

RAINBOW MATCHINGS IN PROPERLY COLORED MULTIGRAPHS*

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Abstract. Aharoni and Berger conjectured that in any bipartite multigraph that is properly edge-colored by n colors with at least $n + 1$ edges of each color there must be a matching that uses each color exactly once. In this paper we consider the same question without the bipartiteness assumption. We show that in any multigraph with edge multiplicities $o(n)$ that is properly edge-colored by n colors with at least $n + o(n)$ edges of each color there must be a matching of size $n - O(1)$ that uses each color at most once.

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1. Introduction. A Latin square of order n is an $n \times n$ array of n symbols in which each symbol occurs exactly once in each row and in each column. A (*partial*) *transversal* of a Latin square is a set of entries that represent each row, column, and symbol (at most) exactly once. The following fundamental open problem on transversals in Latin squares is known as the Ryser–Brualdi–Stein conjecture.¹ (The best known bound is $n - O(\log^2 n)$ by Hatami and Shor [5].)

CONJECTURE 1.1 (Ryser–Brualdi–Stein). *Every Latin square of order n has a partial transversal of size $n - 1$.*

It is not hard to see that Conjecture 1.1 is equivalent to saying that any proper n -edge-coloring of the complete bipartite graph $K_{n,n}$ has a rainbow matching (i.e., a matching with no repeated color) of size $n - 1$. Aharoni and Berger [1] conjectured the following generalization.

CONJECTURE 1.2 (Aharoni and Berger). *Let G be a bipartite multigraph that is properly edge-colored with n colors and has at least $n + 1$ edges of each color. Then G has a rainbow matching using every color.*

Pokrovskiy [6] showed that this conjecture is asymptotically true, in that the conclusion holds if there are at least $n + o(n)$ edges of each color. In this paper we consider the same question without the bipartiteness assumption. We obtain a result somewhat analogous to Pokrovskiy’s, although we require an additional (mild) assumption on edge multiplicities (which is perhaps just an artifact of our proof), and we have to allow a constant number of unused colors. A bipartite version of our result (without the additional assumption on edge multiplicities) was obtained by Barát,

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¹Ryser [8] conjectured that the number of transversals has the same parity as n , so any Latin square of odd order has a transversal (see [10]). For any even n the addition table of a cyclic group is a Latin square with no transversal. Brualdi made Conjecture 1.1 (see, e.g., [3]); Stein [9] independently made a stronger conjecture, recently disproved by Pokrovskiy and Sudakov [7].

Gyárfás, and Sárközy [2].² We give explicit constants in the statement for the sake of concreteness, but we did not attempt to optimize these.

THEOREM 1.1. *Let $0 < \varepsilon < 10^{-3}$, $k = 2^{20/\varepsilon}$, and $n > k^2$. Suppose G is a multigraph with edge multiplicities at most n/k that is properly edge-colored with n colors so that every color appears at least $(1 + \varepsilon)n$ times. Then G has a rainbow matching of size $n - k$.*

Our proof of Theorem 1.1 is algorithmic. Given any rainbow matching M of size less than $n - k$, we construct a hierarchy of edges e in M for which we can find another rainbow matching M' of the same size which does not contain e , or indeed any edge with the same color as e . We find M' by a sequence of switches using edges at lower levels in hierarchy. The switching method is robust, in that there are many choices at each step, and so it is possible to avoid any constant size set of vertices or edges that were altered by previous switches. To analyze the algorithm, we suppose for contradiction that it does not find a switch to increase the matching, and then prove several structural properties of G that culminate in a counting argument to give the required contradiction.

We describe and analyze our algorithm in section 3, after briefly summarizing our notation in the next section. Then we give the proof of Theorem 1.1 in section 4. In the conclusion we compare our work to a recent result of Gao et al. [4].

2. Notation. For convenient reference we summarize our notation in this section.

Let G be a properly edge-colored multigraph. We write $c(e)$ for the color of an edge e . We also say that e is a c -edge, where $c(e) = c$. Similarly, for any set C of colors, we say that e is a C -edge if $c(e) \in C$.

Now suppose M is a rainbow matching in G . We let $C(M)$ denote the set of colors used by M . For $c \in C(M)$ we let m_c be the edge of M of color c . If $e = uv \in M$, we say that u and v are *twins*, and write $u = t(v)$ (and so $v = t(u)$). For $S \subseteq V(M)$ we write $T(S) = \{t(v) : v \in S\}$. These definitions depend on M , but this will be fixed throughout, so we suppress it in our notation.

Any edge that goes between $V(G) \setminus V(M)$ and $V(M)$ is called *external*.

We say that M *avoids* a set S of vertices if $V(M) \cap S = \emptyset$. We say that M *avoids* a set C of colors if $C(M) \cap C = \emptyset$. We say that M *fixes* a set E of edges if $E \subseteq M$.

Suppose M and M' are rainbow matchings in G . We say M and M' are λ -close if $|M'| = |M|$ and $|M \Delta M'| \leq \lambda$. Note that this is nonstandard terminology. (Standard usage would not include the condition $|M'| = |M|$.)

For the remainder of the paper we fix ε , k , n , and G as in the statement of Theorem 1.1. We also use an auxiliary variable $\alpha := \varepsilon/16$. We fix a rainbow matching M in G of maximum size and write $V_0 = V(G) \setminus V(M)$. Let C_0 be the set of colors not used by M . We note that by maximality there are no C_0 -edges contained in V_0 . We suppose for a contradiction that $|C_0| > k$.

3. The reachability algorithm. We will describe an algorithm which builds a set of colors that we call *reachable*. Informally, these are the colors c in M such that we can robustly replace c by some unused color $c_0 \in C_0$ and obtain another rainbow matching M' with $|M'| = |M|$. To achieve this we may need to make several other changes to the matching, but if the number of changes is bounded by some constant,

²We thank an anonymous referee for this comment, which requires some thought, as it is not immediately apparent that their result implies the bipartite version of ours.

then we think of c as reachable. The changes will consist of several flips along disjoint M -alternating paths (as in the classical matching algorithms).

As a first step, let us show that there are many edges in M which we call “flexible,” meaning they have several options to be directly replaced by a C_0 -edge, with no further changes to the matching. This property is not as strong as the reachability property that we will define below, but in the analysis of reachability we will need to know that there are many flexible edges.

We let E_0 be the set of edges in M which can be oriented from tail to head so that the number of external C_0 -edges incident to the tail is at least $\alpha|C_0|$. (If both orientations work, then we choose one arbitrarily.) We let S_0 denote the set of heads of edges in E_0 (so the set of tails is the set $T(S_0)$ of twins of S_0). Then $F = C(E_0)$ is the set of *flexible* colors.

For any $v \in S_0$ there are at least $\alpha|C_0|$ choices of a C_0 -edge $t(v)u$ with $u \in V_0$, and $M' = M \setminus \{vt(v)\} \cup \{t(v)u\}$ is a rainbow matching with $|M'| = |M|$. Note however that u may be the same for all such C_0 -edges.

LEMMA 3.1. $|F| \geq \alpha n$.

Proof. For every $c \in C_0$, we note that there are at least εn external c -edges. Indeed, the total number of c -edges is at least $n + \varepsilon n$, there are none in V_0 , and at most n are contained in $V(M)$, as $|M| < n$ and the coloring is proper. Thus we have at least $\varepsilon n \cdot |C_0|$ external C_0 -edges. Note that any vertex in $V(M)$ is incident to at most $|C_0|$ of these edges, and by definition of F , any vertex of $V(M) \setminus (S_0 \cup T(S_0))$ is incident to at most $\alpha|C_0|$ of them. Thus $\varepsilon n \cdot |C_0| \leq 2|F| \cdot |C_0| + 2n \cdot \alpha|C_0|$, so $|F| \geq (\frac{\varepsilon}{2} - \alpha)n \geq \alpha n$. \square

For more options of switches we need longer paths. For example, if an edge e of M is incident to many external F -edges, we may be able to use one of these edges instead of e , and then remove the corresponding F -edge from M using its incident C_0 -edges. We will be able to implement this idea for edges satisfying the following condition. For $c \in F$, we classify external c -edges as *good* or *bad*, where e is good if m_c (the c -edge in M) is incident to at least $\alpha|C_0|/2$ external C_0 -edges which share no endpoint with e .

Now we formulate our algorithm.

ALGORITHM 3.1. We iteratively define sets E_i of “ i -reachable” edges for $i \geq 1$. Each edge in E_i will be assigned an orientation, from tail to head. We let V_i denote the set of heads of edges in E_i (so the set of tails is the set $T(V_i)$ of twins of V_i). We write $R_i = C(E_i)$ for the set of i -reachable colors.

We let E_1 be the set of edges in M that can be oriented from tail to head such that the tail is incident to at least $\alpha|F|$ good F -edges.

For $i > 1$ we let E_i be the set of edges in $M \setminus \cup_{1 \leq j < i} E_j$ that can be oriented from tail t to head such that for some $1 \leq j < i$ the number of R_j -edges tu with $u \in \cup_{j=0}^{i-1} V_j$ is at least $\alpha|R_j|$. (Note that for all $i \geq 1$ if both orientations work, then we choose one arbitrarily.)

We stop if $|E_i| < \alpha n$.

Let q be such that the algorithm stops with $|E_{q+1}| < \alpha n$. As $|M| < n$, we have $q < 1/\alpha$.

Let R be the set of reachable colors, that is, $R = \cup_{i=1}^q R_i$. We also write $V_{reach} = \cup_{i=1}^q V_i$.

Our next lemma shows that the first step of the algorithm finds some linear proportion of reachable colors.

LEMMA 3.2. $|R_1| \geq \alpha n$.

Proof. First we show that there are few F -edges contained in V_0 .

CLAIM 3.3. *For any color $c \in F$, there can be at most one c -edge contained in V_0 .*

Proof. Indeed, fix $c \in F$, and suppose there exist two such edges, say, e_1 and e_2 . Let $m_c = ut(u)$, where $u \in S_0$. Fix any external C_0 -edge e_3 incident to $t(u)$ (by definition of S_0) and suppose without loss of generality that $e_2 \cap e_3 = \emptyset$. (We have $e_1 \cap e_2 = \emptyset$, as the coloring is proper.) Then $M' = M \setminus \{ut(u)\} \cup \{e_2, e_3\}$ is a larger rainbow matching, which contradicts our choice of M . \square

We deduce that there are at least $\varepsilon|F|n/2$ external F -edges. Indeed, the total number of F -edges is at least $(1+\varepsilon)n|F|$, of which at most $|F|$ are in V_0 (by Claim 3.3), and at most $|F|n$ are contained in $V(M)$ (as $|M| < n$ and the coloring is proper).

CLAIM 3.4. *There are at least $\varepsilon|F|n/4$ good external F -edges.*

Proof. It suffices to show that for each $c \in F$ there are at most $2/\alpha$ bad c -edges. (Then the number of bad c -edges is at most $2|F|/\alpha < \varepsilon|F|n/4$.) To see this, fix $c \in F$, let $m_c = ut(u)$, where $u \in S_0$, and consider any external c -edge vw that is bad with $w \in V_0$. There are at least $\alpha|C_0|$ external C_0 -edges incident to $t(u)$ (by definition of S_0), of which at most $\alpha|C_0|/2$ do not contain w (by definition of “bad”), so $t(u)w$ has multiplicity at least $\alpha|C_0|/2$ in C_0 -edges. As the coloring is proper, there are at most $|C_0|$ (external) C_0 -edges incident to $t(u)$, and each bad external c -edge determines a unique w incident to at least $\alpha|C_0|/2$ of them, so the claim follows. \square

On the other hand, by definition of R_1 there are at most $2|R_1||F| + 2n \cdot \alpha|F|$ external F -edges, so by Claim 3.4

$$\varepsilon|F|n/4 \leq 2|R_1||F| + 2n \cdot \alpha|F|,$$

giving $|R_1| \geq \varepsilon n/8 - \alpha n = \alpha n$. \square

Our next definition and associated lemma justify our description of the colors in R as “reachable.” The lemma shows that they can be replaced in the matching, maintaining its size and being rainbow, while making only constantly many changes and avoiding certain proscribed sets of vertices, edges, and colors.

DEFINITION 3.5. *For $1 \leq i \leq q$ we say that M is $(i, \lambda, \lambda', a)$ -robust if for any rainbow matching M' which is λ -close to M , any $c \in R$ such that $m_c = vt(v) \in M'$ with $v \in \cup_{j=1}^i V_j$, and any $E_{\text{fix}} \subseteq M' \setminus \{m_c\}$, any $V_{\text{avoid}} \subseteq V \setminus V(M')$, and $C_{\text{avoid}} \subseteq C \setminus C(M')$, each of size at most a , there is a rainbow matching M'' in G such that*

- (1) $c \notin C(M'')$ and $v \notin V(M'')$,
- (2) M'' is λ' -close to M ,
- (3) M'' fixes E_{fix} and avoids C_{avoid} and V_{avoid} .

We will apply the following statement only in the case $i = q$, but we formulate it for all i to facilitate an inductive proof. We write $f(i) = 3 \cdot 2^{i-1} - 2$.

LEMMA 3.6. *Let $1 \leq i \leq q$ and $0 \leq \lambda \leq 2^{15/\varepsilon - i/15}$. Then M is $(i, \lambda, \lambda + f(i), 2(q - i + 1))$ -robust.*

Proof. Let M' , c , v , E_{fix} , V_{avoid} , and C_{avoid} be given as in Definition 3.5. We will show by induction on i that there is a rainbow matching M'' satisfying (1), (2), and (3) of Definition 3.5.

First we consider the base case $i = 1$. We consider the set \mathcal{T} of all pairs $(vt(v)w, ut(u)z)$ of 2-edge paths, where $t(v)w$ is a good F -edge, w and z are distinct and in V_0 , and we have $c(t(v)w) = c(ut(u))$ and $c(t(u)z) \in C_0$. If $vt(v), ut(u)$ also appear in M' (a priori they are from M), we will obtain M'' from M' by removing $vt(v)$ and $ut(u)$ and adding $t(v)w$ and $t(u)z$. Note that any such M'' is $(\lambda + 4)$ -close to M , as required for (2). There are various constraints that need to be satisfied for M'' to be a rainbow matching satisfying (1) and (3); we will see below that \mathcal{T} is large enough so that we can pick an element of \mathcal{T} such that the matching derived from it satisfies these constraints.

First we show that $|\mathcal{T}| \geq \alpha^3 |C_0| n / 2$. To see this, note that as $v \in V_1$ there are at least $\alpha |F|$ good external F -edges incident to $t(v)$. For each such edge $t(v)w$, there is an edge in M of the same color, which we can label as $ut(u)$ so that there are at least $\alpha |C_0| / 2$ external C_0 -edges $t(u)z$ with $z \neq w$. Hence, by Lemma 3.1 we get that $|\mathcal{T}| \geq \alpha |F| \cdot \alpha |C_0| / 2 \geq \alpha^3 |C_0| n / 2$.

Next we estimate how many configurations in \mathcal{T} are forbidden by the various constraints described above.

We start by considering the constraints on w , namely, $w \in V'_0$ and $w \notin V_{\text{avoid}}$. These forbid at most $2\lambda + 2q$ choices of w . For each such w , there are at most n/k choices of an F -edge $t(v)w$, which determines $ut(u) \in M$ with $c(ut(u)) = c(t(v)w)$, and then there are at most $|C_0|$ choices for a C_0 -edge $t(u)z$. Thus we forbid at most $(2\lambda + 2q)|C_0|n/k$ configurations.

We also have the same constraints on z , i.e., $z \in V'_0$ and $z \notin V_{\text{avoid}}$. Again, this forbids at most $2\lambda + 2q$ choices of z . For each such z , there are at most $|C_0|$ choices for a C_0 -edge $t(u)z$, which determines $ut(u) \in M$, and then at most one edge $t(v)w$ with the same color. Thus we forbid at most $(2\lambda + 2q)|C_0|$ configurations.

Now we consider the constraints on $ut(u)$, namely, $ut(u) \in M'$ and $ut(u) \notin E_{\text{fix}}$. These forbid at most $\lambda + 2q$ choices of $ut(u)$. For each such $ut(u)$, there is at most one edge $t(v)w$ with the same color, and at most $|C_0|$ choices for $t(u)z$, so we forbid at most $(\lambda + 2q)|C_0|$ configurations.

For $t(v)w$ we have the constraint $c(t(v)w) \in C(M')$, which forbids at most λ edges $t(v)w$. This fixes $ut(u) \in M$ with the same color, and then there are at most $|C_0|$ choices for $t(u)z$, so we forbid at most $\lambda |C_0|$ configurations.

Finally, for $t(u)z$ we have the constraint $c(t(u)z) \notin C_{\text{avoid}} \cup (C(M') \setminus C(M))$. There are at most $2q + \lambda$ choices for $c' \in C_{\text{avoid}} \cup (C(M') \setminus C(M))$, then at most n choices for a c' -edge $t(u)z$, which determines $ut(u)$ and then at most one edge $t(v)w$ with the same color, so we forbid at most $(2q + \lambda)n$ configurations.

The total number of forbidden configurations is at most $2(\lambda + m)|C_0|(2 + n/k) + (2q + \lambda)n < \alpha^3 |C_0| n / 2 \leq |\mathcal{T}|$, as $\lambda/k \leq 2^{-5/\varepsilon} < 10^{-6}\varepsilon^3$, so $2(\lambda + m)|C_0|(2 + n/k) < \alpha^3 |C_0| n / 4$ and $\alpha^3 |C_0| n / 4 > \alpha^3 kn / 4 > (2q + \lambda)n$. We can therefore pick a configuration in \mathcal{T} that satisfies all the above constraints, which completes the proof of the base case.

Now suppose $i > 1$. As $v \in V_i$, there is $1 \leq j < i$ such that the number of R_j -edges $t(v)w$ with $w \in \cup_{j=0}^{i-1} V_j$ is at least $\alpha |R_j| \geq \alpha^2 n$. (Note that $|R_j| = |E_j| \geq \alpha n$; otherwise the algorithm would have stopped.) We will consider two cases according to whether most of these edges go to V_0 . In each case we note by the induction hypothesis that M is $(j, \lambda, \lambda + f(j), 2(q - j + 1))$ -robust.

Suppose first that at least $\alpha^2 n/2$ of these R_j -edges $t(v)w$ have $w \in V_0$. Then we can fix such an edge $t(v)w$ with $c' = c(t(v)w) \in R_j$ such that $w \in V_0' \setminus V_{\text{avoid}}$ and $m_{c'} \in M' \setminus E_{\text{fix}}$. (The first condition forbids at most $(2q + 2\lambda)n/k$ choices and the second at most $2q + \lambda$ choices.)

By induction hypothesis, we can apply Definition 3.5 applied to $m_{c'}$ in M' with sets $E'_{\text{fix}} := E_{\text{fix}} \cup \{vt(v)\}$, $V'_{\text{avoid}} := V_{\text{avoid}} \cup \{w\}$, and $C'_{\text{avoid}} := C_{\text{avoid}}$, each of size at most $2(q - i + 1) + 1 < 2(q - j + 1)$, and we obtain a rainbow matching M_1 which is $(\lambda + f(j))$ -close to M , satisfies (3), contains $v(t)v$, and avoids the color c' and vertex w . As $\lambda + f(j) + 2 \leq \lambda + f(i)$, we see that $M'' = M_1 \setminus \{vt(v)\} \cup \{t(v)w\}$ is as required to complete the proof in this case.

It remains to consider the case that at least $\alpha^2 n/2$ of the R_j -edges $t(v)u$ have $u \in \cup_{j=1}^{i-1} V_j$. We can fix such an edge $t(v)u$ with $c(t(v)u) = c_1 \in R_j$ such that $c(ut(u)) = c_2 \in R_{j'}$ with $1 \leq j' \leq i - 1$ and $ut(u) \in M' \setminus E_{\text{fix}}$ (this forbids at most $(2q + \lambda)n/k$ choices), and $m_{c_1} \in M' \setminus E_{\text{fix}}$ (which forbids at most $2q + \lambda$ choices).

By induction hypothesis, we can apply Definition 3.5 to m_{c_1} in M' with sets $E'_{\text{fix}} := E_{\text{fix}} \cup \{vt(v), ut(u)\}$, $V'_{\text{avoid}} := V_{\text{avoid}}$, and $C'_{\text{avoid}} := C_{\text{avoid}}$, each of size at most $2(q - i + 1) + 2 \leq 2(q - j + 1)$, and we obtain a rainbow matching M_1 which is $(\lambda + f(j))$ -close to M , satisfies (3), contains $v(t)v$ and $ut(u)$, and avoids the color c_1 .

By induction hypothesis, we can apply Definition 3.5 to $m_{c_2} = ut(u) \in M_1$ with sets $E''_{\text{fix}} := E_{\text{fix}} \cup \{vt(v)\}$, $V''_{\text{avoid}} := V_{\text{avoid}}$, and $C''_{\text{avoid}} := C_{\text{avoid}} \cup \{c_1\}$, noting that $\lambda + f(j) \leq 2^{15/\varepsilon - i/15} + 2^{1+1/\alpha} \leq 2^{15/\varepsilon - j'/15}$, as $\alpha = \varepsilon/12$, $j' \leq i - 1$, $i \leq m < 1/\alpha$, and $\varepsilon < 10^{-3}$. Thus we obtain a rainbow matching M_2 which is $(\lambda + f(j) + f(j'))$ -close to M , satisfies (3), contains $v(t)v$, and avoids the color c_1 and the vertex u . As $\lambda + f(j) + f(j') + 2 \leq \lambda + f(i)$, we see that $M'' = M_2 \setminus \{vt(v)\} \cup \{t(v)u\}$ is as required to complete the proof. \square

4. Structure and counting. Continuing with the proof strategy outlined above, we now use the maximality assumption on M to prove the following structural properties of G , which we will then combine with a counting argument to obtain a contradiction to the assumption that $|C_0| > k$, thus completing the proof of Theorem 1.1.

In the proof of the following lemma we repeatedly use Lemma 3.6 in the case $i = q$, i.e., that M is $(q, \lambda, \lambda + f(q), 2)$ -robust for any $0 \leq \lambda \leq 2^{15/\varepsilon - q/15}$. Here we recall that $R = \cup_{i=1}^q R_i$ and $q < 1/\alpha = 12/\varepsilon$, so Definition 3.5 is applicable whenever $0 \leq \lambda \leq 2^{14/\varepsilon}$.

LEMMA 4.1.

- (C1) Any $v \in V_{\text{reach}}$ is not incident to an external R -edge.
- (C2) Any two $u, v \in V_{\text{reach}}$ are not adjacent by an R -edge.
- (C3) There are no R -edges contained in V_0 .

Proof. (C1): Suppose v is incident to an external R -edge, say, $w \in V_0$ and $c = c(vw) \in R$. We can apply Lemma 3.6 to $vt(v)$ in $M' = M$ with sets $E_{\text{fix}} = \{m_c\}$ and $V_{\text{avoid}} = \{w\}$, $C_{\text{avoid}} = \emptyset$ and obtain a rainbow matching M_1 that is $f(q)$ -close to M , contains m_c and avoids v and w . As $f(q) < 2^{14/\varepsilon}$, we can apply Lemma 3.6 again to m_c in M_1 with sets $E_{\text{fix}} = C_{\text{avoid}} = \emptyset$ and $V_{\text{avoid}} = \{v, w\}$, obtaining a rainbow matching M_2 that is $2f(q)$ -close to M and avoids the vertices v, w and the color c . Now $M_2 \cup \{uw\}$ is a larger rainbow matching than M , which is a contradiction.

(C2): Suppose for a contradiction that we have $u, v \in V_{\text{reach}}$ and an edge uv with $c(uv) = c_3 \in R$. Let $c_1 = c(vt(v))$ and $c_2 = c(ut(u))$. Then c_1 and c_2 are also in R .

Applying Lemma 3.6 to $vt(v)$ in $M' = M$ with sets $E_{fix} = \{ut(u), m_{c_3}\}$ and $V_{avoid} = C_{avoid} = \emptyset$, we obtain a rainbow matching M_1 that is $f(q)$ -close to M , contains $ut(u)$ and m_{c_3} , and avoids v .

Applying Lemma 3.6 again to $ut(u)$ in M_1 with sets $E_{fix} = \{m_{c_3}\}$, $V_{avoid} = \{v\}$, and $C_{avoid} = \emptyset$, we obtain a rainbow matching M_2 that is $2f(q)$ -close to M , contains m_{c_3} , and avoids u and v .

Applying Lemma 3.6 a final time to m_{c_3} in M_2 with $V_{avoid} = \{u, v\}$ and $E_{fix} = C_{avoid} = \emptyset$, noting that $2f(q) < 2^{14/\varepsilon}$, we obtain a rainbow matching M_3 that is $3f(q)$ -close to M and avoids the vertices u and v and the color c_3 .

Now $M_3 \cup \{uv\}$ is a larger rainbow matching than M , which is a contradiction.

(C3): We omit the proof as it is very similar to that of the first two statements. \square

Proof of Theorem 1.1. Recalling that M is a maximum rainbow matching in G , we write $V_0 = V(G) \setminus V(M)$ and suppose for a contradiction that $|M| < n - k$. Recalling also that the reachable colors are $R = \cup_{i=1}^q R_i$, we write $V_{reach} = \cup_{i=1}^q V_i$, and the algorithm stopped with $|V_{q+1}| < \alpha n$.

As every color appears at least $(1 + \varepsilon)n$ times the number of R -edges is at least $|R|(1 + \varepsilon)n$. By Lemma 4.1 they are all incident to $V(M) \setminus V_{reach}$. Write

$$V^* = T(V_{reach}) \cup V_{q+1} \cup T(V_{q+1}) \quad \text{and} \quad V' = V(M) \setminus (V_{reach} \cup V^*).$$

At most $|V^*||R| < (|R| + 2\alpha n)|R|$ of the R -edges are incident to V^* , so at least $|R|(1 + \varepsilon)n - (|R| + 2\alpha n)|R| > |R|(|V'| + \varepsilon n)/2$ are incident to V' but not to V^* .

By definition of our algorithm, for each v in V' the number of R -edges vu with $u \in V_0 \cup V_{reach}$ is at most $\alpha|R|$. All remaining R -edges are contained in V' , and they number at least $|R|(|V'| + \varepsilon n)/2 - \alpha|R||V'| > |R||V'|/2$. But the coloring is proper, so there are at most $|V'|/2$ edges of any color contained in V' . This contradiction completes the proof. \square

5. Conclusion. A similar result to Theorem 1.1 was very recently obtained independently by Gao et al. [4]. Their result is closer than ours to the spirit of Conjecture 1.2, as they obtain a full rainbow matching (whereas we allow a constant number of unused colors). However, our results are incomparable, as they need a stronger bound on edge multiplicities, namely, $\sqrt{n}/\log^2 n$. Our algorithm is deterministic, whereas theirs is randomized, and the analysis of our algorithm is simpler. In any case, the two approaches are very different, so we think that it will be valuable to pursue both sets of ideas, and perhaps also the approach of Pokrovskiy, in making further progress on Conjecture 1.2.

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