

On Saturated k -Sperner Systems

Natasha Morrison Jonathan A. Noel Alex Scott

Mathematical Institute
University of Oxford
Woodstock Road
Oxford OX2 6GG, UK.
`{morrison,noel,scott}@maths.ox.ac.uk`

Abstract

Given a set X , a collection $\mathcal{F} \subseteq \mathcal{P}(X)$ is said to be k -Sperner if it does not contain a chain of length $k + 1$ under set inclusion and it is *saturated* if it is maximal with respect to this property. Gerbner et al. [11] conjectured that, if $|X|$ is sufficiently large with respect to k , then the minimum size of a saturated k -Sperner system $\mathcal{F} \subseteq \mathcal{P}(X)$ is 2^{k-1} . We disprove this conjecture by showing that there exists $\varepsilon > 0$ such that for every k and $|X| \geq n_0(k)$ there exists a saturated k -Sperner system $\mathcal{F} \subseteq \mathcal{P}(X)$ with cardinality at most $2^{(1-\varepsilon)k}$.

A collection $\mathcal{F} \subseteq \mathcal{P}(X)$ is said to be an *oversaturated k -Sperner system* if, for every $S \in \mathcal{P}(X) \setminus \mathcal{F}$, $\mathcal{F} \cup \{S\}$ contains more chains of length $k + 1$ than \mathcal{F} . Gerbner et al. [11] proved that, if $|X| \geq k$, then the smallest such collection contains between $2^{k/2-1}$ and $O\left(\frac{\log k}{k} 2^k\right)$ elements. We show that if $|X| \geq k^2 + k$, then the lower bound is best possible, up to a polynomial factor.

Keywords: minimum saturation; set systems; antichains

1 Introduction

Given a set X , a collection $\mathcal{F} \subseteq \mathcal{P}(X)$ is a *Sperner system* or an *antichain* if there do not exist $A, B \in \mathcal{F}$ such that $A \subsetneq B$. More generally, a collection

$\mathcal{F} \subseteq \mathcal{P}(X)$ is a *k-Sperner system* if there does not exist a subcollection $\{A_1, \dots, A_{k+1}\} \subseteq \mathcal{F}$ such that $A_1 \subsetneq \dots \subsetneq A_{k+1}$. Such a subcollection $\{A_1, \dots, A_{k+1}\}$ is called a *(k + 1)-chain*. We say that a *k-Sperner system* is *saturated* if, for every $S \in \mathcal{P}(X) \setminus \mathcal{F}$, we have that $\mathcal{F} \cup \{S\}$ contains a *(k + 1)-chain*. A collection $\mathcal{F} \subseteq \mathcal{P}(X)$ is a *oversaturated k-Sperner system*¹ if, for every $S \in \mathcal{P}(X) \setminus \mathcal{F}$, we have that the number of *(k + 1)-chains* in $\mathcal{F} \cup \{S\}$ is greater than the number of *(k + 1)-chains* in \mathcal{F} . Thus a family is a saturated *k-Sperner system* if and only if it is an oversaturated *k-Sperner system* that contains no *(k + 1)-chains*.

For a set X of cardinality n , the problem of determining the maximum size of a saturated *k-Sperner system* in $\mathcal{P}(X)$ is well understood. In the case $k = 1$, Sperner's Theorem [17] (see also [4]), says that every antichain in $\mathcal{P}(X)$ contains at most $\binom{n}{\lfloor n/2 \rfloor}$ elements, and this bound is attained by the collection consisting of all subsets of X with cardinality $\lfloor n/2 \rfloor$. Erdős [6] generalised Sperner's Theorem by proving that the largest size of a *k-Sperner system* in $\mathcal{P}(X)$ is the sum of the k largest binomial coefficients $\binom{n}{i}$. In this paper, we are interested in determining the minimum size of a saturated *k-Sperner system* or an oversaturated *k-Sperner system* in $\mathcal{P}(X)$. These problems were first studied by Gerbner, Keszegh, Lemons, Palmer, Pálvölgyi and Patkós [11].

Given integers n and k , let $\text{sat}(n, k)$ denote the minimum size of a saturated *k-Sperner system* in $\mathcal{P}(X)$ where $|X| = n$. It was shown in [11] that $\text{sat}(n, k) = \text{sat}(m, k)$ if n and m are sufficiently large with respect to k . We can therefore define

$$\text{sat}(k) := \lim_{n \rightarrow \infty} \text{sat}(n, k).$$

We are motivated by the following conjecture of [11].

Conjecture 1 (Gerbner et al. [11]). *For all k , $\text{sat}(k) = 2^{k-1}$.*

Gerbner et al. [11] observed that their conjecture is true for $k = 1, 2, 3$. They also proved that $2^{k/2-1} \leq \text{sat}(k) \leq 2^{k-1}$ for all k , where the upper bound is implied by the following construction.

¹In [11], this is called a *weakly saturated k-Sperner system*. Since there is another notion of weak saturation in the literature (see, for instance, Bollobás [3]), we have chosen to use a different term to avoid possible confusion.

Construction 2 (Gerbner et al. [11]). Let Y be a set such that $|Y| = k - 2$ and let H be a non-empty set disjoint from Y . Let $X = Y \cup H$ and define

$$\mathcal{G} := \mathcal{P}(Y) \cup \{S \cup H : S \in \mathcal{P}(Y)\}.$$

It is easily verified that $\mathcal{G} \subseteq \mathcal{P}(X)$ is a saturated k -Sperner system of cardinality 2^{k-1} .

In this paper, we disprove Conjecture 1 by establishing the following:

Theorem 3. *There exists $\varepsilon > 0$ such that, for all k , $\text{sat}(k) \leq 2^{(1-\varepsilon)k}$.*

We remark that the value of ε that can be deduced from our proof is approximately $\left(1 - \frac{\log_2(15)}{4}\right) \approx 0.023277$. The proof of Theorem 3 comes in two parts. First, we give an infinite family of saturated 6-Sperner systems of cardinality 30 which shows that $\text{sat}(6) \leq 30 < 2^5$. We then provide a method which, under certain conditions, allows us to combine a saturated k_1 -Sperner system of small order and a saturated k_2 -Sperner system of small order to obtain a saturated $(k_1 + k_2 - 2)$ -Sperner system of small order. By repeatedly applying this method, we are able to prove Theorem 3 for general k . As it turns out, our method yields the bound $\text{sat}(k) < 2^{k-1}$ for every $k \geq 6$. For completeness, we will prove that $\text{sat}(k) = 2^{k-1}$ for $k \leq 5$, and so $k = 6$ is the first value of k for which Conjecture 1 is false.

Similar techniques show that $\text{sat}(k)$ satisfies a submultiplicativity condition, which leads to the following result.

Theorem 4. *For ε as in Theorem 3, there exists $c \in [1/2, 1 - \varepsilon]$ such that $\text{sat}(k) = 2^{(1+o(1))ck}$.*

Naturally, we wonder about the correct value of c in Theorem 4.

Problem 5. *Determine the constant c for which $\text{sat}(k) = 2^{(1+o(1))ck}$.*

We are also interested in oversaturated k -Sperner systems. Given integers n and k , let $\text{osat}(n, k)$ denote the minimum size of an oversaturated k -Sperner system in $\mathcal{P}(X)$ where $|X| = n$. As we will prove in Lemma 7, $\text{osat}(n, k) = \text{osat}(m, k)$ provided that n and m are sufficiently large with respect to k . Similarly to $\text{sat}(k)$, we define $\text{osat}(k) := \lim_{n \rightarrow \infty} \text{osat}(n, k)$. Gerbner et al. [11] proved that if $|X| \geq k$, then an oversaturated k -Sperner

system in $\mathcal{P}(X)$ of minimum size has between $2^{k/2-1}$ and $O\left(\frac{\log(k)}{k}2^k\right)$ elements. Together with Lemma 7, this implies

$$2^{k/2-1} \leq \text{osat}(k) \leq O\left(\frac{\log(k)}{k}2^k\right).$$

We show that the lower bound gives the correct asymptotic behaviour, up to a polynomial factor.

Theorem 6. *For every integer k and set X with $|X| \geq k^2 + k$ there exists an oversaturated k -Sperner system $\mathcal{F} \subseteq \mathcal{P}(X)$ such that $|\mathcal{F}| = O(k^5 2^{k/2})$. In particular,*

$$\text{osat}(k) = 2^{(1/2+o(1))k}.$$

In Section 2, we prove some preliminary results which will be used throughout the paper. In particular, we provide methods for constructing and decomposing saturated k -Sperner systems. In Section 3 we show that certain types of saturated k_1 -Sperner and k_2 -Sperner systems can be combined to produce a saturated $(k_1 + k_2 - 2)$ -Sperner system, and use this to prove Theorems 3 and 4. Finally, in Section 4, we provide a method for constructing oversaturated k -Sperner systems of small cardinality in order to prove Theorem 6.

Minimum saturation has been studied extensively in the context of graphs [1, 2, 5, 10, 12, 13, 18, 19, 20] and hypergraphs [7, 14, 15, 16]. Such problems are typically of the following form: for a fixed (hyper)graph H , determine the minimum size of a (hyper)graph G on n vertices which does not contain a copy of H and for which adding any edge $e \notin G$, yields a (hyper)graph which contains a copy of H . This line of research was initiated in a paper of Erdős, Hajnal and Moon [8]. For more background on minimum saturation problems for graphs, we refer the reader to the survey of Faudree, Faudree and Schmitt [9].

2 Preliminaries

Given a collection $\mathcal{F} \subseteq \mathcal{P}(X)$, we say that a set $A \subseteq X$ is an *atom* for \mathcal{F} if A is maximal with respect to the property that

$$\text{for every set } S \in \mathcal{F}, S \cap A \in \{\emptyset, A\}. \quad (1)$$

We say that an atom A with $|A| \geq 2$ is *homogeneous* for \mathcal{F} . Gerbner et al. [11] proved that if n, m are sufficiently large with respect to k , then $\text{sat}(n, k) = \text{sat}(m, k)$. Using a similar approach, we extend this result to $\text{osat}(n, k)$.

Lemma 7. *Fix k . If $n, m > 2^{2^{k-1}}$, then $\text{sat}(n, k) = \text{sat}(m, k)$ and $\text{osat}(n, k) = \text{osat}(m, k)$.*

Proof. Fix $n > 2^{2^{k-1}}$ and let X be a set of cardinality n . Suppose that $\mathcal{F} \subseteq \mathcal{P}(X)$ is an oversaturated k -Sperner system of cardinality at most 2^{k-1} . We know that such a family exists by Construction 2. We will show that, for sets X_1 and X_2 such that $|X_1| = n - 1$ and $|X_2| = n + 1$, there exists $\mathcal{F}_1 \subseteq \mathcal{P}(X_1)$ and $\mathcal{F}_2 \subseteq \mathcal{P}(X_2)$ such that

- (a) $|\mathcal{F}_1| = |\mathcal{F}_2| = |\mathcal{F}|$,
- (b) \mathcal{F}_1 and \mathcal{F}_2 have the same number of $(k + 1)$ -chains as \mathcal{F} ,
- (c) \mathcal{F}_1 and \mathcal{F}_2 are oversaturated k -Sperner systems.

We observe that this is enough to prove the lemma. Indeed, by taking \mathcal{F} to be a saturated k -Sperner system or an oversaturated k -Sperner system in $\mathcal{P}(X)$ of minimum order, we will have that

$$\max\{\text{sat}(n - 1, k), \text{sat}(n + 1, k)\} \leq \text{sat}(n, k) \text{ and}$$

$$\max\{\text{osat}(n - 1, k), \text{osat}(n + 1, k)\} \leq \text{osat}(n, k).$$

Since $n > 2^{2^{k-1}}$ was arbitrary, the result will follow by induction.

We prove the following claim.

Claim 8. *Given a set X and a collection $\mathcal{F} \subseteq \mathcal{P}(X)$, if $|X| > 2^{|\mathcal{F}|}$, then there is a homogeneous set for \mathcal{F} .*

Proof. We observe that every atom A for \mathcal{F} corresponds to a subcollection $\mathcal{F}_A := \{S \in \mathcal{F} : A \subseteq S\}$ of \mathcal{F} such that $\mathcal{F}_A \neq \mathcal{F}_{A'}$ whenever $A \neq A'$. This implies that the number of atoms for \mathcal{F} is at most $2^{|\mathcal{F}|}$. Therefore, since $|X| > 2^{|\mathcal{F}|}$, there must be a homogeneous set H for \mathcal{F} . \square

By Claim 8 and the fact that $|X| > 2^{2^{k-1}} \geq 2^{|\mathcal{F}|}$, there exists a homogeneous set H for \mathcal{F} . Let $x_1 \in H$ and $x_2 \notin X$ and define $X_1 := X \setminus \{x_1\}$ and $X_2 := X \cup \{x_2\}$. Let

$$\mathcal{F}_1 := \{S \in \mathcal{F} : S \cap H = \emptyset\} \cup \{S \setminus \{x_1\} : S \in \mathcal{F}_H\}, \text{ and}$$

$$\mathcal{F}_2 := \{S \in \mathcal{F} : S \cap H = \emptyset\} \cup \{S \cup \{x_2\} : S \in \mathcal{F}_H\}.$$

Since H is homogeneous for \mathcal{F} , there does not exist a pair of sets in \mathcal{F} which differ only on x_1 . Thus, for $i \in \{1, 2\}$ there is a natural bijection from \mathcal{F}_i to \mathcal{F} which preserves set inclusion. Hence, (a) and (b) hold. Now, let $i \in \{1, 2\}$ and $T_i \in \mathcal{P}(X_i) \setminus \mathcal{F}_i$ and define

$$T := (T_i \setminus (H \cup \{x_2\})) \cup \{x_1\}.$$

Then $T \in \mathcal{P}(X) \setminus \mathcal{F}$ since H is a non-singleton atom and $T \cap H = \{x_1\}$, and so there exists $A_1, \dots, A_k \in \mathcal{F}$ and $t \in \{0, \dots, k\}$ such that

$$A_1 \subsetneq \dots \subsetneq A_t \subsetneq T \subsetneq A_{t+1} \subsetneq \dots \subsetneq A_k.$$

Since $T \cap H \neq H$, we must have $A_j \cap H = \emptyset$ for $j \leq t$ and so $A_1, \dots, A_t \in \mathcal{F}_i$ and $A_1 \subsetneq \dots \subsetneq A_t \subsetneq T_i$. Also, since $T \cap H \neq \emptyset$, we have $A_j \cap H = H$ for $j \geq t+1$. Setting $A'_j := (A_j \cup \{x_2\}) \cap X_i$, we see that $A'_j \in \mathcal{F}_i$ for $j \geq t+1$ and that $T_i \subsetneq A'_{t+1} \subsetneq \dots \subsetneq A'_k$. Thus, (c) holds. \square

The rest of the results of this section are concerned with the structure of saturated k -Sperner systems. The next lemma, which is proved in [11], implies that for any saturated k -Sperner system there can be at most one homogeneous set. We include a proof for completeness.

Lemma 9 (Gerbner et al. [11]). *If $\mathcal{F} \subseteq \mathcal{P}(X)$ is a saturated k -Sperner system and H_1 and H_2 are homogeneous for \mathcal{F} , then $H_1 = H_2$.*

Proof. Suppose to the contrary that H_1 and H_2 are homogeneous for \mathcal{F} and that $H_1 \neq H_2$. Then, since each of H_1 and H_2 are maximal with respect to (1), we have that $H_1 \cup H_2$ is not homogeneous for \mathcal{F} . Therefore, there is a set $S \in \mathcal{F}$ which contains some, but not all, of $H_1 \cup H_2$. Without loss of generality, we have $S \cap H_1 = H_1$ and $S \cap H_2 = \emptyset$ since H_1 and H_2 are homogeneous for \mathcal{F} . Now, pick $x \in H_1$ and $y \in H_2$ arbitrarily and define

$$T := (S \setminus \{x\}) \cup \{y\}.$$

Clearly T cannot be in \mathcal{F} since $T \cap H_1 = H_1 \setminus \{x\}$ and H_1 is homogeneous for \mathcal{F} . Since \mathcal{F} is saturated, there must exist sets $A_1, \dots, A_k \in \mathcal{F}$ and $t \in \{0, \dots, k\}$ such that

$$A_1 \subsetneq \dots \subsetneq A_t \subsetneq T \subsetneq A_{t+1} \subsetneq \dots \subsetneq A_k.$$

Since H_1 and H_2 are homogeneous for \mathcal{F} , and neither H_1 nor H_2 is contained in T , we get that $A_t \subsetneq T \setminus (H_1 \cup H_2) \subseteq S$. Similarly, $A_{t+1} \supsetneq S$. However, this implies that $\{A_1, \dots, A_k\} \cup \{S\}$ is a $(k+1)$ -chain in \mathcal{F} , a contradiction. \square

By Lemma 9, if \mathcal{F} is a saturated k -Sperner system for which there exists a homogeneous set, then the homogeneous set must be unique. Throughout the paper, it will be useful to distinguish the elements of \mathcal{F} which contain the homogeneous set from those that do not.

Definition 10. Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a saturated k -Sperner system and let H be homogeneous for \mathcal{F} . We say that a set $S \in \mathcal{F}$ is *large* if $H \subseteq S$ or *small* if $S \cap H = \emptyset$. Let $\mathcal{F}^{\text{large}}$ and $\mathcal{F}^{\text{small}}$ denote the collection of large and small sets of \mathcal{F} , respectively. Thus, $\mathcal{F} = \mathcal{F}^{\text{small}} \cup \mathcal{F}^{\text{large}}$.

Lemma 11. Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a saturated antichain with homogeneous set H . Then every set $S \in \mathcal{P}(X) \setminus \mathcal{A}$ either contains a set in $\mathcal{A}^{\text{small}}$ or is contained in a set of $\mathcal{A}^{\text{large}}$.

Proof. Suppose, to the contrary, that $S \in \mathcal{P}(X) \setminus \mathcal{A}$ does not contain a set of $\mathcal{A}^{\text{small}}$ and is not contained in a set of $\mathcal{A}^{\text{large}}$. Since \mathcal{A} is saturated, we get that either

- (a) there exists $A \in \mathcal{A}^{\text{large}}$ such that $A \subsetneq S$, or
- (b) there exists $B \in \mathcal{A}^{\text{small}}$ such that $S \subsetneq B$.

Suppose that (a) holds. Let $y \in S \setminus A$ and $x \in H$ and define $T := (A \setminus \{x\}) \cup \{y\}$. Since H is homogeneous for \mathcal{A} and $T \cap H = H \setminus \{x\}$, we must have $T \notin \mathcal{A}$. Also, since H is homogeneous for \mathcal{A} , any set $T' \in \mathcal{A}$ containing T would have to contain $T \cup \{x\} \supsetneq A$. Therefore, since \mathcal{A} is an antichain, no such set T' can exist. Thus, there is a set $T'' \in \mathcal{A}$ such that $T'' \subsetneq T \subseteq S$. Since H is homogeneous for \mathcal{A} and $T \cap H \neq H$, we get that $T'' \in \mathcal{A}^{\text{small}}$, contradicting our assumption on S .

Note that we are also done in the case that (b) holds by considering the saturated antichain $\{X \setminus A : A \in \mathcal{A}\}$ and applying the argument of the previous paragraph. \square

2.1 Constructing and Decomposing Saturated k -Sperner Systems

There is a natural way to partition a k -Sperner system $\mathcal{F} \subseteq \mathcal{P}(X)$ into a sequence of k pairwise disjoint antichains. Specifically, for $0 \leq i \leq k-1$, let \mathcal{A}_i be the collection of all minimal elements of $\mathcal{F} \setminus \left(\bigcup_{j < i} \mathcal{A}_j\right)$ under inclusion.

We say that $(\mathcal{A}_i)_{i=0}^{k-1}$ is the *canonical decomposition* of \mathcal{F} into antichains.

In this section we provide conditions under which a sequence of k pairwise disjoint saturated antichains can be united to obtain a saturated k -Sperner system. Later we will prove a partial converse: if $\mathcal{F} \subseteq \mathcal{P}(X)$ is a saturated k -Sperner system with a homogeneous set, then every antichain of the canonical decomposition of \mathcal{F} is saturated. We also provide an example which shows that this is not necessarily the case if we remove the condition that \mathcal{F} has a homogeneous set.

Definition 12. We say that a sequence $(\mathcal{D}_i)_{i=0}^t$ of subsets of $\mathcal{P}(X)$ is *layered* if, for $1 \leq i \leq t$, every $D \in \mathcal{D}_i$ strictly contains some $D' \in \mathcal{D}_{i-1}$ as a subset.

Note that the canonical decomposition of any set system is layered.

Lemma 13. *If $(\mathcal{A}_i)_{i=0}^t$ is a layered sequence of pairwise disjoint saturated antichains, then every $A \in \mathcal{A}_i$ is strictly contained in some $B \in \mathcal{A}_{i+1}$*

Proof. Let $A \in \mathcal{A}_i$. Since \mathcal{A}_{i+1} is a saturated antichain disjoint from \mathcal{A}_i , there exists some $B \in \mathcal{A}_{i+1}$ such that either $B \subsetneq A$ or $A \subsetneq B$. In the latter case we are done, so suppose $B \subsetneq A$. Since $(\mathcal{A}_i)_{i=0}^t$ is layered, there exists some $A' \in \mathcal{A}_i$ such that $A' \subsetneq B$. Hence we have $A' \subsetneq B \subsetneq A$, contradicting the fact that \mathcal{A}_i is an antichain and completing the proof. \square

Lemma 14. *If $(\mathcal{A}_i)_{i=0}^{k-1}$ is a layered sequence of pairwise disjoint saturated antichains in $\mathcal{P}(X)$, then $\mathcal{F} := \bigcup_{i=0}^{k-1} \mathcal{A}_i$ is a saturated k -Sperner system.*

Proof. Clearly, \mathcal{F} is a k -Sperner system since $\mathcal{A}_0, \dots, \mathcal{A}_{k-1}$ are antichains. Let $S \in \mathcal{P}(X) \setminus \mathcal{F}$ be arbitrary and define $t = \max\{i : S \supsetneq A \text{ for some } A \in \mathcal{A}_i\}$. Then, if $t \geq 0$, S strictly contains some set $A_t \in \mathcal{A}_t$. As $(\mathcal{A}_i)_{i=0}^{k-1}$ is layered, for $0 \leq i \leq t-1$, there exist sets $A_i \in \mathcal{A}_i$ such that

$$A_0 \subsetneq \dots \subsetneq A_t \subsetneq S.$$

Now, since \mathcal{A}_{t+1} is a saturated antichain and S does not contain a set of \mathcal{A}_{t+1} , there must exist $A_{t+1} \in \mathcal{A}_{t+1}$ such that $S \subsetneq A_{t+1}$. By Lemma 13, we see that for $t+2 \leq i \leq k-1$ there exists $A_i \in \mathcal{A}_i$ such that

$$S \subsetneq A_{t+1} \subsetneq \cdots \subsetneq A_{k-1}.$$

Thus $\{A_0, \dots, A_{k-1}\} \cup \{S\}$ is a $(k+1)$ -chain, as desired. \square

In Lemma 14, we require the sequence $(\mathcal{A}_i)_{i=0}^{k-1}$ of saturated antichains to be layered. As it turns out, if each antichain \mathcal{A}_i has a homogeneous set, then $(\mathcal{A}_i)_{i=0}^{k-1}$ is layered if and only if $(\mathcal{A}_i^{\text{small}})_{i=0}^{k-1}$ is layered.

Lemma 15. *Let $(\mathcal{A}_i)_{i=0}^{k-1}$ be a sequence of pairwise disjoint saturated antichains in $\mathcal{P}(X)$, each of which has a homogeneous set. Then $(\mathcal{A}_i)_{i=0}^{k-1}$ is layered if and only if $(\mathcal{A}_i^{\text{small}})_{i=0}^{k-1}$ is layered.*

Proof. Suppose that $(\mathcal{A}_i)_{i=0}^{k-1}$ is layered and, for some $i \geq 0$, let $A \in \mathcal{A}_{i+1}^{\text{small}}$ be arbitrary. We show that A contains a set of $\mathcal{A}_i^{\text{small}}$. Otherwise, since $(\mathcal{A}_i)_{i=0}^{k-1}$ is layered, we get that there is some $B \in \mathcal{A}_i^{\text{large}}$ such that $B \subsetneq A$. Therefore, since \mathcal{A}_i is an antichain, A cannot be contained in an element of $\mathcal{A}_i^{\text{large}}$. By Lemma 11 and the fact that \mathcal{A}_i and \mathcal{A}_{i+1} are disjoint, we get that A contains a set of $\mathcal{A}_i^{\text{small}}$, as desired.

Now, suppose that $(\mathcal{A}_i^{\text{small}})_{i=0}^{k-1}$ is layered. Given $i \geq 0$ and $S \in \mathcal{A}_{i+1}^{\text{large}}$, we show that S contains a set of \mathcal{A}_i , which will complete the proof. If not, then since \mathcal{A}_i is saturated and disjoint from \mathcal{A}_{i+1} , there must exist $T \in \mathcal{A}_i$ such that $S \subsetneq T$. Since \mathcal{A}_{i+1} is an antichain, T cannot be contained in a set of $\mathcal{A}_{i+1}^{\text{large}}$. Therefore, by Lemma 11, there is a set $A \in \mathcal{A}_{i+1}^{\text{small}}$ contained in T . However, since $(\mathcal{A}_i^{\text{small}})_{i=0}^{k-1}$ is layered, there exists $A' \in \mathcal{A}_i^{\text{small}}$ such that $A' \subsetneq A$. But then, $A' \subsetneq T$, which contradicts the assumption that \mathcal{A}_i is an antichain. The result follows. \square

It is natural to wonder whether a converse to Lemma 14 is true. That is: *if \mathcal{F} is a saturated k -Sperner system, can we decompose \mathcal{F} into a layered sequence of k pairwise disjoint saturated antichains?* The following example shows that this is not always the case.

Example 16. Let $X := \{x_1, x_2, x_3\}$, $Y := \{y_1, y_2, y_3\}$ and $Z := X \cup Y$. We define

$$\mathcal{B}_0 := \{\{x_i, x_j\} : i \neq j\} \cup \{\{x_i, y_i\} : i \in \{1, 2, 3\}\} \cup \{\{x_k, y_i, y_j\} : i, j, k \text{ distinct}\} \cup \{Y\},$$

$$\mathcal{B}_1 := \{X, \{x_1, x_2, y_1\}, \{x_1, x_3, y_3\}, \{x_2, x_3, y_2\}, \{x_1, y_1, y_3\}, \{x_2, y_1, y_2\}, \{x_3, y_2, y_3\}, \\ \{x_1, x_2, y_2, y_3\}, \{x_1, x_3, y_1, y_2\}, \{x_2, x_3, y_1, y_3\}\}.$$

Then $(\mathcal{B}_i)_{i=0}^1$ is a layered sequence of disjoint antichains. In fact, $(\mathcal{B}_i)_{i=0}^1$ is the canonical decomposition of $\mathcal{F} := \mathcal{B}_0 \cup \mathcal{B}_1$. Clearly \mathcal{B}_1 is not saturated as $\mathcal{B}_1 \cup \{Y\}$ is an antichain. We claim that \mathcal{F} is a saturated 2-Sperner system.

Consider any $S \in \mathcal{P}(Z) \setminus \mathcal{F}$. We will show that $\mathcal{F} \cup \{S\}$ contains a 3-chain. It is easy to check that every element of $\mathcal{B}_0 \setminus \{Y\}$ is contained in a set of \mathcal{B}_1 . Hence if S is contained in some set $B \in \mathcal{B}_0 \setminus \{Y\}$, then $\mathcal{F} \cup \{S\}$ contains a 3-chain. In particular, this completes the proof when $|S| \in \{0, 1, 2\}$. Similarly, since $(\mathcal{B}_i)_{i=0}^1$ is layered, if S contains some set $B \in \mathcal{B}_1$, then $\mathcal{F} \cup \{S\}$ contains a 3-chain. Therefore, we are done if $|S| \in \{4, 5, 6\}$.

It remains to consider the case that $|S| = 3$. Since $X, Y \in \mathcal{F}$, we must have $|S \cap Y| = 2$, or $|S \cap X| = 2$. If $|S \cap Y| = 2$, we have $S \in \{\{x_1, y_1, y_2\}, \{x_2, y_2, y_3\}, \{x_3, y_1, y_3\}\}$. This implies that S is contained in a set $B \in \mathcal{B}_1$ and contains a set $B' \in \mathcal{B}_0 \cap \mathcal{P}(X)$. If $|S \cap X| = 2$, then S contains some set $\{x_i, x_j\} \in \mathcal{B}_0$. Also, it is easily verified that S is contained in a set of \mathcal{B}_1 . Thus, \mathcal{F} is a saturated 2-Sperner system.

However, for saturated k -Sperner systems with a homogeneous set, the converse to Lemma 14 does hold; we can partition \mathcal{F} into a layered sequence of k pairwise disjoint saturated antichains.

Lemma 17. *Let $\mathcal{F} \in \mathcal{P}(X)$ be a saturated k -Sperner system with homogeneous set H and canonical decomposition $(\mathcal{A}_i)_{i=0}^{k-1}$. Then \mathcal{A}_i is saturated for all i .*

Proof. Fix i and let $S \in \mathcal{P}(X) \setminus \mathcal{A}_i$. Let $x \in H$ and define

$$T := (S \setminus H) \cup \{x\}.$$

Then $T \notin \mathcal{F}$ since $T \cap H = \{x\}$ and H is homogeneous for \mathcal{F} . Therefore, there exists $\{A_0, \dots, A_{k-1}\} \subseteq \mathcal{F}$ and $t \in \{0, \dots, k\}$ such that

$$A_0 \subsetneq \dots \subsetneq A_{t-1} \subsetneq T \subsetneq A_t \subsetneq \dots \subsetneq A_{k-1}.$$

By definition of the canonical decomposition, we must have $A_j \in \mathcal{A}_j$ for all j . Also, since H is homogeneous for \mathcal{F} and $T \cap H \notin \{\emptyset, H\}$, we must have $A_{t-1} \subseteq T \setminus H \subseteq S$ and $A_t \supseteq T \cup H \supseteq S$. Therefore,

$$A_0 \subsetneq \dots \subsetneq A_{t-1} \subseteq S \subseteq A_t \subsetneq \dots \subsetneq A_{k-1}.$$

Since $S \neq A_i$, we must have either $A_i \subsetneq S$ or $S \subsetneq A_i$. Therefore, \mathcal{A}_i is saturated for all i . \square

3 Combining Saturated k -Sperner Systems

Our first goal in this section is to prove that, under certain conditions, a saturated k_1 -Sperner system $\mathcal{F}_1 \subseteq \mathcal{P}(X_1)$ and a saturated k_2 -Sperner system $\mathcal{F}_2 \subseteq \mathcal{P}(X_2)$ can be combined to yield a saturated $(k_1 + k_2 - 2)$ -Sperner system in $\mathcal{P}(X_1 \cup X_2)$. We apply this result to prove Theorem 3. Afterwards, we prove that $\text{sat}(k) = 2^{k-1}$ for $k \leq 5$. We conclude the section with a proof of Theorem 4.

Lemma 18. *Let X_1 and X_2 be disjoint sets. For $i \in \{1, 2\}$, let $\mathcal{F}_i \subseteq \mathcal{P}(X_i)$ be a saturated k_i -Sperner system which contains $\{\emptyset, X_i\}$ and let $H_i \subseteq X_i$ be homogeneous for \mathcal{F}_i . If \mathcal{G} is the set system on $\mathcal{P}(X_1 \cup X_2)$ defined by*

$$\mathcal{G} := \{A \cup B : A \in \mathcal{F}_1^{\text{small}}, B \in \mathcal{F}_2^{\text{small}}\} \cup \{S \cup T : S \in \mathcal{F}_1^{\text{large}}, T \in \mathcal{F}_2^{\text{large}}\},$$

then \mathcal{G} is a saturated $(k_1 + k_2 - 2)$ -Sperner system which contains $\{\emptyset, X_1 \cup X_2\}$ and $H_1 \cup H_2$ is homogeneous for \mathcal{G} .

Proof. It is clear that \mathcal{G} contains $\{\emptyset, X_1 \cup X_2\}$ and that $H_1 \cup H_2$ is homogeneous for \mathcal{G} . We show that \mathcal{G} is a saturated $(k_1 + k_2 - 2)$ -Sperner system.

First, let us show that \mathcal{G} does not contain a chain of length $k_1 + k_2 - 1$. Suppose that $\{A_1, \dots, A_r\}$ is an r -chain in \mathcal{G} . We can assume that $A_1 = \emptyset$ and $A_r = X_1 \cup X_2$. Define

$$I_1 := \{i : A_i \cap X_1 \subsetneq A_{i+1} \cap X_1\}, \text{ and}$$

$$I_2 := \{i : A_i \cap X_2 \subsetneq A_{i+1} \cap X_2\}.$$

Clearly, $I_1 \cup I_2 = \{1, \dots, r-1\}$. Also, for $i \in \{1, 2\}$, since \mathcal{F}_i is a k_i -Sperner system, we must have $|I_i| \leq k_i - 1$. Let t be the maximum index such that $A_t \cap X_1 \in \mathcal{F}_1^{\text{small}}$. Note that t exists and is less than r since $A_1 = \emptyset$ and $A_r = X_1 \cup X_2$. By construction of \mathcal{G} , $A_t \cap X_2$ is a small set for \mathcal{F}_2 and, for $i \in \{1, 2\}$, $A_{t+1} \cap X_i$ is a large set for \mathcal{F}_i . This implies that $t \in I_1 \cap I_2$ and so

$$r - 1 = |I_1 \cup I_2| = |I_1| + |I_2| - |I_1 \cap I_2| \leq k_1 + k_2 - 3$$

as required.

Now, let $S \in \mathcal{P}(X_1 \cup X_2) \setminus \mathcal{G}$. We show that $\mathcal{G} \cup \{S\}$ contains a $(k_1 + k_2 - 1)$ -chain. Fix $x_1 \in H_1$ and $x_2 \in H_2$ and define

$$T := (S \setminus (H_1 \cup H_2)) \cup \{x_1, x_2\}.$$

For $i \in \{1, 2\}$, let $T_i := T \cap X_i$. Then $T_i \notin \mathcal{F}_i$ since $T_i \cap H_i = \{x_i\}$. Therefore, there exists $A_1^i, \dots, A_{k_i}^i \in \mathcal{F}_i$ and $t_i \in \{1, \dots, k_i - 1\}$ such that

$$\emptyset = A_1^i \subsetneq \dots \subsetneq A_{t_i}^i \subsetneq T_i \subsetneq A_{t_i+1}^i \subsetneq \dots \subsetneq A_{k_i}^i = X_i$$

Note that $A_j^i \in \mathcal{F}_i^{\text{small}}$ for $j \leq t_i$ and $A_j^i \in \mathcal{F}_i^{\text{large}}$ for $j \geq t_i + 1$. This implies that $A_{t_1}^1 \cup A_{t_2}^2 \subsetneq S$ and $A_{t_1+1}^1 \cup A_{t_2+1}^2 \supsetneq S$. Therefore,

$$\begin{aligned} & A_1^1 \cup A_1^2 \subsetneq A_1^1 \cup A_2^2 \subsetneq \dots \subsetneq A_1^1 \cup A_{t_2}^2 \subsetneq A_2^1 \cup A_{t_2}^2 \subsetneq \dots \subsetneq A_{t_1}^1 \cup A_{t_2}^2 \subsetneq S \\ & \subsetneq A_{t_1+1}^1 \cup A_{t_2+1}^2 \subsetneq A_{t_1+1}^1 \cup A_{t_2+2}^2 \subsetneq \dots \subsetneq A_{t_1+1}^1 \cup A_{k_2}^2 \subsetneq A_{t_1+2}^1 \cup A_{k_2}^2 \subsetneq \dots \subsetneq A_{k_1}^1 \cup A_{k_2}^2 \\ & \text{and so } \mathcal{G} \cup \{S\} \text{ contains a } (k_1 + k_2 - 1)\text{-chain. The result follows. } \square \end{aligned}$$

Remark 19. If $\mathcal{F}_1, \mathcal{F}_2$ and \mathcal{G} are as in Lemma 18, then

$$|\mathcal{G}| = |\mathcal{F}_1^{\text{small}}| |\mathcal{F}_2^{\text{small}}| + |\mathcal{F}_1^{\text{large}}| |\mathcal{F}_2^{\text{large}}|.$$

3.1 Proof of Theorem 3

We apply Lemma 18 to prove Theorem 3. The first part of the proof of Theorem 3 is to exhibit an infinite family of saturated 6-Sperner systems with cardinality $30 < 2^5$.

Proposition 20. *For any set X such that $|X| \geq 8$, there is a saturated 6-Sperner system $\mathcal{F} \subseteq \mathcal{P}(X)$ with a homogeneous set such that $|\mathcal{F}^{\text{small}}| = |\mathcal{F}^{\text{large}}| = 15$.*

Proof. Let X be a set such that $|X| \geq 8$. Let x_1, x_2, y_1, y_2, w and z be distinct elements of X and define $H := X \setminus \{x_1, x_2, y_1, y_2, w, z\}$. We apply Lemma 14 to construct a saturated 6-Sperner system $\mathcal{F} \subseteq \mathcal{P}(X)$ of order 30. Naturally, we define $\mathcal{A}_0 = \{\emptyset\}$ and $\mathcal{A}_5 = \{X\}$. Also, define

$$\mathcal{A}_1 := \{\{x_1\}, \{x_2\}, \{y_1\}, \{y_2\}, \{w\}, H \cup \{y_2, z\}\}, \text{ and}$$

$$\mathcal{A}_4 := \{X \setminus A : A \in \mathcal{A}_1\}.$$

It is easily observed that \mathcal{A}_1 and \mathcal{A}_4 are saturated antichains. We define \mathcal{A}_2 and \mathcal{A}_3 by first specifying their small sets. Define

$$\mathcal{A}_2^{\text{small}} := \{\{x_i, y_j\} : 1 \leq i, j \leq 2\} \cup \{\{w, z\}\}, \text{ and}$$

$$\mathcal{A}_3^{\text{small}} := \{\{x_1, y_1, w\}, \{x_1, y_1, z\}, \{x_2, y_2, w\}, \{x_2, y_2, z\}\}.$$

Given any collection $\mathcal{B} \subseteq \mathcal{P}(X)$, a set $S \subseteq X$ is said to be *stable* for \mathcal{B} if S does not contain an element of \mathcal{B} . For $i = 2, 3$, define $\mathcal{A}_i^{\text{large}}$ to be the collection consisting of all maximal stable sets of $\mathcal{A}_i^{\text{small}}$ and let $\mathcal{A}_i := \mathcal{A}_i^{\text{small}} \cup \mathcal{A}_i^{\text{large}}$. Note that every element of $\mathcal{A}_i^{\text{large}}$ contains H . It is clear that \mathcal{A}_i is an antichain for $i = 2, 3$. Moreover, \mathcal{A}_i is saturated since every set $A \in \mathcal{P}(X)$ either contains an element of $\mathcal{A}_i^{\text{small}}$ or is contained in an element of $\mathcal{A}_i^{\text{large}}$.

One can easily verify that $(\mathcal{A}_i^{\text{small}})_{i=0}^5$ is layered. Therefore, by Lemma 15, $(\mathcal{A}_i)_{i=0}^5$ is a layered sequence of pairwise disjoint saturated antichains. By Lemma 14, $\mathcal{F} := \bigcup_{i=0}^5 \mathcal{A}_i$ is a saturated 6-Sperner system. Also,

$$|\mathcal{F}^{\text{small}}| = \sum_{i=0}^5 |\mathcal{A}_i^{\text{small}}| = (1 + 5 + 9 + 0) = 15, \text{ and}$$

$$|\mathcal{F}^{\text{large}}| = \sum_{i=0}^5 |\mathcal{A}_i^{\text{large}}| = (0 + 9 + 5 + 1) = 15,$$

as desired. \square

We remark that the construction in Proposition 20 is similar to one which was used in [11] to prove that $\text{sat}(k, k) \leq \frac{15}{16} 2^{k-1}$ for every $k \geq 6$.

For the proof of Theorem 3 we require that

$$\text{sat}(k) \leq 2 \text{sat}(k-1). \quad (2)$$

This was proved in [11] using the fact that if $\mathcal{F} \subseteq \mathcal{P}(X)$ is a saturated $(k-1)$ -Sperner system and $y \notin X$, then $\mathcal{F} \cup \{A \cup \{y\} : A \in \mathcal{F}\}$ is a saturated k -Sperner system in $\mathcal{P}(X \cup \{y\})$.

Proof of Theorem 3. First, we prove that the result holds when k is of the form $4j+2$ for some $j \geq 1$. In this case, we repeatedly apply Lemma 18 and Proposition 20 to obtain a saturated k -Sperner system \mathcal{F} on an arbitrarily large ground set X such that

$$|\mathcal{F}^{\text{small}}| + |\mathcal{F}^{\text{large}}| = 15^j + 15^j = 2 \cdot 15^j.$$

Therefore, if $k = 4j+2$, then $\text{sat}(k) \leq 2 \cdot 15^j$.

For k of the form $4j+2+s$ for $j \geq 1$ and $1 \leq s \leq 3$, apply (2) to obtain $\text{sat}(k) \leq 2^s \text{sat}(4j+2) \leq 2^{s+1} \cdot 15^j$. Thus, we are done by setting ε slightly smaller than $\left(1 - \frac{\log_2(15)}{4}\right)$. \square

3.2 Bounding $\text{sat}(k)$ From Below

One can easily deduce from the proof of Theorem 3 that $\text{sat}(k) < 2^{k-1}$ for all $k \geq 6$. For completeness, we prove that $\text{sat}(k) = 2^{k-1}$ for $k \leq 5$.

Proposition 21. *If $k \leq 5$, then $\text{sat}(k) = 2^{k-1}$.*

Proof. Fix $k \leq 5$. The upper bound follows from Construction 2, and so it suffices to prove that $\text{sat}(k) \geq 2^{k-1}$. Let X be a set with $n := |X| > 2^{2^{k-1}}$ and let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a saturated k -Sperner system of minimum order. By Claim 8 and the fact that $|X| > 2^{2^{k-1}} \geq 2^{|\mathcal{F}|}$, there is a homogeneous set H for \mathcal{F} .

Let $(\mathcal{A}_i)_{i=0}^{k-1}$ be the canonical decomposition of \mathcal{F} . By Lemma 17, we get that \mathcal{A}_i is a saturated antichain for each i . Also, since $(\mathcal{A}_i)_{i=0}^{k-1}$ is layered, by Lemma 13 we see that

$$\text{every element of } \mathcal{A}_i \text{ has cardinality between } i \text{ and } n - k + i + 1. \quad (3)$$

Our goal is to show that for $k \leq 5$, every saturated antichain \mathcal{A}_i which satisfies (3) must contain at least $\binom{k-1}{i}$ elements. Clearly this is enough to complete the proof of the proposition. Note that it suffices to prove this for $i < \frac{k}{2}$ since $\{X \setminus A : A \in \mathcal{A}_i\}$ is a saturated antichain in which every set has size between $k - i - 1$ and $n - i$. Since $k \leq 5$, this means that we need only check the cases $i = 0, 1, 2$. In the case $i = 0$, we obtain $|\mathcal{A}_0| \geq 1 = \binom{k-1}{0}$ trivially.

Next, consider the case $i = 1$. Let A be the largest set in \mathcal{A}_1 such that $H \subseteq A$. Then, by (3), we must have $|A| \leq n - k + 2$ and so $|X \setminus A| \geq k - 2$. Fix an element x of H and, for each $y \in X \setminus A$, define $A_y := (A \setminus \{x\}) \cup \{y\}$. Since \mathcal{A}_1 is saturated, H is homogeneous for \mathcal{F} , and A is the largest set in \mathcal{A}_1 containing H , there must be a set $B_y \in \mathcal{A}_1$ such that $B_y \subsetneq A_y$. However, since \mathcal{A}_1 is an antichain, $B_y \not\subseteq A$, and so $B_y \setminus A = \{y\}$. In particular, $B_y \neq B_{y'}$ for $y \neq y'$. Therefore, $|\mathcal{A}_1| \geq |\{A\} \cup \{B_y : y \in X \setminus A\}| \geq 1 + |X \setminus A| \geq k - 1 = \binom{k-1}{1}$, as desired.

Thus, we are finished except for the case $i = 2$ and $k = 5$. Suppose to the contrary that $|\mathcal{A}_2| < \binom{4}{2} = 6$. We begin by proving the following claim.

Claim 22. *For every vertex $y \in X \setminus H$, there is a set $S_y \in \mathcal{A}_2^{\text{large}}$ containing y .*

Proof. Let $x \in H$ be arbitrary and consider the set $T := \{x, y\}$. Then T is not contained in \mathcal{A}_2 since H is homogeneous for \mathcal{F} . Also, no strict subset

of T is in \mathcal{A}_2 by (3). Since \mathcal{A}_2 is saturated, there must be some $S_y \in \mathcal{A}_2^{\text{large}}$ containing T , which completes the proof. \square

Let us argue that $|\mathcal{A}_2^{\text{large}}| \geq 3$. By (3), each set $A \in \mathcal{A}_2^{\text{large}}$ has at most $n - 2$ elements. So, by Claim 22, if $|\mathcal{A}_2^{\text{large}}| < 3$, then it must be the case that $\mathcal{A}_2^{\text{large}} = \{A_1, A_2\}$ where $A_1 \cup A_2 = X$. Therefore, since each of $|A_1|$ and $|A_2|$ is at most $n - 2$, we can pick $\{w_1, w_2\} \subseteq A_1 \setminus A_2$ and $\{z_1, z_2\} \subseteq A_2 \setminus A_1$. Given $x \in H$ and $1 \leq i, j \leq 2$, we have that $\{x, w_i, z_j\} \notin \mathcal{A}_2$ since H is homogeneous for \mathcal{F} . Note that $\{x, w_i, z_j\}$ is not contained in either A_1 or A_2 , and so by Lemma 11 and (3) we must have $\{w_i, z_j\} \in \mathcal{A}_2$. However, this implies that $|\mathcal{A}_2| \geq |\{\{w_i, z_j\} : 1 \leq i, j \leq 2\} \cup \{A_1, A_2\}| = 6$, a contradiction.

So, we get that $|\mathcal{A}_2^{\text{large}}| \geq 3$. Note that $\{X \setminus A : A \in \mathcal{A}_2\}$ is also a saturated antichain in which every set has cardinality between 2 and $n - 2$. Thus, we can apply the argument of the previous paragraph to obtain $|\mathcal{A}_2^{\text{small}}| \geq 3$. Therefore, $|\mathcal{A}_2| = |\mathcal{A}_2^{\text{small}}| + |\mathcal{A}_2^{\text{large}}| \geq 6$, which completes the proof. \square

It is possible that a similar approach may prove fruitful for improving the lower bound on $\text{sat}(k)$ from $2^{k/2-1}$ to $2^{(1+o(1))ck}$ for some $c > 1/2$. That is, one may first decompose a saturated k -Sperner system $\mathcal{F} \subseteq \mathcal{P}(X)$ of minimum size into its canonical decomposition $(\mathcal{A}_i)_{i=0}^{k-1}$ and then bound the size of $|\mathcal{A}_i|$ for each i individually. Since there are only k antichains in the decomposition and the bound on $|\mathcal{F}|$ that we are aiming for is exponential in k , one could obtain a fairly tight lower bound on $\text{sat}(k)$ by focusing on a single antichain of the decomposition. Setting $i = \lfloor \frac{k}{2} \rfloor$ in (3), we see that it would be sufficient to prove that there exists $c > 1/2$ such that every saturated antichain \mathcal{A} with a homogeneous set such that every element of \mathcal{A} has cardinality between $\lfloor \frac{k}{2} \rfloor$ and $n - \lceil \frac{k}{2} \rceil + 1$ must satisfy $|\mathcal{A}| \geq 2^{(1+o(1))ck}$.

3.3 Asymptotic Behaviour of $\text{sat}(k)$

To prove Theorem 4, we require the following fact, which is proved in [11].

Lemma 23 (Gerbner et al. [11]). *For any $n \geq k \geq 1$ and set X with $|X| = n$ there is a saturated k -Sperner system $\mathcal{F} \subseteq \mathcal{P}(X)$ such that $|\mathcal{F}| = \text{sat}(n, k)$ and $\{\emptyset, X\} \subseteq \mathcal{F}$.*

Proof. Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a saturated k -Sperner system such that $|\mathcal{F}| = \text{sat}(n, k)$. We let $(\mathcal{A}_i)_{i=0}^{k-1}$ denote the canonical decomposition of \mathcal{F} and define

$$\mathcal{F}' := (\mathcal{F} \setminus (\mathcal{A}_0 \cup \mathcal{A}_{k-1})) \cup \{\emptyset, X\}.$$

It is clear that $\mathcal{F}' \subseteq \mathcal{P}(X)$ is a saturated k -Sperner system and $|\mathcal{F}'| \leq |\mathcal{F}| = \text{sat}(n, k)$, which proves the result. \square

Proof of Theorem 4. We show that, for all k, ℓ ,

$$\text{sat}(k + \ell) \leq 4 \text{sat}(k) \text{sat}(\ell). \quad (4)$$

Letting $f(k) := 4 \text{sat}(k)$, we see that (4) implies that $f(k + \ell) \leq f(k)f(\ell)$ for every k, ℓ . It follows by Fekete's Lemma that $f(k)^{1/k}$ converges, and so $\text{sat}(k)^{1/k}$ converges as well.

For $n > 2^{2^{k+\ell-2}}$, let X and Y be disjoint sets of size n and let $\mathcal{F}_k \subseteq \mathcal{P}(X)$ and $\mathcal{F}_\ell \subseteq \mathcal{P}(Y)$ be saturated k -Sperner and ℓ -Sperner systems of cardinalities $\text{sat}(k)$ and $\text{sat}(\ell)$, respectively. By Claim 8, we can assume that \mathcal{F}_k and \mathcal{F}_ℓ have homogeneous sets and, by Lemma 23, we can assume that $\{\emptyset, X\} \subseteq \mathcal{F}_k$ and $\{\emptyset, Y\} \subseteq \mathcal{F}_\ell$. We apply Lemma 18 and Remark 19 to obtain a saturated $(k + \ell - 2)$ -Sperner system $\mathcal{G} \subseteq \mathcal{P}(X \cup Y)$ of order at most $|\mathcal{F}_k||\mathcal{F}_\ell| = \text{sat}(k) \text{sat}(\ell)$. Therefore, by (2), we have

$$\text{sat}(k + \ell) \leq 4 \text{sat}(k + \ell - 2) \leq 4|\mathcal{G}| \leq 4 \text{sat}(k) \text{sat}(\ell)$$

as required. \square

4 Oversaturated k -Sperner Systems

In this section we construct oversaturated k -Sperner systems of small order. We first state a lemma, from which Theorem 6 follows, and then prove the lemma itself.

Lemma 24. *Given $k \geq 1$, let X be a set of cardinality $k^2 + k$. Then for all t such that $1 \leq t \leq k^2 + k$ there exist non-empty collections $\mathcal{F}_t, \mathcal{G}_t \subseteq \mathcal{P}(X)$ that have the following properties:*

- (a) *For every $F \in \mathcal{F}_t$ and $G \in \mathcal{G}_t$, $|F| + |G| \geq k$,*
- (b) *$|\mathcal{F}_t| + |\mathcal{G}_t| = O(k^2 2^{k/2})$,*

(c) For every $S \subseteq X$ such that $|S| = t$, there exists some $F \in \mathcal{F}_t$ and some $G \in \mathcal{G}_t$ such that $F \subsetneq S$ and $G \cap S = \emptyset$.

We apply Lemma 24 to prove Theorem 6.

Proof of Theorem 6. First, let X be a set of cardinality $k^2 + k$. For $t \in \{1, \dots, k^2 + k\}$, let \mathcal{F}_t and \mathcal{G}_t be as in Lemma 24. For each $F \in \mathcal{F}_t \cup \mathcal{G}_t$, choose $F_1, \dots, F_i \in \mathcal{P}(X)$ such that

$$F_1 \subsetneq \dots \subsetneq F_i \subsetneq F$$

where $i := \min\{k - 1, |F|\}$. We let $\mathcal{C}_F := F \cup \{F_1, \dots, F_i\}$ and define

$$\mathcal{G} := \bigcup_{1 \leq t \leq k^2 + k} (\{T : T \in \mathcal{C}_F \text{ for some } F \in \mathcal{F}_t\} \cup \{X \setminus T : T \in \mathcal{C}_G \text{ for some } G \in \mathcal{G}_t\}).$$

For each $t \leq k^2 + k$ and $F \in \mathcal{F}_t \cup \mathcal{G}_t$, we have $|\mathcal{C}_F| \leq k$. Thus, by Property (b) of Lemma 24,

$$|\mathcal{G}| \leq \sum_{t=1}^{k^2+k} k(|\mathcal{F}_t| + |\mathcal{G}_t|) = O(k^5 2^{k/2}).$$

We will now show that for any $S \in \mathcal{P}(X) \setminus \mathcal{G}$ there is a $(k + 1)$ -chain in $\mathcal{G} \cup \{S\}$ containing S , which will imply that \mathcal{G} is an oversaturated k -Sperner system. Let $S \subseteq X$ and define $t := |S|$. By Property (c) of Lemma 24, there exists $F \in \mathcal{F}_t$ such that $F \subsetneq S$ and $G \in \mathcal{G}_t$ such that $G \cap S = \emptyset$. This implies that $S \subsetneq X \setminus G$. By Property (a) of Lemma 24 we get that

$$\mathcal{C}_F \cup \{X \setminus T : T \in \mathcal{C}_G\} \cup \{S\}$$

contains a $(k + 1)$ -chain in $\mathcal{G} \cup \{S\}$ containing S .

Now, suppose that $|X| > k^2 + k$. Let $Y \subseteq X$ such that $|Y| = k^2 + k$ and define $H := X \setminus Y$. As above, let $\mathcal{G} \subseteq \mathcal{P}(Y)$ be an oversaturated k -Sperner system of cardinality at most $O(k^5 2^{k/2})$. Define $\mathcal{G}' \subseteq \mathcal{P}(X)$ as follows:

$$\mathcal{G}' := \{T : T \in \mathcal{G}\} \cup \{T \cup H : T \in \mathcal{G}\}.$$

Consider any set $S \in \mathcal{P}(X) \setminus \mathcal{G}'$. Let $S' = S \cap Y$. We have, by definition of \mathcal{G} , that there is a $(k + 1)$ -chain \mathcal{C} in $\mathcal{G} \cup \{S'\}$ containing S' . Adding H to every superset of S' in \mathcal{C} and replacing S' by S in \mathcal{C} gives us a $(k + 1)$ -chain in $\mathcal{G}' \cup \{S\}$ containing S . The result follows. \square

To prove Lemma 24, we use a probabilistic approach.

Proof of Lemma 24. Throughout the proof, we assume that k is sufficiently large and let X be a set of cardinality $k^2 + k$. Let $1 \leq t \leq k^2 + k$ be given. We can assume that $t \leq \frac{k^2 + k}{2}$ since, otherwise, we can simply define $\mathcal{F}_t := \mathcal{G}_{k^2 + k - t}$ and $\mathcal{G}_t := \mathcal{F}_{k^2 + k - t}$. We divide the proof into two cases depending on the size of t .

Case 1: $t \leq \frac{k^2 + k}{8}$.

We define $\mathcal{F}_t := \{\emptyset\}$ and let \mathcal{G}_t be a uniformly random collection of $2^{k/2}$ subsets of X , each of cardinality k . Given $S \subseteq X$ of cardinality t , the probability that S is not disjoint from any set of \mathcal{G}_t is

$$\begin{aligned} \left(1 - \prod_{i=0}^{k-1} \left(\frac{k^2 + k - t - i}{k^2 + k - i}\right)\right)^{2^{k/2}} &\leq \left(1 - \left(\frac{k^2 - t}{k^2}\right)^k\right)^{2^{k/2}} \leq \left(1 - \left(\frac{7}{8} - \frac{1}{8k}\right)^k\right)^{2^{k/2}} \\ &\leq e^{-\left(\frac{7}{8} - \frac{1}{8k}\right)^k 2^{k/2}} < e^{-(1.1)^k}. \end{aligned}$$

Therefore, the expected number of subsets of X of cardinality t which are not disjoint from any set of \mathcal{G}_t is at most $\binom{k^2 + k}{t} e^{-(1.1)^k}$, which is less than 1. Thus, with non-zero probability, every $S \subseteq X$ of cardinality t is disjoint from some set in \mathcal{G}_t .

Case 2: $\frac{k^2 + k}{8} < t \leq \frac{k^2 + k}{2}$.

Define $p := \frac{t}{k^2 + k}$ and let a be the rational number such that $ak = \left\lfloor \frac{-k \log \sqrt{2}}{\log(p)} + 1 \right\rfloor$. Then, since $\frac{1}{8} \leq p \leq \frac{1}{2}$, we have

$$1/6 \leq a \leq 1/2 + 1/k < 4/7. \quad (5)$$

Now, let \mathcal{F}_t be a collection of $\lceil 8e^8 k^2 2^{k/2} \rceil$ subsets of X , each of cardinality ak , chosen uniformly at random with replacement. Similarly, let \mathcal{G}_t be a collection of $\lceil e^2 k^2 2^{k/2} \rceil$ subsets of X , each of cardinality $(1 - a)k$, chosen uniformly at random with replacement. We show that, with non-zero probability, every $S \subseteq X$ of size t contains a set of \mathcal{F}_t and is disjoint from a set of \mathcal{G}_t .

Given $S \subseteq X$ of size $t = p(k^2 + k)$, the probability that S does not contain a set of \mathcal{F}_t is at most

$$\begin{aligned} \left(1 - \prod_{i=0}^{ak-1} \left(\frac{p(k^2 + k) - i}{k^2 + k - i}\right)\right)^{|\mathcal{F}_t|} &\leq \left(1 - \left(\frac{p(k^2 + k) - k}{k^2}\right)^{ak}\right)^{|\mathcal{F}_t|} \\ &= \left(1 - \left(1 - \frac{1-p}{pk}\right)^{ak} p^{ak}\right)^{|\mathcal{F}_t|}. \end{aligned} \quad (6)$$

Observe that $\left(1 - \frac{1-p}{pk}\right) \geq e^{-\frac{2(1-p)}{pk}}$ for large enough k . So, $\left(1 - \frac{1-p}{pk}\right)^{ak} \geq e^{\frac{-2a(1-p)}{p}}$ which is at least e^{-8} since $a < 4/7$ and $p \geq 1/8$. Thus, the expression in (6) is at most

$$(1 - e^{-8} p^{ak})^{|\mathcal{F}_t|} \leq e^{-e^{-8} p^{ak} |\mathcal{F}_t|} \leq e^{-e^{-8} p^{ak} (8e^8 k^2 2^{k/2})} = e^{-p^{ak} 8k^2 2^{k/2}}.$$

Using our choice of a and the fact that $p \geq 1/8$, we can bound the exponent by

$$p^{ak} 8k^2 2^{k/2} \geq p^{\left(-\frac{\log \sqrt{2}}{\log(p)} + \frac{1}{k}\right)k} 8k^2 2^{k/2} = p 8k^2 \geq k^2.$$

Therefore, the expected number of subsets of X of size t which do not contain a set of \mathcal{F}_t is at most

$$\binom{k^2 + k}{t} e^{-k^2} < 2^{k^2+k} e^{-k^2}$$

which is less than 1. Thus, with positive probability, every subset of X of cardinality t contains a set of \mathcal{F}_t .

The proof that, with positive probability, every set of cardinality t is disjoint from a set of \mathcal{G}_t is similar; we sketch the details. First, let us note that

$$a \geq \frac{-\log \sqrt{2}}{\log(p)} \geq 1 + \frac{\log \sqrt{2}}{\log(1-p)} \quad (7)$$

since $p \leq 1/2$. For a fixed set $S \subseteq X$ of size $t = p(k^2 + k)$, the probability that S is not disjoint from any set of \mathcal{G}_t is at most

$$\left(1 - \prod_{i=0}^{(1-a)k-1} \left(\frac{(1-p)(k^2 + k) - i}{k^2 + k - i}\right)\right)^{|\mathcal{G}_t|} \leq \left(1 - \left(\frac{(1-p)(k^2 + k) - k}{k^2}\right)^{(1-a)k}\right)^{|\mathcal{G}_t|}$$

$$= \left(1 - \left(1 - \frac{p}{(1-p)k} \right)^{(1-a)k} (1-p)^{(1-a)k} \right)^{|\mathcal{G}_t|} \quad (8)$$

Now, $\left(1 - \frac{p}{(1-p)k} \right) \geq e^{\frac{-2p}{(1-p)k}}$ for large enough k . So, $\left(1 - \frac{p}{(1-p)k} \right)^{(1-a)k} \geq e^{\frac{-2(1-a)p}{(1-p)}}$, which is at least e^{-2} since $a \geq 1/6$ and $\frac{1}{8} \leq p \leq \frac{1}{2}$. Therefore, the expression in (8) is at most

$$\begin{aligned} (1 - e^{-2}(1-p)^{(1-a)k})^{|\mathcal{G}_t|} &\leq e^{-e^{-2}(1-p)^{(1-a)k}|\mathcal{G}_t|} \leq e^{-e^{-2}(1-p)^{(1-a)k}(e^2 k^2 2^{k/2})} \\ &= e^{-(1-p)^{(1-a)k} k^2 2^{k/2}}. \end{aligned}$$

By (7), we can bound the exponent by

$$(1-p)^{(1-a)k} k^2 2^{k/2} \geq (1-p)^{\left(\frac{-\log \sqrt{2}}{\log(1-p)} \right)k} k^2 2^{k/2} \geq k^2.$$

As with \mathcal{F}_t , we get that the expected number of sets of cardinality t which are not disjoint from a set of \mathcal{G}_t is less than one. The result follows. \square

Acknowledgements

The first two authors would like to thank Antonio Girão for many stimulating discussions, one of which lead to the discovery of Example 16.

References

- [1] T. Bohman, M. Fonoferova, and O. Pikhurko, *The saturation function of complete partite graphs*, J. Comb. **1** (2010), no. 2, 149–170.
- [2] B. Bollobás, *On a conjecture of Erdős, Hajnal and Moon*, Amer. Math. Monthly **74** (1967), 178–179.
- [3] B. Bollobás, *Weakly k -saturated graphs*, Beiträge zur Graphentheorie (Kolloquium, Manebach, 1967), Teubner, Leipzig, 1968, pp. 25–31.
- [4] B. Bollobás, *Combinatorics*, Cambridge University Press, Cambridge, 1986, Set systems, hypergraphs, families of vectors and combinatorial probability.

- [5] Y.-C. Chen, *All minimum C_5 -saturated graphs*, J. Graph Theory **67** (2011), no. 1, 9–26.
- [6] P. Erdős, *On a lemma of Littlewood and Offord*, Bull. Amer. Math. Soc. **51** (1945), 898–902.
- [7] P. Erdős, Z. Füredi, and Z. Tuza, *Saturated r -uniform hypergraphs*, Discrete Math. **98** (1991), no. 2, 95–104.
- [8] P. Erdős, A. Hajnal, and J. W. Moon, *A problem in graph theory*, Amer. Math. Monthly **71** (1964), 1107–1110.
- [9] J. R. Faudree, R. J. Faudree, and J. R. Schmitt, *A Survey of Minimum Saturated Graphs*, Electron. J. Combin. **18** (2011), Dynamic Survey 19, 36 pp. (electronic).
- [10] W. Gan, D. Korándi, and B. Sudakov, *$K_{s,t}$ -saturated bipartite graphs*, arXiv: 1402.2471, preprint, February 2014.
- [11] D. Gerbner, B. Keszegh, N. Lemons, C. Palmer, D. Pálvölgyi, and B. Patkós, *Saturating Sperner Families*, Graphs Combin. **29** (2013), no. 5, 1355–1364.
- [12] J. R. Johnson and T. Pinto, *Saturated Subgraphs of the Hypercube*, arXiv:1406.1766, preprint, June 2014.
- [13] L. Kászonyi and Z. Tuza, *Saturated graphs with minimal number of edges*, J. Graph Theory **10** (1986), no. 2, 203–210.
- [14] O. Pikhurko, *The minimum size of saturated hypergraphs*, Combin. Probab. Comput. **8** (1999), no. 5, 483–492.
- [15] O. Pikhurko, *Asymptotic evaluation of the sat-function for r -stars*, Discrete Math. **214** (2000), no. 1-3, 275–278.
- [16] O. Pikhurko, *Results and open problems on minimum saturated hypergraphs*, Ars Combin. **72** (2004), 111–127.
- [17] E. Sperner, *Ein Satz über Untermengen einer endlichen Menge*, Math. Z. **27** (1928), no. 1, 544–548.

- [18] Z. Tuza, *C_4 -saturated graphs of minimum size*, Acta Univ. Carolin. Math. Phys. **30** (1989), no. 2, 161–167, 17th Winter School on Abstract Analysis (Srní, 1989).
- [19] W. Wessel, *Über eine Klasse paarer Graphen. I. Beweis einer Vermutung von Erdős, Hajnal und Moon*, Wiss. Z. Techn. Hochsch. Ilmenau **12** (1966), 253–256.
- [20] W. Wessel, *Über eine Klasse paarer Graphen. II. Bestimmung der Minimalgraphen*, Wiss. Z. Techn. Hochsch. Ilmenau **13** (1967), 423–426.