Indifference Pricing in a Basis Risk Model with Stochastic Volatility

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Abstract

The aim of this dissertation is to study exponential indifference pricing in a basis risk model of one tradable asset and one correlated non-tradable asset in which a claim on the non-tradable asset is hedged using the tradable asset. We extend this to incorporate stochastic volatilities for both assets, driven by a common stochastic factor, and look for the corresponding indifference price characterisation under such a model. We would also look at the optimal portfolio in hedging the claim on the non-tradable asset, the residual risk process and the payoff decomposition of the claim involving the indifference price process and a local martingale. Towards the end of the discussion, we would outline a procedure which one could use to obtain numerical results for the indifference price under this model.

Contents

1 Introduction **1**

2 Indifference Price Representation in Basis Risk Model with Stochastic Volatility **6**

2.1 Model Setting . . . 6

2.2 Primal Approach for Indifference Price PDE . . . 8

2.2.1 Problem without non-tradable asset . . . 8

2.2.2 Problem with random terminal endowment . . . 10

2.3 Dual Approach for Indifference Price PDE . . . 13

2.3.1 Problem without non-tradable asset . . . 14

2.3.2 Problem with random terminal endowment . . . 16

2.4 Marginal Utility Based Price . . . 19

2.5 Optimal Hedging Strategy . . . 20

2.6 Residual Risk . . . 21

2.7 Payoff Decomposition . . . 22

3 Methods for Explicit Computation **24**

3.1 Numerical Approach . . . 24

3.2 Asymptotic Expansion . . . 25

4 Conclusion **27**

Bibliography **28**

Chapter 1

Introduction

In a complete market, every claim written on any underlying asset can be perfectly replicated by constructing a hedging portfolio through dynamic trading in the underlying asset. Therefore, assuming no arbitrage in the market, the price of the claim is equivalent to the cost associated with constructing the corresponding hedging portfolio. In fact the 2nd Fundamental Theorem of Asset Pricing (see for example F. Delbaen, W. Schachermayer [6]) tells us that the pricing measure is unique in such a market and hence the price of any claim is unique regardless of the agent's risk preferences and so on. This idea of fair pricing via perfect replication and no arbitrage was developed by F. Black and M. Scholes [2] and R.C. Merton [19] who showed how to replicate the payoff of an option using a risk free asset and the underlying asset under a continuous time model where the underlying asset follows a geometric Brownian motion. The theory is elegant in the sense that it is possible to find explicit pricing formulae for some of the most liquidly traded options in the market. But reality is more complicated than the model can cater for; transactions incur costs, trading may be restricted, options may be written on non-tradable underlyings and asset prices may jump as well as exhibit features such as stochastic volatility. All of these features can alter the completeness of the market and it has become increasingly important to look for pricing beyond the complete market framework in order to cater for these limitations in reality.

In an incomplete market, the pricing measures consistent with no arbitrage are not unique. Equivalently, pricing of claims depends upon how individuals value the associated risk involved in taking a position in the claim. Hence there is no preference-free way to price contingent claims in an incomplete market. It is natural that a reasonable pricing method should incorporate the agent's risk appetite into the pricing formulation. A popular approach that achieves this was introduced by S.D. Hodges and A.

Neuberger [13] who developed the notion of *utility indifference pricing*. The idea is to provide a compensation (the indifference ask price, or selling price) to the writer of a claim so that he is indifferent in terms of achievable expected utility whether selling the claim or not. A symmetric definition of the indifference bid price, or buying price, is also available.

The method relies on solving two utility maximization problems (one with the claim and one without the claim) and define a so called indifference (ask) price p by the relation $u^0(x) = u^C(x+p)$, where $u^C(x)$ denotes the maximal expected utility given initial capital x with a short position in the claim C and $C = 0$ denotes the same quantity in the absence of the claim. Optimal investment problems to maximize expected utility of terminal wealth (so in the absence of any radom endowment due to claims) were solved by R.C. Merton [18] for lognormal asset price processes using the Dynamic Programming Principle and the Hamilton-Jacobi-Bellman (HJB) equation. Some problems with random endowment can also be solved in this way, but this is the exception rather than the rule. One example where explicit solutions are available is the class of basis risk models [4] which form the main object of study in this dissertation. Depending on the utility function used, indifference pricing can incorporate any desired risk aversion characteristics, and may or may not exhibit dependence on an agent's initial endowment. A prominent example in which indifference prices are wealth independent is when the agent's utility function is exponential, as in this dissertation. A survey of indifference pricing and its characterization can be found in V. Henderson and D. Hobson [12].

Early work on pricing claims in incomplete markets focused on the so called quadratic hedging approaches [7], [8], [27]. In this approach, a quadratic criterion which (in one way or another) measures the impact of not perfectly replicating a claim payoff, and hence measures the associated risk, is minimized. For instance, one might minimize the variance of the terminal hedging error, defined as $\mathrm{Var}(X_T - C)$, where X_T is the terminal wealth at time T (the maturity time of a European claim C) achieved by dynamic trading of fundamental securities such as stocks. This is called variance optimal hedging [26].

Another approach, local risk minimization (LRM), initiated by H. Föllmer, M. Schweizer and D. Sondermann ([7], [8]) essentially minimizes the variance of a 'cost process' which, when added to a self-financing trading strategy, guarantees perfect replication. A feature of this pricing and hedging mechanism is that it essentially ignores unhedgeable risk. A result of this is that the equivalent local martingale measure (ELMM) associated with pricing a claim via LRM is the so called minimal martingale measure. This is the ELMM which converts tradable asset prices to local martingales but does not change any Brownian motions orthogonal to those driving the tradable stocks. Although the technicalities of LRM can be quite involved, it turns out that finding a local risk minimizing strategy is equivalent to finding the so called Föllmer-Sondermann-Schweizer decomposition of the claim ([7], [14]). This is a decomposition of the form $C = c + \int_0^T \xi_t dS_t + L_T$, where c is some constant, S_t is the vector of stock prices, ξ_t is a predictable process and L_t is a local martingale orthogonal to S_t (this represents the ignored unhedgeable component of risk). Then ξ_t is the locally risk minimizing strategy.

One critique of quadratic approaches is that they give equal weight to both positive and negative deviations from perfect replication. One could argue that it is better to treat profits differently to losses. This motivates using a genuine utility maximization objective instead. Moreover, it is now well known ([15], [17], [22]) that the zero risk aversion limit of exponential indifference pricing corresponds to a quadratic hedging criterion. Furthermore, an alternative approach to all these is via BSDE.

The term 'basis risk' refers to the risk associated with imperfect hedging arising from hedging a claim on a non-tradable asset with a correlated tradable asset. M.H.A. Davis [4] (this first appeared in 2000 in preprint form) applied pricing via utility maximization to a basis risk model, where both the tradable and non-tradable assets followed geometric Brownian motions and looked for the optimal hedging strategy via a duality approach ([5], [16]). M.H.A. Davis obtained an approximate formula for the indifference prices of a claim on a non-tradable asset with price process Y_t. The formula was of the form of an asymptotic expansion. Given $Y_t = y$ for some $t \in [0, T]$, the indifference pricing function $p(t, y)$ was of the form (suppose the interest rate $r = 0$)

$$p(t,y) = \mathbb{E}^{\mathbb{Q}^M}[C(Y_T)|Y_t = y] + \frac{1}{2}\gamma(1-\rho^2)\mathrm{Var}^{\mathbb{Q}^M}[C(Y_T)|Y_t = y] + O\left((\gamma(1-\rho^2))^2\right). \tag{1.1}$$

In fact, this representation turned out to be a special case of an expectation representation for the indifference price, that was later developed by other authors ([11],

[20], [23]) using a technique developed by T. Zariphopoulou [28] in the context of optimal investment in stochastic volatility model. In [28], a non-linear transformation was applied to the value function $u(t,x,y)$ of a control problem, in which (x,y) represents the realization of wealth and stochastic volatility at time $t \in [0,T]$. The transformation was of the form $u(t,x,y) = U(x)(F(t,y))^{\delta}$ for some function F and constant δ, where $U(x)$ is the utility function. It turns out the δ can be chosen to yield a linear PDE for F. The so called 'distortion power' method also works for basis risk models with constant parameters and this was exploited in a number of papers.

Under such a constant parameter model, it was shown in ([11], [20], [23]) that the value function of the utility maximization problem with a random endowment, u^C, can be represented as

$$u^C(t,x,y) = -e^{-\gamma x - \frac{1}{2}(\lambda^S)^2(T-t)} \left(\mathbb{E}^{\mathbb{Q}^M} \left[e^{\gamma(1-\rho^2)C(Y_T)} \middle| Y_t = y \right] \right)^{\frac{1}{1-\rho^2}} \tag{1.2}$$

where $\gamma > 0$ is the risk aversion of the agent, λ^S is the Sharpe ratio of S_t. This formula follows via the distortion technique and we shall see an example of this technique in the next section. A Taylor expansion of the exponential inside the expectation in (1.2) leads to (1.1), but the formula (1.2) was not available to M.H.A. Davis in 2000. An important duality result was proven by Delbaen et al [5] when they established a relationship between utility maximization and minimizing entropy measure. Although a similar result was proven by R. Rouge and N. El Karoui [25], they only proved it in a Brownian filtration while Delbaen et al proved it in a more general setting. This result provides an alternative characterization for the indifference price which is perhaps more elegant and useful in high dimensional problems. Based upon the previous works, M. Monoyios [20] derived a perturbative representation for the indifference price and tested the result numerically in 2004. He compared the optimal and the 'naive' Black Scholes delta together with their hedge for a put option under a constant parameter lognormal basis risk model. Later in two subsequent papers [21], [22] published in 2007 and 2010 respectively, he examined the corresponding valuation and hedging of claims under a basis risk model with random drift that arises from incomplete information on the value of the drift.

As an extension to all the previous works, we shall be considering a new three-factor model with a tradable asset S_t and a non-tradable asset Y_t, both having a volatility that depends explicitly on a stochastic factor Z_t. [11], [20], [23] considered a basis risk model with no stochastic volatility; [1], [21] considered a model with a tradable asset and stochastic volatility. All of the attempts so far have been looking at variations of a two-factor model and this extension has not been studied previously. In the next chapter, we will derive a PDE for the indifference price of a claim on the non-tradable asset via both the primal and dual approach. We will then use the PDE to look at the marginal utility based price, optimal hedge, residual risk process and payoff decomposition. In Chapter 3, we will outline some methods to compute the indifference price. In the final chapter, we will talk about some further works that can be performed as future research.

Chapter 2

Indifference Price Representation in Basis Risk Model with Stochastic Volatility

2.1 Model Setting

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ be a filtered probability space, where $\mathbb{P}$ is the physical measure that we write the dynamics of our stochastic model in and $\mathcal{F}_t = \sigma\{(B_s^S, B_s^Y, B_s^Z) : 0 \leq s \leq t\}$ for $t \in [0, T]$ is the σ-algebra generated by three correlated Brownian motions, each driving a different stochastic process in our model. Throughout the following discussion, we assume the interest rate $r = 0$. From now on, write $B_t^{j,\mathbb{P}} \equiv B_t^j$ for any j.

Let S_t, Y_t be the price processes of the tradable and non-tradable assets respectively. We have the following dynamics:

$$dS_t = \sigma^S(Z_t)S_t(\lambda^S(Z_t)dt + dB_t^S) \tag{2.1}$$

$$dY_t = \sigma^Y(Z_t)Y_t(\lambda^Y(Z_t)dt + dB_t^Y) \tag{2.2}$$

where

$$\lambda^i(Z_t) := \frac{\mu^i}{\sigma^i(Z_t)}, \quad \text{for } i = S, Y$$

are the Sharpe ratios of the corresponding assets. We allow ourselves to write for brevity $\lambda_t^i \equiv \lambda^i(Z_t)$ and $\sigma_t^i \equiv \sigma^i(Z_t)$ for $i = S, Y$ in subsequent context. Here we assume the physical drifts μ^S, μ^Y of both assets are constant but the volatilities depend on a stochastic factor Z_t which has its own dynamics. Note that since S_t is tradable, $\lambda^S(Z_t)$ corresponds to the amount of drift adjusted to the $\mathbb{P}$ Brownian motion driving S in order to change from the physical measure $\mathbb{P}$ to any risk neutral measure $\mathbb{Q}$.

We allow the stochastic factor process Z_t to have a general but stationary dynamics (one that does not explicitly have time dependence)

$$dZ_s = a(Z_s)ds + b(Z_s)dB_s^Z \quad \text{with } Z_t = z \tag{2.3}$$

for $s \in [t, T]$, where the coefficient functions a and b are defined such that the above equation admits a unique strong solution.

Consider a self financing portfolio containing the traded asset S. Let X_s be the wealth process of the portfolio at any given time $s \in [t, T]$ and θ_s be the holding of the tradable asset in the portfolio. Let x be the initial wealth, then the dynamics of X_s are

$$dX_s = \theta_s dS_s = \sigma^S(Z_s)\pi_s(\lambda^S(Z_s)ds + dB_s^S) \quad \text{with } X_t = x \tag{2.4}$$

where $\pi_s := \theta_s S_s$ is the amount of wealth invested in S_s at any given time s. We take π_s to be our control variable and we demand it to satisfy the admissibility conditions, i.e. π_s is $\mathcal{F}_s$-measurable and $\mathbb{E}\int_0^T(\sigma^S(Z_s)\pi_s)^2 ds < \infty$ almost surely. Denote $\mathcal{A}(x, y, z)$ to be the set of admissible controls with initial wealth x, and the initial realizations of the non-tradable asset price and stochastic volatility to be y and z respectively.

With the dynamics defined in (2.1), (2.2), (2.3), it would be rather trivial if all three Brownian motions are independent and that we know trading in S_t would have no hope in hedging a claim in Y_t. For this reason, we postulate the correlation between these Brownian motions as follows and find the necessary condition in order to write them in terms of independent Brownian motions under $\mathbb{P}$. Let

$$\langle B^S, B^Y \rangle_t = \rho t, \tag{2.5}$$

$$\langle B^S, B^Z \rangle_t = \kappa t, \tag{2.6}$$

$$\langle B^Y, B^Z \rangle_t = \nu t, \tag{2.7}$$

where ρ, κ, ν are constants which lie in [-1,1].

Let B^1,B^2,B^3 be 3 independent $\mathbb{P}$ Brownian motions, then we can write B^S,B^Y,B^Z as linear combinations of these 3 Brownian motions with appropriate coefficients. Set $B^S = B^1$, then we would have $B^Y = \rho B^1 + \sqrt{1-\rho^2}B^2$.

Now let $B^Z = \alpha B^1 + \beta B^2 + \delta B^3$, then the quadratic variation of the Brownian motion B^Z imposes a constraint on these parameters, i.e.

$$d\langle B^Z \rangle_t = (\alpha^2 + \beta^2 + \delta^2)dt = dt \Rightarrow \alpha^2 + \beta^2 + \delta^2 = 1$$

In other words, we can remove δ by setting $\delta = \sqrt{1-\alpha^2-\beta^2}$.

Considering the cross variation of B^S and B^Z, we have

$$d\langle B^S, B^Z\rangle_t = d\langle B^1, \alpha B^1 + \beta B^2 + \delta B^3\rangle = \alpha dt$$

since B^1,B^2,B^3 are independent and the cross variation between independent Brownian motions is zero. We can then conclude from (2.6) that $\alpha = \kappa$. Now from the correlation between B^Y and B^Z, we can deduce the value for β that is consistent with our model, i.e.

$$\begin{aligned} d\langle B^Y, B^Z\rangle_t &= d\langle \rho B^1 + \sqrt{1-\rho^2}B^2, \kappa B^1 + \beta B^2 + \sqrt{1-\kappa^2-\beta^2}B^3\rangle_t \\ &= (\kappa\rho + \beta\sqrt{1-\rho^2})dt \\ &= \nu dt \quad \text{using (2.7)} \\ &\Rightarrow \beta = \frac{\nu - \kappa\rho}{\sqrt{1-\rho^2}} \end{aligned}$$

To conclude the above calculation, we have the following:

$$\left\{ \begin{array}{ll} B^S & = B^1, \\ B^Y & = \rho B^1 + \sqrt{1-\rho^2}B^2, \\ B^Z & = \kappa B^1 + \beta B^2 + \sqrt{1-\kappa^2-\beta^2}B^3. \end{array} \right. \tag{2.8}$$

Throughout the following discussion, we shall be using an exponential utility function, $U(x) = -e^{-\gamma x}$, with $\gamma > 0$ being the agent's risk aversion, when looking for a representation of the indifference price of a claim on Y. This is because exponential utility function has the feature that the initial wealth x can be factored out and hence we can remove one dimension in the problem.

2.2 Primal Approach for Indifference Price PDE

2.2.1 Problem without non-tradable asset

Suppose we first look at the model without the non-tradable asset Y. Later we shall consider a variant of this problem involving an additional random endowment of an European claim with payoff $C(Y_T)$ at terminal time T. An agent with risk aversion γ wants to maximize his utility by dynamic trading, then his primal value function, denoted by u^0, is defined as

$$u^0(t,x,z) = \sup_{\pi \in \mathcal{A}(x,z)} \mathbb{E}\left[-e^{-\gamma X_T} | X_t = x, Z_t = z\right] \tag{2.9}$$

A classical approach to solve this is via the Dynamic Programming Principle (DPP), $u^0(t, X_t, Z_t)$ is a supermartingale under any admissible strategy, and a martingale for the optimal strategy, under the physical measure $\mathbb{P}$. By applying Ito's formula to u^0, we have

$$\begin{aligned} du^0(t, X_t, Z_t) = &\left(u_t^0 + \lambda_t^S \sigma_t^S \pi_t u_x^0 + \frac{1}{2}(\sigma_t^S \pi_t)^2 u_{xx}^0 + a_t u_z^0 + \frac{1}{2} b_t^2 u_{zz}^0 + b_t \sigma_t^S \pi_t \kappa u_{xz}^0 \right) dt \\ &+ \sigma_t^S \pi_t u_x^0 dB_t^S + b_t u_z^0 dB_t^Z, \end{aligned}$$

where the arguments of u^0 on the RHS are omitted for brevity. We know the drift of u^0 must be non-positive for all admissible π_t. In fact we recover the martingale property when π_t is optimal. Assuming π_t is Markov, i.e. we have a feedback control, the DPP leads to the HJB equation for u^0:

$$\sup_{\pi \in \mathcal{A}(x,z)} \left[u_t^0 + \lambda_t^S \sigma_t^S \pi_t u_x^0 + \frac{1}{2}(\sigma_t^S \pi_t)^2 u_{xx}^0 + a_t u_z^0 + \frac{1}{2} b_t^2 u_{zz}^0 + b_t \sigma_t^S \pi_t \kappa u_{xz}^0 \right] = 0 \quad (2.10)$$

with $u^0(T, x, z) = -e^{-\gamma x}$. Concavity of utility function ensures concavity of the primal value function which in turns implies $u_{xx}^0 < 0$. Note that the PDE is a quadratic in terms of π_t with negative leading order coefficient, so there exists a well defined supremum for π_t and this is the optimal portfolio.

Denote the optimal portfolio (without the claim) to be $\pi_t^{0,*} := \pi^0(t, X_t^*, Z_t)$, where X_t^* is the optimal wealth process, then the function $\pi^0 : [0, T] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ satisfies

$$\pi^0(t, x, z) = -\frac{\lambda_t^S u_x^0 + b_t \kappa u_{xz}^0}{\sigma_t^S u_{xx}^0} \quad (2.11)$$

Substituting this π back into the HJB equation (2.10), then we have

$$u_t^0 + \mathcal{L}_Z u^0 - \frac{1}{2} \frac{(\lambda^S(z) u_x^0 + b(z) \kappa u_{xz}^0)^2}{u_{xx}^0} = 0 \quad (2.12)$$

with $u^0(T, x, z) = -e^{-\gamma x}$, where $\mathcal{L}_Z$ is the generator of Z under $\mathbb{P}$, defined by

$$\mathcal{L}_Z g = a(z) g_z + \frac{1}{2} b(z)^2 g_{zz}$$

for any twice differentiable $g : [0, T] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$.

Note that this is a non-linear PDE and it is not clear how one could go about solving it. However, T. Zariphopoulou showed in her paper [28] that instead of solving the non-linear HJB equation directly, one could introduce a specific power transformation, so called 'distortion power', to the primal value function such that the resulting equation is linear. In other words, let $u^0(t,x,z) = -e^{-\gamma x}[h(t,z)]^\eta$, then we know there exists an η such that the equation for h is linear under this transformation. In fact the η that achieves this is $\frac{1}{1-\kappa^2}$.

Moreover, the equation that $h(t,z)$ satisfies is

$$h_t + (a(z) - \lambda^S(z)b(z)\kappa)h_z + \frac{1}{2}b^2(z)h_{zz} - \frac{1}{2}(1-\kappa^2)(\lambda^S(z))^2 h = 0 \tag{2.13}$$

with $h(T,z) = 1$ (c.f. a similar equation derived by F.E. Benth and K.H. Karlsen [1]). This equation will become useful when we derive the indifference price PDE later.

By Feynman-Kac Theorem, we can write this PDE problem as a conditional expectation under the minimal martingale measure $\mathbb{Q}^M$ that will be defined later (2.26) and a particular 'interest rate'. Hence we obtain an explicit solution for the primal value function u^0, i.e.

$$u^0(t,x,z) = -e^{-\gamma x}\mathbb{E}^{\mathbb{Q}^M}\left[e^{-\frac{1}{2}(1-\kappa^2)\int_t^T(\lambda_u^S)^2 du} \middle| Z_t = z\right]^{\frac{1}{1-\kappa^2}} \tag{2.14}$$

with

$$\begin{aligned} h(t,z) &= \mathbb{E}^{\mathbb{Q}^M}\left[e^{-\frac{1}{2}(1-\kappa^2)\int_t^T(\lambda_u^S)^2 du} \middle| Z_t = z\right], \\ dZ_t &= \big(a(z) - \lambda^S(z)b(z)\kappa\big)\, dt + b(z)dB_t^{Z,\mathbb{Q}^M} \end{aligned} \tag{2.15}$$

for some $\mathbb{Q}^M$-Brownian motions.

2.2.2 Problem with random terminal endowment

Suppose now the non-tradable asset Y is present in the market and an agent with risk aversion γ is selling a claim C written on this asset, then his primal value function is defined as

$$u^C(t,x,y,z) = \sup_{\pi\in\mathcal{A}(x,y,z)} \mathbb{E}\left[-e^{-\gamma(X_T - C(Y_T))} | X_t = x, Y_t = y, Z_t = z\right] \tag{2.16}$$

Using the same argument as before, i.e. arguing by the Dynamic Programming Principle and supermartingale property of u^C for sub-optimal portfolio π, applying Ito's formula to u^C, we have

$$\begin{aligned} du^C(t,X_t,Y_t,Z_t) = \bigg(& u_t^C + \lambda_t^S\sigma_t^S\pi_t u_x^C + \frac{1}{2}(\sigma_t^S\pi_t)^2 u_{xx}^C + \lambda_t^Y\sigma_t^Y Y_t u_y^C + \frac{1}{2}(\sigma_t^Y Y_t)^2 u_{yy}^C \\ & + a_t u_z^C + \frac{1}{2}b_t^2 u_{zz}^C + \sigma_t^S\sigma_t^Y\rho\pi_t Y_t u_{xy}^C + b_t\nu\sigma_t^Y Y_t u_{yz}^C + b_t\sigma_t^S\pi_t\kappa u_{xz}^C \bigg) dt \\ & + \sigma_t^S\pi_t u_x^C dB_t^S + \sigma_t^Y Y_t u_y^C dB_t^Y + b_t u_z^C dB_t^Z. \end{aligned}$$

Assuming a feedback control, we obtain the following results:

Denote the optimal portfolio (with the claim) to be $\pi_t^{C,*} := \pi^C(t, X_t^*, Y_t, Z_t)$, where X_t^* is the optimal wealth process, then the function $\pi^C : [0,T] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ satisfies

$$\pi^C(t,x,y,z) = -\frac{\lambda_t^S u_x^C + b_t \kappa u_{xz}^C + \sigma_t^Y y \rho u_{xy}^C}{\sigma_t^S u_{xx}^C} \tag{2.17}$$

Then the HJB equation for u^C becomes

$$u_t^C + \mathcal{L}_{Y,Z} u^C - \frac{1}{2}\frac{(\lambda^S(z) u_x^C + b(z)\kappa u_{xz}^C + \sigma^Y(z) y \rho u_{xy}^C)^2}{u_{xx}^C} = 0 \tag{2.18}$$

with $u^C(T,x,y,z) = -e^{-\gamma(x - C(y))}$, where $\mathcal{L}_{Y,Z}$ is the generator of Y and Z under $\mathbb{P}$, defined by

$$\mathcal{L}_{Y,Z} g = \lambda^Y(z)\sigma^Y(z) y g_y + \frac{1}{2}(\sigma^Y(z)y)^2 g_{yy} + a(z) g_z + \frac{1}{2} b^2(z) g_{zz} + b(z)\nu\sigma^Y(z) y g_{yz}$$

for any twice differentiable $g : [0,T] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$.

Now let $u^C(t,x,y,z) = -e^{-\gamma x} f(t,y,z)$. Suppose we try to mimic the 'distortion power' method, we let $f(t,y,z) = F(t,y,z)^\eta$, would we be able to find a constant η such that we could obtain a linear equation for F just as what we would expect from the previous case?

Using this relationship $f(t,y,z) = F(t,y,z)^\eta$, we obtain from (2.18), after working out the corresponding derivatives F_y, F_z and so on, the following equation (omitting terminal condition)

$$\begin{aligned}
&\eta F^{\eta-1}\Big[F_t + \lambda^Y \sigma^Y y F_y + \frac{1}{2}(\sigma^Y y)^2 (F_{yy} + (\eta-1)F^{-1}F_y^2) + aF_z \\
&+\frac{1}{2} b^2 (F_{zz} + (\eta-1)F^{-1}F_z^2) + b\nu\sigma^Y y (F_{yz} + (\eta-1)F^{-1}F_yF_z)\Big] \\
&-\frac{1}{2F^\eta}(\rho\eta\sigma^Y y F^{\eta-1} F_y + b\kappa\eta F^{\eta-1}F_z + \lambda^S F^\eta)^2 = 0
\end{aligned}$$

Try to linearize equation by setting the coefficient of the non-linear terms to zero, i.e.

Coefficient of $F_y^2 = 0 \quad \Rightarrow \frac{1}{2}(\sigma^Y y)^2 F^{\eta-2}[\eta(\eta-1) - \rho^2\eta^2]F_y^2 = 0 \quad \Rightarrow \eta = 0$ or $\dfrac{1}{1-\rho^2}$

Coefficient of $F_z^2 = 0 \quad \Rightarrow \frac{1}{2} b^2 F^{\eta-2}[\eta(\eta-1) - \kappa^2\eta^2]F_z^2 = 0 \quad \Rightarrow \eta = 0$ or $\dfrac{1}{1-\kappa^2}$

Suppose $\eta \neq 0$, then the two conditions above yield a necessary condition $\rho = \kappa$ in order for such an η to exist.

$$\text{Coefficient of } F_y F_z = 0 \quad \Rightarrow F^{\eta-2}[\eta(\eta-1)b\nu\sigma^Y y - \eta^2 b\kappa\rho\sigma^Y y]F_y F_z = 0 \quad \Rightarrow (\eta-1)\nu = \rho\kappa\eta$$
$$\Rightarrow \nu = 1 \quad (\text{using } \rho = \kappa \text{ and definition of } \eta)$$

We can see Y, Z are perfectly correlated and hence the problem reduces back to just the stochastic volatility model. This calculation explicitly shows that the 'distortion power' method fails in our model. In fact it is well known that such a method fails for higher dimensional (> 2) problem [24] and here we only reconfirm such a result.

So now we work with $u^C(t, x, y, z) = -e^{-\gamma x} f(t, y, z)$, by computing the derivatives and substituting into (2.18), we have

- $\mathcal{L}_{Y,Z} u^C = -e^{-\gamma x} \mathcal{L}_{Y,Z} f$
- $\dfrac{(\lambda^S(z)u_x^C + b(z)\kappa u_{xz}^C + \sigma^Y(z)y\rho u_{xy}^C)^2}{u_{xx}^C} = \dfrac{-e^{-\gamma x}}{f}(\lambda^S(z)f + b(z)\kappa f_z + \sigma^Y(z)y\rho f_y)^2$

so the PDE for f (dividing by f, assuming $f \neq 0$) is

$$\frac{f_t}{f} + \frac{1}{f}\mathcal{L}_{Y,Z} f - \frac{1}{2}\left(\lambda^S(z) + b(z)\kappa\frac{f_z}{f} + \sigma^Y(z)y\rho\frac{f_y}{f}\right)^2 \tag{2.19}$$
$$f(T, y, z) = e^{\gamma C(y)}$$

We shall see below why it is useful to divide the PDE by f.

Definition 2.2.1 (Indifference ask price)**.** The indifference ask price is denoted by $p(t, Y_t, Z_t)$, where the function $p : [0, T] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ satisfies

$$u^0(t, x, z) = u^C(t, x + p(t, y, z), y, z) \tag{2.20}$$

where u^0, u^C are the primal value functions defined as above.

Using the definition above together with h (2.15) and f defined in the previous section, we can now find an expression for the indifference price, i.e.

$$-e^{-\gamma(x+p(t,y,z))} f(t, y, z) = -e^{-\gamma x} h(t, z)^{\frac{1}{1-\kappa^2}}$$
$$\Rightarrow \quad p(t, y, z) = \frac{1}{\gamma}\left(\log f(t, y, z) - \frac{1}{1-\kappa^2}\log h(t, z)\right) \tag{2.21}$$

Note the indifference price p depends on the logarithm of the two quantities that we defined before, so any derivative of p would involve $\frac{h_*}{h}$ or $\frac{f_*}{f}$ terms and this justifies the comment above regarding dividing the PDE for f by f. Since we have got both the PDEs for h (2.13) and f (2.19), we can readily work out a corresponding PDE for the indifference price p, i.e.

$$p_t+\mathcal{L}^{\tilde{\mathbb{Q}}}_{Y,Z}p+\gamma\left(\frac{1}{2}(1-\rho^2)(\sigma^Y(z)y)^2p_y^2+b(z)\sigma^Y(z)y\beta\sqrt{1-\rho^2}p_yp_z+\frac{1}{2}(1-\kappa^2)b^2(z)p_z^2\right)=0 \tag{2.22}$$

with $p(T,y,z)=C(y)$, where $\mathcal{L}^{\tilde{\mathbb{Q}}}_{Y,Z}$ is the generator of Y and Z under the measure $\tilde{\mathbb{Q}}$ defined by

$$\begin{aligned}\mathcal{L}^{\tilde{\mathbb{Q}}}_{Y,Z}g =&\sigma^Y(z)y\left(\lambda^Y(z)-\rho\lambda^S(z)+\frac{b(z)}{1-\kappa^2}\beta\sqrt{1-\rho^2}\frac{h_z}{h}\right)g_y+\frac{1}{2}(\sigma^Y(z)y)^2g_{yy}\\&+\left(a(z)-\lambda^S(z)b(z)\kappa+b^2(z)\frac{h_z}{h}\right)g_z+\frac{1}{2}b^2(z)g_{zz}+b(z)\nu\sigma^Y(z)yg_{yz}\end{aligned} \tag{2.23}$$

for any twice differentiable $g:[0,T]\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$.

The definition of this measure $\tilde{\mathbb{Q}}$ may seem rather peculiar at the first sight. However, after we introduced the dual approach, we will see that this $\tilde{\mathbb{Q}}$ is in fact equivalent to the minimal entropy measure $\mathbb{Q}^E$ that we are about to define and the two results would agree with each other.

2.3 Dual Approach for Indifference Price PDE

As we see in the previous section, deriving indifference price PDE via the primal approach involves lots of tedious calculations in working out derivatives. In this section, we shall be looking to derive the same result via the so called 'Dual Approach' which is more elegant and succinct.

We shall start with some further definitions and settings.

Definition 2.3.1 (Relative entropy). The relative entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$, denoted by $\mathcal{H}(\mathbb{Q}|\mathbb{P})$, is defined as ([5], [10])

$$\mathcal{H}(\mathbb{Q}|\mathbb{P})=\begin{cases}\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\log\frac{d\mathbb{Q}}{d\mathbb{P}}\right] & \text{if } \mathbb{Q}\ll\mathbb{P}\\ +\infty & \text{otherwise}\end{cases}$$

Define $\mathbb{M}_f$ to be the set of martingale measures with finite relative entropy [5] and assume this is non-empty. From now on we work with $\mathbb{Q} \in \mathbb{M}_f$.

A general duality theory proven by Delbaen et al [5] established an equivalence between utility maximization and minimizing entropy measures. The general result states that the primal value function has a representation described as follows:

$$u^i(t, x, y, z) = -\exp(-\gamma x - H^i(t, y, z)), \quad \text{for } i = 0, C \tag{2.24}$$

where u^0, u^C are the primal value functions and H^0, H^C arise from two dual optimization problems that will be defined later. In fact this result allows us to characterize the indifference price p in terms of H^0 and H^C, i.e.

$$p(t, y, z) = -\frac{1}{\gamma}\left(H^C(t, y, z) - H^0(t, z)\right) \tag{2.25}$$

2.3.1 Problem without non-tradable asset

Consider the market without the existence of the non tradable asset Y, so the market consists of a tradable asset under a general stochastic volatility model.

Define a measure $\mathbb{Q}$ with density

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}\left(-\lambda^S \cdot B^S - \psi \cdot B^{\perp}\right)_T := \Gamma_T,$$

with

$$(\lambda^S \cdot B^S)_t \equiv \int_0^t \lambda^S(Z_u) dB^S_u, \quad (\psi \cdot B^{\perp})_t \equiv \int_0^t \psi_u dB^{\perp}_u,$$

where B^S and $B^{\perp}$ are independent Brownian motions and $\mathcal{E}(\cdot)$ denotes the stochastic exponential. We demand the arbitrary process ψ_t to be admissible, meaning we need ψ_t to be $\mathcal{F}_t$-measurable and $\mathbb{E}\int_0^T \psi_t^2 dt < \infty$ almost surely. Note that the second condition is automatically satisfied since we are working in the realm of finite relative entropy measures.

By Girsanov Theorem, we know that there are $\mathbb{Q}$-Brownian motions $B^{S,\mathbb{Q}}$ and $B^{\perp,\mathbb{Q}}$ given by

$$B^{S,\mathbb{Q}}_t = B^S_t + \int_0^t \lambda^S_u du, \quad B^{\perp,\mathbb{Q}}_t = B^{\perp}_t + \int_0^t \psi_u du$$

under such a change of measure. Hence by rewriting the dynamics for S, Z under $\mathbb{Q}$, we have

$$
\begin{aligned}
dS_t &= \sigma^S(Z_t) S_t dB_t^{S,\mathbb{Q}} \\
dZ_t &= \left[a(Z_t) - b(Z_t)\left(\kappa\lambda^S(Z_t) + \sqrt{1-\kappa^2}\psi_t\right)\right] dt + b(Z_t) dB_t^{Z,\mathbb{Q}}
\end{aligned}
$$

Define the minimal martingale measure $\mathbb{Q}^M$ to be the measure that we set $\psi \equiv 0$ in the density Γ so that

$$
\frac{d\mathbb{Q}^M}{d\mathbb{P}} = \mathcal{E}\left(-\lambda^S \cdot B^S\right)_T, \tag{2.26}
$$

with

$$
dZ_t = \left[a(Z_t) - b(Z_t)\kappa\lambda^S(Z_t)\right] dt + b(Z_t) dB_t^{Z,\mathbb{Q}^M}.
$$

In fact this is the minimal change to the Brownian motions so that the tradable asset S_t is a martingale (assume appropriate integrability) under the change of measure.

Define the minimal entropy measure $\mathbb{Q}^E$ to be the measure that achieves the infimum of relative entropy $\mathcal{H}(\mathbb{Q}|\mathbb{P})$.

Define H^0 to be the value function of the following optimization problem, i.e.

$$
\begin{aligned}
H^0(t,z) &:= \inf_{\psi} \mathbb{E}^{\mathbb{Q}}\left[\log \frac{\Gamma_T}{\Gamma_t}\middle| Z_t = z\right] \\
&= \inf_{\psi} \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{2}\int_t^T (\lambda_u^S)^2 + (\psi_u)^2 du\middle| Z_t = z\right]
\end{aligned} \tag{2.27}
$$

then we can see that $H^0(0,\cdot)$ is the quantity that corresponds exactly to the minimal relative entropy $\mathcal{H}(\mathbb{Q}^E|\mathbb{P})$ by definition. This optimization problem can be treated as a new control problem over the new feedback control ψ and the corresponding HJB equation is

$$
\inf_{\psi} \mathbb{E}^{\mathbb{Q}}\left[H_t^0 + \mathcal{L}_Z^{\mathbb{Q}} H^0 + \frac{1}{2}\left((\lambda_t^S)^2 + (\psi_t)^2\right)\middle| Z_t = z\right] = 0
$$

with $H^0(T,z) = 0$, where $\mathcal{L}_Z^{\mathbb{Q}}$ is the generator of Z under $\mathbb{Q}$, defined by

$$
\mathcal{L}_Z^{\mathbb{Q}} g = \left[a(z) - b(z)\left(\kappa\lambda^S(z) + \sqrt{1-\kappa^2}\psi_t\right)\right] g_z + \frac{1}{2} b^2(z) g_{zz}
$$

for any twice differentiable $g : [0,T] \times \mathbb{R}^+ \to \mathbb{R}$.

Optimizing over ψ in the above control problem, we have

$$\psi_t^* = b(Z_t)\sqrt{1-\kappa^2}H_z^0(t, Z_t) \tag{2.28}$$

Note that the ψ that achieves the infimum in (2.27) is the one that minimizes relative entropy, so in fact ψ^* defines the minimal entropy measure $\mathbb{Q}^E$ via $\frac{d\mathbb{Q}^E}{d\mathbb{P}} = \mathcal{E}\left(-\lambda^S \cdot B^S - \psi^E \cdot B^\perp\right)_T$. From now on, we write $\psi^E = \psi^*$ to stress its relationship with $\mathbb{Q}^E$. Substituting this choice of ψ into the generator and we get the generator of Z under the minimal entropy measure $\mathbb{Q}^E$, i.e.

$$\begin{aligned}\mathcal{L}_Z^{\mathbb{Q}^E} g &= \left[a(z) - b(z)\left(\kappa\lambda^S(z) + \sqrt{1-\kappa^2}\psi_t^E\right)\right] g_z + \frac{1}{2}b^2(z)g_{zz} \\ &= \left[a(z) - b(z)\kappa\lambda^S(z) - b^2(z)(1-\kappa^2)H_z^0\right] g_z + \frac{1}{2}b^2(z)g_{zz} \end{aligned}\tag{2.29}$$

for any twice differentiable $g : [0,T] \times \mathbb{R}^+ \to \mathbb{R}$.

Furthermore, after substituting ψ^E back into the HJB equation, we obtain the following equation for H^0:

$$H_t^0 + \mathcal{L}_Z^{\mathbb{Q}^M} H^0 - \frac{1}{2}(1-\kappa^2)b^2(z)(H_z^0)^2 + \frac{1}{2}(\lambda^S(z))^2 = 0 \tag{2.30}$$

where $\mathbb{Q}^M$ is defined as before in (2.26).

2.3.2 Problem with random terminal endowment

Suppose now we have the non-tradable asset Y in the market and the market is operating under three independent Brownian motions B^1, B^2, B^3 described in Section 2.1.

Define a new measure $\mathbb{Q}$ with density

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}\left(-\lambda^S \cdot B^1 - \psi^2 \cdot B^2 - \psi^3 \cdot B^3\right)_T := \Gamma_T$$

with the definition of $\cdot$ defined in the previous section.

Rewrite the dynamics for Y, Z under $\mathbb{Q}$ using Girsanov Theorem, we have

$$\begin{aligned}dY_t &= \sigma^Y(Z_t)Y_t\left[(\lambda^Y(Z_t) - \rho\lambda^S(Z_t) - \sqrt{1-\rho^2}\psi_t^2)dt + dB_t^{Y,\mathbb{Q}}\right], \\ dZ_t &= \left[a(Z_t) - b(Z_t)\left(\kappa\lambda^S(Z_t) + \beta\psi_t^2 + \sqrt{1-\kappa^2-\beta^2}\psi_t^3\right)\right] dt + b(Z_t)dB_t^{Z,\mathbb{Q}}\end{aligned}$$

Recall the definition in (2.27), but we now have

$$H^0(t,z) = \inf_{\psi^2,\psi^3} \mathbb{E}^{\mathbb{Q}} \left[\frac{1}{2} \int_t^T (\lambda_u^S)^2 + (\psi_u^2)^2 + (\psi_u^3)^2 du \middle| Z_t = z \right]$$

since Γ_t is apparently different under this change of measure. As we shall see though, this problem will indeed reduce to the problem of Section 2.3.1 and the solution for H^0 will be the same. Note that unlike the previous case, this is a 2-dimensional optimization problem which involves optimizing over both ψ^2 and ψ^3. Treating the above problem as a control problem over the 2-dimensional feedback control (ψ^2, ψ^3) and the optimal controls are

$$\psi_t^{2,E} = b(Z_t)\beta H_z^0(t, Z_t) \tag{2.31}$$

$$\psi_t^{3,E} = b(Z_t)\sqrt{1-\kappa^2-\beta^2} H_z^0(t, Z_t) \tag{2.32}$$

as the optimizers of ψ^2, ψ^3 correspond to achieving the minimal relative entropy. Now we can clearly see a relationship between $\psi^{2,E}, \psi^{3,E}$ and ψ^E (2.28), i.e.

$$\sqrt{1-\kappa^2}\psi^E = \beta\psi^{2,E} + \sqrt{1-\kappa^2-\beta^2}\psi^{3,E} \tag{2.33}$$

In fact, this relationship can be seen by equating the drift of Z under the minimal entropy measure $\mathbb{Q}^E$ and the coefficient of the z derivative in the generator defined in (2.29).

Define H^C to be the value function of the following optimization problem, i.e.

$$\begin{aligned} H^C(t,y,z) &:= \inf_{\psi^2,\psi^3} \mathbb{E}^{\mathbb{Q}} \left[\log \frac{\Gamma_T}{\Gamma_t} - \gamma C(Y_T) \middle| Y_t = y, Z_t = z \right] \\ &= \inf_{\psi^2,\psi^3} \mathbb{E}^{\mathbb{Q}} \left[\frac{1}{2} \int_t^T (\lambda_u^S)^2 + (\psi_u^2)^2 + (\psi_u^3)^2 du - \gamma C(Y_T) \middle| Y_t = y, Z_t = z \right] \end{aligned}$$

Similar to the previous problem, we can treat this optimization problem as a control problem over the 2 dimensional feedback control $\Psi = (\psi^2, \psi^3)$ and obtain the HJB equation

$$\inf_{\Psi} \mathbb{E}^{\mathbb{Q}} \left[H_t^C + \mathcal{L}_{Y,Z}^{\mathbb{Q}} H^C + \frac{1}{2}\left((\lambda_t^S)^2 + (\psi_t^2)^2 + (\psi_t^3)^2 \right) \middle| Y_t = y, Z_t = z \right] = 0$$

with $H^C(T,y,z) = -\gamma C(y)$, where $\mathcal{L}_{Y,Z}^{\mathbb{Q}}$ is the generator of Y and Z under $\mathbb{Q}$, defined by

$$\begin{aligned} \mathcal{L}_{Y,Z}^{\mathbb{Q}} g =& \sigma^Y(z) y \left(\lambda^Y(z) - \rho\lambda^S(z) - \sqrt{1-\rho^2}\psi_t^2 \right) g_y + \frac{1}{2}(\sigma^Y(z)y)^2 g_{yy} \\ &+ \left[a(z) - b(z)\left(\kappa\lambda^S(z) + \beta\psi_t^2 + \sqrt{1-\kappa^2-\beta^2}\psi_t^3 \right) \right] g_z + \frac{1}{2} b^2(z) g_{zz} \\ &+ b(z)\nu\sigma^Y(z) y g_{yz} \end{aligned}$$

for any twice differentiable $g : [0, T] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$.

Optimizing over ψ^2 and ψ^3 gives

$$\begin{aligned}\psi_t^{2,*} &= \sigma^Y(Z_t)y\sqrt{1-\rho^2}H_y^C(t, Y_t, Z_t) + b(Z_t)\beta H_z^C(t, Y_t, Z_t)\\ \psi_t^{3,*} &= b(Z_t)\sqrt{1-\kappa^2-\beta^2}H_z^C(t, Y_t, Z_t)\end{aligned}$$

Substituting $\psi^{2,*}$ and $\psi^{3,*}$ back into the problem and we obtain the following equation for H^c:

$$H_t^C + \mathcal{L}_{Y,Z}^{\mathbb{Q}^M} H^C - \left[\frac{1}{2}(1-\rho^2)(\sigma^Y y)^2(H_y^C)^2 + b\sigma^Y y\beta\sqrt{1-\rho^2}H_y^C H_z^C + \frac{1}{2}(1-\kappa^2)b^2(H_z^C)^2\right] + \frac{1}{2}(\lambda^S)^2 = 0 \tag{2.34}$$

where $\mathbb{Q}^M$ is defined as before in (2.26).

Using the characterization of the indifference price in (2.25) and the PDE we obtained for H^0, H^C in (2.30), (2.34), we can derive the PDE representation for the indifference price p.

Lemma 2.3.1 (Indifference ask price PDE)**.** The indifference ask price $p(t, y, z)$, under the model described as in Section 2.1, satisfies the following PDE:

$$p_t + \mathcal{L}_{Y,Z}^{\mathbb{Q}^E} p + \gamma\left(\frac{1}{2}(1-\rho^2)(\sigma^Y(z)y)^2 p_y^2 + b(z)\sigma^Y(z)y\beta\sqrt{1-\rho^2}p_y p_z + \frac{1}{2}(1-\kappa^2)b^2(z)p_z^2\right) = 0 \tag{2.35}$$

$$p(T, y, z) = C(y)$$

where $\mathcal{L}_{Y,Z}^{\mathbb{Q}^{\mathbb{E}}}$ is the generator of Y and Z under the minimal entropy measure $\mathbb{Q}^E$, defined by

$$\begin{aligned}\mathcal{L}_{Y,Z}^{\mathbb{Q}^E} g =& \sigma^Y(z)y\left(\lambda^Y(z) - \rho\lambda^S(z) - b(z)\beta\sqrt{1-\rho^2}H_z^0\right)g_y + \frac{1}{2}(\sigma^Y(z)y)^2 g_{yy}\\ &+ \left[a(z) - b(z)\kappa\lambda^S(z) - b^2(z)(1-\kappa^2)H_z^0\right]g_z + \frac{1}{2}b^2(z)g_{zz}\\ &+ b(z)\nu\sigma^Y(z)y g_{yz}\end{aligned}$$

for any twice differentiable $g : [0, T] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and H^0 defined as in (2.27).

So how would the PDE we derive via the primal approach compare to the one that we just derive via the dual approach? The connection of the two approaches lies within the relationship between h and H^0 through the definition via the primal value function u^0. By definition,

$$
\begin{aligned}
-e^{-\gamma x} h(t,z)^{\frac{1}{1-\kappa^2}} &= u^0(t,x,z) = -e^{-\gamma x - H^0(t,z)} \\
\Rightarrow H^0(t,z) &= -\frac{1}{1-\kappa^2} \log h(t,z) \qquad (2.36) \\
\Rightarrow \frac{h_z(t,z)}{h(t,z)} &= -(1-\kappa^2) H^0_z(t,z)
\end{aligned}
$$

If we now go back to the definition of the generator under $\tilde{\mathbb{Q}}$ in (2.23), then we should see

$$
\begin{aligned}
\frac{b(z)}{1-\kappa^2}\beta\sqrt{1-\rho^2}\frac{h_z}{h} &= -b(z)\beta\sqrt{1-\rho^2}H^0_z \\
b^2(z)\frac{h_z}{h} &= -b^2(z)(1-\kappa^2)H^0_z
\end{aligned}
$$

so in fact the generator under $\tilde{\mathbb{Q}}$ is the same as the generator under $\mathbb{Q}^E$. Therefore we must have $\tilde{\mathbb{Q}} = \mathbb{Q}^E$. Now we have demonstrated that both approaches do indeed arrive at the same conclusion, we will try to use this representation in the following sections.

2.4 Marginal Utility Based Price

When we let $\gamma \to 0$, we can see that all contribution from the non-linear terms disappear and we recover a linear PDE, i.e.

$$
\hat{p}_t + \mathcal{L}^{\mathbb{Q}^E}_{Y,Z}\hat{p} = 0, \quad \hat{p}(T,y,z) = C(y)
$$

The indifference price here is called the marginal utility based price (MUBP), denoted by $\hat{p}$. It was first introduced in mathematical finance by M.H.A Davis [3]. It can be interpreted as the price that an agent would trade an infinitely small position of the claim. Given the linearity, we can solve the PDE using standard approach provided we are given a particular volatility structure and a stochastic volatility model. Alternatively, we can write this as an expectation over the minimal entropy measure $\mathbb{Q}^E$, i.e.

$$
\hat{p}(t,y,z) = \lim_{\gamma\to 0} p(t,y,z) = \mathbb{E}^{\mathbb{Q}^E}\left[C(Y_T)|Y_t = y, Z_t = z\right] \qquad (2.37)
$$

and do Monte Carlo simulation under such a measure. The dynamics of Y and Z are given by putting the corresponding ψ^Es into the dynamics described in Section 2.3.2.

2.5 Optimal Hedging Strategy

The optimal hedging strategy π^H for a short position in the claim C satisfies $\pi^H = \pi^{C,*} - \pi^{0,*}$, with $\pi^{C,*}$ and $\pi^{0,*}$ defined as in (2.11), (2.17). Using the definitions in (2.11), (2.17) together with the dual representation of the primal value function as in (2.24) and the corresponding characterization of the indifference price (2.25), we can write the derivatives of the primal value functions into derivatives of the indifference price p. The optimal hedging strategy $\pi^H(t, Y_t, Z_t)$ is such that the function $\pi^H : [0,T] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is given by

$$\begin{aligned}\pi^H(t,y,z) &= \pi^{C,*}(t,y,z) - \pi^{0,*}(t,z)\\ &= -\frac{1}{\sigma^S(z)\gamma}\left[b(z)\kappa(H_z^C - H_z^0) + \rho\sigma^Y(z)y(H_y^C - H_y^0)\right]\\ &= \kappa\frac{b(z)}{\sigma^S(z)}\frac{\partial p}{\partial z}(t,y,z) + \rho\frac{\sigma^Y(z)}{\sigma^S(z)}y\frac{\partial p}{\partial y}(t,y,z) \qquad (2.38)\end{aligned}$$

Lemma 2.5.1 (Optimal hedging strategy)**.** The optimal holding θ_t^H of S_t at any time $t \in [0,T]$ is given by

$$\theta_t^H = \left(\kappa\frac{b(Z_t)}{\sigma^S(Z_t)S_t}\frac{\partial p}{\partial z}(t,Y_t,Z_t) + \rho\frac{\sigma^Y(Z_t)}{\sigma^S(Z_t)}\frac{Y_t}{S_t}\frac{\partial p}{\partial y}(t,Y_t,Z_t)\right). \qquad (2.39)$$

Similar calculations were done by M. Monoyios [21], M. Musiela and T. Zariphopoulou [23], but their optimal hedges depended only on the y derivative of p. The correction term that involves the z derivative of p in our case represents the extra hedge required induced by the stochastic volatility factor.

Furthermore, motivated by the above representation (2.39) and the corresponding result in [22], we are tempted to postulate a formula, that we shall not prove here, of the optimal holding θ_t^H in higher dimensional models, under exponential utility maximization. Suppose there exists a tradable asset S_t and n non-tradable assets $N_t^1, N_t^2, \ldots, N_t^n$ in the market and the indifference price p is a function of t and all of the assets. Then the optimal holding θ_t^H of S_t is given by

$$\theta_t^H = \left(\frac{\partial p}{\partial s}(t,S_t,N_t) + \sum_{i=1}^n \rho_{si}\frac{\sigma^{N^i}}{\sigma^S}\frac{\partial p}{\partial n^i}(t,S_t,N_t)\right),$$

where $N_t = (N_t^1, N_t^2, \ldots, N_t^n)$ is the vector of non-tradable assets, σ^S, σ^{N^i} are the volatilities of S_t and N_t^i respectively, and ρ_{si} is the constant correlation between S_t and N_t^i.

2.6 Residual Risk

Residual risk is the risk remaining after we optimally hedge the claim on the non-tradable asset with the tradable asset. In complete market, the residual risk is zero as every claim can be replicated perfectly but it is not the case in incomplete market primarily due to the fact that some assets cannot be directly traded. The fact that these assets are not perfectly correlated exposes both the holder/writer of claims written on such assets to additional unhedgeable risk.

Define the residual risk process R_t to be $R_t = X_t^H - p(t, Y_t, Z_t)$ under the physical measure $\mathbb{P}$, where X_t^H is the optimal wealth process generated by optimal hedging as described in the previous section and $p(t, Y_t, Z_t)$ is the indifference price process. Applying Ito's formula to the indifference price process, we have

$$\begin{aligned} dp(t,&Y_t, Z_t) = \left(p_t + \sigma^Y Y_t \lambda^Y p_y + \frac{1}{2}(\sigma^Y Y_t)^2 p_{yy} + a p_z + \frac{1}{2} b^2 p_{zz} + b\nu\sigma^Y Y_t p_{yz} \right) dt \\ &+ \left(\rho\sigma^Y Y_t p_y + b\kappa p_z\right) dB_t^1 + \left(\sigma^Y Y_t \sqrt{1-\rho^2} p_y + b\beta p_z\right) dB_t^2 + b\sqrt{1-\kappa^2-\beta^2} p_z dB_t^3 \end{aligned}$$

and hence we compute the dynamics of R_t as follows:

$$\begin{aligned} dR_t =& dX_t^H - dp(t, Y_t, Z_t) \\ =& \theta_t^H dS_t - dp(t, Y_t, Z_t) \quad \text{(using self-financing property)} \\ =& \left(\sigma^Y Y_t \lambda^S p_y + b\kappa\lambda^S p_z\right) dt + \left(\rho\sigma^Y Y_t p_y + b\kappa p_z\right) dB_t^1 - dp(t, Y_t, Z_t) \\ =& \left(\gamma \left[\frac{1}{2}(1-\rho^2)(\sigma^Y Y_t)^2 p_y^2 + b\sigma^Y Y_t \beta \sqrt{1-\rho^2} p_y p_z + \frac{1}{2}(1-\kappa^2) b^2 p_z^2 \right] \right. \\ & \left. -\sigma^Y Y_t b\beta\sqrt{1-\rho^2} H_z^0 p_y - b^2(1-\kappa^2) H_z^0 p_z \right) dt \\ & - \left(\sigma^Y Y_t \sqrt{1-\rho^2} p_y + b\beta p_z\right) dB_t^2 - b\sqrt{1-\kappa^2-\beta^2} p_z dB_t^3 \\ & \text{(using the PDE representation in (2.35))} \end{aligned}$$

Note that we do not see the dependence of B^1 or B^S in the expression as constructing an optimal portfolio in S_t would completely eliminate the risk arises from that component. But the remaining part associated with the Brownian motions B^2, B^3 are the unhedgeable risk components as dynamic trading in S_t would have no chance to eliminate the risk(randomness) driven by those independent Brownian motions.

Suppose we want to recover the complete market, since Y_t, Z_t are not traded, we need them to be perfectly correlated with S_t, i.e. $\rho, \kappa = \pm 1$. Then we know they will all be driven by the same single Brownian motion B^1 and hence $\beta = 0$. Hence every term in the expression of dR_t is zero and R_t is constant. $R_T = X_T^H - p(T, Y_T, Z_T) = 0$ and therefore the residual risk is zero in such a market. Also note that when $\gamma \to 0$, $p \to \hat{p}$, this process is exactly the residual risk process under local risk minimization.

2.7 Payoff Decomposition

Using the dynamics of the indifference price in the previous section, derived from its PDE representation (2.35), one can also derive a payoff decomposition for the claim C in terms of the indifference price, the optimal hedge and a local martingale, denoted by L_t, under the minimal entropy measure $\mathbb{Q}^E$.

Consider the unhedgeable risk component as in the previous section,

$$\begin{aligned}
&\left(\sigma^Y Y_t \sqrt{1-\rho^2} p_y + b\beta p_z\right) dB_t^2 + b\sqrt{1-\kappa^2-\beta^2} p_z dB_t^3 \\
=&\left(\sigma^Y Y_t \sqrt{1-\rho^2} p_y + b\beta p_z\right) (dB_t^{2,\mathbb{Q}^E} - b\beta H_z^0 dt) \\
&+ b\sqrt{1-\kappa^2-\beta^2} p_z (dB_t^{3,\mathbb{Q}^E} - b\sqrt{1-\kappa^2-\beta^2} H_z^0 dt) \\
=&\left(\sigma^Y Y_t \sqrt{1-\rho^2} p_y + b\beta p_z\right) dB_t^{2,\mathbb{Q}^E} + b\sqrt{1-\kappa^2-\beta^2} p_z dB_t^{3,\mathbb{Q}^E} \\
&- \left(\sigma^Y Y_t b\beta \sqrt{1-\rho^2} H_z^0 p_y + b^2(1-\kappa^2) H_z^0 p_z\right) dt \\
=& dL_t - \left(\sigma^Y Y_t b\beta \sqrt{1-\rho^2} H_z^0 p_y + b^2(1-\kappa^2) H_z^0 p_z\right) dt
\end{aligned}$$

where the local martingale L_t under $\mathbb{Q}^E$ is defined as

$$L_t = \int_0^t \left(\sigma^Y Y_u \sqrt{1-\rho^2} p_y + b\beta p_z\right) dB_u^{2,\mathbb{Q}^E} + \int_0^t b\sqrt{1-\kappa^2-\beta^2} p_z dB_u^{3,\mathbb{Q}^E} \tag{2.40}$$

Note that the dt part of the unhedgeable risk component is part of the generator acting on p under the minimal entropy measure $\mathbb{Q}^E$ and L_t is orthogonal to the Brownian motion generating the tradable asset S_t.

With this definition of L_t and the indifference price PDE (2.35), we can now rewrite the dynamics of p as

$$dp(t, Y_t, Z_t) = \theta_t^H dS_t + dL_t - \frac{1}{2}\gamma d\langle L\rangle_t$$

Integrating this expression and we will get the payoff decomposition for the claim C, i.e.

$$C(Y_T) = p(t, Y_t, Z_t) + \int_t^T \theta_u^H dS_u + (L_T - L_t) - \frac{1}{2}\gamma(\langle L\rangle_T - \langle L\rangle_t) \tag{2.41}$$

as $p(T, Y_T, Z_T) = C(Y_T)$, where θ_t^H is the holding of S_t in the optimal hedge.

Let $\gamma \to 0$, then we recover the Föllmer-Schweizer decomposition ([7], [14]), i.e.

$$C(Y_T) = \hat{p}(t, Y_t, Z_t) + \int_t^T \hat{\theta}_u^H dS_u + (\hat{L}_T - \hat{L}_t) \tag{2.42}$$

where $\hat{p}$ is the marginal utility based price, $\hat{\theta}^H$ is the marginal hedge and $\hat{L}_t$ is the local martingale defined as in (2.40) with $p = \hat{p}$. We can see here that there is a link between the utility maximization approach and the quadratic approach via local risk minimization.

Furthermore, recall (2.37), by taking conditional expectation of the payoff decomposition (2.41), we have

$$\begin{aligned}
\hat{p}(t, y, z) &= \mathbb{E}^{\mathbb{Q}^E}\left[p(t, Y_t, Z_t) + \int_t^T \theta_u^H dS_u + (L_T - L_t) - \frac{1}{2}\gamma(\langle L\rangle_T - \langle L\rangle_t)\bigg| Y_t = y, Z_t = z\right] \\
&= p(t, y, z) - \frac{1}{2}\gamma\mathbb{E}^{\mathbb{Q}^E}\left[\langle L\rangle_T - \langle L\rangle_t | Y_t = y, Z_t = z\right]
\end{aligned}$$

since both $\int_0^t \theta_u^H dS_u$ and L_t are local martingales under the minimal entropy measure $\mathbb{Q}^E$ and so the $\mathbb{Q}^E$ expectations are zero. Rearrange this formula and we get another indifference price representation in terms of the marginal utility based price and a quadratic term, i.e.

$$p(t, y, z) = \hat{p}(t, y, z) + \frac{1}{2}\gamma\mathbb{E}^{\mathbb{Q}^E}\left[\langle L\rangle_T - \langle L\rangle_t | Y_t = y, Z_t = z\right]. \tag{2.43}$$

The correction term corresponds to the extra charge added to the marginal price due to utility maximization.

Chapter 3

Methods for Explicit Computation

3.1 Numerical Approach

Able to solve analytically the PDE representation for the indifference price that we derived before would be ideal in the sense that we would then have incorporated every details we build into the model into the formulation, calibration would be convenient and optimal hedge can be easily computed. However, due to the the nature of the model, i.e. the high dimension of the problem, the non-linearity of the equation and the dependence of explicitness of volatility model and structure, finding such an analytical representation has been proven to be a very difficult task to perform, if even possible. Alternatively, we can look into numerical approaches that utilize numerical methods such as Monte Carlo simulation for estimating expectations and finite difference methods for solving PDEs, as it is often good enough to find good numerical approximations. Here I would outline some of the major steps to find the indifference price numerically based on the theoretical result that we covered.

Before going on to finding indifference price, we need some basic ingredients. Firstly we have to choose and specify an appropriate stochastic volatility model Z_t, for example Heston, that we will be working on and a feasible volatility structure σ^S, σ^Y, for example we need a structure that is consistent with the HJM drift condition if we were pricing fixed income claims, that built into the dynamics of the two assets. Closed form solution for Z_t would certainly ease computation, but otherwise we will have to estimate values for Z at different times via simulation type methods and input these into the volatility structure.

Secondly, there is a very important quantity that we would need to compute and this is the value function of the relative entropy minimization problem, denoted by

$H^0(t,z)$. This quantity is important in the sense that all of ψ^2, ψ^3, ψ which define the minimal entropy measure that appears in the indifference price PDE depend upon the z derivative of this quantity. In the constant parameter case with no non-tradable asset or stochastic factor, this quantity is just a deterministic function of time. In our model, this can be obtained by solving a non-linear PDE or simulating Z_t under the minimal martingale measure $\mathbb{Q}^M$ (2.30) with the model inputs we obtained from the previous step. In fact, a simpler way is to do this is to solve the linear equation for h in (2.13) and transform the values to H^0 knowing its relationship with H^0 (2.36). Linear equation can readily be solved by finite difference methods. Once this quantity has been determined numerically (or analytically under particular model), we can work out the z derivatives on the (t,z) plane and hence determine the appropriate ψ that allows us to define the minimal entropy measure $\mathbb{Q}^E$. Now we have all the ingredients to solve the non-linear PDE for the indifference price. We can see the complication in carrying out such a computation as it involves multiple simulations and transformations, we shall see in the next section that an asymptotic expansion might help in computing numerical approximations.

However, further simplification is sometimes possible. For example if the agent selling the claim has very small risk aversion, then the indifference price PDE we are solving can be approximated as being linear and we can readily use standard methods to compute the so called marginal utility based price. Recall the section on payoff decomposition, then we know we have an alternative way (2.43) to compute the indifference price by working out approximation to the expectation of a stochastic integral together with the marginal utility based price.

3.2 Asymptotic Expansion

Motivated by the representation in (2.43), we would like to look at asymptotic expansion of the indifference price in order of the risk aversion parameter γ. Suppose we have an asymptotic expansion of the form

$$p(t,y,z) = \hat{p}(t,y,z) + \gamma p^{(1)}(t,y,z) + \gamma^2 p^{(2)}(t,y,z) + O(\gamma^3),$$

then by substituting into the indifference price PDE (2.35) and equating coefficients in order of γ, we would be able to get some kind of PDE representations for $p^{(1)}, p^{(2)}$ that might be simpler to solve. For example,

$$\hat{p}_t + \mathcal{L}^{\mathbb{Q}^E}_{Y,Z}\hat{p} = 0,$$

$$p_t^{(1)}+\mathcal{L}_{Y,Z}^{\mathbb{Q}^E}p^{(1)}+\frac{1}{2}(1-\rho^2)(\sigma^Y(z)y)^2\hat{p}_y^2+b(z)\sigma^Y(z)y\beta\sqrt{1-\rho^2}\hat{p}_y\hat{p}_z+\frac{1}{2}(1-\kappa^2)b^2(z)\hat{p}_z^2=0.$$

These PDEs are to be solved iteratively, i.e. one would need the values for $p^{(1)}$ to solve the equation for $p^{(2)}$ and so on, but we can see they are easier to handle than the original fully non-linear equation. J.P. Fouque, G. Papanicolaou and K.R. Sircar looked into using asymptotics to price derivatives in a stochastic volatility model [9] based on the fact that common models have 'fast mean-reversion speed'. The content of the book provided some inspiration as to how the method would work in the model we are considering, but now we shall not look into the details and shall leave it to future research.

Chapter 4

Conclusion

As seen in the previous discussion, we have derived a PDE representation for the indifference ask price and the optimal hedging strategy under a basis risk model with stochastic volatility. Moreover, we utilized the result to look at residual risk and payoff decomposition for the indifference price. Comparing the results from our model with the ones obtained from the classical two assets model, we see indeed they have a great degree of similarities. In both models, all non-linear terms in the indifference price PDE representation are attached to the risk aversion parameter γ and the generator is under the minimal entropy measure. Also, the results for the payoff decomposition and the asymptotic expansion are identical in both cases except that the local martingale L_t, which appears in both expressions, has an extra orthogonal component in our model. These results are in line with our expectation and we are pleased to see some consistency here.

Although it is a common practice to look at exponential utility due to its ability to simplify problem in computing indifference price, it is worth noting that using such utility function removes the initial wealth dependence from the indifference price that is a desirable feature to have. As a result, it would be a good idea to consider power utility, but one can imagine the computational complexity that arose when we are considering a 4 dimensional indifference price. Also, the use of power utility in the maximization problem might pose difficulties in pricing short position claims. In addition to all the technicalities, one thing that we have not considered, or in fact assumed, is the existence of a smooth solution to the PDE representation. It is shown by H. Pham in his paper [24] in 2002 that such a solution exists provided certain conditions are met.

In terms of future work, one should compute prices and optimal hedges numerically based on the representations that we derived and perhaps following the procedure that was outlined in the previous chapter. Then one could simulate residual risk and see the distribution of the hedging error. M. Monoyios showed in [20] that the optimal hedge obtained from utility indifference pricing outperforms the usual Black-Scholes hedge, assuming the claim is perfectly replicable, for a put option. Therefore it will be interesting to see whether our work would have any merit despite the excessive complication in deriving the representations when we compare the results that we derived to the ones obtained from the model that we assume volatility is not stochastic. Another reasonable extension is to look at the corresponding results under a model with stochastic correlation, since we are interested in hedging a claim on a non-tradable asset with a correlated tradable asset and by no means the correlation between the assets should be constant as time evolves. It will be interesting to see any further work that builds on these extensions in the future.

Bibliography

[1] F.E. Benth, K.H. Karlsen, *A PDE representation of the density of the minimal entropy martingale measure in stochastic volatility markets*, Stochastics, 77(2), 109-137, (2005)

[2] F. Black, M. Scholes, *The pricing of options and corporate liabilities*, Journal of Political Economy, 81(3), 637-654, (1973)

[3] M.H.A. Davis, *Option pricing in incomplete markets*, in Mathematics of Derivative Securities, M.A.H. Dempster and S.R. Pliska, eds, New York: Cambridge University Press (1997)

[4] M.H.A. Davis, *Option hedging with basis risk*, in Y. Kabanov, R. Liptser and J. Stoyanov, eds, 'From Stochastic Calculus to Mathematical Finance', Springer, pp. 169-187 (2006)

[5] F. Delbaen, P. Grandits, T. Rheinläinder, D. Sampieri, M. Schweizer, Ch. Stricker, *Exponential hedging and entropic penalties*, Mathematical Finance, 12:99-124, (2002)

[6] F. Delbaen, W. Schachermayer, *The mathematics of arbitrage*, Springer Finance, (2006)

[7] H. Föllmer and M. Schweizer, *Hedging of contingent claims under incomplete information*, in M.H.A. Davis and R.J. Elliot, editors, Applied Stochastic Analysis, 389-414. Gordon and Breach, London, (1990)

[8] H. Föllmer and D. Sondermann, *Hedging of nonredundant contingent claims*, in Contributions to mathematical economics, North-Holland, Amsterdam, 205-223, (1986)

[9] J.P. Fouque, G. Papanicolaou, K.R. Sircar, *Derivatives in financial markets with stochastic volatility*, Cambridge: Cambridge University Press, (2000)

[10] M. Frittelli, *The Minimal Entropy Martingale Measure and the Valuation Problem in Incomplete Markets*, Mathematical Finance, Vol. 10/1, 39-52, (2000)

[11] V. Henderson, *Valuation of claims on non-traded assets using utility maximization*, Mathematical Finance, 12(4):351-373, (2002)

[12] V. Henderson, D. Hobson, *Utility Indifference Pricing - An Overview*, in R. Carmona (ed.), *Indifference Pricing: Theory and Applications*, Princeton University Press, (2009)

[13] S.D. Hodges, A. Neuberger, *Optimal replication of contingent claims under transaction costs*, Review of Futures Markets 8, 222-239, (1989)

[14] H. Hulley, T.A. McWalter, *Quadratic hedging of basis risk*, Quantitative Finance Research Centre, Research paper 225 (2008)

[15] J. Kallsen, T. Rheinländer, *Asymptotic utility-based pricing and hedging for exponential utility*, Statistics & Decisions, 28, 17-36, (2011)

[16] I. Karatzas, J.P. Lehoczky, S.E. Shreve, G-L. Xu, *Martingale and duality methods for utility maximization in an incomplete market*, SIAM Journal of Control and Optimisation, 29:702-730 (1991)

[17] D. Kramkov, M. Sîrbu, *Asymptotic analysis of utility-based hedging strategies for small number of contingent claims*, Stochastic Processes and their Applications, 117, 1606-1620, (2007)

[18] R.C. Merton, *Lifetime portfolio selection under uncertainty: the continuous-time case*, Rev. Econom. Statist., 51:247-257, (1969)

[19] R.C. Merton, *Theory of rational option pricing*, BELL Journal of Economics, 4, 141-183, (1973)

[20] M. Monoyios, *Performance of utility-based strategies for hedging basis risk*, Quantitative Finance, 4, 245-255, (2004)

[21] M. Monoyios, *Optimal hedging and parameter uncertainty*, IMA Journal of Management Mathematics, 18, 331-351, (2007)

[22] M. Monoyios, *Utility based valuation and hedging of basis risk with partial information*, Applied Mathematical Finance, 17, 519-551, (2010)

[23] M. Musiela, T. Zariphopoulou, *An example of indifference prices under exponential preferences*, Finance and Stochastics **8**, 229-239, (2004)

[24] H. Pham, *Smooth solutions to optimal investment models with stochastic volatilities and portfolio constraints*, Applied Mathematics & Optimization, 46(1), 55-78, (2002)

[25] R. Rouge, N. El Karoui, *Pricing via utility maximization and entropy*, Mathematical Finance, 10:259-276, (2000)

[26] M. Schweizer, *Mean-variance hedging for general claims*, Annals of Applied Probability, 2(1), 171-179 (1992)

[27] M. Schweizer, *A guided tour through quadratic hedging approaches*, in E. Jouini, J. Cvitanić and M. Musiela, eds, 'Option Pricing, Interest Rates and Risk Management', Cambridge University Press, Chapter 15, 538-574

[28] T. Zariphopoulou, *A solution approach to valuation with unhedgeable risks*, Finance and Stochastics **5**, 61-82, (2001)