

TIME-HOMOGENEOUS DIFFUSION WITH A GIVEN MARGINAL SUBORDINATED BY DIFFERENT TIME CHANGES

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CONTENTS

Introduction	2
1 Time-Homogeneous Diffusions With a Given Marginal At A Random Time	4
1.1 Main Results In Cox, Hobson And Oblój's Paper	4
1.1.1 A Heuristic Argument	5
1.1.2 Application	5
1.2 Parallel Result In P. Carr's Paper	6
2 Estimation of σ And Path Simulation	8
2.1 Estimation of σ	8
2.2 Asymptotic Behaviour For Extreme Values Of K	10
2.2.1 Bachelier Model	11
2.2.2 Black Scholes Model	11
2.2.3 Heston Model	12
2.3 Path Simulation	12
2.3.1 Time Changes	12
2.3.2 Simulation Of Stock Paths	14
2.4 Verification And Improvement	17
2.4.1 Verification	17
2.4.2 Observations And Improvements	19
3 Prices Of European Call Options With Maturity $T_1 < t^*$	22
3.1 Implementation	22
3.2 Analysis And Interpretation	22
4 Exotic Option Pricing	26
4.1 One-touch Options	26
4.2 Barrier Options	28
4.3 Variance Swaps	30

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Lastly, I would like to devote my deepest gratitudes to my parents for their continuous love and support.

INTRODUCTION

MOTIVATION

In two recent papers P. Carr (see [2]) as well as Cox, Hobson and Obłój (see [1]) have looked at time-homogeneous diffusions with a given marginal stopped at a random time. Both of them are interested in the following problem: suppose μ is a given distribution on $\mathbb{R}$ and τ is a random time, we would like to find a function $\sigma: \mathbb{R} \rightarrow \mathbb{R}^+$ such that

$$X_\tau = \int_0^\tau \sigma(X_s) dW_s \sim \mu, \quad (1)$$

where (W_t) is an independent Brownian Motion. Cox, Hobson and Obłój have found that, in the case where τ is exponentially distributed, this problem has a unique fully explicit solution. (see [1])

Its application in finance is immediate. Suppose we have a financial derivative with maturity t^* . Its underlying stock price can be modelled as $S_t = X_{g(t)}$ for all $t \in [0, t^*]$, where $dX_t = \sigma(X_t) dW_t$ and g is an independent subordinator with $g(t^*) \sim \tau$ which is exponentially distributed. Since μ , the probability distribution of the stock at maturity t^* can be fully characterized by the set of call prices with different strikes but the same maturity and underlying via the formula

$$\rho(K) = e^{rt^*} C''(K), \quad (2)$$

where ρ is the probability density function (PDF) associated with μ , we can therefore fully and explicitly calibrate the model to market prices. (see [1]) In P. Carr's paper, he develops an example with $g \sim \Gamma_t(t^*, \theta)$ which is a Γ -subordinator. We can verify that $g(t^*)$ is exponentially distributed. With the help of forward-backward Kolmogorov equations combined with Laplace transform properties, he manages to give an explicit and exact solution of calibrating the model to one arbitrarily given smile of an arbitrarily given maturity. However, since g is a pure jump process, (S_t) is also a pure jump and we will lose the continuity of the stock paths. (see [2])

If we try to understand in more details how P. Carr deduces the formula of σ in the case of Γ -subordinator, we will find the only thing we need is to let $g(t^*)$ be exponentially distributed. As a result, we can partially alleviate the assumption of Γ -subordinator and try

some continuous time changes g where $g(t^*)$ are exponentially distributed. One advantage is that these models will preserve the continuity of stock paths. Although both local variance gamma model (the model described in [2]) and continuous time change models should give identical prices for European options with maturity t^* , which are consistent with market prices, due to very different intrinsic characteristics (pure jump or continuous process), they should behave differently in terms of path-dependent options as well as European options with maturity $T_1 < t^*$.

Therefore, the objective of our thesis is to analyse in more details time-homogeneous diffusions with a given marginal at a random time and take a look at some derivative prices given by models subordinated by different time changes. It will be interesting to see how price differences can reflect differences in stock dynamics for models subordinated by different time changes.

MAJOR STEPS

The thesis is structured as follows: In the first part, we give a quick review of some main results in P. Carr's and Cox, Hobson and Oblój's papers, which are the starting point for our thesis. Secondly, based on the methodology given by these two papers, we can calibrate our model to market prices (Instead of taking real market data, we will use Bachelier, Black Scholes and Heston models to generate prices.) and therefore generate stock paths. Calibration can be verified by comparing initial data (given by the corresponding Bachelier, Black Scholes and Heston models) with simulated results. Next, using the simulated stock paths, we can price path-dependent options (such as barrier options, variance swaps and one touches) as well as European options with maturity $T_1 < t^*$. Some interpretation will be given to associate price differences with differences in stock dynamics for models subordinated by different time changes.

Throughout the thesis, t^* or T (used in figures) denotes the maturity of the financial contracts.

TIME-HOMOGENEOUS DIFFUSIONS WITH A GIVEN MARGINAL AT A RANDOM TIME

1.1 MAIN RESULTS IN COX, HOBSON AND OBLÓJ'S PAPER

In one recent paper, (see [1]) Cox, Hobson and Oblój are interested in the following problem: suppose μ is a given distribution on $\mathbb{R}$ and τ is a random time. We would like to know whether it is possible to find a time-homogeneous martingale diffusion, (X_t) , independent of τ , such that $X_\tau \sim \mu$. In the case where μ is regular enough, the problem is equivalent to find a function $\sigma: \mathbb{R} \rightarrow \mathbb{R}^+$ such that

$$X_\tau = \int_0^\tau \sigma(X_s) dW_s \sim \mu, \quad (1.1)$$

where (W_t) is an independent Brownian Motion. Interestingly, an explicit and exact solution is found where τ is distributed exponentially with parameter 1. The main result is presented in Theorem 3.1 of the paper:

Theorem 1. Let μ be a probability measure with $\int |x| \mu(dx) < \infty$ and $\int x \mu(dx) = x_0$. Let $u_\mu(x) = \int_{\mathbb{R}} |x - y| \mu(dy)$. l_-^μ and l_+^μ denote respectively the lower and the upper bounds of the support of μ . Define a measure m via

$$m(dx) = \frac{\mu(dx)}{u_\mu(x) - |x - x_0|}, \quad (1.2)$$

for $x \in (l_-^\mu, l_+^\mu)$,

$$m([y, x_0)) = m([x_0, x]) = \infty, \quad (1.3)$$

for $y \leq l_-^\mu \leq l_+^\mu \leq x$.

Let (X_t) be the generalised diffusion associated with m and τ be an $\mathcal{F}_0$ -measurable $\mathbb{P}^{x_0}$ -exponential random variable independent of (X_t) . Then, under $\mathbb{P}^{x_0}$, $X_\tau \sim \mu$ and $(X_{\min(t, \tau)})$ is a uniformly integrable martingale.

Remark 1.1.1. The details of generalised diffusions are given in [18]. In the case where μ is regular enough, which is the case in our thesis, the process (X_t) is a diffusion on natural scale described by its speed measure $m(dx) = \sigma(x)^{-2} dx$.

1.1.1 A HEURISTIC ARGUMENT

A heuristic argument can be given to justify Theorem 1. (see [1] for details)

Direct application of Itô's formula gives

$$f(X_\tau) - f(X_0) = \int_0^\tau f'(X_s) dX_s + \frac{1}{2} \int_0^\tau f''(X_s) \sigma^2(X_s) ds \quad (1.4)$$

If we let $f(z) = |z - x|$ and take expectation of both sides, we can obtain

$$\mathbb{E}[|X_\tau - x| - |x_0 - x|] = \sigma(x)^2 \mathbb{E}\left[\int_0^\tau \delta_x(X_s) ds\right] dx, \quad (1.5)$$

where δ_x is the delta function at x .

Since $X_\tau \sim \mu$ where τ is exponentially distributed with parameter 1, we have

$$\mathbb{P}(X_\tau \in (x, x + dx)) = \mathbb{E}(\text{time in } (x, x + dx)) * (\text{Rate of stopping at } x), \quad (1.6)$$

where $\mathbb{P}(X_\tau \in (x, x + dx)) = \mu(dx)$ and $(\text{Rate of stopping at } x) = 1$.

As $\mathbb{E}(\text{time in } (x, x + dx)) = \mathbb{E}[\int_0^\tau \delta_x(X_s) ds] dx$, σ should be taken to solve

$$\sigma(x)^2 \mu(dx) = \mathbb{E}[|X_\tau - x| - |x - x_0|] dx \quad (1.7)$$

Using the definition of $u_\mu(x) := \int_{\mathbb{R}} |x - y| \mu(dy)$ which is equal to $\mathbb{E}(|X_\tau - x|)$, we can find the results stated in Theorem 1.

1.1.2 APPLICATION

Theorem 1 gives a practical method of calculating σ as long as the probability distribution μ is provided. In the real financial world, μ can be deduced from the set of call prices of same maturity, same underlying and different strikes. The explicit formula can be derived as followed:

In a world with zero interest rate (The justification of this assumption will be given in Remark 2.2.2),

$$C_0(S_0, t^*, K) = \int_0^\infty (s - K)^+ \mu(ds) \quad (1.8)$$

Second derivation w.r.t. K gives

$$\frac{\partial^2}{\partial K^2} C_0(S_0, t^*, K) dK = \mu(dK) \quad (1.9)$$

According to the definition of u_μ ,

$$u_\mu(K) = \int_0^\infty |K - y| \mu(dy) \quad (1.10)$$

$$= \int_0^K (K - y) \mu(dy) + \int_K^\infty (y - K) \mu(dy) \quad (1.11)$$

$$= \int_0^\infty (K - y) \mu(dy) - \int_K^\infty (K - y) \mu(dy) + \int_K^\infty (y - K) \mu(dy) \quad (1.12)$$

$$(1.13)$$

Since the riskless interest rate $r=0$, (S_t) is a martingale and

$$\mathbb{E}(S_{t^*}) = \int_0^\infty y \mu(dy) = S_0 \quad (1.14)$$

As a result,

$$u_\mu(K) = K - S_0 + 2C_0(S_0, t^*, K) \quad (1.15)$$

And

$$u_\mu(K) - |K - S_0| = 2(C_0(S_0, t^*, K) - (S - K)^+) \quad (1.16)$$

Hence σ should be given to solve

$$\frac{\sigma^2(K)}{2} = \frac{C_0(S_0, t^*, K) - (S_0 - K)^+}{\frac{\partial^2}{\partial K^2} C_0(S_0, t^*, K)} \quad (1.17)$$

1.2 PARALLEL RESULT IN P. CARR'S PAPER

In P. Carr's paper, the underlying stock price S is assumed to follow

$$S_t = X_{\Gamma_t}, t \geq 0, \quad (1.18)$$

where X is a driftless time-homogeneous diffusion generated by

$$dX_s = \sigma(X_s) dW_s, s \geq 0 \quad (1.19)$$

and (Γ_t) is a Γ -process with parameters t^* and θ .

By using forward-backward Kolmogorov equations with Laplace transform properties, it can be deduced that σ should satisfy

$$\frac{\sigma^2(K) \theta t^*}{2} = \frac{C_0(S_0, t^*, K) - (S_0 - K)^+}{\frac{\partial^2}{\partial K^2} C_0(S_0, t^*, K)} \quad (1.20)$$

Remark 1.2.1. Let's take a quick review of Γ -process. (see [5] for details)

A Γ -process with parameters $t^* > 0$ and $\theta > 0$ is an increasing, pure jump, Lévy process whose Lévy density is given by

$$k_\Gamma(t) = \frac{e^{-\frac{t}{\theta t^*}}}{t^* t}, t > 0 \quad (1.21)$$

with parameters $t^* > 0$ and $\theta > 0$.

The marginal distribution of a Γ -process at time $t \geq 0$ is a Γ -distribution with PDF

$$\mathbb{Q}\{\Gamma_t \in ds\} = \frac{1}{(\theta t^*)^{\frac{t}{t^*}} \Gamma(\frac{t}{t^*})} s^{\frac{t}{t^*}-1} e^{-\frac{s}{\theta t^*}}, s > 0, t > 0, \quad (1.22)$$

with parameters $\theta > 0$ and $t^* > 0$.

Remark 1.2.2. When $t = t^*$, the PDF of (Γ_t) becomes

$$\mathbb{Q}\{\Gamma_{t^*} \in ds\} = \frac{1}{\theta t^*} e^{-\frac{s}{\theta t^*}}, s > 0, t > 0. \quad (1.23)$$

Hence (Γ_{t^*}) is exponentially distributed with parameter θt^* . We can therefore notice that both of the two papers ([1] and [2]) solve the same problem. It can be verified from Equations (1.17) and (1.20) that they give consistent results. If we understand well how P. Carr deduces the formula of σ in the case of Γ -process, we will find the only thing we really need is to let τ be exponentially distributed. As a result, we can alleviate the assumption of Γ -process and try some continuous time changes where τ are exponentially distributed. These time changes are interesting since they preserve the continuity of stock paths. In the following sections of the thesis, $g: [0, t^*] \rightarrow [0, \infty)$ is an increasing function considered as the random time-change of the model, where $g(t^*) \sim \tau$ is distributed exponentially with parameter θt^* .

ESTIMATION OF σ AND PATH SIMULATION2.1 ESTIMATION OF σ

Instead of using real market data, we use Bachelier, Black Scholes and Heston models to generate market prices of European calls with the same maturity t^* , same underlying (S_t) and different strikes.

As is mentioned in the previous chapter, σ should be taken to solve

$$\frac{\sigma^2(K)\theta t^*}{2} = \frac{C_0(S_0, t^*, K) - (S_0 - K)^+}{\frac{\partial^2}{\partial K^2} C_0(S_0, t^*, K)} \quad (2.1)$$

Therefore, we can rely on prices given by Bachelier, Black Scholes and Heston models to deduce three types of σ profiles. (see Figures 2.1, 2.2 and 2.3)

In the next passage, we would like to understand analytically the asymptotic behaviour of $\sigma(K)$ when K tends to extreme values and compare them with calibrated results.

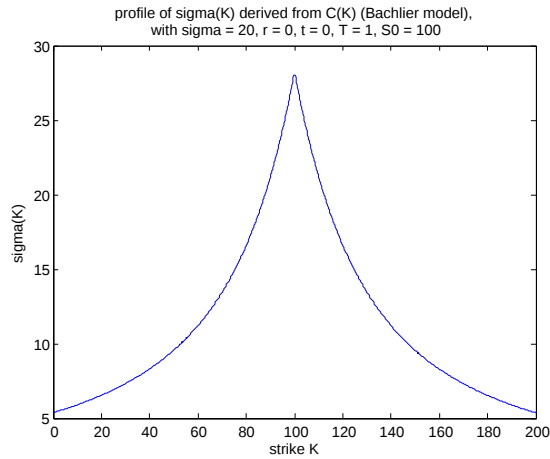
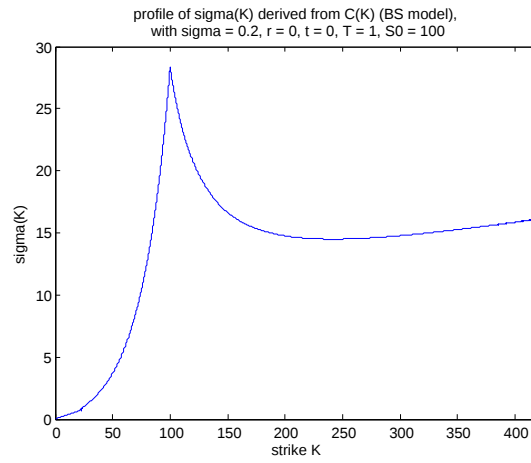
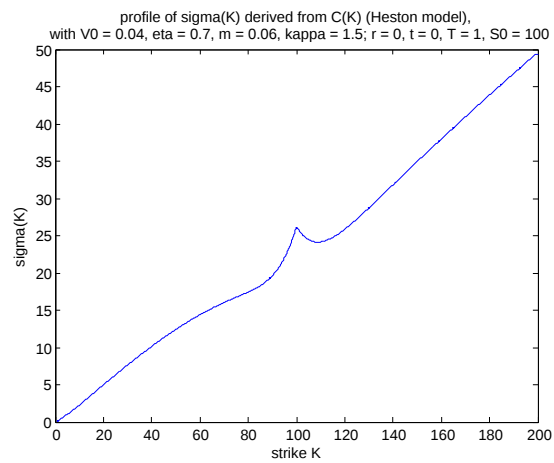


Figure 2.1: σ derived from call prices given by Bachelier Model

Figure 2.2: σ derived from call prices given by Black Scholes ModelFigure 2.3: σ derived from call prices given by Heston Model

2.2 ASYMPTOTIC BEHAVIOUR FOR EXTREME VALUES OF K

Equation (2.1) can be written as

$$\frac{\sigma^2(K)\theta t^*}{2} = \frac{C_0(S_0, t^*, K) - (S_0 - K)^+}{\frac{\partial^2}{\partial K^2} C_0(S_0, t^*, K)} \quad (2.2)$$

$$= \frac{\int_K^\infty (y - K)\mu(dy) - (S_0 - K)^+}{\rho(K)}, \quad (2.3)$$

where μ is the probability distribution of S_{t^*} and ρ is the probability density function (PDF) of S_{t^*} .

In order to get asymptotic behaviour for extreme values of K , we can use L'Hôpital's rule, according to which for extreme values of K , we have

$$\frac{\sigma^2(K)\theta t^*}{2} \sim -\frac{\int_K^\infty \mu(dy)}{\rho'(K)} \quad (2.4)$$

$$\sim \frac{\rho(K)}{\rho''(K)} \quad (2.5)$$

Remark 2.2.1. Depending on different tail behaviours for the PDF of S_{t^*} , the behaviours of $\sigma(K)$ for extreme values of K are different. For example,

a) If S_{t^*} is distributed exponentially with parameter 1, then $\frac{\sigma^2(K)\theta t^*}{2} \rightarrow 1$ for $K \rightarrow \pm\infty$. As a result, $\sigma(K)$ tends to a constant for $K \rightarrow \pm\infty$.

b) In the next three subsections, we will take a look at different asymptotic behaviours of $\sigma(K)$ respectively given by Bachelier, Black Scholes and Heston models.

Remark 2.2.2. As is stated in the previous chapter, in our thesis, we assume

$$S_t = X_{g(t)}, t \geq 0, \quad (2.6)$$

where (X_t) is a driftless time-homogeneous diffusion generated by

$$dX_s = \sigma(X_s)dW_s, s \geq 0 \quad (2.7)$$

and g is an independent time change.

Therefore, (S_t) inherits the martingale property from (X_t) .

In order to be consistent with this property, while using Bachelier, Black Scholes and Heston models to generate initial prices, we set the riskless interest rate to be zero, so that these models can be adapted to forward price of assets which also exhibits the martingale property. This remark justifies the use of zero interest rate in Section 1.1.2.

2.2.1 BACHELIER MODEL

Under Bachelier Model, the dynamics of S_t is

$$dS_t = \sigma dW_t \quad (2.8)$$

with initial stock value S_0 .

As a result,

$$\rho(K) = \frac{1}{\sigma\sqrt{2\pi t^*}} e^{-\frac{(K-S_0)^2}{2\sigma^2 t^*}} \quad (2.9)$$

and

$$\rho''(K) = \frac{1}{\sigma\sqrt{2\pi t^*}} \left(\frac{(K-S_0)^2}{\sigma^2 t^*} - 1 \right) e^{-\frac{(K-S_0)^2}{2\sigma^2 t^*}} \quad (2.10)$$

Hence, $\sigma(K) \sim C|S_0 - K|^{-1}$ (where C is a positive constant) for $K \rightarrow \pm\infty$

2.2.2 BLACK SCHOLES MODEL

Under Black Scholes Model, the dynamics of S_t is

$$dS_t = \sigma S_t dW_t \quad (2.11)$$

with initial stock value S_0 .

As a result,

$$\rho(K) = \frac{1}{\sigma K \sqrt{2\pi t^*}} e^{-\frac{(\log(\frac{K}{S_0}) + \frac{1}{2}\sigma^2 t^*)^2}{2\sigma^2 t^*}} \quad (2.12)$$

and

$$\rho''(K) = \frac{2((4a + 3b)\mathbf{log}(K) + 2\mathbf{log}^2(K) + 2a^2 + 3ab + b(b-1))e^{-\frac{(\log(\frac{K}{S_0}) + \frac{1}{2}\sigma^2 t^*)^2}{2\sigma^2 t^*}}}{\sigma\sqrt{2\pi t^*} b^2 K^3}, \quad (2.13)$$

where $a = \frac{1}{2}\sigma^2 t^* - \mathbf{log} S_0$ and $b = 2\sigma^2 t^*$.

Hence, for $\mathbf{log}(K) \rightarrow \pm\infty$,

$$\sigma(K) \sim C \frac{K^2}{\mathbf{log}^2(K)}, \quad (2.14)$$

with C a positive constant.

For $K \rightarrow \infty$, K grows faster than $\mathbf{log}(K)$. Hence

$$\sigma(K) \rightarrow \infty \quad (2.15)$$

For $K \rightarrow 0$, K tends to 0 and $\mathbf{log}(K)$ tends to $-\infty$. Hence

$$\sigma(K) \rightarrow 0 \quad (2.16)$$

2.2.3 HESTON MODEL

Under Heston Model, the dynamics of S_t is

$$\begin{cases} dS_t = \sqrt{V_t} S_t dW_t^1 \\ dV_t = -\kappa(V_t - m)dt + \eta\sqrt{V_t}dW_t^\rho \end{cases} \quad (2.17)$$

with initial stock value S_0 and initial variance V_0 . We also have $\langle W^1, W^\rho \rangle_t = \rho t$. The asymptotic behaviour of σ in Heston model is more complicated. [4] gives asymptotic behaviour for $\rho(K)$, but further study is needed to find asymptotic behaviour for $\rho''(K)$. Due to limit of time, we cannot offer an analytical result here.

Remark 2.2.3. Not surprisingly, we observe that, analytical asymptotic behaviours of $\sigma(K)$ for extreme values of K are consistent with the calibrated results. (see Figures 2.1, 2.2). In Figure 2.2, we can observe an increasing tendency for large K , which, according to analytical calculation, should tend to ∞ while K tends to ∞ . One might argue that we should take a bigger scale for K in order to visualize the tendency more clearly. However, for large K , $C_0(S_0, t^*, K) - (S_0 - K)^+ \rightarrow 0$ and $\frac{\partial^2}{\partial K^2} C_0(S_0, t^*, K) \rightarrow 0$, and a big numerical error occur when we divide the first one by the second one.

2.3 PATH SIMULATION

Once we get the profile of σ , we can therefore generate stock paths by using different time changes.

2.3.1 TIME CHANGES

In P. Carr's paper, (see [2]) he develops an exact and explicit calibration methodology under the assumption of Γ -time change. Unfortunately, since a Γ -process is a pure jump, we have to abandon the continuity of the stock price. If we go into more details of his proof, we will find that assumption of Γ -time change is not necessary and that we only need $g(t^*)$ to be exponentially distributed. This means we can preserve the continuity of stock paths by trying some continuous time change g which satisfies:

- $g: [0, t^*] \rightarrow [0, +\infty)$ is increasing with $g(0)=0$.
- $g(t^*)$ is exponentially distributed with parameter θt^* .

Remark 2.3.1. In [1], we consider the cases where τ is exponentially distributed with parameter 1. But in order to be consistent with [2], we will also look at the cases where the parameter in the exponential distribution is different from 1, which means $\theta t^* \neq 1$.

Dynamics of stock paths are different under different time changes. In the following passage, we will try three types of time changes

- $g(t) = \Gamma_t(t^*, \theta)$ which is an independent Γ -process defined in Remark 1.2.1. (see Figure 2.4)

We try three values of θ : $\theta = 1$, $\theta = 1000$ and $\theta = 0.001$.

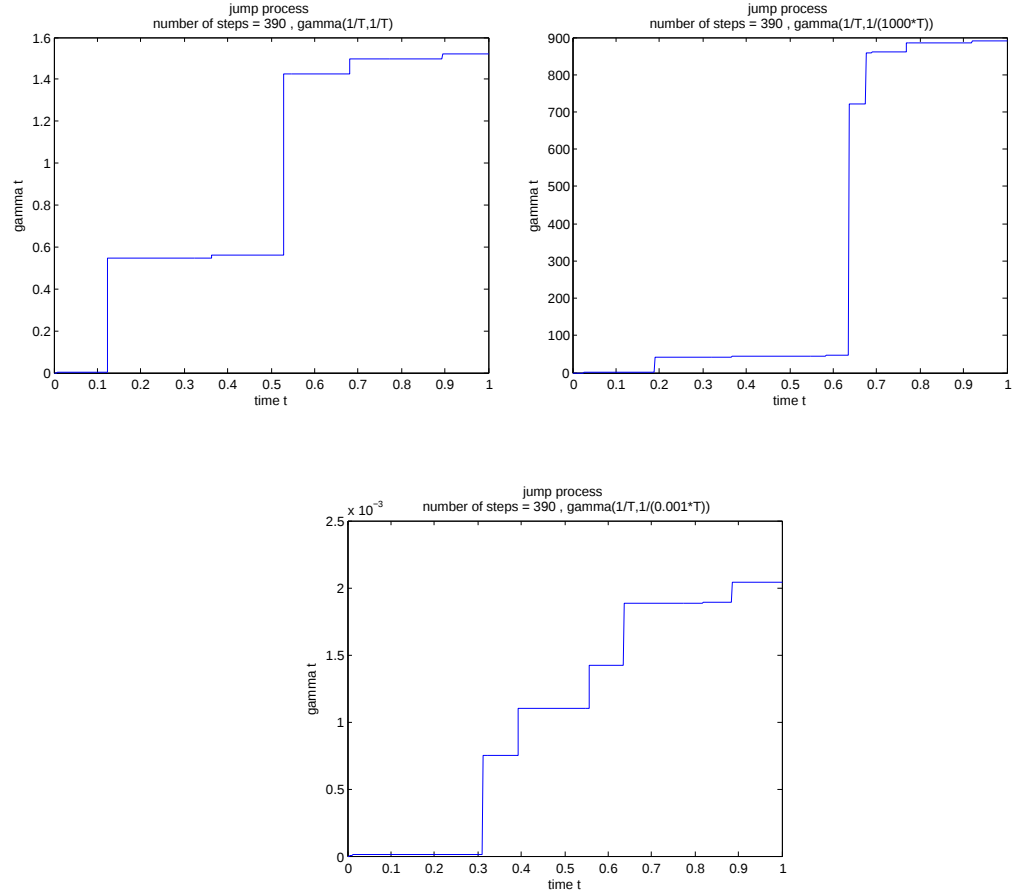


Figure 2.4: Simulated trajectories of Γ -processes with $t^* = 1$; $\theta = 1$ (left), $\theta = 1000$ (right) and $\theta = 0.001$ (bottom)

Remark 2.3.2. We can observe from Figure 2.4 that $\Gamma_t(t^*, \theta)$ behaves differently with different θ . In fact, θ controls the mean jump size and the smaller θ is, the longer it takes to reach a fixed positive value. However, if we take a look at Figure 2.7, we will find the differences are not so obvious in the generated stock paths. This is because we also need to take into consideration different time scales and the fact that the diffusion processes will be different under models with different parameters.

For instance, if the time change process encounters a big jump, σ might fall into a small value, the entire variance $\sigma^2 \Delta g$ might be close to the case where time change process doesn't move a lot and σ remains a big value.

- $g(t) = \min(\tau, \frac{t}{t^* - t})$, (see Figures 2.5) where τ is an exponentially distributed random variable with PDF

$$\mathbb{Q}\{\tau \in ds\} = \frac{e^{-\frac{s}{\theta t^*}}}{\theta t^*} \mathbf{1}_{s>0}. \quad (2.18)$$

It is legitimate to use the letter τ since for $t=t^*$, $\frac{t}{t^* - t} = \infty$ and we have $g(t^*) \sim \tau$.

We try two values of θ : $\theta = 1$ and $\theta = 1000$.

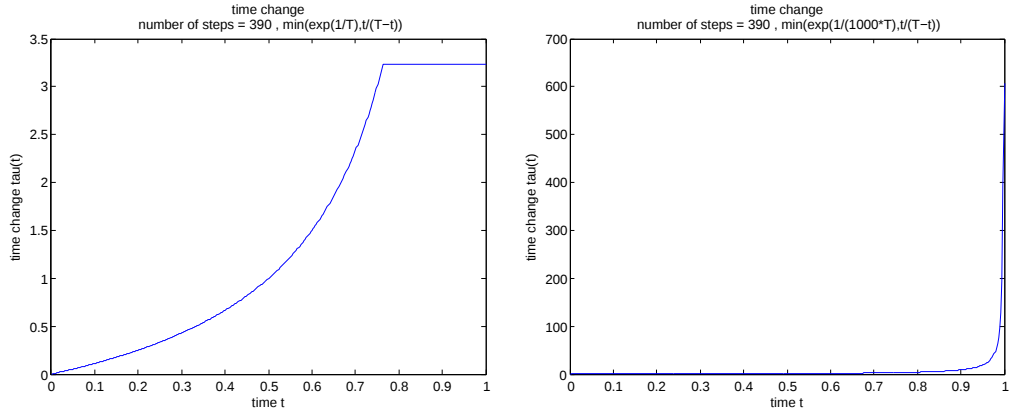


Figure 2.5: Simulated trajectories of $\min(\tau, \frac{t}{t^* - t})$ with $t^* = 1$; $\theta = 1$ (left), $\theta = 1000$ (right)

- $g(t) = \min(\tau, e^{-\frac{0.5}{t} + \frac{0.1}{t^* - t}})$ with τ defined as previously. (see Figures 2.6)

It is legitimate to use the letter τ since for $t = t^*$, $e^{-\frac{0.5}{t} + \frac{0.1}{t^* - t}} = \infty$ and we have $g(t^*) \sim \tau$.

We try two values of θ : $\theta = 1$ and $\theta = 1000$.

2.3.2 SIMULATION OF STOCK PATHS

Now we are ready to simulate $(S_{t_1}, \dots, S_{t_n})$ for n fixed times $t_1, \dots, t_n$ where (S_t) satisfies

$$S_t = X_{g(t)}, \quad (2.19)$$

$$dX_t = \sigma(X_t) dW_t \quad (2.20)$$

To achieve this, we use the following algorithm

- Simulate increments of the subordinator: $\Delta g_i = g(t_i) - g(t_{i-1})$ where $g(t_0) = 0$.

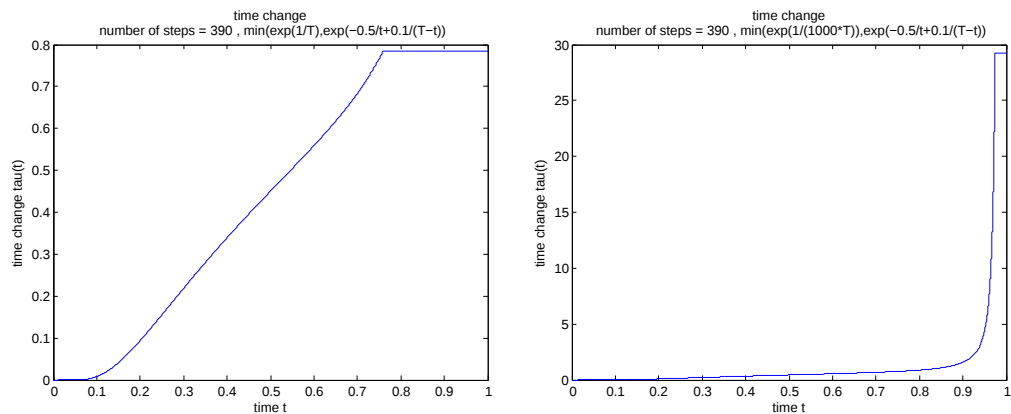


Figure 2.6: Simulated trajectories of $\min(\tau, e^{-\frac{0.5}{t} + \frac{0.1}{t^* - t}})$ with $t^* = 1$; $\theta = 1$ (left), $\theta = 1000$ (right)

- Simulate n independent standard normal random variables $N_1, \dots, N_n$. Set $\Delta S_i = \sigma(S_{i-1})\sqrt{\Delta g_i}N_i$.

Then the discretized trajectory is given by $S_{t_i} = \sum_{k=1}^i \Delta S_{t_k}$.

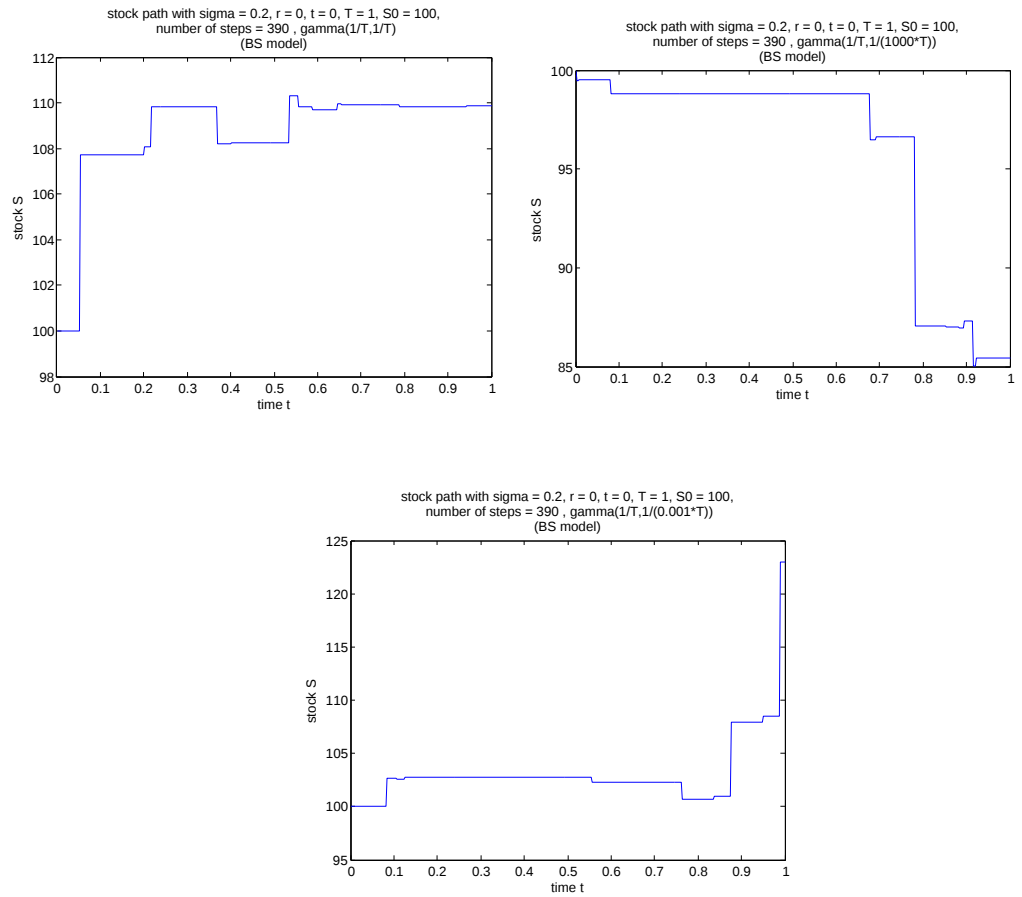


Figure 2.7: Simulated trajectories of stock process (time change: (Γ_t)) with $t^* = 1$; $\theta = 1$ (left), $\theta = 1000$ (right) and $\theta = 0.001$ (bottom)

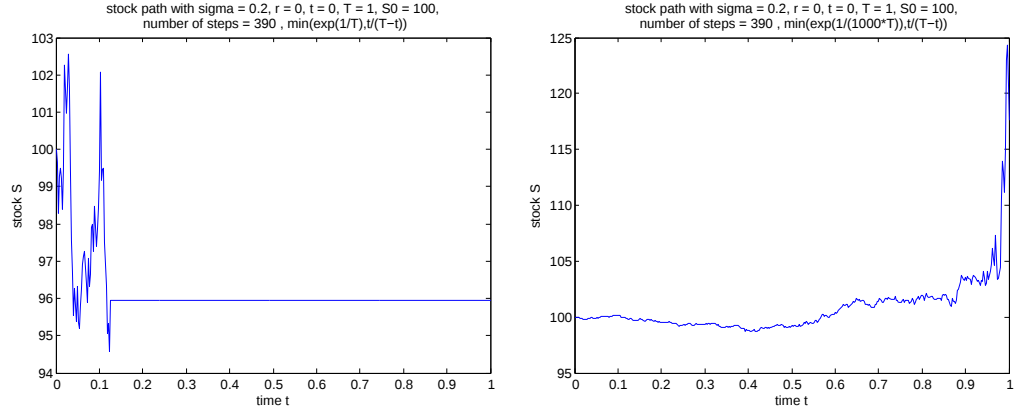


Figure 2.8: Simulated trajectories of stock process (time change: $\min(\tau, \frac{t}{t^* - t})$) with $t^* = 1$; $\theta = 1$ (left) and $\theta = 1000$ (right)

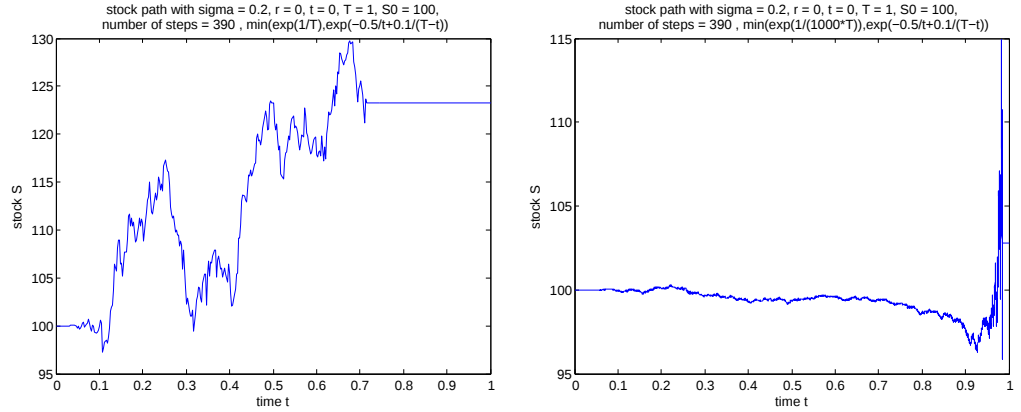


Figure 2.9: Simulated trajectories of stock process (time change: $\min(\tau, e^{-\frac{0.5}{t} + \frac{0.1}{t^* - t}})$) with $t^* = 1$; $\theta = 1$ (left) and $\theta = 1000$ (right)

2.4 VERIFICATION AND IMPROVEMENT

2.4.1 VERIFICATION

The generated stock paths can be used to get histograms of S_{t^*} as well as the entire set of call prices of the same maturity and underlying. Theoretically, these results should be respectively identical with the PDF of S_{t^*} and call prices given by the initial models.

Initially the number of time steps is set to be 390. (see Figures from 2.10 to 2.15) The

figures on the left present numerical histograms of S_{t^*} via simulation (blue bar) as well as PDF of S_{t^*} given by the initial models (Bachelier, BS or Heston) (red line). The figures on the right present simulated call prices (blue line) as well as prices given by the initial models (Bachelier, BS or Heston) (green cross). Here we present results given five by BS model (see Figures from 2.10 to 2.14) and one by Heston model (see Figure 2.15).

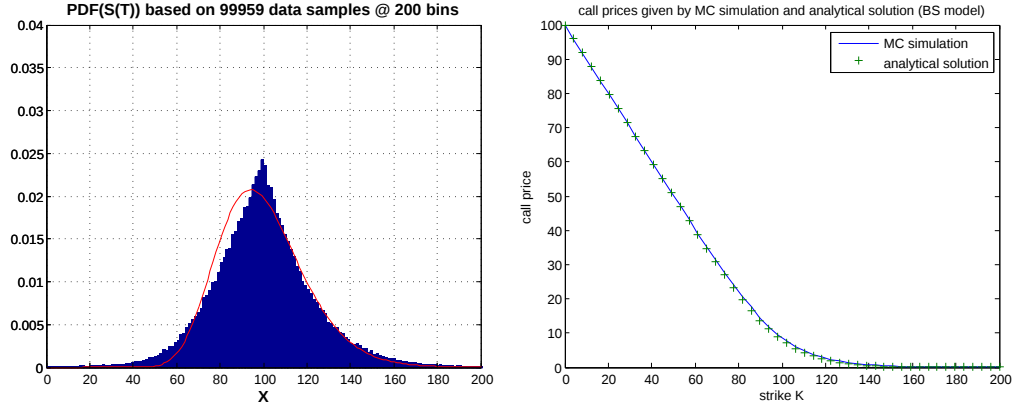


Figure 2.10: Simulated histogram of $S(t^*)$ (left) and call price (right). Initial data given by BS model with $\sigma=0.2$; $r = 0$; $t^*=1$; $S_0=100$; $\theta = 1$. Time change $g(t)=\Gamma_t(t^*, \theta)$

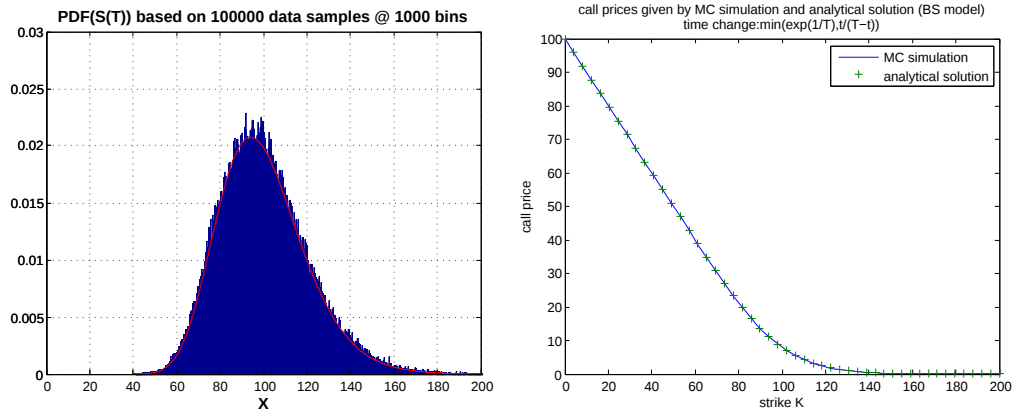


Figure 2.11: Simulated histogram of $S(t^*)$ (left) and call price (right). Initial data given by BS model with $\sigma=0.2$; $r = 0$; $t^*=1$; $S_0=100$; $\theta = 1$. Time change $g(t)=\min(\tau, \frac{t}{t^*-t})$ where τ is exponentially distributed with parameter 1.

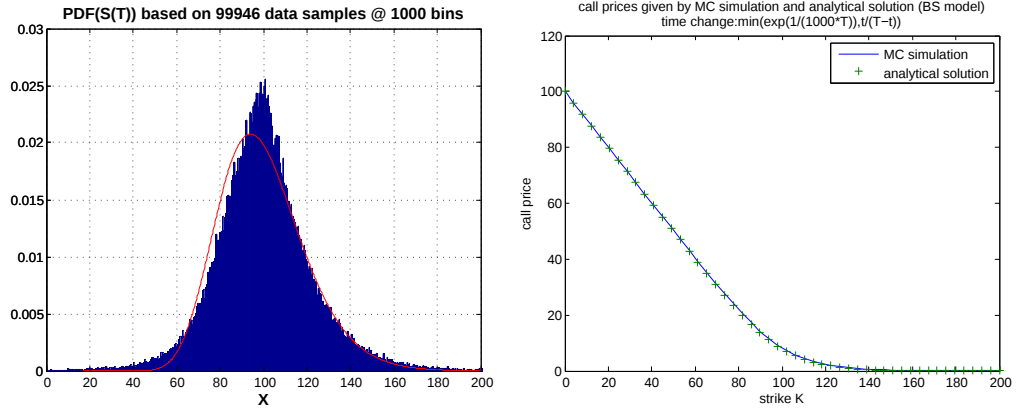


Figure 2.12: Simulated histogram of $S(t^*)$ (left) and call price (right). Initial data given by BS model with $\sigma=0.2$; $r = 0$; $t^*=1$; $S_0=100$; $\theta = 1000$. Time change $g(t)=\min(\tau, \frac{t}{t^*-t})$ where τ is exponentially distributed with parameter 1000.

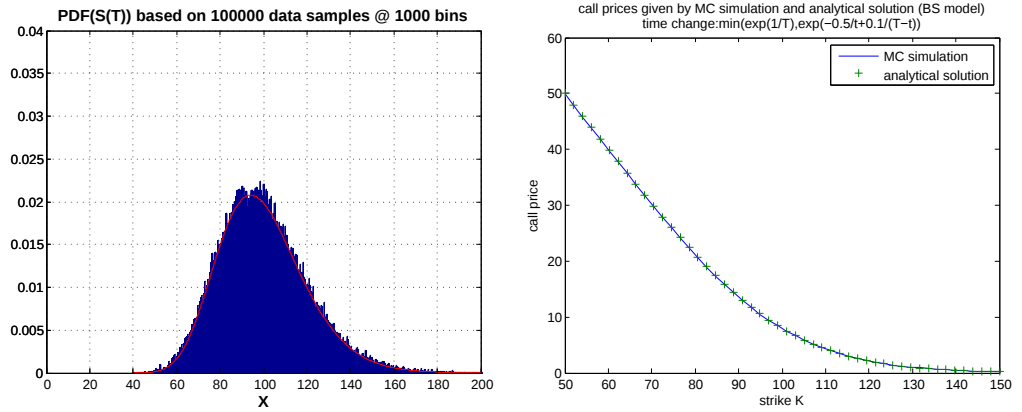


Figure 2.13: Simulated histogram of $S(t^*)$ (left) and call price (right). Initial data given by BS model with $\sigma=0.2$; $r = 0$; $t^*=1$; $S_0=100$; $\theta = 1$. Time change $g(t)=\min(\tau, -\frac{0.5}{t} + \frac{0.1}{T-t})$ where τ is exponentially distributed with parameter 1.

2.4.2 OBSERVATIONS AND IMPROVEMENTS

In Figures 2.11 and 2.13, simulated histograms of S_{t^*} are almost consistent with PDFs of S_{t^*} given by the initial models. In the rest of cases, a slight inconsistency is observed. Errors can be explained by discretization of time steps. In the case of Γ -time change, if g has a big jump at time t_0 , a considerable error will occur if we approximate the change of the process at t_0 by $X_{g(t_0^+)} - X_{g(t_0^-)} = \sigma(X_{g(t_0^-)})(W_{g(t_0^+)} - W_{g(t_0^-)})$ since σ changes significantly

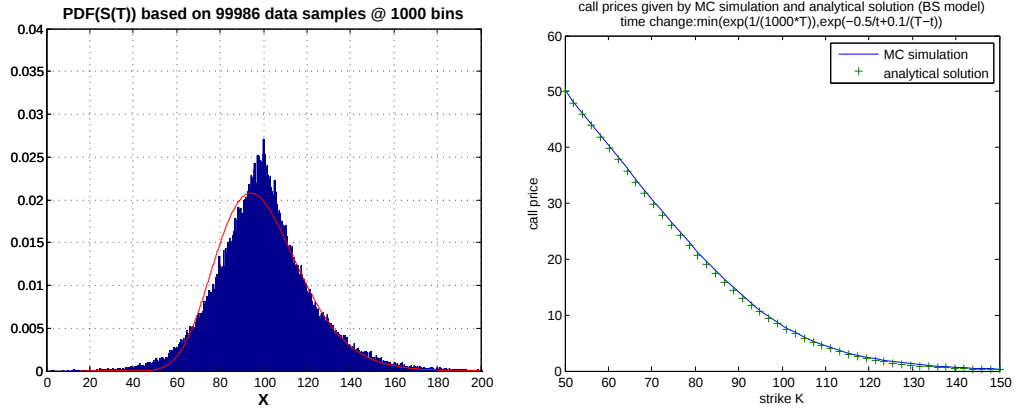


Figure 2.14: Simulated histogram of $S(t^*)$ (left) and call price (right). Initial data given by BS model with $\sigma=0.2$; $r = 0$; $t^*=1$; $S_0=100$; $\theta = 1000$. Time change $g(t)=\min(\tau, -\frac{0.5}{t} + \frac{0.1}{T-t})$ where τ is exponentially distributed with parameter 1000.

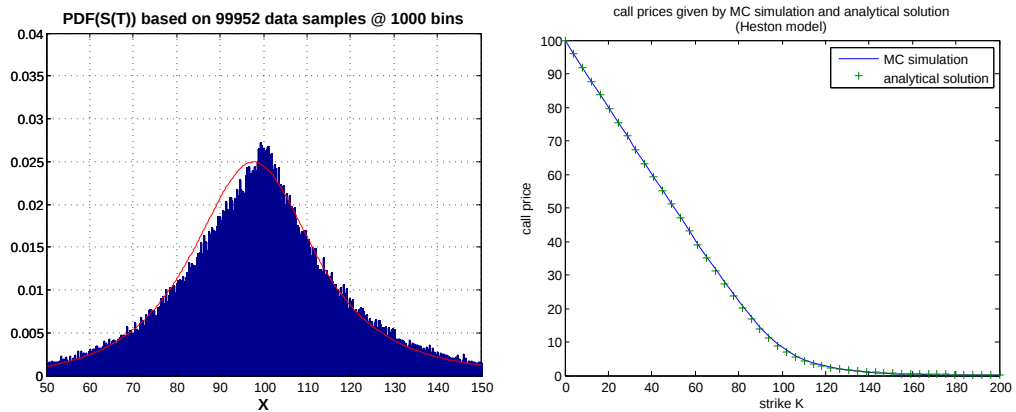


Figure 2.15: Simulated histogram of $S(t^*)$ (left) and call price (right). Initial data given by Heston model with $r = 0$; $t^*=1$; $S_0=100$; $V_0 = 0.04$; $\eta = 0.7$; mean = 0.06; $\kappa = 1.5$; $\theta = 1$. Time change $g(t)=\Gamma_t(t^*, \theta)$.

during $[X_{g(t-)}, X_{g(t+)}]$. In the case of continuous time changes, this error can be reduced by taking more time steps. The required number of time steps varies with θ , since the time scale changes with θ and to maintain the same accuracy of Δt we need to take more steps. For $\theta = 1$, 390 time steps are enough to give a satisfactory result. When it comes to $\theta = 1000$, time horizon becomes longer and we need to take more steps to ensure the same accuracy. For instance, 39000 time steps give a satisfactory approximation where the outline of the simulated histograms of S_{t^*} become almost coincide with PDFs of S_{t^*} given

by the initial models. (see Figures 2.16 and 2.17) Therefore, in the rest of the thesis, except special mentioning, we will use 39000 time steps to generate stock paths.

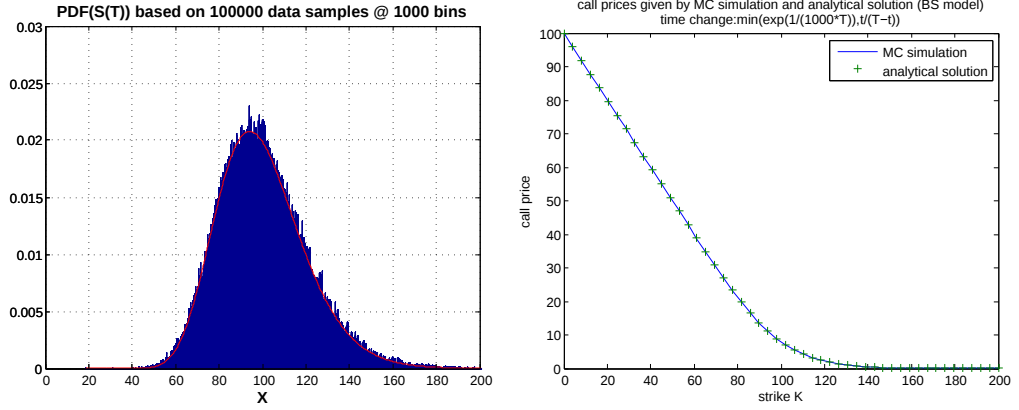


Figure 2.16: Improvement: Simulated histogram of $S(t^*)$ (left) and call price (right). Initial data given by BS model with $\sigma=0.2$; $r = 0$; $t^*=1$; $S_0=100$; $\theta = 1000$. Time change $g(t)=\min(\tau, \frac{t}{t^*-t})$ where τ is exponentially distributed with parameter 1000.

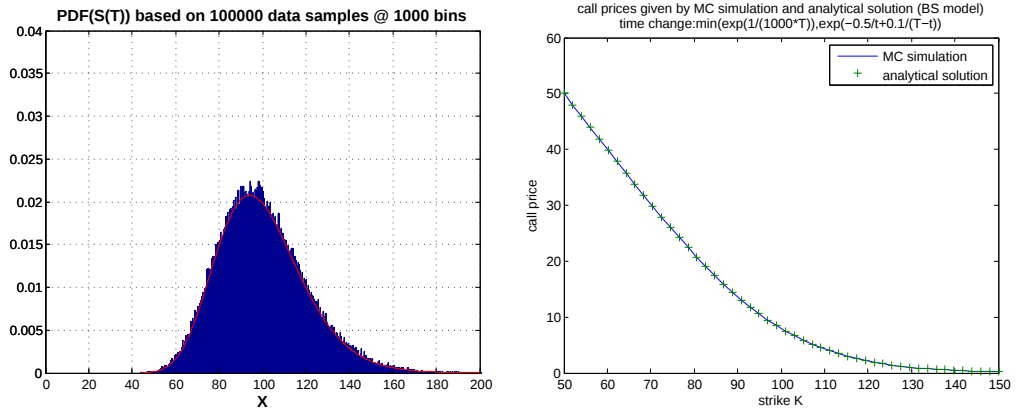


Figure 2.17: Improvement: Simulated histogram of $S(t^*)$ (left) and call price (right). Initial data given by BS model with $\sigma=0.2$; $r = 0$; $t^*=1$; $S_0=100$; $\theta = 1000$. Time change $g(t)=\min(\tau, -\frac{0.5}{t} + \frac{0.1}{T-t})$ where τ is exponentially distributed with parameter 1000.

CHAPTER 3

PRICES OF EUROPEAN CALL OPTIONS WITH MATURITY $T_1 < t^*$

In the previous chapter, we calibrate the subordinated models to market data so that, at estimation time $t=0$, simulated European call prices with maturity t^* are consistent with call prices given by the initial models (Bachelier, Black Scholes or Heston). However, since the dynamics of the stock process behaves differently under different time changes, prices of European call with maturity $0 < T_1 < t^*$ should be different for different models. Hence the objective of this chapter is to analyse these options and associate price differences with intrinsic model differences.

3.1 IMPLEMENTATION

We use the seven time changes defined in Section 2.3.1 to price European calls and numerical results are summarized in Figures 3.1 and 3.2. Initial data are respectively generated by Black Scholes model (see Figure 3.1) and Heston model (see Figure 3.2). On the left side, we present intermediate European call prices generated by the seven time changes mentioned above. The figures on the right side are just extracts of the left ones, corresponding to cases with continuous time changes. We would like to do so in order to give a clearer presentation for prices with continuous time changes, with which we will give a more detailed analysis in the following section.

For each case, we give results corresponding to three time-to-maturity: $T-t=2$ weeks; $T-t=1$ month; $T-t=6$ months.

3.2 ANALYSIS AND INTERPRETATION

It is well known that, all other parameters being equal, a higher accumulated volatility will lead to higher prices of European options. Since all subordinated models discussed here have the same accumulated volatility for the period $[0, t^*]$, we can conclude that they will give the same prices for European call prices with maturity t^* . However, due to very

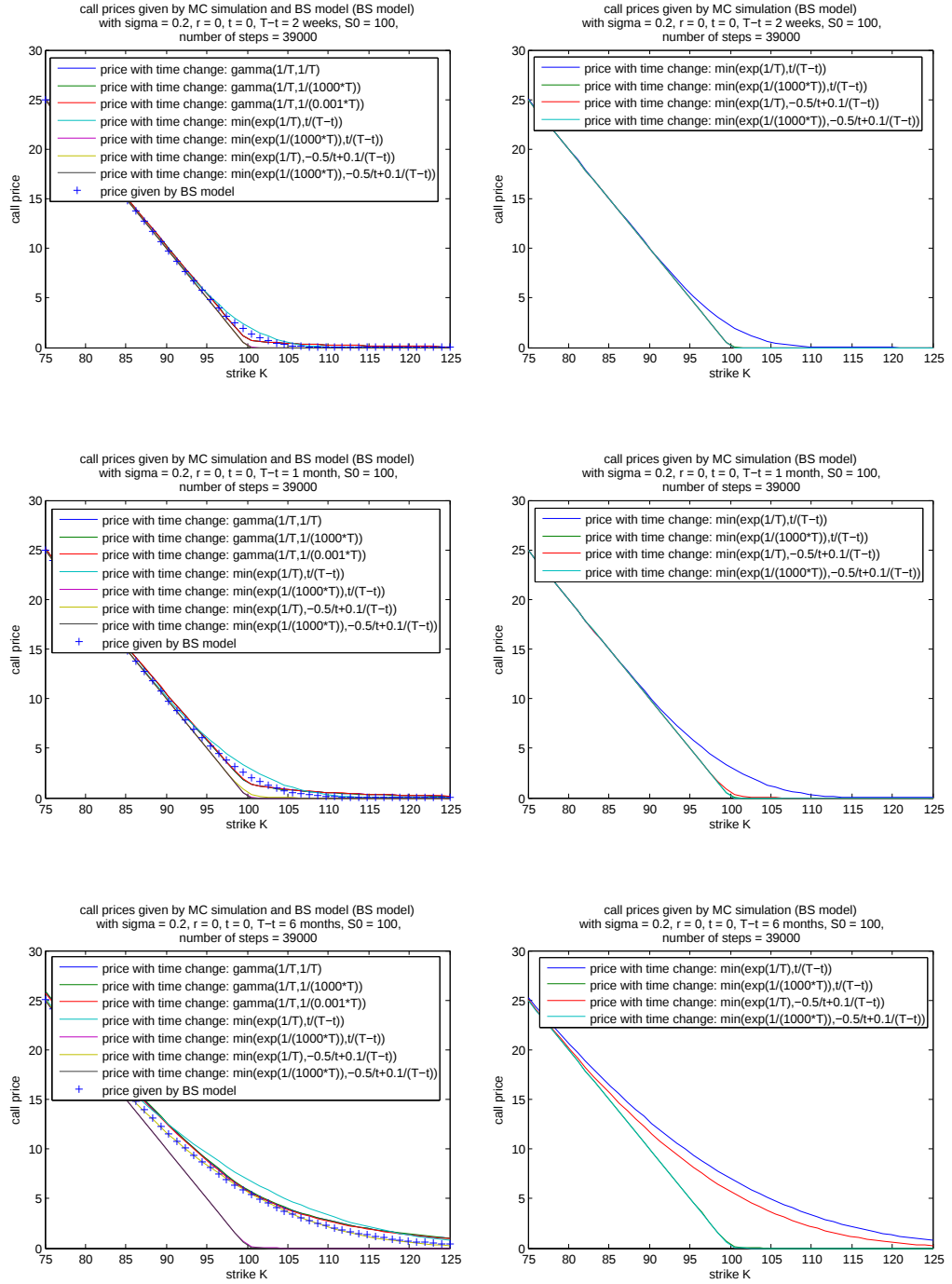


Figure 3.1: Intermediate European call prices (Initial data given by Black Scholes model) with $\sigma=0.2$, $r=0$, $S_0=100$, $T-t=2$ weeks(top), $T-t=1$ month(middle) or $T-t=6$ months(bottom). Left side: call prices with seven different time changes. Right side: an extract of the left side where only cases with continuous time changes are presented.

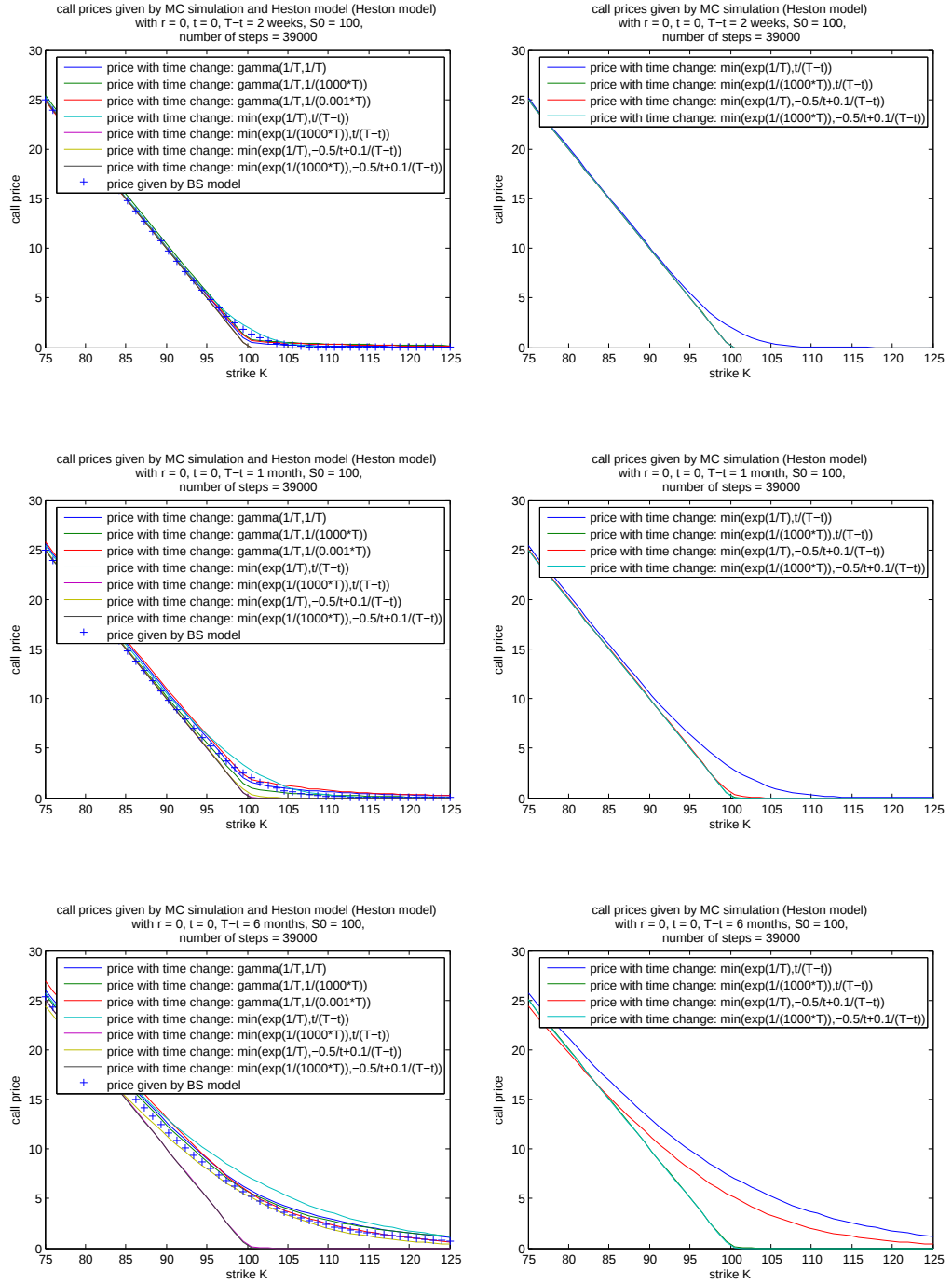


Figure 3.2: Intermediate European call prices (Initial data given by Heston model) with $V_0=0.04$, $\eta=0.7$, $m=0.06$, $\kappa=1.5$, $r=0$, $S_0=100$, $T-t=2$ weeks(top), $T-t=1$ month(middle) or $T-t=6$ months(bottom). Left side: call prices with seven different time changes. Right side: an extract of the left side where only cases with continuous time changes are presented.

different intermediate dynamics of S_t , the call prices with maturity $T_1 < t^*$ behave will behave differently.

From Figures 3.1 and 3.2, we can observe that, with the time change $g(t)=\min(\tau, \frac{t}{t^*-t})$, prices with $\theta = 1$ (deep blue line) are always higher than prices with $\theta = 1000$ (green line). Similarly, with the time change $g(t)=\min(\tau, e^{-\frac{0.5}{t}+\frac{0.1}{t^*-t}})$, prices with $\theta = 1$ (red line) are always higher than prices with $\theta = 1000$ (light blue line).

Some explanations can be given if we take a glance at simulated stock paths in the cases of continuous time changes ($g(t)=\min(\tau, \frac{t}{t^*-t})$) (see Figure 2.8) and $g(t)=\min(\tau, e^{-\frac{0.5}{t}+\frac{0.1}{t^*-t}})$ (see Figure 2.9). When $\theta = 1$, most of the volatility is accumulated at the early stage of $[0, t^*]$; when it comes to $\theta = 1000$, the majority is accumulated at the late stage. For a European option with maturity t^* , it doesn't make a difference since we will consider the accumulated volatility over the period $[0, t^*]$ as a whole. However, for an option with maturity $T_1 < t^*$, only the volatility accumulated at an early stage will be contributed to the pricing. That is why for European options with maturity $T_1 < t^*$ in the case of continuous time changes, models with $\theta = 1$ always give higher prices than models with $\theta = 1000$.

EXOTIC OPTION PRICING

In this chapter, relied on the stock paths generated previously, we are going to price one-touch options, barrier options and variance swaps.

4.1 ONE-TOUCH OPTIONS

In our thesis, we use the one-touch options with final payoff $\mathbf{1}_{\min S_t \geq B}$. It is well known that, under the framework of Black Scholes model, the price of a one-touch call option is (see [3] for details)

$$C_0^{one-touch,BS}(S_0, t^*, B) = C_0^{digital,BS}(S_0, t^*, B) - \left(\frac{S_0}{B}\right)^{1-\frac{2r}{\sigma^2}} C_0^{digital,BS}\left(\frac{B^2}{S_0}, t^*, B\right) \quad (4.1)$$

Numerically, in order to get simulated price of one-touch options, we treat the drift and volatility as being approximately constant within each time step $[g(t_i), g(t_{i+1})]$, the probability of having crossed the barrier within time-step i is

$$P_i = e^{-\frac{2(\hat{S}_{t_i} - B)^+(\hat{S}_{t_{i+1}} - B)^+}{\sigma(\hat{S}_{t_i})^2(g(t_{i+1}) - g(t_i))}} \quad (4.2)$$

Probability at end of not having crossed barrier is $\prod_{i=1}^M (1 - P_i)$ and so the payoff is

$$f = e^{-rT} \prod_{i=1}^M (1 - P_i) \quad (4.3)$$

Numerical results are summarized in Table 4.1 (Black Scholes model) and Table 4.2 (Heston model). We present prices given by the initial models, simulated prices as well as differences between the two.

Remark 4.1.1. It can be observed in Tables 4.1 and 4.2 that, for each case, models with different continuous time changes give very close prices for one-touch options. In fact, it can be proved analytically that, given the same dynamics of X and probability distribution of $g(t^*)$, all continuous time changes g give the same prices for one-touch options.

Proof. The price of the one-touch option is given by

$$C_0^{one-touch}(S_0, t^*, B) = \mathbb{Q}(M_{t^*} > B), \quad (4.4)$$

where $M_t = \min_{0 \leq u \leq t} S_u$.

Since $S_t = X_{g(t)}$, where g is a continuous time change, we have

$$M_{t^*} = M_{g(t^*)}^X, \quad (4.5)$$

where $M_t^X = \min_{0 \leq u \leq t} X_u$.

As a result, we have

$$C_0^{one-touch}(S_0, t^*, B) = \mathbb{Q}(M_{g(t^*)}^X > B), \quad (4.6)$$

which only depend on the dynamics of X and the probability distribution of $g(t^*)$. $\square$

It should be noticed that these results can only be applied to continuous time changes, without which we do not have Equation (4.5).

Table 4.1: One-touch options with $\sigma=0.2$; $r=0$; $T-t=1$; $B=90$; $S_0=100$; initial data given by Black Scholes model; number of time steps=39000; number of simulated paths=100000

time change	Black Scholes price	simulation price	difference
$\Gamma_t(1, 1)$	0.3626	0.4237	0.0611
$\Gamma_t(1, 1000)$	0.3626	0.4243	0.0617
$\Gamma_t(1, 0.001)$	0.3626	0.4242	0.0616
$\min(\exp(1), \frac{t}{T-t})$	0.3626	0.4397	0.078
$\min(\exp(1000), \frac{t}{T-t})$	0.3626	0.4406	0.071
$\min(\exp(1), -\frac{0.5}{t} + \frac{0.1}{T-t})$	0.3626	0.4408	0.0782
$\min(\exp(1), -\frac{0.5}{t} + \frac{0.1}{T-t})$	0.3626	0.4367	0.0741

4.2 BARRIER OPTIONS

In our thesis, we use the down-and-out call options with final payoff $(S_T - K)^+ \mathbf{1}_{\min S_t \geq B}$. It is well known that, under the framework of Black Scholes model, the price of a down-and-out call option is (see [3] for details)

$$C_0^{DO,BS}(S_0, t^*, K, B) = C_0^{European,BS}(S_0, t^*, K) - \left(\frac{S_0}{B}\right)^{1-\frac{2r}{\sigma^2}} C_0^{European,BS}\left(\frac{B^2}{S_0}, t^*, K\right) \quad (4.7)$$

In order to get simulated price of barrier calls, we use the same algorithm as the one for one-touch options, but with the final payoff

$$f = e^{-rT} (\hat{S}_{t^*} - K)^+ \prod_{i=1}^M (1 - P_i) \quad (4.8)$$

Numerical results are summarized in Figure 4.1 (Black Scholes model with Γ - time change) and Figure 4.2 (Heston model with time change $\min(\exp, \frac{t}{t^*-t})$). Due to computational complexity, here we set time steps to be 390 instead of 39000.

Remark 4.2.1. In Figures 4.1 and 4.2, we present down-and-out call prices given by MC simulation $C^{DO,MC}$ and initial models $C^{DO,IM}$ (left) as well as $C^{DO,MC} - C^{DO,IM}$ (right). For $K \rightarrow +\infty$, both simulated and initial model prices tend to zero and so will $C^{DO,MC} - C^{DO,IM}$. For $K < B$, an affine relationship can be observed between $C^{DO,MC} - C^{DO,IM}$ and K . This can be justified analytically.

Proof. For $K < B$,

$$(S_{t^*} - K)^+ \mathbf{1}_{\min S_t \geq B} = (S_{t^*} - K) \mathbf{1}_{\min S_t \geq B} \quad (4.9)$$

Table 4.2: One-touch options with $r = 0$; $T-t=1$; $S_0=100$; $V_0 = 0.04$; $\eta = 0.7$; $\text{mean} = 0.06$; $\kappa = 1.5$; $\theta = 1$; initial data given by Heston model; number of time steps=39000; number of simulated paths=100000

time change	Heston price	simulation price	difference
$\Gamma_t(1, 1)$	0.3695	0.4355	0.066
$\Gamma_t(1, 1000)$	0.3695	0.4298	0.0603
$\Gamma_t(1, 0.001)$	0.3695	0.4328	0.0633
$\min(\exp(1), \frac{t}{T-t})$	0.3695	0.4407	0.0712
$\min(\exp(1000), \frac{t}{T-t})$	0.3695	0.4369	0.0674
$\min(\exp(1), -\frac{0.5}{t} + \frac{0.1}{T-t})$	0.3626	0.4414	0.0719
$\min(\exp(1), -\frac{0.5}{t} + \frac{0.1}{T-t})$	0.3626	0.4394	0.0699

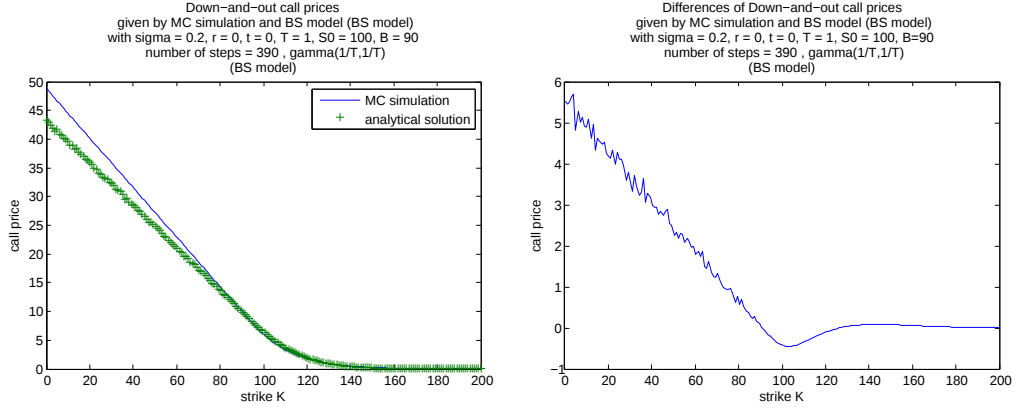


Figure 4.1: Left: Down-and-out call prices given by MC simulation and BS model; Right: Difference of down-and-out call prices given by MC simulation and down-and-out call prices given by BS. Initial data given by BS model with $\sigma=0.2$; $r=0$; $t^*=1$; $S_0=100$; $\theta=1$. Time change $g(t)=\Gamma_t(t^*, \theta)$

As a result,

$$C_0^{DO,IM}(S_0, t^*, K_1, B) - C_0^{DO,IM}(S_0, t^*, K_2, B) = \mathbb{E}^{IM}((K_2 - K_1)\mathbf{1}_{\min S_t \geq B}) \quad (4.10)$$

and

$$C_0^{DO,MC}(S_0, t^*, K_1, B) - C_0^{DO,MC}(S_0, t^*, K_2, B) = \mathbb{E}^{MC}((K_2 - K_1)\mathbf{1}_{\min S_t \geq B}) \quad (4.11)$$

Using the definition of one-touch options combined with Equation (4.10)-Equation (4.11), we can give

$$D_1 - D_2 = (K_1 - K_2)(C_0^{one-touch,IM}(S_0, t^*, B) - C_0^{one-touch,MC}(S_0, t^*, B)), \quad (4.12)$$

where

$$D_1 = C_0^{DO,MC}(S_0, t^*, K_1, B) - C_0^{DO,IM}(S_0, t^*, K_1, B), \quad (4.13)$$

$$D_2 = C_0^{DO,MC}(S_0, t^*, K_2, B) - C_0^{DO,IM}(S_0, t^*, K_2, B) \quad (4.14)$$

□

Hence, for $K < B$, there is an affine relationship between $C^{DO,MC} - C^{DO,IM}$ and K , with slope $C^{one-touch,IM} - C^{one-touch,MC}$. This can be verified numerically. Take the example where initial data is given by BS model with $\sigma = 0.2$, $r=0$, $T-t=1$, $S_0=100$. The time change is $\Gamma_t(1,1)$. The barrier B is 90. As is shown in the right figure of Figure 4.1, the slope of the difference = -0.6, which is consistent with $C^{one-touch,BS} - C^{one-touch,MC} =$

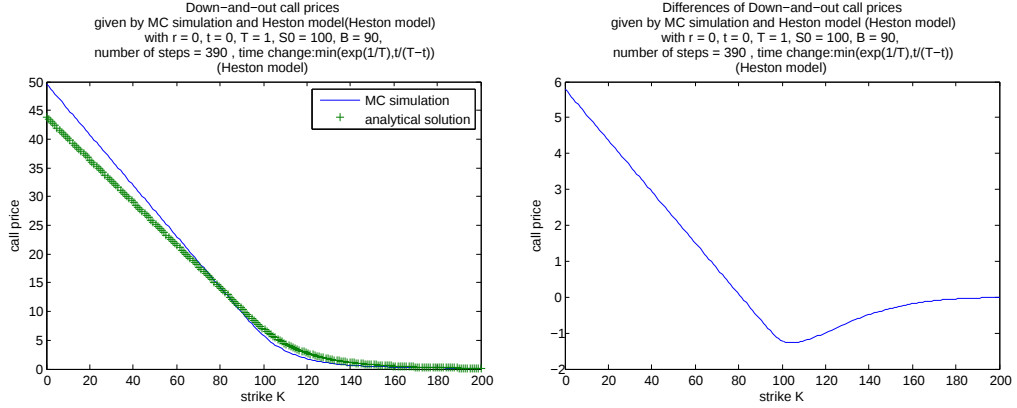


Figure 4.2: Left: Down-and-out call prices given by MC simulation and Heston model; Right: Difference of down-and-out call prices given by MC simulation and down-and-out call prices given by Heston model. Initial data given by Heston model with $r = 0$; $t^* = 1$; $S_0 = 100$; $V_0 = 0.04$; $\eta = 0.7$; mean = 0.06; $\kappa = 1.5$; $\theta = 1$. Time change $g(t) = \min(\exp(\theta t^*), \frac{t}{T-t})$

0.0611 shown in Table 4.1. Similar results can be observed in the example where initial data is given by Heston model with $r = 0$, $T-t=1$, $S_0=100$, $V_0 = 0.04$, $\eta = 0.7$, mean = 0.06, $\kappa = 1.5$, $\theta = 1$ where time change is $g(t) = \min(\exp(\theta t^*), \frac{t}{T-t})$ (see Figure 4.2 and Table 4.2). We can find the the slop of the difference $= -0.7$ (see Figure 4.2), which is consistent with $C^{one-touch, Heston} - C^{one-touch, MC} = 0.0712$ shown in Table 4.2.

4.3 VARIANCE SWAPS

Let us denote $Y_t = \log(\frac{S_t}{S_0})$ and $[Y]_t$ the quadratic variation of (Y_t) . A variance swap is a contract with payoff $[Y]_{t^*} - K_{var}$ in which the constant K_{var} is chosen so that there is no cash flow when the contract is initiated. Therefore the fair value of K_{var} should be $\mathbb{E}^{\mathbb{Q}}([Y]_{t^*})$. For instance, under Black Scholes Model,

$$Y_t = \log\left(\frac{S_t}{S_0}\right) \quad (4.15)$$

$$= \left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t \quad (4.16)$$

So we have

$$[Y]_t = \sigma^2 t \quad (4.17)$$

And the price given by BS model should be

$$K_{var} = [Y]_T \quad (4.18)$$

$$= \sigma^2 T \quad (4.19)$$

The objective of our thesis is to determine K_{var} under models with different time changes.

Theoretically, the value of K_{var} can be given by

$$K_{var} = \mathbb{E}^{\mathbb{Q}}\left(\lim_{N \rightarrow \infty} \sum_{i=1}^{N-1} \log^2\left(\frac{S_{t_{i+1}}}{S_{t_i}}\right)\right) \quad (4.20)$$

This motivates us to use its numerical approximation given by

$$K_{var} \approx \mathbb{E}^{\mathbb{Q}}\left(\sum_{i=1}^{N-1} \log^2\left(\frac{S_{t_{i+1}}}{S_{t_i}}\right)\right) \quad (4.21)$$

with a large $N=39000$.

Numerical results are summarized in Tables 4.3 and 4.4.

Remark 4.3.1. From Tables 4.3 and 4.4, we see that, given the same probability distribution of $S(t^*)$, models with continuous time changes give close prices for K_{var} , which are consistent with the prices given by initial models. Some analytical explanations can be given (see p135-137 of [16] for details):

Proof. Assume (S_t) follows

$$S_t = X_{g(t)}, \quad (4.22)$$

where

$$dX_t = \sigma(X_t)dW_t \quad (4.23)$$

and g is a continuous time change.

We have therefore

$$\log(S_{t^*}) = \log(X_{g(t^*)}) \quad (4.24)$$

$$= \log(S_0) + \int_0^{g(t^*)} \frac{dX_u}{X_u} - \frac{1}{2} \int_0^{g(t^*)} \frac{d[X]_u}{X_u^2} \quad (4.25)$$

$$= \log(S_0) + \int_0^{g(t^*)} \frac{dX_u}{X_u} - \frac{1}{2} \int_0^{g(t^*)} \frac{\sigma^2(X_u)}{X_u^2} du \quad (4.26)$$

Hence, we have

$$[\log X]_{g(t^*)} = \int_0^{g(t^*)} \left(\frac{1}{X_u}\right)^2 d[X]_u \quad (4.27)$$

$$= \int_0^{g(t^*)} \frac{\sigma^2(X_u)}{X_u^2} du \quad (4.28)$$

$$= -2\log \frac{S_{t^*}}{S_0} + 2 \int_0^{g(t^*)} \frac{dX_u}{X_u} \quad (4.29)$$

The initial price of K_{var} is therefore given by

$$K_{var} = \mathbb{E}^{\mathbb{Q}}[\mathbf{log} S]_{t^*} \quad (4.30)$$

$$= \mathbb{E}^{\mathbb{Q}}[\mathbf{log} X]_{g(t^*)} \quad (4.31)$$

$$= -2\mathbb{E}^{\mathbb{Q}}[\mathbf{log} \frac{S_{t^*}}{S_0}] + 2\mathbb{E}^{\mathbb{Q}}[\int_0^{g(t^*)} \frac{dX_u}{X_u}] \quad (4.32)$$

$$= -2\mathbb{E}^{\mathbb{Q}}[\mathbf{log} \frac{S_{t^*}}{S_0}], \quad (4.33)$$

since $\int_0^{g(t^*)} \frac{dX_u}{X_u}$ is centered. □

Hence, with a continuous time change, the variance swap strike K_{var} is equal to -2 * price of a log-contract with payoff $\mathbf{log} \frac{S_{t^*}}{S_0}$.

It should be noticed that this conclusion cannot be valid without the assumption of continuous time changes, since we need the time changes to be continuous in order to pass from Equation (4.30) to Equation (4.31).

Table 4.3: Variance swaps with $\sigma=0.2$; $r=0$; $T-t=1$; $S_0=100$; initial data given by Black Scholes model; number of time steps=39000; number of simulated paths=100000

time change	Black Scholes price	simulation price
$\Gamma_t(1, 1)$	0.04	0.0679
$\Gamma_t(1, 1000)$	0.04	0.0654
$\Gamma_t(1, 0.001)$	0.04	0.0676
$\min(\exp(1), \frac{t}{T-t})$	0.04	0.0403
$\min(\exp(1000), \frac{t}{T-t})$	0.04	0.0405
$\min(\exp(1), -\frac{0.5}{t} + \frac{0.1}{T-t})$	0.04	0.0403
$\min(\exp(1), -\frac{0.5}{t} + \frac{0.1}{T-t})$	0.04	0.0401

Table 4.4: Variance swaps with $r = 0$; $T-t=1$; $S_0=100$; $V_0 = 0.04$; $\eta = 0.7$; $\text{mean} = 0.06$; $\kappa = 1.5$; initial data given by Heston model; number of time steps=39000; number of simulated paths=100000

time change	Heston price	simulation price
$\Gamma_t(1, 1)$	0.05	0.0654
$\Gamma_t(1, 1000)$	0.05	0.0686
$\Gamma_t(1, 0.001)$	0.05	0.0705
$\min(\exp(1), \frac{t}{T-t})$	0.05	0.051
$\min(\exp(1000), \frac{t}{T-t})$	0.05	0.0525
$\min(\exp(1), -\frac{0.5}{t} + \frac{0.1}{T-t})$	0.05	0.0513
$\min(\exp(1), -\frac{0.5}{t} + \frac{0.1}{T-t})$	0.05	0.0521

CONCLUSION

In this thesis, we have studied the process for the (forward) price of an asset which satisfies

$$S_t = X_{g(t)}, 0 \leq t \leq t^*, \quad (4.34)$$

where X is a driftless time-homogeneous diffusion generated by

$$dX_s = \sigma(X_s)dW_s, s \geq 0 \quad (4.35)$$

and g is a random time change with $g(t^*) \sim \tau$ which is exponentially distributed with parameters θt^* .

The profile of σ can be explicitly and exactly calibrated to the initial prices (generated by Bachelier, Black Scholes and Heston models in our thesis) of call options with the same maturity t^* , the same underlying and different strikes via:

$$\frac{\sigma^2(K)\theta t^*}{2} = \frac{C_0(S_0, t^*, K) - (S_0 - K)^+}{\frac{\partial^2}{\partial K^2} C_0(S_0, t^*, K)} \quad (4.36)$$

Given the same initial prices and the same probability distribution of τ , models with different time changes give the same prices for European options of the same maturity t^* which are consistent with prices given by the initial models. However, exotic options as well as European calls with a shorter maturity behave differently between pure jump subordinator and continuous subordinator.

In our thesis, there have been several occasions where we observe that models with different continuous time changes behave similarly or even almost identically. For instance, we have seen that, given the same initial prices and the same probability distribution of τ , models with different continuous time changes give almost the same price in terms of one-touches, barrier options and variance swaps. This has been proved analytically in our thesis but can also be explained intuitively. In fact, the only important thing for pricing these options is the continuous dynamics of (X_t) from 0 until t^* . Since the dynamics of (X_t) can be fully characterized by σ and σ for all continuous time changes is deduced from the same Equation (4.36), we can conclude that prices of one-touches, barrier options and variance swaps should always be same for different continuous time changes, as long as we have the same initial prices and the same probability distribution of τ .

When it comes to pure jump subordinator, we give an example of Γ -time change with three different parameters $\theta = 1$, $\theta = 1000$ and $\theta = 0.001$. Although the jump behaves quite differently with different parameters, the generated stock paths have not shown a dramatic difference. This is because the differences in time scales and diffusion processes will also influence the dynamics of stock paths.

We have also priced European calls with a maturity T_1 shorter than t^* . These options can be used as a measure of volatility accumulated during $[0, T_1]$.

There are many possible directions for future research. For instance, one can try Asian options with different time changes. Or, so far, with the derivatives that we have examined, we have always observed similar behaviours for models subordinated by different continuous time changes. So it would be interesting to seek some exotic options which can explore the differences in dynamics for models with different continuous time changes.

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