$. Then*

$$\Delta u(\sigma, r) = r^{\alpha-2} (\Delta_\Sigma v - \alpha(\alpha + m - 2)v), \quad (2)$$

where Δ, Δ_Σ are the Laplacians on (C'_Σ, g) and (Σ, g_Σ) . Hence, u is harmonic on C'_Σ if and only if v is an eigenfunction of Δ_Σ with eigenvalue $\alpha(\alpha + m - 2)$.

Now for our later work we should also consider harmonic functions u on cones with more general scaling behaviour under dilations than homogeneous of order α . For example, \mathbb{R}^2 with its Euclidean metric is the Riemannian cone on S^1 , and $\log r$ is harmonic on $\mathbb{R}^2 \setminus \{0\}$. We shall show that in dimension $m > 2$, harmonic functions cannot scale like $r^\alpha (\log r)^k$ for $k > 0$.

Proposition 2.4 *In the situation of Definition 2.2, suppose $m > 2$. Then there do not exist any harmonic functions u on $C'_\Sigma = \Sigma \times (0, \infty)$ of the form*

$$u(\sigma, r) = r^\alpha (\log r)^k v_k(\sigma) + r^\alpha (\log r)^{k-1} v_{k-1}(\sigma) + \cdots + r^\alpha v_0(\sigma), \quad (3)$$

where $\alpha \in \mathbb{R}, k > 0$ and $v_k, v_{k-1}, \dots, v_0 \in C^2(\Sigma)$ with $v_k \neq 0$.

Proof. Suppose that u in (3) is harmonic. By applying infinitesimal dilations we see that $r \frac{\partial u}{\partial r}$ is also harmonic, and so

$$r \frac{\partial u}{\partial r} - \alpha u = r^\alpha (\log r)^{k-1} k v_k(\sigma) + r^\alpha (\log r)^{k-2} (k-1) v_{k-1}(\sigma) + \cdots + r^\alpha v_1(\sigma)$$

is harmonic. So if there exist harmonic u of the form (3) for k , there also exist such u for $k-1$. Thus by induction, it is sufficient to prove the case $k=1$.

Suppose for a contradiction that u is harmonic of the form (3) with $k = 1$ and $v_1 \neq 0$ in $C^2(\Sigma)$. A calculation similar to Lemma 2.3 shows that

$$\begin{aligned} \Delta u(\sigma, r) &= r^{\alpha-2} \log r (\Delta_\Sigma v_1 - \alpha(\alpha + m - 2)v_1) \\ &\quad + r^{\alpha-2} (\Delta_\Sigma v_0 - \alpha(\alpha + m - 2)v_0 - (2\alpha + m - 2)v_1). \end{aligned}$$

Thus, as u is harmonic we have

$$\Delta_\Sigma v_1 = \alpha(\alpha + m - 2)v_1 \quad \text{and} \quad \Delta_\Sigma v_0 = \alpha(\alpha + m - 2)v_0 + (2\alpha + m - 2)v_1.$$

Integrating $v_0(\Delta_\Sigma - \alpha(\alpha + m - 2))v_1$ over Σ by parts we get

$$\begin{aligned} 0 &= \int_\Sigma v_0(\Delta_\Sigma - \alpha(\alpha + m - 2))v_1 \, dV_g = \int_\Sigma v_1(\Delta_\Sigma - \alpha(\alpha + m - 2))v_0 \, dV_g \\ &= \int_\Sigma v_1(2\alpha + m - 2)v_1 = (2\alpha + m - 2)\|v_1\|_{L^2}^2. \end{aligned}$$

Thus $2\alpha + m - 2 = 0$, as $v_1 \neq 0$, so $\alpha = \frac{1}{2}(2 - m)$. But then $\Delta_\Sigma v_1 = -\frac{1}{4}(m - 2)^2 v_1$ and $v_1 \neq 0$, so that $-\frac{1}{4}(m - 2)^2$ is an eigenvalue of Δ_Σ . As $m > 2$ this contradicts the fact that eigenvalues of Δ_Σ are nonnegative. \square

Here is some more notation.

Definition 2.5 In the situation of Definition 2.2, suppose $m > 2$ and define

$$\mathcal{D}_\Sigma = \{\alpha \in \mathbb{R} : \alpha(\alpha + m - 2) \text{ is an eigenvalue of } \Delta_\Sigma\}. \quad (4)$$

By Lemma 2.3, an equivalent definition is that \mathcal{D}_Σ is the set of $\alpha \in \mathbb{R}$ for which there exists a nonzero homogeneous harmonic function u of order α on C'_Σ . By properties of the spectrum of Δ_Σ , it follows that \mathcal{D}_Σ is a countable, discrete subset of \mathbb{R} .

Define $m_\Sigma : \mathcal{D}_\Sigma \rightarrow \mathbb{N}$ by taking $m_\Sigma(\alpha)$ to be the multiplicity of the eigenvalue $\alpha(\alpha + m - 2)$ of Δ_Σ , or equivalently the dimension of the vector space of homogeneous harmonic functions u of order α on C'_Σ . Define $N_\Sigma : \mathbb{R} \rightarrow \mathbb{Z}$ by

$$N_\Sigma(\delta) = - \sum_{\alpha \in \mathcal{D}_\Sigma \cap (\delta, 0)} m_\Sigma(\alpha) \text{ if } \delta < 0, \text{ and } N_\Sigma(\delta) = \sum_{\alpha \in \mathcal{D}_\Sigma \cap [0, \delta]} m_\Sigma(\alpha) \text{ if } \delta \geq 0. \quad (5)$$

Then N_Σ is monotone increasing and upper semicontinuous, and is discontinuous exactly on \mathcal{D}_Σ , increasing by $m_\Sigma(\alpha)$ at each $\alpha \in \mathcal{D}_\Sigma$. As the eigenvalues of Δ_Σ are nonnegative, we see that $\mathcal{D}_\Sigma \cap (2 - m, 0) = \emptyset$ and $N_\Sigma \equiv 0$ on $(2 - m, 0)$.

2.2 Weighted Banach spaces

We will need the following tool, a smoothed out version of the distance from the singular set $\{x_1, \dots, x_n\}$ in X .

Definition 2.6 Let (X, d) be a compact Riemannian manifold with conical singularities $\{x_1, \dots, x_n\}$, and use the notation of Definition 2.1. Define a *radius*

function ρ on X' to be a smooth function $\rho : X' \rightarrow (0, 1]$ such that $\rho(y) = d(x_i, y)$ whenever $0 < d(x_i, y) \leq \frac{1}{2}\epsilon$ and $i = 1, \dots, n$, and $\rho(y) = 1$ when $d(x_i, y) \geq \epsilon$ for all $i = 1, \dots, n$. Radius functions always exist.

For $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$, define a function ρ^β on X' by $\rho^\beta(y) = \rho(y)^{\beta_i}$ whenever $0 < d(x_i, y) < \epsilon$ for some $i = 1, \dots, n$ and $\rho^\beta(y) = 1$ when $d(x_i, y) \geq \epsilon$ for all $i = 1, \dots, n$. Then ρ^β is well-defined and smooth on X' , and equals ρ^{β_i} near x_i in X' . If $\beta, \gamma \in \mathbb{R}^n$, write $\beta \geq \gamma$ if $\beta_i \geq \gamma_i$ and $\beta > \gamma$ if $\beta_i > \gamma_i$ for $i = 1, \dots, n$. If $\beta \in \mathbb{R}^n$ and $a \in \mathbb{R}$, write $\beta + a = (\beta_1 + a, \dots, \beta_n + a)$ in \mathbb{R}^n .

Now we define some Banach spaces of functions on X' .

Definition 2.7 Let (X, d) be a compact Riemannian m -manifold with conical singularities x_1, \dots, x_n , and use the notation of Definition 2.1. Let ρ be a radius function on X' . For $\beta \in \mathbb{R}^n$ and $k \geq 0$ define $C_\beta^k(X')$ to be the space of continuous functions f on X' with k continuous derivatives, such that $|\rho^{-\beta+j}\nabla^j f|$ is bounded on X' for $j = 0, \dots, k$. Define the norm $\|\cdot\|_{C_\beta^k}$ on $C_\beta^k(X')$ by

$$\|f\|_{C_\beta^k} = \sum_{j=0}^k \sup_{X'} |\rho^{-\beta+j}\nabla^j f|. \quad (6)$$

Then $C_\beta^k(X')$ is a Banach space. Define $C_\beta^\infty(X') = \bigcap_{k \geq 0} C_\beta^k(X')$.

For $p \geq 1$, $\beta \in \mathbb{R}^n$ and $k \geq 0$ define the *weighted Sobolev space* $L_{k,\beta}^p(X')$ to be the set of functions f on X' that are locally integrable and k times weakly differentiable, and for which the norm

$$\|f\|_{L_{k,\beta}^p} = \left(\sum_{j=0}^k \int_{X'} |\rho^{-\beta+j}\nabla^j f|^p \rho^{-m} dV_g \right)^{1/p} \quad (7)$$

is finite. Then $L_{k,\beta}^p(X')$ is a Banach space, and $L_{k,\beta}^2(X')$ a Hilbert space.

We call these *weighted Banach spaces* since the norms are locally weighted by a power of ρ . Roughly speaking, if f lies in $L_{k,\beta}^p(X')$ or $C_\beta^k(X')$ then f grows at most like ρ^{β_i} near x_i as $\rho \rightarrow 0$, and so the multi-index $\beta = (\beta_1, \dots, \beta_n)$ should be interpreted as an *order of growth*. Similarly, $\nabla^j f$ grows at most like ρ^{β_i-j} near x_i for $j = 1, \dots, k$. The vector spaces $L_{k,\beta}^p(X')$ and $C_\beta^k(X')$ are independent of the choice of radius function ρ . Different choices of ρ give equivalent norms.

Our spaces $L_{k,\beta}^p(X')$ are part of the scheme of Lockhart and McOwen [16], [17]. They consider a larger class of metrics, called *admissible metrics* on manifolds with ends [16, §2], and they use two weight functions z, ρ rather than one. In the notation of [16, §4], our space $L_{k,\beta}^p(X')$ coincides with Lockhart's space $W_{k,\delta,a}^p(X')$ if $\beta = a - \delta$. Definition 2.7 is actually based on Bartnik [3, §1] for asymptotically Euclidean manifolds.

The Banach space dual of $L_{0,\beta}^p(X')$ is another space of the same form.

Lemma 2.8 *In the situation above, let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\beta \in \mathbb{R}^n$. Then the map $\langle \cdot, \cdot \rangle : L_{0,\beta}^p(X') \times L_{0,-\beta-m}^q(X') \rightarrow \mathbb{R}$ given by $\langle u, v \rangle = \int_{X'} uv dV_g$*

is well-defined and continuous and defines a dual pairing, so that $L_{0,\beta}^p(X')$, $L_{0,-\beta-m}^q(X')$ are the Banach space duals of each other.

Proof. Let $u \in L_{0,\beta}^p(X')$ and $v \in L_{0,-\beta-m}^q(X')$. Then (7) and $\frac{1}{p} + \frac{1}{q} = 1$ imply that $u\rho^{-\beta-m/p} \in L^p(X')$ and $v\rho^{\beta+m/p} \in L^q(X')$ with

$$\|u\rho^{-\beta-m/p}\|_{L^p} = \|u\|_{L_{0,\beta}^p} \quad \text{and} \quad \|v\rho^{\beta+m/p}\|_{L^q} = \|v\|_{L_{0,-\beta-m}^q}.$$

Here $L^p(X')$, $L^q(X')$ are the usual, unweighted Lebesgue spaces on X' . Since $\frac{1}{p} + \frac{1}{q} = 1$, Hölder's inequality gives $uv \in L^1(X')$ with

$$\|uv\|_{L^1} \leq \|u\rho^{-\beta-m/p}\|_{L^p} \cdot \|v\rho^{\beta+m/p}\|_{L^q} = \|u\|_{L_{0,\beta}^p} \cdot \|v\|_{L_{0,-\beta-m}^q}.$$

So $\langle u, v \rangle = \int_{X'} uv \, dV_g$ exists, and $|\langle u, v \rangle| \leq \|uv\|_{L^1} \leq \|u\|_{L_{0,\beta}^p} \cdot \|v\|_{L_{0,-\beta-m}^q}$. Thus $\langle \cdot, \cdot \rangle$ is well-defined and continuous. The last part follows from the well-known fact that $L^p(X') \times L^q(X') \rightarrow \mathbb{R}$ is a dual pairing. \square

Here is a weighted version of the *Sobolev Embedding Theorem* and the *Konradov Theorem*, giving (compact) inclusions between these spaces.

Theorem 2.9 *In the situation above, suppose $k \geq l \geq 0$ are integers, $p, q > 1$ and $\beta, \gamma \in \mathbb{R}^n$. Then*

- (a) *If $\frac{1}{p} \leq \frac{1}{q} + \frac{k-l}{m}$ and $\beta \geq \gamma$, then $L_{k,\beta}^p(X') \hookrightarrow L_{l,\gamma}^q(X')$ is a continuous inclusion. If $\frac{1}{p} < \frac{1}{q} + \frac{k-l}{m}$ and $\beta > \gamma$, this inclusion is compact.*
- (b) *If $\frac{1}{p} < \frac{k-l}{m}$ and $\beta \geq \gamma$, then $L_{k,\beta}^p(X') \hookrightarrow C_\gamma^l(X')$ is a continuous inclusion. If $\frac{1}{p} < \frac{k-l}{m}$ and $\beta > \gamma$, this inclusion is compact.*

Proof. Part (a) follows from [16, Th. 4.8 & Th. 4.9] once the notation is disentangled. Inclusion in (b) when $\beta = \gamma$ is proved by Bartnik [3, Th. 1.2, eq. (1.9)] in \mathbb{R}^n using a scaling argument on annuli, and then generalized to asymptotically Euclidean manifolds. The same method works in our case, where instead of the annulus $A_R = B_{2R} \setminus \overline{B}_R$ in \mathbb{R}^n we substitute $\Sigma_i \times (R, 2R)$ in C_{Σ_i} . The rest of (b) follows from (a). \square

2.3 Elliptic regularity on weighted spaces

Let (X, d) be a compact Riemannian manifold with conical singularities, and use the notation above. Let $\Delta = d^*d$ be the Laplacian on functions. We will study the map

$$\Delta_{k,\beta}^p = \Delta : L_{k,\beta}^p(X') \rightarrow L_{k-2,\beta-2}^p(X') \quad (8)$$

for $p > 1$, $k \geq 2$ and $\beta \in \mathbb{R}^n$. As a shorthand we will refer to this map as $\Delta_{k,\beta}^p$. We will show that under certain conditions on β it is *Fredholm*, and describe its kernel and cokernel.

Here is an *elliptic regularity result* for $\Delta_{k,\beta}^p$.

Theorem 2.10 *Let (X, d) be a compact Riemannian manifold with conical singularities. Then for all $p > 1$, $k \geq 2$ and $\beta \in \mathbb{R}^n$ there exists $C > 0$ such that if $u \in L_{0,\beta}^p(X')$ lies in L_2^p locally and $v \in L_{k-2,\beta-2}^p(X')$ with $\Delta u = v$ then $u \in L_{k,\beta}^p(X')$ and $\|u\|_{L_{k,\beta}^p} \leq C(\|u\|_{L_{0,\beta}^p} + \|v\|_{L_{k-2,\beta-2}^p})$.*

Proof. Gilbarg and Trudinger [6, Th. 9.19] show that u lies in L_k^p locally in X' , and the result then follows from Lockhart [16, Th. 3.7]. \square

Recall that a continuous linear map between Banach spaces is *Fredholm* if it has finite-dimensional kernel and cokernel.

Theorem 2.11 *Let (X, d) be a compact Riemannian m -manifold with conical singularities x_1, \dots, x_n . Then for all $p > 1$, $k \geq 2$ and $\beta \in \mathbb{R}^n$, the map $\Delta_{k,\beta}^p$ is Fredholm if and only if $\beta_i \notin \mathcal{D}_{\Sigma_i}$ for all $i = 1, \dots, n$, where \mathcal{D}_{Σ_i} is defined in (4), that is, if β lies in the subset*

$$(\mathbb{R} \setminus \mathcal{D}_{\Sigma_1}) \times \dots \times (\mathbb{R} \setminus \mathcal{D}_{\Sigma_n}) \quad (9)$$

in \mathbb{R}^n . Equivalently, $\Delta_{k,\beta}^p$ is Fredholm if and only if for all $i = 1, \dots, n$, there exists no nonzero homogeneous harmonic function u on C'_{Σ_i} with rate β_i .

Proof. Translating our problem into his notation, Lockhart [16, Th. 5.2] shows that $\Delta_{k,\beta}^p$ is Fredholm if and only if $\beta_i \notin \mathcal{D}_{\Sigma_i}$ for $i = 1, \dots, n$, where \mathcal{D}_{Σ_i} is a countable, discrete subset of \mathbb{R} . Following the definition of \mathcal{D}_{Σ_i} back through [16, Th. 3.7], [17, Th. 6.2] and [17, p. 416-7] we eventually find that it is given by (4). In fact, [17, p. 416-7] defines \mathcal{D}_{Σ_i} as the imaginary part of the spectrum of a complex eigenvalue problem, but as the spectrum of Δ_{Σ_i} is real and nonnegative, it reduces to (4). The final part follows from Definition 2.5. \square

We study the dependence of the kernel of $\Delta_{k,\beta}^p$ on p, k and β .

Theorem 2.12 *Let (X, d) be a compact Riemannian m -manifold with conical singularities x_1, \dots, x_n , and let $p > 1$, $k \geq 2$ and $\beta \in \mathbb{R}^n$. Then $\text{Ker}(\Delta_{k,\beta}^p)$ is independent of $k \geq 2$, and is a finite-dimensional subspace of $C_{\beta}^{\infty}(X')$. If also β lies in (9) then $\text{Ker}(\Delta_{k,\beta}^p)$ is independent of $p > 1$, and depends only on (X, d) and the connected component of (9) containing β .*

Proof. If $u \in L_{2,\beta}^p(X')$ and $\Delta u = 0$ then Theorem 2.10 with $v = 0$ shows that $u \in L_{k,\beta}^p(X')$ for any $k \geq 2$. Part (b) of Theorem 2.9 then implies that $u \in C_{\beta}^l(X')$ for all $l \geq 0$, and so $u \in C_{\beta}^{\infty}(X')$. Thus the kernel of $\Delta : L_{k,\beta}^p(X') \rightarrow L_{k-2,\beta-2}^p(X')$ is independent of k and lies in $C_{\beta}^{\infty}(X')$. Finite-dimensionality follows from [16, Cor. 5.6].

When $n = 1$ and β lies in (9), Lockhart and McOwen show [17, Lem. 7.3] that the kernel of $\Delta_{k,\beta}^p$ is independent of $p > 1$, and depends only on the connected component of β in $\mathbb{R} \setminus \mathcal{D}_{\Sigma_1}$, [17, Lem. 7.1]. These are easily generalized to the case $n > 1$ as in [17, §8]. \square

Here is an integration by parts formula in weighted Sobolev spaces.

Lemma 2.13 *In the situation above, let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\beta \in \mathbb{R}^n$. Then for all $u \in L_{2,\beta}^p(X')$ and $v \in L_{2,-\beta+2-m}^q(X')$ we have*

$$\langle \Delta u, v \rangle = \int_{X'} (\Delta u)v \, dV_g = \int_{X'} (du \cdot dv) \, dV_g = \int_{X'} u(\Delta v) \, dV_g = \langle u, \Delta v \rangle. \quad (10)$$

Proof. First suppose u, v are smooth with compact support in X' . Then (10) is immediate by integration by parts. But the smooth functions with compact support are dense in $L_{2,\beta}^p(X')$ and $L_{2,-\beta+2-m}^q(X')$ by [16, Cor. 4.5], and Lemma 2.8 shows that $(u, v) \mapsto \langle \Delta u, v \rangle$, $(u, v) \mapsto \int_{X'} (du \cdot dv) \, dV_g$ and $(u, v) \mapsto \langle u, \Delta v \rangle$ are continuous maps $L_{2,\beta}^p(X') \times L_{2,-\beta+2-m}^q(X') \rightarrow \mathbb{R}$, so the result follows. \square

Now we can describe the cokernel of $\Delta_{k,\beta}^p$ when it is Fredholm.

Theorem 2.14 *In the situation of Theorem 2.11, suppose $\Delta_{k,\beta}^p$ is Fredholm. Define $q > 1$ by $\frac{1}{p} + \frac{1}{q} = 1$. Then $u \in L_{k-2,\beta-2}^p(X')$ lies in the image of $\Delta_{k,\beta}^p$ if and only if $\langle u, v \rangle = 0$ for all v in the kernel of $\Delta_{2,-\beta+2-m}^q$. Hence, the cokernel of $\Delta_{k,\beta}^p$ is isomorphic to the dual of the kernel of $\Delta_{2,-\beta+2-m}^q$.*

Proof. If $u = \Delta w$ for $w \in L_{k,\beta}^p(X')$ and $v \in \text{Ker}(\Delta_{2,-\beta+2-m}^q)$ then $\langle u, v \rangle = \langle \Delta w, v \rangle = \langle w, \Delta v \rangle = 0$ by Lemma 2.13. This proves the ‘only if’ part. To prove the ‘if’ part, first suppose $k = 2$, and let $f : L_{0,\beta-2}^p(X') \rightarrow \mathbb{R}$ be a linear map vanishing on the image of $\Delta_{2,\beta}^p$.

As $\Delta_{2,\beta}^p$ is Fredholm this image is closed and has finite codimension, so that f is *continuous*. Thus f defines an element of the Banach space dual of $L_{0,\beta-2}^p(X')$. Lemma 2.8 then gives a unique $v \in L_{0,-\beta+2-m}^q(X')$ such that $\int_{X'} uv \, dV_g = \langle u, v \rangle = f(u)$ for all $u \in L_{0,\beta-2}^p(X')$. As $f = 0$ on the image of $\Delta_{2,\beta}^p$, this shows that

$$\int_{X'} v(\Delta w) \, dV_g = 0 \quad \text{for all } w \in L_{2,\beta}^p(X'). \quad (11)$$

By an elliptic regularity result of Morrey [22, Th. 6.4.3, p. 246], if v is L^q locally for $q > 1$ and (11) holds for all compactly-supported smooth w (which automatically lie in $L_{2,\beta}^p(X')$), then v is L_l^q locally for all $l \geq 0$. Thus v is smooth, by the Sobolev Embedding Theorem, and integration by parts shows that v is harmonic. Theorem 2.10 then proves that $v \in L_{2,-\beta+2-m}^q(X')$. So v lies in the kernel of $\Delta_{2,-\beta+2-m}^q$.

We have shown that any linear map $f : L_{0,\beta-2}^p(X') \rightarrow \mathbb{R}$ vanishing on the image of $\Delta_{2,\beta}^p$ is of the form $f(u) = \langle u, v \rangle$ for some v in the kernel of (11). As the image of $\Delta_{2,\beta}^p$ has finite codimension, this proves the ‘if’ part when $k = 2$.

For the $k > 2$ case, suppose that $u \in L_{k-2,\beta-2}^p(X')$ and $\langle u, v \rangle = 0$ for all v in the kernel of $\Delta_{2,-\beta+2-m}^q$. Then $u \in L_{0,\beta-2}^p(X')$, so by the $k = 2$ case we have

$u = \Delta w$ for some $w \in L_{2,\beta}^p(X')$. Theorem 2.10 then implies that $w \in L_{k,\beta}^p(X')$, so u lies in the image of $\Delta_{k,\beta}^p$, and the proof is complete. \square

The *index* of a Fredholm operator P is $\text{ind}(P) = \dim \text{Ker}(P) - \dim \text{Coker}(P)$.

Theorem 2.15 *Let (X, d) be a compact Riemannian m -manifold for $m > 2$ with conical singularities x_1, \dots, x_n , and let $p > 1$, $k \geq 2$ and β lie in (9), so that $\Delta_{k,\beta}^p$ is Fredholm. Then in the notation of Definition 2.5, we have*

$$\text{ind}(\Delta_{k,\beta}^p) = - \sum_{i=1}^n N_{\Sigma_i}(\beta_i). \quad (12)$$

Proof. First we prove the special case with $\beta_i = \frac{1}{2}(2-m)$ for $i = 1, \dots, n$. From Definition 2.5 we have $\mathcal{D}_{\Sigma_i} \cap (2-m, 0) = \emptyset$ and $N_{\Sigma_i} \equiv 0$ on $(2-m, 0)$, so β lies in (9) and $N_{\Sigma_i}(\beta_i) = 0$ for $i = 1, \dots, n$ as $m > 2$. Theorems 2.12 and 2.14 then show that $\text{Coker}(\Delta_{k,\beta}^p) \cong \text{Ker}(\Delta_{k,\beta}^p)^*$. Hence $\text{ind}(\Delta_{k,\beta}^p) = 0$. As $N_{\Sigma_i}(\beta_i) = 0$ for $i = 1, \dots, n$ this proves (12) when $\beta_i = \frac{1}{2}(2-m)$.

Now Lockhart and McOwen prove formulae for how $\text{ind}(\Delta_{k,\beta}^p)$ changes as β varies in (9). They do this for the $n = 1$ case in [17, Th. 6.2], and for all n but with exactly rather than asymptotically conical metrics near x_i in [17, Th. 8.1]; their proof is easily generalized to the asymptotically conical case.

In the $n = 1$ case, [17, Th. 6.2] shows that if $\beta < \gamma$ lie in $\mathbb{R} \setminus \mathcal{D}_{\Sigma_1}$ then

$$\text{ind}(\Delta_{k,\beta}^p) - \text{ind}(\Delta_{k,\gamma}^p) = N(\beta, \gamma), \quad \text{where} \quad N(\beta, \gamma) = \sum_{\alpha \in \mathcal{D}_{\Sigma_1} \cap (\beta, \gamma)} d(\alpha).$$

Here $d(\alpha)$ is defined on [17, p. 416], and is effectively the dimension of the space of all harmonic functions on C'_{Σ_1} of the form (3). But Proposition 2.4 shows that such functions are homogeneous of order α as $m > 2$, so $d(\alpha) = m_{\Sigma}(\alpha)$ in our notation. Thus $N(\beta, \gamma) = N_{\Sigma_1}(\gamma) - N_{\Sigma_1}(\beta)$.

Similarly, for $n \geq 1$ we find using [17, Th. 8.1] that if β, γ lie in (9) then

$$\text{ind}(\Delta_{k,\beta}^p) - \text{ind}(\Delta_{k,\gamma}^p) = \sum_{i=1}^n N_{\Sigma_i}(\gamma_i) - \sum_{i=1}^n N_{\Sigma_i}(\beta_i).$$

Combining this with the case $\gamma_i = \frac{1}{2}(2-m)$ for $i = 1, \dots, n$ proved above yields (12). \square

We identify $\text{Ker}(\Delta_{k,\beta}^p)$ in simple cases.

Lemma 2.16 *Let (X, d) be a compact Riemannian m -manifold with conical singularities x_1, \dots, x_n , and let $p > 1$, $k \geq 2$ and $\beta \in \mathbb{R}^n$. Then*

- (a) *If $\beta_i > 0$ for $i = 1, \dots, n$ then $\text{Ker}(\Delta_{k,\beta}^p) = \{0\}$.*
- (b) *If $\beta_i \in (2-m, 0)$ for $i = 1, \dots, n$ then $\text{Ker}(\Delta_{k,\beta}^p) = \langle 1 \rangle$.*

Proof. Let $u \in \text{Ker}(\Delta_{k,\beta}^p)$, so that $u \in C_\beta^\infty(X')$ by Theorem 2.12. If $\beta_i > 0$ this implies that $u(y) \rightarrow 0$ as $y \rightarrow x_i$ for $i = 1, \dots, n$. Applying the *maximum principle* [6, §3] then shows that $u = 0$, proving (a). For (b), first let $\beta_i = \frac{1}{2}(2-m)$ for $i = 1, \dots, n$, and suppose $u \in \text{Ker}(\Delta_{k,\beta}^p)$. Then Lemma 2.13 gives

$$0 = \int_{X'} u(\Delta u) dV_g = \int_{X'} |du|^2 dV_g.$$

Thus $du = 0$, so u is constant, and $u \in \langle 1 \rangle$ as X' is connected.

Conversely, $1 \in \text{Ker}(\Delta_{k,\beta}^p)$ as $\beta_i < 0$, so $\text{Ker}(\Delta_{k,\beta}^p) = \langle 1 \rangle$. Now $(2-m, 0)^n$ is a connected subset of (9) containing $(\frac{1}{2}(2-m), \dots, \frac{1}{2}(2-m))$, by Definition 2.5. Thus Theorem 2.12 shows that $\text{Ker}(\Delta_{k,\beta}^p)$ is independent of β for $\beta_i \in (2-m, 0)$, and part (b) follows. \square

The next inequality is in effect a lower bound for the positive eigenvalues of the Laplacian Δ on X' . Here $\|u\|_{L^2}$ is the *unweighted* L^2 -norm $(\int_{X'} u^2 dV)^{1/2}$.

Theorem 2.17 *Let (X, d) be a compact Riemannian m -manifold for $m > 2$ with conical singularities x_1, \dots, x_n , and suppose $X' = X \setminus \{x_1, \dots, x_n\}$ is connected. Then there exists $C > 0$ such that whenever $u \in C^2(X')$ is compactly-supported with $\int_{X'} u dV_g = 0$ we have $\|u\|_{L^2} \leq C \|du\|_{L^2} \leq C^2 \|\Delta u\|_{L^2}$.*

Proof. Let $\beta_i = \frac{1}{2}(2-m)$ for $i = 1, \dots, n$. Then β lies in (9) by Definition 2.5, so $\Delta_{2,\beta}^2$ is Fredholm by Theorem 2.11, and Theorem 2.14 and part (b) of Lemma 2.12 show that $\Delta_{2,\beta}^2$ has kernel and cokernel $\langle 1 \rangle$. Therefore

$$\Delta_{2,\beta}^2 : \{u \in L_{2,\beta}^2(X') : \int_{X'} u dV_g = 0\} \rightarrow V \quad (13)$$

is a continuous vector space isomorphism between Banach spaces, where

$$V = \{v \in L_{0,\beta-2}^2(X') : \int_{X'} v dV_g = 0\}$$

is a Hilbert space, with the $L_{0,\beta-2}^2$ norm.

By the Open Mapping Theorem it follows that (13) has a continuous inverse, P say. Let $\iota : L_{2,\beta}^2(X') \rightarrow L_{0,\beta-2}^2(X')$ be the inclusion. Then ι is continuous and compact, by part (a) of Theorem 2.9. Hence $\iota \circ P : V \rightarrow V$ is a *continuous, injective, compact, linear automorphism* of the Hilbert space V .

We are interested in the ordinary, unweighted L^2 -norm $\|\cdot\|_{L^2}$ on functions. Define $\gamma_i = -\frac{1}{2}m$ for $i = 1, \dots, n$. Then the power of ρ in (7) used to define $L_{0,\gamma}^2(X')$ is trivial, so that $L^2(X') = L_{0,\gamma}^2(X')$ with $\|\cdot\|_{L^2} = \|\cdot\|_{L_{0,\gamma}^2}$. As $L_{2,\beta}^2(X') \hookrightarrow L_{0,\gamma}^2(X')$ is a continuous inclusion by Theorem 2.9, we see that $\iota \circ P$ maps $V \rightarrow V \cap L^2(X')$.

Now $\iota \circ P$ is an inverse for the Laplacian Δ , and Δ is *self-adjoint* w.r.t. the L^2 inner product (though not w.r.t. the $L_{0,\beta-2}^2$ inner product). It easily follows that the restriction of $\iota \circ P$ to $V \cap L^2(X')$ is self-adjoint w.r.t. the L^2 inner product on $V \cap L^2(X')$.

We can now apply the theory of compact self-adjoint operators on Hilbert spaces. As $\iota \circ P$ is compact it has a countable set of eigenvalues converging to zero, with finite multiplicity. As $\iota \circ P$ is injective, the eigenvalues are nonzero. Thus all eigenspaces lie in $\iota \circ P(V) \subset V \cap L^2(X')$. As $\iota \circ P$ is self-adjoint in the L^2 inner product the eigenvalues are all real, there are no nilpotency phenomena, and eigenvectors for distinct eigenvalues are L^2 -orthogonal.

Therefore there exists a sequence $(e_i)_{i=1}^{\infty}$ of eigenvectors of $\iota \circ P$ in $V \cap L^2(X')$, such that $\iota \circ P(e_i) = \lambda_i$ for $\lambda_i \in \mathbb{R} \setminus \{0\}$ with $\lambda_i \rightarrow 0$ as $i \rightarrow \infty$, and $(e_i)_{i=1}^{\infty}$ is orthonormal in the L^2 inner product, and $(e_i)_{i=1}^{\infty}$ is a basis for the Hilbert space V . As $\iota \circ P$ is an inverse for Δ we see that $\Delta e_i = \lambda_i^{-1} e_i$. Thus

$$0 < \int_{X'} |de_i|^2 dV_g = \int_{X'} e_i \Delta e_i dV_g = \lambda_i^{-1} \|e_i\|_{L^2}^2 = \lambda_i^{-1},$$

integrating by parts using Lemma 2.13. So $\lambda_i > 0$ for all i .

Now let $u \in C^2(X')$ be compactly-supported with $\int_{X'} u dV_g = 0$. Then $u \in V \cap L^2(X')$ and $\Delta u \in V \cap L^2(X')$. Thus $\Delta u = \sum_{i=1}^{\infty} x_i e_i$, where $x_i = \langle \Delta u, e_i \rangle_{L^2} \in \mathbb{R}$. As $\iota \circ P$ is an inverse for Δ we have $u = \iota \circ P(\Delta u)$, so $u = \sum_{i=1}^{\infty} \lambda_i x_i e_i$. Hence

$$\|u\|_{L^2}^2 = \sum_{i=1}^{\infty} \lambda_i^2 x_i^2, \quad \|du\|_{L^2}^2 = \langle u, \Delta u \rangle_{L^2} = \sum_{i=1}^{\infty} \lambda_i x_i^2 \quad \text{and} \quad \|\Delta u\|_{L^2}^2 = \sum_{i=1}^{\infty} x_i^2.$$

Since $\lambda_i > 0$ with $\lambda_i \rightarrow 0$ as $i \rightarrow \infty$ this implies that $\|u\|_{L^2} \leq C \|du\|_{L^2} \leq C^2 \|\Delta u\|_{L^2}$ with $C = \min_{i=1}^{\infty} (\lambda_i^{-1})$. This completes the proof. \square

2.4 Homology and cohomology

Next we discuss homology and cohomology of manifolds with conical singularities. For a general reference on (co)homology of manifolds, see for instance Bredon [4]. If Y is a manifold, write $H^k(Y, \mathbb{R})$ for the k^{th} *de Rham cohomology group* and $H_{\text{cs}}^k(Y, \mathbb{R})$ for the k^{th} *compactly-supported de Rham cohomology group* of Y . If Y is compact then $H^k(Y, \mathbb{R}) = H_{\text{cs}}^k(Y, \mathbb{R})$.

Let Y be a topological space, and $Z \subset Y$ a subspace. Write $H_k(Y, \mathbb{R})$ for the k^{th} *real singular homology group* of Y , and $H_k(Y; Z, \mathbb{R})$ for the k^{th} *real singular relative homology group* of (Y, Z) . When Y is a manifold and Z a submanifold, we may define $H_k(Y, \mathbb{R})$ and $H_k(Y; Z, \mathbb{R})$ using *smooth* simplices, as in [4, §V.5]. Then the pairing between (singular) homology and (de Rham) cohomology is defined at the chain level by integrating k -forms over k -simplices.

Suppose Y is a compact m -manifold with boundary, so that ∂Y is a compact $(m-1)$ -manifold and $Y^\circ = Y \setminus \partial Y$ is an m -manifold without boundary, which is noncompact if $\partial Y \neq \emptyset$. Then there is a natural long exact sequence

$$\cdots \rightarrow H_{\text{cs}}^k(Y^\circ, \mathbb{R}) \rightarrow H^k(Y^\circ, \mathbb{R}) \rightarrow H^k(\partial Y, \mathbb{R}) \rightarrow H_{\text{cs}}^{k+1}(Y^\circ, \mathbb{R}) \rightarrow \cdots \quad (14)$$

Note that $H^k(Y^\circ, \mathbb{R}) = H^k(Y, \mathbb{R})$. Suppose Y is oriented. Then by Poincaré–Lefschetz duality there are isomorphisms

$$H_k(Y; \partial Y, \mathbb{R})^* \cong H_{\text{cs}}^k(Y^\circ, \mathbb{R}) \cong H_{m-k}(Y^\circ, \mathbb{R}) \cong H^{m-k}(Y^\circ, \mathbb{R})^*. \quad (15)$$

If X is a compact Riemannian manifold with conical singularities x_1, \dots, x_n then $X' = X \setminus \{x_1, \dots, x_n\}$ is the interior of a compact manifold \bar{X}' with boundary $\partial\bar{X}'$ the disjoint union $\coprod_{i=1}^n \Sigma_i$. Thus (14) gives an exact sequence

$$\dots \rightarrow H_{\text{cs}}^k(X', \mathbb{R}) \rightarrow H^k(X', \mathbb{R}) \rightarrow \bigoplus_{i=1}^n H^k(\Sigma_i, \mathbb{R}) \rightarrow H_{\text{cs}}^{k+1}(X', \mathbb{R}) \rightarrow \dots \quad (16)$$

If X' is oriented then (15) gives isomorphisms

$$H_k(X; \{x_1, \dots, x_n\}, \mathbb{R})^* \cong H_{\text{cs}}^k(X', \mathbb{R}) \cong H_{m-k}(X', \mathbb{R}) \cong H^{m-k}(X', \mathbb{R})^*, \quad (17)$$

as $H_{m-k}(X; \{x_1, \dots, x_n\}, \mathbb{R}) \cong H_{m-k}(\bar{X}'; \partial\bar{X}', \mathbb{R})$ by excision.

Since $H_k(\{x_1, \dots, x_n\}, \mathbb{R}) = 0$ for $k \neq 0$, the long exact sequence

$$\dots \rightarrow H_k(X, \mathbb{R}) \rightarrow H_k(X; \{x_1, \dots, x_n\}, \mathbb{R}) \rightarrow H_{k-1}(\{x_1, \dots, x_n\}, \mathbb{R}) \rightarrow \dots$$

implies that $H_k(X, \mathbb{R}) \cong H_k(X; \{x_1, \dots, x_n\}, \mathbb{R})$ for $k > 1$. Therefore (17) gives

$$H_{\text{cs}}^k(X', \mathbb{R}) \cong H_k(X, \mathbb{R})^* \quad \text{for all } k > 1. \quad (18)$$

We can now study $\Delta_{k,\lambda}^p$ when λ_i is small and positive.

Proposition 2.18 *Suppose (X, d) is a compact Riemannian m -manifold for $m > 2$ with conical singularities x_1, \dots, x_n , and use the notation above. Let $\mathcal{K}_{X'} \subset C^\infty(X')$ be a vector space of smooth functions constant on S_i for $i = 1, \dots, n$, such that $v \mapsto [dv]$ is an isomorphism from $\mathcal{K}_{X'}$ to the kernel of the map $H_{\text{cs}}^1(X', \mathbb{R}) \rightarrow H^1(X', \mathbb{R})$ in (16). Let $p > 1$, $k \geq 2$ and $\lambda_i > 0$ with $(0, \lambda_i] \cap \mathcal{D}_{\Sigma_i} = \emptyset$ for $i = 1, \dots, n$. Then*

$$\Delta : L_{k,\lambda}^p(X') \oplus \mathcal{K}_{X'} \rightarrow \{w \in L_{k-2,\lambda-2}^p(X') : \int_{X'} w \, dV_g = 0\}, \quad (19)$$

given by $(u, v) \mapsto \Delta_{k,\lambda}^p u + \Delta v$, is an isomorphism of topological vector spaces.

Proof. Since $\mathcal{K}_{X'}$ is isomorphic to the kernel of $H_{\text{cs}}^1(X', \mathbb{R}) \rightarrow H^1(X', \mathbb{R})$, equation (16) gives an exact sequence

$$0 \rightarrow H^0(X', \mathbb{R}) \rightarrow \bigoplus_{i=1}^n H^0(\Sigma_i, \mathbb{R}) \rightarrow \mathcal{K}_{X'} \rightarrow 0,$$

and thus $\dim \mathcal{K}_{X'} = \sum_{i=1}^n b^0(\Sigma_i) - 1$ as X' is connected. As $\lambda_i \notin \mathcal{D}_{\Sigma_i}$, Theorem 2.15 shows that $\Delta_{k,\lambda}^p$ is Fredholm with

$$\text{ind}(\Delta_{k,\lambda}^p) = \sum_{i=1}^n N_{\Sigma_i}(\lambda_i) = \sum_{i=1}^n N_{\Sigma_i}(0) = \sum_{i=1}^n b^0(\Sigma_i) = 1 + \dim \mathcal{K}_{X'}. \quad (20)$$

Here $N_{\Sigma_i}(\lambda_i) = N_{\Sigma_i}(0)$ as $(0, \lambda_i] \cap \mathcal{D}_{\Sigma_i} = \emptyset$ and N_{Σ_i} is upper semicontinuous and locally constant on $\mathbb{R} \setminus \mathcal{D}_{\Sigma_i}$. Also $N_{\Sigma_i}(0)$ is the multiplicity of the eigenvalue 0 of Δ_{Σ_i} by Definition 2.5, which is $b^0(\Sigma_i)$.

By integrating by parts as in (10) we see that $\int_{X'} \Delta(u+v) dV_g = 0$ for $u \in L_{k,\lambda}^p(X')$ and $v \in \mathcal{K}_{X'}$, so Δ does map into the given r.h.s. in (19). Now (19) modifies the Fredholm map $\Delta_{k,\lambda}^p$, increasing the dimension of its domain by $\dim \mathcal{K}_{X'}$, and decreasing the dimension of its range by 1. Therefore from (20) we see that (19) is Fredholm with index 0. Thus (19) is an isomorphism of topological vector spaces if and only if it is injective.

Suppose (u, v) lies in the kernel of (19). Then $\Delta(u+v) \equiv 0$, so multiplying by $u+v$ and integrating by parts as in (10) shows that $\int_{X'} |du+dv|^2 dV_g = 0$, so $d(u+v) = 0$ and $u+v \equiv c$ for some $c \in \mathbb{R}$. Now $u \in C_\lambda^0(X')$ by Theorem 2.9, so that $u(x) \rightarrow 0$ as $x \rightarrow x_i$, and v is constant on S_i .

Taking $x \rightarrow x_i$ shows that $v \equiv c$ on S_i for all i , so $v - c$ is compactly supported. But then $[dv] = [d(v-c)] = 0$ in $H_{\text{cs}}^1(X', \mathbb{R})$, so $v = 0$ as $v \mapsto [dv]$ is an isomorphism with a subspace of $H_{\text{cs}}^1(X', \mathbb{R})$. Hence $c = 0$, so $u = 0$, and (19) is injective. This completes the proof. \square

2.5 Hodge theory

Hodge theory for a compact Riemannian manifold (Y, g) shows that each class in $H^k(Y, \mathbb{R})$ is represented by a unique k -form α with $d\alpha = d^*\alpha = 0$. Here is an analogue of this on X' for $k = 1$, with decay conditions.

Theorem 2.19 *Let X be a compact Riemannian m -manifold for $m > 2$ with conical singularities at x_1, \dots, x_n , and let $X', \epsilon, \Sigma_i, \phi_i, S_i$ and ν_i be as in Definition 2.1, \mathcal{D}_{Σ_i} as in Definition 2.5, and ρ as in Definition 2.6. Define*

$$Y_{X'} = \{\alpha \in C^\infty(T^*X') : d\alpha = 0, d^*\alpha = 0, |\nabla^k \alpha| = O(\rho^{-1-k}) \text{ for } k \geq 0\}. \quad (21)$$

Then $\pi : Y_{X'} \rightarrow H^1(X', \mathbb{R})$ given by $\pi : \alpha \mapsto [\alpha]$ is an isomorphism. Furthermore:

- (a) Fix $\alpha \in Y_{X'}$. By Hodge theory there exists a unique $\gamma_i \in C^\infty(T^*\Sigma_i)$ with $d\gamma_i = d^*\gamma_i = 0$ for $i = 1, \dots, n$, such that the image of $\pi(\alpha)$ under the map $H^1(X', \mathbb{R}) \rightarrow \bigoplus_{i=1}^n H^1(\Sigma_i, \mathbb{R})$ of (16) is $([\gamma_1], \dots, [\gamma_n])$. There exist unique $T_i \in C^\infty(\Sigma_i \times (0, \epsilon))$ for $i = 1, \dots, n$ such that

$$\phi_i^*(\alpha) = \pi_i^*(\gamma_i) + dT_i \quad \text{on } \Sigma_i \times (0, \epsilon) \text{ for } i = 1, \dots, n, \text{ and} \quad (22)$$

$$\begin{aligned} \nabla^k T_i(\sigma, r) = O(r^{\lambda_i - k}) & \quad \text{as } r \rightarrow 0, \text{ for all } k \geq 0 \text{ and} \\ \lambda_i \in (0, \nu_i) \text{ with } (0, \lambda_i] \cap \mathcal{D}_{\Sigma_i} = \emptyset. & \quad (23) \end{aligned}$$

- (b) Suppose $\gamma_i \in C^\infty(T^*\Sigma_i)$ with $d\gamma_i = d^*\gamma_i = 0$ for $i = 1, \dots, n$, and the image of $([\gamma_1], \dots, [\gamma_n])$ under $\bigoplus_{i=1}^n H^1(\Sigma_i, \mathbb{R}) \rightarrow H_{\text{cs}}^2(X', \mathbb{R})$ in (16) is $[\beta]$ for some exact 2-form β on X' supported on $X' \setminus (S_1 \cup \dots \cup S_n)$. Then there exists $\alpha \in C^\infty(T^*X')$ with $d\alpha = \beta$, $d^*\alpha = 0$ and $|\nabla^k \alpha| = O(\rho^{-1-k})$ for $k \geq 0$, such that (22) and (23) hold for $T_i \in C^\infty(\Sigma_i \times (0, \epsilon))$.

- (c) Let $f \in C^\infty(X')$ with $|\nabla^k f| = O(\rho^{\nu-2-k})$ for $k \geq 0$ and $\int_{X'} f dV = 0$. Then there exists a unique exact 1-form α on X' with $d^*\alpha = f$ and $|\nabla^k \alpha| = O(\rho^{-1-k})$ for $k \geq 0$, such that (22) and (23) hold for $\gamma_i = 0$ and $T_i \in C^\infty(\Sigma_i \times (0, \epsilon))$.

Proof. Clearly $Y_{X'}$ is a vector space and π is linear. We must show that π is injective and surjective. Suppose $\alpha \in Y_{X'}$ and $[\alpha] = 0 \in H^1(X', \mathbb{R})$. Then $\alpha = d\theta$ for $\theta \in C^\infty(X')$. Using $\nabla^k \theta = \nabla^{k-1} \alpha$ for $k > 0$, and integrating $|\alpha| = O(\rho^{-1})$ to estimate $|\theta|$, gives

$$\theta = O(1 + |\log \rho|) \quad \text{and} \quad \nabla^k \theta = O(\rho^{-k}) \quad \text{for all } k \geq 0. \quad (24)$$

Now $d^*d\theta = 0$. Multiplying this by θ and integrating over X' by parts, using (24) and arguing as in Lemma 2.13, we can show that $\int_{X'} |d\theta|^2 dV_g = 0$. Thus $\alpha = d\theta = 0$, so if $[\alpha] = 0$ then $\alpha = 0$, and π is injective.

Next we show π is surjective, and at the same time prove part (a). Let $\eta \in H^1(X', \mathbb{R})$. By Hodge theory there exists a unique $\gamma_i \in C^\infty(T^*\Sigma_i)$ with $d\gamma_i = d^*\gamma_i = 0$ for $i = 1, \dots, n$, such that the image of η under $H^1(X', \mathbb{R}) \rightarrow \bigoplus_{i=1}^n H^1(\Sigma_i, \mathbb{R})$ in (16) is $([\gamma_1], \dots, [\gamma_n])$.

Choose a smooth, closed 1-form γ on X' with $[\gamma] = \eta$ and $\phi_i^*(\gamma) = \pi_i^*(\gamma_i)$, where $\pi_i : \Sigma_i \times (0, \epsilon) \rightarrow \Sigma_i$ is the obvious projection. Note that the condition $d(x_i, x_j) > 2\epsilon$ for $i \neq j$ in part (b) of Definition 2.1 implies that the closures $\bar{S}_1, \dots, \bar{S}_n$ are disjoint in X , and using this we can show that γ exists.

As $\phi_i^*(\gamma) = \pi_i^*(\gamma_i)$ we can regard γ as independent of r on $S_i \cong \Sigma_i \times (0, \epsilon) \ni (\sigma, r)$. Since the metric g on $\Sigma_i \times (0, \epsilon)$ is approximately the cone metric by (1), we find that $|\gamma| = O(\rho^{-1})$, and more generally

$$|\nabla^k \gamma| = O(\rho^{-1-k}) \quad \text{for } k \geq 0. \quad (25)$$

This suggests that $d^*\gamma = O(\rho^{-2})$. However, because $d^*\gamma_i = 0$ on Σ_i we have $d^*(\pi_i^*(\gamma_i)) = 0$ on $\Sigma_i \times (0, \epsilon)$, computing d^* w.r.t. the cone metric on $\Sigma_i \times (0, \epsilon)$. Since g approximates the cone metric on S_i , calculation using (1) shows that

$$|\nabla^k(d^*\gamma)| = O(\rho^{\nu-2-k}) \quad \text{for } k \geq 0, \quad (26)$$

where $\nu = (\nu_1, \dots, \nu_n)$. Thus $d^*\gamma \in C_{\nu-2}^\infty(X')$.

Choose $\lambda_i \in (0, \nu_i)$ with $(0, \lambda_i] \cap \mathcal{D}_{\Sigma_i} = \emptyset$ for $i = 1, \dots, n$. Let $p > 1$ and $k \geq 2$. Then (26) implies that $d^*\gamma \in L_{k-2, \lambda-2}^p(X')$. Integrating by parts as in (10) shows that $\int_{X'} (d^*\gamma) dV_g = \int_{X'} 1(d^*\gamma) dV_g = \int_{X'} (d1) \cdot \gamma dV_g = 0$, using $m > 2$ and (25) for $k = 0, 1$.

Thus $d^*\gamma$ lies in the r.h.s. of (19), and by Proposition 2.18 there exist unique $u \in L_{k, \lambda}^p(X')$ and $v \in \mathcal{K}_{X'}$ with $d^*\gamma = \Delta u + \Delta v$. As $\Delta u = d^*\gamma - \Delta v$ and $d^*\gamma, \Delta v \in L_{k-2, \lambda-2}^p(X')$ for all $k \geq 2$, Theorem 2.10 shows that $u \in L_{k, \lambda}^p(X')$ for all $k \geq 2$, so that $u \in C_\lambda^\infty(X')$ by Theorem 2.9.

Define $\alpha = \gamma - du - dv$. Then $d\alpha = 0$ and $d^*\alpha = d^*\gamma - \Delta u - \Delta v = 0$. As γ satisfies (25), $u \in C_\lambda^\infty(X')$ with $\lambda_i > 0$ and dv is compactly-supported, we see that $|\nabla^k \alpha| = O(\rho^{-1-k})$ for all $k \geq 0$. Hence $\alpha \in Y_{X'}$ and $[\alpha] = [\gamma] = \eta$, so $\pi : Y_{X'} \rightarrow H^1(X', \mathbb{R})$ is surjective. This proves the first part of the theorem. Define $T_i = -\phi_i^*(u)$. Then (22) holds as $\alpha = \gamma - du - dv$, $dv = 0$ on S_i and $\phi_i^*(\gamma) = \pi_i^*(\gamma_i)$, and (23) holds as $u \in C_\lambda^\infty(X')$ whenever $\lambda_i \in (0, \nu_i)$ with $(0, \lambda_i] \cap \mathcal{D}_{\Sigma_i} = \emptyset$. This proves part (a).

For part (b), let γ_i and β be as in the theorem. Choose $\gamma \in C^\infty(T^*X')$ with $\phi_i^*(\gamma) = \pi_i^*(\gamma_i)$ for $i = 1, \dots, n$. Then $d\gamma$ is supported in $X' \setminus (S_1 \cup \dots \cup S_n)$ as

$d\gamma_i = 0$. By construction $[d\gamma] \in H_{\text{cs}}^2(X', \mathbb{R})$ is the image of $([\gamma_1], \dots, [\gamma_n])$ under $\bigoplus_{i=1}^n H^1(\Sigma_i, \mathbb{R}) \rightarrow H_{\text{cs}}^2(X', \mathbb{R})$, so $[d\gamma] = [\beta]$.

Thus $\beta = d\gamma + d\delta$, for some compactly-supported 1-form δ on X' . Since β, γ are supported on $X' \setminus (S_1 \cup \dots \cup S_n)$ we can choose δ supported there too. As in part (a) there exist unique $u \in C_{\lambda}^{\infty}(X')$ and $v \in \mathcal{K}_{X'}$ with $\Delta u + \Delta v = d^*(\gamma + \delta)$. Then $\alpha = \gamma + \delta - du - dv$ and $T_i = -\phi_i^*(u)$ satisfy the conditions in (b).

For part (c), choose $\lambda_i \in (0, \nu_i)$ with $(0, \lambda_i] \cap \mathcal{D}_{\Sigma_i} = \emptyset$ for $i = 1, \dots, n$. Let $p > 1$ and $k \geq 2$. Then $|\nabla^j f| = O(\rho^{\nu-2-j})$ and $\int_{X'} f dV = 0$ imply that f lies in the r.h.s. of (19). So by Proposition 2.18 there exist unique $u \in L_{k, \lambda}^p(X')$ and $v \in \mathcal{K}_{X'}$ with $\Delta u + \Delta v = d^*f$. As u, v are independent of $k \geq 2$, Theorem 2.9 shows that $u \in C_{\lambda}^{\infty}(X')$. Thus $\alpha = du + dv$ and $T_i = \phi_i^*(u)$ satisfy the conditions in (c). This completes the proof. \square

3 Special Lagrangian geometry

We now introduce special Lagrangian submanifolds (SL m -folds) in two different geometric contexts. First, in §3.1, we define SL m -folds in \mathbb{C}^m . Then §3.2 discusses SL m -folds in *almost Calabi–Yau m -folds*, compact Kähler manifolds with a holomorphic volume form which generalize Calabi–Yau manifolds.

Then §3.3 defines *special Lagrangian m -folds with conical singularities* in almost Calabi–Yau m -folds, which are the subject of the paper. Some references for §3.1–§3.2 are Harvey and Lawson [7] and the author [10].

3.1 Special Lagrangian submanifolds in \mathbb{C}^m

We begin by defining *calibrations* and *calibrated submanifolds*, following Harvey and Lawson [7].

Definition 3.1 Let (M, g) be a Riemannian manifold. An *oriented tangent k -plane* V on M is a vector subspace V of some tangent space $T_x M$ to M with $\dim V = k$, equipped with an orientation. If V is an oriented tangent k -plane on M then $g|_V$ is a Euclidean metric on V , so combining $g|_V$ with the orientation on V gives a natural *volume form* vol_V on V , which is a k -form on V .

Now let φ be a closed k -form on M . We say that φ is a *calibration* on M if for every oriented k -plane V on M we have $\varphi|_V \leq \text{vol}_V$. Here $\varphi|_V = \alpha \cdot \text{vol}_V$ for some $\alpha \in \mathbb{R}$, and $\varphi|_V \leq \text{vol}_V$ if $\alpha \leq 1$. Let N be an oriented submanifold of M with dimension k . Then each tangent space $T_x N$ for $x \in N$ is an oriented tangent k -plane. We say that N is a *calibrated submanifold* if $\varphi|_{T_x N} = \text{vol}_{T_x N}$ for all $x \in N$.

It is easy to show that calibrated submanifolds are automatically *minimal submanifolds* [7, Th. II.4.2]. Here is the definition of special Lagrangian submanifolds in \mathbb{C}^m , taken from [7, §III].

Definition 3.2 Let \mathbb{C}^m have complex coordinates (z_1, \dots, z_m) , and define a

metric g' , a real 2-form ω' and a complex m -form Ω' on \mathbb{C}^m by

$$\begin{aligned} g' &= |dz_1|^2 + \cdots + |dz_m|^2, & \omega' &= \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + \cdots + dz_m \wedge d\bar{z}_m), \\ & & \text{and } \Omega' &= dz_1 \wedge \cdots \wedge dz_m. \end{aligned} \quad (27)$$

Then $\operatorname{Re} \Omega'$ and $\operatorname{Im} \Omega'$ are real m -forms on \mathbb{C}^m . Let L be an oriented real submanifold of \mathbb{C}^m of real dimension m . We say that L is a *special Lagrangian submanifold* of \mathbb{C}^m , or *SL m -fold* for short, if L is calibrated with respect to $\operatorname{Re} \Omega'$, in the sense of Definition 3.1.

Harvey and Lawson [7, Cor. III.1.11] give the following alternative characterization of special Lagrangian submanifolds:

Proposition 3.3 *Let L be a real m -dimensional submanifold of \mathbb{C}^m . Then L admits an orientation making it into an SL submanifold of \mathbb{C}^m if and only if $\omega'|_L \equiv 0$ and $\operatorname{Im} \Omega'|_L \equiv 0$.*

Thus SL m -folds are *Lagrangian submanifolds* in $\mathbb{R}^{2m} \cong \mathbb{C}^m$ satisfying the extra condition that $\operatorname{Im} \Omega'|_L \equiv 0$, which is how they get their name.

3.2 Almost Calabi–Yau m -folds and SL m -folds

We shall define special Lagrangian submanifolds not just in Calabi–Yau manifolds, as usual, but in the much larger class of *almost Calabi–Yau manifolds*.

Definition 3.4 Let $m \geq 2$. An *almost Calabi–Yau m -fold* is a quadruple (M, J, ω, Ω) such that (M, J) is a compact m -dimensional complex manifold, ω is the Kähler form of a Kähler metric g on M , and Ω is a non-vanishing holomorphic $(m, 0)$ -form on M .

We call (M, J, ω, Ω) a *Calabi–Yau m -fold* if in addition ω and Ω satisfy

$$\omega^m/m! = (-1)^{m(m-1)/2} (i/2)^m \Omega \wedge \bar{\Omega}. \quad (28)$$

Then for each $x \in M$ there exists an isomorphism $T_x M \cong \mathbb{C}^m$ that identifies g_x, ω_x and Ω_x with the flat versions g', ω', Ω' on \mathbb{C}^m in (27). Furthermore, g is Ricci-flat and its holonomy group is a subgroup of $\operatorname{SU}(m)$.

This is not the usual definition of a Calabi–Yau manifold, but is essentially equivalent to it.

Definition 3.5 Let (M, J, ω, Ω) be an almost Calabi–Yau m -fold, and N a real m -dimensional submanifold of M . We call N a *special Lagrangian submanifold*, or *SL m -fold* for short, if $\omega|_N \equiv \operatorname{Im} \Omega|_N \equiv 0$. It easily follows that $\operatorname{Re} \Omega|_N$ is a nonvanishing m -form on N . Thus N is orientable, with a unique orientation in which $\operatorname{Re} \Omega|_N$ is positive.

Again, this is not the usual definition of SL m -fold, but is essentially equivalent to it. Suppose (M, J, ω, Ω) is an almost Calabi–Yau m -fold, with metric g . Let $\psi : M \rightarrow (0, \infty)$ be the unique smooth function such that

$$\psi^{2m} \omega^m/m! = (-1)^{m(m-1)/2} (i/2)^m \Omega \wedge \bar{\Omega}, \quad (29)$$

and define \tilde{g} to be the conformally equivalent metric $\psi^2 g$ on M . Then $\text{Re } \Omega$ is a *calibration* on the Riemannian manifold (M, \tilde{g}) , and SL m -folds N in (M, J, ω, Ω) are calibrated with respect to it, so that they are minimal with respect to \tilde{g} .

If M is a Calabi–Yau m -fold then $\psi \equiv 1$ by (28), so $\tilde{g} = g$, and an m -submanifold N in M is special Lagrangian if and only if it is calibrated w.r.t. $\text{Re } \Omega$ on (M, g) , as in Definition 3.2. This recovers the usual definition of special Lagrangian m -folds in Calabi–Yau m -folds.

3.3 Special Lagrangian m -folds with conical singularities

Now we can define *conical singularities* of SL m -folds.

Definition 3.6 Let (M, J, ω, Ω) be an almost Calabi–Yau m -fold for $m > 2$, and define $\psi : M \rightarrow (0, \infty)$ as in (29). Suppose X is a compact singular SL m -fold in M with singularities at distinct points $x_1, \dots, x_n \in X$, and no other singularities.

Fix isomorphisms $v_i : \mathbb{C}^m \rightarrow T_{x_i} M$ for $i = 1, \dots, n$ such that $v_i^*(\omega) = \omega'$ and $v_i^*(\Omega) = \psi(x_i)^m \Omega'$, where ω', Ω' are as in (27). Let C_1, \dots, C_n be SL cones in \mathbb{C}^m with isolated singularities at 0. For $i = 1, \dots, n$ let $\Sigma_i = C_i \cap \mathcal{S}^{2m-1}$, and let $\mu_i \in (2, 3)$ with

$$(2, \mu_i] \cap \mathcal{D}_{\Sigma_i} = \emptyset, \quad \text{where } \mathcal{D}_{\Sigma_i} \text{ is defined in (4)}. \quad (30)$$

Then we say that X has a *conical singularity* at x_i , with *rate* μ_i and *cone* C_i for $i = 1, \dots, n$, if the following holds.

By Darboux’s Theorem [19, Th. 3.15] there exist embeddings $\Upsilon_i : B_R \rightarrow M$ for $i = 1, \dots, n$ satisfying $\Upsilon_i(0) = x_i$, $d\Upsilon_i|_0 = v_i$ and $\Upsilon_i^*(\omega) = \omega'$, where B_R is the open ball of radius R about 0 in \mathbb{C}^m for some small $R > 0$. Define $\iota_i : \Sigma_i \times (0, R) \rightarrow B_R$ by $\iota_i(\sigma, r) = r\sigma$ for $i = 1, \dots, n$.

Define $X' = X \setminus \{x_1, \dots, x_n\}$. Then there should exist a compact subset $K \subset X'$ such that $X' \setminus K$ is a union of open sets S_1, \dots, S_n with $S_i \subset \Upsilon_i(B_R)$, whose closures $\bar{S}_1, \dots, \bar{S}_n$ are disjoint in X . For $i = 1, \dots, n$ and some $R' \in (0, R]$ there should exist a smooth $\phi_i : \Sigma_i \times (0, R') \rightarrow B_R$ such that $\Upsilon_i \circ \phi_i : \Sigma_i \times (0, R') \rightarrow M$ is a diffeomorphism $\Sigma_i \times (0, R') \rightarrow S_i$, and

$$|\nabla^k(\phi_i - \iota_i)| = O(r^{\mu_i - 1 - k}) \quad \text{as } r \rightarrow 0 \text{ for } k = 0, 1. \quad (31)$$

Here $\nabla, |\cdot|$ are computed using the cone metric $\iota_i^*(g')$ on $\Sigma_i \times (0, R')$.

We will show in Theorem 5.1 that if (31) holds for $k = 0, 1$ then we can choose a natural ϕ_i for which (31) holds for all $k \geq 0$. Thus the number of derivatives required in (31) makes little difference, and we choose $k = 0, 1$ to make the definition as weak as possible. We will also show in Theorem 5.5 that if (31) holds for *some* choice of rates μ_i satisfying the conditions of the definition, then it holds for *all* choices of rates μ_i satisfying the conditions, for the ϕ_i in Theorem 5.1. Thus the choice of rates μ_i again makes little difference.

We restrict to $m > 2$ in Definition 3.6 for two reasons. Firstly, the only SL cones C in \mathbb{C}^2 are finite unions of SL planes \mathbb{R}^2 in \mathbb{C}^2 intersecting only at

0. Therefore, SL 2-folds with conical singularities are actually *nonsingular* as immersed 2-folds, so there is really no point in studying them. Secondly, parts of the analysis in §2 do not hold when $m = 2$, in particular Proposition 2.4 and Theorem 2.15. Therefore, in the rest of the paper we shall suppose $m > 2$.

Here are the reasons for the conditions on μ_i in Definition 3.6:

- We need $\mu_i > 2$, or else (31) does not force X to approach C_i near x_i .
- The definition involves a choice of $\Upsilon_i : B_R \rightarrow M$. If we replace Υ_i by a different choice $\tilde{\Upsilon}_i$ then we should replace ϕ_i by $\tilde{\phi}_i = (\tilde{\Upsilon}_i^{-1} \circ \Upsilon_i) \circ \phi_i$ near 0 in B_R . Calculation shows that as $\Upsilon_i, \tilde{\Upsilon}_i$ agree up to second order, we have $|\nabla^k(\tilde{\phi}_i - \phi_i)| = O(r^{2-k})$.

Therefore if $\mu_i \leq 3$ then (31) for ϕ_i is equivalent to (31) for $\tilde{\phi}_i$, and the definition is independent of the choice of Υ_i . However, if $\mu_i > 3$ then the definition would depend on the choice of Υ_i , which we do not want. We also exclude $\mu_i = 3$ for technical reasons, to prevent $O(r^{2-k})$ terms from Υ_i dominating $\nabla^k(\phi_i - \iota_i)$, so we require $\mu_i < 3$.

- If we omit condition (30) then the proof of Theorem 5.5 below would fail. Also, extra obstructions would appear in the deformation theory of compact SL m -folds with conical singularities studied in [11].

To avoid proliferation of indices we have chosen R, R' above to be independent of $i = 1, \dots, n$. This is valid as we may take $R = \min(R_1, \dots, R_n)$, and so on. We will do this without remark for other variables in later proofs.

4 Lagrangian Neighbourhood Theorems

Let N be a real m -manifold. Then its tangent bundle T^*N has a canonical symplectic form $\hat{\omega}$, defined as follows. Let (x_1, \dots, x_m) be local coordinates on N . Extend them to local coordinates $(x_1, \dots, x_m, y_1, \dots, y_m)$ on T^*N such that (x_1, \dots, y_m) represents the 1-form $y_1 dx_1 + \dots + y_m dx_m$ in $T^*_{(x_1, \dots, x_m)}N$. Then $\hat{\omega} = dx_1 \wedge dy_1 + \dots + dx_m \wedge dy_m$.

Identify N with the zero section in T^*N . Then N is a *Lagrangian submanifold* of T^*N . The *Lagrangian Neighbourhood Theorem* [19, Th. 3.33], due to Weinstein [26], shows that any compact Lagrangian submanifold N in a symplectic manifold looks locally like the zero section in T^*N .

Theorem 4.1 *Let (M, ω) be a symplectic manifold and $N \subset M$ a compact Lagrangian submanifold. Then there exists an open tubular neighbourhood U of the zero section N in T^*N , and an embedding $\Phi : U \rightarrow M$ with $\Phi|_N = \text{id} : N \rightarrow N$ and $\Phi^*(\omega) = \hat{\omega}$, where $\hat{\omega}$ is the canonical symplectic structure on T^*N .*

We shall need the following variation of this, which may be deduced from the proof of a result of Weinstein [26, Th. 7.1] on *Lagrangian foliations*.

Theorem 4.2 *Let (M, ω) be a $2m$ -dimensional symplectic manifold and $N \subset M$ an embedded m -dimensional submanifold. Suppose $\{L_x : x \in N\}$ is a smooth family of embedded, noncompact Lagrangian submanifolds in M parametrized by $x \in N$, such that for each $x \in N$ we have $x \in L_x$, and $T_x L_x \cap T_x N = \{0\}$.*

*Then there exists an open neighbourhood U of the zero section N in T^*N such that the fibres of the natural projection $\pi : U \rightarrow N$ are connected, and a unique embedding $\Phi : U \rightarrow M$ with $\Phi(\pi^{-1}(x)) \subset L_x$ for each $x \in N$, $\Phi|_N = \text{id} : N \rightarrow N$ and $\Phi^*(\omega) = \hat{\omega} + \pi^*(\omega|_N)$, where $\hat{\omega}$ is the canonical symplectic structure on T^*N .*

In particular, if N is compact and Lagrangian in Theorem 4.2 then making U smaller we can suppose it is a tubular neighbourhood, and then U, Φ satisfy the conditions of Theorem 4.1. The important point is that in Theorem 4.1, the subsets $L_x = \Phi(\pi^{-1}(x))$ form a smooth family of noncompact Lagrangian submanifolds of M , and L_x intersects N transversely at x . Theorem 4.2 says that any such family $\{L_x : x \in N\}$ locally comes from a unique Lagrangian neighbourhood map Φ .

The goal of this section is to extend Theorem 4.1 to SL cones C in \mathbb{C}^m and to SL m -folds X with conical singularities in an almost Calabi–Yau m -fold M . As this involves *noncompact* Lagrangian m -folds C', X' , we need to impose suitable asymptotic conditions on the Lagrangian neighbourhood at the noncompact ends of C', X' . Throughout we suppose $m > 2$.

4.1 Dilation-equivariant neighbourhoods of cones

We first extend Theorem 4.1 to SL cones in \mathbb{C}^m . Most of the theorem is notation, not requiring proof. We have to extend from a compact N to the noncompact $\Sigma \times (0, \infty)$, and include equivariance properties under dilations on \mathbb{C}^m .

Theorem 4.3 *Let C be an SL cone in \mathbb{C}^m with isolated singularity at 0, and set $\Sigma = C \cap S^{2m-1}$. Define $\iota : \Sigma \times (0, \infty) \rightarrow \mathbb{C}^m$ by $\iota(\sigma, r) = r\sigma$, with image $C \setminus \{0\}$. For $\sigma \in \Sigma$, $\tau \in T_\sigma^* \Sigma$, $r \in (0, \infty)$ and $u \in \mathbb{R}$, let (σ, r, τ, u) represent the point $\tau + u dr$ in $T_{(\sigma, r)}^*(\Sigma \times (0, \infty))$. Identify $\Sigma \times (0, \infty)$ with the zero section $\tau = u = 0$ in $T^*(\Sigma \times (0, \infty))$. Define an action of $(0, \infty)$ on $T^*(\Sigma \times (0, \infty))$ by*

$$t : (\sigma, r, \tau, u) \mapsto (\sigma, tr, t^2\tau, tu) \quad \text{for } t \in (0, \infty), \quad (32)$$

so that $t^(\hat{\omega}) = t^2\hat{\omega}$, for $\hat{\omega}$ the canonical symplectic structure on $T^*(\Sigma \times (0, \infty))$.*

Then there exists an open neighbourhood U_C of $\Sigma \times (0, \infty)$ in $T^(\Sigma \times (0, \infty))$ invariant under (32) given by*

$$U_C = \{(\sigma, r, \tau, u) \in T^*(\Sigma \times (0, \infty)) : |(\tau, u)| < 2\zeta r\} \quad \text{for some } \zeta > 0, \quad (33)$$

where $|\cdot|$ is calculated using the cone metric $\iota^(g')$ on $\Sigma \times (0, \infty)$, and an embedding $\Phi_C : U_C \rightarrow \mathbb{C}^m$ with $\Phi_C|_{\Sigma \times (0, \infty)} = \iota$, $\Phi_C^*(\omega') = \hat{\omega}$ and $\Phi_C \circ t = t\Phi_C$ for all $t > 0$, where t acts on U_C as in (32) and on \mathbb{C}^m by multiplication.*

Proof. For each $(\sigma, r) \in \Sigma \times (0, \infty)$, define $L_{(\sigma, r)}$ to be the unique affine subspace \mathbb{R}^m in \mathbb{C}^m passing through $r\sigma$ and normal to C there. Then $L_{(\sigma, r)}$ is a *Lagrangian plane* in \mathbb{C}^m , as C is Lagrangian. This defines a family $\{L_{(\sigma, r)} : (\sigma, r) \in \Sigma \times (0, \infty)\}$ of Lagrangian submanifolds of \mathbb{C}^m with $r\sigma \in L_{(\sigma, r)}$ and $T_{r\sigma}L_{(\sigma, r)} \cap T_{r\sigma}C' = \{0\}$. We can therefore apply Theorem 4.2.

We have defined dilation actions of \mathbb{R}_+ on $T^*(\Sigma \times (0, \infty))$ and \mathbb{C}^m , and it is easy to see that we may choose U to be *dilation-invariant*, and then Φ is *dilation-equivariant*, in the sense that $\Phi \circ t = t \Phi$. It remains to show that we can take U to be U_C in (33) for some $\zeta > 0$. This is true if $U_C \subset U$. As U, U_C are both dilation-invariant, it is enough for $U_C \subset U$ to hold on the hypersurface $r = 1$, that is, over the compact subset $\Sigma \times \{1\}$. The existence of some small $\zeta > 0$ with $U_C \subset U$ then follows by compactness. \square

Theorem 4.3 can also be proved by applying the *Legendrian Neighbourhood Theorem* to Σ in \mathcal{S}^{2m-1} . This is the analogue of Theorem 4.1 for Legendrian submanifolds in contact manifolds, and is described briefly in [19, p. 107].

4.2 Distinguished coordinates on X' near x_i

We shall use Theorem 4.3 to construct a particular choice of ϕ_i in Definition 3.6.

Theorem 4.4 *Let (M, J, ω, Ω) , $\psi, X, n, x_i, v_i, C_i, \Sigma_i, \mu_i, R, \Upsilon_i$ and ι_i be as in Definition 3.6. Theorem 4.3 gives $\zeta > 0$, neighbourhoods U_{C_i} of $\Sigma_i \times (0, \infty)$ in $T^*(\Sigma_i \times (0, \infty))$ and embeddings $\Phi_{C_i} : U_{C_i} \rightarrow \mathbb{C}^m$ for $i = 1, \dots, n$.*

Then for sufficiently small $R' \in (0, R]$ there exist unique closed 1-forms η_i on $\Sigma_i \times (0, R')$ for $i = 1, \dots, n$ written $\eta_i(\sigma, r) = \eta_i^1(\sigma, r) + \eta_i^2(\sigma, r)dr$ for $\eta_i^1(\sigma, r) \in T_\sigma^\Sigma_i$ and $\eta_i^2(\sigma, r) \in \mathbb{R}$, and satisfying $|\eta_i(\sigma, r)| < \zeta r$ and*

$$|\nabla^k \eta_i| = O(r^{\mu_i - 1 - k}) \quad \text{as } r \rightarrow 0 \text{ for } k = 0, 1, \quad (34)$$

computing $\nabla, |\cdot|$ using the cone metric $\iota_i^(g')$, such that the following holds.*

Define $\phi_i : \Sigma_i \times (0, R') \rightarrow B_R$ by $\phi_i(\sigma, r) = \Phi_{C_i}(\sigma, r, \eta_i^1(\sigma, r), \eta_i^2(\sigma, r))$. Then $\Upsilon_i \circ \phi_i : \Sigma_i \times (0, R') \rightarrow M$ is a diffeomorphism $\Sigma_i \times (0, R') \rightarrow S_i$ for open sets S_1, \dots, S_n in X' with $\bar{S}_1, \dots, \bar{S}_n$ disjoint, and $K = X' \setminus (S_1 \cup \dots \cup S_n)$ is compact. Also ϕ_i satisfies (31), so that R', ϕ_i, S_i, K satisfy Definition 3.6.

Proof. As X has a conical singularity at x_i it follows from (31) that near 0 in B_R we can write $\Upsilon_i^*(X')$ as the image under Φ_{C_i} of the graph of a smooth 1-form η_i on $\Sigma_i \times (0, R')$ for small $R' \in (0, R]$. This just means that $\Upsilon_i^*(X')$ intersects the Lagrangian ball $\Phi_{C_i}(T_{(\sigma, r)}^*(\Sigma_i \times (0, \infty)) \cap U_{C_i})$ transversely in exactly one point for $(\sigma, r) \in \Sigma_i \times (0, R')$, and we define η_i such that this point is $\Phi_{C_i}(\eta_i(\sigma, r))$. Since $\omega|_{X'} \equiv 0$ and $\Upsilon_i^*(\omega) = \omega', \Phi_{C_i}^*(\omega') = \hat{\omega}$ we see that $\hat{\omega}$ restricted to the graph of η_i in $T^*(\Sigma_i \times (0, R'))$ is zero. By a well-known fact in symplectic geometry, this implies that η_i is *closed*.

Now define $\phi_i : \Sigma_i \times (0, R') \rightarrow B_R$ by $\phi_i(\sigma, r) = \Phi_{C_i}(\sigma, r, \eta_i^1(\sigma, r), \eta_i^2(\sigma, r))$. Then ϕ_i is an embedding, and by definition $\Upsilon_i \circ \phi_i$ maps $\Sigma_i \times (0, R') \rightarrow X'$.

Define $S_i = \Upsilon_i \circ \phi_i(\Sigma_i \times (0, R'))$ and $K = X \setminus (S_1 \cup \dots \cup S_n)$. Making R' smaller if necessary we can arrange that $\bar{S}_1, \dots, \bar{S}_n$ are disjoint. Then $\Upsilon_i \circ \phi_i$ is a diffeomorphism $\Sigma_i \times (0, R') \rightarrow S_i$, and S_i is an open set in X' , and K is the complement of open neighbourhoods of x_1, \dots, x_n in the compact space X , so K is compact.

We have not yet shown that ϕ_1, \dots, ϕ_n satisfy (31). By Definition 3.6 there must exist some ϕ'_1, \dots, ϕ'_n satisfying the conditions, including (31). Then ϕ_1, \dots, ϕ_n are obtained from ϕ'_1, \dots, ϕ'_n by a kind of *projection*. What happens is that $\phi_i(\sigma, r) = \phi'_i(\sigma', r')$ if $(\sigma, r), (\sigma', r')$ are close in $\Sigma_i \times (0, R')$ and $\phi'_i(\sigma', r')$ lies in the affine normal subspace to C_i at (σ, r) .

For small $R'' \in (0, R']$ define $\Xi_i : \Sigma_i \times (0, R'') \rightarrow \Sigma_i(0, R')$ by $\Xi_i(\sigma', r') = (\sigma, r)$. Then (31) for ϕ'_i implies that

$$\nabla^k(\Xi_i - \text{id}) = O((r')^{\mu_i - 1 - k}) \quad \text{as } r' \rightarrow 0 \text{ for } k = 0, 1. \quad (35)$$

But $\phi_i = \phi'_i \circ \Xi_i^{-1}$ for small r , so combining (35) and (31) for ϕ'_i implies (31) for ϕ_i .

Equation (31) for ϕ_i and properties of Φ_{C_i} easily imply (34). Finally, as $\mu_i > 2$ by Definition 3.6, equation (34) implies that $|\eta_i| = o(r)$ for small r . Therefore, making R' smaller if necessary, we can suppose that $|\eta_i(\sigma, r)| < \zeta r$ for $(\sigma, r) \in \Sigma_i \times (0, R')$. \square

We can integrate the 1-forms η_i in Theorem 4.4.

Lemma 4.5 *In Theorem 4.4 we have $\eta_i = dA_i$ for $i = 1, \dots, n$, where $A_i : \Sigma_i \times (0, R') \rightarrow \mathbb{R}$ is given by $A_i(\sigma, r) = \int_0^r \eta_i^2(\sigma, s) ds$ and satisfies*

$$|\nabla^k A_i| = O(r^{\mu_i - k}) \quad \text{as } r \rightarrow 0 \text{ for } k = 0, 1, 2, \quad (36)$$

computing ∇ and $|\cdot|$ using the cone metric $\iota^*(g')$.

Proof. From (34) we deduce that $|\nabla^k \eta_i^2| = O(r^{\mu_i - 1 - k})$ as $r \rightarrow 0$ for $k = 0, 1$. Integrating this and using $\mu_i > 2$ shows that $A_i(\sigma, r) = \int_0^r \eta_i^2(\sigma, s) ds$ is well-defined and (36) holds for $k = 0, 1$. The dr component in dA_i is η_i^2 , so that $\eta_i - dA_i$ is a closed 1-form on $\Sigma_i \times (0, R')$ with no dr component, and is therefore independent of r . But (34) for $k = 0$ and (36) for $k = 1$ imply that $\eta_i - dA_i = O(r^{\mu_i - 1})$ in the cone metric on $\Sigma_i \times (0, R')$, so $\eta_i - dA_i = O(r^{\mu_i - 2})$ in the cylinder metric, and taking the limit $r \rightarrow 0$ gives $\eta_i - dA_i = 0$ as $\mu_i > 2$. Hence $\eta_i = dA_i$, and (34) for $k = 1$ then yields (36) for $k = 2$. \square

4.3 A Lagrangian Neighbourhood Theorem for X'

Here is an analogue of Theorem 4.1 for SL m -folds X with conical singularities. We construct a Lagrangian neighbourhood of X' compatible with the distinguished coordinates of Theorem 4.4. The theorem will be an important tool in [11, 12, 13], where we study *deformations* and *desingularizations* of X .

Theorem 4.6 *Suppose (M, J, ω, Ω) is an almost Calabi–Yau m -fold and X a compact SL m -fold in M with conical singularities at x_1, \dots, x_n . Let the notation $\psi, v_i, C_i, \Sigma_i, \mu_i, R, \Upsilon_i$ and ι_i be as in Definition 3.6, and let $\zeta, U_{C_i}, \Phi_{C_i}, R', \eta_i, \eta_i^1, \eta_i^2, \phi_i, S_i$ and K be as in Theorem 4.4.*

*Then making R' smaller if necessary, there exists an open tubular neighbourhood $U_{X'} \subset T^*X'$ of the zero section X' in T^*X' , such that under $d(\Upsilon_i \circ \phi_i) : T^*(\Sigma_i \times (0, R')) \rightarrow T^*X'$ for $i = 1, \dots, n$ we have*

$$(d(\Upsilon_i \circ \phi_i))^*(U_{X'}) = \{(\sigma, r, \tau, u) \in T^*(\Sigma_i \times (0, R')) : |(\tau, u)| < \zeta r\}, \quad (37)$$

and there exists an embedding $\Phi_{X'} : U_{X'} \rightarrow M$ with $\Phi_{X'}|_{X'} = \text{id} : X' \rightarrow X'$ and $\Phi_{X'}^(\omega) = \hat{\omega}$, where $\hat{\omega}$ is the canonical symplectic structure on T^*X' , such that*

$$\Phi_{X'} \circ d(\Upsilon_i \circ \phi_i)(\sigma, r, \tau, u) \equiv \Upsilon_i \circ \Phi_{C_i}(\sigma, r, \tau + \eta_i^1(\sigma, r), u + \eta_i^2(\sigma, r)) \quad (38)$$

for all $i = 1, \dots, n$ and $(\sigma, r, \tau, u) \in T^(\Sigma_i \times (0, R'))$ with $|(\tau, u)| < \zeta r$. Here $|(\tau, u)|$ is computed using the cone metric $\iota_i^*(g')$ on $\Sigma_i \times (0, R')$.*

Proof. Let us regard (37) and (38) as *definitions* of $U_{X'}$ and $\Phi_{X'}$ over the subset S_i of X' for $i = 1, \dots, n$. Since $|\eta_i(\sigma, r)| < \zeta r$ by Theorem 4.4, $|(\tau, u)| < \zeta r$ in (37) and $\Phi_{C_i}(\sigma, r, \tau', u')$ is defined provided $|(\tau', u')| < 2\zeta r$ by (33), we see that $\Phi_{C_i}(\dots)$ is well-defined in (38).

Making R' smaller if necessary, we can ensure that $\Phi_{C_i}(\dots)$ lies in B_R , and so (38) makes sense and $U_{X'}, \Phi_{X'}$ are well-defined over S_i . As $\Upsilon_i^*(\omega) = \omega'$, $\Phi_{C_i}^*(\omega') = \hat{\omega}$ and η_i is closed, it easily follows that $\Phi_{X'}^*(\omega) = \hat{\omega}$ on these regions of $U_{X'}$. Also $\Phi_{X'}$ is an embedding on these regions, as Υ_i and Φ_{C_i} are, and is the identity on each S_i , by definition of ϕ_i in Theorem 4.4. It remains to extend $U_{X'}$ and $\Phi_{X'}$ over the compact subset K in X' .

For $x \in S_i$ define $L_x = \Phi_{X'}(T_x^*X' \cap U_{X'})$, where $U_{X'}, \Phi_{X'}$ are defined over S_i as above. As $\Phi_{X'}$ is an embedding with $\Phi_{X'}^*(\omega) = \hat{\omega}$ we see that L_x is an open Lagrangian ball in M which meets X' transversely at x , and depends smoothly on x . Extend this family $\{L_x : x \in S_i, i = 1, \dots, n\}$ to a family $\{L_x : x \in X'\}$ such that L_x is an open Lagrangian ball in M which meets X' transversely at x , and depends smoothly on x . This is possible by standard symplectic geometry techniques, as the extension is over a compact set K .

Now apply Theorem 4.2 to the family $\{L_x : x \in X'\}$. This gives an open neighbourhood U of X' in T^*X' , and a map $\Phi : U \rightarrow M$ with $\Phi|_{X'} = \text{id} : X' \rightarrow X'$ and $\Phi^*(\omega) = \hat{\omega}$. By the local uniqueness of Φ in Theorem 4.2 we see that Φ and $\Phi_{X'}$ defined above coincide where they are both defined.

Therefore we can take U to be $U_{X'}$ and Φ to be $\Phi_{X'}$ as defined above over S_i , for $i = 1, \dots, n$. Choose an open tubular neighbourhood $U_{X'}$ of X' in U , which coincides with the previous definition of $U_{X'}$ over S_i . This is possible as U is open and it only remains to choose $U_{X'}$ over the compact set K . Let $\Phi_{X'}$ be the restriction of Φ to $U_{X'} \subseteq U$. Then $U_{X'}, \Phi_{X'}$ satisfy all the conditions of the theorem. \square

4.4 Extending to families of almost Calabi–Yau m -folds

In [11, 13] we will study SL m -folds not just in one almost Calabi–Yau m -fold (M, J, ω, Ω) , but in a smooth family of them.

Definition 4.7 Let (M, J, ω, Ω) be an almost Calabi–Yau m -fold. A *smooth family of deformations* of (M, J, ω, Ω) is a connected open set $\mathcal{F} \subset \mathbb{R}^d$ for $d \geq 0$ with $0 \in \mathcal{F}$ called the *base space*, and a smooth family $\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\}$ of almost Calabi–Yau structures on M with $(J^0, \omega^0, \Omega^0) = (J, \omega, \Omega)$.

We now extend the Lagrangian neighbourhood of an SL m -fold X with conical singularities in (M, J, ω, Ω) constructed in Theorem 4.6 to a smooth family of similar neighbourhoods of X in $(M, J^s, \omega^s, \Omega^s)$ for small s . If $\omega^s|_{X'}$ is not exact then we cannot deform X' to a Lagrangian m -fold in (M, ω^s) . Therefore we replace the condition $\Phi_{X'}^*(\omega) = \hat{\omega}$ in Theorem 4.6 by $(\Phi_{X'}^s)^*(\omega^s) = \hat{\omega} + \pi^*(\nu^s)$, where ν^s is a compactly-supported closed 2-form on X' .

Theorem 4.8 Let (M, J, ω, Ω) be an almost Calabi–Yau m -fold and X a compact SL m -fold in M with conical singularities at x_1, \dots, x_n . Let the notation $R, \Upsilon_i, \zeta, \Phi_{C_i}, R', \eta_i, \eta_i^1, \eta_i^2, \phi_i, S_i, K$ be as in Theorem 4.4, and let $U_{X'}, \Phi_{X'}$ be as in Theorem 4.6. Suppose $\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\}$ is a smooth family of deformations of (M, J, ω, Ω) with base space $\mathcal{F} \subset \mathbb{R}^d$. Define $\psi^s : M \rightarrow (0, \infty)$ for $s \in \mathcal{F}$ as in (29), but using ω^s, Ω^s .

Then making R, R' and $U_{X'}$ smaller if necessary, for some connected open $\mathcal{F}' \subseteq \mathcal{F}$ with $0 \in \mathcal{F}'$ and all $s \in \mathcal{F}'$ there exist

- (a) isomorphisms $v_i^s : \mathbb{C}^m \rightarrow T_{x_i}M$ for $i = 1, \dots, n$ with $v_i^0 = v_i$, $(v_i^s)^*(\omega^s) = \omega'$ and $(v_i^s)^*(\Omega) = \psi^s(x_i)^m \Omega'$,
- (b) embeddings $\Upsilon_i^s : B_R \rightarrow M$ for $i = 1, \dots, n$ with $\Upsilon_i^0 = \Upsilon_i$, $\Upsilon_i^s(0) = x_i$, $d\Upsilon_i^s|_0 = v_i^s$ and $(\Upsilon_i^s)^*(\omega^s) = \omega'$,
- (c) a closed 2-form $\nu^s \in C^\infty(\Lambda^2 T^*X')$ supported in $K \subset X'$ with $\nu^0 = 0$, and
- (d) an embedding $\Phi_{X'}^s : U_{X'} \rightarrow M$ with $\Phi_{X'}^0 = \Phi_{X'}$ and $(\Phi_{X'}^s)^*(\omega^s) = \hat{\omega} + \pi^*(\nu^s)$,

all depending smoothly on $s \in \mathcal{F}'$ with

$$\Phi_{X'}^s \circ d(\Upsilon_i \circ \phi_i)(\sigma, r, \tau, u) \equiv \Upsilon_i^s \circ \Phi_{C_i}(\sigma, r, \tau + \eta_i^1(\sigma, r), u + \eta_i^2(\sigma, r)) \quad (39)$$

for all $s \in \mathcal{F}'$, $i = 1, \dots, n$ and $(\sigma, r, \tau, u) \in T^*(\Sigma_i \times (0, R'))$ with $|(\tau, u)| < \zeta r$.

Proof. Clearly, for some open neighbourhood \mathcal{F}_1 of 0 in \mathcal{F} we can extend v_i to a smooth family v_i^s for $s \in \mathcal{F}_1$ satisfying part (a). By a version of Darboux’s Theorem [19, Th. 3.15] for smooth families of symplectic manifolds, making R smaller if necessary, for some open neighbourhood \mathcal{F}_2 of 0 in \mathcal{F}_1 we can extend $\Upsilon_i : B_R \rightarrow M$ to a smooth family of embeddings $\Upsilon_i^s : B_R \rightarrow M$ for $s \in \mathcal{F}_2$ satisfying (b). We then make R' smaller if necessary so that Theorem 4.6 holds.

Next, for some open neighbourhood \mathcal{F}_3 of 0 in \mathcal{F}_2 , choose a smooth family of embeddings $\chi^s : X' \rightarrow M$ for $s \in \mathcal{F}_2$ with $\chi^0 = \text{id} : X' \rightarrow X' \subset M$ such that

$$\chi^s \circ \Upsilon_i \circ \phi_i \equiv \Upsilon_i^s \circ \phi_i \quad \text{on } \Sigma_i \times (0, R') \text{ for } i = 1, \dots, n \text{ and } s \in \mathcal{F}_3. \quad (40)$$

That is, we define χ^s to be $\Upsilon_i^s \circ \Upsilon_i^{-1}$ on S_i for $i = 1, \dots, n$, and then extend χ^s smoothly to an embedding on K , the rest of X' . This is possible for s near 0, as K is compact.

Now define $\nu^s = (\chi^s)^*(\omega^s) \in C^\infty(\Lambda^2 T^* X')$ for $s \in \mathcal{F}_3$. As χ^s, ω^s depend smoothly on s so does ν^s , and as $\chi^0 = \text{id}$ and $\omega^0|_{X'} = \omega|_{X'} = 0$ we have $\nu^0 = 0$. Also, as $(\Upsilon_i^s)^*(\omega^s) = \omega'$ we see from (40) that $\nu^s = (\chi^s)^*(\omega^s)$ is independent of s on S_i , the image of $\Upsilon_i \circ \phi_i$, so that $\nu^s = \nu^0 = 0$ on S_i , and ν^s is supported on $K = X' \setminus (S_1 \cup \dots \cup S_n)$. This gives part (c).

Define $\Phi_{X'}^0 = \Phi_{X'}$. As in the proof of Theorem 4.6, regard (39) as a *definition* of $\Phi_{X'}^s$ on $\pi^*(S_i) \subset U_{X'}$ for $s \in \mathcal{F}_3$. This is well-defined, and depends smoothly on s . Since $(\Upsilon_i^s)^*(\omega^s) = \omega'$ is independent of s , we see from (39) that $(\Phi_{X'}^s)^*(\omega^s)$ is independent of s on $\pi^*(S_i) \subset U_{X'}$. Thus on $\pi^*(S_i) \subset U_{X'}$ we have

$$(\Phi_{X'}^s)^*(\omega^s) = (\Phi_{X'}^0)^*(\omega^0) = \Phi_{X'}^*(\omega) = \hat{\omega} = \hat{\omega} + \pi^*(\nu^s), \quad (41)$$

since $\nu^s \equiv 0$ on S_i . It only remains to extend $\Phi_{X'}^s$ over $\pi^*(K) \subset U_{X'}$ for $s \neq 0$.

Generalizing the proof of Theorem 4.6, for $x \in X'$ and $s = 0$, or for $x \in S_i$ and $s \in \mathcal{F}_3$, define $L_x^s = \Phi_{X'}^s(T_x^* X' \cap U_{X'})$, where $\Phi_{X'}^s$ is defined in these regions as above. Since $(\Phi_{X'}^s)^*(\omega^s) = \hat{\omega} + \pi^*(\nu^s)$ wherever $\Phi_{X'}^s$ is defined and $\hat{\omega} + \pi^*(\nu^s)$ vanishes on $T_x^* X$, we see that L_x^s is an open Lagrangian ball in (M, ω^s) which meets $\chi^s(X')$ transversely at $\chi^s(x)$, and depends smoothly on x, s .

Extend this family $\{L_x^0 : x \in X'\} \cup \{L_x^s : x \in S_i, i = 1, \dots, n, s \in \mathcal{F}_3\}$ to a family $\{L_x^s : x \in X', s \in \mathcal{F}_4\}$ for some open neighbourhood \mathcal{F}_4 of 0 in \mathcal{F}_3 , such that L_x^s is an open Lagrangian ball in (M, ω^s) containing $\chi^s(x)$, which meets $\chi^s(X')$ transversely at $\chi^s(x)$, and depends smoothly on x, s . This is possible by standard symplectic geometry techniques, as the extension is over x in a compact set K and for small s .

Now apply Theorem 4.2 to the family $\{L_x^s : x \in X'\}$ in (M, ω^s) for $s \in \mathcal{F}_4$, replacing X by $\chi^s(X')$. Arguing as Theorem 4.6, we get a tubular neighbourhood $U_{X'}^s$ of X' in $T^* X'$ with $U_{X'}^s \cap \pi^*(S_i) = U_{X'} \cap \pi^*(S_i)$ and an embedding $\Phi_{X'}^s : U_{X'}^s \rightarrow M$ with $\Phi_{X'}^s|_{X'} = \chi^s$, satisfying (39). The formula $\Phi^*(\omega) = \hat{\omega} + \pi^*(\omega|_N)$ in Theorem 4.2 yields $(\Phi_{X'}^s)^*(\omega^s) = \hat{\omega} + \pi^*(\nu^s)$, as we have to prove, since $\omega^s|_{\chi^s(X')} = (\chi^s)^*(\omega^s) = \nu^s$.

As the L_x^s, χ^s and ω^s depend smoothly on s , so does $\Phi_{X'}^s$. It remains to show that we may take the domain $U_{X'}^s \subset T^* X'$ of $\Phi_{X'}^s$ to be $U_{X'}$, independent of s . We can achieve this by making $U_{X'}^s$ smaller if necessary, though keeping it defined by (37) over S_i , and restricting s to a small connected open neighbourhood \mathcal{F}' of 0 in \mathcal{F}_4 . This completes the proof. \square

In the notation of §2.4, the 2-forms ν^s in Theorem 4.8 define classes $[\nu^s]$ in $H_{\text{cs}}^2(X', \mathbb{R})$. We investigate these classes, and the freedom to choose ν^s .

Theorem 4.9 *In the situation of Theorem 4.8, under the isomorphism (18), the class $[\nu^s] \in H_{\text{cs}}^2(X', \mathbb{R})$ is identified with the map $H_2(X, \mathbb{R}) \rightarrow \mathbb{R}$ given by $\gamma \mapsto \iota_*(\gamma) \cdot [\omega^s]$, where $\iota : X \rightarrow M$ is the inclusion, $\iota_* : H_2(X, \mathbb{R}) \rightarrow H_2(M, \mathbb{R})$ the induced map, and $[\omega^s] \in H^2(M, \mathbb{R})$. Thus $[\nu^s]$ depends only on X, M and $[\omega^s] \in H^2(M, \mathbb{R})$.*

Let $V \cong H_{\text{cs}}^2(X', \mathbb{R})$ be a vector space of smooth closed 2-forms on X' supported in K representing $H_{\text{cs}}^2(X', \mathbb{R})$. Then making \mathcal{F}' smaller if necessary, we can choose Υ_i^s, ν^s and $\Phi_{X'}^s$ in Theorem 4.8 so that $\nu^s \in V$ for all $s \in \mathcal{F}'$. In particular, if $[\nu^s] = 0$ in $H_{\text{cs}}^2(X', \mathbb{R})$ then we can choose $\nu^s = 0$.

Proof. As $\nu^s = (\chi^s)^*(\omega^s)$, for $\gamma \in H_2(X, \mathbb{R})$ we have

$$\gamma \cdot [\nu^s] = \chi_*^s(\gamma) \cdot [\omega^s] = \iota_*(\gamma) \cdot [\omega^s],$$

since $\chi^s, \iota : X \rightarrow M$ are isotopic as \mathcal{F}' is connected, and so $\chi_*^s(\gamma) = \iota_*(\gamma)$. This proves the first part, and thus $[\nu^s]$ depends only on X, M and $[\omega^s]$.

Now let $\mathcal{F}', \Upsilon_i^s, \chi^s, \nu^s, \Phi_{X'}^s$ be as in Theorem 4.8, and let V be a vector space of closed 2-forms on X' supported in K representing $H_{\text{cs}}^2(X', \mathbb{R})$. We shall show that making \mathcal{F}' smaller if necessary, we can modify $\chi^s, \nu^s, \Phi_{X'}^s$ to alternative choices $\tilde{\chi}^s, \tilde{\nu}^s, \tilde{\Phi}_{X'}^s$ with $\tilde{\nu}^s \in V$, keeping the same choice of Υ_i^s .

For each $s \in \mathcal{F}'$, let $\tilde{\nu}^s$ be the unique element of V with $[\tilde{\nu}^s] = [\nu^s]$ in $H_{\text{cs}}^2(X', \mathbb{R})$. Then $\tilde{\nu}^s$ depends smoothly on s , as $[\nu^s]$ does. As $[\nu^s - \tilde{\nu}^s] = 0$ in $H_{\text{cs}}^2(X', \mathbb{R})$ there exist compactly-supported 1-forms β^s on X' with $d\beta^s = \nu^s - \tilde{\nu}^s$. Since $\tilde{\nu}^s, \nu^s$ are supported in K we can choose β^s supported in K . We can also choose β^s to depend smoothly on $s \in \mathcal{F}'$, as $\tilde{\nu}^s, \nu^s$ do, and choose $\beta^0 = 0$, as $\tilde{\nu}^0 = \nu^0 = 0$.

As $\beta^0 = 0$ and β^s depends smoothly on s and is supported in K , making \mathcal{F}' smaller if necessary we can suppose that the graph $\Gamma(\beta^s)$ of β^s lies in $U_{X'} \subset T^*X'$ for all $s \in \mathcal{F}'$. Define $\tilde{\chi}^s : X' \rightarrow M$ by $\tilde{\chi}^s = \Phi_{X'}^s \circ \beta^s$, regarding β^s as a map $X' \rightarrow \Gamma(\beta^s) \subset U_{X'}$ in the obvious way. Then $\tilde{\chi}^s$ depends smoothly on $s \in \mathcal{F}'$ as $\beta^s, \Phi_{X'}^s$ do, and $\tilde{\chi}^0 = \chi^0 = \text{id}$ as $\beta^0 = 0$, and $\tilde{\chi}^s = \chi^s$ on S_i as $\beta^s = 0$ on S_i , so $\tilde{\chi}^s$ satisfies (40).

As $(\Phi_{X'}^s)^*(\omega^s) = \hat{\omega} + \pi^*(\nu^s)$, we see that $(\tilde{\chi}^s)^*(\omega^s) = (\beta^s)^*(\hat{\omega} + \pi^*(\nu^s))$, where β^s maps $X' \rightarrow \Gamma(\beta^s) \subset U_{X'}$. But $(\beta^s)^*(\hat{\omega}) = -d\beta^s$ by a well-known fact in symplectic geometry, and $(\beta^s)^*(\pi^*(\nu^s)) = \nu^s$ as $\pi \circ \beta^s = \text{id} : X' \rightarrow X'$, so

$$(\tilde{\chi}^s)^*(\omega^s) = -d\beta^s + \nu^s = -(\nu^s - \tilde{\nu}^s) + \nu^s = \tilde{\nu}^s.$$

Thus in the proof of Theorem 4.8 we are free to choose $\tilde{\chi}^s$ instead of χ^s such that $(\tilde{\chi}^s)^*(\omega^s) = \tilde{\nu}^s$ lies in V . The rest of the proof of Theorem 4.8 then shows that we can choose $\tilde{\Phi}_{X'}^s$ consistently with $\tilde{\chi}^s, \tilde{\nu}^s$. Finally, if $[\nu^s] = 0$ in $H_{\text{cs}}^2(X', \mathbb{R})$ then $\tilde{\nu}^s = 0$ in V , so we can choose $\nu^s = 0$. \square

5 The asymptotic behaviour of X near x_i

We shall now show that the asymptotic condition (31) in Definition 3.6 can be strengthened in two ways: we can make (31) hold for all $k \geq 0$ rather than

just $k = 0, 1$, and we can improve the asymptotic decay rates μ_i . On the way we will prove that compact SL m -folds X with conical singularities are automatically Riemannian manifolds with conical singularities in the sense of §2, so we deduce some analytic and Hodge theoretic results on X' . Throughout we suppose $m > 2$.

5.1 Regularity of higher derivatives

We shall use the special Lagrangian condition to show that (31), (34) and (36) hold for all $k \geq 0$ for the ϕ_i constructed in Theorem 4.4. Note that this is *not* true for arbitrary ϕ_1, \dots, ϕ_n satisfying Definition 3.6.

Theorem 5.1 *In the situation of Theorem 4.4 and Lemma 4.5 we have*

$$\begin{aligned} |\nabla^k(\phi_i - \iota_i)| &= O(r^{\mu_i - 1 - k}), \quad |\nabla^k \eta_i| = O(r^{\mu_i - 1 - k}) \quad \text{and} \\ |\nabla^k A_i| &= O(r^{\mu_i - k}) \quad \text{as } r \rightarrow 0 \text{ for all } k \geq 0 \text{ and } i = 1, \dots, n. \end{aligned} \quad (42)$$

Here ∇ and $|\cdot|$ are computed using the cone metric $\iota_i^*(g')$.

Proof. Let α be a smooth 1-form on $\Sigma_i \times (0, R')$ with $|\alpha(\sigma, r)| < \zeta r$, written $\alpha(\sigma, r) = \alpha^1(\sigma, r) + \alpha^2(\sigma, r)dr$ for $\alpha^1(\sigma, r) \in T_\sigma^* \Sigma_i$ and $\alpha^2(\sigma, r) \in \mathbb{R}$. Define a map $\Theta_\alpha : \Sigma_i \times (0, R') \rightarrow B_R$ by $\Theta_\alpha(\sigma, r) = \Phi_{C_i}(\sigma, r, \alpha^1(\sigma, r), \alpha^2(\sigma, r))$. Define a smooth real function $F_i(\alpha)$ on $\Sigma_i \times (0, R')$ by

$$F_i(\alpha) dV = \psi(x_i)^{-m} (\Upsilon_i \circ \Theta_\alpha)^*(\text{Im } \Omega), \quad (43)$$

where dV is the volume form of $\iota_i^*(g')$ on $\Sigma_i \times (0, R')$. This defines a function F_i from smooth 1-forms α on $\Sigma_i \times (0, R')$ with $|\alpha| < \zeta r$ to smooth functions on $\Sigma_i \times (0, R')$.

The value of $F_i(\alpha)$ at $(\sigma, r) \in \Sigma_i \times (0, R')$ depends on $\Upsilon_i \circ \Theta_\alpha$ and $d(\Upsilon_i \circ \Theta_\alpha)$ at (σ, r) , which depend on both α and $\nabla \alpha$ at (σ, r) . Hence $F_i(\alpha)$ depends pointwise on both α and $\nabla \alpha$, rather than just α . Define a map

$$Q_i : \left\{ (\sigma, r, y, z) : (\sigma, r) \in \Sigma_i \times (0, R'), \quad y \in T_{(\sigma, r)}^*(\Sigma_i \times (0, R')), \right. \\ \left. |y| < \zeta r, \quad z \in \otimes^2 T_{(\sigma, r)}^*(\Sigma_i \times (0, R')) \right\} \rightarrow \mathbb{R} \quad (44)$$

$$\text{by } Q_i(\sigma, r, \alpha(\sigma, r), \nabla \alpha(\sigma, r)) = (d^* \alpha + F_i(\alpha))[(\sigma, r)] \quad (45)$$

for all 1-forms α on $\Sigma_i \times (0, R')$ with $|\alpha(\sigma, r)| < \zeta r$ when $(\sigma, r) \in \Sigma_i \times (0, R')$. This is well-defined, as F_i has the right pointwise dependence in (45), and the 1-forms α sweep out the domain of Q_i in (44).

Let Ω' be as in (27), and rewrite (43) as

$$F_i(\alpha) dV = \Theta_\alpha^*(\psi(x_i)^{-m} \Upsilon_i^*(\text{Im } \Omega) - \text{Im } \Omega') + \Theta_\alpha^*(\text{Im } \Omega'). \quad (46)$$

As $\Upsilon_i^*(\text{Im } \Omega) = \psi(x_i)^m \text{Im } \Omega'$ at $0 \in B_R$ by Definition 3.6 and $\Upsilon_i^*(\text{Im } \Omega)$ is smooth we see that

$$\psi(x_i)^{-m} \Upsilon_i^*(\text{Im } \Omega) - \text{Im } \Omega' = O(r) \quad \text{on } B_R. \quad (47)$$

Now consider the map $\alpha \mapsto \Theta_\alpha^*(\text{Im } \Omega')$. When $\alpha = 0$ this is the pull-back of $\text{Im } \Omega'|_{C'_i}$, which is zero as C'_i is special Lagrangian. So $\Theta_0^*(\text{Im } \Omega') = 0$.

Following [20, p. 721-2] or [11, Prop. 2.10], we find that the linearization of $\Theta_\alpha^*(\text{Im } \Omega')$ in α at $\alpha = 0$ is $-(d^*\alpha) dV$. Thus we see that

$$\Theta_\alpha^*(\text{Im } \Omega') = -(d^*\alpha) dV + O(r^{-2}|\alpha|^2 + |\nabla\alpha|^2) \quad (48)$$

for $r^{-1}|\alpha|, |\nabla\alpha|$ small, using the dilation-equivariance properties of Θ_α to determine the powers of r in $O(r^{-2}|\alpha|^2 + r^0|\nabla\alpha|^2)$. Combining (45)–(48) gives

$$Q_i(\sigma, r, y, z) = O(r + r^{-2}|y|^2 + |z|^2) \quad \text{when } |y| = O(r) \text{ and } |z| = O(1). \quad (49)$$

When $\alpha = \eta_i = dA_i$ we have $\Theta_\alpha = \phi_i$, so $\Upsilon_i \circ \Theta_\alpha$ maps $\Sigma_i \times (0, R') \rightarrow S_i \subset X'$. Thus $(\Upsilon_i \circ \Theta_\alpha)^*(\text{Im } \Omega) = 0$ as X' is special Lagrangian, and $F_i(dA_i) = 0$. Hence (45) gives

$$\Delta A_i - Q_i(\sigma, r, dA_i(\sigma, r), \nabla^2 A_i(\sigma, r)) = 0 \quad (50)$$

for $(\sigma, r) \in \Sigma_i \times (0, R')$. This is a *second-order nonlinear elliptic equation* on A_i . We shall use *elliptic regularity results* for (50) to prove (42).

For $t \in (0, R']$ define $\delta^t : \Sigma_i \times (\frac{1}{2}, 1) \rightarrow \Sigma_i \times (0, R')$ by $\delta^t(\sigma, r) = (\sigma, tr)$. Define $Q_i^t(\sigma, r, y, z) = t^{2-\mu_i} Q_i(\sigma, tr, t^{\mu_i} \delta_*^t(y), t^{\mu_i} \delta_*^t(z))$, where

$$Q_i^t : \left\{ (\sigma, r, y, z) : (\sigma, r) \in \Sigma_i \times \left(\frac{1}{2}, 1\right), \quad y \in T_{(\sigma, r)}^*(\Sigma_i \times \left(\frac{1}{2}, 1\right)), \right. \\ \left. |y| < t^{2-\mu_i} \zeta r, \quad z \in \otimes^2 T_{(\sigma, r)}^*(\Sigma_i \times \left(\frac{1}{2}, 1\right)) \right\} \rightarrow \mathbb{R}. \quad (51)$$

Define functions $A_i^t : \Sigma_i \times (\frac{1}{2}, 1) \rightarrow \mathbb{R}$ by

$$A_i^t(\sigma, r) = t^{-\mu_i} A_i(\sigma, tr). \quad (52)$$

Then (50) implies that for $(\sigma, r) \in \Sigma_i \times (\frac{1}{2}, 1)$ we have

$$\Delta A_i^t - Q_i^t(\sigma, r, dA_i^t(\sigma, r), \nabla^2 A_i^t(\sigma, r)) = 0. \quad (53)$$

Also (36) shows that for some $C > 0$ independent of i, t we have

$$|\nabla^k A_i^t| \leq C \quad \text{on } \Sigma_i \times \left(\frac{1}{2}, 1\right) \text{ for } k = 0, 1, 2 \text{ and } t \in (0, R']. \quad (54)$$

From (49), noting that $|\delta_*^t(y)| = t^{-1}|y|$ and $|\delta_*^t(z)| = t^{-2}|z|$, we find that

$$Q_i^t(\sigma, r, y, z) = O(t^{3-\mu_i} + t^{\mu_i-2}|y|^2 + t^{\mu_i-2}|z|^2)$$

when $|y| = O(t^{2-\mu_i})$ and $|z| = O(t^{2-\mu_i})$. Thus $Q_i^t \rightarrow 0$ as $t \rightarrow 0$ uniformly on compact subsets of the domain in (51), since $2 < \mu_i < 3$. Furthermore, one can show that all derivatives of Q_i^t converge to 0 uniformly on compact subsets as $t \rightarrow 0$. Therefore for small t , equation (53) approximates the much simpler *linear* elliptic equation $\Delta A_i^t = 0$.

Now Ivanov [8] studies nonlinear elliptic equations $F(x, u, du, \nabla^2 u) = 0$ for x in a bounded domain S in \mathbb{R}^n and $u \in C^4(S)$, where $F(x, u, v, w)$ is a smooth function of its arguments. When $T \subset S^\circ$ is an interior domain, $|\nabla^k u| \leq C$ for $k = 0, 1, 2$ and F is *close to quasilinear*, in the sense that the second derivatives of F in the w variables are small compared to other constants depending on S, T, C and the first and second derivatives of F , he proves [8, Th. 2.2] a priori interior estimates for the Hölder $C^{k+2, \alpha}$ norm of u on T , depending on the same constants and the $C^{k, \alpha}$ norm of F on a compact subset of its domain.

This generalizes immediately to interior estimates on Riemannian manifolds. Thus we can apply it to (53) with $S = \Sigma_i \times (\frac{1}{2}, 1)$, $T = \Sigma_i \times (\frac{2}{3}, \frac{3}{4})$, $u = A_i^t$ and C as in (54). For small t , say when $t \leq \kappa$ for $\kappa \in (0, R']$, equation (53) is ‘close to quasilinear’ in the appropriate sense, and Ivanov’s result applies *uniformly in t* . Hence there exist constants $C_k > 0$ for $k \geq 0$ such that

$$|\nabla^k A_i^t| \leq C_k \quad \text{on } \Sigma_i \times (\frac{2}{3}, \frac{3}{4}) \text{ for all } k \geq 0 \text{ and } t \in (0, \kappa]. \quad (55)$$

Combining (52) and (55) proves that $|\nabla^k A_i| = O(r^{\mu_i - k})$ as $r \rightarrow 0$ for all $k \geq 0$. As $\eta_i = dA_i$ by Lemma 4.5, it immediately follows that $|\nabla^k \eta_i| = O(r^{\mu_i - 1 - k})$ as $r \rightarrow 0$ for all $k \geq 0$. Finally $|\nabla^k(\phi_i - \iota_i)| = O(r^{\mu_i - 1 - k})$ follows from relationship between η_i and ϕ_i in Theorem 4.4, and the dilation equivariance properties of Φ_{C_i} . This completes the proof. \square

5.2 Treating X as a manifold with conical singularities

From Theorem 5.1 it follows that g on X satisfies (1) with $\nu_i = \mu_i - 2$. Therefore SL m -folds with conical singularities fit into the framework of §2.

Theorem 5.2 *Suppose (M, J, ω, Ω) is an almost Calabi–Yau m -fold and X a compact SL m -fold in M with conical singularities at x_1, \dots, x_n with rates $\mu_1, \dots, \mu_n > 2$, as in Definition 3.6. Then X with the induced metric d is a Riemannian manifold with conical singularities in the sense of Definition 2.1, with $\nu_i = \mu_i - 2 > 0$.*

There are a few small notational differences between §2 and §3.3. For instance, ϕ_i in §2 is replaced by $\Upsilon_i \circ \phi_i$ in §3.3, ϵ in §2 is replaced by R' in §3.3, and S_i is defined to be $\{y \in X : 0 < d(x_i, y) < \epsilon\}$ in §2 and the image of $\Upsilon_i \circ \phi_i$ in §3.3. These differences are all entirely superficial, so we will ignore them.

We can now use the analysis of §2 to prove elliptic regularity results on X' . However, rather than studying the Laplacian Δ on X we consider the operator $P : f \mapsto d^*(\psi^m df)$, as this is what we will need in [11].

Theorem 5.3 *Let (M, J, ω, Ω) be an almost Calabi–Yau m -fold, and define $\psi : M \rightarrow (0, \infty)$ as in (29). Suppose X is a compact SL m -fold in M with conical singularities at x_1, \dots, x_n with cones C_i and rates μ_i . Define the Banach spaces $L_{k, \beta}^p(X')$ as in §2.2. Let $p > 1$ and $k \geq 2$, and for $\beta \in \mathbb{R}^n$ define $P_\beta : L_{k, \beta}^p(X') \rightarrow L_{k-2, \beta-2}^p(X')$ by $P_\beta(f) = d^*(\psi^m df)$. Then*

(a) P_β is Fredholm if and only if $\beta \in (\mathbb{R} \setminus \mathcal{D}_{\Sigma_1}) \times \cdots \times (\mathbb{R} \setminus \mathcal{D}_{\Sigma_n})$, and then

$$\text{ind}(P_\beta) = - \sum_{i=1}^n N_{\Sigma_i}(\beta_i). \quad (56)$$

(b) If $\beta_i > 0$ for all i then P_β is injective.

Proof. Define $\check{g} = \psi^{2m/(m-2)}g$, a Riemannian metric on X' conformally equivalent to g . This is well-defined as $m > 2$. Since $\psi|_{X'} \in C_0^\infty(X')$ and $\psi(x) \rightarrow \psi(x_i) > 0$ as $x \rightarrow x_i$ for $i = 1, \dots, n$, one can show as in Theorem 5.2 that \check{g} induces a metric \check{d} on X and (X, \check{d}) is a Riemannian manifold with conical singularities at x_1, \dots, x_n .

Furthermore, (X, \check{d}) has the same cones C_i and rates ν_i as does (X, d) induced by g . The cones C_i do not change because as Riemannian cones they are rescaled by a homothety multiplying distances by $\psi(x_i)^{m/(m-2)}$, but this gives the same Riemannian cone. As vector spaces of functions $L_{k,\beta}^p(X')$ and $L_{k-2,\beta-2}^p(X')$ are the same for g and \check{g} , with equivalent norms.

Write $d_g^*, d_{\check{g}}^*$ for d^* computed using g, \check{g} respectively. Let $\check{\Delta} = d_{\check{g}}^*d$ be the Laplacian of \check{g} on functions. An elementary calculation shows that

$$d_g^*(\psi^m df) = \psi^{m^2/(m-2)} \check{\Delta} f$$

for twice differentiable functions f on X' . Thus $P_\beta = \psi^{m^2/(m-2)} \check{\Delta}_{k,\beta}^p$.

Now multiplication by $\psi^{m^2/(m-2)}$ gives an automorphism of $L_{k-2,\beta-2}^p(X')$. So P_β is Fredholm, or injective, if and only if $\check{\Delta}_{k,\beta}^p$ is. Therefore (a) follows from Theorems 2.11 and 2.15, and (b) from part (a) of Lemma 2.16. \square

By a similar proof we modify Theorem 2.19, giving a result needed in [11, 13].

Theorem 5.4 *Let (M, J, ω, Ω) be an almost Calabi–Yau m -fold, and define $\psi : M \rightarrow (0, \infty)$ as in (29). Suppose X is a compact SL m -fold in M with conical singularities at x_1, \dots, x_n , and let $X', K, R', \Sigma_i, \Upsilon_i, \phi_i, S_i$ and μ_i be as in Definition 2.7, \mathcal{D}_{Σ_i} as in Definition 2.5, and ρ as in Definition 2.6. Define*

$$Y_{X'} = \left\{ \alpha \in C^\infty(T^*X') : d\alpha = 0, \quad d^*(\psi^m \alpha) = 0, \right. \\ \left. |\nabla^k \alpha| = O(\rho^{-1-k}) \text{ for } k \geq 0 \right\}. \quad (57)$$

Then $\pi : Y_{X'} \rightarrow H^1(X', \mathbb{R})$ given by $\pi : \alpha \mapsto [\alpha]$ is an isomorphism. Furthermore:

(a) Fix $\alpha \in Y_{X'}$. By Hodge theory there exists a unique $\gamma_i \in C^\infty(T^*\Sigma_i)$ with $d\gamma_i = d^*\gamma_i = 0$ for $i = 1, \dots, n$, such that the image of $\pi(\alpha)$ under the map $H^1(X', \mathbb{R}) \rightarrow \bigoplus_{i=1}^n H^1(\Sigma_i, \mathbb{R})$ of (16) is $([\gamma_1], \dots, [\gamma_n])$. There exist unique $T_i \in C^\infty(\Sigma_i \times (0, R'))$ for $i = 1, \dots, n$ such that

$$(\Upsilon_i \circ \phi_i)^*(\alpha) = \pi_i^*(\gamma_i) + dT_i \quad \text{on } \Sigma_i \times (0, R') \text{ for } i = 1, \dots, n, \text{ and} \quad (58)$$

$$\nabla^k T_i(\sigma, r) = O(r^{\nu_i - k}) \quad \text{as } r \rightarrow 0, \text{ for all } k \geq 0 \text{ and} \\ \nu_i \in (0, \mu_i - 2) \text{ with } (0, \nu_i] \cap \mathcal{D}_{\Sigma_i} = \emptyset. \quad (59)$$

- (b) Suppose $\gamma_i \in C^\infty(T^*\Sigma_i)$ with $d\gamma_i = d^*\gamma_i = 0$ for $i = 1, \dots, n$, and the image of $([\gamma_1], \dots, [\gamma_n])$ under $\bigoplus_{i=1}^n H^1(\Sigma_i, \mathbb{R}) \rightarrow H_{\text{cs}}^2(X', \mathbb{R})$ in (16) is $[\beta]$ for some exact 2-form β on X' supported on K . Then there exists $\alpha \in C^\infty(T^*X')$ with $d\alpha = \beta$, $d^*(\psi^m\alpha) = 0$ and $|\nabla^k\alpha| = O(\rho^{-1-k})$ for $k \geq 0$, such that (58) and (59) hold for $T_i \in C^\infty(\Sigma_i \times (0, R'))$.
- (c) Let $f \in C^\infty(X')$ with $|\nabla^k f| = O(\rho^{\mu-4-k})$ for $k \geq 0$ and $\int_{X'} f dV = 0$. Then there exists a unique exact 1-form α on X' with $d^*(\psi^m\alpha) = f$ and $|\nabla^k\alpha| = O(\rho^{-1-k})$ for $k \geq 0$, such that (58) and (59) hold for $\gamma_i = 0$ and $T_i \in C^\infty(\Sigma_i \times (0, R'))$.

5.3 Improving the rates of convergence μ_i

We shall use the analysis results of §2 to show that we can improve the rate μ_i of the conical singularity x_i in X to all possibilities allowed by Definition 3.6.

Theorem 5.5 *In the situation of Theorem 4.4 and Lemma 4.5 suppose $\mu'_i \in (2, 3)$ with $(2, \mu'_i] \cap \mathcal{D}_{\Sigma_i} = \emptyset$ for $i = 1, \dots, n$. Then*

$$\begin{aligned} |\nabla^k(\phi_i - \iota_i)| &= O(r^{\mu'_i-1-k}), & |\nabla^k\eta_i| &= O(r^{\mu'_i-1-k}) \quad \text{and} \\ |\nabla^k A_i| &= O(r^{\mu'_i-k}) \quad \text{as } r \rightarrow 0 \text{ for all } k \geq 0 \text{ and } i = 1, \dots, n. \end{aligned} \quad (60)$$

Hence X has conical singularities at x_i with cone C_i and rate μ'_i , for all possible rates μ'_i allowed by Definition 3.6. Therefore, the definition of conical singularities is essentially independent of the choice of rate μ_i .

Proof. Define a smooth function $A : X' \rightarrow \mathbb{R}$ by

$$A(\Upsilon_i \circ \phi_i(\sigma, r)) = A_i(\sigma, r) \quad \text{on } S_i \text{ for } i = 1, \dots, n, \quad (61)$$

and extend A smoothly over $K = X' \setminus (S_1 \cup \dots \cup S_n)$. Then $A \in C_{\mu'}^\infty(X')$ by (42). Let Q'_i and g_i be the push-forwards of Q_i in (44)–(45) and $\iota_i^*(g')$ from $\Sigma_i \times (0, R')$ to S_i under $\Upsilon_i \times \phi_i$. Then (50) implies that

$$d_{g_i}^* dA[x] = Q'_i(x, dA(x), \nabla^2 A(x)) \quad \text{for all } x \in S_i.$$

Here $d_{g_i}^*$ is computed using the exactly conical metric g_i on S_i , rather than the asymptotically conical metric g . Rearranging yields

$$\Delta A[x] = Q'_i(x, dA(x), \nabla^2 A(x)) + (d_g^* - d_{g_i}^*)dA[x] \quad (62)$$

for all $x \in S_i$, where $\Delta = d_g^* d$ is the Laplacian of g .

We shall prove the theorem by using an inductive argument to improve the decay rate of A and its derivatives step by step until we show that $A \in C_{\mu'}^\infty(X')$ for all μ' satisfying the conditions of the theorem. The next two lemmas will be needed for the ‘inductive step’.

Lemma 5.6 *Let $\lambda_i \in (2, 3)$ and define $\hat{\lambda}_i = \min(3, 2\lambda_i - 2)$ for $i = 1, \dots, n$. Then if $A \in C_{\hat{\lambda}}^\infty(X')$, then $\Delta A \in C_{\hat{\lambda}-2}^\infty(X')$.*

Proof. Suppose $A \in C_{\hat{\lambda}}^\infty(X')$. Then from (49) we find that

$$Q'_i(x, dA(x), \nabla^2 A(x)) = O(\rho) + O(\rho^{2\lambda_i-4}) + O(\rho^{2\hat{\lambda}_i-4}) \quad \text{for } x \in S_i.$$

As the asymptotic behaviour of g on S_i as $\rho \rightarrow 0$ depends on $\eta_i = dA_i = dA$ we have $g - g_i = O(\rho^{\lambda_i-2})$ on S_i . Thus $(d_g^* - d_{g_i}^*)dA = O(\rho^{2\lambda_i-4})$. Combining these with (62) gives $\Delta A = O(\rho) + O(\rho^{2\lambda-4}) = O(\rho^{\hat{\lambda}-2})$, by definition of $\hat{\lambda}$. The argument easily extends to derivatives of ΔA , and so $\Delta A \in C_{\hat{\lambda}-2}^\infty(X')$. \square

Lemma 5.7 *Suppose $\lambda_i, \hat{\lambda}_i \in (2, 3)$ with $(2, \lambda_i] \cap \mathcal{D}_{\Sigma_i} = (2, \hat{\lambda}_i] \cap \mathcal{D}_{\Sigma_i} = \emptyset$ for $i = 1, \dots, n$. Let $p > 1$ and $k \geq 2$. Then if $A \in L_{k, \hat{\lambda}}^p(X')$ and $\Delta A \in L_{k-2, \hat{\lambda}-2}^p(X')$, then $A \in L_{k, \hat{\lambda}}^p(X')$.*

Proof. Define $q > 1$ by $\frac{1}{p} + \frac{1}{q} = 1$. By Theorem 2.14, ΔA is orthogonal to $\text{Ker}(\Delta_{2, -\lambda+2-m}^q)$ as $A \in L_{k, \hat{\lambda}}^p(X')$. But the conditions on $\lambda, \hat{\lambda}$ imply that $\lambda, \hat{\lambda}$ lie in the same connected component of (9), and therefore $-\lambda+2-m, -\hat{\lambda}+2-m$ also lie in the same connected component of (9). Hence $\text{Ker}(\Delta_{2, -\lambda+2-m}^q) = \text{Ker}(\Delta_{2, -\hat{\lambda}+2-m}^q)$ by Theorem 2.12.

Therefore ΔA lies in $L_{k-2, \hat{\lambda}-2}^p(X')$ by assumption and is orthogonal to $\text{Ker}(\Delta_{2, -\hat{\lambda}+2-m}^q)$. So by Theorem 2.14, ΔA lies in the image of $\Delta_{k, \hat{\lambda}}^p$, and $\Delta A = \Delta A'$ for some $A' \in L_{k, \hat{\lambda}}^p(X')$. Thus $\Delta(A - A') = 0$ and $A - A'$ is harmonic. But $A - A' = O(\rho^2)$ as $\lambda_i, \hat{\lambda}_i > 2$, so $(A - A')(x) \rightarrow 0$ as $x \rightarrow x_i$ in X' . Hence using the maximum principle [6, §3] we see that $A - A' \equiv 0$, giving $A = A'$ and $A \in L_{k, \hat{\lambda}}^p(X')$. \square

Now we can prove the theorem. As $A \in C_\mu^\infty(X')$ from above, Lemma 5.6 shows that $\Delta A \in C_{\hat{\mu}-2}^\infty(X')$, where $\hat{\mu}_i = \min(3, 2\mu_i - 2)$ for $i = 1, \dots, n$. Note that $\hat{\mu}_i > \mu_i$ as $\mu_i \in (2, 3)$. Therefore if $p > 1$, $k \geq 2$ and $2 < \lambda_i < \mu_i$, $2 < \hat{\lambda}_i < \hat{\mu}_i$ for $i = 1, \dots, n$ we see that $A \in L_{k, \hat{\lambda}}^p(X')$ and $\Delta A \in L_{k-2, \hat{\lambda}-2}^p(X')$, since $C_\mu^\infty(X') \subset L_{k, \hat{\lambda}}^p(X')$ and $C_{\hat{\mu}-2}^\infty(X') \subset L_{k-2, \hat{\lambda}-2}^p(X')$.

Hence Lemma 5.7 shows that for all $\hat{\lambda}_i \in (2, \hat{\mu}_i)$ with $(2, \hat{\lambda}_i] \cap \mathcal{D}_{\Sigma_i} = \emptyset$ for $i = 1, \dots, n$ we have $A \in L_{k, \hat{\lambda}}^p(X')$. As this holds for all $k \geq 2$, Theorem 2.9 then proves that $A \in C_{\hat{\lambda}}^\infty(X')$. Thus starting with $A \in C_\mu^\infty(X')$ we have shown that $A \in C_{\hat{\lambda}}^\infty(X')$ for all $\hat{\lambda} = (\hat{\lambda}_1, \dots, \hat{\lambda}_n)$ with $2 < \hat{\lambda}_i < \hat{\mu}_i$ and $(2, \hat{\lambda}_i] \cap \mathcal{D}_{\Sigma_i} = \emptyset$.

Since $\hat{\mu}_i > \mu_i$, this is an improvement in the rate of convergence of A . If $\hat{\mu}_i = 3$ or $(2, \hat{\mu}_i] \cap \mathcal{D}_{\Sigma_i} \neq \emptyset$ then we have proved what we want for convergence of A on S_i . Otherwise $\hat{\mu}_i = 2\mu_i - 2$, so that $\hat{\mu}_i - 2 = 2(\mu_i - 2)$. Applying the same argument j times, we find that either we prove what we want for the convergence of A on S_i , or else $A \in C_{\hat{\lambda}}^\infty(X')$ for all $\hat{\lambda}$ with $2 < \hat{\lambda}_i < \tilde{\mu}_i < 3$ with $\tilde{\mu}_i - 2 = 2^j(\mu_i - 2)$.

If $2^j(\mu_i - 2) \geq 1$ this gives $\tilde{\mu}_i \geq 3$, a contradiction, so the process must terminate, and therefore for all μ' satisfying the conditions of the theorem we have $A \in C_{\mu'}^\infty(X')$. Equation (61) then gives $|\nabla^k A_i| = O(r^{\mu'_i - k})$ as $r \rightarrow 0$ for all $k \geq 0$ and $i = 1, \dots, n$, the final equation of (60). The first two equations of (60) then follow as for (42). This completes the proof of Theorem 5.5. \square

6 Geometric Measure Theory and tangent cones

We now review some *Geometric Measure Theory*, and apply it to special Lagrangian geometry. An introduction to the subject is provided by Morgan [21] and an in-depth (but dated) treatment by Federer [5], and Harvey and Lawson [7, §II] relate Geometric Measure Theory to calibrated geometry.

Geometric Measure Theory studies measure-theoretic generalizations of submanifolds called *integral currents*, which may be very singular, and is particularly powerful for *minimal* submanifolds. We shall distinguish between submanifolds or currents which are *volume-minimizing* (local minima of the volume functional), and those which are *minimal* (stationary points of the volume functional). Stronger results are available for the volume-minimizing case.

We can consider *special Lagrangian integral currents*, a natural class of singular SL m -folds with strong compactness properties, which are automatically volume-minimizing. Our main result, Theorem 6.8, says that if the *tangent cones* of an SL integral current T satisfy a certain condition then T is actually an SL m -fold with conical singularities, in the sense of §3.3. Throughout we suppose $m > 2$.

6.1 Introduction to Geometric Measure Theory

Let (M, g) be a complete Riemannian manifold. One defines a class of m -dimensional *rectifiable currents* in M , which are measure-theoretic generalizations of compact, oriented m -submanifolds N with boundary ∂N in M , with integer *multiplicities*. Here N with multiplicity k is like k copies of N superimposed, and changing the orientation of N changes the sign of the multiplicity. This enables us to add and subtract submanifolds.

If T is an m -dimensional rectifiable current, one can define the *volume* $\text{vol}(T)$ of T , by Hausdorff m -measure. If φ is a compactly-supported m -form on M then one can define $\int_T \varphi$. Thus we can regard T as a *current*, that is, an element $\varphi \mapsto \int_T \varphi$ of the dual space $(\mathcal{D}^m)^*$ of the vector space \mathcal{D}^m of smooth compactly-supported m -forms on M . This induces a topology on the space of rectifiable currents in M .

Let T be a m -dimensional rectifiable current, and define an $(m-1)$ -current ∂T by $\partial T \cdot \alpha = \int_T d\alpha$ for $\alpha \in \mathcal{D}^{m-1}$. We call T an *integral current* if ∂T is a rectifiable current. By [21, 5.5], [5, 4.2.17], integral currents have strong *compactness properties*.

Harvey and Lawson [7, §II] discuss calibrated geometry and Geometric Measure Theory. They show that on a Riemannian manifold (M, g) with calibration

φ one can define *integral φ -currents*, that is, integral currents which are calibrated w.r.t. φ , and that they are *volume-minimizing* in their homology class.

In particular, as in §3 SL m -folds in \mathbb{C}^m and in an almost Calabi–Yau manifold M may be defined as calibrated submanifolds, using the conformally rescaled metric \tilde{g} on M . Therefore we can define *special Lagrangian integral currents* in \mathbb{C}^m and in almost Calabi–Yau manifolds M , and they are *volume-minimizing currents* w.r.t. an appropriate metric.

6.2 Tangent cones

Next we discuss *tangent cones* of volume-minimizing integral currents, a generalization of tangent spaces of submanifolds, as in [21, 9.7]. Define the *interior* T° of T to be $T \setminus \partial T$ (that is, $\text{supp } T \setminus \text{supp } \partial T$).

Definition 6.1 An integral current C in \mathbb{R}^n is called a *cone* if $C = tC$ for all $t > 0$, where $t : \mathbb{R}^n \rightarrow \mathbb{R}^n$ acts by dilations in the obvious way. Let T be an integral current in \mathbb{R}^n , and let $\mathbf{x} \in T^\circ$. We say that C is a *tangent cone* to T at \mathbf{x} if there exists a decreasing sequence $r_1 > r_2 > \dots$ tending to zero such that $r_j^{-1}(T - \mathbf{x})$ converges to C as an integral current as $j \rightarrow \infty$.

More generally, if (M, g) is a complete Riemannian n -manifold, T is an integral current in M , and $x \in T^\circ$, then one can define a *tangent cone* C to T at x , which is an integral current cone in the Euclidean vector space $T_x M$. Identifying M with \mathbb{R}^n near x using a coordinate system, the two notions of tangent cone coincide.

The next result follows from Morgan [21, p. 94-95], Federer [5, 5.4.3] and Harvey and Lawson [7, Th. II.5.15].

Theorem 6.2 *Let (M, g) be a complete Riemannian manifold, and T a volume-minimizing integral current in M . Then for all $x \in T^\circ$, there exists a tangent cone C to T at x . Moreover C is itself a volume-minimizing integral current in $T_x M$ with $\partial C = \emptyset$, and if T is calibrated with respect to a calibration φ on (M, g) , then C is calibrated with respect to the constant calibration $\varphi|_x$ on $T_x M$.*

Note that the theorem does *not* claim that the tangent cone C is unique, and in fact it is an important open question whether a volume-minimizing integral current has a unique tangent cone at each point of T° . However, Leon Simon [23, 24], improving an earlier result of Allard and Almgren [2], shows that if some tangent cone C is nonsingular and multiplicity 1 away from 0, then C is the unique tangent cone, and T converges to C in a C^1 sense. For later use we model the result on the notation of Definition 3.6.

Theorem 6.3 *Let C be an m -dimensional oriented minimal cone in \mathbb{R}^n with $C' = C \setminus \{0\}$ nonsingular, so that $\Sigma = C' \cap \mathcal{S}^{n-1}$ is a compact, oriented, nonsingular, embedded, minimal $(m-1)$ -submanifold of \mathcal{S}^{n-1} . Define $\iota : \Sigma \times (0, \infty) \rightarrow C' \subset \mathbb{R}^n$ by $\iota(\sigma, r) = r\sigma$. Let (M, g) be a complete Riemannian n -manifold and $x \in M$. Fix an isometry $v : \mathbb{R}^n \rightarrow T_x M$, and choose an embedding*

$\Upsilon : B_R \rightarrow M$ with $\Upsilon(0) = x$ and $d\Upsilon|_0 = v$, where B_R is the ball of radius $R > 0$ about $0 \in \mathbb{R}^n$.

Suppose that T is a minimal integral current in M with $x \in T^\circ$, and that $v_*(C)$ is a tangent cone to T at x with multiplicity 1. Then $v_*(C)$ is the unique tangent cone to T at x . Furthermore there exists $R' \in (0, R]$ and an embedding $\phi : \Sigma \times (0, R') \rightarrow B_{R'} \subseteq B_R$ with

$$|\phi(\sigma, r)| \equiv r, \quad |\phi - \iota| = o(r) \quad \text{and} \quad |\nabla(\phi - \iota)| = o(1) \quad \text{as } r \rightarrow 0, \quad (63)$$

such that $T \cap (\Upsilon(B_{R'}) \setminus \{x\})$ is the embedded submanifold $\Upsilon \circ \phi(\Sigma \times (0, R'))$, with multiplicity 1.

Proof. This follows from [23, Cor., p. 564] and [24, Th. 5.7], which are equivalent results, the latter more explicit. Simon claims only that ϕ is C^2 rather than smooth, but smoothness follows from standard regularity results for minimal submanifolds. \square

We define *Jacobi fields* on Σ , following Lawson [15, p. 46-52].

Definition 6.4 Let Σ be a compact, minimal submanifold in the unit sphere \mathcal{S}^{n-1} in \mathbb{R}^n . Let ν be the normal bundle of Σ in \mathcal{S}^{n-1} , so that $T\mathcal{S}^{n-1}|_\Sigma = \nu \oplus T\Sigma$ is an orthogonal splitting. Let g and g_Σ be the Riemannian metrics on \mathcal{S}^{n-1} and Σ induced by the Euclidean metric on \mathbb{R}^n .

Let ∇^ν be the connection on ν defined by projecting the Levi-Civita connection of g on $T\mathcal{S}^{n-1}|_\Sigma$ to ν . Let $\Delta^\nu : C^\infty(\nu) \rightarrow C^\infty(\nu)$ be the Laplacian $(\nabla^\nu)^*\nabla^\nu$ defined using ∇^ν, g and g_Σ . Define maps $\mathcal{R}, \mathcal{B} : C^\infty(\nu) \rightarrow C^\infty(\nu)$ by

$$\mathcal{R}(w)^i = \pi^\nu(R^i_{jkl}g^{jk}w^l) \quad \text{and} \quad \mathcal{B}(w)^a = B^a_{bc}B^i_{jk}g^{bj}g^{ck}g_{il}w^l,$$

using the index notation for tensors, where R^i_{jkl} is the Riemann curvature of g , $B^i_{jk} \in C^\infty(\nu \otimes S^2T^*\Sigma)$ is the second fundamental form of Σ in \mathcal{S}^{n-1} , and π^ν is the orthogonal projection from $T\mathcal{S}^{n-1}$ to ν .

We call a normal vector field $w \in C^\infty(\nu)$ to Σ in \mathcal{S}^{n-1} a *Jacobi field* if

$$\Delta^\nu w - \mathcal{R}(w) + \mathcal{B}(w) = 0. \quad (64)$$

Jacobi fields are zeroes of the linearization at Σ of the Euler–Lagrange equation for the volume of submanifolds Σ' in \mathcal{S}^{n-1} . Therefore a Jacobi field is an *infinitesimal deformation of Σ as a minimal submanifold*, a null direction of the second variation of volume for submanifolds.

In particular, the Lie algebra $\mathfrak{so}(n)$ of isometries of \mathcal{S}^n clearly induce infinitesimal deformations of Σ as a minimal submanifold, and so Jacobi fields. Regarding $v \in \mathfrak{so}(n)$ as a vector field on \mathcal{S}^{n-1} , the corresponding Jacobi field on Σ is $w = \pi^\nu(v|_\Sigma)$. However, for some Σ not all Jacobi fields come from $\mathfrak{so}(n)$ in this way. Note that as Σ is compact and (64) is an elliptic equation, the Jacobi fields form a finite-dimensional vector space.

Now by Allard and Almgren [2, p. 215], or equivalently by Adams and Simon [1, Th. 1], if the Jacobi fields on Σ satisfy a condition then we can strengthen the rate of convergence in (63).

Theorem 6.5 *Let C be an m -dimensional oriented minimal cone in \mathbb{R}^n with $C' = C \setminus \{0\}$ nonsingular, and set $\Sigma = C \cap \mathcal{S}^{n-1}$. Suppose that Σ satisfies*

- (*) *Each Jacobi field w of Σ in \mathcal{S}^{n-1} exponentiates to a smooth 1-parameter family $\{\Sigma_t : t \in (-\epsilon, \epsilon)\}$ of minimal submanifolds in \mathcal{S}^{n-1} for $\epsilon > 0$, with $\Sigma_0 = \Sigma$ and velocity w at $t = 0$.*

Then for some $\mu > 2$, the map ϕ of Theorem 6.3 satisfies

$$|\phi - \iota| = O(r^{\mu-1}) \quad \text{and} \quad |\nabla(\phi - \iota)| = O(r^{\mu-2}) \quad \text{as } r \rightarrow 0. \quad (65)$$

Adams and Simon [1, Th. 1(ii)] also study the case when condition (*) does not hold, and prove:

Theorem 6.6 *Let C be an m -dimensional oriented minimal cone in \mathbb{R}^n with $C' = C \setminus \{0\}$ nonsingular, and set $\Sigma = C \cap \mathcal{S}^{n-1}$. Suppose that condition (*) of Theorem 6.5 does not hold, and also that a certain sign condition [1, p. 232] holds for some Jacobi field.*

Then there exist large families of minimal integral currents T in \mathbb{R}^n with $0 \in T^\circ$ such that C is a tangent cone to T at 0 with multiplicity 1, and for some $\alpha \in (0, 1]$ the map ϕ in Theorem 6.3 with $\Upsilon = \text{id} : B_R \rightarrow B_R \subset \mathbb{R}^n$ decays exactly at rate

$$|\phi - \iota| = O(r|\log r|^{-\alpha}) \quad \text{and} \quad |\nabla(\phi - \iota)| = O(|\log r|^{-\alpha}) \quad \text{as } r \rightarrow 0. \quad (66)$$

Here is what we mean by the ‘sign condition’ above. If condition (*) fails then there exists an integer $p > 2$ and a nonzero homogeneous degree p real polynomial P on the Jacobi fields. If $P(w) > 0$ then we can construct minimal integral currents T near 0 in \mathbb{R}^n for which

$$\phi(\sigma, r) = \iota(\sigma, r) + r|\log r|^{-1/(p-2)}w(\sigma) + \text{lower order terms} \quad (67)$$

as $r \rightarrow 0$. Thus (66) holds exactly for $\alpha = 1/(p-2)$. If $P(w) < 0$ then we can instead construct *Asymptotically Conical* minimal integral currents T near ∞ in \mathbb{R}^n for which (67) holds as $r \rightarrow \infty$. We need $P(w) > 0$ for some Jacobi field w , which is automatic when p is odd, and hence in the ‘most generic’ case $p = 3$.

6.3 Tangent cones of special Lagrangian m -folds

We shall now specialize the results of §6.2 to the case when T is a *special Lagrangian integral current* in an almost Calabi–Yau m -fold (M, J, ω, Ω) . Our aim is to prove that if the tangent cones of T satisfy certain conditions then T satisfies Definition 3.6, and so is an *SL m -fold with conical singularities*.

By restricting to special Lagrangian currents we can strengthen Theorem 6.5, as condition (*) need not hold for all Jacobi fields w , but only for those

which represent infinitesimal deformations of C as a *special Lagrangian cone*, rather than as a minimal cone.

Definition 6.7 Let C be an SL cone in \mathbb{C}^m with $C' = C \setminus \{0\}$ nonsingular, and set $\Sigma = C \cap \mathcal{S}^{2m-1}$. Then Σ is a compact, nonsingular, minimal Legendrian submanifold of \mathcal{S}^{2m-1} . Define $\iota : \Sigma \times (0, \infty) \rightarrow \mathbb{C}^m$ by $\iota(\sigma, r) = r\sigma$, with image C' . Let $g_\Sigma = g'|_\Sigma$ be the metric on Σ and Δ_Σ the Laplacian on Σ .

Suppose $v \in C^\infty(\Sigma)$ is an eigenfunction of Δ_Σ with eigenvalue $2m$. Then $u : r\sigma \mapsto r^2v(\sigma)$ is a homogeneous harmonic function on C' of order 2, by Lemma 2.3. Thus du is a homogeneous closed and coclosed 1-form on C' of order 1. Let $\nu \rightarrow C'$ be the normal bundle of C' in \mathbb{C}^m . Then $\nu \cong T^*C'$ by the usual isomorphism. So du corresponds to a homogeneous section of ν of order 1, which is an *infinitesimal deformation of C as an SL cone*.

Define w_v to be the restriction of this section to $\Sigma \subset C'$. Then w_v is a smooth section of the normal bundle of Σ in \mathcal{S}^{2m-1} , and is a *Jacobi field* on Σ in the sense of Definition 6.4. Define a *special Lagrangian Jacobi field* to be a Jacobi field w_v on Σ constructed from a Δ_Σ $2m$ -eigenfunction $v \in C^\infty(\Sigma)$ in this way.

We call C *Jacobi integrable* if it satisfies the condition

- (**) Each special Lagrangian Jacobi field w_v of Σ in \mathcal{S}^{2m-1} exponentiates to a smooth 1-parameter family $\{\Sigma_t : t \in (-\epsilon, \epsilon)\}$ for $\epsilon > 0$ with $\Sigma_0 = \Sigma$ and velocity w_v at $t = 0$, where $\Sigma_t = C_t \cap \mathcal{S}^{2m-1}$ for C_t a special Lagrangian cone in \mathbb{C}^m .

That is, each special Lagrangian Jacobi field should be integrable.

Each element x of the Lie algebra $\mathfrak{su}(m)$, regarded as vector field on \mathcal{S}^{2m-1} , induces an infinitesimal deformation of C as a special Lagrangian cone, so that $\pi^\nu(x|_\Sigma)$ is a special Lagrangian Jacobi field w_v on Σ . The corresponding eigenfunction $v \in C^\infty(\Sigma)$ is the restriction to Σ of the unique moment map $\mu : \mathbb{C}^m \rightarrow \mathbb{R}$ of x with $\mu(0) = 0$. Now Jacobi fields w_v constructed from $x \in \mathfrak{su}(m)$ in this way automatically satisfy (**), as $\exp(tx) \in \text{SU}(m)$ for $t \in \mathbb{R}$, so $C_t = \exp(tx)C$ is a special Lagrangian cone, and $\Sigma_t = C_t \cap \mathcal{S}^{2m-1}$ satisfies the conditions.

Define C to be *rigid* if all special Lagrangian Jacobi fields w_v on Σ come from $\mathfrak{su}(m)$ as above. Then C *rigid implies C Jacobi integrable*, from above. There is a simple test for rigidity: let G be the Lie subgroup of $\text{SU}(m)$ preserving C , and \mathfrak{g} the Lie algebra of G . Then the special Lagrangian Jacobi fields on Σ from $\mathfrak{su}(m)$ are a vector space isomorphic to $\mathfrak{su}(m)/\mathfrak{g}$, with dimension $m^2 - 1 - \dim G$. Therefore C is *rigid if and only if the multiplicity of the eigenvalue $2m$ of Δ_Σ is $m^2 - 1 - \dim G$* , that is, if $m_\Sigma(2) = m^2 - 1 - \dim G$ in the notation of Definition 2.5. This may be taken as an alternative definition of rigidity.

Now we can prove the main result of this section.

Theorem 6.8 *Let (M, J, ω, Ω) be an almost Calabi–Yau m -fold and define $\psi : M \rightarrow (0, \infty)$ as in (29). Let $x \in M$ and fix an isomorphism $v : \mathbb{C}^m \rightarrow T_x M$ with $v^*(\omega) = \omega'$ and $v^*(\Omega) = \psi(x)^m \Omega'$, where ω', Ω' are as in (27).*

Suppose that T is a special Lagrangian integral current in M with $x \in T^\circ$, and that $v_*(C)$ is a multiplicity 1 tangent cone to T at x , where C is a Jacobi integrable special Lagrangian cone in \mathbb{C}^m , in the sense of Definition 6.7. Then T has a conical singularity at x , in the sense of Definition 3.6.

Suppose that T is a special Lagrangian integral current in M with $\partial T = \emptyset$, and that every singular point of T has a Jacobi integrable multiplicity 1 special Lagrangian tangent cone. Then T is a compact SL m -fold in M with conical singularities, in the sense of Definition 3.6.

Proof. Let $(M, J, \omega, \Omega), x, v$ and T be as in the first part of the theorem, and choose an embedding $\Upsilon : B_R \rightarrow M$ with $\Upsilon(0) = x$, $d\Upsilon|_0 = v$ and $\Upsilon^*(\omega) = \omega'$, as in Definition 3.6. Then Theorem 6.3 applies, and gives $R' \in (0, R]$ and an embedding $\phi : \Sigma \times (0, R') \rightarrow B_R$ satisfying (63) such that $\Upsilon \circ \phi$ parametrizes T near x .

We would like to apply Theorem 6.5 to deduce that ϕ satisfies (65). Now following the proof of Theorem 6.5 in [1], we find that *either* (65) holds, *or* we can construct a Jacobi field w from T by a limiting process, which does not satisfy (*). Since T is special Lagrangian it turns out that w must be a *special Lagrangian Jacobi field*, and so does not satisfy (**).

But as C is Jacobi integrable, condition (**) holds for all such w . Therefore ϕ satisfies (65) for some $\mu > 2$. Making μ smaller if necessary we can suppose $\mu \in (2, 3)$ and μ satisfies (30). Then (65) is equivalent to (31), so T satisfies Definition 3.6 near x , and has a conical singularity at x with identification v , cone C and rate μ . This completes the first part of the theorem.

For the second part, note that by the first part every singular point of T is a conical singularity, and so is isolated. Thus by compactness of M there are only finitely many singular points x_1, \dots, x_n of T , and it quickly follows that T is a compact SL m -fold with conical singularities. \square

This is a *weakening* of Definition 3.6, in that if T satisfies the apparently much weaker condition of having a certain kind of tangent cone at x , then T actually has a conical singularity at x .

Finally we discuss singularities x of SL m -folds X modelled on multiplicity one SL cones C with $C \setminus \{0\}$ nonsingular, but where C is not Jacobi integrable. Then Theorem 6.3 shows that X can be parametrized near x using a map $\phi : \Sigma \times (0, R') \rightarrow B_R$ satisfying (63).

However, Theorem 6.5 suggests that the asymptotic behaviour we should expect of ϕ , at least for X suitably generic, is exactly that of (66) for some $\alpha \in (0, 1]$. This does not satisfy (31), and so such singular points will *not* be conical singularities in our sense.

This indicates that for SL cones C which are not Jacobi integrable, Definition 3.6 is actually *too strong*, in that there should exist examples of singular SL m -folds with tangent cone C which are not covered by Definition 3.6, since the decay conditions in (31) are too strict. Nevertheless, we will continue to use Definition 3.6 in the sequels [11, 12, 13, 14], because without it we will be unable to use the powerful analytic framework of §2.

7 Asymptotically Conical SL m -folds

Let C be an SL cone in \mathbb{C}^m with an isolated singularity at 0. Sections 3–6 considered SL m -folds with conical singularities, which are asymptotic to C at 0. We now discuss *Asymptotically Conical* SL m -folds L in \mathbb{C}^m , which are asymptotic to C at infinity. Here is the definition.

Definition 7.1 Let C be a closed SL cone in \mathbb{C}^m with isolated singularity at 0 for $m > 2$, and let $\Sigma = C \cap \mathcal{S}^{2m-1}$, so that Σ is a compact, nonsingular $(m-1)$ -manifold, not necessarily connected. Let g_Σ be the metric on Σ induced by the metric g' on \mathbb{C}^m in (27), and r the radius function on \mathbb{C}^m . Define $\iota : \Sigma \times (0, \infty) \rightarrow \mathbb{C}^m$ by $\iota(\sigma, r) = r\sigma$. Then the image of ι is $C \setminus \{0\}$, and $\iota^*(g') = r^2 g_\Sigma + dr^2$ is the cone metric on $C \setminus \{0\}$.

Let L be a closed, nonsingular SL m -fold in \mathbb{C}^m . We call L *Asymptotically Conical (AC)* with *rate* $\lambda < 2$ and *cone* C if there exists a compact subset $K \subset L$ and a diffeomorphism $\varphi : \Sigma \times (T, \infty) \rightarrow L \setminus K$ for some $T > 0$, such that

$$|\nabla^k(\varphi - \iota)| = O(r^{\lambda-1-k}) \quad \text{as } r \rightarrow \infty \text{ for } k = 0, 1. \quad (68)$$

Here $\nabla, |\cdot|$ are computed using the cone metric $\iota^*(g')$.

This is very similar to Definition 3.6, and in fact there are strong similarities between the theories of SL m -folds with conical singularities and of Asymptotically Conical SL m -folds. Note that we do *not* impose any condition on λ analogous to (30), although we could. We continue to assume $m > 2$.

We begin in §7.1 by defining *cohomological invariants* $Y(L), Z(L)$ of L in $H^*(\Sigma, \mathbb{R})$, which have no parallel in the conical singularities case. Then §7.2 and §7.3 develop the analogues of parts of §4 and §5 for AC SL m -folds. The *deformation theory* of AC SL m -folds is studied by Marshall [18], and *examples* of AC SL m -folds will be discussed in [14, §6.4].

7.1 Cohomological invariants of AC SL m -folds

Let L be an AC SL m -fold in \mathbb{C}^m with cone C , and set $\Sigma = C \cap \mathcal{S}^{2m-1}$. Using the notation of §2.4, as in (15) there is a long exact sequence

$$\dots \rightarrow H_{\text{cs}}^k(L, \mathbb{R}) \rightarrow H^k(L, \mathbb{R}) \rightarrow H^k(\Sigma, \mathbb{R}) \rightarrow H_{\text{cs}}^{k+1}(L, \mathbb{R}) \rightarrow \dots \quad (69)$$

We shall define *cohomological invariants* $Y(L), Z(L)$ of L .

Definition 7.2 Let L be an AC SL m -fold in \mathbb{C}^m with cone C , and let $\Sigma = C \cap \mathcal{S}^{2m-1}$. As $\omega', \text{Im } \Omega'$ in (27) are closed forms with $\omega'|_L \equiv \text{Im } \Omega'|_L \equiv 0$, they define classes in the relative de Rham cohomology groups $H^k(\mathbb{C}^m; L, \mathbb{R})$ for $k = 2, m$. But for $k > 1$ we have the exact sequence

$$0 = H^{k-1}(\mathbb{C}^m, \mathbb{R}) \rightarrow H^{k-1}(L, \mathbb{R}) \xrightarrow{\cong} H^k(\mathbb{C}^m; L, \mathbb{R}) \rightarrow H^k(\mathbb{C}^m, \mathbb{R}) = 0.$$

Define $Y(L) \in H^1(\Sigma, \mathbb{R})$ to be the image of $[\omega']$ in $H^2(\mathbb{C}^m; L, \mathbb{R}) \cong H^1(L, \mathbb{R})$ under the map $H^1(L, \mathbb{R}) \rightarrow H^1(\Sigma, \mathbb{R})$ of (69), and $Z(L) \in H^{m-1}(\Sigma, \mathbb{R})$ to be

the image of $[\text{Im } \Omega']$ in $H^m(\mathbb{C}^m; L, \mathbb{R}) \cong H^{m-1}(L, \mathbb{R})$ under $H^{m-1}(L, \mathbb{R}) \rightarrow H^{m-1}(\Sigma, \mathbb{R})$ in (69).

Here are some conditions for $Y(L)$ or $Z(L)$ to be zero.

Proposition 7.3 *Let L be an AC SL m -fold in \mathbb{C}^m with cone C and rate λ , and let $\Sigma = C \cap \mathcal{S}^{2m-1}$. If $\lambda < 0$ or $b^1(L) = 0$ then $Y(L) = 0$. If $\lambda < 2 - m$ or $b^0(\Sigma) = 1$ then $Z(L) = 0$.*

Proof. Let $C, \Sigma, K, T, \varphi, \iota$ be as in Definition 7.1. Suppose $Y(L) \neq 0$. Then there exists $\gamma \in H_1(\Sigma, \mathbb{Z})$ with $Y(L) \cdot \gamma \neq 0$. Choose a closed 1-chain δ in Σ with $[\delta] = \gamma$. Let $r > T$. Then $\delta \times \{r\}$ is a closed 1-chain in $\Sigma \times (T, \infty)$, and so $\varphi(\delta \times \{r\}), \iota(\delta \times \{r\})$ are closed 1-chains in \mathbb{C}^m .

Suppose S is a 2-chain in \mathbb{C}^m with $\partial S = \varphi(\delta \times \{r\})$. Then one can show using Definition 7.2 that $\int_S \omega' = Y(L) \cdot \gamma$. Also, if T is a 2-chain in \mathbb{C}^m with $\partial T = \iota(\delta \times \{r\})$ then $\int_T \omega' = 0$. This is because the answer is independent of T , and we can choose T inside the cone C , so that $\omega'|_T \equiv 0$ as C is Lagrangian.

Therefore if U is a 2-chain in \mathbb{C}^m with $\partial U = \varphi(\delta \times \{r\}) - \iota(\delta \times \{r\})$ then $\int_U \omega' = Y(L) \cdot \gamma \neq 0$. Now $|\int_U \omega'| \leq \text{vol}(U)$ as ω' is a calibration. But by (68) we can choose U with $\text{vol}(U) = O(r^\lambda)$ for large r . Thus $|\int_U \omega'| = O(r^\lambda)$ for large r , and also $\int_U \omega' \neq 0$ is independent of r . Together these force $\lambda \geq 0$. Hence if $\lambda < 0$ then $Y(L) = 0$. The $\lambda < 2 - m$ case is similar, using $\text{Im } \Omega'$ instead of ω' , and is left as an exercise.

If $b^1(L) = 0$ then $H^1(L, \mathbb{R}) = 0$, so $Y(L) = 0$, as it lies in the image of $H^1(L, \mathbb{R}) \rightarrow H^1(\Sigma, \mathbb{R})$ from Definition 7.2. If $b^0(\Sigma) = 1$ then $H_{m-1}(\Sigma, \mathbb{R}) = \langle [\Sigma] \rangle$. Now Σ is a boundary in L , so the map $H_{m-1}(\Sigma, \mathbb{R}) \rightarrow H_{m-1}(L, \mathbb{R})$ is zero, and the dual map $H^{m-1}(L, \mathbb{R}) \rightarrow H^{m-1}(\Sigma, \mathbb{R})$ also zero. But $Z(L)$ lies in the image of this, so $Z(L) = 0$. \square

7.2 Lagrangian Neighbourhood Theorems

Next we give versions of parts of §4 for AC SL m -folds rather than SL m -folds with conical singularities. Here is an analogue of Theorem 4.4.

Theorem 7.4 *Let C be an SL cone in \mathbb{C}^m with isolated singularity at 0, and set $\Sigma = C \cap \mathcal{S}^{2m-1}$. Define $\iota : \Sigma \times (0, \infty) \rightarrow \mathbb{C}^m$ by $\iota(\sigma, r) = r\sigma$. Let $\zeta, U_C \subset T^*(\Sigma \times (0, \infty))$ and $\Phi_C : U_C \rightarrow \mathbb{C}^m$ be as in Theorem 4.3.*

Suppose L is an AC SL m -fold in \mathbb{C}^m with cone C and rate $\lambda < 2$. Then there exists a compact $K \subset L$ and a diffeomorphism $\varphi : \Sigma \times (T, \infty) \rightarrow L \setminus K$ for some $T > 0$ satisfying (68), and a closed 1-form χ on $\Sigma \times (T, \infty)$ written $\chi(\sigma, r) = \chi^1(\sigma, r) + \chi^2(\sigma, r)dr$ for $\chi^1(\sigma, r) \in T_\sigma^ \Sigma$ and $\chi^2(\sigma, r) \in \mathbb{R}$, satisfying*

$$\begin{aligned} |\chi(\sigma, r)| &< \zeta r, \quad \varphi(\sigma, r) \equiv \Phi_C(\sigma, r, \chi^1(\sigma, r), \chi^2(\sigma, r)) \\ \text{and } |\nabla^k \chi| &= O(r^{\lambda-1-k}) \quad \text{as } r \rightarrow \infty \text{ for } k = 0, 1, \end{aligned} \tag{70}$$

computing $\nabla, |\cdot|$ using the cone metric $\iota^(g')$.*

Proof. As L is Asymptotically Conical with cone C it follows from (68) that near infinity in \mathbb{C}^m we can write L as the image under Φ_C of the graph of a 1-form χ on $\Sigma \times (T, \infty)$ for large $T > 0$. This just means that L intersects the Lagrangian ball $\Phi_C(T_{(\sigma,r)}^*(\Sigma \times (0, \infty)) \cap U_C)$ in exactly one point for $(\sigma, r) \in \Sigma \times (T, \infty)$, and we define χ such that this point is $\Phi_C(\chi(\sigma, r))$.

Now define $\varphi : \Sigma \times (T, \infty) \rightarrow L$ by $\varphi(\sigma, r) = \Phi_C(\sigma, r, \chi^1(\sigma, r), \chi^2(\sigma, r))$ and $K = L \setminus \text{Image } \varphi$. Then K is compact and $\varphi : \Sigma \times (T, \infty) \rightarrow L \setminus K$ is a diffeomorphism. These T, φ, K are a valid choice of T, φ, K for L in Definition 7.1. In particular, φ satisfies (68). One can show this by starting with T', φ', K' satisfying Definition 7.1, regarding φ as obtained from φ' by a kind of projection, and showing that $|\nabla^k(\varphi' - \varphi)| = O(r^{\lambda-1-k})$ as $r \rightarrow \infty$ for $k = 0, 1$.

As $\omega'|_L \equiv 0$ and $\Phi_C^*(\omega') = \hat{\omega}$ we see that $\hat{\omega}$ restricted to the graph of χ is zero. By a well-known fact in symplectic geometry, this implies that χ is *closed*. Equation (68) and the properties of Φ_C imply that $|\nabla^k \chi| = O(r^{\lambda-1-k})$ as $r \rightarrow \infty$ for $k = 0, 1$. As $\lambda < 2$ this gives $|\chi| = o(r)$, and so by making T, K larger if necessary we can suppose that $|\chi(\sigma, r)| < \zeta r$ for $(\sigma, r) \in \Sigma \times (T, \infty)$. This completes the proof. \square

Here is the analogue of Theorem 4.6. Its proof is a straightforward modification of that of Theorem 4.6, and we leave it as an exercise.

Theorem 7.5 *Suppose L is an AC SL m -fold in \mathbb{C}^m with cone C . Let $\Sigma, \iota, \zeta, U_C, \Phi_C, K, T, \varphi, \chi, \chi^1, \chi^2$ be as in Theorem 7.4. Then making T, K larger if necessary, there exists an open tubular neighbourhood $U_L \subset T^*L$ of the zero section L in T^*L , such that under $d\varphi : T^*(\Sigma \times (T, \infty)) \rightarrow T^*L$ we have*

$$(d\varphi)^*(U_L) = \{(\sigma, r, \tau, u) \in T^*(\Sigma \times (T, \infty)) : |(\tau, u)| < \zeta r\}, \quad (71)$$

and there exists an embedding $\Phi_L : U_L \rightarrow \mathbb{C}^m$ with $\Phi_L|_L = \text{id} : L \rightarrow L$ and $\Phi_L^*(\omega') = \hat{\omega}$, where $\hat{\omega}$ is the canonical symplectic structure on T^*L , such that

$$\Phi_L \circ d\varphi(\sigma, r, \tau, u) \equiv \Phi_C(\sigma, r, \tau + \chi^1(\sigma, r), u + \chi^2(\sigma, r)) \quad (72)$$

for all $(\sigma, r, \tau, u) \in T^*(\Sigma \times (T, \infty))$ with $|(\tau, u)| < \zeta r$, computing $|\cdot|$ using $\iota^*(g')$.

We can decompose χ in Theorem 7.4, in a similar way to Lemma 4.5.

Proposition 7.6 *In the situation of Theorem 7.4 we have $[\chi] = Y(L)$ in $H^1(\Sigma \times (T, \infty), \mathbb{R}) \cong H^1(\Sigma, \mathbb{R})$, where $Y(L)$ is as in Definition 7.2. Let γ be the unique 1-form on Σ with $d\gamma = d^*\gamma = 0$ and $[\gamma] = Y(L) \in H^1(\Sigma, \mathbb{R})$, which exists by Hodge theory. Then we may write $\chi = \pi^*(\gamma) + dE$, where $\pi : \Sigma \times (T, \infty) \rightarrow \Sigma$ is the projection and $E \in C^\infty(\Sigma \times (T, \infty))$, such that*

- (a) *If $\lambda < 0$ then $Y(L) = \gamma = 0$ and E is given by $E(\sigma, r) = -\int_r^\infty \chi^2(\sigma, s) ds$ and satisfies $|\nabla^k E| = O(r^{\lambda-k})$ for $k = 0, 1, 2$ as $r \rightarrow \infty$.*
- (b) *If $\lambda = 0$ then $|E| = O(|\log r|)$ and $|\nabla^k E| = O(r^{-k})$ for $k = 1, 2$.*

(c) If $\lambda > 0$ then $|\nabla^k E| = O(r^{\lambda-k})$ for $k = 0, 1, 2$ as $r \rightarrow \infty$.

Here we compute $\nabla, |\cdot|$ using the cone metric $\iota^*(g')$ on $\Sigma \times (T, \infty)$.

Proof. The proof that $[\chi] = Y(L)$ is similar to Proposition 7.3, and we leave it as an exercise. Let γ be as in the proposition. Then $\pi^*(\gamma)$ is a closed 1-form on $\Sigma \times (T, \infty)$ with $[\pi^*(\gamma)] = Y(L) = [\chi] \in H^1(\Sigma \times (T, \infty), \mathbb{R})$. Thus $\chi - \pi^*(\gamma)$ is an exact 1-form, and we may write $\chi - \pi^*(\gamma) = dE$ for some $E \in C^\infty(\Sigma \times (T, \infty))$, unique up to addition of constants.

For part (a), if $\lambda < 0$ then $Y(L) = 0$ by Proposition 7.3, so $\gamma = 0$. By (70) we see that $E'(\sigma, r) = -\int_r^\infty \chi^2(\sigma, s) ds$ is well-defined. The dr component in dE' is χ^2 , so that $\chi - dE'$ is a closed 1-form on $\Sigma \times (T, \infty)$ with no dr component, and is therefore independent of r . But (70) implies that $\chi - dE' = O(r^{\lambda-1})$ in the cone metric on $\Sigma \times (T, \infty)$, so $\chi - dE' = O(r^\lambda)$ in the cylinder metric, and taking the limit $r \rightarrow \infty$ gives $\chi - dE' = 0$ as $\lambda < 0$. Thus we may take $E = E'$, and (70) then yields $|\nabla^k E| = O(r^{\lambda-k})$ as $r \rightarrow \infty$ for $k = 0, 1, 2$.

For parts (b) and (c), using $|\nabla^k \pi^*(\gamma)| = O(r^{-1-k})$, equation (70) and $\chi = \pi^*(\gamma) + dE$, we find that if $\lambda \geq 0$ then $|\nabla^k E| = O(r^{\lambda-k})$ for $k = 1, 2$. But

$$E(\sigma, r) = E(\sigma, T+1) + \int_{T+1}^r \frac{dE}{dr}(\sigma, s) ds \quad (73)$$

for $r \geq T+1$, and $|\frac{dE}{dr}(\sigma, s)| \leq |\nabla E(\sigma, s)| = O(s^{\lambda-1})$. Substituting this into (73) gives $|E| = O(|\log r|)$ for $\lambda = 0$ and $|E| = O(r^\lambda)$ for $\lambda > 0$, which completes the proof. \square

7.3 The asymptotic behaviour of L at infinity

Finally we give analogues of the material of §5 for AC SL m -folds. Here is the analogue of Theorem 5.1. Stephen Marshall also has an independent proof.

Theorem 7.7 *In the situation of Theorem 7.4 and Proposition 7.6 we have*

$$\begin{aligned} |\nabla^k(\varphi - \iota)| &= O(r^{\lambda-1-k}), \quad |\nabla^k \chi| = O(r^{\lambda-1-k}) \quad \text{for all } k \geq 0 \\ \text{and } |\nabla^k E| &= O(r^{\lambda-k}) \quad \text{for all } k \geq 1 \text{ as } r \rightarrow \infty. \end{aligned} \quad (74)$$

Here $\nabla, |\cdot|$ are computed using the cone metric $\iota^*(g')$ on $\Sigma \times (T, \infty)$.

Proof. We modify the proof of Theorem 5.1. Let α be a smooth 1-form on $\Sigma \times (T, \infty)$ with $|\alpha(\sigma, r)| < \zeta r$, written $\alpha(\sigma, r) = \alpha^1(\sigma, r) + \alpha^2(\sigma, r) dr$ for $\alpha^1(\sigma, r) \in T_\sigma^* \Sigma$ and $\alpha^2(\sigma, r) \in \mathbb{R}$. Define a map $\Theta_\alpha : \Sigma \times (T, \infty) \rightarrow \mathbb{C}^m$ by $\Theta_\alpha(\sigma, r) = \Phi_c(\sigma, r, \alpha^1(\sigma, r), \alpha^2(\sigma, r))$. Define a smooth real function $F(\alpha)$ on $\Sigma \times (T, \infty)$ by $F(\alpha) dV = \Theta_\alpha^*(\text{Im } \Omega')$, where dV is the volume form of $\iota^*(g')$ on $\Sigma \times (T, \infty)$. As in (44)–(45), define

$$Q : \{(\sigma, r, y, z) : (\sigma, r) \in \Sigma \times (T, \infty), \quad y \in T_{(\sigma, r)}^*(\Sigma \times (T, \infty)), \quad (75)$$

$$|y| < \zeta r, \quad z \in \otimes^2 T_{(\sigma, r)}^*(\Sigma \times (T, \infty))\} \rightarrow \mathbb{R}$$

$$\text{by } Q(\sigma, r, \alpha(\sigma, r), \nabla \alpha(\sigma, r)) = (d^* \alpha + F(\alpha))[(\sigma, r)] \quad (76)$$

for all 1-forms α on $\Sigma \times (T, \infty)$ with $|\alpha(\sigma, r)| < \zeta r$ when $(\sigma, r) \in \Sigma \times (T, \infty)$. Then F, Q are well-defined, and analogous to F_i, Q_i in the proof of Theorem 5.1.

As $d^*\pi^*(\gamma) = 0$ on $\Sigma \times (T, \infty)$, the proofs of (49) and (50) give

$$Q(\sigma, r, y, z) = O(r^{-2}|y|^2 + |z|^2) \quad \text{when } |y| = O(r) \text{ and } |z| = O(1), \text{ and} \quad (77)$$

$$\Delta E(\sigma, r) - Q(\sigma, r, \pi^*(\gamma)(\sigma, r) + dE(\sigma, r), \nabla \pi^*(\gamma) + \nabla^2 E(\sigma, r)) = 0. \quad (78)$$

Adapting the rest of the proof of Theorem 5.1 now proves the result. Note that the $\pi^*(\gamma)$ terms in (78) are zero when $\lambda < 0$, and when $\lambda \geq 0$ they can be absorbed into the other estimates, and do not cause a problem. \square

From (74) we deduce that on $\Sigma \times (T, \infty)$ we have

$$|\nabla^k(\varphi^*(g') - \iota^*(g'))| = O(r^{\lambda-2-k}) \quad \text{as } r \rightarrow \infty \text{ for all } k \geq 0, \quad (79)$$

computing $\nabla, |\cdot|$ using $\iota^*(g')$. This is an analogue of equation (1). As in Theorem 5.2, equation (79) implies that (L, g') is an *Asymptotically Conical Riemannian manifold* with cone C and rate $\lambda-2 < 0$, in a sense analogous to Definition 2.1. Therefore we can develop a theory of *analysis on AC SL m -folds*, similar to §2. Here is the analogue of Definitions 2.6 and 2.7.

Definition 7.8 Let L be an AC SL m -fold in \mathbb{C}^m , as in Definition 7.1. Define the *radius function* $\rho : L \rightarrow [1, \infty)$ by $\rho(x) = (1 + |x|^2)^{1/2}$. For $\beta \in \mathbb{R}$ and $k \geq 0$ define $C_\beta^k(L)$ to be the space of continuous functions f on L with k continuous derivatives, such that $|\rho^{-\beta+j}\nabla^j f|$ is bounded on L for $j = 0, \dots, k$. Define the norm $\|\cdot\|_{C_\beta^k}$ on $C_\beta^k(L)$ by $\|f\|_{C_\beta^k} = \sum_{j=0}^k \sup_L |\rho^{-\beta+j}\nabla^j f|$. Then $C_\beta^k(L)$ is a Banach space. Define $C_\beta^\infty(L) = \bigcap_{k \geq 0} C_\beta^k(L)$.

For $p \geq 1$, $\beta \in \mathbb{R}$ and $k \geq 0$ define the *weighted Sobolev space* $L_{k,\beta}^p(L)$ to be the set of functions f on L that are locally integrable and k times weakly differentiable, and for which the norm $\|f\|_{L_{k,\beta}^p} = (\sum_{j=0}^k \int_L |\rho^{-\beta+j}\nabla^j f|^p \rho^{-m} dV_g)^{1/p}$ is finite. Then $L_{k,\beta}^p(L)$ is a Banach space, and $L_{k,\beta}^2(L)$ a Hilbert space.

We can now develop the theory of §2.2–§2.3 for these spaces. This is done in detail by Marshall [18, §4]. The basic references [16, 17] apply to Asymptotically Conical Riemannian manifolds just as for Riemannian manifolds with conical singularities. Theorem 2.9 holds for $C_\beta^k(L), L_{k,\beta}^p(L)$ except that the directions of the inequalities $\beta \geq \gamma, \beta > \gamma$ must be reversed. As in (8), let $\Delta = d^*d$ be the Laplacian on L , and for $p > 1, k \geq 2$ and $\beta \in \mathbb{R}$ write $\Delta_{k,\beta}^p$ for the map

$$\Delta_{k,\beta}^p = \Delta : L_{k,\beta}^p(L) \rightarrow L_{k-2,\beta-2}^p(L). \quad (80)$$

Then we can prove the following condensation of the analogue of §2.3:

Theorem 7.9 *Let L be an AC SL m -fold in \mathbb{C}^m , with cone C , and set $\Sigma = C \cap \mathcal{S}^{2m-1}$. Let \mathcal{D}_Σ and N_Σ be as in Definition 2.5. Let $p > 1, k \geq 2$ and $\beta \in \mathbb{R}$, and define $q > 1$ by $\frac{1}{p} + \frac{1}{q} = 1$. Let $\Delta_{k,\beta}^p$ be as in (80). Then*

- (a) $\Delta_{k,\beta}^p$ is Fredholm if and only if $\beta \notin \mathcal{D}_\Sigma$.

- (b) If $\Delta_{k,\beta}^p$ is Fredholm then $\text{ind}(\Delta_{k,\beta}^p) = N_\Sigma(\beta)$.
- (c) $\text{Ker}(\Delta_{k,\beta}^p)$ is a finite-dimensional subspace of $C_\beta^\infty(L)$, and independent of k . If $\beta \notin \mathcal{D}_\Sigma$ then $\text{Ker}(\Delta_{k,\beta}^p)$ is independent of p , and depends only on the connected component of $\mathbb{R} \setminus \mathcal{D}_\Sigma$ containing β . If $\beta < 0$ then $\text{Ker}(\Delta_{k,\beta}^p) = 0$.
- (d) Suppose $\Delta_{k,\beta}^p$ is Fredholm. Then $u \in L_{k-2,\beta-2}^p(L)$ lies in the image of $\Delta_{k,\beta}^p$ if and only if $\int_L uv \, dV = 0$ for all $v \in \text{Ker}(\Delta_{2,-\beta+2-m}^q)$.

We study the vector space V of bounded harmonic functions on L . A similar result is proved by Marshall [18, §5.1.3].

Theorem 7.10 *Suppose L is an AC SL m -fold in \mathbb{C}^m , with cone C . Let Σ, T and φ be as in Theorem 7.4. Let $l = b^0(\Sigma)$, and $\Sigma^1, \dots, \Sigma^l$ be the connected components of Σ . Let V be the vector space of bounded harmonic functions on L . Then $\dim V = l$, and for each $\mathbf{c} = (c^1, \dots, c^l) \in \mathbb{R}^l$ there exists a unique $v^\mathbf{c} \in V$ such that for all $j = 1, \dots, l$, $k \geq 0$ and $\beta \in (2 - m, 0)$ we have*

$$\nabla^k(\varphi^*(v^\mathbf{c}) - c^j) = O(|\mathbf{c}|r^{\beta-k}) \quad \text{on } \Sigma^j \times (T, \infty) \text{ as } r \rightarrow \infty. \quad (81)$$

Note also that $V = \{v^\mathbf{c} : \mathbf{c} \in \mathbb{R}^l\}$ and $v^{(1, \dots, 1)} \equiv 1$.

Proof. Let \mathcal{D}_Σ and N_Σ be as in Definition 2.5, and choose $p > 1$ and $0 < \gamma < \min(\mathcal{D}_\Sigma \cap (0, \infty))$. Let $v \in V$. Then $v \in L_{0,\gamma}^p(L)$ as v is bounded and $\gamma > 0$. Also v is smooth, as it is harmonic. By the analogue of Theorem 2.10 for AC SL m -folds we find that $v \in L_{k,\gamma}^p(L)$ for all $k \geq 0$. Fix $k \geq 2$. As $\Delta v = 0$ we have $v \in \text{Ker}(\Delta_{k,\gamma}^p)$, so that $V \subseteq \text{Ker}(\Delta_{k,\gamma}^p)$.

Part (a) of Theorem 2.12 shows that $\Delta_{k,\gamma}^p$ is Fredholm as $\gamma \notin \mathcal{D}_\Sigma$, so

$$\text{ind}(\Delta_{k,\gamma}^p) = N_\Sigma(\gamma) = N_\Sigma(0) = b^0(\Sigma) = l, \quad (82)$$

by part (b) of Theorem 2.12. Here $N_\Sigma(\gamma) = N_\Sigma(0)$ as $\mathcal{D}_\Sigma \cap (0, \gamma] = \emptyset$ and N_Σ is upper semicontinuous and discontinuous exactly on \mathcal{D}_Σ , and $N_\Sigma(0) = m_\Sigma(0) = b^0(\Sigma)$ is the multiplicity of the eigenvalue 0 of Δ_Σ . Now $\Delta_{k,\gamma}^p$ is surjective by parts (c) and (d) of Theorem 7.9. Thus $\dim \text{Ker}(\Delta_{k,\gamma}^p) = l$ by (82) and $V \subseteq \text{Ker}(\Delta_{k,\gamma}^p)$, which proves that $\dim V \leq l$.

Let $\mathbf{c} = (c^1, \dots, c^l) \in \mathbb{R}^l$ and $\beta \in (2 - m, 0)$, and choose a smooth function $\hat{v}^\mathbf{c}$ on L with $\hat{v}^\mathbf{c} \equiv c^j$ on $\varphi(\Sigma^j \times (T+1, \infty))$ for $j = 1, \dots, l$. Clearly this is possible. Then $\Delta \hat{v}^\mathbf{c}$ is smooth and compactly-supported on L , so $\Delta \hat{v}^\mathbf{c} \in L_{k-2,\beta-2}^p(L)$. Now as $\beta \in (2 - m, 0)$, $k \geq 2$ and $p > 1$ parts (a), (c) and (d) of Theorem 7.9 show that $\Delta_{k,\beta}^p : L_{k,\beta}^p(L) \rightarrow L_{k-2,\beta-2}^p(L)$ is an isomorphism.

Hence there exists a unique $\tilde{v}^\mathbf{c} \in L_{k,\beta}^p(L)$ with $\Delta \tilde{v}^\mathbf{c} = \Delta \hat{v}^\mathbf{c}$. This $\tilde{v}^\mathbf{c}$ is independent of $k \geq 2$ and β , so $\tilde{v}^\mathbf{c} \in C_\beta^\infty(L)$ by the analogue of Theorem 2.9, for all $\beta \in (2 - m, 0)$. Define $v^\mathbf{c} = \hat{v}^\mathbf{c} - \tilde{v}^\mathbf{c}$. Then $\Delta v^\mathbf{c} = \Delta \hat{v}^\mathbf{c} - \Delta \tilde{v}^\mathbf{c} = 0$, and (81) holds as $\hat{v}^\mathbf{c} \equiv c^j$ on $\varphi(\Sigma^j \times (T+1, \infty))$ and $\tilde{v}^\mathbf{c} \in C_\beta^\infty(L)$. Thus $v^\mathbf{c}$ is harmonic and bounded, so $v^\mathbf{c} \in V$.

All possible such functions $v^{\mathbf{c}}$ for $\mathbf{c} \in \mathbb{R}^l$ generate a vector subspace of V , of dimension at least l . But $\dim V \leq l$ from above. Hence each $v^{\mathbf{c}}$ is unique, and $V = \{v^{\mathbf{c}} : \mathbf{c} \in \mathbb{R}^l\}$, and $\dim V = l$, as we have to prove. Finally, $1 \in V$, and clearly $v^{(1, \dots, 1)} = 1$. \square

We finish by proving a version of Theorem 5.5 for AC SL m -folds, improving the rate of convergence λ .

Theorem 7.11 *Let L be an AC SL m -fold in \mathbb{C}^m with cone C and rate λ . Set $\Sigma = C \cap \mathcal{S}^{2m-1}$, and let $\mathcal{D}_\Sigma, N_\Sigma$ be as in Definition 2.5. Let ι, T, φ, χ be as in Theorem 7.4, and $Y(L), \gamma, E$ as in Proposition 7.6. Then*

(a) *Suppose λ, λ' lie in the same connected component of $\mathbb{R} \setminus \mathcal{D}_\Sigma$. Then*

$$\begin{aligned} |\nabla^k(\varphi - \iota)| &= O(r^{\lambda'-1-k}), & |\nabla^k \chi| &= O(r^{\lambda'-1-k}) \quad \text{and} \\ |\nabla^k E| &= O(r^{\lambda'-k}) \quad \text{as } r \rightarrow \infty \text{ for all } k \geq 0. \end{aligned} \quad (83)$$

Hence L is an AC SL m -fold with rate λ' . In particular, if $\lambda \in (2-m, 0)$ then L is an AC SL m -fold with rate λ' for all $\lambda' \in (2-m, 0)$.

(b) *Suppose $0 \leq \lambda < \min(\mathcal{D}_\Sigma \cap (0, \infty))$. Then adding a constant to E if necessary, for all $\lambda' \in (\max(-2, 2-m), 0)$ we have*

$$|\nabla^k E| = O(r^{\lambda'-k}) \quad \text{as } r \rightarrow \infty \text{ for all } k \geq 0. \quad (84)$$

Thus if $Y(L) = 0 = \gamma$ then L is an AC SL m -fold with rate λ' , and if $Y(L) \neq 0 \neq \gamma$ then L is an AC SL m -fold with rate 0.

Proof. Use the notation of the rest of this section. Define a smooth function $\tilde{E} : L \rightarrow \mathbb{R}$ by $\tilde{E}(\varphi(\sigma, r)) = E(\sigma, r)$ on $L \setminus K$, and extend \tilde{E} smoothly over K . Then $\tilde{E} \in C_\lambda^\infty(L)$ by (74). Write g_L for the metric $g'|_L$ on L , and g_C for the cone metric $\varphi_*(\iota^*(g'))$ on $L \setminus K \cong \Sigma \times (T, \infty)$. Let d_L^*, d_C^* be the d^* operators w.r.t. g_L, g_C on $L, L \setminus K$. Let $\Delta_L = d_L^* d$ be the Laplacian on L . Let Q be as in (75)–(76), and Q' the push-forward of Q to $L \setminus K$ under φ . Let $\tilde{\gamma}$ be the push-forward of $\pi^*(\gamma)$ to $L \setminus K$ under φ .

Following the proof of (62), equation (78) implies that for $x \in L \setminus K$

$$\Delta_L \tilde{E} = Q'(x, \tilde{\gamma} + d\tilde{E}(x), \nabla \tilde{\gamma} + \nabla^2 \tilde{E}(x)) + (d_L^* - d_C^*) d\tilde{E}[x]. \quad (85)$$

From (74), (77) and similar estimates on the derivatives of Q we deduce that

$$|\nabla^k(\Delta_L \tilde{E})| = \begin{cases} O(\rho^{-4-k}) + O(\rho^{2\lambda-4-k}), & \gamma \neq 0, \\ O(\rho^{2\lambda-4-k}), & \gamma = 0, \end{cases} \quad (86)$$

as $|\nabla^k \tilde{\gamma}| = O(\rho^{-1-k})$. Since $\lambda \geq 0$ when $\gamma \neq 0$, this gives $\Delta_L \tilde{E} \in C_{2\lambda-4}^\infty(L)$, which is the analogue of Lemma 5.6. Here is the analogue of Lemma 5.7.

Lemma 7.12 *Let $p > 1$, $k \geq 2$ and λ, λ' lie in the same connected component of $\mathbb{R} \setminus \mathcal{D}_\Sigma$. If $\tilde{E} \in L_{k,\lambda}^p(L)$ and $\Delta_L \tilde{E} \in L_{k-2,\lambda'-2}^p(L)$, then $\tilde{E} \in L_{k,\lambda'}^p(L)$.*

Proof. This is trivial for $\lambda' \geq \lambda$, so suppose $\lambda' < \lambda$. As $\lambda \in \mathbb{R} \setminus \mathcal{D}_\Sigma$ part (a) of Theorem 7.9 shows that $\Delta_{k,\lambda}^p$ is Fredholm, and thus part (d) that $\Delta_L \tilde{E}$ is L^2 -orthogonal to $\text{Ker}(\Delta_{2,-\lambda+2-m}^q)$. But λ, λ' lie in the same connected component of $\mathbb{R} \setminus \mathcal{D}_\Sigma$, and \mathcal{D}_Σ is preserved by the involution $\beta \mapsto -\beta + 2 - m$ by (4), so $-\lambda + 2 - m, -\lambda' + 2 - m$ lie in the same connected component of $\mathbb{R} \setminus \mathcal{D}_\Sigma$.

Hence $\text{Ker}(\Delta_{2,-\lambda+2-m}^q) = \text{Ker}(\Delta_{2,-\lambda'+2-m}^q)$ by part (c), so $\Delta_L \tilde{E}$ lies in $L_{k-2,\lambda'-2}^p(L)$ by assumption and is L^2 -orthogonal to $\text{Ker}(\Delta_{2,-\lambda'+2-m}^q)$. Thus $\Delta_L \tilde{E}$ lies in the image of $\Delta_{k,\lambda'}^p$ by part (d) of Theorem 7.9.

Therefore $\Delta_L \tilde{E} = \Delta_L E'$ for some $E' \in L_{k,\lambda'}^p(L)$. Then $E' \in L_{k,\lambda}^p(L)$ as $\lambda' < \lambda$, so $\tilde{E} - E' \in \text{Ker}(\Delta_{k,\lambda}^p)$. But $\text{Ker}(\Delta_{k,\lambda}^p) = \text{Ker}(\Delta_{k,\lambda'}^p)$ by part (c), so both E' and $\tilde{E} - E'$ lie in $L_{k,\lambda'}^p(L)$, and $\tilde{E} \in L_{k,\lambda'}^p(L)$. \square

We can now use the method of Theorem 5.5 to decrease the rate λ by an inductive process. Applying Lemma 7.12 repeatedly j times as in the proof of Theorem 5.5 shows that (83) holds for all λ' in the same connected component of $\mathbb{R} \setminus \mathcal{D}_\Sigma$ as λ with $\lambda' - 2 > 2^j(\lambda - 2)$. But $2^j(\lambda - 2) \rightarrow -\infty$ as $j \rightarrow \infty$ since $\lambda < 2$, so this proves part (a) of Theorem 7.11.

Now suppose that L has rate λ with $0 \leq \lambda < \min(\mathcal{D}_\Sigma \cap (0, \infty))$, as in part (b). Then (a) implies that L has rate μ for all μ with $0 < \mu < \min(\mathcal{D}_\Sigma \cap (0, \infty))$. Thus (86) gives $\Delta_L \tilde{E} \in C_{2\mu-4}^\infty(L)$. Therefore $\Delta_L \tilde{E} \in C_{\lambda'-2}^\infty(L)$ for all $\lambda' > -2$, and $\Delta_L \tilde{E} \in L_{k-2,\lambda'-2}^p(L)$ for all $p > 1, k \geq 2$ and $\lambda' > -2$.

Let $p > 1, k \geq 2$ and $\lambda' \in (\max(-2, 2 - m), 0)$. Then $\Delta_L \tilde{E} \in L_{k-2,\lambda'-2}^p(L)$, and $\lambda' > 2 - m$ implies $-\lambda' + 2 - m < 0$, so that $\text{Ker}(\Delta_{2,-\lambda'+2-m}^q) = 0$ by part (c) of Theorem 7.9. Thus $\Delta_L \tilde{E}$ is trivially L^2 -orthogonal to $\text{Ker}(\Delta_{2,-\lambda'+2-m}^q)$. Also $\Delta_{k,\lambda'}^p$ is Fredholm by part (a) of Theorem 7.9, as $\lambda' \in (2 - m, 0)$ and $\mathcal{D}_\Sigma \cap (2 - m, 0) = \emptyset$. Therefore part (d) of Theorem 7.9 shows that $\Delta_L \tilde{E}$ lies in the image of $\Delta_{k,\lambda'}^p$, so $\Delta_L \tilde{E} = \Delta_L E'$ for some $E' \in L_{k,\lambda'}^p(L)$.

As $\lambda' < \lambda$ we have $E' \in L_{k,\lambda}^p(L)$, and so $\tilde{E} - E' \in \text{Ker}(\Delta_{k,\lambda}^p)$. Increasing λ if $\lambda = 0$ we may take $0 < \lambda < \min(\mathcal{D}_\Sigma \cap (0, \infty))$, so that $\lambda \notin \mathcal{D}_\Sigma$ and $N_\Sigma(\lambda) = m_\Sigma(0) = b^0(\Sigma)$. Parts (a), (c) and (d) then show that $\Delta_{k,\lambda}^p$ is Fredholm and surjective with $\text{ind}(\Delta_{k,\lambda}^p) = N_\Sigma(\lambda)$.

Hence $\dim \text{Ker}(\Delta_{k,\lambda}^p) = b^0(\Sigma)$. If $b^0(\Sigma) = 1$ then $\text{Ker}(\Delta_{k,\lambda}^p) = \langle 1 \rangle$, the constant functions on L . More generally, if $b^0(\Sigma) = l$ then L has l ends at infinity, and elements of $\text{Ker}(\Delta_{k,\lambda}^p)$ are harmonic functions on L which are asymptotic to $O(\rho^{2-m})$ at infinity to a constant c_i for $i = 1, \dots, l$ on each of the l ends. The values of c_1, \dots, c_l parametrize $\text{Ker}(\Delta_{k,\lambda}^p) \cong \mathbb{R}^l$.

Now the function $E \in C^\infty(\Sigma \times (T, \infty))$ was defined in Proposition 7.6 to satisfy $dE = \chi$. Thus, if $b^0(\Sigma) = l$ then E is unique up to the addition of a constant on each of the l components of $\Sigma \times (T, \infty)$. By choosing these constants

appropriately we can set to zero the constants c_1, \dots, c_l that $\tilde{E} - E'$ is asymptotic to on the l ends of L . Then $\tilde{E} - E' = 0$, as c_1, \dots, c_l parametrize $\text{Ker}(\Delta_{k,\lambda}^p)$.

Thus adding a constant to E if necessary we have $\tilde{E} = E'$, so $\tilde{E} \in L_{k,\lambda'}^p(L)$. As this holds for all $p > 1$ and $k \geq 2$ we have $\tilde{E} \in C_{\lambda'}^\infty(L)$ by the Asymptotically Conical version of Theorem 2.9. But $\varphi^*(\tilde{E}) = E$ on $\Sigma \times (T, \infty)$, so this implies (84). The last part is immediate. This concludes the proof of Theorem 7.11. \square

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