

Blow-ups in non-reductive GIT and moduli spaces of unstable vector bundles over curves



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Chapter 1

Introduction

The thesis aims to use non-reductive geometric invariant theory (GIT) to study moduli problems involving vector bundles of fixed Harder-Narasimhan type on a nonsingular complex projective curve, or more generally pure sheaves on a projective scheme over an algebraically closed field of characteristic 0.

The construction of moduli spaces for such sheaves can be reduced to the construction of non-reductive GIT quotients of subschemes of Quot schemes. Quot schemes are projective schemes which are in general not reduced, and so far non-reductive GIT has been developed only for actions on projective varieties. So our first task in this thesis is to put the relevant existing non-reductive GIT into a scheme-theoretic framework and extend the results to this setting. In particular a scheme-theoretic version (Condition 3.3.2) is provided of the condition that “semistability coincides with stability” (or “ss=s”) for the action on a projective scheme X of a semi-direct product $\hat{U} = U \rtimes_{\lambda} \mathbb{G}_m$ where U is a unipotent group graded by a one-parameter subgroup $\lambda : \mathbb{G}_m \rightarrow \hat{U}$ of $\hat{U}$. It is shown that under this condition an open subscheme of X has a geometric quotient $X//\hat{U}$ which is itself a projective scheme (possibly non-reduced), and that under a weaker condition (Condition 4.2.1) it is possible to blow X up along a sheaf of ideals so that the condition “ss=s” holds. In contrast to the construction of [Bér+16] which involves a sequence of blow-ups, we only require one blow-up to ensure that “ss=s” holds. Finally, it is shown that this weaker condition always holds for sheaves of Harder-Narasimhan length two, and holds for sheaves of Harder-Narasimhan length greater than two under some additional hypotheses. Moduli spaces of sheaves of Harder-Narasimhan length 2 have been studied in [BMN09] and [BS22] using

methods different from GIT, and the dimension of the space of *unipotent endomorphisms* is also considered as an important discrete invariant.

To describe some of the contents in more detail, let $\mathbb{k}$ be an algebraically closed field of characteristic zero. Let $\text{Sch}/\mathbb{k}$ be the category of schemes of finite type over $\mathbb{k}$. Let C be a nonsingular projective curve over $\mathbb{k}$. Let τ be a Harder-Narasimhan type of length 2 and let $\delta \in \mathbb{N}$ be a natural number. In this thesis, we pose a moduli problem

$$\mathbf{M}'_{\tau,\delta} : (\text{Sch}/\mathbb{k})^{\text{op}} \rightarrow \text{Set} \quad (1.1)$$

classifying (τ, δ) -stable vector bundles on C and construct the coarse moduli space of $\mathbf{M}'_{\tau,\delta}$. To this end, we develop existing non-reductive geometric invariant theory for algebraic schemes over $\mathbb{k}$, and apply it to a non-reductive GIT problem to produce a coarse moduli space. Unlike the moduli spaces in [Jac21], our coarse moduli space of $\mathbf{M}'_{\tau,\delta}$ is not automatically reduced by construction.

In [Jac21] Definition 7.1.2, a vector bundle E on C is called τ -stable, if E has Harder-Narasimhan type τ and $E \not\cong E_1 \oplus E_2$ and each $E_i := \text{HN}_i(E)/\text{HN}_{i-1}(E)$ is stable for $i = 1, 2$, where $\{\text{HN}_i(E)\}_{i=0}^2$ is the Harder-Narasimhan filtration of E . In [HJ21] Definition 6.1, Hoskins and Jackson called (τ, δ) a *refined Harder-Narasimhan type of length 2*, and the vector bundle E on C is called (τ, δ) -stable, if E is τ -stable and $\dim \text{Hom}(E_2, E_1) = \delta$. For a vector bundle E of type τ , we refer to its space of *unipotent endomorphisms* as the vector space $\text{Hom}(E_2, E_1)$. One key idea behind these new stability conditions is that we do not only fix the Harder-Narasimhan type τ , but also fix dimensions of spaces of *unipotent endomorphisms*, which are all regarded as discrete invariants. Moduli of (τ, δ) -vector bundles were studied in [Jac21] Theorem 7.1.3. However, Jackson did not associate the moduli spaces with moduli functors and his construction was in the category of varieties¹.

To define the moduli functor $\mathbf{M}'_{\tau,\delta}$, flat families of (τ, δ) -stable vector bundles are considered. For $T \in \text{Sch}/\mathbb{k}$, a *flat family of (τ, δ) -stable vector bundles parameterised by T* , denoted by $(E, \{E_{\leq i}\}_{i=0}^2)$, consists of the following data:

¹The condition on irreducibility for varieties can be removed for existing non-reductive GIT. See [HJ21] Section 5.4 for a discussion. However, the condition on reducedness are not easily removed, which will be treated in Chapter 3.

- $0 = E_{\leq 0} \subseteq E_{\leq 1} \subseteq E_{\leq 2} = E$ is a filtration of coherent sheaves on $C_T := C \times_{\mathbb{k}} T$;
- $E_i := E_{\leq i}/E_{\leq i-1}$ is flat over T for $i = 1, 2$;
- for any closed point $t \in T$, the fibre E_t is τ -stable with Harder-Narasimhan filtration $\{E_{\leq i,t}\}_{i=0}^2$;
- for any closed subscheme $T' \hookrightarrow T$, the *sheaf of unipotent endomorphisms* of the restricted family $(E|_{T'}, \{E_{\leq i}|_{T'}\}_{i=0}^2)$, defined as

$$(C_{T'} \rightarrow T')_* \mathcal{H}om_{\mathcal{O}_{C_{T'}}}(E_2|_{T'}, E_1|_{T'}) \quad (1.2)$$

is locally free of rank δ on T' .

This notion generalises the concept of (τ, δ) -stable vector bundles in the sense that a flat family of (τ, δ) -stable vector bundles parameterised by $\text{Spec}(\mathbb{k})$ is exactly a (τ, δ) -stable vector bundle.

A flat family of (τ, δ) -stable vector bundles parameterised by T is not simply a flat family $(E, \{E_{\leq i}\}_{i=0}^2)$ such that closed fibres are (τ, δ) -stable vector bundles. One reason is that the condition on closed points $t \in T$ that $\dim \text{Hom}(E_{2,t}, E_{1,t}) = \delta$ is not an open condition but only a locally closed condition. In this thesis, we generalise the notion of unipotent endomorphisms such that they are defined for a flat family of vector bundles. It turns out that spaces of unipotent endomorphisms of vector bundles generalise to coherent sheaves of unipotent endomorphisms of families. The condition on unipotent endomorphisms we impose for a flat family of (τ, δ) -stable vector bundles is that the sheaf of unipotent endomorphisms of any pull-back of this family is locally free of rank δ .

The moduli functor $\mathbf{M}'_{\tau, \delta}$ is defined as follows

$$\begin{aligned} \mathbf{M}'_{\tau, \delta} : (\text{Sch}/\mathbb{k})^{\text{op}} &\rightarrow \text{Set} \\ T &\mapsto \left\{ \begin{array}{c} \text{flat families of } (\tau, \delta)\text{-stable} \\ \text{vector bundles parameterised by } T \end{array} \right\} / \text{isomorphisms} \end{aligned} \quad (1.3)$$

and for a morphism $g : T' \rightarrow T$ in $\text{Sch}/\mathbb{k}$, the map $\mathbf{M}'_{\tau, \delta}(g)$ is the map sending the isomorphism class of $(E, \{E_{\leq i}\}_{i=0}^2)$ to the class of $(g_C^* E, \{g_C^* E_{\leq i}\}_{i=0}^2)$ where $g_C : C_{T'} \rightarrow C_T$ is the morphism $g_C := 1_C \times g$.

We will apply non-reductive geometric invariant theory to construct the coarse moduli space of $\mathbf{M}'_{\tau,\delta}$. In [Sim94], GIT for reductive group actions was applied to the construction of moduli spaces of semistable sheaves. For unstable sheaves, non-reductive GIT problems have been set up, and their moduli spaces have been studied in [HK12], [Jac18], [Jac21] and [HJ21]. New stability conditions have been considered for unstable sheaves, such as τ -stability (Definition 7.12, [Jac21] and Definition 4.18, [HJ21]) and *refined Harder-Narasimhan type* (Definition 6.1, [HJ21]). However, in [Jac18], [Jac21] and [HJ21], moduli functors and families of τ -stable or (τ, δ) -stable vector bundles were not considered. In this thesis, we will set up and study a non-reductive GIT problem similar to that in [Jac21] for the moduli space of $\mathbf{M}'_{\tau,\delta}$. GIT for non-reductive group actions has been studied in [DK07], [Bér+16], [BK17], [BDK18], [Bér+18b], [Bér+18a] and [HJ21], and many results in non-reductive GIT have been applied to the study of moduli of unstable curves in [Jac18], and the moduli of unstable sheaves in [Jac21], [HJ21]. The non-reductive GIT problem we set up for the moduli of $\mathbf{M}'_{\tau,\delta}$ involves an action of a parabolic group on a projective scheme, in the category $\text{Sch}/\mathbb{k}$. However in the references listed above, non-reductive GIT was only considered in the category of varieties. This motivates us to develop the existing non-reductive GIT, especially the so-called U-hat theory in [Bér+16], [Bér+18b], for actions on projective schemes.

The version of non-reductive GIT developed in this thesis has the potential to produce non-reduced quotients. The moduli spaces constructed in [Jac18], [Jac21] and [HJ21] are necessarily reduced since the non-reductive GIT they applied in the construction only produces quotient varieties. Whether the moduli space is reduced is not a tautology for our moduli space of $\mathbf{M}'_{\tau,\delta}$ and needs further research, in contrast to the moduli spaces in [Jac18], [Jac21] and [HJ21], where the answer is always “yes” if we ask this question.

The structure of the thesis is as follows.

Chapter 2 In this chapter, definitions and theorems of the classic reductive geometry invariant theory are reviewed. Most of the contents can be found in [MFK94] and [Hos16]. Besides the classic material, we introduce some new definitions concerning Lie algebras

and infinitesimal actions. For a group action $G \curvearrowright X$ in $\text{Sch}/\mathbb{k}$, there is a morphism $\phi : \Omega_{X/\mathbb{k}} \rightarrow \mathfrak{g}^* \otimes_{\mathbb{k}} \mathcal{O}_X$, which encodes the information of the action of the Lie algebra $\mathfrak{g}$ on X . The coherent sheaf $\mathcal{Q} := \text{coker}(\phi)$ on X plays an important role in later chapters. We also define *T-valued infinitesimal transformations in G* and when they *stabilise T-valued points in X* for $T \in \text{Sch}/\mathbb{k}$. For $f : T \rightarrow X$, we have a characterisation (Proposition 2.2.9) of the condition that $f^*\mathcal{Q}$ is locally free of rank r in terms of T' -valued infinitesimal stabilisers for $T' \in \text{Sch}/T$.

Chapter 3 In this chapter, the so-called U-hat theorem in [Bér+18b] and [Bér+16] is generalised for $\hat{U}$ actions on projective schemes in $\text{Sch}/\mathbb{k}$ to Theorem 3.3.8 and Lemma 3.4.2.

The group $\hat{U} := U \rtimes_{\lambda} \mathbb{G}_m$ is a semi-direct product of a unipotent group U and a multiplicative group $\mathbb{G}_m$ via a *grading 1PS* $\lambda : \mathbb{G}_m \rightarrow \text{Aut}(U)$, where a *grading 1PS* refers to $\lambda : \mathbb{G}_m \rightarrow \text{Aut}(U)$ such that the induced action $\lambda \curvearrowright \mathfrak{u} := \text{Lie}(U)$ has all weights positive. Let $w_1 > \dots > w_n$ be the λ -weights on $\mathfrak{u} := \text{Lie}(U)$. For $1 \leq i \leq n$, let $\mathfrak{u}_i := \mathfrak{u}_{\lambda \geq w_i}$ denote the sum of weight subspaces whose λ -weights are greater than or equal to w_i . Let $U_i \subseteq U$ be the subgroup whose Lie algebra is $\mathfrak{u}_i$.

Let X be a projective scheme in $\text{Sch}/\mathbb{k}$ with an ample line bundle L on X . Let $\hat{U} \curvearrowright (X, L)$. Let $X_{\min}^0 \subseteq X$ be the non-vanishing locus of $\bigoplus_{d>0} H^0(X, L^d)_{\lambda=\max}$, where $H^0(X, L^d)_{\lambda=\max} \subseteq H^0(X, L^d)$ is the weight subspace of maximal λ -weight. Let $Z_{\min}^0 \hookrightarrow X_{\min}^0$ be the *fixed point closed subscheme with respect to the action of λ* in the sense of [Fog73]. Assume $Z_{\min}^0 \neq \emptyset$.

When X is reduced, the U-hat theorem in [Bér+16] requires the so-called “semistability coincides with stability” condition (“ss=s”), which is that $\dim \text{Stab}_{U_i}(x)$ is constant on $x \in X_{\min}^0$ for $1 \leq i \leq n$ (Remark 7.12, [Bér+16]). When X is not assumed reduced, it turns out the condition “ss=s” needs to be replaced by the condition (Condition 3.3.2) that $\text{coker}(\Omega_{X/\mathbb{k}} \rightarrow \mathfrak{u}_i \otimes_{\mathbb{k}} \mathcal{O}_X)$ is locally free of constant rank on $X_{\min}^0$ for $1 \leq i \leq n$. These two conditions coincide when X is reduced (Lemma 3.3.4).

With the generalised condition “ss=s” for non-reduced X assumed, the proofs in Section 7, [Bér+16] can be easily adapted for our U-hat theorem (Theorem 3.3.8).

For the construction of the moduli space of $\mathbf{M}'_{\tau,\delta}$, we consider a linearised action on (X, L) of the group $U \rtimes (T \times K)$, where U is unipotent, T is a torus and K is semisimple. Assume there exists an *internal grading 1PS*, i.e. a 1PS $\lambda : \mathbb{G}_m \rightarrow T$ such that the composition $\mathbb{G}_m \xrightarrow{\lambda} T \rightarrow \text{Aut}(U)$ is a grading 1PS. With some assumptions specific to the GIT problem associated to the moduli problem, we prove Lemma 3.4.2 about quotients by $U \rtimes (T \times K)$.

Chapter 4 In this chapter, we prove that when Condition 3.3.2 (“ss=s”) is not satisfied for $\hat{U} \curvearrowright (X, L)$, but a weaker condition (Condition 4.2.1) is satisfied, we can perform an equivariant blow-up to get “ss=s”. We call this condition the *Weak Unipotent Upstairs Stabiliser Condition* (WUU), where the name is from [HJ21], Definition 4.42. For reduced X , Section 8 in [Bér+16] described a sequence of equivariant blow-ups for “ss=s” under the condition that for $1 \leq i \leq n$, the minimum of $\dim \text{Stab}_{U_i}(z)$ for $z \in Z_{\min}^0$ equals the minimum of $\dim \text{Stab}_{U_i}(x)$ for $x \in X_{\min}^0$. The blow-up described in this chapter is different from that in [Bér+16] even when X is reduced. Here we only need one blow-up rather than a sequence of them.

Since X is not assumed reduced, the centre of the blow-up is not determined by the underlying closed subset. We describe the centre scheme-theoretically and prove our result (Theorem 4.4.1) in a more algebraic manner than [Bér+16].

Chapter 5 In this chapter, we describe a scheme-theoretic stratification according to dimensions of unipotent stabilisers (Theorem 5.1.2). Let U be a unipotent group acting on $X \in \text{Sch}/\mathbb{k}$. Let $\{\mathfrak{u}_i\}_{i=0}^n$ be a filtration of $\mathfrak{u} := \text{Lie}(U)$. Let $\delta = (d_1, \dots, d_n) \in \mathbb{N}^n$. There is a locally closed subset $S_\delta \subseteq X$ such that closed points in S_δ are $x \in X$ with $\dim \text{Stab}_{\mathfrak{u}_i}(x) = d_1 + \dots + d_i$ for $1 \leq i \leq n$. We show that there exists a locally closed subscheme structure on S_δ such that $\mathcal{Q}_i := \text{coker}(\Omega_{X/\mathbb{k}} \rightarrow \mathfrak{u}_i^* \otimes_{\mathbb{k}} \mathcal{O}_X)$ pulls back to a locally free sheaf of rank $d_1 + \dots + d_i$ on S_δ for $1 \leq i \leq n$. The subscheme $S_\delta \subseteq X$ is U -invariant. Moreover, the pull-back $\mathcal{Q}_i|_{S_\delta}$ equals the cokernel of $\Omega_{S_\delta/\mathbb{k}} \rightarrow \mathfrak{u}_i^* \otimes_{\mathbb{k}} \mathcal{O}_{S_\delta}$. If X is projective and $\hat{U} \curvearrowright X$ is linearised with respect to L , then under some hypothesis (Condition 5.1.6), the scheme-theoretic image $\overline{S_\delta}$ of $S_\delta \hookrightarrow X$, with the restricted line bundle $L|_{\overline{S_\delta}}$, satisfies Condition 4.2.1. So we can apply results from previous chapters and prove Theorem 5.2.1.

Chapter 6 In this final chapter, we start to consider the moduli problem $\mathbf{M}'_{\tau, \delta}$.

Let C be a nonsingular projective curve over $\mathbb{k}$ with a degree one line bundle $\mathcal{L}$. Our results can also be generalised to higher dimensional projective schemes. For the time being we assume τ is a general Harder-Narasimhan type of length $n \in \mathbb{N}_+$. Let $P(t), P_i(t)$ be the Hilbert polynomials such that if E is a vector bundle of type τ with Harder-Narasimhan filtration $\{\text{HN}_i(E)\}_{i=0}^n$, then $P(t)$ is the Hilbert polynomial of E and $P_i(t)$ is the Hilbert polynomial of $\text{HN}_i(E)/\text{HN}_{i-1}(E)$.

Let $m \in \mathbb{N}_+$ be such that $\mathcal{L}^m$ is very ample. For $l \gg 0$, let $V_i := \mathbb{k}^{P_i(lm)}$ for $1 \leq i \leq n$ and let $V = V_1 \oplus \cdots \oplus V_n$. Let $P_\tau \subseteq \text{SL}(V)$ be the parabolic subgroup stabilising the filtration $\{V_{\leq i}\}_{i=0}^n$, where $V_{\leq i} := V_1 \oplus \cdots \oplus V_i$.

Let $V(-lm)$ denote the coherent sheaf $V \otimes_{\mathbb{k}} \mathcal{L}^{-lm}$ on C , and similarly for $V_i(-lm)$ and $V_{\leq i}(-lm)$. Vector bundles of type τ are bounded, and then any of them can be represented by a closed point in $\text{Quot}_{V(-lm)/C/\mathbb{k}}^{P(t), \mathcal{L}}$ up to $\text{SL}(V)$ -action. There exist locally closed subspaces $Y_\tau^s \subseteq Y_\tau^{\text{ss}} \subseteq \text{Quot}_{V(-lm)/C/\mathbb{k}}^{P(t), \mathcal{L}}$ such that vector bundles of type τ are represented by P_τ -orbits of closed points in Y_τ^{ss} , and if E is a vector bundle of type τ such that $\text{HN}_i(E)/\text{HN}_{i-1}(E)$ is stable for $1 \leq i \leq n$, then its P_τ -orbit is in Y_τ^s . The subspaces $Y_\tau^{(s)s} \subseteq \text{Quot}_{V(-lm)/C/\mathbb{k}}^{P(t), \mathcal{L}}$ were described in [HK12], [Jac18], [Jac21] and [HJ21] to study moduli of vector bundles of type τ .

In the references listed above, $Y_\tau^s \subseteq Y_\tau^{\text{ss}}$ are only defined as locally closed subsets in the Zariski topology and are equipped with the induced reduced subscheme structures for the application of non-reductive GIT. We provide moduli-theoretic descriptions of $Y_\tau^{(s)s} \subseteq \text{Quot}_{V(-lm)/C/\mathbb{k}}^{P(t), \mathcal{L}}$ and then subscheme structures are determined. The author believes it is the natural way to define $Y_\tau^{(s)s}$ moduli-theoretically, since the natural way to define $\text{Quot}_{V(-lm)/C/\mathbb{k}}^{P(t), \mathcal{L}}$ is moduli-theoretic. The advantage is that we have a moduli-theoretic description of scheme-valued points of $Y_\tau^{(s)s}$, which eventually motivates our definition of flat families of (τ, δ) -stable vector bundles. The disadvantage is that without necessarily being reduced, we need non-reductive GIT for non-reduced schemes, and many point-wise statements need to be carefully adapted, such as the condition “ss=s” in [Bér+16].

Let $U_\tau \subseteq P_\tau$ be the unipotent radical, and let $K \subseteq P_\tau$ be its semi-simple subgroup. There exists a homomorphism $T := (\mathbb{G}_m)^{n-1} \rightarrow P_\tau$ such that $U_\tau \rtimes (T \times K) \rightarrow P_\tau$ is

surjective with finite kernel. We can safely replace P_τ by $U_\tau \rtimes (T \times K)$ in the non-reductive GIT problem. Choose an internal grading 1PS $\lambda : \mathbb{G}_m \rightarrow T$. There is an induced filtration of $\mathfrak{u} := \text{Lie}(U_\tau)$ of length $N \in \mathbb{N}_+$

$$0 = \mathfrak{u}_0 \subseteq \mathfrak{u}_1 \subseteq \cdots \subseteq \mathfrak{u}_N = \mathfrak{u} \quad (1.4)$$

where $w_1 > \cdots > w_N$ are all the weights of $\lambda \curvearrowright \mathfrak{u}$ and $\mathfrak{u}_i := \mathfrak{u}_{\lambda \geq w_i}$, the sum of weight spaces whose λ -weights are no less than w_i .

We can stratify Y_τ^s according to dimensions of unipotent stabilisers with respect to $\{\mathfrak{u}_i\}_{i=0}^N$ (Theorem 5.1.2). Let $\coprod_{\delta \in \mathbb{N}^N} S_\delta(Y_\tau^s) \rightarrow Y_\tau^s$ be the stratification, where $S_\delta(Y_\tau^s) \subseteq Y_\tau^s$ for $\delta = (d_1, \dots, d_N) \in \mathbb{N}^N$ is a locally closed subscheme whose closed points are $y \in Y_\tau^s$ such that $\dim \text{Stab}_{\mathfrak{u}_i}(y) = d_1 + \cdots + d_i$ for $1 \leq i \leq n$. This stratification is a scheme-theoretic version of the decomposition of Y_τ^s according to dimensions of unipotent stabilisers in [Jac21] and [HJ21].

Fix $\delta \in \mathbb{N}^N$. By Theorem 5.2.1, there is a P_τ -invariant open subscheme $Y_{\tau,\delta}^s \subseteq S_\delta(Y_\tau^s)$ whose closed points are $y \in S_\delta(Y_\tau^s)$ corresponding to non-split vector bundles, i.e. closed points in $Y_{\tau,\delta}^s$ correspond to (τ, δ) -stable vector bundles up to the P_τ -action. Also $Y_{\tau,\delta}^s \rightarrow Y_{\tau,\delta}^s/P_\tau$ is a quasi-projective universal geometric quotient by P_τ , which is the coarse moduli space of $\mathbf{M}'_{\tau,\delta}$.

We can not always apply Theorem 5.2.1, since Condition 5.1.6 does not hold in general. However, when the Harder-Narasimhan type has length 2, and more generally when $\mu_i - \mu_{i+2} > 2g - 2$ for $1 \leq i < i+2 \leq n$, we have Condition 5.1.6. So we can construct the moduli spaces for Harder-Narasimhan type of length 2 and some special longer Harder-Narasimhan types.

Chapter 2

Affine algebraic groups and reductive Geometric Invariant Theory

This chapter review some related material in reductive GIT, including definitions for groups, group actions, notions of quotients and the main theorem in reductive GIT. Most contents can be found in [MFK94] and [Hos16].

Besides, some new definitions and lemmas about Lie algebras and scheme-valued infinitesimal transformations are presented in Section 2.2, which will play a role in Chapter 5 and Chapter 6.

2.1 Affine algebraic groups

2.1.1 Definitions and examples

Definition 2.1.1. An *affine algebraic group* G is an affine scheme of finite type over $\mathbb{k}$ with morphisms

$$\mu : G \times_{\mathbb{k}} G \rightarrow G, \quad e : \mathrm{Spec}(\mathbb{k}) \rightarrow G, \quad \iota : G \rightarrow G \quad (2.1)$$

satisfying:

- associativity:

$$\begin{array}{ccccc}
 G \times_{\mathbb{k}} G \times_{\mathbb{k}} G & \xrightarrow{\mu \times 1} & G \times_{\mathbb{k}} G & & \\
 \downarrow 1 \times \mu & & \downarrow \mu & & \\
 G \times_{\mathbb{k}} G & \xrightarrow{\mu} & G & &
 \end{array}$$
- unit:

$$\begin{array}{ccccc}
 G \times_{\mathbb{k}} \mathrm{Spec}(\mathbb{k}) & \xrightarrow{1 \times e} & G \times_{\mathbb{k}} G & \xleftarrow{e \times 1} & \mathrm{Spec}(\mathbb{k}) \times_{\mathbb{k}} G \\
 \searrow \cong & & \downarrow \mu & & \swarrow \cong \\
 & & G & &
 \end{array}$$

$$\begin{array}{ccccc}
G & \xrightarrow{1 \times \iota} & G \times_{\mathbb{k}} G & \xleftarrow{\iota \times 1} & G \times_{\mathbb{k}} G \\
\downarrow & & \downarrow \mu & & \downarrow \\
\text{Spec}(\mathbb{k}) & \xrightarrow{e} & G & \xleftarrow{e} & \text{Spec}(\mathbb{k}).
\end{array}$$

• inverse:

We call the morphism $\mu : G \times_{\mathbb{k}} G \rightarrow G$ the *multiplication*, the morphism $e : \text{Spec}(\mathbb{k}) \rightarrow G$ the *unit*, and the morphism $\iota : G \rightarrow G$ the *inverse*.

Since $G \cong \text{Spec}(\mathcal{O}(G))$ is affine, the morphisms μ, e, ι are equivalent to the following ring maps

$$\mu^* : \mathcal{O}(G) \rightarrow \mathcal{O}(G) \otimes_{\mathbb{k}} \mathcal{O}(G), \quad e^* : \mathcal{O}(G) \rightarrow \mathbb{k}, \quad \iota^* : \mathcal{O}(G) \rightarrow \mathcal{O}(G) \quad (2.2)$$

and we call μ^*, e^*, ι^* the *co-multiplication*, *co-unit*, *co-inverse* respectively.

Definition 2.1.2. Let G, H be affine algebraic groups with multiplications μ_G, μ_H respectively. A *group homomorphism* $f : G \rightarrow H$ is a morphism of schemes over $\mathbb{k}$ such that the diagram commutes

$$\begin{array}{ccc}
G \times_{\mathbb{k}} G & \xrightarrow{\mu_G} & G \\
\downarrow f \times f & & \downarrow f \\
H \times_{\mathbb{k}} H & \xrightarrow{\mu_H} & H.
\end{array} \quad (2.3)$$

A *closed subgroup* $H \subseteq G$ is a subscheme such that the inclusion is a group homomorphism.

Example 2.1.3. Many groups are affine algebraic groups.

- A *torus* is $T := \text{Spec}(\mathbb{k}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}])$ with co-multiplication, co-unit and co-inverse maps

$$\begin{aligned}
\mu^* : \mathcal{O}(T) &\rightarrow \mathcal{O}(T) \otimes_{\mathbb{k}} \mathcal{O}(T), & x_i &\mapsto x_i \otimes x_i, & x_i^{-1} &\mapsto x_i^{-1} \otimes x_i^{-1} \\
e^* : \mathcal{O}(T) &\rightarrow \mathbb{k}, & x_i, x_i^{-1} &\mapsto 1 \\
\iota^* : \mathcal{O}(T) &\rightarrow \mathcal{O}(T), & x_i &\mapsto x_i^{-1}, & x_i^{-1} &\mapsto x_i.
\end{aligned} \quad (2.4)$$

In particular the *multiplicative group* $\mathbb{G}_m$ is a torus of dimension 1, and $T \cong (\mathbb{G}_m)^n$.

- An *additive group of dimension n* is $U := \text{Spec}(\mathbb{k}[x_1, \dots, x_n])$ with maps

$$\begin{aligned}
\mu^* : \mathcal{O}(U) &\rightarrow \mathcal{O}(U) \otimes_{\mathbb{k}} \mathcal{O}(U), & x_i &\mapsto 1 \otimes x_i + x_i \otimes 1 \\
e^* : \mathcal{O}(U) &\rightarrow \mathbb{k}, & x_i &\mapsto 0 \\
\iota^* : \mathcal{O}(U) &\rightarrow \mathcal{O}(U), & x_i &\mapsto -x_i.
\end{aligned} \quad (2.5)$$

In particular, the *additive group of $\mathbb{k}$* $\mathbb{G}_a$ is an additive group of dimension 1, and $U \cong (\mathbb{G}_a)^n$.

- Let $n \in \mathbb{N}_+$. The *general linear group of degree n* is the affine scheme GL_n with affine coordinate ring $\mathcal{O}(\mathrm{GL}_n) = \mathbb{k}[x_{ij} : 1 \leq i, j \leq n]_{\det(x_{ij})}$, which is the localisation of $\mathbb{k}[x_{ij} : 1 \leq i, j \leq n]$ with respect to the multiplicative set $\{1, \det(x_{ij}), \det(x_{ij})^2, \dots\}$.

The ring maps defining the group structure are

$$\begin{aligned}
\mu^* : \mathcal{O}(\mathrm{GL}_n) &\rightarrow \mathcal{O}(\mathrm{GL}_n) \otimes_{\mathbb{k}} \mathcal{O}(\mathrm{GL}_n), & x_{ij} &\mapsto \sum_{k=1}^n x_{ik} \otimes x_{kj} \\
e^* : \mathcal{O}(\mathrm{GL}_n) &\rightarrow \mathbb{k}, & x_{ij} &\mapsto \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \\
\iota^* : \mathcal{O}(\mathrm{GL}_n) &\rightarrow \mathcal{O}(\mathrm{GL}_n) \\
\begin{pmatrix} \iota^*(x_{11}) & \cdots & \iota^*(x_{1n}) \\ \vdots & \ddots & \vdots \\ \iota^*(x_{n1}) & \cdots & \iota^*(x_{nn}) \end{pmatrix} &= \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}^{-1}.
\end{aligned} \tag{2.6}$$

For a vector space V of dimension n , choose a basis $(v_1, \dots, v_n)$ of V and a dual basis $(x_1, \dots, x_n)$ of V^* . Then $x_i \otimes v_j$ is a linear function on $V \otimes_{\mathbb{k}} V^* \cong \mathrm{End}(V)$. Identify the symbol x_{ij} with $x_i \otimes v_j$ and then $\mathrm{GL}_n \cong \mathrm{GL}(V)$ defines the *general linear group of V* .

- The *special linear group of degree n* is the subgroup $\mathrm{SL}_n \hookrightarrow \mathrm{GL}_n$ cut out by the equation $\det(x_{ij}) - 1 = 0$. Similarly we can define $\mathrm{SL}(V) \hookrightarrow \mathrm{GL}(V)$ for a finite dimensional vector space V .
- The *unipotent group of upper triangular $n \times n$ -matrices* is the subgroup $\mathrm{U}_n \hookrightarrow \mathrm{GL}_n$ cut out by equations

$$\begin{aligned}
x_{ij} &= 0, & 1 \leq j < i \leq n \\
x_{ii} - 1 &= 0, & 1 \leq i \leq n
\end{aligned} \tag{2.7}$$

which is a subgroup of SL_n .

Proposition 2.1.4. *Recall $\mathbb{k}$ has characteristic zero. The following are some properties of affine algebraic groups over $\mathbb{k}$:*

- *Any affine algebraic group over $\mathbb{k}$ is smooth (Cartier's theorem).*
- *For an affine algebraic group G , its irreducible components are connected components, and the irreducible component containing $e \in G$ is a closed subgroup $G^0 \hookrightarrow G$.*

- Affine algebraic groups are linear, in the sense that any affine algebraic group is isomorphic to a closed subgroup of some GL_n .

2.1.2 Actions and linearisations

Definition 2.1.5. Let X be a scheme of finite type over $\mathbb{k}$ and let G be an affine algebraic group over $\mathbb{k}$ with multiplication $\mu : G \times_{\mathbb{k}} G \rightarrow G$ and unit $e : \mathrm{Spec}(\mathbb{k}) \rightarrow G$. A (left) algebraic action of G on X is a morphism

$$\sigma : G \times_{\mathbb{k}} X \rightarrow X \quad (2.8)$$

satisfying:

- associativity:
$$\begin{array}{ccc} G \times_{\mathbb{k}} G \times_{\mathbb{k}} G & \xrightarrow{\mu \times 1} & G \times_{\mathbb{k}} X \\ \downarrow 1 \times \sigma & & \downarrow \sigma \\ G \times_{\mathbb{k}} X & \xrightarrow{\sigma} & X; \end{array}$$
- unit:
$$\begin{array}{ccc} \mathrm{Spec}(\mathbb{k}) \times_{\mathbb{k}} X & \xrightarrow{e \times 1} & G \times_{\mathbb{k}} X \\ & \searrow \cong & \downarrow \sigma \\ & & X. \end{array}$$

We call $\sigma : G \times_{\mathbb{k}} X \rightarrow X$ the *action morphism*.

Definition 2.1.6. Let V be a finite dimensional vector space over $\mathbb{k}$, and let G be an affine algebraic group over $\mathbb{k}$. A *representation* of G on V is a group homomorphism $G \rightarrow \mathrm{GL}(V)$.

Definition 2.1.7. Let X, Y be schemes of finite type over $\mathbb{k}$ and let G be an affine algebraic group over $\mathbb{k}$ with multiplication $\mu : G \times_{\mathbb{k}} G \rightarrow G$. Let $\sigma_X : G \times_{\mathbb{k}} X \rightarrow X$ and $\sigma_Y : G \times_{\mathbb{k}} Y \rightarrow Y$ be actions of G on X, Y respectively. A morphism $f : X \rightarrow Y$ is called *G -equivariant* if the diagram commutes

$$\begin{array}{ccc} G \times_{\mathbb{k}} X & \xrightarrow{\sigma_X} & X \\ \downarrow 1 \times f & & \downarrow f \\ G \times_{\mathbb{k}} Y & \xrightarrow{\sigma_Y} & Y. \end{array} \quad (2.9)$$

Definition 2.1.8. Let G be an affine algebraic group over $\mathbb{k}$ acting on a scheme X of finite type over $\mathbb{k}$ by $\sigma : G \times_{\mathbb{k}} X \rightarrow X$. Let $x : \mathrm{Spec}(\mathbb{k}) \rightarrow X$ be a $\mathbb{k}$ -point on X . We define:

- the *orbit* of $x \in X$, denoted by $G.x \subseteq X$, is the set-theoretic image of $\sigma_x : G \rightarrow X$, where σ_x is the composition $G \times_{\mathbb{k}} \mathrm{Spec}(\mathbb{k}) \xrightarrow{1 \times x} G \times_{\mathbb{k}} X \xrightarrow{\sigma} X$;

- the *stabiliser* of $x \in X$, denoted by $G_x \subseteq G$, is the fibre of σ_x at x

$$\begin{array}{ccc} G_x & \hookrightarrow & G \\ \downarrow & & \downarrow \sigma_x \\ \text{Spec}(\mathbb{k}) & \xhookrightarrow{x} & X. \end{array} \quad (2.10)$$

Here are some properties of orbits and stabilisers. Statements and proofs can be found in Section 3, [Hos16].

Proposition 2.1.9. • *Every orbit $G.x \subseteq G$ is locally closed, and $G.x \subseteq \overline{G.x}$ is open dense.*

- *The boundary of an orbit $\overline{G.x} \setminus G.x$ is a union of orbits of strictly lower dimensions.*
- *Orbits of minimal dimensions are closed. Every orbit contains a closed orbit in its closure.*
- *The dimensions have $\dim G = \dim G.x + \dim G_x$.*
- *The function on closed points $x \mapsto \dim G_x$ is upper semi-continuous.*

When G acts on an affine scheme $\text{Spec}(A)$ the action morphism can be described by the corresponding ring map. We call such ring maps *dual actions*, and it is natural to define *invariant functions* for dual actions.

Definition 2.1.10. Let G be an affine algebraic group over $\mathbb{k}$ with co-multiplication map $\mu^* : \mathcal{O}(G) \rightarrow \mathcal{O}(G) \otimes_{\mathbb{k}} \mathcal{O}(G)$ and co-unit map $e^* : \mathcal{O}(G) \rightarrow \mathbb{k}$. Let V be a vector space over $\mathbb{k}$. Let A be a $\mathbb{k}$ -algebra. A *dual action* of G on V (resp. on A) is a morphism of $\mathbb{k}$ -vector spaces (resp. $\mathbb{k}$ -algebras)

$$\sigma^* : V \rightarrow \mathcal{O}(G) \otimes_{\mathbb{k}} V \quad (\text{resp. } \sigma^* : A \rightarrow \mathcal{O}(G) \otimes_{\mathbb{k}} A) \quad (2.11)$$

satisfying:

$$\begin{array}{ccc} V & \xrightarrow{\sigma^*} & \mathcal{O}(G) \otimes_{\mathbb{k}} V \\ \downarrow \sigma^* & & \downarrow \mu^* \otimes 1 \\ \mathcal{O}(G) \otimes_{\mathbb{k}} V & \xrightarrow{1 \otimes \sigma^*} & \mathcal{O}(G) \otimes_{\mathbb{k}} \mathcal{O}(G) \otimes_{\mathbb{k}} V \end{array} \quad (\text{resp. similar diagram for } A);$$

- the composition is the identity $V \xrightarrow{\sigma^*} \mathcal{O}(G) \otimes_{\mathbb{k}} V \xrightarrow{e^* \otimes 1} V$ (resp. similar for A).

We also call that G acts *dually* on V (resp. on A).

Definition 2.1.11. Let $\sigma^* : V \rightarrow \mathcal{O}(G) \otimes_{\mathbb{k}} V$ be a dual action of G on V . A subspace $W \subseteq V$ is called *invariant* if $\sigma^*(W) \subseteq \mathcal{O}(G) \otimes_{\mathbb{k}} W$. A vector $v \in V$ is called *invariant* if $\sigma^*(v) = 1 \otimes v$. For a dual action of G on a $\mathbb{k}$ -algebra A , we regard it as a dual action on a vector space and define invariant subspaces and invariant vectors in the same way.

Let G be an affine algebraic group and let X be a scheme of finite type over $\mathbb{k}$. Let $p_0 : G \times_{\mathbb{k}} X \rightarrow G$ and $p_1 : G \times_{\mathbb{k}} X \rightarrow X$ be projections. Similarly let

$$\begin{aligned} p_{01} : G \times_{\mathbb{k}} G \times_{\mathbb{k}} X &\rightarrow G \times_{\mathbb{k}} G, & (g, h, x) &\mapsto (g, h) \\ p_{12} : G \times_{\mathbb{k}} G \times_{\mathbb{k}} X &\rightarrow G \times_{\mathbb{k}} X, & (g, h, x) &\mapsto (h, x) \end{aligned} \tag{2.12}$$

be projections.

Definition 2.1.12. Let G act on X . Let $\mathcal{F}$ be a quasi-coherent sheaf on X . A G -*linearisation* of $\mathcal{F}$ is an isomorphism $\Phi : \sigma^* \mathcal{F} \rightarrow p_1^* \mathcal{F}$ of quasi-coherent sheaves on $G \times_{\mathbb{k}} X$, satisfying:

$$(\mu \times 1)^* \Phi = (p_{12}^* \Phi) \circ (1 \times \sigma)^* \Phi \tag{2.13}$$

where $\mu : G \times_{\mathbb{k}} G \rightarrow G$ is the multiplication and $\sigma : G \times_{\mathbb{k}} X \rightarrow X$ is the action morphism.

We also call that $\mathcal{F}$ is G -*linearised* or G linearises $\mathcal{F}$.

Example 2.1.13. Let G act on X . There is a naturally induced G -linearisation of $\mathcal{O}_X$. With the natural identification $p_1^* \mathcal{O}_X \cong \mathcal{O}_{G \times_{\mathbb{k}} X}$ and the adjunction $\sigma^* \dashv \sigma_*$ we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_{G \times_{\mathbb{k}} X}}(\sigma^* \mathcal{O}_X, p_1^* \mathcal{O}_X) &\cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \sigma_* p_1^* \mathcal{O}_X) \\ &\cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \sigma_* \mathcal{O}_{G \times_{\mathbb{k}} X}). \end{aligned} \tag{2.14}$$

The linearisation $\Phi : \sigma^* \mathcal{O}_X \rightarrow p_1^* \mathcal{O}_X$ is then corresponding to $\sigma^\# : \mathcal{O}_X \rightarrow \sigma_* \mathcal{O}_{G \times_{\mathbb{k}} X}$.

Example 2.1.14. Let G act trivially on $\mathrm{Spec}(\mathbb{k})$. Then $\sigma = p_1$ since there is only one morphism $G \times_{\mathbb{k}} \mathrm{Spec}(\mathbb{k}) \rightarrow \mathrm{Spec}(\mathbb{k})$. Let V be a vector space over $\mathbb{k}$, defining a quasi-coherent sheaf on $\mathrm{Spec}(\mathbb{k})$. A G -linearisation of V is an isomorphism of $\mathcal{O}(G)$ -modules

$$\Phi : \mathcal{O}(G) \otimes_{\mathbb{k}} V \rightarrow \mathcal{O}(G) \otimes_{\mathbb{k}} V \tag{2.15}$$

which is determined by the $\mathbb{k}$ -linear map

$$\phi : V \rightarrow \mathcal{O}(G) \otimes_{\mathbb{k}} V. \quad (2.16)$$

The co-cycle condition $(\mu \times 1)^*\Phi = (p_{12}^*\Phi) \circ (1 \times \sigma)^*\Phi$ is equivalent to that ϕ defines a dual action of G on V .

Definition 2.1.15. Let G act on X . Let $\mathcal{F}, \mathcal{F}'$ be quasi-coherent sheaves on X with G -linearisations

$$\Phi : \sigma^*\mathcal{F} \rightarrow p_1^*\mathcal{F}, \quad \Phi' : \sigma^*\mathcal{F}' \rightarrow p_1^*\mathcal{F}'. \quad (2.17)$$

A *morphism of G -linearisations* is a morphism $\phi : \mathcal{F} \rightarrow \mathcal{F}'$ respecting G -linearisations

$$\begin{array}{ccc} \sigma^*\mathcal{F} & \xrightarrow{\Phi} & p_1^*\mathcal{F} \\ \downarrow \sigma^*\phi & & \downarrow p_1^*\phi \\ \sigma^*\mathcal{F}' & \xrightarrow{\Phi'} & p_1^*\mathcal{F}'. \end{array} \quad (2.18)$$

From existing linearisations, we can construct new linearisations:

- Let $\phi : \mathcal{F} \rightarrow \mathcal{F}'$ be a morphism of G -linearisations. Then $\ker(\phi), \operatorname{coker}(\phi), \operatorname{im}(\phi)$ have natural G -linearisations.
- Let $\mathcal{F}, \mathcal{F}'$ be G -linearised over X . Then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}'$, $\operatorname{Sym}_{\mathcal{O}_X}^n \mathcal{F}$ and $\bigwedge_{\mathcal{O}_X}^n \mathcal{F}$ are G -linearised.
- Let $f : X \rightarrow Y$ be G -equivariant. Let $\mathcal{F}$ be G -linearised on Y . Then $f^*\mathcal{F}$ is G -linearised.
- Let $f : X \rightarrow Y$ be G -equivariant. Let $\mathcal{F}$ be G -linearised on X . Since we only consider schemes of finite type over $\mathbb{k}$, the higher direct images $R^i f_* \mathcal{F}$ for $i \geq 0$ are quasi-coherent on Y and they have natural G -linearisations.
- A special case of direct images. Let $X \rightarrow \operatorname{Spec}(\mathbb{k})$ be a scheme of finite type over $\mathbb{k}$. Let G act on X and let $\mathcal{F}$ be a G -linearisation. Then $H^0(X, \mathcal{F})$ admits a dual action of G (cf. Example 2.1.14).

- Let G act on X and let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$ -algebra. Let $\Phi : \sigma^* \mathcal{A} \rightarrow p_1^* \mathcal{A}$ be a G -linearisation of $\mathcal{A}$. Assume Φ is a morphism of $\mathcal{O}_{G \times_{\mathbb{k}} X}$ -algebras. Then there is an action of G on $\underline{\mathrm{Spec}}_X(\mathcal{A})$ such that $\underline{\mathrm{Spec}}_X(\mathcal{A}) \rightarrow X$ is G -equivariant.
- Similarly as above, if $\mathcal{A}$ is a G -linearised quasi-coherent graded $\mathcal{O}_X$ -algebra, then $\underline{\mathrm{Proj}}_X(\mathcal{A})$ has an action of G and $\underline{\mathrm{Proj}}_X(\mathcal{A}) \rightarrow X$ is G -equivariant.

The following constructs G -linearisations of very ample line bundles over projective schemes, and all such linearisations are constructed this way. Let V be a finite dimensional vector space with a dual action of G . Then G acts on $\mathbb{P}(V)$ and linearises $\mathcal{O}_{\mathbb{P}(V)}(1)$. For any G -invariant closed subscheme $i : X \hookrightarrow \mathbb{P}(V)$, the very ample line bundle $i^* \mathcal{O}_{\mathbb{P}(V)}(1)$ on X is G -linearised. Conversely, let X be a projective scheme over $\mathbb{k}$ and let $\mathcal{L}$ be a very ample line bundle on X . Then $i : X \hookrightarrow \mathbb{P}(V)$ for $V := H^0(X, \mathcal{L})^*$. Let G act on X and linearise $\mathcal{L}$. Then $V^* = H^0(X, \mathcal{L})$ as the direct image of $\mathcal{L}$ along $X \rightarrow \mathrm{Spec}(\mathbb{k})$ is G -linearised, i.e. there is a dual action of G on V^* . Then G acts on $\mathbb{P}(V)$ and linearises $\mathcal{O}(1)_{\mathbb{P}(V)}(1)$. Moreover the closed immersion $i : X \rightarrow \mathbb{P}(V)$ is G -equivariant and the isomorphism $i^* \mathcal{O}_{\mathbb{P}(V)}(1) \cong \mathcal{L}$ is an isomorphism of G -linearised line bundles.

2.1.3 Reductive groups and unipotent groups

Recall that $\mathbb{k}$ has characteristic zero.

Definition 2.1.16. An affine algebraic group G over $\mathbb{k}$ is *unipotent* if one of the following equivalent conditions holds:

- Every non-trivial finite dimensional linear representation $\rho : G \rightarrow \mathrm{GL}(V)$ has a non-zero G -invariant vector;
- Every non-trivial finite dimensional linear representation $\rho : G \rightarrow \mathrm{GL}(V)$ has a basis such that the image of ρ is contained in the upper triangular matrices with diagonals 1;
- G is isomorphic to a subgroup of U_n , the unipotent group of upper triangular matrices in GL_n ;

- Every $g \in G(k)$ is unipotent, i.e. there exists a faithful representation $\rho : G \rightarrow \mathrm{GL}_n$ such that $\rho(g)$ is unipotent.

Definition 2.1.17. An affine algebraic group G over $\mathbb{k}$ is *reductive* if one of the following equivalent conditions holds:

- Every finite dimensional linear representation $\rho : G \rightarrow \mathrm{GL}_n$ is completely reducible;
 - If V is a finite dimensional G -representation, and $W \subseteq V$ is a G -invariant subspace, then there exists a G -invariant subspace $W' \subseteq V$ and $V = W \oplus W'$;
 - If $V \rightarrow W$ is a surjective G -equivariant linear map of finite dimensional G -representations, then the induced map on subspaces of invariant vectors $V^G \rightarrow W^G$ is surjective;
 - If V is a finite dimensional G -representation, and $v \in V$ a non-zero G -invariant vector, then there exists a G -invariant linear function $f : V \rightarrow \mathbb{k}$ such that $f(v) \neq 0$.
- If $\rho : G \rightarrow \mathrm{GL}(V)$ is a finite dimensional linear representation and $v \in V$ is a non-zero G -invariant vector, then there exists a G -invariant non-constant homogeneous polynomial $f \in \mathrm{Sym}^\bullet V^*$ such that $f(v) \neq 0$;
- Every unipotent normal algebraic subgroup is trivial.

Remark 2.1.18. For positive characteristics, the above conditions are not equivalent. The first condition is the strongest, which defines *linearly reductive groups*. The second condition defines *geometric reductive groups* and the last defines *reductive groups*. They coincide in characteristic zero.

Definition 2.1.19. An affine algebraic group G over $\mathbb{k}$ is *semisimple* if:

- G is connected;
- $G \neq \{e\}$;
- the only connected closed normal Abelian subgroup of G is $\{e\}$.

Theorem 2.1.20 (Hilbert, Mumford, Nagata, Popov). *An affine algebraic group over $\mathbb{k}$ is reductive if and only if for every dual action of G on a finitely generated $\mathbb{k}$ -algebra, the G -invariant sub-algebra A^G is finitely generated.*

The following are some properties of dual actions of reductive groups, details in [MFK94].

Proposition 2.1.21. *Let G be a reductive group and let A be a $\mathbb{k}$ -algebra. Let G act dually on A . Let $A^G \subseteq A$ be the sub-algebra of G -invariant elements. The following are true.*

- *Let $I \subseteq A$ be a G -invariant ideal. Then*

$$I = (I \cap A^G)A, \quad A^G / (I \cap A^G) \cong (A/I)^G. \quad (2.19)$$

- *Let $\{I_\alpha\}$ be a collection of G -invariant ideals of A . Then*

$$\left(\sum_{\alpha} I_{\alpha} \right) \cap A^G = \sum_{\alpha} (I_{\alpha} \cap A^G). \quad (2.20)$$

- *Let $A^G \rightarrow R$ be a A^G -algebra. Then $R \rightarrow R \otimes_{A^G} A$ is injective and*

$$R \cong (R \otimes_{A^G} A)^G. \quad (2.21)$$

2.2 Lie algebras and infinitesimal stabilisers

Let G be an affine algebraic group over $\mathbb{k}$. Let X be a scheme of finite type over $\mathbb{k}$. Let G act on X with the action morphism $\sigma : G \times_{\mathbb{k}} X \rightarrow X$, the unit morphism $e : \text{Spec}(\mathbb{k}) \rightarrow G$ and the inverse morphism $\iota : G \rightarrow G$.

2.2.1 Lie algebras and derivations

For Lie groups, their Lie algebras are their tangent spaces at their identity elements. For affine algebraic groups, we define Lie algebras similarly.

For the group G , define the dual of the Lie algebra $\mathfrak{g}^*$ as the cotangent space of G at e

$$\mathfrak{g}^* := \mathfrak{m}_e / \mathfrak{m}_e^2, \quad \mathfrak{m}_e := \ker (e^* : \mathcal{O}(G) \rightarrow \mathbb{k}) \quad (2.22)$$

and define the Lie algebra $\mathfrak{g}$ as the dual vector space $\mathfrak{g} := \text{Hom}_{\mathbb{k}}(\mathfrak{g}^*, \mathbb{k})$.

Let $\xi \in \mathfrak{g}$ be a vector. It induces a $\mathbb{k}$ -derivation $\xi : \mathcal{O}(G) \rightarrow \mathcal{O}(G)$ as the following composition

$$\begin{array}{c} \mathcal{O}(G) \xrightarrow{\mu^*} \mathcal{O}(G) \otimes_{\mathbb{k}} \mathcal{O}(G) \xrightarrow{q \otimes 1} \mathfrak{m}_e / \mathfrak{m}_e^2 \otimes_{\mathbb{k}} \mathcal{O}(G) \xrightarrow{(-\xi) \otimes 1} \mathcal{O}(G) \\ \searrow \hspace{10em} \nearrow \\ \hspace{10em} \xi \end{array} \quad (2.23)$$

where $q : \mathcal{O}(G) \rightarrow \mathfrak{m}_e / \mathfrak{m}_e^2$ is the $\mathbb{k}$ -linear map $f \mapsto \overline{f - f(e)}$. It is easy to see:

- $\xi : \mathcal{O}(G) \rightarrow \mathcal{O}(G)$ is a $\mathbb{k}$ -derivation;
- The map $\mathfrak{g} \rightarrow \text{Der}_{\mathbb{k}}(\mathcal{O}(G))$ is $\mathbb{k}$ -linear and preserves Lie brackets.

Recall that G acts on X via $\sigma : G \times_{\mathbb{k}} X \rightarrow X$. Consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & G \times_{\mathbb{k}} X \\ & \searrow \delta & \downarrow \rho \\ & & X \times_{\mathbb{k}} X \end{array}, \quad \begin{cases} \gamma : x \mapsto (e, x) \\ \rho : (u, x) \mapsto (u.x, x) \\ \delta : x \mapsto (x, x). \end{cases} \quad (2.24)$$

The following is exact by [Sta22, Lemma 01UX]

$$\rho^* \Omega_{X \times_{\mathbb{k}} X / \mathbb{k}} \longrightarrow \Omega_{G \times_{\mathbb{k}} X / \mathbb{k}} \longrightarrow \Omega_{\rho} \longrightarrow 0 \quad (2.25)$$

where $\Omega_{\rho} := \Omega_{G \times_{\mathbb{k}} X / X \times_{\mathbb{k}} X}$. The above exact sequence on $G \times_{\mathbb{k}} X$ pulls back to the following exact sequence on X along γ

$$\delta^* \Omega_{X \times_{\mathbb{k}} X / \mathbb{k}} \longrightarrow \gamma^* \Omega_{G \times_{\mathbb{k}} X / \mathbb{k}} \longrightarrow \gamma^* \Omega_{\rho} \longrightarrow 0. \quad (2.26)$$

The morphisms γ, δ are locally closed immersions. Let $\mathcal{C}_{\gamma}, \mathcal{C}_{\delta}$ denote the co-normal sheaves of γ, δ respectively. See [Sta22, Section 01R1] for co-normal sheaves. Consider the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{C}_{\delta} & \longrightarrow & \delta^* \Omega_{X \times_{\mathbb{k}} X / \mathbb{k}} & \longrightarrow & \Omega_{X / \mathbb{k}} \longrightarrow 0 \\ & & \downarrow (d\rho)^* & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{C}_{\gamma} & \longrightarrow & \gamma^* \Omega_{G \times_{\mathbb{k}} X / \mathbb{k}} & \longrightarrow & \Omega_{X / \mathbb{k}} \longrightarrow 0 \end{array} \quad (2.27)$$

where $(d\rho)^* : \mathcal{C}_{\delta} \rightarrow \mathcal{C}_{\gamma}$ is induced from the commutative triangle $\rho \circ \gamma = \delta$ and rows are exact by [Sta22, Lemma 0474]. By the Snake Lemma, the cokernels of the left and middle

vertical arrows are isomorphic. From sequence (2.26), we have that $\gamma^*\Omega_\rho$ is the cokernel of the middle vertical arrow. Then the following sequence is exact

$$\mathcal{C}_\delta \xrightarrow{(\mathrm{d}\rho)^*} \mathcal{C}_\gamma \longrightarrow \gamma^*\Omega_\rho \longrightarrow 0. \quad (2.28)$$

By [Sta22, Lemma 08S2], we have $\mathcal{C}_\gamma \cong \mathfrak{g}^* \otimes_{\mathbb{k}} \mathcal{O}_X$ and $\mathcal{C}_\delta \cong \Omega_{X/\mathbb{k}}$. Then the following sequence is exact

$$\Omega_{X/\mathbb{k}} \xrightarrow{(\mathrm{d}\rho)^*} \mathfrak{g}^* \otimes_{\mathbb{k}} \mathcal{O}_X \longrightarrow \gamma^*\Omega_\sigma \longrightarrow 0. \quad (2.29)$$

Definition 2.2.1. Let G be an affine algebraic group over $\mathbb{k}$. Let $\mathfrak{g}$ be its Lie algebra. Let X be a scheme of finite type over $\mathbb{k}$. Let G act on X . We call the morphism $(\mathrm{d}\rho)^* : \Omega_{X/\mathbb{k}} \rightarrow \mathfrak{g}^* \otimes_{\mathbb{k}} \mathcal{O}_X$ defined above the *infinitesimal action* of G on X , or the *infinitesimal action* of $\mathfrak{g}$ on X .

Let $\xi \in \mathfrak{g}$. Then it defines a $\mathbb{k}$ -derivation $\xi \in \mathrm{Der}_{\mathbb{k}}(\mathcal{O}_X) \cong \mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/\mathbb{k}}, \mathcal{O}_X)$

$$\begin{array}{ccc} \Omega_{X/\mathbb{k}} & \xrightarrow{(\mathrm{d}\rho)^*} & \mathfrak{g}^* \otimes_{\mathbb{k}} \mathcal{O}_X \xrightarrow{\xi \otimes 1} \mathcal{O}_X \\ & \searrow \xi \nearrow & \end{array} \quad (2.30)$$

Let $(\xi_1, \dots, \xi_n)$ be a basis of $\mathfrak{g}$ and let $(u_1, \dots, u_n)$ be the dual basis of $\mathfrak{g}^*$. Each ξ_i defines a $\mathbb{k}$ -derivation $\xi_i : \mathcal{O}_X \rightarrow \mathcal{O}_X$. Then the morphism $(\mathrm{d}\rho)^*$ has the form

$$(\mathrm{d}\rho)^* : \Omega_{X/\mathbb{k}} \rightarrow \mathfrak{g}^* \otimes_{\mathbb{k}} \mathcal{O}_X, \quad a \, \mathrm{d}b \mapsto \sum_{i=1}^n u_i \otimes a(\xi_i \cdot b). \quad (2.31)$$

2.2.2 Infinitesimal stabilisers

Let T be a scheme of finite type over $\mathbb{k}$. A T -valued point of X is a morphism $f : T \rightarrow X$. In this subsection, we define T -valued infinitesimal transformations in G and T -valued infinitesimal stabilisers of T -valued points.

Let $\mathbb{k}[\epsilon] := \mathbb{k}[t]/t^2$ be the algebra of dual numbers. For $T \in \mathrm{Sch}/\mathbb{k}$, let $T[\epsilon] := T \times_{\mathbb{k}} \mathrm{Spec}(\mathbb{k}[\epsilon])$. The underlying topological spaces of T and $T[\epsilon]$ are the same. We can think of $T[\epsilon]$ as a scheme on the topological space $|T|$ with the structure sheaf

$$\mathcal{O}_{T[\epsilon]} := \mathcal{O}_T \otimes_{\mathbb{k}} \mathbb{k}[\epsilon] \cong \mathcal{O}_T \oplus \epsilon \mathcal{O}_T. \quad (2.32)$$

There is a natural closed immersion $T \hookrightarrow T[\epsilon]$, corresponding to $\mathcal{O}_{T[\epsilon]} \xrightarrow{\epsilon=0} \mathcal{O}_T$.

Definition 2.2.2. Let G be an affine algebraic group over $\mathbb{k}$. A T -valued infinitesimal transformation in G is a morphism $\rho : T[\epsilon] \rightarrow G$ satisfying the diagram

$$\begin{array}{ccc} T & \hookrightarrow & T[\epsilon] \\ \downarrow & & \downarrow \rho \\ \mathrm{Spec}(\mathbb{k}) & \xhookrightarrow{e} & G. \end{array} \quad (2.33)$$

Definition 2.2.3. Let G act on X . Let $f : T \rightarrow X$ be a T -valued point of X , and let $\rho : T[\epsilon] \rightarrow G$ be a T -valued infinitesimal transformation in G . We say that the T -valued infinitesimal transformation ρ stabilises the T -valued point f if the following diagram commutes

$$\begin{array}{ccc} T[\epsilon] & & \\ (\rho, f \circ \mathrm{pr}_T) \downarrow & \searrow f \circ \mathrm{pr}_T & \\ G \times_{\mathbb{k}} X & \xrightarrow{\sigma} & X \end{array} \quad (2.34)$$

where $\mathrm{pr}_T : T[\epsilon] \rightarrow T$ is the projection.

Remark 2.2.4. If we take $T = \mathrm{Spec}(\mathbb{k})$, then T -valued points of X are exactly closed points, and T -valued infinitesimal transformations in G are vectors in the Lie algebra $\mathfrak{g}$. For a T -valued point $x \in X$ and a T -valued infinitesimal transformation $\xi \in \mathfrak{g}$, the definition of when ξ stabilises x in Definition 2.2.3 is equivalent to the requirement that the pull-back of $\xi : \Omega_{X/\mathbb{k}} \rightarrow \mathcal{O}_X$ along $\{x\} \hookrightarrow X$ is zero.

The set of all T -valued infinitesimal transformations and the set of all T -valued infinitesimal transformations stabilising a certain T -valued point of X are both $\Gamma(T, \mathcal{O}_T)$ -modules by the following lemma.

Lemma 2.2.5. *Let G be an affine algebraic group over $\mathbb{k}$ acting on $X \in \mathrm{Sch}/\mathbb{k}$. Then:*

- *For $T \in \mathrm{Sch}/\mathbb{k}$, the set of all T -valued infinitesimal transformations in G forms a $\Gamma(T, \mathcal{O}_T)$ -module;*
- *For $(f : T \rightarrow X) \in \mathrm{Sch}/X$, the set of all T -valued infinitesimal transformations stabilising f is a sub-module of the set of all T -valued infinitesimal transformations.*

Proof. Let $Y \in \mathrm{Sch}/\mathbb{k}$. A morphism $\rho : T[\epsilon] \rightarrow Y$ is equivalent to a pair (g, ξ) for a morphism $g : T \rightarrow Y$ and a $\mathbb{k}$ -derivation $\xi : \mathcal{O}_Y \rightarrow g_* \mathcal{O}_T$. The relation between ρ and

(g, ξ) is that $g : T \rightarrow Y$ is the composition $T \hookrightarrow T[\epsilon] \xrightarrow{\rho} Y$ and $\rho^\# : \mathcal{O}_Y \rightarrow \rho_* \mathcal{O}_{T[\epsilon]}$ equals $\rho^\# = g^\# \oplus \epsilon \xi$, where we identify $\rho_* \mathcal{O}_{T[\epsilon]} \cong g_* \mathcal{O}_T \oplus \epsilon g_* \mathcal{O}_T$. Therefore a T -valued infinitesimal transformation $\rho : T[\epsilon] \rightarrow G$ is equivalent to (e, ξ) for $e : T \rightarrow \operatorname{Spec}(\mathbb{k}) \xrightarrow{e} G$ and a $\mathbb{k}$ -derivation $\xi : \mathcal{O}_G \rightarrow e_* \mathcal{O}_T$. The $\Gamma(T, \mathcal{O}_T)$ -module on the set of all $\mathbb{k}$ -derivations $\xi : \mathcal{O}_G \rightarrow e_* \mathcal{O}_T$ induces a $\Gamma(T, \mathcal{O}_T)$ -module on the set of all T -valued infinitesimal transformations. This proves the first statement.

Let $\rho_i : T[\epsilon] \rightarrow G$ be a T -valued infinitesimal transformations for $i = 1, 2$. For each i , the morphism $(\rho_i, f \circ \operatorname{pr}_T) : T[\epsilon] \rightarrow G \times_{\mathbb{k}} X$ corresponds to the pair $((e, f), \xi_i)$ for a morphism $(e, f) : T \rightarrow G \times_{\mathbb{k}} X$ and a $\mathbb{k}$ -derivation $\xi_i : \mathcal{O}_{G \times_{\mathbb{k}} X} \rightarrow (e, f)_* \mathcal{O}_T$. The T -valued infinitesimal transformation ρ_i stabilises $f : T \rightarrow X$ if and only if the following $\mathbb{k}$ -derivation is zero

$$\mathcal{O}_X \xrightarrow{\sigma^\#} \sigma_* \mathcal{O}_{G \times_{\mathbb{k}} X} \xrightarrow{\sigma_*(\xi_i)} \sigma_*(e, f)_* \mathcal{O}_T \cong f_* \mathcal{O}_T. \quad (2.35)$$

Then $\rho_1 + \rho_2$ also stabilises f , since $\sigma_*(\xi_1 + \xi_2) \circ \sigma^\# = \sigma_*(\xi_1) \circ \sigma^\# + \sigma_*(\xi_2) \circ \sigma^\# = 0$, and $(\rho_1 + \rho_2, f \circ \operatorname{pr}_T) : T[\epsilon] \rightarrow G \times_{\mathbb{k}} X$ corresponds to $((e, f), \xi_1 + \xi_2)$. For any $a \in \Gamma(T, \mathcal{O}_T)$, the morphism $(a\rho_1, f \circ \operatorname{pr}_T) : T[\epsilon] \rightarrow G \times_{\mathbb{k}} X$ corresponds to $((e, f), a\xi_1)$. Then $\sigma_*(a\xi_1) \circ \sigma^\# = 0$, i.e. $a\rho_1$ stabilises f . This proves that the set of all T -valued infinitesimal transformations in G stabilising f is a sub-module. $\square$

Lemma 2.2.6. *Consider the following diagram in $\operatorname{Sch}/\mathbb{k}$ such that $i : Z \rightarrow Y$ is an immersion and $j : T \hookrightarrow T[\epsilon]$ is the natural closed immersion:*

$$\begin{array}{ccc} T & \xhookrightarrow{j} & T[\epsilon] \\ \downarrow g & \swarrow \text{---} & \downarrow \rho \\ Z & \xrightarrow{i} & Y. \end{array} \quad (2.36)$$

Then $\rho : T[\epsilon] \rightarrow Y$ factors through $i : Z \rightarrow Y$ if and only if the morphism $g^* \mathcal{C}_{Z/Y} \rightarrow \epsilon \mathcal{O}_T$ in [Sta22, Lemma 01R4] is zero, where $\epsilon \mathcal{O}_T \cong \mathcal{C}_{T/T[\epsilon]}$.

Proof. Assume without loss of generality that T, Z, Y are affine schemes, and $i : Z \hookrightarrow Y$ is a closed immersion. Let $T = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(R)$ and $Z = \operatorname{Spec}(R/I)$. Let $\varphi : R/I \rightarrow A$ correspond to $g : T \rightarrow Z$, and let $\psi : R \rightarrow A[\epsilon]$ corresponds to $\rho : T[\epsilon] \rightarrow Y$.

Since $\rho \circ j = i \circ g$, the right square in the following diagram commutes

$$\begin{array}{ccccccc}
0 & \longrightarrow & I & \longrightarrow & R & \longrightarrow & R/I \longrightarrow 0 \\
& & \downarrow \psi|_I & & \downarrow \psi & & \downarrow \varphi \\
0 & \longrightarrow & \epsilon A & \longrightarrow & A[\epsilon] & \longrightarrow & A \longrightarrow 0
\end{array} \tag{2.37}$$

and thus $\psi|_I : I \rightarrow \epsilon A$ exists so that the left square commutes.

The morphism in [Sta22, Lemma 01R4] is the following

$$\eta : I/I^2 \otimes_{R/I, \varphi} A \rightarrow \epsilon A, \quad \bar{h} \otimes 1 \mapsto \psi|_I(h). \tag{2.38}$$

Then we have $\eta = 0$ if and only if $\psi(I) = 0$.

The morphism $\rho : T[\epsilon] \rightarrow Y$ factors as $T[\epsilon] \rightarrow Z \rightarrow Y$ if and only if the ring map $\psi : R \rightarrow A[\epsilon]$ factors as $R \rightarrow R/I \rightarrow A[\epsilon]$, which is equivalent to $\psi(I) = 0$, and thus equivalent to $\eta = 0$. This completes the proof. $\square$

Lemma 2.2.7. *Let $\sigma : G \times_{\mathbb{k}} X \rightarrow X$ be an action. Let $f : T \rightarrow X$ be a T -valued point. Let $\mathcal{Q} := \text{coker}(\Omega_{X/\mathbb{k}} \xrightarrow{\phi} \mathfrak{g}^* \otimes_{\mathbb{k}} \mathcal{O}_X)$. Then we have the following commutative diagram of $\Gamma(T, \mathcal{O}_T)$ -modules*

$$\begin{array}{ccc}
\{\rho : \rho \text{ is a } T\text{-valued infinitesimal transformation}\} & \xrightarrow{\cong} & \text{Hom}_{\mathcal{O}_T}(\mathfrak{g}^* \otimes_{\mathbb{k}} \mathcal{O}_T, \epsilon \mathcal{O}_T) \\
\uparrow \cup \! \! \uparrow & & \uparrow \cup \! \! \uparrow \\
\left\{ \begin{array}{l} \rho : \rho \text{ is a } T\text{-valued infinitesimal transformation} \\ \text{and } \rho \text{ stabilises } f \end{array} \right\} & \xrightarrow{\cong} & \text{Hom}_{\mathcal{O}_T}(f^* \mathcal{Q}, \epsilon \mathcal{O}_T)
\end{array} \tag{2.39}$$

where both horizontal maps are bijections and the top is given by [Sta22, Lemma 01R4].

Moreover, the maps are natural for $T \in \text{Sch}/X$.

Proof. It is easy to see the maps are $\Gamma(T, \mathcal{O}_T)$ -linear and are natural for $T \in \text{Sch}/X$.

We construct the inverse of the top map. Let $\eta : \mathfrak{g}^* \otimes_{\mathbb{k}} \mathcal{O}_T \rightarrow \epsilon \mathcal{O}_T$. Over T , we have a map of $\Gamma(T, \mathcal{O}_T)$ -modules

$$\eta_T : \mathfrak{g}^* \otimes_{\mathbb{k}} \Gamma(T, \mathcal{O}_T) \rightarrow \epsilon \Gamma(T, \mathcal{O}_T). \tag{2.40}$$

Consider the ring map $\rho^* : \mathcal{O}(G) \rightarrow \Gamma(T, \mathcal{O}_T)[\epsilon]$ such that for $h \in \mathcal{O}(G)$

$$\rho^*(h) = h(e) \oplus \eta_T(\overline{h - h(e)} \otimes 1) \in \Gamma(T, \mathcal{O}_T) \oplus \epsilon \Gamma(T, \mathcal{O}_T) \cong \Gamma(T[\epsilon], \mathcal{O}_{T[\epsilon]}) \tag{2.41}$$

where $h - h(e) \in \mathfrak{m}_e := \ker(e^*)$ represents $\overline{h - h(e)} \in \mathfrak{m}_e/\mathfrak{m}_e^2 \cong \mathfrak{g}^*$. The ring map ρ^* corresponds to a morphism $\rho : T[\epsilon] \rightarrow G$. It is easy to check that the map $\eta \mapsto \rho$ provides the required inverse.

Let $\rho : T[\epsilon] \rightarrow G$ be a T -valued infinitesimal transformation. The following diagram commutes

$$\begin{array}{ccccc}
 T & \hookrightarrow & T[\epsilon] & & \\
 \downarrow f & & \downarrow (\rho, f \circ \text{pr}_T) & & \\
 X & \xrightarrow{\gamma} & G \times_{\mathbb{k}} X & & \\
 \swarrow & \searrow & \swarrow & \searrow & \\
 \text{Spec}(\mathbb{k}) & \xrightarrow{e} & G & & X \xrightarrow{\delta} X \times_{\mathbb{k}} X
 \end{array}
 \quad (2.42)$$

where $\gamma : X \rightarrow G \times_{\mathbb{k}} X$ is the pull-back of $e : \text{Spec}(\mathbb{k}) \hookrightarrow G$ in the bottom left parallelogram, $\delta : X \rightarrow X \times_{\mathbb{k}} X$ is the diagonal morphism, $\text{pr}_T : T[\epsilon] \rightarrow T$ is the projection and $\text{pr} : G \times_{\mathbb{k}} X \rightarrow X$ is the projection.

Note that $\phi : \Omega_{X/\mathbb{k}} \rightarrow \mathfrak{g}^* \otimes_{\mathbb{k}} \mathcal{O}_X$ is the morphism of co-normal sheaves when [Sta22, Lemma 01R4] is applied to the bottom right parallelogram. Consider the top square and the bottom right parallelogram; we have by [Sta22, Lemma 01R4]

$$f^* \Omega_{X/\mathbb{k}} \xrightarrow{f^* \phi} \mathfrak{g}^* \otimes_{\mathbb{k}} \mathcal{O}_T \longrightarrow \epsilon \mathcal{O}_T \quad (2.43)$$

where the morphism $\mathfrak{g}^* \otimes_{\mathbb{k}} \mathcal{O}_T \rightarrow \epsilon \mathcal{O}_T$ above is from the top square.

There is another morphism $\mathfrak{g}^* \otimes_{\mathbb{k}} \mathcal{O}_T \rightarrow \epsilon \mathcal{O}_T$ associated to ρ , the one given by the bijective map

$$\{\rho : \rho \text{ is a } T\text{-valued infinitesimal transformation}\} \cong \text{Hom}_{\mathcal{O}_T}(\mathfrak{g}^* \otimes_{\mathbb{k}} \mathcal{O}_T, \epsilon \mathcal{O}_T) \quad (2.44)$$

where $\mathfrak{g}^* \otimes_{\mathbb{k}} \mathcal{O}_T \rightarrow \epsilon \mathcal{O}_T$ associated to ρ is from the square

$$\begin{array}{ccc}
 T & \hookrightarrow & T[\epsilon] \\
 \downarrow & & \downarrow \rho \\
 \text{Spec}(\mathbb{k}) & \xrightarrow{e} & G
 \end{array}$$

, which is the composition of the top and bottom left parallelogram in diagram (2.42). Since the bottom left parallelogram is Cartesian with southwest arrows flat, by [Sta22, Lemma 0473], these two morphisms $\mathfrak{g}^* \otimes_{\mathbb{k}} \mathcal{O}_T \rightarrow \epsilon \mathcal{O}_T$ are the same.

We have that ρ stabilises $f : T \rightarrow X$ if and only if the dashed arrow below exists in the following diagram

$$\begin{array}{ccc} T & \hookrightarrow & T[\epsilon] \\ \downarrow f & \swarrow & \downarrow (\sigma \circ (\rho, f \circ \text{pr}_T), f \circ \text{pr}_T) \\ X & \xrightarrow{\delta} & X \times_{\mathbb{k}} X \end{array} \quad (2.45)$$

where the square (2.45) is the composition of the top and bottom right parallelogram in diagram 2.42. By Lemma 2.2.6, we have that ρ stabilises f if and only if $f^* \Omega_{X/\mathbb{k}} \rightarrow \epsilon \mathcal{O}_T$ is zero, i.e. the following composition is zero:

$$f^* \Omega_{X/\mathbb{k}} \xrightarrow{f^* \phi} \mathfrak{g}^* \otimes_{\mathbb{k}} \mathcal{O}_T \longrightarrow \epsilon \mathcal{O}_T. \quad (2.46)$$

Therefore along the bijective map

$$\{\rho : \rho \text{ is a } T\text{-valued infinitesimal transformation}\} \cong \text{Hom}_{\mathcal{O}_T}(\mathfrak{g}^* \otimes_{\mathbb{k}} \mathcal{O}_T, \epsilon \mathcal{O}_T) \quad (2.47)$$

the images of those ρ which stabilise f constitute the kernel of

$$\text{Hom}_{\mathcal{O}_T}(\mathfrak{g}^* \otimes_{\mathbb{k}} \mathcal{O}_T) \rightarrow \text{Hom}_{\mathcal{O}_T}(f^* \Omega_{X/\mathbb{k}}, \mathcal{O}_T) \quad (2.48)$$

which is $\text{Hom}_{\mathcal{O}_T}(f^* \mathcal{Q}, \mathcal{O}_T)$. This completes the proof. $\square$

Lemma 2.2.8. *Let S be a scheme. Let*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \quad (2.49)$$

be an exact sequence of locally free sheaves on S . Assume $\mathcal{F}, \mathcal{G}$ are finite locally free of the same rank. Then the injective morphism $\mathcal{F} \rightarrow \mathcal{G}$ is an isomorphism.

Proof. Without loss of generality, assume $S = \text{Spec}(R)$ is an affine scheme and $\mathcal{F}, \mathcal{G}, \mathcal{H}$ are free sheaves. Let

$$0 \longrightarrow R^{\oplus r} \xrightarrow{\varphi} R^{\oplus r} \longrightarrow R^{\oplus J} \quad (2.50)$$

be the corresponding exact sequence of R -modules, where $r \in \mathbb{N}$ is the rank of $\mathcal{F}$ and $\mathcal{G}$.

By Theorem 5.10, [Con13], the following is injective

$$\wedge^r \varphi : \bigwedge^r (R^{\oplus r}) \rightarrow \bigwedge^r (R^{\oplus r}). \quad (2.51)$$

We can identify $\bigwedge^r(R^{\oplus r}) \cong R$ and then $\wedge^r \varphi$ is the scalar multiplication by $\det(\varphi) \in R$. This shows that $\det(\varphi)$ is not a zero-divisor.

We have that $\operatorname{coker}(\varphi) \subseteq R^{\oplus J}$ and the sub-module $\operatorname{coker}(\varphi)$ is annihilated by $\det(\varphi) \in R$, which is not a zero-divisor. This proves $\operatorname{coker}(\varphi) = 0$ and then φ is an isomorphism. $\square$

Proposition 2.2.9. *Let G be an affine algebraic group over $\mathbb{k}$. Let $X \in \operatorname{Sch}/\mathbb{k}$ and let G act on X . Let $\mathcal{Q} := \operatorname{coker}(\phi)$ for $\phi : \Omega_{X/\mathbb{k}} \rightarrow \mathfrak{g}^* \otimes_{\mathbb{k}} \mathcal{O}_X$. Let $(f : T \rightarrow X) \in \operatorname{Sch}/X$ be a T -valued point of X . Let $r \in \mathbb{N}$. Then the following are equivalent:*

(1) $f^* \mathcal{Q}$ is locally free of rank r on T ;

(2) For any closed subscheme $T' \hookrightarrow T$, if $f' : T' \hookrightarrow T \xrightarrow{f} X$ denotes the composition, then the pre-sheaf on T' , which associates the following set to an open subset $U' \subseteq T'$

$$U' \mapsto \left\{ \rho : \begin{array}{l} \rho \text{ is a } U' \text{-valued infinitesimal transformation in } G \\ \rho \text{ stabilises } f'|_{U'} \end{array} \right\} \quad (2.52)$$

is a locally free sheaf of rank r .

Proof. Choose a morphism $(f' : T' \rightarrow X) \rightarrow (f : T \rightarrow X)$ in Sch/X . By Lemma 2.2.5 and Lemma 2.2.7, the pre-sheaf in (2) equals the coherent sheaf $\mathcal{H}om_{\mathcal{O}_{T'}}((f')^* \mathcal{Q}, \mathcal{O}_{T'})$. Therefore (2) is equivalent to that for all $(f' : T' \rightarrow X) \rightarrow (f : T \rightarrow X)$ in Sch/X such that $T' \rightarrow T$ is a locally closed immersion, the coherent sheaf $\mathcal{H}om_{\mathcal{O}_{T'}}((f')^* \mathcal{Q}, \mathcal{O}_{T'})$ is locally free of rank r on T' .

Assume (1). Since $f^* \mathcal{Q}$ is locally free of rank r , its pull-back $(f')^* \mathcal{Q}$ is locally free of rank r on T' , so is its dual $\mathcal{H}om_{\mathcal{O}_{T'}}((f')^* \mathcal{Q}, \mathcal{O}_{T'})$. This proves (2).

Assume (2). For $m \in \mathbb{Z}$, let $Z_m \hookrightarrow T$ be the closed subscheme associated to $\operatorname{Fit}_m(f^* \mathcal{Q}) \subseteq \mathcal{O}_T$, the m th Fitting ideal of $f^* \mathcal{Q}$. By [Sta22, Lemma 05P8], the locally closed subscheme $T_m := Z_{m-1} \setminus Z_m$ of T satisfies that $f^* \mathcal{Q}|_{T_m}$ is locally free of rank m . The dual sheaf $\mathcal{H}om_{\mathcal{O}_{T_m}}(f^* \mathcal{Q}|_{T_m}, \mathcal{O}_{T_m})$ is then locally free of rank m . However (2) assumes that $\mathcal{H}om_{\mathcal{O}_{T_m}}(f^* \mathcal{Q}|_{T_m}, \mathcal{O}_{T_m})$ is locally free of rank r . Then $T_m = \emptyset$ if $m \neq r$. In particular, set theoretically $Z_r = \bigsqcup_{m > r} T_m = \emptyset$, i.e. $\operatorname{Fit}_r(f^* \mathcal{Q}) = \mathcal{O}_T$.

By [Sta22, Lemma 0C3F], the equality $\mathrm{Fit}_r(f^*\mathcal{Q}) = \mathcal{O}_T$ implies that T has an open cover $T = \bigcup_\alpha U_\alpha$ such that $f^*\mathcal{Q}|_{U_\alpha}$ can be generated by r sections on each U_α . Let $U \subseteq T$ be such an open subscheme and choose a local presentation of $f^*\mathcal{Q}$

$$\mathcal{O}_U^{\oplus J} \xrightarrow{\psi} \mathcal{O}_U^{\oplus r} \longrightarrow f^*\mathcal{Q}|_U \longrightarrow 0. \quad (2.53)$$

If we can show $\psi = 0$, then $f^*\mathcal{Q}|_U$ is free of rank r , and then $f^*\mathcal{Q}$ is locally free of rank r , which is (1). Since ψ is a morphism between locally free sheaves, for $\psi = 0$, it suffices to prove $\psi^* : \mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_U^{\oplus r}, \mathcal{O}_U) \rightarrow \mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_U^{\oplus J}, \mathcal{O}_U)$, the induced morphism between dual sheaves, is zero.

The following is exact

$$0 \longrightarrow \mathcal{H}om_{\mathcal{O}_U}(f^*\mathcal{Q}|_U, \mathcal{O}_U) \longrightarrow \mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_U^{\oplus r}, \mathcal{O}_U) \xrightarrow{\psi^*} \mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_U^{\oplus J}, \mathcal{O}_U). \quad (2.54)$$

The sheaf $\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_U^{\oplus r}, \mathcal{O}_U)$ is locally free of rank r , and (2) assumes that the sheaf $\mathcal{H}om_{\mathcal{O}_U}(f^*\mathcal{Q}|_U, \mathcal{O}_U)$ is also locally free of rank r . By Lemma 2.2.8, we have that

$$\mathcal{H}om_{\mathcal{O}_U}(f^*\mathcal{Q}|_U, \mathcal{O}_U) \cong \mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_U^{\oplus r}, \mathcal{O}_U). \quad (2.55)$$

Therefore $\psi^* = 0$, and by the argument above, this proves (1). $\square$

2.3 Notions of quotients

Let G be an affine algebraic group over $\mathbb{k}$ acting on a scheme X of finite type over $\mathbb{k}$. Let $\mathrm{Sch}/\mathbb{k}$ be the category of schemes of finite type over $\mathbb{k}$ with morphisms over $\mathbb{k}$. References for this section are [MFK94], and [Hos16].

Definition 2.3.1. A *categorical quotient* by G of X is a G -invariant morphism $\phi : X \rightarrow Y$ in $\mathrm{Sch}/\mathbb{k}$ satisfying the universal property that for all G -invariant morphism $\varphi : X \rightarrow Z$, there exists a unique morphism $\psi : Y \rightarrow Z$ such that $\psi \circ \phi = \varphi$.

Definition 2.3.2. A *good quotient* by G of X is a morphism $\phi : X \rightarrow Y$ such that:

- ϕ is G -invariant, affine and surjective;

- the pull-back of functions $\phi^\# : \mathcal{O}_Y \rightarrow \phi_* \mathcal{O}_X$ factors through an isomorphism $\mathcal{O}_Y \cong (\phi_* \mathcal{O}_X)^G$;
- for any G -invariant closed subset $W \subseteq X$, its image $\phi(W) \subseteq Y$ is closed;
- for any pair of disjoint G -invariant closed subsets $W_1, W_2 \subseteq X$, their images are disjoint $\phi(W_1) \cap \phi(W_2) = \emptyset$.

Definition 2.3.3. A *geometric quotient* by G of X is a good quotient $\phi : X \rightarrow Y$ such that each fibre $\phi^{-1}(y) \subseteq X$ is a single orbit for all closed points $y \in Y$.

Definition 2.3.4. A categorical quotient (resp. good quotient, geometric quotient) is called *universal* if any base change is also a categorical quotient (resp. good quotient, geometric quotient).

The following are some properties.

Lemma 2.3.5. • A good quotient is a categorical quotient (Proposition 3.30, [Hos16]);

- If $\phi : X \rightarrow Y$ is a good quotient by G with closed orbits, then ϕ is a geometric quotient (Corollary 3.32, [Hos16]);
- If $\phi : X \rightarrow Y$ is a good quotient by G , then $\phi(x_1) = \phi(x_2)$ if and only if $\overline{G.x_1} = \overline{G.x_2}$ for any closed points $x_1, x_2 \in X$ (Corollary 3.32, [Hos16]);
- Good quotients and geometric quotients are stable under base change along open immersions (Corollary 3.33, [Hos16]);
- Good quotients and geometric quotients are local in the target, in the sense that if $\phi : X \rightarrow Y$ is G -invariant, and there is an open cover $\{U_i\}$ of Y such that ϕ pulls back to a good quotient (resp. geometric quotient) along $U_i \rightarrow Y$ for each i , then ϕ is a good quotient (resp. geometric quotient) (Remark 3.34, [Hos16]).

2.4 Reductive GIT

Assume G is a (linearly) reductive group over $\mathbb{k}$. Let X be a projective scheme over $\mathbb{k}$ and let L be an ample line bundle on X . Let G act on X linearly with respect to L . For $n \in \mathbb{N}$ and $f \in H^0(X, L^n)$, let $X_f \subseteq X$ denote the nonvanishing locus of f .

Definition 2.4.1 ([MFK94] Definition 1.7, Definition 1.8). For the linear action of G on X with respect to L , a closed point $x \in X$ is called *semistable* if there exists $n \in \mathbb{N}_+$ and a G -invariant section $f \in H^0(X, L^n)^G$ such that $x \in X_f$. A closed point $x \in X$ is called *stable* if there exists $n \in \mathbb{N}_+$ and a G -invariant section $f \in H^0(X, L^n)^G$ such that:

- $x \in X_f$;
- the action of G on X_f has closed orbits and finite stabilisers.

Let $X^{G,L,s}$ denote the set of stable points and let $X^{G,L,ss}$ denote the set of semistable points.

Remark 2.4.2. In [MFK94], what we call stable points are called *properly stable*.

It is easy to see for any $m \in \mathbb{N}_+$ we have

$$X^{G,L,s} = X^{G,L^m,s}, \quad X^{G,L,ss} = X^{G,L^m,ss}. \quad (2.56)$$

In fact, for an ample line bundle L on X , the open subsets $X^{G,L,s}$ and $X^{G,L,ss}$ depend only on the fractional linearisation class of L , in the sense of [Tha96].

Theorem 2.4.3 ([MFK94] Theorem 1.10). *Let G be a linearly reductive group over $\mathbb{k}$. Let X be a projective scheme over $\mathbb{k}$ and let L be an ample line bundle on X . Let G act on X linearly with respect to L . Then:*

- $X^{G,L,s} \subseteq X^{G,L,ss}$ are G -invariant open subsets of the topological space of closed points of X in the Zariski topology;
- the subring of G -invariant sections $\bigoplus_{d \geq 0} H^0(X, L^d)^G$ is finitely generated;
- the inclusion $\bigoplus_{d \geq 0} H^0(X, L^d)^G \subseteq \bigoplus_{d \geq 0} H^0(X, L^d)$ induces a morphism

$$\phi : X^{G,L,ss} \rightarrow X//G := \text{Proj} \left(\bigoplus_{d \geq 0} H^0(X, L^d)^G \right) \quad (2.57)$$

which is a projective universal good quotient;

- the morphism $\phi : X^{G,L,ss} \rightarrow X//G$ restricts to a quasi-projective universal geometric quotient

$$\phi|_{X^{G,L,s}} : X^{G,L,s} \rightarrow X^{G,L,s}/G; \quad (2.58)$$

- the quasi-projective scheme $X^{G,L,s}/G$ is open in the projective scheme $X//G$ and the following diagram is Cartesian

$$\begin{array}{ccc} X^{G,L,s} & \xrightarrow{\subseteq} & X^{G,L,ss} \\ \downarrow \phi|_{X^{G,L,s}} & & \downarrow \phi \\ X^{G,L,s}/G & \xrightarrow{\subseteq} & X//G. \end{array} \quad (2.59)$$

We can replace the ample line bundle L by L^m for sufficiently large $m \in \mathbb{N}_+$ such that there exists a closed immersion $i : X \hookrightarrow \mathbb{P}(V)$ for $V = H^0(X, L^m)^*$ such that $i^* \mathcal{O}_{\mathbb{P}(V)}(1) \cong L^m$. The linearisation of G on L induces a linear action of G on $\mathbb{P}(V)$ with respect to $\mathcal{O}_{\mathbb{P}(V)}(1)$ such that $i : X \hookrightarrow \mathbb{P}(V)$ is G -equivariant. So we can define $\mathbb{P}(V)^{G, \mathcal{O}(1), (s)s}$. By [MFK94] Theorem 1.19, these open subsets of (semi)stable points satisfy

$$X^{G,L,(s)s} = i^{-1}(\mathbb{P}(V)^{G, \mathcal{O}(1), (s)s}). \quad (2.60)$$

The following is called the *Hilbert-Mumford criterion*, with which we can determine (semi)stable points without knowing the subring of invariant sections $\bigoplus_{d \geq 0} H^0(X, L^d)^G$.

Theorem 2.4.4 ([MFK94] Theorem 2.1). *Let G be a linearly reductive group. Let X be a projective scheme and let L be an ample line bundle on X . Let G act on X linearly with respect to L . Let $x \in X$ be a closed point. Then the following are equivalent:*

- $x \in X^{G,L,s}$;
- for any non-trivial 1PS $\lambda : \mathbb{G}_m \rightarrow G$, the weight of λ acting on the fibre of L at $x_0 := \lim_{t \rightarrow 0} \lambda(t).x$ is positive.

and the following are equivalent:

- $x \in X^{G,L,ss}$;
- for any 1PS $\lambda : \mathbb{G}_m \rightarrow G$, the weight of λ acting on the fibre of L at $x_0 := \lim_{t \rightarrow 0} \lambda(t).x$ is non-negative.

Let T be a torus. Define

$$\mathfrak{t}_{\mathbb{Q}}^* := \text{Hom}(T, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Q}. \quad (2.61)$$

Let V be a finite dimensional vector space and let $\rho : T \rightarrow \text{GL}(V)$ be a representation. By [Bor91] Proposition 8.4, the representation ρ is diagonalisable, i.e. there is a decomposition

$$V = \bigoplus_{\alpha \in \text{Hom}(T, \mathbb{G}_m)} V_{\alpha} \quad (2.62)$$

$$V_{\alpha} := \{v \in V : \rho(t)(v) = \alpha(t)v \text{ for } t \in T\}.$$

We call $\alpha \in \text{Hom}(T, \mathbb{G}_m)$ a *weight* of the representation if $V_{\alpha} \neq 0$. For any $v \in V$, we can decompose it according to $V = \bigoplus_{\alpha} V_{\alpha}$

$$v = \sum_{\alpha} v_{\alpha}, \quad v_{\alpha} \in V_{\alpha} \text{ for all } \alpha \in \text{Hom}(T, \mathbb{G}_m). \quad (2.63)$$

For $v \in V \setminus \{0\}$, define the set $\text{weight}_T(v) \subseteq \text{Hom}(T, \mathbb{G}_m)$

$$\text{weight}_T(v) := \{\alpha \in \text{Hom}(T, \mathbb{G}_m) : v_{\alpha} \neq 0\} \quad (2.64)$$

which is a non-empty finite set.

By equation (2.60), we can easily know $X^{G,L,(s)s}$ from $\mathbb{P}(V)^{G,\mathcal{O}(1),(s)s}$. The following is a variation of the Hilbert-Mumford criterion for linearisations on $(\mathbb{P}(V), \mathcal{O}(1))$.

Theorem 2.4.5 ([Dol03] Theorem 9.2, Theorem 9.3). *Let G be a linearly reductive group. Let $T \subseteq G$ be a maximal torus (Cartan subgroup). Let V be a finite dimensional vector space. Let G act on $\mathbb{P}(V)$ linearly with respect to $\mathcal{O}(1)$. Let $[v] \in \mathbb{P}(V)$ denote the one-dimensional subspace generated by $v \in V \setminus \{0\}$. Then $[v] \in \mathbb{P}(V)$ is (semi)stable with respect to the G -linearisation if and only if $g.[v] \in \mathbb{P}(V)$ is (semi)stable with respect to the T -linearisation for all $g \in G$.*

Moreover the following are equivalent:

- $[v] \in \mathbb{P}(V)^{T,\mathcal{O}(1),s};$
- $0 \in \mathfrak{t}_{\mathbb{Q}}^*$ is in the interior of the convex hull of $\text{weight}_T(v) \subseteq \mathfrak{t}_{\mathbb{Q}}^*;$

and the following are equivalent:

- $[v] \in \mathbb{P}(V)^{T,\mathcal{O}(1),ss};$
- $0 \in \mathfrak{t}_{\mathbb{Q}}^*$ is in the convex hull of $\text{weight}_T(v) \subseteq \mathfrak{t}_{\mathbb{Q}}^*.$

Chapter 3

Non-reductive Geometric Invariant Theory

Let $\mathbb{k}$ be an algebraically closed field of characteristic zero. An affine algebraic group H can be written as $H = U \rtimes R$, where $U \trianglelefteq H$ is the unipotent radical and $R \cong H/U$ is the reductive Levi factor. Given a (linear) action of H , a natural strategy for constructing a quotient by H is to first construct a quotient by U and then construct a quotient by R . Geometric invariant theory for reductive linearisations has been well developed, so we focus more on quotients by U in this chapter.

Let U be a unipotent group with Lie algebra $\mathfrak{u}$. A *grading one-parameter subgroup* is $\lambda : \mathbb{G}_m \rightarrow \text{Aut}(U)$ such that the weight of λ on $\mathfrak{u}$ are all positive. We call U *graded* if there exists a grading 1PS. Given a grading 1PS λ , let $\hat{U}$ denote the semi-direct product $U \rtimes_{\lambda} \mathbb{G}_m$, or $U \rtimes \lambda$ alternatively. For a Levi decomposition $H \cong U \rtimes R$, the unipotent radical U is called *internally graded* if there exists a 1PS $\lambda : \mathbb{G}_m \rightarrow Z(R)$ such that the composition $\mathbb{G}_m \xrightarrow{\lambda} Z(R) \rightarrow \text{Aut}(U)$ is a grading 1PS, where $Z(R) \subseteq R$ is the centre of R . Geometric invariant theory for linearisations of $\hat{U}$ on projective varieties was studied in [Bér+18b], [BK17], [BDK18], [Bér+16], [HJ21], with conditions on dimensions of U -stabilisers or conditions on an extension of the U -action to a suitable reductive group.

The so-called $\hat{U}$ -theorem and the so-called “semistability coincides with stability” condition appear in [Bér+18b] and [Bér+16]. Let X be a projective *variety* and let L be an ample line bundle on X . Let $\hat{U} \curvearrowright (X, L)$. Let $X_{\min}^0 \subseteq X$ be the non-vanishing locus of $\bigoplus_{d>0} H^0(X, L^d)_{\lambda=\max}$ and let $Z_{\min} \hookrightarrow X$ be the vanishing locus of $\bigoplus_{d>0} H^0(X, L^d)_{\lambda<\max}$.

In [Bér+18b], under the condition “ss=s” (Condition 3.1.10), it is proved that $X_{\min}^0 \setminus UZ_{\min}$ is open and it has a projective geometric quotient by $\hat{U}$. In [Bér+16], the condition “ss=s” is replaced by a weaker version that $\text{Stab}_U(z) = \{e\}$ for all closed points $z \in Z_{\min}$. These two versions of “ss=s” are actually equivalent for $\hat{U}$ -linearisations on projective varieties (Corollary 3.1.13 below).

Let $w_1 > \dots > w_n$ be the positive weights of $\lambda \curvearrowright \mathfrak{u}$. Let $U_i \subseteq U$ be such that $\text{Lie}(U_i) = \mathfrak{u}_{\lambda \geq w_i}$, which is the sum of weight spaces whose λ -weights are greater than or equal to w_i . As suggested in [Bér+16] Remark 2.8 and Remark 7.12, the condition “ss=s” can be further relaxed to the condition that the minimal dimension of U_i -stabilisers on $Z_{\min}$ and the minimal dimension of U_i -stabilisers on $X_{\min}^0$ coincide for all $1 \leq i \leq n$. We split this relaxation into two steps. The first relaxation of “ss=s” is the condition that the dimension of U_i -stabilisers are constant on $X_{\min}^0$ for $1 \leq i \leq n$, which is assumed in this chapter. The second relaxation is that the minimal dimensions of U_i -stabilisers on $Z_{\min}$ and $X_{\min}^0$ coincide, which requires a blow-up to satisfy the first relaxation of “ss=s” above, which is assumed in the next chapter.

In this chapter, we follow ideas in [Bér+16] Section 7, and generalise the so-called $\hat{U}$ -hat theorem from projective *varieties* to projective *schemes* over $\mathbb{k}$, and complete the proof of the results stated in [Bér+16] Remark 2.8 and Remark 7.12 (See Theorem 3.3.8, Theorem 3.3.9). The scheme-theoretic version of “ss=s” is Condition 3.3.2 (We call it the ‘constant dimension of relative stabilisers’ or CDRS condition), which says cokernels of certain infinitesimal actions are locally free. More precisely, for an action $G \curvearrowright Y$ over $\mathbb{k}$, the infinitesimal action is a morphism of sheaves $\phi : \Omega_{Y/\mathbb{k}} \rightarrow \mathfrak{g}^* \otimes_{\mathbb{k}} \mathcal{O}_Y$, and for a closed point $y \in Y$, we have $\text{coker}(\phi) \otimes_{\mathcal{O}_Y} \kappa(y) \cong \text{Stab}_{\mathfrak{g}}(y)^*$. Therefore the condition that $\dim \text{Stab}_{\mathfrak{g}}$ is constant can be replaced by requiring that $\text{coker}(\phi)$ is locally free of constant rank. If $X_{\min}^0$ is reduced, then Condition 3.3.2 is equivalent to the condition that U_i -stabilisers have constant dimensions on $X_{\min}^0$.

Fix $\hat{U}$ and denote the grading 1PS by $\lambda : \mathbb{G}_m \rightarrow \text{Aut}(U)$. Recall the grading 1PS λ on U induces a filtration of subgroups $\{U_i\}_{i=0}^n$, where $U_i \subseteq U$ corresponds to $\mathfrak{u}_{\lambda \geq w_i} \subseteq \mathfrak{u}$ for $w_1 > \dots > w_n$ being the weights of λ on $\mathfrak{u}$. Let X be a projective scheme over $\mathbb{k}$ and let

L be an ample line bundle on X . Consider a linearisation $\hat{U} \curvearrowright (X, L)$. Let $X_{\min}^0 \subseteq X$ be the non-vanishing locus of $\bigoplus_{d>0} H^0(X, L^d)_{\lambda=\max}$, and let $Z_{\min} \hookrightarrow X_{\min}^0$ be the *fixed point subscheme* with respect to $\lambda \curvearrowright X_{\min}^0$ in the sense of [Fog73]. Assume $X_{\min}^0 \neq \emptyset$. Note that without assuming that X is reduced, it is possible that $X_{\min}^0 = \emptyset$ when sections in $\bigoplus_{d>0} H^0(X, L^d)_{\lambda=\max}$ are all nilpotent.

Our strategy is to construct quotients by U_i/U_{i-1} successively. For this process, we assume the CDRS condition (Condition 3.3.2), which generalises the condition “ss=s” scheme-theoretically. With this condition, we prove that the universal geometric quotient by U of $X_{\min}^0$ exists and the quotient is quasi-projective (Theorem 3.3.8). Moreover $X_{\min}^0 \setminus UZ_{\min}$ is a $\hat{U}$ -invariant open subscheme of X and there exists a projective universal geometric quotient $X//\hat{U}$ of $X_{\min}^0 \setminus UZ_{\min}$ by $\hat{U}$ (Theorem 3.3.9).

3.1 Fitting ideals and stabilisers

In this section, we introduce some useful exact sequences for actions by U (Lemma 3.1.4). We can recover unipotent stabilisers of $\mathbf{k}$ -points from these sequences (Lemma 3.1.5). Therefore conditions on unipotent stabilisers can be formulated in terms of these exact sequences. As a result, two versions of the condition “semistability coincides with stability” for $\hat{U}$ -linearisations, one in [Bér+18b], another in [Bér+16], are proved equivalent (Corollary 3.1.13).

3.1.1 Fitting ideals

This section gives definitions and properties of Fitting ideals ([Sta22, Section 07Z6], [Sta22, Section 0C3C]).

Let R be a commutative ring and let M be a finite R -module. There exists a presentation of M

$$R^{\oplus J} \longrightarrow R^{\oplus n} \longrightarrow M \longrightarrow 0 \quad (3.1)$$

where $n \in \mathbb{N}$ is the number of generators, and J is the cardinality of generators of the kernel of $R^{\oplus n} \rightarrow M$. The R -linear map $R^{\oplus J} \rightarrow R^{\oplus n}$ is represented by a matrix $(a_{ij})_{\substack{1 \leq i \leq n \\ j \in J}}$

$$R^{\oplus J} \rightarrow R^{\oplus n}, \quad e_j \mapsto \sum_{i=1}^n a_{ij} e_i. \quad (3.2)$$

Definition 3.1.1. For $k \in \mathbb{Z}$, the k th Fitting ideal $\text{Fit}_k(M) \subseteq R$ of M is the ideal generated by $(n-k) \times (n-k)$ -minors of the matrix $(a_{ij})_{1 \leq i \leq n, j \in J}$. The ideals $\text{Fit}_k(M)$ are independent of the presentation.

Let S be a scheme, and let $\mathcal{F}$ be a quasi-coherent sheaf of finite type over S .

Definition 3.1.2. For $k \in \mathbb{Z}$, the k th Fitting ideal $\text{Fit}_k(\mathcal{F}) \subseteq \mathcal{O}_S$ is the quasi-coherent sheaf of ideals such that for any affine open $U \subseteq S$

$$\text{Fit}_k(\mathcal{F})(U) \cong \text{Fit}_k(\mathcal{F}(U)). \quad (3.3)$$

The Fitting ideals form increasing chains

$$\begin{aligned} 0 = \text{Fit}_{-1}(M) &\subseteq \cdots \subseteq \text{Fit}_{k-1}(M) \subseteq \text{Fit}_k(M) \subseteq \cdots \subseteq R \\ 0 = \text{Fit}_{-1}(\mathcal{F}) &\subseteq \cdots \subseteq \text{Fit}_{k-1}(\mathcal{F}) \subseteq \text{Fit}_k(\mathcal{F}) \subseteq \cdots \subseteq \mathcal{O}_S. \end{aligned} \quad (3.4)$$

The following proposition lists some properties of Fitting ideals.

Proposition 3.1.3. *Let S, T be schemes. Let $\mathcal{F}, \mathcal{F}'$ be quasi-coherent sheaves of finite type over S . Then:*

- $\mathcal{F}$ can be generated by k sections in a neighbourhood of $s \in S$ if and only if $\text{Fit}_k(\mathcal{F})_s = \mathcal{O}_{S,s}$ ([Sta22, Lemma 0C3F]);
- The closed subset $\text{Supp}(\mathcal{F})$ is cut out by $\text{Fit}_0(\mathcal{F})$ ([Sta22, Lemma 0CYX]);
- $\mathcal{F}$ is locally free of rank k if and only if $\text{Fit}_{k-1}(\mathcal{F}) = 0$ and $\text{Fit}_k(\mathcal{F}) = \mathcal{O}_S$ ([Sta22, Lemma 0C3G]);
- If $f : T \rightarrow S$ is a morphism, then $\text{Fit}_k(f^*\mathcal{F}) = f^{-1}\text{Fit}_k(\mathcal{F})\mathcal{O}_T$ ([Sta22, Lemma 0C3D]);
- If $\mathcal{F} \rightarrow \mathcal{F}'$ is surjective, then $\text{Fit}_k(\mathcal{F}) \subseteq \text{Fit}_k(\mathcal{F}')$ ([Sta22, Lemma 07ZA]).

3.1.2 Infinitesimal stabilisers

Let U be a unipotent group with Lie algebra $\mathfrak{u}$. Let $Y = \operatorname{Spec}(A)$ be an affine scheme of finite type over $\mathbb{k}$. Let U act on Y . Recall the *infinitesimal action* of U on Y is defined in Definition 2.2.1 as a morphism on Y , corresponding to a map of A -modules

$$\Omega_{A/\mathbb{k}} \rightarrow \mathfrak{u}^* \otimes_{\mathbb{k}} A. \quad (3.5)$$

Let $\xi \in \mathfrak{u}$ and let $y \in Y$ be a $\mathbb{k}$ -point. Let $y^* : A \rightarrow \mathbb{k}$ be the ring map corresponding to $y \in Y$. We say $\xi \in \mathfrak{u}$ *stabilises* $y \in Y$ if the following composition is zero

$$A \xrightarrow{d} \Omega_{A/\mathbb{k}} \longrightarrow \mathfrak{u}^* \otimes A \xrightarrow{\xi \otimes y^*} \mathbb{k} \quad (3.6)$$

where $d : A \rightarrow \Omega_{A/\mathbb{k}}$ is the derivation.

Let $\mathfrak{v} \subseteq \mathfrak{u}$ be a subspace. Define A -modules $K(\mathfrak{v}), Q(\mathfrak{v})$ as the kernel and cokernel

$$0 \longrightarrow K(\mathfrak{v}) \longrightarrow \Omega_{A/\mathbb{k}} \longrightarrow \mathfrak{v}^* \otimes_{\mathbb{k}} A \longrightarrow Q(\mathfrak{v}) \longrightarrow 0 \quad (3.7)$$

where the morphism $\Omega_{A/\mathbb{k}} \rightarrow \mathfrak{v}^* \otimes_{\mathbb{k}} A$ is the composition $\Omega_{A/\mathbb{k}} \rightarrow \mathfrak{u}^* \otimes_{\mathbb{k}} A \rightarrow \mathfrak{v}^* \otimes_{\mathbb{k}} A$.

Let $\mathfrak{w} \subseteq \mathfrak{v}$ be subspaces of $\mathfrak{u}$. For $y \in Y$ a $\mathbb{k}$ -point, define

$$\begin{aligned} \operatorname{Stab}_{\mathfrak{v}}(y) &:= \{\xi \in \mathfrak{v} : \xi \text{ stabilises } y\} \subseteq \mathfrak{v} \\ \operatorname{Stab}_{(\mathfrak{v}:\mathfrak{w})}(y) &:= \frac{\operatorname{Stab}_{\mathfrak{v}}(y)}{\operatorname{Stab}_{\mathfrak{w}}(y)} \subseteq \mathfrak{v}/\mathfrak{w}. \end{aligned} \quad (3.8)$$

We call $\operatorname{Stab}_{\mathfrak{v}}(y)$ the *stabiliser of y in $\mathfrak{v}$* , and we call $\operatorname{Stab}_{(\mathfrak{v}:\mathfrak{w})}(y)$ the *stabiliser of y in $\mathfrak{v}$ relative to $\mathfrak{w}$* .

Lemma 3.1.4. *Let U be a unipotent group. Let $Y = \operatorname{Spec}(A)$ be an affine scheme of finite type over $\mathbb{k}$. Let U act on Y . Let $\mathfrak{w} \subseteq \mathfrak{v}$ be subspaces of $\mathfrak{u}$. Then there is an exact sequence*

$$0 \longrightarrow K(\mathfrak{v}) \longrightarrow K(\mathfrak{w}) \longrightarrow (\mathfrak{v}/\mathfrak{w})^* \otimes A \longrightarrow Q(\mathfrak{v}) \longrightarrow Q(\mathfrak{w}) \longrightarrow 0. \quad (3.9)$$

Moreover if $Q(\mathfrak{w})$ is a flat A -module, then for any ring map $A \rightarrow B$, the following is exact

$$K(\mathfrak{w}) \otimes_A B \longrightarrow (\mathfrak{v}/\mathfrak{w})^* \otimes_A B \longrightarrow Q(\mathfrak{v}) \otimes_A B \longrightarrow Q(\mathfrak{w}) \otimes_A B \longrightarrow 0. \quad (3.10)$$

Proof. Consider the following diagram with splitting exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \Omega_{A/\mathbb{k}} & \xlongequal{\quad} & \Omega_{A/\mathbb{k}} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \phi \\
0 & \longrightarrow & (\mathfrak{v}/\mathfrak{w})^* \otimes A & \longrightarrow & \mathfrak{v}^* \otimes A & \longrightarrow & \mathfrak{w}^* \otimes A \longrightarrow 0.
\end{array} \tag{3.11}$$

Apply the Snake lemma to get the long exact sequence in the lemma.

Base change of the above diagram along $A \rightarrow B$ also has exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \Omega_{A/\mathbb{k}} \otimes_A B & \xlongequal{\quad} & \Omega_{A/\mathbb{k}} \otimes_A B \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \phi \otimes 1 \\
0 & \longrightarrow & (\mathfrak{v}/\mathfrak{w})^* \otimes B & \longrightarrow & \mathfrak{v}^* \otimes B & \longrightarrow & \mathfrak{w}^* \otimes B \longrightarrow 0.
\end{array} \tag{3.12}$$

Then by the Snake lemma the following is exact:

$$\ker(\phi \otimes 1) \longrightarrow (\mathfrak{v}/\mathfrak{w})^* \otimes B \longrightarrow Q(\mathfrak{v}) \otimes_A B \longrightarrow Q(\mathfrak{w}) \otimes_A B \longrightarrow 0. \tag{3.13}$$

Since $Q(\mathfrak{w}) = \text{coker } \phi$ is a flat A -module, the kernel of $\phi \otimes 1 : \Omega_{A/\mathbb{k}} \otimes_A B \rightarrow \mathfrak{w}^* \otimes B$ is the image of $K(\mathfrak{w}) \otimes_A B \rightarrow \Omega_{A/\mathbb{k}} \otimes_A B$, i.e. there is a surjective map $K(\mathfrak{w}) \otimes_A B \rightarrow \ker(\phi \otimes 1)$.

So the following is exact:

$$K(\mathfrak{w}) \otimes_A B \longrightarrow (\mathfrak{v}/\mathfrak{w})^* \otimes_A B \longrightarrow Q(\mathfrak{v}) \otimes_A B \longrightarrow Q(\mathfrak{w}) \otimes_A B \longrightarrow 0. \tag{3.14}$$

□

Lemma 3.1.5. *Let U be a unipotent group. Let $Y = \text{Spec}(A)$ be an affine scheme of finite type over $\mathbb{k}$. Let U act on Y . Let $\mathfrak{w} \subseteq \mathfrak{v}$ be subspaces of $\mathfrak{u}$. Let $y \in Y$ be a $\mathbb{k}$ -point. Then*

(1) *The stabiliser of y in $\mathfrak{v}$ is*

$$\text{Stab}_{\mathfrak{v}}(y) = \text{im}(y^* \phi_{\mathfrak{v}})^{\perp} \subseteq \mathfrak{v} \tag{3.15}$$

where $\phi_{\mathfrak{v}} : \Omega_{A/\mathbb{k}} \rightarrow \mathfrak{v}^* \otimes A$ is the infinitesimal action of $\mathfrak{v}$ on Y ;

(2) *If $Q(\mathfrak{w})$ is a flat A -module, then*

$$\text{Stab}_{(\mathfrak{v}:\mathfrak{w})}(y) = \text{im}(y^* \phi_{(\mathfrak{v}:\mathfrak{w})})^{\perp} \subseteq \mathfrak{v}/\mathfrak{w} \tag{3.16}$$

where $\phi_{(\mathfrak{v}:\mathfrak{w})} : K(\mathfrak{w}) \rightarrow (\mathfrak{v}/\mathfrak{w})^* \otimes A$ is the map in Lemma 3.1.4.

Proof. Let $y^\# : A \rightarrow \mathbb{k}$ be the ring map corresponding to $y : \text{Spec}(\mathbb{k}) \rightarrow Y$.

We first prove $\text{Stab}_{\mathfrak{v}}(y) = \text{im}(y^*\phi_{\mathfrak{v}})^\perp$. For $\xi \in \mathfrak{v}$, it stabilises $y \in Y$ if and only if $(\xi \otimes y^\#) \circ \phi_{\mathfrak{v}} = 0$, where $\xi : \mathfrak{v}^* \rightarrow \mathbb{k}$ is the linear map on $\mathfrak{v}^*$. Consider the following diagram with surjective vertical arrows

$$\begin{array}{ccccc} \Omega_{A/\mathbb{k}} & \xrightarrow{\phi_{\mathfrak{v}}} & \mathfrak{v}^* \otimes A & \xrightarrow{\xi \otimes y^\#} & \mathbb{k} \\ \downarrow 1 \otimes y^\# & & \downarrow 1 \otimes y^\# & & \parallel \\ \Omega_{A/\mathbb{k}} \otimes_{A, y^\#} \mathbb{k} & \xrightarrow{y^*\phi_{\mathfrak{v}}} & \mathfrak{v}^* & \xrightarrow{\xi} & \mathbb{k}. \end{array} \quad (3.17)$$

Then $(\xi \otimes y^\#) \circ \phi_{\mathfrak{v}} = 0$ if and only if $\xi \circ (y^*\phi_{\mathfrak{v}}) = 0$, i.e. $\xi \in \text{im}(y^*\phi_{\mathfrak{v}})^\perp$. This proves (1).

When $Q(\mathfrak{w})$ is a flat A -module, the following sequence is exact by Lemma 3.1.4

$$y^*K(\mathfrak{w}) \xrightarrow{y^*\phi_{(\mathfrak{v}:\mathfrak{w})}} (\mathfrak{v}/\mathfrak{w})^* \longrightarrow y^*Q(\mathfrak{v}) \longrightarrow y^*Q(\mathfrak{w}) \longrightarrow 0. \quad (3.18)$$

We have that $\mathfrak{v}^* \rightarrow y^*Q(\mathfrak{v})$ and $\mathfrak{v}^* \rightarrow (\text{Stab}_{\mathfrak{v}}(y))^*$ are isomorphic quotients since they have the same kernel $\text{im}(y^*\phi_{\mathfrak{v}}) = \text{Stab}_{\mathfrak{v}}(y)^\perp \subseteq \mathfrak{v}^*$ by (1). Similarly for $\mathfrak{w}$. Therefore the above exact sequence becomes

$$y^*K(\mathfrak{w}) \xrightarrow{y^*\phi_{(\mathfrak{v}:\mathfrak{w})}} (\mathfrak{v}/\mathfrak{w})^* \longrightarrow (\text{Stab}_{\mathfrak{v}}(y))^* \longrightarrow (\text{Stab}_{\mathfrak{w}}(y))^* \longrightarrow 0 \quad (3.19)$$

that is the following is exact

$$y^*K(\mathfrak{w}) \xrightarrow{y^*\phi_{(\mathfrak{v}:\mathfrak{w})}} (\mathfrak{v}/\mathfrak{w})^* \longrightarrow (\text{Stab}_{(\mathfrak{v}:\mathfrak{w})}(y))^* \longrightarrow 0. \quad (3.20)$$

It is easy to check the linear map $(\mathfrak{v}/\mathfrak{w})^* \rightarrow (\text{Stab}_{(\mathfrak{v}:\mathfrak{w})}(y))^*$ is dual to the inclusion $\text{Stab}_{(\mathfrak{v}:\mathfrak{w})}(y) \subseteq \mathfrak{v}/\mathfrak{w}$. Then the above exact sequence is equivalent to (2). $\square$

Lemma 3.1.6. *Let U be a unipotent group. Let $Y = \text{Spec}(A)$ be an affine scheme of finite type over $\mathbb{k}$. Let U act on Y . Let $\mathfrak{w} \subseteq \mathfrak{v}$ be subspaces of $\mathfrak{u}$. Let $\phi_{(\mathfrak{v}:\mathfrak{w})} : K(\mathfrak{w}) \rightarrow (\mathfrak{v}/\mathfrak{w})^* \otimes A$. Let $y \in Y$ be a $\mathbb{k}$ -point and let $k \in \mathbb{Z}$. Then the following are equivalent:*

- (1) $\dim \text{coker}(y^*\phi_{(\mathfrak{v}:\mathfrak{w})}) > k$;
- (2) $\text{Fit}_k(\phi_{(\mathfrak{v}:\mathfrak{w})}) \subseteq \ker y^\#$, where $\text{Fit}_k(\phi_{(\mathfrak{v}:\mathfrak{w})})$ denotes the k th Fitting ideal of $\text{coker}(\phi_{(\mathfrak{v}:\mathfrak{w})})$ and $y^\# : A \rightarrow \mathbb{k}$ corresponds to the $\mathbb{k}$ -point $y \in Y$;

Moreover, when $Q(\mathfrak{w})$ is flat, the above are equivalent to the following:

(3) $\dim \text{Stab}_{(\mathfrak{v}:\mathfrak{w})}(y) > k$.

Proof. It is easy to see that (1), (2) and (3) are equivalent when $k \leq -1$ or $k \geq \dim(\mathfrak{v}/\mathfrak{w})$. Moreover when $Q(\mathfrak{w})$ is flat, we have that $\dim \text{Stab}_{(\mathfrak{v}:\mathfrak{w})}(y) = \dim \text{coker}(y^* \phi_{(\mathfrak{v}:\mathfrak{w})})$, so (1) and (3) are equivalent. Therefore we only need to prove the equivalence of (1) and (2) when $0 \leq k < \dim(\mathfrak{v}/\mathfrak{w})$.

Let $r := \dim \mathfrak{v}/\mathfrak{w}$. Choose a basis of $(\mathfrak{v}/\mathfrak{w})^*$; this basis is naturally a set of generators of $(\mathfrak{v}/\mathfrak{w})^* \otimes A$. Choose a set of generators of $K(\mathfrak{w})$. Let M be a matrix with entries in A representing $\phi_{(\mathfrak{v}:\mathfrak{w})} : K(\mathfrak{w}) \rightarrow (\mathfrak{v}/\mathfrak{w})^* \otimes A$ with respect to these generators. Then $y^\#(M)$ is a matrix with entries in $\mathbb{k}$, representing $y^* \phi_{(\mathfrak{v}:\mathfrak{w})}$, and $\text{Fit}_k(\phi_{(\mathfrak{v}:\mathfrak{w})}) \subseteq A$ is the ideal generated by $(r - k) \times (r - k)$ -minors of M . So $\text{Fit}_k(\phi_{(\mathfrak{v}:\mathfrak{w})}) \subseteq \ker y^\#$ if and only if any $(r - k) \times (r - k)$ -minor of $y^\#(M)$ is zero, i.e. $y^\#(M)$ has rank less than $r - k$. The rank of $y^\#(M)$ is the dimension of $\text{im}(y^* \phi_{(\mathfrak{v}:\mathfrak{w})})$. This proves $\dim \text{im}(y^* \phi_{(\mathfrak{v}:\mathfrak{w})}) < r - k$ if and only if $\text{Fit}_k(\phi) \subseteq \ker y^\#$. $\square$

Remark 3.1.7. The lemma shows that $\text{Fit}_k(\phi_{(\mathfrak{v}:\mathfrak{w})}) \subseteq A$ cuts out the closed subset of $\mathbb{k}$ -points where $\dim \text{Stab}_{(\mathfrak{v}:\mathfrak{w})}(y) > k$ when $Q(\mathfrak{w})$ is flat. In particular when $\mathfrak{w} = 0$, we have $Q(\mathfrak{w}) = 0$. The ideal $\text{Fit}_k(\phi_{\mathfrak{v}})$ cuts out the closed subset of $\mathbb{k}$ -points where $\dim \text{Stab}_{\mathfrak{v}}(y) > k$. These ideals define closed subscheme structures on the closed subsets.

Corollary 3.1.8. *Let U be a unipotent group. Let $Y = \text{Spec}(A)$ be an affine scheme of finite type over $\mathbb{k}$. Let U act on Y . Let $\mathfrak{w} \subseteq \mathfrak{v}$ be subspaces of $\mathfrak{u}$. Let $\phi_{(\mathfrak{v}:\mathfrak{w})} : K(\mathfrak{w}) \rightarrow (\mathfrak{v}/\mathfrak{w})^* \otimes A$. Assume $Q(\mathfrak{w})$ is a flat A -module. Let $k \in \mathbb{N}$. Consider the following conditions:*

(1) $\text{coker}(\phi_{(\mathfrak{v}:\mathfrak{w})})$ is locally free of rank k ;

(2) $\dim \text{Stab}_{(\mathfrak{v}:\mathfrak{w})}(y) = k$ for all $\mathbb{k}$ -points $y \in Y$.

Then (1) $\implies$ (2). When Y is reduced, they are equivalent.

Proof. By [Sta22, Theorem 00FV], the *nilpotent radical* of A is equal to its *Jacobson radical*. The condition that Y is reduced is therefore equivalent to $\text{rad}(A) = 0$, where $\text{rad}(A) \subseteq A$ denotes the Jacobson radical, which is defined as the intersection of all maximal ideals.

Since $Q(\mathfrak{w})$ is flat, we have that (2) is equivalent to the following by Lemma 3.1.6:

$$(2') \text{ Fit}_{k-1}(\phi_{(\mathfrak{v}:\mathfrak{w})}) \subseteq \text{rad}(A) \text{ and } \text{Fit}_k(\phi_{(\mathfrak{v}:\mathfrak{w})}) = A.$$

By Proposition 3.1.3, we have that (1) is equivalent to

$$(1') \text{ Fit}_{k-1}(\phi_{(\mathfrak{v}:\mathfrak{w})}) = 0 \text{ and } \text{Fit}_k(\phi_{(\mathfrak{v}:\mathfrak{w})}) = A.$$

Obviously $(1') \implies (2')$, i.e. $(1) \implies (2)$. When Y is reduced, we have $\text{rad}(A) = 0$ and then $(1') \iff (2')$, i.e. $(1) \iff (2)$. $\square$

In our non-reductive GIT, we often consider a filtration $\{\mathfrak{u}_i\}_{i=0}^n$ of the Lie algebra $\mathfrak{u}$ of the unipotent radical, and require the condition that $\text{coker}(\phi_{(\mathfrak{u}_i:\mathfrak{u}_{i-1})})$ is locally free of constant rank for $1 \leq i \leq n$. Here is a corollary about this condition. In particular, the condition is equivalent to a condition on unipotent stabilisers of $\mathbb{k}$ -points when Y is reduced.

Corollary 3.1.9. *Let U be a unipotent group. Let $Y = \text{Spec}(A)$ be an affine scheme of finite type over $\mathbb{k}$. Let U act on Y . Let the following be a filtration of subspaces of $\mathfrak{u}$*

$$0 = \mathfrak{u}_0 \subseteq \mathfrak{u}_1 \subseteq \cdots \subseteq \mathfrak{u}_n = \mathfrak{u}. \quad (3.21)$$

Recall that $K(\mathfrak{u}_i), Q(\mathfrak{u}_i)$ are the kernel and cokernel of $\Omega_{A/\mathbb{k}} \rightarrow (\mathfrak{u}_i)^* \otimes A$. For $1 \leq i \leq n$, let $\phi_i : K(\mathfrak{u}_{i-1}) \rightarrow (\mathfrak{u}_i/\mathfrak{u}_{i-1})^* \otimes A$ denote the morphism. Let $(k_1, \dots, k_n) \in \mathbb{N}^n$. Then the following are equivalent:

- (1) $\text{coker}(\phi_i)$ is locally free of rank k_i for each $1 \leq i \leq n$;
- (2) $Q(\mathfrak{u}_i)$ is locally free of rank $k_1 + \cdots + k_i$ for each $1 \leq i \leq n$.

The above conditions imply the following:

- (3) $\dim \text{Stab}_{\mathfrak{u}_i}(y) = k_1 + \cdots + k_i$ for $1 \leq i \leq n$ and all $\mathbb{k}$ -points $y \in Y$.

When Y is reduced, all three conditions above are equivalent.

Proof. By Corollary 3.1.8, we have that $(2) \implies (3)$ in general and $(2) \iff (3)$ when Y is reduced.

We then prove the equivalence of (1) and (2). Let $1 \leq i \leq n$. By Lemma 3.1.4, the following is exact

$$0 \longrightarrow \text{coker}(\phi_i) \longrightarrow Q(\mathfrak{u}_i) \longrightarrow Q(\mathfrak{u}_{i-1}) \longrightarrow 0. \quad (3.22)$$

By [Sta22, Lemma 00HM], when $Q(\mathfrak{u}_{i-1})$ is flat, the flatness of $\text{coker}(\phi_i)$ is equivalent to the flatness of $Q(\mathfrak{u}_i)$. Initially we have that $Q(\mathfrak{u}_0) = 0$ is flat. Then inductively we can prove that $\text{coker}(\phi_i)$ are flat for all $1 \leq i \leq n$ if and only if $Q(\mathfrak{u}_i)$ are flat for all $1 \leq i \leq n$.

By [Sta22, Lemma 05P2], finite locally free A -modules are equivalent to finite flat A -modules. Ranks of $\text{coker}(\phi_i)$ and $Q(\mathfrak{u}_i)$ can be read from the short exact sequences for $1 \leq i \leq n$. $\square$

3.1.3 “Semistability coincides with stability”

Let $\lambda : \mathbb{G}_m \rightarrow \text{Aut}(U)$ be a grading 1PS with weights $w_1 > \dots > w_n$. Let $\mathfrak{u}$ be the Lie algebra of U and let $\mathfrak{u}_{\lambda=w_i} \subseteq \mathfrak{u}$ be the weight space of weight w_i . Let X be a projective scheme over $\mathbb{k}$ and let L be an ample line bundle on X . Let $\hat{U}$ act on X linearly with respect to L . Let $X_{\min}^0 \subseteq X$ be the non-vanishing locus of $\bigoplus_{d>0} H^0(X, L^d)_{\lambda=\max}$ and let $Z_{\min} \hookrightarrow X_{\min}^0$ be the fixed point subscheme with respect to λ in the sense of [Fog73]. Then $Z_{\min} \hookrightarrow X$ is the vanishing locus of $\bigoplus_{d>0} H^0(X, L^d)_{\lambda<\max}$.

For the $\hat{U}$ -linearisation on (X, L) , we will call the following condition “semistability coincides with stability” (“ss=s”), which is the condition required in [Bér+18b] with a slight modification. Note that in [Bér+16], the name “semistability coincides with stability” refers to a different condition.

Condition 3.1.10 (cf. Definition 2.1, [Bér+18b]). There exists a positive integer $n \in \mathbb{N}_+$ such that $H^0(X, L^n)$ generates $\bigoplus_{d \geq 0} H^0(X, L^{nd})$ and the representation of $\hat{U}$ on $H^0(X, L^n)$ satisfies that if $\xi, \mathfrak{u}'$ satisfy the following conditions:

- (1) $\mathfrak{u}' \subseteq \mathfrak{u}$ is a λ -invariant Lie ideal;
- (2) $\xi \in \mathfrak{u} \setminus \mathfrak{u}'$ is an weight vector of λ ;
- (3) the weights of λ on $\mathfrak{u}'$ are all greater than or equal to the weight of ξ .

then

$$H^0(X, L^n)_{\lambda=\max} \subseteq \xi \cdot H^0(X, L^n)^{\mathfrak{u}'}. \quad (3.23)$$

Remark 3.1.11. Definition 2.1 in [Bér+18b] says that if $\xi, \mathbf{u}'$ satisfy (1) and (2) above, then $H^0(X, L^n)_{\lambda=\max} \subseteq \xi.H^0(X, L^n)^{\mathbf{u}'}$. Therefore Definition 2.1 in [Bér+18b] implies Condition 3.1.10. However, they both suffice to prove Theorem 2, the main theorem in [Bér+18b].

Consider the diagram

$$\begin{array}{ccc} X_{\min}^0 & \xrightarrow{\gamma} & U \times X_{\min}^0 \\ & \searrow \delta & \downarrow \sigma \\ & & X_{\min}^0 \times X_{\min}^0 \end{array}, \quad \begin{cases} \gamma : x \mapsto (e, x) \\ \sigma : (u, x) \mapsto (u.x, x) \\ \delta : x \mapsto (x, x). \end{cases} \quad (3.24)$$

By Definition 2.2.1, the following sequence is exact

$$\Omega_{X_{\min}^0/\mathrm{Spec}(\mathbb{k})} \longrightarrow \mathbf{u}^* \otimes \mathcal{O}_{X_{\min}^0/\mathrm{Spec}(\mathbb{k})} \longrightarrow \gamma^* \Omega_{\sigma} \longrightarrow 0. \quad (3.25)$$

Proposition 3.1.12. *Let X be a projective scheme over $\mathbb{k}$ and let L be an ample line bundle on X . Let $\hat{U} \curvearrowright (X, L)$. The following are equivalent:*

- (1) “ $ss=s$ ”;
- (2) $\gamma^* \Omega_{\sigma} = 0$.

Proof. Let $m \in \mathbb{N}_+$ be any integer such that $H^0(X, L^m)$ generates $\bigoplus_{d \geq 0} H^0(X, L^{md})$. Then $X_{\min}^0$ is the union of the open affine subschemes X_f ’s for $f \in H^0(X, L^m)_{\max}$. The following is exact

$$\Omega_{X_{\min}^0/\mathrm{Spec}(\mathbb{k})} \longrightarrow \mathbf{u}^* \otimes \mathcal{O}_{X_{\min}^0/\mathrm{Spec}(\mathbb{k})} \longrightarrow \gamma^* \Omega_{\sigma} \longrightarrow 0 \quad (3.26)$$

whose local version is

$$\Omega_{\mathcal{O}(X_f)/\mathbb{k}} \longrightarrow \mathbf{u}^* \otimes \mathcal{O}(X_f) \longrightarrow \gamma^* \Omega_{\sigma}(X_f) \longrightarrow 0. \quad (3.27)$$

So the condition $\gamma^* \Omega_{\sigma} = 0$ is equivalent to the surjectivity of $\Omega_{\mathcal{O}(X_f)/\mathbb{k}} \rightarrow \mathbf{u}^* \otimes \mathcal{O}(X_f)$ for all $f \in H^0(X, L^m)_{\max}$.

For (1) $\implies$ (2), assume “ $ss=s$ ”. Let $m \in \mathbb{N}_+$ be such that $H^0(X, L^m)$ generates $\bigoplus_{d \geq 0} H^0(X, L^{md})$ and $H^0(X, L^m)_{\max} \subseteq \xi.H^0(X, L^m)^{\mathbf{u}'}$ for all $\xi, \mathbf{u}'$ satisfying conditions in Condition 3.1.10. We want to show the following map is surjective

$$\begin{aligned} \phi : \Omega_{\mathcal{O}(X_f)/\mathbb{k}} &\rightarrow \mathbf{u}^* \otimes \mathcal{O}(X_f) \\ a \, db &\mapsto \sum_{i,j} u_j^{(i)} \otimes a(\xi_j^{(i)}.b) \end{aligned} \quad (3.28)$$

where $(\xi_1^{(i)}, \dots, \xi_{r_i}^{(i)})$ is a basis of $\mathfrak{u}_{w_i}$, the weight space of $\mathfrak{u}$ of weight w_i , and $(u_1^{(i)}, \dots, u_{r_i}^{(i)})$ is the set of basis of $(\mathfrak{u}_{w_i})^*$ dual to $(\xi_1^{(i)}, \dots, \xi_{r_i}^{(i)})$.

Take $\xi := \xi_j^{(i)}$ and

$$\mathfrak{u}' := \text{span}\{\xi_{j'}^{(i)} : 1 \leq j' \leq r_i, j' \neq j\} + \bigoplus_{i' < i} \mathfrak{u}_{w_{i'}}. \quad (3.29)$$

Then $\xi, \mathfrak{u}'$ satisfies items in Condition 3.1.10, so $H^0(X, L^m)_{\max} \subseteq \xi \cdot H^0(X, L^m)^{\mathfrak{u}'}$. In particular for $f \in H^0(X, L^m)_{\max}$, there exists $g \in H^0(X, L^m)$, fixed by $\mathfrak{u}'$, and $\xi_j^{(i)} \cdot g = f$. Without loss of generality assume $g \in H^0(X, L^m)_{\max - w_i}$, since we can decompose g according to weights and replace g by its eigen-component in $H^0(X, L^m)_{\max - w_i}$. Then $\frac{g}{f} \in \mathcal{O}(X_f)_{-w_i}$ and

$$\begin{aligned} d\left(\frac{g}{f}\right) &\mapsto \sum_{i', j'} u_{j'}^{(i')} \otimes \frac{\xi_{j'}^{(i')} \cdot g}{f} \\ &= u_j^{(i)} \otimes \frac{\xi_j^{(i)} \cdot g}{f} + \sum_{i' > i} \sum_{j'=1}^{r_{i'}} u_{j'}^{(i')} \otimes \frac{\xi_{j'}^{(i')} \cdot g}{f} \\ &= u_j^{(i)} \otimes 1 + \sum_{i' > i} \sum_{j'=1}^{r_{i'}} u_{j'}^{(i')} \otimes \frac{\xi_{j'}^{(i')} \cdot g}{f}. \end{aligned} \quad (3.30)$$

The first equality is because that $\xi_{j'}^{(i')} \cdot g = 0$ when $i' < i$ or $\begin{cases} i' = i \\ j' \neq j \end{cases}$, and the second is $\xi_j^{(i)} \cdot g = f$. The above for $1 \leq j \leq r_i$ together show that

$$(\mathfrak{u}_{w_i})^* \otimes \mathcal{O}(X_f) \subseteq \text{im}(\phi) + \sum_{i' > i} (\mathfrak{u}_{w_{i'}})^* \otimes \mathcal{O}(X_f). \quad (3.31)$$

In particular $(\mathfrak{u}_{w_n})^* \otimes \mathcal{O}(X_f) \subseteq \text{im}(\phi)$, and if $(\mathfrak{u}_{w_{i'}})^* \otimes \mathcal{O}(X_f) \subseteq \text{im}(\phi)$ for all $i' > i$, then $(\mathfrak{u}_{w_i})^* \otimes \mathcal{O}(X_f) \subseteq \text{im}(\phi)$. By induction, this proves $(\mathfrak{u}_{w_i})^* \otimes \mathcal{O}(X_f) \subseteq \text{im}(\phi)$ for all $1 \leq i \leq n$, i.e. ϕ is surjective. This proves (1) $\implies$ (2).

For (2) $\implies$ (1), assume $\gamma^* \Omega_\sigma = 0$. Let $m \in \mathbb{N}_+$ be such that $H^0(X, L^m)$ generates $\bigoplus_{d \geq 0} H^0(X, L^{md})$. For any non-nilpotent $f \in H^0(X, L^m)_{\max}$, the following map is surjective

$$\begin{aligned} \phi : \Omega_{\mathcal{O}(X_f)/\mathbb{k}} &\rightarrow \mathfrak{u}^* \otimes \mathcal{O}(X_f) \\ a \, db &\mapsto \sum_{i, j} u_j^{(i)} \otimes a(\xi_j^{(i)} \cdot b) \end{aligned} \quad (3.32)$$

where $(\xi_1^{(i)}, \dots, \xi_{r_i}^{(i)})$ is a basis of $\mathfrak{u}_{w_i}$, and $(u_1^{(i)}, \dots, u_{r_i}^{(i)})$ is the dual basis of $(\mathfrak{u}_{w_i})^*$.

For $u_j^{(i)}$, there exists $\omega_j^{(i)} \in \Omega_{\mathcal{O}(X_f)/\mathbb{k}}$ such that $\phi(\omega_j^{(i)}) = u_j^{(i)} \otimes 1$. Write

$$\omega_j^{(i)} = \sum_{\mu} a_{j,\mu}^{(i)} db_{j,\mu}^{(i)}, \quad a_{j,\mu}^{(i)}, b_{j,\mu}^{(i)} \in \mathcal{O}(X_f). \quad (3.33)$$

Recall $\phi(\omega_j^{(i)}) = u_j^{(i)} \otimes 1$, that is

$$\delta^{ii'} \delta_{jj'} = \sum_{\mu} a_{j,\mu}^{(i)} (\xi_{j'}^{(i')} \cdot b_{j,\mu}^{(i)}), \quad 1 \leq i, i' \leq n, \quad 1 \leq j \leq r_i, \quad 1 \leq j' \leq r_{i'}. \quad (3.34)$$

Let $\lambda : \mathbb{G}_m \rightarrow \hat{U}$ be the grading 1PS. Then λ acts on $\mathcal{O}(X_f)$. Decompose $a_{j,\mu}^{(i)}, b_{j,\mu}^{(i)}$ according to λ -weights

$$\begin{aligned} a_{j,\mu}^{(i)} &= a_{j,\mu,\lambda=0}^{(i)} + a_{j,\mu,\lambda<0}^{(i)} \\ b_{j,\mu}^{(i)} &= b_{j,\mu,\lambda=-w_i}^{(i)} + b_{j,\mu,\lambda<-w_i}^{(i)} + b_{j,\mu,\lambda>-w_i}^{(i)}. \end{aligned} \quad (3.35)$$

Then

$$\begin{aligned} \delta_{jj'} &= \sum_{\mu} a_{j,\mu}^{(i)} (\xi_{j'}^{(i)} \cdot b_{j,\mu}^{(i)}) \\ &= \sum_{\mu} (a_{j,\mu,\lambda=0}^{(i)} + a_{j,\mu,\lambda<0}^{(i)}) \xi_{j'}^{(i)} \cdot (b_{j,\mu,\lambda=-w_i}^{(i)} + b_{j,\mu,\lambda<-w_i}^{(i)} + b_{j,\mu,\lambda>-w_i}^{(i)}) \\ &= \sum_{\mu} a_{j,\mu,\lambda=0}^{(i)} (\xi_{j'}^{(i)} \cdot b_{j,\mu,\lambda=-w_i}^{(i)}) + \text{err} \end{aligned} \quad (3.36)$$

where $\text{err} \in \mathcal{O}(X_f)_{<0}$. Then $\delta_{jj'} = \xi_{j'}^{(i)} \cdot h_j^{(i)}$ for

$$h_j^{(i)} := \sum_{\mu} a_{j,\mu,\lambda=0}^{(i)} b_{j,\mu,\lambda=-w_i}^{(i)} \in \mathcal{O}(X_f)_{-w_i}. \quad (3.37)$$

Since $h_j^{(i)} \in \mathcal{O}(X_f)_{-w_i}$, it is fixed by $\mathfrak{u}_j^{(i)}$ for

$$\mathfrak{u}_j^{(i)} := \text{span}\{\xi_{j'}^{(i)} : 1 \leq j' \leq r_i, \quad j' \neq j\} + \sum_{i' < i} \mathfrak{u}_{w_{i'}}. \quad (3.38)$$

For $h_j^{(i)}$, we can write $h_j^{(i)} = \frac{g_j^{(i)}}{f^N}$ for $N \in \mathbb{N}_+$ and $g_j^{(i)} \in H^0(X, L^{mN})_{\max - w_i}$. We have:

- $\xi_j^{(i)} \cdot g_j^{(i)} - f^N$ is annihilated by a power of f ;
- for any $\xi \in \mathfrak{u}_j^{(i)}$, the section $\xi \cdot g_j^{(i)} \in H^0(X, L^{mN})_{\max}$ is annihilated by a power of f .

We can enlarge N and assume $\xi_j^{(i)} \cdot g_j^{(i)} = f^N$ and $g_j^{(i)} \in H^0(X, L^{mN})^{u_j^{(i)}}$. We can enlarge N further such that it is uniform for all $1 \leq i \leq n$ and $1 \leq j \leq r_i$. In other words we have

$$f^N \in \xi_j^{(i)} \cdot H^0(X, L^{mN})^{u_j^{(i)}}, \quad 1 \leq i \leq n, 1 \leq j \leq r_i. \quad (3.39)$$

Choose a basis $f_1, \dots, f_p$ of $H^0(X, L^m)_{\max}$, then there exists N_t for each $1 \leq t \leq p$ such that $f_t^{N_t} \in \xi_j^{(i)} \cdot H^0(X, L^{mN_t})^{u_j^{(i)}}$. Take $N = \max\{N_1, \dots, N_p\}$ and then $f_t^N \in \xi_j^{(i)} \cdot H^0(X, L^{mN})^{u_j^{(i)}}$ for any $1 \leq t \leq p$ and any i, j . This implies that $H^0(X, L^{mN})_{\max} \subseteq \xi_j^{(i)} \cdot H^0(X, L^{mN})^{u_j^{(i)}}$ for all i, j , which is the condition “ss=s”. $\square$

Corollary 3.1.13. *Let X be a projective scheme over $\mathbb{k}$ and let L be an ample line bundle on X . Let $\hat{U} \curvearrowright (X, L)$. Then the following conditions are equivalent:*

- (1) “ss=s”;
- (2) $\gamma^* \Omega_\sigma = 0$;
- (3) $\text{Fit}_0(\gamma^* \Omega_\sigma) = \mathcal{O}_{X_{\min}^0}$;
- (4) $\text{Stab}_U(z) = \{e\}$ for all $\mathbb{k}$ -points $z \in Z_{\min}$.

Proof. It is by definition that (2) $\iff$ (3). Proposition 3.1.12 proves (1) $\iff$ (2).

The application of Lemma 3.1.6 to $k = 0$ and $\mathfrak{v} = 0$ shows (3) is equivalent to that $\text{Stab}_U(x) = \{e\}$ for all $\mathbb{k}$ -points $x \in X_{\min}^0$. In particular (3) $\implies$ (4).

By the upper semi-continuity of $\dim \text{Stab}_U(x)$, we have that (4) implies that $\text{Stab}_U(x) = \{e\}$ for all $\mathbb{k}$ -points $x \in X_{\min}^0$, which is equivalent to (3). This proves (4) $\implies$ (3). $\square$

Remark 3.1.14. When “ss=s” fails, there exists closed point $z \in Z_{\min}$ such that $\text{Stab}_U(z) \neq \{e\}$. The locus in $Z_{\min}$ where $\text{Stab}_U(z) \neq \{e\}$ is cut out by the sheaf of ideals $\text{Fit}_0(\gamma^* \Omega_\sigma)$. This gives a subscheme structure on the locus.

3.2 U-hat theory for affine schemes

Let U be a unipotent group with Lie algebra $\mathfrak{u}$ and let $\lambda : \mathbb{G}_m \rightarrow \text{Aut}(U)$ be a grading 1PS with weights $w_1 > \dots > w_n$. Consider a λ -invariant filtration of Lie ideals

$$0 = \mathfrak{u}_0 \subseteq \mathfrak{u}_1 \subseteq \dots \subseteq \mathfrak{u}_n = \mathfrak{u} \quad (3.40)$$

where $\mathfrak{u}_i := \mathfrak{u}_{\lambda \geq w_i}$ is the direct sum of λ -weight spaces whose weight is greater than or equal to w_i . Then each $\mathfrak{u}_i/\mathfrak{u}_{i-1}$ is Abelian, since it is graded with one weight w_i . Let $U_i \subseteq \hat{U}$ be the normal Lie subgroup corresponding to $\mathfrak{u}_i \subseteq \mathfrak{u}$.

Let $Y := \text{Spec}(A)$ be an affine scheme of finite type over $\mathbb{k}$. Let $\hat{U}$ act on Y . We prove under certain conditions, a universal geometric quotient of Y by U exists and it is affine of finite type over $\mathbb{k}$.

If $\mathbb{G}_m$ acts on Y , with the dual action $\rho^* : A \rightarrow \mathbb{k}[t, t^{-1}] \otimes A$, then the weight space of weight $w \in \mathbb{Z}$ is defined as $A_w := \{a \in A : \rho^*(a) = t^{-w} \otimes a\}$. If the action of $\mathbb{G}_m$ on Y is from $\mathbb{G}_m \rightarrow U \rtimes \lambda$, then denote by $A_{\lambda=w}$ the weight space.

Define $K_0, K_1, \dots, K_n$ as the kernels

$$0 \longrightarrow K_i \longrightarrow \Omega_{A/\mathbb{k}} \longrightarrow (\mathfrak{u}_i)^* \otimes A. \quad (3.41)$$

Consider the following diagram for $1 \leq i \leq n$

$$\begin{array}{ccccccc} & & & K_{i-1} & & & \\ & & \swarrow \phi_i & \downarrow & \searrow 0 & & \\ 0 & \longrightarrow & (\mathfrak{u}_i/\mathfrak{u}_{i-1})^* \otimes A & \longrightarrow & (\mathfrak{u}_i)^* \otimes A & \longrightarrow & (\mathfrak{u}_{i-1})^* \otimes A \longrightarrow 0 \end{array} \quad (3.42)$$

where the dashed arrow exists and denote it by $\phi_i : K_{i-1} \rightarrow (\mathfrak{u}_i/\mathfrak{u}_{i-1})^* \otimes A$.

Condition 3.2.1. Let λ grade U with weights $w_1 > \dots > w_n$, and let $\{\mathfrak{u}_i\}_{i=0}^n$ be the filtration $\mathfrak{u}_i := \mathfrak{u}_{\lambda \geq w_i}$. Let $\hat{U} \curvearrowright Y$. This condition for the action $\hat{U} \curvearrowright Y$ assumes that

- $A = \bigoplus_{w \leq 0} A_{\lambda=w}$;
- For each $1 \leq i \leq n$, there exists a surjective linear map $\mathfrak{u}_i/\mathfrak{u}_{i-1} \rightarrow \mathfrak{s}_i$ such that $(\mathfrak{s}_i)^* \otimes A$ is isomorphic to the cokernel of $K_{i-1} \rightarrow (\mathfrak{u}_i/\mathfrak{u}_{i-1})^* \otimes A$ in the following sense

$$\begin{array}{ccccc} & & (\mathfrak{s}_i)^* \otimes A & & \\ & & \downarrow & \searrow \cong & \\ K_{i-1} & \longrightarrow & (\mathfrak{u}_i/\mathfrak{u}_{i-1})^* \otimes A & \longrightarrow & \text{coker} \longrightarrow 0. \end{array} \quad (3.43)$$

Remark 3.2.2. When Y is further assumed reduced, the second requirement above is equivalent to that for $1 \leq i \leq n$ and all $\mathbb{k}$ -points $x \in Y$, the composition $\text{Stab}_{(\mathfrak{u}_i:\mathfrak{u}_{i-1})}(x) \rightarrow \mathfrak{u}_i/\mathfrak{u}_{i-1} \rightarrow \mathfrak{s}_i$ is an isomorphism, which implies that for $1 \leq i \leq n$, the dimensions of $\text{Stab}_{(\mathfrak{u}_i:\mathfrak{u}_{i-1})}(x)$ are constant for $\mathbb{k}$ -points $x \in Y$.

3.2.1 Length one filtrations

To construct the quotient of Y by U , the first step is to construct the quotient of Y by $U_1 \subseteq \hat{U}$. It is equivalent to consider the case when the filtration has length one. In this subsection, Condition 3.2.1 in the length one case for $\hat{U} \curvearrowright Y$ is always assumed:

- λ acts on $\mathfrak{u}$ with one positive weight;
- $A = \bigoplus_{w \leq 0} A_{\lambda=w}$;
- There exists a quotient $\mathfrak{u} \rightarrow \mathfrak{s}$ such that

$$\begin{array}{ccccccc} & & \mathfrak{s}^* \otimes A & & & & \\ & & \downarrow & \searrow \cong & & & \\ \Omega_{A/\mathbb{k}} & \longrightarrow & \mathfrak{u}^* \otimes A & \longrightarrow & \text{coker} & \longrightarrow & 0. \end{array} \quad (3.44)$$

Let $w_1 \in \mathbb{N}_+$ be the unique weight of λ on $\mathfrak{u}$. Let $\mathfrak{u}' \subseteq \mathfrak{u}$ be the kernel

$$0 \longrightarrow \mathfrak{u}' \longrightarrow \mathfrak{u} \longrightarrow \mathfrak{s} \longrightarrow 0 \quad (3.45)$$

and choose a splitting $\mathfrak{u} = \mathfrak{u}' \oplus \mathfrak{s}$. Let $(\xi_1, \dots, \xi_r)$ be a basis of $\mathfrak{u}$ such that $(\xi_1, \dots, \xi_{r-k})$ is a basis of $\mathfrak{u}'$. Let $(u_1, \dots, u_r)$ be the basis of $\mathfrak{u}^*$ dual to $(\xi_1, \dots, \xi_r)$.

Lemma 3.2.3. *Assume Condition 3.2.1. Let $\mathfrak{u}'$ be as above. Let $U' \subseteq U$ be the subgroup corresponding to $\mathfrak{u}' \subseteq \mathfrak{u}$. Then $A^U = A^{U'}$, where $A^{U'} \subseteq A$ denotes the subring of U' -invariant functions.*

Proof. Since unipotent groups are connected, a function $f \in A$ is U' -invariant if and only if it is $\mathfrak{u}'$ -invariant, i.e. $A^{U'} = A^{\mathfrak{u}'}$ and $A^U = A^{\mathfrak{u}}$ for

$$\begin{aligned} A^{\mathfrak{u}'} &:= \{f \in A : \langle \xi \otimes 1, \phi(df) \rangle = 0 \text{ for all } \xi \in \mathfrak{u}'\} \\ A^{\mathfrak{u}} &:= \{f \in A : \langle \xi \otimes 1, \phi(df) \rangle = 0 \text{ for all } \xi \in \mathfrak{u}\} \end{aligned} \quad (3.46)$$

where $\phi : \Omega_{A/\mathbb{k}} \rightarrow \mathfrak{u} \otimes A$ is the infinitesimal action.

Let $d : A \rightarrow \Omega_{A/\mathbb{k}}$ be the derivation. Let $i : \mathfrak{u}' \rightarrow \mathfrak{u}$ be the inclusion and let $q : \mathfrak{u} \rightarrow \mathfrak{s}$ be the quotient. Let $i^* : \mathfrak{u}^* \rightarrow (\mathfrak{u}')^*$ and $q^* : \mathfrak{s}^* \rightarrow \mathfrak{u}^*$ be the dual maps.

Consider the following diagram with arrows labelled as ϕ, φ

$$\begin{array}{ccccc} & \mathfrak{s}^* \otimes A & & & \\ & \downarrow q^* \otimes 1 & \searrow \cong & & \\ \Omega_{A/\mathbb{k}} & \xrightarrow{\phi} & \mathfrak{u}^* \otimes A & \xrightarrow{\varphi} & \text{coker} \longrightarrow 0. \end{array} \quad (3.47)$$

We prove $\ker((i^* \otimes 1) \circ \phi) \subseteq \ker \phi$. Let $\omega \in \ker((i^* \otimes 1) \circ \phi) \subseteq \Omega_{A/\mathbb{k}}$. Then $\phi(\omega) \in \ker(i^* \otimes 1) = \text{im}(q^* \otimes 1)$. There exists $g \in \mathfrak{u}^* \otimes A$ such that $\phi(\omega) = (q^* \otimes 1)(g)$. Since $\varphi \circ (q^* \otimes 1)$ is an isomorphism and $\varphi \circ (q^* \otimes 1)(g) = \varphi \circ \phi(\omega) = 0$, we have $g = 0$. Then $\phi(\omega) = (q^* \otimes 1)(g) = 0$. This proves $\ker((i^* \otimes 1) \circ \phi) \subseteq \ker \phi$.

Let $f \in A^{\mathfrak{u}'}$, i.e. $(i^* \otimes 1) \circ \phi(df) = 0$. Since $\ker((i^* \otimes 1) \circ \phi) \subseteq \ker \phi$, we have $\phi(df) = 0$, i.e. $f \in A^{\mathfrak{u}}$. This proves $A^{\mathfrak{u}'} \subseteq A^{\mathfrak{u}}$. Since $\mathfrak{u}' \subseteq \mathfrak{u}$, we have $A^{\mathfrak{u}} \subseteq A^{\mathfrak{u}'}$. $\square$

Lemma 3.2.4. *Assume Condition 3.2.1. Let $w_1, \mathfrak{u}', \{\xi_i\}$ and $\{u_i\}$ be as above. Then there exist $f_1, \dots, f_{r-k} \in A_{\lambda=-w_1}$ such that*

$$\xi_i \cdot f_j = \delta_{ij}, \quad 1 \leq i, j \leq r-k. \quad (3.48)$$

Proof. For any $w \in \mathbb{Z}$, and any $\xi \in \mathfrak{u}$, we have $\xi \cdot A_{\lambda=w} \subseteq A_{\lambda=w+w_1}$, where w_1 is the λ -weight on $\mathfrak{u}$. In particular elements in $A_{\lambda=0}$ are fixed by $\mathfrak{u}$.

Consider diagram (3.47). We have $\mathfrak{u}^* \otimes A = \text{im} \phi + \text{im}(q^* \otimes 1)$. In particular, for $1 \leq i \leq r-k$, there exists

$$\sum_{\mu} a_{i,\mu} db_{i,\mu} \in \Omega_{A/\mathbb{k}} \quad (3.49)$$

such that

$$u_i \otimes 1 - \sum_{i'=1}^r u_{i'} \otimes \left(\sum_{\mu} a_{i,\mu} (\xi_{i'} \cdot b_{i,\mu}) \right) \in \mathfrak{s}^* \otimes A. \quad (3.50)$$

Recall the surjective linear map $i^* : \mathfrak{u}^* \rightarrow (\mathfrak{u}')^*$ dual to the inclusion $i : \mathfrak{u}' \rightarrow \mathfrak{u}$. Apply $i^* \otimes 1$ to the above equation, and compare the coefficients of $u_{i'} \otimes 1$

$$\delta_{ii'} = \sum_{\mu} a_{i,\mu} (\xi_{i'} \cdot b_{i,\mu}), \quad 1 \leq i, i' \leq r-k. \quad (3.51)$$

Decompose $a_{i,\mu}, b_{i,\mu}$ according to λ -weights

$$\begin{aligned} a_{i,\mu} &= a_{i,\mu,\lambda=0} + a_{i,\mu,\lambda<0} \\ b_{i,\mu} &= b_{i,\mu,\lambda>-w_1} + b_{i,\mu,\lambda=-w_1} + b_{i,\mu,\lambda<-w_1}. \end{aligned} \quad (3.52)$$

Then

$$\begin{aligned}\delta_{ii'} &= \sum_{\mu} a_{i,\mu}(\xi_{i'}.b_{i,\mu}) \\ &= \sum_{\mu} a_{i,\mu,\lambda=0}(\xi_{i'}.b_{i,\mu,\lambda=-w_1}) + \sum_{\mu} a_{i,\mu,\lambda=0}(\xi_{i'}.b_{i,\mu,\lambda<-w_1}) + \sum_{\mu} a_{i,\mu,\lambda<0}(\xi_{i'}.b_{i,\mu}).\end{aligned}\quad (3.53)$$

Compare the components of λ -weight zero

$$\delta_{ii'} = \sum_{\mu} a_{i,\mu,\lambda=0}(\xi_{i'}.b_{i,\mu,\lambda=-w_1}) = \xi_{i'} \cdot \left(\sum_{\mu} a_{i,\mu,\lambda=0} b_{i,\mu,\lambda=-w_1} \right). \quad (3.54)$$

Let $f_i := \sum_{\mu} a_{i,\mu,\lambda=0} b_{i,\mu,\lambda=-w_1} \in A_{\lambda=-w_1}$ and the lemma is proved. $\square$

The following lemma is essentially from Lemma 4.7.5, [Dix96], adapted to a set of commuting derivations.

Lemma 3.2.5 (cf. Lemma 7.2, [Bér+16], Lemma 4.7.5, [Dix96]). *Let R be an algebra over $\mathbb{k}$ with a dual action of $(\mathbb{G}_a)^r$. Let $\xi_1, \dots, \xi_r$ be commuting derivations on R generating the dual action of $(\mathbb{G}_a)^r$. Assume there exists $f_1, \dots, f_r \in R$ such that $\xi_i.f_j = \delta_{ij}$, then:*

- $f_1, \dots, f_r$ are algebraically independent over $R^{(\mathbb{G}_a)^r}$;
- $R = R^{(\mathbb{G}_a)^r}[f_1, \dots, f_r]$;
- $\text{Spec}(R) \rightarrow \text{Spec}(R^{(\mathbb{G}_a)^r})$ is a trivial $(\mathbb{G}_a)^r$ -quotient

$$\begin{array}{ccc} \text{Spec}(R) & \xrightarrow{\cong} & (\mathbb{G}_a)^r \times_{\text{Spec}(\mathbb{k})} \text{Spec}(R^{(\mathbb{G}_a)^r}) \\ & \searrow & \downarrow \text{pr} \\ & & \text{Spec}(R^{(\mathbb{G}_a)^r}). \end{array} \quad (3.55)$$

Proof. Let $I \subseteq R$ be the ideal generated by $f_1, \dots, f_r$. Let $\alpha : R \rightarrow R/I$ be the quotient map. Consider the following linear map

$$\begin{aligned}\phi : R &\rightarrow (R/I)[t_1, \dots, t_r] \\ g &\mapsto \sum_{n \in \mathbb{N}^r} \frac{1}{n!} \alpha(\xi^n.g) t^n\end{aligned} \quad (3.56)$$

where

$$\begin{aligned}n &:= (n_1, \dots, n_r) \\ n! &:= n_1! \cdots n_r! \\ \xi^n.g &:= \xi_1^{n_1} \cdots \xi_r^{n_r}.g \\ t^n &:= t_1^{n_1} \cdots t_r^{n_r}.\end{aligned} \quad (3.57)$$

We need to show that the summation defining ϕ is essentially finite. For any $g \in R$, there exists a finite dimensional invariant subspace $V \subseteq R$ containing g . Since $(\mathbb{G}_m)^r$ is unipotent, and V is a finite dimensional representation, $\xi^n|_V = 0$ if $|n| := n_1 + \cdots + n_r > \dim V$. So $\xi^n \cdot g = 0$ for all but finite $n \in \mathbb{N}^r$.

The linear map ϕ preserves multiplication. For $f, g \in R$, we have

$$\begin{aligned} \phi(fg) &= \sum_{n \in \mathbb{N}^r} \frac{1}{n!} \alpha(\xi^n \cdot (fg)) t^n \\ &= \sum_{n \in \mathbb{N}^r} \frac{1}{n!} t^n \alpha \left(\sum_{0 \leq m \leq n} \binom{n}{m} (\xi^{n-m} \cdot f) (\xi^m \cdot g) \right) \\ &= \sum_{u, v \in \mathbb{N}^r} \frac{1}{u!} \frac{1}{v!} t^{u+v} \alpha(\xi^u \cdot f) \alpha(\xi^v \cdot g) \\ &= \phi(f) \phi(g). \end{aligned} \tag{3.58}$$

So ϕ is a map of $\mathbb{k}$ -algebras. It is easy to check $\phi(f_i) = t_i$.

We show ϕ is an isomorphism by constructing its inverse. Consider

$$\begin{aligned} \varphi' : R[t_1, \dots, t_r] &\rightarrow R \\ gt^m &\mapsto \sum_{n \in \mathbb{N}^r} \frac{(-1)^{|n|}}{n!} (\xi^n \cdot g) f_1^{m_1+n_1} \cdots f_r^{m_r+n_r}. \end{aligned} \tag{3.59}$$

By the same argument for ϕ , we have that the summation defining φ' is finite, and φ' is a ring map. It is easy to see $\varphi'(f_i) = 0$ for $i = 1, \dots, r$. So φ' factors through $R[t_1, \dots, t_r] \rightarrow (R/I)[t_1, \dots, t_r]$, inducing

$$\varphi : (R/I)[t_1, \dots, t_r] \rightarrow R \tag{3.60}$$

which is the inverse of ϕ .

Via the isomorphism $R \cong (R/I)[t_1, \dots, t_r]$, the derivation ξ_i on R corresponds to the derivation $\frac{\partial}{\partial t_i}$ on $(R/I)[t_1, \dots, t_r]$. Then

$$R^{(\mathbb{G}_a)^r} \cong R/I, \quad g \mapsto \alpha(g). \tag{3.61}$$

The isomorphism $\varphi : R/I[t_1, \dots, t_r] \cong R$ then becomes

$$R^{(\mathbb{G}_a)^r}[t_1, \dots, t_r] \cong R, \quad gt^n \mapsto g f_1^{n_1} \cdots f_r^{n_r}; \tag{3.62}$$

that is

$$R = R^{(\mathbb{G}_a)^r}[f_1, \dots, f_r] \tag{3.63}$$

and $f_1, \dots, f_r$ are algebraically independent over $R^{(\mathbb{G}_a)^r}$.

There is an isomorphism $\mathbb{k}[t_1, \dots, t_r] \cong \mathcal{O}((\mathbb{G}_a)^r)$. The left action of $(\mathbb{G}_a)^r$ on itself induces a dual action of $(\mathbb{G}_a)^r$ on $\mathcal{O}((\mathbb{G}_a)^r)$. Through the identification $\mathcal{O}((\mathbb{G}_a)^r) \cong \mathbb{k}[t_1, \dots, t_r]$, the induced dual action of $(\mathbb{G}_a)^r$ on $\mathbb{k}[t_1, \dots, t_r]$ is generated by derivations $\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_r}$.

The following isomorphism is $(\mathbb{G}_a)^r$ -equivariant

$$\begin{aligned} R &\cong R^{(\mathbb{G}_a)^r}[t_1, \dots, t_r] \\ &\cong \mathbb{k}[t_1, \dots, t_r] \otimes_{\mathbb{k}} R^{(\mathbb{G}_a)^r} \\ &\cong \mathcal{O}((\mathbb{G}_a)^r) \otimes_{\mathbb{k}} R^{(\mathbb{G}_a)^r}. \end{aligned} \tag{3.64}$$

Geometrically, there is a $(\mathbb{G}_a)^r$ -equivariant isomorphism

$$\begin{array}{ccc} \mathrm{Spec}(R) & \xrightarrow{\cong} & (\mathbb{G}_a)^r \times_{\mathrm{Spec}(\mathbb{k})} \mathrm{Spec}(R^{(\mathbb{G}_a)^r}) \\ & \searrow & \downarrow \mathrm{pr} \\ & & \mathrm{Spec}(R^{(\mathbb{G}_a)^r}). \end{array} \tag{3.65}$$

□

Lemma 3.2.6. *Assume Condition 3.2.1. The inclusion $A^U \rightarrow A$ induces a universal geometric quotient by U*

$$\mathrm{Spec}(A) \rightarrow \mathrm{Spec}(A^U). \tag{3.66}$$

Proof. By Lemma 3.2.4, there exist $f_1, \dots, f_{r-k} \in A_{\lambda=-w_1}$ such that $\xi_i \cdot f_j = \delta_{ij}$ for $1 \leq i, j \leq r-k$. Since $\mathfrak{u}$ is Abelian, the subspace $\mathfrak{u}' \subseteq \mathfrak{u}$ is a Lie sub-algebra. Let $U' \subseteq U$ be the subgroup corresponding to $\mathfrak{u}' \subseteq \mathfrak{u}$. The derivations $\xi_1, \dots, \xi_{r-k}$ on A generate the dual action of U' on A . By Lemma 3.2.3 and Lemma 3.2.5, we have:

- $f_1, \dots, f_{r-k}$ are algebraically independent over A^U ;
- $A = A^U[f_1, \dots, f_{r-k}]$;
- $\mathrm{Spec}(A) \rightarrow \mathrm{Spec}(A^U)$ is a trivial U' -quotient.

Let $A^U \rightarrow B$ be any A^U -algebra. Consider the Cartesian diagram

$$\begin{array}{ccc} \mathrm{Spec}(B') & \longrightarrow & \mathrm{Spec}(A) \\ \downarrow \pi & & \downarrow \\ \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(A^U) \end{array} \tag{3.67}$$

where

$$B' = B \otimes_{A^U} A. \quad (3.68)$$

We first show that $B \rightarrow B'$ is injective and $B = (B')^U$. As a pull back of a trivial U' -quotient, the morphism $\pi : \text{Spec}(B') \rightarrow \text{Spec}(B)$ is a trivial U' -quotient. In particular, π is a geometric quotient by U' . Since π is surjective, $B \rightarrow B'$ is injective. Since π is a good quotient, $(B')^{U'} = B$. Consider U -invariant elements on both sides, and we have

$$(B')^U = B \quad (3.69)$$

since B is acted trivially by U .

We then show $\pi : \text{Spec}(B') \rightarrow \text{Spec}(B)$ is a geometric quotient by U . The conditions to check are:

- π is affine surjective and U -invariant. By the above argument, π is surjective, and $B = (B')^U$, which implies that π is U -invariant.
- For any principal affine open $V := \text{Spec}(B_f) \rightarrow \text{Spec}(B)$ with $f \in B$, we can replace $A^U \rightarrow B$ by $A^U \rightarrow B_f$ and apply the above argument to claim $B_f \subseteq B_f \otimes_{A^U} A \cong B'_f$ and

$$B_f = (B_f \otimes_{A^U} A)^U \cong (B'_f)^U \quad (3.70)$$

i.e.

$$\mathcal{O}_{\text{Spec}(B)}(V) \cong \mathcal{O}_{\text{Spec}(B')}(\pi^{-1}(V))^U. \quad (3.71)$$

- For any U -invariant closed subset $W \subseteq \text{Spec}(B')$, it is also U' -invariant, so $\pi(W)$ is closed since π is a good quotient by U' .
- For any pair of disjoint U -invariant closed subsets $W_1, W_2 \subseteq \text{Spec}(B')$, they are also U' -invariant, so $\pi(W_1) \cap \pi(W_2) = \emptyset$.
- For any closed point $z \in \text{Spec}(B)$, the fibre $\pi^{-1}(z) \hookrightarrow \text{Spec}(B')$ is a union of U -orbits, since π is U -invariant. Besides, $\pi^{-1}(z)$ is a single U' -orbit, since π is a geometric quotient by U' . So there is only one U -orbit in $\pi^{-1}(z)$.

We see that any affine base change of $\mathrm{Spec}(A) \rightarrow \mathrm{Spec}(A^U)$ is a geometric quotient by U . Since arbitrary base change is covered by affine base changes as open subschemes, and geometric quotients are local on target, we claim any base change of $\mathrm{Spec}(A) \rightarrow \mathrm{Spec}(A^U)$ is a geometric quotient by U . This proves that $\mathrm{Spec}(A) \rightarrow \mathrm{Spec}(A^U)$ is a universal geometric quotient by U . $\square$

Theorem 3.2.7. *Let $Y = \mathrm{Spec}(A)$ be an affine scheme of finite type over $\mathbb{k}$. Let $\lambda : \mathbb{G}_m \rightarrow \mathrm{Aut}(U)$ be a grading 1PS of one weight. Let $\hat{U} \curvearrowright Y$. Assume Condition 3.2.1 for the action $\hat{U} \curvearrowright Y$. Then:*

- (1) $Y/U := \mathrm{Spec}(A^U)$ is affine of finite type over $\mathbb{k}$;
- (2) $Y \rightarrow Y/U$ is a universal geometric quotient by U ;
- (3) There exists an affine space $\mathbb{A}^m$ such that $Y \cong \mathbb{A}^m \times Y/U$ over Y/U , and $m = \dim \mathfrak{u} - \dim \mathfrak{s}$;
- (4) The map $\Omega_{A/\mathbb{k}} \rightarrow \mathfrak{u}^* \otimes A$ induces an injective map $\Omega_{A/A^U} \rightarrow \mathfrak{u}^* \otimes A$ such that the following diagram commutes

$$\begin{array}{ccc} \Omega_{A/\mathbb{k}} & \xrightarrow{\quad} & \mathfrak{u}^* \otimes A \\ \downarrow & \nearrow \subseteq & \\ \Omega_{A/A^U} & & \end{array} \quad (3.72)$$

Proof. Let $w_1 \in \mathbb{N}_+$ be the unique λ -weight on $\mathfrak{u}$. Let $\mathfrak{u} \rightarrow \mathfrak{s}$ be the quotient map in Condition 3.2.1. Let $\mathfrak{u}' \subseteq \mathfrak{u}$ be the kernel of $\mathfrak{u} \rightarrow \mathfrak{s}$. Choose a decomposition $\mathfrak{u} = \mathfrak{u}' \oplus \mathfrak{s}$. Choose a basis $(\xi_1, \dots, \xi_r)$ of $\mathfrak{u}$ such that $\xi_j \in \mathfrak{u}'$ for $1 \leq j \leq r - k$ and $\xi_j \in \mathfrak{s}$ for $r - k < j \leq r$, where $r = \dim \mathfrak{u}$, $k = \dim \mathfrak{s}$.

Let $U' \subseteq U$ be the subgroup with Lie algebra $\mathrm{Lie}(U') = \mathfrak{u}'$. Consider the action of U' on $Y = \mathrm{Spec}(A)$. We have that $\xi_1, \dots, \xi_{r-k}$ are commuting derivations on A generating the dual action of U' on A . By Lemma 3.2.4, there exist $f_1, \dots, f_{r-k} \in A_{\lambda=-w_1}$ such that $\xi_i \cdot f_j = \delta_{ij}$ for $1 \leq i, j \leq r - k$. By Lemma 3.2.3, Lemma 3.2.5 and Lemma 3.2.6, we have:

- $f_1, \dots, f_{r-k} \in A$ are algebraically independent over A^U ;

- $A = A^U[f_1, \dots, f_{r-k}]$;
- $\text{Spec}(A) \rightarrow \text{Spec}(A^U)$ is a universal geometric quotient by U .

Then (1) and (2) are immediate.

For (3), let $x_1, \dots, x_{r-k}$ be free variables, and let

$$S = \mathbb{k}[x_1, \dots, x_{r-k}] \quad (3.73)$$

be a polynomial ring over $\mathbb{k}$. Then the following map is an isomorphism

$$A^U \otimes_{\mathbb{k}} S \rightarrow A, \quad g \otimes p(x_1, \dots, x_{r-k}) \mapsto gp(f_1, \dots, f_{r-k}). \quad (3.74)$$

Let $\mathbb{A}^m := \text{Spec}(S)$. Then $m = r - k = \dim \mathfrak{u} - \dim \mathfrak{s}$. Geometrically there is an isomorphism over Y/U

$$\begin{array}{ccc} Y & \xrightarrow{\cong} & \mathbb{A}^m \times Y/U \\ & \searrow & \downarrow \text{pr} \\ & & Y/U \end{array} \quad (3.75)$$

which proves (3).

For (4), consider the diagram

$$\begin{array}{ccccccc} \Omega_{A^U/\mathbb{k}} \otimes_{A^U} A & \xrightarrow{\phi} & \Omega_{A/\mathbb{k}} & \longrightarrow & \Omega_{A/A^U} & \longrightarrow & 0 \\ & \searrow \psi \circ \phi & \downarrow \psi & & \nearrow \varphi & & \\ & & \mathfrak{u}^* \otimes A & & & & \end{array} \quad (3.76)$$

where the top row is exact by [Sta22, Lemma 00RS] and $\psi \circ \phi = 0$ by direct check. Therefore φ exists.

We have

$$\Omega_{A/A^U} = \bigoplus_{j=1}^{r-k} A \, df_j \quad (3.77)$$

since $A = A^U[f_1, \dots, f_{r-k}]$ and $f_1, \dots, f_{r-k}$ are algebraically independent over A^U . Let

$$\omega := \sum_{j=1}^{r-k} a_j \, df_j \in \Omega_{A/A^U} \quad (3.78)$$

be an element in $\ker \varphi$. It suffices to prove $a_j = 0$ for $1 \leq j \leq r - k$. For any $1 \leq j' \leq r - k$, the coefficient of $u_{j'} \otimes 1$ of $\varphi(\omega)$ equals

$$0 = \sum_{j=1}^{r-k} a_j (\xi_{j'} \cdot f_j) = a_{j'} \quad (3.79)$$

where we have used $\xi_{j'} \cdot f_j = \delta_{j'j}$. This proves (4). $\square$

3.2.2 Induction for general filtrations

Consider the general case when $\lambda : \mathbb{G}_m \rightarrow \text{Aut}(U)$ has weights $w_1 > \cdots > w_n$. We will prove in this subsection that if the action $\hat{U} \curvearrowright Y = \text{Spec}(A)$ satisfies Condition 3.2.1, then the induced action $(U/U_1) \rtimes \lambda \curvearrowright Y/U_1$ satisfies Condition 3.2.1. This will allow us to iteratively apply Theorem 3.2.7 (See Theorem 3.2.9 below).

Proposition 3.2.8. *If $\hat{U} \curvearrowright Y := \text{Spec}(A)$ satisfies Condition 3.2.1, then so does the induced action $(U/U_1) \rtimes \lambda \curvearrowright Y/U_1 := \text{Spec}(A^{U_1})$.*

Proof. If Condition 3.2.1 is satisfied for $\hat{U} \curvearrowright Y$, then this condition is also satisfied for $U_1 \rtimes \lambda \curvearrowright Y$. Apply Theorem 3.2.7 to $U_1 \rtimes \lambda \curvearrowright Y$ to conclude:

- $Y/U_1 := \text{Spec}(A^{U_1})$ is affine of finite type over $\mathbb{k}$;
- $Y \rightarrow Y/U_1$ is a universal geometric quotient by U_1 ;
- $Y \cong \mathbb{A}^m \times Y/U_1$ over Y/U_1 ;
- $\Omega_{A/A^{U_1}} \rightarrow (\mathfrak{u}_1)^* \otimes A$ is injective.

The actions $\lambda \curvearrowright U/U_1$ and $(U/U_1) \rtimes \lambda \curvearrowright Y/U_1$ are induced naturally. We then check Condition 3.2.1 for the action $(U/U_1) \rtimes \lambda \curvearrowright Y/U_1$. The following are immediate:

- λ is a grading 1PS on U/U_1 with weights $w_2 > \cdots > w_n$;
- The filtration $\{\mathfrak{u}_i/\mathfrak{u}_1\}_{i=1}^n$ is such that $\mathfrak{u}_i/\mathfrak{u}_1 = (\mathfrak{u}/\mathfrak{u}_1)_{\lambda \geq w_i}$;
- $A^{U_1} = \bigoplus_{w \leq 0} (A^{U_1})_{\lambda=w}$.

We check the last statement in Condition 3.2.1. Let $\mathfrak{u}_i/\mathfrak{u}_{i-1} \rightarrow \mathfrak{s}_i$ be the quotient in Condition 3.2.1 for $1 \leq i \leq n$. We will prove that φ is an isomorphism in the following diagram for $2 \leq i \leq n$

$$\begin{array}{ccccccc}
 & & (\mathfrak{s}_i)^* \otimes A^{U_1} & & & & \\
 & & \downarrow & \searrow \varphi & & & \\
 K'_{i-1} & \longrightarrow & (\mathfrak{u}_i/\mathfrak{u}_{i-1})^* \otimes A^{U_1} & \longrightarrow & \text{coker}' & \longrightarrow & 0
 \end{array} \tag{3.80}$$

where coker' is the cokernel and $K'_i := \ker(\Omega_{A^{U_1}/\mathbb{k}} \rightarrow (\mathfrak{u}_i/\mathfrak{u}_{i-1})^* \otimes A^{U_1})$ for $1 \leq i \leq n$.

By Theorem 3.2.7, the inclusion $A^{U_1} \rightarrow A$ is a polynomial ring, so faithfully flat. Then φ is an isomorphism if and only if its base change along $A^{U_1} \rightarrow A$ is an isomorphism. The base change of diagram (3.80) is

$$\begin{array}{ccccccc} & & (\mathfrak{s}_i)^* \otimes A & & & & \\ & & \downarrow & \searrow \varphi \otimes 1 & & & \\ K'_{i-1} \otimes_{A^{U_1}} A & \longrightarrow & (\mathfrak{u}_i/\mathfrak{u}_{i-1})^* \otimes A & \longrightarrow & \text{coker}' \otimes_{A^{U_1}} A & \longrightarrow & 0. \end{array} \quad (3.81)$$

Condition 3.2.1 assumes that ϕ is an isomorphism in the diagram

$$\begin{array}{ccccccc} & & (\mathfrak{s}_i)^* \otimes A & & & & \\ & & \downarrow & \searrow \phi & & & \\ K_{i-1} & \longrightarrow & (\mathfrak{u}_i/\mathfrak{u}_{i-1})^* \otimes A & \longrightarrow & \text{coker} & \longrightarrow & 0 \end{array} \quad (3.82)$$

where coker is the cokernel such that the horizontal sequence is exact.

Consider the following diagram with exact rows:

$$\begin{array}{ccccccc} \Omega_{A^{U_1}/\mathbb{k}} \otimes_{A^{U_1}} A & \longrightarrow & \Omega_{A/\mathbb{k}} & \longrightarrow & \Omega_{A/A^{U_1}} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \text{in} & & \\ 0 & \longrightarrow & (\mathfrak{u}_{i-1}/\mathfrak{u}_1)^* \otimes A & \longrightarrow & (\mathfrak{u}_{i-1})^* \otimes A & \longrightarrow & (\mathfrak{u}_1)^* \otimes A \longrightarrow 0 \end{array} \quad (3.83)$$

where the exactness of the top row is by [Sta22, Lemma 00RS]. By Theorem 3.2.7, the right vertical arrow is injective. Since $A^{U_1} \rightarrow A$ is flat, the kernel of the left vertical arrow is $K'_{i-1} \otimes_{A^{U_1}} A$. The following is exact by the Snake lemma

$$K'_{i-1} \otimes_{A^{U_1}} A \longrightarrow K_{i-1} \longrightarrow 0. \quad (3.84)$$

This surjective map fills the following diagram

$$\begin{array}{ccccccc} & & (\mathfrak{s}_i)^* \otimes A & & & & \\ & & \downarrow & \searrow \varphi \otimes 1 & & & \\ K'_{i-1} \otimes_{A^{U_1}} A & \longrightarrow & (\mathfrak{u}_i/\mathfrak{u}_{i-1})^* \otimes A & \longrightarrow & C \otimes_{A^{U_1}} A & \longrightarrow & 0 \\ \downarrow & & \parallel & \searrow \phi & \downarrow \text{in} & & \\ K_{i-1} & \longrightarrow & (\mathfrak{u}_i/\mathfrak{u}_{i-1})^* \otimes A & \longrightarrow & C' & \longrightarrow & 0 \end{array} \quad (3.85)$$

so an isomorphism between cokernels is induced. Therefore $\varphi \otimes 1$ is an isomorphism since ϕ is. This proves the last statement in Condition 3.2.1, and hence completes the proof. $\square$

Theorem 3.2.9. *Let $Y = \operatorname{Spec}(A)$ be an affine scheme of finite type over $\mathbb{k}$. Assume Condition 3.2.1 for the action $\hat{U} \curvearrowright Y$. Then:*

- (1) $Y/U := \operatorname{Spec}(A^U)$ is affine of finite type over $\mathbb{k}$;
- (2) $Y \rightarrow Y/U$ is a universal geometric quotient by U ;
- (3) There exists an affine space $\mathbb{A}^m$ such that $Y \cong \mathbb{A}^m \times Y/U$ over Y/U .

Proof. For $1 \leq i \leq n$, we prove by induction on i the following:

- (1) $Y/U_i := \operatorname{Spec}(A^{U_i})$ is of finite type over $\mathbb{k}$;
- (2) $Y \rightarrow Y/U_i$ is a universal geometric quotient by U_i ;
- (3) There exists an affine space $\mathbb{A}^{m_i}$ such that $Y \cong \mathbb{A}^{m_i} \times Y/U_i$ over Y/U_i ;
- (4) The action $(U/U_i) \rtimes \lambda \curvearrowright Y/U_i$ satisfies Condition 3.2.1.

For $i = 1$, Proposition 3.2.8 proves (1) – (4). Assume the above are proved for some $1 \leq i < n$. Apply Theorem 3.2.7 to the action $(U_{i+1}/U_i) \rtimes \lambda \curvearrowright Y/U_i$ to conclude:

- (1') $Y/U_{i+1} := \operatorname{Spec}\left((A^{U_i})^{U_{i+1}/U_i}\right) = \operatorname{Spec}(A^{U_{i+1}})$ is of finite type over $\mathbb{k}$;
- (2') $Y/U_i \rightarrow Y/U_{i+1}$ is a universal geometric quotient by U_{i+1}/U_i , so $Y \rightarrow Y/U_{i+1}$ is a universal geometric quotient by U_{i+1} as the composition of $Y \rightarrow Y/U_i$ and $Y/U_i \rightarrow Y/U_{i+1}$;
- (3') There exists an affine $\mathbb{A}^{m'_{i+1}}$ such that $Y/U_i \cong \mathbb{A}^{m'_{i+1}} \times Y/U_{i+1}$ over Y/U_{i+1} , so $Y \cong \mathbb{A}^{m_i} \times Y/U_i \cong \mathbb{A}^{m_i+m'_{i+1}} \times Y/U_{i+1}$.

Apply Proposition 3.2.8 to the action $U/U_i \rtimes \lambda \curvearrowright Y/U_i$ to conclude:

- (4') The action $(U/U_{i+1}) \rtimes \lambda \curvearrowright Y/U_{i+1}$ satisfies Condition 3.2.1.

This completes the induction and the proof. □

3.3 U-hat theory for projective schemes

Let $\lambda : \mathbb{G}_m \rightarrow \text{Aut}(U)$ be a grading 1PS on U with weights $w_1 > \dots > w_n$ on $\mathfrak{u}$. Let

$$0 = \mathfrak{u}_0 \subseteq \mathfrak{u}_1 \subseteq \dots \subseteq \mathfrak{u}_n = \mathfrak{u}, \quad \mathfrak{u}_i := \mathfrak{u}_{\lambda \geq w_i} \quad (3.86)$$

be the corresponding filtration of Lie ideals of $\mathfrak{u}$ by λ -weights. Let $U_i \subseteq \hat{U}$ be the normal subgroup whose Lie algebra is $\mathfrak{u}_i$.

Let X be a projective scheme over $\mathbb{k}$, and let L be an ample line bundle over X . Let $\hat{U}$ act linearly on X with respect to L . Then $\bigoplus_{d \geq 0} H^0(X, L^d)$ is finitely generated ([Sta22, Lemma 0B5T]), and $\hat{U}$ acts on each $H^0(X, L^d)$.

For $d \in \mathbb{N}_+$ and $w \in \mathbb{Z}$, let $H^0(X, L^d)_{\lambda=w} \subseteq H^0(X, L^d)$ be the weight space for λ of weight w . When $w \in \mathbb{Z}$ is not a weight, we can define the subspace $H^0(X, L^d)_{\lambda=w}$ as zero. Let $H^0(X, L^d)_{\lambda=\max}$ denote the λ -weight space whose weight equals the maximal weight on $H^0(X, L^d)$. Let $H^0(X, L^d)_{\lambda < \max}$ denote the sum of λ -weight spaces whose weights are less than the maximal weight on $H^0(X, L^d)$. Then $H^0(X, L^d) = H^0(X, L^d)_{\lambda=\max} \oplus H^0(X, L^d)_{\lambda < \max}$, and $H^0(X, L^d)_{\lambda=\max} \neq 0$ if $H^0(X, L^d) \neq 0$.

Let $X_{\min}^0 \subseteq X$ be the open subscheme where $\bigoplus_{d > 0} H^0(X, L^d)_{\lambda=\max}$ does not vanish. For the action of λ on $X_{\min}^0$, let $Z_{\min} \hookrightarrow X_{\min}^0$ be the *fixed point subscheme* in the sense of Fogarty ([Fog73]). It is easy to see that $Z_{\min} \hookrightarrow X$ is the closed subscheme associated to the ideal $\bigoplus_{d > 0} H^0(X, L^d)_{\lambda < \max}$. Let $m \in \mathbb{N}_+$ be such that $H^0(X, L^m)$ generates $\bigoplus_{d \geq 0} H^0(X, L^{md})$. Then $X_{\min}^0 \subseteq X$ has an affine open covering

$$X_{\min}^0 = \bigcup X_f, \quad f \in H^0(X, L^m)_{\lambda=\max}. \quad (3.87)$$

Remark 3.3.1. If all sections $\bigoplus_{d > 0} H^0(X, L^d)_{\lambda=\max}$ are nilpotent, then $X_{\min}^0 = \emptyset$ by definition. We can assume there exists a section $f \in H^0(X, L^d)_{\lambda=\max}$ for some $d > 0$ and f is not nilpotent for $X_{\min}^0 \neq \emptyset$, or we regard statements as vacuous truth when $X_{\min}^0 = \emptyset$.

Since λ is a grading 1PS on U , sections of maximal λ -weights are U -invariant, i.e. $H^0(X, L^d)_{\lambda=\max} \subseteq H^0(X, L^d)^U$ for all $d \geq 0$. Then for $f \in H^0(X, L^m)_{\lambda=\max}$, the open subscheme $X_f \subseteq X$ is $\hat{U}$ -invariant.

The ring of functions on X_f is $\mathcal{O}(X_f) := \left(\bigoplus_{d \geq 0} H^0(X, L^{md}) \right)_{(f)}$. There is an induced action of λ on $\mathcal{O}(X_f)$. Since f has maximal λ -weight in $H^0(X, L^m)$, the λ -weights on $\mathcal{O}(X_f)$ are ≤ 0 , i.e.

$$\mathcal{O}(X_f) = \bigoplus_{w \leq 0} \mathcal{O}(X_f)_{\lambda=w}. \quad (3.88)$$

The closed subscheme $Z_{\min, f} \hookrightarrow X_f$ corresponds to $\mathcal{O}(X_f) \rightarrow \mathcal{O}(X_f)/\mathcal{O}(X_f)_{\lambda < 0}$. Then $\mathcal{O}(Z_{\min, f}) \cong \mathcal{O}(X_f)/\mathcal{O}(X_f)_{\lambda < 0} \cong \mathcal{O}(X_f)_{\lambda=0}$. The closed immersion $Z_{\min, f} \hookrightarrow X_f$ has a left inverse $X_f \rightarrow Z_{\min, f}$, corresponding to $\mathcal{O}(X_f)_{\lambda=0} \rightarrow \mathcal{O}(X_f)$. They glue over X_f to a morphism $p : X_{\min}^0 \rightarrow Z_{\min}$. On $\mathbb{k}$ -points, the morphism p sends $x \in X_{\min}^0$ to $p(x) := \lim_{t \rightarrow 0} \lambda(t).x$.

3.3.1 The CDRS condition

The condition Constant-Dimension-of-Relative-Stabilisers (CDRS) is a scheme-theoretic version of the condition in [Bér+16] Remark 2.8 (and Remark 7.1, Remark 7.12).

For the action $\hat{U} \curvearrowright (X, L)$, consider the diagram

$$\begin{array}{ccc} X_{\min}^0 & \xrightarrow{\gamma} & U \times X_{\min}^0 \\ & \searrow \delta & \downarrow \sigma \\ & & X_{\min}^0 \times X_{\min}^0 \end{array}, \quad \begin{cases} \gamma : & x \mapsto (e, x) \\ \sigma : & (u, x) \mapsto (u.x, x) \\ \delta : & x \mapsto (x, x). \end{cases} \quad (3.89)$$

where the fibre products are over $\mathbb{k}$.

Recall the infinitesimal action of U on $X_{\min}^0$ defined in Definition 2.2.1 is the following morphism

$$\Omega_{X_{\min}^0/\mathrm{Spec}(\mathbb{k})} \rightarrow \mathfrak{u}^* \otimes \mathcal{O}_{X_{\min}^0}. \quad (3.90)$$

For $0 \leq i \leq n$, consider the following composition of morphisms of sheaves

$$\Omega_{X_{\min}^0/\mathrm{Spec}(\mathbb{k})} \longrightarrow \mathfrak{u}^* \otimes \mathcal{O}_{X_{\min}^0} \longrightarrow (\mathfrak{u}_i)^* \otimes \mathcal{O}_{X_{\min}^0} \quad (3.91)$$

and let $\mathcal{K}_i$ be the kernel of the composition

$$0 \longrightarrow \mathcal{K}_i \longrightarrow \Omega_{X_{\min}^0/\mathrm{Spec}(\mathbb{k})} \longrightarrow (\mathfrak{u}_i)^* \otimes \mathcal{O}_{X_{\min}^0}. \quad (3.92)$$

There exists a morphism for each $1 \leq i \leq n$

$$\phi_i : \mathcal{K}_{i-1} \rightarrow (\mathfrak{u}_i/\mathfrak{u}_{i-1})^* \otimes \mathcal{O}_{X_{\min}^0}. \quad (3.93)$$

We write down the local version of the above morphism of quasi-coherent sheaves. Let $m \in \mathbb{N}_+$ be such that $H^0(X, L^m)$ generates $\bigoplus_{d \geq 0} H^0(X, L^{md})$. Then $X_{\min}^0$ is the non-vanishing locus of $H^0(X, L^m)_{\lambda=\max}$. Let $f \in H^0(X, L^m)_{\lambda=\max}$ be a non-nilpotent section. Then $X_f \subseteq X_{\min}^0$ is an open affine subscheme invariant under U and λ . On X_f , the morphism $\Omega_{X_{\min}^0/\mathrm{Spec}(\mathbb{k})} \rightarrow \mathfrak{u}^* \otimes \mathcal{O}_{X_{\min}^0}$ is induced by the following map of $\mathcal{O}(X_f)$ -modules

$$\Omega_{\mathcal{O}(X_f)/\mathbb{k}} \rightarrow \mathfrak{u}^* \otimes \mathcal{O}(X_f), \quad a \, db \mapsto \sum_j u_j \otimes a \, (\xi_j \cdot b) \quad (3.94)$$

where $\{u_j\}$ is a basis of $\mathfrak{u}^*$ and $\{\xi_j\}$ is the basis of $\mathfrak{u}$ dual to $\{u_j\}$.

We will call the following condition Constant-Dimension-of-Relative-Stabilisers.

Condition 3.3.2. Let $\hat{U} \curvearrowright (X, L)$. Let $\lambda : \mathbb{G}_m \rightarrow \mathrm{Aut}(U)$ be a grading 1PS on $\mathfrak{u}$ with weights $w_1 > \dots > w_n$ on $\mathfrak{u}$. Let $\{\mathfrak{u}_i\}_{i=0}^n$ be the filtration $\mathfrak{u}_i := \mathfrak{u}_{\lambda \geq w_i}$. This condition for the action $\hat{U} \curvearrowright (X, L)$ assumes that $\mathrm{coker}(\phi_i)$ for ϕ_i in (3.93) is locally free of constant rank on $X_{\min}^0$ for all $1 \leq i \leq n$.

Remark 3.3.3. The rank of $\mathrm{coker}(\phi_i)$ is $\dim \mathrm{Stab}_{\mathfrak{u}_i} - \dim \mathrm{Stab}_{\mathfrak{u}_{i-1}}$ at $\mathbb{k}$ -points.

Lemma 3.3.4. Let $\hat{U} \curvearrowright (X, L)$. Let $\{\mathfrak{u}_i\}_{i=0}^n$ be a filtration of $\mathfrak{u}$. Let $\lambda : \mathbb{G}_m \rightarrow \mathrm{Aut}(U)$ be a grading 1PS on U . Let $1 \leq i \leq n$ and $r \in \mathbb{N}$. When X is reduced, the following are equivalent:

- (1) $\mathrm{coker}(\phi_i)$ for ϕ_i in (3.93) is locally free of rank r on $X_{\min}^0$;
- (2) $\dim \mathrm{Stab}_{(\mathfrak{u}_i; \mathfrak{u}_{i-1})}(x) = r$ for all $\mathbb{k}$ -points $x \in X_{\min}^0$.

Proof. Let $m \in \mathbb{N}_+$ be such that $H^0(X, L^m)$ generates $\bigoplus_{d \geq 0} H^0(X, L^{md})$. Let $f \in H^0(X, L^m)_{\lambda=\max}$ be a nonzero section. We have seen that f is U -invariant and then $X_f \subseteq X_{\min}^0$ is a $\hat{U}$ -invariant open affine subset. The two conditions above are local, so we only need to consider what happens on X_f .

Let $\mathrm{Fit}_k(\phi_i)$ be the k th Fitting ideal of $\mathrm{coker}(\phi_i)$ for $k \in \mathbb{Z}$. Then (1) is equivalent to $\mathrm{Fit}_{r-1}(\phi_i) = 0$ and $\mathrm{Fit}_r(\phi_i) = \mathcal{O}_{X_{\min}^0}$ by Proposition 3.1.3. The local version of (1) on X_f is equivalent to $\mathrm{Fit}_{r-1}(\phi_i)(X_f) = 0$ and $\mathrm{Fit}_r(\phi_i)(X_f) = \mathcal{O}(X_f)$.

For (1) $\implies$ (2), assume $\text{Fit}_{r-1}(\phi_i)(X_f) = 0$ and $\text{Fit}_r(\phi_i)(X_f) = \mathcal{O}(X_f)$. Let $x \in X_f$ be a $\mathbb{k}$ -point and let $x^* : \mathcal{O}(X_f) \rightarrow \mathbb{k}$ be the corresponding map. By Lemma 3.1.6, we have $r - 1 < \dim \text{Stab}_{(u_i : u_{i-1})}(x) \leq r$, i.e. (2) holds.

Conversely, assume $\dim \text{Stab}_{(u_i : u_{i-1})}(x) = r$ for all $\mathbb{k}$ -points $x \in X_f$. Since X is reduced, the zero ideal in $\mathcal{O}(X_f)$ is a radical ideal, which equals the intersection of all maximal ideals by [Sta22, Theorem 00FV]. By Lemma 3.1.6, we have $\text{Fit}_r(\phi_i)(X_f) \not\subseteq \ker x^*$ and $\text{Fit}_{r-1}(\phi_i)(X_f) \subseteq \ker x^*$. By Hilbert's Nullstellensatz, the ideals $\ker x^*$ are exactly the maximal ideals in $\mathcal{O}(X_f)$. Then $\text{Fit}_r(\phi_i)(X_f)$ is not contained in any maximal ideal, which implies $\text{Fit}_r(\phi_i)(X_f) = \mathcal{O}(X_f)$. Besides, $\text{Fit}_{r-1}(\phi_i)(X_f)$ is contained in the intersection of all maximal ideals, which is the zero ideal. This proves (1). $\square$

3.3.2 The Grassmannian

Let W be a finite dimensional vector space over $\mathbb{k}$. Let $r := \dim W$ and let $0 \leq k \leq r$ be an integer. Let $\text{Sch}/\mathbb{k}$ be the category of schemes of finite type over $\mathbb{k}$.

Consider the functor

$$\mathbf{Grass}(W, k) : (\text{Sch}/\mathbb{k})^{\text{op}} \rightarrow \text{Set} \quad (3.95)$$

which associates to a scheme $S \in \text{Sch}/\mathbb{k}$ the set $\mathbf{Grass}(W, k)(S)$ of isomorphism classes of surjections

$$q : W \otimes_{\mathbb{k}} \mathcal{O}_S \rightarrow \mathcal{Q} \quad (3.96)$$

where $\mathcal{Q}$ is a locally free sheaf of rank k on S . Given $f : T \rightarrow S$ a morphism, let $\mathbf{Grass}(W, k)(f) : \mathbf{Grass}(W, k)(S) \rightarrow \mathbf{Grass}(W, k)(T)$ be the map sending the class of $q : W \otimes \mathcal{O}_S \rightarrow \mathcal{Q}$ to the class of $f^*q : W \otimes \mathcal{O}_T \rightarrow f^*\mathcal{Q}$.

It is well known (cf. [Sta22, Section 089R]) that the functor above is represented by a projective scheme over $\mathbb{k}$. Denote the representative scheme by $\text{Grass}(W, k)$.

The closed points of $\text{Grass}(W, k)$ are surjective linear maps $W \rightarrow Q$ with $\dim Q = k$. Let $V \subseteq W$ be a subspace of dimension k . Consider the subset $\text{Grass}(W, k)_V \subseteq \text{Grass}(W, k)$ of quotient spaces $W \rightarrow Q$ such that $V \subseteq W \rightarrow Q$ induces an isomorphism $V \cong Q$. Then $\text{Grass}(W, k)_V$ is an open subset. The open subscheme $\text{Grass}(W, k)_V \subseteq \text{Grass}(W, k)$ defines an open sub-functor

$$\mathbf{Grass}(W, k)_V : (\text{Sch}/\mathbb{k})^{\text{op}} \rightarrow \text{Set} \quad (3.97)$$

which associates $S \in \text{Sch}/\mathbb{k}$ to the subset $\mathbf{Grass}(W, k)_V(S) \subseteq \mathbf{Grass}(W, k)(S)$ of classes of surjections $q : W \otimes \mathcal{O}_S \rightarrow \mathcal{Q}$ such that the composition $V \otimes \mathcal{O}_S \rightarrow W \otimes \mathcal{O}_S \rightarrow \mathcal{Q}$ is an isomorphism.

Lemma 3.3.5. *Let G be a reductive group over $\mathbb{k}$. Let W be a representation of G , inducing an action of G on $\text{Grass}(W, k)$. Let $\text{Grass}(W, k)^G \hookrightarrow \text{Grass}(W, k)$ denote the fixed point subscheme. Then there exist finitely many sub-representations $V_i \subseteq W$ with $\dim V_i = k$ such that $\text{Grass}(W, k)^G \subseteq \bigcup_i \text{Grass}(W, k)_{V_i}$.*

Proof. Consider closed points, i.e. $\mathbb{k}$ -points. A $\mathbb{k}$ -point in $\text{Grass}(W, k)$ is a quotient $q : W \rightarrow Q$ with $\dim Q = k$. It is fixed by G if and only if $\ker(q) \subseteq W$ is invariant under G . Since G is reductive, there exists a sub-representation $V \subseteq W$ and $V \oplus \ker(q) = W$. Then $\dim V = k$ and $[q] \in \text{Grass}(W, k)_V$. We only need a finite number of such V_i since $\text{Grass}(W, k)^G$ is quasi-compact, as a subscheme of a Noetherian scheme. $\square$

3.3.3 Statement and proof of the first $\hat{U}$ -theorem

In this subsection, we will state and prove our first $\hat{U}$ -theorem, Theorem 3.3.8.

Lemma 3.3.6. *For any $m \in \mathbb{N}_+$, let $f \in H^0(X, L^m)_{\lambda=\max}$, and let $Y \subseteq X_{\min}^0$ be an open subset. If Y is λ -invariant and $Z_{\min, f} \subseteq Y$, then $X_f \subseteq Y$.*

Proof. The open affine X_f is λ -invariant. Let $I \subseteq \mathcal{O}(X_f)$ be the radical ideal cutting out $X_f \setminus Y$. Then I is λ -invariant. Since $X_f \setminus Y$ and $Z_{\min, f}$ are disjoint, we have $\mathcal{O}(X_f) = I + \mathcal{O}(X_f)_{\lambda < 0}$. There exists $g \in I$ and $h \in \mathcal{O}(X_f)_{\lambda < 0}$ such that $1 = g + h$. The weight zero component of g is $g_{\lambda=0} = 1$. Since I is λ -invariant and $g \in I$, we have $1 = g_{\lambda=0} \in I$. So $X_f \setminus Y = \emptyset$. $\square$

Lemma 3.3.7. *Assume Condition 3.3.2 for the action $\hat{U} \curvearrowright (X, L)$ with the grading 1PS $\lambda : \mathbb{G}_m \rightarrow \text{Aut}(U)$. Then there exists $m \in \mathbb{N}_+$ and $f_1, \dots, f_r \in H^0(X, L^m)_{\lambda=\max}$ such that:*

- $X_{\min}^0 = \bigcup_{k=1}^r X_{f_k}$ is a covering by $\hat{U}$ -invariant affine open subsets;
- Condition 3.2.1 is satisfied for the action $\hat{U} \curvearrowright X_f$ for $f \in \{f_1, \dots, f_r\}$.

Proof. For $1 \leq i \leq n$, let k_i be the rank of $\text{coker}(\phi_i)$, and let $W_i := (\mathbf{u}_i/\mathbf{u}_{i-1})^*$. Then the surjective morphism

$$q_i : W_i \otimes \mathcal{O}_{X_{\min}^0} \rightarrow \text{coker}(\phi_i) \quad (3.98)$$

defines an isomorphism class in $\text{Grass}(W_i, k_i)(X_{\min}^0)$, corresponding to a morphism

$$\Theta_i : X_{\min}^0 \rightarrow \text{Grass}(W_i, k_i). \quad (3.99)$$

The product over $\mathbb{k}$ of the morphisms over $1 \leq i \leq n$ is a morphism

$$\Theta : X_{\min}^0 \rightarrow \prod_{i=1}^n \text{Grass}(W_i, k_i). \quad (3.100)$$

Let $\iota : Z_{\min} \hookrightarrow X_{\min}^0$. The morphism q_i is λ -equivariant, so Θ_i and $\Theta_i \circ \iota$ are λ -equivariant. In particular $\Theta_i \circ \iota$ is λ -equivariant and λ acts trivially on $Z_{\min}$. Then $\Theta_i \circ \iota$ factors through the fixed point subscheme $\text{Grass}(W_i, k_i)^\lambda \hookrightarrow \text{Grass}(W_i, k_i)$.

By Lemma 3.3.5, there exist a finite number of λ -invariant subspaces $V_{ij} \subseteq W_i$ for $j = 1, \dots, m_i$ such that $\text{Grass}(W_i, k_i)^\lambda \subseteq \bigcup_{j=1}^{m_i} \text{Grass}(W_i, k_i)_{V_{ij}}$. Then

$$\prod_{i=1}^n \text{Grass}(W_i, k_i)^\lambda \subseteq \bigcup_{j_1, \dots, j_n} \prod_{i=1}^n \text{Grass}(W_i, k_i)_{V_{i,j_i}}. \quad (3.101)$$

We then have a covering of $Z_{\min}$ by open subsets invariant under λ :

$$\begin{aligned} Z_{\min} &= \bigcup_{j_1, \dots, j_n} (\Theta \circ \iota)^{-1} \left(\prod_{i=1}^n \text{Grass}(W_i, k_i)_{V_{i,j_i}} \right) \\ &= \bigcup_{j_1, \dots, j_n} Z_{\min} \cap \Theta^{-1} \left(\prod_{i=1}^n \text{Grass}(W_i, k_i)_{V_{i,j_i}} \right). \end{aligned} \quad (3.102)$$

Since $V_{ij} \subseteq W_i$ is λ -invariant and Θ_i is λ -equivariant for each $1 \leq i \leq n$ and $1 \leq j \leq m_i$, each preimage $\Theta^{-1} \left(\prod_{i=1}^n \text{Grass}(W_i, k_i)_{V_{i,j_i}} \right)$ is λ -invariant open.

This covering can be reduced and refined to a finite covering

$$Z_{\min} = \bigcup_{k=1}^r Z_{\min, f_k} \quad (3.103)$$

such that $f_1, \dots, f_r \in H^0(X, L^m)_{\lambda=\max}$ for some $m \in \mathbb{N}_+$ and each $Z_{\min, f_k}$ is contained in some $\Theta^{-1} \left(\prod_{i=1}^n \text{Grass}(W_i, k_i)_{V_{i,j_i}} \right)$. The following is an open affine covering of $X_{\min}^0$

$$X_{\min}^0 = \bigcup_{k=1}^r X_{f_k} \quad (3.104)$$

and by Lemma 3.3.6, each X_{f_k} is contained in some $\Theta^{-1}\left(\prod_{i=1}^n \text{Grass}(W_i, k_i)_{V_{i,j_i}}\right)$.

Let f be one of $f_1, \dots, f_r$ and assume $X_f \subseteq \Theta^{-1}\left(\prod_{i=1}^n \text{Grass}(W_i, k_i)_{V_{i,j_i}}\right)$ for some $j_1, \dots, j_n$. Then X_f is $\hat{U}$ -invariant and we show that Condition 3.2.1 is satisfied for the action $\hat{U} \curvearrowright X_f$.

The inclusion $\Theta_i(X_f) \subseteq \text{Grass}(W_i, k_i)_{V_{i,j_i}}$ is equivalent to the following diagram

$$\begin{array}{ccccc} & & V_{i,j_i} \otimes \mathcal{O}(X_f) & & \\ & & \downarrow & \searrow \cong & \\ \mathcal{K}_{i-1}(X_f) & \xrightarrow{\phi_i(X_f)} & W_i \otimes \mathcal{O}(X_f) & \longrightarrow & \text{coker}(\phi_i)(X_f) \longrightarrow 0. \end{array} \quad (3.105)$$

The quotient map $\mathfrak{u}_i/\mathfrak{u}_{i-1} \rightarrow \mathfrak{s}_i$ in Condition 3.2.1 can be the quotient map $\mathfrak{u}_i/\mathfrak{u}_{i-1} \rightarrow (V_{i,j_i})^*$.

These together show Condition 3.2.1 for the action $\hat{U} \curvearrowright X_f$. $\square$

Theorem 3.3.8. *Let $\lambda : \mathbb{G}_m \rightarrow \text{Aut}(U)$ be a grading 1PS on U with weights $w_1 > \dots > w_n$. Let $\{\mathfrak{u}_i\}_{i=0}^n$ be the filtration $\mathfrak{u}_i := \mathfrak{u}_{\lambda \geq w_i}$. Let X be a projective scheme over $\mathbb{k}$ and let L be an ample line bundle over X . Let $\hat{U}$ act on X linearly with respect to L . Assume that the CDRS condition (Condition 3.3.2) holds for the action $\hat{U} \curvearrowright (X, L)$. Then the universal geometric quotient of $X_{\min}^0$ by U exists*

$$\phi : X_{\min}^0 \rightarrow X_{\min}^0/U \quad (3.106)$$

and $X_{\min}^0/U$ is quasi-projective of finite type over $\mathbb{k}$.

Moreover, there exists $N \in \mathbb{N}_+$ such that

- the linear system $W := H^0(X, L^N)^U$ is base point free on $X_{\min}^0$;
- the morphism $X_{\min}^0 \rightarrow \mathbb{P}(W^*)$ factors through a closed immersion

$$X_{\min}^0/U \hookrightarrow \mathbb{P}(W^*)_{\min}^0 \quad (3.107)$$

where $\mathbb{P}(W^*)_{\min}^0 \subseteq \mathbb{P}(W^*)$ is the non-vanishing locus of $H^0(X, L^N)_{\lambda=\max}$

$$\begin{array}{ccc} X_{\min}^0 & \longrightarrow & \mathbb{P}(W^*) \\ \downarrow \phi & & \uparrow \cup \\ X_{\min}^0/U & \hookrightarrow & \mathbb{P}(W^*)_{\min}^0. \end{array} \quad (3.108)$$

Proof. Condition 3.3.2 is unchanged if L is replaced by a tensor power. Assume without loss of generality that $H^0(X, L)$ generates $\bigoplus_{d \geq 0} H^0(X, L^d)$.

By Lemma 3.3.7, there exists $m \in \mathbb{N}_+$ and non-zero sections

$$f_1, \dots, f_r \in H^0(X, L^m)_{\lambda=\max} \quad (3.109)$$

satisfying:

- $X_{\min}^0 = \bigcup_{k=1}^r X_{f_k}$;
- the action $\hat{U} \curvearrowright X_{f_k}$ satisfies Condition 3.2.1.

By Theorem 3.2.7, each X_{f_k} has a universal geometric quotient by U

$$\phi_k : X_{f_k} \rightarrow Y_k := X_{f_k}/U \quad (3.110)$$

and each Y_k is an affine scheme of finite type over $\mathbb{k}$.

The function $f_{k'}/f_k \in \mathcal{O}(X_{f_k})$ is U -invariant, so it corresponds to a function $g_{k'k} \in \mathcal{O}(Y_k) \cong \mathcal{O}(X_{f_k})^U$. The pull back of ϕ_k along the open immersion $Y_k \setminus \mathbb{V}(g_{k'k}) \rightarrow Y_k$ is also a geometric quotient, i.e.

$$X_{f_k} \setminus \mathbb{V}(f_{k'}/f_k) \rightarrow Y_k \setminus \mathbb{V}(g_{k'k}) \quad (3.111)$$

is a geometric quotient. Since $X_{f_k} \setminus \mathbb{V}(f_{k'}/f_k) = X_{f_{k'}} \setminus \mathbb{V}(f_k/f_{k'})$, their quotients by U are isomorphic by a unique isomorphism

$$Y_k \setminus \mathbb{V}(g_{k'k}) \cong Y_{k'} \setminus \mathbb{V}(g_{kk'}). \quad (3.112)$$

The affine schemes $\{Y_k\}$ then glue to a scheme Y , and the morphisms $\phi_k : X_{f_k} \rightarrow Y_k$ glue to a morphism $\phi : X_{\min}^0 \rightarrow Y$, which is a universal geometric quotient by U . Moreover, Y is a scheme of finite type over $\mathbb{k}$.

Since $Z_{\min} = \bigcup_{k=1}^r Z_{\min, f_k}$, for all $m' \gg 1$ we have

$$H^0(X, L^{mm'})_{\lambda=\max} = \sum_{k=1}^r f_k H^0(X, L^{m(m'-1)})_{\lambda=\max}. \quad (3.113)$$

Let $1 \leq k \leq r$. Since $\mathcal{O}(X_{f_k})^U$ is finitely generated over $\mathbb{k}$, there exists $m'_k \in \mathbb{N}_+$ such that $\frac{1}{f_k^{m'_k}} H^0(X, L^{mm'_k})^U$ generates $\mathcal{O}(X_{f_k})^U$. We can enlarge m'_k to $m' \in \mathbb{N}_+$ uniform for

all $1 \leq k \leq r$ and further assume (3.113), i.e. $H^0(X, L^{mm'})_{\lambda=\max} \subseteq \langle f_1, \dots, f_r \rangle$. The generating subspace $\frac{1}{f_k^{m'}} H^0(X, L^{mm'})^U \subseteq \mathcal{O}(X_{f_k})^U$ induces a surjective ring map

$$\mathrm{Sym}^\bullet(W)_{(f_k^{m'})} \rightarrow \mathcal{O}(X_{f_k})^U \quad (3.114)$$

where $W := H^0(X, L^{mm'})^U$ and the symmetric product is over $\mathbb{k}$.

The linear system $W = H^0(X, L^{mm'})^U$ is base point free on $X_{\min}^0$ since

$$\mathbb{V}(W) \cap X_{\min}^0 \subseteq \mathbb{V}(H^0(X, L^{mm'})_{\lambda=\max}) \cap X_{\min}^0 = \emptyset. \quad (3.115)$$

Then there is a morphism $X_{\min}^0 \rightarrow \mathbb{P}(W^*)$, which factors through $X_{\min}^0/U \rightarrow \mathbb{P}(W^*)$, since it is U -invariant

$$\begin{array}{ccc} X_{\min}^0 & \longrightarrow & \mathbb{P}(W^*) \\ \downarrow \phi & \nearrow & \\ X_{\min}^0/U & & \end{array} \quad (3.116)$$

We then prove that $X_{\min}^0/U \hookrightarrow \mathbb{P}(W^*)_{\min}^0$ is a closed immersion.

Observe that for any $g \in H^0(X, L^{m(m'-1)})_{\lambda=\max}$, the open affine subscheme $X_{f_k g} \subseteq X_{f_k}$ is $\hat{U}$ -invariant and Condition 3.2.1 is satisfied for the action $\hat{U} \curvearrowright X_{f_k g}$. Therefore $X_{f_k g} \rightarrow X_{f_k g}/U$ is a universal geometric quotient by U and $\mathcal{O}(X_{f_k g})^U$ is finitely generated by Theorem 3.2.9. We show that $\mathcal{O}(X_{f_k g})^U$ is generated by $\frac{1}{f_k g} W$ for any $g \in H^0(X, L^{m(m'-1)})_{\lambda=\max}$. Any element in $\mathcal{O}(X_{f_k g})$ is of the form $\frac{h}{(f_k g)^s}$ for $s \in \mathbb{N}_+$ and $h \in H^0(X, L^{mm's})$. We can enlarge $s \in \mathbb{N}_+$ such that $(f_k g)^s$ is U -invariant. Assume $\frac{h}{(f_k g)^s}$ is U -invariant. Then $\xi \cdot h \in H^0(X, L^{mm's})$ is annihilated by a power of $f_k g$ for all $\xi \in \mathfrak{u}$. We can test on a basis of $\mathfrak{u}$, and choose the power of $f_k g$ uniformly. There exists $s' \in \mathbb{N}_+$ such that $(f_k g)^{s'} h \in H^0(X, L^{mm'(s+s')})^U$. Then $\frac{(f_k g)^{s'} h}{f_k^{m'(s+s')}} \in \mathcal{O}(X_{f_k})^U$, which is in the image of $\mathrm{Sym}^\bullet(W)_{(f_k^{m'})}$, i.e.

$$\frac{(f_k g)^{s'} h}{f_k^{m'(s+s')}} = \frac{\sum_{i_1, \dots, i_t} w_{i_1} \cdots w_{i_t}}{(f_k^{m'})^t} \quad (3.117)$$

for some $t \in \mathbb{N}_+$ and $w_{i_1}, \dots, w_{i_t} \in W$. We also have

$$g^{m'} \in H^0(X, L^{mm'(m'-1)})_{\lambda=\max} = \prod_{i=1}^{m'-1} H^0(X, L^{mm'})_{\lambda=\max} \subseteq \prod_{i=1}^{m'-1} W \quad (3.118)$$

so $g^{m'} = \sum_{j_1, \dots, j_{m'-1}} w'_{j_1} \cdots w'_{j_{m'-1}}$ for some $w'_{j_1}, \dots, w'_{j_{m'-1}} \in W$.

Then in $\mathcal{O}(X_{f_k g})^U$

$$\begin{aligned}
\frac{h}{(f_k g)^s} &= \left(\frac{f_k^{m'}}{f_k g} \right)^{s+s'} \frac{(f_k g)^{s'} h}{f_k^{m'(s+s')}} \\
&= \left(\frac{f_k^{m'}}{f_k g} \right)^{s+s'} \frac{\sum_{i_1, \dots, i_t} w_{i_1} \cdots w_{i_t}}{(f_k^{m'})^t} \\
&= \left(\frac{f_k^{m'}}{f_k g} \right)^{s+s'} \frac{(\sum_{i_1, \dots, i_t} w_{i_1} \cdots w_{i_t}) g^{m't}}{(f_k g)^{m't}} \\
&= \left(\frac{f_k^{m'}}{f_k g} \right)^{s+s'} \frac{(\sum_{i_1, \dots, i_t} w_{i_1} \cdots w_{i_t}) (\sum_{j_1, \dots, j_{m'-1}} w'_{j_1} \cdots w'_{j_{m'-1}})^t}{(f_k g)^{m't}}
\end{aligned} \tag{3.119}$$

which is generated by $\frac{1}{f_k g} W$. This completes the proof that $\frac{1}{f_k g} W \subseteq \mathcal{O}(X_{f_k g})^U$ is a generating subspace, i.e. the following ring map is surjective

$$\mathrm{Sym}^\bullet(W)_{(f_k g)} \rightarrow \mathcal{O}(X_{f_k g})^U. \tag{3.120}$$

The surjective ring map above is equivalent to the closed immersion

$$X_{f_k g}/U \hookrightarrow \mathbb{P}(W^*)_{f_k g}. \tag{3.121}$$

The union of $\mathbb{P}(W^*)_{f_k g}$ over $k = 1, \dots, r$ and $g \in H^0(X, L^{m(m-1)})_{\lambda=\max}$ is the non-vanishing locus of $H^0(X, L^{mm'})_{\lambda=\max} \subseteq W$, since we have assumed (3.113). The closed immersions above then glue over $1 \leq k \leq r$ to a closed immersion

$$X_{\min}^0/U \hookrightarrow \mathbb{P}(W^*) \setminus \mathbb{V}(H^0(X, L^{mm'})_{\lambda=\max}) = \mathbb{P}(W^*)_{\min}^0. \tag{3.122}$$

□

3.3.4 Quotienting by $\hat{U}$

The residual linearisation of λ can be twisted to get a projective geometric quotient of X by $\hat{U}$ when the CDRS condition (Condition 3.3.2) holds.

Theorem 3.3.9 ([Bér+18b]). *Let X be a projective scheme over $\mathbb{k}$ and let L be an ample line bundle over X . Let $\hat{U}$ act on X linearly with respect to L . Assume CDRS (Condition 3.3.2) holds for $\hat{U} \curvearrowright (X, L)$. Assume $X_{\min}^0 \neq \emptyset$. Then the following is $\hat{U}$ -invariant open in $X_{\min}^0$*

$$X_{\min}^0 \setminus U Z_{\min} \tag{3.123}$$

which admits a projective universal geometric quotient by $\hat{U}$.

Proof. With Condition 3.3.2, by Theorem 3.3.8, the universal geometric quotient of $X_{\min}^0$ by U exists

$$\phi : X_{\min}^0 \rightarrow X_{\min}^0/U \quad (3.124)$$

and there exists $N \in \mathbb{N}_+$ such that for $W := H^0(X, L^N)^U$, there is a locally closed immersion

$$\varphi : X_{\min}^0/U \rightarrow \mathbb{P}(W^*). \quad (3.125)$$

Let $\bar{\varphi} : \overline{X_{\min}^0/U} \hookrightarrow \mathbb{P}(W^*)$ be the scheme theoretic image of φ . By [Sta22, Lemma 01R8], the underlying closed subset of $\overline{X_{\min}^0/U} \subseteq \mathbb{P}(W^*)$ is the closure of $\varphi : X_{\min}^0/U \rightarrow \mathbb{P}(W^*)$. Since $X_{\min}^0/U \hookrightarrow \mathbb{P}(W^*)_{\min}^0$ is closed, we have $X_{\min}^0/U = \overline{X_{\min}^0/U} \cap \mathbb{P}(W^*)_{\min}^0$.

The action of $\hat{U}$ on $H^0(X, L^N)$ induces an action of λ on W . Then $\lambda \curvearrowright (\mathbb{P}(W^*), \mathcal{O}(1))$ and $\lambda \curvearrowright (\overline{X_{\min}^0/U}, \bar{\varphi}^* \mathcal{O}(1))$.

By Theorem 1.19, [MFK94], for any ample linearisation M of λ on $\mathbb{P}(W^*)$

$$\overline{X_{\min}^0/U}^{\lambda, \bar{\varphi}^* M, (s)s} = \overline{X_{\min}^0/U} \cap \mathbb{P}(W^*)^{\lambda, M, (s)s}. \quad (3.126)$$

The linearisation M will be a twist of $\mathcal{O}(1)$, such that $\mathbb{P}(W^*)^{\lambda, M, (s)s} \subseteq \mathbb{P}(W^*)_{\min}^0$.

Characters of λ are integers. Let $\text{Hom}(\lambda, \mathbb{G}_m)$ be the Abelian group of characters. Call elements in $\text{Hom}(\lambda, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Q}$ *rational characters*. If $\theta \in \text{Hom}(\lambda, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a rational character, then we denote the twisted linearisation by $\mathcal{O}(1)_{\theta}$.

Let $\max(\lambda) \in \mathbb{Z}$ denote the maximal weight of λ on $H^0(X, L)$. Let $\max(\lambda, W)$ and $2\text{nd max}(\lambda, W)$ denote the maximal and next maximal weights of λ on W . Then we have $\max(\lambda, W) = N \max(\lambda)$. When $\epsilon > 0$ is small and $\max(\lambda) - \epsilon < -\theta < \max(\lambda)$, we have

$$2\text{nd max}(\lambda, W) < -N\theta < \max(\lambda, W) \quad (3.127)$$

then

$$\mathbb{P}(W^*)^{\lambda, \mathcal{O}(1)_{N\theta}, ss} = \mathbb{P}(W^*)^{\lambda, \mathcal{O}(1)_{N\theta}, s} = \mathbb{P}(W^*)_{\min}^0. \quad (3.128)$$

For such θ , we have

$$\overline{X_{\min}^0/U}^{\lambda, L_{\theta}, (s)s} = X_{\min}^0/U \cap \mathbb{P}(W^*)_{\min}^0 \setminus \mathbb{P}((W_{\lambda=\max})^*). \quad (3.129)$$

The preimage of $\overline{X_{\min}^0/U}^{\lambda, L_{\theta}, (s)s} \subseteq X_{\min}^0/U$ along $\phi : X_{\min}^0 \rightarrow X_{\min}^0/U$ is

$$X_{\min}^0 \setminus UZ_{\min} \quad (3.130)$$

which is $\hat{U}$ -invariant open. As a composition of universal geometric quotients the following is a universal geometric quotient

$$X_{\min}^0 \setminus UZ_{\min} \rightarrow (X_{\min}^0 \setminus UZ_{\min})/\hat{U}. \quad (3.131)$$

For projectivity, observe that

$$(X_{\min}^0 \setminus UZ_{\min})/\hat{U} \cong \overline{X_{\min}^0/U}^{\lambda, L_{\theta}, (s)s} / \lambda = \overline{X_{\min}^0/U} // \lambda. \quad (3.132)$$

□

3.4 Two lemmas for further application

In this section, we consider a group $U \rtimes (T \times K)$, where $U \neq \{e\}$ is unipotent, T is a torus and K is semisimple, and $T \times K$ acts on U via $T \times K \rightarrow \text{Aut}(U)$. Let $\lambda : \mathbb{G}_m \rightarrow T$ be an internal grading 1PS, i.e. the composition $\mathbb{G}_m \xrightarrow{\lambda} T \rightarrow \text{Aut}(U)$ is a grading 1PS. Let X be a projective scheme over $\mathbb{k}$ and let L be an ample line bundle on X . Let $U \rtimes (T \times K) \curvearrowright (X, L)$. In this section, we denote $X_{\min}^0$ by $X_{\lambda-\min}^0$ and similarly for $Z_{\lambda-\min}$ to emphasise the dependence on the internal 1PS λ . Let $Z_{\lambda-\min}^{K, (s)s} \subseteq Z_{\lambda-\min}$ denote the (semi)stable locus of $K \curvearrowright (Z_{\lambda-\min}, L|_{Z_{\lambda-\min}})$. Assume $Z_{\lambda-\min}^{K, s} \neq \emptyset$.

Let $r \in \mathbb{N}$ be the dimension of T . The existence of an internal grading 1PS implies $r \geq 1$. We fix an identification $T \cong (\mathbb{G}_m)^r$. For $1 \leq i \leq r$, let $\alpha_i : T \rightarrow \mathbb{G}_m$ be the projection to the i th factor. Then $\alpha_1, \dots, \alpha_r \in \text{Hom}(T, \mathbb{G}_m)$ are characters of T . For simplicity denote

$$\begin{aligned} \mathfrak{t}_{\mathbb{Q}} &:= \text{Hom}(\mathbb{G}_m, T) \otimes_{\mathbb{Z}} \mathbb{Q} \\ \mathfrak{t}_{\mathbb{Q}}^* &:= \text{Hom}(T, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Q}. \end{aligned} \quad (3.133)$$

There is a $\mathbb{Q}$ -bilinear form

$$\langle -, - \rangle : \mathfrak{t}_{\mathbb{Q}}^* \times \mathfrak{t}_{\mathbb{Q}} \rightarrow \mathbb{Q}, \quad (\alpha, \beta) \mapsto \langle \alpha, \beta \rangle \quad (3.134)$$

extending the pairing $\text{Hom}(T, \mathbb{G}_m) \times \text{Hom}(\mathbb{G}_m, T) \rightarrow \mathbb{Z}$.

Define a cone in $\mathfrak{t}_{\mathbb{Q}} \cong \mathbb{Q}^r$

$$\text{Cone}_+ := \{\beta \in \mathfrak{t}_{\mathbb{Q}} : \langle \alpha_i, \beta \rangle \geq 0 \text{ for all } 1 \leq i \leq r\} \quad (3.135)$$

For a subset $I \subseteq \{1, \dots, r\}$, define an *open face* of Cone_+

$$\sigma_I := \left\{ \beta \in \mathfrak{t}_{\mathbb{Q}} : \begin{array}{l} \langle \alpha_i, \beta \rangle = 0 \text{ for all } i \in I \\ \langle \alpha_i, \beta \rangle > 0 \text{ for all } i \notin I \end{array} \right\}. \quad (3.136)$$

Then we have:

- $\text{Cone}_+ = \bigsqcup_{I \subseteq \{1, \dots, r\}} \sigma_I$;
- $\overline{\sigma_I} = \bigsqcup_{I' \supseteq I} \sigma_{I'}$, where the topology is the Euclidean topology on $\mathfrak{t}_{\mathbb{Q}} \cong \mathbb{Q}^r$;
- The *minimal face* is $\sigma_{\{1, \dots, r\}} = \{0\}$ and the *principal face* is $\sigma_{\emptyset}$, which is dense in Cone_+ .

Lemma 3.4.1. *Let $T \times K \curvearrowright (\mathbb{P}(V), \mathcal{O}(1))$. Let $T \cong (\mathbb{G}_m)^r$ and let $\alpha_i, \text{Cone}_+, \sigma_I$ be as above. Let $\lambda : \mathbb{G}_m \rightarrow T$ be an arbitrary 1PS of T . Assume that when $\lambda \in \sigma_{\emptyset}$, the retraction $p_{\lambda} : \mathbb{P}(V)_{\lambda-\min}^0 \rightarrow \mathbb{P}(V_{\lambda-\min})$ does not depend on λ , in the sense that if $\lambda, \lambda' \in \sigma_{\emptyset}$, then*

$$\mathbb{P}(V)_{\lambda-\min}^0 = \mathbb{P}(V)_{\lambda'-\min}^0, \quad \mathbb{P}(V_{\lambda-\min}) = \mathbb{P}(V_{\lambda'-\min}), \quad p_{\lambda} = p_{\lambda'}. \quad (3.137)$$

We can denote $p_{\lambda} : \mathbb{P}(V)_{\lambda-\min}^0 \rightarrow \mathbb{P}(V_{\lambda-\min})$ when $\lambda \in \sigma_{\emptyset}$ by

$$p_{\sigma_{\emptyset}} : \mathbb{P}(V)_{\sigma_{\emptyset}-\min}^0 \rightarrow \mathbb{P}(V_{\sigma_{\emptyset}-\min}). \quad (3.138)$$

Then:

- (1) *There exists $\theta \in \text{Hom}(T, \mathbb{G}_m)$ such that θ is the T -weight on fibres of $\mathcal{O}(-1)$ at closed points $x \in \mathbb{P}(V_{\sigma_{\emptyset}-\min})$. The weight θ is minimal with respect to $\alpha_1, \dots, \alpha_r$ in the sense that for any T -weight $\tau \in \text{Hom}(T, \mathbb{G}_m)$ on fibres of $\mathcal{O}(-1)$ at T -fixed closed points $x \in \mathbb{P}(V)^T$, we have $\tau - \theta \in \mathbb{Q}_{\geq 0}\alpha_1 + \dots + \mathbb{Q}_{\geq 0}\alpha_r$.*
- (2) *When $\lambda \in \text{Cone}_+$, the retraction $p_{\lambda} : \mathbb{P}(V)_{\lambda-\min}^0 \rightarrow \mathbb{P}(V_{\lambda-\min})$ only depends on the open face σ_I containing λ , i.e. if $\lambda, \lambda' \in \sigma_I$, then*

$$\mathbb{P}(V)_{\lambda-\min}^0 = \mathbb{P}(V)_{\lambda'-\min}^0, \quad \mathbb{P}(V_{\lambda-\min}) = \mathbb{P}(V_{\lambda'-\min}), \quad p_{\lambda} = p_{\lambda'}. \quad (3.139)$$

We can denote $p_{\lambda} : \mathbb{P}(V)_{\lambda-\min}^0 \rightarrow \mathbb{P}(V_{\lambda-\min})$ when $\lambda \in \sigma_I$ by

$$p_{\sigma_I} : \mathbb{P}(V)_{\sigma_I-\min}^0 \rightarrow \mathbb{P}(V_{\sigma_I-\min}). \quad (3.140)$$

(3) For any permutation $\mu : \{1, \dots, r\} \rightarrow \{1, \dots, r\}$, let $I_{\mu,i} := \{\mu(1), \dots, \mu(i)\}$ for $0 \leq i \leq r$, and write $\sigma_{\mu,i} := \sigma_{I_{\mu,i}}$ for simplicity. Then there exists $\rho \in \text{Hom}(T, \mathbb{G}_m)$ such that ρ is positive on $\sigma_\emptyset$ and for sufficiently small $\epsilon \in \mathbb{Q}_{>0}$ and the rational character $\theta_\epsilon := \theta + \epsilon\rho$, the following is $T \times K$ -invariant open in the stable locus

$$\left\{ x \in \mathbb{P}(V) : \begin{array}{l} x \in \mathbb{P}(V)_{\sigma_\emptyset - \min}^0, p_{\sigma_\emptyset}(x) \in \mathbb{P}(V_{\sigma_\emptyset - \min})^{K,s} \\ p_{\sigma_{\mu,i-1}}(x) \neq p_{\sigma_{\mu,i}}(x) \text{ for } 1 \leq i \leq r \end{array} \right\} \subseteq \mathbb{P}(V)^{T \times K, \mathcal{O}(1)_{\theta_\epsilon, s}}. \quad (3.141)$$

Proof. We first prove (1). For any 1PS $\lambda : \mathbb{G}_m \rightarrow T$ such that $\lambda \in \sigma_\emptyset$, we have $V_{\sigma_\emptyset - \min} = V_{\lambda - \min}$. We claim that the T -invariant subspace $V_{\sigma_\emptyset - \min}$ has only one T -weight. Otherwise let $\theta, \theta' \in \text{Hom}(T, \mathbb{G}_m)$ be T -weights on $V_{\sigma_\emptyset - \min} = V_{\lambda - \min}$. Then

$$\langle \theta, \lambda \rangle = \min \{ \langle \tau, \lambda \rangle : \tau \in \text{Hom}(T, \mathbb{G}_m) \text{ is a weight of } T \curvearrowright V \} = \langle \theta', \lambda \rangle \quad (3.142)$$

and the equality $\langle \theta, \lambda \rangle = \langle \theta', \lambda \rangle$ holds for all $\lambda \in \sigma_\emptyset$, which implies $\theta = \theta'$. The unique weight θ on $V_{\sigma_\emptyset - \min}$ is the T -weight on fibres of $\mathcal{O}(-1)$ at closed points $x \in \mathbb{P}(V_{\sigma_\emptyset - \min})$.

If $\tau \in \text{Hom}(T, \mathbb{G}_m)$ is a T -weight on fibres of $\mathcal{O}(-1)$ at some T -fixed closed points $x \in \mathbb{P}(V)^T$, then τ is a weight for $T \curvearrowright V$. By the construction of θ , we have $\langle \theta, \lambda \rangle \leq \langle \tau, \lambda \rangle$ for all $\lambda \in \sigma_\emptyset$. Since $\sigma_\emptyset \subseteq \text{Cone}_+$ is dense, we have $\langle \theta, \lambda \rangle \leq \langle \tau, \lambda \rangle$ for $\lambda \in \text{Cone}_+$. This is equivalent to $\tau - \theta \in \mathbb{Q}_{\geq 0}\alpha_1 + \dots + \mathbb{Q}_{\geq 0}\alpha_r$.

Next we prove (2). For any $\lambda \in \text{Cone}_+ = \overline{\sigma_\emptyset}$, we have

$$\langle \theta, \lambda \rangle = \min \{ \langle \tau, \lambda \rangle : \tau \in \text{Hom}(T, \mathbb{G}_m) \text{ is a weight of } T \curvearrowright V \}. \quad (3.143)$$

For $v \in V$, decompose v according to T -weights $v = \sum_{\tau \in \text{Hom}(T, \mathbb{G}_m)} v_\tau$ where $v_\tau \in V_\tau$. Let $\lambda \in \sigma_I$. Then we have the following description of closed points

$$\begin{aligned} \mathbb{P}(V) \setminus \mathbb{P}(V)_{\lambda - \min}^0 &= \{ [v] \in \mathbb{P}(V) : v_\tau = 0 \text{ if } \langle \tau - \theta, \lambda \rangle = 0 \} \\ &= \{ [v] \in \mathbb{P}(V) : v_\tau = 0 \text{ if } \tau - \theta \in \sigma_I^\perp \} \\ \mathbb{P}(V_{\lambda - \min}) &= \{ [v] \in \mathbb{P}(V) : v_\tau = 0 \text{ if } \langle \tau - \theta, \lambda \rangle \neq 0 \} \\ &= \{ [v] \in \mathbb{P}(V) : v_\tau = 0 \text{ if } \tau - \theta \notin \sigma_I^\perp \}. \end{aligned} \quad (3.144)$$

The closed points of $\mathbb{P}(V)_{\lambda - \min}^0$ and $\mathbb{P}(V_{\lambda - \min})$ only depend on σ_I . Since $\mathbb{P}(V)_{\lambda - \min}^0 \subseteq \mathbb{P}(V)$ is open, the subscheme is determined by the closed points. By [FN77], the fixed point subscheme $\mathbb{P}(V_{\lambda - \min}) = (\mathbb{P}(V)_{\lambda - \min}^0)^\lambda$ is smooth and then the scheme structure is

determined by the closed points. Therefore both $\mathbb{P}(V)_{\lambda-\min}^0$ and $\mathbb{P}(V_{\lambda-\min})$ only depend on σ_I .

The morphism $p_\lambda : \mathbb{P}(V)_{\lambda-\min}^0 \rightarrow \mathbb{P}(V_{\lambda-\min})$ is a rational map $p_\lambda : \mathbb{P}(V) \dashrightarrow \mathbb{P}(V_{\lambda-\min})$ corresponding to the map of homogeneous rings

$$\mathrm{Sym}^\bullet \left(\bigoplus_{\tau-\theta \in \sigma_I^\perp} (V_\tau)^* \right) \subseteq \mathrm{Sym}^\bullet V^* \quad (3.145)$$

which only depends on σ_I and $\mathbb{P}(V)_{\lambda-\min}^0$ is the maximal domain of definition, thus independent of $\lambda \in \sigma_I$. This proves that $p_\lambda : \mathbb{P}(V)_{\lambda-\min}^0 \rightarrow \mathbb{P}(V_{\lambda-\min})$ only depends on the open face σ_I containing λ .

Then we prove (3). Without loss of generality, assume $\mu : \{1, \dots, r\} \rightarrow \{1, \dots, r\}$ is the trivial permutation. This can be achieved by choosing a different identification $T \cong (\mathbb{G}_m)^r$. In this case $I_{\mu,i} = \{1, \dots, i\}$ and denote $\sigma_i := \sigma_{I_{\mu,i}}$ for $0 \leq i \leq r$. Let $\sigma_i^\perp \subseteq \mathfrak{t}_{\mathbb{Q}}^*$ be the subspace of $\mathfrak{t}_{\mathbb{Q}}^*$ where σ_i vanishes. Then we have a complete flag

$$0 = \sigma_0^\perp \subsetneq \sigma_1^\perp \subsetneq \dots \subsetneq \sigma_r^\perp = \mathfrak{t}_{\mathbb{Q}}^*. \quad (3.146)$$

It is easy to see $\alpha_i \in \sigma_i^\perp \setminus \sigma_{i-1}^\perp$ for $1 \leq i \leq r$.

Choose a Cartan subgroup $T_K \subseteq K$ and let $r_K := \dim T_K$. Denote

$$\mathfrak{t}_{K,\mathbb{Q}}^* := \mathrm{Hom}(T_K, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^{r_K}. \quad (3.147)$$

Let $S \subseteq \mathfrak{t}_{\mathbb{Q}}^* \oplus \mathfrak{t}_{K,\mathbb{Q}}^*$ be the finite set of weights

$$S := \{(\tau, \kappa) \in \mathfrak{t}_{\mathbb{Q}}^* \oplus \mathfrak{t}_{K,\mathbb{Q}}^* : (\tau, \kappa) \text{ is a weight of } T \times T_K \curvearrowright V\}. \quad (3.148)$$

By (1) we have $S \cap (\{\theta\} \times \mathfrak{t}_{K,\mathbb{Q}}^*) \neq \emptyset$.

Consider the Euclidean topology on $\mathfrak{t}_{K,\mathbb{Q}}^*$ and $\mathfrak{t}_{\mathbb{Q}}^* \oplus \mathfrak{t}_{K,\mathbb{Q}}^*$. For a subset $S' \subseteq S$, denote by $\mathrm{CH}(S') \subseteq \mathfrak{t}_{\mathbb{Q}}^* \oplus \mathfrak{t}_{K,\mathbb{Q}}^*$ the convex hull of S' , and $\mathrm{IntCH}(S') \subseteq \mathrm{CH}(S')$ the interior of $\mathrm{CH}(S')$ as a topological subspace of $\mathfrak{t}_{\mathbb{Q}}^* \oplus \mathfrak{t}_{K,\mathbb{Q}}^*$. For a subset $S' \subseteq S \cap (\{\theta\} \times \mathfrak{t}_{K,\mathbb{Q}}^*)$, denote by $\mathrm{Int}'\mathrm{CH}(S') \subseteq \mathrm{CH}(S')$ the interior of $\mathrm{CH}(S')$ as a subspace of $\{\theta\} \times \mathfrak{t}_{K,\mathbb{Q}}^* \cong \mathfrak{t}_{K,\mathbb{Q}}^*$.

The finite nature of S suggests the existence of $\gamma_0, \dots, \gamma_{r_K} \in \mathfrak{t}_{K,\mathbb{Q}}^*$ satisfying:

- $\gamma_0, \dots, \gamma_{r_K} \in \mathfrak{t}_{K,\mathbb{Q}}^*$ are in general position in the sense that $\gamma_1 - \gamma_0, \dots, \gamma_{r_K} - \gamma_0$ form a basis of $\mathfrak{t}_{K,\mathbb{Q}}^*$;

- $0 \in \mathfrak{t}_{K, \mathbb{Q}}^*$ is the barycentre of $\gamma_0, \dots, \gamma_{r_K}$;
- If $S' \subseteq S \cap (\{\theta\} \times \mathfrak{t}_{K, \mathbb{Q}}^*)$ and $(\theta, 0) \in \text{Int}'\text{CH}(S')$, then $(\theta, \gamma_i) \in \text{Int}'\text{CH}(S')$ for $0 \leq i \leq r_K$.

Let

$$R := \left\{ M \in \mathbb{Q}^{r \times r} : \begin{array}{l} \text{there exists } (\tau_1, \kappa_1), \dots, (\tau_r, \kappa_r) \in S \\ \tau_i - \theta \in \sigma_i^\perp \setminus \sigma_{i-1}^\perp \text{ for } 1 \leq i \leq r \\ (\tau_1 - \theta, \dots, \tau_r - \theta) = (\alpha_1, \dots, \alpha_r)M \end{array} \right\}. \quad (3.149)$$

Then:

- R is a finite set. Since $\alpha_1, \dots, \alpha_r$ form a basis of $\mathfrak{t}_{\mathbb{Q}}^*$, the matrix M in $(\tau_1 - \theta, \dots, \tau_r - \theta) = (\alpha_1, \dots, \alpha_r)M$ is determined by $\{\tau_i - \theta\}_{i=1}^r$. There are only finitely many $\{\tau_i - \theta\}_{i=1}^r$, since S is finite.
- Matrices in R are invertible upper triangular. The condition that $\tau_i - \theta \in \sigma_i^\perp \setminus \sigma_{i-1}^\perp$ for $1 \leq i \leq r$ implies that $\tau_1 - \theta, \dots, \tau_r - \theta$ form a basis of $\mathfrak{t}_{\mathbb{Q}}^*$. Since $\{\alpha_i\}_{i=1}^r$ and $\{\tau_i - \theta\}_{i=1}^r$ are two bases compatible with the complete flag $\{\sigma_i^\perp\}_{i=0}^r$, the matrix M in $(\tau_1 - \theta, \dots, \tau_r - \theta) = (\alpha_1, \dots, \alpha_r)M$ is invertible and upper triangular.
- Matrices in R have positive diagonals. By (1) we have $\tau_i - \theta \in \mathbb{Q}_{\geq 0}\alpha_1 + \dots + \mathbb{Q}_{\geq 0}\alpha_r$, so diagonals of matrices $M \in R$ are all non-negative, thus positive since M is invertible upper triangular.

Since R is a finite set of invertible upper triangular matrices with positive diagonals, we can choose $y_1 \gg \dots \gg y_r \in \mathbb{N}_+$ such that

$$M^{-1} \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix} \in (\mathbb{Q}_{>0})^r, \quad \text{for all } M \in R. \quad (3.150)$$

We choose the character $\rho \in \text{Hom}(T, \mathbb{G}_m)$ as $\rho = y_1\alpha_1 + \dots + y_r\alpha_r$. It is easy to see ρ is positive on $\sigma_\emptyset$.

Define a norm $\| - \|_\infty$ on $\mathbb{Q}^r$ by

$$\|(b_1, \dots, b_r)^t\|_\infty := \max\{|b_1|, \dots, |b_r|\}, \quad (b_1, \dots, b_r)^t \in \mathbb{Q}^r. \quad (3.151)$$

The norm $\| - \|_\infty$ on $\mathbb{Q}^n$ induces a norm on $\text{End}(\mathbb{Q}^r)$ by

$$\|A\|_\infty := \sup \left\{ \frac{\|Av\|_\infty}{\|v\|_\infty} : v \in \mathbb{Q}^r, v \neq 0 \right\}, \quad A \in \text{End}(\mathbb{Q}^r). \quad (3.152)$$

Let $N := r \max \{ \|M^{-1}\|_\infty : M \in R \} \cdot \|(y_1, \dots, y_r)^\dagger\|_\infty \in \mathbb{Q}_{\geq 0}$. Let $\text{pr} : \mathfrak{t}_{\mathbb{Q}}^* \oplus \mathfrak{t}_{K, \mathbb{Q}}^* \rightarrow \mathfrak{t}_{K, \mathbb{Q}}^*$ be the projection. There exists $\delta > 0$ such that

$$\delta \cdot N \cdot \text{pr}(\text{CH}(S)) \subseteq \text{interior of the convex hull of } \{\gamma_0, \dots, \gamma_{r_K}\}. \quad (3.153)$$

We claim that if $\epsilon \in \mathbb{Q}$ and $0 < \epsilon < \delta$, then

$$\left\{ x \in \mathbb{P}(V) : \begin{array}{l} x \in \mathbb{P}(V)_{\sigma_\emptyset - \min}^0, \quad p_{\sigma_\emptyset}(x) \in \mathbb{P}(V_{\sigma_\emptyset - \min})^{K, s} \\ p_{\sigma_{i-1}}(x) \neq p_{\sigma_i}(x) \text{ for } 1 \leq i \leq r \end{array} \right\} \subseteq \mathbb{P}(V)^{T \times K, \mathcal{O}(1)_{\theta_\epsilon, s}}. \quad (3.154)$$

The left-hand side is K -invariant, so by the Hilbert-Mumford criterion, it suffices to show that the left-hand side is in $\mathbb{P}(V)^{T \times T_K, \mathcal{O}(1)_{\theta_\epsilon, s}}$.

Let $x = [v] \in \mathbb{P}(V)$ be a closed point in the left-hand side. Decompose $v \in V$ according to $T \times T_K$ -weights $v = \sum_{(\tau, \kappa) \in S} v_{(\tau, \kappa)}$ for $v_{(\tau, \kappa)} \in V_{(\tau, \kappa)}$. Let $\text{weight}(v) \subseteq S$ be the set of (τ, κ) such that $v_{(\tau, \kappa)} \neq 0$. Then:

- $\text{weight}(v) \cap (\{\theta\} \times \mathfrak{t}_{K, \mathbb{Q}}^*) \neq \emptyset$ since $[v] \in \mathbb{P}(V)_{\sigma_\emptyset - \min}^0$.
- $(\theta, 0) \in \text{Int}'\text{CH}(\text{weight}(g.v) \cap (\{\theta\} \times \mathfrak{t}_{K, \mathbb{Q}}^*))$ for all $g \in K$, since $p_{\sigma_\emptyset}([v]) \in \mathbb{P}(V_{\sigma_\emptyset})^{K, s}$.

By the last property on $\gamma_0, \dots, \gamma_{r_K}$ we have

$$(\theta, \gamma_i) \in \text{Int}'\text{CH}(\text{weight}(v) \cap (\{\theta\} \times \mathfrak{t}_{K, \mathbb{Q}}^*)), \quad 0 \leq i \leq r_K. \quad (3.155)$$

- There exist $(\tau_1, \kappa_1), \dots, (\tau_r, \kappa_r) \in \text{weight}(v)$ such that $\tau_i - \theta \in \sigma_i^\perp \setminus \sigma_{i-1}^\perp$ for $1 \leq i \leq r$.

This is because

$$p_{\sigma_i}([v]) = \left[\sum_{\substack{(\tau, \kappa) \in S \\ \tau - \theta \in \sigma_i^\perp}} v_{(\tau, \kappa)} \right] \quad (3.156)$$

and $p_{\sigma_i}([v]) \neq p_{\sigma_{i-1}}([v])$ for $1 \leq i \leq r$.

We prove that $[v] \in \mathbb{P}(V)^{T \times T_K, \mathcal{O}(1)_{\theta_\epsilon, s}}$. By the Hilbert-Mumford criterion, it suffices to show that $(\theta_\epsilon, 0) \in \text{IntCH}(\text{weight}(v))$. The following vectors in $\mathbb{P}(V)^{T \times K, \mathcal{O}(1)_{\theta_\epsilon, s}}$ are in general position

$$(\theta, \gamma_0), \dots, (\theta, \gamma_{r_K}), \quad (\tau_1, \kappa_1), \dots, (\tau_r, \kappa_r) \quad (3.157)$$

i.e. $(0, \gamma_1 - \gamma_0), \dots, (0, \gamma_{r_K} - \gamma_0), (\tau_1 - \theta, \kappa_1 - \gamma_0), \dots, (\tau_r - \theta, \kappa_r - \gamma_0)$ form a basis of $\mathfrak{t}_{\mathbb{Q}}^* \oplus \mathfrak{t}_{K, \mathbb{Q}}^*$. This is because $(0, \gamma_1 - \gamma_0), \dots, (0, \gamma_{r_K} - \gamma_0)$ form a basis of $0 \oplus \mathfrak{t}_{K, \mathbb{Q}}^*$ and $(\tau_1 - \theta, \kappa_1 - \gamma_0), \dots, (\tau_r - \theta, \kappa_r - \gamma_0)$ project along $\mathfrak{t}_{\mathbb{Q}}^* \oplus \mathfrak{t}_{K, \mathbb{Q}}^* \rightarrow \mathfrak{t}_{\mathbb{Q}}^*$ to a basis of $\mathfrak{t}_{\mathbb{Q}}^*$. Note that $(\theta, \gamma_i) \in \text{CH}(\text{weight}(v))$ by (3.155) and $(\tau_i, \kappa_i) \in \text{CH}(\text{weight}(v))$ obviously.

There exist unique $a_0, \dots, a_{r_K}, b_1, \dots, b_r \in \mathbb{Q}$ and $M \in R$ such that

$$\begin{cases} \sum_{i=0}^{r_K} a_i + \sum_{i=1}^r b_i = 1 \\ \sum_{i=0}^{r_K} a_i(\theta, \gamma_i) + \sum_{i=1}^r b_i(\tau_i, \kappa_i) = (\theta_{\epsilon}, 0) \\ (\tau_1 - \theta, \dots, \tau_r - \theta) = (\alpha_1, \dots, \alpha_r)M \end{cases} \quad (3.158)$$

and then

$$\begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix} = \epsilon M^{-1} \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix}, \quad \sum_{i=0}^{r_K} a_i \gamma_i = - \sum_{i=1}^r b_i \kappa_i. \quad (3.159)$$

By (3.150), we have that $b_1, \dots, b_r > 0$. Also by (3.153)

$$\begin{aligned} \sum_{i=0}^{r_K} a_i \gamma_i &= - \sum_{i=1}^r b_i \kappa_i \\ &\subseteq \sum_{i=1}^r \|(b_1, \dots, b_r)^t\|_{\infty} \cdot \text{pr}(\text{CH}(S)) \\ &\subseteq \epsilon \cdot r \cdot \|M^{-1}\|_{\infty} \cdot \|(y_1, \dots, y_r)^t\|_{\infty} \cdot \text{pr}(\text{CH}(S)) \\ &\subseteq \delta \cdot N \cdot \text{pr}(\text{CH}(S)) \\ &\subseteq \text{interior of the convex hull of } \{\gamma_0, \dots, \gamma_{r_K}\} \end{aligned} \quad (3.160)$$

which implies $a_0, \dots, a_{r_K} > 0$. Then $0 < a_i, b_i < 1$ for each a_i, b_i . This shows

$$\begin{aligned} (\theta_{\epsilon}, 0) &\in \text{IntCH}\{(\theta, \gamma_i), (\tau_j, \kappa_j) : 0 \leq i \leq r_K, 1 \leq j \leq r\} \\ &\subseteq \text{IntCH}(\text{weight}(v)) \end{aligned} \quad (3.161)$$

where the inclusion is due to $(\theta, \gamma_i), (\tau_j, \kappa_j) \in \text{CH}(\text{weight}(v))$. By the Hilbert-Mumford criterion, this proves $[v] \in \mathbb{P}(V)^{T \times T_K, \mathcal{O}(1)_{\theta_{\epsilon}, s}}$ and thus (3). $\square$

Lemma 3.4.2. *Let X be a projective scheme over $\mathbb{k}$ and let L be an ample line bundle on X . Let $U \rtimes (T \times K)$ act on X linearly with respect to L . Let $r := \dim T$ and let $\alpha_i, \text{Cone}_+, \sigma_I$ be as above. Assume:*

- A 1PS $\lambda : \mathbb{G}_m \rightarrow T$ grades U if and only if $\lambda \in \sigma_{\emptyset}$;

- When $\lambda \in \sigma_\emptyset$ is a grading 1PS, the retraction $p_\lambda : X_{\lambda-\min}^0 \rightarrow Z_{\lambda-\min}$ does not depend on λ , in the sense that there exists

$$p_{\sigma_\emptyset} : X_{\sigma_\emptyset-\min}^0 \rightarrow Z_{\sigma_\emptyset-\min} \quad (3.162)$$

and $X_{\lambda-\min}^0 = X_{\sigma_\emptyset-\min}^0$, $Z_{\lambda-\min} = Z_{\sigma_\emptyset-\min}$ and $p_\lambda = p_{\sigma_\emptyset}$ for all 1PS $\lambda \in \sigma_\emptyset$;

- There exists a 1PS $\lambda \in \sigma_\emptyset$ such that CDRS (Condition 3.3.2) holds for $U \rtimes \lambda \curvearrowright (X, L)$.

Then:

- (1) There exists $\theta \in \text{Hom}(T, \mathbb{G}_m)$ such that θ is the T -weight on fibres of L^* at closed points $x \in Z_{\sigma_\emptyset-\min} \subseteq X^T$;
- (2) For a 1PS $\lambda \in \text{Cone}_+$, the retraction $p_\lambda : X_{\lambda-\min}^0 \rightarrow Z_{\lambda-\min}$ only depends on the open face σ_I containing λ , i.e. there exists

$$p_{\sigma_I} : X_{\sigma_I-\min}^0 \rightarrow Z_{\sigma_I-\min} \quad (3.163)$$

equal to p_λ for $\lambda \in \sigma_I$;

- (3) Let $\sigma_i := \sigma_{\{1, \dots, i\}}$ for $0 \leq i \leq r$. Then the following set of closed points is $U \rtimes (T \times K)$ -invariant open

$$\left\{ x \in X_{\sigma_\emptyset-\min}^0 : \begin{array}{l} p_{\sigma_\emptyset}(x) \in (Z_{\sigma_\emptyset-\min})^{K,L,s} \\ U.p_{\sigma_{i-1}}(x) \neq U.p_{\sigma_i}(x) \text{ for } 1 \leq i \leq r \end{array} \right\} \quad (3.164)$$

which admits a quasi-projective universal geometric quotient by $U \rtimes (T \times K)$.

Proof. Choose a 1PS $\lambda \in \sigma_\emptyset$ such that $U \rtimes \lambda \curvearrowright (X, L)$ satisfies Condition 3.3.2. By Theorem 3.3.8, the universal geometric quotient of $X_{\sigma_\emptyset-\min}^0 = X_{\lambda-\min}^0$ by U exists

$$\phi : X_{\sigma_\emptyset-\min}^0 \rightarrow X_{\sigma_\emptyset-\min}^0/U \quad (3.165)$$

and there exists $N \in \mathbb{N}_+$ and a closed immersion $X_{\sigma_\emptyset-\min}^0/U \hookrightarrow \mathbb{P}(W^*)_{\lambda-\min}^0$ for $W := H^0(X, L^N)^U$. The assumptions in this lemma verify the assumptions in Lemma 3.4.1 for

$$T \times K \curvearrowright (\mathbb{P}(H^0(X, L^N)^*), \mathcal{O}(1)), \quad T \times K \curvearrowright (P(W^*), \mathcal{O}(1)). \quad (3.166)$$

Then (1), (2) follow by Lemma 3.4.1.

For (3), note that

$$\begin{aligned}
& \left\{ x \in X_{\sigma_\emptyset - \min}^0 : \begin{array}{l} p_{\sigma_\emptyset}(x) \in (Z_{\sigma_\emptyset - \min})^{K,L,s} \\ U.p_{\sigma_i}(x) \neq U.p_{\sigma_i}(x) \text{ for } 1 \leq i \leq r \end{array} \right\} \\
&= \phi^{-1} \left(X_{\sigma_\emptyset - \min}^0 / U \cap \left\{ x \in \mathbb{P}(W^*)_{\sigma_\emptyset - \min}^0 : \begin{array}{l} p'_{\sigma_\emptyset}(x) \in \mathbb{P}((W_{\sigma_\emptyset - \min})^*)^{K,s} \\ p'_{\sigma_{i-1}}(x) \neq p'_{\sigma_i}(x) \text{ for } 1 \leq i \leq r \end{array} \right\} \right)
\end{aligned} \tag{3.167}$$

where $p'_{\sigma_i} : \mathbb{P}(W^*)_{\sigma_i - \min}^0 \rightarrow \mathbb{P}((W_{\sigma_i - \min})^*)$. This proves (3). $\square$

Chapter 4

Blow-up

Let $\hat{U} \curvearrowright (X, L)$ and $w_1 > \cdots > w_n$ and $\{\mathbf{u}_i\}_{i=0}^n$ be as in last chapter. In this chapter, we do not assume the CDRS condition (Condition 3.3.2) and describe a blow-up procedure to achieve CDRS so that Theorem 3.3.8 and Theorem 3.3.9 apply to the induced action on the blow-up of X . A weaker condition is needed for this process (*Weak Unipotent Upstairs condition*, Condition 4.2.1), and the main result is Theorem 4.4.1 and Corollary 4.4.2.

4.1 Examples with few grading weights

The general construction involves a tedious calculation (Lemma 4.3.3). It may help to motivate the construction there by first considering some more tractable examples. In this section, we consider the simple situations when there are one or two grading weights for λ on U . Consider the situation:

- $\lambda : \mathbb{G}_m \rightarrow \text{Aut}(U)$ is a grading 1PS on U with grading weights $w_1 > \cdots > w_n$ for $n = 1$ or 2 ;
- $Y = \text{Spec}(A)$ is a reduced affine scheme of finite type over $\mathbb{k}$;
- $\hat{U}$ acts on Y such that the action of λ on A has all weights ≤ 0 (cf. the first requirement in Condition 3.2.1).

The dimensions of unipotent stabilisers are not assumed constant, and the blow-up is to achieve this. Consider the exact sequence

$$K_{i-1} \xrightarrow{\phi_i} (\mathbf{u}_i/\mathbf{u}_{i-1})^* \otimes A \longrightarrow \text{coker}(\phi_i) \longrightarrow 0 \quad (4.1)$$

where $K_{i-1} \subseteq \Omega_{A/\mathbb{k}}$ is the kernel of $\Omega_{A/\mathbb{k}} \rightarrow \mathfrak{u}_{i-1}^* \otimes A$. For simplicity, write $\text{Fit}_k(\phi_i)$ for $\text{Fit}_k(\text{coker}(\phi_i))$. A suitable $\hat{U}$ -equivariant blow-up $\tilde{Y} \rightarrow Y$ with an ample linearisation will satisfy the condition that on $\tilde{Y}_{\min}^0$, the dimensions of $\mathfrak{u}_i$ -stabilisers are constant for each $1 \leq i \leq n$. By Proposition 3.1.3 and Corollary 3.1.8, the condition that $\mathfrak{u}_i$ -stabilisers have constant dimensions is equivalent to that $\text{coker}(\tilde{\phi}_i)$'s are locally free of constant rank. We will verify that $\text{Fit}_{k_i-1}(\tilde{\phi}_i) = 0$ and $\text{Fit}_{k_i}(\tilde{\phi}_i) = \mathcal{O}_{\tilde{Y}_{\min}^0}$ for each $1 \leq i \leq n$ for some $k_1, \dots, k_n \in \mathbb{N}$. This condition is equivalent to $\dim \text{Stab}_{\mathfrak{u}_i}(y) = k_1 + \dots + k_i$ for $\mathbb{k}$ -points $y \in \tilde{Y}_{\min}^0$ and $1 \leq i \leq n$ by Proposition 3.1.3 and Corollary 3.1.9.

Let $k_1, \dots, k_n \in \mathbb{N}$ be such that

$$k_i := \min\{k \in \mathbb{N} : \text{Fit}_k(\phi_i) \neq 0\}, \quad i = 1, \dots, n. \quad (4.2)$$

Recall that $Y = \text{Spec}(A)$ is a reduced scheme of finite type over $\mathbb{k}$. If we assume $\prod_{i=1}^n \text{Fit}_{k_i} \neq 0$, then the non-vanishing locus of $\prod_{i=1}^n \text{Fit}_{k_i}(\phi_i)$ is non-empty. This non-vanishing locus equals the intersection of the non-vanishing loci of $\text{Fit}_{k_i}(\phi_i)$ for $i = 1, \dots, n$. For $1 \leq i \leq n$, the non-vanishing locus of $\text{Fit}_{k_i}(\phi_i)$ is the largest open subscheme such that the restriction of $\text{coker}(\phi_i)$ is locally free of rank k_i by [Sta22, Lemma 05P8]. Therefore the non-vanishing locus of $\prod_{i=1}^n \text{Fit}_{k_i}(\phi_i)$ is the largest open subscheme such that $\text{coker}(\phi_i)$ is locally free of rank k_i for each $i = 1, \dots, n$. By Corollary 3.1.9, the closed points of the non-vanishing locus of $\prod_{i=1}^n \text{Fit}_{k_i}(\phi_i)$ are exactly $y \in Y$ such that

$$\dim \text{Stab}_{\mathfrak{u}_i}(y) = k_1 + \dots + k_i, \quad i = 1, \dots, n. \quad (4.3)$$

4.1.1 One grading weight

Assume there is only one grading weight, i.e. $n = 1$. Let $w > 0$ be the grading weight. Let $r = \dim U$ and choose a basis $(\xi_1, \dots, \xi_r)$ of $\mathfrak{u}$ with its dual basis $(u_1, \dots, u_r)$ of $\mathfrak{u}^*$. Let $\phi : \Omega_{A/\mathbb{k}} \rightarrow \mathfrak{u}^* \otimes A$ be the morphism of coherent sheaves and denote by $\text{Fit}_k(\phi) \subseteq A$ the k th Fitting ideal of $\text{coker}(\phi)$. Let $k \in \mathbb{N}$ be such that $\text{Fit}_k(\phi) \neq 0$ and $\text{Fit}_{k-1}(\phi) = 0$.

We need an assumption on $\text{Fit}_k(\phi)$, that is $\text{Fit}_k(\phi) \not\subseteq A_{\lambda < 0}$, which is equivalent to that $\dim \text{Stab}_U(z) = k$ for some $\mathbb{k}$ -points $z \in \text{Spec}(A/A_{\lambda < 0})$ (cf. Condition 4.2.1). Let $I \subseteq A$ be the ideal

$$I := \text{Fit}_k(\phi) + A_{\lambda < 0} = \text{Fit}_k(\phi)_{\lambda=0} \oplus A_{\lambda < 0}. \quad (4.4)$$

Let $J \subseteq A$ be the kernel of the composition $A \rightarrow \mathcal{O}(U) \otimes A \rightarrow \mathcal{O}(U) \otimes (A/I)$. The closed subscheme associated to $J \subseteq A$ is the scheme theoretic image of $U \times \mathbb{V}(I) \rightarrow Y$, where $\mathbb{V}(I) \hookrightarrow Y$ is the closed subscheme associated to $I \subseteq A$.

Remark 4.1.1. The $\mathbb{k}$ -points in $\mathbb{V}(I)$ are exactly $z \in \mathbb{V}(A_{<0})$ such that $\dim \text{Stab}_U(z) > k$. By [Sta22, Lemma 01R8], the closed subset $\mathbb{V}(J)$ is the closure of the U -sweep of $\mathbb{V}(I)$.

We choose the centre $C \hookrightarrow Y$ of the blow-up as the closed subscheme associated to $J \subseteq A$. The extreme situation when $J = 0$ does not happen, since always $0 \neq \text{Fit}_k(\phi)_{\lambda=0} \subseteq J$.

The ideal $J \subseteq A$ is $\hat{U}$ -invariant and $J_{\lambda=\max} = J_{\lambda=0} = \text{Fit}_k(\phi)_{\lambda=0}$. Let $\tilde{Y} \rightarrow Y$ be the blow-up. Then $\tilde{Y}$ is constructed by

$$\begin{aligned} \text{Bl}_J(A) &:= \bigoplus_{d \geq 0} J^d = A \oplus J \oplus J^2 \oplus \dots \\ \tilde{Y} &:= \text{Proj}(\text{Bl}_J(A)) \end{aligned} \tag{4.5}$$

where $\text{Bl}_J(A)$ is called the *blowup algebra associated to $J \subseteq A$* . The degree on $\text{Bl}_J(A)$ is the natural one from the direct sum. Let $E \hookrightarrow \tilde{Y}$ be the exceptional divisor with its associated line bundle $\mathcal{O}(E)$. The line bundle $\mathcal{O}(1)$ from the Proj-construction equals $\mathcal{O}(-E)$ and it is relatively ample very. Since Y is affine, we have that $\mathcal{O}(1)$ is ample. The blow-up is $\hat{U}$ -equivariant, and then $\hat{U}$ acts on $\tilde{Y}$ and linearises $\mathcal{O}(1)$. With this ample linearisation, we can define $(\tilde{Y})_{\min}^0$, which is the union of the following open affine subschemes

$$\begin{aligned} (\tilde{Y})_{\min}^0 &= \bigcup_{a \in J_{\lambda=\max}} \tilde{Y}_a \\ \tilde{Y}_a &:= \text{Spec}\left(A\left[\frac{J}{a}\right]\right) \\ A\left[\frac{J}{a}\right] &:= \left(\text{Bl}_J(A)\right)_{(a)} \end{aligned} \tag{4.6}$$

where $A\left[\frac{J}{a}\right]$ is the homogeneous localisation of $\text{Bl}_J(A)$ at $a \in J = \text{Bl}_J(A)_1$, called the *affine blowup algebra*.

Let $a \in J_{\lambda=\max} = \text{Fit}_k(\phi)_{\lambda=0}$ be of the following form

$$a = \det \begin{pmatrix} \xi_1 \cdot f_1 & \cdots & \xi_1 \cdot f_{r-k} \\ \vdots & \ddots & \vdots \\ \xi_{r-k} \cdot f_1 & \cdots & \xi_{r-k} \cdot f_{r-k} \end{pmatrix} \tag{4.7}$$

for $f_1, \dots, f_{r-k} \in A_{\lambda=-w}$. We prove for such $a \in J_{\lambda=\max}$

$$\text{Fit}_{k-1}(\tilde{\phi}_a) = 0, \quad \text{Fit}_k(\tilde{\phi}_a) = \mathcal{O}(\tilde{Y}_a) \tag{4.8}$$

where $\tilde{\phi}_a$ is the morphism

$$\Omega_{\mathcal{O}(\tilde{Y}_a)/\mathbb{k}} \xrightarrow{\tilde{\phi}_a} \mathfrak{u}^* \otimes \mathcal{O}(\tilde{Y}_a) \longrightarrow \text{coker}(\tilde{\phi}_a) \longrightarrow 0 \quad (4.9)$$

and $\text{Fit}_k(\tilde{\phi}_a)$ denotes the k th Fitting ideal of $\text{coker}(\tilde{\phi}_a)$.

Lemma. $\text{Fit}_{k-1}(\tilde{\phi}_a) = 0$.

Proof. There is a surjective morphism

$$A\left[\frac{J}{a}\right] \otimes_{\mathbb{k}} d\left(\frac{J}{a}\right) \rightarrow \Omega_{\mathcal{O}(\tilde{Y}_a)/\mathbb{k}}, \quad \frac{g}{a^m} \otimes d\left(\frac{h}{a}\right) \mapsto \frac{g}{a^m} d\left(\frac{h}{a}\right) \quad (4.10)$$

where $g \in J^m, h \in J$.

Let $h_1, \dots, h_{r-k+1} \in J$. Then $\tilde{\phi}_a$ on $d\left(\frac{h_1}{a}\right), \dots, d\left(\frac{h_{r-k+1}}{a}\right)$ is represented by the following matrix

$$\tilde{\phi}_a\left(d\left(\frac{h_1}{a}\right), \dots, d\left(\frac{h_{r-k+1}}{a}\right)\right) = (u_1 \otimes 1, \dots, u_r \otimes 1) \begin{pmatrix} \frac{\xi_1 \cdot h_1}{a} & \dots & \frac{\xi_1 \cdot h_{r-k+1}}{a} \\ \vdots & \ddots & \vdots \\ \frac{\xi_r \cdot h_1}{a} & \dots & \frac{\xi_r \cdot h_{r-k+1}}{a} \end{pmatrix}. \quad (4.11)$$

To prove $\text{Fit}_k(\tilde{\phi}_a) = 0$, it suffices to show all $(r-k+1) \times (r-k+1)$ -minors of the above matrix are zero. Without loss of generality we consider the top $(r-k+1) \times (r-k+1)$ -minor, which is of the form $\frac{\det(M)}{a^{r-k+1}}$, where $\det(M)$ is the determinant of the following square matrix

$$M = \begin{pmatrix} \xi_1 \cdot h_1 & \dots & \xi_1 \cdot h_{r-k+1} \\ \vdots & \ddots & \vdots \\ \xi_{r-k+1} \cdot h_1 & \dots & \xi_{r-k+1} \cdot h_{r-k+1} \end{pmatrix}. \quad (4.12)$$

Since $\text{Fit}_{k-1}(\phi) = 0$, we have $\det(M) = 0$. This proves $\text{Fit}_{k-1}(\tilde{\phi}_a) = 0$. $\square$

Lemma. $\text{Fit}_k(\tilde{\phi}_a) = \mathcal{O}(\tilde{Y}_a)$.

Proof. We need to prove $1 \in \text{Fit}_k(\tilde{\phi}_a)$. For $1 \leq i \leq r-k$, let

$$b_i := \det \begin{pmatrix} \xi_1 \cdot f_1 & \dots & \xi_1 \cdot f_{r-k} \\ \vdots & \dots & \vdots \\ f_1 & \dots & f_{r-k} \\ \vdots & \dots & \vdots \\ \xi_{r-k} \cdot f_1 & \dots & \xi_{r-k} \cdot f_{r-k} \end{pmatrix} \in A \quad (4.13)$$

where $(f_1, \dots, f_{r-k})$ occupies the i th row. We see that $b_i \in A_{\lambda=-w}$.

It is easy to see $\xi_i \cdot b_j = \delta_{ij}a$ for $1 \leq i, j \leq r - k$. Moreover, we have $U(\mathfrak{u}) \cdot b_j \subseteq aA \subseteq I$. By Corollary 4.5.4 in Appendix 4.5.2, the map $A \rightarrow \mathcal{O}(U) \otimes (A/I)$ sends b_j to zero, i.e. $b_j \in J$. Then $d\left(\frac{b_j}{a}\right) \in \Omega_{\mathcal{O}(\tilde{Y}_a)/\mathbb{k}}$.

The map $\tilde{\phi}_a : \Omega_{\mathcal{O}(\tilde{Y}_a)/\mathbb{k}} \rightarrow \mathfrak{u}^* \otimes \mathcal{O}(\tilde{Y}_a)$ on $d\left(\frac{b_1}{a}\right), \dots, d\left(\frac{b_{r-k}}{a}\right)$ is represented by the following matrix

$$\tilde{\phi}_a\left(d\left(\frac{b_1}{a}\right), \dots, d\left(\frac{b_{r-k}}{a}\right)\right) = (u_1 \otimes 1, \dots, u_r \otimes 1) \begin{pmatrix} I_{r-k} \\ M' \end{pmatrix} \quad (4.14)$$

where I_{r-k} is the identity matrix of size $(r - k) \times (r - k)$, and $M' \in \mathcal{O}(\tilde{Y}_a)^{k \times (r-k)}$. Then the top $(r - k) \times (r - k)$ -minor is $1 \in \text{Fit}_k(\tilde{\phi}_a)$. $\square$

In the case when there is one grading weight, when proving $\text{Fit}_k(\tilde{\phi}_a) = \mathcal{O}(\tilde{Y}_a)$, we construct $b_1, \dots, b_{r-k} \in A$ with the following properties:

- (1) $b_1, \dots, b_{r-k} \in J$, so $d\left(\frac{b_j}{a}\right) \in \Omega_{\mathcal{O}(\tilde{Y}_a)/\mathbb{k}}$, the domain of definition of $\tilde{\phi}_a$;
- (2) $\xi_i \cdot b_j = \delta_{ij}a$ for $1 \leq i, j \leq r - k$, which realises I_{r-k} as a sub-matrix of size $(r - k) \times (r - k)$ of the matrix representing $\tilde{\phi}_a$.

In the general case with multiple grading weights, property (1) above, $b_j \in J$, becomes much more complicated, while property (2) and $\text{Fit}_{k-1}(\tilde{\phi}_i) = 0$ can be verified in essentially the same way. Part of the difficulty of (1) is the difficulty of describing scheme theoretic images. The blow-up centre is such a scheme theoretic image, and the condition $b_j \in J$ is that b_j pulls back to a zero function on the centre. Another non-trivial part for (1) is to guess what is the form of b_j in general.

4.1.2 Two grading weights

We focus on the construction of b_j and the condition $b_j \in J$ for cases with 2 grading weights.

Assume the condition that $\prod_{i=1}^2 \text{Fit}_{k_i}(\phi_i) \not\subseteq A_{\lambda < 0}$.

Let $r_i := \dim \mathfrak{u}_{\lambda=w_i}$. Choose a basis $(\xi_1^{(i)}, \dots, \xi_{r_i}^{(i)})$ of $\mathfrak{u}_{\lambda=w_i}$ and let $(u_1^{(i)}, \dots, u_{r_i}^{(i)})$ be its dual basis on $\mathfrak{u}_{\lambda=w_i}^*$. Let $I := \prod_{i=1}^2 \text{Fit}_{k_i}(\phi_i) + A_{\lambda < 0}$ and let $J \subseteq A$ be the kernel of $A \rightarrow \mathcal{O}(U) \otimes (A/I)$.

It is easy to see $\prod_{i=1}^2 \text{Fit}_{k_i}(\phi_i)$ is generated by $a \in \prod_{i=1}^2 \text{Fit}_{k_i}(\phi_i)$ of the form $a = a^{(1)}a^{(2)}$ for $a^{(i)} \in \text{Fit}_{k_i}(\phi_i)$

$$a^{(i)} := \det \begin{pmatrix} \xi_1^{(i)} \cdot f_1^{(i)} & \cdots & \xi_1^{(i)} \cdot f_{r_i-k_i}^{(i)} \\ \vdots & \ddots & \vdots \\ \xi_{r_i-k_i}^{(i)} \cdot f_1^{(i)} & \cdots & \xi_{r_i-k_i}^{(i)} \cdot f_{r_i-k_i}^{(i)} \end{pmatrix} \quad (4.15)$$

$$f_1^{(i)}, \dots, f_{r_i-k_i}^{(i)} \in A_{\lambda=-w_i}.$$

Recall

$$\begin{aligned} \tilde{\phi}_{1,a} : \Omega_{\mathcal{O}(\tilde{Y}_a)/\mathbb{k}} &\rightarrow \mathfrak{u}_1^* \otimes \mathcal{O}(\tilde{Y}_a) \\ \tilde{\phi}_{2,a} : \tilde{K}_{1,a} &\rightarrow (\mathfrak{u}_2/\mathfrak{u}_1)^* \otimes \mathcal{O}(\tilde{Y}_a) \cong \mathfrak{u}_{\lambda=w_2}^* \otimes \mathcal{O}(\tilde{Y}_a) \end{aligned} \quad (4.16)$$

where $\tilde{K}_{1,a} := \ker(\tilde{\phi}_{1,a}) \subseteq \Omega_{\mathcal{O}(\tilde{Y}_a)/\mathbb{k}}$. Let $b_\nu^{(2)}$ for $\nu = 1, \dots, r_2 - k_2$ be

$$b_\nu^{(2)} := a^{(1)} \cdot \det \begin{pmatrix} \xi_1^{(2)} \cdot f_1^{(2)} & \cdots & \xi_1^{(2)} \cdot f_{r_2-k_2}^{(2)} \\ \vdots & \cdots & \vdots \\ f_1^{(2)} & \cdots & f_{r_2-k_2}^{(2)} \\ \vdots & \cdots & \vdots \\ \xi_{r_2-k_2}^{(2)} \cdot f_1^{(2)} & \cdots & \xi_{r_2-k_2}^{(2)} \cdot f_{r_2-k_2}^{(2)} \end{pmatrix} \quad (4.17)$$

where $(f_1^{(2)}, \dots, f_{r_2-k_2}^{(2)})$ occupies the ν th row. It is easy to check:

- (1) $b_\nu^{(2)} \in A_{\lambda=-w_2}$;
- (2) $\xi_\mu^{(2)} \cdot b_\nu^{(2)} = \delta_{\mu\nu} a$ for $1 \leq \mu, \nu \leq r_2 - k_2$, and $\xi_\mu^{(2)} \cdot b_\nu^{(2)} \in a^{(1)} \cdot \text{Fit}_{k_2}(\phi_2)$ for $1 \leq \mu \leq r_2$;
- (3) $\xi_\mu^{(1)} \cdot b_\nu^{(2)} = 0$ for $1 \leq \mu \leq r_1$ and $1 \leq \nu \leq r_2 - k_2$.

The last two conditions above show that $b_\nu^{(2)} \in J$ (Corollary 4.5.4). Then $d(\frac{b_\nu^{(2)}}{a}) \in \Omega_{\mathcal{O}(\tilde{Y}_a)/\mathbb{k}}$, and since it has λ -weight $-w_2$, we have $d(\frac{b_\nu^{(2)}}{a}) \in \tilde{K}_{1,a}$. We can write down the matrix representing $\tilde{\phi}_{2,a}$ on $d(\frac{b_1^{(2)}}{a}), \dots, d(\frac{b_{r_2-k_2}^{(2)}}{a})$ and the top sub-matrix of size $(r_2 - k_2) \times (r_2 - k_2)$ is $I_{r_2-k_2}$ by condition (2) above. This proves $\text{Fit}_{k_2}(\tilde{\phi}_{2,a}) = \mathcal{O}(\tilde{Y}_a)$.

We need to show $\text{Fit}_{k_1}(\tilde{\phi}_{1,a}) = \mathcal{O}(\tilde{Y}_a)$. Before we prove it, we prove an identity.

Lemma. *The following identity holds*

$$(\xi_\mu^{(2)} \cdot f) a^{(2)} = \sum_{\mu'=1}^{r_2-k_2} (\xi_\mu^{(2)} \cdot b_{\mu'}^{(2)}) (\xi_{\mu'}^{(2)} \cdot f), \quad f \in A_{\lambda=-w_2}, \quad 1 \leq \mu \leq r_2. \quad (4.18)$$

Proof. For $f \in A_{\lambda=-w_2}$ and $1 \leq \mu \leq r_2$, consider the following matrix of size $(r_2 - k_2 + 1) \times (r_2 - k_2 + 1)$

$$\begin{pmatrix} \xi_1^{(2)} \cdot f_1^{(2)} & \cdots & \xi_1^{(2)} \cdot f_{r_2-k_2}^{(2)} & \xi_1^{(2)} \cdot f \\ \vdots & \ddots & \vdots & \vdots \\ \xi_{r_2-k_2}^{(2)} \cdot f_1^{(2)} & \cdots & \xi_{r_2-k_2}^{(2)} \cdot f_{r_2-k_2}^{(2)} & \xi_{r_2-k_2}^{(2)} \cdot f \\ \xi_\mu^{(2)} \cdot f_1^{(2)} & \cdots & \xi_\mu^{(2)} \cdot f_{r_2-k_2}^{(2)} & \xi_\mu^{(2)} \cdot f \end{pmatrix}. \quad (4.19)$$

Since $f \in A_{\lambda=-w_2}$, we have $df \in K_1 = \ker(\phi_1)$. Then the above matrix has determinant zero, since $\text{Fit}_{k_2-1}(\phi_2) = 0$. Its determinant can be expanded through the last column

$$0 = (\xi_\mu^{(2)} \cdot f) a^{(2)} - \sum_{\mu'=1}^{r_2-k_2} (\xi_{\mu'}^{(2)} \cdot f) \det \begin{pmatrix} \xi_1^{(2)} \cdot f_1^{(2)} & \cdots & \xi_1^{(2)} \cdot f_{r_2-k_2}^{(2)} \\ \vdots & \cdots & \vdots \\ \xi_{\mu'}^{(2)} \cdot f_1^{(2)} & \cdots & \xi_{\mu'}^{(2)} \cdot f_{r_2-k_2}^{(2)} \\ \vdots & \cdots & \vdots \\ \xi_{r_2-k_2}^{(2)} \cdot f_1^{(2)} & \cdots & \xi_{r_2-k_2}^{(2)} \cdot f_{r_2-k_2}^{(2)} \end{pmatrix} \leftarrow \mu'\text{th row} \quad (4.20)$$

where the summand labelled by μ' is the determinant $a^{(2)}$ with the μ' -th row replaced.

Equation (4.18) is the compact form of the above equation. $\square$

For $1 \leq \nu \leq r_1 - k_1$, let $\beta_\nu^{(1)}$ be

$$\beta_\nu^{(1)} := \det \begin{pmatrix} \xi_1^{(1)} \cdot f_1^{(1)} & \cdots & \xi_1^{(1)} \cdot f_{r_1-k_1}^{(1)} \\ \vdots & \cdots & \vdots \\ f_1^{(1)} & \cdots & f_{r_1-k_1}^{(1)} \\ \vdots & \cdots & \vdots \\ \xi_{r_1-k_1}^{(1)} \cdot f_1^{(1)} & \cdots & \xi_{r_1-k_1}^{(1)} \cdot f_{r_1-k_1}^{(1)} \end{pmatrix} \quad (4.21)$$

where $(f_1^{(1)}, \dots, f_{r_1-k_1}^{(1)})$ occupies the ν th row. For $1 \leq \nu \leq r_1 - k_1$, let $b_\nu^{(1)}$ be

$$b_\nu^{(1)} := \beta_\nu^{(1)} a^{(2)} - \frac{w_2}{w_1} \sum_{\rho=1}^{r_2-k_2} (\xi_\rho^{(2)} \cdot \beta_\nu^{(1)}) b_\rho^{(2)}. \quad (4.22)$$

We have:

- $b_\nu^{(1)} \in A_{\lambda=-w_1}$;
- $\xi_\mu^{(1)} \cdot b_\nu^{(1)} = \xi_\mu^{(1)} \cdot (\beta_\nu^{(1)} a^{(2)}) \in \prod_{i=1}^2 \text{Fit}_{k_i}(\phi_i)$ for $1 \leq \mu \leq r_1$ and $1 \leq \nu \leq r_1 - k_1$;
- In particular, $\xi_\mu^{(1)} \cdot b_\nu^{(1)} = \delta_{\mu\nu} a$ for $1 \leq \mu, \nu \leq r_1 - k_1$.

If we can prove $b_\nu^{(1)} \in J$, then $d(\frac{b_\nu^{(1)}}{a}) \in \Omega_{\mathcal{O}(\tilde{Y}_a)/\mathbb{k}}$, and the matrix representing $\tilde{\phi}_{1,a}$ on $d(\frac{b_1^{(1)}}{a}), \dots, d(\frac{b_{r_1-k_1}^{(1)}}{a})$ is

$$\tilde{\phi}_{1,a}\left(d\left(\frac{b_1^{(1)}}{a}\right), \dots, d\left(\frac{b_{r_1-k_1}^{(1)}}{a}\right)\right) = (u_1^{(1)} \otimes 1, \dots, u_{r_1}^{(1)} \otimes 1) \begin{pmatrix} I_{r_1-k_1} \\ M' \end{pmatrix} \quad (4.23)$$

where $M' \in \mathcal{O}(\tilde{Y}_a)^{k_1 \times (r_1-k_1)}$. Then $1 \in \text{Fit}_{k_1}(\tilde{\phi}_{1,a})$, i.e. $\text{Fit}_{k_1}(\tilde{\phi}_{1,a}) = \mathcal{O}(\tilde{Y}_a)$.

The final step is to verify $b_\nu^{(1)} \in J$. By Corollary 4.5.4 in Appendix 4.5.2, it suffices to check:

- $\xi_\mu^{(1)} \cdot b_\nu^{(1)} \in \prod_{i=1}^2 \text{Fit}_{k_i}(\phi_i)$ for $1 \leq \mu \leq r_1$, which is seen above;
- if $w_1/w_2 \in \mathbb{N}_+$, then $\xi_{\mu_1}^{(2)} \cdots \xi_{\mu_m}^{(2)} \cdot b_\nu^{(1)} \in \prod_{i=1}^2 \text{Fit}_{k_i}(\phi_i)$ for all $1 \leq \mu_1, \dots, \mu_m \leq r_1$.

We have $m \geq 2$ since $w_1 > w_2$. When $m = 2$, we have

$$\begin{aligned} \xi_{\mu_1}^{(2)} \xi_{\mu_2}^{(2)} \cdot b_\nu^{(1)} &= (\xi_{\mu_1}^{(2)} \xi_{\mu_2}^{(2)} \cdot \beta_\nu^{(1)}) a^{(2)} \\ &\quad - \frac{1}{2} \sum_{\rho=1}^{r_2-k_2} \left((\xi_{\mu_1}^{(2)} \xi_{\mu_2}^{(2)} \cdot \beta_\nu^{(1)}) (\xi_{\mu_2}^{(2)} \cdot b_\rho^{(2)}) + (\xi_{\mu_2}^{(2)} \xi_{\mu_1}^{(2)} \cdot \beta_\nu^{(1)}) (\xi_{\mu_1}^{(2)} \cdot b_\rho^{(2)}) \right) \end{aligned} \quad (4.24)$$

where we have used that $\xi_{\mu_1}^{(2)} \xi_{\mu_2}^{(2)} \cdot (\xi_\rho^{(2)} \cdot \beta_\nu^{(1)}) = 0$ and $\xi_{\mu_1}^{(2)} \xi_{\mu_2}^{(2)} \cdot b_\rho^{(2)} = 0$.

Apply equation (4.18) to $(\xi_{\mu_1}^{(2)} \xi_{\mu_2}^{(2)} \cdot \beta_\nu^{(1)}) a^{(2)}$

$$(\xi_{\mu_1}^{(2)} \xi_{\mu_2}^{(2)} \cdot \beta_\nu^{(1)}) a^{(2)} = \sum_{\rho=1}^{r_2-k_2} (\xi_{\mu_1}^{(2)} \cdot b_\rho^{(2)}) (\xi_\rho^{(2)} \xi_{\mu_2}^{(2)} \cdot \beta_\nu^{(1)}). \quad (4.25)$$

Then

$$\begin{aligned} \xi_{\mu_1}^{(2)} \xi_{\mu_2}^{(2)} \cdot b_\nu^{(1)} &= \frac{1}{2} \sum_{\rho=1}^{r_2-k_2} ([\xi_\rho^{(2)}, \xi_{\mu_2}^{(2)}] \cdot \beta_\nu^{(1)}) (\xi_{\mu_1}^{(2)} \cdot b_\rho^{(2)}) \\ &\quad + \frac{1}{2} \sum_{\rho=1}^{r_2-k_2} (\xi_\rho^{(2)} \xi_{\mu_2}^{(2)} \cdot \beta_\nu^{(1)}) (\xi_{\mu_1}^{(2)} \cdot b_\rho^{(2)}) - \frac{1}{2} \sum_{\rho=1}^{r_2-k_2} (\xi_{\mu_1}^{(2)} \xi_\rho^{(2)} \cdot \beta_\nu^{(1)}) (\xi_{\mu_2}^{(2)} \cdot b_\rho^{(2)}). \end{aligned} \quad (4.26)$$

We can exchange μ_1 and μ_2 to get a similar expression for $\xi_{\mu_2}^{(2)} \xi_{\mu_1}^{(2)} \cdot b_\nu^{(1)}$. Then

$$\begin{aligned} 2\xi_{\mu_1}^{(2)} \xi_{\mu_2}^{(2)} \cdot b_\nu^{(1)} &= [\xi_{\mu_1}^{(2)}, \xi_{\mu_2}^{(2)}] \cdot b_\nu^{(1)} + (\xi_{\mu_1}^{(1)} \xi_{\mu_2}^{(1)} + \xi_{\mu_2}^{(1)} \xi_{\mu_1}^{(1)}) \cdot b_\nu^{(1)} \\ &= [\xi_{\mu_1}^{(2)}, \xi_{\mu_2}^{(2)}] \cdot b_\nu^{(1)} \\ &\quad + \sum_{\rho=1}^{r_2-k_2} \left(([\xi_\rho^{(2)}, \xi_{\mu_2}^{(2)}] \cdot \beta_\nu^{(1)}) (\xi_{\mu_1}^{(2)} \cdot b_\rho^{(2)}) + ([\xi_\rho^{(2)}, \xi_{\mu_1}^{(2)}] \cdot \beta_\nu^{(1)}) (\xi_{\mu_2}^{(2)} \cdot b_\rho^{(2)}) \right). \end{aligned} \quad (4.27)$$

Since $w_1 = 2w_2$, we have $[\xi_\rho^{(2)}, \xi_{\mu_1}^{(2)}] \in \mathfrak{u}_{\lambda=w_1}$. We see that $\xi_{\mu_1}^{(2)} \xi_{\mu_2}^{(2)} b_\nu^{(1)} \in \prod_{i=1}^2 \text{Fit}_{k_i}(\phi_i)$. This proves $b_\nu^{(1)} \in J$ when $w_1 = 2w_2$.

When $w_1/w_2 = m \geq 3$, we have that $\mathfrak{u}$ is Abelian, and apply equation (4.18) to get

$$\begin{aligned}
\xi_{\mu_1}^{(2)} \cdots \xi_{\mu_m}^{(2)} b_\nu^{(1)} &= (\xi_{\mu_1}^{(2)} \cdots \xi_{\mu_m}^{(2)} \cdot \beta_\nu^{(1)}) a^{(2)} \\
&\quad - \frac{1}{m} \sum_{\rho=1}^{r_2-k_2} \sum_{j=1}^m (\xi_{\mu_1}^{(2)} \cdots \widehat{\xi_{\mu_j}^{(2)}} \cdots \xi_{\mu_m}^{(2)} \xi_\rho^{(2)} \cdot \beta_\nu^{(1)}) (\xi_{\mu_j}^{(2)} \cdot b_\rho^{(2)}) \\
&= \frac{1}{m} \sum_{j=1}^m (\xi_{\mu_j}^{(2)} \xi_{\mu_1}^{(2)} \cdots \widehat{\xi_{\mu_j}^{(2)}} \cdots \xi_{\mu_m}^{(2)} \cdot \beta_\nu^{(1)}) a^{(2)} \\
&\quad - \frac{1}{m} \sum_{\rho=1}^{r_2-k_2} \sum_{j=1}^m (\xi_{\mu_1}^{(2)} \cdots \widehat{\xi_{\mu_j}^{(2)}} \cdots \xi_{\mu_m}^{(2)} \xi_\rho^{(2)} \cdot \beta_\nu^{(1)}) (\xi_{\mu_j}^{(2)} \cdot b_\rho^{(2)}) \\
&= \frac{1}{m} \sum_{j=1}^m \sum_{\rho=1}^{r_2-k_2} (\xi_\rho^{(2)} \xi_{\mu_1}^{(2)} \cdots \widehat{\xi_{\mu_j}^{(2)}} \cdots \xi_{\mu_m}^{(2)} \cdot \beta_\nu^{(1)}) (\xi_{\mu_j}^{(2)} \cdot b_\rho^{(2)}) \\
&\quad - \frac{1}{m} \sum_{\rho=1}^{r_2-k_2} \sum_{j=1}^m (\xi_{\mu_1}^{(2)} \cdots \widehat{\xi_{\mu_j}^{(2)}} \cdots \xi_{\mu_m}^{(2)} \xi_\rho^{(2)} \cdot \beta_\nu^{(1)}) (\xi_{\mu_j}^{(2)} \cdot b_\rho^{(2)}) \\
&= 0 \in \prod_{i=1}^2 \text{Fit}_{k_i}(\phi_i).
\end{aligned} \tag{4.28}$$

This proves $b_\nu^{(1)} \in J$.

In the case with 2 grading weights, the construction is no longer simple. In Lemma 4.3.3, the general construction of $b_\nu^{(i)}$ has to be formulated inductively, and equation (4.18) in the general situation is Lemma 4.3.2. The assumption $\prod_{i=0}^2 \text{Fit}_{k_i}(\phi_i) \not\subseteq A_{\lambda < 0}$ generalises to Condition 4.2.1.

4.2 The Weak Unipotent Upstairs condition

In [HJ21], the so-called *Weak Unipotent Upstairs Assumption* (Assumption 4.42, [HJ21]) was assumed for the sequence of blow-ups described in that paper. In this subsection, we describe a similar assumption for our blow-up procedure and call it the *Weak Unipotent Upstairs Condition* as well.

Recall $\hat{U} = U \rtimes_\lambda \mathbb{G}_m$ for a grading 1PS $\lambda : \mathbb{G}_m \rightarrow \text{Aut}(U)$ with grading weights $w_1 > \cdots > w_n$. For a linear action $\hat{U} \curvearrowright (X, L)$, the open subscheme $X_{\min}^0 \subseteq X$ is defined

as the non-vanishing locus of $\bigoplus_{d>0} H^0(X, L^d)_{\lambda=\max}$. Recall the notation

$$\begin{aligned}\mathcal{K}_i &:= \ker \left(\Omega_{X_{\min}^0/\mathbb{k}} \rightarrow (\mathbf{u}_i)^* \otimes_{\mathbb{k}} \mathcal{O}_{X_{\min}^0} \right) \\ \phi_i : \mathcal{K}_{i-1} &\rightarrow (\mathbf{u}_i/\mathbf{u}_{i-1})^* \otimes_{\mathbb{k}} \mathcal{O}_{X_{\min}^0} \\ \text{Fit}_k(\phi_i) &:= \text{Fit}_k(\text{coker}(\phi_i)).\end{aligned}\tag{4.29}$$

We call the following condition *Weak Unipotent Upstairs* (WUU). The name comes from Assumption 4.42, [HJ21].

Condition 4.2.1 (the Weak Unipotent Upstairs Condition). Let $\hat{U} \curvearrowright (X, L)$, $\{\mathbf{u}_i\}_{i=0}^n$ and ϕ_i be as above. Let $k_i := \min\{k \in \mathbb{N} : \text{Fit}_k(\phi_i) \neq 0\}$. This condition for the action $\hat{U} \curvearrowright (X, L)$ assumes that there exists a closed point $z \in Z_{\min}$ such that $\dim \text{Stab}_{\mathbf{u}_i}(z) = k_1 + \dots + k_i$ for all $1 \leq i \leq n$.

Remark 4.2.2. Condition 3.3.2 implies Condition 4.2.1 by Corollary 3.1.9.

Lemma 4.2.3. Let $\hat{U} \curvearrowright (X, L)$ with the grading 1PS $\lambda : \mathbb{G}_m \rightarrow \text{Aut}(U)$ with grading weights $w_1 > \dots > w_n$. Let $\{\mathbf{u}_i\}_{i=0}^n$ be the filtration of $\mathbf{u}$ such that $\mathbf{u}_i := \mathbf{u}_{\lambda \geq w_i}$. Let $k_i := \min\{k \in \mathbb{N} : \text{Fit}_k(\phi_i) \neq 0\}$ for $1 \leq i \leq n$. Then for each $1 \leq i \leq n$, the following subsets of closed points of $X_{\min}^0$ are the same:

$$\begin{aligned}S_i^{(1)} &:= \left\{ x \in X_{\min}^0 : x \text{ is a closed point in the non-vanishing locus of } \prod_{i'=1}^i \text{Fit}_{k_{i'}}(\phi_{i'}) \right\} \\ S_i^{(2)} &:= \left\{ x \in X_{\min}^0 : \begin{array}{l} x \text{ is a closed point and for } i' = 1, \dots, i \\ \dim \text{Stab}_{\mathbf{u}_{i'}}(x) \leq k_1 + \dots + k_{i'} \end{array} \right\} \\ S_i^{(3)} &:= \left\{ x \in X_{\min}^0 : \begin{array}{l} x \text{ is a closed point and for } i' = 1, \dots, i \\ \dim \text{Stab}_{\mathbf{u}_{i'}}(x) = k_1 + \dots + k_{i'} \end{array} \right\}\end{aligned}\tag{4.30}$$

Moreover the subsets are all non-empty if $X_{\min}^0$ is integral, i.e. reduced and irreducible.

Proof. If $X_{\min}^0$ is reduced, then the non-vanishing locus of $\text{Fit}_{k_i}(\phi_i)$ is a non-empty open subset for each $1 \leq i \leq n$. Finite intersections of non-empty open subsets are non-empty if $X_{\min}^0$ is irreducible. This proves $S_i^{(1)} \neq \emptyset$ for all $1 \leq i \leq n$ if $X_{\min}^0$ is integral.

For every non-nilpotent $f \in H^0(X, L^m)_{\lambda=\max}$, we will prove $S_i^{(1)} \cap X_f = S_i^{(2)} \cap X_f = S_i^{(3)} \cap X_f$ by induction on $i = 1, \dots, n$.

Since X is of finite type over $\mathbb{k}$, open subsets are determined uniquely by their closed points ([Sta22, Lemma 02J6], [Sta22, Lemma 005Z]). We will also denote the non-vanishing locus of $\prod_{i'=1}^i \text{Fit}_{k_{i'}}(\phi_{i'})$ by $S_i^{(1)}$, which is open in $X_{\min}^0$.

Initially consider when $i = 1$. Let $x : \text{Spec}(\mathbb{k}) \rightarrow X_f$ be a closed point in $X_f \subseteq X_{\min}^0$ and let $x^\# : \mathcal{O}(X_f) \rightarrow \mathbb{k}$ be the associated ring map. Then $x \notin S_1^{(1)}$ if and only if $\text{Fit}_{k_1}(\phi_1)(X_f) \subseteq \ker x^\#$, which is equivalent to the condition that $\dim \text{Stab}_{u_1}(x) > k_1$ by Lemma 3.1.6, i.e. $x \notin S_1^{(2)}$. This proves $S_1^{(1)} \cap X_f = S_1^{(2)} \cap X_f$. Obviously $S_1^{(3)} \subseteq S_1^{(2)}$. Assume the closed point x is in $S_1^{(2)} \setminus S_1^{(3)}$. Let $l_1 := \dim \text{Stab}_{u_1}(x) < k_1$. Then $\text{Fit}_{l_1}(\phi_1) = 0$ since k_1 is the smallest integer such that $\text{Fit}_k(\phi_1) \neq 0$. We have $\text{Fit}_{l_1}(\phi_1)(X_f) \subseteq \ker x^\#$ and then $l_1 = \dim \text{Stab}_{u_1}(x) > l_1$ by Lemma 3.1.6, which is a contradiction. This proves $S_1^{(2)} \cap X_f = S_1^{(3)} \cap X_f$, so $S_1^{(1)} \cap X_f = S_1^{(2)} \cap X_f = S_1^{(3)} \cap X_f$.

Assume $S_i^{(1)} \cap X_f = S_i^{(2)} \cap X_f = S_i^{(3)} \cap X_f$ for some $1 \leq i < n$. If $S_i^{(1)} \cap X_f = S_i^{(2)} \cap X_f = S_i^{(3)} \cap X_f = \emptyset$, then $S_{i+1}^{(1)} \cap X_f = S_{i+1}^{(2)} \cap X_f = S_{i+1}^{(3)} \cap X_f = \emptyset$. Without loss of generality, we assume $S_i^{(1)} \cap X_f \neq \emptyset$. We have that $\text{coker}(\phi_{i'})|_{S_i^{(1)} \cap X_f}$ is locally free of rank $k_{i'}$ for each $1 \leq i' \leq i$ by Proposition 3.1.3. By Corollary 3.1.9, we have that $Q(u_{i'})|_{S_i^{(1)} \cap X_f}$ is locally free of rank $k_1 + \dots + k_{i'}$ for each $1 \leq i' \leq i$, where $Q(u_{i'})$ is the cokernel of $\Omega_{X_{\min}^0/\mathbb{k}} \rightarrow (u_i)^* \otimes_{\mathbb{k}} \mathcal{O}_{X_{\min}^0}$. In particular $Q(u_i)|_{S_i^{(1)} \cap X_f}$ is flat. Let $x : \text{Spec}(\mathbb{k}) \rightarrow X_f$ be a closed point. If $x \in S_i^{(1)}$, then the condition that $\dim \text{Stab}_{(u_{i+1}:u_i)}(x) \leq k_{i+1}$ is equivalent to the condition that $\text{Fit}_{k_{i+1}}(\phi_{i+1})(X_f) \not\subseteq \ker x^\#$ by Lemma 3.1.6. Then we have

$$\begin{aligned}
x \in S_{i+1}^{(2)} \quad & \text{if and only if} \quad \begin{cases} x \in S_i^{(1)} = S_i^{(2)} = S_i^{(3)} \\ \dim \text{Stab}_{u_{i+1}}(x) \leq k_1 + \dots + k_i + k_{i+1} \end{cases} \\
& \text{if and only if} \quad \begin{cases} x \in S_i^{(1)} = S_i^{(2)} = S_i^{(3)} \\ \dim \text{Stab}_{(u_{i+1}:u_i)}(x) \leq k_{i+1} \end{cases} \\
& \text{if and only if} \quad \begin{cases} x \in S_i^{(1)} = S_i^{(2)} = S_i^{(3)} \\ \text{Fit}_{k_{i+1}}(\phi_{i+1})(X_f) \not\subseteq \ker x^\# \end{cases} \\
& \text{if and only if} \quad x \in S_{i+1}^{(1)}.
\end{aligned} \tag{4.31}$$

This proves $S_{i+1}^{(1)} \cap X_f = S_{i+1}^{(2)} \cap X_f$.

Obviously we have $S_{i+1}^{(3)} \subseteq S_{i+1}^{(2)}$. Let $x : \text{Spec}(\mathbb{k}) \rightarrow X_f$ be a closed point in $S_{i+1}^{(2)} \setminus S_{i+1}^{(3)}$. Then $x \in S_i^{(3)}$ and $\dim \text{Stab}_{(u_{i+1}:u_i)}(x) < k_{i+1}$. As above, we have that $Q(u_i)|_{S_i^{(1)} \cap X_f}$ is flat and $x \in S_i^{(1)}$. Let $l_{i+1} := \dim \text{Stab}_{(u_{i+1}:u_i)}(x) < k_{i+1}$. We have that $\text{Fit}_{l_{i+1}}(\phi_{i+1}) = 0$ since

k_{i+1} is the smallest integer such that $\text{Fit}_k(\phi_{i+1}) \neq 0$. In particular $\text{Fit}_{l_{i+1}}(\phi_{i+1})(X_f) \subseteq \ker x^\#$. Then $\dim \text{Stab}_{(u_{i+1}:u_i)}(x) > l_{i+1}$ by Lemma 3.1.6, which is a contradiction. This proves $S_{i+1}^{(2)} \cap X_f = S_{i+1}^{(3)} \cap X_f$, so $S_{i+1}^{(1)} \cap X_f = S_{i+1}^{(2)} \cap X_f = S_{i+1}^{(3)} \cap X_f$. $\square$

4.3 Centre of the blow-up

Assume Condition 4.2.1. Let $\mathcal{I}_{k_1, \dots, k_n} := \prod_{i=1}^n \text{Fit}_{k_i}(\phi_i) \subseteq \mathcal{O}_{X_{\min}^0}$. By Lemma 4.2.3, Condition 4.2.1 is equivalent to the condition that $Z_{\min} \not\subseteq \mathbb{V}(\mathcal{I}_{k_1, \dots, k_n})$. The blow-up centre $\overline{C} \hookrightarrow X$ is a closed subscheme supported on the closure of the U -sweep of $Z_{\min} \cap \mathbb{V}(\mathcal{I}_{k_1, \dots, k_n})$. In this subsection $\overline{C} \hookrightarrow X$ will be defined and $C = \overline{C} \cap X_{\min}^0$, its intersection with the open subscheme $X_{\min}^0 \subseteq X$ will be studied locally.

Let $m \in \mathbb{N}_+$ be such that $H^0(X, L^m)$ generates $\bigoplus_{d \geq 0} H^0(X, L^{md})$. Let $f \in H^0(X, L^m)$ be a non-nilpotent section of maximal λ -weight. Then $X_f \subseteq X_{\min}^0$ is an affine open subscheme. On X_f , let $I_f := \mathcal{I}_{k_1, \dots, k_n}(X_f) + \mathcal{O}(X_f)_{\lambda < 0}$. Then $I_f \subseteq \mathcal{O}(X_f)$ cuts out $Z_{\min, f} \cap \mathbb{V}(\mathcal{I}_{k_1, \dots, k_n})$. If $Z_{\min, f} \not\subseteq \mathbb{V}(\mathcal{I}_{k_1, \dots, k_n})$, then $\mathcal{O}(X_f)_{\lambda < 0} \subsetneq I_f$, i.e. $(I_f)_{\lambda = \max} = (I_f)_{\lambda = 0}$. Else $Z_{\min, f} \subseteq \mathbb{V}(\mathcal{I}_{k_1, \dots, k_n})$, and then $I_f = \mathcal{O}(X_f)_{\lambda < 0}$. Let $J_f \subseteq \mathcal{O}(X_f)$ be the kernel of the composition

$$\mathcal{O}(X_f) \rightarrow \mathcal{O}(U) \otimes \mathcal{O}(X_f) \rightarrow \mathcal{O}(U) \otimes (\mathcal{O}(X_f)/I_f) \quad (4.32)$$

where $\mathcal{O}(X_f) \rightarrow \mathcal{O}(U) \otimes \mathcal{O}(X_f)$ is the co-action map of $U \curvearrowright X_f$. Let $C_f \hookrightarrow X_f$ be the closed subscheme associated to the ideal $J_f \subseteq \mathcal{O}(X_f)$. The ideals $J_f \subseteq \mathcal{O}(X_f)$ for $f \in H^0(X, L^m)_{\lambda = \max}$ glue to a sheaf of ideals $\mathcal{J} \subseteq \mathcal{O}_{X_{\min}^0}$. The closed subschemes $C_f \hookrightarrow X_f$ glue to a closed subscheme $C \hookrightarrow X_{\min}^0$, which is the closed subscheme associated to $\mathcal{J}$. Let $\overline{C} \hookrightarrow X$ be the scheme theoretic image of $C \hookrightarrow X_{\min}^0 \subseteq X$, which is the blow-up centre.

Remark 4.3.1. Let $Z \hookrightarrow X_{\min}^0$ be the scheme theoretic intersection of $Z_{\min} \hookrightarrow X_{\min}^0$ and $\mathbb{V}(\mathcal{I}_{k_1, \dots, k_n}) \hookrightarrow X_{\min}^0$, where the subscheme structure on $\mathbb{V}(\mathcal{I}_{k_1, \dots, k_n})$ is associated to $\mathcal{I}_{k_1, \dots, k_n}$. Consider the morphism $U \times Z \rightarrow X_{\min}^0$, such that it sends a $\mathbb{k}$ -point (u, z) to $u \cdot z \in X_{\min}^0$. Then $C \hookrightarrow X_{\min}^0$ is the scheme theoretic image of this morphism. By [Sta22, Lemma 01R8], the closed subset of C is the closure of the image, i.e. the U -sweep of Z . Also by [Sta22, Lemma 01R8], scheme theoretic images can be constructed locally. Then the scheme theoretic image of $U \times Z_f \rightarrow X_f$ is $C_f \hookrightarrow X_f$.

Let $\pi : \tilde{X} \rightarrow X$ be the blow-up of X along $\overline{C} \hookrightarrow X$. Let $E \hookrightarrow X$ be the exceptional divisor. For any $x, y \in \mathbb{N}_+$, the line bundle $\pi^* L^x \otimes \mathcal{O}(-yE)$ is ample on $\tilde{X}$. Naturally $\hat{U}$ acts on $\tilde{X}$ linearly with respect to $\pi^* L^x \otimes \mathcal{O}(-yE)$. Then $(\tilde{X})_{\min}^0$ is defined. Blow-ups are constructed locally. Define the *blowup algebra associated to* $J_f \subseteq \mathcal{O}(X_f)$

$$\mathrm{Bl}_{J_f}(\mathcal{O}(X_f)) := \bigoplus_{k \in \mathbb{N}} (J_f)^k \quad (4.33)$$

where $(J_f)^k$ has degree k . For a degree one element $a \in J_f$, define the *affine blowup algebra* as the homogeneous localisation at a

$$\mathcal{O}(X_f) \left[\frac{J_f}{a} \right] := \left(\mathrm{Bl}_{J_f}(\mathcal{O}(X_f)) \right)_{(a)}. \quad (4.34)$$

Let $\tilde{X}_f \rightarrow X_f$ be the blow-up of X_f along $C_f \hookrightarrow X_f$. It has an open affine covering

$$\tilde{X}_f = \bigcup_{a \in J_f} \tilde{X}_{f,a}, \quad \tilde{X}_{f,a} := \mathrm{Spec} \left(\mathcal{O}(X_f) \left[\frac{J_f}{a} \right] \right). \quad (4.35)$$

The open subscheme $(\tilde{X})_{\min}^0 \subseteq \tilde{X}$ is the union of the following affine open subschemes

$$(\tilde{X})_{\min}^0 = \bigcup_{\substack{f \in H^0(X, L^m)_{\lambda=\max} \\ a \in (J_f)_{\lambda=0}}} \tilde{X}_{f,a}. \quad (4.36)$$

The open subscheme $(\tilde{X})_{\min}^0 \subseteq \tilde{X}$ is covered by those $\tilde{X}_{f,a}$ such that $f \in H^0(X, L^m)_{\lambda=\max}$ with $Z_{\min, f} \not\subseteq \mathbb{V}(\mathcal{I}_{k_1, \dots, k_n})$ and $a = a^{(1)} \cdots a^{(n)} \in (J_f)_{\lambda=0}$ for $a^{(i)} \in \mathrm{Fit}_{k_i}(\phi_i)(X_f)_{\lambda=0}$. We focus on one such $\tilde{X}_{f,a}$. For each $1 \leq i \leq n$ choose a suitable basis $(\xi_1^{(i)}, \dots, \xi_{r_i}^{(i)})$ of $\mathfrak{u}_{\lambda=w_i}$. Assume without loss of generality that

$$a^{(i)} = \det \begin{pmatrix} \xi_1^{(i)} \cdot f_1^{(i)} & \cdots & \xi_1^{(i)} \cdot f_{r_i-k_i}^{(i)} \\ \vdots & \ddots & \vdots \\ \xi_{r_i-k_i}^{(i)} \cdot f_1^{(i)} & \cdots & \xi_{r_i-k_i}^{(i)} \cdot f_{r_i-k_i}^{(i)} \end{pmatrix} \in \mathrm{Fit}_{k_i}(\phi_i)(X_f)_{\lambda=0} \quad (4.37)$$

where $f_j^{(i)} \in \mathcal{O}(X_f)_{\lambda=-w_i}$.

Let $U(\mathfrak{u})$ denote the *universal enveloping algebra* of $\mathfrak{u}$. For $1 \leq i \leq n$ and $1 \leq \mu_i \leq r_i - k_i$, define a linear map

$$E_{\mu_i}^{(i)} : U(\mathfrak{u}) \rightarrow \mathcal{O}(X_f)$$

$$A \mapsto \det \begin{pmatrix} \xi_1^{(i)} \cdot f_1^{(i)} & \cdots & \xi_1^{(i)} \cdot f_{r_i-k_i}^{(i)} \\ \vdots & \cdots & \vdots \\ A \cdot f_1^{(i)} & \cdots & A \cdot f_{r_i-k_i}^{(i)} \\ \vdots & \cdots & \vdots \\ \xi_{r_i-k_i}^{(i)} \cdot f_1^{(i)} & \cdots & \xi_{r_i-k_i}^{(i)} \cdot f_{r_i-k_i}^{(i)} \end{pmatrix} \quad (4.38)$$

where $(A.f_1^{(i)}, \dots, A.f_{r_i-k_i}^{(i)})$ occupies the μ_i th row. For $A, B \in U(\mathfrak{u})$, we have $A.E_{\mu_i}^{(i)}(B) = E_{\mu_i}^{(i)}(AB)$, since all entries in the matrix above are in $\mathcal{O}(X_f)_{\lambda=0} \subseteq \mathcal{O}(X_f)^U$ except the μ_i th row.

Lemma 4.3.2. *Let $f, a^{(i)}, \xi_j^{(i)}$ be as above. Let $1 \leq i \leq n$ and $h \in \mathcal{O}(X_f)_{\lambda=-w_i}$ and $A \in \mathfrak{u}_{\lambda=w_i}$. Then we have the following equation*

$$\sum_{\mu_i=1}^{r_i-k_i} (\xi_{\mu_i}^{(i)} \cdot h) E_{\mu_i}^{(i)}(A) = (A.h) a^{(i)}. \quad (4.39)$$

Proof. Since $f_j^{(i)}, h \in \mathcal{O}(X_f)_{\lambda=-w_i}$, we have

$$df_1^{(i)}, \dots, df_{r_i-k_i}^{(i)}, dh \in \mathcal{K}_{i-1}(X_f). \quad (4.40)$$

Then the determinant of the following $(r_i - k_i + 1) \times (r_i - k_i + 1)$ matrix is in $\text{Fit}_{k_i-1}(\phi_i)$

$$M := \begin{pmatrix} \xi_1^{(i)} \cdot f_1^{(i)} & \cdots & \xi_1^{(i)} \cdot f_{r_i-k_i}^{(i)} & \xi_1^{(i)} \cdot h \\ \vdots & \ddots & \vdots & \vdots \\ \xi_{r_i-k_i}^{(i)} \cdot f_1^{(i)} & \cdots & \xi_{r_i-k_i}^{(i)} \cdot f_{r_i-k_i}^{(i)} & \xi_{r_i-k_i}^{(i)} \cdot h \\ A.f_1^{(i)} & \cdots & A.f_{r_i-k_i}^{(i)} & A.h \end{pmatrix} \quad (4.41)$$

Expand $\det(M)$ along the right column

$$\det(M) = (A.h) a^{(i)} - \sum_{\mu_i=1}^{r_i-k_i} (\xi_{\mu_i}^{(i)} \cdot h) E_{\mu_i}^{(i)}(A). \quad (4.42)$$

By the definition of k_i we have $\text{Fit}_{k_i-1}(\phi_i) = 0$, which implies that $\det(M) = 0$. This completes the proof. $\square$

Lemma 4.3.3. *Let $f, a^{(i)}, \xi_j^{(i)}, I_f, J_f$ be as above. Then for $1 \leq i \leq n$ and $1 \leq \mu \leq r_i - k_i$, there exists $b_\mu^{(i)} \in \mathcal{O}(X_f)_{\lambda=-w_i}$ satisfying:*

- (1) $\xi_\mu^{(i)} \cdot b_\nu^{(i)} = w_i \delta_{\mu\nu} \prod_{i'=i}^n a^{(i')}$ for $1 \leq i \leq n$ and $1 \leq \mu, \nu \leq r_i - k_i$;
- (2) $X.b_\mu^{(i)} \in \prod_{i'=i}^n \text{Fit}_{k_{i'}}(\phi_{i'})(X_f)_{\lambda=0}$ for all $1 \leq i \leq n$ and $X \in U(\mathfrak{u})_{\lambda=-w_i}$;
- (3) $\left(\prod_{i' < i} a^{(i')} \right) \cdot b_\mu^{(i)} \in J_f$ for $1 \leq i \leq n$ and $1 \leq \mu \leq r_i - k_i$.

Proof. If (2) is proved, then $X \cdot \left(\prod_{i' < i} a^{(i')} \cdot b_{\mu}^{(i)} \right) \in I_f$ for all $X \in U(\mathfrak{u})$. By Lemma 4.5.4 in Appendix 4.5.2, it follows that $\prod_{i' < i} a^{(i')} \cdot b_{\mu}^{(i)} \in J_f$, which is (3).

For $1 \leq \mu_n \leq r_n - k_n$, we regard $w_n \in U(\mathfrak{u})$ as a scalar and define

$$b_{\mu_n}^{(n)} := E_{\mu_n}^{(n)}(w_n). \quad (4.43)$$

We construct $b_{\mu_i}^{(i)}$ inductively for $i = n, \dots, 1$. Assume $b_{\mu_{i'}}^{(i')}$ is defined for $i < i' \leq n$ and $1 \leq \mu_{i'} \leq r_{i'} - k_{i'}$. Define $b_{\mu_i}^{(i)}$ as

$$b_{\mu_i}^{(i)} := E_{\mu_i}^{(i)}(w_i) \cdot \prod_{i'=i+1}^n a^{(i')} - \sum_{i'=i+1}^n \sum_{\mu_{i'}=1}^{r_{i'}-k_{i'}} E_{\mu_i}^{(i)}(\xi_{\mu_{i'}}^{(i')}) \cdot \prod_{i < i'' < i'} a^{(i'')} \cdot b_{\mu_{i'}}^{(i')}. \quad (4.44)$$

We see $b_{\mu_i}^{(i)} \in \mathcal{O}(X_f)_{\lambda=-w_i}$ and (1) is checked immediately.

By the Poincaré-Birkhoff-Witt theorem, the following elements form a basis of $U(\mathfrak{u})$

$$(\xi_1^{(1)})^{p_1^{(1)}} \dots (\xi_{r_1}^{(1)})^{p_{r_1}^{(1)}} \dots (\xi_1^{(n)})^{p_1^{(n)}} \dots (\xi_{r_n}^{(n)})^{p_{r_n}^{(n)}} \in U(\mathfrak{u}), \quad p_j^{(i)} \in \mathbb{N}. \quad (4.45)$$

For simplicity denote

$$\begin{aligned} \xi^p &:= (\xi_1^{(1)})^{p_1^{(1)}} \dots (\xi_{r_1}^{(1)})^{p_{r_1}^{(1)}} \dots (\xi_1^{(n)})^{p_1^{(n)}} \dots (\xi_{r_n}^{(n)})^{p_{r_n}^{(n)}}, & (\xi^{(i)})^{p^{(i)}} &:= (\xi_1^{(i)})^{p_1^{(i)}} \dots (\xi_{r_i}^{(i)})^{p_{r_i}^{(i)}} \\ p &:= (p^{(1)}, \dots, p^{(n)}), & p^{(i)} &:= (p_1^{(i)}, \dots, p_{r_i}^{(i)}) \in \mathbb{N}^{r_i}, & |p^{(i)}| &:= p_1^{(i)} + \dots + p_{r_i}^{(i)}. \end{aligned} \quad (4.46)$$

For (2) it suffices to consider $X = \xi^p \in U(\mathfrak{u})_{\lambda=-w_i}$.

We construct a family of functions $\beta_{\mu_i}^{(i)}(\xi^p) \in \mathcal{O}(X_f)_{\lambda=0}$ for $1 \leq i \leq n$ and $1 \leq \mu_i \leq r_i - k_i$ and $\xi^p \in U(\mathfrak{u})_{\lambda=w_i}$ inductively by

$$\begin{aligned} \beta_{\mu_n}^{(n)}(\xi^p) &:= w_{\max(p)} E_{\mu_n}^{(n)}([\xi^p]) \\ \beta_{\mu_i}^{(i)}(\xi^p) &:= w_{\max(p)} E_{\mu_i}^{(i)}([\xi^p]) \cdot \prod_{i'=i+1}^n a^{(i')} \\ &\quad - \sum_{i'=i+1}^n \sum_{\substack{0 \leq q \leq p \\ \text{weight}(\xi^q)=w_{i'}} \sum_{\mu_{i'}=1}^{r_{i'}-k_{i'}} \binom{p}{q} E_{\mu_i}^{(i)}([\xi^{p-q} \xi_{\mu_{i'}}^{(i')}]) \cdot \prod_{i < i'' < i'} a^{(i'')} \cdot \beta_{\mu_{i'}}^{(i')}(\xi^q) \end{aligned} \quad (4.47)$$

where $w_{\max(p)} := w_{\max\{j: p^{(j)} \neq 0\}}$ and $[-] : U(\mathfrak{u}) \rightarrow \mathfrak{u}$ is the complete bracket defined in Appendix 4.5.1, which respects λ -weights. It is easy to prove inductively that $\beta_{\mu_i}^{(i)}(\xi^p) \in \prod_{i'=i}^n \text{Fit}_{k_{i'}}(\phi_{i'})(X_f)_{\lambda=0}$. We then prove for $1 \leq i \leq n$ and $1 \leq \mu_i \leq r_i - k_i$ and $\xi^p \in U(\mathfrak{u})_{\lambda=w_i}$

$$\xi^p \cdot b_{\mu_i}^{(i)} = \beta_{\mu_i}^{(i)}(\xi^p). \quad (4.48)$$

Let $1 \leq i < n$, and assume (4.48) is proved for $i' > i$. Let $\xi^p \in U(\mathfrak{u})_{\lambda=w_i}$, i.e. $\sum_{i'=1}^n w_{i'} |p^{(i')}| = w_i$. We calculate $\xi^p \cdot b_{\mu_i}^{(i)}$

$$\begin{aligned} \xi^p \cdot b_{\mu_i}^{(i)} &= \xi^p \cdot \left(E_{\mu_i}^{(i)}(w_i) \cdot \prod_{i'=i+1}^n a^{(i')} - \sum_{i'=i+1}^n \sum_{\mu_{i'}=1}^{r_{i'}-k_{i'}} E_{\mu_i}^{(i)}(\xi_{\mu_{i'}}^{(i')}) \cdot \prod_{i < i'' < i'} a^{(i'')} \cdot b_{\mu_{i'}}^{(i')} \right) \\ &= E_{\mu_i}^{(i)}(w_i \xi^p) \cdot \prod_{i' > i} a^{(i')} - \xi^p \cdot \left(\sum_{i'=i+1}^n \sum_{\mu_{i'}=1}^{r_{i'}-k_{i'}} E_{\mu_i}^{(i)}(\xi_{\mu_{i'}}^{(i')}) \cdot \prod_{i < i'' < i'} a^{(i'')} \cdot b_{\mu_{i'}}^{(i')} \right) \end{aligned} \quad (4.49)$$

where we have used that $a^{(i)} \in \mathcal{O}(X_f)_{\lambda=0}$ is U -invariant, and $X \cdot E_{\mu_i}^{(i)}(Y) = E_{\mu_i}^{(i)}(XY)$ for all $X, Y \in U(\mathfrak{u})$. Write $\xi^p \cdot b_{\mu_i}^{(i)} = T_1 - T_2$ for

$$\begin{aligned} T_1 &:= E_{\mu_i}^{(i)}(w_i \xi^p) \cdot \prod_{i' > i} a^{(i')} \\ T_2 &:= \xi^p \cdot \left(\sum_{i'=i+1}^n \sum_{\mu_{i'}=1}^{r_{i'}-k_{i'}} E_{\mu_i}^{(i)}(\xi_{\mu_{i'}}^{(i')}) \cdot \prod_{i < i'' < i'} a^{(i'')} \cdot b_{\mu_{i'}}^{(i')} \right). \end{aligned} \quad (4.50)$$

For the term T_1 we have by Lemma 4.5.1

$$\begin{aligned} T_1 &\equiv E_{\mu_i}^{(i)}(w_i \xi^p) \cdot \prod_{i' > i} a^{(i')} \\ &= \sum_{0 < q \leq p} \binom{p}{q} w_{\max(q)} E_{\mu_i}^{(i)}([\xi^q] \xi^{p-q}) \cdot \prod_{i' > i} a^{(i')} \\ &= w_{\max(p)} E_{\mu_i}^{(i)}([\xi^p]) \cdot \prod_{i' > i} a^{(i')} + \sum_{0 < q < p} \binom{p}{q} w_{\max(q)} E_{\mu_i}^{(i)}([\xi^q] \xi^{p-q}) \cdot \prod_{i' > i} a^{(i')} \\ &= w_{\max(p)} E_{\mu_i}^{(i)}([\xi^p]) \cdot \prod_{i' > i} a^{(i')} \\ &\quad + \sum_{i'=i+1}^n \sum_{\substack{0 \leq q \leq p \\ \text{weight}(\xi^q)=w_{i'}}} \binom{p}{q} w_{\max(q)} E_{\mu_i}^{(i)}([\xi^q] \xi^{p-q}) \cdot \prod_{i'' > i} a^{(i'')} \\ &= T_{1,1} + T_{1,2} \end{aligned} \quad (4.51)$$

where

$$\begin{aligned} T_{1,1} &:= w_{\max(p)} E_{\mu_i}^{(i)}([\xi^p]) \cdot \prod_{i' > i} a^{(i')} \\ T_{1,2} &:= \sum_{\substack{i' > i \\ 0 \leq q \leq p \\ \text{weight}(\xi^q)=w_{i'}}} \binom{p}{q} w_{\max(q)} E_{\mu_i}^{(i)}([\xi^q] \xi^{p-q}) \cdot \prod_{i'' > i} a^{(i'')}. \end{aligned} \quad (4.52)$$

For the term T_2 we have

$$\begin{aligned}
T_2 &\equiv \xi^p \cdot \left(\sum_{i'=i+1}^n \sum_{\mu_{i'}=1}^{r_{i'}-k_{i'}} E_{\mu_i}^{(i)}(\xi_{\mu_{i'}}^{(i')}) \cdot \prod_{i < i'' < i'} a^{(i'')} \cdot b_{\mu_{i'}}^{(i')} \right) \\
&= \sum_{i'=i+1}^n \sum_{0 \leq q \leq p} \sum_{\mu_{i'}=1}^{r_{i'}-k_{i'}} \binom{p}{q} E_{\mu_i}^{(i)}(\xi_{\mu_{i'}}^{p-q} \xi_{\mu_{i'}}^{(i')}) \cdot \prod_{i < i'' < i'} a^{(i'')} \cdot \beta_{\mu_{i'}}^{(i')}(\xi^q) \\
&= \sum_{i'=i+1}^n \sum_{\substack{0 \leq q \leq p \\ \text{weight}(\xi^q)=w_{i'}}} \sum_{\mu_{i'}=1}^{r_{i'}-k_{i'}} \binom{p}{q} E_{\mu_i}^{(i)}(\xi_{\mu_{i'}}^{p-q} \xi_{\mu_{i'}}^{(i')}) \cdot \prod_{i < i'' < i'} a^{(i'')} \cdot \beta_{\mu_{i'}}^{(i')}(\xi^q) \\
&= \sum_{\substack{i' > i \\ 0 \leq q \leq p \\ \text{weight}(\xi^q)=w_{i'}}} \sum_{\mu_{i'}=1}^{r_{i'}-k_{i'}} \binom{p}{q} E_{\mu_i}^{(i)} \left(\sum_{0 \leq r \leq p-q} \binom{p-q}{r} [\xi_{\mu_{i'}}^{p-q-r} \xi_{\mu_{i'}}^{(i')}] \xi^r \right) \cdot \prod_{i < i'' < i'} a^{(i'')} \cdot \beta_{\mu_{i'}}^{(i')}(\xi^q) \\
&= \sum_{\substack{i' > i \\ 0 \leq q \leq p \\ \text{weight}(\xi^q)=w_{i'}}} \sum_{\mu_{i'}=1}^{r_{i'}-k_{i'}} \binom{p}{q} E_{\mu_i}^{(i)}([\xi_{\mu_{i'}}^{p-q} \xi_{\mu_{i'}}^{(i')}]) \cdot \prod_{i < i'' < i'} a^{(i'')} \cdot \beta_{\mu_{i'}}^{(i')}(\xi^q) \\
&\quad + \sum_{\substack{i' > i \\ 0 \leq q \leq p \\ \text{weight}(\xi^q)=w_{i'}}} \sum_{\mu_{i'}=1}^{r_{i'}-k_{i'}} \binom{p}{q} E_{\mu_i}^{(i)}(\xi_{\mu_{i'}}^{(i')} \xi^{p-q}) \cdot \prod_{i < i'' < i'} a^{(i'')} \cdot \beta_{\mu_{i'}}^{(i')}(\xi^q) \\
&\quad + \sum_{\substack{i' > i \\ 0 \leq q \leq p \\ \text{weight}(\xi^q)=w_{i'}}} \sum_{\mu_{i'}=1}^{r_{i'}-k_{i'}} \binom{p}{q} E_{\mu_i}^{(i)} \left(\sum_{0 < r < p-q} \binom{p-q}{r} [\xi_{\mu_{i'}}^{p-q-r} \xi_{\mu_{i'}}^{(i')}] \xi^r \right) \cdot \prod_{i < i'' < i'} a^{(i'')} \cdot \beta_{\mu_{i'}}^{(i')}(\xi^q)
\end{aligned} \tag{4.53}$$

where we have used:

- the generalised Leibniz rule

$$\xi^p \cdot (fg) = \sum_{0 \leq q \leq p} \binom{p}{q} (\xi^{p-q} \cdot f)(\xi^q \cdot g); \tag{4.54}$$

- the induction hypothesis for $i' > i$

$$\xi^q \cdot b_{\mu_{i'}}^{(i')} = \beta_{\mu_{i'}}^{(i')}(\xi^q); \tag{4.55}$$

- if $\text{weight}(\xi^q) > w_{i'}$, then $\xi^q \cdot b_{\mu_{i'}}^{(i')} = 0$, since $b_{\mu_{i'}}^{(i')} \in \mathcal{O}(X_f)_{\lambda=-w_{i'}}$;
- if $\text{weight}(\xi^q) < w_{i'}$, then $\text{weight}(\xi^{p-q}) > w_i - w_{i'}$, and then $E_{\mu_i}^{(i)}(\xi_{\mu_{i'}}^{p-q} \xi_{\mu_{i'}}^{(i')}) = 0$, since $E_{\mu_i}^{(i)}(\xi_{\mu_{i'}}^{(i')}) \in \mathcal{O}(X_f)_{-w_i+w_{i'}}$;

- the identity in Lemma 4.5.2

$$\xi^{p-q}\xi_{\mu_{i'}}^{(i')} = \sum_{0 \leq r \leq p-q} \binom{p-q}{r} [\xi^{p-q-r}\xi_{\mu_{i'}}^{(i')}] \xi^r. \quad (4.56)$$

We write $T_2 = T_{2,1} + T_{2,2} + T_{2,3}$ for

$$\begin{aligned} T_{2,1} &:= \sum_{\substack{i' > i \\ 0 \leq q \leq p \\ \text{weight}(\xi^q) = w_{i'}} \sum_{\mu_{i'}=1}^{r_{i'}-k_{i'}} \binom{p}{q} E_{\mu_i}^{(i)}([\xi^{p-q}\xi_{\mu_{i'}}^{(i')}] \cdot \prod_{i < i'' < i'} a^{(i'')} \cdot \beta_{\mu_{i'}}^{(i')}(\xi^q) \\ T_{2,2} &:= \sum_{\substack{i' > i \\ 0 \leq q \leq p \\ \text{weight}(\xi^q) = w_{i'}} \sum_{\mu_{i'}=1}^{r_{i'}-k_{i'}} \binom{p}{q} E_{\mu_i}^{(i)}(\xi_{\mu_{i'}}^{(i')}\xi^{p-q}) \cdot \prod_{i < i'' < i'} a^{(i'')} \cdot \beta_{\mu_{i'}}^{(i')}(\xi^q) \\ T_{2,3} &:= \sum_{\substack{i' > i \\ 0 \leq q \leq p \\ \text{weight}(\xi^q) = w_{i'}} \sum_{\mu_{i'}=1}^{r_{i'}-k_{i'}} \binom{p}{q} E_{\mu_i}^{(i)} \left(\sum_{0 < r < p-q} \binom{p-q}{r} [\xi^{p-q-r}\xi_{\mu_{i'}}^{(i')}] \xi^r \right) \cdot \prod_{i < i'' < i'} a^{(i'')} \cdot \beta_{\mu_{i'}}^{(i')}(\xi^q). \end{aligned} \quad (4.57)$$

Observe that $\beta_{\mu_i}^{(i)}(\xi^p) = T_{1,1} - T_{2,1}$. Then (4.48) is equivalent to $T_{1,2} - T_{2,2} - T_{2,3} = 0$.

For $T_{2,3}$ by a change of indices we have

$$\begin{aligned} &T_{2,3} \\ &\equiv \sum_{\substack{i' > i \\ 0 \leq q \leq p \\ \text{weight}(\xi^q) = w_{i'}} \sum_{\mu_{i'}=1}^{r_{i'}-k_{i'}} \binom{p}{q} E_{\mu_i}^{(i)} \left(\sum_{0 < r < p-q} \binom{p-q}{r} [\xi^{p-q-r}\xi_{\mu_{i'}}^{(i')}] \xi^r \right) \cdot \prod_{i < i'' < i'} a^{(i'')} \cdot \beta_{\mu_{i'}}^{(i')}(\xi^q) \\ &= \sum_{\substack{i' > i \\ 0 \leq q \leq p \\ \text{weight}(\xi^q) = w_{i'}} \sum_{q < s < p} \sum_{\mu_{i'}=1}^{r_{i'}-k_{i'}} \binom{p}{q} \binom{p-q}{p-s} E_{\mu_i}^{(i)}([\xi^{s-q}\xi_{\mu_{i'}}^{(i')}] \xi^{p-s}) \cdot \prod_{i < i'' < i'} a^{(i'')} \cdot \beta_{\mu_{i'}}^{(i')}(\xi^q) \\ &= \sum_{\substack{i' > i \\ 0 \leq q \leq p \\ \text{weight}(\xi^q) = w_{i'}} \sum_{\substack{i < i'' < i' \\ q \leq s \leq p \\ \text{weight}(\xi^s) = w_{i''}}} \sum_{\mu_{i'}=1}^{r_{i'}-k_{i'}} \binom{p}{q} \binom{p-q}{p-s} E_{\mu_i}^{(i)}([\xi^{s-q}\xi_{\mu_{i'}}^{(i')}] \xi^{p-s}) \cdot \prod_{i < i'' < i'} a^{(i'')} \cdot \beta_{\mu_{i'}}^{(i')}(\xi^q) \end{aligned} \quad (4.58)$$

$$\begin{aligned}
&= \sum_{\substack{i'' > i \\ 0 \leq s \leq p \\ \text{weight}(\xi^s) = w_{i''}}} \sum_{\substack{i' > i'' \\ 0 \leq q \leq s \\ \text{weight}(\xi^q) = w_{i'}} \sum_{\mu_{i'}=1}^{r_{i'}-k_{i'}} \binom{p}{q} \binom{p-q}{p-s} E_{\mu_i}^{(i)}([\xi^{s-q} \xi_{\mu_{i'}}^{(i')}] \xi^{p-s}) \cdot \prod_{i < i'' < i'} a^{(i'')} \cdot \beta_{\mu_{i'}}^{(i')}(\xi^q) \\
&= \sum_{\substack{i' > i \\ 0 \leq q \leq p \\ \text{weight}(\xi^q) = w_{i'}} \sum_{\substack{i'' > i' \\ 0 \leq r \leq q \\ \text{weight}(\xi^r) = w_{i''}}} \sum_{\mu_{i''}=1}^{r_{i''}-k_{i''}} \binom{p}{q} \binom{q}{r} E_{\mu_i}^{(i)}([\xi^{q-r} \xi_{\mu_{i''}}^{(i'')}] \xi^{p-q}) \cdot \prod_{i < i''' < i''} a^{(i''')} \cdot \beta_{\mu_{i''}}^{(i'')}(\xi^r).
\end{aligned} \tag{4.59}$$

For $T_{2,2}$ by the inductive definition of $\beta_{\mu_{i'}}^{(i')}(\xi^q)$, we have

$$\begin{aligned}
&T_{2,2} \\
&\equiv \sum_{\substack{i' > i \\ 0 \leq q \leq p \\ \text{weight}(\xi^q) = w_{i'}} \sum_{\mu_{i'}=1}^{r_{i'}-k_{i'}} \binom{p}{q} E_{\mu_i}^{(i)}(\xi_{\mu_{i'}}^{(i')} \xi^{p-q}) \cdot \prod_{i < i'' < i'} a^{(i'')} \cdot \beta_{\mu_{i'}}^{(i')}(\xi^q) \\
&= \sum_{\substack{i' > i \\ 0 \leq q \leq p \\ \text{weight}(\xi^q) = w_{i'}} \sum_{\mu_{i'}=1}^{r_{i'}-k_{i'}} \binom{p}{q} E_{\mu_i}^{(i)}(\xi_{\mu_{i'}}^{(i')} \xi^{p-q}) \cdot \prod_{i < i'' < i'} a^{(i'')} \\
&\quad \times \left(- \sum_{i'' > i'} \sum_{\substack{0 \leq r \leq q \\ \text{weight}(\xi^r) = w_{i''}}} \sum_{\mu_{i''}=1}^{r_{i''}-k_{i''}} \binom{q}{r} E_{\mu_{i'}}^{(i')}([\xi^{q-r} \xi_{\mu_{i''}}^{(i'')}] \cdot \prod_{i' < i''' < i''} a^{(i''')} \cdot \beta_{\mu_{i''}}^{(i'')}(\xi^r) \right) \\
&= \sum_{\substack{i' > i \\ 0 \leq q \leq p \\ \text{weight}(\xi^q) = w_{i'}} \sum_{\mu_{i'}=1}^{r_{i'}-k_{i'}} \binom{p}{q} w_{\max(q)} E_{\mu_i}^{(i)}(\xi_{\mu_{i'}}^{(i')} \xi^{p-q}) E_{\mu_{i'}}^{(i')}([\xi^q]) \cdot \prod_{\substack{i'' > i \\ i'' \neq i'}} a^{(i'')} \\
&\quad - \sum_{\substack{i' > i \\ 0 \leq q \leq p \\ \text{weight}(\xi^q) = w_{i'}} \sum_{\mu_{i'}=1}^{r_{i'}-k_{i'}} \sum_{\substack{i'' > i' \\ 0 \leq r \leq q \\ \text{weight}(\xi^r) = w_{i''}}} \sum_{\mu_{i''}=1}^{r_{i''}-k_{i''}} \binom{p}{q} \binom{q}{r} E_{\mu_i}^{(i)}(\xi_{\mu_{i'}}^{(i')} \xi^{p-q}) E_{\mu_{i'}}^{(i')}([\xi^{q-r} \xi_{\mu_{i''}}^{(i'')}] \cdot \prod_{\substack{i' < i''' < i'' \\ i''' \neq i'}} a^{(i''')} \cdot \beta_{\mu_{i''}}^{(i'')}(\xi^r) \Bigg).
\end{aligned} \tag{4.60}$$

By Lemma 4.3.2, we have the following equations

$$\begin{aligned}
&\sum_{\mu_{i'}=1}^{r_{i'}-k_{i'}} E_{\mu_i}^{(i)}(\xi_{\mu_{i'}}^{(i')} \xi^{p-q}) E_{\mu_{i'}}^{(i')}([\xi^q]) = E_{\mu_i}^{(i)}([\xi^q] \xi^{p-q}) a^{(i')} \\
&\sum_{\mu_{i'}=1}^{r_{i'}-k_{i'}} E_{\mu_i}^{(i)}(\xi_{\mu_{i'}}^{(i')} \xi^{p-q}) E_{\mu_{i'}}^{(i')}([\xi^{q-r} \xi_{\mu_{i''}}^{(i'')}] = E_{\mu_i}^{(i)}([\xi^{q-r} \xi_{\mu_{i''}}^{(i'')}] \xi^{p-q}) a^{(i')}.
\end{aligned} \tag{4.61}$$

Then for $T_{2,2}$ we have

$$\begin{aligned}
& T_{2,2} \\
&= \sum_{\substack{i' > i \\ 0 \leq q \leq p \\ \text{weight}(\xi^q) = w_{i'}} \binom{p}{q} w_{\max(q)} E_{\mu_i}^{(i)}([\xi^q] \xi^{p-q}) \cdot \prod_{i'' > i} a^{(i'')} \\
&\quad - \sum_{\substack{i' > i \\ 0 \leq q \leq p \\ \text{weight}(\xi^q) = w_{i'}} \sum_{\substack{i'' > i' \\ 0 \leq r \leq q \\ \text{weight}(\xi^r) = w_{i''}}} \sum_{\mu_{i''}=1}^{r_{i''}-k_{i''}} \binom{p}{q} \binom{q}{r} E_{\mu_i}^{(i)}([\xi^{q-r} \xi_{\mu_{i''}}^{(i'')}] \xi^{p-q}) \cdot \prod_{i < i''' < i''} a^{(i''')} \cdot \beta_{\mu_{i''}}^{(i'')}(\xi^r) \\
&= T_{1,2} - T_{2,3}.
\end{aligned} \tag{4.62}$$

This completes the proof. $\square$

4.4 Statement and proof of the main theorem

We will state and prove Theorem 4.4.1 and Corollary 4.4.2. Let $\hat{U} \curvearrowright (X, L)$ with grading weights $w_1 > \dots > w_n$. Let $\{\mathbf{u}_i\}_{i=0}^n$ be the filtration $\mathbf{u}_i := \mathbf{u}_{\lambda \geq w_i}$. On $X_{\min}^0$, consider coherent sheaves and morphisms

$$\begin{aligned}
\mathcal{K}_i &= \ker \left(\Omega_{X_{\min}^0 / \text{Spec}(\mathbb{k})} \rightarrow (\mathbf{u}_i)^* \otimes \mathcal{O}_{X_{\min}^0} \right) \\
\phi_i : \mathcal{K}_{i-1} &\rightarrow (\mathbf{u}_i / \mathbf{u}_{i-1})^* \otimes \mathcal{O}_{X_{\min}^0} \\
\text{Fit}_k(\phi_i) &:= \text{Fit}_k(\text{coker}(\phi_i)).
\end{aligned} \tag{4.63}$$

Let $(k_1, \dots, k_n) \in \mathbb{N}^n$ be such that

$$k_i := \min\{k \in \mathbb{Z} : \text{Fit}_k(\phi_i) \neq 0\}, \quad i = 1, \dots, n \tag{4.64}$$

and let $\mathcal{I}_{k_1, \dots, k_n} := \prod_{i=1}^n \text{Fit}_{k_i}(\phi_i)$, which is a coherent sheaf of ideals on $X_{\min}^0$.

Assume WUU (Condition 4.2.1), i.e. $Z_{\min} \not\subseteq \mathbb{V}(\mathcal{I}_{k_1, \dots, k_n})$. Let $Z \hookrightarrow X_{\min}^0$ be the scheme theoretic intersection $Z_{\min} \cap \mathbb{V}(\mathcal{I}_{k_1, \dots, k_n})$. Let $C \hookrightarrow X_{\min}^0$ be the scheme theoretic image of $U \times Z \rightarrow X_{\min}^0$. Let $\overline{C} \hookrightarrow X$ be the scheme theoretic image of $C \hookrightarrow X_{\min}^0 \subseteq X$. Let $\pi : \tilde{X} \rightarrow X$ be the blow-up of X along $\overline{C} \hookrightarrow X$. For any $x, y \in \mathbb{N}_+$, consider the ample line bundle $\tilde{L} := \pi^* L^x \otimes \mathcal{O}(-yE)$ on $\tilde{X}$. There is an induced action and linearisation $\hat{U} \curvearrowright (\tilde{X}, \tilde{L})$. Then similar constructions apply to $\tilde{X}_{\min}^0 := (\tilde{X})_{\min}^0$

$$\begin{aligned}
\tilde{\mathcal{K}}_i &:= \ker \left(\Omega_{\tilde{X}_{\min}^0 / \text{Spec}(\mathbb{k})} \rightarrow (\mathbf{u}_i)^* \otimes \mathcal{O}_{\tilde{X}_{\min}^0} \right) \\
\tilde{\phi}_i : \tilde{\mathcal{K}}_{i-1} &\rightarrow (\mathbf{u}_i / \mathbf{u}_{i-1})^* \otimes \mathcal{O}_{\tilde{X}_{\min}^0} \\
\text{Fit}_k(\tilde{\phi}_i) &:= \text{Fit}_k(\text{coker}(\tilde{\phi}_i)).
\end{aligned} \tag{4.65}$$

Theorem 4.4.1. *Let X be a projective scheme over $\mathbb{k}$ and let L be an ample line bundle on X . Let $\hat{U} \curvearrowright (X, L)$ and let $\lambda : \mathbb{G}_m \rightarrow \text{Aut}(U)$ be a grading 1PS on U . Assume that the WUU condition (Condition 4.2.1) holds. Let $\pi : \tilde{X} \rightarrow X$ be the blow-up described above with the ample line bundle $\tilde{L}$. Let $(k_1, \dots, k_n)$, $\tilde{\phi}_i$ be as above. Then $\text{coker}(\tilde{\phi}_i)$ is locally free of rank k_i on $\tilde{X}_{\min}^0$ for $1 \leq i \leq n$.*

Proof. The statement is local. Let $m \in \mathbb{N}_+$ be a large integer such that $H^0(X, L^m)$ generates $\bigoplus_{d \geq 0} H^0(X, L^{md})$. Let $f \in H^0(X, L^m)_{\lambda=\max}$ such that $Z_{\min, f} \not\subseteq \mathbb{V}(\mathcal{I}_{k_1, \dots, k_n})$. Let $I_f := \mathcal{I}_{k_1, \dots, k_n}(X_f) + \mathcal{O}(X_f)_{\lambda < 0} \subseteq \mathcal{O}(X_f)$ and let $J_f \subseteq \mathcal{O}(X_f)$ be the kernel of $\mathcal{O}(X_f) \rightarrow \mathcal{O}(U) \otimes (\mathcal{O}(X_f)/I_f)$. Then J_f is the ideal associated to $C_f \hookrightarrow X_f$.

A set of generators of $\text{Fit}_{k_i}(\phi_i)(X_f)_{\lambda=0}$ as an ideal of $\mathcal{O}(X_f)_{\lambda=0}$ consists of $(r_i - k_i) \times (r_i - k_i)$ -minors of the following matrices

$$\begin{pmatrix} \langle \xi_1^{(i)}, \omega_1^{(i)} \rangle & \cdots & \langle \xi_1^{(i)}, \omega_{r_i - k_i}^{(i)} \rangle \\ \vdots & \ddots & \vdots \\ \langle \xi_{r_i}^{(i)}, \omega_1^{(i)} \rangle & \cdots & \langle \xi_{r_i}^{(i)}, \omega_{r_i - k_i}^{(i)} \rangle \end{pmatrix}, \quad \omega_1^{(i)}, \dots, \omega_{r_i - k_i}^{(i)} \in \mathcal{K}_{i-1}(X_f)_{\lambda = -w_i}. \tag{4.66}$$

Choose an $(r_i - k_i) \times (r_i - k_i)$ -minor of the above matrix. We can change the basis of $\mathbf{u}_{\lambda=w_i}$ and then assume this $(r_i - k_i) \times (r_i - k_i)$ -minor has the form

$$a^{(i)} = \det \begin{pmatrix} \langle \xi_1^{(i)}, \omega_1^{(i)} \rangle & \cdots & \langle \xi_1^{(i)}, \omega_{r_i - k_i}^{(i)} \rangle \\ \vdots & \ddots & \vdots \\ \langle \xi_{r_i - k_i}^{(i)}, \omega_1^{(i)} \rangle & \cdots & \langle \xi_{r_i - k_i}^{(i)}, \omega_{r_i - k_i}^{(i)} \rangle \end{pmatrix}. \tag{4.67}$$

If $\omega_j^{(i)} = \sum_{\mu} a_{j, \mu}^{(i)} db_{j, \mu}^{(i)}$, then let $f_j^{(i)} := \sum_{\mu} (a_{j, \mu}^{(i)})_{\lambda=0} (b_{j, \mu}^{(i)})_{\lambda = -w_i} \in \mathcal{O}(X_f)_{\lambda = -w_i}$. We then have every $df_j^{(i)} \in \mathcal{K}_{i-1}(X_f)_{\lambda = -w_i}$ and

$$a^{(i)} = \det \begin{pmatrix} \xi_1^{(i)} \cdot f_1^{(i)} & \cdots & \xi_1^{(i)} \cdot f_{r_i - k_i}^{(i)} \\ \vdots & \ddots & \vdots \\ \xi_{r_i - k_i}^{(i)} \cdot f_1^{(i)} & \cdots & \xi_{r_i - k_i}^{(i)} \cdot f_{r_i - k_i}^{(i)} \end{pmatrix}. \tag{4.68}$$

Let $a = a^{(1)} \cdots a^{(n)}$ with $a^{(i)} \in \text{Fit}_{k_i}(\phi_i)_{\lambda=0}$ of the above form for $f_j^{(i)} \in \mathcal{O}(X_f)_{\lambda=-w_i}$. Then $a \in (J_f)_{\lambda=0}$ and such elements generate $(J_f)_{\lambda=0}$, i.e. $\tilde{X}_{\min}^0 \cap \tilde{X}_f$ is covered by such $\tilde{X}_{f,a}$'s.

The theorem is proved if we can show that on $\tilde{X}_{f,a}$, for $1 \leq i \leq n$

$$\text{Fit}_{k_i-1}(\tilde{\phi}_i)(\tilde{X}_{f,a}) = 0, \quad \text{Fit}_{k_i}(\tilde{\phi}_i)(\tilde{X}_{f,a}) = \mathcal{O}(\tilde{X}_{f,a}). \quad (4.69)$$

There is a natural surjective morphism

$$\mathcal{O}(\tilde{X}_{f,a}) \otimes_{\mathbb{k}} d\left(\frac{J_f}{a}\right) \longrightarrow \Omega_{\mathcal{O}(\tilde{X}_{f,a})/\mathbb{k}} \longrightarrow 0. \quad (4.70)$$

Let $\tilde{\omega}_1^{(i)}, \dots, \tilde{\omega}_{r_i-k_i+1}^{(i)} \in \tilde{\mathcal{K}}_{i-1}(X_f)$. By the above surjective morphism, we can write $\tilde{\omega}_j^{(i)} = \sum_{\mu} \frac{h_{j,\mu}^{(i)}}{a^N} d\left(\frac{g_{j,\mu}^{(i)}}{a}\right)$ for $N \in \mathbb{N}_+$, $h_{j,\mu}^{(i)} \in (J_f)^N$ and $g_{j,\mu}^{(i)} \in J_f$. The fact that $\tilde{\omega}_j^{(i)} \in \tilde{\mathcal{K}}_{i-1}(\tilde{X}_{f,a})$ implies that there exists $M \in \mathbb{N}_+$ such that $\omega_j^{(i)} := a^M \sum_{\mu} h_{j,\mu}^{(i)} dg_{j,\mu}^{(i)} \in \mathcal{K}_{i-1}(X_f)$.

The matrix representing $\tilde{\phi}_i$ on $(\tilde{\omega}_1^{(i)}, \dots, \tilde{\omega}_{r_i-k_i+1}^{(i)})$ is

$$\begin{aligned} & \tilde{\phi}_i(\tilde{\omega}_1^{(i)}, \dots, \tilde{\omega}_{r_i-k_i+1}^{(i)}) \\ &= (u_1^{(i)} \otimes 1, \dots, u_{r_i}^{(i)} \otimes 1) \begin{pmatrix} \sum_{\mu} \frac{h_{1,\mu}^{(i)}}{a^N} \frac{\xi_1^{(i)} \cdot g_{1,\mu}^{(i)}}{a} & \cdots & \sum_{\mu} \frac{h_{r_i-k_i+1,\mu}^{(i)}}{a^N} \frac{\xi_1^{(i)} \cdot g_{r_i-k_i+1,\mu}^{(i)}}{a} \\ \vdots & \ddots & \vdots \\ \sum_{\mu} \frac{h_{1,\mu}^{(i)}}{a^N} \frac{\xi_{r_i}^{(i)} \cdot g_{1,\mu}^{(i)}}{a} & \cdots & \sum_{\mu} \frac{h_{r_i-k_i+1,\mu}^{(i)}}{a^N} \frac{\xi_{r_i}^{(i)} \cdot g_{r_i-k_i+1,\mu}^{(i)}}{a} \end{pmatrix} \\ &= (u_1^{(i)} \otimes 1, \dots, u_{r_i}^{(i)} \otimes 1) \begin{pmatrix} \frac{\langle \xi_1^{(i)}, \omega_1^{(i)} \rangle}{a^{M+N+1}} & \cdots & \frac{\langle \xi_1^{(i)}, \omega_{r_i-k_i+1}^{(i)} \rangle}{a^{M+N+1}} \\ \vdots & \ddots & \vdots \\ \frac{\langle \xi_{r_i}^{(i)}, \omega_1^{(i)} \rangle}{a^{M+N+1}} & \cdots & \frac{\langle \xi_{r_i}^{(i)}, \omega_{r_i-k_i+1}^{(i)} \rangle}{a^{M+N+1}} \end{pmatrix}. \end{aligned} \quad (4.71)$$

Let

$$\begin{aligned} \tilde{A} &:= \begin{pmatrix} \frac{\langle \xi_1^{(i)}, \omega_1^{(i)} \rangle}{a^{M+N+1}} & \cdots & \frac{\langle \xi_1^{(i)}, \omega_{r_i-k_i+1}^{(i)} \rangle}{a^{M+N+1}} \\ \vdots & \ddots & \vdots \\ \frac{\langle \xi_{r_i}^{(i)}, \omega_1^{(i)} \rangle}{a^{M+N+1}} & \cdots & \frac{\langle \xi_{r_i}^{(i)}, \omega_{r_i-k_i+1}^{(i)} \rangle}{a^{M+N+1}} \end{pmatrix} \\ A &:= \begin{pmatrix} \langle \xi_1^{(i)}, \omega_1^{(i)} \rangle & \cdots & \langle \xi_1^{(i)}, \omega_{r_i-k_i+1}^{(i)} \rangle \\ \vdots & \ddots & \vdots \\ \langle \xi_{r_i}^{(i)}, \omega_1^{(i)} \rangle & \cdots & \langle \xi_{r_i}^{(i)}, \omega_{r_i-k_i+1}^{(i)} \rangle \end{pmatrix}. \end{aligned} \quad (4.72)$$

Let $\tilde{m}$ be an $(r_i - k_i + 1) \times (r_i - k_i + 1)$ -minor of $\tilde{A}$. Then it is of the form

$$\tilde{m} = \frac{m}{a^{(r_i-k_i+1)(M+N+1)}} \quad (4.73)$$

where $m \in (J_f)^{(r_i-k_i+1)(M+N+1)}$ is the $(r_i - k_i + 1) \times (r_i - k_i + 1)$ -minor of A with the same columns and rows as $\tilde{m}$. By the definition of k_i , any $(r_i - k_i + 1) \times (r_i - k_i + 1)$ -minor of the matrix A is zero. In particular, the minor m is zero, so

$$\tilde{m} = \frac{m}{a^{(r_i-k_i+1)(M+N+1)}} = 0 \quad (4.74)$$

which proves $\text{Fit}_{k_i-1}(\tilde{\phi}_i)(\tilde{X}_{f,a}) = 0$.

We then prove $\text{Fit}_{k_i}(\tilde{\phi}_i)(\tilde{X}_{f,a}) = \mathcal{O}(\tilde{X}_{f,a})$. By Lemma 4.3.3, for $1 \leq i \leq n$ and $1 \leq \mu \leq r_i - k_i$, there exists $c_\mu^{(i)} \in (J_f)_{\lambda=-w_i}$ such that

$$\xi_\mu^{(i)} \cdot c_\nu^{(i)} = w_i \delta_{\mu\nu} a, \quad 1 \leq i \leq n, \quad 1 \leq \mu \leq r_i - k_i. \quad (4.75)$$

Then $d(\frac{c_\mu^{(i)}}{a}) \in (\Omega_{\mathcal{O}(\tilde{X}_{f,a})/\mathbb{k}})_{\lambda=-w_i} \subseteq \tilde{\mathcal{K}}_{i-1}(\tilde{X}_{f,a})$. For each $1 \leq i \leq n$ the matrix representing $\tilde{\phi}_i$ on $(d(\frac{c_1^{(i)}}{a}), \dots, d(\frac{c_{r_i-k_i}^{(i)}}{a}))$ is

$$\begin{aligned} & \tilde{\phi}_i \left(d\left(\frac{c_1^{(i)}}{a}\right), \dots, d\left(\frac{c_{r_i-k_i}^{(i)}}{a}\right) \right) \\ &= (u_1^{(i)} \otimes 1, \dots, u_{r_i}^{(i)} \otimes 1) \begin{pmatrix} w_i & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & w_i \\ & & A' \end{pmatrix}. \end{aligned} \quad (4.76)$$

Then the top $(r_i - k_i) \times (r_i - k_i)$ -minor of the matrix is $w_i^{r_i-k_i} \in \text{Fit}_{k_i}(\tilde{\phi}_i)(\tilde{X}_{f,a})$. Therefore $\text{Fit}_{k_i}(\tilde{\phi}_i)(\tilde{X}_{f,a}) = \mathcal{O}(\tilde{X}_{f,a})$. This completes the proof. $\square$

Corollary 4.4.2. *Let X be a projective scheme over $\mathbb{k}$ and let L be an ample line bundle on X . Let $\hat{U} \curvearrowright (X, L)$ and let $\lambda : \mathbb{G}_m \rightarrow \text{Aut}(U)$ be a grading 1PS. Assume WUU (Condition 4.2.1) holds. Then there exists a $\hat{U}$ -equivariant blow-up $\pi : \tilde{X} \rightarrow X$ and the action $\hat{U} \curvearrowright \tilde{X}$ lifts to an ample line bundle $\tilde{L}$, such that:*

- $\tilde{X}_{\min}^0 \rightarrow \tilde{X}_{\min}^0/U$ is a quasi-projective universal geometric quotient by U ;
- $\tilde{X}_{\min}^0 \setminus U\tilde{Z}_\lambda \rightarrow (\tilde{X}_{\min}^0 \setminus U\tilde{Z}_\lambda)/\hat{U}$ is a projective universal geometric quotient by $\hat{U}$.

Proof. This is obvious from Theorem 4.4.1, Theorem 3.3.8 and Theorem 3.3.9. $\square$

4.5 Appendix

4.5.1 Associative algebra and complete brackets

Let A be an associative algebra. Let $m \in \mathbb{N}_+$. Define the *complete bracket of m elements* by

$$[-]_m : \prod_{i=1}^m A \rightarrow A \quad (4.77)$$

$$[a_1, \dots, a_m]_m := \text{ad}_{a_1} \cdots \text{ad}_{a_{m-1}}(a_m) = \left[a_1, [a_2, \dots, [a_{m-1}, a_m] \cdots] \right].$$

When $m = 1$, the map $[-]_1 : A \rightarrow A$ is the identity map.

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{k}$ and let $U(\mathfrak{g})$ be its universal enveloping algebra. Let $(\xi_1, \dots, \xi_r)$ be an ordered basis of $\mathfrak{g}$. Then $\{\xi_1^{p_1} \cdots \xi_r^{p_r} : (p_1, \dots, p_r) \in \mathbb{N}^r\}$ is a basis of $U(\mathfrak{g})$ by the Poincaré-Birkhoff-Witt theorem. We can define the *complete bracket with respect to the basis $(\xi_1, \dots, \xi_n)$* by

$$[-] : U(\mathfrak{g}) \rightarrow \mathfrak{g} \quad (4.78)$$

$$[\xi^p] := \left[\underbrace{\xi_1, \dots, \xi_1}_{p_1}, \dots, \underbrace{\xi_r, \dots, \xi_r}_{p_r} \right]_{p_1 + \dots + p_r}.$$

Moreover if there is a representation of $\mathbb{G}_m$ on $\mathfrak{g}$, which induces a representation of $\mathbb{G}_m$ on $U(\mathfrak{g})$, and each ξ_i in the basis $(\xi_1, \dots, \xi_r)$ is a weight vector, then $[-] : U(\mathfrak{g}) \rightarrow \mathfrak{g}$ preserves the weights, i.e. it is a morphism of $\mathbb{G}_m$ -representations.

The following two lemmas can be proved by induction, and we omit the proofs.

Lemma 4.5.1. *Let $n \in \mathbb{N}_+$ and let $\mathbb{k}\langle y_1, \dots, y_n \rangle$ be the free associative algebra over $\mathbb{k}$ generated by n variables. Let $w_1, \dots, w_n \in \mathbb{k}$ be scalars in $\mathbb{k}$. Let $k_1, \dots, k_n \in \mathbb{N}$. Then*

$$\left(\sum_{i=1}^n k_i w_i \right) y^k = \sum_{0 < s \leq k} \binom{k}{s} w_{\max\{i:s_i \neq 0\}} [y^s] y^{k-s} \quad (4.79)$$

where

$$s := (s_1, \dots, s_n) \neq (0, \dots, 0), \quad s_i \leq k_i, \text{ for } i = 1, \dots, n$$

$$\binom{k}{s} := \binom{k_1}{s_1} \cdots \binom{k_n}{s_n}$$

$$y^s := y_1^{s_1} \cdots y_n^{s_n} \quad (4.80)$$

$$[y^s] := \left[\underbrace{y_1, \dots, y_1}_{s_1}, \dots, \underbrace{y_n, \dots, y_n}_{s_n} \right]_{s_1 + \dots + s_n}$$

Lemma 4.5.2. *Let $n \in \mathbb{N}$ and let $\mathbb{k}\langle x_1, \dots, x_n, y \rangle$ be the free associative algebra over $\mathbb{k}$ generated by $n + 1$ variables. Let $k := (k_1, \dots, k_n) \in \mathbb{N}^n$. Then*

$$x^k y = \sum_{0 \leq s \leq k} \binom{k}{s} [x^{k-s} y] x^s \quad (4.81)$$

where

$$[x^{k-s} y] := \left[\underbrace{x_1, \dots, x_1}_{k_1-s_1}, \dots, \underbrace{x_n, \dots, x_n}_{k_n-s_n}, y \right]_{1+\sum_{i=1}^n (k_i-s_i)}. \quad (4.82)$$

□

4.5.2 Unipotent action on affine schemes

Let U be an affine algebraic unipotent group over $\mathbb{k}$. Let A be a commutative $\mathbb{k}$ -algebra and let U acts on A dually by $\varphi : A \rightarrow \mathcal{O}(U) \otimes A$, where the tensor product is over $\mathbb{k}$. Identify $\mathcal{O}(U) \cong \mathbb{k}[u_1, \dots, u_n]$ and let $\mu^* : \mathbb{k}[u] \rightarrow \mathbb{k}[u] \otimes \mathbb{k}[u]$ be the co-multiplication. Let $U(\mathfrak{u})$ be the universal enveloping algebra of $\mathfrak{u} := \text{Lie}(U)$. The aim is to prove the following proposition.

Proposition 4.5.3. *For $f \in A$, if*

$$A \rightarrow \mathbb{k}[u] \otimes A, \quad f \mapsto \sum_{\alpha \in \mathbb{N}^n} u^\alpha \otimes f_\alpha \quad (4.83)$$

then $f_\alpha \in U(\mathfrak{u}).f$ for all α .

Before the proof we state a useful corollary.

Corollary 4.5.4. *Let U be a unipotent group over $\mathbb{k}$ and let A be a $\mathbb{k}$ -algebra of finite type. Let $\varphi : A \rightarrow \mathcal{O}(U) \otimes A$ be a dual action of U on A . Let $I \subseteq A$ be an ideal. Then the following are equivalent for $a \in A$:*

- (1) $X.a \in I$ for all $X \in U(\mathfrak{u})$;
- (2) the map $A \rightarrow \mathcal{O}(U) \otimes A \rightarrow \mathcal{O}(U) \otimes (A/I)$ sends $a \in A$ to zero.

Proof. Identify $\mathcal{O}(U) \cong \mathbb{k}[u_1, \dots, u_n]$. For $a \in A$ and $\alpha \in \mathbb{N}^n$, let $a_\alpha \in A$ be such that

$$A \rightarrow \mathcal{O}(U) \otimes A, \quad a \mapsto \sum_{\alpha \in \mathbb{N}^n} u^\alpha \otimes a_\alpha. \quad (4.84)$$

Assume (1). We have $U(\mathfrak{u}).a \subseteq I$, and then $a_\alpha \in I$ by Proposition 4.5.3, which proves (2).

Assume (2). Let $q : A \rightarrow A/I$ be the quotient ring map. Let $J \subseteq A$ be the kernel of $A \rightarrow \mathcal{O}(U) \otimes (A/I)$. To prove (1), it suffices to show:

- $J \subseteq I$;
- $J \subseteq A$ is U -invariant.

Because if we assume they are proved, then for $a \in J$ and $X \in U(\mathfrak{u})$ we have $X.a \in J$ since J is U -invariant, and then $X.a \in I$ since $J \subseteq I$, which is (1).

We prove that $J \subseteq I$. Consider the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\varphi} & \mathcal{O}(U) \otimes A & \xrightarrow{1 \otimes q} & \mathcal{O}(U) \otimes (A/I) \\ & \searrow 1 & \downarrow e^* \otimes 1 & & \downarrow e^* \otimes 1 \\ & & A & \xrightarrow{q} & A/I \end{array} \quad (4.85)$$

where $e^* : \mathcal{O}(U) \rightarrow \mathbb{k}$ is the co-unit map. Since $J = \ker((1 \otimes q) \circ \varphi)$, we have $q(J) = q \circ 1(J) = (e^* \otimes 1) \circ (1 \otimes q) \circ \varphi(J) = 0$, i.e. $J \subseteq I$.

We prove that $J \subseteq A$ is U -invariant. Consider the following diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & \mathcal{O}(U) \otimes A \\ \downarrow \varphi & & \downarrow 1 \otimes \varphi \\ \mathcal{O}(U) \otimes A & \xrightarrow{\mu^* \otimes 1} & \mathcal{O}(U) \otimes \mathcal{O}(U) \otimes A \\ \downarrow 1 \otimes q & & \downarrow 1 \otimes 1 \otimes q \\ \mathcal{O}(U) \otimes (A/I) & \xrightarrow{\mu^* \otimes 1} & \mathcal{O}(U) \otimes \mathcal{O}(U) \otimes (A/I). \end{array} \quad (4.86)$$

Since $J = \ker((1 \otimes q) \circ \varphi)$, by the diagram we have $\varphi(J) \subseteq \ker((1 \otimes 1 \otimes q) \circ (1 \otimes \varphi))$. The composition of the right vertical arrows, $(1 \otimes 1 \otimes q) \circ (1 \otimes \varphi)$, is the base change of $(1 \otimes q) \circ \varphi$ along $\mathbb{k} \rightarrow \mathcal{O}(U)$. By the flatness of $\mathbb{k} \rightarrow \mathcal{O}(U)$, we have $\mathcal{O}(U) \otimes J = \ker((1 \otimes 1 \otimes q) \circ (1 \otimes \varphi))$. So $\varphi(J) \subseteq \mathcal{O}(U) \otimes J$, which proves that J is U -invariant. $\square$

The action of $\mathfrak{u}$ on A is such that for $\xi \in \mathfrak{u}$, the derivation $\xi : A \rightarrow A$ is the following composition

$$\begin{array}{c} A \xrightarrow{\varphi} \mathbb{k}[u] \otimes A \xrightarrow{\psi} \mathfrak{m}/\mathfrak{m}^2 \otimes A \xrightarrow{(-\xi) \otimes 1} A \\ \searrow \xi \quad \quad \quad \nearrow \end{array} \quad (4.87)$$

where $\mathfrak{m} = (u_1, \dots, u_n) \subseteq \mathbb{k}[u]$ is the maximal ideal associated to the identity, and ψ is the linear map

$$\psi : \mathbb{k}[u] \rightarrow \mathfrak{m}/\mathfrak{m}^2, \quad p(u) \mapsto \overline{p(u) - p(0)}. \quad (4.88)$$

The action of $U(\mathfrak{u})$ on A is induced from that of $\mathfrak{u} \curvearrowright A$.

There exists $c_{\beta, \gamma}^\alpha \in \mathbb{k}$ for $\alpha, \beta, \gamma \in \mathbb{N}^n$ such that

$$\mu^* : \mathbb{k}[u] \rightarrow \mathbb{k}[u] \otimes \mathbb{k}[u], \quad u^\alpha \mapsto \sum_{\beta, \gamma} c_{\beta, \gamma}^\alpha u^\beta \otimes u^\gamma. \quad (4.89)$$

They satisfy the following properties:

- Since μ^* respects multiplication, we have

$$c_{\beta, \gamma}^{\alpha_1 + \alpha_2} = \sum_{\substack{\beta_1 + \beta_2 = \beta \\ \gamma_1 + \gamma_2 = \gamma}} c_{\beta_1, \gamma_1}^{\alpha_1} c_{\beta_2, \gamma_2}^{\alpha_2}; \quad (4.90)$$

- The existence of identity in U implies

$$c_{0, \gamma}^\alpha = \delta_\gamma^\alpha, \quad c_{\beta, 0}^\alpha = \delta_\beta^\alpha. \quad (4.91)$$

Define a norm on $\mathbb{N}^n$ via $|\alpha| := \alpha_1 + \dots + \alpha_n$ for $\alpha \in \mathbb{N}^n$. For $1 \leq i \leq n$, let $e_i \in \mathbb{N}^n$ be the vector whose i th entry is 1 and other entries are 0.

Lemma 4.5.5. *If $c_{\beta, \gamma}^\alpha \neq 0$, then $|\alpha| \leq |\beta| + |\gamma|$.*

Proof. We prove $c_{\beta, \gamma}^\alpha = 0$ if $|\alpha| > |\beta| + |\gamma|$ by induction on $|\alpha|$. When $|\alpha| = 1$, we have $|\beta| + |\gamma| = 0$, i.e. $\beta = \gamma = 0$. Then $c_{\beta, \gamma}^\alpha = c_{0, 0}^\alpha = \delta_0^\alpha = 0$. Assume $c_{\beta, \gamma}^\alpha = 0$ if $k \geq |\alpha| > |\beta| + |\gamma|$ for some $k \in \mathbb{N}_+$. If $k+1 = |\alpha| > |\beta| + |\gamma|$, write $\alpha = \alpha' + e_i$ for some $|\alpha'| = k$ and $1 \leq i \leq n$. Then

$$\begin{aligned} c_{\beta, \gamma}^\alpha &= c_{\beta, \gamma}^{\alpha' + e_i} \\ &= \sum_{\substack{\beta' + \beta'' = \beta \\ \gamma' + \gamma'' = \gamma}} c_{\beta', \gamma'}^{\alpha'} c_{\beta'', \gamma''}^{e_i}. \end{aligned} \quad (4.92)$$

If there exists a nonzero summand

$$c_{\beta', \gamma'}^{\alpha'} \neq 0, \quad c_{\beta'', \gamma''}^{e_i} \neq 0 \quad (4.93)$$

then by the induction hypothesis for $|\alpha'| = k$ and $|e_i| = 1$

$$|\alpha'| \leq |\beta'| + |\gamma'|, \quad 1 \leq |\beta''| + |\gamma''| \quad (4.94)$$

which implies $|\alpha| = |\alpha'| + |e_j| \leq |\beta| + |\gamma|$, a contradiction. Then $c_{\beta,\gamma}^\alpha = 0$ as a sum of zeros. $\square$

Lemma 4.5.6. *If $|\alpha| = |\beta| + 1$ and $1 \leq j \leq n$, then the following are equivalent:*

$$(1) \quad c_{\beta,e_j}^\alpha = 1 + \max\{k \in \mathbb{N} : ke_j \leq \beta\};$$

$$(2) \quad c_{\beta,e_j}^\alpha \neq 0;$$

$$(3) \quad \alpha = \beta + e_j.$$

Proof. We prove the equivalence by induction on $|\alpha|$. When $|\alpha| = 1$, we have $\beta = 0$ and then $c_{0,e_j}^\alpha = \delta_{e_j}^\alpha$, which proves the equivalence.

Assume for some $k \in \mathbb{N}_+$ the equivalence is proved for $|\alpha| \leq k$. Let $|\alpha| = k + 1$. Obviously (1) $\implies$ (2).

For (2) $\implies$ (3), assume $c_{\beta,e_j}^\alpha \neq 0$. Write $\alpha = \alpha' + e_i$ for some $|\alpha'| = k$ and $1 \leq i \leq n$.

Then

$$\begin{aligned} 0 \neq c_{\beta,e_j}^\alpha &= \sum_{\beta' + \beta'' = \beta} c_{\beta',e_j}^{\alpha'} c_{\beta'',0}^{e_i} + \sum_{\beta' + \beta'' = \beta} c_{\beta',0}^{\alpha'} c_{\beta'',e_j}^{e_i} \\ &= \sum_{\beta' + \beta'' = \beta} c_{\beta',e_j}^{\alpha'} \delta_{\beta''}^{e_i} + \sum_{\beta' + \beta'' = \beta} \delta_{\beta'}^{\alpha'} c_{\beta'',e_j}^{e_i}. \end{aligned} \quad (4.95)$$

If there exist $\beta' + \beta'' = \beta$ such that $c_{\beta',e_j}^{\alpha'} \delta_{\beta''}^{e_i} \neq 0$, then

$$\beta'' = e_i, \quad |\beta'| = |\beta| - |\beta''| = k - 1. \quad (4.96)$$

So $k = |\alpha'| = |\beta'| + 1$, which implies $\alpha' = \beta' + e_j$ by the induction hypothesis. So $\alpha = \alpha' + e_i = \beta' + e_j + \beta'' = \beta + e_j$, which is (3).

Else, there exist $\beta' + \beta'' = \beta$ such that $\delta_{\beta'}^{\alpha'} c_{\beta'',e_j}^{e_i} \neq 0$, and then $\alpha' = \beta'$. In particular $|\beta''| = |\beta| - |\beta'| = 0$, i.e. $\beta'' = 0$. Then $c_{\beta'',e_j}^{e_i} = \delta_{e_j}^{e_i} \neq 0$, i.e. $e_i = e_j$. We have $\alpha = \alpha' + e_i = \beta' + e_j = \beta + e_j$, which is (3).

For (3) $\implies$ (1), assume $\alpha = \beta + e_j$. Then

$$\begin{aligned}
c_{\beta, e_j}^\alpha &= c_{\beta, e_j}^{\beta + e_j} \\
&= \sum_{\beta' + \beta'' = \beta} c_{\beta', e_j}^\beta c_{\beta'', 0}^{e_j} + \sum_{\beta' + \beta'' = \beta} c_{\beta', 0}^\beta c_{\beta'', e_j}^{e_j} \\
&= \sum_{\beta' + \beta'' = \beta} c_{\beta', e_j}^\beta \delta_{\beta''}^{e_j} + \sum_{\beta' + \beta'' = \beta} \delta_{\beta'}^\beta c_{\beta'', e_j}^{e_j} \\
&= \sum_{\beta' + \beta'' = \beta} c_{\beta', e_j}^\beta \delta_{\beta''}^{e_j} + 1.
\end{aligned} \tag{4.97}$$

If $e_j \leq \beta$, then

$$c_{\beta, e_j}^\alpha = c_{\beta - e_j, e_j}^\beta + 1 \tag{4.98}$$

and $c_{\beta - e_j, e_j}^\beta = 1 + \max\{k \in \mathbb{N} : ke_j \leq \beta - e_j\} = \max\{k \in \mathbb{N} : ke_j \leq \beta\}$ by the induction hypothesis, which proves (1). Else we have $e_j \not\leq \beta$, and then $c_{\beta, e_j}^\alpha = 1$, which is (1). $\square$

We can now prove Proposition 4.5.3.

Proof of Proposition 4.5.3. We prove this by induction on $|\alpha|$. When $\alpha = 0$, we have $f_0 = f$.

When $|\alpha| = 1$, we have $f_{e_i} = -\xi_i \cdot f \in U(\mathfrak{u}) \cdot f$.

Assume for some $k \in \mathbb{N}_+$, we have $f_\alpha \in U(\mathfrak{u}) \cdot f$ when $|\alpha| \leq k$. Let $|\alpha| = k + 1$.

Consider the ring map

$$\varphi_{k+1} : A \rightarrow \underbrace{\mathbb{k}[u] \otimes \cdots \otimes \mathbb{k}[u]}_{k+1} \otimes A \tag{4.99}$$

which maps $f \in A$ to

$$\varphi_{k+1}(f) = \sum_{\beta_1, \dots, \beta_{k+1}} \sum_{\gamma_1, \dots, \gamma_{k+1}} c_{0, \gamma_1}^{\beta_1} c_{\beta_1, \gamma_2}^{\beta_2} c_{\beta_2, \gamma_3}^{\beta_3} \cdots c_{\beta_k, \gamma_{k+1}}^{\beta_{k+1}} u^{\gamma_1} \otimes \cdots \otimes u^{\gamma_{k+1}} \otimes f_{\beta_{k+1}}. \tag{4.100}$$

Write $\alpha = e_{i_1} + \cdots + e_{i_{k+1}}$ for $1 \leq i_1, \dots, i_{k+1} \leq n$. Consider the map X

$$\begin{array}{ccc}
\mathbb{k}[u] \otimes \cdots \otimes \mathbb{k}[u] \otimes A & \xlongequal{\quad} & \mathbb{k}[u] \otimes \cdots \otimes \mathbb{k}[u] \otimes A \\
\downarrow X & & \downarrow \psi \otimes \cdots \otimes \psi \otimes 1 \\
& & \mathfrak{m}/\mathfrak{m}^2 \otimes \cdots \otimes \mathfrak{m}/\mathfrak{m}^2 \otimes A \\
& & \downarrow (-\xi_{i_1}) \otimes \cdots \otimes (-\xi_{i_{k+1}}) \otimes 1 \\
A & \xlongequal{\quad} & A
\end{array} \tag{4.101}$$

which maps $\varphi_{k+1}(f)$ to $\xi_{i_{k+1}} \cdots \xi_{i_1} \cdot f \in U(\mathfrak{u}) \cdot f$ by the definition of $U(\mathfrak{u}) \curvearrowright A$. We can evaluate $X \circ \varphi_{k+1}(f)$ using the expression of $\varphi_{k+1}(f)$

$$\begin{aligned} & X \circ \varphi_{k+1}(f) \\ &= X \left(\sum_{\beta_1, \dots, \beta_{k+1}} \sum_{\gamma_1, \dots, \gamma_{k+1}} c_{0, \gamma_1}^{\beta_1} c_{\beta_1, \gamma_2}^{\beta_2} \cdots c_{\beta_k, \gamma_{k+1}}^{\beta_{k+1}} u^{\gamma_1} \otimes \cdots \otimes u^{\gamma_{k+1}} \otimes f_{\beta_{k+1}} \right) \\ &= \sum_{\beta, \gamma} \left(\prod_{j=1}^{k+1} c_{\beta_{j-1}, \gamma_j}^{\beta_j} \right) \left(\prod_{j=1}^{k+1} \langle -\xi_{i_j}, \psi(u^{\gamma_j}) \rangle \right) f_{\beta_{k+1}} \end{aligned} \quad (4.102)$$

where $\beta_0 = 0$. We have

$$\langle -\xi_{i_j}, \psi(u^{\gamma_j}) \rangle = -\delta_{e_{i_j}}^{\gamma_j}, \quad j = 1, \dots, k+1 \quad (4.103)$$

so

$$\sum_{\beta} \left(\prod_{j=1}^{k+1} c_{\beta_{j-1}, e_{i_j}}^{\beta_j} \right) f_{\beta_{k+1}} = (-1)^{k+1} X \circ \varphi_{k+1}(f) \in U(\mathfrak{u}) \cdot f. \quad (4.104)$$

If $\beta_1, \dots, \beta_{k+1}$ are such that $\prod_{j=1}^{k+1} c_{\beta_{j-1}, e_{i_j}}^{\beta_j} \neq 0$, then

$$\beta_1 = e_{i_1}, \quad c_{\beta_{j-1}, e_{i_j}}^{\beta_j} \neq 0 \text{ for } j \geq 2. \quad (4.105)$$

By Lemma 4.5.5, we have $|\beta_j| \leq |\beta_{j-1}| + 1$ for $j \geq 2$. In particular $|\beta_{k+1}| \leq k+1$, and $|\beta_{k+1}| = k+1$ implies $|\beta_j| = j$ for all $j = 1, \dots, k+1$. By the induction hypothesis $f_{\beta_{k+1}} \in U(\mathfrak{u}) \cdot f$ if $|\beta_{k+1}| \leq k$. Then

$$\sum_{\substack{\beta_1, \dots, \beta_{k+1} \\ |\beta_j| = j \text{ for all } j}} \left(\prod_{j=1}^{k+1} c_{\beta_{j-1}, e_{i_j}}^{\beta_j} \right) f_{\beta_{k+1}} \in U(\mathfrak{u}) \cdot f \quad (4.106)$$

By Lemma 4.5.6, when $|\beta_j| = j$ for all j , we have $c_{\beta_{j-1}, e_{i_j}}^{\beta_j} \neq 0$ if and only if $\beta_j = \beta_{j-1} + e_{i_j}$. Therefore only one summand above is non-zero

$$\left(\prod_{j=1}^{k+1} c_{\sigma_{j-1}, e_{i_j}}^{\sigma_j} \right) f_{\sigma_{k+1}} \in U(\mathfrak{u}) \cdot f \quad (4.107)$$

where $\sigma_j = e_{i_1} + \cdots + e_{i_j}$. The coefficient $\prod_{j=1}^{k+1} c_{\sigma_{j-1}, e_{i_j}}^{\sigma_j}$ is nonzero since $c_{\sigma_{j-1}, e_{i_j}}^{\sigma_j} \neq 0$ by Lemma 4.5.6. Notice that $\sigma_{k+1} = e_{i_1} + \cdots + e_{i_{k+1}} = \alpha$. We have

$$f_{\alpha} \in U(\mathfrak{u}) \cdot f \quad (4.108)$$

which completes the induction. $\square$

Chapter 5

Stratifications by unipotent stabilisers

Let $\mathbb{k}$ be an algebraically closed field of characteristic zero. For a connected affine algebraic group H over $\mathbb{k}$, there exists a group of the form $U \rtimes (T \times K)$ with U unipotent, T a torus and K semisimple, and a surjective homomorphism $U \rtimes (T \times K) \rightarrow H$ whose kernel is a finite central subgroup of $U \rtimes (T \times K)$. From a GIT point of view, we can without loss of generality replace the group H by $U \rtimes (T \times K)$.

In this chapter, we consider groups of the form $U \rtimes (T \times K)$. Let $\lambda : \mathbb{G}_m \rightarrow T$ be an internal grading 1PS with grading weights $w_1 > \dots > w_n$. Let $\{\mathfrak{u}_i\}_{i=0}^n$ be the increasing filtration with $\mathfrak{u}_0 = 0$ and $\mathfrak{u}_i := \mathfrak{u}_{\lambda \geq w_i}$ is the sum of weight spaces whose λ -weights are no less than w_i for $1 \leq i \leq n$. Each $\mathfrak{u}_i$ is $U \rtimes (T \times K)$ -invariant.

Let X be a projective scheme over $\mathbb{k}$ and let L be an ample line bundle over X . Let $U \rtimes (T \times K)$ act on X linearly with respect to L . Let $X_{\lambda-\min}^0 \subseteq X$ be the non-vanishing locus of $\bigoplus_{d>0} H^0(X, L^d)_{\lambda=\max}$ and let $Z_{\lambda-\min} \hookrightarrow X_{\lambda-\min}^0$ be the fixed point subscheme for the λ -action on $X_{\lambda-\min}^0$. Let $Z_{\lambda-\min}^{K,(s)s} \subseteq Z_{\lambda-\min}$ denote the (semi)stable locus for $K \curvearrowright (Z_{\lambda-\min}, L|_{Z_{\lambda-\min}})$. Assume $Z_{\lambda-\min}^{K,s} \neq \emptyset$.

In this chapter we consider the quotient of X by $U \rtimes (T \times K)$. To meet the various conditions on unipotent stabilisers, the CDRS condition (Condition 3.3.2) and the WUU condition (Condition 4.2.1), we first stratify an open subscheme of X , such that on each stratum, the unipotent stabilisers have constant dimensions. Next for each stratum, we try to apply the blow-up construction (Theorem 4.4.1) and non-reductive GIT (Theorem 3.3.8).

However, without further assumptions, not all strata satisfy WUU (Condition 4.2.1), the condition required for blow-up. Therefore we impose Condition 5.1.6 for $U \rtimes (T \times K) \curvearrowright (X, L)$, which is verified for the non-reductive GIT construction of sheaves of a Harder-Narasimhan type of length 2 in [Jac21], and for some special longer Harder-Narasimhan types on curves (Condition 6.6.2 in next chapter).

5.1 Constructions and assumptions

In this section, we prove the stratification theorem (Theorem 5.1.2), and describe the assumption (Condition 5.1.6) needed for further application.

5.1.1 The stratification

Let U be a unipotent group over $\mathbb{k}$ with Lie algebra $\mathfrak{u}$. Choose a filtration of Lie ideals

$$0 = \mathfrak{u}_0 \subseteq \mathfrak{u}_1 \subseteq \cdots \subseteq \mathfrak{u}_n = \mathfrak{u}. \quad (5.1)$$

Let $r_i := \dim(\mathfrak{u}_i/\mathfrak{u}_{i-1})$. Assume $n \geq 1$ and $r_i \geq 1$ for all $1 \leq i \leq n$. Let X be a scheme of finite type over $\mathbb{k}$, not necessarily affine nor projective. Let U act on X .

The infinitesimal action (See Definition 2.2.1) of $\mathfrak{u}$ on X is given by the morphism $\Omega_{X/\mathbb{k}} \rightarrow \mathfrak{u}^* \otimes_{\mathbb{k}} \mathcal{O}_X$ of coherent sheaves on X . The infinitesimal action of $\mathfrak{u}_i$ is the composition $\Omega_{X/\mathbb{k}} \rightarrow \mathfrak{u}^* \otimes_{\mathbb{k}} \mathcal{O}_X \rightarrow (\mathfrak{u}_i)^* \otimes_{\mathbb{k}} \mathcal{O}_X$. Let $\mathcal{K}_i, \mathcal{Q}_i$ be the kernel and cokernel of $\Omega_{X/\mathbb{k}} \rightarrow (\mathfrak{u}_i)^* \otimes_{\mathbb{k}} \mathcal{O}_X$

$$\begin{aligned} \mathcal{K}_i &:= \ker(\Omega_{X/\mathbb{k}} \rightarrow (\mathfrak{u}_i)^* \otimes_{\mathbb{k}} \mathcal{O}_X) \\ \mathcal{Q}_i &:= \operatorname{coker}(\Omega_{X/\mathbb{k}} \rightarrow (\mathfrak{u}_i)^* \otimes_{\mathbb{k}} \mathcal{O}_X). \end{aligned} \quad (5.2)$$

Then there is an exact sequence for $1 \leq i \leq n$

$$0 \longrightarrow \mathcal{K}_i \longrightarrow \mathcal{K}_{i-1} \xrightarrow{\phi_i} (\mathfrak{u}_i/\mathfrak{u}_{i-1})^* \otimes_{\mathbb{k}} \mathcal{O}_X \xrightarrow{\varphi_i} \mathcal{Q}_i \longrightarrow \mathcal{Q}_{i-1} \longrightarrow 0. \quad (5.3)$$

The following is exact for $1 \leq i \leq n$ since $\operatorname{coker}(\phi_i) \cong \operatorname{im}(\varphi_i)$

$$0 \longrightarrow \operatorname{coker}(\phi_i) \longrightarrow \mathcal{Q}_i \longrightarrow \mathcal{Q}_{i-1} \longrightarrow 0. \quad (5.4)$$

Motivated by CDRS (Condition 3.3.2), it is desirable that $\operatorname{coker}(\phi_i)$ are locally free of constant rank for $1 \leq i \leq n$. On Noetherian schemes, flat coherent sheaves and finite locally

free sheaves coincide (cf. [Sta22, Lemma 00NX]), and when $\mathcal{Q}_{i-1}$ is flat, flatness of $\mathcal{Q}_i$ and flatness of $\text{coker}(\phi_i)$ coincide (cf. [Sta22, Lemma 00HM]). Therefore if we have that $\mathcal{Q}_i$ are locally free for $1 \leq i \leq n$, so are $\text{coker}(\phi_i)$.

The formation of $\mathcal{Q}_i$ is stable under base change along U -equivariant immersions, however $\text{coker}(\phi_i)$ is not.

Lemma 5.1.1. *Let U be a unipotent group and let $\{\mathfrak{u}_i\}_{i=0}^n$ be an increasing filtration of Lie ideals. Let $f : X \rightarrow X'$ be a U -equivariant immersion of schemes of finite type over $\mathbb{k}$. For $0 \leq i \leq n$, let $\mathcal{Q}_i := \text{coker}(\Omega_{X/\mathbb{k}} \rightarrow (\mathfrak{u}_i)^* \otimes_{\mathbb{k}} \mathcal{O}_X)$ and $\mathcal{Q}'_i := \text{coker}(\Omega_{X'/\mathbb{k}} \rightarrow (\mathfrak{u}_i)^* \otimes_{\mathbb{k}} \mathcal{O}_{X'})$. Then $f^* \mathcal{Q}_i \cong \mathcal{Q}'_i$.*

Proof. By [Sta22, Lemma 01UZ], the following is surjective

$$f^* \Omega_{X'/\mathbb{k}} \longrightarrow \Omega_{X/\mathbb{k}} \longrightarrow 0. \quad (5.5)$$

Consider the diagram

$$\begin{array}{ccccccc} f^* \Omega_{X'/\mathbb{k}} & \longrightarrow & (\mathfrak{u}_i)^* \otimes_{\mathbb{k}} \mathcal{O}_X & \longrightarrow & f^* \mathcal{Q}'_i & \longrightarrow & 0 \\ \downarrow & & \parallel & & \downarrow & & \\ \Omega_{X/\mathbb{k}} & \longrightarrow & (\mathfrak{u}_i)^* \otimes_{\mathbb{k}} \mathcal{O}_X & \longrightarrow & \mathcal{Q}_i & \longrightarrow & 0 \end{array} \quad (5.6)$$

where the left square commutes since f is U -equivariant, and then the right vertical arrow exists. The top row is exact as a pull-back of an exact sequence. Since the left vertical arrow is surjective, by the Five Lemma, we have that the right vertical arrow is an isomorphism, i.e. $f^* \mathcal{Q}'_i \cong \mathcal{Q}_i$. $\square$

For fixed $1 \leq i \leq n$ and $k \in \mathbb{Z}$, let $Z_k(\mathcal{Q}_i) \hookrightarrow X$ be the closed subscheme associated to the k th Fitting ideal $\text{Fit}_k(\mathcal{Q}_i) \subseteq \mathcal{O}_X$. Then we have a decreasing chain of closed subschemes

$$X = Z_{-1}(\mathcal{Q}_i) \supseteq Z_0(\mathcal{Q}_i) \supseteq \cdots \supseteq Z_{r_1+\cdots+r_i}(\mathcal{Q}_i) = \emptyset. \quad (5.7)$$

Let $S_k(\mathcal{Q}_i) := Z_{k-1}(\mathcal{Q}_i) \setminus Z_k(\mathcal{Q}_i)$, which is an open subscheme of $Z_{k-1}(\mathcal{Q}_i)$, thus a locally closed subscheme of X .

For $\delta := (d_1, \dots, d_n) \in \mathbb{N}^n$, define

$$S_\delta := \bigcap_{i=1}^n S_{d_1+\cdots+d_i}(\mathcal{Q}_i) \cong S_{d_1}(\mathcal{Q}_1) \times_X \cdots \times_X S_{d_1+\cdots+d_n}(\mathcal{Q}_n) \quad (5.8)$$

which is a locally closed subscheme of X .

Theorem 5.1.2. *Let U be a unipotent group and let $\{\mathfrak{u}_i\}_{i=0}^n$ be an increasing filtration of Lie ideals of $\mathfrak{u}$. Let X be a scheme of finite type over $\mathbb{k}$. Let U act on X . Let $\mathcal{Q}_i$ be as above. Let $S_\delta \rightarrow X$ be immersions described above for $\delta \in \mathbb{N}^n$. Then:*

- (1) *Each S_δ is U -invariant;*
- (2) *The morphism $\coprod_{\delta \in \mathbb{N}^n} S_\delta \rightarrow X$ is a bijective monomorphism in $\text{Sch}/\mathbb{k}$;*
- (3) *Topologically $\overline{S_\delta} \subseteq \bigsqcup_{\delta' \geq \delta} S_{\delta'}$, where $(d'_1, \dots, d'_n) \geq (d_1, \dots, d_n)$ if and only if $d'_i \geq d_i$ for all $1 \leq i \leq n$;*
- (4) *For any $f : T \rightarrow X$ in $\text{Sch}/\mathbb{k}$, the sheaf $f^* \mathcal{Q}_i$ is locally free of rank $d_1 + \dots + d_i$ for $1 \leq i \leq n$ if and only if $f : T \rightarrow X$ factors as $T \rightarrow S_\delta \rightarrow X$.*

Proof. For (1), it suffices to show $Z_k(\mathcal{Q}_i)$ is U -invariant for all $1 \leq i \leq n$ and $0 \leq k \leq r_1 + \dots + r_i$. There is a natural U -linearisation on $\Omega_{X/\mathbb{k}}$. Since $\mathfrak{u}_i \subseteq \mathfrak{u}$ is invariant under conjugation by U , there is a U -linearisation on $(\mathfrak{u}_i)^* \otimes_{\mathbb{k}} \mathcal{O}_X$ such that $\Omega_{X/\mathbb{k}} \rightarrow (\mathfrak{u}_i)^* \otimes_{\mathbb{k}} \mathcal{O}_X$ is U -equivariant. Then $\mathcal{Q}_i$ as the cokernel has a U -linearisation, and then its Fitting ideals are U -invariant, so is the associated closed subschemes, i.e. $Z_k(\mathcal{Q}_i)$ is U -invariant.

For (2), we have that $S_\delta = \emptyset$ for all but finitely many $\delta \in \mathbb{N}^n$, because the related chains of Fitting ideals have finite lengths bounded uniformly by $\dim U$. In particular $\coprod_{\delta \in \mathbb{N}^n} S_\delta$ is in $\text{Sch}/\mathbb{k}$. Also, set theoretically $X = \bigsqcup_{k=0}^{r_1+\dots+r_i} S_k(\mathcal{Q}_i)$ for all $1 \leq i \leq n$. The common refinement of such decompositions over $1 \leq i \leq n$ is $X = \bigsqcup_{\delta \in \mathbb{N}^n} S_\delta$. The morphism is a monomorphism by [Sta22, Lemma 01L6].

For (3), we have $\overline{S_d(\mathcal{Q}_i)} \subseteq Z_{d-1}(\mathcal{Q}_i) = \bigsqcup_{d' \geq d} S_{d'}(\mathcal{Q}_i)$ and for $\delta = (d_1, \dots, d_n) \in \mathbb{N}^n$

$$\overline{S_\delta} = \overline{\bigcap_{i=1}^n S_{d_i}(\mathcal{Q}_i)} \subseteq \bigcap_{i=1}^n \overline{S_{d_i}(\mathcal{Q}_i)} \subseteq \bigcap_{i=1}^n \bigcup_{d'_i \geq d_i} S_{d'_i}(\mathcal{Q}_i) = \bigcup_{\delta' \geq \delta} S_{\delta'} \quad (5.9)$$

and the union is disjoint by (2).

For (4), by [Sta22, Lemma 05P8], we have that $f^* \mathcal{Q}_i$ is locally free of rank $d_1 + \dots + d_i$ if and only if $f : T \rightarrow X$ factors as $T \rightarrow S_{d_1+\dots+d_i}(\mathcal{Q}_i) \rightarrow X$. Consider for all $1 \leq i \leq n$ and we prove (4). $\square$

Remark 5.1.3. We can choose $T = \operatorname{Spec}(\mathbb{k})$ in (4) and detect closed points of S_δ . By Lemma 3.1.6, we see that the $\mathbb{k}$ -points of S_δ for $\delta = (d_1, \dots, d_n)$ are exactly the $\mathbb{k}$ -points $x \in X$ such that $\dim \operatorname{Stab}_{\mathbf{u}_i}(x) = d_1 + \dots + d_i$ for $1 \leq i \leq n$. If we work in the category of reduced schemes of finite type over $\mathbb{k}$, the reduced subscheme structure on S_δ is determined by the underlying topological subspace, and the stratification $X = \bigsqcup_{\delta \in \mathbb{N}^n} S_\delta$ is exactly the stratification given by dimensions of unipotent stabilisers.

Remark 5.1.4. The construction of S_δ only depends on the action $U \curvearrowright X$ and the filtration $\{\mathbf{u}_i\}_{i=0}^n$. If there is an action of $H = U \rtimes R$ on X and the filtration $\{\mathbf{u}_i\}_{i=0}^n$ is H -invariant, then the stratification $\coprod_{\delta \in \mathbb{N}^n} S_\delta \rightarrow X$ is H -equivariant, i.e. each S_δ is H -invariant.

Corollary 5.1.5. *Let $U, \{\mathbf{u}_i\}_{i=0}^n$ and X be as in Theorem 5.1.2. Let $\coprod_{\delta \in \mathbb{N}^n} S_\delta \rightarrow X$ be the stratification in the same theorem. Let $f : T \rightarrow X$ be a morphism in $\operatorname{Sch}/\mathbb{k}$. Let $\delta = (d_1, \dots, d_n) \in \mathbb{N}^n$. Then the following are equivalent:*

- (1) $f : T \rightarrow X$ factors as $T \rightarrow S_\delta \rightarrow X$;
- (2) for any closed subscheme $T' \hookrightarrow T$, if $f' : T' \rightarrow X$ denotes the composition $T' \hookrightarrow T \xrightarrow{f} X$, then for each $1 \leq i \leq n$, the following defines a locally free sheaf of rank $d_1 + \dots + d_i$ on T'

$$V' \mapsto \left\{ \begin{array}{l} \rho : \rho \text{ is a } V'\text{-valued infinitesimal transformation in } U \\ \rho \text{ stabilises } f'|_{V'} \end{array} \right\} \quad (5.10)$$

for open subsets $V' \subseteq T'$

where V' -valued infinitesimal transformations and they stabilising V' -valued points are defined in Definition 2.2.2 and Definition 2.2.3.

Proof. By Theorem 5.1.2, the condition that $f^* \mathcal{Q}_i$ is locally free of rank $d_1 + \dots + d_i$ for each $1 \leq i \leq n$ is equivalent to (1). By Proposition 2.2.9, the condition that $f^* \mathcal{Q}_i$ is locally free of rank $d_1 + \dots + d_i$ for each $1 \leq i \leq n$ is equivalent to (2). Therefore (1) and (2) are equivalent. $\square$

5.1.2 A special assumption

Consider the group $U \rtimes (T \times K)$ with an internal grading 1PS $\lambda : \mathbb{G}_m \rightarrow T$. Let $w_1 > \dots > w_n$ be the weights of $\lambda \curvearrowright \mathfrak{u}$ and let $\{\mathfrak{u}_i\}_{i=0}^n$ be the filtration $\mathfrak{u}_i := \mathfrak{u}_{\lambda \geq w_i}$, where $\mathfrak{u}_{\lambda \geq w_i} \subseteq \mathfrak{u}$ is the sum of λ -weight spaces whose weights are greater than or equal to w_i . Let X be a projective scheme over $\mathbb{k}$ and let L be an ample line bundle over X . Let $U \rtimes (T \times K) \curvearrowright (X, L)$.

Condition 5.1.6. Let $U \rtimes (T \times K) \curvearrowright (X, L)$. Let $r := \dim T$. Choose an identification $T \cong (\mathbb{G}_m)^r$. Let $\alpha_i, \text{Cone}_+, \sigma_I$ be as in Lemma 3.4.1. Let $p_\lambda : X_{\lambda-\min}^0 \rightarrow Z_{\lambda-\min}$ be the retraction. This condition assumes that:

- (1) A 1PS $\lambda' : \mathbb{G}_m \rightarrow T$ grades U if and only if $\lambda' \in \sigma_\emptyset$;
- (2) When $\lambda' \in \sigma_\emptyset$ is a grading 1PS, the retraction $p_{\lambda'} : X_{\lambda'-\min}^0 \rightarrow Z_{\lambda'-\min}$ does not depend on λ' ;
- (3) $Z_{\lambda-\min}^{K,s} \neq \emptyset$;
- (4) $\dim \text{Stab}_{\mathfrak{u}_i}(x) = \dim \text{Stab}_{\mathfrak{u}_i}(p_\lambda(x))$ for $1 \leq i \leq n$ and for closed points $x \in p_\lambda^{-1}(Z^{K,ss})$.

The conditions (1), (2) and (4) above are verified in [Jac21] for the non-reductive GIT problem associated to the moduli problem for sheaves of a Harder-Narasimhan type of length 2. They also hold for some special longer Harder-Narasimhan types on curves (Condition 6.6.2 below).

Apply Theorem 5.1.2 to $U \rtimes (T \times K) \curvearrowright p_\lambda^{-1}(Z_{\lambda-\min}^{K,s})$ and $\{\mathfrak{u}_i\}_{i=0}^n$ to get a stratification

$$\coprod_{\delta \in \mathbb{N}^n} S_\delta \rightarrow p_\lambda^{-1}(Z_{\lambda-\min}^{K,s}) \quad (5.11)$$

where $S_\delta \subseteq X$ for $\delta = (d_1, \dots, d_n)$ is a $U \rtimes (T \times K)$ -invariant locally closed subscheme whose closed points are those $x \in p_\lambda^{-1}(Z_{\lambda-\min}^{K,s})$ with $\dim \text{Stab}_{\mathfrak{u}_i}(x) = d_1 + \dots + d_i$ for $1 \leq i \leq n$.

Let $S_\delta \neq \emptyset$ be a non-empty stratum. Let $\overline{S_\delta} \hookrightarrow X$ be the scheme-theoretic image of $S_\delta \rightarrow X$. For the linearisation $U \rtimes (T \times K) \curvearrowright (\overline{S_\delta}, L|_{\overline{S_\delta}})$, it is easy to verify the following:

- The non-vanishing locus of $\bigoplus_{d>0} H^0(\overline{S}_\delta, (L|_{\overline{S}_\delta})^d)_{\lambda=\max}$ is

$$(\overline{S}_\delta)_{\lambda-\min}^0 = X_{\lambda-\min}^0 \cap \overline{S}_\delta \cong X_{\lambda-\min}^0 \times_X \overline{S}_\delta \quad (5.12)$$

which contains S_δ as an open subscheme.

- The fixed point subscheme of $\lambda \curvearrowright (\overline{S}_\delta)_{\lambda-\min}^0$ is

$$((\overline{S}_\delta)_{\lambda-\min}^0)^\lambda = Z_{\lambda-\min} \cap \overline{S}_\delta \cong Z_{\lambda-\min} \times_X \overline{S}_\delta. \quad (5.13)$$

- Let $((\overline{S}_\delta)_{\lambda-\min}^0)^{\dim \text{Stab} \leq \delta}$ denote the open subset of $(\overline{S}_\delta)_{\lambda-\min}^0$ whose closed points have $\dim \text{Stab}_{\mathbf{u}_i}(x) \leq d_1 + \dots + d_i$ for $1 \leq i \leq n$. As open subschemes of $\overline{S}_\delta$ we have

$$((\overline{S}_\delta)_{\lambda-\min}^0)^{\dim \text{Stab} \leq \delta} = S_\delta. \quad (5.14)$$

- If Condition 5.1.6 holds for $U \rtimes (T \times K) \curvearrowright (X, L)$, then $S_\delta \cap Z_{\lambda-\min}^{K,s} \neq \emptyset$, and we will prove that WUU (Condition 4.2.1) is satisfied for $U \rtimes (T \times K) \curvearrowright (\overline{S}_\delta, L|_{\overline{S}_\delta})$ and the 1PS λ .

Remark 5.1.7. Note that the last statement requires Condition 5.1.6, and the others are true in general. Without Condition 5.1.6, we can work on those strata such that $S_\delta \cap Z_{\lambda-\min}^{K,s} \neq \emptyset$.

Remark 5.1.8. Condition 5.1.6 is independent of λ as long as it grades U . The choice of λ affects the filtration $\mathbf{u}_i := \mathbf{u}_{\lambda \geq w_i}$ and then the stratification.

5.2 Quotients of strata

In this section, let $U \rtimes (T \times K) \curvearrowright (X, L)$, $\lambda : \mathbb{G}_m \rightarrow T$ and $\{\mathbf{u}_i\}$ be as above. We assume Condition 5.1.6.

We will prove that with Condition 5.1.6, the WUU condition (Condition 4.2.1) is satisfied for $U \rtimes (T \times K) \curvearrowright (\overline{S}_\delta, L|_{\overline{S}_\delta})$ and the 1PS λ . Then by Theorem 4.4.1, a blow-up $\widetilde{\overline{S}_\delta} \rightarrow \overline{S}_\delta$ with suitable ample linearisation satisfies CDRS (Condition 3.3.2). Moreover, the centre of the blow-up does not meet the open subscheme $S_\delta = ((\overline{S}_\delta)_{\lambda-\min}^0)^{\dim \text{Stab} \leq \delta}$, so we can regard S_δ as an open subscheme of $(\widetilde{\overline{S}_\delta})_{\lambda-\min}^0$. Theorem 3.3.8 applies to the blow-up and then $(\widetilde{\overline{S}_\delta})_{\lambda-\min}^0$ admits a universal geometric quotient by U . We can restrict to the invariant open subscheme S_δ and claim that $S_\delta \rightarrow S_\delta/U$ is a universal geometric quotient by U . For further quotients by $T \times K$, we apply Lemma 3.4.2. See the following theorem for details.

Theorem 5.2.1. *Let X be a projective scheme over $\mathbb{k}$ and let L be an ample line bundle on X . Let $U \rtimes (T \times K)$ act on X linearly with respect to L . Let $\lambda : \mathbb{G}_m \rightarrow T$ be a grading 1PS with grading weights $w_1 > \dots > w_n$ and let $\mathbf{u}_i := \mathbf{u}_{\lambda \geq w_i}$. Choose an identification $T \cong (\mathbb{G}_m)^r$ and let $\alpha_i, \text{Cone}_+, \sigma_I$ be as in Lemma 3.4.2. Assume Condition 5.1.6. Then:*

(1) *For a 1PS $\lambda' \in \sigma_I$, the retraction $p_{\lambda'} : X_{\lambda'-\min}^0 \rightarrow Z_{\lambda'-\min}$ only depends on σ_I , denoted by $p_{\sigma_I} : X_{\sigma_I-\min}^0 \rightarrow Z_{\sigma_I-\min}$.*

(2) *For closed points $x \in X_{\lambda-\min}^0$, we have $p_{\sigma_I}(x) \in X_{\lambda-\min}^0$ for all σ_I .*

(3) *For each $\delta := (d_1, \dots, d_n) \in \mathbb{N}^n$, the following defines an open subset of closed points in S_δ , which is $U \rtimes (T \times K)$ -invariant and has a universal geometric quotient by $U \rtimes (T \times K)$*

$$\left\{ x \in X_{\lambda-\min}^0 : \begin{array}{l} p_\lambda(x) \in Z_{\lambda-\min}^{K,L,s} \\ \dim \text{Stab}_{\mathbf{u}_i}(x) = d_1 + \dots + d_i \text{ for } 1 \leq i \leq n \\ U \cdot p_{\sigma_i}(x) \neq U \cdot p_{\sigma_{i-1}}(x) \text{ for } 1 \leq i \leq n \end{array} \right\} \quad (5.15)$$

where $\sigma_i := \sigma_{\{1, \dots, i\}}$.

Proof. The statements (1) and (2) are immediate from Lemma 3.4.2.

Stratify $p_\lambda^{-1}(Z_{\lambda-\min}^{K,s})$ with respect to the action of U and the filtration $\{\mathbf{u}_i\}$ by Theorem 5.1.2

$$\coprod_{\delta \in \mathbb{N}^n} S_\delta \rightarrow p_\lambda^{-1}(Z_{\lambda-\min}^{K,s}). \quad (5.16)$$

Let $S_\delta \neq \emptyset$ be a non-empty stratum. Let $\overline{S_\delta} \hookrightarrow X$ be the scheme-theoretic image of $S_\delta \rightarrow X$. Let $(\overline{S_\delta})_{\lambda-\min}^0 \subseteq \overline{S_\delta}$ be the non-vanishing locus of $\bigoplus_{d>0} H^0(\overline{S_\delta}, (L|_{\overline{S_\delta}})^d)_{\lambda=\max}$. Then $S_\delta \subseteq (\overline{S_\delta})_{\lambda-\min}^0$, since $(\overline{S_\delta})_{\lambda-\min}^0 = \overline{S_\delta} \cap X_{\lambda-\min}^0$. Moreover $S_\delta \subseteq p_\lambda^{-1}(Z_{\lambda-\min}^{K,s})$. Let $\mathcal{K}_i, \mathcal{Q}_i$ for $0 \leq i \leq n$ be kernels and cokernels

$$\begin{aligned} \mathcal{K}_i &:= \ker(\Omega_{X/\mathbb{k}} \rightarrow \mathbf{u}_i^* \otimes_{\mathbb{k}} \mathcal{O}_X) \\ \mathcal{Q}_i &:= \text{coker}(\Omega_{X/\mathbb{k}} \rightarrow \mathbf{u}_i^* \otimes_{\mathbb{k}} \mathcal{O}_X). \end{aligned} \quad (5.17)$$

There exists an exact sequence for $1 \leq i \leq n$

$$0 \longrightarrow \mathcal{K}_i \longrightarrow \mathcal{K}_{i-1} \xrightarrow{\phi_i} (\mathbf{u}_i/\mathbf{u}_{i-1})^* \otimes_{\mathbb{k}} \mathcal{O}_X \xrightarrow{\varphi_i} \mathcal{Q}_i \longrightarrow \mathcal{Q}_{i-1} \longrightarrow 0 \quad (5.18)$$

and in particular the following is a short exact sequence

$$0 \longrightarrow \operatorname{coker}(\phi_i) \longrightarrow \mathcal{Q}_i \longrightarrow \mathcal{Q}_{i-1} \longrightarrow 0. \quad (5.19)$$

Similarly for $0 \leq i \leq n$ define $\mathcal{K}_{\bar{\delta},i}, \mathcal{Q}_{\bar{\delta},i}$ on $\bar{S}_\delta$ and $\mathcal{K}_{\delta,i}, \mathcal{Q}_{\delta,i}$ on S_δ as follows

$$\begin{aligned} \mathcal{K}_{\bar{\delta},i} &:= \ker(\Omega_{\bar{S}_\delta/\mathbb{k}} \rightarrow \mathbf{u}_i^* \otimes_{\mathbb{k}} \mathcal{O}_{\bar{S}_\delta}) \\ \mathcal{Q}_{\bar{\delta},i} &:= \operatorname{coker}(\Omega_{\bar{S}_\delta/\mathbb{k}} \rightarrow \mathbf{u}_i^* \otimes_{\mathbb{k}} \mathcal{O}_{\bar{S}_\delta}) \\ \mathcal{K}_{\delta,i} &:= \ker(\Omega_{S_\delta/\mathbb{k}} \rightarrow \mathbf{u}_i^* \otimes_{\mathbb{k}} \mathcal{O}_{S_\delta}) \\ \mathcal{Q}_{\delta,i} &:= \operatorname{coker}(\Omega_{S_\delta/\mathbb{k}} \rightarrow \mathbf{u}_i^* \otimes_{\mathbb{k}} \mathcal{O}_{S_\delta}) \end{aligned} \quad (5.20)$$

and define $\phi_{\bar{\delta},i}, \varphi_{\bar{\delta},i}$ on $\bar{S}_\delta$ and $\phi_{\delta,i}, \varphi_{\delta,i}$ on S_δ as follows

$$\begin{aligned} \mathcal{K}_{\bar{\delta},i-1} &\xrightarrow{\phi_{\bar{\delta},i}} (\mathbf{u}_i/\mathbf{u}_{i-1})^* \otimes_{\mathbb{k}} \mathcal{O}_{\bar{S}_\delta} \xrightarrow{\varphi_{\bar{\delta},i}} \mathcal{Q}_{\bar{\delta},i} \longrightarrow \mathcal{Q}_{\bar{\delta},i-1} \longrightarrow 0 \\ \mathcal{K}_{\delta,i-1} &\xrightarrow{\phi_{\delta,i}} (\mathbf{u}_i/\mathbf{u}_{i-1})^* \otimes_{\mathbb{k}} \mathcal{O}_{S_\delta} \xrightarrow{\varphi_{\delta,i}} \mathcal{Q}_{\delta,i} \longrightarrow \mathcal{Q}_{\delta,i-1} \longrightarrow 0. \end{aligned} \quad (5.21)$$

We first check the WUU condition (Condition 4.2.1) for $U \rtimes (T \times K) \curvearrowright (\bar{S}_\delta, L|_{\bar{S}_\delta})$ and the 1PS λ . Denote the pull-back of $\mathcal{Q}_i$ along $S_\delta \rightarrow X$ by $\mathcal{Q}_i|_{S_\delta}$ and similarly for other immersions. By Lemma 5.1.1, we have

$$\mathcal{Q}_i|_{\bar{S}_\delta} \cong \mathcal{Q}_{\bar{\delta},i}, \quad \mathcal{Q}_{\bar{\delta},i}|_{S_\delta} \cong \mathcal{Q}_{\delta,i}. \quad (5.22)$$

By the universal property of S_δ , we have that $\mathcal{Q}_{\bar{\delta},i}|_{S_\delta} \cong \mathcal{Q}_i|_{S_\delta}$ is locally free of rank $d_1 + \dots + d_i$ for $1 \leq i \leq n$. On $\bar{S}_\delta$, for $1 \leq i \leq n$, consider the Fitting ideals $\operatorname{Fit}_k(\operatorname{coker}(\phi_{\bar{\delta},i})) \subseteq \mathcal{O}_{\bar{S}_\delta}$ for $k \in \mathbb{Z}$, whose associated closed subschemes are denoted by $Z_k(\operatorname{coker}(\phi_{\bar{\delta},i})) \hookrightarrow \bar{S}_\delta$. Since $S_\delta \subseteq \bar{S}_\delta$ is open, the following is exact for $1 \leq i \leq n$

$$0 \longrightarrow \operatorname{coker}(\phi_{\bar{\delta},i})|_{S_\delta} \longrightarrow \mathcal{Q}_{\bar{\delta},i}|_{S_\delta} \longrightarrow \mathcal{Q}_{\bar{\delta},i-1}|_{S_\delta} \longrightarrow 0. \quad (5.23)$$

Then $\operatorname{coker}(\phi_{\bar{\delta},i})|_{S_\delta}$ is finite locally free, since $\mathcal{Q}_{\bar{\delta},i}|_{S_\delta}, \mathcal{Q}_{\bar{\delta},i-1}|_{S_\delta}$ are finite locally free ([Sta22, Lemma 00NX], [Sta22, Lemma 00HM]). The rank of $\operatorname{coker}(\phi_{\bar{\delta},i})|_{S_\delta}$ is d_i . Since Fitting ideals can be calculated locally, we have

$$\operatorname{Fit}_{d_i-1}(\operatorname{coker}(\phi_{\bar{\delta},i}))|_{S_\delta} = 0, \quad \operatorname{Fit}_{d_i}(\operatorname{coker}(\phi_{\bar{\delta},i}))|_{S_\delta} = \mathcal{O}_{S_\delta}. \quad (5.24)$$

Therefore the open immersion $S_\delta \rightarrow \overline{S_\delta}$ factors as $S_\delta \rightarrow Z_{d_i-1}(\text{coker}(\phi_{\overline{\delta},i})) \hookrightarrow \overline{S_\delta}$. Since $\overline{S_\delta}$ is the scheme-theoretic closure of $S_\delta \subseteq \overline{S_\delta}$, we have $Z_{d_i-1}(\text{coker}(\phi_{\overline{\delta},i})) = \overline{S_\delta}$. This proves that on $\overline{S_\delta}$

$$\text{Fit}_{d_i-1}(\text{coker}(\phi_{\overline{\delta},i})) = 0, \quad \text{for } 1 \leq i \leq n. \quad (5.25)$$

It is easy to see

$$d_i := \min \{k \in \mathbb{N} : \text{Fit}_k(\text{coker}(\phi_{\overline{\delta},i})) \neq 0\}, \quad \text{for all } 1 \leq i \leq n$$

$$\left(\prod_{i=1}^n \text{Fit}_{d_i}(\text{coker}(\phi_{\overline{\delta},i})) \right) \Big|_{S_\delta} = \mathcal{O}_{S_\delta}. \quad (5.26)$$

With Condition 5.1.6, we have

$$\emptyset \neq S_\delta \cap Z_{\lambda-\min} \subseteq \overline{S_\delta} \setminus \mathbb{V} \left(\prod_{i=1}^n \text{Fit}_{d_i}(\text{coker}(\phi_{\overline{\delta},i})) \right) \quad (5.27)$$

which verifies the WUU condition (Condition 4.2.1). Then a suitable blow-up $\widetilde{\overline{S_\delta}} \rightarrow \overline{S_\delta}$ with a suitable ample linearisation $\tilde{L}_\delta$ satisfies CDRS (Condition 3.3.2) by Theorem 4.4.1. Moreover, the centre of the blow-up does not meet S_δ .

Assumptions in Lemma 3.4.2 are satisfied for $U \rtimes (T \times K) \curvearrowright (\widetilde{\overline{S_\delta}}, \tilde{L}_\delta)$. Apply Lemma 3.4.2 and we have that the following is $U \rtimes (T \times K)$ -invariant and open in $\widetilde{\overline{S_\delta}}$ and it admits a quasi-projective universal geometric quotients by $U \rtimes (T \times K)$

$$\left\{ x \in (\widetilde{\overline{S_\delta}})^0_{\lambda-\min} : \begin{array}{l} \tilde{p}_\lambda(x) \text{ is } K\text{-stable} \\ U.\tilde{p}_{\sigma_{i-1}}(x) \neq U.\tilde{p}_{\sigma_i}(x) \text{ for } 1 \leq i \leq n \end{array} \right\} \quad (5.28)$$

where $\tilde{p}_{\sigma_i}, \tilde{p}_\lambda$ are retractions for the blow-up. Since S_δ does not meet the blow-up centre, we can view S_δ as a $U \rtimes (T \times K)$ open subscheme of $\widetilde{\overline{S_\delta}}$. The intersection of S_δ and the above subset is open in S_δ whose closed points form the set

$$\left\{ x \in X_{\lambda-\min}^0 : \begin{array}{l} p_\lambda(x) \in Z_{\lambda-\min}^{K,L,s} \\ \dim \text{Stab}_{u_i}(x) = d_1 + \dots + d_i \text{ for } 1 \leq i \leq n \\ U.p_{\sigma_i}(x) \neq U.p_{\sigma_{i-1}}(x) \text{ for } 1 \leq i \leq n \end{array} \right\} \subseteq S_\delta \quad (5.29)$$

which defines an open subscheme of S_δ , and this open subscheme of S_δ admits a universal geometric quotient by $U \rtimes (T \times K)$. This completes the proof. $\square$

Chapter 6

Moduli spaces of unstable sheaves

Let $\mathbb{k}$ be an algebraically closed field of characteristic zero. Let X be a connected projective scheme and let $\mathcal{L}$ be an ample line bundle on X . Let $\text{Sch}/\mathbb{k}$ be the category of schemes of finite type over $\mathbb{k}$. For a Harder-Narasimhan type τ of length 2 and a natural number $\delta \in \mathbb{N}$, we will consider a moduli functor $\mathbf{M}'_{\tau,\delta} : (\text{Sch}/\mathbb{k})^{\text{op}} \rightarrow \text{Set}$ classifying (τ, δ) -stable sheaves (Definition 6.3.10). We will prove that $\mathbf{M}'_{\tau,\delta}$ has a coarse moduli space, which can be constructed using non-reductive GIT developed previously (Theorem 6.5.7). We will extend this construction to some special Harder-Narasimhan types of length greater than 2 (See Condition 6.6.2).

The moduli problem of sheaves of length 2 has been considered by Jackson in [Jac18] and [Jac21], and Brambila-Paz et al in [BMN09]. In [BMN09], moduli stacks for vector bundles of rank 2 over projective curves were described, and their moduli schemes were constructed, using Lange's universal extensions [Lan83]. Using NRGIT, Jackson constructed moduli spaces of sheaves of length two over polarised projective schemes. The NRGIT Jackson applied only produces quotient varieties, however as indicated in [BMN09] Remark 7.8, moduli schemes of vector bundles of rank 2 over curves can be non-reduced in some situations. Our results generalise theirs. We do not restrict to vector bundles of rank 2 over projective curves but can consider pure sheaves of length 2 over polarised projective schemes. Also, our moduli schemes are not necessarily reduced.

6.1 Preliminaries

6.1.1 Representable functors

Let $\text{Sch}/\mathbb{k}$ be the category of schemes of finite type over $\mathbb{k}$. A *presheaf of sets on $\text{Sch}/\mathbb{k}$* is a contravariant functor $\mathbf{F} : (\text{Sch}/\mathbb{k})^{\text{op}} \rightarrow \text{Set}$. Let $\text{PSh}(\text{Sch}/\mathbb{k})$ be the category of presheaves on $\text{Sch}/\mathbb{k}$ where morphisms are natural transformations. Let $h_{(-)} : \text{Sch}/\mathbb{k} \rightarrow \text{PSh}(\text{Sch}/\mathbb{k})$ be the Yoneda functor, sending an object $X \in \text{Sch}/\mathbb{k}$ to the presheaf $h_X := \text{Hom}_{\text{Sch}/\mathbb{k}}(-, X)$. The Yoneda functor is fully faithful.

Definition 6.1.1. Let $\mathbf{M} : (\text{Sch}/\mathbb{k})^{\text{op}} \rightarrow \text{Set}$ be a presheaf. Let $M \in \text{Sch}/\mathbb{k}$ be an object. Then:

- We say that M *represents $\mathbf{M}$* or $\mathbf{M}$ is *represented by M* , if $\mathbf{M} \cong h_M$ in $\text{PSh}(\text{Sch}/\mathbb{k})$, i.e. there exists a bijective map

$$\mathbf{M}(T) \cong h_M(T) = \text{Hom}_{\text{Sch}/\mathbb{k}}(T, M) \quad (6.1)$$

for any $T \in \text{Sch}/\mathbb{k}$ and the family of maps $\{\mathbf{M}(T) \rightarrow h_M(T)\}$ is natural in $T \in \text{Sch}/\mathbb{k}$;

- We say that M *corepresents $\mathbf{M}$* or $\mathbf{M}$ is *corepresented by M* , if there exists a natural transformation $\eta : \mathbf{M} \rightarrow h_M$ such that for every $T \in \text{Sch}/\mathbb{k}$ and every natural transformation $\mathbf{M} \rightarrow h_T$, there exists a unique morphism $f : M \rightarrow T$ such that the diagram commutes

$$\begin{array}{ccc} \mathbf{M} & \longrightarrow & h_T; \\ \downarrow \eta & \nearrow h_f & \\ h_M & & \end{array} \quad (6.2)$$

- $\mathbf{M}$ is called *representable* if there exists $M \in \text{Sch}/\mathbb{k}$ which represents $\mathbf{M}$;
- $\mathbf{M}$ is called *corepresentable* if there exists $M \in \text{Sch}/\mathbb{k}$ which corepresents $\mathbf{M}$.

Definition 6.1.2. Let $\mathbf{M}, \mathbf{N}$ be presheaves of sets on $\text{Sch}/\mathbb{k}$. We call $\mathbf{M}$ a *sub-functor* of $\mathbf{N}$ if there exists a natural transformation $\mathbf{M} \rightarrow \mathbf{N}$ such that on each $T \in \text{Sch}/\mathbb{k}$, the map $\mathbf{M}(T) \rightarrow \mathbf{N}(T)$ is injective. We denote the sub-functor by $\mathbf{M} \subseteq \mathbf{N}$.

For a sub-functor $\mathbf{M} \subseteq \mathbf{N}$, we call it a *sub-functor representable by open (resp. closed, locally closed) subschemes* if for any $T \in \text{Sch}/\mathbb{k}$ and any natural transformation $\xi : h_T \rightarrow \mathbf{N}$,

there exists an open (resp. closed, locally closed) subscheme $T_\xi \subseteq T$ such that the following diagram is Cartesian in $\mathbf{PSh}(\mathrm{Sch}/\mathbb{k})$

$$\begin{array}{ccc} h_{T_\xi} & \xrightarrow{\subseteq} & h_T \\ \downarrow & & \downarrow \xi \\ \mathbf{M} & \xrightarrow{\subseteq} & \mathbf{N}. \end{array} \quad (6.3)$$

Definition 6.1.3. Let $\mathbf{M}$ be a presheaf. A *fine moduli space* of $\mathbf{M}$ is an object $M \in \mathrm{Sch}/\mathbb{k}$ which represents $\mathbf{M}$. A *coarse moduli space* is an object $M \in \mathrm{Sch}/\mathbb{k}$ with a natural transformation $\eta : \mathbf{M} \rightarrow h_M$ such that M corepresents $\mathbf{M}$ with respect to η , and $\eta : \mathbf{M} \rightarrow h_M$ is a isomorphism on $\mathrm{Spec}(\mathbb{k})$

$$\eta_{\mathrm{Spec}(\mathbb{k})} : \mathbf{M}(\mathrm{Spec}(\mathbb{k})) \cong h_M(\mathrm{Spec}(\mathbb{k})) = \mathrm{Hom}_{\mathrm{Sch}/\mathbb{k}}(\mathrm{Spec}(\mathbb{k}), M). \quad (6.4)$$

We see that if M represents $\mathbf{M}$, then M corepresents $\mathbf{M}$. Similarly, fine moduli spaces are coarse moduli spaces, but the inverse implication is not true in general.

6.1.2 Definitions for sheaves

Recall that $\mathbb{k}$ is an algebraically closed field of characteristic zero, X is a connected projective scheme over $\mathbb{k}$ and $\mathcal{L}$ is an ample line bundle on X . We introduce some definitions for coherent sheaves.

Definition 6.1.4. Let $\mathcal{F}$ be a coherent sheaf on X . The *Euler characteristic* of $\mathcal{F}$ is

$$\chi(X, \mathcal{F}) := \sum_{i \in \mathbb{N}} (-1)^i \dim_{\mathbb{k}} H^i(X, \mathcal{F}). \quad (6.5)$$

The *Hilbert polynomial* of $\mathcal{F}$ with respect to $\mathcal{L}$ is

$$P(\mathcal{F}, t) := \chi(X, \mathcal{F} \otimes \mathcal{L}^t) \in \mathbb{Q}[t] \quad (6.6)$$

where $\mathcal{L}^t$ is the t th tensor power of $\mathcal{L}$ over $\mathcal{O}_X$.

Euler characteristics and Hilbert polynomials are additive ([Sta22, Section 0BEI]) in the sense that if

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0 \quad (6.7)$$

is an exact sequence of coherent sheaves, then

$$\chi(X, \mathcal{F}_2) = \chi(X, \mathcal{F}_1) + \chi(X, \mathcal{F}_3), \quad P(\mathcal{F}_2, t) = P(\mathcal{F}_1, t) + P(\mathcal{F}_3, t). \quad (6.8)$$

Definition 6.1.5. Let $\mathcal{F}$ be a coherent sheaf on X . The *support* of $\mathcal{F}$ is a subset of X

$$\text{Supp}(\mathcal{F}) := \{x \in X : \mathcal{F}_x \neq 0\} \quad (6.9)$$

which is a closed subset ([Sta22, Lemma 056J]). The *dimension* of $\mathcal{F}$ is

$$\dim \mathcal{F} := \dim \text{Supp}(\mathcal{F}). \quad (6.10)$$

The dimension of $\mathcal{F}$ equals the degree of its Hilbert polynomial.

Definition 6.1.6. Let $d \in \mathbb{N}$. Let $\mathcal{F}$ be a coherent sheaf on X . It is called *pure of dimension* d if $\dim(\mathcal{F}') = d$ for all non-zero coherent sub-sheaves $0 \neq \mathcal{F}' \subseteq \mathcal{F}$. Let $\alpha_d(\mathcal{F}) \in \mathbb{N}_+$ and $\alpha_{d-1}(\mathcal{F}) \in \mathbb{Z}$ be such that

$$P(\mathcal{F}, t) = \alpha_d(\mathcal{F}) \frac{t^d}{d!} + \alpha_{d-1}(\mathcal{F}) \frac{t^{d-1}}{(d-1)!} + \cdots. \quad (6.11)$$

The *slope* (Definition 1.6.8, [HL10]) of $\mathcal{F}$ is $\hat{\mu}(\mathcal{F}) := \frac{\alpha_{d-1}(\mathcal{F})}{\alpha_d(\mathcal{F})}$. The *reduced Hilbert polynomial* is

$$p(\mathcal{F}, t) := \frac{P(\mathcal{F}, t)}{\alpha_d(\mathcal{F})}. \quad (6.12)$$

Consider the lexicographic order on $\mathbb{Q}[t]$, i.e. $f(t) < g(t)$ if and only if $f(m) < g(m)$ for $m \gg 0$.

Definition 6.1.7. Let $\mathcal{F}$ be pure sheaf of dimension d . It is called *semistable* if $p(\mathcal{F}', t) \leq p(\mathcal{F}, t)$ for any non-zero coherent sub-sheaf $0 \neq \mathcal{F}' \subseteq \mathcal{F}$. The sheaf $\mathcal{F}$ is called *stable* if $p(\mathcal{F}', t) < p(\mathcal{F}, t)$ for any proper coherent sub-sheaf $0 \neq \mathcal{F}' \subsetneq \mathcal{F}$.

Definition 6.1.8. Let $\mathcal{F}$ be a pure sheaf of dimension d . The *Harder-Narasimhan filtration* of $\mathcal{F}$ is an increasing chain of pure sheaves of dimension d

$$0 = \text{HN}_0(\mathcal{F}) \subsetneq \cdots \subsetneq \text{HN}_n(\mathcal{F}) = \mathcal{F} \quad (6.13)$$

such that:

- $\text{HN}_i(\mathcal{F})/\text{HN}_{i-1}(\mathcal{F})$ is semi-stable for $1 \leq i \leq n$;
- $p_1 > \cdots > p_l$ for $p_i(t) := p_i(\text{HN}_i(\mathcal{F})/\text{HN}_{i-1}(\mathcal{F}), t)$.

Theorem 6.1.9 ([HL10] Theorem 1.3.4). *Every pure sheaf has a unique Harder-Narasimhan filtration.*

Definition 6.1.10. Let $\mathcal{F}$ be a semistable sheaf of dimension d . A *Jordan-Hölder filtration* of $\mathcal{F}$ is a filtration

$$0 = \mathrm{JH}_0(\mathcal{F}) \subsetneq \cdots \subsetneq \mathrm{JH}_n(\mathcal{F}) = \mathcal{F} \quad (6.14)$$

such that $\mathrm{JH}_i(\mathcal{F})/\mathrm{JH}_{i-1}(\mathcal{F})$ are stable of the same reduced Hilbert polynomial for $1 \leq i \leq n$.

Remark 6.1.11. Jordan-Hölder filtrations are not unique.

Theorem 6.1.12 ([HL10], Proposition 1.5.2). *Jordan-Hölder filtrations always exist. The associated graded sheaf $\bigoplus_{i=1}^n \mathrm{JH}_i(\mathcal{F})/\mathrm{JH}_{i-1}(\mathcal{F})$ does not depend on the choice of the Jordan-Hölder filtration.*

A polynomial $f \in \mathbb{Q}[t]$ is called *numerical* if $f(\mathbb{Z}) \subseteq \mathbb{Z}$. For a numerical polynomial of degree d , we can define $\alpha_d(f)$ similarly such that $\alpha_d(f) \frac{t^d}{d!}$ is the leading monomial of f .

Definition 6.1.13. A *Harder-Narasimhan type* τ consists of the following data:

- the length $n \in \mathbb{N}_+$;
- the dimension $d \in \mathbb{N}$;
- numerical polynomials $P_1(t), \dots, P_n(t) \in \mathbb{Q}[t]$ of degree d such that

$$p_1(t) > \cdots > p_n(t), \quad p_i(t) := \frac{P_i(t)}{\alpha_d(P_i(t))} \quad (6.15)$$

For $\mathcal{F}$ a pure sheaf of dimension d , we say that its Harder-Narasimhan type is τ if:

- the Harder-Narasimhan filtration of $\mathcal{F}$ has length $n \in \mathbb{N}_+$;
- $P_i(t)$ is the Hilbert polynomial of $\mathrm{HN}_i(\mathcal{F})/\mathrm{HN}_{i-1}(\mathcal{F})$ for $i = 1, \dots, n$

where $\{\mathrm{HN}_i(\mathcal{F})\}_{i=0}^l$ is the Harder-Narasimhan filtration of $\mathcal{F}$.

6.1.3 Definitions for vector bundles on curves

Let C be a connected nonsingular projective curve on $\mathbb{k}$ of genus $g \in \mathbb{N}$. By *vector bundles*, we refer to locally free sheaves of finite rank. For non-zero coherent sheaves $\mathcal{F}$ on C , the following are equivalent:

- $\mathcal{F}$ is a vector bundle;
- $\mathcal{F}$ is torsion free;
- $\mathcal{F}$ is pure of dimension one.

Therefore the definitions for sheaves above apply to vector bundles on the curve C . It turns out the choice of ample line bundles does not make a difference.

Ranks are defined for coherent sheaves of top dimension when we consider general $(X, \mathcal{L})$. On the curve C , we can define ranks for all coherent sheaves. We define *ranks*, *degrees* and *slopes* as follows. Let $\mathcal{F}$ be a coherent sheaf on C . Then:

- the *rank* of $\mathcal{F}$ is

$$\text{rank}(\mathcal{F}) := \dim_{\kappa(\eta)} \mathcal{F}_\eta; \quad (6.16)$$

where $\eta \in C$ is the generic point and $\mathcal{F}_\eta$ is the stalk at η ;

- the *degree* of $\mathcal{F}$ is

$$\deg(\mathcal{F}) := \chi(C, \mathcal{F}) - \text{rank}(\mathcal{F})\chi(C, \mathcal{O}_C); \quad (6.17)$$

- the *slope* of $\mathcal{F}$ when $\text{rank}(\mathcal{F}) \neq 0$ is

$$\mu(\mathcal{F}) := \frac{\deg(\mathcal{F})}{\text{rank}(\mathcal{F})}. \quad (6.18)$$

It is easy to see the functions $\text{rank}, \deg$ are additive. The definitions justify their names on vector bundles. Note that ranks, degrees and slopes are defined without an ample line bundle $\mathcal{L}$.

Remark 6.1.14. We have defined two slopes $\hat{\mu}$ (in Definition 6.1.6) and μ (at equation (6.18)). Over a curve of genus g , they differ by a constant $\hat{\mu} = \mu + 1 - g$.

We calculate these discrete invariants for some coherent sheaves. Let $x \in C$ be a closed point. Let $\mathcal{O}_C(-x) \subseteq \mathcal{O}_C$ denote the sub-sheaf of functions vanishing at $x \in C$. Then $\mathcal{O}_C(-x)$ is a line bundle. Let k_x denote the cokernel of $\mathcal{O}_C(-x) \rightarrow \mathcal{O}_C$

$$0 \longrightarrow \mathcal{O}_C(-x) \longrightarrow \mathcal{O}_C \longrightarrow k_x \longrightarrow 0. \quad (6.19)$$

Then k_x is the skyscraper sheaf at x with value $\mathbb{k}$. Let $\mathcal{O}_C(x)$ denote the dual line bundle of $\mathcal{O}_C(-x)$. The tensor product over $\mathcal{O}_C$ of the above short exact sequence with $\mathcal{O}_C(x)$ is also exact

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_C(x) \longrightarrow k_x \longrightarrow 0. \quad (6.20)$$

From the definition, we can easily write down the discrete invariants of $\mathcal{O}_C$. The discrete invariants of k_x can be calculated by [Sta22, Lemma 0AYT]. Then by additivity we get the discrete invariants of $\mathcal{O}_C(x)$. To sum up we have

$$\begin{aligned} \chi(C, \mathcal{O}_C) &= 1 - g, \quad \text{rank}(\mathcal{O}_C) = 1, \quad \deg(\mathcal{O}_C) = 0 \\ \chi(C, \mathcal{O}_C(x)) &= 2 - g, \quad \text{rank}(\mathcal{O}_C(x)) = 1, \quad \deg(\mathcal{O}_C(x)) = 1 \\ \chi(C, k_x) &= 1, \quad \text{rank}(k_x) = 0, \quad \deg(k_x) = 1. \end{aligned} \quad (6.21)$$

On a general projective scheme X , Hilbert polynomials depend on an ample line bundle, but on the curve C , the choice of line bundle does not affect the Hilbert polynomials, provided the line bundles have the same degree. By [Sta22, Lemma 0B5X], on the curve C , a line bundle is ample if and only if its degree is positive. In particular, the line bundle $\mathcal{O}_C(x)$ is ample.

Let $\mathcal{L}$ be any line bundle of degree one on C . We have $\deg(\mathcal{L}^t) = t$ for $t \in \mathbb{Z}$ ([Sta22, Lemma 0AYX]). By [Sta22, Lemma 0AYV], we have

$$\begin{aligned} \chi(C, \mathcal{L}^t \otimes \mathcal{F}) &= \text{rank}(\mathcal{F}) \deg(\mathcal{L}^t) + \text{rank}(\mathcal{L}^t) \chi(C, \mathcal{F}) \\ &= \text{rank}(\mathcal{F})t + (\deg(\mathcal{F}) + \text{rank}(\mathcal{F}) \chi(C, \mathcal{O}_C)) \\ &= \text{rank}(\mathcal{F})t + \deg(\mathcal{F}) + \text{rank}(\mathcal{F})(1 - g) \end{aligned} \quad (6.22)$$

which is the *Riemann-Roch theorem*. The above function of $t \in \mathbb{Z}$ is a polynomial of degree ≤ 1 , and it is independent of the choice of $\mathcal{L}$. The *Hilbert polynomial* of $\mathcal{F}$ is this polynomial

$$P(\mathcal{F}, t) := \text{rank}(\mathcal{F})t + \deg(\mathcal{F}) + \text{rank}(\mathcal{F})(1 - g) \quad (6.23)$$

which contains the same data as the degree and rank of $\mathcal{F}$. The *reduced Hilbert polynomial* of $\mathcal{F}$ when $\text{rank}(\mathcal{F}) \neq 0$ is

$$p(\mathcal{F}, t) := \frac{P(\mathcal{F}, t)}{\text{rank}(\mathcal{F})} = t + \mu(\mathcal{F}) + 1 - g \quad (6.24)$$

which contains the same data as the slope of $\mathcal{F}$. The notions of stability, semistability are defined for vector bundles on C , i.e. pure sheaves of dimension one (Recall that we only consider locally free sheaves of finite rank). Let E be a non-zero vector bundle over C . The vector bundle E is *semistable* if for any non-zero coherent sub-sheaf $0 \neq F \subseteq E$, we have $\mu(F) \leq \mu(E)$. It is *stable* if for any proper non-zero coherent sub-sheaf $0 \neq F \subsetneq E$, we have $\mu(F) < \mu(E)$. Again, the choice of ample line bundle makes no difference.

For any non-zero vector bundle E over C , the *Harder-Narasimhan filtration* is the unique filtration of sub-bundles

$$0 = \text{HN}_0(E) \subsetneq \text{HN}_1(E) \subsetneq \cdots \subsetneq \text{HN}_n(E) = E \quad (6.25)$$

such that:

- each $\text{HN}_i(E)/\text{HN}_{i-1}(E)$ is a semistable vector bundle;
- $\mu_1 > \cdots > \mu_n$ for $\mu_i := \mu(\text{HN}_i(E)/\text{HN}_{i-1}(E))$.

Let $\mu_{\max} = \mu_1$ and $\mu_{\min} = \mu_n$. In particular, $\mu_{\max}$ is the *maximal slope of sub-bundles*, i.e. $\mu(F) \leq \mu_{\max}$ for all non-zero vector sub-bundles $0 \subsetneq F \subseteq E$, and $\mu_{\min}$ is the *minimal slope of saturated quotient bundles*, i.e. $\mu(Q) \geq \mu_{\min}$ for all nonzero saturated quotients $E \rightarrow Q$.

We reformulate the notion of Harder-Narasimhan type for vector bundles on C in terms of slopes.

Definition 6.1.15. A *Harder-Narasimhan type* τ consists of the following data:

- the length $n \in \mathbb{N}_+$;
- ranks $r_1, \dots, r_n \in \mathbb{N}_+$ and slopes $\mu_1 > \cdots > \mu_n$.

For the type τ of length $n \in \mathbb{N}_+$, define polynomials $P(\tau, t), P_i(\tau, t)$ for $1 \leq i \leq n$

$$\begin{aligned} P_i(\tau, t) &:= r_i t + r_i \mu_i + r_i(1 - g) \in \mathbb{Z}[t], \quad i = 1, \dots, n \\ P(\tau, t) &:= P_1(\tau, t) + \dots + P_n(\tau, t). \end{aligned} \tag{6.26}$$

We say a vector bundle E has Harder-Narasimhan type τ , if the Harder-Narasimhan filtration of E has length n and for $1 \leq i \leq n$

$$\text{rank}(\text{HN}_i(E)/\text{HN}_{i-1}(E)) = r_i, \quad \deg(\text{HN}_i(E)/\text{HN}_{i-1}(E)) = r_i \mu_i \tag{6.27}$$

where $\{\text{HN}_i(E)\}_{i=0}^n$ is the Harder-Narasimhan filtration of E .

The following lists some facts about vector bundles over curves.

Lemma 6.1.16 (Section 8, [Hos16]). *Let C be a nonsingular projective curve of genus g over $\mathbb{k}$. Let E, F be vector bundles over C . Then:*

- *The dualising sheaf over C is a line bundle of degree $2g - 2$;*
- *If $\mu_{\min}(E) > \mu_{\max}(F)$, then $\text{Hom}(E, F) = 0$;*
- *$\mu_{\max}(E \otimes F) = \mu_{\max}(E) + \mu_{\max}(F)$;*
- *If $\mu_{\max}(E) - \mu_{\min}(F) < 2 - 2g$, then $\text{Ext}^1(E, F) = 0$.*

6.2 Quot functors and Quot schemes

In this section, we will introduce Quot functors and sketch their representability by Quot schemes. We will also introduce some special properties of the Quot schemes we need for the moduli of sheaves.

6.2.1 Quot functors and Quot schemes in general

Let $\mathbb{k}$ be an algebraically closed field of characteristic zero. Let S be a scheme of finite type over $\mathbb{k}$. Let Sch/S be the category of schemes of finite type over S . In other sections, we only consider $S = \text{Spec}(\mathbb{k})$.

Let $f : X \rightarrow S$ be a projective morphism of finite type. Let $\mathcal{L}$ be an f -ample line bundle, in the sense that there is a closed immersion $X \hookrightarrow \mathbb{P}_S^N$ and the pull-back of $\mathcal{O}_{\mathbb{P}_S^N}(1)$

is a positive tensor power of $\mathcal{L}$. Let $\mathcal{F}$ be a coherent sheaf on X . Let $P(t) \in \mathbb{Q}[t]$ be a numerical polynomial.

For $T \rightarrow S$ in Sch/S , a *family of quotients of $\mathcal{F}$ parameterised by T with Hilbert polynomial P* is a pair $(\mathcal{E}, q)$ where

- $\mathcal{E}$ is a coherent sheaf on $X_T := X \times_S T$, flat over T , and $\mathcal{E}_t$ on $X \times_S \text{Spec}(\kappa(t))$ has Hilbert polynomial P with respect to $\mathcal{L}$, where $\mathcal{E}_t$ is the pull-back of $\mathcal{E}$ along $X \times_S \text{Spec}(\kappa(t)) \hookrightarrow X \times_S T$;
- $q : \mathcal{F}_T \rightarrow \mathcal{E}$ is surjective, where $\mathcal{F}_T$ on $X \times_S T$ is the pull-back of $\mathcal{F}$ along $X \times_S T \rightarrow X$.

We also call $(\mathcal{E}, q)$ a *family of quotients of $\mathcal{F}$ over T with Hilbert polynomial P* . Define an equivalence relation between such families over T such that two families $(\mathcal{E}, q)$ and $(\mathcal{E}', q')$ are equivalent if there is an isomorphism $\mathcal{E} \rightarrow \mathcal{E}'$ compatible with q, q'

$$\begin{array}{ccc} \mathcal{F}_T & \xrightarrow{q} & \mathcal{E} \\ & \searrow q' & \downarrow \cong \\ & & \mathcal{E}' \end{array} \quad (6.28)$$

For $f : X \rightarrow S$, $\mathcal{F}$, $\mathcal{L}$ and P above, we define the Quot functor

$$\mathbf{Quot}_{\mathcal{F}/X/S}^{P, \mathcal{L}} : (\text{Sch}/S)^{\text{op}} \rightarrow \text{Set} \quad (6.29)$$

such that:

- For $T \in \text{Sch}/S$, the set $\mathbf{Quot}_{\mathcal{F}/X/S}^{P, \mathcal{L}}(T)$ consists of equivalence classes of families $(\mathcal{E}, q)$ over T ;
- For $T' \rightarrow T$ a morphism in Sch/S , the map $\mathbf{Quot}_{\mathcal{F}/X/S}^{P, \mathcal{L}}(T) \rightarrow \mathbf{Quot}_{\mathcal{F}/X/S}^{P, \mathcal{L}}(T')$ sends the class of $(\mathcal{E}, q)$ to the class of its pull-back along $T' \rightarrow T$.

The Grassmannian functor is a special Quot functor. Let W be a finite dimensional vector space over $\mathbb{k}$, and let $0 \leq r \leq \dim W$ be an integer. Then

$$\mathbf{Grass}(W, r) = \mathbf{Quot}_{W/\text{Spec}(\mathbb{k})/\text{Spec}(\mathbb{k})}^{r, \mathcal{O}_{\text{Spec}(\mathbb{k})}}. \quad (6.30)$$

Grothendieck proved that the functor $\mathbf{Quot}_{\mathcal{F}/X/S}^{P,\mathcal{L}}$ is representable by a scheme which is projective over S . Let $\mathbf{Quot}_{\mathcal{F}/X/S}^{P,\mathcal{L}} \in \mathrm{Sch}/S$ be the representative; we call it the *Quot scheme*. For the construction of Quot schemes, we refer to [Nit05], and Section 2 in [HL10].

The Quot scheme $\mathbf{Quot}_{\mathcal{F}/X/S}^{P,\mathcal{L}}$ can be embedded into a Grassmannian. We describe the embedding of $\mathbf{Quot}_{\mathcal{F}/X/S}^{P,\mathcal{L}}$ into a suitable Grassmannian given in [Nit05]. This embedding can be described as a chain of sub-functors, each one is *represented by a closed subscheme*. Here we describe the sub-functors and their inclusions.

- (1) Choose $m \in \mathbb{N}_+$ and $N \in \mathbb{N}_+$ such that $i^* \mathcal{O}_{\mathbb{P}_S^N}(1) \cong \mathcal{L}^m$ for the diagram

$$\begin{array}{ccc} X & \xhookrightarrow{i} & \mathbb{P}_S^N \\ \downarrow f & \swarrow \pi & \\ S & & \end{array} \quad (6.31)$$

Note that these choices are independent of $\mathcal{F}$. Then we have the isomorphism and embedding of functors

$$\mathbf{Quot}_{\mathcal{F}/X/S}^{P(t),\mathcal{L}} \cong \mathbf{Quot}_{\mathcal{F}/X/S}^{P(mt),\mathcal{L}^m} \hookrightarrow \mathbf{Quot}_{i_*\mathcal{F}/\mathbb{P}_S^N/S}^{P(mt),\mathcal{O}_{\mathbb{P}_S^N}(1)}. \quad (6.32)$$

The first isomorphism is obvious and the second arrow sends the class of families $[(\mathcal{E}, q)] \in \mathbf{Quot}_{\mathcal{F}/X/S}^{P(mt),\mathcal{L}^m}(T)$ to the class of the family

$$(i_T)_* q : (i_* \mathcal{F})_T \rightarrow (i_T)_* \mathcal{E} \quad (6.33)$$

which is surjective since $(i_T)_*$ is exact, and we have used the affine base change formula for $(i_T)_* \mathcal{F}_T \cong (i_* \mathcal{F})_T$. Flatness and Hilbert polynomials are easy to check.

- (2) Choose a finite dimensional vector space W over $\mathbb{k}$ and $M \in \mathbb{N}_+$ such that the following is surjective

$$W \otimes_{\mathbb{k}} \mathcal{O}_{\mathbb{P}_S^N} \longrightarrow (i_* \mathcal{F}) \otimes_{\mathcal{O}_{\mathbb{P}_S^N}} \mathcal{O}_{\mathbb{P}_S^N}(M) \longrightarrow 0. \quad (6.34)$$

Then we have the following embedding of functors

$$\mathbf{Quot}_{i_*\mathcal{F}/\mathbb{P}_S^N/S}^{P(mt),\mathcal{O}_{\mathbb{P}_S^N}(1)} \hookrightarrow \mathbf{Quot}_{W \otimes_{\mathbb{k}} \mathcal{O}_{\mathbb{P}_S^N}/\mathbb{P}_S^N/S}^{P(m(t+M)),\mathcal{O}_{\mathbb{P}_S^N}(1)} \quad (6.35)$$

which sends the class of families $[(\mathcal{E}, q)] \in \mathbf{Quot}_{i_*\mathcal{F}/\mathbb{P}_S^N/S}^{P(mt),\mathcal{O}_{\mathbb{P}_S^N}(1)}(T)$ to the class of the following composition

$$W \otimes_{\mathbb{k}} \mathcal{O}_{\mathbb{P}_T^N}(M) \twoheadrightarrow (i_* \mathcal{F})_T \otimes_{\mathcal{O}_{\mathbb{P}_T^N}} \mathcal{O}_{\mathbb{P}_T^N}(M) \twoheadrightarrow \mathcal{E} \otimes_{\mathcal{O}_{\mathbb{P}_T^N}} \mathcal{O}_{\mathbb{P}_T^N}(M). \quad (6.36)$$

(3) Choose $M' \in \mathbb{N}_+$ such that for any $T \in \text{Sch}/S$ and any class of families $[(\mathcal{E}, q)] \in$

$$\mathbf{Quot}_{W \otimes_{\mathbb{k}} \mathcal{O}_{\mathbb{P}_S^N} / \mathbb{P}_S^N / S}^{P(m(t+M)), \mathcal{O}_{\mathbb{P}_S^N}(1)}(T)$$

- for $i > 0$, the higher direct image functor $R^i(\pi_T)_*$ vanishes on $W \otimes_{\mathbb{k}} \mathcal{O}_{\mathbb{P}_T^N}(M')$, $\mathcal{E} \otimes_{\mathcal{O}_{\mathbb{P}_T^N}} \mathcal{O}_{\mathbb{P}_T^N}(M')$ and $\ker(q) \otimes_{\mathcal{O}_{\mathbb{P}_T^N}} \mathcal{O}_{\mathbb{P}_T^N}(M')$, where $q : W \otimes_{\mathbb{k}} \mathcal{O}_{\mathbb{P}_T^N} \rightarrow \mathcal{E}$ and $\pi_T : \mathbb{P}_T^N \rightarrow T$;
- $(\pi_T)_*(\mathcal{E} \otimes_{\mathcal{O}_{\mathbb{P}_T^N}} \mathcal{O}_{\mathbb{P}_T^N}(M'))$ is locally free of rank $P(m(M + M'))$.

Then we have the following embedding of functors

$$\mathbf{Quot}_{W \otimes_{\mathbb{k}} \mathcal{O}_{\mathbb{P}_S^N} / \mathbb{P}_S^N / S}^{P(m(t+M)), \mathcal{O}_{\mathbb{P}_S^N}(1)} \hookrightarrow \mathbf{Grass}_S(W \otimes_{\mathbb{k}} \pi_* \mathcal{O}_{\mathbb{P}_S^N}(M'), P(m(M + M'))) \quad (6.37)$$

which sends the class of families $[(\mathcal{E}, q)] \in \mathbf{Quot}_{W \otimes_{\mathbb{k}} \mathcal{O}_{\mathbb{P}_S^N} / \mathbb{P}_S^N / S}^{P(m(t+M)), \mathcal{O}_{\mathbb{P}_S^N}(1)}(T)$ to the following locally free quotient

$$\begin{aligned} & (W \otimes_{\mathbb{k}} \pi_* \mathcal{O}_{\mathbb{P}_S^N}(M'))_T \\ & \cong W \otimes_{\mathbb{k}} (\pi_T)_* \mathcal{O}_{\mathbb{P}_T^N}(M') \\ & \cong (\pi_T)_*(W \otimes_{\mathbb{k}} \mathcal{O}_{\mathbb{P}_T^N}(M')) \\ & \twoheadrightarrow (\pi_T)_*(\mathcal{E} \otimes_{\mathcal{O}_{\mathbb{P}_T^N}} \mathcal{O}_{\mathbb{P}_T^N}(M')). \end{aligned} \quad (6.38)$$

(4) We have the following isomorphism between Grassmannians

$$\begin{aligned} & \mathbf{Grass}_S(W \otimes_{\mathbb{k}} \pi_* \mathcal{O}_{\mathbb{P}_S^N}(M'), P(m(M + M'))) \\ & \cong \mathbf{Grass}_S(W \otimes_{\mathbb{k}} H \otimes_{\mathbb{k}} \mathcal{O}_S, P(m(M + M'))) \\ & \cong S \times_{\text{Spec}(\mathbb{k})} \mathbf{Grass}(W \otimes_{\mathbb{k}} H, P(m(M + M'))) \end{aligned} \quad (6.39)$$

where $H := H^0(\mathbb{P}_{\mathbb{k}}^N, \mathcal{O}_{\mathbb{P}_{\mathbb{k}}^N}(M'))$. We have used $\pi_* \mathcal{O}_{\mathbb{P}_S^N}(M') \cong H \otimes_{\mathbb{k}} \mathcal{O}_S$, which gives the first isomorphism above.

The existence of above choices (and the fact that such choices make sub-functors representable by closed sub-schemes) is non-trivial and proofs can be found in [Nit05] and Section

2, [HL10]. The embedding of $\mathbf{Quot}_{\mathcal{F}/X/S}^{P(t), \mathcal{L}}$ into the Grassmannian is the composition

$$\begin{aligned}
\mathbf{Quot}_{\mathcal{F}/X/S}^{P(t), \mathcal{L}} &\cong \mathbf{Quot}_{\mathcal{F}/X/S}^{P(mt), \mathcal{L}^m} \\
&\hookrightarrow \mathbf{Quot}_{i_* \mathcal{F}/\mathbb{P}_S^N/S}^{P(mt), \mathcal{O}_{\mathbb{P}_S^N}(1)} \\
&\hookrightarrow \mathbf{Quot}_{W \otimes_{\mathbb{K}} \mathcal{O}_{\mathbb{P}_S^N}/\mathbb{P}_S^N/S}^{P(m(t+M)), \mathcal{O}_{\mathbb{P}_S^N}(1)} \\
&\hookrightarrow \mathbf{Grass}_S(W \otimes_{\mathbb{K}} \pi_* \mathcal{O}_{\mathbb{P}_S^N}(M'), P(m(M+M'))) \\
&\cong S \times_{\mathrm{Spec}(\mathbb{K})} \mathbf{Grass}(W \otimes_{\mathbb{K}} H, P(m(M+M'))).
\end{aligned} \tag{6.40}$$

6.2.2 Quot schemes for moduli of sheaves

Next we consider Quot schemes of the form $\mathbf{Quot}_{V(-m')/X/\mathrm{Spec}(\mathbb{K})}^{P(t), \mathcal{L}}$, which were used in Simpson's construction of moduli space of semistable sheaves ([Sim94]) and many studies of moduli of unstable sheaves ([HK12], [Jac21]). For these Quot schemes, we introduce new embeddings into smaller Grassmannians, very ample line bundles and linearisations.

Let $S = \mathrm{Spec}(\mathbb{K})$, and let X be a projective scheme over $\mathbb{K}$ with an ample line bundle $\mathcal{L}$. Choose $m \in \mathbb{N}_+$ such that $i^* \mathcal{O}_{\mathbb{P}^N}(1) \cong \mathcal{L}^m$ for

$$\begin{array}{ccc}
X & \xleftarrow{i} & \mathbb{P}^N \\
\downarrow f & \swarrow \pi & \\
\mathrm{Spec}(\mathbb{K}) & &
\end{array} \tag{6.41}$$

where $\mathbb{P}^N := \mathbb{P}(H^0(X, \mathcal{L}^m)^*)$ and we assume further that $\pi_* \mathcal{O}_{\mathbb{P}^N}(d) \rightarrow f_*(\mathcal{L}^{md})$ is surjective for each $d \in \mathbb{N}_+$. Let $P(t) \in \mathbb{Q}[t]$ be a numerical polynomial. Let $l \in \mathbb{N}_+$ and let V be a finite dimensional vector space over $\mathbb{K}$. The vector space V can depend on $l, m \in \mathbb{N}_+$, for example $V = \mathbb{K}^{P(lm)}$. Let $\mathcal{F} := V \otimes_{\mathbb{K}} \mathcal{L}^{-lm}$. For simplicity denote $V(-lm) := V \otimes_{\mathbb{K}} \mathcal{L}^{-lm}$. These data define the Quot functor $\mathbf{Quot}_{V(-lm)/X/\mathbb{K}}^{P(t), \mathcal{L}}$. Follow the above steps we can embed it into a Grassmannian:

- (1) The integers $m, N \in \mathbb{N}_+$ are chosen as above. Note that $\mathbb{P}^N = \mathbb{P}(H^0(X, \mathcal{L}^m)^*)$;
- (2) We can choose $W = V$ and $M = l$. Since we have

$$\begin{aligned}
i_* \mathcal{F} &= i_*(V \otimes_{\mathbb{K}} \mathcal{L}^{-lm}) \\
&\cong V \otimes_{\mathbb{K}} i_* i^* \mathcal{O}_{\mathbb{P}^N}(-l) \\
&\cong V \otimes_{\mathbb{K}} i_* \mathcal{O}_X \otimes_{\mathcal{O}_{\mathbb{P}^N}} \mathcal{O}_{\mathbb{P}^N}(-l)
\end{aligned} \tag{6.42}$$

and a surjective morphism $V \otimes_{\mathbb{K}} \mathcal{O}_{\mathbb{P}^N} \twoheadrightarrow V \otimes_{\mathbb{K}} i_* \mathcal{O}_X \cong i_* \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^N}} \mathcal{O}_{\mathbb{P}^N}(l)$.

(3) We choose $M' \in \mathbb{N}_+$ sufficiently large.

(4) Denote $H := H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(M')) = \text{Sym}_{\mathbb{k}}^{M'} H^0(X, \mathcal{L}^m)$. Recall we choose m such that $\pi_* \mathcal{O}_{\mathbb{P}^N}(M') \rightarrow f_*(\mathcal{L}^{mM'})$ is surjective, i.e. $H \rightarrow H^0(X, \mathcal{L}^{mM'})$ is surjective.

With these choices, we have the following closed embedding

$$\mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}} \hookrightarrow \mathbf{Grass}(V \otimes_{\mathbb{k}} H, P(lm + mM')) \quad (6.43)$$

which sends the class of families $[(\mathcal{E}, q)] \in \mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}(T)$ to the class of the quotient

$$(V \otimes_{\mathbb{k}} \pi_* \mathcal{O}_{\mathbb{P}^N}(M'))_T \longrightarrow (f_T)_*(\mathcal{E} \otimes_{\mathcal{O}_{X_T}} \mathcal{L}_T^{lm+mM'}) \longrightarrow 0 \quad (6.44)$$

where the quotient above factors as follows

$$\begin{array}{ccc} (V \otimes_{\mathbb{k}} \pi_* \mathcal{O}_{\mathbb{P}^N}(M'))_T & \longrightarrow & (f_T)_*(\mathcal{E} \otimes_{\mathcal{O}_{X_T}} \mathcal{L}_T^{lm+mM'}) \\ \downarrow & \nearrow & \\ (V \otimes_{\mathbb{k}} f_* \mathcal{L}^{mM'})_T & & \end{array} \quad (6.45)$$

The existence of factorisation provides another embedding into a new Grassmannian

$$\begin{array}{ccc} & \mathbf{Grass}(V \otimes_{\mathbb{k}} \pi_* \mathcal{O}_{\mathbb{P}^N}(M'), P(lm + mM')) & \\ & \nearrow & \uparrow \\ \mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}} & \hookrightarrow \mathbf{Grass}(V \otimes_{\mathbb{k}} f_* \mathcal{L}^{mM'}, P(lm + mM')) & \end{array} \quad (6.46)$$

The Grassmannians are projective, and there are natural choices of very ample line bundles from Plücker embeddings, which then pull back to very ample line bundles on Quot schemes. For simplicity we denote the representatives by Quot and Grass

$$\begin{aligned} \mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}} &\cong \text{Hom}_{\text{Sch}/\mathbb{k}}(-, \text{Quot}) \\ \mathbf{Grass}(V \otimes_{\mathbb{k}} f_* \mathcal{L}^{mM'}, P(lm + mM')) &\cong \text{Hom}_{\text{Sch}/\mathbb{k}}(-, \text{Grass}). \end{aligned} \quad (6.47)$$

There exist *universal elements* for representable functors. Let

$$\begin{aligned} q_{\mathcal{U}} : V(-lm)_{\text{Quot}} &\rightarrow \mathcal{U} \\ q_{\mathcal{W}} : (V \otimes_{\mathbb{k}} f_* \mathcal{L}^{mM'})_{\text{Grass}} &\rightarrow \mathcal{W} \end{aligned} \quad (6.48)$$

represent the universal classes respectively, i.e. the classes

$$\begin{aligned} [(\mathcal{U}, q_{\mathcal{U}})] &\in \mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}(\text{Quot}) \cong \text{Hom}_{\text{Sch}/\mathbb{k}}(\text{Quot}, \text{Quot}) \\ [(\mathcal{W}, q_{\mathcal{W}})] &\in \mathbf{Grass}(V \otimes_{\mathbb{k}} f_* \mathcal{L}^{mM'}, P(lm + mM'))(\text{Grass}) \\ &\cong \text{Hom}_{\text{Sch}/\mathbb{k}}(\text{Grass}, \text{Grass}) \end{aligned} \quad (6.49)$$

corresponding to the identity morphisms of Quot and Grass respectively.

The Plücker embedding gives an embedding $\text{Grass} \hookrightarrow \mathbb{P}^{N'}$ into some projective space $\mathbb{P}^{N'}$. The pull-back of $\mathcal{O}_{\mathbb{P}^{N'}}(1)$ is very ample on Grass, which is isomorphic to

$$\det \mathcal{W} := \bigwedge^{P(lm+mM')} \mathcal{W}. \quad (6.50)$$

The very ample line bundle we choose on Quot is the pull-back of $\det \mathcal{W}$ along $\text{Quot} \hookrightarrow \text{Grass}$, which is isomorphic to the anti-symmetric product of the pull-back of $\mathcal{W}$. The pull-back of $(\mathcal{W}, q_{\mathcal{W}})$ along $\text{Quot} \hookrightarrow \text{Grass}$ is a family in the image of $[(\mathcal{U}, q_{\mathcal{U}})]$ along $\mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}(\text{Quot}) \rightarrow \mathbf{Grass}(V \otimes_{\mathbb{k}} f_* \mathcal{L}^{mM'}, P(lm + mM'))(\text{Quot})$, and the image of $[(\mathcal{U}, q_{\mathcal{U}})]$ is the class of the following

$$(V \otimes_{\mathbb{k}} f_* \mathcal{L}^{mM'})_{\text{Quot}} \longrightarrow (f_{\text{Quot}})_*(\mathcal{U} \otimes_{\mathcal{O}_{X_{\text{Quot}}}} \mathcal{L}_{\text{Quot}}^{lm+mM'}) \longrightarrow 0. \quad (6.51)$$

The pull-back of $\mathcal{W}$ is isomorphic to the quotient above. So the very ample line bundle on Quot is

$$\mathcal{M} := \det \left((f_{\text{Quot}})_*(\mathcal{U} \otimes_{\mathcal{O}_{X_{\text{Quot}}}} \mathcal{L}_{\text{Quot}}^{lm+mM'}) \right). \quad (6.52)$$

Recall the notation that $[(\mathcal{U}, q_{\mathcal{U}})]$ is the universal class of families on Quot, and $f : X \rightarrow \text{Spec}(\mathbb{k})$ is the structure morphism, and for $T \in \text{Sch}/\mathbb{k}$, we denote $X_T := X \times_{\mathbb{k}} T$ and $f_T : X_T \rightarrow T$ and $\mathcal{L}_T := (X_T \rightarrow X)^* \mathcal{L}$ for the related pull-backs.

We then describe an action of $\text{GL}(V)$ on Quot. It is easy to see $\text{GL}(V)$ represents the following functor $\mathbf{GL}(V)$

$$\mathbf{GL}(V) : (\text{Sch}/\mathbb{k})^{\text{op}} \rightarrow \text{Group}, \quad T \mapsto \{g \in \text{End}_{\mathcal{O}_T}(V \otimes_{\mathbb{k}} \mathcal{O}_T) : g \text{ is invertible}\} \quad (6.53)$$

and the maps on morphisms are naturally defined as pull-backs. The action of $\text{GL}(V)$ on Quot is the morphism $\sigma : \text{GL}(V) \times_{\mathbb{k}} \text{Quot} \rightarrow \text{Quot}$ corresponding to the following natural transformation of functors

$$\sigma : \mathbf{GL}(V) \times \mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}} \rightarrow \mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}} \quad (6.54)$$

which on $T \in \text{Sch}/\mathbb{k}$ is a map sending $(g, [\mathcal{E}, q])$ to the class $[\mathcal{E}, g.q]$ for $g.q$ defined as follows

$$\begin{array}{ccc} (V \otimes_{\mathbb{k}} \mathcal{O}_T) \otimes_{\mathcal{O}_T} \mathcal{L}_T^{-lm} & \xrightarrow{q} & \mathcal{E} \longrightarrow 0 \\ \downarrow g \otimes 1_{\mathcal{L}^{-lm}} & & \parallel \\ (V \otimes_{\mathbb{k}} \mathcal{O}_T) \otimes_{\mathcal{O}_T} \mathcal{L}_T^{-lm} & \xrightarrow{g.q} & \mathcal{E} \longrightarrow 0. \end{array} \quad (6.55)$$

The maps on morphisms are pull-backs and it is easy to check the axioms for natural transformations. We also see for each $T \in \text{Sch}/\mathbb{k}$, the natural transformation defines an action of $\mathbf{GL}(V)(T)$ on $\mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t),\mathcal{L}}(T)$ (cf. [Sta22, Definition 022Z]).

Fix a family $(\mathcal{U}, q_{\mathcal{U}})$ in the universal class of $\mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t),\mathcal{L}}$. Consider the diagram of functors

$$\begin{array}{ccc} \text{Hom}_{\text{Sch}/\mathbb{k}}(-, \text{GL}(V) \times_{\mathbb{k}} \text{Quot}) & \xrightarrow{\sigma \circ (-)} & \text{Hom}_{\text{Sch}/\mathbb{k}}(-, \text{Quot}) \\ \downarrow \text{IR} & & \downarrow \text{IR} \\ \mathbf{GL}(V) \times \mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t),\mathcal{L}} & \xrightarrow{\sigma} & \mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t),\mathcal{L}} \end{array} \quad (6.56)$$

Let $T = \text{GL}(V) \times_{\mathbb{k}} \text{Quot}$ and let $\phi : T \rightarrow \text{GL}(V) \times_{\mathbb{k}} \text{Quot}$ be the identity morphism. The image of $\phi \in \text{Hom}_{\text{Sch}/\mathbb{k}}(T, \text{GL}(V) \times_{\mathbb{k}} \text{Quot})$ in $\mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t),\mathcal{L}}(T)$ can be calculated in two ways. Along the top right arrows, the image of ϕ is the class of

$$\sigma_X^* V(-lm)_{\text{Quot}} \xrightarrow{\sigma_X^* q_{\mathcal{U}}} \sigma_X^* \mathcal{U} \longrightarrow 0 \quad (6.57)$$

where $\sigma_X : \text{GL}(V) \times_{\mathbb{k}} X \times_{\mathbb{k}} \text{Quot} \rightarrow X \times_{\mathbb{k}} \text{Quot}$ is the base change of $\sigma : \text{GL}(V) \times_{\mathbb{k}} \text{Quot} \rightarrow \text{Quot}$. Along the bottom left arrows, the image of ϕ is the class of the composition of

$$V \otimes_{\mathbb{k}} p_X^* \mathcal{L}^{-lm} \xrightarrow{(p_T)^* \tau^{-1}(-lm)} V \otimes_{\mathbb{k}} p_X^* \mathcal{L}^{-lm} \xrightarrow{p_{X\text{Quot}}^* q_{\mathcal{U}}} p_{X\text{Quot}}^* \mathcal{U} \quad (6.58)$$

where

- $\tau : V \otimes_{\mathbb{k}} \mathcal{O}_T \rightarrow V \otimes_{\mathbb{k}} \mathcal{O}_T$ corresponds to the projection $T \equiv \text{GL}(V) \times_{\mathbb{k}} \text{Quot} \rightarrow \text{GL}(V)$;
- $p_X, p_T, p_{X\text{Quot}}$ are projection morphisms

$$\begin{aligned} p_X &: \text{GL}(V) \times_{\mathbb{k}} X \times_{\mathbb{k}} \text{Quot} \rightarrow X \\ p_T &: \text{GL}(V) \times_{\mathbb{k}} X \times_{\mathbb{k}} \text{Quot} \rightarrow T \\ p_{X\text{Quot}} &: \text{GL}(V) \times_{\mathbb{k}} X \times_{\mathbb{k}} \text{Quot} \rightarrow X \times_{\mathbb{k}} \text{Quot}. \end{aligned} \quad (6.59)$$

In particular we have an isomorphism

$$p_{X\text{Quot}}^* \mathcal{U} \cong \sigma_X^* \mathcal{U} \quad (6.60)$$

which defines a linearisation of $\text{GL}(V)$ on $\mathcal{U}$. This linearisation on $\mathcal{U}$ and the trivial linearisation on $\mathcal{L}$ induce a linearisation of $\text{GL}(V)$ on the very ample line bundle over Quot

$$\mathcal{M} := \det \left((f_{\text{Quot}})_* (\mathcal{U} \otimes_{\mathcal{O}_{X\text{Quot}}} \mathcal{L}_{\text{Quot}}^{lm+mM'}) \right). \quad (6.61)$$

6.3 Simpson's and Jackson's moduli spaces

In this section, we introduce Simpson's moduli spaces of stable and semistable sheaves ([Sim94]) and Jackson's moduli space of sheaves of Harder-Narasimhan length 2 ([Jac21]). These moduli spaces were constructed by reductive GIT and non-reductive GIT respectively. Simpson's moduli spaces can be interpreted as coarse moduli spaces of naturally defined moduli functors, so these moduli spaces are independent of the way we set up the GIT problem. Jackson has not associated his moduli spaces with moduli functors, so it is not clear whether his moduli spaces depend on associated non-reductive GIT problems.

6.3.1 Moduli of semistable sheaves

Let X be a connected projective scheme over $\mathbb{k}$ with an ample line bundle $\mathcal{L}$. Let $m \in \mathbb{N}_+$ be such that $\mathcal{L}^m$ is very ample. Let $i : X \hookrightarrow \mathbb{P}^N$ be a closed immersion into a projective space such that $i^*\mathcal{O}_{\mathbb{P}^N}(1) \cong \mathcal{L}^m$. Let $P(t) \in \mathbb{Q}[t]$ be a numerical polynomial. For $j \in \mathbb{Z}$ and a coherent sheaf $\mathcal{F}$ on X , we denote $\mathcal{F}(j) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^j$. Moduli spaces of semistable pure sheaves on X of fixed Hilbert polynomials were constructed by Simpson in [Sim94]. A more recent reference is [HL10].

Let $\text{Sch}/\mathbb{k}$ be the category of schemes of finite type over $\mathbb{k}$. For $T \in \text{Sch}/\mathbb{k}$, a *family on T of (semi)stable sheaves of Hilbert polynomial $P(t)$* is a coherent sheaf $\mathcal{F}$ on $X_T := X \times_{\mathbb{k}} T$ such that $\mathcal{F}$ is flat over T and $\mathcal{F}_s$ is a (semi)stable sheaf on X of Hilbert polynomial $P(t)$ for every closed point $s \in T$, where $\mathcal{F}_s$ is the pull-back of $\mathcal{F}$ along $X \cong X \times_{\mathbb{k}} \text{Spec}(\kappa(s)) \hookrightarrow X_T$. Two families $\mathcal{F}, \mathcal{F}'$ on T are called equivalent if there exists a line bundle $\mathcal{A}$ on T such that $\mathcal{F} \cong \mathcal{F}' \otimes_{\mathcal{O}_{X_T}} (f_T)^*\mathcal{A}$, where $f_T : X_T \rightarrow T$ is the pull-back of $f : X \rightarrow \text{Spec}(\mathbb{k})$. With the notions of families and their equivalence relations, we consider the moduli functors

$$\begin{aligned} \mathbf{M}_{X/\mathbb{k}}^{P(t), \mathcal{L}, \text{ss}} : (\text{Sch}/\mathbb{k})^{\text{op}} &\rightarrow \text{Set} \\ \mathbf{M}_{X/\mathbb{k}}^{P(t), \mathcal{L}, s} : (\text{Sch}/\mathbb{k})^{\text{op}} &\rightarrow \text{Set} \end{aligned} \tag{6.62}$$

which associate to $T \in \text{Sch}/\mathbb{k}$ the sets

$$\mathbf{M}_{X/\mathbb{k}}^{P(t), \mathcal{L}, (s)s}(T) := \left\{ \begin{array}{c} \text{families on } T \text{ of (semi)stable sheaves} \\ \text{of Hilbert polynomial } P(t) \end{array} \right\} / \text{equivalence relation} \tag{6.63}$$

and the functors on morphisms are naturally pull-backs of coherent sheaves. These functors are both corepresentable and $\mathbf{M}_{X/\mathbb{k}}^{P(t), \mathcal{L}, s}$ has a coarse moduli space.

Definition 6.3.1. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^N$. Let $r \in \mathbb{Z}$ be an integer. The sheaf $\mathcal{F}$ is called *r-regular with respect to $\mathcal{O}_{\mathbb{P}^N}(1)$* if

$$H^j(\mathbb{P}^N, \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^N}} \mathcal{O}_{\mathbb{P}^N}(r-j)) = 0, \quad j > 0. \quad (6.64)$$

The *Mumford-Castelnuovo regularity* is $\text{reg}(\mathcal{F}) := \min\{r \in \mathbb{Z} : \mathcal{F} \text{ is } r\text{-regular}\}$. When $\mathcal{F}$ is a coherent sheaf on X , it is called *r-regular with respect to $i : X \hookrightarrow \mathbb{P}^N$* if $i_*\mathcal{F}$ is r -regular and its *Mumford-Castelnuovo regularity* is $\text{reg}(\mathcal{F}) := \text{reg}(i_*\mathcal{F})$.

Recall $\mathcal{L}^m \cong i^*\mathcal{O}_{\mathbb{P}^N}(1)$. We refer to [HL10] Lemma 1.7.2 (cf. [Mum66]) for the following properties for coherent sheaves $\mathcal{F}$ on X :

- If $\mathcal{F}$ is r -regular, and $r' \geq r$, then $\mathcal{F}$ is r' -regular;
- If $\mathcal{F}$ is r -regular, then $\mathcal{F}(mr)$ is globally generated and $H^j(X, \mathcal{F}(mr)) = 0$ for $j > 0$;
- If $\mathcal{F}$ is r -regular and $n \geq 0$, then the following is surjective

$$H^0(X, \mathcal{F}(mr)) \otimes_{\mathbb{k}} H^0(X, \mathcal{L}^{mn}) \rightarrow H^0(X, \mathcal{F}(m(r+n))). \quad (6.65)$$

Definition 6.3.2 ([HL10] Definition 1.7.5). A collection of isomorphism classes of coherent sheaves $\{\mathcal{F}_i\}_{i \in I}$ on X is *bounded* if there exists a scheme T of finite type over $\mathbb{k}$ and a coherent sheaf $\mathcal{F}$ on $X \times_{\mathbb{k}} T$ such that for each $i \in I$, there exists a $\mathbb{k}$ -point $t \in T$ such that $\mathcal{F}_i \cong \mathcal{F}_t$, where $\mathcal{F}_t$ is the pull-back of $\mathcal{F}$ along $X \cong X \times_{\mathbb{k}} \text{Spec}(\kappa(t)) \hookrightarrow X \times_{\mathbb{k}} T$.

A bounded collection has a uniform regularity by the following lemma.

Lemma 6.3.3 ([HL10] Lemma 1.7.6). *Let $\{\mathcal{F}_i\}_{i \in I}$ be a collection of coherent sheaves on X . The following are equivalent:*

- $\{\mathcal{F}_i\}_{i \in I}$ is bounded;
- There are finitely many Hilbert polynomials in $\{\mathcal{F}_i\}_{i \in I}$, and $\max_{i \in I} \text{reg}(\mathcal{F}_i) < \infty$.

The collection of semistable sheaves on X of Hilbert polynomial $P(t)$ is bounded. A more general boundedness result is the following.

Theorem 6.3.4 ([Sim94] Theorem 1.1, [HL10] Theorem 3.3.7). *Let X be a projective scheme of finite type over $\mathbb{k}$ with an ample line bundle $\mathcal{L}$. Let $P(t) \in \mathbb{Q}[t]$ be a numerical polynomial of degree $d := \deg P(t)$. Let $b \in \mathbb{Q}$. Consider the collection of coherent sheaves such that each $\mathcal{F}$ in this collection satisfies:*

- $\mathcal{F}$ is pure of dimension d ;
- $\mathcal{F}$ has Hilbert polynomial $P(t)$;
- $\hat{\mu}(\mathcal{F}') \leq b$ for all non-zero coherent sub-sheaf $0 \neq \mathcal{F}' \subseteq \mathcal{F}$, where $\hat{\mu}$ is defined in Definition 6.1.6.

Then this collection is bounded.

Theorem 6.3.4 applies to the collection of semistable sheaves with Hilbert polynomial $P(t)$, so we conclude they are bounded. By Lemma 6.3.3, their regularities are uniformly bounded above. For $l \gg 0$, all semistable sheaves $\mathcal{F}$ of Hilbert polynomial $P(t)$ are l -regular. Then:

- $\dim H^0(X, \mathcal{F}(lm)) = P(lm)$;
- $H^0(X, \mathcal{F}(lm)) \otimes_{\mathbb{k}} \mathcal{L}^{-lm} \rightarrow \mathcal{F}$ is surjective.

Let $V = \mathbb{k}^{P(lm)}$. There is a non-canonical linear isomorphism $V \cong H^0(X, \mathcal{F}(lm))$. With one such isomorphism, we have a surjective morphism

$$q : V(-lm) \rightarrow \mathcal{F} \tag{6.66}$$

which represents a closed point in $\text{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$.

Let $R \subseteq \text{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$ be the open subscheme whose closed points are classes $[(\mathcal{F}, q)]$ such that:

- (1) $\mathcal{F}$ is semistable;
- (2) the induced map $V \cong H^0(X, V) \rightarrow H^0(X, \mathcal{F}(lm))$ is an isomorphism.

Let $R^s \subseteq R$ be the open subset of classes $[(\mathcal{F}, q)]$ such that $\mathcal{F}$ is stable. Condition (2) above is open since the function $[(\mathcal{F}, q)] \mapsto \dim H^0(X, \ker(q)(lm))$ is upper semi-continuous and the condition that $\dim H^0(X, \ker(q)(lm)) = 0$ is equivalent to the condition (2). Condition (1) above is open by [HL10] Proposition 2.3.1. This justifies the claim that $R \subseteq \text{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$ is open, and similarly $R^s \subseteq R$ is open.

Recall that there is a very ample line bundle $\mathcal{M}$ on $\text{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$ and the group $\text{GL}(V)$ acts on $\text{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$ linearly with respect to $\mathcal{M}$. We consider the linear action of the subgroup $\text{SL}(V)$. It is easy to see R is invariant under $\text{SL}(V)$. Let $\bar{R} \hookrightarrow \text{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$ be the scheme-theoretic image of $R \rightarrow \text{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$, which is a subscheme on the closure of R by [Sta22, Lemma 01R8]. The linearisation on $\mathcal{M}$ pulls back to a linearisation $\mathcal{M}|_{\bar{R}}$. We can apply reductive GIT for the linear action of $\text{SL}(V)$ on $\bar{R}$ with respect to $\mathcal{M}|_{\bar{R}}$. Note that $\mathcal{M}$ depends on two large numbers $l, M' \in \mathbb{N}_+$.

Theorem 6.3.5 ([HL10] Theorem 4.3.3, Theorem 4.3.4). *For $M' \gg l \gg 0$, we have*

- *GIT (semi)stable points are isomorphism classes $[(\mathcal{F}, q)] \in \bar{R}$ such that $\mathcal{F}$ is (semi)stable and $q : V(-lm) \rightarrow \mathcal{E}$ induces an isomorphism $V \cong H^0(X, \mathcal{E}(lm))$, i.e.*

$$\bar{R}^{\text{SL}(V), \mathcal{M}, \text{ss}} = R, \quad \bar{R}^{\text{SL}(V), \mathcal{M}, \text{s}} = R^s; \quad (6.67)$$

- *Two semistable points $[(\mathcal{F}, q)], [(\mathcal{F}', q')] \in R$ are (GIT) S-equivalent if and only if $\text{gr}(\mathcal{F}) \cong \text{gr}(\mathcal{F}')$, where $\text{gr}(\mathcal{F})$ denotes the associated graded sheaf of the Jordan-Hölder filtration of $\mathcal{F}$ (Definition 6.1.10);*
- *A semistable point $[(\mathcal{F}, q)] \in R$ has closed $\text{SL}(V)$ -orbit in R if and only if $\mathcal{F}$ is polystable, i.e. $\mathcal{F}$ is a direct sum of stable sheaves;*
- *$\bar{R} // \text{SL}(V)$ corepresents $\mathbf{M}_{X/\mathbb{k}}^{P(t), \mathcal{L}, \text{ss}}$ and $R^s // \text{SL}(V)$ corepresents $\mathbf{M}_{X/\mathbb{k}}^{P(t), \mathcal{L}, \text{s}}$. Moreover $R^s // \text{SL}(V)$ is a coarse moduli space of $\mathbf{M}_{X/\mathbb{k}}^{P(t), \mathcal{L}, \text{s}}$.*

Remark 6.3.6. The coarse moduli space of $\mathbf{M}_{X/\mathbb{k}}^{P(t), \mathcal{L}, \text{s}}$ has a natural compactification, which is the corepresentative of $\mathbf{M}_{X/\mathbb{k}}^{P(t), \mathcal{L}, \text{ss}}$, which is independent of the GIT problem.

Remark 6.3.7. If there exists a semistable sheaf $\mathcal{F}$ of Hilbert polynomial $P(t)$ that is not polystable, then $\mathbf{M}_{X/\mathbb{k}}^{P(t), \mathcal{L}, \text{ss}}$ does not have a coarse moduli space, since $\eta : \mathbf{M}_{X/\mathbb{k}}^{P(t), \mathcal{L}, \text{ss}} \rightarrow h_{\bar{R} // \text{SL}(V)}$ is not bijective on $\text{Spec}(\mathbb{k})$.

6.3.2 Moduli of unstable sheaves

Let τ be a Harder-Narasimhan type of length $n \in \mathbb{N}_+$. Let $P_i(t)$ for $1 \leq i \leq n$ be the Hilbert polynomials associated to τ in the sense that if $\mathcal{E}$ is a pure sheaf of type τ on X , and $\{\text{HN}_i(\mathcal{E})\}_{i=0}^n$ is its Harder-Narasimhan filtration, then $P_i(t)$ is the Hilbert polynomial of $\text{HN}_i(\mathcal{E})/\text{HN}_{i-1}(\mathcal{E})$ for $1 \leq i \leq n$. We have that $P(t) := \sum_{i=1}^n P_i(t)$ is the Hilbert polynomial of $\mathcal{E}$. Let $V_i := \mathbb{k}^{P_i(lm)}$ for $1 \leq i \leq n$ and let $V := V_1 \oplus \cdots \oplus V_n$. Denote $V_{\leq i} := V_1 \oplus \cdots \oplus V_i$. Let $P_\tau \subseteq \text{SL}(V)$ be the parabolic subgroup of $g \in \text{SL}(V)$ stabilising the partial flag $\{V_{\leq i}\}_{i=0}^n$.

It turns out that it is useful to refine the classification problem according to dimensions of unipotent endomorphisms. In [Jac21] Definition 7.12 and [HJ21] Definition 4.18, the so-called τ -stability of pure sheaves was defined. In [HJ21] Definition 6.1, the so-called *refined Harder-Narasimhan type* was defined, which encodes information on dimensions of unipotent endomorphisms.

Theorem 6.3.4 shows that sheaves of fixed Harder-Narasimhan type are bounded. In the proof of [HK12] Proposition 6.8, a locally closed subscheme $Y_\tau^{\text{ss}} \subseteq \text{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$ was described as a parameter space of sheaves of type τ . Closed points in Y_τ^{ss} are exactly isomorphism classes $[(\mathcal{E}, q)] \in \text{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$ such that each $\mathcal{E}_i$ is semistable with Hilbert polynomial $P_i(t)$ for $1 \leq i \leq n$, where $\mathcal{E}_i := q(V_{\leq i}(-lm))/q(V_{\leq i-1}(-lm))$. In the same manner, one can define locally closed subschemes Y_τ^s and Y_τ such that $Y_\tau^s \subseteq Y_\tau^{\text{ss}} \subseteq Y_\tau \subseteq \text{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$. Closed points in $Y_\tau \subseteq \text{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$ are classes $[(\mathcal{E}, q)]$ such that $\mathcal{E}_i$ has Hilbert polynomial $P_i(t)$, and closed points in Y_τ^s are classes $[(\mathcal{E}, q)] \in Y_\tau^{\text{ss}}$ such that $\mathcal{E}_i$ is stable. The locally closed subsets $Y_\tau^{(s)s} \subseteq Y_\tau$ were studied in [HK12], [Jac18], [Jac21] and [HJ21], where moduli spaces of τ -stable sheaves were constructed as parabolic GIT quotients of Y_τ^s .

We will recover the locally closed subschemes $Y_\tau^{(s)s} \subseteq Y_\tau \subseteq \text{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$ and actions of P_τ on them from a more functorial perspective in this subsection. From this functorial perspective, we do not only see closed points, but also view T -valued points of $Y^{(s)s} \subseteq Y_\tau$ as families of quotients over T .

For Y_τ , we describe the associated functor of points $\mathbf{Y}_\tau \subseteq \mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$ as a *sub-functor represented by locally closed subschemes* (See Definition 6.1.2 for representable sub-functors). Let $T \in \text{Sch}/\mathbb{k}$ be a scheme and let $[(\mathcal{E}, q)] \in \mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}(T)$ be an isomorphism class and choose a quotient in this class

$$V(-lm)_T \xrightarrow{q} \mathcal{E} \longrightarrow 0. \quad (6.68)$$

Define the following notation for $1 \leq i \leq n$

$$\begin{aligned} \mathcal{E}_{\leq i} &:= q(V_{\leq i}(-lm)_T) \subseteq \mathcal{E} \\ \mathcal{E}_i &:= \mathcal{E}_{\leq i} / \mathcal{E}_{\leq i-1}. \end{aligned} \quad (6.69)$$

There is a morphism $q_i : V_i(-lm)_T \rightarrow \mathcal{E}_i$ such that the following diagram commutes

$$\begin{array}{ccccccc} 0 & \longrightarrow & V_{\leq i-1}(-lm)_T & \longrightarrow & V_{\leq i}(-lm)_T & \longrightarrow & V_i(-lm)_T \longrightarrow 0 \\ & & \downarrow q_{\leq i-1} & & \downarrow q_{\leq i} & & \downarrow q_i \\ 0 & \longrightarrow & \mathcal{E}_{\leq i-1} & \longrightarrow & \mathcal{E}_{\leq i} & \longrightarrow & \mathcal{E}_i \longrightarrow 0 \end{array} \quad (6.70)$$

where $q_{\leq i} := q|_{V_{\leq i}(-lm)_T}$ and then the Five lemma shows the surjectivity of q_i .

The subset $\mathbf{Y}_\tau(T)$ consists of $[(\mathcal{E}, q)]$ such that each $\mathcal{E}_i$ is flat over T with Hilbert polynomial $P_i(t)$ for $1 \leq i \leq n$. By [Sta22, Lemma 00HM], we have that $\mathcal{E}_{\leq i-1}$ is flat if $\mathcal{E}_i$ and $\mathcal{E}_{\leq i}$ are flat. Since $\mathcal{E} = \mathcal{E}_{\leq n}$ is flat over T , inductively we see each $\mathcal{E}_{\leq i}$ is flat over T if $[(\mathcal{E}, q)] \in \mathbf{Y}_\tau(T)$. Since flatness and Hilbert polynomials are preserved under pull-backs, we have that $\mathbf{Y}_\tau$ defines a sub-functor of $\mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$.

The following theorem of *flattening stratification* is important in the construction of Quot schemes. Here we apply it to show that $\mathbf{Y}_\tau \subseteq \mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$ is represented by locally closed subschemes.

Theorem 6.3.8 (Theorem 2.1.5, [HL10]). *Let $X \rightarrow T$ be a projective morphism of schemes of finite type over $\mathbb{k}$. Let $\mathcal{L}$ be a line bundle on X which is ample relative to $X \rightarrow T$. Let $\mathcal{F}$ be a coherent sheaf on X . Then:*

- *The set of Hilbert polynomials is finite*

$$\mathcal{P} := \left\{ P(t) \in \mathbb{Q}[t] : \begin{array}{l} P(t) \text{ is the Hilbert polynomial of } \mathcal{F} \\ \text{at some closed point of } T \end{array} \right\}; \quad (6.71)$$

- For each $P(t) \in \mathcal{P}$ the following functor is represented by a locally closed subscheme $T_P \subseteq T$

$$\mathbf{Flat}_{\mathcal{F}/X/T}^{P(t), \mathcal{L}} : (\text{Sch}/T)^{\text{op}} \rightarrow \text{Set}, \quad T' \mapsto \begin{cases} \{\text{pt}\}, & \mathcal{F}_{T'} \text{ is flat over } T' \\ & \text{with Hilbert polynomial } P(t) \\ \emptyset, & \text{otherwise} \end{cases} \quad (6.72)$$

where Sch/T is the category of schemes of finite type over T , and $\mathcal{F}_{T'}$ is the pull-back of $\mathcal{F}$ along $T' \rightarrow T$;

- The morphism $\coprod_{P \in \mathcal{P}} T_P \rightarrow T$ is bijective on points.

Lemma 6.3.9. $\mathbf{Y}_\tau \subseteq \mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$ is representable by locally closed subschemes (Definition 6.1.2).

Proof. Let $T \in \text{Sch}/\mathbb{k}$ and consider the following diagram in $\text{PSh}(\text{Sch}/\mathbb{k})$

$$\begin{array}{ccc} & h_T & \\ & \downarrow \xi & \\ \mathbf{Y}_\tau & \xrightarrow{\subseteq} & \mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}} \end{array} \quad (6.73)$$

where $\xi : h_T \rightarrow \mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$ is identified with a class $[(\mathcal{E}, q)] \in \mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}(T)$ by the Yoneda lemma.

Recall $\mathcal{E}_{\leq i} := q(V_{\leq i}(-lm)_T)$ and $\mathcal{E}_i := \mathcal{E}_{\leq i} / \mathcal{E}_{\leq i-1}$. For each $1 \leq i \leq n$ consider the functor $\mathbf{Flat}_{\mathcal{E}_i/X_T/T}^{P_i(t), \mathcal{L}_T}$ defined in Theorem 6.3.8, which is represented by a locally closed subscheme $T_{\xi, P_i} \subseteq T$. Let

$$T_\xi := T_{\xi, P_1} \times_T T_{\xi, P_2} \times_T \cdots \times_T T_{\xi, P_n} \quad (6.74)$$

which is a locally closed subscheme of T , since immersions are stable under base change and composition.

For $T' \rightarrow T$ in $\text{Sch}/\mathbb{k}$, denote $\mathcal{E}|_{T'}$ the pull-back of $\mathcal{E}$ along $X_{T'} \rightarrow X_T$, and similar for $\mathcal{E}_i, \mathcal{E}_{\leq i}$. We have

$$\begin{aligned} & T' \rightarrow T \text{ factors through } T' \rightarrow T_\xi \rightarrow T \\ \text{if and only if } & T' \rightarrow T \text{ factors through } T_{\xi, P_i} \text{ for each } 1 \leq i \leq n \\ \text{if and only if } & \begin{cases} \mathcal{E}_i|_{T'} \text{ is flat over } T' \\ \text{with Hilbert polynomial } P_i(t) \end{cases} \quad \text{for } 1 \leq i \leq n \\ \text{if and only if } & (T' \rightarrow T)^* \xi \in \mathbf{Y}_\tau(T'). \end{aligned} \quad (6.75)$$

The above equivalences show the following diagram is Cartesian

$$\begin{array}{ccc}
h_{T_\xi} & \xrightarrow{\subseteq} & h_T \\
\downarrow & & \downarrow \xi \\
\mathbf{Y}_\tau & \xrightarrow{\subseteq} & \mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}
\end{array} \tag{6.76}$$

which proves the representability of $\mathbf{Y}_\tau \subseteq \mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$ by locally closed subschemes. $\square$

Recall that $\mathbf{Quot}$ represents $\mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$. Let $T = \mathbf{Quot}$ be the $\mathbf{Quot}$ scheme and let $\xi : h_T \rightarrow \mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$ correspond to the universal family $[(\mathcal{U}, q_{\mathcal{U}})]$. Then the locally closed subscheme $T_\xi \subseteq T$ represents the sub-functor $\mathbf{Y}_\tau$, which defines the locally closed subscheme $Y_\tau \subseteq \mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$.

Consider $\prod_{i=1}^n \mathbf{Quot}_{V_i(-lm)/X/\mathbb{k}}^{P_i(t), \mathcal{L}}$, where the fibre product is over $\mathbb{k}$. There is a morphism $\prod_{i=1}^n \mathbf{Quot}_{V_i(-lm)/X/\mathbb{k}}^{P_i(t), \mathcal{L}} \rightarrow \mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$, whose associated natural transformation sends $\prod_{i=1}^n [(\mathcal{E}_i, q_i)]$, a product of isomorphism classes of families over $T \in \text{Sch}/\mathbb{k}$, to $\left[\left(\bigoplus_{i=1}^n \mathcal{E}_i, \bigoplus_{i=1}^n q_i \right) \right]$. This morphism is a monomorphism, and it is proper ([Sta22, Lemma 01W6]), thus a closed immersion ([Sta22, Lemma 04XV]). The closed immersion $\prod_{i=1}^n \mathbf{Quot}_{V_i(-lm)/X/\mathbb{k}}^{P_i(t), \mathcal{L}} \hookrightarrow \mathbf{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$ factors through the locally closed subscheme Y_τ , which is easily seen from the associated functors of points. There is a retraction $p_\tau : Y_\tau \rightarrow \prod_{i=1}^n \mathbf{Quot}_{V_i(-lm)/X/\mathbb{k}}^{P_i(t), \mathcal{L}}$ such that the following diagram commutes

$$\begin{array}{ccc}
\prod_{i=1}^n \mathbf{Quot}_{V_i(-lm)/X/\mathbb{k}}^{P_i(t), \mathcal{L}} & \xrightarrow{\subseteq} & Y_\tau \\
& \searrow = & \downarrow p_\tau \\
& & \prod_{i=1}^n \mathbf{Quot}_{V_i(-lm)/X/\mathbb{k}}^{P_i(t), \mathcal{L}}
\end{array} \tag{6.77}$$

where p_τ sends an isomorphism class $[(\mathcal{E}, q)]$ of families over $T \in \text{Sch}/\mathbb{k}$ to the isomorphism class of $\left[\left(\bigoplus_{i=1}^n \mathcal{E}_i, \bigoplus_{i=1}^n q_i \right) \right]$, where $\mathcal{E}_i$ and q_i are defined in equation (6.69) and equation (6.70).

Consider $(\mathbb{G}_m)^n \rightarrow \text{GL}(V)$ such that $(t_1, \dots, t_n) \mapsto t_1 1_{V_1} \oplus \dots \oplus t_n 1_{V_n} \in \text{GL}(V)$. Then $(\mathbb{G}_m)^n$ acts on Y_τ . With this $(\mathbb{G}_m)^n$ -action, we have that $\prod_{i=1}^n \mathbf{Quot}_{V_i(-lm)/X/\mathbb{k}}^{P_i(t), \mathcal{L}}$ is the *fixed point subscheme* $(Y_\tau)^{(\mathbb{G}_m)^n} \hookrightarrow Y_\tau$ in the sense of [Fog73]. It is easy to see if we consider

$\lambda : \mathbb{G}_m \rightarrow (\mathbb{G}_m)^n$ given by $t \mapsto (t^{w_1}, \dots, t^{w_n})$ such that $w_1 > \dots > w_n$, then the fixed point subscheme of the action of λ is also $\prod_{i=1}^n \text{Quot}_{V_i(-lm)/X/\mathbb{k}}^{P_i(t), \mathcal{L}}$

$$\prod_{i=1}^n \text{Quot}_{V_i(-lm)/X/\mathbb{k}}^{P_i(t), \mathcal{L}} \cong (Y_\tau)^{(\mathbb{G}_m)^n} = (Y_\tau)^\lambda. \quad (6.78)$$

We then describe the open subschemes $Y_\tau^s \subseteq Y_\tau^{ss} \subseteq Y_\tau$. For $1 \leq i \leq n$, let $R_i^{(s)s} \subseteq \text{Quot}_{V_i(-lm)/X/\mathbb{k}}^{P_i(t), \mathcal{L}}$ be the open subscheme whose closed points are classes of quotients $(\mathcal{E}_i, q_i)$ such that $\mathcal{E}_i$ is a (semi)stable pure sheaf on X and $q : V_i(-lm) \rightarrow \mathcal{E}_i$ induces an isomorphism $V_i \cong H^0(X, \mathcal{E}_i(lm))$ (cf. Theorem 6.3.5). We define $Y_\tau^{(s)s}$ as the fibre product indicated in the diagram below

$$\begin{array}{ccc} Y_\tau^{(s)s} & \xrightarrow{\subseteq} & Y_\tau \\ \downarrow p_\tau^{(s)s} & & \downarrow p_\tau \\ \prod_{i=1}^n R_i^{(s)s} & \xrightarrow{\subseteq} & \prod_{i=1}^n \text{Quot}_{V_i(-lm)/X/\mathbb{k}}^{P_i(t), \mathcal{L}} \end{array} \quad (6.79)$$

We have that $Y_\tau^{(s)s} \subseteq Y_\tau$ is open. So $p_\tau^{(s)s} : Y_\tau^{(s)s} \rightarrow \prod_{i=1}^n R_i^{(s)s}$ are also retractions as base changes of p_τ . In particular, p_τ and $p_\tau^{(s)s}$ are surjective.

The action of P_τ on Y_τ is induced from the action of $\text{SL}(V)$ on $\text{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$. The open subschemes $Y_\tau^s \subseteq Y_\tau^{ss}$ of Y_τ are invariant under P_τ , i.e. P_τ acts on $Y_\tau^{(s)s}$.

In [Jac18] and [Jac21], Jackson considered the induced reduced subscheme structure on Y_τ^s for a Harder-Narasimhan type τ of length 2, and stratify this reduced subscheme further according to the dimension of the stabiliser in the unipotent radical of P_τ . Non-reductive GIT could be applied to each stratum and the following theorem was proved.

Definition 6.3.10 (cf. [Jac21] Definition 7.1.2, [HJ21] Definition 6.1). Let X be a connected projective scheme over $\mathbb{k}$ with an ample line bundle $\mathcal{L}$. Let τ be a Harder-Narasimhan type of length 2. Let $\delta \in \mathbb{N}$. A pure sheaf $\mathcal{E}$ on X is called (τ, δ) -stable if:

- $\mathcal{E}$ has Harder-Narasimhan type τ ;
- if $\{\mathcal{E}_{\leq i}\}_{i=0}^2$ is the Harder-Narasimhan filtration, then $\mathcal{E}_i := \mathcal{E}_{\leq i}/\mathcal{E}_{\leq i-1}$ is stable for $i = 1, 2$;
- $\mathcal{E} \not\cong \mathcal{E}_1 \oplus \mathcal{E}_2$;

- $\dim \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{E}_2, \mathcal{E}_1) = \delta$.

Theorem 6.3.11 ([Jac21] Theorem 7.1.3). *Let X be a projective scheme over $\mathbb{k}$ with an ample line bundle $\mathcal{L}$. Let τ be a Harder-Narasimhan type of length 2 and let $\delta \in \mathbb{N}$. Then there exists a quasi-projective moduli space $M_{\tau, \delta}$ of (τ, δ) -stable sheaves on X , in the sense that closed points in $M_{\tau, \delta}$ are exactly isomorphism classes of (τ, δ) -stable sheaves.*

Remark 6.3.12. The moduli spaces $M_{\tau, \delta}$ above are reduced schemes since they were constructed by the classic U-hat theorem (cf. Theorem 2.20, [Bér+16]), which only produces reduced quotient schemes.

6.4 On the boundaries of Y_τ and Y_τ^s

Let τ be a Harder-Narasimhan type of length $n \in \mathbb{N}_+$. Let $P_i(t), P(t)$ be the Hilbert polynomials associated to τ . For $1 \leq i \leq n$, let $P_{\leq i}(t) := P_1(t) + \dots + P_i(t)$. Recall that $P_\tau \subseteq \operatorname{SL}(V)$ is the parabolic subgroup stabilising the flag $\{V_{\leq i}\}_{i=0}^n$ (See the first paragraph of Section 6.3.2). Recall that $Y_\tau \subseteq \operatorname{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$ is the locally closed subscheme that represents the sub-functor $\mathbf{Y}_\tau$ (See Lemma 6.3.9). There is a retraction $p_\tau : Y_\tau \rightarrow \prod_{i=1}^n \operatorname{Quot}_{V_i(-lm)/X/\mathbb{k}}^{P_i(t), \mathcal{L}}$ (See diagram (6.77)), and $Y_\tau^s \subseteq Y_\tau$ is the preimage of $\prod_{i=1}^n R_i^s \subseteq \prod_{i=1}^n \operatorname{Quot}_{V_i(-lm)/X/\mathbb{k}}^{P_i(t), \mathcal{L}}$ under p_τ (See diagram (6.79)).

Let $U_\tau \subseteq P_\tau$ be the unipotent radical and let $K_\tau \subseteq P_\tau$ be its semisimple part $K_\tau = \prod_{i=1}^n \operatorname{SL}(V_i)$. Let $T_\tau = (\mathbb{G}_m)^{n-1}$ and let $T_\tau \rightarrow P_\tau$ be the following homomorphism

$$(1, \dots, t_i, \dots, 1) \mapsto \begin{pmatrix} 1_{V_1} & & & & \\ & \ddots & & & \\ & & t_i^{\frac{P_{i+1}}{(P_i, P_{i+1})}} 1_{V_i} & & \\ & & & t_i^{\frac{-P_i}{(P_i, P_{i+1})}} 1_{V_{i+1}} & \\ & & & & \ddots & \\ & & & & & 1_{V_n} \end{pmatrix} \quad (6.80)$$

where $P_i := P_i(lm) = \dim V_i$ and $(P_i, P_{i+1}) := \gcd(P_i, P_{i+1})$. Then $U_\tau \rtimes (T_\tau \times K_\tau) \rightarrow P_\tau$ is surjective with finite kernel.

We want to consider the P_τ -quotient of Y_τ^s . However, Y_τ^s is not projective and our non-reductive GIT requires a linear action on a projective scheme. Therefore we need to consider

the P_τ -action on a suitable projective completion $\overline{Y_\tau^s}$. We will take $\overline{Y_\tau^s}$ to be the scheme-theoretic image of $Y_\tau^s \rightarrow \text{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$. The main result of this section is Proposition 6.4.2, which says that we do not need to worry about the boundary of $Y_\tau^s \subseteq \overline{Y_\tau^s}$.

For $T \in \text{Sch}/\mathbb{k}$ and a coherent sheaf $\mathcal{F}$ on X , recall the notation

$$\begin{aligned} X_T &:= X \times_{\mathbb{k}} T \\ f_T : X_T &\rightarrow T \quad \text{the projection} \\ \mathcal{F}_T &\quad \text{the pull-back of } \mathcal{F} \text{ along } X_T \rightarrow X. \end{aligned} \tag{6.81}$$

For a family $(\mathcal{E}, q)$ of quotients parameterised by T , recall the notation

$$\begin{aligned} \mathcal{E}_{\leq i} &:= q(V_{\leq i}(-lm)_T) \\ \mathcal{E}_i &:= \mathcal{E}_{\leq i} / \mathcal{E}_{\leq i-1} \\ q_i : V_i(-lm)_T &\rightarrow \mathcal{E}_i \quad \text{the induced surjective morphism.} \end{aligned} \tag{6.82}$$

Recall that there exist closed immersions

$$\begin{aligned} \text{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}} &\hookrightarrow \text{Grass}(V \otimes_{\mathbb{k}} H, P(lm + mM')) \\ &\hookrightarrow \mathbb{P} \left(\left(\bigwedge^{P(lm+mM')} (V \otimes_{\mathbb{k}} H) \right)^* \right) \end{aligned} \tag{6.83}$$

where $H := H^0(X, \mathcal{L}^{mM'}) = f_*(\mathcal{L}^{mM'})$. Note that points in the Grassmannian are quotients while points in the projective space are one-dimensional subspaces. Recall that $\mathcal{M}$ denotes a very ample line bundle on $\text{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$, which is the pull back of $\mathcal{O}(1)$ along the immersions above.

Let $\overline{Y_\tau} \hookrightarrow \text{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$ be the scheme-theoretic image of $Y_\tau \rightarrow \text{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$.

Proposition 6.4.1. *For sufficiently large $M' \gg l \gg 0$, and for any internal grading 1PS $\lambda : \mathbb{G}_m \rightarrow T_\tau$, there exists a closed subscheme*

$$\mathbb{P}^{N'} \hookrightarrow \mathbb{P} \left(\left(\bigwedge^{P(lm+mM')} (V \otimes_{\mathbb{k}} H) \right)^* \right) \tag{6.84}$$

such that:

- (1) $\mathbb{P}^{N'}$ is a T_τ -invariant linear subspace;
- (2) the immersion $Y_\tau \rightarrow \mathbb{P} \left(\left(\bigwedge^{P(lm+mM')} (V \otimes_{\mathbb{k}} H) \right)^* \right)$ factors through $\mathbb{P}^{N'}$, and so does $\overline{Y_\tau}$, i.e. there exists a closed immersion $\overline{Y_\tau} \hookrightarrow \mathbb{P}^{N'}$;

$$(3) \quad (\mathbb{P}^{N'})_{\lambda-\min}^0 \cap \overline{Y_\tau} = Y_\tau.$$

Proof. Only in this proof, for simplicity we denote

$$P := P(lm + mM'), \quad P_i := P_i(lm + mM'), \quad P_{\leq i} := P_{\leq i}(lm + mM') \quad (6.85)$$

and we simplify $(-) \otimes_{\mathbb{k}} (-)$ to $(-) \otimes (-)$

We first construct the linear subspace $\mathbb{P}^{N'}$, which is determined by a quotient space of $\bigwedge^P(V \otimes H)$. We have

$$\begin{aligned} \bigwedge^P(V \otimes H) &= \bigwedge^P \left(\bigoplus_{i=1}^n (V_i \otimes H) \right) \\ &= \bigoplus_{(m_1, \dots, m_n) \in \Lambda} \bigwedge^{m_1}(V_1 \otimes H) \otimes \dots \otimes \bigwedge^{m_n}(V_n \otimes H) \end{aligned} \quad (6.86)$$

where the direct sum is over the index set Λ

$$\Lambda := \{(m_1, \dots, m_n) \in \mathbb{N}^n : m_1 + \dots + m_n = P(lm + mM')\}. \quad (6.87)$$

Let $\Lambda_0 \subseteq \Lambda$ be the subset

$$\Lambda_0 := \left\{ (m_1, \dots, m_n) \in \Lambda : \begin{array}{l} \text{there exist } 1 \leq i \leq n \text{ such that} \\ m_1 + \dots + m_i > P_{\leq i}(lm + mM') \end{array} \right\}. \quad (6.88)$$

and let $W \subseteq \bigwedge^P(V \otimes H)$ be the following subspace

$$W := \bigoplus_{(m_1, \dots, m_n) \in \Lambda_0} \bigwedge^{m_1}(V_1 \otimes H) \otimes \dots \otimes \bigwedge^{m_n}(V_n \otimes H). \quad (6.89)$$

Then the following quotient space of $\bigwedge^P(V \otimes H)$

$$\bigwedge^P(V \otimes H) \longrightarrow \bigwedge^P(V \otimes H) / W \longrightarrow 0 \quad (6.90)$$

defines a linear subspace

$$\mathbb{P}^{N'} \hookrightarrow \mathbb{P} \left(\left(\bigwedge^P(V \otimes H) \right)^* \right). \quad (6.91)$$

Since $W \subseteq \bigwedge^P(V \otimes H)$ is invariant under the torus T_τ , we have that $\mathbb{P}^{N'}$ is invariant under T_τ . This proves (1).

Next we prove (2). Assume first we have proved that $Y_\tau \rightarrow \mathbb{P}\left((\wedge^P(V \otimes H))^*\right)$ factors through $\mathbb{P}^{N'}$, then we have the following commutative diagram

$$\begin{array}{ccc} Y_\tau & \longrightarrow & \text{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}} \\ \downarrow & & \downarrow \\ \mathbb{P}^{N'} & \hookrightarrow & \mathbb{P}\left((\wedge^P(V \otimes H))^*\right). \end{array} \quad (6.92)$$

It is easy to see that if we have morphisms $T \rightarrow Z_1 \hookrightarrow Z_2$ where $Z_1 \hookrightarrow Z_2$ is a closed immersion, then the scheme-theoretic image of $T \rightarrow Z_1$ is canonically isomorphic to the scheme-theoretic image of $T \rightarrow Z_2$. Apply this observation to the diagram above. Then $\overline{Y}_\tau$ is the scheme-theoretic image of all three morphisms from Y_τ in the diagram. In particular, we have $\overline{Y}_\tau \hookrightarrow \mathbb{P}^{N'}$, which is the second part of (2).

The first part of (2) is equivalent to the following

(2') for any $T \in \text{Sch}/\mathbb{k}$ and any $T \rightarrow Y_\tau$, the composition $T \rightarrow Y_\tau \rightarrow \mathbb{P}\left((\wedge^P(V \otimes H))^*\right)$ factors through $\mathbb{P}^{N'}$.

Let $T \in \text{Sch}/\mathbb{k}$ and let $T \rightarrow Y_\tau$ be a morphism. Let $(\mathcal{E}, q)$ be a family of quotients over T whose classifying morphism is $T \rightarrow Y_\tau$. The composition $T \rightarrow Y_\tau \rightarrow \mathbb{P}\left((\wedge^P(V \otimes H))^*\right)$ is the classifying morphism of the following surjective morphism

$$\wedge^P(V \otimes H) \otimes \mathcal{O}_T \xrightarrow{\wedge^P \rho} \wedge^P(f_T)_*(\mathcal{E}(lm + mM')) \longrightarrow 0 \quad (6.93)$$

where

- $\mathcal{E}(lm + mM') := \mathcal{E} \otimes_{\mathcal{O}_{X_T}} \mathcal{L}_T^{lm+mM'}$;
- $\rho : (V \otimes H) \otimes \mathcal{O}_T \rightarrow (f_T)_*(\mathcal{E}(lm + mM'))$ is the morphism

$$\rho := (f_T)_*\left(q \otimes_{\mathcal{O}_{X_T}} 1_{\mathcal{L}_T^{lm+mM'}}\right); \quad (6.94)$$

- $(f_T)_*(\mathcal{E}(lm + mM'))$ is locally free of rank $P = P(lm + mM')$ on T when M' is sufficiently large (See step (3) in Section 6.2.1).

The morphism $T \rightarrow \mathbb{P}\left(\left(\bigwedge^P(V \otimes H)\right)^*\right)$ factors through $\mathbb{P}^{N'}$ if and only if the following composition is zero

$$W \otimes \mathcal{O}_T \xrightarrow{\subseteq} \bigwedge^P(V \otimes H) \otimes \mathcal{O}_T \xrightarrow{\bigwedge^P \rho} \bigwedge^P(f_T)_*(\mathcal{E}(lm + mM')). \quad (6.95)$$

Since the isomorphism class of $(\mathcal{E}, q)$ corresponds to a morphism to Y_τ , we have that $\mathcal{E}_{\leq i}$ is flat over T with Hilbert polynomial $P_{\leq i}(t)$. We can enlarge M' and assume that $(f_T)_*(\mathcal{E}_{\leq i}(lm + mM'))$ is locally free of rank $P_{\leq i}$ for all $1 \leq i \leq n$ and all $T \in \text{Sch}/\mathbb{k}$ and all families of quotients $(\mathcal{E}, q)$ over T (Again see step (3) in Section 6.2.1).

Let $(m_1, \dots, m_n) \in \Lambda_0$. Let $1 \leq j \leq n$ be such that $m_1 + \dots + m_j > P_{\leq j}$. Choose

$$\begin{aligned} s_1^{(1)}, \dots, s_{m_1}^{(1)} &\in V_1 \otimes H \\ &\vdots \\ s_1^{(j)}, \dots, s_{m_j}^{(j)} &\in V_j \otimes H \\ S^{(>j)} &\in \bigwedge_{m_{j+1}}^{m_{j+1}} (V_{j+1} \otimes H) \otimes \dots \otimes \bigwedge_{m_n}^{m_n} (V_n \otimes H). \end{aligned} \quad (6.96)$$

Then vectors of the following form span the vector space $\bigwedge^{m_1}(V_1 \otimes H) \otimes \dots \otimes \bigwedge^{m_n}(V_n \otimes H)$

$$v := (s_1^{(1)} \wedge \dots \wedge s_{m_1}^{(1)}) \otimes \dots \otimes (s_1^{(j)} \wedge \dots \wedge s_{m_j}^{(j)}) \otimes S^{(>j)}. \quad (6.97)$$

If we consider all $(m_1, \dots, m_n) \in \Lambda_0$, then vectors of the above form span W . Therefore to show $\left(\bigwedge^P \rho\right)(W \otimes \mathcal{O}_T) = 0$, it suffices to show that $v \otimes 1 \in W \otimes \mathcal{O}_T$ maps to zero under $\bigwedge^P \rho$.

The image of $v \otimes 1$ under $\bigwedge^P \rho$ is

$$\left(\rho(s_1^{(1)} \otimes 1) \wedge \dots \wedge \rho(s_{m_1}^{(1)} \otimes 1)\right) \wedge \dots \wedge \left(\rho(s_1^{(j)} \otimes 1) \wedge \dots \wedge \rho(s_{m_j}^{(j)} \otimes 1)\right) \wedge (\wedge \rho)^{(>j)} \quad (6.98)$$

where $(\wedge \rho)^{(>j)} := \left(\bigwedge^{P-(m_1+\dots+m_j)} \rho\right)(S^{(>j)} \otimes 1)$. We have

$$\left(\rho(s_1^{(1)} \otimes 1) \wedge \dots \wedge \rho(s_{m_1}^{(1)} \otimes 1)\right) \wedge \dots \wedge \left(\rho(s_1^{(j)} \otimes 1) \wedge \dots \wedge \rho(s_{m_j}^{(j)} \otimes 1)\right) = 0 \quad (6.99)$$

since it is a section of the sheaf

$$\bigwedge_{m_1+\dots+m_j}^{m_1+\dots+m_j} (f_T)_*(\mathcal{E}_{\leq j}(lm + mM')) = 0 \quad (6.100)$$

which is the zero sheaf since $(f_T)_*(\mathcal{E}_{\leq j}(lm+mM'))$ is locally free of rank $P_{\leq j} < m_1 + \dots + m_j$. This proves (2)', thus (2).

For (3), let $\lambda : \mathbb{G}_m \rightarrow P_\tau$ be an internal grading 1PS which factors through $T_\tau \rightarrow P_\tau$. Then there exist integers $w_1 > \dots > w_n$ such that $w_1 \dim V_1 + \dots + w_n \dim V_n = 0$ and $\lambda(t) = t^{w_1} 1_{V_1} \oplus \dots \oplus t^{w_n} 1_{V_n}$. By [Sta22, Lemma 01QV] and [Sta22, Lemma 01R8], we have that $Y_\tau \subseteq \overline{Y_\tau}$ is an open subscheme. Then both sides of (3) are open subschemes of $\overline{Y_\tau}$, so it suffices to show that they contain the same closed points.

Recall that

$$\begin{aligned} H^0(\mathbb{P}^{N'}, \mathcal{O}(1)) &\cong \bigwedge^P (V \otimes H) / W \\ &\cong \bigoplus_{(m_1, \dots, m_n) \in \Lambda \setminus \Lambda_0} \bigwedge^{m_1} (V_1 \otimes H) \otimes \dots \otimes \bigwedge^{m_n} (V_n \otimes H) \end{aligned} \quad (6.101)$$

The weights for $\lambda \curvearrowright H^0(\mathbb{P}^{N'}, \mathcal{O}(1))$ are

$$\sum_{i=1}^n w_i m_i \dim V_i \dim H, \quad (m_1, \dots, m_n) \in \Lambda \setminus \Lambda_0. \quad (6.102)$$

When $(m_1, \dots, m_n) \in \Lambda \setminus \Lambda_0$, we have $m_1 + \dots + m_i \leq P_{\leq i}$ for $1 \leq i \leq n$ and

$$\begin{aligned} &\sum_{i=1}^n w_i m_i \dim V_i \dim H \\ &= w_n P \dim V \dim H + \sum_{i=1}^{n-1} (w_i - w_{i+1}) (m_1 + \dots + m_i) \dim V_{\leq i} \dim H \\ &\leq w_n P \dim V \dim H + \sum_{i=1}^{n-1} (w_i - w_{i+1}) P_{\leq i} \dim V_{\leq i} \dim H \\ &= \sum_{i=1}^n w_i P_i \dim V_i \dim H. \end{aligned} \quad (6.103)$$

So the maximal λ -weight on $H^0(\mathbb{P}^{N'}, \mathcal{O}(1))$ is $\sum_{i=1}^n w_i P_i \dim V_i \dim H$ and

$$H^0(\mathbb{P}^{N'}, \mathcal{O}(1))_{\lambda=\max} \cong \bigwedge^{P_1} (V_1 \otimes H) \otimes \dots \otimes \bigwedge^{P_n} (V_n \otimes H). \quad (6.104)$$

For one direction of inclusion $Y_\tau \subseteq (\mathbb{P}^{N'})_{\lambda=\min}^0 \cap \overline{Y_\tau}$, let $[(\mathcal{E}, q)] \in Y_\tau$ be a closed point. By [HL10] Lemma 4.4.3, we have when considered in $\text{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$

$$\lim_{t \rightarrow 0} \lambda(t) \cdot [(\mathcal{E}, q)] = \left[\left(\bigoplus_{i=1}^n \mathcal{E}_i, \bigoplus_{i=1}^n q_i \right) \right] \in \prod_{i=1}^n \text{Quot}_{V_i(-lm)/X/\mathbb{k}}^{P_i(t), \mathcal{L}}. \quad (6.105)$$

The limit point is still in Y_τ and its image in $\mathbb{P}^{N'}$ represents the one-dimensional quotient

$$\begin{aligned} \bigwedge^P (V \otimes H) / W &\rightarrow \bigwedge^P \left(\bigoplus_{i=1}^n H^0(X, \mathcal{E}_i(lm + mM')) \right) \\ &\cong \bigwedge^{P_1} H^0(X, \mathcal{E}_1(lm + mM')) \otimes \cdots \otimes \bigwedge^{P_n} H^0(X, \mathcal{E}_n(lm + mM')). \end{aligned} \quad (6.106)$$

The point $\lim \lambda(t) \cdot [(\mathcal{E}, q)]$ is in $(\mathbb{P}^{N'})_{\lambda-\min}^0$ if and only if the following composition is not zero

$$\begin{aligned} &\bigwedge^{P_1} (V_1 \otimes H) \otimes \cdots \otimes \bigwedge^{P_n} (V_n \otimes H) \\ &\subseteq \bigwedge^P (V \otimes H) / W \\ &\rightarrow \bigwedge^{P_1} H^0(X, \mathcal{E}_1(lm + mM')) \otimes \cdots \otimes \bigwedge^{P_n} H^0(X, \mathcal{E}_n(lm + mM')). \end{aligned} \quad (6.107)$$

The composition above is surjective since it is the tensor product of exterior products of surjective morphisms

$$V_i \otimes H := V_i \otimes H^0(X, \mathcal{L}^{mM'}) \rightarrow H^0(X, \mathcal{E}_i(lm + mM')), \quad 1 \leq i \leq n. \quad (6.108)$$

This proves $\lim \lambda(t) \cdot [(\mathcal{E}, q)] \in (\mathbb{P}^{N'})_{\lambda-\min}^0$ and this shows $[(\mathcal{E}, q)] \in (\mathbb{P}^{N'})_{\lambda-\min}^0$. This proves $Y_\tau \subseteq (\mathbb{P}^{N'})_{\lambda-\min}^0 \cap \overline{Y_\tau}$.

For the other direction $(\mathbb{P}^{N'})_{\lambda-\min}^0 \cap \overline{Y_\tau} \subseteq Y_\tau$, let $[(\mathcal{E}, q)] \in \text{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}}$ be a closed point such that $[(\mathcal{E}, q)] \in (\mathbb{P}^{N'})_{\lambda-\min}^0 \cap \overline{Y_\tau}$. The point $[(\mathcal{E}, q)] \in \mathbb{P} \left(\left(\bigwedge^P (V \otimes H) \right)^* \right)$ corresponds to the one-dimensional quotient

$$\pi : \bigwedge^P (V \otimes H) \rightarrow \bigwedge^P H^0(X, \mathcal{E}(lm + mM')) \quad (6.109)$$

and $\pi(W) = 0$ since $[(\mathcal{E}, q)] \in \mathbb{P}^{N'}$.

The following collection of coherent sheaves on X is bounded (Definition 6.3.2)

$$\bigcup_{i=1}^n \left\{ \mathcal{F} : \mathcal{F} = \mathcal{E}_i \text{ or } \mathcal{F} = \mathcal{E}_{\leq i} \text{ for } [(\mathcal{E}, q)] \in \text{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}} \right\} \quad (6.110)$$

By Lemma 6.3.3, the set of Hilbert polynomials of the above collection is finite, i.e. the following set is finite

$$\mathcal{P} := \bigcup_{i=1}^n \left\{ P(\mathcal{F}, t) : \mathcal{F} = \mathcal{E}_i \text{ or } \mathcal{F} = \mathcal{E}_{\leq i} \text{ for } [(\mathcal{E}, q)] \in \text{Quot}_{V(-lm)/X/\mathbb{k}}^{P(t), \mathcal{L}} \right\}. \quad (6.111)$$

Also by Lemma 6.3.3, regularities of $\mathcal{E}_i$ and $\mathcal{E}_{\leq i}$ are bounded. We can enlarge M' such that

- higher cohomology groups of $\mathcal{E}_{\leq i}(lm + mM')$ and $\mathcal{E}_i(lm + mM')$ are zero;
- the following map is order-preserving and bijective

$$\mathcal{P} \rightarrow \mathbb{N}, \quad P(\mathcal{F}, t) \mapsto P(\mathcal{F}, lm + mM'). \quad (6.112)$$

Let $m_i := \dim H^0(X, \mathcal{E}_i(lm + mM'))$ for $1 \leq i \leq n$. When M' is large, higher cohomology groups of $\mathcal{E}_i(lm + mM')$ vanish, and then for $1 \leq i \leq n$

$$\begin{aligned} m_i &= P(\mathcal{E}_i, lm + mM') \\ m_1 + \cdots + m_i &= P(\mathcal{E}_{\leq i}, lm + mM'). \end{aligned} \quad (6.113)$$

We claim that $m_1 + \cdots + m_i \geq P_{\leq i}$ for $1 \leq i \leq n$. Otherwise the following composition is zero

$$\begin{aligned} & \bigwedge^{P_1} (V_1 \otimes H) \otimes \cdots \otimes \bigwedge^{P_n} (V_n \otimes H) \\ & \subseteq \bigwedge^P (V \otimes H) \\ & \xrightarrow{\pi} \bigwedge^P H^0(X, \mathcal{E}(lm + mM')). \end{aligned} \quad (6.114)$$

which contradicts that $[(\mathcal{E}, q)] \in (\mathbb{P}^{N'})_{\lambda-\min}^0$.

Either $m_i = P_i$ for $1 \leq i \leq n$ or $(m_1, \dots, m_n) \in \Lambda_0$. We exclude the case when $(m_1, \dots, m_n) \in \Lambda_0$. Otherwise assume $(m_1, \dots, m_n) \in \Lambda_0$. There exist $s_j^{(i)} \in V_i \otimes H$ for $1 \leq i \leq n$ and $1 \leq j \leq m_i$ such that for all $1 \leq i \leq n$

$$s_1^{(1)}, \dots, s_{m_1}^{(1)}, \dots, s_1^{(i)}, \dots, s_{m_i}^{(i)} \in V_{\leq i} \otimes H \quad (6.115)$$

are sent to a basis of $H^0(X, \mathcal{E}_{\leq i}(lm + mM'))$. In particular the images of $s_j^{(i)}$ for $1 \leq i \leq n$ and $1 \leq j \leq m_i$ in $H^0(X, \mathcal{E}(lm + mM'))$ form a basis. Then the following composition is non-zero

$$\begin{aligned} & \bigwedge^{m_1} (V_1 \otimes H) \otimes \cdots \otimes \bigwedge^{m_n} (V_n \otimes H) \\ & \subseteq \bigwedge^P (V \otimes H) \\ & \xrightarrow{\pi} \bigwedge^P H^0(X, \mathcal{E}(lm + mM')) \end{aligned} \quad (6.116)$$

which contradicts $\pi(W) = 0$. Therefore we have

$$P(\mathcal{E}_i, lm + mM') = P_i(lm + mM'). \quad (6.117)$$

Since $P(\mathcal{E}_i, t), P_i(t) \in \mathcal{P}$, we have $P(\mathcal{E}_i, t) = P_i(t)$ and this shows $[(\mathcal{E}, q)] \in Y_\tau$. This proves $(\mathbb{P}^{N'})^0_{\lambda-\min} \cap \overline{Y_\tau} \subseteq Y_\tau$, thus (3). $\square$

Let $\overline{Y_\tau^s} \hookrightarrow \overline{Y_\tau}$ be the scheme-theoretic image of $Y_\tau^s \rightarrow \overline{Y_\tau}$.

Proposition 6.4.2. *Assume $\prod_{i=1}^n R_i^s \neq \emptyset$. For $M' \gg l \gg 0$, and any internal grading 1PS $\lambda : \mathbb{G}_m \rightarrow \mathbb{T}_\tau$, we have the following:*

$$(1) \quad (\overline{Y_\tau^s})^0_{\lambda-\min} = \overline{Y_\tau^s} \cap Y_\tau;$$

$$(2) \quad \text{Let } Z_{\lambda-\min}(Y_\tau^s) := \left((\overline{Y_\tau^s})^0_{\lambda-\min} \right)^\lambda. \text{ Then}$$

$$\begin{aligned} Z_{\lambda-\min}(Y_\tau^s) &= \overline{Y_\tau^s} \cap \prod_{i=1}^n \text{Quot}_{V_i(-lm)/X/\mathbb{k}}^{P_i(t), \mathcal{L}} \\ \prod_{i=1}^n R_i^s &\subseteq Z_{\lambda-\min}(Y_\tau^s) \hookrightarrow \prod_{i=1}^n \overline{R_i}; \end{aligned} \quad (6.118)$$

$$(3) \quad \text{Let } Z_{\lambda-\min}^{K_\tau, s}(Y_\tau^s) \subseteq Z_{\lambda-\min}(Y_\tau^s) \text{ denote the stable locus for the } K_\tau\text{-linearisation. Then}$$

$$Z_{\lambda-\min}^{K_\tau, s}(Y_\tau^s) = \prod_{i=1}^n R_i^s, \quad p_\tau^{-1}(Z_{\lambda-\min}^{K_\tau, s}(Y_\tau^s)) = Y_\tau^s. \quad (6.119)$$

Proof. We consider the ample line bundle $\mathcal{M}|_{\overline{Y_\tau^s}}$ on $\overline{Y_\tau^s}$, which equals the pull-back of $\mathcal{O}(1)$ along $\overline{Y_\tau^s} \hookrightarrow \mathbb{P}\left(\left(\bigwedge^P(V \otimes H)\right)^*\right)$, for $P := P(lm + mM')$. By Proposition 6.4.1, there is a linear subspace $\mathbb{P}^{N'} \hookrightarrow \mathbb{P}\left(\left(\bigwedge^P(V \otimes H)\right)^*\right)$ containing $\overline{Y_\tau}$. Then $\mathcal{M}|_{\overline{Y_\tau^s}} \cong \mathcal{O}_{\mathbb{P}^{N'}}(1)|_{\overline{Y_\tau^s}}$.

For (1), both sides are open in $\overline{Y_\tau^s}$, so we can check that the closed points coincide. We have that $\emptyset \neq \prod_{i=1}^n R_i^s \subseteq \overline{Y_\tau^s} \cap (\mathbb{P}^{N'})^0_{\lambda-\min}$, and then

$$(\overline{Y_\tau^s})^0_{\lambda-\min} = \overline{Y_\tau^s} \cap (\mathbb{P}^{N'})^0_{\lambda-\min}. \quad (6.120)$$

By Proposition 6.4.1, $\overline{Y_\tau} \cap (\mathbb{P}^{N'})^0_{\lambda-\min} = Y_\tau$ and then

$$\begin{aligned} (\overline{Y_\tau^s})^0_{\lambda-\min} &= \overline{Y_\tau^s} \cap (\mathbb{P}^{N'})^0_{\lambda-\min} \\ &= \overline{Y_\tau^s} \cap \overline{Y_\tau} \cap (\mathbb{P}^{N'})^0_{\lambda-\min} \\ &= \overline{Y_\tau^s} \cap Y_\tau \end{aligned} \quad (6.121)$$

which proves (1).

For (2), let $Z_{\lambda-\min}(Y_\tau^s) := \left((\overline{Y_\tau^s})_{\lambda-\min}^0 \right)^\lambda$ denote the fixed subscheme with respect to λ for an internal grading 1PS $\lambda : \mathbb{G}_m \rightarrow T_\tau$. Recall that $(Y_\tau)^\lambda = \prod_{i=1}^n \text{Quot}_{V_i(-lm)/X/\mathbb{k}}^{P_i(t), \mathcal{L}}$ and the intersection $\overline{Y_\tau^s} \cap (Y_\tau)^\lambda$ is defined as the fibre product over $\overline{Y_\tau}$. For the first equality, it suffices to show existence of following morphisms over $\overline{Y_\tau}$

$$\begin{aligned} Z_{\lambda-\min}(Y_\tau^s) &\rightarrow \overline{Y_\tau^s} \cap (Y_\tau)^\lambda \\ \overline{Y_\tau^s} \cap (Y_\tau)^\lambda &\rightarrow Z_{\lambda-\min}(Y_\tau^s). \end{aligned} \tag{6.122}$$

It is easy that the morphism $Z_{\lambda-\min}(Y_\tau^s) \rightarrow \overline{Y_\tau^s} \cap (Y_\tau)^\lambda$ exists. The existence of $\overline{Y_\tau^s} \cap (Y_\tau)^\lambda \rightarrow Z_{\lambda-\min}(Y_\tau^s)$ is from that the morphism $\overline{Y_\tau^s} \cap (Y_\tau)^\lambda \rightarrow (Y_\tau^s)_{\lambda-\min}^0$ is λ -equivariant with the trivial λ -action on $\overline{Y_\tau^s} \cap (Y_\tau)^\lambda$. This proves $Z_{\lambda-\min}(Y_\tau^s) = \overline{Y_\tau^s} \cap (Y_\tau)^\lambda$. Note that the equality is scheme-theoretic.

The morphism $\prod_{i=1}^n R_i^s \rightarrow Z_{\lambda-\min}(Y_\tau^s)$ is from $\prod_{i=1}^n R_i^s \rightarrow (Y_\tau^s)_{\lambda-\min}^0$, which is λ -equivariant with trivial action on the source. It is an open immersion, since the composition $\prod_{i=1}^n R_i^s \rightarrow Z_{\lambda-\min}(Y_\tau^s) \rightarrow \prod_{i=1}^n \text{Quot}_{V_i(-lm)/X/\mathbb{k}}^{P_i(t), \mathcal{L}}$ is an open immersion.

We then construct the morphism $Z_{\lambda-\min}(Y_\tau^s) \rightarrow \prod_{i=1}^n \overline{R_i}$. By [Sta22, Lemma 01R8], $\overline{Y_\tau^s} \cap Y_\tau \hookrightarrow Y_\tau$ is the scheme-theoretic image of $Y_\tau^s \rightarrow Y_\tau$. We have that the morphism $Y_\tau^s \rightarrow Y_\tau$ factors through $p_\tau^{-1}\left(\prod_{i=1}^n \overline{R_i}\right) \hookrightarrow Y_\tau$, so the dashed arrow below exists by the universal property of scheme-theoretic image

$$\begin{array}{ccccc} Y_\tau^s & \xrightarrow{\subseteq} & \overline{Y_\tau^s} \cap Y_\tau & \hookrightarrow & Y_\tau \\ & \searrow & \downarrow & \swarrow & \\ & & p_\tau^{-1}\left(\prod_{i=1}^n \overline{R_i}\right) & & \end{array} \tag{6.123}$$

We obtain a morphism $Z_{\lambda-\min}(Y_\tau^s) \rightarrow \prod_{i=1}^n \overline{R_i}$ by taking fixed point subschemes with respect to λ of $\overline{Y_\tau^s} \cap Y_\tau \rightarrow p_\tau^{-1}\left(\prod_{i=1}^n \overline{R_i}\right)$. It is easy to see the morphism is a closed immersion over $\overline{Y_\tau}$. This proves (2).

For (3), the second equality follows immediately from the first. Recall that $K_\tau = \prod_{i=1}^n \text{SL}(V_i)$. We have

$$\left(\prod_{i=1}^n \overline{R_i} \right)^{K_\tau, s} = \prod_{i=1}^n \overline{R_i}^{\text{SL}(V_i), s} = \prod_{i=1}^n R_i^s \tag{6.124}$$

where the first equality is a formula on stable loci of product linearisations (See [HK12] Lemma 6.6, Proposition 6.7), and the second equality is from Theorem 6.3.5 when $M' \gg l \gg 0$. By [MFK94], Theorem 1.19 and (2) proved above, we have

$$Z_{\lambda-\min}^{\mathbf{K}_{\tau,s}}(Y_{\tau}^s) = Z_{\lambda-\min}(Y_{\tau}^s) \cap \left(\prod_{i=1}^n \overline{R_i} \right)^{\mathbf{K}_{\tau,s}} = \prod_{i=1}^n R_i^s \quad (6.125)$$

which proves (3). □

6.5 Sheaves of length 2

Let τ be a Harder-Narasimhan type of length 2. Let $P(t), P_i(t) \in \mathbb{Q}[t]$ for $1 \leq i \leq 2$ be the Hilbert polynomials associated to τ , in the sense that if $\mathcal{E}$ is a sheaf of type τ , then $P(t)$ is its Hilbert polynomial and $P_i(t)$ is the Hilbert polynomial of $\mathrm{HN}_i(\mathcal{E})/\mathrm{HN}_{i-1}(\mathcal{E})$. Recall that X is a connected projective scheme over $\mathbb{k}$ and $\mathcal{L}$ is an ample line bundle on X .

Recall the linear action of $U_{\tau} \rtimes (T_{\tau} \times K_{\tau})$ on $\overline{Y_{\tau}^s}$ with respect to $\mathcal{M}|_{\overline{Y_{\tau}^s}}$. Let $\lambda : \mathbb{G}_m \rightarrow T_{\tau}$ be a grading 1PS. We will assume that there exists a (τ, δ) -stable sheaf (Definition 6.5.4) for some $\delta \in \mathbb{N}$. In the case when τ has length 2, we have:

- the torus T_{τ} has dimension 1.
- U_{τ} is Abelian and the adjoint action of T_{τ} on $\mathfrak{u} := \mathrm{Lie}(U_{\tau})$ has one weight, so any internal grading 1PS $\lambda : \mathbb{G}_m \rightarrow T_{\tau}$ has one weight on $\mathfrak{u}$, inducing the trivial filtration of $\mathfrak{u}$;

Lemma 6.5.1. *Assume there exists a (τ, δ) -stable sheaf. For $M' \gg l \gg 0$, Condition 5.1.6 is satisfied for*

$$U_{\tau} \rtimes (T_{\tau} \times K_{\tau}) \curvearrowright (\overline{Y_{\tau}^s}, \mathcal{M}|_{\overline{Y_{\tau}^s}}). \quad (6.126)$$

Proof. There are four requirements in Condition 5.1.6. Note that a 1PS $\lambda' : \mathbb{G}_m \rightarrow P_{\tau}$ which factors through $T_{\tau} \rightarrow P_{\tau}$ grades U_{τ} if and only if there exist $w_1, w_2 \in \mathbb{Z}$ such that $w_1 \dim V_1 + w_2 \dim V_2 = 0$ and $w_1 > w_2$ and $\lambda'(t) = t^{w_1} 1_{V_1} \oplus t^{w_2} 1_{V_2}$. For such grading 1PS λ' , the retraction $p_{\lambda'}$ equals p_{τ} , which does not depend on λ' . Then (1) and (2) in Condition 5.1.6 are immediate.

The existence of a (τ, δ) -stable sheaf implies that R_1^s and R_2^s are non-empty, and by Proposition 6.4.2, when $M' \gg l \gg 0$

$$Z_{\lambda-\min}^{K_\tau, s}(Y_\tau^s) \neq \emptyset \quad (6.127)$$

which verifies (3) in Condition 5.1.6.

The last requirement (4) in Condition 5.1.6 comes from [Jac21] Corollary 7.2.6. $\square$

We can apply Theorem 5.2.1 to the linear action of $U_\tau \rtimes (T_\tau \times K_\tau)$ on $\overline{Y_\tau^s}$ with respect to $\mathcal{M}|_{\overline{Y_\tau^s}}$. Let the stratification in Theorem 5.2.1 be

$$\prod_{\delta \in \mathbb{N}} S_\delta(Y_\tau^s) \rightarrow Y_\tau^s \quad (6.128)$$

where we have applied Proposition 6.4.2 to get the identity $p_\lambda^{-1}(Z_{\lambda-\min}^{K_\tau, s}(Y_\tau^s)) = Y_\tau^s$.

Let $Y_{\tau, \delta}^s \subseteq S_\delta(Y_\tau^s)$ be the subset of closed points $x \in S_\delta(Y_\tau^s)$ such that $U_\tau \cdot x \neq U_\tau \cdot p_\tau(x)$, which defines a $U_\tau \rtimes (T_\tau \times K_\tau)$ -invariant open subset by Theorem 5.2.1. Therefore $Y_{\tau, \delta}^s$ has a canonical open subscheme structure. Also by Theorem 5.2.1, there exists a quasi-projective universal geometric quotient

$$Y_{\tau, \delta}^s \rightarrow Y_{\tau, \delta}^s / P_\tau. \quad (6.129)$$

Similar to Y_τ and Y_τ^s , the schemes $S_\delta(Y_\tau^s)$ and $Y_{\tau, \delta}^s$ also have moduli-theoretic descriptions. We will describe their functors of points (Lemma 6.5.3) in Section 6.5.1. The relation between the quotient $Y_{\tau, \delta}^s \rightarrow Y_{\tau, \delta}^s / P_\tau$ and the moduli functor $\mathbf{M}'_{\tau, \delta}$ (Lemma 6.5.5) will be discussed in Section 6.5.2, where the main result (Theorem 6.5.7) is proved in Section 6.5.3.

6.5.1 Unipotent endomorphisms

Let $U_\tau \subseteq P_\tau$ be the unipotent radical and let $\mathfrak{u}$ denote its Lie algebra. Then $\mathfrak{u} := \text{Hom}_{\mathbb{k}}(V_2, V_1)$. The filtration induced by λ is the trivial one of length one.

We will apply Corollary 5.1.5 and describe $S_\delta(Y_\tau^s)$ and $Y_{\tau, \delta}^s$ moduli-theoretically. Let $T \in \text{Sch}/\mathbb{k}$. Recall that a morphism $T \rightarrow Y_\tau^s$ is equivalent to an isomorphism class $[(\mathcal{E}, q)]$ such that

$$q : V(-lm)_T \rightarrow \mathcal{E} \quad (6.130)$$

is surjective, each $\mathcal{E}_i$ is flat over T with Hilbert polynomial $P_i(t)$, and fibres of each $\mathcal{E}_i$ at closed points of T are stable.

If we have a filtration in an Abelian category $\mathcal{A}$

$$0 = E_{\leq 0} \subseteq E_{\leq 1} \subseteq \cdots \subseteq E_{\leq n} = E \quad (6.131)$$

then we call $\varphi \in \text{End}_{\mathcal{A}}(E)$ a *unipotent endomorphism* if $\varphi(E_{\leq i}) \subseteq E_{\leq i-1}$ for $i = 1, \dots, n$.

Recall Definition 2.2.2 and Definition 2.2.3 for scheme-valued infinitesimal transformations and when they stabilise scheme-valued points.

Lemma 6.5.2. *When $l \gg 0$, we have the following commutative diagram of $\Gamma(T, \mathcal{O}_T)$ -modules for any $T \in \text{Sch}/\mathbb{k}$ and any T -valued point $[(\mathcal{E}, q)]$ in Y_τ^s*

$$\begin{array}{ccc} \left\{ \begin{array}{l} T\text{-valued infinitesimal} \\ \text{transformations in } U_\tau \end{array} \right\} & \xrightarrow{\cong} & \left\{ \begin{array}{l} \eta : \eta \in \text{End}_{\mathcal{O}_T}(V \otimes_{\mathbb{k}} \mathcal{O}_T) \text{ and for } i = 1, 2 \\ \eta(V_{\leq i} \otimes_{\mathbb{k}} \mathcal{O}_T) \subseteq V_{\leq i-1} \otimes_{\mathbb{k}} \mathcal{O}_T \end{array} \right\} \\ \uparrow \cup \! \! \! \uparrow & & \uparrow \cup \! \! \! \uparrow \\ \left\{ \begin{array}{l} \rho : \rho \text{ is a } T\text{-valued} \\ \text{infinitesimal} \\ \text{transformation in } U_\tau \\ \text{and } \rho \text{ stabilises } [(\mathcal{E}, q)] \end{array} \right\} & \xrightarrow{\cong} & \left\{ \begin{array}{l} \varphi : \varphi \in \text{End}_{\mathcal{O}_{X_T}}(\mathcal{E}) \\ \varphi(\mathcal{E}_{\leq i}) \subseteq \mathcal{E}_{\leq i-1} \text{ for } i = 1, 2 \end{array} \right\} \end{array} \quad (6.132)$$

where the horizontal maps are isomorphisms and vertical maps are injective. Moreover, the top map is natural in $T \in \text{Sch}/\mathbb{k}$ and the bottom map is natural in $T \in \text{Sch}/Y_\tau^s$.

Proof. The left vertical map in diagram (6.132) is the inclusion. For the right vertical arrow, let $\varphi : \mathcal{E} \rightarrow \mathcal{E}$ be $\mathcal{O}_{X_T}$ -linear such that $\varphi(\mathcal{E}_{\leq i}) \subseteq \mathcal{E}_{\leq i-1}$ for $i = 1, 2$. For a coherent sheaf $\mathcal{F}$ on X_T , let $\mathcal{F}(lm)$ denote $\mathcal{F} \otimes_{\mathcal{O}_{X_T}} \mathcal{L}_T^{lm}$.

Consider the diagram on X_T for $1 \leq i, i' \leq 2$ with $\varphi(\mathcal{E}_{\leq i'}) \subseteq \mathcal{E}_{\leq i}$

$$\begin{array}{ccc} V_{\leq i'}(-lm)_T & & V_{\leq i}(-lm)_T \\ \downarrow q_{\leq i'} & & \downarrow q_{\leq i} \\ \mathcal{E}_{\leq i'} & \xrightarrow{\varphi_{ii'}} & \mathcal{E}_{\leq i} \end{array} \quad (6.133)$$

where $q_{\leq i}$ and $\varphi_{ii'}$ are the truncated morphisms.

Apply the functor $\mathcal{F} \mapsto \mathcal{F}(lm) \mapsto (f_T)_*(\mathcal{F}(lm))$ to the diagram above to obtain the following diagram

$$\begin{array}{ccc} V_{\leq i'} \otimes_{\mathbb{k}} \mathcal{O}_T & \xrightarrow{\eta_{ii'}} & V_{\leq i} \otimes_{\mathbb{k}} \mathcal{O}_T \\ \downarrow (f_T)_*(q_{\leq i'}(lm)) & & \downarrow (f_T)_*(q_{\leq i}(lm)) \\ (f_T)_*(\mathcal{E}_{\leq i'}(lm)) & \xrightarrow{(f_T)_*(\varphi_{ii'}(lm))} & (f_T)_*(\mathcal{E}_{\leq i}(lm)). \end{array} \quad (6.134)$$

When $l \gg 0$ is large, we have

- $(f_T)_*(\mathcal{E}_j(lm))$ is locally free of rank $P_j(lm)$ for $j = 1, 2$;
- $(f_T)_*(\mathcal{E}_{\leq j}(lm))$ is locally free of rank $P_{\leq j}(lm)$ for $j = 1, 2$

for all $T \in \text{Sch}/\mathbb{k}$ and $[(\mathcal{E}, q)]$ (See step (3) in Section 6.2.1). So $(f_T)_*(q_{\leq i}(lm))$ is a morphism between locally free sheaves of the same rank $P_{\leq i}(lm)$. For any closed point $\{t\} \hookrightarrow T$, the morphism $(f_T)_*(q_{\leq i}(lm))$ pulls back to an isomorphism $V_{\leq i} \cong H^0(X, \mathcal{E}_{\leq i}(lm)|_t)$. By [Sta22, Lemma 00O0], we have that $(f_T)_*(q_{\leq i}(lm))$ is an isomorphism since its pull-backs at closed points are isomorphisms. Similarly $(f_T)_*(q_{\leq i'}(lm))$ is an isomorphism. Therefore the morphism $\eta_{ii'} : V_{\leq i'} \otimes_{\mathbb{k}} \mathcal{O}_T \rightarrow V_{\leq i} \otimes_{\mathbb{k}} \mathcal{O}_T$ exists in diagram (6.134). Consider for all $1 \leq i, i' \leq 2$ such that $\varphi(\mathcal{E}_{\leq i'}) \subseteq \mathcal{E}_{\leq i}$. These $\{\eta_{ii'}\}$ glue to a morphism $\eta : V \otimes_{\mathbb{k}} \mathcal{O}_T \rightarrow V \otimes_{\mathbb{k}} \mathcal{O}_T$, which satisfies $\eta(V_{\leq i'} \otimes_{\mathbb{k}} \mathcal{O}_T) \subseteq V_{\leq i} \otimes_{\mathbb{k}} \mathcal{O}_T$ if $\varphi(\mathcal{E}_{\leq i'}) \subseteq \mathcal{E}_{\leq i}$. The association $\varphi \mapsto \eta$ defines the right vertical map in diagram (6.132).

The right vertical map in diagram (6.132) is injective, since if $\varphi \mapsto \eta$, then φ, η satisfy the following diagram

$$\begin{array}{ccc} V(-lm)_T & \xrightarrow{\eta(-lm)} & V(-lm)_T \\ \downarrow q & & \downarrow q \\ \mathcal{E} & \xrightarrow{\varphi} & \mathcal{E}. \end{array} \quad (6.135)$$

where $\eta(-lm) := ((f_T)^*\eta) \otimes_{\mathcal{O}_{X_T}} 1_{\mathcal{L}_T^{-lm}}$.

For the top map in diagram (6.132), let $\rho : T[\epsilon] \rightarrow U_\tau$ be a T -valued infinitesimal transformation in U_τ . A morphism $\rho : T[\epsilon] \rightarrow U_\tau$ is equivalent to a morphism of coherent sheaves on $T[\epsilon]$

$$u : V \otimes_{\mathbb{k}} \mathcal{O}_{T[\epsilon]} \rightarrow V \otimes_{\mathbb{k}} \mathcal{O}_{T[\epsilon]} \quad (6.136)$$

such that:

- (1) $u(V_{\leq i} \otimes_{\mathbb{k}} \mathcal{O}_{T[\epsilon]}) \subseteq V_{\leq i} \otimes_{\mathbb{k}} \mathcal{O}_{T[\epsilon]}$, and we denote $u_{\leq i} : V_{\leq i} \otimes_{\mathbb{k}} \mathcal{O}_{T[\epsilon]} \rightarrow V_{\leq i} \otimes_{\mathbb{k}} \mathcal{O}_{T[\epsilon]}$;
- (2) the induced morphism $u_i : V_i \otimes_{\mathbb{k}} \mathcal{O}_{T[\epsilon]} \rightarrow V_i \otimes_{\mathbb{k}} \mathcal{O}_{T[\epsilon]}$ is the identity

$$\begin{array}{ccccccc} 0 & \longrightarrow & V_{\leq i-1} \otimes_{\mathbb{k}} \mathcal{O}_{T[\epsilon]} & \xrightarrow{\subseteq} & V_{\leq i} \otimes_{\mathbb{k}} \mathcal{O}_{T[\epsilon]} & \longrightarrow & V_i \otimes_{\mathbb{k}} \mathcal{O}_{T[\epsilon]} \longrightarrow 0 \\ & & \downarrow u_{\leq i-1} & & \downarrow u_{\leq i} & & \downarrow u_i=1 \\ 0 & \longrightarrow & V_{\leq i-1} \otimes_{\mathbb{k}} \mathcal{O}_{T[\epsilon]} & \xrightarrow{\subseteq} & V_{\leq i} \otimes_{\mathbb{k}} \mathcal{O}_{T[\epsilon]} & \longrightarrow & V_i \otimes_{\mathbb{k}} \mathcal{O}_{T[\epsilon]} \longrightarrow 0. \end{array} \quad (6.137)$$

Note that the underlying topological spaces of T and $T[\epsilon]$ are the same, and $\mathcal{O}_{T[\epsilon]} \cong \mathcal{O}_T \oplus \epsilon \mathcal{O}_T$ as $\mathcal{O}_T$ -modules. The morphism $u : V \otimes_{\mathbb{k}} \mathcal{O}_{T[\epsilon]} \rightarrow V \otimes_{\mathbb{k}} \mathcal{O}_{T[\epsilon]}$ is equivalent to two $\mathcal{O}_T$ -linear morphisms

$$\mu : V \otimes_{\mathbb{k}} \mathcal{O}_T \rightarrow V \otimes_{\mathbb{k}} \mathcal{O}_T, \quad \eta : V \otimes_{\mathbb{k}} \mathcal{O}_T \rightarrow V \otimes_{\mathbb{k}} \mathcal{O}_T \quad (6.138)$$

in the sense that according to the decomposition $V \otimes_{\mathbb{k}} \mathcal{O}_{T[\epsilon]} \cong V \otimes_{\mathbb{k}} \mathcal{O}_T \oplus \epsilon V \otimes_{\mathbb{k}} \mathcal{O}_T$, we have

$$u = \begin{pmatrix} \mu & \\ \epsilon \eta & \mu_\epsilon \end{pmatrix} \quad (6.139)$$

where $\mu_\epsilon : \epsilon V \otimes_{\mathbb{k}} \mathcal{O}_T \rightarrow \epsilon V \otimes_{\mathbb{k}} \mathcal{O}_T$ is the morphism equivalent to μ .

For a morphism $\rho : T[\epsilon] \rightarrow U_\tau$ to be an infinitesimal transformation, the diagram

$$\begin{array}{ccc} T & \xrightarrow{\quad} & T[\epsilon] \\ \downarrow & & \downarrow \rho \\ \text{Spec}(\mathbb{k}) & \xrightarrow{e} & U_\tau \end{array}$$

has to commute, which is equivalent to the following condition for u :

$$(3) \quad \mu = 1_{V \otimes_{\mathbb{k}} \mathcal{O}_T}, \text{ i.e. } u = \begin{pmatrix} 1 & \\ \epsilon \eta & 1_\epsilon \end{pmatrix}.$$

Condition (3) for u says that u is determined by η , so the association $\rho \mapsto \eta$ is injective.

With (3), condition (1) and (2) together for u are equivalent to the following

$$\eta(V_{\leq i} \otimes_{\mathbb{k}} \mathcal{O}_T) \subseteq V_{\leq i-1} \otimes_{\mathbb{k}} \mathcal{O}_T, \quad i = 1, 2. \quad (6.140)$$

Therefore the association $\rho \mapsto \eta$ defines the top map in diagram (6.132), which is bijective from its construction.

For the bottom map in diagram (6.132), let $\rho : T[\epsilon] \rightarrow U_\tau$ be a T -valued infinitesimal transformation in U_τ and assume ρ is mapped to $\eta : V \otimes_{\mathbb{k}} \mathcal{O}_T \rightarrow V \otimes_{\mathbb{k}} \mathcal{O}_T$ along the top map in diagram (6.132). We have $\eta(V_{\leq i} \otimes_{\mathbb{k}} \mathcal{O}_T) \subseteq V_{\leq i-1} \otimes_{\mathbb{k}} \mathcal{O}_T$ for $i = 1, 2$.

Let $h : T \rightarrow Y_\tau^s$ be the morphism corresponding to $[(\mathcal{E}, q)]$. Then $\rho : T[\epsilon] \rightarrow U_\tau$ stabilises $[(\mathcal{E}, q)]$ if and only if the following diagram commutes

$$\begin{array}{ccc} T[\epsilon] & \xrightarrow{\text{pr}_T} & T \\ \downarrow (\rho, h \circ \text{pr}_T) & & \downarrow h \\ U_\tau \times_{\mathbb{k}} Y_\tau^s & \xrightarrow{\sigma} & Y_\tau^s \end{array} \quad (6.141)$$

where $\sigma : U_\tau \times_{\mathbb{k}} Y_\tau^s \rightarrow Y_\tau^s$ is the action morphism of $U_\tau \curvearrowright Y_\tau^s$. There are two families of quotients parameterised by $T[\epsilon]$ corresponding to two morphisms $h \circ \text{pr}_T$ and $\sigma \circ (\rho, h \circ \text{pr}_T)$. The diagram above commutes if and only if $h \circ \text{pr}_T = \sigma \circ (\rho, h \circ \text{pr}_T)$, if and only if the two families over $T[\epsilon]$ are equivalent.

The family over $T[\epsilon]$ corresponding to $h \circ \text{pr}_T$ is the following

$$V(-lm)_T \oplus \epsilon V(-lm)_T \xrightarrow{\begin{pmatrix} q \\ q_\epsilon \end{pmatrix}} \mathcal{E} \oplus \epsilon \mathcal{E} \longrightarrow 0 \quad (6.142)$$

where $q_\epsilon : \epsilon V(-lm)_T \rightarrow \epsilon \mathcal{E}$ is equivalent to q . The family corresponding to $\sigma \circ (\rho, h \circ \text{pr}_T)$ is the composition

$$V(-lm)_T \oplus \epsilon V(-lm)_T \xrightarrow{\begin{pmatrix} 1 & \\ -\epsilon \eta(-lm) & 1_\epsilon \end{pmatrix}} V(-lm)_T \oplus \epsilon V(-lm)_T \xrightarrow{\begin{pmatrix} q \\ q_\epsilon \end{pmatrix}} \mathcal{E} \oplus \epsilon \mathcal{E} \quad (6.143)$$

where $\eta(-lm) : V(-lm)_T \rightarrow V(-lm)_T$ denotes $\eta(-lm) := ((f_T)^* \eta) \otimes_{\mathcal{O}_{X_T}} 1_{\mathcal{L}_T^{-lm}}$. It is easy to see the two families are equivalent if and only if $\eta(-lm)(\ker q) \subseteq \ker q$, which is equivalent to the existence of $\varphi : \mathcal{E} \rightarrow \mathcal{E}$ such that the following diagram commutes

$$\begin{array}{ccc} V(-lm)_T & \xrightarrow{\eta(-lm)} & V(-lm)_T \\ \downarrow q & & \downarrow q \\ \mathcal{E} & \xrightarrow{\varphi} & \mathcal{E} \end{array} \quad (6.144)$$

which is equivalent to that η is in the image of the right vertical map in diagram (6.132), i.e. the association $\rho \mapsto \varphi$ defines the bottom map in diagram (6.132).

We have now defined all the maps in diagram (6.132), with horizontal maps being bijective and vertical maps being injective. It is easy to check that they are all $\Gamma(T, \mathcal{O}_T)$ -linear and natural in $T \in \text{Sch}/\mathbb{k}$ and $T \in \text{Sch}/Y_\tau^s$ respectively. $\square$

Recall that the locally closed subscheme $S_\delta(Y_\tau^s) \rightarrow Y_\tau^s$ is defined in equation (6.128), and the open subscheme $Y_{\tau, \delta}^s \subseteq S_\delta(Y_\tau^s)$ is defined in the paragraph following equation (6.128). For a coherent sheaf $\mathcal{E}$ on $X_T := X \times_{\mathbb{k}} T$ and a morphism $T' \rightarrow T$, let $\mathcal{E}|_{T'}$ denote the pull-back of $\mathcal{E}$ along $X_{T'} \rightarrow X_T$.

Lemma 6.5.3. *The scheme $S_\delta(Y_\tau^s)$ represents the functor which sends $T \in \text{Sch}/\mathbb{k}$ to the set of isomorphism classes $[(\mathcal{E}, q)]$ such that:*

- (1) $q : V(-lm)_T \rightarrow \mathcal{E}$ is a surjective morphism of coherent sheaves on X_T , and $\mathcal{E}_i$ is flat over T with Hilbert polynomial $P_i(t)$ for $i = 1, 2$;
- (2) fibres of $\mathcal{E}_i$ at closed points are stable for $i = 1, 2$;
- (3) for any closed subscheme $T' \hookrightarrow T$, the coherent sheaf

$$(f_{T'})_* \mathcal{H}om_{\mathcal{O}_{X_{T'}}}(\mathcal{E}_2|_{T'}, \mathcal{E}_1|_{T'}) \quad (6.145)$$

is locally free of rank δ on T' .

The open subscheme $Y_{\tau, \delta}^s \subseteq S_\delta(Y_\tau^s)$ represents the sub-functor which sends $T \in \text{Sch}/\mathbb{k}$ to the set of isomorphism classes $[(\mathcal{E}, q)]$ such that (1), (2), (3) above hold and:

- (4) fibres of $\mathcal{E}$ at closed points $t \in T$ are non-split, in the sense that $\mathcal{E}_t \not\cong \mathcal{E}_1|_t \oplus \mathcal{E}_2|_t$.

Proof. Recall in Section 6.3.2, the schemes $Y_\tau^s \subseteq Y_\tau$ are defined moduli-theoretically. Let $T \in \text{Sch}/\mathbb{k}$. Then (1) is equivalent to that $[(\mathcal{E}, q)]$ corresponds to a T -valued point in Y_τ , and (2) is equivalent to that $[(\mathcal{E}, q)]$ corresponds to a T -valued point in Y_τ^s .

By Corollary 5.1.5, the morphism $T \rightarrow Y_\tau^s$ factors as $T \rightarrow S_\delta(Y_\tau^s) \rightarrow Y_\tau^s$ if and only if the following holds:

- (3') for any closed subscheme $T' \hookrightarrow T$, the following defines a locally free sheaf of rank δ on T'

$$W' \mapsto \left\{ \begin{array}{l} \rho : \rho \text{ is a } W'\text{-valued infinitesimal transformation in } U_\tau \\ \rho \text{ stabilises } W' \subseteq T' \hookrightarrow T \rightarrow Y_\tau^s \end{array} \right\} \quad (6.146)$$

for open subsets $W' \subseteq T'$.

By Lemma 6.5.2, the sheaf in (3') is isomorphic to the following sheaf on T'

$$\begin{aligned} W' &\mapsto \left\{ \begin{array}{l} \varphi : \varphi \in \text{End}_{\mathcal{O}_{X_{W'}}}(\mathcal{E}|_{W'}) \text{ and for } i = 1, 2 \\ \varphi((\mathcal{E}|_{W'})_{\leq i}) \subseteq (\mathcal{E}|_{W'})_{\leq i-1} \end{array} \right\} \\ &\cong \text{Hom}_{\mathcal{O}_{X_{W'}}}((\mathcal{E}|_{W'})_{\leq 2}/(\mathcal{E}|_{W'})_{\leq 1}, (\mathcal{E}|_{W'})_{\leq 1}) \\ &\cong \text{Hom}_{\mathcal{O}_{X_{W'}}}(\mathcal{E}_2|_{W'}, \mathcal{E}_1|_{W'}) \\ &\cong \mathcal{H}om_{\mathcal{O}_{X_{T'}}}(\mathcal{E}_2|_{T'}, \mathcal{E}_1|_{T'})((f_{T'})^{-1}(W')). \end{aligned} \quad (6.147)$$

So (3) and (3') are equivalent. This proves the moduli description of $S_\delta(Y_\tau^s)$.

The morphism $T \rightarrow S_\delta(Y_\tau^s)$ factors as $T \rightarrow Y_{\tau, \delta}^s \subseteq S_\delta(Y_\tau^s)$ if and only if any closed point $t \in T$ is mapped into $Y_{\tau, \delta}^s$, which is equivalent to:

(4') fibres of $\mathcal{E}$ at closed points $t \in T$, denoted by $\mathcal{E}|_t$, satisfy the condition that the following two closed points in Y_τ^s are not in one U_τ -orbit

$$[(\mathcal{E}|_t, q|_t)], \quad \left[\left(\bigoplus_{i=1}^2 \mathcal{E}_i|_t, \bigoplus_{i=1}^2 q_i|_t \right) \right] \in Y_\tau^s. \quad (6.148)$$

If (4) $\iff$ (4'), then the moduli description of $Y_{\tau,\delta}^s$ is proved, hence the lemma.

Obviously we have (4) $\implies$ (4'). Conversely, it suffices to prove that if $[(\mathcal{E}, q)] \in Y_\tau^s$ is a closed point such that $\mathcal{E} \cong \bigoplus_{i=1}^2 \mathcal{E}_i$, then $\left[\left(\bigoplus_{i=1}^2 \mathcal{E}_i, \bigoplus_{i=1}^2 q_i \right) \right]$ is in the U_τ -orbit of $[(\mathcal{E}, q)]$. Let $[(\mathcal{E}, q)] \in Y_\tau^s$ be such a closed point. Then the following diagram commutes

$$\begin{array}{ccccccc} 0 & \longrightarrow & V_1(-lm) & \longrightarrow & V(-lm) & \longrightarrow & V_2(-lm) \longrightarrow 0 \\ & & \downarrow q_1 & & \downarrow q & & \downarrow q_2 \\ 0 & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}_2 \longrightarrow 0 \\ & & \parallel & & \downarrow \mathbb{R} & & \parallel \\ 0 & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \mathcal{E}_1 \oplus \mathcal{E}_2 & \longrightarrow & \mathcal{E}_2 \longrightarrow 0 \\ & & q_1 \uparrow & & q_1 \oplus q_2 \uparrow & & q_2 \uparrow \\ 0 & \longrightarrow & V_1(-lm) & \longrightarrow & V(-lm) & \longrightarrow & V_2(-lm) \longrightarrow 0. \end{array} \quad (6.149)$$

Apply the functor $\mathcal{F} \mapsto \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{lm} \mapsto f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{lm})$ to the diagram above, and we have the following

$$\begin{array}{ccccccc} 0 & \longrightarrow & V_1 & \longrightarrow & V & \longrightarrow & V_2 \longrightarrow 0 \\ & & \downarrow f_*(q_1(lm)) & & \downarrow f_*(q(lm)) & & \downarrow f_*(q_2(lm)) \\ 0 & \longrightarrow & f_*(\mathcal{E}_1(lm)) & \longrightarrow & f_*(\mathcal{E}(lm)) & \longrightarrow & f_*(\mathcal{E}_2(lm)) \longrightarrow 0 \\ & & \parallel & & \downarrow \mu & & \parallel \\ 0 & \longrightarrow & f_*(\mathcal{E}_1(lm)) & \longrightarrow & f_*(\mathcal{E}_1(lm)) \oplus f_*(\mathcal{E}_2(lm)) & \longrightarrow & f_*(\mathcal{E}_2(lm)) \longrightarrow 0 \\ & & f_*(q_1(lm)) \uparrow & & f_*(q_1(lm)) \oplus f_*(q_2(lm)) \uparrow & & f_*(q_2(lm)) \uparrow \\ 0 & \longrightarrow & V_1 & \longrightarrow & V & \longrightarrow & V_2 \longrightarrow 0 \end{array} \quad (6.150)$$

where $\mathcal{E}(lm) := \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{L}^{lm}$ and $q(lm) := q \otimes 1_{\mathcal{L}^{lm}}$ and similarly for $\mathcal{E}_i(lm)$, $q_i(lm)$. We denote by $\mu : f_*(\mathcal{E}(lm)) \rightarrow f_*(\mathcal{E}_1(lm)) \oplus f_*(\mathcal{E}_2(lm))$ the map induced from $\mathcal{E} \rightarrow \mathcal{E}_1 \oplus \mathcal{E}_2$. We have that $f_*(q(lm))$ and $f_*(q_i(lm))$ are isomorphisms.

Let $g : V \rightarrow V$ be the composition

$$g := (f_*(q_1(lm))^{-1} \oplus f_*(q_2(lm))^{-1}) \circ \mu \circ f_*(q(lm)). \quad (6.151)$$

Then $g \in \mathrm{GL}(V)$ and the following diagram commutes

$$\begin{array}{ccc} V(-lm) & \xrightarrow{g \otimes 1_{\mathcal{L}^{-lm}}} & V(-lm) \\ \downarrow q & & \downarrow q_1 \oplus q_2 \\ \mathcal{E} & \xrightarrow{\cong} & \mathcal{E}_1 \oplus \mathcal{E}_2 \end{array} \quad (6.152)$$

i.e. $g \cdot [(\mathcal{E}, q)] = \left[\left(\bigoplus_{i=1}^2 \mathcal{E}_i, \bigoplus_{i=1}^2 q_i \right) \right]$. Finally we have $g \in \mathrm{U}_\tau$, since the following diagram commutes

$$\begin{array}{ccccccc} 0 & \longrightarrow & V_1 & \longrightarrow & V & \longrightarrow & V_2 \longrightarrow 0 \\ & & \parallel & & \downarrow g & & \parallel \\ 0 & \longrightarrow & V_1 & \longrightarrow & V & \longrightarrow & V_2 \longrightarrow 0. \end{array} \quad (6.153)$$

This proves $(4) \iff (4')$. $\square$

6.5.2 The moduli functor $\mathbf{M}'_{\tau, \delta}$

Recall that (τ, δ) -stability is defined in Definition 6.3.10. We view a (τ, δ) -stable sheaf $\mathcal{E}$ as a family parameterised by $\mathrm{Spec}(\mathbb{k})$. To define the moduli functor $\mathbf{M}'_{\tau, \delta}$, we need the notion of *families of (τ, δ) -stable sheaves parameterised by T* for $T \in \mathrm{Sch}/\mathbb{k}$. Recall that $f : X \rightarrow \mathrm{Spec}(\mathbb{k})$ is the structure morphism.

Definition 6.5.4. Let τ be a Harder-Narasimhan type of length 2 and let $\delta \in \mathbb{N}$. Let $P(t), P_i(t)$ be the Hilbert polynomials associated to τ . Let $T \in \mathrm{Sch}/\mathbb{k}$. A *family of (τ, δ) -stable sheaves on X parameterised by T* is $(\mathcal{E}, \{\mathcal{E}_{\leq i}\}_{i=0}^2)$ such that:

- (1) $\mathcal{E}$ is a coherent sheaf on $X_T := X \times_{\mathbb{k}} T$, and $\{\mathcal{E}_{\leq i}\}_{i=0}^2$ is a length 2 filtration

$$0 = \mathcal{E}_{\leq 0} \subseteq \mathcal{E}_{\leq 1} \subseteq \mathcal{E}_{\leq 2} = \mathcal{E}; \quad (6.154)$$

- (2) Each $\mathcal{E}_i := \mathcal{E}_{\leq i} / \mathcal{E}_{\leq i-1}$ is flat over T with Hilbert polynomial $P_i(t)$;

- (3) For any closed point $t \in T$, the fibre $\mathcal{E}_t$ is (τ, δ) -stable;

- (4) For any closed subscheme $T' \hookrightarrow T$, the following sheaf

$$(f_{T'})_* \mathcal{H}om_{\mathcal{O}_{X_{T'}}}(\mathcal{E}_2|_{T'}, \mathcal{E}_1|_{T'}) \quad (6.155)$$

is locally free of rank δ on T' .

We call $(f_{T'})_* \mathcal{H}om_{\mathcal{O}_{X_{T'}}}(\mathcal{E}_2|_{T'}, \mathcal{E}_1|_{T'})$ in (4) the *sheaf of unipotent endomorphisms* of the pull-back family $(\mathcal{E}|_{T'}, \{\mathcal{E}_{\leq i}|_{T'}\}_{i=0}^2)$.

Two families $(\mathcal{E}, \{\mathcal{E}_{\leq i}\}_{i=0}^2)$ and $(\mathcal{E}', \{\mathcal{E}'_{\leq i}\}_{i=0}^2)$ parameterised by T are called *isomorphic* if there is a filtration-preserving isomorphism $\mathcal{E} \cong \mathcal{E}'$ on X_T .

The definitions above obviously generalise (τ, δ) -stable sheaves and their isomorphisms to families. The moduli functor $\mathbf{M}'_{\tau, \delta}$ is defined as follows

$$\mathbf{M}'_{\tau, \delta} : (\text{Sch}/\mathbb{k})^{\text{op}} \rightarrow \text{Set} \quad (6.156)$$

which sends $T \in \text{Sch}/\mathbb{k}$ to the set

$$\mathbf{M}'_{\tau, \delta}(T) := \left\{ \begin{array}{l} \text{isomorphism classes of families of} \\ (\tau, \delta)\text{-stable sheaves parameterised by } T \end{array} \right\} \quad (6.157)$$

and for a morphism $g : T' \rightarrow T$, the map

$$\mathbf{M}'_{\tau, \delta}(g) : \mathbf{M}'_{\tau, \delta}(T) \rightarrow \mathbf{M}'_{\tau, \delta}(T') \quad (6.158)$$

sends the isomorphism class of $(\mathcal{E}, \{\mathcal{E}_{\leq i}\}_{i=0}^2)$ to the isomorphism class of the pull-back family $(\mathcal{E}|_{T'}, \{\mathcal{E}_{\leq i}|_{T'}\}_{i=0}^2)$, where $\mathcal{E}|_{T'} := (1_X \times g)^* \mathcal{E}$. It is easy to check that $\mathbf{M}'_{\tau, \delta}$ defines a functor.

Let $m \in \mathbb{N}_+$ be such that $H^0(X, \mathcal{L}^m)$ generates $\bigoplus_{d \geq 0} H^0(X, \mathcal{L}^{md})$. Let $l \gg 0$. Consider $Y_{\tau, \delta}^s$ defined in the paragraph following equation 6.128. There is a fully faithful embedding $\text{Sch}/\mathbb{k} \rightarrow \text{PSh}(\text{Sch}/\mathbb{k})$, and we will not distinguish between a scheme and its Yoneda functor. We will define a P_τ -invariant natural transformation

$$\rho : Y_{\tau, \delta}^s \rightarrow \mathbf{M}'_{\tau, \delta}. \quad (6.159)$$

Let $T \in \text{Sch}/\mathbb{k}$, and let $T \rightarrow Y_{\tau, \delta}^s$ be a morphism. By Lemma 6.5.3, the morphism $T \rightarrow Y_{\tau, \delta}^s$ corresponds to an isomorphism class $[(\mathcal{E}, q)]$ satisfying (1) – (4) in Lemma 6.5.3. For the family of quotients $(\mathcal{E}, q)$ over T , define $\mathcal{E}_{\leq i} := q(V_{\leq i}(-lm)) \subseteq \mathcal{E}$. Then the isomorphism class of $(\mathcal{E}, \{\mathcal{E}_{\leq i}\}_{i=0}^2)$ is an element in $\mathbf{M}'_{\tau, \delta}(T)$. This defines a map

$$\rho_T : \text{Hom}_{\text{Sch}/\mathbb{k}}(T, Y_{\tau, \delta}^s) \rightarrow \mathbf{M}'_{\tau, \delta}(T) \quad (6.160)$$

It is easy to see ρ_T is $\text{Hom}_{\text{Sch}/\mathbb{k}}(T, P_\tau)$ -invariant and it is natural in $T \in \text{Sch}/\mathbb{k}$. This defines a P_τ -invariant natural transformation $\rho : Y_{\tau, \delta}^s \rightarrow \mathbf{M}'_{\tau, \delta}$.

Since $\rho : Y_{\tau,\delta}^s \rightarrow \mathbf{M}'_{\tau,\delta}$ is P_τ -invariant, it factors through the quotient presheaf

$$\underline{Y_{\tau,\delta}^s} / \underline{P_\tau} : (\text{Sch}/\mathbb{k})^{\text{op}} \rightarrow \text{Set}, \quad T \mapsto \frac{\text{Hom}_{\text{Sch}/\mathbb{k}}(T, Y_{\tau,\delta}^s)}{\text{Hom}_{\text{Sch}/\mathbb{k}}(T, P_\tau)} \quad (6.161)$$

i.e. there exists a unique natural transformation θ such that the following diagram in $\text{PSh}(\text{Sch}/\mathbb{k})$ commutes

$$\begin{array}{ccc} Y_{\tau,\delta}^s & \xrightarrow{\rho} & \mathbf{M}'_{\tau,\delta} \\ \downarrow \pi & \nearrow \theta & \\ \underline{Y_{\tau,\delta}^s} / \underline{P_\tau} & & \end{array} \quad (6.162)$$

A *Zariski covering* in $\text{Sch}/\mathbb{k}$ is a collection of morphisms $\{T_j \rightarrow T\}_{j \in J}$ such that each $T_j \rightarrow T$ is an open immersion and every point of T is in one of the images. A Zariski covering is nothing but an open covering in the Zariski topology.

Lemma 6.5.5. *When $l \gg 0$ is large, the natural transformation $\theta : \underline{Y_{\tau,\delta}^s} / \underline{P_\tau} \rightarrow \mathbf{M}'_{\tau,\delta}$ satisfies:*

- (1) $\theta : \underline{Y_{\tau,\delta}^s} / \underline{P_\tau} \rightarrow \mathbf{M}'_{\tau,\delta}$ is injective.
- (2) For any $T \in \text{Sch}/\mathbb{k}$ and any $x \in \mathbf{M}'_{\tau,\delta}(T)$, there exists a Zariski covering $\{T_j \rightarrow T\}_{j \in J}$ such that $x|_{T_j} \in \mathbf{M}'_{\tau,\delta}(T_j)$ is in the image of $\theta_{T_j} : (\underline{Y_{\tau,\delta}^s} / \underline{P_\tau})(T_j) \rightarrow \mathbf{M}'_{\tau,\delta}(T_j)$ for each $j \in J$, where $x|_{T_j}$ is the pull-back of x along $T_j \rightarrow T$;
- (3) For all $Z \in \text{Sch}/\mathbb{k}$ the following map is injective

$$(-) \circ \theta : \text{Hom}_{\text{PSh}(\text{Sch}/\mathbb{k})}(\mathbf{M}'_{\tau,\delta}, Z) \rightarrow \text{Hom}_{\text{PSh}(\text{Sch}/\mathbb{k})}(\underline{Y_{\tau,\delta}^s} / \underline{P_\tau}, Z). \quad (6.163)$$

Proof. For $i = 1, 2$ the collection of stable sheaves of Hilbert polynomial $P_i(t)$ on X is bounded and then regularities of such stable sheaves are bounded by Lemma 6.3.3. Let $l \gg 0$ be larger than the regularity of any stable sheaf of Hilbert polynomial in $\{P_1(t), P_2(t)\}$. We will prove the lemma for this l . For $T \in \text{Sch}/\mathbb{k}$, recall that $f_T : X_T \rightarrow T$ is the projection. Denote $\mathcal{E}(lm) := \mathcal{E} \otimes_{\mathcal{O}_{X_T}} \mathcal{L}_T^{lm}$. For any morphism $T' \rightarrow T$ in $\text{Sch}/\mathbb{k}$, denote by $\mathcal{E}|_{T'}$ the pull-back of $\mathcal{E}$ along $X_{T'} \rightarrow X_T$.

For (1), let $T \in \text{Sch}/\mathbb{k}$. It suffices to prove that if $\rho_T(h) = \rho_T(h')$ for two elements $h, h' \in \text{Hom}_{\text{Sch}/\mathbb{k}}(T, Y_{\tau,\delta}^s)$, then h, h' are in one $\text{Hom}_{\text{Sch}/\mathbb{k}}(T, P_\tau)$ -orbit. By Lemma 6.5.3,

elements in $\text{Hom}_{\text{Sch}/\mathbb{k}}(T, Y_{\tau, \delta}^s)$ correspond to isomorphism classes $[(\mathcal{E}, q)]$ satisfying (1) – (4) in Lemma 6.5.3. Let $[(\mathcal{E}, q)]$ and $[(\mathcal{E}', q')]$ be two isomorphism classes corresponding to h and h' respectively

$$q : V(-lm)_T \rightarrow \mathcal{E}, \quad q' : V(-lm)_T \rightarrow \mathcal{E}'. \quad (6.164)$$

The equality $\rho_T(h) = \rho_T(h')$ is equivalent to that $\mathcal{E} \cong \mathcal{E}'$ on X_T and the isomorphism preserves the filtrations $\{\mathcal{E}_{\leq i}\}_{i=0}^2$ and $\{\mathcal{E}'_{\leq i}\}_{i=0}^2$.

Consider the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & V_{\leq i-1} \otimes_{\mathbb{k}} \mathcal{O}_T & \longrightarrow & V_{\leq i} \otimes_{\mathbb{k}} \mathcal{O}_T & \longrightarrow & V_i \otimes_{\mathbb{k}} \mathcal{O}_T \longrightarrow 0 \\ & & \downarrow (f_T)_*(q_{\leq i-1}(lm)) & & \downarrow (f_T)_*(q_{\leq i}(lm)) & & \downarrow (f_T)_*(q_i(lm)) \\ 0 & \longrightarrow & (f_T)_*(\mathcal{E}_{\leq i-1}(lm)) & \longrightarrow & (f_T)_*(\mathcal{E}_{\leq i}(lm)) & \longrightarrow & (f_T)_*(\mathcal{E}_i(lm)) \longrightarrow 0. \end{array} \quad (6.165)$$

Recall that l is larger than the regularity of $\mathcal{E}_i$ for $i = 1, 2$. By [Nit05] Theorem 3.7, we have that $(f_T)_*(\mathcal{E}_{\leq i}(lm))$ and $(f_T)_*(\mathcal{E}_i(lm))$ are locally free of ranks $P_{\leq i}(lm)$ and $P_i(lm)$ on T for $i = 1, 2$ respectively. Therefore $(f_T)_*(q_{\leq i}(lm))$ and $(f_T)_*(q_i(lm))$ in diagram (6.165) are morphisms between locally free sheaves. For any closed point $t \in T$, we have that $[(\mathcal{E}|_t, q|_t)]$ corresponds to a closed point in $Y_{\tau, \delta}^s$, so

$$\begin{aligned} (f_{\{t\}})_*(q_i|_t(lm)) : V_i &\rightarrow (f_{\{t\}})_*(\mathcal{E}_i|_t(lm)) \\ (f_{\{t\}})_*(q_{\leq i}|_t(lm)) : V_{\leq i} &\rightarrow (f_{\{t\}})_*(\mathcal{E}_{\leq i}|_t(lm)) \end{aligned} \quad (6.166)$$

are isomorphisms, i.e. fibres at $t \in T$ of $(f_T)_*(q_{\leq i}(lm))$ and $(f_T)_*(q_i(lm))$ in diagram (6.165) are isomorphisms. This proves that $(f_T)_*(q_{\leq i}(lm))$ and $(f_T)_*(q_i(lm))$ are isomorphisms by [Sta22, Lemma 00O0]

$$\begin{aligned} (f_T)_*(q_{\leq i}(lm)) : V_{\leq i} \otimes_{\mathbb{k}} \mathcal{O}_T &\cong (f_T)_*(\mathcal{E}_{\leq i}(lm)) \\ (f_T)_*(q_i(lm)) : V_i \otimes_{\mathbb{k}} \mathcal{O}_T &\cong (f_T)_*(\mathcal{E}_i(lm)). \end{aligned} \quad (6.167)$$

Similarly we have that $(f_T)_*(q'_{\leq i}(lm))$ and $(f_T)_*(q'_i(lm))$ are isomorphisms for $i = 1, 2$.

Consider the morphism $g : V \otimes_{\mathbb{k}} \mathcal{O}_T \rightarrow V \otimes_{\mathbb{k}} \mathcal{O}_T$ defined as the following composition on T

$$\begin{array}{ccc} V \otimes_{\mathbb{k}} \mathcal{O}_T & \xrightarrow{(f_T)_*(q(lm))} (f_T)_*(\mathcal{E}(lm)) \xrightarrow{(f_T)_*(q'(lm))^{-1}} & V \otimes_{\mathbb{k}} \mathcal{O}_T. \\ & \searrow g \nearrow & \end{array} \quad (6.168)$$

It is easy to check:

- the morphism $g : V \otimes_{\mathbb{k}} \mathcal{O}_T \rightarrow V \otimes_{\mathbb{k}} \mathcal{O}_T$ preserves the filtration $\{V_{\leq i}\}_{i=0}^2$, thus it defines a morphism $g \in \text{Hom}_{\text{Sch}/\mathbb{k}}(T, \text{P}_{\text{GL}, \tau})$, where $\text{P}_{\text{GL}, \tau} \subseteq \text{GL}(V)$ is the parabolic subgroup stabilising the flag $\{V_{\leq i}\}_{i=0}^2$;
- the following diagram commutes

$$\begin{array}{ccc} V(-lm)_T & \xrightarrow{g(-lm)} & V(-lm)_T \\ \downarrow q & & \downarrow q' \\ \mathcal{E} & \xlongequal{\quad} & \mathcal{E} \end{array} \quad (6.169)$$

that is $g.[(\mathcal{E}, q)] = [(\mathcal{E}', q')]$.

Thus $h, h' \in \text{Hom}_{\text{Sch}/\mathbb{k}}(T, Y_{\tau, \delta}^s)$ are in one $\text{Hom}_{\text{Sch}/\mathbb{k}}(T, \text{P}_{\text{GL}, \tau})$ -orbit, which is a $\text{Hom}_{\text{Sch}/\mathbb{k}}(T, \text{P}_{\tau})$ -orbit, since the central $\mathbb{G}_m \subseteq \text{GL}(V)$ acts trivially. This proves (1).

For (2), let $T \in \text{Sch}/\mathbb{k}$ and let $x \in \mathbf{M}'_{\tau, \delta}(T)$. Let $(\mathcal{E}, \{\mathcal{E}_{\leq i}\}_{i=0}^2)$ represent the isomorphism class $x \in \mathbf{M}'_{\tau, \delta}(T)$. Note that $\mathcal{E}$ and $\mathcal{E}_{\leq i}$ are coherent sheaves on X_T and $\mathcal{E}_i := \mathcal{E}_{\leq i}/\mathcal{E}_{\leq i-1}$ is flat over T with Hilbert polynomial $P_i(t)$.

By [Nit05] Theorem 3.7, we have that:

- $(f_T)_*(\mathcal{E}_{\leq i}(lm))$ and $(f_T)_*(\mathcal{E}_i(lm))$ are locally free of ranks $P_{\leq i}(lm)$ and $P_i(lm)$ on T for $i = 1, 2$ respectively;
- $R^j(f_T)_*(\mathcal{E}_{\leq i}(lm)) = 0$ and $R^j(f_T)_*(\mathcal{E}_i(lm)) = 0$ for $i = 1, 2$ and $j > 0$;
- $(f_T)^*(f_T)_*(\mathcal{E}_{\leq i}(lm)) \rightarrow \mathcal{E}_{\leq i}(lm)$ and $(f_T)^*(f_T)_*(\mathcal{E}_i(lm)) \rightarrow \mathcal{E}_i(lm)$ are surjective for $i = 1, 2$.

We choose a Zariski covering $\{T_j \rightarrow T\}_{j \in J}$ such that $(f_T)_*(\mathcal{E}_i(lm))|_{T_j}$ is free of rank $P_i(lm)$ for $i = 1, 2$. Then $(f_T)_*(\mathcal{E}_{\leq i}(lm))|_{T_j}$ is free of rank $P_{\leq i}(lm)$ for $i = 1, 2$. On each T_j , we can choose generating global sections compatible with the filtration $\{(f_T)_*(\mathcal{E}_{\leq i}(lm))\}_{i=0}^2$ and obtain isomorphisms in the following diagram

$$\begin{array}{ccccc} 0 & \xrightarrow{\subseteq} & V_{\leq 1} \otimes_{\mathbb{k}} \mathcal{O}_{T_j} & \xrightarrow{\subseteq} & V_{\leq 2} \otimes_{\mathbb{k}} \mathcal{O}_{T_j} \xlongequal{\quad} V \otimes_{\mathbb{k}} \mathcal{O}_{T_j} \\ & & \downarrow \mathbb{R} & & \downarrow \mathbb{R} \\ 0 & \xrightarrow{\subseteq} & (f_T)_*(\mathcal{E}_{\leq 1}(lm))|_{T_j} & \xrightarrow{\subseteq} & (f_T)_*(\mathcal{E}_{\leq 2}(lm))|_{T_j} \xlongequal{\quad} (f_T)_*(\mathcal{E}(lm))|_{T_j} . \end{array} \quad (6.170)$$

On each T_j and $i = 1, 2$ we have a surjective morphism

$$\begin{aligned}
V_{\leq i}(-lm)_{T_j} &:= (f_{T_j})^*(V_{\leq j} \otimes_{\mathbb{k}} \mathcal{O}_{T_j}) \otimes_{\mathcal{O}_{X_{T_j}}} \mathcal{L}_{T_j}^{-lm} \\
&\cong (f_{T_j})^*((f_T)_*(\mathcal{E}_{\leq i}(lm))|_{T_j}) \otimes_{\mathcal{O}_{X_{T_j}}} \mathcal{L}_{T_j}^{-lm} \\
&\cong (f_T)^*(f_T)_*(\mathcal{E}_{\leq i}(lm))|_{T_j} \otimes_{T_j} \mathcal{L}_{T_j}^{-lm} \\
&\twoheadrightarrow \mathcal{E}_{\leq i}(lm)|_{T_j} \otimes_{T_j} \mathcal{L}_{T_j}^{-lm} \\
&\cong \mathcal{E}_{\leq i}|_{T_j}.
\end{aligned} \tag{6.171}$$

When $i = 2$, let $q_j : V(-lm)_{T_j} \rightarrow \mathcal{E}|_{T_j}$ denote the surjective morphism. Then we have $q_j(V_{\leq i}(-lm)_{T_j}) = \mathcal{E}_{\leq i}|_{T_j}$ for $i = 1, 2$. Therefore $[(\mathcal{E}|_{T_j}, q_j)]$ defines a morphism $T_j \rightarrow Y_\tau$. It is easy to check that it actually defines a morphism $h_j : T_j \rightarrow Y_{\tau, \delta}^s$, for example by Lemma 6.5.3. We have that ρ_{T_j} maps h_j to $x|_{T_j} = [(\mathcal{E}|_{T_j}, \{\mathcal{E}_{\leq i}^2\}_{i=0})] \in \mathbf{M}'_{\tau, \delta}(T_j)$. Since the image of ρ_{T_j} and the image of θ_{T_j} coincide, we have that $x|_{T_j}$ is in the image of θ_{T_j} . This proves (2).

For (3), let $Z \in \text{Sch}/\mathbb{k}$ and let $\mu, \nu : \mathbf{M}'_{\tau, \delta} \rightarrow Z$ be two natural transformations such that $\mu \circ \theta = \nu \circ \theta$, which implies $\mu \circ \rho = \nu \circ \rho$. The equality $\mu = \nu$ unfolds as follows

$$\mu_T(x) = \nu_T(x) \in \text{Hom}_{\text{Sch}/\mathbb{k}}(T, Z), \quad \text{for all } T \in \text{Sch}/\mathbb{k}, x \in \mathbf{M}'_{\tau, \delta}(T). \tag{6.172}$$

Let $T \in \text{Sch}/\mathbb{k}$ and let $x \in \mathbf{M}'_{\tau, \delta}(T)$. By (2), there is a Zariski covering $\{T_j \rightarrow T\}_{j \in J}$ such that $x|_{T_j} \in \mathbf{M}'_{\tau, \delta}(T_j)$ is in the image of θ_{T_j} , which equals the image of ρ_{T_j} . Let $h_j \in \text{Hom}_{\text{Sch}/\mathbb{k}}(T_j, Y_{\tau, \delta}^s)$ be a morphism such that $\rho_{T_j}(h_j) = x|_{T_j}$ for each $j \in J$. Since $Z \in \text{Sch}/\mathbb{k}$, the following is injective

$$\text{Hom}_{\text{Sch}/\mathbb{k}}(T, Z) \rightarrow \prod_{j \in J} \text{Hom}_{\text{Sch}/\mathbb{k}}(T_j, Z). \tag{6.173}$$

Therefore to prove $\mu_T(x) = \nu_T(x)$ in $\text{Hom}_{\text{Sch}/\mathbb{k}}(T, Z)$, it suffices to prove $\mu_T(x)|_{T_j} = \nu_T(x)|_{T_j}$ in $\text{Hom}_{\text{Sch}/\mathbb{k}}(T_j, Z)$ for all $j \in J$. We have

$$\begin{aligned}
\mu_T(x)|_{T_j} &= \mu_{T_j}(x|_{T_j}) \\
&= \mu_{T_j}(\rho_{T_j}(h_j)) \\
&= (\mu \circ \rho)_{T_j}(h_j) \\
&= (\nu \circ \rho)_{T_j}(h_j) \\
&= \nu_T(x)|_{T_j}.
\end{aligned} \tag{6.174}$$

This proves (3). The argument also indicates that in (3) we can replace $Z \in \text{Sch}/\mathbb{k}$ by separated presheaves on the site $(\text{Sch}/\mathbb{k})_{\text{zar}}$. $\square$

6.5.3 Construction of the moduli

Let $m \in \mathbb{N}_+$ be such that $H^0(X, \mathcal{L}^m)$ generates $\bigoplus_{d \geq 0} H^0(X, \mathcal{L}^{md})$. Let $l \gg 0$ be large. Recall that $q : Y_{\tau, \delta}^s \rightarrow Y_{\tau, \delta}^s / P_\tau$ is a geometric quotient by Theorem 5.2.1, thus a categorical quotient. Recall the following diagram in $\text{PSh}(\text{Sch}/\mathbb{k})$ in Lemma 6.5.5

$$\begin{array}{ccc} Y_{\tau, \delta}^s & \xrightarrow{\rho} & \mathbf{M}'_{\tau, \delta} \\ \downarrow \pi & \nearrow \theta & \\ \underline{Y_{\tau, \delta}^s} / \underline{P_\tau} & & \end{array} \quad (6.175)$$

It is easy to see the geometric quotient $q : Y_{\tau, \delta}^s \rightarrow Y_{\tau, \delta}^s / P_\tau$ factors uniquely through π , i.e. that there exists a unique morphism ζ in $\text{PSh}(\text{Sch}/\mathbb{k})$

$$\zeta : \underline{Y_{\tau, \delta}^s} / \underline{P_\tau} \rightarrow Y_{\tau, \delta}^s / P_\tau \quad (6.176)$$

such that the following diagram commutes

$$\begin{array}{ccc} Y_{\tau, \delta}^s & & \\ \downarrow \pi & \searrow q & \\ \underline{Y_{\tau, \delta}^s} / \underline{P_\tau} & \xrightarrow{\zeta} & Y_{\tau, \delta}^s / P_\tau. \end{array} \quad (6.177)$$

In the construction of moduli of stable sheaves, the connection between GIT quotients and co-representatives of moduli functors was shown in [HL10] Lemma 4.3.1. The following is the analogue in the construction of moduli of unstable sheaves.

Lemma 6.5.6. *When $l \gg 0$, the natural transformation $\zeta : \underline{Y_{\tau, \delta}^s} / \underline{P_\tau} \rightarrow Y_{\tau, \delta}^s / P_\tau$ factors through $\theta : \underline{Y_{\tau, \delta}^s} / \underline{P_\tau} \rightarrow \mathbf{M}'_{\tau, \delta}$ uniquely, i.e. there exists a unique $\eta : \mathbf{M}'_{\tau, \delta} \rightarrow Y_{\tau, \delta}^s / P_\tau$ such that the following diagram in $\text{PSh}(\text{Sch}/\mathbb{k})$ commutes*

$$\begin{array}{ccc} & & \mathbf{M}'_{\tau, \delta} \\ & \nearrow \theta & \downarrow \eta \\ \underline{Y_{\tau, \delta}^s} / \underline{P_\tau} & \xrightarrow{\zeta} & Y_{\tau, \delta}^s / P_\tau. \end{array} \quad (6.178)$$

Proof. Uniqueness of η comes from Lemma 6.5.5 (3).

We now construct the natural transformation $\eta : \mathbf{M}'_{\tau,\delta} \rightarrow Y_{\tau,\delta}^s/P_\tau$. Let $T \in \text{Sch}/\mathbb{k}$ and let $x \in \mathbf{M}'_{\tau,\delta}(T)$. By Lemma 6.5.5 we can find a Zariski covering $\{T_j \rightarrow T\}_{j \in J}$ such that there exists $g_j \in (\underline{Y_{\tau,\delta}^s}/\underline{P_\tau})(T_j)$ and $\theta_{T_j}(g_j) = x|_{T_j}$ for each $j \in J$. If we denote $h_j := \zeta_{T_j}(g_j) \in \text{Hom}_{\text{Sch}/\mathbb{k}}(T_j, Y_{\tau,\delta}^s/P_\tau)$ for $j \in J$, then the relations of g_j, h_j and $x|_{T_j}$ are as follows

$$\begin{array}{ccc}
 & \mathbf{M}'_{\tau,\delta}(T_j) & \\
 \theta_{T_j} \nearrow & & \nwarrow x|_{T_j} \\
 (\underline{Y_{\tau,\delta}^s}/\underline{P_\tau})(T_j) & \xrightarrow{\zeta_{T_j}} \text{Hom}_{\text{Sch}/\mathbb{k}}(T_j, Y_{\tau,\delta}^s/P_\tau) & g_j \longmapsto h_j.
 \end{array} \tag{6.179}$$

Note that the equality $\theta_{T_j}(g_j) = x|_{T_j}$ uniquely determines $g_j \in (\underline{Y_{\tau,\delta}^s}/\underline{P_\tau})(T_j)$, since θ_{T_j} is injective by Lemma 6.5.5 (1).

Let $j, j' \in J$ be two indices. We have

$$\theta_{T_j \cap T_{j'}}(g_j|_{T_j \cap T_{j'}}) = \theta_{T_j}(g_j)|_{T_j \cap T_{j'}} = x|_{T_j \cap T_{j'}} = \theta_{T_j \cap T_{j'}}(g_{j'}|_{T_j \cap T_{j'}}). \tag{6.180}$$

Since θ is injective by Lemma 6.5.5, we have $g_j|_{T_j \cap T_{j'}} = g_{j'}|_{T_j \cap T_{j'}}$. Then $h_j|_{T_j \cap T_{j'}} = h_{j'}|_{T_j \cap T_{j'}}$. This checks the gluing condition for $\{h_j : T_j \rightarrow Y_{\tau,\delta}^s/P_\tau\}_{j \in J}$. So there is a morphism $h \in \text{Hom}_{\text{Sch}/\mathbb{k}}(T, Y_{\tau,\delta}^s/P_\tau)$ such that $h|_{T_j} = h_j$ for $j \in J$. Note that the construction of h relies on a Zariski covering $\{T_j \rightarrow T\}_{j \in J}$, however the morphism $h : T \rightarrow \text{Hom}_{\text{Sch}/\mathbb{k}}(T, Y_{\tau,\delta}^s/P_\tau)$ does not depends on the covering, since any finer Zariski covering defines the same element. Then the following map is well defined

$$\eta_T : \mathbf{M}'_{\tau,\delta}(T) \rightarrow \text{Hom}_{\text{Sch}/\mathbb{k}}(T, Y_{\tau,\delta}^s/P_\tau), \quad x \mapsto (h : T \rightarrow Y_{\tau,\delta}^s/P_\tau). \tag{6.181}$$

It is easy to check:

- $\eta_T \circ \theta_T = \zeta_T$ for all $T \in \text{Sch}/\mathbb{k}$;
- η_T is natural in $T \in \text{Sch}/\mathbb{k}$.

The lemma is proved. □

This is the main theorem on the coarse moduli space of $\mathbf{M}'_{\tau,\delta}$.

Theorem 6.5.7. *Let X be a projective scheme over $\mathbb{k}$ and let $\mathcal{L}$ be an ample line bundle on X . Let τ be a Harder-Narasimhan type of length 2 and let $\delta \in \mathbb{N}$. Then the moduli functor $\mathbf{M}'_{\tau,\delta}$ has a quasi-projective coarse moduli space.*

Proof. We will prove that $Y_{\tau,\delta}^s/P_\tau$ is the coarse moduli space, and the natural transformation $\eta : \mathbf{M}'_{\tau,\delta} \rightarrow Y_{\tau,\delta}^s/P_\tau$ in Lemma 6.5.6 defines the co-representative.

On $\text{Spec}(\mathbb{k})$, consider the following diagram in Lemma 6.5.6

$$\begin{array}{ccc} & & \mathbf{M}'_{\tau,\delta}(\text{Spec}(\mathbb{k})) \\ & \nearrow \theta_{\mathbb{k}} & \downarrow \eta_{\mathbb{k}} \\ (\underline{Y_{\tau,\delta}^s}/\underline{P_\tau})(\text{Spec}(\mathbb{k})) & \xrightarrow{\zeta_{\mathbb{k}}} & \text{Hom}_{\text{Sch}/\mathbb{k}}(\text{Spec}(\mathbb{k}), Y_{\tau,\delta}^s/P_\tau) \end{array} \quad (6.182)$$

where we have simplified $\eta_{\text{Spec}(\mathbb{k})}, \theta_{\text{Spec}(\mathbb{k})}, \zeta_{\text{Spec}(\mathbb{k})}$ to $\eta_{\mathbb{k}}, \theta_{\mathbb{k}}, \zeta_{\mathbb{k}}$. Since $\text{Spec}(\mathbb{k})$ has only one point, there is no non-trivial Zariski covering. By Lemma 6.5.5, the map $\theta_{\mathbb{k}}$ is both injective and surjective, thus bijective. Since $q : Y_{\tau,\delta}^s \rightarrow Y_{\tau,\delta}^s/P_\tau$ is a geometric quotient by P_τ (Theorem 5.2.1), the map $\zeta_{\mathbb{k}}$ above is bijective. Then $\eta_{\mathbb{k}}$ is bijective.

Next we prove that $\eta : \mathbf{M}'_{\tau,\delta} \rightarrow Y_{\tau,\delta}^s/P_\tau$ corepresents $\mathbf{M}'_{\tau,\delta}$. Let $Z \in \text{Sch}/\mathbb{k}$ and let $\xi : \mathbf{M}'_{\tau,\delta} \rightarrow Z$ be a natural transformation. Consider the diagram in $\text{PSh}(\text{Sch}/\mathbb{k})$

$$\begin{array}{ccc} & & \mathbf{M}'_{\tau,\delta} \xrightarrow{\xi} Z \\ & \nearrow \theta & \downarrow \eta \quad \searrow h \\ \underline{Y_{\tau,\delta}^s}/\underline{P_\tau} & \xrightarrow{\zeta} & Y_{\tau,\delta}^s/P_\tau \end{array} \quad (6.183)$$

Since $q : Y_{\tau,\delta}^s \rightarrow Y_{\tau,\delta}^s/P_\tau$ is a categorical quotient by P_τ , the natural transformation $\zeta : \underline{Y_{\tau,\delta}^s}/\underline{P_\tau} \rightarrow Y_{\tau,\delta}^s/P_\tau$ corepresents $\underline{Y_{\tau,\delta}^s}/\underline{P_\tau}$. Then there exists a unique morphism $h : Y_{\tau,\delta}^s/P_\tau \rightarrow Z$ such that $h \circ \zeta = \xi \circ \theta$.

We have

$$(h \circ \eta) \circ \theta = h \circ (\eta \circ \theta) = h \circ \zeta = \xi \circ \theta \quad (6.184)$$

so $h \circ \eta = \xi$ since the following map is injective by Lemma 6.5.5

$$(-) \circ \theta : \text{Hom}_{\text{PSh}(\text{Sch}/\mathbb{k})}(\mathbf{M}'_{\tau,\delta}, Z) \rightarrow \text{Hom}_{\text{PSh}(\text{Sch}/\mathbb{k})}(\underline{Y_{\tau,\delta}^s}/\underline{P_\tau}, Z). \quad (6.185)$$

If $h, h' : Y_{\tau,\delta}^s/P_\tau \rightarrow Z$ are two morphisms such that $h \circ \eta = h' \circ \eta = \xi$, then $h \circ \zeta = \xi \circ \theta = h' \circ \zeta$. Since ζ corepresents $\underline{Y_{\tau,\delta}^s}/\underline{P_\tau}$, we have $h = h'$.

We have proved that for any $Z \in \text{Sch}/\mathbb{k}$ and any $\xi : \mathbf{M}'_{\tau,\delta} \rightarrow Z$, there exists a unique morphism $h : Y_{\tau,\delta}^s/P_\tau \rightarrow Z$ such that $h \circ \eta = \xi$. This shows that η corepresents $\mathbf{M}'_{\tau,\delta}$. $\square$

6.6 Moduli spaces of vector bundles on curves

In this section, we will try to generalise the results to Harder-Narasimhan types of general lengths and finally prove Theorem 6.6.16. For simplicity in this section we only consider vector bundles on curves, i.e. the base scheme is a nonsingular projective curve C over $\mathbb{k}$. Let $g \in \mathbb{N}$ be the genus of C . By vector bundles, we refer to locally free sheaves of finite rank. On C , vector bundles are pure sheaves of dimension 1. In this section, the notation E , rather than $\mathcal{E}$, is preferred for vector bundles. Let E be a vector bundle on C . Denote its Harder-Narasimhan filtration by

$$0 = \mathrm{HN}_0(E) \subseteq \mathrm{HN}_1(E) \subseteq \cdots \subseteq \mathrm{HN}_n(E) = E. \quad (6.186)$$

Some discrete invariants are:

$$\begin{aligned} r_i(E) &:= \mathrm{rank}(\mathrm{HN}_i(E)/\mathrm{HN}_{i-1}(E)), \quad 1 \leq i \leq n \\ d_i(E) &:= \deg(\mathrm{HN}_i(E)/\mathrm{HN}_{i-1}(E)), \quad 1 \leq i \leq n \\ \mu_i(E) &:= d_i(E)/r_i(E), \quad 1 \leq i \leq n. \end{aligned} \quad (6.187)$$

However, our previous methods relied on some properties specific to Harder-Narasimhan types of length 2. For general lengths, we need some assumptions on the Harder-Narasimhan type (Condition 6.6.2). One important property for Harder-Narasimhan types of length 2 is that if E is a vector bundle of length 2, then the dimension of its space of *unipotent endomorphisms* only depends on the associated graded sheaf $\bigoplus_{i=1}^2 \mathrm{HN}_i(E)/\mathrm{HN}_{i-1}(E)$

$$\begin{aligned} &\{\varphi \in \mathrm{End}_{\mathcal{O}_C}(E) : \varphi(\mathrm{HN}_i(E)) \subseteq \mathrm{HN}_{i-1}(E) \text{ for } i = 1, 2\} \\ &\cong \mathrm{Hom}_{\mathcal{O}_C}(\mathrm{HN}_2(E)/\mathrm{HN}_1(E), \mathrm{HN}_1(E)). \end{aligned} \quad (6.188)$$

This was proved in [Jac21] Corollary 7.2.6, followed by Remark 7.2.7 on its failure for general length Harder-Narasimhan types. This was also discussed in the text following [HJ21] Proposition 3.8 and our assumption on the Harder-Narasimhan type is motivated by this discussion.

Let E be a vector bundle on C with Harder-Narasimhan filtration $\{\mathrm{HN}_i(E)\}_{i=0}^n$. We will consider the following condition for E .

Condition 6.6.1. Let E be a vector bundle on C whose Harder-Narasimhan filtration has length $n \in \mathbb{N}_+$. This condition for E assumes

$$\begin{aligned} \text{Ext}^1(\text{HN}_j(E)/\text{HN}_{j'}(E), \text{HN}_i(E)/\text{HN}_{i'}(E)) &= 0 \\ \text{for all } 0 \leq i' \leq i < j' \leq j \leq n. \end{aligned} \quad (6.189)$$

There is a condition on the Harder-Narasimhan type τ , which is easy to check.

Condition 6.6.2. Let τ be a Harder-Narasimhan type of vector bundles over C . Let $n \in \mathbb{N}_+$ be the length of τ , and let $\mu_1 > \cdots > \mu_n$ be the associated slopes of τ . This condition for τ assumes

$$\mu_i - \mu_{i+2} > 2g - 2, \quad \text{for all } 1 \leq i < i+2 \leq n. \quad (6.190)$$

Lemma 6.6.3. *If τ is a Harder-Narasimhan type satisfying Condition 6.6.2, then vector bundles of type τ satisfy Condition 6.6.1.*

Proof. Let $n \in \mathbb{N}_+$ be the length of τ and let $\mu_1 > \cdots > \mu_n$ be the associated slopes. Let E be a vector bundle of type τ with Harder-Narasimhan filtration $\{\text{HN}_i(E)\}_{i=0}^n$.

Let $0 \leq i' \leq i < j' \leq j \leq n$. Assume $i' < i$ and $j' < j$ for non-triviality. For

$$\text{Ext}^1(\text{HN}_j(E)/\text{HN}_{j'}(E), \text{HN}_i(E)/\text{HN}_{i'}(E)) = 0 \quad (6.191)$$

by Lemma 6.1.16, it suffices to show

$$\mu_{\max}(\text{HN}_j(E)/\text{HN}_{j'}(E)) - \mu_{\min}(\text{HN}_i(E)/\text{HN}_{i'}(E)) < 2 - 2g. \quad (6.192)$$

The Harder-Narasimhan filtration of $\text{HN}_j(E)/\text{HN}_{j'}(E)$ is

$$0 = \text{HN}_{j'}(E)/\text{HN}_{j'}(E) \subsetneq \text{NH}_{j'+1}(E)/\text{HN}_{j'}(E) \subsetneq \cdots \subsetneq \text{HN}_j(E)/\text{HN}_{j'}(E). \quad (6.193)$$

So $\mu_{\max}(\text{HN}_j(E)/\text{HN}_{j'}(E)) = \mu_{j'+1}$. Similarly we have $\mu_{\min}(\text{HN}_i(E)/\text{HN}_{i'}(E)) = \mu_i$. We have

$$\begin{aligned} \mu_{\min}(\text{HN}_i(E)/\text{HN}_{i'}(E)) - \mu_{\max}(\text{HN}_j(E)/\text{HN}_{j'}(E)) &= \mu_i - \mu_{j'+1} \\ &\geq \mu_i - \mu_{i+2} \\ &> 2g - 2. \end{aligned} \quad (6.194)$$

This completes the proof. $\square$

6.6.1 (τ, δ) -stability

Let τ be a Harder-Narasimhan type of length $n \in \mathbb{N}_+$. In the length 2 case, we considered the trivial filtration of the Lie algebra of U_τ . In general, the filtration of the Lie algebra of U_τ can not be trivial and the choice of filtrations will affect the definition of (τ, δ) -stability.

There is a partially ordered set $(S, \leq)$ only depending on $n \in \mathbb{N}_+$, such that filtrations of the Lie algebra of U_τ correspond to increasing chains in $(S, \leq)$.

Definition 6.6.4. The partially ordered set $(S, \leq)$ consists of the following data:

- S is the set of functions $s : \{1, \dots, n\} \rightarrow \{0, \dots, n-1\}$ such that s is non-decreasing and $s(i) < i$ for all $1 \leq i \leq n$;
- $s \leq s'$ in S if and only if $s(i) \leq s'(i)$ for all $1 \leq i \leq n$.

The partially ordered set $(S, \leq)$ has unique minimal and maximal elements

$$s_{\min}(x) = 0, \quad s_{\max}(x) = x - 1, \quad \text{for all } x \in \{1, \dots, n\}. \quad (6.195)$$

Definition 6.6.5. We call a non-trivial increasing chain of $(S, \leq)$

$$s_{\min} = s_0 \leq s_1 \leq \dots \leq s_N = s_{\max} \quad (6.196)$$

admissible if there exist $\beta_1 > \dots > \beta_n$ in $\mathbb{Q}$ such that:

- the length of the chain equals the number of positive differences among $\beta_1 > \dots > \beta_n$, i.e.

$$N = \#\{\beta_i - \beta_j : 1 \leq i < j \leq n\}; \quad (6.197)$$

- If $w_1 > \dots > w_N$ are the rational numbers in $\{\beta_i - \beta_j : 1 \leq i < j \leq n\}$, then for $1 \leq i \leq n$, $0 \leq j \leq n-1$ and $1 \leq k \leq N$ we have

$$s_k(i) = j \quad \text{if and only if} \quad \beta_{j+1} < \beta_i + w_k \leq \beta_j \quad (6.198)$$

where we set $\beta_0 = +\infty$.

Remark 6.6.6. Fix $n \in \mathbb{N}_+$ and $(S, \leq)$. Let $\beta_1 > \cdots > \beta_n$ be rational numbers. Then they determine a non-trivial increasing chain uniquely by equation (6.198). A chain is admissible if and only if it arises this way.

Admissible chains in $(S, \leq)$ can describe filtrations of Lie algebras of unipotent groups.

Example 6.6.7. Let $n \in \mathbb{N}_+$ and $(S, \leq)$ be as above. Let $\beta_1 > \cdots > \beta_n$ be integers. Let $s_0 \leq s_1 \leq \cdots \leq s_N$ be the admissible chain determined by $\beta_1 > \cdots > \beta_n$. Let V be a vector space with a decomposition $V = V_1 \oplus \cdots \oplus V_n$ into non-zero subspaces. Consider the representation $\lambda : \mathbb{G}_m \rightarrow \mathrm{GL}(V)$ such that the weight on V_i is β_i . Let

$$\mathfrak{u} := \{\varphi \in \mathrm{End}(V) : \varphi(V_{\leq i}) \subseteq V_{\leq i-1}, \text{ for all } 1 \leq i \leq n\}. \quad (6.199)$$

Then λ induces a representation on $\mathfrak{u}$ with weight $\beta_i - \beta_j$ on $\mathrm{Hom}(V_j, V_i) \subseteq \mathfrak{u}$, and $w_1 > \cdots > w_N$ are exactly the weights of $\mathbb{G}_m$ on $\mathfrak{u}$. Let $\mathfrak{u}_k := \mathfrak{u}_{\lambda \geq w_k} \subseteq \mathfrak{u}$ denote the sum of weight spaces whose λ -weights are greater than or equal to w_k . Then

$$\mathfrak{u}_k = \{\varphi \in \mathrm{End}(V) : \varphi(V_{\leq i}) \subseteq V_{\leq s_k(i)} \text{ for all } 1 \leq i \leq n\}. \quad (6.200)$$

The following definition makes sense for filtrations of length n in any Abelian category.

Definition 6.6.8. Let $\mathcal{A}$ be an Abelian category. Let $E \in \mathcal{A}$ be an object with a filtration of length n

$$0 = E_{\leq 0} \subseteq E_{\leq 1} \subseteq \cdots \subseteq E_{\leq n} = E. \quad (6.201)$$

Let $s \in S$. Define

$$\mathrm{End}(E)_s := \{\varphi \in \mathrm{End}_{\mathcal{A}}(E) : \varphi(E_{\leq i}) \subseteq E_{\leq s(i)}, \text{ for all } 1 \leq i \leq n\}. \quad (6.202)$$

We call $\varphi \in \mathrm{End}_{\mathcal{A}}(E)_s$ a *unipotent endomorphism of E of shape s* .

When E is a vector bundle on C whose Harder-Narasimhan filtration has length $n \in \mathbb{N}_+$, we define $\mathrm{End}(E)_s$ with the Abelian category of coherent sheaves on C and the Harder-Narasimhan filtration of E .

When E is coherent sheaf on $T \in \mathrm{Sch}/\mathbb{k}$ with a filtration of coherent sheaves $\{E_{\leq i}\}_{i=0}^n$ on T , we define a coherent sheaf $\mathcal{E}nd_{\mathcal{O}_T}(E)_s$ on T , whose module of sections on an open subset $W \subseteq T$ is

$$\mathcal{E}nd_{\mathcal{O}_T}(E)_s(W) := \mathrm{End}_{\mathcal{O}_W}(E|_W)_s \quad (6.203)$$

where $\text{End}_{\mathcal{O}_W}(E|_W)_s$ is defined in the Abelian category of coherent sheaves on W with respect to the filtration $\{E_{\leq i}|_W\}_{i=0}^n$.

Recall τ is a Harder-Narasimhan type of length n . The partially ordered set $(S, \leq)$ is defined for $n \in \mathbb{N}_+$.

Definition 6.6.9. Let $s_0 \precneq s_1 \precneq \cdots \precneq s_N$ be an admissible chain of length $N \in \mathbb{N}_+$ in $(S, \leq)$. Let $\delta = (d_1, \dots, d_N) \in \mathbb{N}^N$. A vector bundle E on C is called (τ, δ) -stable if:

- E has Harder-Narasimhan type τ ;
- if $\{E_{\leq i}\}_{i=0}^n$ is the Harder-Narasimhan filtration, then $E_i := E_{\leq i}/E_{\leq i-1}$ is stable for $1 \leq i \leq n$;
- $E_{\leq i-1} \subseteq E_{\leq i}$ is non-split for $1 \leq i-1 < i \leq n$;
- $\dim \text{End}(E)_{s_k} = d_1 + \cdots + d_k$ for $1 \leq k \leq N$.

Remark 6.6.10. When $n = 2$, the set S has two elements $s_{\min} \precneq s_{\max}$ and there is only one admissible chain in $(S, \leq)$ of length $N = 1$. The definition of (τ, δ) -stability above reduces to Definition 6.3.10.

6.6.2 Conditions in Theorem 5.2.1

Recall the surjective homomorphism $U_\tau \rtimes (T_\tau \times K_\tau) \rightarrow P_\tau$ defined at the beginning of Section 6.4. We want to apply Theorem 5.2.1 to the linear action of $U_\tau \rtimes (T_\tau \times K_\tau)$ on $\overline{Y}_\tau^s$ with respect to $\mathcal{M}|_{\overline{Y}_\tau^s}$. In this subsection, we will

- fix the filtration $\{u_i\}_{i=0}^N$ of $u := \text{Lie}(U_\tau)$ associated to an internal grading 1PS $\lambda : \mathbb{G}_m \rightarrow T_\tau$;
- write down explicitly α_i, σ_I , which are defined in Section 3.4;
- verify Condition 5.1.6 (cf. Lemma 6.5.1);
- describe $Y_{\tau, \delta}^s$ and $S_\delta(Y_\tau^s)$ moduli-theoretically (cf. Lemma 6.5.3).

Most proofs will be omitted if they are obvious modifications of proofs in the length 2 case.

Choose an admissible chain $s_0 \preceq s_1 \preceq \cdots \preceq s_N$ in $(S, \leq)$. By Definition 6.6.5 and Example 6.6.7, we can find $\lambda : \mathbb{G}_m \rightarrow T_\tau$ such that:

- λ is a grading 1PS, i.e. the weights of $\lambda \curvearrowright \mathfrak{u} := \text{Lie}(U_\tau)$ are all positive;
- there are N weights of $\lambda \curvearrowright \mathfrak{u}$;
- if $w_1 > \cdots > w_N$ are the N weights of $\lambda \curvearrowright \mathfrak{u}$, then

$$\mathfrak{u}_{\lambda \geq w_k} = \text{End}(V)_{s_k} \quad (6.204)$$

where $\mathfrak{u}_{\lambda \geq w_k} \subseteq \mathfrak{u}$ is the sum of weight spaces whose λ -weights are greater than or equal to w_k .

We write down α_i, σ_I for Theorem 5.2.1, which are defined in Section 3.4. By definition $T_\tau = (\mathbb{G}_m)^{n-1}$, thus

$$\alpha_i : T_\tau \rightarrow \mathbb{G}_m, \quad (t_1, \dots, t_{n-1}) \rightarrow t_i, \quad \text{for } 1 \leq i \leq n-1. \quad (6.205)$$

Recall that $\mathfrak{t}_{\mathbb{Q}} := \text{Hom}(\mathbb{G}_m, T_\tau) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^{n-1}$ and for $I \subseteq \{1, \dots, n-1\}$

$$\begin{aligned} \sigma_I &= \left\{ \gamma \in \mathfrak{t}_{\mathbb{Q}} : \begin{array}{l} \langle \alpha_i, \gamma \rangle = 0, \text{ for all } i \in I \\ \langle \alpha_i, \gamma \rangle > 0, \text{ for all } i \notin I \end{array} \right\} \\ &\cong \left\{ (\gamma_1, \dots, \gamma_{n-1}) \in \mathbb{Q}^{n-1} : \begin{array}{l} \gamma_i = 0, \text{ for all } i \in I \\ \gamma_i > 0, \text{ for all } i \notin I \end{array} \right\}. \end{aligned} \quad (6.206)$$

In particular for $0 \leq i \leq n-1$, the open faces $\sigma_i := \sigma_{\{1, \dots, i\}} \subseteq \mathbb{Q}^{n-1}$ are

$$\begin{aligned} \sigma_0 &= \{(\gamma_1, \dots, \gamma_{n-1}) \in \mathbb{Q}^{n-1} : \gamma_1, \dots, \gamma_{n-1} > 0\} \\ \sigma_i &= \left\{ (\gamma_1, \dots, \gamma_{n-1}) \in \mathbb{Q}^{n-1} : \begin{array}{l} \gamma_1 = \cdots = \gamma_i = 0 \\ \gamma_{i+1}, \dots, \gamma_{n-1} > 0 \end{array} \right\}, \quad 1 \leq i \leq n-1 \end{aligned} \quad (6.207)$$

If $\lambda' : \mathbb{G}_m \rightarrow T_\tau$ is a 1PS such that $\lambda' \in \sigma_i$, then the composition $\mathbb{G}_m \xrightarrow{\lambda'} T_\tau \rightarrow P_\tau$ is of the form

$$\mathbb{G}_m \rightarrow P_\tau, \quad t \mapsto t^{\beta_1} 1_{V_1} \oplus \cdots \oplus t^{\beta_n} 1_{V_n} \quad (6.208)$$

for some $(\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$ satisfying $\beta_1 = \cdots = \beta_{i+1} > \cdots > \beta_n$ and $\beta_1 \dim V_1 + \cdots + \beta_n \dim V_n = 0$. We can calculate $\lim_{t \rightarrow 0} \lambda'(t) \cdot [(E, q)]$ for closed points $[(E, q)] \in Y_\tau^s$ by

[HL10] Lemma 4.4.3. It turns out the limits only depend on σ_i , so the retraction p_{σ_i} in Theorem 5.2.1 is

$$p_{\sigma_i}([(E, q)]) = \left[\left(E_{\leq i+1} \oplus E_{i+2} \oplus \cdots \oplus E_n, q_{\leq i+1} \oplus q_{i+2} \oplus \cdots \oplus q_n \right) \right]. \quad (6.209)$$

The following lemma describes when $U_{\tau} \cdot p_{\sigma_i}([(E, q)]) = U_{\tau} \cdot p_{\sigma_{i-1}}([(E, q)])$. The proof comes from [HJ21] Lemma 4.14 and Proposition 4.16.

Lemma 6.6.11. *Let $[(E, q)] \in Y_{\tau}^s$ be a closed point. Let $1 \leq i < i+1 \leq n$. The following are equivalent:*

(1) $E_{\leq i} \subseteq E_{\leq i+1}$ splits;

(2) $U_{\tau} \cdot p_{\sigma_i}([(E, q)]) = U_{\tau} \cdot p_{\sigma_{i-1}}([(E, q)])$, i.e. the following quotients represent two closed points in one U_{τ} -orbit

$$\begin{aligned} q_{\leq i} \oplus q_{i+1} \oplus \cdots \oplus q_n : V(-lm) &\rightarrow E_{\leq i-1} \oplus E_i \oplus \cdots \oplus E_n \\ q_{\leq i+1} \oplus q_{i+2} \oplus \cdots \oplus q_n : V(-lm) &\rightarrow E_{\leq i} \oplus E_{i+1} \oplus \cdots \oplus E_n \end{aligned} \quad (6.210)$$

Proof. Assume (1), which is equivalent to that $E_{\leq i+1} \rightarrow E_{i+1}$ splits, i.e. there exists a morphism $\sigma : E_{i+1} \rightarrow E_{\leq i+1}$ such that the composition $E_{i+1} \xrightarrow{\sigma} E_{\leq i+1} \rightarrow E_{i+1}$ is the identity. We have the following diagram

$$\begin{array}{ccccc} & & 1_{V_{i+1}(-lm)} & & \\ & \searrow & \text{---} & \swarrow & \\ V_{i+1}(-lm) & \xrightarrow{f_*(\sigma(lm)) \otimes 1_{\mathcal{L}(-lm)}} & V_{\leq i+1}(-lm) & \twoheadrightarrow & V_{i+1}(-lm) \\ \downarrow q_{i+1} & & \downarrow q_{\leq i+1} & & \downarrow q_{i+1} \\ E_{i+1} & \xrightarrow{\sigma} & E_{\leq i+1} & \twoheadrightarrow & E_{i+1} \\ & \swarrow & 1_{E_{i+1}} & \searrow & \end{array} \quad (6.211)$$

Then $f_*(\sigma(lm)) : V_{i+1} \rightarrow V_1 \oplus \cdots \oplus V_{i+1}$ is of the form

$$f_i(\sigma(lm)) = \begin{pmatrix} \mu \\ 1_{V_{i+1}} \end{pmatrix}, \quad \mu : V_{i+1} \rightarrow V_{\leq i}. \quad (6.212)$$

Let

$$g = \begin{pmatrix} 1_{V_{\leq i}} & \mu & 0 \\ & 1_{V_{i+1}} & 0 \\ & & 1_{V_{> i+1}} \end{pmatrix} \in U_{\tau}. \quad (6.213)$$

Then

$$\begin{array}{ccc}
V(-lm) & \xrightarrow{g(-lm)} & V(-lm) \\
\downarrow q_{\leq i} \oplus q_{i+1} \oplus \cdots \oplus q_n & & \downarrow q_{\leq i+1} \oplus q_{i+2} \oplus \cdots \oplus q_n \\
E_{\leq i} \oplus E_{i+1} \oplus \cdots \oplus E_n & \xrightarrow{\cong} & E_{\leq i+1} \oplus E_{i+2} \oplus \cdots \oplus E_n
\end{array} \quad (6.214)$$

where the bottom morphism is the direct sum of the identity morphism of $E_{i+2} \oplus \cdots \oplus E_n$ and

$$(\iota, \sigma) : E_{\leq i} \oplus E_{i+1} \rightarrow E_{\leq i+1}, \quad \iota : E_{\leq i} \subseteq E_{\leq i+1}. \quad (6.215)$$

This proves (1) $\implies$ (2).

Assume (2). There exists

$$g = \begin{pmatrix} \mu_{V_{\leq i}} & \mu & \mu' \\ & 1_{V_{i+1}} & \mu'' \\ & & \mu_{V_{> i+1}} \end{pmatrix} \in U_\tau \quad (6.216)$$

such that $g(-lm) : V(-lm) \rightarrow V(-lm)$ maps $\ker(q_{\leq i} \oplus q_{i+1} \oplus \cdots \oplus q_n)$ in $\ker(q_{\leq i+1} \oplus q_{i+2} \oplus \cdots \oplus q_n)$. In particular $g(-lm)$ maps $\ker(q_{i+1}) \subseteq V_i(-lm)$ in $\ker(q_{\leq i+1})$, i.e. $g(-lm)|_{V_{i+1}(-lm)} : V_{i+1}(-lm) \rightarrow V_{\leq i+1}(-lm)$ descends to a morphism $\sigma : E_{i+1} \rightarrow E_{\leq i+1}$. It is easy to check that $\sigma : E_{i+1} \rightarrow E_{\leq i+1}$ splits the quotient $E_{\leq i+1} \rightarrow E_{i+1}$. This proves (2) $\implies$ (1). $\square$

Recall Definition 2.2.2 and Definition 2.2.3 for the notions of T -valued infinitesimal transformations and when they stabilise T -valued points.

Lemma 6.6.12. *Let $s \in S$. Let $U_{\tau,s} \subseteq U_\tau$ be the subgroup whose Lie algebra is $\text{End}(V)_s$ with respect to the filtration $\{V_{\leq i}\}_{i=0}^n$. When $l \gg 0$, the following diagram of $\Gamma(T, \mathcal{O}_T)$ -modules commutes for any $T \in \text{Sch}/\mathbb{k}$ and any T -valued point $[(E, q)]$ in Y_τ^s*

$$\begin{array}{ccc}
\left\{ \begin{array}{l} T\text{-valued infinitesimal} \\ \text{transformations in } U_{\tau,s} \end{array} \right\} & \xrightarrow{\cong} & \text{End}(V \otimes_{\mathbb{k}} \mathcal{O}_T)_s \\
\uparrow \cup & & \uparrow \cup \\
\left\{ \begin{array}{l} \rho : \rho \text{ is a } T\text{-valued} \\ \text{infinitesimal} \\ \text{transformation in } U_{\tau,s} \\ \text{and } \rho \text{ stabilises } [(E, q)] \end{array} \right\} & \xrightarrow{\cong} & \text{End}(E)_s
\end{array} \quad (6.217)$$

where $\text{End}(V \otimes_{\mathbb{k}} \mathcal{O}_T)_s$ and $\text{End}(E)_s$ are defined for the filtrations $\{V_{\leq i} \otimes_{\mathbb{k}} \mathcal{O}_T\}_{i=0}^n$ and $\{E_{\leq i}\}_{i=0}^n$. Moreover, the horizontal maps are bijective and vertical maps are injective, and all maps are natural in $[(E, q)] \in \text{Sch}/Y_\tau^s$.

Proof. The lemma can be proved as Lemma 6.5.2. $\square$

We can now verify Condition 5.1.6.

Lemma 6.6.13. *Let τ be a Harder-Narasimhan type of length n satisfying Condition 6.6.2. Let $s_0 \preceq s_1 \preceq \cdots \preceq s_N$ be an admissible chain in $(S, \leq)$. Assume there exists a (τ, δ) -stable vector bundle for some $\delta \in \mathbb{N}^N$. Then for $M' \gg l \gg 0$, Condition 5.1.6 is satisfied for any internal grading 1PS $\lambda : \mathbb{G}_m \rightarrow T_\tau$ and*

$$U_\tau \rtimes (T_\tau \times K_\tau) \curvearrowright (\overline{Y_\tau^s}, \mathcal{M}|_{\overline{Y_\tau^s}}). \quad (6.218)$$

Proof. The proof of Lemma 6.5.1 can be adapted to obtain a proof of this lemma, except that for (4) in Condition 5.1.6, we can not cite [Jac21]. The retraction p_λ does not depend on λ by [HL10] Lemma 4.4.3. When restricted on $Y_\tau^s \subseteq (\overline{Y_\tau^s})_{\lambda-\min}^0$, we have $p_\lambda = p_\tau$.

By Lemma 6.6.12, we have that for a closed point $y := [(E, q)] \in Y_\tau^s$

$$\text{Stab}_{\text{End}(V)_s}(y) \cong \text{End}(E)_s. \quad (6.219)$$

The vector bundle E has Harder-Narasimhan type τ . Then E satisfies Condition 6.6.1 by Lemma 6.6.3. By Proposition 6.6.19 in Appendix 6.6.4, we have

$$\dim \text{Stab}_{\text{End}(V)_s}(y) = \dim \text{Stab}_{\text{End}(V)_s}(p_\tau(y)) \quad (6.220)$$

which proves (4) in Condition 5.1.6. Note that Condition 6.6.2 is only used in Proposition 6.6.19. $\square$

With every condition verified, we can apply Theorem 5.2.1. Let the stratification in Theorem 5.2.1 be

$$\coprod_{\delta \in \mathbb{N}^N} S_\delta(Y_\tau^s) \rightarrow Y_\tau^s \quad (6.221)$$

where we have applied Proposition 6.4.2 to obtain the identity $p_\lambda^{-1}(Z_{\lambda-\min}^{K_\tau, s}(Y_\tau^s)) = Y_\tau^s$. Also by Theorem 5.2.1, there is an open subscheme $Y_{\tau, \delta}^s \subseteq S_\delta(Y_\tau^s)$ whose closed points are $x \in S_\delta(Y_\tau^s)$ such that $U_\tau \cdot p_{\sigma_i}(x) \neq U_\tau \cdot p_{\sigma_{i-1}}$, which has been described in Lemma 6.6.11.

Here is the moduli-theoretic descriptions of $Y_{\tau, \delta}^s$ and $S_\delta(Y_\tau^s)$.

Lemma 6.6.14. *Let τ be a Harder-Narasimhan type of length $n \in \mathbb{N}_+$. Let $P_i(t), P(t)$ be the Hilbert polynomials associated to τ . Assume Condition 6.6.2 for τ . Choose an admissible chain $s_0 \preceq s_1 \preceq \cdots \preceq s_N$ in $(S, \leq)$ and choose $\delta = (d_1, \dots, d_N) \in \mathbb{N}^N$. Let $m, l \gg 0$. Then the scheme $S_\delta(Y_\tau^s)$ represents the functor which sends $T \in \text{Sch}/\mathbb{k}$ to the set of isomorphism classes $[(E, q)]$ such that:*

- (1) $q : V(-lm)_T \rightarrow E$ is a surjective morphism of coherent sheaves on C_T , and E_i is flat over T with Hilbert polynomial $P_i(t)$ for $i = 1, \dots, n$, where $E_i := E_{\leq i}/E_{\leq i-1}$ and $E_{\leq i} := q(V_{\leq i}(-lm)_T)$;
- (2) fibres of E_i at closed points of T are stable for $i = 1, \dots, n$;
- (3) for any closed subscheme $T' \hookrightarrow T$, and for all $1 \leq i \leq N$, the coherent sheaf

$$(C_{T'} \rightarrow T')_* \mathcal{E}nd_{\mathcal{O}_{C_{T'}}}(E|_{T'})_{s_i} \quad (6.222)$$

is locally free of rank $d_1 + \cdots + d_i$ on T' .

The open subscheme $Y_{\tau, \delta}^s \subseteq S_\delta(Y_\tau^s)$ represents the sub-functor which sends $T \in \text{Sch}/\mathbb{k}$ to the set of isomorphism classes $[(E, q)]$ such that (1), (2), (3) above hold and:

- (4) for any closed point $t \in T$ and any $1 \leq i < i+1 \leq n$, the inclusion $E_{\leq i}|_t \subseteq E_{\leq i+1}|_t$ does not split.

Proof. The lemma is proved in the same way as Lemma 6.5.3, and we need Lemma 6.6.11 for (4). □

6.6.3 The moduli functor $\mathbf{M}'_{\tau, \delta}$

We define *flat families of (τ, δ) -stable vector bundles parameterised by T* and the moduli functor $\mathbf{M}'_{\tau, \delta}$. All contents are generalisations of those in Section 6.5.2 and Section 6.5.3.

Definition 6.6.15. Let τ be a Harder-Narasimhan type of length $n \in \mathbb{N}_+$. Let $P_i(t), P(t)$ be the Hilbert polynomials associated to τ . Choose an admissible chain $s_0 \preceq s_1 \preceq \cdots \preceq s_N$ in $(S, \leq)$ and choose $\delta = (d_1, \dots, d_N) \in \mathbb{N}^N$. Let $T \in \text{Sch}/\mathbb{k}$. A *family of (τ, δ) -stable vector bundles on C parameterised by T* is $(E, \{E_{\leq i}\}_{i=0}^n)$ such that:

- E is a coherent sheaf on $C_T := C \times_{\mathbb{k}} T$ and $\{E_{\leq i}\}_{i=0}^n$ is a length n filtration

$$0 = E_{\leq 0} \subsetneq E_{\leq 1} \subsetneq \cdots \subsetneq E_{\leq n} = E; \quad (6.223)$$

- Each $E_i := E_{\leq i}/E_{\leq i-1}$ is flat over T with Hilbert polynomial $P_i(t)$;
- For any closed point $t \in T$, the fibre $E|_t$ is a (τ, δ) -stable vector bundle on $C \cong C \times_{\mathbb{k}} \text{Spec}(\kappa(t))$;
- For any closed subscheme $T' \hookrightarrow T$, and for any $1 \leq i \leq N$, the following sheaf is locally free of rank $d_1 + \cdots + d_i$ on T'

$$(C_{T'} \rightarrow T')_* \mathcal{E}nd_{\mathcal{O}_{C_{T'}}}(E|_{T'})_{s_i}. \quad (6.224)$$

Two families $(E, \{E_{\leq i}\}_{i=0}^n)$ and $(E', \{E'_{\leq i}\}_{i=0}^n)$ parameterised by T are called *isomorphic* if there is a filtration-preserving isomorphism $E \cong E'$ on C_T .

Let τ, δ and $\{s_i\}_{i=0}^N$ be as in Definition 6.6.15. The moduli functor $\mathbf{M}'_{\tau, \delta}$ is defined as follows

$$\mathbf{M}'_{\tau, \delta} : (\text{Sch}/\mathbb{k})^{\text{op}} \rightarrow \text{Set} \quad (6.225)$$

which sends $T \in \text{Sch}/\mathbb{k}$ to the set

$$\mathbf{M}'_{\tau, \delta}(T) := \left\{ \begin{array}{c} \text{isomorphism classes of families of} \\ (\tau, \delta)\text{-stable vector bundles parameterised by } T \end{array} \right\} \quad (6.226)$$

and for a morphism $g : T' \rightarrow T$ the map

$$\mathbf{M}'_{\tau, \delta}(g) : \mathbf{M}'_{\tau, \delta}(T') \rightarrow \mathbf{M}'_{\tau, \delta}(T) \quad (6.227)$$

sends the isomorphism class of $(E, \{E_{\leq i}\}_{i=0}^n)$ to the isomorphism class of the pull-back family $(E|_{T'}, \{E_{\leq i}|_{T'}\}_{i=0}^n)$, where $E|_{T'} := (1_X \times g)^* E$. It is easy to check that $\mathbf{M}'_{\tau, \delta}$ defines a functor.

When $m, l \gg 0$, and Condition 6.6.2 holds for τ , similar to the length 2 case, we have:

- $Y_{\tau, \delta}^s$ has a quasi-projective universal geometric quotient by P_{τ} by Theorem 5.2.1

$$q : Y_{\tau, \delta}^s \rightarrow Y_{\tau, \delta}^s / P_{\tau}; \quad (6.228)$$

- The following diagrams in $\text{PSh}(\text{Sch}/\mathbb{k})$ commute

$$\begin{array}{ccc}
Y_{\tau,\delta}^s & & Y_{\tau,\delta}^s \xrightarrow{\rho} \mathbf{M}'_{\tau,\delta} \\
\downarrow \pi & \searrow q & \downarrow \pi \nearrow \theta \\
Y_{\tau,\delta}^s / \underline{P}_\tau & \xrightarrow{\zeta} & Y_{\tau,\delta}^s / \underline{P}_\tau
\end{array}
\quad
\begin{array}{ccc}
Y_{\tau,\delta}^s & \xrightarrow{\rho} & \mathbf{M}'_{\tau,\delta} \\
\downarrow \pi & \nearrow \theta & \downarrow \eta \\
Y_{\tau,\delta}^s / \underline{P}_\tau & \xrightarrow{\zeta} & Y_{\tau,\delta}^s / \underline{P}_\tau
\end{array}
\quad (6.229)$$

In essentially the same way, we can prove that $\eta : \mathbf{M}'_{\tau,\delta} \rightarrow Y_{\tau,\delta}^s / \underline{P}_\tau$ is a coarse moduli space.

Theorem 6.6.16. *Let C be a nonsingular projective curve of genus g over $\mathbb{k}$. Let τ be a Harder-Narasimhan type of length $n \in \mathbb{N}_+$. Assume Condition 6.6.2 for τ . Choose an admissible chain $s_0 \preceq s_1 \preceq \cdots \preceq s_N$ in $(S, \leq)$ and choose $\delta = (d_1, \dots, d_N) \in \mathbb{N}^N$. Then the functor $\mathbf{M}'_{\tau,\delta}$ has a quasi-projective coarse moduli space.*

Proof. The statements of Lemma 6.5.5, Lemma 6.5.6 are still true here, and proofs are essentially the same. The proof of Theorem 6.5.7 is purely categorical and works to prove this theorem as well. $\square$

6.6.4 Appendix

The Appendix aims to prove Proposition 6.6.19.

Recall $(S, \leq)$ is the set of all functions $s : \{1, \dots, n\} \rightarrow \{0, \dots, n-1\}$ such that s is non-decreasing and $s(i) < i$ for all $1 \leq i \leq n$. Let $P := \{(s, t) \in S \times S : s \leq t\}$. The set P is endowed with the partial order such that $(s, t) \leq (s', t')$ if and only if $s \leq s'$ and $t \leq t'$. Define an equivalence relation on P such that $(s, t) \sim (s', t')$ if and only if

- $t - s = t' - s'$;
- for $1 \leq i \leq n$, if $t(i) - s(i) = t'(i) - s'(i) > 0$, then $s(i) = s'(i)$ and $t(i) = t'(i)$.

Lemma 6.6.17. *In the set P , every equivalence class has a lower bound in the class.*

Proof. An explicit construction of the lower bound $(s_0, t_0) \in P$ of the equivalence class of $(s, t) \in P$ is given by

$$t_0(x) = \begin{cases} 0, & 1 \leq x \leq i_1 \\ t(i_p), & i_p \leq x < i_{p+1} \\ t(i_k), & i_k \leq x \leq n \end{cases} \quad (6.230)$$

where $i_1 < \cdots < i_k$ are those $i \in \{1, \dots, n\}$ such that $s(i) < t(i)$. The function s_0 is determined by $t_0 - s_0 = t - s$. $\square$

Recall that $\text{End}(E)_s$ is defined for $s \in S$ and vector bundles E on C whose Harder-Narasimhan filtration has length n . It is easy to see:

- If $s \leq s'$ in S , then $\text{End}(E)_s \subseteq \text{End}(E)_{s'}$;
- If $(s, t) \leq (s', t')$ in P , then there is a natural injective map $\frac{\text{End}(E)_t}{\text{End}(E)_s} \rightarrow \frac{\text{End}(E)_{t'}}{\text{End}(E)_{s'}}$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{End}(E)_s & \longrightarrow & \text{End}(E)_t & \longrightarrow & \frac{\text{End}(E)_t}{\text{End}(E)_s} \longrightarrow 0 \\ & & \downarrow \text{Id} & & \downarrow \text{Id} & & \downarrow \\ 0 & \longrightarrow & \text{End}(E)_{s'} & \longrightarrow & \text{End}(E)_{t'} & \longrightarrow & \frac{\text{End}(E)_{t'}}{\text{End}(E)_{s'}} \longrightarrow 0. \end{array} \quad (6.231)$$

Lemma 6.6.18. *Assume Condition 6.6.1 for E . Let $(s, t), (s', t') \in P$. If $(s, t) \leq (s', t')$ and $(s, t) \sim (s', t')$, then the following map is a linear isomorphism*

$$\frac{\text{End}(E)_t}{\text{End}(E)_s} \rightarrow \frac{\text{End}(E)_{t'}}{\text{End}(E)_{s'}}. \quad (6.232)$$

Proof. For injectivity, it suffices to prove

$$\text{End}(E)_{s'} \cap \text{End}(E)_t \subseteq \text{End}(E)_s. \quad (6.233)$$

Let $\varphi \in \text{End}(E)_{s'} \cap \text{End}(E)_t$. Let $i \in \{1, \dots, n\}$. If $s(i) < t(i)$, then $s(i) = s'(i)$ since $(s, t) \sim (s', t')$, and then $\varphi(E_{\leq i}) \subseteq E_{\leq s'(i)} = E_{\leq s(i)}$. Else $s(i) = t(i)$, then $\varphi(E_{\leq i}) \subseteq E_{\leq t(i)} = E_{\leq s(i)}$. In both cases, $\varphi(E_{\leq i}) \subseteq E_{\leq s(i)}$, i.e. $\varphi \in \text{End}(E)_s$.

For surjectivity, it suffices to prove

$$\text{End}(E)_{t'} \subseteq \text{End}(E)_t + \text{End}(E)_{s'}. \quad (6.234)$$

It is helpful to prove the following first

$$\text{End}(E)_t = \sum_{i=2}^n \text{End}(E)_{h_{t(i),i}}, \quad \text{for all } t \in S \quad (6.235)$$

where $h_{i,j} \in S$ is defined for $0 \leq i < j \leq n$ as

$$h_{i,j} : \{1, \dots, n\} \rightarrow \{0, \dots, n-1\}, \quad x \mapsto \begin{cases} 0, & x < j \\ i, & x \geq j. \end{cases} \quad (6.236)$$

For $t \in S$, and $1 \leq i \leq n$ define the truncation functions

$$\begin{aligned} \tau_{\leq i}(t) : \{1, \dots, n\} &\rightarrow \{0, \dots, n-1\}, & j &\mapsto \begin{cases} t(x), & x \leq i \\ t(i), & x > i \end{cases} \\ \tau_{> i}(t) : \{1, \dots, n\} &\rightarrow \{0, \dots, n-1\}, & j &\mapsto \begin{cases} 0, & x \leq i \\ t(x), & x > i. \end{cases} \end{aligned} \quad (6.237)$$

The following properties are straightforward:

- $\tau_{\leq i}(t)$ and $\tau_{> i}(t)$ are functions in S , so $\tau_{\leq i}$ and $\tau_{> i}$ define maps $S \rightarrow S$;
- $\tau_{> i} \circ \tau_{> i-1} = \tau_{> i}$ for $2 \leq i \leq n$;
- $\tau_{\leq i} \circ \tau_{> i-1}(t) = h_{t(i), i}$ for $2 \leq i \leq n$.

For $t \in S$ and $1 \leq i \leq n$, apply $\text{Hom}(-, E_{\leq t(i)})$ to the short exact sequence

$$0 \longrightarrow E_{\leq i} \longrightarrow E \longrightarrow E/E_{\leq i} \longrightarrow 0 \quad (6.238)$$

to get the exact sequence

$$\longrightarrow \text{Hom}(E, E_{\leq t(i)}) \longrightarrow \text{Hom}(E_{\leq i}, E_{\leq t(i)}) \longrightarrow \text{Ext}^1(E/E_{\leq i}, E_{\leq t(i)}). \quad (6.239)$$

We have that $\text{Hom}(E, E_{\leq t(i)}) \rightarrow \text{Hom}(E_{\leq i}, E_{\leq t(i)})$ is surjective since Condition 6.6.1 assumes $\text{Ext}^1(E/E_{\leq i}, E_{\leq t(i)}) = 0$. For any $\varphi \in \text{End}(E)_t$, we have $\varphi|_{E_{\leq i}} \in \text{Hom}(E_{\leq i}, E_{\leq t(i)})$. There exists $\psi \in \text{Hom}(E, E_{\leq t(i)})$ such that $\varphi|_{E_{\leq i}} = \psi|_{E_{\leq i}}$. We consider $\psi : E \rightarrow E_{\leq t(i)}$ as a morphism $\psi : E \rightarrow E$ via the inclusion $E_{\leq t(i)} \rightarrow E$. We show $\psi \in \text{End}(E)_{\tau_{\leq i}(t)}$. When $x \leq i$, since $\psi|_{E_{\leq i}} = \varphi|_{E_{\leq i}}$, we have $\psi(E_{\leq x}) = \varphi(E_{\leq x}) \subseteq E_{\leq t(x)}$, and when $x > i$, we have $\psi(E_{\leq x}) \subseteq \psi(E) \subseteq E_{\leq t(i)}$. This proves $\psi \in \text{End}(E)_{\tau_{\leq i}(t)}$. We next show $\varphi - \psi \in \text{End}(E)_{\tau_{> i}(t)}$. When $x \leq i$, we have $(\varphi - \psi)(E_{\leq x}) = 0$, and when $x > i$, we have

$$(\varphi - \psi)(E_{\leq x}) \subseteq \varphi(E_{\leq x}) + \psi(E) \subseteq E_{\leq t(x)} + E_{\leq t(i)} = E_{\leq t(x)}. \quad (6.240)$$

This proves $\varphi - \psi \in \text{End}(E)_{\tau_{> i}(t)}$. The above argument together with $\tau_{\leq i}(t) \leq t$ and $\tau_{> i}(t) \leq t$ prove the following identity

$$\text{End}(E)_t = \text{End}(E)_{\tau_{\leq i}(t)} + \text{End}(E)_{\tau_{> i}(t)}. \quad (6.241)$$

Replace t by $\tau_{>i-1}(t)$ in the above equation and we have

$$\begin{aligned} \text{End}(E)_{\tau_{>i-1}(t)} &= \text{End}(E)_{\tau_{\leq i} \circ \tau_{>i-1}(t)} + \text{End}(E)_{\tau_{>i} \circ \tau_{>i-1}(t)} \\ &= \text{End}(E)_{h_{t(i),i}} + \text{End}(E)_{\tau_{>i}(t)}. \end{aligned} \quad (6.242)$$

Consider the following equations

$$\begin{aligned} \text{End}(E)_t &= \text{End}(E)_{h_{t(2),2}} + \text{End}(E)_{\tau_{>2}(t)} \\ \text{End}(E)_{\tau_{>2}(t)} &= \text{End}(E)_{h_{t(3),3}} + \text{End}(E)_{\tau_{>3}(t)} \\ &\dots \\ \text{End}(E)_{\tau_{>n-1}(t)} &= \text{End}(E)_{h_{t(n),n}} + \text{End}(E)_0 \end{aligned} \quad (6.243)$$

which sum to

$$\text{End}(E)_t = \sum_{i=2}^n \text{End}(E)_{h_{t(i),i}}. \quad (6.244)$$

Now we prove $\text{End}(E)_{t'} \subseteq \text{End}(E)_t + \text{End}(E)_{s'}$. Since $\text{End}(E)_{t'} = \sum_{i=2}^n \text{End}(E)_{h_{t'(i),i}}$, it suffices to show

$$\text{End}(E)_{h_{t'(i),i}} \subseteq \text{End}(E)_t + \text{End}(E)_{s'}, \quad \text{for all } 2 \leq i \leq n. \quad (6.245)$$

When $s'(i) = t'(i)$, we have $h_{t'(i),i} = h_{s'(i),i} \leq s'$ and then

$$\text{End}(E)_{h_{t'(i),i}} \subseteq \text{End}(E)_{s'} \subseteq \text{End}(E)_t + \text{End}(E)_{s'}. \quad (6.246)$$

When $s'(i) < t'(i)$, since $(s, t) \sim (s', t')$ we have $t(i) = t'(i)$, so $h_{t'(i),i} = h_{t(i),i} \leq t$ and then

$$\text{End}(E)_{h_{t'(i),i}} \subseteq \text{End}(E)_t \subseteq \text{End}(E)_t + \text{End}(E)_{s'}. \quad (6.247)$$

This completes the proof. $\square$

Proposition 6.6.19. *Let E be a vector bundle on C with Harder-Narasimhan filtration $\{E_{\leq i}\}_{i=0}^n$. Assume Condition 6.6.1 for E . Let $s \in S$. Then*

$$\dim \text{End}(E)_s = \dim \text{End}(\text{gr}(E))_s. \quad (6.248)$$

Proof. Recall the notation $E_i := E_{\leq i}/E_{\leq i-1}$ for $1 \leq i \leq n$. The first step is to consider when $s = h_{i,j} \in S$ for $0 \leq i < j \leq n$, where

$$h_{i,j}(x) = \begin{cases} 0, & x < j \\ i, & x \geq j. \end{cases} \quad (6.249)$$

By definition

$$\text{End}(E)_{h_{i,j}} = \{\varphi \in \text{End}(E) : \varphi(E) \subseteq E_{\leq i}, \varphi(E_{\leq j-1}) = 0\} \cong \text{Hom}(E/E_{\leq j-1}, E_{\leq i}). \quad (6.250)$$

We show by induction on i that

$$\dim \text{Hom}(E/E_{\leq j-1}, E_{\leq i}) = \sum_{i'=1}^i \dim \text{Hom}(E/E_{\leq j-1}, E_{i'}). \quad (6.251)$$

When $i = 0, 1$, the equation is immediate. Assume the equation holds for $i - 1$ for $1 \leq i < j \leq n$. The following is exact since $\text{Ext}^1(E/E_{\leq j-1}, E_{\leq i-1}) = 0$ by Condition 6.6.1

$$\begin{aligned} 0 \longrightarrow \text{Hom}(E/E_{\leq j-1}, E_{\leq i-1}) &\longrightarrow \text{Hom}(E/E_{\leq j-1}, E_{\leq i}) \\ &\longrightarrow \text{Hom}(E/E_{\leq j-1}, E_i) \longrightarrow 0. \end{aligned} \quad (6.252)$$

Then

$$\begin{aligned} \dim \text{Hom}(E/E_{\leq j-1}, E_{\leq i}) &= \dim \text{Hom}(E/E_{\leq j-1}, E_i) + \dim \text{Hom}(E/E_{\leq j-1}, E_{\leq i-1}) \\ &= \dim \text{Hom}(E/E_{\leq j-1}, E_i) + \sum_{i'=1}^{i-1} \dim \text{Hom}(E/E_{\leq j-1}, E_{i'}) \\ &= \sum_{i'=1}^i \dim \text{Hom}(E/E_{\leq j-1}, E_{i'}) \end{aligned} \quad (6.253)$$

which completes the induction.

We then show for all $1 \leq i < j \leq n$ the following equation by induction on j

$$\dim \text{Hom}(E/E_{\leq j-1}, E_i) = \sum_{j'=j}^n \dim \text{Hom}(E_{j'}, E_i). \quad (6.254)$$

When $j = n$, the equation holds immediately. Assume the equation holds for $j + 1$ for $1 \leq i < j < n$. The following is exact since $\text{Ext}^1(E/E_{\leq j}, E_i) = 0$ by Condition 6.6.1

$$0 \longrightarrow \text{Hom}(E/E_{\leq j}, E_i) \longrightarrow \text{Hom}(E/E_{\leq j-1}, E_i) \longrightarrow \text{Hom}(E_j, E_i) \longrightarrow 0. \quad (6.255)$$

Then

$$\begin{aligned} \dim \text{Hom}(E/E_{\leq j-1}, E_i) &= \dim \text{Hom}(E_j, E_i) + \dim \text{Hom}(E/E_{\leq j}, E_i) \\ &= \dim \text{Hom}(E_j, E_i) + \sum_{j'=j+1}^n \dim \text{Hom}(E_{j'}, E_i) \\ &= \sum_{j'=j}^n \dim \text{Hom}(E_{j'}, E_i) \end{aligned} \quad (6.256)$$

which completes the induction.

With the two equations proved we have for $0 \leq i < j \leq n$

$$\begin{aligned}
\dim \operatorname{Hom}(E/E_{\leq j-1}, E_{\leq i}) &= \sum_{i'=1}^i \dim \operatorname{Hom}(E/E_{\leq j-1}, E_{i'}) \\
&= \sum_{i'=1}^i \sum_{j'=j}^n \dim \operatorname{Hom}(E_{j'}, E_{i'}) \\
&= \dim \operatorname{Hom}\left(\bigoplus_{j'=j}^n E_{j'}, \bigoplus_{i'=1}^i E_{i'}\right)
\end{aligned} \tag{6.257}$$

that is

$$\dim \operatorname{End}(E)_{h_{i,j}} = \dim \operatorname{End}(\operatorname{gr}(E))_{h_{i,j}}. \tag{6.258}$$

The next step is to consider general $s \in S$. We prove by induction on $\max\{s(x) : 1 \leq x \leq n\}$. When $\max\{s(x) : 1 \leq x \leq n\} = 0$, we have $s = 0$ and $\dim \operatorname{End}(E)_s = \dim \operatorname{End}(\operatorname{gr}(E))_s = 0$. Assume for some $1 \leq i < n - 1$ we have proved $\dim \operatorname{End}(E)_s = \dim \operatorname{End}(\operatorname{gr}(E))_s$ if $s \in S$ and $\max\{s(x) : 1 \leq x \leq n\} \leq i$. Let $s \in S$ be such that $\max\{s(x) : 1 \leq x \leq n\} = i + 1$. Let $j := \min\{x : s(x) = i + 1\}$. Then

$$i + 1 < j \leq n, \quad s(j - 1) < s(j) = i + 1. \tag{6.259}$$

The function $h_{i+1,j} \in S$ is defined and $h_{i+1,j} \leq s$. Consider $s' : \{1, \dots, n\} \rightarrow \mathbb{Z}$ defined by

$$s' = s - h_{i+1,j} + h_{i,j}. \tag{6.260}$$

Then we have $(h_{i,j}, s') \in P$ and $\max\{s'(x) : 1 \leq x \leq n\} \leq i$.

We claim that $(h_{i+1,j}, s) \sim (h_{i,j}, s')$ in P . By the definition of s' we have $s - h_{i+1,j} = s' - h_{i,j}$. For $1 \leq x \leq n$, if $h_{i+1,j}(x) < s(x)$, then $x < j$ and then $s'(x) = s(x)$. This proves $(h_{i+1,j}, s) \sim (h_{i,j}, s)$. It is easy to see $(h_{i,j}, s') \leq (h_{i+1,j}, s)$ in P . By Lemma 6.6.18, the right vertical arrow in the diagram is an isomorphism

$$\begin{array}{ccccccc}
0 & \longrightarrow & \operatorname{End}(E)_{h_{i,j}} & \longrightarrow & \operatorname{End}(E)_{s'} & \longrightarrow & \frac{\operatorname{End}(E)_{s'}}{\operatorname{End}(E)_{h_{i,j}}} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \cong \\
0 & \longrightarrow & \operatorname{End}(E)_{h_{i+1,j}} & \longrightarrow & \operatorname{End}(E)_s & \longrightarrow & \frac{\operatorname{End}(E)_s}{\operatorname{End}(E)_{h_{i+1,j}}} \longrightarrow 0.
\end{array} \tag{6.261}$$

Then

$$\begin{aligned}
\dim \operatorname{End}(E)_s &= \dim \operatorname{End}(E)_{h_{i+1,j}} + \dim \operatorname{End}(E)_{s'} - \dim \operatorname{End}(E)_{h_{i,j}} \\
&= \dim \operatorname{End}(\operatorname{gr}(E))_{h_{i+1,j}} + \dim \operatorname{End}(\operatorname{gr}(E))_{s'} - \dim \operatorname{End}(\operatorname{gr}(E))_{h_{i,j}} \quad (6.262) \\
&= \dim \operatorname{End}(\operatorname{gr}(E))_s
\end{aligned}$$

where the first equality is from the diagram, and the second equality is from the induction hypothesis on $\operatorname{End}(E)_{s'}$ and the equation (6.258), and the third equality holds since $\operatorname{gr}(E)$ also satisfies Condition 6.6.1 and we can replace E by $\operatorname{gr}(E)$ in the first equality. This completes the induction and the lemma is proved. $\square$

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