On the Analysis of Isometric Immersions of Riemannian Manifolds

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Abstract

This thesis is a study of problems related to isometric immersions of Riemannian manifolds in Euclidean space. We address three main questions: the weak continuity of geometric invariants along sequences of immersions of Riemannian manifolds; the construction of isometric immersions by weak compactness methods; and the validity and regularity of the Gauß equation.

First, we investigate the validity of Cartan's equations for $\mathrm{W}^{1,p}$ coframes on surfaces, for all $1 \leqslant p \leqslant \infty$, and employ this to derive a version of the Gauß equation valid for $\mathrm{W}^{2,p}$ immersed surfaces in $\mathbf{R}^3$. Under some additional regularity hypotheses, a distributional formulation of the Gauß equation on immersed surfaces in $\mathbf{R}^3$ is proved, and as a corollary, a new local regularity result is established for isometric immersions of positively curved surfaces.

Investigating the weak continuity properties of immersions of Riemannian manifolds, we first prove a general weak continuity result for curvatures of connections with L^p bounds on principal bundles. As a corollary, it is proved that when $p > 2$, curvatures are weakly continuous for the weak $\mathrm{W}^{2,p}$ convergence of immersions. In some cases, we also recover the weak continuity of the general Gauß equation when $p = 2$.

Finally, we give a general viscosity framework for constructing isometric immersions in prescribed target spaces under natural boundedness assumptions in L^p spaces. Assuming this set-up, we prove a new weak compactness theorem for approximate solutions of the Gauß–Codazzi–Ricci equations.

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Finally, it seems both proper and fitting to borrow W. B. Yeats' words, who captured so well the spirit of the place.

> I wonder anybody does anything at Oxford but dream and remember, the place is so beautiful. One almost expects the people to sing instead of speaking. It is all like an opera.[1]

---

[1] W. B. Yeats.

Statement of Originality

I declare that the contents of this thesis are, to the best of my knowledge, original and my own work, except where otherwise indicated, cited, or commonly known, nor has any part of this thesis been submitted for a degree at another university.

Examiners

Prof. Camillo De Lellis	Prof. Jan Kristensen
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Tous les géomètres seraient donc fins s'ils avaient la vue bonne, car ils ne raisonnent pas faux sur les principes qu'ils connaissent. Et les esprits fins seraient géomètres s'ils pouvaient plier leur vue vers les principes inaccoutumés de géométrie.

B. Pascal, *Pensées*

Contents

Index of notations

$\mathbf{N}, \mathbf{Z}, \mathbf{R}, \mathbf{C}$	set of natural numbers, integers, real numbers and complex numbers
(M, g)	smooth manifold M equipped with a Riemannian metric g
K, K_g	on a two-dimensional manifold, Gauß curvature of the metric g
$(\mathbf{R}^N, e)$	Euclidean N-dimensional space equipped with the standard Euclidean metric
$(\mathbf{S}^N, \sigma)$	N-sphere equipped with the standard round metric
$\mathbb{G}_{k,n}$	Grassmanian of k-dimensional subspaces of an n-dimensional real vector space
$\mathbf{GL}(n), \mathbf{O}(n), \mathbf{SO}(n)$	general linear, orthogonal and special orthogonal respectively, groups of n by n matrices
$\mathfrak{gl}(n), \mathfrak{so}(n)$	Lie Algebra of n by n matrices, respectively skew-symmetric n by n matrices
$\binom{n}{k}$	Binomial factor $n!/k!(n-k)!$, $n, k \in \mathbf{N}$
$J(n)$	Janet dimension, $J(n) = \binom{n}{2}$, $n \in \mathbf{N}$
TM, T^*M	tangent bundle, respectively cotangent bundle of a manifold M
UV	unit bundle associated to a vector bundle $V \to M$ with a fiberwise scalar product (*e.g.* UTM, where M is a Riemannian manifold)
NM	normal bundle of M viewed as a submanifold $u(M)$ of $\mathbf{R}^N$
$\mathscr{D}(u(M))$	Darboux bundle associated to an immersion $u : M \to \mathbf{R}^N$
$\mathscr{F}V$	Frame bundle of a vector bundle $V \to E$
$\mathscr{L}(V, W)$	Linear maps between two vector spaces V and W
$C(E)$	set of continuous functions on a domain E
$C_0(E)$	set of continuous functions on a domain E vanishing at infinity
$C^\alpha(E)$	α-Hölder continuous functions on E, $0 < \alpha < 1$
$\mathrm{Lip}(E)$	Lipschitz continuous functions on E
$\mathrm{Lip}_0(E)$	Lipschitz continuous functions on E vanishing at infinity
$C^k(E)$	k-times differentiable functions on E, $k \in \mathbf{N}$
$C^\infty(E)$	smooth real-valued functions on E
$C_c^\infty(E)$	smooth compactly supported real-valued functions on E
$C^\omega(E)$	analytic real-valued functions on E
$\Gamma(E, P)$	smooth sections of the fiber bundle $P \to E$
$(P)_x$	fiber at $x \in E$ for a fiber bundle $P \to E$

$\Omega^k(E)$	smooth k-alternating sections of the bundle $T^*E \to E$ (i.e. k-forms), $k \in \mathbf{N}$
$\Omega^k(E, \mathfrak{g})$	smooth $\mathfrak{g}$-valued k-forms, where $\mathfrak{g}$ is a Lie Algebra.
u_*	push-forward of a differential map u
u^*	pull-back of a differential map u
$\sharp, ^\flat$	musical isomorphism transforming a (p,q) tensor into a $(p+1, q-1)$ tensor, respectively into a $(p-1, q+1)$ tensor
d	exterior derivative (acting on differential forms)
$*$	Hodge dual operator
$dV_g = *1$	volume form associated with a metric g
δ	co-differential. On k-forms $\delta = (-1)^k *^{-1} d*$
$\triangle = (d+\delta)^2$	Hodge Laplacian acting on differential forms
$\triangle_g$	Laplace–Beltrami operator on (M, g) acting on functions
$\not\triangle = \triangle_\sigma$	Laplace–Beltrami operator on the sphere $(\mathbf{S}^N, \sigma)$
D_A	covariant derivative associated to a connection ∇_A
D_A^*	covariant co-derivative associated to a connection ∇_A
$\Omega_A = \nabla_A \circ \nabla_A$	curvature 2-form associated with the connection ∇_A
$\mathbf{Ad}(P)$	Adjoint bundle of a principal G-bundle $P \to M$, where G is a Lie group
$\mathrm{L}^p(E, V)$	Lebesgue space of p-integrable sections of the vector bundle $V \to E$, $1 \leqslant p \leqslant \infty$
$\mathrm{W}^{k,p}(E, V)$	Sobolev space, completion of $\Gamma(\overline{E}, V)$
$\mathrm{W}_0^{k,p}(E, V)$	Sobolev space, completion of compactly supported sections $\Gamma(E, V)$
$\mathcal{M}(E, T^*E)$	Space of covector-valued measures on the domain E
$\mathbf{BV}(E)$	Spaces of functions on E of bounded variation
$\mathfrak{g} = T_{\mathbf{1}}G$	Lie Algebra associated to a Lie group G
$\mathcal{A}(P)$	space of smooth connections on a principal bundle $P \to M$
$\mathcal{A}^{k,p}(P)$	space of Sobolev connections on a bundle $P \to M$
$\Subset$	compactly contained in...
$\langle \cdot, \cdot \rangle_g$	scalar product induced by the metric g
$\lvert \cdot \rvert_g$	norm induced by $\langle \cdot, \cdot \rangle_g$
$\langle\!\langle \cdot, \cdot \rangle\!\rangle$	duality pairing between a distribution and a test function
$\mathcal{M}(E)$	real-valued signed Radon measures on E
$\lVert\mu\rVert(E)$	for a (signed) Radon measure μ, total mass of μ on E
$[\mu]^{\mathrm{ac}}, [\mu]^{\mathrm{sing}}$	absolutely continuous part, respectively singular part of a measure μ with respect to the Lebesgue measure
$\mathcal{H}^m$	m-Hausdorff measure
$\det \nabla^2 f$	Hessian determinant of the real-valued function f
$\mathrm{Det}\, \nabla^2 f$	Distributional Hessian determinant of the real-valued function f
$\mathcal{M}_f(\cdot)$	Alexandrov measure associated with the convex function f
$\mathrm{div}, \mathrm{div}_g$	Euclidean divergence operator, respectively divergence associated with a metric g
rot	curl operator
χ_E	characteristic function of the set E.

Chapter 1

Introduction

> "Bekanntlich setzt die Geometrie den Begriff des Raumes..."[1]

Riemannian manifolds are a central object of interest in differential geometry. Recall that a Riemannian manifold (M, g) is a smooth manifold M equipped with a metric g, which is a scalar product on the fibers of the tangent bundle. These objects generalise in an intrinsic manner the differential study of curves and surfaces in space, a point of view which historically was the earlier, classical perspective of Euler, Lagrange, Gauß and others. For instance, in Euclidean space $\mathbf{R}^3$, the Euclidean structure induces a Riemannian metric on embedded surfaces, and one may study their geometry in the abstract, without needing to consider a specific embedding in space.

The notion of Riemannian manifold was formalised around the turn of the twentieth century, following Riemann's work; since then, an extensively studied question has been whether every Riemannian manifold can be isometrically realised as a submanifold in Euclidean space.

This thesis is a contribution to the study of isometric immersions from the point of view of non-linear partial differential equations. A natural setting to study nonlinear PDE is Sobolev spaces and weak convergence, which are the main focuses of our study.

Contrary to other problems of differential geometry such as minimal surfaces or harmonic maps, the isometric immersion problem is not variational. Properties of isometric immersions and embeddings depend significantly and in a non-trivial manner on their regularity. A central issue in isometric immersions is the role and properties of curvatures, which are the main objects of our investigation.

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[1] Bernhard Riemann, "Über die Hypothesen, welche der Geometrie zu Grunde liegen", [150]

1.1 Isometric immersions: focal points

We start by reviewing the basic equations describing isometric immersions. The reader is referred to Chapter 3 for a complete presentation of the general framework.

1.1.1 The equations of isometric immersions

Our starting point will always be a smooth, compact, Riemannian n-manifold without boundary (M, g). An immersion $u \in C^1(M, \mathbf{R}^N)$ into some Euclidean space $\mathbf{R}^N$ is isometric if the Euclidean metric e on $\mathbf{R}^N$ pulls back to the metric g on M. In other words, we require $u^*e = g$, and this equation is the basic constraint of the problem. Choosing local coordinates on M, this is a system of $n(n-1)/2$ non-linear, first-order PDE

$$\sum_{k=1}^{N} \frac{\partial u^k}{\partial x^i} \frac{\partial u^k}{\partial x^j} = g_{ij}, \qquad 1 \leqslant i, j \leqslant n. \tag{1.1}$$

We refer to this equation as the *isometry constraint*.

Depending on the regularity class of solutions, the isometry constraint has markedly different behaviours, which we now discuss. First recall that by Whitney's theorems, every n-dimensional Riemannian manifold admits an immersion in $\mathbf{R}^{2n-1}$ (see §2.1.1). Since certainly there is no hope of constructing isometric immersions when the set of immersions at large is empty (the *topological obstruction*), one looks to construct isometric immersions in $\mathbf{R}^{2n-1}$. Nash's C^1 theorem [134] asserts that this is indeed possible, but the resulting isometric immersion will only be C^1 and in no further C^k space for $1 < k \in \mathbf{N}$ in general. A regularity improvement due to [43] shows that the immersion may in fact always be taken to have some Hölder differentiability. Nash's result is surprising in view of the fact that there are $n(n-1)/2$ components in the metric tensor, but only $2n-1$ components of u, so that in some sense the C^1 theorem gives a solution to an over-determined problem. Following Gromov [77], a possible interpretation of this fact is that the isometry constraint alone is *flexible*, and that there is no further obstruction to the solvability of the isometry constraint in C^1 than the topological one.

Looking for isometric immersions of higher regularity, additional constraints emerge in the form of the Gauß–Codazzi–Ricci equations: not only does the immersion u have to satisfy the isometry constraint (1.1), but additionally the extrinsic and intrinsic curvatures have to agree.

The main intrinsic invariant of Riemannian geometry is the Riemann curvature tensor, which is entirely determined by the Riemannian metric. Cartan's method of moving frames is our dominant viewpoint in this thesis. Choosing a local orthonormal coframe $\theta^1, \dots, \theta^n$, we define a connection form ω and a curvature form Ω by *Cartan's structure equations*,

$$\begin{aligned} d\theta^i &= \omega^i_j \wedge \theta^j, & \omega^i_j &= -\omega^j_i \\ d\omega^i_j + \omega^i_k \wedge \omega^k_j &= \Omega^i_j, & \Omega^i_j &= -\Omega^j_i. \end{aligned}$$

On a Riemannian manifold, the curvature form is related to the Riemann curvature tensor by

$$\Omega^i_j = \frac{1}{2} R^i_{jkm} \theta^k \wedge \theta^m =: \mathbf{Rm}(\theta^k \wedge \theta^m). \tag{1.3}$$

Classical theorems characterise the local geometry of a Riemannian manifold in terms of its curvature. In the smooth regime, the curvature therefore induces further constraints (*curvature constraints*).

The extrinsic geometry of an immersed manifold is described by the second fundamental form $\mathbf{II}$, which is a tensor defined as the difference between the connection on $\mathbf{R}^N$ and the Levi-Civita connection on (M, g). The second fundamental form and the Riemann curvature tensor have to satisfy a set of compatibility equations, the Gauß–Codazzi–Ricci system. From a philosophical perspective, it links the intrinsic and extrinsic geometries of a manifold.

Several formalisms are available to describe the Gauß–Codazzi–Ricci system. Following Cartan's moving frame method, let $u : M^n \to \mathbf{R}^N$ be an isometric immersion, and let $(\theta^1, \dots, \theta^n, \theta^{n+1}, \dots, \theta^N)$ be an adapted moving coframe on $u(M)$ (also called a Darboux coframe for the immersion—see Chapter 3), in the sense that $\theta^1, \dots, \theta^n$ is tangent to M. To a coframe is associated a connection form; the adaptedness condition splits the connection into a tangential connection $\omega^\top$, a normal connection $\omega^\perp$ and a second fundamental form term $\omega^{\mathbf{II}}$, which obey the following equations:

$$\begin{aligned} d\theta^i &= (\omega^\top)^i_j \wedge \theta^j + ({}^t\omega^{\mathbf{II}})^i_k \wedge \theta^k, & & 1 \leqslant i, j \leqslant n, \quad n+1 \leqslant k \leqslant N \\ d\theta^i &= (\omega^\perp)^i_j \wedge \theta^j - (\omega^{\mathbf{II}})^i_k \wedge \theta^k, & & n+1 \leqslant i, j \leqslant N, \quad 1 \leqslant k \leqslant n. \end{aligned}$$

The Gauß–Codazzi–Ricci system is the following system:

$$ {}^t\omega^{\mathbf{II}} \wedge \omega^{\mathbf{II}} = \tau^* \, \mathbf{Rm}, \tag{1.4a}$$

$$ d\omega^{\mathbf{II}} = \omega^\top \wedge \omega^{\mathbf{II}} + \omega^\perp \wedge \omega^{\mathbf{II}}, \tag{1.4b}$$

$$ d\omega^\perp = -\omega^\perp \wedge \omega^\perp + \omega^{\mathbf{II}} \wedge {}^t\omega^{\mathbf{II}}. \tag{1.4c}$$

The first equation (1.4a) is the generalised Gauß equation, while the other two (the Codazzi equations (1.4b) and the Ricci equations (1.4c)) can be considered compatibility conditions for the Gauß equation. We refer the reader to Chapter 3 for the notation and derivation of these equations. Here **Rm** is the curvature tensor on (M,g), i.e. the intrinsic curvature tensor on M, given by (1.3); $\tau : u(M) \subset \mathbf{R}^N \to M$ is the projection onto M.

1.1.2 Some problems about isometric immersions

The contributions of this thesis are concerned with the analytic properties of such systems. In its general form, the Gauß–Codazzi–Ricci equations are a semi-linear, mixed-type system with quadratic non-linearities. At the moment, there is no general theory of existence and non-existence for the curvature constraints.

Special cases of the Gauß–Codazzi–Ricci equations, or other related equations, have been studied, and are relevant to elasticity, general relativity, and many other fields. In particular, the case of surfaces of positive or negative curvature immersed in $\mathbf{R}^3$ has received considerable attention. For instance, a question raised by Weyl [176] asked to show that every positively curved metric on the sphere $\mathbf{S}^2$ may be realised as the boundary of a convex body in $\mathbf{R}^3$ (this was answered positively by Nirenberg [137]). Another example is a question raised by Yau [178] to find sufficient conditions for the realisation of complete surfaces with negative curvature. This was motivated by a famous theorem of Hilbert [91, 118] asserting that a surface of constant negative curvature cannot be smoothly isometrically immersed in $\mathbf{R}^3$. All such methods depend heavily on additional properties of the curvature constraints in each special case: *e.g.* when immersing a surface with positive curvature in $\mathbf{R}^3$, the curvature equation forms an elliptic system of Monge–Ampère type.

Fundamental theorems due to Nash [134, 135, 136] have also been known for over half a century, but much still remains to be understood about isometric immersions and their regularity. In the subsequent paragraphs, we review several specific problems in the theory of isometric immersions, which are essential motivations for the results of the thesis.

Weak convergence methods

This thesis is part of an effort to establish a weak convergence framework to study and describe isometric immersions and embeddings. Weak convergence methods refer to a broad programme to tackle non-linear PDE (see *e.g.* [65] and the references therein). These methods are flexible enough that they may used in a variety of contexts, including mixed-type systems, which motivates developing the theory for isometric immersions. Such methods were first applied in this context to treat the special case of surfaces of negative curvature in $\mathbf{R}^3$ in [35]. There are significant difficulties in extending the authors' approach, either to the mixed-type case, or higher dimensions and codimensions.

The basic problem that we hope to be able to address with such methods is the existence of isometric immersions in prescribed target spaces. An avenue for existence is to seek isometric immersions as solutions of the Gauß–Codazzi–Ricci equations (1.4a), (1.4b), (1.4c). By the realisation theorem, this implies the existence of an isometric immersion with the prescribed tangential and normal connection form, and second fundamental form.

One of the questions to be addressed when implementing a weak convergence framework is the weak compactness of approximate solutions to the system under suitable bounds. Recall that in our description of isometric immersions, the main invariants of the theory are the Riemann curvature tensor, which is prescribed by the data, and the second fundamental form. Thus, one seeks to construct solutions of the Gauß–Codazzi–Ricci equations by taking limits of approximate solutions with second fundamental form bounded in some L^p space. Such bounds are commonly obtained through approximation schemes or viscosity methods.

Earlier efforts to address this question [33, 36] required additional a priori boundedness assumptions on the tangential and normal connections. We address the problem of removing such assumptions, and thus obtaining a general weak compactness framework, in Chapter 8.

Flexibility and Rigidity for the Gauß equation

Another important problem related to isometric immersions is the question of flexibility and rigidity (see [43, 53, 77] and the references therein).

We mention a connection with fluid dynamics, and a source of inspiration for some of our results. Onsager's conjecture (now a theorem, [22, 42, 99]) is concerned with conservation of energy for the motion equations of incompressible fluids

$$\begin{aligned} \partial_t u + \operatorname{div}_x(u \otimes u) + \nabla p &= 0, \\ \operatorname{div} u &= 0. \end{aligned}$$

Onsager [140] proposed that α-Hölder continuous solutions of Euler's equations conserve energy if $\alpha > 1/3$ and may dissipate energy for $\alpha < 1/3$. In other words, regular enough solutions of Euler's equations satisfy an additional conservation law

$$\frac{d}{dt}\|u(t)\|_{\mathrm{L}^2_x} = 0.$$

This dichotomy is seemingly of the same nature as the isometric immersion problem, as in both cases anomalous solutions can be constructed via convex integration. Further connections between isometric immersions and fluid dynamics are discussed in Section 7.A.

In the case of isometric immersions, the "dichotomy" between rigidity and flexibility is twofold. First, Nash's theorems demonstrate that the isometry constraint is not continuous for the weak* topology of $\mathrm{W}^{1,\infty}$ (see Chapter 7). A natural question is to understand under what conditions the isometry constraint is weakly continuous, and if that is the case, whether the curvature constraint is also weakly continuous. This mirrors the question of weak continuity for Euler's equations, see *e.g.* [54, 58, 59, 60], and many other problems in non-linear PDE, where some equations admit "wild" solutions which may be constructed by a process similar to Nash's C^1 construction (see *e.g.* [52, 128, 129] and the references therein). This particular problem is addressed in Chapter 7; we refer to it as *weak continuity of the constraint equations.*

Secondly, Nash's C^1 theorem shows the existence of low regularity "wild" isometric immersions, for which the curvature constraints do not make sense. Determining the exact threshold between this regime and the "rigid" regime where curvature can be made sense of is an open problem [53, 77]. In the case of surfaces in $\mathbf{R}^3$, which so far has received the most attention, the threshold is conjectured to be $\alpha = 1/2$ for Hölder spaces $C^{1,\alpha}$.

More generally, recalling that curvature constraints are corollaries of the Cartan framework (see Chapter 3), one may formulate a more general problem, namely, to determine the validity of Cartan's structure equations for low regularity coframes. In the context of Sobolev spaces $\mathrm{W}^{1,p}$, this question is addressed by the results of Chapter 4.

A special case of interest is the case of surfaces with positive curvature. Indeed, when the curvature is positive, classically the Gauß–Codazzi equations form an elliptic system. A long-standing question [7] is to make sense of this in low regularity, and in particular to prove regularity theorems. Such results have been studied for some regularity classes of isometric immersions, see *e.g.* [43, 95].

Relevant to these problems is the work of Chapter 5, where we prove a distributional version of the Gauß equation for a class of Sobolev isometric immersions. Additionally, when the curvature is positive, we also prove a regularity result. Though we do not settle the aforementioned conjecture, our result highlights the key role played by the fractional 1/2-differentiability.

1.2 Main results

We now describe the results obtained in the thesis. All theorems indexed with letters are the author's work. Precise statements are given anew at the beginning of each chapter.

1.2.1 Validity of Cartan's equations

First, we address the validity question. The properties of isometric immersions depend considerably on their regularity. In particular, Nash's C^1 isometric immersions do not satisfy the Gauß–Codazzi–Ricci system; in fact, even writing the system for such low regularity objects is problematic.

In view of the quadratic non-linearities, it is natural to expect the Cartan equations to hold for $\mathrm{W}^{1,2}$ coframes (and this is indeed the case, see Proposition 4.17), and therefore the Gauß–Codazzi–Ricci equations to hold for $\mathrm{W}^{2,2}$ immersions. On the other hand, one does not expect the Gauß–Codazzi–Ricci equations to hold for isometric immersions that would be only C^1, as Nash's theorem [134] shows. A natural question is therefore to investigate intermediate regimes, *e.g.* the validity of the Gauß equation for immersions that are also in some Sobolev space larger than $\mathrm{W}^{2,2}$.

From the point of view of coframes, a necessary preliminary is the validity of Cartan's equation for rough coframes. We focus on the case of two-dimensional surfaces, where a smooth coframe satisfies, in matrix notation,

$$\begin{aligned} d\theta &= \omega \wedge \theta, \\ d\omega &= K\theta \wedge \theta, \end{aligned}$$

where ω is the connection form and K is the Gauß curvature. When θ is merely $\mathrm{W}^{1,p}$, the curvature equation can only be expected to be satisfied in the sense of distributions. Moreover, when $p < 2$, we have to allow for the equation to only be satisfied up to a defect term. Our first result makes this precise (see Chapter 4, Theorem 4.1):

THEOREM A. *Let (M, g) a smooth Riemannian surface, K the Gauß curvature, and $E \subset M$ a domain. Consider an orthonormal coframe $\theta^1, \theta^2 \in \mathrm{W}^{1,p}(E, T^*E)$,* *with* $p \geqslant 1$. *Then there exists a connection form* $\omega \in \mathrm{L}^p(E, T^*E)$ *such that*

$$d\theta^1 = \omega \wedge \theta^2, \qquad d\theta^2 = -\omega \wedge \theta^1.$$

Moreover, defining the defect distribution $\mathfrak{s}$ *of the coframe as*

$$\langle\!\langle\, \mathfrak{s}, \phi \,\rangle\!\rangle = \int_E \omega \wedge d\phi - \int_E \phi K \theta^1 \wedge \theta^2, \qquad \forall \phi \in C_c^\infty(E),$$

there exists countably many points P_j *and* N_j *in* $\bar{E}$ *such that*

$$\begin{aligned} &\sum_{j \in \mathbf{N}} \rho(P_j, N_j) < \infty, \\ &\mathfrak{s} = 2\pi \sum_{j \in \mathbf{N}} \delta_{P_j} - \delta_{N_j}, \end{aligned} \tag{1.5}$$

and these points may be chosen in such a way that $\sum_{j \in \mathbf{N}} \rho(P_j, N_j)$ *is arbitrarily close to the operator norm of* $(2\pi)^{-1}\mathfrak{s}$*, as an element of* $\mathrm{Lip}_0(E)^*$.

Finally, if $p \geqslant 2$ *or if the coframe is continuous then* $\mathfrak{s} = 0$.

In the statement of Theorem A, $\rho(\cdot, \cdot)$ is a suitable pseudo-distance on $\bar{E}$ (see Chapter 4).

Distributions of the form given by (1.5) have been well-studied, as they are relevant to a number of other problems, such as Ginzburg–Landau equations, liftings of manifold-valued Sobolev mappings, or harmonic maps: the reader is referred to *e.g.* [11, 13, 15, 17, 19, 98, 145, 160] and the references therein.

In the higher-dimensional case, Cartan's second equations include a quadratic non-linearity in the connection form:

$$\begin{aligned} &d\theta = \omega \wedge \theta, \\ &d\omega + \omega \wedge \omega = \mathbf{Rm}(\theta \wedge \theta). \end{aligned}$$

Here θ is an $\mathbf{SO}(N)$-valued one-form, ω is $\mathfrak{so}(N)$-valued, and $\mathbf{Rm}$ is the curvature endomorphism associated to the metric g.

Cartan's equations are shown to hold in the sense of distributions for $\mathrm{W}^{1,2}$ coframes (Proposition 4.17). The Gauß–Codazzi–Ricci equations being consequences of Cartan's equations (see Chapter 3), this also proves their validity for adapted $\mathrm{W}^{1,2}$ coframes.

1.2.2 Towards the rigidity-flexibility dichotomy

Recall that if $u : M \to \mathbf{R}^3$ is a regular enough isometric embedding, then its components must satisfy the Darboux equation, which is another form of the Gauß equation. Let us assume that in local coordinates (x^1, x^2), u is given as a graph, i.e. $u = (x^1, x^2, f(x^1, x^2))$. Then f verifies

$$\det \nabla^2 f = K(1 + |\nabla f|^2)^2.$$

We interpret this equation as the analogue of conservation of energy for isometric immersions. It is seen to hold for $\mathrm{W}^{2,2}$ immersions by a direct approximation argument (see *e.g.* [95]). In search of the 'rigid-flexible' dichotomy for isometric immersions, we investigate the Gauß equation in spaces of lower regularity. Note that in larger regularity classes than $\mathrm{W}^{2,2}$, the pointwise Hessian determinant $\det \nabla^2 f$ may fail to be integrable; however we may still give meaning to the distributional Hessian determinant $\mathrm{Det}\, \nabla^2 f$. This notion of determinant was introduced by Ball [2] in the context of elasticity. Using the coframe method and Theorem A, in Chapter 5 we prove:

THEOREM B. *Let (M, g) be a smooth Riemannian surface, K its Gauß curvature, and $E \subset M$ an open domain. Let $u : E \to \mathbf{R}^3$ be a locally isometric embedding, locally parametrised by a function $f \in \mathrm{W}^{2,p}(D) \cap \mathrm{W}^{1+\alpha,q}(D)$, in such a way that $u^3 = f(u^1, u^2)$, and D is the domain of f. Then the following hold: if p, q and α satisfy*

$$\frac{2}{q} + \frac{1}{p} \leqslant 1, \qquad 1 \leqslant p < 2, \qquad 4 < q \leqslant \infty, \qquad \alpha > \frac{1}{2}, \tag{R}$$

then the Gauß equation holds in the sense of distributions, i.e.

$$\mathrm{Det}\, \nabla^2 f = (K \circ (u^1, u^2)^{-1})(1 + |\nabla f|^2)^2.$$

Moreover, in this case, the Gauß equation also holds almost everywhere, i.e. *for a.e. $x \in D$,*

$$\det \nabla^2 f(x) = (K \circ (u^1, u^2)^{-1})(x)(1 + |\nabla f(x)|^2)^2.$$

A similar motivation led the authors of [43] to study a "weak" form of the Gauß equation for surfaces. Specifically, when the embedding is $C^{1,\alpha}$ for $\alpha > 2/3$, the Gauß map of an embedded surface in $\mathbf{R}^3$ satisfies an integral characterisation of the Gauß equation.

Recall that for a regular enough isometric immersion, the Gauß equation is elliptic when $K > 0$. It is natural to ask whether this is still the case in the class of regularity of Theorem B. In particular, natural questions to consider are whether $K > 0$ implies the convexity of the surface, and whether this implies additional regularity for the isometric immersion considered.

Extending a classical theorem of Hadamard, several works [43, 95, 173] have investigated whether positive curvature implied convexity and regularity under various regularity hypotheses; however the regularity class considered in Theorem B is not covered by any of these theorems. Using Pogorelov's theory of bounded extrinsic curvature [144], we show:

THEOREM C. *With the same notations and hypotheses as in Theorem B, if in addition* $\inf_E K > 0$*, then* f *is smooth and the Gauß equation holds classically in* D.

Along with similar results [43, 95, 144, 173], Theorem C suggests to think of the regularity class given by condition (R), where the Gauß equation holds, as "rigid", compared to the "flexible" regime of the Nash–Kuiper constructions.

1.2.3 Convergence of isometric immersions

A map $u : M \to \mathbf{R}^N$ is said to a be a weak immersion if it is a bi-Lipschitz homeomorphism onto its image, and additionally $u \in \mathrm{W}^{2,p}(M, \mathbf{R}^N)$. It is isometric if $\langle du, du \rangle = g$ almost everywhere. Such immersions are said to converge weakly if they converge in the weak topology of $\mathrm{W}^{2,p}$ and the weak* topology of $\mathrm{W}^{1,\infty}$.

One of the corollaries of Nash's theorems [134, 135] is that the isometry constraint is not weakly* continuous in $\mathrm{W}^{1,\infty}$, even if all the sequence of isometric immersions are smooth. Seeking classes of isometric immersions for which the isometry constraint is weakly continuous therefore requires some higher-order control. Our next theorem, which is Theorem 7.4 in Chapter 7, addresses this question: if the generalised mean curvatures are uniformly controlled, the isometry constraint is weakly continuous:

THEOREM D. *Let* (M, g) *be a closed Riemannian manifold, and consider a sequence of weak isometric immersions* ${}_\varepsilon : M \to \mathbf{R}^N$ *such that*

$$\sup_{k \in \mathbf{N}} \|A_\varepsilon\|(M) < \infty, \tag{1.6}$$

where $A = tr(g^{-1}\mathbf{II})$ *is the vector-valued mean curvature. Finally, let* u *be the weak* limit of the* u_ε *(up to a non-relabelled subsequence). Then* u *is an isometric immersion.*

The boundedness hypothesis (1.6) of Theorem D naturally arises in a number of geometric context, *e.g.* conformal immersions or the Weyl problem. In Chapter 7, we use Theorem D to construct convex solutions to the Weyl problem with non-negative curvature. Our approach contrasts with previous works on the subject [81] in that our a priori estimate relies only on the Gauß–Bonnet theorem rather than higher-order estimates on the Gauß curvature (see Theorem 7.10).

We now turn to the curvatures. Under uniform $\mathrm{W}^{2,p}$ bounds, the isometry constraint is always weakly continuous; in Chapter 7 we investigate the curvature constraints and prove the following:

THEOREM E. *Let M be a closed, smooth manifold, and let $g^\varepsilon \to g$ be a sequence of smooth Riemannian metrics on M converging together with all its first and second derivatives. Let u_ε be a sequence of immersions such that $\langle du_\varepsilon, du_\varepsilon\rangle = g^\varepsilon$, and assume that*

$$\sup_{\varepsilon>0} \|\mathbf{II}_\varepsilon\|_{\mathrm{L}^p} < \infty, \qquad p > 2,$$

where $\mathbf{II}^\varepsilon$ is the second fundamental form of u_ε. Finally, assume that there exists a $\mathrm{W}^{2,p}$ immersion such that up to a subsequence $u_\varepsilon \rightharpoonup u$ weakly in $\mathrm{W}^{2,p}(M, \mathbf{R}^N)$.

Then u satisfies $\langle du, du\rangle = g$ and on an arbitrary parallelisable subset of M, any orthonormal coframe on (M, g) may be extended into an orthonormal Darboux coframe satisfying the Gauß–Codazzi–Ricci equations (1.4a), (1.4b), (1.4c).

We show additionally that the Gauß equation ${}^t\omega^{\mathbf{II}} \wedge \omega^{\mathbf{II}} = \tau^* \mathbf{Rm}$ is weakly continuous along sequences of isometric immersions weakly converging in $\mathrm{W}^{2,2}$ (see Theorem 7.2).

1.2.4 Connections with L^p bounds

Weak continuity theorems such as Theorem E are consequences of a general fact about the curvature of an abstract connection on a principal bundle. More precisely, the curvatures of a sequence of connections with uniform L^p bounds converge in the sense of distributions, provided we assume a very weak bound on the curvatures:

THEOREM F. *Let (M, g) be a closed smooth Riemannian manifold, $P \to M$ be a principal G-bundle, and $\nabla \in \mathcal{A}(P)$ a smooth reference connection.*

Let $\nabla^\varepsilon = \nabla + A^\varepsilon$ be a sequence of connections on P, Ω^ε the associated curvature forms, and assume that

$$\sup_{\varepsilon>0} \|A^\varepsilon\|_{\mathrm{L}^p} < \infty, \quad p > 2,$$
$$\sup_{\varepsilon>0} \|\Omega_\varepsilon\|(M) < \infty.$$

Finally let $\nabla_A = \nabla + A$ be an L^p connection such that $A^\varepsilon \rightharpoonup A$ in L^p up to a non-relabelled subsequence. Then $\Omega_\varepsilon \overset{*}{\rightharpoonup} \Omega_A$ in the sense of distributions, i.e. for all $\phi \in \Gamma(M, \mathbf{Ad}(P) \otimes \wedge^2 T^* M)$,

$$\lim_{\varepsilon\to 0} \int_M \langle A^\varepsilon, \delta\phi\rangle + \frac{1}{2}\langle [A^\varepsilon \wedge A^\varepsilon], \phi\rangle = \int_M \langle A, \delta\phi\rangle + \frac{1}{2}\langle [A \wedge A], \phi\rangle.$$

The notation $\|\Omega^\varepsilon\|(M)$ refers to the total mass of Ω^ε, viewed as a measure. Theorem F is a div-curl-type result, and in some sense can be considered to extend the classical div-curl theorems [130, 131, 153, 163].

We note that Theorem F may be applied to a broad range of problems, provided one is able to recover a uniform L^p bound on connections. One example of such problems is the Yang–Mills equations, a variational theory of mathematical physics. For a connection ∇_A on a principal bundle $P \to M$, the Yang–Mills energy functional is given by

$$\nabla_A \mapsto \mathcal{YM}(A) := \|\Omega_A\|^2_{\mathrm{L}^2(M)}.$$

Critical points of this functional are called Yang–Mills connections and satisfy the Yang-Mills equation

$$D_A^* \Omega_A = 0.$$

The analysis of Yang–Mills equations is a rich field with far-reaching implications (*e.g.* [61, 70]).

As an application of Theorem F we obtain a compactness result for Yang–Mills connections with bounded L^p norm. Though related, this result is quite different to Uhlenbeck's gauge-fixing and compactness theorems [168]. In particular, our estimates are gauge-free.

1.2.5 Weak compactness of approximate solutions to the Gauß–Codazzi–Ricci equations

We now turn to the existence question, using the weak compactness method. As mentioned earlier, we seek an isometric immersion as a weak limit of a sequences of solutions of the Gauß–Codazzi–Ricci equations. For a sequence of exact solutions, Theorem E lets us extract a further solution of the Gauß–Codazzi–Ricci system from a sequence of connections with a uniform L^p bound. However, in many applications (even for surfaces), this situation is unrealistic. A more natural setting is to consider approximate solutions, and only assume bounds on the second fundamental form. The isometry constraint remains continuous along sequences with L^p bounded second fundamental form, as Theorem D shows; on the other hand, it is no longer possible to apply the weak continuity theorem (Theorem E).

We call the connections

$$\omega_k = \begin{pmatrix} \omega_k^\top & {}^t\omega_k^{\mathbf{II}} \\ -\omega_k^{\mathbf{II}} & \omega_k^\perp \end{pmatrix},$$

defined on the bundle $TM \oplus NM$, a sequence of approximate solutions of the Gauß–Codazzi–Ricci equations if there exist two-forms η_k such that

$$d\omega_k + \omega_k \wedge \omega_k = \eta_k \overset{*}{\rightharpoonup} 0,$$

where the RHS is assumed to converge in a suitably weak sense (see Chapter 8 for our exact hypothesis).

Our next theorem shows that it is possible to control the tangential connection form $\omega_k^\top$ and normal connection form $\omega_k^\perp$ in terms of the second fundamental form (see Section 8.1 for our precise set of assumptions).

THEOREM G. *Let (M, g) be a closed Riemannian manifold, $V \to M$ a vector bundle over M, $p > n = \dim(M)$. We assume that there exists a sequence of tensors $\mathbf{II}_k$ uniformly bounded in L^p and a sequence of connection forms $\omega_k^\perp$ on V forming a sequence of approximate solutions to the Gauß–Codazzi–Ricci equations. Then there exists a $\mathrm{W}^{2,p}$ isometric immersion $u : (M, g) \to \mathbf{R}^N$, unique up to rigid motions of $\mathbf{R}^N$, such that its second fundamental form is $\mathbf{II}$.*

Using weak compactness methods to tackle the existence question was first proposed in [35]. The subsequent paper [36] proved a local version of the above theorem, under additional boundedness hypotheses. Theorem G generalises the main results of [33, 36] using new arguments. The methods used in Chapter 8, where Theorem G is proved, may also be generalised in a variety of other related contexts.

We comment briefly on the set of techniques employed to prove this theorem. An approximate solution of the Gauß–Codazzi–Ricci equation corresponds to a choice of coframe on the product bundle $TM \oplus NM$ which just fails to be flat. Such a coframe generates a connection on the bundle of coframes. A celebrated theorem due to Uhlenbeck [168] shows that a sequence of connections with uniformly bounded curvature in a suitable L^p norm is gauge-pre-compact. Theorem G essentially states that a similar fact holds while respecting the additional Darboux (or adaptedness) structure.

Several generalisations of Theorems E, F, and G can be obtained in the case where (M, g) is not necessarily a closed Riemannian manifold. For the sake of brevity these questions are not included in the thesis.

1.3 Structure of the thesis

The remainder of this thesis is organised in seven chapters.

Chapters 2 and 3 are prolegomenous in nature. Chapter 2 is a survey of some of the main results in the field of isometric immersions, starting with Nash's theorems, which may be considered "fundamental" theorems in the field. The remainder of the chapter surveys known results on several specific problems of isometric immersions. Chapter 3 then derives the basic equations which are studied throughout the remainder of the thesis. The presentation focuses on Cartan's moving frames method.

In the next two chapters, we focus our attention on two-dimensional manifolds. Chapter 4 investigates the validity of Cartan's equations for coframes of Sobolev regularity, and proves Theorem A. First, Section 4.1 treats the case where the coframe is assumed to be continuous. Section 4.2 then investigates the case where the continuity hypothesis is removed. Section 4.3 treats the higher-dimensional case when $p \geqslant 2$.

Then, Chapter 5 applies the results of Chapter 4 to isometric immersions of surfaces in $\mathbf{R}^3$, with the aim of proving Theorem B. We first obtain a version of the Gauß equation for surfaces in Section 5.2. In the case where the surface has positive curvature, we obtain a regularity result in Section 5.3, which is Theorem C.

Chapter 6 presents Theorem F. The key point of the argument is a div-curl estimate, which is obtained in a very general context in Section 6.1. The proof of Theorem F occupies Section 6.2, which is then applied to two problems: first, the Yang–Mills equations in Section 6.3, and second, immersions of Riemannian manifolds, in the next chapter.

Chapter 7 focuses on the convergence of isometric immersions of Riemannian manifolds in the weak topology of $\mathrm{W}^{2,p}$ spaces. Theorem E, on the weak continutiy of curvatures, is proved in Section 7.1, and Theorem D, on the weak* continuity of the isometry constraint, is proved in Section 7.2. We close this chapter by reviewing a parallel between isometric immersions and transonic flow first observed by Chen, Slemrod and Wang [35], which was also a motivation for the results of this chapter.

The final chapter, Chapter 8, focuses on obtaining a weak convergence theorem for the curvatures of isometric immersions, under the sole hypothesis of L^p control on the second fundamental form. The main result, Theorem G, is proved in Section 8.4.

The thesis includes two appendices for reference purposes. Appendix A collects some facts about Sobolev spaces and embeddings theorems on Riemannian manifolds, used throughout the thesis. Finally, Appendix B fixes basic notions and notations of the theory of principal bundles.

Chapters 4 and 6 are self-contained. Chapter 5 only uses a particular case of the main result of Chapter 4. Chapters 7 and 8 rely on the div-curl lemmas proved in Chapter 6.

Chapter 2

An Invitation to Isometric Immersions

> Die Beziehung zwischen “Idee” und “Wirklichkeit” is hier also die denkbar vollkommenste.[1]

Even though, seemingly, the first paper to explicitly consider local isometric immersions of a Riemannian manifold in Euclidean space is due to Schläfli [156], the ‘isometric embedding question’ certainly has its roots in B. Riemann’s work. In his 1854 habilitation thesis [150], he proposed the abstract notion of Riemannian manifold; a later work of his from 1861 (see [149]) studies precisely the question of when a Riemannian metric is locally Euclidean (see Section 2.5), though he stops short of asking whether Riemannian manifolds may be “Euclidean in higher dimensions”, that is, isometrically immersed in some larger Euclidean space $\mathbf{R}^N$.

This chapter surveys selected aspects of the question from the point of view of partial differential equations. We do not have the ambition to present an exhaustive account of the many developments of the last century and a half on the subject. The reader is referred to [53, 79, 80, 84, 146, 161] for detailed expositions of a number of topics in the theory of isometric embeddings. Instead, this chapter embraces subjectivity and follows particular threads that are motivations for the work of the subsequent chapters. Quoting Gromov’s words [79],

> The terrain of isometric embeddings and the fields surrounding this terrain are vast and craggy with valleys separated by ridges of unreachable mountains; people cultivating their personal gardens in these “valleys” [are] only vaguely aware of what happens away from their domains.

---

[1] Hermann Weyl, [176]

This chapter is organised as follows. In Section 2.1, we review the basic definitions as well as Nash's theorems, which are the main results on existence of isometric immersions and embeddings in the large. The next sections review long-studied problems pertaining to isometric embeddings: the local problem is reviewed in Section 2.2; some specific results about global isometric immersions and embeddings, chiefly of surfaces in $\mathbf{R}^3$ are surveyed in Section 2.3. The notion of uniqueness, or rigidity, for isometric embeddings is discussed in Section 2.4. The special case of equidimensional isometric immersions is reviewed in Section 2.5. As a conclusion, some further generalisations of the theorems of this chapter are mentioned in Section 2.6.

2.1 Nash's theorems

This section reviews Nash's isometric embeddings theorems, which address the existence problem in the large. We start with Whitney's results on immersions and embeddings, Theorem 2.1, addressing the topological obstructions to the existence of isometric immersions. A precise definition of the concept of isometric immersion and embedding is given in §2.1.2. Paragraphs §2.1.3, §2.1.4 and §2.1.5 present the main theorems.

2.1.1 Whitney's theorems

Let M be a smooth manifold. A map $u \in C^1(M, \mathbf{R}^N)$ is called an immersion if du is injective at every point of M; if moreover the map u itself is injective, then u is said to be an embedding.

Given a smooth manifold M, the set of immersions, respectively embeddings, of M into $\mathbf{R}^N$ is non-empty in general provided that N is large enough (clearly $N \geqslant \dim(M)$ is a necessary condition). This is the content of Whitney's theorems (see *e.g.* [161] and the references therein).

THEOREM 2.1 (Whitney)**.** *Let M be a smooth manifold, and let $n = \dim(M)$. Then there exists a smooth immersion of M into $\mathbf{R}^{2n-1}$, and a smooth embedding of M into $\mathbf{R}^{2n}$.*

In this level of generality, the result is sharp, as the example of the projective plane $\mathbf{PR}^n$ shows. Specific classes of submanifolds may admit immersions and embedding in lower-dimensional Euclidean spaces than the one prescribed by Whitney's theorem: *e.g.* the sphere $\mathbf{S}^n$ can be embedded in $\mathbf{R}^{n+1}$.

Morrey [123] later proved that if M is an analytic n-manifold, there exists an analytic embedding of M into $\mathbf{R}^{2n+1}$.

2.1.2 Definition of isometric immersions and embeddings

Let M be a smooth manifold. A Riemannian metric on M is a covariant, symmetric, positive definite 2-tensor, given in local coordinates by

$$g = g_{ij} dx^i \otimes dx^j.$$

The tensor g defines an inner product on the fibers of the tangent bundle TM, and, by composition with musical isomorphisms, it defines an inner product for all (p, q)-tensors on M.

A locally isometric immersion is an immersion $u \in C^1(M, \mathbf{R}^N)$ such that $u^*e = g$, where e is the Euclidean metric on $\mathbf{R}^N$. In local coordinates, this condition, referred to as the isometry condition, takes the form

$$\left\langle \frac{\partial u}{\partial x^i}, \frac{\partial u}{\partial x^j} \right\rangle = g_{ij}.$$

To be precise, let $p \in M$, and let $X, Y \in T_pM$ be two tangent vectors at the point p. The isometry condition is equivalent to

$$g_p(X, Y) = \langle u_*X, u_*Y \rangle = \langle du(X), du(Y) \rangle. \tag{2.1}$$

Alternatively, finding an isometric immersion $u \in C^1(M, \mathbf{R}^N)$ is equivalent to finding N scalar-valued functions u^i, $1 \leqslant i \leqslant N$, such that

$$g = (du^1)^2 + \cdots + (du^N)^2.$$

We recall that when given two metrics spaces (X, d) and (Y, d'), a continuous map $u : X \to Y$ is an isometry if $u^*d' = d$, in the sense that for any two points $x_1, x_2 \in X$, we have

$$d(x_1, x_2) = d'(u(x_1), u(x_2)).$$

In particular, u is Lipschitz. A Riemannian metric induces a metric structure on the manifold M. If u is a C^1 isometric immersion, then it is locally distance-preserving.

In the sequel we shall need to deal with maps u that are not C^1 in general. A map $u \in \mathrm{W}^{1,\infty}(M, \mathbf{R}^N)$ induces a bilinear form $u^*e \in \mathrm{L}^\infty(M, TM \otimes TM)$ defined as above by equation (2.1). Note that $u : (M, g) \to (M, u^*e)$ is not necessarily a distance-preserving isometry (see [77, 78]).

Remark 2.1. *A locally isometric immersion need not be a global isometry. For instance, a flat cylinder is locally isometric to the plane; however they are clearly not globally isometric, as the flat cylinder possesses closed geodesics, whereas the plane does not. In this thesis, we are exclusively concerned with locally isometric immersions.*

2.1.3 Nash's C^1 theorem

Parallel to Whitney's theorem, the isometric embedding problem asks whether every Riemannian manifold can be seen as a submanifold of some Euclidean space with the metric inherited from the Euclidean structure.

There are known obstructions to the construction of smooth isometric embeddings: we refer the reader to the survey presented in [161, V.13] or [84, Part III]. Nash's first theorem [134] states that if there exists an immersion in $\mathbf{R}^N$ for $N \in \mathbf{N}$, then there exists a C^1 isometric immersion in $\mathbf{R}^N$ too. From this point of view, there are no other obstruction for the existence of C^1 immersion than the topological one.

THEOREM 2.2. *Let (M, g) be a smooth Riemannian n-manifold. Then there exists an isometric immersion $u \in C^1(M, \mathbf{R}^{2n})$, and an isometric embedding in $C^1(M, \mathbf{R}^{2n+1})$.*

The isometric immersion of Theorem 2.2 is obtained as a perturbation of Whitney's immersion by adding "corrugations", that is, smooth oscillatory perturbations which eventually converge to a C^1 map.

Theorem 2.2 has been much generalised. Kuiper [112] lowered the minimal codimension required for Nash's original construction. Gromov [77] introduced the framework of convex integration as a more general approach for the corrugation construction mentioned above. Convex integration has many applications in partial differential equations and beyond (see [53, 77, 162] and the references therein).

A question laid out in [77, §2.4.12] is whether convex integration produces solutions of better regularity than C^1, for instance, additional Hölder continuity. This was partially addressed, first by Borisov [8, 9], Källen [104], and more recently by Conti, De Lellis and Székelyhidi [43]—for further results and improvements, see [49, 50]. The following theorem is a condensed statement of the main results of [43, 112, 134].

THEOREM 2.3 (Nash–Kuiper)**.** *Let (M, g) be a compact smooth Riemannian manifold,* $\dim(M) = n < N \in \mathbf{N}$*, and let $v \in C^1(M, \mathbf{R}^N)$ be an immersion such that*

$v^*e < g$ in the sense of bilinear forms. Then v can be uniformly (in C^0) approximated by $C^{1,\alpha}(M, \mathbf{R}^N)$ isometric immersions, for

$$\alpha < \frac{1}{1 + 2(n+1)J(n)} = \frac{1}{1 + n(n+1)^2}.$$

If v is an embedding, the previous statement also holds for $C^{1,\alpha}$ isometric embeddings approximating v.

Here $J(n)$ is the Janet dimension (see Section 2.2). Theorem 2.3 implies Theorem 2.2 by combining it with Whitney's theorem (Theorem 2.1). Determining the optimal α for which Theorem 2.3 holds is still an open problem. We note that for surfaces, one may take advantage of the existence of conformal patches to have $\alpha < 1/5$ (see [50]).

Codimension and regularity may be traded one for the other. Källen [104] showed that if one is willing to consider higher codimension, then better Hölder continuity may be expected. The gain in regularity in [43, 104] is obtained by introducing a smoothing operator in the convex integration process of Nash, Kuiper and Gromov.

Remark 2.2. *Smoothness of the metric tensor g is not required for any of the theorem of this section. Theorem 2.2 holds for C^0 metrics. If $g \in C^\beta$ for $0 < \beta < 2$, $\beta \notin \mathbf{N}$, then the map u obtained in Theorem 2.3 is at most of class $C^{1,\beta/2}$.*

2.1.4 Nash's C^∞ and C^ω theorems

The regularity of the immersion given by Theorem 2.3 cannot be improved to C^2 or above. Nash [135] proved another existence theorem for isometric immersions with smooth regularity.

THEOREM 2.4. *Let (M, g) be a smooth Riemannian manifold. Then there exists an $N \in \mathbf{N}$ and a smooth isometric embedding $u \in C^\infty(M, \mathbf{R}^N)$.*

Nash later gave a statement for the analytic category in [136] (see [80] for the non-compact case):

THEOREM 2.5. *Let (M, g) be an analytic Riemannian manifold. Then there exists an $N \in \mathbf{N}$ and an analytic isometric embedding $u \in C^\omega(M, \mathbf{R}^N)$.*

In Theorem 2.4, the embedding obtained is smooth, contrary to Theorem 2.3, but we may have to enlarge considerably the dimension of the target space. Sharper dimensional conditions than Nash's original calculations were subsequently obtained in [77, 80, 82, 102]. Up to further increasing the dimension N, one may even require that the group of isometries of M coincides with that of $\mathbf{R}^N$ [119] but this result does not provide lower bound on N in terms of n.

The minimal dimension for which it is always possible to embed smoothly and isometrically a (compact) Riemannian manifold is still an open question. The state of the art [82] for compact manifolds without boundary is

$$N_{\min} = \frac{n(n+1)}{2} + \max\{2n, n+5\}\,. \tag{2.2}$$

The reader is also referred to the exposition in [84, Chapter 1]. Naturally, there are classes of Riemannian manifolds for which this bound is not optimal, such as convex 2-spheres, which can always be smoothly isometrically embedded in $\mathbf{R}^3$ (see Theorem 2.9), whereas for $n = 2$, (2.2) gives $N_{\min} = 10$. One might hope to improve this dimension if some mild topological requirements are imposed on the manifold M. In [79], it is conjectured that $N = 1 + n(n+1)/2$ for parallelisable manifolds. Some known results in prescribed target spaces are reviewed in Section 2.3.

Nash's theorem 2.4 is also valid for Hölder continuous metrics $g \in C^\beta$ for $\beta > 2$ (see [102]): any C^β metric may be realised as a C^β isometric embedding in some Euclidean space. Thus C^2 appears as a threshold between the C^1 theorems of the previous sections and the "smooth" theorems of this section.

2.1.5 Homotopy and approximations

Theorems 2.2 and 2.4 may appear somewhat unrelated to each other. Gromov [77, 80] gave a very general framework which unifies both theorems into the h-principle—which for concision shall not be discussed in these pages. Gromov's reformulation of Nash's theorems may be summarised in the following theorem (from [80]). For simplicity, we restrict ourselves to the case of compact manifolds M.

For a Riemannian metric g, an immersion $u : M \to \mathbf{R}^N$ is said to be *short* if $u^*e < g$ in the sense of bilinear forms.

THEOREM 2.6. *Let (M, g) be a closed smooth Riemannian manifold, $n = \dim(M)$, and $u : M \to \mathbf{R}^N$ a strictly short immersion.*

1. *if $N \geqslant n + 1$, u may be uniformly approximated by C^1 isometric immersions. Furthermore any two isometric immersions that are homotopic through short immersions may be deformed into one another by C^1 isometric immersions.*

2. *if $N \geqslant J(n)+n+5$, u may be uniformly approximated by C^∞ isometric immersions. Furthermore any two isometric immersions that are homotopic through short immersions may be deformed into one another by C^∞ isometric immersions.*

3. *The same statements hold if « immersions » is replaced by « embedding ».*

The C^1 part of the theorem gives solutions to a severely overdetermined system; while the C^∞ is an underdetermined system, as there are $n(n-1)/2$ independent components of the metric tensor.

2.2 The local problem (Schläfli's problem)

The results of the previous section are global. A related question is the existence of local isometric embeddings. More precisely, given a point $p \in (M, g)$ in a Riemannian manifold, is there a neighborhood $E \subset M$ of p and an isometric embedding $u : E \to \mathbf{R}^N$?

Counting the parameters of the metric tensor g_{ij}, which is a symmetric two-tensor of dimension n, suggests that one should take $N = \binom{n+1}{2} = \frac{1}{2}n(n+1)$. This is called the Janet dimension $J(n)$, and it is an open problem (known as the Schläfli problem) whether every smooth Riemannian metric admits a smooth local isometric embedding into $\mathbf{R}^{J(n)}$.

In the analytic category, a celebrated theorem due to Elie Cartan [29] states that this is exactly possible.

THEOREM 2.7 (Cartan)**.** *Let (M, g) be an analytic Riemannian manifold. Then for all points $p \in M$, there exists a neighborhood $E \ni p$ and a local analytic isometric embedding $u \in C^\omega(E, \mathbf{R}^{J(n)})$.*

Remark 2.3. *The dimension $J(n)$ is optimal in general; see [80, §1.2.10]. We also note that $\mathbf{R}^{J(n)}$ may be replaced by any analytic Riemannian manifold of dimension $J(n)$ in the statement of Theorem 2.7.*

For a PDE-based proof using the Cauchy–Kowalevski theorem, the reader is referred to [80, 84]. Cartan's original proof [29] relied on exterior differential systems. A modern exposition of the proof is in [3] (see also [72]).

Cartan's method may be summarised as showing that the constraint equations of isometric embeddings (see Chapter 3) are involutive. In order to do so, the isometry constraint alone is not sufficient and one needs to consider a suitable prolongation of the system. From the point of view of exterior differential systems, the Gauß equation then appears naturally when trying to complete the isometry constraint into an involutive system. Although this work only deals with the C^ω regularity class, we regard this as strong motivation to investigate the analytical structure of the Gauß–Codazzi–Ricci equations.

2.2.1 The smooth case

Cartan's theorem is not known for the C^∞ category, even in the case $n = 2$. To the best of the author's knowledge, at this level of generality the optimal result is due to Gromov and Rohlin [80]:

THEOREM 2.8 (Gromov, Rohlin)**.** *Let (M, g) be an smooth Riemannian manifold. Then for all points $p \in M$, there exists a neighborhood $E \ni p$ and a local smooth isometric embedding $u \in C^\infty(E, \mathbf{R}^{J(n)+n})$.*

Schläfli's question has been studied for specific values of n. For $n = 2$, the Janet dimension is $J(2) = 3$. Schläfli's problem then reduces to the study of a single, mixed-type scalar equation of Monge–Ampère type

$$\det \nabla^2 u = K|g|(1 - |\nabla u|^2). \tag{2.3}$$

Equation (2.3) is known as the Darboux equation. When $K > 0$ or $K < 0$, the problem is elliptic, respectively hyperbolic, and the result can be obtained by implicit function theorem. The question is in general still open when K changes sign. Further results are known under specific hypotheses on K: see [84] for a detailed exposition of works due to Q. Han, J.-X. Hong and C. S. Lin and others, and [85, 105] for further recent work when the curvature is allowed to change sign. A common strategy of these results is to construct a local solution for equation (2.3) via a perturbation argument, using the Nash–Moser implicit function theorem.

When $n = 3$, the Janet dimension is $J(3) = 6$. Schläfli's question has been studied in [133] under a non-degeneracy hypothesis on the curvature. Related existence results are given in [21, 74], again using the Nash–Moser technique, and in [177]. Another approach is given in [30], where the isometry constraint is recast as a non-linear hyperbolic system, and then solved by means of the Nash–Moser theorem.

For higher dimensions, very little seems to be known besides the aforementioned result of Gromov and Rohlin. We note that the cases of $M^3 \hookrightarrow \mathbf{R}^6$ and $M^4 \hookrightarrow \mathbf{R}^{10}$ have been studied specifically in [21, 74]. An obstacle for treating higher-dimensional cases from the point of view of exterior differential systems is to understand the characteristic variety of the system (see [3, 21] for the relevant definitions). It was conjectured in [21] that the characteristic variety is not generically smooth if $n \geqslant 5$; this question has been investigated in [86].

2.3 Global problems

When looking at specific families of Riemannian manifolds, one may seek smooth isometric immersions or embeddings in “better” spaces (i.e. Euclidean spaces with fewer dimension) than those prescribed by Nash’s theorems (see Section 2.1). One possible way to attack this global problem in prescribed spaces is to construct isometric immersions by solving the compatibility equations for such specific families of Riemannian manifolds. Natural classes of metrics to look to in order to apply such a method are the classes where one may prove some uniqueness result. From this viewpoint, convex surfaces are a natural candidate, as the Gauß–Codazzi equations are elliptic when curvature is positive. We delay the discussion of uniqueness and rigidity theorems until the next section.

2.3.1 The Weyl problem

Let $\mathbf{S}^2$ be the smooth 2-sphere. We consider a smooth Riemannian metric g on $\mathbf{S}^2$, the Gauß curvature of which is strictly positive $K > 0$. If $(\mathbf{S}^2, g)$ is isometrically embedded in $\mathbf{R}^3$, a celebrated theorem of Hadamard shows that it is a convex smooth surface in $\mathbf{R}^3$, i.e. it bounds a convex body of $\mathbf{R}^3$. Weyl [176] asked the reciprocal question: when does there exist a convex isometric embedding $u : (\mathbf{S}^2, g) \to (\mathbf{R}^3, e)$?

A complete solution was given by Nirenberg [137], and independently by Pogorelov [143]. Subsequent generalisations have been concerned with the case where the curvature is allowed to vanish locally [81, 94, 97, 113]. [97, 132] also gave examples of metrics with non-negative curvature that admit no global smooth isometric embeddings.

Remark 2.4. *A necessary condition for the convexity of the target $u(\mathbf{S}^2) \subset \mathbf{R}^3$ is that the metric g be non-negatively curved. Without the assumption of non-negative curvature, [75] constructed a metric on $\mathbf{S}^2$ that has no smooth isometric embeddings in $\mathbf{R}^3$. In [80, Appendix 8], it is showed that any Riemannian manifold diffeomorphic to $\mathbf{S}^2$ is representable in $\mathbf{R}^7$.*

Nirenberg [137] used the continuity method to solve a non-linear elliptic PDE, equivalent to the Gauß equation: letting $\rho = \frac{1}{2}\langle u, u\rangle$, u is a convex isometric embedding of $(\mathbf{S}^2, g)$ in $\mathbf{R}^3$ (containing the origin of $\mathbf{R}^3$) if and only if ρ is a convex function solving

$$\det(\nabla^2\rho - g) = K|g|(2\rho - |\nabla\rho|^2). \tag{2.4}$$

This equation is related to (2.3). When u is convex and $K > 0$, equation (2.4) is a Monge–Ampère-type equation of elliptic type. The reader is referred to [84, Chapter 9] for a detailed study of this equation, and a proof of Theorem 2.9 below.

A metric g on $\mathbf{S}^2$ is said to be *realisable* if there exists an isometric embedding $u : \mathbf{S}^2 \to \mathbf{R}^3$. Nirenberg's theorem states:

THEOREM 2.9. *Let* $(\mathbf{S}^2, g)$ *be a smooth Riemannian manifold with* $K > 0$. *Then* g *is smoothly realisable in* $\mathbf{R}^3$, i.e. *there exists a smooth isometric embedding* u *of* $(\mathbf{S}^2, g)$ *in* $\mathbf{R}^3$.

The proof makes use of the conformal structure of surfaces, and the continuity method to tackle equation (2.4). Let σ be the standard round metric on $\mathbf{S}^2$ (inherited from $\mathbf{R}^3$ by the standard embedding of $\mathbf{S}^2$ into $\mathbf{R}^3$). By the uniformisation theorem there exists a smooth $\phi \in C^\infty(\mathbf{S}^2)$ such that $g = e^{2\phi}\sigma$. Let us define $g^\varepsilon = e^{2\varepsilon\phi}\sigma$ for $0 \leqslant \varepsilon \leqslant 1$. Observe that

$$K^\varepsilon = e^{-2\varepsilon\phi}(1 - \varepsilon \triangle_\sigma \phi) = \varepsilon e^{2(1-\varepsilon)\phi} K_g + (1 - \varepsilon)e^{-2\varepsilon\phi} > 0.$$

Here K^ε is the Gauß curvature of g^ε, K_g that of g, and $\triangle_\sigma$ is the Laplace–Beltrami operator with respect to the metric σ.

Clearly, when $\varepsilon = 0$, the metric $g^0 = \sigma$ is realisable in $\mathbf{R}^3$. After the work of Weyl [176], Nirenberg [137] gave a complete argument showing that the set of ε for which g^ε is realisable is both open and closed.

There has been related work on the question of isometrically embedding a “polar cap”, i.e. a disk with positive Gauß curvature and positive geodesic curvature on its boundary. The reader is referred to [84, Chapter 11].

Another class of surfaces for which one might expect to construct isometric immersions, in view of some uniqueness property, is the class of Alexandrov–Nirenberg surfaces (see §2.4.1)—for recent progress on this question, see [83].

2.3.2 Surfaces with negative curvature

As mentionned earlier, there are obstructions to the existence of a global isometric embedding of a surface in $\mathbf{R}^3$. There are non-existence results due to Hilbert [91], and subsequently generalised by Efimov [62, 118], showing that there are no smooth isometric embedding of complete surfaces with constant negative curvature in $\mathbf{R}^3$.

A natural question is whether a decay condition on the curvature is a sufficient condition to construct isometric immersions or embeddings of negatively curved complete surfaces. This question was raised by Yau [178], and positive results were subsequently obtained, assuming a decay rate on the curvature at infinity, in [92, 93] (see also [41]). An argument for optimal decay rate has been proposed in [26].

In [25, 27, 35], the constraint equations are treated as a system of conservation laws; using compensated compactness together with the method of invariant regions, the authors construct $C^{1,1}$ isometric immersions for certain classes of metrics with negative curvature.

2.4 Rigidity problems

Nash's isometric embedding theorems (see §2.1) may be read as non-uniqueness statements: any short immersion may be uniformly approximated by arbitrarily close isometric immersions, C^1 in codimension 1, and even smooth provided the codimension is high enough. A natural question is to study uniqueness in the class of isometric immersions in a given target space.

This leads naturally to considering the extrinsic geometry of the immersion in addition to the internal geometry, which is entirely determined by the metric tensor. Reconciling the extrinsic and intrinsic geometries of the manifold leads to the Gauß–Codazzi–Ricci equations (see Chapter 3).

Two isometric embeddings whose second fundamental form and normal connection agree are identical up to rigid motions. This fact is commonly referred to as rigidity. From this point of view, the Nash–Kuiper isometric immersions, for which one cannot define an extrinsic geometry (a second fundamental form), cannot be expected to satisfy any such rigidity property.

The Gauß equation expresses the Riemann curvature tensor as a quadratic form of the second fundamental form. In [3] it is proved that for a generic, analytic isometric embedding with low enough codimension, the second fundamental form is entirely determined by the Riemann curvature tensor and its derivatives.

2.4.1 Uniqueness and rigidity of convex surfaces

In some cases, the intrinsic geometry of (M, g) may constrain the extrinsic geometry in such a way that any two solutions of the compatibility equations have to agree, and therefore, the class of (smooth enough) isometric immersions of (M, g) into a prescribed target space is rigid. Two classes of Riemannian manifolds for which this happens are convex surfaces, and equi-dimensional isometric immersions (which are discussed in Section 2.5).

Recall that by virtue of Hadamard's theorem, if $u : (\mathbf{S}^2, g) \to (\mathbf{R}^3, e)$ is a smooth isometric embedding and $K > 0$, then $u(\mathbf{S}^2)$ is a convex surface, *i.e.* it is a the boundary of a convex subset of $\mathbf{R}^3$. The converse is also true: if $u : (\mathbf{S}^2, g) \to (\mathbf{R}^3, e)$ is a convex isometric embedding and $K > 0$, then $u(\mathbf{S}^2)$ is smooth.

The following statement is classical, and is due (in various forms) to Cohn-Vossen, Herglotz, Alexandrov, Pogorelov and Sabitov (see [84, 144] and the references therein).

THEOREM 2.10. *Let $(\mathbf{S}^2, g)$ be a smooth Riemannian metric on $\mathbf{S}^2$ with $K \geqslant 0$, and $u, v : \mathbf{S}^2 \to \mathbf{R}^3$ be convex C^1 isometric embeddings of $(\mathbf{S}^2, g)$ into $(\mathbf{R}^3, e)$. Then u and v are smooth and differ by a rigid motion of $\mathbf{R}^3$.*

A larger class of rigid surfaces is the class of Alexandrov–Nirenberg surfaces, which one may think of as "torus-like". A closed surface in $\mathbf{R}^3$ is said to be Alexandrov–Nirenberg if it satisfies

(A) letting K be its Gauß curvature, there holds

$$\int_{\{x \in M | K(x) > 0\}} K dV_g = 4\pi.$$

(B) whenever $K(x) = 0$, $dK(x) \neq 0$.

(C) each component of $K^{-1}(]-\infty, 0[)$ contains at most one closed asymptotic curve.

Note that condition (C) is extrinsic, whereas conditions (A) and (B) are intrinsic. An example of a Alexandrov–Nirenberg surface is the standard embedding of the (non-flat) torus $\mathbb{T}^2$ into $\mathbf{R}^3$. Note also that convex surfaces are Alexandrov–Nirenberg, as hypotheses (B) and (C) are empty, and (A) is simply the content of the Gauß–Bonnet theorem, and therefore always holds on surfaces with positive Gauß curvature.

Nirenberg [138] proved the following theorem.

THEOREM 2.11. *Alexandrov–Nirenberg surfaces are rigid, in the sense that any two such smooth isometric immersions differ by a rigid motion of $\mathbf{R}^3$.*

It is conjectured in [138] that conditions (B) and (C) are superfluous for this theorem to hold.

Note that C^2 isometric immersions of a domain with positive curvature are necessarily locally convex. By contrast, recall that by the Nash–Kuiper theorem, there exists many C^1 isometric immersions of such domains which are not convex. A natural question is whether there exists a threshold between the "rigid" isometric immersions of the sphere, which are convex, and the "flexible" ones, which cannot be.

2.4.2 A digression into fluid dynamics

A similar dichotomy occurs in the mathematics of fluid dynamics. A mathematical description of an ideal, inviscid, incompressible fluid is given by the Euler equations

$$\begin{aligned} \partial_t u + \operatorname{div}_x(u \otimes u) + \nabla p &= 0, \\ \operatorname{div} u &= 0. \end{aligned}$$

Here $u = u(t,x) \in \mathbf{R}^3$ is the velocity field of a fluid at time $t > 0$ and position $x \in E \subset \mathbf{R}^3$, and $p = p(t,x)$ is the pressure field. First written by Euler [63], their analysis is a subject of active and extensive research. In particular, Onsager [140] was studying hydrodynamics and turbulence, and conjectured a threshold in Hölder regularity between "conservative" solutions and "dissipative" solutions of the Euler equations. Thanks to [22, 42, 52, 99], the conjecture is now a theorem.

A vector field $u = u(t,x)$ is said to be a weak solutions of Euler's incompressible flow equations if for all solennoidal $\phi(t,x) \in C^\infty([0,T] \times E, \mathbf{R}^3)$, there holds

$$\int_0^T \int_E \left(\langle u, \partial_t \phi \rangle + u \otimes u : \nabla \phi \right) dx dt = \int_E \langle u(0,x), \phi(0,x) \rangle dx.$$

A solution u is said to be *conservative* if it conserves energy, that is,

$$\frac{d}{dt} \|u(t)\|_{\mathrm{L}^2(E)} = 0.$$

Note that a solution need not be classically differentiable in time, so that this statement is interpreted as

$$\int_E |u(t,x)|^2 dx = \int_E |u(0,x)|^2 dx \qquad \forall t > 0. \tag{2.5}$$

A solution which fails to satisfy (2.5) is termed *non-conservative.*

THEOREM 2.12 (Onsager's conjecture). 1. *If $u \in C_t C_x^\alpha$ is a weak solution for $\alpha > 1/3$, then u is conservative.*

2. *There exists non-conservative solutions* $u \in C_t C_x^\alpha$ *for all* $\alpha < 1/3$.

The first part of the theorem is due to [42] and may be interpreted as a rigidity statement. The first results towards the second part of the theorem are due to Scheffer [155] and Schnirelman [158, 159]. Following De Lellis and Székelyhidi's successful treatment of the Euler equations as a differential inclusion [52], the "flexible" part of Theorem 2.12 was proved in [22, 99] (the statements differ between these two works).

2.4.3 Dichotomy for isometric embeddings

Motivated in part by the (recent) successful resolution of the Onsager conjecture, much work has been dedicated to investigating the rigidity–flexibility dichotomy for isometric embeddings. For surfaces in $\mathbf{R}^3$, a loose "meta"-conjecture is that there is a critical threshold $\alpha_* = 1/2$ such that any isometric embedding $u \in C^{1,\alpha}(M^2, \mathbf{R}^3)$ with $\alpha > \alpha_*$ is "rigid"; whereas for each $\alpha < \alpha_*$ there exist "flexible" Nash–Kuiper-like isometric embeddings in $C^{1,\alpha}(M^2, \mathbf{R}^3)$.

A precise meaning may be given to this statement for some classes of surfaces. Let g be a smooth metric on the sphere $\mathbf{S}^2$ with $K_g > 0$. It is conjectured [49, 53] that any isometric embedding $u \in C^{1,\alpha}(\mathbf{S}^2, \mathbf{R}^3)$ with $\alpha > \alpha_*$ is convex (and hence satisfies the Cohn-Vossen rigidity—see §2.4.1); whereas for each $\alpha < \alpha_*$ there exists "flexible" Nash–Kuiper-like isometric embeddings in $C^{1,\alpha}(\mathbf{S}^2, \mathbf{R}^3)$ that are not convex. Borisov [7] proposed an argument for rigidity when $\alpha > 2/3$ (see also [144, 170]); [43] contains another argument for the same result. In a related work [95], the authors build on unpublished work of Šverák [173] to show that $\mathrm{W}^{2,2}$ isometric immersions of positively curved spheres are convex.

If $D \subset \mathbf{R}^2$ is a domain with a smooth metric g with vanishing curvature $K_g = 0$, one may similarly conjecture that for $\alpha > \alpha_*$, any isometric embedding $u \in C^{1,\alpha}(D, \mathbf{R}^3)$ is developable, whereas for each $\alpha < \alpha_*$ there exists "flexible" Nash–Kuiper-like isometric embeddings in $C^{1,\alpha}(\mathbf{S}^2, \mathbf{R}^3)$ that are not developable. For recent progress on this question, the reader is referred to [51] where the rigidity statement is studied for $\alpha > 2/3$, thus paralleling the aforementioned case of positively curved spheres. Other works [103, 126, 141] have investigated isometric immersions of $\mathrm{W}^{2,2}$ regularity with vanishing curvature. It is proved that such immersions are in fact $C^{1,1/2}$ and developable.

Remark 2.5. *The Hölder regularity of the methods put forth in [43] to establish flexibility or rigidity worsens with dimension: flexibility (Theorem 2.3) in known for*

$1/\alpha > 1 + 2(n+1)J(n)$, and the calculations used in [43] for the rigidity part yield heuristically $1/\alpha < 1 + 2J(n)$.

2.5 Equidimensional isometric immersions

Let (M, g) be a Riemannian manifold, and assume that on a neighborhood $V \subset M$ of a point $p \in M$, the Riemann curvature tensor vanishes. Let $n = \dim(M)$. When is V locally isometric to an open subset of $\mathbf{R}^n$?

This question was raised by Riemann in his habilitation thesis [150]. We view V as a subset of $\mathbf{R}^n$ equipped with the metric $g = g_{ij} dx^i \otimes dx^j$ in the coordinates of $\mathbf{R}^n$; the question then becomes whether there exists a map $u : V \to \mathbf{R}^n$ such that $\langle \partial_i u, \partial_j u \rangle = g_{ij}$. In the smooth category, Frobenius' theorem yields the well-known integrability condition $\mathbf{Rm}_g = 0$ for this system, where $\mathbf{Rm}_g$ is the Riemann tensor of g (see *e.g.* [161]). In other words, a metric with vanishing Riemann curvature tensor is locally Euclidean. A result of Mardare [117] allows to extend this result to the case where $g \in \mathrm{W}^{1,p}$ for $p > n$, and the equation $\mathbf{Rm}_g = 0$ holds in the sense of distributions.

As in Section 2.4, we may also ask about rigidity and flexibility. The basic rigidity result is originally due to Liouville. Now, many versions and generalisations of Liouville's theorem are known in a variety of contexts (see *e.g.* [101]). We state one relating specifically to the isometric immersion question.

THEOREM 2.13. *Let $u : (M, g) \to \mathbf{R}^n$ be a $\mathrm{W}^{1,n}(M, \mathbf{R}^n)$ map satisfying*

$$\begin{aligned} &\langle du, du \rangle = g, \\ &\det du > 0 \qquad a.e. \end{aligned}$$

Then u is smooth and unique up to rigid motions of $\mathbf{R}^n$.

The main part of the proof consists in showing that the map u is harmonic. Regularity theory them implies smoothness. This theorem is an aggregate of many results. For the convenience of the reader, a proof is provided.

Proof. The problem is local, so that we work without loss of generality in a neighborhood $E \subset M$ which is a coordinate patch. Thus we view $E \subset \mathbf{R}^n$ as equipped with the metric $g = g_{ij} dx^i \otimes dx^j$. By a well-known result of Vodopaynov and Goldstein [69, 169, 172], since $\det du > 0$, $u \in C(E, \mathbf{R}^n)$ and it satisfies the Lusin condition.

Writing $u = (u^1, \dots, u^n)$, we claim that u^k is weakly harmonic for all $1 \leqslant k \leqslant n$; then by regularity for harmonic maps, it follows that u is smooth. To show the claim, let us work in local coordinates on $E \subset M$, and let $\phi \in C_c^\infty(E)$ be an arbitrary compactly supported test function. The claim that u^k is harmonic is equivalent to

$$\int_E \langle du^k, d\phi \rangle dV_g = 0.$$

Here the scalar product $\langle \cdot, \cdot \rangle$ is the scalar product on T^*M induced by g. Since $u^*e = g$, we apply the change of variable formula (see [69]) to calculate

$$\begin{aligned}
\int_E \langle du^k, d\phi \rangle_g dV_g &= \int_E g^{ij}(x)\partial_i u^k(x)\partial_j \phi(x)\sqrt{|g|(x)}dx \\
&= \int_{u(E)} u^*\langle du^k, d\phi \rangle (u^* dV_g) \\
&= \int_{u(E)} e^{ij}\partial_i (u \circ u^{-1})^k \partial_j (\phi \circ u^{-1})(x)dx \\
&= \int_{u(E)} \partial_k (\phi \circ u^{-1})(x)dx \\
&= 0.
\end{aligned}$$

Thus the functions u^k are weakly harmonic. It is known that continuous harmonic maps are smooth (see *e.g.* [90]). Q.E.D.

The sense-preserving condition $\det du > 0$ plays for this particular problem the same role as the convexity condition for the Weyl problem: it implies a topological property (u being a homeomorphism) that makes the problem elliptic. However, "anomalous" isometric immersions may be constructed in $\mathrm{W}^{1,\infty}$: by a result of Gromov [77, 78], there exists equidimensional isometric immersions in $\mathrm{W}^{1,\infty}$ that are not homeomorphisms. For further results, the reader is referred to [107], which shows in particular that equidimensional isometric immersions are residual, and hence dense, among short immersions.

2.6 Generalisations

Let us conclude this survey chapter by remarking that many aspects of the discussion of this chapter may be generalised to immersions and embeddings into more general targets—instead of $\mathbf{R}^N$. Nash's theorems 2.2 and 2.4 may be generalised to immersions and embeddings into arbitrary manifolds, as does the local isometric embedding theorems of Section 2.2; the Weyl problem, discussed in Section 2.3, admits an extension to manifolds of constant sectional curvature.

The first theorem generalises Theorem 2.6; the reader is referred to [77, 80]. An immersion $u : (M, g) \to (\widetilde{M}, h)$ between smooth Riemannian manifolds is said to be *short* if $u^*h < g$ in the sense of bilinear forms.

THEOREM 2.14. *Let $(\widetilde{M}, h)$ be a smooth Riemannian manifold with dimension $N = \dim(\widetilde{M})$, and let (M, g) be a compact Riemannian manifold with dimension $n = \dim(M)$.*

1. *if $N \geqslant n + 1$, any short immersion may be uniformly approximated by C^1 isometric immersions.*
2. *if $N \geqslant J(n) + 3n + 5$, any short immersion may be uniformly approximated by C^∞ isometric immersions.*
3. *the previous two statements remain valid if "embedding" replaces "immersions".*

A generalisation of the Weyl problem to more general target manifolds was obtained by Pogorelov and Alexandrov [144].

THEOREM 2.15. *Let $(\widetilde{M}, h)$ be a 3-manifold of constant sectional curvature $\kappa \in \mathbf{R}$, and let $(\mathbf{S}^2, g)$ be a Riemannian manifold with Gauß curvature $K > \kappa$. Then there exists an isometric embedding $u : (\mathbf{S}^2, g) \to (\widetilde{M}, h)$.*

Further generalisations that are not considered here are the cases where $(\widetilde{M}, h)$ is no longer assumed to be a Riemannian manifold, but rather a semi-Riemannian manifold. In other words, h may be degenerate. Despite considerable interest and relevance of this case in the study of general relativity, many basic questions remain open in the semi-Riemannian case.

For further questions about isometric immersions or embeddings of Riemannian manifolds, or related objects, the reader is referred to *e.g.* [77, 79, 80, 84, 90, 178].

Chapter 3

The Basic Equations

> ...It is clear that [Cartan] had then all the essential ideas of the method of moving frames, one of his favorite subjects in later years, which has not been fully exploited even now.[1]

The main objective of this chapter is to write and justify the Gauß–Codazzi–Ricci equations (3.11). It also provides a concise introduction to Cartan's moving frames method and Darboux frames. Though the formalism of bundles was not available then, there is no doubt that the general equations written in §3.2.3 were already known of Elie Cartan [28]. A standard reference is [161]. Our presentation is taylored towards the isometric immersion problem and the later treatment of this thesis, but the Cartan moving frame method has been successfully used in numerous problems of geometry (*e.g.* [20, 38, 39, 76]) and in applications (see *e.g.* non-linear elasticity [179]).

The Gauß–Codazzi–Ricci system constitutes the main link between the primary intrinsic invariant of the manifold, the Riemann curvature, and the extrinsic invariant, the second fundamental form.

The general notions of frame bundle, connection form and curvature form in the context of Riemannian manifolds are introduced in Section 3.1. Next, we aim to write and prove the Gauß–Codazzi–Ricci equations: in §3.2.1, the basic definition of normal bundle are recalled, and the notion of adapted (or Darboux) coframe is introduced in §3.2.2. Finally, the Gauß–Codazzi–Ricci equations in this framework are stated in §3.2.3. Alternate but equivalent formulations to the Gauß–Codazzi–Ricci could be considered (see *e.g.* [33, 161, 165]). For reference purposes, Section

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[1] Shiing-Shen Chern, Claude Chevalley, *Elie Cartan and his mathematical work*, Bull. Am. Soc. 1952

3.3 contains these equations as well as equivalent statements for the specific case of immersions $M^2 \hookrightarrow \mathbf{R}^3$, which is the focus of Chapters 4 and 5. Finally, in Section 3.4, the compatibility equations are stated when the ambient space $\mathbf{R}^N$ is replaced by an arbitrary Riemannian manifold $(\widetilde{M}, h)$.

3.1 The coframe bundle $\mathscr{F}(T^*M)$

Let M be a smooth manifold, and $V \to M$ a vector bundle of rank $N \in \mathbf{N}$. The frame bundle $\mathscr{F}V$ is the bundle defined as

$$\mathscr{F}V := \{(x, \theta^1, \dots, \theta^N) \,\big|\, x \in M, \theta^i \in (V)_x, \operatorname{span}(\theta^1, \dots, \theta^N) = (V)_x\}. \tag{3.1}$$

Here, $(V)_x$ is the fiber at point x, it is a vector space of dimension N, and definition (3.1) states that the list $\theta^1, \dots, \theta^N$ forms a basis of $(V)_x$ at x. This is a principal $\mathbf{GL}(N)$-bundle, where N is the rank of the bundle V. In a local trivialisation of V, $\mathbf{GL}(N)$ acts by multiplication: there is an isomorphism

$$\begin{aligned} \mathbf{GL}(N) &\to (\mathscr{F}V)_x \\ A = (A_{ij}) &\mapsto (x, A_{ij}\theta^i). \end{aligned}$$

Remark 3.1. *Throughout this chapter (and this thesis), the strong Einstein convention is in use: a pair of identical lower and upper indices is summed over. When need be, we shall explicitly write the interval over which indices range.*

In the sequel, the reference to the base point x shall usually be implicit. In index-free notation, an element $A \in \mathbf{GL}(N)$ acts over a frame θ by composition $A\theta$.

The coframe bundle of a manifold is the frame bundle of its co-tangent bundle $\mathscr{F}(T^*M)$. From definition (3.1), a family of 1-forms $\theta^1, \dots, \theta^n \in \Omega^1(E)$ is a coframe on E if the list $\theta^1, \dots, \theta^n$ is a basis of T^*E at every point of $E \subset M$. Note that a globally defined coframe need not exist in general, unless the underlying manifold is parallelisable.

Remark 3.2. *The procedure described above constructs a principal* **GL***-bundle from a vector bundle. In the language of categories, we may regard $\mathscr{F}$ as a functor from vector bundles to principal bundles.*

Recall that a connection in a principal G-bundle may be represented locally by a $\mathfrak{g}$-valued one-form, where $\mathfrak{g}$ is the Lie algebra of the Lie group G (see Appendix B). Thereby, geometric structures on coframes translate into algebraic properties of the connection form. Riemannian metrics are the main geometric structure under consideration, and we investigate metric torsion-free connections.

When M is a Riemannian manifold, the fibers of the tangent bundle TM are equipped with a scalar product, denoted by g or $\langle \cdot, \cdot, \rangle_g$, which we may extend to tensors of arbitrary order, it being understood that it is composed with the required musical isomorphisms when necessary. The coframe $\theta^1, \dots, \theta^n$ is an orthonormal coframe if it satisfies

$$\langle \theta^i, \theta^j \rangle_g = \delta_{ij}, \qquad 1 \leqslant i, j \leqslant n = \dim M. \tag{3.2}$$

In other words, for any pair of tangent vectors $X, Y \in T_x M$,

$$g_x(X, Y) = \sum_{i=1}^{n} \theta^i_x(X) \theta^i_x(Y).$$

In coordinate-free notation, we shall write $g = \operatorname{Tr} \theta \otimes \theta$.

Given an orthonormal coframe, there exists a unique skew-symmetric matrix-valued 1-form $\omega = \omega^i_j \in \Omega^1(E)$, $1 \leqslant i, j \leqslant n$ such that

$$d\theta^i = \omega^i_j \wedge \theta^j, \qquad \omega^i_j = -\omega^j_i. \tag{3.3}$$

This form is the connection form. Its curvature form is the unique 2-form $\Omega^i_j \in \Omega^2(E)$, $1 \leqslant i, j \leqslant n$ such that

$$\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j, \qquad \Omega^i_j = -\Omega^j_i. \tag{3.4}$$

Equations (3.3) and (3.4) are the first and second structure equations. The curvature form satisfies

$$\Omega^i_j = \frac{1}{2} R^i_{jkm} \theta^k \wedge \theta^m, \tag{3.5}$$

where R^i_{jkm} is the Riemann curvature tensor. Similarly, the connection form may be related to the Christoffel symbols by

$$\omega^i_j = \Gamma^i_{jk} \theta^k.$$

Remark 3.3. *In this thesis, we always work with oriented Riemannian manifolds, and unless stated otherwise, the term 'frame bundle' shall always refer to the orthonormal frame bundle, whose structure group is (a subgroup of)* $\mathbf{SO}(N)$.

In index-free notations, we let $\mathscr{F}(T^*M) \to M$ be the coframe bundle of M. This is the $\mathbf{SO}(n)$-principal bundle given by the set of (local) orthonormal coframes on M. Let $E \subset M$ be a parallelisable domain; a coframe θ is a section of $\mathscr{F}(T^*M)$.

The coframe θ defines a connection form ω as a section of the adjoint bundle $\mathbf{Ad}(\mathscr{F}T^*M) \otimes T^*M$. In a local trivialisation, this bundle is $\mathfrak{so}(n) \otimes T^*M$ and ω is an $\mathfrak{so}(n)$-valued one-form, i.e. ${}^t\omega = -\omega$. Similarly, ω defines a curvature form Ω as a section of $\mathbf{Ad}(\mathscr{F}T^*M) \otimes T^*M \wedge T^*M$, which, in a local trivialisation, is a $\mathfrak{so}(n)$-valued two-form.

Finally, recall that the Riemann curvature tensor R^i_{jkm} may be viewed as an endomorphism on 2-forms, which we denote by $\mathbf{Rm}$. We may restate equations (3.2), (3.3), (3.4) and (3.5) in the following compact form:

$$\begin{cases} \operatorname{Tr} \theta \otimes \theta = g, \\ d\theta = \omega \wedge \theta, \\ \Omega = d\omega + \omega \wedge \omega = \mathbf{Rm}(\theta \wedge \theta), \end{cases} \tag{3.6}$$

henceforth referred to as the Cartan system. In local coordinates, we have $\mathbf{Rm}(\theta \wedge \theta) = \frac{1}{2} R^i_{jkm} \theta^k \wedge \theta^m$. Note that the anti-symmetry relations of equations (3.3) and (3.4) translate into the properties of the Lie algebra $\mathfrak{so}(n)$ (which we recall is the set of anti-symmetric matrices).

In a local trivialisation over $E \subset M$ of $\mathscr{F}(T^*M)$, we may view θ as a $\mathbf{SO}(n)$-valued one form, i.e. $\theta \in \Gamma(E, \mathbf{SO}(n) \otimes T^*M)$, whence $\omega \in \Gamma(E, \mathfrak{so}(n) \otimes T^*M)$ and $\Omega \in \Gamma(E, \mathfrak{so}(n) \otimes \wedge^2 T^*M)$. In the rest of the document, we consider the index-free and component notations as equivalent and use either description whenever most convenient.

3.2 Geometry of immersed manifolds

Our goal is now to write the Gauß–Codazzi–Ricci equations for an immersed Riemannian manifold in $\mathbf{R}^N$.

From now on and for the rest of this chapter we let M be an abstract closed Riemannian manifold, and we assume that there exists a C^1 map $u : M \to \mathbf{R}^N$ which is an immersion. We denote by $u_* : TM \to T\mathbf{R}^N|_{u(M)}$ the tangent map of u, and $u^* : T^*\mathbf{R}^N|_{u(M)} \to T^*M$ the cotangent map.

Remark 3.4. *In this whole document, we write M exclusively for the abstract manifold, and $u(M) \subset \mathbf{R}^N$ for the submanifold of $\mathbf{R}^N$ given by a choice of immersion of M in $\mathbf{R}^N$, it being understood that there may be several immersions which send M to the same submanifold of $\mathbf{R}^N$. Therefore the notation $T\mathbf{R}^N|_{u(M)}$ refers to the restriction of the bundle $T\mathbf{R}^N$ over $u(M) \subset \mathbf{R}^N$.*

3.2.1 The normal bundle NM

Recall that u being an immersion implies that it is a local topological embedding. In particular, for a small enough neighborhood of any given point, u_* is bijective onto its image in $T\mathbf{R}^N$. This image is denoted by $(T\mathbf{R}^N|_{u(M)})^\top$ and we have an isomorphism of bundles $TM \simeq (T\mathbf{R}^N|_{u(M)})^\top$. The orthogonal complement of $(T\mathbf{R}^N|_{u(M)})^\top$ in $T\mathbf{R}^N|_{u(M)}$ is denoted by $(T\mathbf{R}^N|_{u(M)})^\perp$.

We define the normal bundle NM to be the quotient bundle

$$NM = T\mathbf{R}^N|_{u(M)}/TM. \tag{3.7}$$

Let $\pi : T\mathbf{R}^N \to NM$ be the quotient map. As before, we have $NM \simeq (T\mathbf{R}^N)^\perp$. There is a distinguished identification given by the scalar product induced on M by $\mathbf{R}^N$.

Remark 3.5. *The construction of the normal bundle presented here applies to general* C^1 *immersions between two differentiable manifolds* M *and* $\widetilde{M}$. *Note that the normal bundle is well-defined even without a notion of scalar product of either the base or the target manifold (whereas* $(T\widetilde{M}|_{u(M)})^\perp$ *is not).*

The co-normal bundle is defined as $N^*M = T^*\mathbf{R}^N|_{u(M)}/T^*M$. Finally, we recall that a Riemannian metric g on M induces a distinguished isomorphism between TM and T^*M given by $X \mapsto g(X, \cdot)$ for any $X \in TM$.

In the sequel we shall be concerned with differential forms in $\mathbf{R}^N$ that are (co-)tangent, respectively (co-)normal, to $u(M)$. Let $x \in M$; the one-form $\theta \in (T^*\mathbf{R}^N)_{u(x)}$ is normal if for any $X \in (TM)_x$, there holds $\theta(u_*X) = 0$, or equivalently $u^*\theta = 0$, which we shall write $\theta|_{TM} = 0$ by abuse of notation when it holds for all x.

A similar characterisation is possible for tangent forms. Let $\nu : NM \to T\mathbf{R}^N$ be the map given by the identification of the quotient (3.7) with $(T\mathbf{R}^N|_{u(M)})^\perp$. A one-form $\theta \in T\mathbf{R}^N|_{u(M)}$ is tangent to M if $\nu^*\theta = 0$, which we shall refer to as $\theta|_{NM} = 0$.

3.2.2 Adapted coframes and the Darboux bundle $\mathscr{D}^*(u(M))$

A coframe $\theta^1, \ldots, \theta^N \in T^*\mathbf{R}^N$ is adapted to an immersion $u(M^n) \subset \mathbf{R}^N$ if it is orthonormal in $\mathbf{R}^N$ and moreover $\theta^1, \ldots, \theta^n$ are (co-)tangent to $u(M)$. This implies that $\theta^{n+1}, \ldots, \theta^N$ are (co-)normal. Adapted coframes being orthonormal coframes on

$\mathbf{R}^N$, which is flat, we find that $\theta = (\theta^1, \ldots, \theta^N) \in \Gamma(M, \mathscr{F}(T^*\mathbf{R}^N))$ must satisfy

$$\begin{aligned}
\mathrm{Tr}(\theta \otimes \theta) &= e, \\
u^*\theta^k &= 0, \qquad n+1 \leqslant k \leqslant N, \\
\mathrm{Tr}(u^*\theta \otimes u^*\theta) &= g, && \text{(3.8a)} \\
d\theta &= \omega \wedge \theta, \\
\Omega &= d\omega + \omega \wedge \omega = 0. && \text{(3.8b)}
\end{aligned}$$

We recall that the notation $\mathrm{Tr}(\theta \otimes \theta) = e$ expresses the fact that the coframe is orthonormal in $\mathbf{R}^N$, i.e. whenever $X, Y \in T_{u(x)}M$, we have

$$e(X,Y) = \sum_{i=1}^{N} \theta^i_{u(x)}(X)\theta^i_{u(x)}(Y).$$

Letting $Z, W \in T_xM$ be tangent vectors to M, equation (3.8a) states that $u^*\theta$ is orthonormal on M, i.e.

$$g_x(Z,W) = \sum_{i=1}^{N}(u^*\theta^i)_x(Z)(u^*\theta^i)_x(W) = \sum_{i=1}^{n} \theta^i_{u(x)}(u_*Z)\theta^i_{u(x)}(u_*W).$$

Adapted coframes are also referred to as Darboux coframes (see *e.g.* [3, 76]).

Working in a local trivialisation of the frame bundle, we may write

$$\theta = (\theta^1, \ldots, \theta^n; \theta^{n+1}, \ldots, \theta^N) = (\theta^\top; \theta^\perp) \in \mathbf{SO}(n) \oplus \mathbf{SO}(N-n).$$

The condition of being adapted (or Darboux) is equivalent to

$$\theta^\top|_{NM} = 0, \qquad \theta^\perp|_{TM} = 0.$$

The Darboux bundle $\mathscr{D}^*(u(M))$ is the bundle of coframes on $\mathbf{R}^N$ adapted to a particular immersion $u : M \to \mathbf{R}^N$. Similarly one may define the bundle of Darboux frames $\mathscr{D}(u(M)) \subset \mathscr{F}(T\mathbf{R}^N)$.

From the bundle-theoretic point of view, $u : M \to \mathbf{R}^N$ induces a tangent map $u_* : TM \to T\mathbf{R}^N$, and a frame map $\mathscr{F}u_*$ which we shall refer to—in a slight abuse of notation due to the functorial nature of the construction—simply as $u_* : \mathscr{F}(TM) \to \mathscr{F}(T\mathbf{R}^N)$. The restriction operator "$|_{TM}$" considered above is the associated bundle map

$$\mathscr{F}(T^*\mathbf{R}^N) \xrightarrow{u^*} \mathscr{F}(T^*M),$$

If $\theta \in \mathscr{F}(T^*\mathbf{R}^N)$ is adapted, $\tilde{\theta} := u^*\theta$ is an orthonormal coframe on M. In coordinates, for $1 \leqslant i \leqslant n$, $\tilde{\theta}^i := u^*\theta^i$ is an orthonormal coframe on M, and for $n+1 \leqslant k \leqslant N$, we have $u^*\theta^k = 0$.

Let $\tilde{\omega}$ be the connection form associated to $\tilde{\theta}$; by uniqueness, $\tilde{\omega} = u^* \left(\omega^\top\right)$, where in a local trivialisation we write

$$\omega = \begin{pmatrix} \omega^\top & {}^t\omega^{\mathbf{II}} \\ -\omega^{\mathbf{II}} & \omega^\perp \end{pmatrix} \in \mathfrak{so}(N), \tag{3.9}$$

so that

$$\begin{aligned} d\theta^\top &= \omega^\top \wedge \theta^\top + {}^t\omega^{\mathbf{II}} \wedge \theta^\perp \\ d\theta^\perp &= -\omega^{\mathbf{II}} \wedge \theta^\top + \omega^\perp \wedge \theta^\perp. \end{aligned}$$

3.2.3 The Gauß–Codazzi–Ricci equations

From the adaptedness condition $\theta^\perp|_{TM} = 0$, we derive $d\theta^\perp|_{TM} = 0 = -\omega^{\mathbf{II}} \wedge \theta^\top|_{TM}$. Cartan's lemma implies that there exists a symmetric two-tensor on M, which we denote $\mathbf{II} \in \mathbf{Hom}(TM \times TM, NM) \simeq T^*M \otimes T^*M \otimes NM$ and call the second fundamental form, such that

$$\omega^{\mathbf{II}} = \mathbf{II} \cdot \theta^\top.$$

In index notations, $(\omega^{\mathbf{II}})^i_j = \mathbf{II}^i_{jk}\theta^k = \mathbf{II}^i_{kj}\theta^k$, where $1 \leqslant j, k \leqslant n$ but $n+1 \leqslant i \leqslant N$

Writing out equation (3.8b) using the notations (3.9), we find the system of equations given by

$$\begin{cases} d\omega^\top + \omega^\top \wedge \omega^\top - {}^t\omega^{\mathbf{II}} \wedge \omega^{\mathbf{II}} = 0, \\ d\omega^{\mathbf{II}} + \omega^{\mathbf{II}} \wedge \omega^\top + \omega^\perp \wedge \omega^{\mathbf{II}} = 0, \\ d\omega^\perp + \omega^\perp \wedge \omega^\perp - \omega^{\mathbf{II}} \wedge {}^t\omega^{\mathbf{II}} = 0, \end{cases}$$

Finally, recalling the adaptedness condition, we find that

$$\begin{aligned} \Omega^\top :&= d\omega^\top + \omega^\top \wedge \omega^\top \\ &= \tau^*\Omega \\ &= \tau^*\mathbf{Rm}(u^*\theta \wedge u^*\theta), \end{aligned}$$

where $\tau^* : T^*M \to T^*\mathbf{R}^N$ is the pullback of the restriction map $\tau : u(M) \subset \mathbf{R}^N \to M$. This is the Gauß equation, which relates the "intrinsic" curvature given by the Riemann curvature tensor, and the "extrinsic" curvature given by the second fundamental form.

We re-write the whole system of equations for reference. Let $u : M \to \mathbf{R}^N$ be an immersion, and $\theta = (\theta^\top, \theta^\perp) \in \mathscr{D}^*(u(M))$. Spelling out these conditions in equations, we have

$$\begin{cases} \mathrm{Tr}(\theta \otimes \theta) = e, \\ \mathrm{Tr}(u^*\theta \otimes u^*\theta) = g, \\ u^*\theta^\perp = 0, \\ \tilde{\theta} := u^*\theta^\top \in \mathscr{F}(T^*M), \\ d\theta = \omega \wedge \theta. \end{cases} \tag{3.10}$$

The first four equations are the adaptedness conditions. The final equation is simply the first structure equation for the coframe θ, and defines the connection form ω. These conditions imply that the following equations (the Gauß–Codazzi–Ricci equations) are satisfied (recall the convention (3.9)):

$$\begin{cases} {}^t\omega^{\mathbf{II}} \wedge \omega^{\mathbf{II}} = \tau^* \,\mathbf{Rm}, \\ d\omega^{\mathbf{II}} + \omega^{\mathbf{II}} \wedge \omega^\top + \omega^\perp \wedge \omega^{\mathbf{II}} = 0, \\ d\omega^\perp + \omega^\perp \wedge \omega^\perp = \omega^{\mathbf{II}} \wedge {}^t\omega^{\mathbf{II}}, \end{cases} \tag{3.11}$$

The first equation, which is also the main constraint of the whole theory, is the Gauß equation. The second equation is the Codazzi equation, and the final one is the Ricci equation, which in spirit is similar to the Gauß equation in that it relates the "normal curvature" $d\omega^\perp + \omega^\perp \wedge \omega^\perp$ to the second fundamental form.

Summarising, we have shown that any Darboux coframe (i.e. verifying (3.10)) satisfies the Gauß–Codazzi–Ricci equations (3.11). The converse is also true by virtue of the *realisation theorem*: if $(\omega^\top, \omega^{\mathbf{II}}, \omega^\perp)$ satisfy the system (3.11), then there exists an immersion realising M as a submanifold of $\mathbf{R}^N$. This theorem is known as the fundamental theorem of surface theory for the case $M^2 \hookrightarrow \mathbf{R}^3$. For a proof in the general case, see *e.g.* [161, 165]. This fact shall be discussed in further detail in Chapter 8.

3.3 Surfaces in $\mathbf{R}^3$

For reference, we spell out explicitely the Gauß–Codazzi–Ricci equations in the special case of embedded surfaces in three-dimensional Euclidean space $M^2 \hookrightarrow \mathbf{R}^3$. In this case, the normal bundle is a line bundle, and if the manifold is oriented, it is a trivial line bundle. Thus there is no normal connection, and we view the second fundamental form as a scalar-valued symmetric 2-tensor $\mathbf{II} = (\mathbf{II}_{ij})$, $i, j = 1, 2$.

Let $u : M^2 \to \mathbf{R}^3$ be an isometric immersion, and on a parallelisable subset of M, let θ^1, θ^2 be a tangent orthonormal coframe on M. When $n = 2$, Cartan's structure equations reduce to

$$
\begin{aligned}
d\theta^1 &= \omega^1_2 \wedge \theta^2, \\
d\theta^2 &= \omega^2_1 \wedge \theta^1, \\
d\omega^1_2 &= K\theta^1 \wedge \theta^2,
\end{aligned}
$$

where K is the Gauß curvature. Note that the terms that are quadratic in the connection form cancel in the two-dimensional case. Finally, let θ^3 be the covector dual to (a choice of) Gauß map $\nu : M \to \mathbf{R}^3$. It is clear that $u^*\theta^3 = 0$, so that $\theta^1, \theta^2, \theta^3$ is an adapted coframe. We let as usual $d\theta^i = \omega^i_j \wedge \theta^j$ for $1 \leqslant i, j \leqslant 3$, with the anti-symmetry condition $\omega^i_j = -\omega^j_i$.

The Ricci equation is trivial; the Gauß–Codazzi system (3.11) now comprises a single Gauß equation (3.12) and two Codazzi equations, taking the form

$$
\begin{aligned}
\omega^3_1 \wedge \omega^3_2 &= K\theta^1 \wedge \theta^2, \qquad (3.12) \\
d\omega^3_1 &= \omega^2_1 \wedge \omega^3_2, \\
d\omega^3_2 &= \omega^1_2 \wedge \omega^3_1.
\end{aligned}
$$

The second fundamental form may be recovered from the connection form ω and the coframe θ as

$$\mathbf{II} = \omega^3_1 \otimes \theta^1 + \omega^3_2 \otimes \theta^2.$$

We recall that this system is equivalent to the following equations

$$
\begin{aligned}
&\det \mathbf{II} = K \det g, \\
&\nabla_i \mathbf{II}_{jk} = \nabla_k \mathbf{II}_{ij}, \qquad 1 \leqslant i, j, k \leqslant 2.
\end{aligned}
$$

where $\mathbf{II}$ is the second fundamental form and ∇ is the Levi–Civita connection.

3.4 The equations for an arbitrary ambient Riemannian manifold

The method of moving frames developed in this Chapter may be easily adapted to the case where the target manifold $\mathbf{R}^N$ is replaced by more general manifolds (see also the conclusion of Chapter 2). Such manifolds may have non-vanishing curvature, which translates into additional terms in the second structure equation. The Gauß–Codazzi–Ricci equations thus have an extra term which is simply the curvature form of the ambient manifold.

Let us consider the case of a Riemannian manifold $(M,g) \hookrightarrow (\widetilde{M},h)$ isometrically immersed in another Riemannian manifold $(\widetilde{M},h)$. As before, we denote by $\mathbf{Rm}$ the Riemann curvature tensor of g, and $\widetilde{\mathbf{Rm}}$ that of h. Similarly, the curvature form of an orthonormal coframe on (M,g) will be denoted by Ω, and that of an orthonormal coframe on $(\widetilde{M},h)$ by $\widetilde{\Omega}$.

An orthonormal coframe θ on $(\widetilde{M},h)$ is said to be adapted if $u^*\theta$ is an orthonormal coframe on (M,g). Writing out the curvature equations for such coframes, we find that by definition $\widetilde{\Omega} = d\omega + \omega \wedge \omega = \widetilde{\mathbf{Rm}}(\theta \wedge \theta)$ for the connection form on $\widetilde{M}$, i.e. such that $d\theta = \omega \wedge \theta$. Writing out as before the curvature equation on M for the induced frame $u^*\theta$ on (M,g), we find the analogues of system (3.11) for a general curved manifold:

$$\begin{cases} \tau^*\,\mathbf{Rm}(u^*\theta \wedge u^*\theta) = {}^t\omega^{\mathbf{II}} \wedge \omega^{\mathbf{II}} + \widetilde{\Omega}|_{TM\times TM}, \\ d\omega^{\mathbf{II}} + \omega^{\mathbf{II}} \wedge \omega^{\top} + \omega^{\perp} \wedge \omega^{\mathbf{II}} = \widetilde{\Omega}|_{TM\times NM}, \\ d\omega^{\perp} + \omega^{\perp} \wedge \omega^{\perp} = \omega^{\mathbf{II}} \wedge {}^t\omega^{\mathbf{II}} + \widetilde{\Omega}|_{NM\times NM}, \end{cases} \tag{3.13}$$

For clarity, let us write equations (3.13) with indices. Let $\theta^1, \dots, \theta^N$ be an orthonormal coframe on $\widetilde{M}$ such that $\theta^1, \dots, \theta^n$ is an orthonormal coframe tangent to M. Equations (3.13) then reads

$$\begin{aligned} &\Omega^i_j = \omega^i_k \wedge \omega^k_j + \widetilde{\Omega}^i_j, \\ &d\omega^k_i + \omega^k_j \wedge \omega^j_i + \omega^k_j \wedge \omega^j_i = \widetilde{\Omega}^k_i, \\ &d\omega^k_m + \omega^k_l \wedge \omega^l_m = \omega^k_i \wedge \omega^i_m + \widetilde{\Omega}^k_m. \end{aligned}$$

We recall that the strong Einstein convention is used, so that a pair of identical lower and upper indices is summed. Indices have the following ranges: $1 \leqslant i,j \leqslant n$ for the "tangent indices", and $n+1 \leqslant k,m,l \leqslant N$ for the "normal indices".

Chapter 4

Cartan's Equations in Low Sobolev Regularity

We investigate the validity of Cartan's equations for coframes in Sobolev spaces $\mathrm{W}^{1,p}$.

First, we consider the case of surfaces. Let (M,g) be a smooth Riemannian surface (i.e. $\dim(M)=2$) and $E\subset M$ an orientable domain. We denote by K the Gauß curvature of g. Recall that if θ^1,θ^2 is a smooth orthonormal coframe on a domain E, there exists a unique connection form $\omega\in\Omega^1(E)$ such that

$$\begin{cases} d\theta^1=\omega\wedge\theta^2,\\ d\theta^2=-\omega\wedge\theta^1,\\ d\omega=K\theta^1\wedge\theta^2. \end{cases} \tag{4.1}$$

System (4.1) is nothing more than the Cartan equations (3.6) for two-dimensional coframes (see Section 3.3). Recall that in this case, the group $\mathbf{SO}(2)$ is abelian, so that the non-linear terms in the curvature form (3.4) vanish. Moreover, in dimension 2, the Riemann curvature tensor has only one non-trivial component, which is given by the (scalar-valued) Gauß curvature K.

When θ^1,θ^2 are $\mathrm{W}^{1,p}$ coframes with $p\geqslant 1$, there exists an L^p connection form ω satisfying $d\theta^1=\omega\wedge\theta^2$ and $d\theta^2=-\omega\wedge\theta^1$. We say that the second Cartan equation holds in the sense of distributions whenever

$$\int_E \omega\wedge d\phi=\int_E \phi K\theta^1\wedge\theta^2 \qquad \forall\phi\in C_c^\infty(E). \tag{4.2}$$

We note that (4.2) makes sense for connections forms $\omega\in\mathrm{L}^p$ for all $p\geqslant 1$. The aim of this chapter is to prove (4.2) for coframes $\theta^1,\theta^2\in\mathrm{W}^{1,p}\cap\, C(E,T^*E)$ with $p\geqslant 1$, and to consider the validity of (4.2) for less regular coframes.

Specifically, we are interested in describing defects in the Cartan equation. Considering as before θ^1, θ^2 an orthonormal coframe, and ω its connection form, we define the defect distribution $\mathfrak{s}$ of the coframe as

$$\langle\!\langle\, \mathfrak{s}, \phi \,\rangle\!\rangle = \int_E \omega \wedge d\phi - \int_E \phi K \theta^1 \wedge \theta^2, \qquad \forall \phi \in C_c^\infty(E), \tag{4.3}$$

for E a bounded regular domain. It is clear that $\mathfrak{s} \in (\mathrm{Lip}_0(\bar{E}))^*$, where $\mathrm{Lip}_0(E)$ be the closure of $C_c^\infty(E)$ in $\mathrm{Lip}(E)$. When $\mathfrak{s} = 0$, the Cartan equation holds in the sense of distributions. When θ^1, θ^2 is a continuous Sobolev coframe, we prove that $\mathfrak{s} = 0$; on the other hand, equation (4.2) may fail to hold if the continuity assumption is dropped.

We shall show that $\mathfrak{s}$ belongs to a specific class of distributions, the importance of which has been recognised in other contexts, such as the study of harmonic maps, Ginzburg–Landau equations, liftings of Sobolev mappings: see [11, 13, 15, 17, 19, 98, 145, 160].

We equip $\bar{E}$ with the pseudo-distance $\rho(x, y) = \min(d(x, y), d(x, \partial E) + d(y, \partial E))$ where $d(\cdot, \cdot)$ is the Riemannian distance between two points of $\bar{E}$, and ∂E is the boundary of E. Let us introduce the set of distributions

$$\begin{aligned} \mathscr{X}(\bar{E}) := \Big\{ \zeta \in (\mathrm{Lip}_0(E))^* \;\Big|\; & \\ & \exists (P_j)_{j \in J}, (N_j)_{j \in J} \in \bar{E}, \sum_{j \in J} \rho(P_j, N_j) < \infty, \zeta = \sum_{j \in J} \delta_{P_j} - \delta_{N_j} \Big\}, \end{aligned} \tag{4.4}$$

where the index set J is countable. A distribution $\zeta \in \mathscr{X}$ need not be a measure; rather, it is in the dual of Lipschitz functions on E, $\zeta \in (\mathrm{Lip}_0(E))^*$. In fact, a distribution of the form (4.4) is a measure if and only if it has finitely many summands that are Dirac deltas (see [145, 160]). Let us also note that even if ζ may be written as a finite sum of Dirac deltas, the representation of ζ by a sequence of points (P_j, N_j) need not be unique (*cf.* [11, 98]).

For Sobolev coframes, our main result may be summarised by saying that Cartan's second structure equation holds up to an element of $2\pi\mathscr{X}$.

THEOREM 4.1. *Let (M, g) be a Riemann surface, $E \subset M$ a bounded regular domain, K its Gauß curvature, and let $p \geqslant 1$.*

(a) Let $\theta^1, \theta^2 \in \mathrm{W}^{1,p}(E, T^*E)$ be an orthonormal coframe. If the coframe is continuous or $p \geqslant 2$, then $\mathfrak{s} = 0$ (the Cartan equation (4.2) holds without defect).

(b) For $\theta^1, \theta^2 \in \mathrm{W}^{1,p}(E, T^*E)$ an orthonormal coframe, the defect distribution $\mathfrak{s}$ satisfies $(2\pi)^{-1}\mathfrak{s} \in \mathscr{X}(\bar{E})$. Moreover, we may choose points $P_j, N_j \in \bar{E}$ such

that for any $\varepsilon > 0$,

$$\left| \|(2\pi)^{-1}\mathfrak{s}\|_{(\mathrm{Lip}_0(E))^*} - \sum_{j\in J} \rho(P_j, N_j) \right| < \varepsilon,$$

$$\mathfrak{s} = 2\pi \sum_{j\in J} \delta_{P_j} - \delta_{N_j}.$$

Part (b) parallels similar results for the Jacobian determinant, using the theory of minimal connections. The reader is referred see in particular to [11, 15, 17, 19]. In fact, when E is a Euclidean domain, viewing θ^1 as an $\mathbf{R}^2$-valued mapping, we find that $\mathfrak{s} = \mathrm{Det}\,\nabla\theta^1$.

This chapter is organised as follows. The first assertion of Theorem 4.1 about continuous coframes is established in Section 4.1. The argument relies on the existence of certain liftings of $\mathbf{S}^1$-valued maps. The proof of part (b) of Theorem 4.1 is the object of Section 4.2. Finally, in Section 4.3 we examine the case of higher-dimensional $\mathrm{W}^{1,p}$ coframes with $p \geqslant 2$. Proposition 4.17 shows that Cartan's equations hold in the sense of distributions. As a corollary, the compatibility equations hold for a Darboux coframe adapted to an isometric $\mathrm{W}^{2,2}$ immersion.

4.1 Continuous coframes

We first consider the case where the coframe is continuous, and in this case, we establish part (a) of Theorem 4.1. The case where $p = 2$ is dealt with in Lemma 4.3 and 4.4 in Section 4.2.

Lemma 4.2. *Let* (M, g) *be a smooth Riemannian surface,* $E \subset M$ *a domain,* K *the Gauß curvature of* g*. For every orthonormal coframe* $\theta^1, \theta^2 \in \mathrm{W}^{1,p}(E, T^*E)$ *for* $p \geqslant 1$*, there exists a unique connection* 1*-form* $\omega \in \mathrm{L}^p(E, T^*E)$ *such that*

$$\begin{cases} d\theta^1 = \omega \wedge \theta^2, \\ d\theta^2 = -\omega \wedge \theta^1. \end{cases} \tag{4.5}$$

If in addition $\theta^1, \theta^2 \in C(E, T^*E)$*, we have*

$$\int_E \omega \wedge d\phi = \int_E \phi K \theta^1 \wedge \theta^2 \qquad \textit{for all } \phi \in C_c^\infty(E). \tag{4.6}$$

Proof. **Step 1:** orthogonality of θ^1 and θ^2 implies uniqueness of ω: if ω and ω' both satisfy equations (4.5) then $(\omega - \omega') \wedge \theta^1 = 0 = (\omega' - \omega) \wedge \theta^2$, and so $\omega = \omega'$ almost everywhere.

Step 2: we construct the form ω for a coframe $\theta^1, \theta^2 \in \mathrm{W}^{1,p}(E, T^*E)$.

It suffices to construct the form ω on a parallelisable subset of E. Indeed, let E' and E'' be two parallelisable subsets of E such that $E' \cap E'' \neq \varnothing$, and assume that we have constructed ω' and ω'' solving (4.5) on E', respectively E''. Then by uniqueness (step 1) $\omega'|_{E' \cap E''} = \omega''|_{E' \cap E''}$, so that ω' and ω'' extend to a well-defined 1-form ω on $E' \cup E''$.

Equation (4.5) is a linear system; in local coordinates, (4.5) is equivalent to the 1-form $\omega = \omega_1 dx^1 + \omega_2 dx^2$ satisfying

$$\begin{cases} \omega_1 = V^{-1}(\theta^1_1(\theta^1_1)_y + \theta^2_1(\theta^2_1)_y - \theta^1_1(\theta^1_2)_x - \theta^2_1(\theta^2_2)_x), \\ \omega_2 = V^{-1}(-\theta^1_2(\theta^1_2)_x - \theta^2_2(\theta^2_2)_x + \theta^1_2(\theta^1_1)_y + \theta^2_2(\theta^2_1)_y), \end{cases}$$

where $\theta^i = \theta^i_1 dx + \theta^i_2 dy$ for $i = 1, 2$, and $V = \det g = \theta^1_1\theta^2_2 - \theta^1_2\theta^2_1 > 0$. Thus we readily check that $\omega \in \mathrm{L}^p(E, T^*E)$.

Step 3: let us now assume additionally that the coframe is continuous. By orthogonality and continuity, $\theta^2 = *\theta^1$ or $\theta^2 = -*\theta^1$ on all of E (this can also be shown using the Sobolev regularity hypothesis, see Lemma 4.3 below). Without loss of generality, we consider the first case. Let us also assume E to be a parallelisable domain. Let $\eta^1, \eta^2 = *\eta^1 \in \Omega^1(E)$ be a smooth orthonormal coframe on E. We prove that there is $\alpha \in \mathrm{W}^{1,p}(E)$ such that $\omega = \tilde{\omega} + d\alpha$, where $\tilde{\omega}$ is the connection form of η^1, η^2.

Let us now use the notation $\theta^k_j = \langle \theta^k, \eta^j \rangle_g$. We can view $\theta^k \simeq (\theta^k_1, \theta^k_2)$ as an $\mathbf{R}^2$-valued mapping, and since $|\theta|_g = 1$ and η^j is orthonormal, θ^k is an $\mathbf{S}^1$-valued mapping. Continuity of θ^k implies the existence of a continuous lifting $\alpha^k \in C(E)$ such that $\theta^k = \exp(\mathfrak{i}\alpha^k)$, where we identify $\mathbf{R}^2 \simeq \mathbf{C}$, and use complex notation.

We fix any subset $E' \subset E$ such that $\theta^k(E')$ excludes at least one point of $\mathbf{S}^1$. Choosing an appropriate branch of the logarithm, there holds $\alpha^k = -\mathfrak{i}\log\theta^k$. By the chain rule, the following holds in the sense of distributions:

$$\nabla\alpha^k = -\overline{\theta^k}\nabla\theta^k,$$

so that $-\overline{\theta^k}\nabla\theta^k$ is the weak derivative of α^k on E'. Now covering E with two such sets E' and choosing an adapted partition of unity $\sum_j \chi_j = 1$, we have, for an arbitrary $\phi \in C^\infty_c(E)$,

$$\int_E \alpha^k \nabla\phi = \sum_j \int_E \alpha^k \nabla(\phi\chi_j) = \sum_j \int_E \phi\chi_j \overline{\theta^k}\nabla\theta^k = \int_E \phi\overline{\theta^k}\nabla\theta^k.$$

Hence $-\overline{\theta^k}\nabla\theta^k$ is the weak derivative of α^k on E. Noting $|\nabla\theta^k| = |\nabla\alpha^k|$, we have $\|\nabla\alpha^k\|_{\mathrm{L}^p(E)} = \|\nabla\theta^k\|_{\mathrm{L}^p(E)}$.

Consider $\alpha := \alpha^1$ to be the lifting associated with θ^1 as above. We have

$$\begin{aligned}\theta^1 &= \theta^k_j \eta^j = \cos\alpha\eta^1 + \sin\alpha\eta^2,\\ \theta^2 &= *\theta^1 = -\sin\alpha\eta^1 + \cos\alpha\eta^2,\end{aligned}$$

whence we calculate

$$\begin{aligned}d\theta^1 &= -\sin\alpha d\alpha\wedge\eta^1 + \cos\alpha\widetilde{\omega}\wedge\eta^2 + \cos\alpha d\alpha\wedge\eta^2 - \sin\alpha\widetilde{\omega}\wedge\eta^1\\ &= \widetilde{\omega}\wedge\theta^2 + d\alpha\wedge\theta^2.\end{aligned}$$

Similarly,

$$d\theta^2 = -\widetilde{\omega}\wedge\theta^1 - d\alpha\wedge\theta^2.$$

Thus it is readily seen that the form

$$\widetilde{\omega} + d\alpha \in \mathrm{L}^p(E, T^*E)$$

satisfy equations (4.5). By Step 1, $\omega = \tilde{\omega} + d\alpha$.

Step 4: we now obtain equation (4.6).

We start by reducing to the case where E is parallelisable. Indeed, for a fixed $\phi \in C^\infty_c(E)$, $\operatorname{supp}\phi$ is compact, and can be covered by finitely many parallelisable subsets $E = \cup_{j=1}^n E_j$. Assuming that equation (4.5) holds on each of the E_j for compactly supported $\phi_j \in C^\infty_c(E_j)$, we consider a partition of unity $1 = \sum_{j=1}^n \chi_j$ subordinate to E_j and we have,

$$\begin{aligned}\int_E K\theta^1\wedge\theta^2\phi &= \sum_{j=1}^n \int_{E_j} \phi\chi_j K\theta^1\wedge\theta^2\\ &= \sum_{j=1}^n \int_{E_j} \omega\wedge d(\phi\chi_j) = \int_E \omega\wedge d\phi.\end{aligned}$$

So we assume henceforth that E is parallelisable.

Let ϕ be a test function. Then one has, by Step 3,

$$\begin{aligned}\int_E \omega\wedge d\phi &= \int_E d\alpha\wedge d\phi + \int_E \widetilde{\omega}\wedge d\phi\\ &= \int_E d(\alpha d\phi - \phi\widetilde{\omega}) + \int_E \phi d\widetilde{\omega}.\end{aligned}$$

The first integral vanishes by Stokes' theorem, which holds as $\alpha d\phi - \phi\widetilde{\omega} \in \mathrm{W}^{1,p}_0(E, T^*E)$. By the second Cartan structure equation, $d\widetilde{\omega} = *K$ and so $\int_E \phi d\widetilde{\omega} = \int_E *K\phi = \int_E K\theta^1\wedge\theta^2\phi$. This proves equation (4.6). Q.E.D.

4.2 Cartan's equations for Sobolev coframes

We now consider arbitrary coframes in $\mathrm{W}^{1,p}$. Such coframes need not admit $\mathrm{W}^{1,p}$ liftings in $\mathbf{R}$, as the proof of Lemma 4.2; however, liftings in larger functional spaces can be constructed.

In the proof of Lemma 4.2, there are two uses for the continuity of θ^1, θ^2, first to obtain a lifting of the coframe from $\mathbf{SO}(2)$ into $\mathbf{R}$, and second to determine the orientation of the coframe. The $\mathrm{W}^{1,p}$ hypothesis is sufficient to obtain a consistent choice of orientation.

Lemma 4.3. *Let θ^1, θ^2 be an orthonormal $\mathrm{W}^{1,p}(E, T^*E)$ coframe for $p \geqslant 1$. Then $*(\theta^1 \wedge \theta^2)$ is a constant almost everywhere, and so either $\theta^1 = *\theta^2$ or $\theta^2 = *\theta^1$.*

Proof. $\theta^1 \wedge \theta^2 \in \mathrm{W}^{1,p}(E, T^*E \wedge T^*E)$. Moreover, as the coframe is orthonormal, we have $|\theta^1 \wedge \theta^2| = 1$, so that $*(\theta^1 \wedge \theta^2)$ is integer-valued. By *e.g.* [10, Appendix B] and [16], a Sobolev $\mathbf{Z}$-valued function is constant, and so $*\theta^1 \wedge \theta^2$ is constant which defines an orientation on E. Q.E.D.

From the proof of Lemma 4.2 and Lemma 4.3, the following holds at once:

Lemma 4.4. *Let $E \subset (M, g)$ be parallelisable, K as before, and let $\theta^1, \theta^2 = *\theta^1$ be a $\mathrm{W}^{1,p}$ orthonormal coframe, and $p \geqslant 1$. Finally, let $\eta^1, \eta^2 = *\eta^1$ be a smooth orthonormal coframe on E. If there exists a map $\alpha \in \mathrm{W}^{1,p}(E)$ such that*

$$\theta^1 = \cos\alpha\eta^1 + \sin\alpha\eta^2,$$

then the Cartan equation holds in the sense of distributions, i.e.

$$\int_E \omega \wedge d\phi = \int_E \phi K \theta^1 \wedge \theta^2 \qquad \text{for all } \phi \in C_c^\infty(E).$$

In particular, if $p \geqslant 2$, the Cartan equation holds for all orthonormal coframes.

The lifting α depends on the choice of trivialisation η^1, η^2. Note that if $1 \leqslant p < 2$, for a given $\mathrm{W}^{1,p}$ coframe (θ^1, θ^2), there need not exist in general such a map α. This problem has been extensively studied in the literature.

Proof. Let ω be the one-form $\omega \in \mathrm{L}^p(E, T^*E)$ satisfying the first structure equations (4.5). If $p > 2$, the coframe is actually continuous, and the result is just given by Lemma 4.2. If $p = 2$, it is known (by an unpublished result of R. Schoen and K. Uhlenbeck, see [6]) that smooth $\mathbf{S}^1$-valued maps are dense in $\mathrm{W}^{1,2}(E, \mathbf{S}^1)$, and so by [55] there exists $\mathrm{W}^{1,2}(E, \mathbf{R})$ liftings of the maps in $\mathrm{W}^{1,2}(E, \mathbf{S}^1)$. Q.E.D.

Let us now give a first argument to explain how defects arise in the Cartan equations. Recall from the proof of Lemma 4.2 that it suffices to construct the form ω on a parallelisable subset of E. Let η^1, η^2 be a smooth orthonormal coframe on F, and let $\tilde{\omega}$ be its connection form. Thus $\theta|_E$ may be written as a linear combination of η^1 and η^2: we write

$$\theta^i = \theta^i_j \eta^j,$$

and we note that the matrix formed by the coefficients of θ in the frame η^1, η^2 is orthonormal, and its coefficients are $\mathrm{W}^{1,p} \cap \mathrm{L}^\infty$ functions. More generally, two coframes θ and η differ from one another by an element $s \in \mathbf{SO}(2)$, $\theta = s\eta$, which is in the class $\mathrm{W}^{1,p} \cap \mathrm{L}^\infty$ as above. Differentiating $\theta = s\eta$, we find

$$\begin{aligned} d\theta &= ds \wedge \eta + s d\eta \\ &= ds \wedge s^{-1}\theta + s\tilde{\omega} \wedge \eta \\ &= sds \wedge \theta + s^{-1} s\tilde{\omega} \wedge \theta. \end{aligned}$$

and so the connection form is given by $\omega = sds + \tilde{\omega} \in \mathrm{L}^p(E, \mathfrak{so}(2) \otimes T^*E)$.

We now turn our attention to the second Cartan equation. The main issue is that the form sds need not be closed if $s \in \mathrm{W}^{1,p}$ for $p < 2$. Formally, if s is smooth, we have

$$\begin{aligned} d\omega &= d(sds) + d\tilde{\omega} \\ &= ds \wedge ds + K\eta \wedge \eta \\ &= ds \wedge ds + K\theta \wedge \theta. \end{aligned}$$

If $ds \wedge ds = 0$ in the sense of distributions, then the Cartan equation is satisfied without defect, i.e. $\mathfrak{s} = 0$. This holds in particular when $s \in \mathrm{W}^{1,2}$, as Lemma 4.4 shows; however, when $p < 2$ this term is not integrable and a more involved argument is required. Note that $T = d(sds) \in \mathrm{W}^{-1,p}$ is well-defined as a distribution. When θ is continuous, Lemma 4.2 proves that $sds = d\alpha$ is an exact form.

The rest of the section is organised as follows. We first give a very general statement in Lemma 4.5, showing that there exists a measure-valued covector $\mathfrak{m}$ such that

$$\langle\!\langle \mathfrak{s}, \phi \rangle\!\rangle = \int_E d\phi \wedge \mathfrak{m} \qquad \phi \in C^\infty_c(E).$$

The rest of Section 4.2 is dedicated to strenghtening that result by studying the defect term introduced in Lemma 4.5. §4.2.2 shows that the defect $\mathfrak{s}$ does belong to the class $\mathscr{X}(\bar{E})$, defined in (4.4). This is the content of Lemma 4.7, which establishes the first part of assertion (b) in Theorem 4.1. Then, §4.2.3 and §4.2.4 give a proof of the final claim of part (b) of Theorem 4.1 in Lemma 4.15. This last claim relies on the theory of minimal connections, which is presented in detail in §4.2.3.

4.2.1 A first expression for the defect

Lemma 4.5. *Let (M, g) and K as above, and let $E \subset M$ be a parallelisable domain. Consider an orthonormal coframe $\theta^1, \theta^2 \in \mathrm{W}^{1,p}(E, T^*E)$ with $\omega \in \mathrm{L}^p(E, T^*E)$ the connection form constructed in Lemma 4.2. Then there exists a measure-valued covector $\mathfrak{m} \in \mathcal{M}(E, T^*E)$, singular with respect to the volume measure, such that*

$$\int_E \omega \wedge d\phi = \int_E \phi K \theta^1 \wedge \theta^2 + \int_E d\phi \wedge \mathfrak{m} \qquad \forall \phi \in C_c^\infty(E). \tag{4.7}$$

Moreover the support of $\mathfrak{m}$ is $\mathcal{H}^1$-rectifiable.

The meaning of the last term of the RHS of (4.7) is the following. $\mathfrak{m}$ being a measure-valued covector means that in local coordinates x^1, x^2 on E, we can write $\mathfrak{m} = \mu_i dx^i$, where μ_i is a measure on E. Writing $d\phi = \partial_j \phi dx^j$, we have by convention

$$d\phi \wedge \mathfrak{m} = \partial_j \phi \mu_{j+1} dx^j \wedge dx^{j+1},$$

where $j \in \mathbf{Z}/2\mathbf{Z}$. This expression is a measure-valued two-form, which integrates in the following way:

$$\int_E d\phi \wedge \mathfrak{m} = \sum_{j=1,2} \int_E \partial_j \phi d\mu_{j+1}.$$

As a corollary of Lemma 4.5, the Cartan equation is satisfied outside of a null-set.

Before coming to the proof we recall basic notions about measures and functions of bounded variation.

$\mathcal{M}(E)$ is the space of real-valued measures on E; $\mathcal{M}(E, T^*E)$ is the space of measure-valued covectors. Namely, let $x^1, \ldots, x^n$ be local coordinates on E, we have $\mu \in \mathcal{M}(E, T^*E)$ if and only if

$$\mu = \mu_i dx^i, \qquad \mu_i \in \mathcal{M}(E).$$

The set $\mathcal{M}(E, T^*E)$ is independent of the choice of coordinates. For a smooth covector field $\phi = \phi_i dx^i \in \Gamma(E, T^*E)$, we define

$$\int_E \phi \wedge \mu = \int_E \phi^i d\mu_i,$$

and this expression is again coordinate invariant. Note carefully our notation convention: when we write $d\mu$, we mean integration against the measure μ (not any exterior derivative).

$\mathbf{BV}(E)$ is the space of functions $f \in \mathrm{L}^1(E)$ such that

$$\sup \left\{ \int_E f \operatorname{div} \phi dV_g \,\middle|\, \phi \in \Gamma_c(E, TE), |\phi|_g \leqslant 1 \right\} < \infty.$$

Note also that the definition of $\mathbf{BV}(E)$ is independent of the choice of metric on M. Standard arguments (see *e.g.* [66]) imply that if $f \in \mathbf{BV}(E)$, then there exists a $Df = \mu \in \mathcal{M}(E, T^*E)$ such that

$$\int_E f \operatorname{div} \phi dV_g = - \int_E \phi \wedge \mu.$$

Recall the decomposition

$$\begin{aligned} Df &= [Df]^{\mathrm{ac}} + [Df]^{\mathrm{sing}} \\ &= [Df]^{\mathrm{ac}} + D^c f + D^j \alpha \\ &= \bar{D} f + (f_+ - f_-)\nu_J^\flat \mathcal{H}^{n-1}|_J, \end{aligned}$$

where $[Df]^{\mathrm{ac}}$ is the absolutely continuous part of the Radon measure Df with respect to the volume form dV_g; $D^c f$ is the Cantor part; $D^j f$ is the jump part, with $[Df]^{\mathrm{sing}} = D^c f + D^j f$; J is the jump set, ν its measure-theoretical normal; $\mathcal{H}^{n-1}$ is the Hausdorff measure; $\bar{D} f := [Df] + D^c f$ is the diffuse part of Df; $f_\pm$ is the left (resp. right) approximate limit on either side of J (with respect to ν). The statement is well-known in the case of Euclidean space (see [1, Section 3.9]). The same proof translates to Riemannian manifolds.

Proof. Since E is parallelisable, let $\eta^1, \eta^2 = *\eta^1$ be a local smooth orthonormal coframe and $\tilde{\omega}$ its connection form. Without loss of generality, by Lemma 4.3 we can assume $\theta^2 = *\theta^1$. Viewing θ^1 and θ^2 as $\mathbf{S}^1$-valued mappings as in the proof of Lemma 4.2, there exists $\alpha \in \mathbf{BV}(E)$ such that

$$\begin{aligned} \theta^1 &= \cos \alpha \eta^1 + \sin \alpha \eta^2, \\ \theta^2 &= - \sin \alpha \eta^1 + \cos \alpha \eta^2. \end{aligned}$$

This follows from the main result of [46, Theorem 1].

We claim that $\omega = \tilde{\omega} + [d\alpha]^{\mathrm{ac}}$. Here the notation $[d\alpha]^{\mathrm{ac}}$ refers to the 1-form

$$[d\alpha]^{\mathrm{ac}} = [\partial_1 \alpha]^{\mathrm{ac}} dx^1 + [\partial_2 \alpha]^{\mathrm{ac}} dx^2.$$

Assuming this claim for now, we have

$$\begin{aligned}\int_E \omega \wedge d\phi &= \int_E (\tilde{\omega} + [d\alpha]^{\mathrm{ac}}) \wedge d\phi \\ &= \langle\!\langle\, \tilde{\omega} + d\alpha - \mathfrak{m}, (*d\phi)^{\sharp} \,\rangle\!\rangle \\ &= \int_E *K\phi - \int_E \alpha \operatorname{div} \nabla^{\perp}\phi + \int_E d\phi \wedge \mathfrak{m} \\ &= \int_E *K\phi + \int_E d\phi \wedge \mathfrak{m},\end{aligned}$$

where we have defined $\mathfrak{m} = d\alpha - [d\alpha]^{\mathrm{ac}}$.

It remains to prove the claim that $\omega = \tilde{\omega} + [d\alpha]^{\mathrm{ac}}$.

Let $\bar{\alpha}(x)$ be the approximate limit of α at every point not in the jump set J_α (*cf.* [66, Section 5.9] or [1, Section 3.6]). By the chain rule in **BV** (see [1, Theorem 3.96]), for $f \in C^1(E)$, we have

$$D(f \circ \alpha) = f'(\bar{\alpha})\bar{D}\alpha + (f(\alpha_+) - f(\alpha_-))\nu \mathcal{H}^1|_{J_\alpha}.$$

We apply this formula to $f = \cos$ and $f = \sin$ to calculate the derivatives $d\theta^1, d\theta^2$. This yields

$$\begin{aligned}D\cos\alpha &= -\sin\bar{\alpha}\bar{D}\alpha + (\cos\alpha_+ - \cos\alpha_-)\mathcal{H}^1|_{J_\alpha}\nu, \\ D\sin\alpha &= \cos\bar{\alpha}\bar{D}\alpha + (\sin\alpha_+ - \sin\alpha_-)\mathcal{H}^1|_{J_\alpha}\nu.\end{aligned}$$

$D\cos\alpha \in \mathrm{L}^p(E)$ since $\theta \in \mathrm{W}^{1,p}(E, T^*E)$, and similarly for $D\sin\alpha$. Therefore $[D\cos\alpha]^{\mathrm{sing}} = 0 = [D\sin\alpha]^{\mathrm{sing}}$ and thus

$$\begin{aligned}D\cos\alpha &= -\sin\bar{\alpha}[D\alpha]^{\mathrm{ac}}, \\ D\sin\alpha &= \cos\bar{\alpha}[D\alpha]^{\mathrm{ac}};\end{aligned}$$

Moreover the values of α on each side of J_α differ by multiples of 2π:

$$\alpha_+ = \alpha_- + 2k\pi, \quad k \in \mathbf{Z}. \tag{4.8}$$

Since $\theta^1 = \cos\alpha\eta^1 + \sin\alpha\eta^2$ and $\theta^2 = -\sin\alpha\eta^1 + \cos\alpha\eta^2$, we have

$$\begin{aligned}d\theta^1 &= (\tilde{\omega} + d^{\mathrm{ac}}\alpha) \wedge \theta^2, \\ d\theta^2 &= (-\tilde{\omega} - d^{\mathrm{ac}}\alpha) \wedge \theta^1,\end{aligned}$$

whence the connection 1-form ω satisfying equation (4.5) is

$$\omega = \tilde{\omega} + [D\alpha]^{\mathrm{ac}}.$$

The measure $\mathfrak{m}$ only contains the jump part of $D\alpha$ (see [46, Remark 4]), so that the support of $\mathfrak{m}$ is $\mathcal{H}^1$-rectifiable. Q.E.D.

Remark 4.1. *Lemma 4.5 does not define a unique measure* $\mathfrak{m}$ *satisfying the Cartan equation* (4.7)*. For instance, if* $\mathfrak{m}$ *is such a measure and* Γ *is a closed curve in* E *such that* $\Gamma \cap \operatorname{supp} \mathfrak{m} = \varnothing$*, then the measure*

$$\mathfrak{m}' := \mathfrak{m} + \mathcal{H}^1|_\Gamma$$

also satisfies equation (4.7)*.*

To the reader familiar with geometric measure theory, $\mathfrak{s}$ *defines a* 0*-current, arising as the boundary of the* 1*-current* $\mathfrak{m}$*. In general, the failure of a map from an* n*-dimensional manifold to* $\mathbf{S}^1$ *to admit a lifting in* $\mathrm{W}^{1,p}$ *can be represented by a* $(n-2)$*-current (see e.g. [13]).* $\mathfrak{m}$ *is unique up to boundary-less currents* $\mathfrak{t}$ *(i.e. such that* $\partial\mathfrak{t} = \varnothing$*).*

As a motivating example, we consider the case where $\mathfrak{m}$ is supported on a finite number of smooth curves. Recall that J_α is the jump set associated to the **BV** lifting α. Assume that J_α is a disjoint union of curves,

$$J_\alpha = \coprod_{i=1}^{m} \Gamma_i,$$

where Γ_i are embeddings of the interval $I = [0,1]$ into E. We can assume that there exist smooth parametric curves $\gamma_i : I \to E$ such that $\Gamma_i = \gamma_i(I)$. Let $P_i = \gamma_i(0)$, and $Q_i = \gamma_i(1)$. Then there exist integers $c_i = c_i(\gamma_i) \in \mathbf{Z}$ given by (4.8) such that

$$\int_E d\phi \wedge \mathfrak{m} = 2\pi \sum_{i=1}^{m} c_i \left(\phi(P_i) - \phi(Q_i)\right) = \int_E \phi d\mathfrak{s}.$$

Here, we denote by $\mathfrak{s}$ the measure

$$\mathfrak{s} = 2\pi \sum_{i=1}^{m} c_i \left(\delta_{P_i} - \delta_{Q_i}\right). \tag{4.9}$$

If the curves Γ_i are closed, i.e. $\gamma_i(0) = \gamma_i(1)$, we have in fact $\mathfrak{s} = 0$. As noted in Remark 4.1, adding closed curves Γ_i such that $\Gamma_i \cap J_\alpha = \varnothing$ to J_α does not change the Cartan equation (4.7), so that $\mathfrak{m}$ cannot be unique. On the other hand, in this case $\mathfrak{s}$ is unique: this is a consequence of the following general uniqueness statement.

Lemma 4.6. *If there exists a Radon measure* $\mathfrak{s}$ *such that* $\|\mathfrak{s}\|(E) < \infty$ *and*

$$\int_E \omega \wedge d\phi = \int_E \phi K \theta^1 \wedge \theta^2 + \int_E \phi d\mathfrak{s} \quad \forall \phi \in C_c^\infty(E),$$

then $\mathfrak{s}$ *is unique.*

We recall that a Radon measure is a regular Borel measure (see [66]).

Proof. The measure $\mathfrak{s}$ has finite total variation, so that by linearity, the difference of two such $\mathfrak{s}$ would have to vanish in the sense of distributions for any $\phi \in C_c^\infty(E)$. By the Riesz representation theorem, which applies because $\mathfrak{s}$ is Radon and $C_c^\infty(E)$ is dense in $C_0(E)$, two such $\mathfrak{s}$ must therefore coincide. Q.E.D.

Therefore if the points P_i, Q_i are in finite number, the measure $\mathfrak{s}$ given by (4.9) is of finite variation. If in addition $P_i \neq Q_i$ for all i, then they are unique.

4.2.2 Analysis of the defect

We now establish the main claim of Theorem 4.1, namely that modulo a constant, the defect is an element of the class $\mathscr{X}(\bar{E})$, defined in (4.4).

Lemma 4.7. *Let (M, g) be a Riemannian surface, $E \subset M$ a bounded regular domain, $p \geqslant 1$ and $\theta^1, \theta^2 \in \mathrm{W}^{1,p}(E, T^*E)$ an orthonormal coframe, and $\mathfrak{s}$ defined as in (4.3). Then there holds $(2\pi)^{-1}\mathfrak{s} \in \mathscr{X}(\bar{E})$. Moreover, if $\mathfrak{s}$ is a Radon measure, then it is unique.*

First, we obtain another expression for ω. As before, we work on a parallelisable subset $E \subset M$; let $\tilde{\omega}$ be the connection form associated with the coframe $\eta^1, \eta^2 = *\eta^1$. Recall the first structure equations for a coframe θ^1, θ^2. Without loss of generality, we write $\theta := \theta^1$ and $\theta^2 = *\theta$:

$$
\begin{aligned}
d\theta &= \omega \wedge *\theta = *\langle \omega, \theta \rangle, \\
d(*\theta) &= -\omega \wedge \theta = - * \langle \omega, *\theta \rangle.
\end{aligned}
$$

It is clear that writing $\theta = \theta_i \eta^i$ and noting $*\eta^1 = \eta^2$, we have $*\theta = -\theta_2\eta^1 + \theta_1\eta^2$. Thus

$$
\begin{aligned}
d\theta &= d\theta_i \wedge \eta^i + \theta_i d\eta^i \\
&= d\theta_i \wedge \eta^i + \theta_1 \tilde{\omega} \wedge \eta^2 - \theta_2 \tilde{\omega} \wedge \eta^1 \\
&= d\theta_i \wedge \eta^i + \tilde{\omega} \wedge *\theta.
\end{aligned}
$$

For an arbitrary covector field ζ such that $|\zeta| = 1$, we consider the mapping

$$
R_\zeta : \xi \longmapsto \langle \xi, \eta^1 \rangle \zeta + \langle \xi, \eta^2 \rangle (*\zeta),
$$

where $\xi \in T^*E$. The transformation R_ζ rotates the pair (η^1, η^2) to $(\zeta, *\zeta)$. We check that for $\alpha, \beta \in \Gamma(T^*M)$, we have

$$\alpha \wedge \beta = R_\zeta \alpha \wedge R_\zeta \beta.$$

Indeed, it suffices to verify this on the basis η^1, η^2 of T^*M, which is clear in view of the relation $R_\zeta \eta^1 \wedge R_\zeta \eta^2 = \zeta \wedge *\zeta = 1 = \eta^1 \wedge \eta^2$. We apply this observation to calculate the terms $d\theta_i \wedge \eta^i$. First, selecting $R_{(*\theta)}$, we have

$$d\theta_1 \wedge \eta^1 = R_{(*\theta)} d\theta_1 \wedge R_{(*\theta)} \eta^1 = R_{(*\theta)} d\theta_1 \wedge (*\theta).$$

Similarly for the second term, choosing R_θ, there holds $d\theta_2 \wedge \eta^2 = R_\theta d\theta_2 \wedge *\theta$. Combining the previous expressions together, we have the following expression for the connection form

$$d\theta = d\theta_i \wedge \eta^i + \tilde{\omega} \wedge *\theta = (R_\theta d\theta_2 + R_{(*\theta)} d\theta_1 + \tilde{\omega}) \wedge *\theta.$$

Similar calculation for $*\theta$ yields

$$d(*\theta) = -d\theta_2 \wedge \eta^1 + d\theta_1 \wedge \eta^2 - \tilde{\omega} \wedge \theta = -(R_{(*\theta)} d\theta_1 + R_\theta d\theta_2 + \tilde{\omega}) \wedge \theta.$$

Thus the connection form is given by

$$\omega = \tilde{\omega} + R_\theta d\theta_2 + R_{(*\theta)} d\theta_1 =: \tilde{\omega} + D_\theta.$$

Comparing with Lemmas 4.2, D_θ is a closed form when $\theta \in \mathrm{W}^{1,p}(E, T^*E)$, for $p \geqslant 2$. To show Lemma 4.7, we only need to prove that $T_\theta = dD_\theta \in 2\pi\mathscr{Z}$. Properties of T_θ are familiar (see [11, 17]).

Lemma 4.8. *Let (M, g) be a Riemannian surface, $E \subset M$ a bounded regular domain, and $\theta \in \mathrm{W}^{1,p}(E, UT^*M)$ for $p \geqslant 1$. Then T_θ is a well-defined distribution of order one satisfying*

$$|\langle\!\langle T_\theta, \phi \rangle\!\rangle| \leqslant C \|\nabla\theta\|_{\mathrm{L}^1} [\phi]_{\mathrm{Lip}},$$

Proof. We may assume that E is parallelisable, and write $\theta = \theta_i \eta^i$. Recall that $R_\theta d\theta_2 = \langle d\theta_2, \eta^1 \rangle \theta + \langle d\theta_2, \eta_2 \rangle (*\theta)$ by definition, and so taking modulus and noting that $|\theta| = 1$, we have

$$\|R_\theta d\theta_2\|_{\mathrm{L}^1} \leqslant \|d\theta_1\|_{\mathrm{L}^1} \leqslant C \|\nabla\theta\|_{\mathrm{L}^1},$$

and similarly for $R_{(*\theta)} d\theta_1$. Hence we find

$$|\langle\!\langle T_\theta, \phi \rangle\!\rangle| \leqslant \left| \int_E \big(R_{(*\theta)} d\theta_1 + R_\theta d\theta_2\big) \wedge d\phi \right| \leqslant C \|\nabla\theta\|_{\mathrm{L}^1} \|d\phi\|_{\mathrm{L}^\infty} \leqslant C \|\nabla\theta\|_{\mathrm{L}^1} [\phi]_{\mathrm{Lip}}.$$

Q.E.D.

The next lemma follows by a tedious but straightforward application of the chain rule (see also [11, 17]).

Lemma 4.9. *Let (M, g) be a Riemannian surface, $E \subset M$ a bounded, parallelisable, regular domain, η^1, η^2 be a pair of smooth orthonormal coframe, and let $\theta = \theta_i \eta^i$ and $\kappa = \kappa_i \eta^i$ be two unit-length covectors in $\mathrm{W}^{1,p}(E, UT^*M)$ for $p \geqslant 1$. Then*

$$\begin{aligned} T_\theta + T_\kappa &= T_{\theta\kappa} \\ T_{\bar{\theta}} &= -T_\theta. \end{aligned}$$

In the statement of Lemma 4.10, we view θ and κ as $\mathbf{S}^1$-valued mappings, where the unit cotangent bundle UT^*M is trivialised by the coframe (η^1, η^2), i.e. $\theta = \theta_i \eta^i \mapsto (\theta_1, \theta_2)$. We use complex notation, embedding $\mathbf{S}^1$ into $\mathbf{C}$ in the standard way. The notation $\bar{\theta}$ refers to the complex conjugate.

Next, we observe that T_θ is continuous with respect to convergence in $\mathrm{W}^{1,p}$.

Lemma 4.10. *Let (M, g) be a Riemannian surface, $E \subset M$ a bounded, parallelisable domain, and let θ^k be a sequence of orthonormal coframes such that $\theta^k \to \theta$ in $\mathrm{W}^{1,p}(E, UT^*M)$ for $p \geqslant 1$. Then $T_{\theta^k} \overset{*}{\rightharpoonup} T_\theta$ in the sense of distributions.*

Proof. By Lemma 4.9, we may assume that $\theta = 1$ in complex notation, and hence $T_\theta = 0$. Let $\phi \in C_c^\infty(E)$. As $\theta^k \to 1$ in $\mathrm{W}^{1,1}$, we have $\|\nabla \theta^k\|_{\mathrm{L}^1} \to 0$, and so by Lemma 4.8 we have

$$|\langle\!\langle T_{\theta^k}, \phi \rangle\!\rangle| \leqslant C \|\nabla \theta^k\|_{\mathrm{L}^1} [\phi]_{\mathrm{Lip}} \to 0.$$

The result follows by dominated convergence. Q.E.D.

The class of functions $\mathrm{R}^{1,p}(E, \mathbf{S}^1)$ is defined as the set of $\mathrm{W}^{1,p}$ functions that are smooth apart from a finite set of points of the domain E. It is known [6, 11, 12] that $\mathrm{R}^{1,p}(E, \mathbf{S}^1)$ is dense in $\mathrm{W}^{1,p}(E, \mathbf{S}^1)$ for all $1 \leqslant p < \infty$ (for the strong topology of $\mathrm{W}^{1,p}(E, \mathbf{S}^1)$).

Lemma 4.11. *Let $E \subset M$ be an open parallelisable domain, and assume that $\theta \in \mathrm{R}^{1,p}(E, UT^*M)$, $1 \leqslant p < 2$. Then there exists finitely many points $Q_j \in E$ and $c_j \in \mathbf{Z}$ such that*

$$T_\theta = 2\pi \sum_j c_j \delta_{Q_j}.$$

Proof. Let $A := \{x_1, \dots, x_m\} \subset E$ be such that $\theta \in C^\infty(E \setminus A, UT^*M)$. We claim that

$$T_\theta = 2\pi \sum_{x_j \in A} \deg(\theta, x_j)\delta_{x_j}.$$

Step 1: let us first deal with the case $\theta \in C^\infty(E, \mathbf{S}^1)$. Then it is clear that for $\phi \in C_c^\infty(E)$,

$$\langle\!\langle\, T_\theta, \phi \,\rangle\!\rangle = \int_E \left(R_\theta d\theta_2 + R_{(*\theta)} d\theta_1\right) \wedge d\phi = \int_E (\omega - \tilde{\omega}) \wedge d\phi = -\int_E \phi d(\omega - \tilde{\omega}) = 0,$$

as the Cartan equation clearly holds for a smooth coframe $(\theta, *\theta)$.

Step 2: we treat the case where the singular set A is reduced to a single point $A = \{x_0\}$. Let $\phi \in C_c^\infty(E)$, and let $\varepsilon > 0$ be sufficiently small so that $B_\varepsilon(x_0) \subset E$. Then

$$\begin{aligned}\langle\!\langle\, T_\theta, \phi \,\rangle\!\rangle &= \int_E \left(R_{(*\theta)} d\theta_1 + R_\theta d\theta_2\right) \wedge d\phi \\ &= \int_{E \setminus B_\varepsilon(x_0)} \left(R_{(*\theta)} d\theta_1 + R_\theta d\theta_2\right) \wedge d\phi + \int_{B_\varepsilon(x_0)} \left(R_{(*\theta)} d\theta_1 + R_\theta d\theta_2\right) \wedge d\phi.\end{aligned}$$

By Step 1, the first integral vanishes, θ being smooth on $E \setminus B_\varepsilon(x_0)$. To handle the second term, we calculate

$$\int_{B_\varepsilon(x_0)} \left(R_{(*\theta)} d\theta_1 + R_\theta d\theta_2\right) \wedge d\phi = \int_{\partial B_\varepsilon(x_0)} \phi \left(R_{(*\theta)} d\theta_1 + R_\theta d\theta_2\right).$$

Recalling that $\theta \in \mathrm{W}^{1,p}(E, UT^*M)$, it has constant norm 1, and therefore $G := \left(R_{(*\theta)} d\theta_1 + R_\theta d\theta_2\right) \in \mathrm{L}^p(E, T^*M)$.

In order to evaluate the boundary term, we note that it defines a singular distribution of finite order at most 1, whose singular support is contained in $\{x_0\}$. By Schwartz's theorem on finite order singular distributions, there exists coefficients $c_\alpha \in \mathbf{R}$ indexed by the multi-indices $\alpha \in \mathbf{N}^2$, $|\alpha| \leqslant 1$, such that

$$\int_{\partial B_\varepsilon(x_0)} \phi \left(R_{(*\theta)} d\theta_1 + R_{(\theta)} d\theta_2\right) = \sum_{|\alpha| \leqslant 1} c_\alpha \langle\!\langle\, \partial^\alpha \delta_{x_0}, \phi \,\rangle\!\rangle.$$

Take a test function ϕ such that $\phi(x_0) = 0$, and consider a sequence of functions $\phi_k \in C_c^\infty(E \setminus \{x_0\})$ such that $d\phi_k \to d\phi$ and moreover $\|d\phi_k\|_{\mathrm{L}^\infty} \leqslant C$. It is clear that $\langle\!\langle\, T_\theta, \phi_k \,\rangle\!\rangle = 0$; by application of the dominated convergence theorem, one obtains $\langle\!\langle\, T_\theta, \phi \,\rangle\!\rangle = 0$. Thus if $\alpha \neq 0$, $c_\alpha = 0$. Finally, taking a smooth compactly supported test function ϕ such that $\phi = 1$ on $\frac{1}{2}B_j$ and $\phi = 0$ outside of B_j, one easily calculates

$$\int_{\partial B_j} \phi \left(R_{(*\theta)} d\theta_1 + R_{(\theta)} d\theta_2\right) = 2\pi \deg(\theta, \mathbf{S}^1, x_j).$$

Step 3: in the general case, let $A = \{x_1, \dots, x_m\} \subset E$, and let $B_j \subset E$ be small balls such that $B_j \cap A = \{x_j\}$ and $\phi \in C_c^\infty(E)$. Then θ is smooth on $E \setminus \left(\cup_{x_j \in A} B_j\right)$, so that we calculate, as in Step 2,

$$\langle\!\langle\, T_\theta, \phi \,\rangle\!\rangle = \sum_{x_j \in A} \int_{\partial B_j} \phi \left(R_{(*\theta)} d\theta_1 + R_\theta d\theta_2\right).$$

The argument of Step 2 now applies to each boundary term. This yields the desired conclusion. In the sequel we simply write $\deg(\theta, \mathbf{S}^1, x_j) = \deg(\theta, x_j) \in \mathbf{Z}$. Q.E.D.

Remark 4.2. *It is not true in general that* $\sum_j \deg(\theta, x_j) = 0$. *By adding finitely many boundary points* $\{x_{m+1}, \dots, x_k\} \subset \partial E$ *(potentially non-distincts), we may write* $T_\theta = \sum_{j=1}^s \delta_{P_j} - \delta_{Q_j}$, *where* $\{P_1, \dots, P_s, Q_1, \dots, Q_s\} = \{x_1, \dots, x_k\}$.

We are now ready to prove Lemma 4.7.

Proof of Lemma 4.7. Fixing a local trivialisation of UT^*M on E, which is parallelisable, we let $\eta^1, \eta^2 = *\eta^1$ be a smooth orthonormal coframe on T^*E. Then we may write $\theta = \theta_i \eta^i$. Viewing θ as the mapping (θ_1, θ_2), we consider θ as an $\mathbf{S}^1$-valued map.

Recall that the class $\mathrm{R}^{1,1}(E, \mathbf{S}^1)$ is dense in $\mathrm{W}^{1,1}(E, \mathbf{S}^1)$. Let us choose a sequence of functions $\theta^k \in \mathrm{R}^{1,1}(E, \mathbf{S}^1)$ converging to θ in the strong topology of $\mathrm{W}^{1,1}$.

We now choose $\varepsilon = 2^{-1-k}$ for $k \in \mathbf{N}$ and $\|\frac{\theta^{k+1}}{\theta}\|_{(\mathrm{Lip}_0)^*} < \varepsilon$. By Lemma 4.11 and Remark 4.2, there exists finitely points P_j and N_j in $\bar{E}$, non necessarily distinct, such that

$$T_{\theta^{k+1}} - T_{\theta^k} = 2\pi \sum_{j=s_k+1}^{s_{k+1}} (\delta_{P_j} - \delta_{N_j}).$$

By Lemma 4.15, we find

$$\begin{aligned}\sum_{j=s_k+1}^{s_{k+1}} \rho(P_j, N_j) &= \frac{1}{2\pi} \left\| \sum_{j=s_k+1}^{s_{k+1}} (\delta_{P_j} - \delta_{N_j}) \right\|_{(\mathrm{Lip}_0)^*} \\ &= \|T_{\theta^{k+1}} - T_{\theta^k}\|_{(\mathrm{Lip}_0)^*} \\ &\leqslant \|T_{\theta^{k+1}} - T_\theta\|_{(\mathrm{Lip}_0)^*} + \|T_{\theta^k} - T_\theta\|_{(\mathrm{Lip}_0)^*} \\ &\leqslant C 2^{-k},\end{aligned}$$

where we have used Lemma 4.8 in the penultimate line. Therefore, the partial sums $\sum_{j=1}^{s_{k+1}} \rho(P_j, N_j)$ form a Cauchy sequence, and the sum $\sum_j \rho(P_j, N_j) < \infty$ is well-defined. As a corollary, so is the distribution $\sum_j \delta_{P_j} - \delta_{N_j}$.

Applying Lemma 4.10 to the sequence of θ^k, it follows that $T_{\theta^k} \to T_\theta$ in $\mathcal{D}'(E)$, and so

$$T_\theta = 2\pi \sum_j (\delta_{P_j} - \delta_{N_j}).$$

If T_θ is a measure, or equivalently $|\langle\!\langle T_\theta, \phi \rangle\!\rangle| \leqslant C\|\phi\|_{\mathrm{L}^\infty}$, then the summation is finite by Lemma 4.16. In this case, uniqueness follows from the Riesz representation theorem: the measure $\mathfrak{s}$ has finite total variation, so that by linearity, the difference of two such $\mathfrak{s}$ would have to vanish in the sense of distributions for any $\phi \in C_c^\infty(E)$. By the Riesz representation theorem, which applies because $\mathfrak{s}$ is Radon and $C_c^\infty(E)$ is dense in $C_0(E)$, two such $\mathfrak{s}$ must therefore coincide. Q.E.D.

4.2.3 Minimal connections in the sense of Brézis–Coron–Lieb

In this section and the next, we prove the final assertion of Theorem 4.1:

Lemma 4.12. *Let $\varepsilon > 0$. Then there exists a sequence of points P_j, N_j such that*

$$\left| \|(2\pi)^{-1}\mathfrak{s}\|_{(\mathrm{Lip}_0(E))^*} - \sum_{j\in J} \rho(P_j, N_j) \right| < \varepsilon,$$

$$\mathfrak{s} = 2\pi \sum_{j\in J} (\delta_{P_j} - \delta_{N_j}).$$

This statement follows from the more general framework of minimal connections, first introduced by Brézis–Coron–Lieb [15] and Bourgain–Brézis–Mironescu [11], on domains with boundaries. This subsection is dedicated to the basic definitions, given in the general context of metric spaces. §4.2.4 focuses specifically on the case of domains with boundaries. The reader is cautioned that “minimal connections” has a different meaning than the geometric notion of connection form, used elsewhere in this thesis.

Let (X, d) be a metric space. We write $\mathrm{Lip}(X)$ for the space of globally Lipschitz functions on X, and $\mathrm{Lip}_0(X)$ for the subspace of globally Lipschitz functions vanishing at infinity. Namely, a function $f \in \mathrm{Lip}(X)$ is in $\mathrm{Lip}_0(X)$ if for any $\varepsilon > 0$ there exists a compact set $K \subset X$ such that for all $x \in X \setminus K$, $|f(x)| < \varepsilon$. In practice, we shall be concerned with bounded domains and their closure, i.e. compact metric spaces, so that $\mathrm{Lip}_0(X)$ is exactly the set of Lipschitz functions vanishing on the boundary ∂X.

For $f \in \mathrm{Lip}(X)$, we let $[f]_{\mathrm{Lip}}$ be the Lipschitz constant of f, i.e.

$$[f]_{\mathrm{Lip}} := \inf\{k > 0 \,|\, \forall x, y \in X, d(f(x), f(y)) \leqslant k d(x, y)\}.$$

We recall that $[\cdot]_{\mathrm{Lip}}$ defines a semi-norm on $\mathrm{Lip}(X)$ but a norm on $\mathrm{Lip}_0(X)$.

We consider functionals $T \in (\mathrm{Lip}(X))^*$ of the form

$$\langle\!\langle T, \phi \rangle\!\rangle = \sum_{j \in \mathbf{N}} (\phi(P_j) - \phi(N_j)), \tag{4.10}$$

where P_j and N_j are sequences of points of X such that $\sum_{j\in\mathbf{N}} d(P_j, N_j) < \infty$. We write functionals of the form (4.10) as $T = \sum_{j\in\mathbf{N}} \delta_{P_j} - \delta_{N_j}$.

Associated to the sequence of points $(P_j, N_j)_{j\in\mathbf{N}}$, its minimal connection $L((P_j, N_j)_{j\in\mathbf{N}})$ is defined as in [11] as

$$L((P_j, N_j)_{j\in\mathbf{N}}) = \inf_{\tilde{N}_j} \left\{ \sum_{j\in\mathbf{N}} d(P_j, \tilde{N}_j) < \infty \right\}, \tag{4.11}$$

where the sequences $\tilde{N}_j$ range over all possible sequences of points in X such that

$$\sum_{j\in\mathbf{N}} d(P_j, N_j) < \infty, \tag{4.12a}$$

$$\sum_{j\in\mathbf{N}} (\delta_{P_j} - \delta_{N_j}) = \sum_{j\in\mathbf{N}} (\delta_{P_j} - \delta_{\tilde{N}_j}). \tag{4.12b}$$

This last equality is to be understood in the sense of distributions: it means that for every $\phi \in \mathrm{Lip}(X)$ there holds

$$\sum_{j\in\mathbf{N}} (\phi(P_j) - \phi(N_j)) = \sum_{j\in\mathbf{N}} (\phi(P_j) - \phi(\tilde{N}_j)).$$

Such a sequence of $\tilde{N}_j$ is called a replacement of N_j. For instance, if the sequence of points (P_j, N_j) is finite, then

$$L((P_j, N_j)_{j=1,\dots,m}) = \inf_{\sigma\in S_m} \left\{ \sum_{j=1}^{m} d(P_j, N_{\sigma(j)}) < \infty \right\},$$

and the replacements of N_j are simply the permutations $N_{\sigma(j)}$. Note that when there are infinitely many points (P_j, N_j), the infimum (4.11) is not necessarily achieved by any replacement of the N_j (see [145]).

We wish to relate the minimal connection of a sequence of points to the operator norm of functionals of the form (4.11). From the definition, it is clear that if (P_j, N_j) are a sequence such that $\sum_j d(P_j, N_j) < \infty$, then for all $\phi \in \mathrm{Lip}_0(X) \subset \mathrm{Lip}(X)$,

$$\left| \sum_j (\phi(P_j) - \phi(N_j)) \right| \leqslant \sum_j |\phi(P_j) - \phi(\tilde{N}_j)| \leqslant [\phi]_{\mathrm{Lip}(X)} \sum_j d(P_j, \tilde{N}_j)$$

for any sequence of $\tilde{N}_j$ such that (4.12a) and (4.12b) hold; hence for any $\varepsilon > 0$,

$$\left|\sum_j (\phi(P_j) - \phi(N_j))\right| \leqslant [\phi]_{\mathrm{Lip}(X)}(L((P_j, N_j)) + \varepsilon),$$

from whence it follows that

$$\|T\|_{(\mathrm{Lip}_0(X))^*} \leqslant L((P_j, N_j)). \tag{4.13}$$

The reader is cautioned that the reverse inequality does not hold in general (but it is true that $\|T\|_{\mathrm{Lip}^*} = L((P_j, N_j))$, see [11]). To obtain the reverse inequality requires carefully designing the metric spaces involved.

4.2.4 Minimal connections on a metric domain with boundary

We consider the complete metric space $(\bar{E}, d_g(\cdot, \cdot))$, where d_g is the geodesic distance induced by the Riemannian metric g on M. We recall that $\mathrm{Lip}_0(E)$ denotes the set of Lipschitz functions on E which vanish on ∂E, and that $[\cdot]_{\mathrm{Lip}(E)}$ is a norm on $\mathrm{Lip}_0(E)$.

We recall that we defined the pseudo-metric

$$\rho(x, y) = \min\left(d_g(x, y), d_g(x, \partial E) + d_g(y, \partial E)\right)$$

on $\bar{E}$. Noting that $x \sim y \Leftrightarrow \rho(x, y) = 0$ is an equivalence relation for points $x, y \in \bar{E}$, we may define the quotient $\bar{E}/\sim$. In the quotient, the boundary ∂E is identified with a single point $[\partial E]$. Thus we use the suggestive notation $\bar{E}/\sim =: E/\partial E$. The quotient space is a compact topological space which is equipped with a metric $d'(\cdot, \cdot)$ given by ρ on the base space $\bar{E}$. We note that the quotient map $(\bar{E}, d) \twoheadrightarrow (\bar{E}/\partial E, d')$ is distance-decreasing.

When given a sequence of points $(P_j, N_j)_{j\in\mathbf{N}}$ on $\bar{E}$, the notation $L'((P_j, N_j)_{j\in\mathbf{N}})$ refers to the minimal connection of the points $([P_j], [N_j])_{j\in\mathbf{N}}$ in the quotient space $(\bar{E}/\partial E, d')$.

The following lemma is clear.

Lemma 4.13. $\{f \in \mathrm{Lip}(\bar{E}/\partial E) | f([\partial E]) = 0\} \simeq \mathrm{Lip}_0(E)$.

The next Lemma is inspired by (and partially adapted from) [15].

Lemma 4.14. *Let $(P_j, N_j) \subset \bar{E}$ be a finite set of points, $j = 1, \ldots, m$. Then*

$$L'((P_j, N_j)_{j=1,\ldots,m}) = \|T\|_{\mathrm{Lip}_0^*}.$$

We note that in view of inequality (4.13), it suffices to show that there exists a function $\zeta \in \mathrm{Lip}_0(E)$ such that

$$\langle\!\langle T, \zeta \rangle\!\rangle = \sum_{j=1}^{m} \zeta(P_j) - \zeta(N_j) = L'((P_j, N_j)_{j=1,\dots,m}),$$

where L' is the minimal connection with respect to the quotient distance on $\bar{E}/\partial E$.

Proof. **Step 1:** By Lemma 4.2 of [15], there exists real numbers $\alpha_j, \beta_j \in \mathbf{R}$, $j = 1, \dots, m$, such that

$$\begin{aligned} \sum_{j=1}^{m} (\alpha_j - \beta_j) &= \min_{\sigma \in S_m} \sum_{j=1}^{m} \rho(P_j, N_{\sigma(j)}), \\ |\alpha_i - \alpha_j| &\leqslant \rho(P_i, P_j) \\ |\alpha_i - \beta_j| &\leqslant \rho(P_i, N_j), \\ |\beta_i - \beta_j| &\leqslant \rho(N_i, N_j), \end{aligned}$$

for $1 \leqslant i, j \leqslant m$.

In view of the later steps, we note that if α_j, β_j satisfy these constraints, so do the numbers $\alpha_j + \xi, \beta_j + \xi$ for any $\xi \in \mathbf{R}$.

Step 2: Let $x_1, \dots, x_l$ be the set of distinct points of $\{[P_1], \dots, [P_m], [N_1], \dots, [N_m]\}$. If there are no points among the P_i, N_i that are in ∂E, we add a point $x_0 = [\partial E]$ and we define $\zeta_0 = 0$.

In other words, we have defined a mapping $\tau : \{1, \dots, 2m\} \to \{1, \dots, l\}$ by

$$\begin{aligned} [P_i] &= x_{\tau(i)} \qquad 1 \leqslant i \leqslant m, \\ [N_{i-m}] &= x_{\tau(i)} \qquad m+1 \leqslant i \leqslant 2m. \end{aligned}$$

We note that by Step 1, there are well-defined real numbers $\zeta_k \in \mathbf{R}$, $k = 1, \dots, l$, given by $\zeta_{\tau(i)} = \alpha_{\tau(i)}$ for $1 \leqslant i \leqslant m$, and $\zeta_{\tau(i)} = \beta_{\tau(i)}$ for $m + 1 \leqslant i \leqslant 2m$. The ζ_i are well-defined: let $i \neq j$ such that $\tau(i) = \tau(j)$; we need to verify that $\zeta_{\tau(i)} = \zeta_{\tau(j)}$. Let us assume that $i, j \leqslant m$ (the other cases are done similarly). Since $\tau(i) = \tau(j)$, we have $[P_i] = [P_j]$, which is to say $\rho(P_i, P_j) = 0$ and hence $\alpha_i = \alpha_j$. Thus we may unambiguously define $\zeta_{\tau(i)} = \alpha_i = \alpha_j$.

Step 3: Without loss of generality, we may assune, up to translation by a finite real number, that $\zeta_i < 0$ whenever $i > 0$. We define the function

$$\zeta(x) = \max\{\zeta_i - d'(x, x_i) \,\big|\, i = 0, 1, \dots, l\}, \quad x \in \bar{E}/\partial E.$$

The function ζ is clearly Lipschitz with $[\zeta]_{\mathrm{Lip}} \leqslant 1$. Moreover, $\zeta(x_i) = \zeta_i$ for all $i = 0, \ldots, m$: indeed, fix such an index $1 \leqslant i \leqslant l$. By definition, $\zeta(x_i) \geqslant \zeta_i$. Moreover, for $i, j > 0$, by construction $d'(x_i, x_j) \geqslant |\zeta_i - \zeta_j|$ and so $\zeta_j - d'(x_i, x_j) \leqslant \zeta_i$. Since $\zeta(x) \leqslant \zeta_0 = 0$, we have $\zeta([\partial E]) = 0$.

Step 4: we note that there exist $c_j \in \mathbf{Z}$ such that

$$\sum_{i=1}^{m}(\delta_{P_i} - \delta_{N_i}) = \sum_{k=1}^{l} c_j \delta_{x_k},$$

where we understand the right-hand side as acting on functions on the quotient $\bar{E}/\partial E$, and the LHS as acting on $\mathrm{Lip}(\bar{E})$. We calculate

$$\begin{aligned}\langle\!\langle T, \zeta \rangle\!\rangle &= \langle\!\langle \sum_{j=0}^{l} c_j \delta_{x_j}, \zeta \rangle\!\rangle = \sum_{j=0}^{l} c_j \zeta_j = \sum_{j=1}^{l} c_j \zeta_j = \sum_{i=1}^{m} (\alpha_i - \beta_i) \\ &= L'((P_i, N_i)_{i=1,\ldots,m}).\end{aligned}$$

Noting $\zeta \in \mathrm{Lip}_0(\bar{E})$, we conclude that $\|T\|_{(\mathrm{Lip}_0(\bar{E}))^*} = L'((P_i, N_i)_{i=1,\ldots,m})$. Q.E.D.

We use the case of finite sequences proved in Lemma 4.14 to prove the result for infinite sequences. The following lemma corresponds to Lemma 12′ of [11]:

Lemma 4.15. *Let $(P_j, N_j)_{j\in\mathbf{N}}$ be two sequences of points and such that $\sum_{j\in\mathbf{N}} \rho(P_j, N_j) < \infty$, and $T = \sum_{j\in\mathbf{N}} \delta_{P_j} - \delta_{N_j}$. Then*

$$L'((P_j, N_j)_{j\in\mathbf{N}}) = \|T\|_{(\mathrm{Lip}_0(E))^*}.$$

Recalling the definition of L' as the infimum of the norm of all possible representations of a distribution of the form $T = \sum_{j\in\mathbf{N}} \delta_{P_j} - \delta_{N_j}$, Lemma 4.12 follows at once from Lemma 4.15. The proof is provided for completeness, but follows the lines of [11].

Proof. Let $\phi \in \mathrm{Lip}_0(E)$ be such that $[\phi]_{\mathrm{Lip}} \leqslant 1$. By construction, ϕ descends to a function $\tilde{\phi} \in \mathrm{Lip}(\bar{E}/\partial E)$ with $\tilde{\phi}([\partial E]) = 0$. Then we have

$$\sum_{j\in\mathbf{N}} |\tilde{\phi}(P_j) - \tilde{\phi}(N_j)| \leqslant \sum_{j\in\mathbf{N}} d'(P_j, N_j).$$

Substituting any arbitrary other replacement of the N_j, we find that

$$\sum_{j\in\mathbf{N}} |\tilde{\phi}(P_j) - \tilde{\phi}(N_j)| \leqslant L'((P_j, N_j)_{j\in\mathbf{N}}),$$

and so $\|T\|_{\mathrm{Lip}_0^*} \leqslant L'((P_j, N_j)_{j\in\mathbf{N}})$.

We now prove the reverse inequality. Let $\varepsilon > 0$ and let $m \in \mathbf{N}$ be such that

$$\sum_{j>m} d'(P_j, N_j) \leqslant \varepsilon.$$

By Lemma 4.14, let $\sigma \in S_N$ be such that

$$L'((P_j, N_j)_{j=1,\dots,m}) = \sum_{j=1}^{m} d'(P_j, N_{\sigma(j)}) = \|T\|_{\mathrm{Lip}_0^*}.$$

Extending σ to all $\mathbf{N}$ by $\sigma(i) = i$ whenever $i > m$, we find that $\sum_j \delta_{P_j} - \delta_{N_j} = \sum_j \delta_{P_j} - \delta_{N_{\sigma(j)}}$.

Then we have

$$\begin{aligned} L'((P_j, N_j)) &\leqslant \sum_{j\in\mathbf{N}} d'(P_j, N_{\sigma(j)}) \\ &= \sum_{j=1}^{m} d'(P_j, N_{\sigma(j)}) + \sum_{j>m} d'(P_j, N_j) \\ &\leqslant \sup_{[\phi]_{\mathrm{Lip}}\leqslant 1} \left| \sum_{j=1}^{m} \phi(P_j) - \phi(N_{\sigma(j)}) \right| + \varepsilon \\ &\leqslant \sup_{[\phi]_{\mathrm{Lip}}\leqslant 1} \left(\left| \sum_{j>0} \phi(P_j) - \phi(N_{\sigma(j)}) \right| + \left| \sum_{j>m} \phi(P_j) - \phi(N_j) \right| \right) + \varepsilon \\ &\leqslant \|T\|_{\mathrm{Lip}_0^*} + 2\varepsilon. \end{aligned}$$

As $\varepsilon > 0$ is arbitrary, we deduce the conclusion.

Q.E.D.

The case where T is a finite sum of Dirac masses is very special. The following lemma is due to [160] when X is compact, and [145] for the general case.

Lemma 4.16. *Let (X, d) be a complete metric space, $(P_j, Q_j)_{j\in\mathbf{N}}$ two sequences of points in X and such that $\sum_{j\in\mathbf{N}} d(P_j, Q_j) < \infty$, and $T = \sum_{j\in\mathbf{N}} \delta_{P_j} - \delta_{Q_j}$. If T is a measure on X, i.e. $T \in (C(X))^*$, then there exists finitely many distinct points $x_j \in X$ and integers $c_j \in \mathbf{Z}$ such that*

$$T = \sum_{j=1}^{m} c_j \delta_{x_j}.$$

It follows in particular from this last lemma that if the defect $\mathfrak{s}$ is actually a measure, it is a sum of finitely many Diracs, supported on a finite set S, which is therefore closed. Hence the Cartan equation holds in the sense of distributions on $E \setminus S$. However, in general, the support of a distribution belonging to the class $\mathscr{L}(\bar{E})$ need not be closed.

4.3 Cartan's equations in higher dimensions

We now examine the higher-dimensional case, and show that Cartan's equation hold for $\mathrm{W}^{1,p}$ coframes, $p \geqslant 2$, irrespective of the dimension.

Proposition 4.17. *Let (M,g) be a smooth Riemannian manifold, $\theta^1, \ldots, \theta^n$ a local $\mathrm{W}^{1,2}$ orthonormal coframe. Then there exists a L^2 $\mathfrak{so}(n)$-valued connection form $\omega = (\omega^i_j)$, and for any $\phi \in C^\infty_c(E)$, there holds*

$$-\int_E \omega \wedge d\phi + \int_E \phi\omega \wedge \omega = \int_E \phi\, \mathbf{Rm}(\theta \wedge \theta),$$

where $\mathbf{Rm} : \wedge^2 T^*M \to \wedge^2 T^*M$ is the Riemann curvature endomorphism acting on two-forms.

Let $\theta^1, \ldots, \theta^n$ be a local orthonormal $\mathrm{W}^{1,2}$ coframe on a domain $E \subset M$, which we view as a map $\theta \in \mathrm{W}^{1,2}(E, \mathbf{SO}(n) \otimes T^*M|_E)$. Then there exists a unique L^2 map $\omega \in \mathrm{L}^2(E, \mathfrak{so}(n) \otimes T^*M|_E)$ such that

$$d\theta = \omega \wedge \theta.$$

Here $\mathfrak{so}(n)$ is the Lie algebra of skew-symmetric matrices.

Furthermore, if θ were smooth, so would ω and the induced curvature form $d\omega + \omega \wedge \omega \in \Gamma(E, \mathfrak{so}(n) \otimes T^*M|_E \wedge T^*M|_E)$ would verify, in coordinates,

$$d\omega^i_j + \omega^i_k \wedge \omega^k_j = \Omega^i_j = \frac{1}{2} R^i_{jkm} \theta^k \wedge \theta^m,$$

which is nothing else than the higher-dimensional analogue of the Cartan equation (see Chapter 3). We give a proof based on the same approximation argument as Lemmas 4.2, 4.5 and 4.7.

Proof. We recall that for all n the space $\mathbf{SO}(n)$ has vanishing second homotopy group $\pi_2(\mathbf{SO}(n)) = 0$ for all n. Therefore, by a deep theorem due to Béthuel and Hang–Lin [5, 87], smooth maps are dense in $\mathrm{W}^{1,2}(E, \mathbf{SO}(n))$. Fixing a local trivialisation of the coframe bundle, one obtains a sequence of maps θ^ε and the associated connection forms ω^ε such that

$$\begin{aligned} \theta^\varepsilon &\to \theta && \text{in } \mathrm{W}^{1,2}(E, \mathbf{SO}(n) \otimes T^*M|_E) \\ \omega^\varepsilon &\to \omega && \text{in } \mathrm{L}^2(E, \mathfrak{so}(n) \otimes T^*M|_E). \end{aligned}$$

Furthermore, each ω^ε satisfies $d\omega^\varepsilon + \omega^\varepsilon \wedge \omega^\varepsilon = \Omega$. The latter equation passes to limits in the sense of distributions. Q.E.D.

Thus, in general, if $p \geqslant 2$, $\mathrm{W}^{1,2}$ coframes do not exhibit (curvature) defects. When $n > 2$ such coframes are not necessarily continuous.

When $n = 2$, the group $\mathbf{SO}(2)$ is abelian, and so curvature is a linear operator of the connection form, whereas in higher dimension the non-linear term $\omega \wedge \omega$ need not make sense in $\mathrm{L}^1_{\mathrm{loc}}$ if ω is in $\mathrm{W}^{1,p}$ for $p < 2$.

4.3.1 Gauß–Codazzi–Ricci equations in $\mathrm{W}^{1,2}$

Recall (see Chapter 3) that for an immersed Riemannian manifold (M, g) in $\mathbf{R}^N$, the Gauß–Codazzi–Ricci equations are but a mere consequence of the validity of Cartan's equations in both (M, g) and $\mathbf{R}^N$.

Let $u : (M, g) \to \mathbf{R}^N$ be a $\mathrm{W}^{2,2}$ isometric immersion with bounded first derivatives. Recall that on a domain $u(E) \subset u(M) \subset \mathbf{R}^N$, an orthonormal coframe $\theta = (\theta^1, \dots, \theta^N) \in \mathrm{W}^{1,2}(E, \mathbf{SO}(N) \otimes T^*M|_E)$ of $\mathbf{R}^N$ is Darboux if for $1 \leqslant i \leqslant n$, $u^*\theta^i \in \mathrm{W}^{1,2}(E, T^*M|_E)$ forms an orthonormal coframe on E, viewed as a Riemannian subdomain of (M, g). Proposition 4.17 shows that Cartan's equation is then satisfied on (M, g) for $u^*\theta$; similarly θ is a $\mathrm{W}^{1,2}$ orthonormal coframe of $\mathbf{R}^N$, so that Cartan's equation $d\omega + \omega \wedge \omega = 0$ is satisfied in the sense of distributions. Now recall that the Gauß–Codazzi–Ricci equations are obtained from writing

$$\omega = \begin{pmatrix} \omega^{\top} & {}^t\omega^{\mathbf{II}} \\ -\omega^{\mathbf{II}} & \omega^{\perp} \end{pmatrix},$$

and Cartan's equations on (M, g) and $\mathbf{R}^N$. Overall we have obtained:

Corollary 4.18. *Let* $u \in \mathrm{W}^{2,2}(M, \mathbf{R}^N) \cap \mathrm{W}^{1,\infty}(M, \mathbf{R}^N)$ *be an isometric immersion of* (M, g) *into* $\mathbf{R}^N$, $\theta^1, \dots, \theta^N$ *be a local* $\mathrm{W}^{1,2}$ *Darboux coframe on a domain* $u(E) \subset u(M)$. *Let*

$$\omega = \begin{pmatrix} \omega^{\top} & {}^t\omega^{\mathbf{II}} \\ -\omega^{\mathbf{II}} & \omega^{\perp} \end{pmatrix} \in \mathrm{L}^2(u(E), \mathfrak{so}(N) \otimes T^*\mathbf{R}^N|_{u(E)}),$$

be its connection form. Then the Gauß–Codazzi–Ricci equations hold in the sense of distributions.

Chapter 5

Validity and Regularity of the Gauß Equation

We study the validity of the Gauß equation for embeddings $u : M \to \mathbf{R}^3$ for a given Riemannian surface (M, g) into Euclidean space. Such an embedding map u is locally isometric if $u^* e = g$, i.e. the metric induced on M by the Euclidean metric on $\mathbf{R}^3$ coincides with g. Here the pull-back notation means that in local coordinates,

$$\langle \partial_i u, \partial_j u \rangle = g_{ij}.$$

Restricting our attention to a small open subset $E \subset M$, $u(E)$ can be represented graphically by

$$u(E) = \{(x, y, f(x, y)) \,\big|\, (x, y) \in D\},$$

where $D = \{(u^1(\xi), u^2(\xi)) \,\big|\, \xi \in E\} \subset \mathbf{R}^2$ and $f \in C^1(D)$. If u is a C^2 locally isometric embedding, it follows that $f \in C^2$; in addition, f satisfies the Gauß equation

$$\det \nabla^2 f = (K \circ (u^1, u^2)^{-1})(1 + |\nabla f|^2)^2. \tag{5.1}$$

Here K is the Gauß curvature of g on E.

The study of the Gauß equation is a classical topic, relevant in many problems of differential geometry. Since the work of Nash and Kuiper [112, 134, 135] (see also [43]), it is known that C^1 isometric embeddings may have very different properties than smoother ones. It is therefore a natural question to understand the validity of the Gauß equation for less regular isometric embeddings. In this case, $\det \nabla^2 f$ needs to be defined appropriately.

We study the validity and properties of the Gauß equation when the embedding is in some Sobolev class; we assume throughout that $u \in \mathrm{W}^{2,p}(M^2, \mathbf{R}^3)$. If $p \geqslant 2$,

$\det \nabla^2 f$ is an integrable $\mathrm{L}^{p/2}$ function, and equation (5.1) is seen to hold by a direct approximation argument. If $p < 2$, $\det \nabla^2 f$ is only a measurable function which may be not integrable. However, under the additional assumption that f is Lipschitz, one can define a distributional Hessian determinant $\operatorname{Det} \nabla^2 f$ for $p \geqslant 1$ (see [2]; we collect the definition and some technical results in Appendix 5.A). Note that weaker notions of Hessian determinants may be defined in suitable functional spaces: see *e.g.* [100]. $\operatorname{Det} \nabla^2 f$ and $\det \nabla^2 f$ need not agree in general if $p < 2$, as standard examples show (see *e.g.* [2, 101]). In fact, there exist examples [100, 115] of isometric embeddings of a Euclidean disk in $\mathbf{R}^3$ of $\mathrm{W}^{2,p}$ regularity for all $p < 2$ such that (5.1) holds almost everywhere, but the distributional Jacobian is not equal to the RHS of (5.1).

Our first result establishes the validity of the Gauß equation in another regularity class:

THEOREM 5.1. *Let (M, g) be a smooth Riemannian surface, K its Gauß curvature. Let $u : E \subset M \to \mathbf{R}^3$ be a locally isometric embedding, locally parametrised by a function $f \in \mathrm{W}^{2,p} \cap \mathrm{W}^{1+\alpha,q}$, in such a way that $u^3 = f(u^1, u^2)$, and D is the domain of f. Then the following hold: if p, q and α satisfy*

$$\frac{2}{q} + \frac{1}{p} \leqslant 1, \qquad 1 \leqslant p < 2, \qquad 4 < q \leqslant \infty, \qquad \alpha > \frac{1}{2}, \tag{R}$$

then the Gauß equation holds in the sense of distributions, i.e.

$$\operatorname{Det} \nabla^2 f = (K \circ (u^1, u^2)^{-1})(1 + |\nabla f|^2)^2. \tag{5.2}$$

Moreover, in this case, the Gauß equation also holds almost everywhere, i.e. for a.e. $x \in D$,

$$\det \nabla^2 f(x) = (K \circ (u^1, u^2)^{-1})(x)(1 + |\nabla f(x)|^2)^2.$$

Let us briefly comment on the hypotheses of the theorem. We recall that by Sobolev embeddings (see *e.g.* [56]), condition (R) implies that f is actually $C^{1,\alpha - 2/q}$. In particular, the distributional determinant $\operatorname{Det} \nabla^2 f$ is well-defined. If $p = 2$, f is in $\mathrm{W}^{3/2,4}$ by Sobolev embedding; moreover, $\det \nabla^2 f = \operatorname{Det} \nabla^2 f \in \mathrm{L}^1(D)$ and the Gauß equation is seen to hold by a standard approximation argument, as mentioned earlier. Thus, in this case, the assumption on α and q may be dropped.

When $p = 1$, condition (R) mandates f to have a $C^{1,\alpha}$ representative, for $\alpha > 1/2$. For $1 < p < 2$, the statement of (R) can be interpreted as interpolating between the spaces $\mathrm{W}^{2,1} \cap C^{1,\alpha}$ and $\mathrm{W}^{2,2}$.

The study of the Gauß equation in Sobolev regularity has been studied in the literature: see *e.g.* [115, 116, 141, 173]. In particular, in [115], a suitable version of the Gauß equation is proved for $\mathrm{W}^{2,1}$ isometric immersions of Euclidean domains into Euclidean spaces.

As an application of Theorem 5.1, using the theory of bounded extrinsic curvature due to Pogorelov [144] and ideas from [43], we obtain a regularity result in the case of positive Gauß curvature. In particular, we show that f is convex and (5.2) is an elliptic Monge–Ampère equation.

THEOREM 5.2. *With the same notations and hypotheses as Theorem 5.1, if in addition* $\inf_E K > 0$*, then* f *is smooth, and the Gauß equation holds classically in* D*. In particular, the Hessian of* f *is either positive definite in all of* D *or negative definite in all of* D*.*

The Gauß equation is used to prove the regularity of isometric embeddings of positively curved spheres in $\mathbf{R}^3$: this has been studied extensively in the literature, see [43, 49, 95, 173]. In Hölder regularity, when $\alpha > 2/3$, $C^{1,\alpha}$ isometric embeddings of positively curved spheres are smooth: see [7, 8, 43, 170, 171]. For $\alpha < 1/5$, there exist $C^{1,\alpha}$ isometric embeddings of positively curved spheres that are not convex and not smooth, see [43, 50] and the references therein. In Sobolev regularity, it is also known that for $p \geqslant 2$, $\mathrm{W}^{2,p}$ isometric embeddings of positively curved spheres are smooth: see [95, 173].

It is an open question whether the conclusions of Theorems 5.1 and 5.2 hold when condition (R) is replaced by the weaker assumption of $\mathrm{W}^{2,1} \cap C^1$ regularity. In our proof, we use condition (R) to obtain certain cancellations. For example, in Lemma 5.8, we show that expressions of the form $h dv^\varepsilon \wedge dv$ vanish in the limit, where v^ε is a suitable regularising sequence for $v \in \mathrm{W}^{1,p} \cap \mathrm{W}^{\alpha,q}(D)$ and $h \in \mathrm{W}^{\beta,r}(D)$; see also Lemmas 5.7, 5.11 and 5.12 for the related cancellations. If similar cancellations hold in the limit for $v \in \mathrm{W}^{1,1} \cap C^0$, then Theorems 5.1 and 5.2 would hold for $\mathrm{W}^{2,1} \cap C^1$ isometric embeddings.

This chapter is organised as follows. In Section 5.1, we obtain different versions of Gauß' equation, first for rotationally symmetric surfaces, and second for graphical surfaces. In the case of rotationally symmetric surfaces, validity of the Gauß equation implies regularity of the embedding. Section 5.2 is dedicated to the proof of Theorem 5.1. Section 5.3 contains the proof of Theorem 5.2.

As an addendum to the chapter, we have collected in Appendix 5.A several proofs and facts about Hessian determinants which are used through this chapter.

5.1 Coframes on surfaces of $\mathbf{R}^3$

Let (M, g) be a Riemannian surface. We study coframes on M arising from an immersion u of M into $\mathbf{R}^3$ with low Sobolev regularity. First, as a motivation for ulterior developments, §5.1.1 focuses on the special case of rotational symmetry, and we show that if a smooth manifold is isometrically immersed in $\mathbf{R}^3$ as a rotationally symmetric surface, generated by an absolutely continuous curve, then this immersion is necessarily smooth. The argument relies on proving a version of the Gauß equation on surfaces using the machinery of rough coframes of Chapter 4.

In §5.1.2, we obtain a similar equation on surfaces parametrised locally as graphs. The rest of the chapter is then occupied by proving equivalent versions of the Gauß equation and a regularity result in the case of positive Gauß curvature.

5.1.1 Rotationally symmetric surfaces

Let (M, g) be a smooth Riemannian manifold equipped with an $\mathbf{S}^1$-equivariant metric, and assume that there is an equivariant C^1 isometric immersion $u : (M, g) \to \mathbf{R}^3$, in the following sense: the action

$$S_\alpha : (x^1, x^2, x^3) \mapsto (x^1, x^2 \cos\alpha, x^3 \sin\alpha), \qquad \alpha \in \mathbf{R},$$

leaves $u(M) \subset \mathbf{R}^3$ invariant, and moreover, there exists a mapping $\rho : \mathbf{S}^1 \to \mathbf{R}$ such that to any element $\sigma \in \mathbf{S}^1$ acting on (M, g) corresponds an element $\rho(\sigma)$ such that $u(\sigma\xi) = S_{\rho(\sigma)}u(\xi)$ for all $\xi \in M$. Thus we may write $u(M)$ as the rotation of a curve γ around the x^1-axis. Finally, we assume in addition that γ is absolutely continuous (in particular, it is rectifiable).

To fix ideas, let $D = [0, 1] \times [0, 2\pi[$, and let us parametrise $u(M)$ by the map $F : D \subset \mathbf{R}^2 \to \mathbf{R}^3$ given by

$$F(s, \alpha) = (\gamma_1(s), \gamma_2(s)\cos\alpha, \gamma_2(s)\sin\alpha),$$

where we assume the curve $\gamma = (\gamma_1(s), \gamma_2(s)) \in \mathrm{W}^{2,1}(0, 1; \mathbf{R}^2) \hookrightarrow C^1(]0, 1[, \mathbf{R}^2)$ to be parametrised by arc-length.

The induced metric on D by $\mathbf{R}^3$ is given by $ds^2 + \gamma_2^2(s)d\alpha^2$. We select the orthonormal coframe given by

$$\begin{cases} \zeta^1 = ds, \\ \zeta^2 = \gamma_2(s)d\alpha. \end{cases}$$

Away from the x^1-axis, i.e. when $\gamma_2(s) \neq 0$, ζ^1, ζ^2 is orthonormal. Calculating the connection form ω associated to ζ^1, ζ^2 yields

$$\omega = \gamma_2'(s) d\alpha.$$

Let us write $v = u \circ F^{-1} : M \to D$, which is a $C^1 \cap \mathrm{W}^{2,1}$ diffeomorphism onto its image. Pulling back ζ^1, ζ^2 onto M, we obtain an orthonormal coframe $\theta^i = v^* \zeta^i$ on M, whose connection form is $v^*\omega$, as we have

$$\begin{aligned} d\theta^1 &= d(v^*\zeta^1) \\ &= v^* d\zeta^1 \\ &= v^*\omega \wedge v^*\zeta^2 \\ &= v^*\omega \wedge \theta^2, \end{aligned}$$

and similarly for θ^2. Now θ^1, θ^2 is a continuous $\mathrm{W}^{1,1}$ coframe on $E = v^{-1}(D) \subset M$. By Theorem 4.1, we conclude

$$\int_E v^*\omega \wedge d\phi = \int_E \phi K \theta^1 \wedge \theta^2, \quad \forall \phi \in C_c^\infty(E). \tag{5.3}$$

It is not difficult to see that equation (5.3) holds for all $\phi \in C_c^1(E)$.

The special structure of rotationally symmetric surfaces, in combination with the Gauß equation (5.3) may be used to show that F has improved regularity.

Proposition 5.3. *Let (M, g), K, $\gamma = (\gamma_1, \gamma_2) \in \mathrm{W}^{2,1}(I, \mathbf{R}^2)$ as above, and let u be an isometric embedding of (M, g) into $\mathbf{R}^3$ given by the rotation of γ around the x^1-axis. Then u is smooth away from the x^1-axis.*

Proof of Proposition 5.3. We choose test functions $\psi \in C_c^\infty(D)$ that are $\mathbf{S}^1$-invariant, i.e., viewed in the (s, α) coordinates, we choose $\psi = \psi(s)$ to not depend on α and such that $\psi(s, \alpha) = 0$ for all $s \in I$ such that $\gamma_2(s) = 0$. Then $\phi = v^*\psi = \psi \circ v \in C_c^1(E)$. Applying Lemma 4.2 as in (5.3), we have

$$\begin{aligned} \int_E v^*\phi K \theta^1 \wedge \theta^2 &= \int_D \gamma_2'(s) d\alpha \wedge d\phi \\ &= \int_D \gamma_2'(s) d\alpha \wedge \phi'(s) ds \\ &= 2\pi \int_0^1 \gamma_2'(s) \phi'(s) ds. \end{aligned}$$

On the other side,

$$\begin{aligned}\int_E v^*\phi K\theta^1\wedge\theta^2 &= \int_D \phi(v^{-1})^*K\zeta^1\wedge\zeta^2\\ &= \int_0^1 \gamma_2(s)\phi(s)\left(\int_0^{2\pi}(v^{-1})^*Kd\alpha\right)ds\\ &=: \int_0^1 \gamma_2(s)\phi(s)\tilde{K}(s)ds.\end{aligned}$$

Thus we get the ODE

$$\int_0^1 \tilde{K}(s)\phi(s)\gamma_2(s)ds = 2\pi\int_0^1 \gamma_2'(s)\phi'(s)ds, \tag{5.4}$$

or, in other words,

$$2\pi\gamma_2'' = \tilde{K}\gamma_2$$

in the sense of distributions. Regularity for ODE yields $\gamma_2\in C^2$; but now

$$1 = \gamma_1'^2 + \gamma_2'^2,$$

whence γ must be C^2 away from the x^1-axis.

Recall that $F = (\gamma_1, \gamma_2\cos\alpha, \gamma_2\sin\alpha)$. Then F itself is C^2 away from the x^1 axis, and by the Myers–Steenrod theorem [164], so is u; hence so that $\tilde{K}$ is C^2. Bootstrapping yields smoothness of u. Q.E.D.

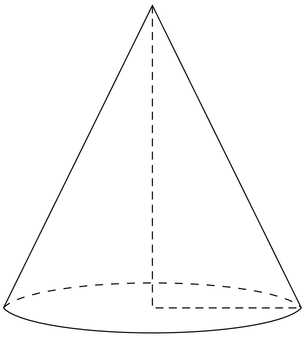

Figure 5.1: A Lipschitz isometric immersion of a flat disk in $\mathbf{R}^3$. The cone is a $W^{2,p}$ immersion for all $p < 2$.

If in addition we require the surface to be C^1, then the same reasonning shows that equation (5.4) holds everywhere; moreover in this case, the surface is smooth. The condition that a rotationally symmetric surface be C^1 is in this case a non-degeneracy condition on the curve γ: for all $s\in I$ such that $\gamma_2(s)=0$, we have $\gamma_1'(s)=0$. As γ is arc-length parametrised, this forces $\gamma_2'(s)=1$. Recall that γ is C^1; thus the points where $\gamma_2(s) = 0$ are isolated. Equation (5.4) holds for test functions supported away from the x^1-axis. By continuity the equation holds on isolated singularity points; C^2 regularity on all of the interval follows.

If u is not necessarily C^1, singularities may form on the points where γ intersects the line $x^2 = x^3 = 0$. We recall the following classical counter-example (see Figure 5.1). Let $I = [-1, 1]$ and for $s \in I$, consider $\gamma(s) = \frac{1}{\sqrt{2}}(s, s)$. Then F is a $\mathrm{W}^{2,p}$ surface for all $p < 2$ and in fact is given as the graph of the function $(r\cos\theta, r\sin\theta) \mapsto r$, smooth everywhere apart from the origin where it is Lipschitz but not C^1. Moreover the form $\omega = 2^{-1/2} d\alpha$ is globally defined. However, it fails to satisfy the second structural equation and satisfies instead the equation

$$\int_E \omega \wedge d\phi = \int_E *K\phi + 2\pi\phi(0).$$

The reader is invited to compare this situation to the main result of Chapter 4.

5.1.2 Surfaces given as graphs

Let us consider a C^1 embedded surface Σ in $\mathbf{R}^3$. By the implicit function theorem, one can locally parametrise it as the graph of a C^1 function over a open subset $D \subset \mathbf{R}^2$, say $f : D \to \mathbf{R}$, so that locally $\Sigma = \{(x, y, f(x, y)) \,\big|\, (x, y) \in D\}$.

Moreover, we assume that Σ is the image of some smooth manifold M under a C^1 isometric embedding $u : M \to \mathbf{R}^3$. Then $\Sigma = \{(x, y, f(x, y)) \in \mathbf{R}^3 \,\big|\, (x, y) \in D\} = \{(u^1(\xi), u^2(\xi), u^3(\xi)) \in \mathbf{R}^3 \,\big|\, \xi \in E\}$ for some open set $E \subset M$.

We first note the following well-known fact that under these hypothesis, if u is $C^1 \cap \mathrm{W}^{2,1}$, then so is f. By definition, as u is a C^1 embedding, u is a diffeomorphism onto its image. Let D be the diffeomorphic image of E under the map $\xi \mapsto (u^1(\xi), u^2(\xi))$. We denote this map by $\bar{u}$ and its inverse by $\bar{u}^{-1}$.

Next, we seek an equation relating the Gauß curvature and the function f.

We adopt the following short-hand notations:

$$\begin{aligned} V^2 &= 1 + f_x^2 + f_y^2, \\ X^2 &= 1 + f_x^2, \\ Y^2 &= 1 + f_y^2, \\ Z &= f_x f_y, \end{aligned}$$

so that we have the relation $X^2 Y^2 = V^2 + Z^2$.

Proposition 5.4. *Let (M, g) be a smooth Riemannian manifold, K its Gauß curvature. Let $E \subset M$, $u \in \mathrm{W}^{2,1} \cap C^1(E, \mathbf{R}^3)$ be a local isometric embedding, parametrised locally graphically by $f \in \mathrm{W}^{2,1} \cap C^1(D, \mathbf{R})$. Then the following equation holds in the sense of distributions.*

$$-d\left(\frac{1}{VY^2} f_x df_y\right) = *(\bar{u}^{-1})^* K,$$

that is, for $\phi \in C^1_c(D)$, we have

$$\int_D \frac{1}{VY^2} f_x df_y \wedge d\phi = \int_E *K\bar{u}^*\phi. \tag{5.6}$$

Proposition 5.4 follows directly from Lemmas 5.5 and 5.6. The proof consists of explicitly calculating a particular connection one-form on Σ that relates the embedding map f to the connection form (Lemma 5.5); then we exploit the Cartan framework of Chapter 4 to obtain an equation involving the Gauß curvature. Since coframes on $T^*\Sigma$ pull back to T^*E (Lemma 5.6) by the map u, we obtain the result of Proposition 5.4.

Lemma 5.5. *Let $f \in \mathrm{W}^{2,p}(D) \cap \mathrm{W}^{1,\infty}(D)$, where $D \subset \mathbf{R}^2$ and $p \geqslant 1$, and let $\Sigma \subset \mathbf{R}^3$ be the graph of f. Denote by (x, y) the coordinate system on D, and consider the metric h induced by the Euclidean metric of $\mathbf{R}^3$ on Σ. Then the connection form $\hat{\omega}$ associated to the orthonormal coframe*

$$\zeta^1 = \frac{dx}{|dx|_g}, \qquad \zeta^2 = *\zeta^1,$$

is given by

$$\hat{\omega} = -\frac{1}{(1 + f_y^2)\sqrt{1 + f_x^2 + f_y^2}} f_x df_y \in \mathrm{L}^p(D, T^*M).$$

Remark 5.1. *A similar calculation can be done for surfaces in $\mathbf{R}^n$. In the proofs of Theorems 5.1 and 5.2, we only use the case $n = 3$; thus we restrict our scope to the special case of surfaces in $\mathbf{R}^3$.*

Proof. The proof is a direct computation on the target manifold Σ. Letting h be the metric induced on D by $\mathbf{R}^3$, we have

$$\begin{aligned} h &= (1 + f_x^2)dx^2 + 2f_x f_y dxdy + (1 + f_y^2)dy^2 \\ &= X^2 dx^2 + 2Z dxdy + Y^2 dy^2. \end{aligned}$$

We want to obtain an expression for $\hat{\omega}$, which by definition is the unique 1-form such that

$$\begin{cases} d\zeta^1 = \hat{\omega} \wedge \zeta^2, \\ d\zeta^2 = -\hat{\omega} \wedge \zeta^1. \end{cases} \tag{5.7}$$

We denote by r, s the components of $\hat{\omega}$, so that $\hat{\omega} = r dx + s dy$. Inverting system (5.7), which is linear, we find the following expressions for r and s:

$$\begin{cases} r = V^{-1}(\zeta_1^1(\zeta_1^1)_y + \zeta_1^2(\zeta_1^2)_y - \zeta_1^1(\zeta_2^1)_x - \zeta_1^2(\zeta_2^2)_x), \\ s = V^{-1}(-\zeta_2^1(\zeta_2^1)_x - \zeta_2^2(\zeta_2^2)_x + \zeta_2^1(\zeta_1^1)_y + \zeta_2^2(\zeta_1^2)_y), \end{cases} \tag{5.8}$$

where we write $\zeta^i = \zeta_1^i dx + \zeta_2^i dy$, $i = 1, 2$. By orthogonality and Pythagoras' theorem, we have

$$\begin{aligned} X^2 &= |dx|_h^2 \\ &= \langle dx, \zeta^1 \rangle_h^2 + \langle dx, \zeta^2 \rangle_h^2 \\ &= (\zeta_1^1)^2 + (\zeta_2^2)^2. \end{aligned}$$

Hence,

$$\begin{aligned} \zeta_1^1(\zeta_1^1)_y + \zeta_1^2(\zeta_1^2)_y &= \frac{1}{2}\partial_y \left[(\zeta_1^1)^2 + (\zeta_2^2)^2\right] \\ &= \frac{1}{2}\partial_y(X^2) = f_x f_{xy}. \end{aligned}$$

Similarly, we find that

$$-\zeta_2^1(\zeta_2^1)_x - \zeta_2^2(\zeta_2^2)_x = -f_y f_{xy}.$$

Inserting both expressions into (5.8), we have

$$\begin{cases} r = V^{-1}(f_x f_{xy} - \zeta_1^1(\zeta_2^1)_x - \zeta_1^2(\zeta_2^2)_x), \\ s = V^{-1}(-f_y f_{xy} + \zeta_2^1(\zeta_1^1)_y + \zeta_2^2(\zeta_1^2)_y). \end{cases} \tag{5.9}$$

For the particular coframe considered in the statement of the lemma, one can calculate its components as

$$\begin{cases} \zeta^1 = \dfrac{V}{Y} dx \\ \zeta^2 = -\dfrac{Z}{Y} dx - Y dy. \end{cases}$$

By calculation we have

$$d\zeta^2 = -\left(\frac{Z_y}{Y} - \frac{Y_y Z}{Y^2}\right) dx \wedge dy - Y_x dy \wedge dx;$$

Noting that $\partial_x (Y)^2 = 2YY_x$, we have that

$$\zeta_2^2(\zeta_1^2)_y = (f_x f_{yy} + f_y f_{xy}) + Z\frac{f_y f_{yy}}{Y^2},$$

and similarly

$$\zeta_1^2(\zeta_2^2)_x = -Z\frac{f_y f_{xy}}{Y^2}.$$

Inserting in equation (5.9), and noting $\zeta_2^1 = 0$, we have

$$\begin{aligned} r &= \frac{1}{V}\left(f_x f_{xy} - Z\frac{f_y f_{xy}}{Y^2}\right) \\ &= -\frac{1}{V}\left(f_x f_{xy}\left(1 - \frac{f_y^2}{Y^2}\right)\right) = -\frac{f_x f_{xy}}{VY^2}. \end{aligned}$$

Similarly,

$$\begin{aligned} s &= \frac{1}{V}\left(-f_y f_{xy} + f_x f_{yy} + f_y f_{xy} - Z\frac{f_y f_{yy}}{Y^2}\right) \\ &= -\frac{1}{V}\left(f_x f_{yy}\left(1 - \frac{f_y^2}{Y^2}\right)\right) = -\frac{f_x f_{yy}}{VY^2}. \end{aligned}$$

Thus

$$\hat{\omega} = -\frac{1}{VY^2}\left(f_x f_{xy} dx + f_x f_{yy} dy\right) = -\frac{1}{VY^2} f_x df_y.$$

Q.E.D.

Lemma 5.6. *Let $u : (M, g) \to \mathbf{R}^3$ be a locally isometric embedding parametrised locally over $E \subset M$ by a function $f \in C^1(D) \cap \mathrm{W}^{2,p}(D)$ over a domain $D = \{(u^1(\xi), u^2(\xi)) \in \mathbf{R}^2 \,|\, \xi \in E\}$. Consider $\hat{\omega} \in \mathrm{L}^p(D, T^*D)$ to be the connection form constructed in Lemma 5.5. For $\psi \in C_c^\infty(D)$, we have*

$$\int_D \hat{\omega} \wedge d\psi = \int_E *K\bar{u}^*\psi.$$

Here $\bar{u}^*\psi = \psi \circ (u^1, u^2)$ is a $C_c^1(E)$ test function on E.

Proof. Pulling back the coframes ζ^1, ζ^2 from the proof of Lemma 5.5, we define $\theta^1 = \bar{u}^*\zeta^1$ and $\theta^2 = \bar{u}^*\zeta^2$. Since $u \in \mathrm{W}^{2,p} \cap C^1$ and $d\theta^k = d(\bar{u}^*\zeta^k) = \bar{u}^* d\zeta^k \in \mathrm{L}^p(E, T^*E)$, we readily see that $\theta^1, \theta^2 \in \mathrm{W}^{1,p}(E, T^*E) \cap C(E, T^*E)$. Moreover, $\bar{u}$ is an isometry between (D, h) (where h is as in the proof of Lemma 5.5) and (E, g), whence θ^1, θ^2 is an orthonormal coframe. We now calculate the connection form of θ^1, θ^2. Since by definition, $\hat{\omega}$ satisfies

$$\begin{cases} d\zeta^1 = \hat{\omega} \wedge \zeta^2, \\ d\zeta^2 = -\hat{\omega} \wedge \zeta^1, \end{cases}$$

we have that

$$d\theta^1 = d(\bar{u}^*\zeta^1) = \bar{u}^*(d\zeta^1) = \bar{u}^*(\hat{\omega} \wedge \zeta^2) = \bar{u}^*\hat{\omega} \wedge \theta^2,$$

and similarly for θ^2. By Lemma 4.2, we obtain for every $\phi \in C_c^\infty(E)$ the equation

$$\int_E u^*\hat{\omega} \wedge d\phi = \int_E *K\phi.$$

By density and continuity, this formula holds for $\phi \in C_c^1(E)$. Let $\psi \in C_c^\infty(D)$; then $\bar{u}^*\psi \in C_c^1(E)$ is a test function. By change of variable, one has

$$\int_D \hat{\omega} \wedge d\psi = \int_E \bar{u}^*(\hat{\omega} \wedge d\psi) = \int_E \bar{u}^*\hat{\omega} \wedge d(\bar{u}^*\psi) = \int_E *K(\bar{u}^*\psi).$$

The function $\bar{u}^*\psi$ has compact support inside E since u is a diffeomorphism into $\operatorname{supp}\psi$. Q.E.D.

Remark 5.2. *Selecting the coframe* $\zeta^1 = -*\zeta^2$, $\quad \zeta^2 = \frac{dy}{|dy|_g}$, *the same calculation as in Lemma 5.5 may be run, yielding another version of Cartan's equation:*

$$-d\left(\frac{1}{VX^2} f_y df_x\right) = *(\bar{u}^{-1})^*K, \tag{5.10}$$

equation (5.10) being understood in the sense of distributions. It follows from Lemma 5.6 that all such expressions for the connection form differ by a closed form.

5.2 Proof of Theorem 5.1

We consider a locally isometric embedding $u : E \subset M \to \mathbf{R}^3$ parametrised by a function $f : D \to \mathbf{R}$. For $\xi \in E \subset M$, we write without loss of generality $u(\xi) = (u^1(\xi), u^2(\xi), u^3(\xi)) = (\bar{u}(\xi), f(\bar{u}(\xi))$ and note that if u is C^1, the map $\bar{u}$ is a C^1 diffeomorphism between E and D.

We aim to derive the equation $\det \nabla^2 f = \bar{K}(1 + |\nabla f|^2)^2$, where $\bar{K} = (\bar{u}^{-1})^*K$ is the pull-back of the Gauß curvature on D. Our main result is Theorem 5.1, the statement of which is recalled hereafter.

In order to prove Theorem 5.1, we now aim to derive the equation $\det \nabla^2 f = \bar{K}(1 + |\nabla f|^2)^2$, where $\bar{K} = (\bar{u}^{-1})^*K$ is the pull-back of the Gauß curvature on D. To illustrate its proof, we first provide an argument for the smooth case, which also gives a proof for $p = 2$.

Recall that by Proposition 5.4, equation (5.6) holds on D. Letting as before $V^2 = 1 + f_x^2 + f_y^2$, $X^2 = 1 + f_x^2$ and $Y^2 = 1 + f_y^2$, we have

$$*(\bar{u}^{-1})^*K = -\frac{1}{2} d\left(\frac{1}{VY^2} f_x df_y - \frac{1}{VX^2} f_y df_x\right).$$

As $p = 2$, the product rule for $\mathrm{W}^{1,2}$ functions applies and we have

$$*(\bar{u}^{-1})^*K = \frac{1}{2}\left\{d\left(\frac{1}{VY^2}f_x\right)\wedge df_y - d\left(\frac{1}{VX^2}f_y\right)\wedge df_x\right\}.$$

We test the above equation with the test function $\phi = V^3\psi$. Noting the vanishing of all terms of the form

$$dY^2 \wedge df_y = 0 \tag{5.11}$$

as L^1 functions, we have the following calculation:

$$d\left(\frac{1}{VY^2}\right)\wedge df_y = -\frac{Y^2}{V^2Y^4}dV\wedge df_y = \frac{1}{2V^3Y^2}d(V^2)\wedge df_y = \frac{2f_x}{V^3Y^2}df_x\wedge df_y.$$

Hence we obtain

$$\begin{aligned}\int_D V^4K\psi &= \int_D \psi\frac{V^2}{X^2+Y^2}df_x\wedge df_y + \int_D \psi\frac{1}{X^2+Y^2}df_x\wedge df_y\\ &= \int_D \psi df_x\wedge df_y\\ &= \int_D \psi \det\nabla^2 f.\end{aligned}$$

Hence from now on we focus on the case $1 \leqslant p < 2$. The main argument is given in §5.2.2. We rely on the structure of the equation to obtain cancellations for the proof. The relevant estimates are proved in detail in §5.2.1.

5.2.1 Cancellation estimates

When $p < 2$, quadratic terms are no longer integrable. In particular, the cancellation (5.11) is unclear, as the product is not even an L^1 function. We deal with this issue by approximation. Our main observation is that anti-commutativity is still preserved in the limit under our regularity hypothesis (R). By inspection of the proof, one needs to understand whether $dv \wedge dv^{\varepsilon}$, where v^{ε} is a suitable regularisation of v, converge to zero in a suitable sense.

Lemma 5.8 addresses this question and it will be used in the proof of Theorem 5.1. Lemma 5.7 is a preliminary estimate of independent interest which is needed for the proof of Lemma 5.8.

Lemma 5.7. *Let* $0 < \alpha, \beta < 1$ *and* $v \in \mathrm{W}^{1,p}(D), h \in \mathrm{W}^{\beta,q}(D), w \in \mathrm{W}^{\alpha,r}(D)$ *on a domain* $D \subset \mathbf{R}^2$*, such that*

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 1, \qquad 1 \leqslant p, q, r \leqslant \infty.$$

In addition, let η be a compactly supported bump function, with $\operatorname{supp}\eta \subset B_1(0)$ *and unit mass* $\int \eta = 1$*, and define* $\eta_\varepsilon(x) = \varepsilon^{-2}\eta(x/\varepsilon)$*, for* $\varepsilon > 0$*. Then*

$$\left|\int_D \int_D (h(x) - h(y))(w(x) - w(y))\langle \nabla\eta_\varepsilon(x-y), \nabla v(x)\rangle dxdy\right| \\ \leqslant C\|h\|_{\mathrm{W}^{\beta,q}}\|w\|_{\mathrm{W}^{\alpha,r}}\|\nabla v\|_{\mathrm{L}^p}\varepsilon^{\alpha+\beta-1}.$$

Proof. We estimate the terms of the integral using fractional Sobolev norms. First we note that if $|x-y| > \varepsilon$ then the integrand vanishes. Therefore the support of the integrand is contained in a band of width ε around the diagonal $x = y$. We now view the integrand as a function on $D \times D$. Applying Hölder's inequality, we get

$$\begin{aligned}
&\left|\iint_{D\times D} (h(x) - h(y))(w(x) - w(y))\langle \nabla\eta_\varepsilon(x-y), \nabla v(x)\rangle dxdy\right| \\
&\qquad \leqslant \iint_{D\times D} |h(x) - h(y)||w(x) - w(y)||\langle \nabla\eta_\varepsilon(x-y), \nabla v(x)\rangle| dxdy \\
&\qquad \leqslant \Bigg\{ \left(\iint_{D\times D} |\langle \nabla\eta_\varepsilon(x-y), \nabla v(x)\rangle|^p dxdy\right)^{1/p} \\
&\qquad\qquad \varepsilon^{\frac{2+\beta q}{q}} \left(\iint_{D\times D} \frac{|h(x)-h(y)|^q}{|x-y|^{2+\beta q}} dxdy\right)^{1/q} \\
&\qquad\qquad \varepsilon^{\frac{2+\alpha r}{r}} \left(\iint_{D\times D} \frac{|w(x)-w(y)|^r}{|x-y|^{2+\alpha r}} dxdy\right)^{1/r} \Bigg\} \\
&\qquad \leqslant \varepsilon^{\gamma}\|h\|_{\beta,q}\|w\|_{\mathrm{W}^{\alpha,r}} \left(\iint_{D\times D} |\langle \nabla\eta_\varepsilon(x-y), \nabla v(x)\rangle|^p dxdy\right)^{1/p}
\end{aligned}$$

Here $\gamma = \frac{2+\beta q}{q} + \frac{2+\alpha r}{r}$. The final integral can be estimated as follows:

$$\begin{aligned}
\iint_{D\times D} |\langle \nabla\eta_\varepsilon(x-y), \nabla v(x)\rangle|^p dxdy &\leqslant \iint_{D\times D} |\nabla\eta_\varepsilon(x-y)|^p |\nabla v(x)|^p dxdy \\
&= \int_D |\nabla v(x)|^p \left(\int_D |\nabla\eta_\varepsilon(x-y)|^p dy\right) dx \\
&= \int_D |\nabla v(x)|^p \left(\int_{B_\varepsilon(x)} |\nabla\eta_\varepsilon(x-y)|^p dy\right) dx \\
&\leqslant C\varepsilon^{2-3p}\|\nabla v\|_{\mathrm{L}^p}^p.
\end{aligned}$$

Inserting back, we get

$$\left|\iint_{D\times D} (h(x) - h(y))(w(x) - w(y))\langle \nabla\eta_\varepsilon(x-y), \nabla v(x)\rangle dxdy\right| \\ \leqslant C\varepsilon^{\gamma+2/p-3}\|h\|_{\mathrm{W}^{\beta,q}}\|w\|_{\mathrm{W}^{\alpha,r}}\|\nabla v\|_{\mathrm{L}^p}.$$

Finally, we calculate $\gamma + 2/p - 3 = \frac{2}{q} + \frac{2}{r} + \frac{2}{p} + \beta + \alpha - 3 = \alpha + \beta - 1$. When $p = \infty$, we find straight away that

$$\left|\iint_{D\times D}(h(x)-h(y))(w(x)-w(y))\langle\nabla\eta_\varepsilon(x-y),\nabla v(x)\rangle dxdy\right|$$
$$\leqslant C\varepsilon^{\gamma-3}\|h\|_{\mathrm{W}^{\beta,q}}\|w\|_{\mathrm{W}^{\alpha,r}}\|\nabla v\|_{\mathrm{L}^\infty} = C\varepsilon^{\alpha+\beta-1}\|h\|_{\mathrm{W}^{\beta,q}}\|w\|_{\mathrm{W}^{\alpha,r}}\|\nabla v\|_{\mathrm{L}^\infty}.$$

Q.E.D.

Lemma 5.8. *Let $D \subset \mathbf{R}^2$ open, bounded, and consider two functions $h \in \mathrm{W}^{\beta,q}(D)$ and $v \in \mathrm{W}^{1,p} \cap \mathrm{W}^{\alpha,r}(D)$, where*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1, \qquad 1 \leqslant p,q,r \leqslant \infty$$
$$\alpha r > 2,$$
$$\beta q > 2.$$

Let $\eta \in C_c^\infty(B_1(0)), \int_{B_1}\eta = 1$, and for small enough $\varepsilon > 0$, consider the family of mollifiers $\eta_\varepsilon = \varepsilon^{-2}\eta(\varepsilon^{-1}x)$, and denote

$$v^\varepsilon(x) = \int_D v(y)\eta_\varepsilon(x-y)dy.$$

Then

$$\left|\int_D h dv \wedge dv^\varepsilon\right| \leqslant C\varepsilon^{\alpha+\beta-1}\|v\|_{\mathrm{W}^{\alpha,r}}\|h\|_{\mathrm{W}^{\beta,q}}\|\nabla v\|_{\mathrm{L}^p}.$$

Proof. Recall that v^ε is a smooth, compactly supported function on $\mathbf{R}^2$. Its derivative is given by

$$\nabla v^\varepsilon(x) = \int_D v(y)\nabla_x\eta_\varepsilon(x-y)dy,$$

We note that for $\varepsilon > 0$ sufficiently small such that $\mathbf{B}_\varepsilon(x) \subset D$,

$$\int_D \eta_\varepsilon(x-y)dy = \int_{B_\varepsilon(x)}\eta_\varepsilon(x-y)dy = \int_{\mathbf{R}^2}\eta,$$

which is a constant, so that

$$\nabla_x\int_D \eta_\varepsilon(x-y)dy = 0$$

and hence

$$\nabla v^\varepsilon(x) = -\int_D (v(x)-v(y))\,\nabla_x\eta_\varepsilon(x-y)dy.$$

Therefore,

$$\begin{aligned}\int_D h(x)\langle\nabla v^\varepsilon(x),\nabla^\perp v(x)\rangle dx &= \int_D\int_D h(x)(v(x)-v(y))\langle\nabla_y\eta_\varepsilon(x-y),\nabla_x^\perp v(x)\rangle dydx\\ &= \int_D\int_D (h(x)-h(y))(v(x)-v(y))\langle\nabla_x\eta_\varepsilon(x-y),\nabla_x^\perp v(x)\rangle dxdy\\ &\quad+\int_D\int_D h(y)v(x)\langle\nabla_x\eta_\varepsilon(x-y),\nabla_x^\perp v(x)\rangle dxdy\\ &\quad-\int_D\int_D h(y)v(y)\langle\nabla_x\eta_\varepsilon(x-y),\nabla_x^\perp v(x)\rangle dxdy\\ &= I_1+I_2+I_3.\end{aligned}$$

We claim that $I_2=0=I_3$. Indeed, $\operatorname{rot}\nabla=0$, and so by integration by part,

$$\int_D\langle\nabla_x\eta_\varepsilon(x-y),\nabla_x^\perp v(x)\rangle dx=0,$$

as the function η_ε is smooth. By Fubini's theorem,

$$\begin{aligned}I_3 &= -\int_D\int_D h(y)v(y)\langle\nabla_x\eta_\varepsilon(x-y),\nabla_x^\perp v(x)\rangle dxdy\\ &= -\int_D h(y)v(y)\left(\int_D\langle\nabla_x\eta_\varepsilon(x-y),\nabla_x^\perp v(x)\rangle dx\right)dy=0.\end{aligned}$$

For I_2, since $\operatorname{rot}\nabla=0$ and $v\in \mathrm{W}^{1,p}\cap\mathrm{L}^\infty$ we have

$$\int_D\langle\nabla_x\eta_\varepsilon(x-y),\nabla^\perp v^2(x)\rangle dx=0,$$

and thus

$$\begin{aligned}I_2 &= \int_D\int_D h(y)v(x)\langle\nabla_x\eta_\varepsilon(x-y),\nabla^\perp v(x)\rangle dxdy\\ &= \frac{1}{2}\int_D\int_D h(y)\langle\nabla_x\eta_\varepsilon(x-y),\nabla^\perp v^2(x)\rangle dxdy\\ &= \int_D h(y)\left(\int_D\langle\nabla_x\eta_\varepsilon(x-y),\nabla^\perp v^2(x)\rangle dx\right)dy=0.\end{aligned}$$

Finally, we can estimate I_1 as follows:

$$\begin{aligned}|I_1| &= \left|\int_D\int_D (h(x)-h(y))(v(x)-v(y))\langle\nabla\eta_\varepsilon(x-y),\nabla^\perp v(x)\rangle dxdy\right|\\ &\leqslant C\varepsilon^{\alpha+\beta-1}\|h\|_{\mathrm{W}^{\beta,q}}\|v\|_{\mathrm{W}^{\alpha,r}}\|\nabla^\perp v\|_{\mathrm{L}^p},\end{aligned}$$

by Lemma 5.7. Q.E.D.

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Remark 5.3. *The bound on I_1 in the last part of Lemma 5.8 is the only part of the proof that relies on a fractional Sobolev estimate to obtain the conclusion, whereas the vanishing of I_2 and I_3 only require $v \in \mathrm{W}^{1,1} \cap \mathrm{L}^\infty$ to show the identities*

$$\int_D \langle \nabla_x^\perp \eta_\varepsilon(x-y), \nabla v(x) \rangle dx = 0, \qquad \int_D \langle \nabla_x \eta_\varepsilon(x-y), \nabla^\perp v^2(x) \rangle dx = 0.$$

Some effort has been spent trying to weaken the regularity requirement of Lemma 5.8. So far this has failed.

5.2.2 Proof of Theorem 5.1

We come to the proof of Theorem 5.1 in the case $1 \leqslant p < 2$. Recall that under hypothesis (R) the function f is then C^1.

Our argument is divided in five steps. First we test the equations obtained from Proposition 5.4 with a suitable test function. Contrary to the case $p = 2$, $\phi = V^3\psi$ cannot be used as a test function as its derivative lacks the required integrability; we proceed by mollification. Step 2 identifies the relevant term for obtaining the distributional Hessian, while Step 3 and 4 show that various remainder terms do vanish in the limit. Lemma 5.8 is used to obtain the latter estimates. Step 5 proves the remaining claim of Theorem 5.1, *i.e.* that the Gauß equation holds pointwise almost everywhere, by means of a regularisation argument presented in Lemma 5.9.

Proof of Theorem 5.1. First note that if strict inequality holds in the first part of (R), we can select $\tilde{p} < p$ such that

$$\frac{2}{q} + \frac{1}{\tilde{p}} = 1.$$

Replacing p by $\tilde{p}$, we may assume that (R) is an equality.

Step 1: by Proposition 5.4 and Remark 5.2, for every $\phi \in C_c^\infty(D)$.

$$\int_D \frac{1}{VY^2} f_x df_y \wedge d\phi = \int_E *K\bar{u}^*\phi,$$
$$\int_D \frac{1}{VX^2} f_y df_x \wedge d\phi = \int_E *K\bar{u}^*\phi,$$

where $V^2 = 1 + |\nabla f|^2$, $Y^2 = 1 + f_y^2$, and $\bar{u} = (u^1, u^2)$. Summing and averaging both equations, we get

$$\frac{1}{2}\int_D \left(\frac{1}{VY^2} f_x df_y + \frac{1}{VX^2} f_y df_x \right) \wedge d\phi = \int_E *K\bar{u}^*\phi. \tag{5.14}$$

We proceed by selecting the test function $\phi = (V^\varepsilon)^3\psi$, where $\psi \in C_c^\infty(D)$, and $V^\varepsilon = (1 + |\nabla f^\varepsilon|^2)^{1/2}$, where $f^\varepsilon = f * \eta_\varepsilon$ is an approximation by convolution with a mollifier $\eta_\varepsilon(x) = \varepsilon^{-2}\eta(x/\varepsilon)$, where $\eta \in C_c^\infty(\mathbf{R}^2)$, $\eta = 0$ for $|x| \geqslant 1$ and $\int_{\{|x|\leqslant 1\}} \eta = 1$. Note that $V^\varepsilon \to V = (1+|\nabla f|^2)^{1/2}$ uniformly on compacts.

Inserting $\phi = (V^\varepsilon)^3\psi$ into (5.14), we get

$$\frac{1}{2}\int_D \left(\frac{1}{VY^2} f_x df_y + \frac{1}{VX^2} f_y df_x\right) \wedge d((V^\varepsilon)^3\psi) = \int_E *K\bar{u}^*((V^\varepsilon)^3\psi).$$

By uniform convergence of V^ε to V, we have

$$\lim_{\varepsilon\to 0}\int_E *K\bar{u}^*\left((V^\varepsilon)^3\psi\right) = \int_E *K\bar{u}^*\left((V)^3\psi\right) = \int_D ((\bar{u}^{-1})^*K)V^4\psi = \int_D \bar{K}V^4\psi.$$

To prove the theorem, it suffices to show

$$\lim_{\varepsilon\to 0}\int_D \left(\frac{1}{VY^2} f_x df_y + \frac{1}{VX^2} f_y df_x\right) \wedge d((V^\varepsilon)^3\psi) = \int_D \mathrm{Adj}(\nabla^2 f)\nabla f \cdot \nabla\psi.$$

Step 2: from now on, and for convenience of writing, we drop the ε superscript notation in favour of a tilde above the quantities $\tilde{X}, \tilde{Y}$ and $\tilde{V}$ that are built from f^ε rather than f.

We calculate

$$d\phi = d\left((V^\varepsilon)^3\psi\right) = \psi d(\widetilde{V}^3) + \widetilde{V}^3 d\psi = \frac{3}{2}\psi\widetilde{V}d(\widetilde{V}^2) + \widetilde{V}^3 d\psi.$$

This yields

$$\begin{aligned}\frac{1}{2}\int_D d\phi \wedge \left(\frac{f_y}{VX^2}df_x + \frac{f_x}{VY^2}df_y\right) = \\ &\frac{3}{4}\int_D \frac{\widetilde{V}}{V}\psi\left(d\widetilde{X}^2 + d\widetilde{Y}^2\right) \wedge \left(\frac{f_y}{X^2}df_x + \frac{f_x}{Y^2}df_y\right) \\ &+ \frac{1}{2}\int_D \widetilde{V}^3 d\psi \wedge \left(\frac{f_y}{VX^2}df_x + \frac{f_x}{VY^2}df_y\right). \qquad (5.15)\end{aligned}$$

The first integral in the RHS of equation (5.15) can be split as

$$\begin{aligned}\frac{3}{4}\int_D \frac{\widetilde{V}}{V}\psi\left(\frac{f_y}{X^2}d\widetilde{X}^2 \wedge df_x + \frac{f_x}{Y^2}d\widetilde{Y}^2 \wedge df_y\right) \\ + \frac{3}{4}\int_D \frac{\widetilde{V}}{V}\psi\left(\frac{f_y}{X^2}d\widetilde{Y}^2 \wedge df_x + \frac{f_x}{Y^2}d\widetilde{X}^2 \wedge df_y\right) =: I_1 + I_2.\end{aligned}$$

Noting the identity $\widetilde{V}^2 = \widetilde{X}^2 + \widetilde{Y}^2 - 1$, the second integral in the RHS of (5.15) is

$$\begin{aligned}
\frac{1}{2}\int_D \widetilde{V}^3 d\psi \wedge \left(\frac{f_y}{VX^2}df_x + \frac{f_x}{VY^2}df_y\right) &= \frac{1}{2}\int_D \frac{\widetilde{V}}{V} d\psi \wedge \widetilde{V}^2 \left(\frac{f_y}{VX^2}df_x + \frac{f_x}{VY^2}df_y\right) \\
&= \frac{1}{2}\int_D \frac{\widetilde{V}}{V} d\psi \wedge \left(\frac{\widetilde{X}^2}{X^2} f_y df_x + \frac{\widetilde{Y}^2}{Y^2} f_x df_y\right) \\
&\quad + \frac{1}{2}\int_D \frac{\widetilde{V}}{V} d\psi \wedge \left(\frac{\widetilde{f}_y^2 f_y df_x}{X^2} + \frac{\widetilde{f}_y^2 f_y df_x}{X^2}\right) \\
&=: I_3 + I_4.
\end{aligned}$$

We claim that

$$I_3 \to \frac{1}{2}\int_D d\psi \wedge (f_y df_x + f_x df_y) = \frac{1}{2}\int_D \mathrm{Adj}(\nabla^2 f)\nabla f \cdot \nabla\psi.$$

To see this, it is sufficient to look only at the first term of the integral I_3. We have

$$\frac{\widetilde{V}}{V}\frac{\widetilde{X}^2}{X^2} = 1 + \rho_\varepsilon \to 1,$$

uniformly as $\varepsilon \to 0$. So the error term ρ_ε is uniformly controlled in ε, we have

$$\int_D \frac{\widetilde{V}}{V}\frac{\widetilde{X}^2}{X^2} d\psi \wedge f_y df_x = \int_D d\psi \wedge f_y df_x + \int_D \rho_\varepsilon d\psi \wedge f_y df_x \to \int_D d\psi \wedge f_y df_x,$$

as by Hölder's inequality,

$$\left|\int_D \left(\frac{\widetilde{V}}{V}\frac{\widetilde{X}^2}{X^2} - 1\right) d\psi \wedge f_y df_x\right| \leqslant \left\|\frac{\widetilde{V}}{V}\frac{\widetilde{X}^2}{X^2} - 1\right\|_{\mathrm{L}^\infty(D)} \|d\psi \wedge f_y df_x\|_{\mathrm{L}^1} \to 0.$$

The rest of the proof consists in proving that the remaining terms in equation (5.15) all vanish as $\varepsilon \to 0$.

Step 3: we verify that as $\varepsilon \to 0$, $I_1 \to 0$. Indeed, $\widetilde{X}^2 = 1 + \widetilde{f}_x^2$, and so $d\widetilde{X}^2 = d\widetilde{f}_x^2$. Moreover, $\widetilde{V}/V \to 1$ uniformly. Thus we are left with

$$\int_D h\psi d\widetilde{f}_x^2 \wedge df_x^2,$$

where the function h is defined to be

$$h = \frac{3}{4}\frac{\widetilde{V}}{V}\frac{f_y}{X^2} \in C^{\alpha - 2/q}(D) \cap \mathrm{W}^{1,p}(D).$$

Recall that h depends on ε. We have

$$\left|\int_D h\psi d\widetilde{f}_x^2 \wedge df_x^2\right| \leqslant C\varepsilon^{2\alpha-1}\|h\|_{\mathrm{W}^{\alpha,q}}\|v\|_{\mathrm{W}^{\alpha,q}}\|\nabla v\|_{\mathrm{L}^p}.$$

by Lemma 5.8. This vanishes as $\varepsilon \to 0$ as $2\alpha > 1$.

Step 4: we claim that $I_2 + I_4 \to 0$. To see this, first notice that up to an error term of uniform size ε (which therefore vanishes by Lemma 5.8), I_4 is equal to

$$I_5 = \frac{1}{2}\int_D \frac{\widetilde{V}}{V} d\psi \wedge \left(\frac{\widetilde{f_y}^3}{\widetilde{X}^2} df_x + \frac{\widetilde{f_x}^3}{\widetilde{Y}^2} df_y \right).$$

Integrating by parts and noting that $d^2 = 0$, we have

$$\begin{aligned}
-\int_D \frac{\widetilde{V}}{V} d\psi \wedge \left(\frac{\widetilde{f_y}^3}{\widetilde{X}^2} df_x + \frac{\widetilde{f_x}^3}{\widetilde{Y}^2} df_x \right) &= \int_D \psi \frac{\widetilde{V}}{V} d\left(\frac{\widetilde{f_y}^3}{\widetilde{X}^2} \right) \wedge df_x \\
&\quad + \int_D \psi \frac{\widetilde{f_y}^3}{\widetilde{X}^2} d\left(\frac{\widetilde{V}}{V} \right) \wedge df_x \\
&\quad + \int_D \psi \frac{\widetilde{V}}{V} d\left(\frac{\widetilde{f_x}^3}{\widetilde{Y}^2} \right) \wedge df_y \\
&\quad + \int_D \psi \frac{\widetilde{f_x}^3}{\widetilde{Y}^2} d\left(\frac{\widetilde{V}}{V} \right) \wedge df_y \\
&= \frac{3}{2}\int_D \psi \frac{\widetilde{V}}{V} \frac{\widetilde{f_y}}{\widetilde{X}^2} d\widetilde{Y}^2 \wedge df_x + 2\int_D \psi \frac{\widetilde{V}}{V} \frac{\widetilde{f_y}^3}{\widetilde{X}^4} d\widetilde{X}^2 \wedge df_x \\
&\quad + \frac{3}{2}\int_D \psi \frac{\widetilde{V}}{V} \frac{\widetilde{f_x}}{\widetilde{Y}^2} d\widetilde{X}^2 \wedge df_y + 2\int_D \psi \frac{\widetilde{V}}{V} \frac{\widetilde{f_x}^3}{\widetilde{Y}^4} d\widetilde{Y}^2 \wedge df_y \\
&\quad + \int_D \psi d\left(\frac{\widetilde{V}}{V} \right) \wedge \left(\frac{\widetilde{f_x}^3}{\widetilde{Y}^2} df_y + \frac{\widetilde{f_y}^3}{\widetilde{X}^2} df_x \right) \\
&= J_1 + J_2 + J_3 + J_4 + J_5
\end{aligned}$$

As in Step 3, we now have that $J_i \to 0$ for $i = 2, 4, 5$, by Lemma 5.8. In addition, up to an error term $o(1)$ as $\varepsilon \to 0$, we have $J_1 + J_3 = I_2$, so that overall $I_2 + I_4 \to 0$.

Step 5: the final claim that the Gauß equation holds pointwise follows from a regularisation argument for the Jacobian determinant, specifically, Lemma 5.9.

Lemma 5.9. *Let* $\Omega \subset \mathbf{R}^2$ *be an open, bounded regular set,* $p \geqslant 1$*, and* $f \in \mathrm{W}^{2,p} \cap \mathrm{W}^{1,\infty}(\Omega)$*. If* $\operatorname{Det} \nabla^2 f \in \mathrm{L}^1(\Omega)$ *then* $\det \nabla^2 f = \operatorname{Det} \nabla^2 f$*.*

Applying Lemma 5.9 and the Morrey inequality to the first claim of Theorem 5.1, we prove the final claim of Step 5 of the proof of Theorem 5.1. Q.E.D.

Results such as Lemma 5.9 are known in the literature: see [47, 100, 125]. As the authors were unable to locate a proof of the precise statement of Lemma 5.9, the details of the proof of are presented in Appendix 5.A.

5.3 Convexity and proof of Theorem 5.2

We investigate the properties of the embedding if the underlying manifold has positive curvature. Here we show the analogue of the classical fact that an embedded surface with positive Gauß curvature is convex.

Let us sketch the proof of Theorem 5.2. Starting with an embedded surface of positive curvature in $\mathbf{R}^3$, and having obtained the Gauß equation in Theorem 5.1, we aim to use the positivity of the Gauß curvature to show convexity of the embedding map. A step of this proof is to verify that the immersed surface contains densely many regular points that are elliptic points, in the sense of Pogorelov [144] (see the precise definitions in §5.3.4. This occupies the core of this section.

Several ingredients enter the proof. First, we obtain an integral characterisation of the Gauß curvature:

THEOREM 5.10. *Let* (M, g) *a smooth Riemannian surface,* K *its Gauß curvature. We consider a locally isometric embedding* $u : E \subset M \to \mathbf{R}^3$ *parametrised by a function* $f : D \to \mathbf{R}$*. For* $\xi \in E \subset M$*, we write without loss of generality* $u(\xi) = (u^1(\xi), u^2(\xi), u^3(\xi)) = (\bar{u}(\xi), f(\bar{u}(\xi))$ *and we assume further that* f *satisfies condition* (R) *and* K *is non-negative. Then there holds*

$$\int_{\mathbf{R}^2} \deg(\nabla f, B, y)\psi(y)dy = \int_B \psi(\nabla f(x))\bar{K}(x)dx, \qquad \forall \psi \in C_c^1(\mathbf{R}^2 \setminus \nabla f(\partial B)),$$

We note that this theorem implies in particular that the degree of ∇f is integrable. The reader is invited to compare the result to [48] and [139], where some integrability results are obtained for the Brouwer degree. In the context of the proof of Theorem 5.2, Theorem 5.10 is used to show that regular points are elliptic.

Second, we show that if the Gauß curvature K is strictly positive, the immersion map f cannot be developable on a set of non-zero measure (see Lemma 5.20). This is a delicate point that requires an analysis of some properties of singular Sobolev mappings. Lemma 5.20 implies that the set of regular points is dense. The proof of Theorem 5.2 can then be completed using some ideas from [43, 144].

We recall that there are several notions of Hessian determinant (see Appendix 5.A). We denote by $\det \nabla^2 f$ the pointwise determinant; if $f \in \mathrm{W}^{2,p}(D)$ for $D \subset \mathbf{R}^2$, $\det \nabla^2 f \in \mathrm{L}^{p/2}$ is a measurable function but may not be integrable when $p < 2$. We denote by $\mathrm{Det}\, \nabla^2 f$ the distributional determinant, which is defined as a distribution as

$$\langle\!\langle\, \mathrm{Det}\, \nabla^2 f, \psi \,\rangle\!\rangle = \int_D \nabla\psi \, \mathrm{Adj}(\nabla^2 f)\nabla f.$$

Finally, if f is convex, we denote by $\mathcal{M}_f$ the Alexandrov measure of f (see Appendix 5.A.4 for the definition).

This section is organised as follows. First, the main regularisation estimates needed for the proof of Theorem 5.10 are proved in §5.3.1, and the proof itself is presented in §5.3.2. In §5.3.3, some properties of singular Sobolev mappings are studied to obtain Lemma 5.20. Finally, the proof of Theorem 5.2 is given in §5.3.4.

5.3.1 Regularisation estimates

First let us recall the following standard mollification estimates (see *e.g.* [56, 65]). Let D be a domain, $B \subset\subset D$, $\phi \in \mathrm{W}^{\alpha,p}(D)$, and $\phi^\varepsilon = \phi * \eta^\varepsilon$, where η^ε is a Friedrich mollifier. Then for $\varepsilon > 0$ sufficiently small, on B there holds:

$$\|\phi^\varepsilon\|_{\mathrm{L}^p} \leqslant \|\phi\|_{\mathrm{L}^p}, \tag{5.16}$$

$$\|\nabla\phi^\varepsilon\|_{\mathrm{L}^p} \leqslant C\varepsilon^{\alpha-1}[\phi]_{\mathrm{W}^{\alpha,p}}, \tag{5.17}$$

$$\|\phi - \phi^\varepsilon\|_{\mathrm{L}^p} \leqslant \varepsilon^{\alpha}[\phi]_{\mathrm{W}^{\alpha,p}}. \tag{5.18}$$

The next lemma is a regularisation result for Jacobians of mappings satisfying condition (R).

Lemma 5.11. *Let $\Phi \in \mathrm{W}^{1,p} \cap \mathrm{W}^{\alpha,q}(D)$ where p, q, α satisfy condition* (R). *We define as above $\Phi^\varepsilon = \Phi * \eta^\varepsilon$ on $D_\varepsilon \subset D$, and we consider $\psi \in C_c^\infty(B)$ for $B \subset D_\varepsilon$. Then*

$$\int_B \psi(\Phi^\varepsilon) d\Phi_1^\varepsilon \wedge d\Phi_2^\varepsilon \to \int_B \psi(\Phi) d\Phi_1 \wedge d\Phi_2.$$

Proof. Let $\xi = \psi \circ \Phi^\varepsilon$. Since ψ has compact supported, it is globally Lipschitz and therefore

$$|\nabla\xi| \leqslant [\psi]_{\mathrm{W}^{1,\infty}} |\nabla\Phi^\varepsilon|.$$

We apply estimate (5.23) to Φ and $\Psi = \Phi^\varepsilon$: this yields

$$\begin{aligned}
\left|\int_B (\det\nabla\Phi - \det\nabla\Psi)\xi\right| &\leqslant \int_B |\Phi - \Psi||\nabla\xi|(|\nabla\Phi| + |\nabla\Psi|) \\
&\leqslant C\|\Phi - \Phi^\varepsilon\|_{\mathrm{L}^q(B)} \|\nabla\Phi\|_{\mathrm{L}^p(B)} \|\nabla\xi\|_{\mathrm{L}^q(B)} \\
&\leqslant C\varepsilon^{\alpha}[\Phi]_{\mathrm{W}^{\alpha,q}} \|\nabla\Phi\|_{\mathrm{L}^p} \varepsilon^{\alpha-1}[\xi]_{\mathrm{W}^{\alpha,q}} \\
&\leqslant C\varepsilon^{2\alpha-1}[\Phi]^2_{\mathrm{W}^{\alpha,q}} \|\nabla\Phi\|_{\mathrm{L}^p}.
\end{aligned}$$

Thus as $2\alpha - 1 > 0$ and $\varepsilon \to 0$, the RHS converges to zero. Q.E.D.

We now study the degree of mappings satisfying condition (R). Recall (see *e.g.* [69]) that for a continuous map $\Phi \in C(D, \mathbf{R}^n)$, there exists a notion of topological degree on a bounded open set B such that $\bar{B} \subset D$, the Brouwer degree, which we denote by $\deg(\Phi, B, \cdot)$. If $\Phi \in C^1(D, \mathbf{R}^n)$, for $p \notin \Phi(\partial B) \cup \Phi(Z_\Phi)$,

$$\deg(\Phi, B, p) = \sum_{x \in \Phi^{-1}(p) \cap B} \operatorname{sgn}(\det \nabla\Phi(x)).$$

Here $Z_\Phi \subset D$ is the set of critical points of Φ. Recall that by Sard's theorem, $|Z_\Phi| = 0$ for a C^1 function.

If Φ is a C^1 function on $\bar{B}$, by [66, Lemma 3.2], the counting function $y \mapsto \mathfrak{n}(\Phi, B, y) = \mathcal{H}^0(B \cap \Phi^{-1}(y))$ is measurable. Since it is non-negative, it is integrable, and moreover the area formula [66, Theorem 3.8] holds. Now for any $y \notin Z_\Phi$,

$$\deg(\Phi, B, y) = \mathfrak{n}(\Phi, B_+, y) - \mathfrak{n}(\Phi, B_- \cup B_0, y).$$

Here we write $B_+ = B \cap \{\det \nabla\Phi > 0\}$, respectively $B_- = B \cap \{\det \nabla\Phi < 0\}$ and $B_0 = B \cap \{\det \nabla\Phi = 0\}$. As Z_Φ is a null-set by Sard's theorem, $y \mapsto \deg(\Phi, B, y)$ is measurable, and being the difference of two integrable functions, it is integrable. Moreover it satisfies the change of variable formula, i.e. for all $B \subset D$ such that $|\partial B| = 0$, there holds

$$\int_{\mathbf{R}^n} \deg(\Phi, B, y)\psi(y)dy = \int_B \psi(\Phi(x)) \det \nabla\Phi(x)dx, \qquad \forall \psi \in C^1_c(\mathbf{R}^n \setminus \Phi(\partial B)). \tag{5.19}$$

We now prove an analogue of (5.19) for functions within the regularity class (R).

Lemma 5.12. *Let $D \subset \mathbf{R}^2$ be a bounded domain, and $\Phi = \nabla f$ for $f \in \mathrm{W}^{2,p} \cap \mathrm{W}^{1+\alpha,q}(D)$, for p, q, α satisfying condition* (R), *and $\operatorname{Det} \nabla\Phi \in \mathrm{L}^1(D)$ then for all $B \Subset D$ such that ∂B is Lipschitz, and for all $\psi \in C^\infty_c(\mathbf{R}^2 \setminus \Phi(\partial B))$, the function $z \mapsto \psi(z)\deg(\Phi, B, z)$ is integrable. Moreover*

$$\int_{\mathbf{R}^2} \psi(y) \deg(\Phi, B, y)dy = \int_B \psi(\Phi(x)) \det \nabla\Phi(x)dx. \tag{5.20}$$

Remark 5.4. *The set B is bounded, and hence ∂B is a compact set. Condition* (R) *implies that the map Φ is continuous and hence the image $\Phi(\partial B)$ is compact. As a result, $\mathbf{R}^2 \setminus \Phi(\partial B)$ is open, and therefore the set of smooth functions with compact support in $\mathbf{R}^2 \setminus \Phi(\partial B)$ is non-empty.*

Recall that the statement $\operatorname{Det}\nabla\Phi \in \mathrm{L}^1$ means that there exists a function $\mathfrak{J} \in \mathrm{L}^1(D)$ such that for any test function $\psi \in C_c^\infty(D)$, we have

$$\langle\!\langle \operatorname{Det}\nabla\Phi, \psi \rangle\!\rangle = \int_D \mathfrak{J}\psi.$$

We note that for gradient maps $\Phi = \nabla f$ satisfying condition (R), we have $\operatorname{Det}\nabla\Phi = \det\nabla\Phi$ almost everywhere (by Lemma 5.24). Moreover, as $\mathrm{W}^{\alpha,q}(D) \hookrightarrow C(D)$ for $\alpha q > n = 2$, the degree of Φ is well-defined.

Proof of Lemma 5.12. **Step 1**: Fix a set $B \Subset D_\varepsilon \subset D$ and a test function $\psi \in C_c^\infty(\mathbf{R}^2\backslash\partial B)$. We consider mollifications of Φ by a standard mollifier. Let $\eta \in C_c^\infty(B_1)$ such that $\int_{B_1}\eta = 1$, and consider $\eta_\varepsilon(x) = \varepsilon^{-2}\eta(x/\varepsilon)$. Finally, we define, for all $x \in D_\varepsilon \subset D$ such that $B_\varepsilon(x) \subset D$,

$$\Phi^\varepsilon(x) = \int_{B_\varepsilon} \Phi(y)\eta_\varepsilon(x-y)dy \in C^\infty(D_\varepsilon).$$

Formula (5.20) holds for Φ^ε, for $\varepsilon > 0$. Furthermore, since Φ^ε converges uniformly on compacts to Φ, for $y \in \operatorname{supp}\psi \subset \mathbf{R}^2 \setminus \Phi(\partial B)$, and for $\varepsilon > 0$ sufficiently small,

$$\deg(\Phi^\varepsilon, B, y) = \deg(\Phi, B, y).$$

Hence the mapping $y \mapsto \psi(y)\deg(\Phi, B, y)$ is measurable, and for any test function ψ, we have

$$\int_{\mathbf{R}^2} \psi(y)\deg(\Phi^\varepsilon, B, y)dy = \int_{\mathbf{R}^2} \psi(y)\deg(\Phi, B, y)dy.$$

Remark 5.5. *The above also shows that* $\deg(\Phi, B, \cdot) \in \mathrm{L}^1_{loc}(\mathbf{R}^2 \setminus \Phi(\partial B))$. *Recall that for all* $y \notin \Phi(\overline{B})$, *we have* $\deg(\Phi, B, y) = 0$.

Resuming with the proof of Lemma 5.12, the next step is dedicated to the convergence of the RHS of (5.20).

Step 2: From now on, we write $\psi^\varepsilon = \psi(\Phi^\varepsilon(x)) \in C_c^\infty(D_\varepsilon)$, and we note that as $\operatorname{supp}\psi \Subset \mathbf{R}^2 \setminus \Phi(\partial B)$, the functions $\psi \circ \Phi$ and $\psi \circ \Phi^\varepsilon$ have compact support: the pre-image of a closed set by a continuous function is closed, and a bounded closed set in $\mathbf{R}^2$ is compact by the Heine–Borel property.

Recall that $\det\nabla\Phi = d\Phi_1 \wedge d\Phi_2 \in \mathrm{L}^1(D)$. First we claim that

$$\int_B (\psi^\varepsilon - \psi)d\Phi_1 \wedge d\Phi_2 \to 0.$$

This is clear since $d\Phi_1 \wedge d\Phi_2 = \det\nabla\Phi \in \mathrm{L}^1(B)$, by Theorem 5.1, the regularisation Lemma 5.24, and uniform convergence. Hence

$$\int_B \psi^\varepsilon d\Phi_1 \wedge d\Phi_2 \to \int_B \psi d\Phi_1 \wedge d\Phi_2,$$

Moreover, we recall Lemma 5.11, which holds since condition (R) implies the hypothesis of the Lemma. This shows

$$\int_B \psi^\varepsilon d\Phi_1^\varepsilon \wedge d\Phi_2^\varepsilon \to \int_B \psi d\Phi_1 \wedge d\Phi_2.$$

End of the proof: Returning to the RHS of (5.20), by definition we have

$$\begin{aligned}\int_B \psi(\Phi^\varepsilon(x))\,\mathrm{Det}\,\nabla\Phi^\varepsilon(x)dx &= \int_B \psi(\Phi^\varepsilon(x))\det\nabla\Phi^\varepsilon(x)dx\\ &= \int_B \psi(\Phi^\varepsilon(x))d\Phi_1^\varepsilon(x)\wedge d\Phi_2^\varepsilon(x)dx\\ &\to \int_B \psi(\Phi)d\Phi_1\wedge d\Phi_2 = \int_B \psi(\Phi)\det\nabla\Phi.\end{aligned}$$

Along with Step 1, this completes the proof. Q.E.D.

5.3.2 Proof of Theorem 5.10

We derive improvements of Lemma 5.12 and the proof of Theorem 5.10 under the additional hypothesis that the Jacobian of Φ be non-negative. The next lemma is clear:

Lemma 5.13. *Let Φ satisfy condition (R) and let $B \subset D$ such that $\mathrm{Det}\,\nabla\Phi \in \mathrm{L}^1(B)$ and is non-negative almost everywhere on B. Then $\deg(\Phi, B, \cdot) \geqslant 0$ and $\deg(\Phi, B, \cdot) \in \mathrm{L}^1(\mathbf{R}^2 \setminus \Phi(\partial B))$.*

The non-negativity of the degree follows from Lemma 5.12 by testing against non-negative test functions. One may then apply the monotone convergence theorem and Lemma 5.12 to conclude the L^1 integrability. This lemma contrasts with Remark 5.5 in that we are able to obtain global integrability.

Remark 5.6. *Applying Lemma 5.13 (together with Theorem 5.1) to the mapping $\Phi = \nabla f$ parametrising an embedded surface, this also proves that on a surface with non-negative Gauß curvature, a regular point is elliptic in the sense of Pogorelov (see Lemma 5.21).*

Lemma 5.14. *Let Φ satisfy condition (R) and let $B \subset D$ such that $\mathrm{Det}\,\nabla\Phi \in \mathrm{L}^1(B)$ and is non-negative almost everywhere on B. Then for all $\psi \in \mathrm{L}^\infty(\mathbf{R}^2)$ there holds*

$$\int_{\mathbf{R}^2} \psi(y)\deg(\Phi, B, y)dy = \int_{B\setminus\Phi^{-1}\Phi(\partial B)} \psi(\Phi(x))\det\nabla\Phi(x)dx.$$

Proof. ∂B is a compact set, so that by continuity of Φ, $\Phi(\partial B)$ is also compact. Consider the boundary $\Phi(\partial B) =: C \subset \mathbf{R}^2$. We define the open sets $C_\delta = \{x \in \mathbf{R}^2 \,\big|\, d(x, C) < \delta\}$ for $\delta > 0$ sufficiently small. Clearly $\cap_{\delta>0} C_\delta = C$, and $\mathbf{R}^2 \setminus C_\delta$ is closed. Note additionally that $\overline{B} = B \cup \partial B$ is also a compact set, so that we may assume it is contained in a large enough ball $A \Subset \mathbf{R}^2$.

We now take $\psi^\delta = \chi_{\Phi(B)}(1 - \chi_{C_\delta})$, which converges pointwise to $\chi_{\Phi(B)\setminus\Phi(\partial B)}$ and $\operatorname{supp}\psi^\delta \Subset \mathbf{R}^2 \setminus C_\delta$. By Lemma 5.12, density of step functions in L^∞, linearity, Lemma 5.13 and the dominated convergence theorem, and noting that

$$\int_B \chi_{\Phi(B)\setminus\Phi(\partial B)}(x)\psi(\Phi(x)) \det \nabla\Phi(x) dx = \int_{B\setminus\Phi^{-1}\Phi(\partial B)} \psi(\Phi(x)) \det \nabla\Phi(x) dx,$$

we obtain the conclusion of Lemma 5.14. Q.E.D.

Proof of Theorem 5.10. We apply Lemma 5.14 to the map $\Phi = \nabla f$ where f is the graphical parametrisation of the embedding considered. Noting that by Theorem 5.1,

$$\det \nabla\Phi = \det \nabla^2 f = K \circ (u^1, u^2)^{-1} \geqslant 0$$

almost everywhere, the result follows. Q.E.D.

5.3.3 Mappings with singular gradients

Before proving Theorem 5.2, we study the local structure of a map $\Phi \in \mathrm{W}^{1,p}(B, \mathbf{R}^2) \cap C^0(B, \mathbf{R}^2)$ such that $B \setminus \Phi^{-1}\Phi(\partial B)$ is a null-set. To do so, we assume (as we may, in view of our later application) that Φ has a gradient structure $\Phi = \nabla f$. We shall prove that $\det \nabla\Phi = 0$ on a set of full Lebesgue measure.

Being a null-set, $B \setminus \Phi^{-1}\Phi(\partial B)$ has empty interior $\mathrm{int}\Phi(B) = \varnothing$. The main result of [109, 110] characterises the local structure of such a gradient map $\Phi = \nabla f$.

Before stating the result, let us fix some terminology for this section. A line $L \subset \mathbf{R}^2$ is a one-dimensional affine subspace of $\mathbf{R}^2$. A segment is a bounded connected subset of a line. A pair of non-parallel lines determine four sectors of $\mathbf{R}^2$, which are each identified by the pair of half-lines bounding them.

THEOREM 5.15 (Korobkov). *Let $\Phi = \nabla f$, $f \in C^1(D)$ and $B \subset D$ such that $|\Phi(B)| = 0$. Then for any small open convex subset $V \subset B$ and any $x \in V$, there exists a line L such that $\Phi|_{L\cap V}$ is a constant.*

We shall refer to maps satisfying the conclusion of Theorem 5.15 as being *developable*. A classical theorem of Hartman and Nirenberg [88] shows that immersed C^2

surfaces with vanishing curvatures are developable; in particular, a map parametrising a developable surface is developable according to our definition. Equivalently, the pre-image of a point $x \in \Phi(B)$ contains at least a segment in B.

As in Theorem 5.15, let L be a line, and let $L' = L \cap \overline{B}$ be the maximal segment in B. We now show the following alternative:

Lemma 5.16. *Let Φ be developable, in the sense of Theorem 5.15. Then either Φ is constant on all of L' (which stretches to ∂B), or there exists an open set $V \subset B$ such that $\Phi|_V$ is constant.*

In other words, either L$'$ is a segment of constancy, or it is not but it is contained in a open set of constancy.

Figure 5.4 presents a typical behaviour of Φ on B. For related works, see [106, 116, 141].

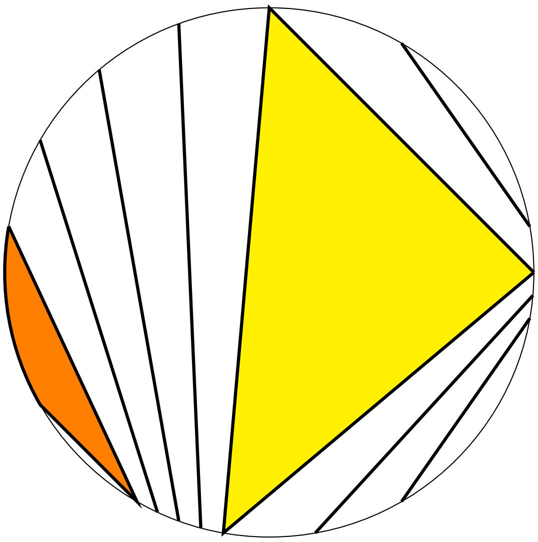

Figure 5.2: An example of typical behaviour of Φ on the ball B, according to Lemma 5.16. The map Φ is constant along the black lines, and across the coloured areas.

Proof of Lemma 5.16. Assume that Φ is not constant on all of L', and let $L'' \subset L'$ be the maximal (connected) segment such that $\Phi|_{L''}$ is constant. As Φ is continuous, L'' is closed. Let y be one of its ends. Then there exists a line L_y and an open convex set $V_y \ni y$ such that $\Phi|_{L_y \cap V_y}$ is a constant.

By maximality and connectedness, $L \neq L_y$, as otherwise $L_y \cap V_y$ would be an extension of $L'' \subset L$. But now by constancy $\Phi|_{L_y \cap V_y} = \Phi|_{L''} = \Phi(y)$.

The line L divides $\mathbf{R}^2$ into two halves, denoted by $\mathbf{R}^2_+$ and $\mathbf{R}^2_-$. It is clear from $L \neq L_y$ that the triangle $\Delta \subset V_y$ formed of the segments $L'' \cap V_y$ and $L_y \cap V_y \cap \mathbf{R}^2_+$

is non-degenerate. Thus, for any point $z \in \Delta$, any line $L_z \ni z$ must intersect either $L'' \cap V_y$ or $L_y \cap V_y \cap \mathbf{R}^2_+$. In any case, by constancy along lines, $\Phi(z) = \Phi(y)$ and so $\Phi|_\Delta = \Phi(y)$ is constant. Thus if L'' does not stretch to the boundary ∂B, then Δ is a non-empty set such that $\Phi|_\Delta$ is constant. Q.E.D.

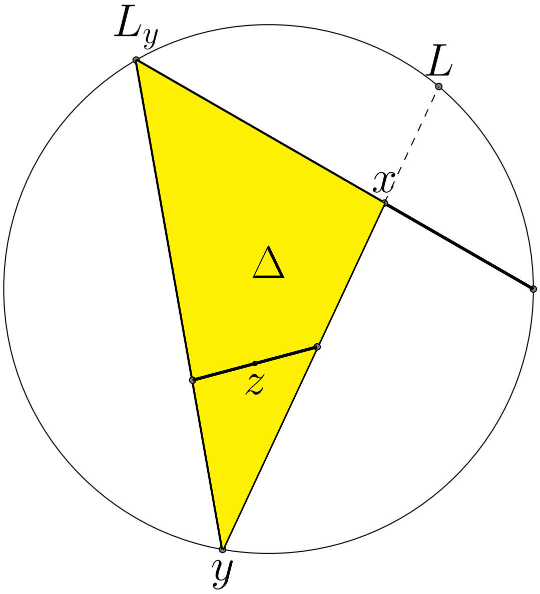

Figure 5.3: Sketch of the proof of Lemma 5.16. Any line L_z containing $z \in \Delta$ must intersect either L_y or L, and hence Φ is constant on Δ.

If L'' stretches to ∂B, then $\Phi|_{L''}$ takes the value it has on the boundary $\partial B \cap L' = \partial B \cap L''$. By connectedness and convexity, either B can be swept by parallel lines, or it contains two segments of constancy that are not parallel but do not intersect in B. Let $\Sigma \subset \mathbf{R}^2$ be the sector spanned by the two half-lines to which belong these two segments.

Lemma 5.17. *Let $\Phi \in \mathrm{W}^{1,p}(B, \mathbf{R}^2)$, and let Σ be a sector of $\mathbf{R}^2$ such that $\Phi|_{\Sigma \cap B}$ is developable (in the sense of Theorem 5.15), but no segment of constancy intersect in $\Sigma \cap B$. Then there exists a $r > 0$ such that $Q =]-r, r[^2 \subset \Sigma \cap B$, and two functions $\phi \in \mathrm{W}^{1,p}(]-4r, 4r[; \mathbf{R}^2)$ and $\gamma(x, y) \in \mathrm{W}^{1,\infty}(Q)$ such that $\Phi|_Q = \phi \circ \gamma$.*

Proof. **Step 1:** up to a rotation of the axis, we may assume that $\Phi|_{\{x=0\}} \in \mathrm{W}^{1,p}(B \cap \Sigma \cap \{x = 0\})$. Let $\phi = \Phi_{\{x=0\}}$.

Up to another rotation, we may also assume that no line of constancy is vertical, i.e. parallel to the $\{x = 0\}$ axis. Therefore each such line of constancy admits a parametrisation as $y = mx + b$. As by hypothesis any point in Σ belongs to at most one line of constancy, the point $(0,b)$ determines the line, and hence the parameter $m = m(b)$.

Step 2: we claim that m is a Lipschitz function.

For a fixed point b on $\{x = 0\} \cap \Sigma$, $m(b) + b$ is the y-coordinate of the intersection of the line $L = \{y = m(b)x + b\}$ and the line $\{x = 1\}$.

Now for a pair of such b, b' and the lines $L' = \{y = m(b')x + b'\}$, if $L \cap L' \neq \varnothing$ then we must have by hypothesis $L \cap L' \cap B = \varnothing$. Let $p \in L \cap L'$. If $p \in \mathbf{R}^2_+ = \{x > 0\}$, it is clear that $|m(b) - m(b')| \leqslant |b - b'|$. If $p \in \mathbf{R}^2_-$, there exists a constant $C > 0$ depending on Σ such that

$$|m(b) - m(b')| \leqslant C|b - b'|,$$

If $L \cap L' = \varnothing$ the two lines are parallel and therefore

$$|m(b) - m(b')| = |b - b'|,$$

hence proving the claim.

Step 3: there exists a Lipschitz function $\gamma(x,y)$ such that $\Phi = \phi \circ \gamma$.

Fix x sufficiently small. We first note that the map

$$b \mapsto m(b)x + b,$$

is injective. This is obvious provided $m(b)x + b \in \Sigma \cap B$, as lines of constancy may not intersect in $\Sigma \cap B$. As by Step 2, m is Lipschitz, the image of this map is closed and connected. Thus we may construct an inverse of this map on its image. Let $\gamma_x(y)$ be this map. we have

$$(x,y) = (x, m(\gamma_x(y))x + \gamma_x(y)).$$

We now claim that $\gamma(x,y) = \gamma_x(y)$ is Lipschitz. For a fixed x, $\gamma_x(\cdot)$ is Lipschitz by Step 2. Now notice that for a pair of points (x,y) and (z,y) for a fixed y, γ satisfies

$$m(\gamma_x(y))x + \gamma_x(y) = m(\gamma_z(y))z + \gamma_z(y).$$

This construction may be extended on all of $\Sigma \cap B$. Q.E.D.

Recall that a function $u : E \to F$ is said to satisfy the N^{-1} property if for every null set $Z \subset F$, its pre-image $u^{-1}(Z) \subset E$ is also a null-set.

Lemma 5.18. *Under the same notations and hypotheses as in Lemma 5.17, γ satisfies the N^{-1} property.*

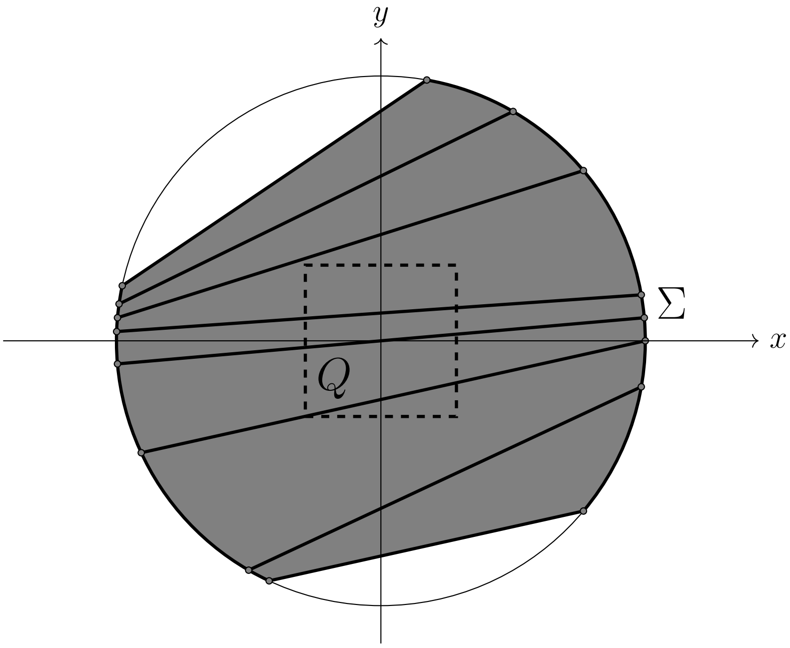

Figure 5.4: The construction of the proofs of Lemma 5.17 and Lemma 5.18.

Proof. Let us consider $[a, b] \subset Q$, and let L and L' be the lines of constancy through $(0, a)$ and $(0, b)$ respectively. We denote by Σ' the sector defined by these two lines. Then $\gamma^{-1}(]a, b[) \subset \Sigma' \cap B$, and hence

$$|\gamma^{-1}(]a, b[)| \leqslant 2|b - a|.$$

Let $Z \subset \mathbf{R}$ be a null-set, and let us consider a cover of Z by disjoint intervals

$$Z \subset \cup_{\alpha \in A} I_\alpha,$$

and such that $\sum_{\alpha \in A} |I_\alpha| < \varepsilon$ for arbitrary $\varepsilon > 0$. Then

$$|\gamma^{-1}(\cup_{\alpha \in A} I_\alpha)| \leqslant \sum_{\alpha \in A} |u^{-1}(I_\alpha)| \leqslant \sum_{\alpha \in A} 2|I_\alpha| \leqslant 2\varepsilon.$$

Therefore $\gamma^{-1}(Z)$ is a nullset.

Q.E.D.

Lemma 5.19. *Assume that* $\Phi \in \mathrm{W}^{1,p}(B, \mathbf{R}^2)$ *is developable in* B *(in the sense of Theorem 5.15). Then* $\det \nabla \Phi = 0$ *almost everywhere.*

Proof. The only case we need to consider is the case where the function is not piecewise constant. Let $U \subset B$ an open set where the function is non-constant. By Lemma 5.16, there is a sector Σ such that $U \subset \Sigma \cap B$. By Lemma 5.17, there exists two functions $\phi \in \mathrm{W}^{1,p}$ and $\gamma \in \mathrm{W}^{1,\infty}$ such that $\Phi|_\Sigma = \phi \circ \gamma$. We now claim that for almost every point in B, there holds

$$\nabla \Phi = \phi'(\gamma) \otimes \nabla \gamma.$$

As an immediate corollary, $\det \nabla \Phi = 0$ pointwise outside of a set of measure 0.

Let us now show the claim. As Φ is $\mathrm{W}^{1,p}$, on almost every line in B it is classically differentiable. We wish to calculate $\nabla \Phi$ in terms of ϕ and γ. Fix a coordinate line, without loss of generality $L_{y_0} = \{(x, y_0) \in \mathbf{R}^2\}$. For almost every such y_0 fixed, $\Phi|_{L_{y_0}} \in \mathrm{W}^{1,p}(L_{y_0}, \mathbf{R}^2)$ (see Appendix A.1).

Let Z be such that ϕ is classically differentiable outside of Z; Z is a nullset and a subset of $\mathbf{R}$. Similarly, let $Z' \subset L_{y_0}$ be such that γ is classically differentiable on $L_{y_0} \setminus Z'$. Then the composition $\phi \circ \gamma$ is classically differentiable on $C := L_{y_0} \setminus (Z' \cup \gamma^{-1}(Z))$ for almost every y_0. The chain rule holds on C; noting that $\gamma^{-1}(Z)$ is a null set by Lemma 5.18, we conclude that for almost every (x, y_0),

$$\partial_x \Phi(x, y_0) = \partial_x \gamma(x, y_0) \phi'(\gamma(x, y_0)).$$

The same reasonning applied to the y-derivative proves the claim. Q.E.D.

5.3.4 Proof of Theorem 5.2

We recall the setting, which is assumed in the statements of Lemmas 5.20, 5.21 and 5.23. We consider a locally isometric embedding $u : E \subset M \to \mathbf{R}^3$ parametrised by a function $f : D \to \mathbf{R}$. For $\xi \in E \subset M$, we write without loss of generality $u(\xi) = (u^1(\xi), u^2(\xi), u^3(\xi)) = (\bar{u}(\xi), f(\bar{u}(\xi))$ and note that if u is C^1, the map $\bar{u}$ is a C^1 diffeomorphism between E and D. The function f is assumed to be in the spaces $\mathrm{W}^{2,p} \cap \mathrm{W}^{1+\alpha,q}$, where p, α, q satisfy condition (R).

The proof of Theorem 5.2 requires to show that the immersed surface is convex, or equivalently that (up to a change of coordinates), the function f is a convex function on its domain. We argue using the theory of bounded extrinsic curvature due to Pogorelov [144], which shows that on neighborhoods of well-behaved points of positive curvature, the surface is indeed convex. Lemma 5.20 first shows that such points are dense, which is required in order to make the surface globally convex.

A point $x \in D$ is said to be a regular point for $\Phi = \nabla f$ if $\Phi^{-1}(\Phi(\{x\}))$ is a finite set. Note that our definition diverges slightly from [144], but up to restricting on a smaller subset, a regular point in the above sense in also regular in the sense of [144].

Lemma 5.20. *For D bounded and Φ as above, the set of regular points of Φ in D is dense.*

The proof of the lemma hinges on the fact that the Gauß map of a surface with bounded extrinsic curvature is well-behaved. Recall that for a C^1 embedding $u : M \to \mathbf{R}^3$, the (vector-valued) Gauß map is defined as

$$\nu = \frac{\partial_1 u \times \partial_2 u}{|\partial_1 u \times \partial_2 u|}.$$

If u is given as the graph of a function $f \in C^1(D)$, then we obtain

$$\nu = \frac{1}{1 + |\nabla f|^2} \begin{pmatrix} -\partial_x f \\ -\partial_y f \\ 1 \end{pmatrix},$$

so that $|\nu| = 1$. Using the notation $\nabla f = \Phi$, we have $\nu = \frac{1}{1+|\Phi|^2} \begin{pmatrix} -\Phi \\ 1 \end{pmatrix}$, so that $\nu(x) = \nu(y) \Leftrightarrow \Phi(x) = \Phi(y)$ for any $x, y \in D$.

Finally, we recall that a surface is said to be of bounded extrinsic curvature if its Gauß map is a **BV** function with finite total mass. A simple fact arising from our regularity hypothesis is that as D is bounded and $\Phi \in \mathrm{W}^{1,1}(D)$, the mapping ν has finite total mass, and so the surface has bounded extrinsic curvature in the sense of Pogorelov.

Proof of Lemma 5.20. By [144], Theorem 3 p. 590, $\int_{\mathbf{S}^2} \mathfrak{n}(\nu, D, a) da < \infty$ for a surface of bounded extrinsic curvature. Here the notation $\mathfrak{n}(\nu, D, a)$ refers to the Banach indicatrix function on D, i.e. the number of pre-images of a for ν in D.

Let $S \subset \mathbf{S}^2$ be the set of points a such that $\nu^{-1}(\{a\}) \subset D$ is infinite. It follows that S is a null set. Furthermore S is in one-to-one correspondence with the set $\tilde{S} = \{a \in \mathbf{R}^2 \,|\, \mathfrak{n}(\Phi, D, a) = \infty\}$. As $\Phi(D)$ is bounded, there is a bi-Lipschitz mapping between $\Phi(D)$ and $\nu(D)$ mapping $\tilde{S}$ onto S. Thus $\tilde{S}$ is a null-set too.

By definition, the set of non-regular points for Φ is given by $\Phi^{-1}(\tilde{S})$. We have $|\nu(\nu^{-1}(S))| = 0$, and so as above $|\Phi(\Phi^{-1}(\tilde{S}))| = 0$.

We now claim that $\Phi^{-1}(\mathbf{R}^2 \setminus \tilde{S})$ is dense in D. If not, there is an interior point contained in a ball $B_{2r} \subset D$ such that $\Phi(B_{2r}) \subset \tilde{S}$. Moreover, the set $\Phi(\overline{B_r}) \subset \Phi(B_{2r})$ being in $\tilde{S}$, it is a null-set: $|\Phi(\overline{B_r})| = 0$. Being a null-set, it cannot have a non-empty interior. But then Φ is developable in the sense of §5.3.3, and hence by Lemma 5.19, we must have $\det \nabla\Phi = 0$ almost everywhere on B, a contradiction with Theorem 5.1, as $K > 0$ by assumption. Q.E.D.

Once density of the regular points in D is established, we may follow the arguments of Pogorelov [144] and Conti-De Lellis-Székelyhidi [43] to establish convexity on the domain.

Lemma 5.21. *Up to a global affine change of coordinates, f is locally convex on D, in the sense that in any convex subset $C \subset D$, $f|_C$ is convex.*

For the convenience of the reader, we sketch the arguments proving Lemma 5.21.

Sketch of proof. By Lemma 5.13 and the hypothesis that $\inf_E K > 0$ (see Remark 5.6), the surface is elliptic in the sense of [144, Chapter IX, §3], i.e., every regular point has degree greater or equal to 1; moreover by [144, Chapter IX, §5, Lemma 2], every elliptic point has a neighborhood which is a convex surface, up to an affine change of coordinates.

As such regular, elliptic points are dense by Lemma 5.20, it follows from Step 2 and 3 of [43, Appendix A] (see also [144, 154]) that the whole neighborhood $u(E) \subset u(M)$ considered is in fact convex, up to a further change of coordinates. In other words, the graph $(x, y, f(x, y))$ is convex up to a local affine change of coordinates. Q.E.D.

Recall that for a convex function, the notion of Alexandrov measure gives a meaning to the Hessian determinant. More precisely, for $D \subset \mathbf{R}^N$ a domain and $f \in C(D)$ a convex function, we recall that its Alexandrov measure $\mathcal{M}_f$ is defined as follows: for all $E \subset D$,

$$\mathcal{M}_f(E) := \left| \bigcup_{x \in E} \partial f(x) \right|$$

Here $\partial f(x)$ refers to the sub-differential of f at the point x, namely the set

$$\partial f(x) = \left\{ p \in \mathbf{R}^N \,\middle|\, \forall y \in D, \langle p, y - x \rangle \leqslant u(y) - u(x) \right\}.$$

The Alexandrov measure is a non-negative Radon measure. When both Alexandrov measure and distributional Hessian are defined, we have the following relation between $\mathcal{M}_f$ and $\operatorname{Det} \nabla f$.

Lemma 5.22. *Let $D \subset \mathbf{R}^N$ be an open domain and $f \in \mathrm{W}^{1,\infty} \cap \mathrm{W}^{2,N-1}(D)$. Assume that f is convex. Then $\operatorname{Det} \nabla^2 f$ is a non-negative Radon measure and moreover $\mathcal{M}_f = \operatorname{Det} \nabla^2 f$.*

The proof of the lemma follows at once from two facts: first, mollifications of a convex function are convex; and second, distributional determinant and Alexandrov measure coincide for smooth, convex functions, so that the equality is preserved when passing to limits.

Lemma 5.23 relates the Alexandrov measure of f to the Gauß curvature.

Lemma 5.23. *Let* $(M,g), u, f, K, \bar{K}$ *be as above, and assume that* f *is convex. Then for* $E \subset D$ *Borel, there holds*

$$\mathcal{M}_f(E) = \int_E \bar{K} V^4,$$

where $V^2 = 1 + |\nabla f|^2$.

Proof. By Theorem 5.1, we have $\operatorname{Det} \nabla^2 f = \bar{K} V^4$, which is non-negative. The result now follows from Lemma 5.26, which states that $\mathcal{M}_f = \operatorname{Det} \nabla^2 f$ in the sense of measures. Q.E.D.

Proof of Theorem 5.2. By Lemma 5.21, up to restricting ourselves to a smaller neighborhood and a change of coordinates, we can assume that f is convex without loss of generality. By Lemma 5.23, the Gauß equation holds in the Alexandrov sense.

Strict convexity follows from the fact that the RHS is uniformly bounded from below (see *e.g.* [67]). Interior regularity for the Monge–Ampère equation implies that, as $\bar{K} V^3 \in C^\alpha$, f must be $C^{2,\alpha}$. But then u is $C^{2,\alpha}$ by the Myers–Steenrod theorem (see [164]). So the RHS is $C^{1,\alpha}$; bootstrapping shows smoothness. Q.E.D.

5.A Appendix: On Hessian determinants

This appendix collects facts about the Jacobian and Hessian determinants, $\det \nabla\Phi$ and $\det \nabla^2 f$ respectively, and their distributional pendants, denoted by $\operatorname{Det}\nabla\Phi$ and $\operatorname{Det}\nabla^2 f$ respectively.

5.A.1 The distributional Jacobian determinant

Let $A = (v_1, \dots, v_N) \in \mathbf{R}^{N\times N}$ be a square matrix, where $v_i \in \mathbf{R}^N$ for $1 \leqslant i \leqslant N$. Identifying—as we may— alternate N-vectors with scalars, we may write

$$\det A = v_1 \wedge \cdots \wedge v_N,$$
$$(\det A)\mathbb{I} = \operatorname{Adj}(A)A.$$

Let $\Omega \subset \mathbf{R}^N$ an open domain. For a smooth vector field $\Phi \in C^\infty(\Omega)$, the Jacobian matrix $A = \nabla\Phi$ is an N by N matrix, with $v_i = \Phi^i$. It is well-known (see *e.g.* [124]) that $\operatorname{div}(\operatorname{Adj}(\nabla\Phi)) = 0$, from which one may write

$$\det\nabla\Phi = \frac{1}{N}\operatorname{Adj}\nabla\Phi\cdot\nabla\Phi = \frac{1}{N}\operatorname{div}(\operatorname{Adj}\nabla\Phi\cdot\Phi).$$

It is an observation of J. Ball [2] that this defines a distributional Jacobian, denoted by $\operatorname{Det}\nabla\Phi$:

$$\langle\!\langle \operatorname{Det}\nabla\Phi, \phi \rangle\!\rangle = \frac{1}{N}\int_\Omega {}^t(\Phi)\operatorname{Adj}(\nabla\Phi)\nabla\phi \qquad \text{for } \phi \in C_c^\infty(\Omega). \tag{5.21}$$

Here $\langle\!\langle \cdot,\cdot \rangle\!\rangle$ refers to the duality pairing. Using differential forms, equation (5.21) can also be written as

$$\langle\!\langle \operatorname{Det}\nabla\Phi, \phi \rangle\!\rangle = -\frac{1}{N}\sum_{i=1}^N \int_\Omega d\Phi^1\wedge\cdots\wedge d\Phi^{i-1}\wedge\Phi^i d\phi\wedge\cdots\wedge d\Phi^N, \tag{5.22}$$

for $\phi \in C_c^\infty(\Omega)$. This distribution is well-defined for $\Phi \in \mathrm{W}^{1,p}(\Omega,\mathbf{R}^N)$ for $p \geqslant \frac{N^2}{N+1}$, or $\Phi \in \mathrm{W}^{1,q}(\Omega,\mathbf{R}^N)\cap\mathrm{L}^\infty(\Omega,\mathbf{R}^N)$ for $q \geqslant N-1$.

5.A.2 Regularisation of the distributional Hessian determinant

First we recall an identity for Jacobians which is used throughout (see*e.g.* [101, Chapter 8]): for two smooth vector-valued functions Φ and Ψ and a test function ξ compactly supported in Ω, we have

$$\left|\int_\Omega(\det\nabla\Phi - \det\nabla\Psi)\xi\right| \leqslant \int_\Omega |\Phi-\Psi||\nabla\xi|(|\nabla\Phi|+|\nabla\Psi|)^{N-1}. \tag{5.23}$$

Applying Hölder's inequality with $p \geqslant N-1$ yields

$$\left|\int_\Omega (\det \nabla\Phi - \det\nabla\Psi)\xi\right| \leqslant (\|\nabla\Phi\|_{\mathrm{L}^p} + \|\nabla\Psi\|_{\mathrm{L}^p})^{N-1}(\|\Phi-\Psi\|_{\mathrm{L}^{\frac{p}{p-N+1}}}\|\nabla\xi\|_{\mathrm{L}^\infty}).$$

The reader is referred to [18] for further inequalities involving the Jacobian determinant.

We now focus on the case where Φ is a gradient field, i.e. there exists a function f such that $\Phi = \nabla f$. As above, the distributional Hessian $\operatorname{Det}\nabla^2 f$ is well-defined for $f \in \mathrm{W}^{2,p}(\Omega)$ and $p \geqslant \frac{N^2}{N+1}$, or for $f \in \mathrm{W}^{2,N-1}\cap\mathrm{W}^{1,\infty}(\Omega)$.

Remark 5.7. *"Weaker" forms of the distributional Hessian may be defined using further cancellations in equation* (5.22) *arising from the gradient structure. The reader is referred to the survey [100].*

A natural question is when the distributional Hessian agrees with the pointwise value of the determinant of the Hessian. By a result of S. Müller [125], if $f \in \mathrm{W}^{2,p}(\Omega)$ for $p \geqslant \frac{N^2}{N+1}$ and moreover $\operatorname{Det}\nabla^2 f \in \mathrm{L}^1(\Omega)$ then $\operatorname{Det}\nabla^2 f = \det\nabla^2 f dx$. However, the arguments of [125] do not cover the case where $f \in \mathrm{W}^{2,N-1}\cap\mathrm{W}^{1,\infty}$. The next lemma may be known but I was unable to locate this precise statement in the literature. It is closely related to [100, 125].

Lemma 5.24. *Let* $f \in \mathrm{W}^{2,N-1}(\Omega)\cap\mathrm{W}^{1,\infty}(\Omega)$. *If* $\operatorname{Det}\nabla^2 f \in \mathrm{L}^1$ *then* $\det\nabla^2 f = \operatorname{Det}\nabla^2 f$ *pointwise almost everywhere. In particular, the pointwise Hessian determinant* $\det\nabla^2 f$ *is integrable.*

This lemma is the analogue of the "det = Det" equality proved in [125] (see also [47]).

Proof. The proof consists in showing that $(\operatorname{Det}\nabla^2 f)^\varepsilon \to \det\nabla^2 f(x_0)$ almost everywhere. Here $(\operatorname{Det}\nabla^2 f)^\varepsilon$ is a regularising sequence given by

$$(\operatorname{Det}\nabla^2 f)^\varepsilon(x_0) = \langle\!\langle \operatorname{Det}\nabla^2 u, \eta_\varepsilon \rangle\!\rangle,$$

for a sequence of mollifiers $y \mapsto \eta_\varepsilon(x_0 - y)$ which is an approximation of identity, i.e. such that as $\varepsilon \to 0$, we have $\eta_\varepsilon \xrightarrow{\mathcal{D}'(\Omega)} \delta_{x_0}$.

Let $x_0 \in \Omega$ be a point of L^{N-1} density for $\nabla^2 f$. Then for small enough balls $B_\varepsilon(x_0)$, there exist quadratic polynomials P_ε of the form

$$P_\varepsilon(x) = c_0 + c_1(x - x_0) + \frac{1}{2}\langle x - x_0, \nabla^2 f(x_0)(x-x_0)\rangle.$$

such that $\sup_{B_\varepsilon} \varepsilon^{-2} |f - P_\varepsilon| < \infty$, by Calderón–Zygmund estimates, as in [23, Theorems 8, 10 and 12] and [100, Theorem 6].

Let $\varepsilon > 0$ be sufficiently small (so that $d(x_0, \partial\Omega) > \varepsilon$), η_ε be a mollifier, and denote $\psi_\varepsilon(y) := \eta_\varepsilon(x_0 - y)$. Testing against ψ, we have on one hand

$$\langle\!\langle \operatorname{Det} \nabla^2 f, \psi \rangle\!\rangle = \operatorname{Det} \nabla^2 f * \eta_\varepsilon,$$

where the convolution is taken in the sense of distributions, and on the other hand, integrating by parts yields

$$\begin{aligned}
&\langle\!\langle \operatorname{Det} \nabla^2 f, \psi_\varepsilon \rangle\!\rangle = \\
&= -\frac{1}{N} \sum_{i=1}^{N} \int_{B_\varepsilon(x_0)} d(\partial_1 f) \wedge \cdots \wedge d(\partial_{i-1} f) \wedge (\partial_i f) d\psi_\varepsilon \wedge d(\partial_{i+1} f) \wedge \cdots \wedge d(\partial_N f) \\
&= \frac{1}{N} \sum_{i=1}^{N} \int_{B_\varepsilon(x_0)} f d(\partial_1 f) \wedge \cdots \wedge d(\partial_{i-1} f) \wedge d(\partial_i \psi_\varepsilon) \wedge \cdots \wedge d(\partial_N f) \\
&=: \frac{1}{N} \sum_{i=1}^{N} \int_{B_\varepsilon(x_0)} f w_\varepsilon^i \\
&= \frac{1}{N} \sum_{i=1}^{N} \left(\int_{B_\varepsilon(x_0)} (f - P_\varepsilon) w_\varepsilon^i + \int_{B_\varepsilon(x_0)} P_\varepsilon w_\varepsilon^i \right) \\
&= \frac{1}{N} \sum_{i=1}^{N} \left(\int_{B_\varepsilon(x_0)} (f - P_\varepsilon) w_\varepsilon^i + \int_{B_\varepsilon(x_0)} \psi_\varepsilon d(\partial_1 f) \wedge \cdots \wedge d(\partial_i P_\varepsilon) \wedge \cdots \wedge d(\partial_N f) \right) \\
&= \frac{1}{N} \sum_{i=1}^{N} \left(\int_{B_\varepsilon(x_0)} (f - P_\varepsilon) w_\varepsilon^i + \int_{\Omega} \psi_\varepsilon d(\partial_1 f) \wedge \cdots \wedge (d\partial_i f)(x_0) \wedge \cdots \wedge d(\partial_N f) \right).
\end{aligned}$$

The second term goes to $\operatorname{Adj}(\nabla^2 f)(x_0) \nabla^2 f(x_0) = \det \nabla^2 f(x_0)$ as $\varepsilon \to 0$. For the first integral, we can estimate

$$\begin{aligned}
\int_{B_\varepsilon(x_0)} |(f - P_\varepsilon) w_\varepsilon^i| &\leqslant \|f - P_\varepsilon\|_{\mathrm{L}^\infty(B_\varepsilon)} \|w_\varepsilon^i\|_{\mathrm{L}^1(B_\varepsilon)} \\
&\leqslant C \|f - P_\varepsilon\|_{\mathrm{L}^\infty(B_\varepsilon)} \|\operatorname{Adj}(\nabla^2 f)\|_{\mathrm{L}^1(B_\varepsilon)} \|\nabla^2 \psi_\varepsilon\|_{\mathrm{L}^\infty(B_\varepsilon)} \\
&\leqslant C \|f - P_\varepsilon\|_{\mathrm{L}^\infty(B_\varepsilon)} \|\nabla^2 f\|_{\mathrm{L}^{N-1}(B_\varepsilon)}^{N-1} \|\nabla^2 \psi_\varepsilon\|_{\mathrm{L}^\infty(B_\varepsilon)}.
\end{aligned}$$

The first term of the product is $o(\varepsilon^{-2})$, by the Calderón–Zygmund theorem; the second is $\mathcal{O}(\varepsilon^N)$; and the third one is $\mathcal{O}(\varepsilon^{-N-2})$. Overall this vanishes as $\varepsilon \to 0$. This yields the conclusion that

$$\operatorname{Det} \nabla^2 f * \eta_\varepsilon = \langle\!\langle \operatorname{Det} \nabla^2 f, \psi_\varepsilon \rangle\!\rangle \to \det \nabla^2 f(x_0).$$

Q.E.D.

5.A.3 Alexandrov measure

Let $D \subset \mathbf{R}^N$ be a domain. If D is convex, we say that $f \in C(D)$ is convex if for all $x, y \in D$ and $0 \leqslant \lambda \leqslant 1$ we have $f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y)$. Otherwise, f is said to be convex if there exists a convex function $\bar{f} \in C(\mathrm{conv}(D))$ such that $\bar{f}|_D = f$. Here $\mathrm{conv}(D)$ is the convex hull of D in $\mathbf{R}^n$.

For a convex function $f \in C(D)$, its Alexandrov measure $\mathcal{M}_f$ is defined as follows. For all $E \subset D$,

$$\mathcal{M}_f(E) := \left| \bigcup_{x \in E} \partial f(x) \right|$$

Here $\partial f(x)$ refers to the sub-differential of f at the point x, namely the set

$$\partial f(x) = \left\{ p \in \mathbf{R}^N \,\middle|\, \forall y \in D, \langle p, y - x \rangle \leqslant u(y) - u(x) \right\}.$$

The Alexandrov measure is a non-negative Radon measure. Finally, we recall the following important fact.

Lemma 5.25. *Let* $(f_k)_{k \in \mathbf{N}} \subset C(D)$ *be a sequence of convex functions converging uniformly on compacts to a convex function* $f \in C(D)$*. Then* $\mathcal{M}_{f_k} \overset{*}{\rightharpoonup} \mathcal{M}_f$ *converges in the weak* topology of measures, i.e. for any* $\phi \in C_c(D)$*,*

$$\lim_{k \to \infty} \int_D \phi d\mathcal{M}_{f_k} = \int_D \phi d\mathcal{M}_f.$$

A proof of this fact can be found in *e.g.* [67].

When both Alexandrov measure and distributional Hessian are defined, we have the following relation between $\mathcal{M}_f$ and $\mathrm{Det}\, \nabla f$.

Lemma 5.26. *Let* $D \subset \mathbf{R}^N$ *be an open domain,* $f \in \mathrm{W}^{1,\infty} \cap \mathrm{W}^{2,N-1}(D)$*. Assume that* f *is convex. Then* $\mathrm{Det}\, \nabla^2 f$ *is a non-negative Radon measure and moreover* $\mathcal{M}_f = \mathrm{Det}\, \nabla^2 f$*.*

Proof. The proof of the lemma follows from two facts: first, mollifications of a convex function are convex; and second, distributional determinant and Alexandrov measure coincide for smooth, convex functions. For completeness, the details of the proof are provided.

Step 1: we give a proof of the first fact. Let $f \in C(D)$, and consider a mollifier $\eta \in C_c^\infty(B_1)$ such that

$$\int_{B_1} \eta(x) dx = 1, \qquad \eta \geqslant 0.$$

Let $D_\varepsilon \subset D$ be the set of points such that $\operatorname{dist}(x, \partial D) > \varepsilon$. Writing $\eta_\varepsilon = \varepsilon^{-N}\eta(\cdot/\varepsilon)$, we define the regularisation f^ε of f as

$$f^\varepsilon(x) = \int_D f(y)\eta_\varepsilon(x-y)dy = \int_{B_\varepsilon(0)} f(x-t)\eta_\varepsilon(t)dt, \qquad x \in D_\varepsilon.$$

If D is not convex, we consider a convex function $\bar{f} \in C(\operatorname{conv}(D))$ such that $\bar{f}|_D = f$. Here $\operatorname{conv}(D)$ is the convex hull of D. We note that if $x \in D_\varepsilon$, $B_\varepsilon \subset D$ and so

$$\bar{f}^\varepsilon(x) = \int_{B_\varepsilon(x)} \bar{f}(y)\eta_\varepsilon(x-y)dy = \int_{B_\varepsilon(x)} f(y)\eta_\varepsilon(x-y)dy = f^\varepsilon(x).$$

Let $0 \leqslant \lambda \leqslant 1$ and $x, y \in D_\varepsilon$. Writing $z_\lambda = \lambda x + (1-\lambda)y$, there holds

$$\begin{aligned}
\bar{f}^\varepsilon(z_\lambda) &= \int_{B_\varepsilon} \bar{f}(z_\lambda - t)\eta_\varepsilon(t)dt \\
&= \int_{B_\varepsilon} \bar{f}(\lambda(x-t) + (1-\lambda)(y-t))\eta_\varepsilon(t)dt \\
&\leqslant \lambda \int_{B_\varepsilon} \bar{f}(x-t)\eta_\varepsilon(t)dt + (1-\lambda)\int_{B_\varepsilon} \bar{f}(y-t)\eta_\varepsilon(t)dt \\
&= \lambda \bar{f}^\varepsilon(x) + (1-\lambda)\bar{f}^\varepsilon(y).
\end{aligned}$$

Therefore the regularisation f^ε of a convex function is convex.
Step 2: we consider the regularisation f^ε of f. Then f^ε is convex and $\mathcal{M}_{f^\varepsilon} = \operatorname{Det}\nabla^2 f^\varepsilon$. Moreover, by Lemma 5.25, we have

$$\mathcal{M}_{f^\varepsilon} \overset{*}{\rightharpoonup} \mathcal{M}_f$$

in the weak* topology of measures on D_ε. In particular, for all $\phi \in C_c^\infty(D)$, there exists an $\varepsilon > 0$ such that $\operatorname{supp}\phi \subset D_\varepsilon$,

$$\lim_{\varepsilon \to 0} \int_{D_\varepsilon} \phi d\mathcal{M}_{f^\varepsilon} = \int_{D_\varepsilon} \phi d\mathcal{M}_f,$$

so that $\mathcal{M}_{f^\varepsilon} \xrightarrow{\mathcal{D}'(D)} \mathcal{M}_f$. On the other hand, it is clear that

$$\operatorname{Det}\nabla^2 f^\varepsilon \xrightarrow{\mathcal{D}'(D)} \operatorname{Det}\nabla^2 f.$$

By uniqueness of limits in $\mathcal{D}'(D)$, $\operatorname{Det}\nabla^2 f = \mathcal{M}_f$. Q.E.D.

5.A.4 Gauß equation in the sense of Alexandrov

We note that to obtain the statement of Lemma 5.23, the regularity condition (R) is superfluous: one can by-pass Theorem 5.1 and obtain directly:

Proposition 5.27. *Let $p \geqslant 1$, $f \in \mathrm{W}^{2,p} \cap C^1(D)$ be a convex graphical parametrisation of a locally isometric embedding with Gauß curvature K, and let $\mathcal{M}_f$ the Alexandrov measure associated to f. Then for all $E \subset D$, $\mathcal{M}_f(E) = \int_E \bar{K} V^4$.*

Lemma 5.23 is a corollary of Proposition 5.27. Furthermore, by Lemma 5.26, if the function is convex, Proposition 5.27 in fact implies Theorem 5.1.

Proof of Proposition 5.27. Proposition 5.4 showed that in the sense of distributions,

$$\operatorname{div}(A\nabla f) = \bar{K}V.$$

Here the matrix A is given by

$$A = V^{-1}\begin{pmatrix} Y^{-2}f_{yy} & -X^{-2}f_{xy} \\ -Y^{-2}f_{xy} & X^{-2}f_{xx} \end{pmatrix},$$

and the operators div and ∇ are the Euclidean operators $\partial_x + \partial_y$ and (∂_x, ∂_y) respectively.

Approximate by smoothing f to f^ε, and define the following:

$$\Xi^\varepsilon = \frac{1}{(1+|\nabla f^\varepsilon|^2)(1+(f^\varepsilon_x)^2)},$$

$$\Upsilon^\varepsilon = \frac{1}{(1+|\nabla f^\varepsilon|^2)(1+(f^\varepsilon_y)^2)},$$

$$A^\varepsilon = \begin{pmatrix} \Upsilon^\varepsilon f^\varepsilon_{yy} & -\Xi^\varepsilon f^\varepsilon_{xy} \\ -\Upsilon^\varepsilon f^\varepsilon_{xy} & \Xi^\varepsilon f^\varepsilon_{xx} \end{pmatrix}.$$

By C^1 regularity of f, we have uniform convergence of the regularisations $\nabla f^\varepsilon \to \nabla f$, and therefore of $\Xi^\varepsilon \to \Xi$ (and similarly for Υ^ε). Moreover by smoothness the following algebraic identity holds (see Section 5.2):

$$\operatorname{div}(A^\varepsilon \nabla f^\varepsilon) = \frac{\det(\nabla^2 f^\varepsilon)}{(V^\varepsilon)^3}.$$

The argument is now similar to that of Lemma 5.24. Recall that f being convex implies that the regularised f^ε are convex. Thus, by convexity we have

$$\det(\nabla^2 f^\varepsilon) \overset{*}{\rightharpoonup} \mathcal{M}_f$$

in the weak* topology of measures. On the other hand, by our regularity hypothesis on f, we have

$$A^\varepsilon \nabla f^\varepsilon \to A\nabla f$$

in $\mathrm{L}^1_{\mathrm{loc}}$. Thus

$$\operatorname{div}(A^\varepsilon \nabla f^\varepsilon) \to \operatorname{div}(A \nabla f)$$

in the sense of distributions. Finally, since $V^\varepsilon \to V$ uniformly on compacts, we have

$$\mathcal{M}_f = \bar{K} V^4.$$

Q.E.D.

Chapter 6

Connections with L^p Bounds

Let $P \to M$ be a principal G-bundle; throughout this chapter we assume that G is a matrix group, i.e. a (Lie) subgroup of **GL**. A central object of study is the set of connections on a bundle. Let $\mathcal{A}(P)$ be the (affine) set of connections on $P \to M$, and recall that for a connection $\nabla_A \in \mathcal{A}(P)$, one can define a curvature $\Omega_A = \nabla_A \circ \nabla_A$. We investigate the behaviour of the curvature form Ω_A along a sequence of connections converging weakly in some Sobolev space.

The main result of this chapter shows that for a sequence of connections with L^p bounds, the curvature tensor passes to limits in the sense of distributions. Before stating the result precisely, let us recall that the space of connections $\mathcal{A}(P)$ on a principal bundle $P \to M$ is affine. We fix a reference smooth connection $\nabla \in \mathcal{A}(P)$. Then for any connection $\nabla_A \in \mathcal{A}(P)$, there exists an $\mathbf{Ad}(P)$-valued one-form A such that

$$\nabla_A - \nabla = A \in \Gamma(M, \mathbf{Ad}(P) \otimes T^*M).$$

Recall that $\mathbf{Ad}(P)$ is the adjoint bundle of P (see Appendix B), whose fiber is the Lie algebra $\mathfrak{g}$ of G. In a local trivialising domain $E \subset M$, we may write $A \in \Gamma(E, \mathfrak{g} \otimes T^*E) \simeq E \times \mathfrak{g} \times \mathbf{R}^n$, which we therefore view as a matrix-valued one-form. The curvature form Ω_A may now be expressed as

$$\Omega_A = \Omega_\nabla + \nabla A + \frac{1}{2}[A \wedge A] \in \Gamma(M, \mathbf{Ad}(P) \otimes \wedge^2 T^*M),$$

where Ω_∇ is the curvature of the reference connection ∇.

A sequence of connections $\nabla^\varepsilon = \nabla + A^\varepsilon$, indexed by $\varepsilon > 0$, is said to converge weakly in L^p if $A^\varepsilon \rightharpoonup A$ weakly in $\mathrm{L}^p(M, \mathbf{Ad}(P) \otimes T^*M)$ (see Appendix A for the relevant definitions of Sobolev spaces).

We now state the main theorem of this chapter:

THEOREM 6.1. *Let (M,g) be a closed smooth Riemannian manifold, $P \to M$ be a principal G-bundle, and $\nabla \in \mathcal{A}(P)$ a smooth reference connection.*

Let $\nabla^\varepsilon = \nabla + A^\varepsilon$ be a sequence of connections on P, Ω^ε the associated curvature forms, and assume that

$$\begin{aligned}&\sup_{\varepsilon>0}\|A^\varepsilon\|_{\mathrm{L}^p(M)} < \infty, \quad p>2,\\ &\sup_{\varepsilon>0}\|\Omega_\varepsilon\|(M) < \infty.\end{aligned} \tag{6.1}$$

Finally let $\nabla_A = \nabla + A$ be an L^p connection such that $A^\varepsilon \rightharpoonup A$ in L^p up to a non-relabelled subsequence. Then $\Omega_\varepsilon \overset{*}{\rightharpoonup} \Omega_A$ in the sense of distributions, i.e. for all $\phi \in \Gamma(M, \mathbf{Ad}(P) \otimes \wedge^2 T^*M)$,

$$\lim_{\varepsilon\to 0}\int_M \langle A^\varepsilon, \delta\phi\rangle + \frac{1}{2}\langle [A^\varepsilon \wedge A^\varepsilon], \phi\rangle = \int_M \langle A, \delta\phi\rangle + \frac{1}{2}\langle [A\wedge A], \phi\rangle.$$

Here the notation $\|\Omega\|(M)$ refers to the total mass of the curvature form.

Remark 6.1. *By the Banach–Alaoglu theorem, the ball is weakly compact in L^p; thus, under hypothesis (6.1), there exists an $A \in \mathrm{L}^p(M, \mathbf{Ad}(P)\otimes T^*M)$ such that $A^\varepsilon \rightharpoonup A$ in the weak topology of L^p, up to a subsequence. We focus on identifying the weak limit of the non-linearity $[A^\varepsilon \wedge A^\varepsilon]$*

The main ingredient of the proof is a div-curl estimate (see Lemma 6.5), which is needed to deal with the limit of the non-linearity $[A^\varepsilon \wedge A^\varepsilon]$.

Theorem 6.1 yields weak continuity results for several systems of equations of differential geometry and physics. As a first application, we give a weak continuity result (Theorem 6.9 below) for Yang–Mills equations under the assumption of uniform L^p bounds.

The second application, which is also the most relevant to the core of the thesis, is the study of the constraint equations for immersions of Riemannian manifolds in Euclidean space. We obtain weak continuity results for the Gauß–Codazzi-Ricci equations generalising the results of [33, 36]. This occupies the next chapter, Chapter 7; the main results are Theorems 7.1 and 7.2.

This chapter is organised as follows. In Section 6.1, we present the required preliminaries and prove a general div-curl lemma, which is then employed in Section 6.2 to prove Theorem 6.1. As an application, in Section 6.3, a compactness theorem for weak Yang–Mills connections is proved (*cf.* Theorem 6.9).

6.1 A general framework for div-curl lemmas

The main results of this section are Lemmas 6.5 and 6.6. These are the main ingredients of the proof of Theorem 6.1. Since the result is inspired by—and extends—many other div-curl lemmas [33, 44, 111, 130, 153], we have embedded its proof in a more abstract framework which highlights the similarities and links between our theorem and other existing results, exposed in the first two pragraphs of this section. The proofs of Lemmas 6.5 and 6.6 are in §6.1.3.

6.1.1 Differential complexes and functional spaces

We start by recalling basic notions about elliptic complexes, which are the natural framework to consider. The material in this section is standard and the reader is referred to *e.g.* [142].

Let M be a closed Riemannian manifold and let $\{E(i) \to M \,\big|\, i \in \mathbf{N}\}$ be a sequence of fiber bundles on M. A complex is a sequence of maps $\{D(i) : \Gamma(M, E(i)) \to \Gamma(M, E(i+1)) \,\big|\, i \in \mathbf{N}\}$. The complex D is differential if $D(i+1) \circ D(i) = 0$, or equivalently if the sequence

$$\cdots \xrightarrow{D(i-1)} \Gamma(M, E(i)) \xrightarrow{D(i)} \Gamma(M, E(i+1)) \xrightarrow{D(i+1)} \Gamma(M, E(i+2)) \xrightarrow{D(i+2)} \cdots$$

is exact.

We are interested in the case where E is a complex of vector bundles and D is a sequence of linear differential operators. We assume that E is also equipped with an inner product. Let $*$ denote the associated Hodge duality operator, and $(*1)$ the volume form. For a element $\phi \in \Gamma(M, E(i))$ and $1 \leqslant p < \infty$, we can define the L^p norms

$$\|\phi\|_{\mathrm{L}^p}^p = \int_M |\phi|^p (*1).$$

The Lebesgue spaces on E are then the completion of $\Gamma(M, E(i))$ with respect to the above norm. We also assume that $\Gamma(M, E(i))$ is equipped with a sequence of fixed, smooth connections $\nabla(i)$, so that one may define Sobolev spaces $\mathrm{W}^{k,p}(M, E(i))$ by acting with ∇, so that *e.g.* the $\mathrm{W}^{1,p}$ Sobolev norms are given by

$$\|\phi\|_{\mathrm{W}^{1,p}}^p = \int_M |\phi|^p (*1) + \int_M |\nabla\phi|^p (*1),$$

and the spaces $\mathrm{W}^{-k,p}$ are defined by duality in the standard way (see Appendix A for further details on Sobolev spaces on vector bundles).

$$\begin{tikzcd}
\pi^* E(i) \arrow[r] \arrow[d] & E(i) \arrow[d] \\
T^*M \arrow[r, "\pi"] & M
\end{tikzcd}$$

Figure 6.1: Pull-back of the bundles $E(i) \to M$ on the cotangent bundle by the fibration $\pi : T^*M \to M$. This diagramme is commutative.

Fix local coordinates on M; in the associated local trivialisations $\{\partial_j \,\big|\, 1 \leqslant j \leqslant \dim(E(i))\}$ of $E(i) \to M$, the operators $D(i)$ take the form

$$D(i) = \sum_{|\alpha| \leqslant k(i)} P^\alpha \partial_\alpha,$$

where $\alpha \in \mathbf{N}^{\dim(E(i))}$ is a multi-index and $P^\alpha = P^\alpha(i) \in \Gamma(M, \mathbf{Hom}(E(i), E(i{+}1))) = \Gamma(M, \mathscr{L}(E(i), E(i{+}1)))$. We use the multi-index notation $\partial_\alpha = \otimes_{1 \leqslant j \leqslant \dim(E(i))} \partial^{\alpha_j}$. The operator $D(i)$ is of order $k(i)$.

Recall that the symbol of $D(i)$, denoted by σ_D, is a vector bundle complex

$$\cdots \xrightarrow{\sigma_D(i-1)} \pi^*(E(i)) \xrightarrow{\sigma_D(i)} \pi^*(E(i+1)) \xrightarrow{\sigma_D(i+1)} \cdots$$

where we write $T^*M \overset{\pi}{\to} M$ for the cotangent bundle, $\pi^* E$ for the vector bundle $\pi^* E(i) \to T^*M$ pulled back over T^*M by the fibration map π (see Figure 6.1). Thus for a fixed $\xi \in T^*_x M$, $\sigma_{D(i)}(\xi) : E(i)_x \to E(i+1)_x$ is a linear map, given as the symbol of the differential operator $D(i)$ in coordinates on $E(i)$. Alternatively, it may be defined directly as follows. Let $f \in \Gamma(M, E(i))$ be such that $f(x) = v$; let $g \in C^\infty(M)$ be such that $g(x) = 0$ and $dg|_x = \xi$. Then $\sigma_{D(i)}(\xi)v = D(i)(g^{k(i)} f / k(i)!)(x) \in E(i{+}1)$ and so $\sigma_{D(i)}(\xi) \in \mathscr{L}(E(i), E(i+1))$. The complex D is elliptic if σ_D is exact.

The L^2 adjoint of D is denoted by D^* and forms a new complex

$$\cdots \xrightarrow{D^*(i+1)} \Gamma(M, E(i+1)) \xrightarrow{D^*(i)} \Gamma(M, E(i)) \xrightarrow{D^*(i-1)} \cdots$$

which is also exact if the complex D was differential; similarly for its symbol σ_{D^*}. The Hodge Laplacian of D is the sequence of operators given by

$$\triangle(i) = D(i-1)D(i-1)^* + D(i)^* D(i) : \Gamma(M, E(i)) \to \Gamma(M, E(i)).$$

It is well-known that if D is an elliptic complex, the Hodge Laplacian is elliptic as $\sigma_\triangle$ is an isomorphism. We also note the following commutation property.

Lemma 6.2. *Let M be a smooth closed Riemannian manifold, (E, D) be an elliptic complex on M, D^* its adjoint complex, and $\triangle$ the associated Hodge Laplacian. Then $\triangle D = D \triangle$. Moreover, letting $\triangle^{-1}$ denote the inverse Laplacian, we have $\triangle^{-1} D = D \triangle^{-1}$.*

Proof. The proof is a simple calculation: we have

$$D \triangle = D(DD^* + D^*D) = (D^*D + DD^*)D = \triangle D.$$

The second assertion follows immediately from the first as follows:

$$\triangle^{-1} D = \triangle^{-1} D \triangle \triangle^{-1} = \triangle^{-1} \triangle D \triangle^{-1} = \triangle^{-1} D.$$

Q.E.D.

For our purpose it is sufficient to consider the case where all the operators $D(i)$ are first-order operators. We note that it is possible to derive the entire theory expounded in this section for general elliptic complex of arbitrary order $k \in \mathbf{N}$, in which case the Hodge Laplacian is in general an operator of order $2k$, and first-order Sobolev spaces need to be replaced with suitable spaces.

6.1.2 The div-curl lemma

By standard elliptic theory, the Hodge Laplacian $\triangle$ of a first-order elliptic complex admits a continuous inverse $\triangle^{-1} : \mathrm{W}^{-1,p}(M, E(i)) \to \mathrm{W}^{1,p}(M, E(i))$, for $1 < p < \infty$. The next observation is essentially due to [153].

Lemma 6.3. *Consider a sequence of sections* α^ε *that is bounded in* L^p *and such that* $D\alpha^\varepsilon$ *is compactly contained in* $\mathrm{W}^{-1,p}$. *Then there exist* $\rho^\varepsilon \in \Gamma(M, E(i))$ *and* $\psi^\varepsilon \in \Gamma(M, E(i-1))$ *satisfy* $\rho^\varepsilon \to \rho$ *and* $\psi^\varepsilon \to \psi$ *strongly in* L^p, *and in addition* $D\psi^\varepsilon \rightharpoonup D\psi$ *weakly in* L^p, *such that* $\alpha^\varepsilon = D\psi^\varepsilon + \rho^\varepsilon$, *and* $\alpha = D\psi + \rho$.

Proof. We note the following algebraic identity:

$$\alpha^\varepsilon = \triangle \triangle^{-1} \alpha^\varepsilon = DD^* \triangle^{-1} \alpha^\varepsilon + D^*D \triangle^{-1} \alpha^\varepsilon.$$

Letting $\psi^\varepsilon = D^* \triangle^{-1} \alpha^\varepsilon$ and $\rho^\varepsilon = D^*D \triangle^{-1} \alpha^\varepsilon$, we have $\alpha^\varepsilon = D\psi^\varepsilon + \rho^\varepsilon$.

By Lemma 6.2, $\rho^\varepsilon = D^*D \triangle^{-1} \alpha^\varepsilon = D^* \triangle^{-1} D\alpha^\varepsilon$. Recalling that $D\alpha^\varepsilon$ is compactly contained in $\mathrm{W}^{-1,p}$, since $\triangle^{-1}$ is continuous, ρ^ε is compactly contained in $\mathrm{L}^p(M, E(i))$, and thus converges strongly to a limit ρ.

On the other hand, ψ^ε is in a bounded set of $\mathrm{W}^{1,p}(M, E(i-1))$, and therefore converges to a limit ψ weakly in $\mathrm{W}^{1,p}(M, E(i-1))$ and strongly in L^p, by compact emebdding. Finally, by uniqueness of weak limits we have

$$\alpha = D\psi + \rho.$$

Q.E.D.

This observation allows us to obtain the following general div-curl lemma for elliptic complexes.

Lemma 6.4. *Let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha^\varepsilon \in \mathrm{L}^p(M, E(i))$ and $\beta^\varepsilon \in \mathrm{L}^q(M, E(i))$ be bounded sequences, such that*

$$\begin{aligned} D\alpha^\varepsilon &\Subset \mathrm{W}^{-1,p}(M, E(i+1)), \\ D^*\beta^\varepsilon &\Subset \mathrm{W}^{-1,q}(M, E(i-1)). \end{aligned}$$

Then for all $\phi \in C_c^\infty(M)$,

$$\int_M \langle \alpha^\varepsilon, \beta^\varepsilon \rangle \phi \to \int_M \langle \alpha, \beta \rangle \phi,$$

The notation $\langle \alpha^\varepsilon, \beta^\varepsilon \rangle \overset{*}{\rightharpoonup} \langle \alpha, \beta \rangle$ is also used in the rest of the text when there is no ambiguity.

Proof. We view the sequence $\alpha^\varepsilon \in \mathrm{L}^p(M, E(i))$ as part of the complex (E, D) and $\beta^\varepsilon \in \Gamma(M, E(i))$ as part of the complex (E, D^*). Applying Lemma 6.3 yields $\alpha^\varepsilon = D\psi^\varepsilon + \rho^\varepsilon, \beta^\varepsilon = D^*\zeta^\varepsilon + \xi^\varepsilon$. We now take $\phi \in C_c^\infty(M)$. Then there holds

$$\begin{aligned} \int_M \langle \alpha^\varepsilon, \beta^\varepsilon \rangle \phi = &\int_M \langle D\psi^\varepsilon, D^*\zeta^\varepsilon \rangle \phi + \int_M \langle D\psi^\varepsilon, \xi^\varepsilon \rangle \phi \\ &+ \int_M \langle \rho^\varepsilon, \xi^\varepsilon \rangle \phi + \int_M \langle \rho^\varepsilon, D^*\zeta^\varepsilon \rangle \phi. \end{aligned}$$

By Hölder's inequality, $\langle \rho^\varepsilon, \xi^\varepsilon \rangle \to \langle \rho, \xi \rangle$ strongly in L^1, so that $\int_M \langle \rho^\varepsilon, \xi^\varepsilon \rangle \phi \to \int_M \langle \rho, \xi \rangle \phi$.

The second terms and fourth terms of the RHS are treated similarly. $D\psi^\varepsilon$ converges weakly in L^p, while ξ^ε converges strongly in L^q. Therefore

$$\int_M \langle D\psi^\varepsilon, \xi^\varepsilon \rangle \phi \to \int_M \langle D\psi, \xi \rangle \phi.$$

Finally, we focus on the first term. Recall that $D(i)$ is a first-order linear differential operator, so that we may write for all $i \in \mathbf{N}$

$$D(i) = \sum_{|\mu| \leqslant 1} P_\mu(i) \partial^\mu.$$

By the Leibniz rule, we have

$$\begin{aligned} D(i+1)(\phi D(i)\psi_\varepsilon) &= \sum_{|\mu| \leqslant 1} \sum_{\kappa \leqslant \mu} \binom{\mu}{\kappa} \partial^\kappa \phi P_\mu(i+1) \partial^{\mu-\kappa} D(i) \psi_\varepsilon \\ &= \phi D(i+1) D(i) \psi_\varepsilon + \sum_{|\mu| \leqslant 1} \sum_{0 \neq \kappa \leqslant \mu} \partial^\kappa \phi P_\mu(i+1) \partial^{\mu-\kappa} D(i) \psi_\varepsilon. \end{aligned}$$

Notice that $\mathfrak{D}$ is an exact complex so that $D(i+1)D(i) = 0$. Thus, we are left with

$$D(i+1)(\phi D(i)\psi_\varepsilon) = \sum_{|\mu|\leqslant 1} \sum_{0\neq\kappa\leqslant\mu} \partial^\kappa \phi P_\mu(i+1)\partial^{\mu-\kappa} D(i)\psi_\varepsilon. \tag{6.2}$$

We note that $D(i)$ is a first-order operator so that, whenever $0 \neq \kappa \leqslant \mu$ and $|\mu| \leqslant 1$, we conclude that $\mu = \kappa$. Then the right-hand side of (6.2) simplifies further to

$$D(i+1)(\phi D(i)\psi_\varepsilon) = \sum_{|\mu|\leqslant 1} \partial^\mu \phi P_\mu(i+1) D(i)\psi_\varepsilon.$$

We now integrate by parts to obtain

$$\begin{aligned} \int_M \langle \phi D(i)\psi_\varepsilon, D^*(i)\zeta_\varepsilon\rangle &= -\int_M \langle D(i)(\phi D(i)\psi_\varepsilon), \zeta_\varepsilon\rangle \\ &= -\sum_{|\mu|\leqslant 1} \int_M \partial^\mu\phi \left\langle P_\mu(i+1)D(i)\psi_\varepsilon, \zeta_\varepsilon\right\rangle . \end{aligned}$$

Recall that, by construction, ζ_ε converges to ζ strongly in L^q. Furthermore, $D(i)\psi_\varepsilon$ converges weakly in L^p. Therefore, we have

$$\begin{aligned} \int_M \langle \phi D(i)\psi_\varepsilon, D^*(i)\zeta_\varepsilon\rangle &= -\sum_{|\mu|\leqslant 1} \int_M \partial^\mu\phi \left\langle P_\mu(i+1)D(i)\psi_\varepsilon, \zeta_\varepsilon\right\rangle \\ &\xrightarrow{\varepsilon\to 0} -\sum_{|\mu|\leqslant 1} \int_M \partial^\mu\phi \left\langle P_\mu(i+1)D(i)\psi, \zeta\right\rangle \\ &= \int_M \langle \phi D(i)\psi, D^*\zeta\rangle, \end{aligned}$$

where we have integrated by parts once more for the last equality. This completes the proof. Q.E.D.

Lemma 6.4 generalises the standard div-curl lemma. Indeed, as an example of the approach of this section, let us consider the complex

$$0 \to \Gamma(M) \xrightarrow{\text{grad}} \Gamma(M, TM) \xrightarrow{\text{rot}} \Gamma(M, TM \otimes TM) \to \cdots$$

whose adjoint complex is

$$\cdots \to \Gamma(M, TM) \xrightarrow{\text{div}} \Gamma(M) \to 0,$$

For instance, if $\dim M = 3$, we have the short exact sequence

$$0 \to \Gamma(M) \xrightarrow{\text{grad}} \Gamma(M, TM) \xrightarrow{\text{rot}} \Gamma(M, TM) \xrightarrow{\text{div}} \Gamma(M) \to 0, \quad \dim M = 3.$$

Applying Lemma 6.4 to these complexes, we obtain the classical div-curl lemma: let u^ε and v^ε be sequences of vector fields bounded in L^p and L^q respectively, for Hölder conjugate indices $1 < p, q < \infty$, such that $\operatorname{div} u^\varepsilon \Subset \mathrm{W}^{-1,p}$ and $\operatorname{rot} v^\varepsilon \Subset \mathrm{W}^{-1,q}$. Then $\langle u^\varepsilon, v^\varepsilon\rangle \overset{*}{\rightharpoonup} \langle u, v\rangle$.

We now focus on the de Rham complex

$$\cdots \xrightarrow{d} \Gamma(M, \wedge^k T^*M) \xrightarrow{d} \Gamma(M, \wedge^{k+1} T^*M) \xrightarrow{d} \cdots \tag{6.3}$$

where $\wedge^k T^*M$ is the alternate k-th power of the cotangent bundle (i.e. k-forms on M), and d is the exterior derivative. We denote by δ the adjoint of d. M being a Riemannian manifold, $\wedge^k T^*M$ is equipped with an inner product.

We obtain a div-curl lemma for products of differential forms. As before, let $\alpha^\varepsilon \in \mathrm{L}^p(M, \wedge^k T^*M)$ and $\beta^\varepsilon \in \mathrm{L}^q(M, \wedge^k T^*M)$, such that $d\alpha^\varepsilon \Subset \mathrm{W}^{-1,p}(M, \wedge^{k+1} T^*M)$ and $\delta\beta^\varepsilon \Subset \mathrm{W}^{-1,q}(M, \wedge^{k-1} T^*M)$. Then, by Lemma 6.4, for all $\phi \in C_c^\infty(M)$,

$$\int_M \langle \alpha^\varepsilon, \beta^\varepsilon\rangle \phi \to \int_M \langle \alpha, \beta\rangle \phi.$$

Note that

$$\int_M \langle \alpha, \beta\rangle \phi = \int_M \phi(\alpha \wedge *\beta),$$

and moreover the confinement hypothesis for $\delta\beta^\varepsilon$ is equivalent to the statement that $d(*\beta^\varepsilon) \Subset \mathrm{W}^{-1,q}(M, \wedge^{n-k+1} T^*M)$. Thus, using the graded structure of the de Rham complex, we see that the following result holds true: let $\alpha^\varepsilon \in \mathrm{L}^p(M, \wedge^k T^*M)$ and $\beta^\varepsilon \in \mathrm{L}^q(M, \wedge^m T^*M)$, such that $d\alpha^\varepsilon \Subset \mathrm{W}^{-1,p}(M, \wedge^{k+1} T^*M)$ and $d\beta^\varepsilon \Subset \mathrm{W}^{-1,q}(M, \wedge^{m+1} T^*M)$. Then for all $\phi \in \Omega^{n-k-m}(M)$ smooth, compactly supported test $(n-k-m)$-form, there holds

$$\int_M \phi \wedge \alpha^\varepsilon \wedge \beta^\varepsilon \to \int_M \phi \wedge \alpha \wedge \beta.$$

This is a special case of the main result of [153]. By induction one obtains the general result of [153], which concerns general alternating products of differential forms.

6.1.3 Lie algebra-valued differential forms

Let $\mathfrak{g}$ be a Lie algebra. We consider the complex of $\mathfrak{g}$-valued differential forms

$$E(i) = \Gamma(M, \mathfrak{g} \otimes \wedge^i T^*M),$$

and the differential given by $D(i) = d$, the exterior differential. $\mathfrak{g}$ is equipped with the Lie bracket, which makes E into a non-associative bundle. Additionally, it is equipped with an inner product.

The situation is quite different from the case of the complex (6.3). Here, anti-commutativity of the Lie algebra $\mathfrak{g}$ implies that $\mathfrak{g}$-valued differential forms need not satisfy alternativity, *e.g.* for A a $\mathfrak{g}$-valued one-form, the product $[A \wedge A]$ need not vanish (see Appendix B).

The main aim of the discussion is the following result, of which we now give a detailed proof.

Lemma 6.5. *Let (M, g) be a Riemannian manifold with $n = \dim M$, let $1 \leqslant \mu_i \leqslant n$ and $1 < p_i < \infty$ such that*

$$\sum_{i=1}^{k} \frac{1}{p_i} = 1, \qquad \sum_{i=1}^{k} \mu_i =: s \leqslant n,$$

and let $\mathfrak{g}$ be a Lie algebra. Assume that $A^i_\varepsilon \in \mathrm{L}^{p_i}_{loc}(M, \mathfrak{g} \otimes \wedge^{\mu_i} T^*M)$ is a sequence of differential forms such that

$$A^i_\varepsilon \rightharpoonup A^i \qquad \text{weakly in } \mathrm{L}^{p_i}_{loc} \text{ as } \varepsilon \to 0, \tag{6.4}$$

$$dA^i_\varepsilon \Subset \mathrm{W}^{-1,p_i}_{loc}. \tag{6.5}$$

Then, for an arbitrary smooth, compactly supported test form $\phi \in \Gamma(M, \mathfrak{g} \otimes \wedge^{n-s} T^*M)$,

$$\lim_{\varepsilon \to 0} \int_M \langle [A^1_\varepsilon \wedge [\cdots \wedge A^k_\varepsilon] \cdots], \phi \rangle = \int_M \langle [A^1 \wedge [\cdots \wedge A^k] \cdots], \phi \rangle. \tag{6.6}$$

Proof. The proof is based on an induction argument. There is nothing to prove when $k = 1$.

Step 1. We now prove (6.6) for $k = 2$. Let A^1_ε and A^2_ε be $\mathfrak{g}$-valued forms of degree μ_1 and μ_2, respectively. The complex $E(i) = \Gamma(M, \mathfrak{g} \otimes \wedge^i T^*M)$, equipped with the exterior differentiation d, is an elliptic complex. Thus, by Lemma 6.3, for $p_1, p_2 \in (1, \infty)$, for each A^i_ε satisfying conditions (6.4)–(6.5), there exist Ψ^i_ε and ρ^i_ε such that

$$\begin{aligned}
&A^i_\varepsilon = d\Psi^i_\varepsilon + \rho^i_\varepsilon, && \\
&\Psi^i_\varepsilon \to \Psi^i && \text{in } \mathrm{L}^{p_i}_{\mathrm{loc}} \\
&d\Psi^i_\varepsilon \rightharpoonup d\Psi^i && \text{in } \mathrm{L}^{p_i}_{\mathrm{loc}} \\
&\rho^i_\varepsilon \to \rho^i && \text{in } \mathrm{L}^{p_i}_{\mathrm{loc}}.
\end{aligned}$$

Thus, as in the proof of Lemma 6.4, every product $[A^i_\varepsilon \wedge A^j_\varepsilon]$ can be decomposed as the sum of a product of form $[d\Psi^i_\varepsilon \wedge d\Psi^j_\varepsilon]$, a product of a weakly-strongly converging sequence of form $[d\Psi^i_\varepsilon \wedge \rho^j_\varepsilon]$ (that converges in $\mathrm{L}^1_{\mathrm{loc}}$), and a strongly convergent product $[\rho^i_\varepsilon \wedge \rho^j_\varepsilon]$. Therefore, the only term we have to deal with is $[d\Psi^1_\varepsilon \wedge d\Psi^2_\varepsilon]$.

Integrating by parts, we obtain that, for an arbitrary smooth, compactly supported test form $\phi \in \Gamma(M, \mathfrak{g} \otimes \wedge^{n-s} T^*M)$,

$$\begin{aligned}\int_M \langle [d\Psi^1_\varepsilon \wedge d\Psi^2_\varepsilon], \phi\rangle &= \int_M \langle d\Psi^1_\varepsilon, *[d\Psi^2_\varepsilon \wedge *\phi]\rangle = \int_M \langle \Psi^1_\varepsilon, \delta * [d\Psi^2_\varepsilon \wedge *\phi]\rangle \\ &= \int_M \langle \Psi^1_\varepsilon, *d[d\Psi^2_\varepsilon \wedge *\phi]\rangle = \int_M \langle [\Psi^1_\varepsilon \wedge d\Psi^2_\varepsilon], \delta\phi\rangle,\end{aligned}$$

by virtue of Stokes' theorem and Jacobi's identity, where δ is the co-differential on M. Using that Ψ^1_ε converges strongly to Ψ^1 in $\mathrm{L}^{p_1}_{\mathrm{loc}}$, we conclude that,

$$\int_M \langle [\Psi^1_\varepsilon \wedge d\Psi^2_\varepsilon], \delta\phi\rangle \to -\int_M \langle [\Psi^1 \wedge d\Psi^2], \delta\phi\rangle = \int_M \langle [d\Psi^1 \wedge d\Psi^2], \phi\rangle \qquad \text{as } \varepsilon \to 0,$$

Step 2. We now deal with the general case. For $\mathfrak{g}$-valued forms $A^i, i = 1, \ldots, k$, each of degree μ_i, assume by induction that the statement holds for the product of $k-1$ forms. By Lemma 6.3, we may write $A^i_\varepsilon = d\Psi^i_\varepsilon + \rho^i_\varepsilon$ for all $i = 1, \ldots, k$. Thus, the product $[A^1_\varepsilon \wedge [\cdots \wedge A^k_\varepsilon]\cdots]$ may be expressed as a sum of products involving $d\Psi^i_\varepsilon$ and ρ^i_ε. In this sum, the only term we have to treat is the product, $Q^\varepsilon := [d\Psi^1_\varepsilon \wedge [\cdots \wedge d\Psi^k_\varepsilon]\cdots]$; all the other terms are treated by the induction hypothesis, the Banach–Alaoglu theorem, and the uniqueness of weak limits.

Note that, for each fixed ε, the form Q_ε is closed, *i.e.*, $dQ_\varepsilon = 0$. Integrating by parts, we obtain that, for an arbitrary smooth, compactly supported test form $\phi \in \Gamma(M, \mathfrak{g} \otimes \wedge^{n-s} T^*M)$,

$$\begin{aligned}\int_M \langle Q_\varepsilon, \phi\rangle &= \int_M \langle d\Psi^1_\varepsilon, *[d\Psi^2_\varepsilon \wedge [\cdots \wedge d\Psi^k_\varepsilon]\cdots] \wedge *\phi]\rangle \\ &= \int_M \langle [\Psi^1_\varepsilon \wedge [d\Psi^2_\varepsilon \wedge [\cdots \wedge d\Psi^k_\varepsilon]\cdots]], \delta\phi\rangle \\ &\to -\int_M \langle [\Psi^1 \wedge [d\Psi^2 \wedge [\cdots \wedge d\Psi^k]\cdots]], \delta\phi\rangle \\ &= \int_M \langle [d\Psi^1 \wedge [d\Psi^2 \wedge [\cdots \wedge d\Psi^k]\cdots]], \phi\rangle\end{aligned}$$

by virtue of Stokes' theorem and Jacobi's identity.

By induction, we complete the proof. Q.E.D.

The hypothesis of compact confinement (6.5) can be relaxed if we assume in addition that the non-linear term is equi-integrable. In the case of the classical div-curl lemma, this observation is due to [44].

Lemma 6.6. *Let* (M, g) *be a Riemannian manifold,* $n = \dim M$*,* $1 \leqslant \mu_i \leqslant n$*, and* $1 \leqslant p_i \leqslant \infty$ *so that*

$$\sum_{i=1}^{k} \frac{1}{p_i} = 1, \qquad \sum_{i=1}^{k} \mu_i := s \leqslant n.$$

Assume that $A^i_\varepsilon \in \mathrm{L}^{p_i}_{loc}(M, \mathfrak{g} \otimes \wedge^{\mu_i} T^* M)$ *be sequences of differential forms such that, as* $\varepsilon \to 0$*,*

$$\begin{aligned} &A^i_\varepsilon \rightharpoonup A^i \qquad \textit{weakly in } \mathrm{L}^{p_i}_{loc}, \\ &dA^i_\varepsilon \Subset \mathrm{W}^{-1,1}_{loc}(M, \mathfrak{g} \otimes \wedge^{\mu_i} T^* M), \\ &[A^1_\varepsilon \wedge [A^2_\varepsilon \wedge [\cdots \wedge A^k_\varepsilon] \cdots]] \quad \textit{is locally equi-integrable.} \end{aligned}$$

Then

$$A^1_\varepsilon \wedge \cdots \wedge A^k_\varepsilon \longrightarrow A^1 \wedge \cdots \wedge A^k \qquad \textit{in the sense of distributions.}$$

The proof follows the structure of that of [44].

Proof. We deal with the case where $k = 2$ and $\mu_1 = \mu_2 = 1$. The general argument follows exactly the same lines as in Lemma 6.5 and [153].

Step 1: by the biting lemma, for each $1 \leqslant i \leqslant k = 2$, there exists subsets $E^\varepsilon_i \subset M$ such that ,

- $|E^\varepsilon_i| \to 0$ as $\varepsilon \to 0$.
- $|A^\varepsilon_i|^{p_i} \chi_{M \setminus E^\varepsilon_i}$ is equi-integrable.

We note that $| \cup_{j=1}^k E^\varepsilon_k| \to 0$.

We consider the modified sequences $B^\varepsilon_i := A^\varepsilon_i \chi_{M \setminus E^\varepsilon_i}$. The B_i converge weakly to $B_i \in \mathrm{L}^{p_i}$, and they are L^{p_i}-equi-integrable.

Step 2: we claim that $dB^\varepsilon_i \Subset \mathrm{W}^{-1,p_i}$ for each $1 \leqslant i \leqslant k$. Indeed, without loss of generality it suffices to prove that if $dB^\varepsilon_1 \to 0$ in $\mathrm{W}^{-1,1}$ then $dB^\varepsilon_1 \to 0$ in W^{-1,p_1}. In view of the fact that B^ε_i are uniformly bounded in L^{p_i}, it is clear that $\|dB^\varepsilon_1\|_{\mathrm{W}^{-1,p_i}} < \infty$ for all ε; it suffices to show that $\|dB^\varepsilon_1\|_{\mathrm{W}^{-1,p_i}} \to 0$, i.e. for any arbitrary compactly supported $\phi \in \Omega^{n-d_1-1}(M, \mathfrak{g})$ with $\|\nabla \phi\|_{\mathrm{L}^{p_2}} \leqslant 1$, there holds $| \int_M [B^\varepsilon_1 \wedge d\phi]| \to 0$ uniformly in ϕ.

To do this, following [44], we let $\kappa > 0$ be a fixed real number (to be chosen), and we consider a compactly supported κ-Lipschitz approximation ψ of ϕ, such that

$$|\{x \in M \,|\, \phi(x) \neq \psi(x)\} \cup \{x \in M \,|\, \nabla\phi(x) \neq \nabla\psi(x)\}| =: |S| \leqslant C(M)\kappa^{-p_2}.$$

One can estimate

$$\begin{aligned}\left|\int_M [B_1^\varepsilon \wedge d\phi]\right| &\leqslant \left|\int_M [B_1^\varepsilon \wedge (d\phi - d\psi)]\right| + \left|\int_M [B_1^\varepsilon \wedge d\psi]\right| \\ &\leqslant \left|\int_S B[_1^\varepsilon \wedge (d\phi - d\psi)]\right| + \|d\psi\|_{\mathrm{L}^\infty} \|dB_i^\varepsilon\|_{\mathrm{W}^{-1,1}} \\ &\leqslant \left(\int_S |B_1^\varepsilon|^{p_1}\right)^{1/p_1} \left(\int_S |d\phi - d\psi|^{p_2}\right)^{1/p_2} + \kappa \|dB_i^\varepsilon\|_{\mathrm{W}^{-1,1}} \\ &\leqslant \left(\int_S |B_1^\varepsilon|^{p_1}\right)^{1/p_1} \left(\|d\phi\|_{\mathrm{L}^{p_2}(S)} + \kappa |S|^{1/p_2}\right) + \kappa \|dB_i^\varepsilon\|_{\mathrm{W}^{-1,1}}.\end{aligned}$$

Since $|B_1^\varepsilon|^{p_1} = |A_1^\varepsilon|^{p_1} \chi_{M \setminus E_1^\varepsilon}$ is equi-integrable, we may assume that there exists a positive function $\rho = \rho(s)$, defined for $s > 0$, which does not depend on ε, satisfies $\rho(s) \to 0$ whenever $s \to 0$ and such that

$$\int_S |B_1^\varepsilon|^{p_1} \leqslant \rho(|S|).$$

Inserting this into the previous line of estimate, we obtain

$$\begin{aligned}\left|\int_M [B_1^\varepsilon \wedge d\phi]\right| &\leqslant \rho(C(M)\kappa^{-p_2})^{1/p_1} \left(\|d\phi\|_{\mathrm{L}^{p_2}(S)} + \kappa |S|^{1/p_2}\right) + \kappa \|dB_i^\varepsilon\|_{\mathrm{W}^{-1,1}} \\ &\leqslant \rho(C(M)\kappa^{-p_2})^{1/p_1} (1 + C(M)^{1/p_2}) + \kappa \|dB_i^\varepsilon\|_{\mathrm{W}^{-1,1}}.\end{aligned}$$

Note that if we let κ go to infinity, then $\rho(C(M)\kappa^{-p_2}) \to 0$; if in addition $1/\kappa$ grows slower than $\|dB_i^\varepsilon\|_{\mathrm{W}^{-1,1}}$, the whole of the RHS goes to zero. Thus we take

$$\kappa = \kappa(\varepsilon) = \frac{1}{\sqrt{\|dB_1^\varepsilon\|_{\mathrm{W}^{-1,1}}}}.$$

This proves the claim.

Step 3: applying Lemma 6.5, we obtain $[B_1^\varepsilon \wedge B_2^\varepsilon] \overset{*}{\rightharpoonup} [B_1 \wedge B_2]$ in the sense of distributions.

Note that by the above, $B_i^\varepsilon \rightharpoonup B_i$ in L^p. Since $|E_i^\varepsilon| \to 0$, $B_i = A_i$ in L^{p_i}. Similarly, $|E_1^\varepsilon \cup E_2^\varepsilon| \to 0$, so that $[B_1^\varepsilon \wedge B_2^\varepsilon] \overset{*}{\rightharpoonup} [B_1 \wedge B_2] = [A_1 \wedge A_2]$. This concludes the proof. Q.E.D.

6.2 Proof of Theorem 6.1

Recall that for two connections ∇ and $\nabla_A = \nabla + A$, their curvatures are related by

$$\Omega_A = \Omega_\nabla + \nabla A + \frac{1}{2}[A \wedge A].$$

Given a sequence of connections $\nabla^\varepsilon = \nabla + A^\varepsilon$, verifying convergence in the sense of distributions of the sequence of associated curvature therefore amounts to showing the convergence of the quadratic term $[A^\varepsilon \wedge A^\varepsilon]$.

At first, one might be tempted to apply Lemma 6.5 to the complex formed by

$$E(i) = \mathbf{Ad}(P) \otimes \wedge^i T^* M, \quad D(i) = \nabla.$$

However, this complex is not exact, unless $\nabla \circ \nabla = 0$, which is to say, unless $\Omega_\nabla = 0$. This in turns implies that the $E(i)$ are trivial bundles. Instead, a localisation argument is necessary.

Proof of Theorem 6.1. **Step 1:** we consider the local result first.

Fix a local trivialisation of the bundle P, so that locally $P \simeq G \times B^n$. The parameters of this trivialisation induce a trivialisation of $\mathbf{Ad}(P)$ as $\mathfrak{g} \times \mathbf{R}^n$. Here $\mathfrak{g}$ is the Lie algebra associated to the Lie group G (recall that P is a principal G-bundle). In this trivialisation, we consider the trivial connection $\widetilde{\nabla} = d$, which acts by exterior differentiation. Then the complex formed by $E(i) = \Gamma(B, \mathfrak{g} \otimes \wedge^i \mathbf{R}^n)$ and $D(i) = d$ is exact and elliptic. Moreover, the trivial bundle $\mathfrak{g} \times \wedge^i \mathbf{R}^n$ is an algebra bundle with internal multiplication given by the Lie bracket.

With respect to $\widetilde{\nabla}$, we can write $\nabla^\varepsilon|_B = d + \tilde{A}^\varepsilon$, and similarly $\nabla_A|_B = d + \tilde{A}$. Since $A^\varepsilon \rightharpoonup A$ weakly in L^p, we have $\tilde{A}^\varepsilon \rightharpoonup \tilde{A}$ in L^p.

Moreover, in this trivialisation, we have

$$\Omega^\varepsilon = \Omega_{\widetilde{\nabla}} + d\tilde{A}^\varepsilon + \frac{1}{2}[\tilde{A}^\varepsilon \wedge \tilde{A}^\varepsilon] = d\tilde{A}^\varepsilon + \frac{1}{2}[\tilde{A}^\varepsilon \wedge \tilde{A}^\varepsilon].$$

It is clear that $d\tilde{A}^\varepsilon \overset{*}{\rightharpoonup} d\tilde{A}$ in the sense of distributions. Moreover, there holds $d\tilde{A}^\varepsilon = \Omega^\varepsilon - \frac{1}{2}[\tilde{A}^\varepsilon \wedge \tilde{A}^\varepsilon]$, and so $d\tilde{A}^\varepsilon$ is in a bounded subset of Radon measures, which compactly embeds into $\mathrm{W}^{-1,q}$ for $q < 1^*$. On the other hand $\tilde{A}^\varepsilon$ is uniformly bounded in L^p and so $d\tilde{A}^\varepsilon$ is bounded in $\mathrm{W}^{-1,p}$. By interpolation, as $q < 1^* \leqslant 2 < p$, we obtain $d\tilde{A}^\varepsilon \Subset \mathrm{W}^{-1,2}$. By Lemma 6.5, we conclude that $[\tilde{A}^\varepsilon \wedge \tilde{A}^\varepsilon] \overset{*}{\rightharpoonup} [\tilde{A} \wedge \tilde{A}]$ in the sense of distributions.

Overall, for a compactly supported $\phi \in \Gamma(B, \mathbf{Ad}(P) \otimes \wedge^2 T^* M)$, we have

$$\begin{aligned}
\int_M \langle \Omega^\varepsilon, \phi \rangle &= \int_B \langle \Omega^\varepsilon, \phi \rangle \\
&= \int_B -\langle \tilde{A}^\varepsilon, \delta\phi \rangle + \frac{1}{2}\langle [\tilde{A}^\varepsilon \wedge \tilde{A}^\varepsilon], \phi \rangle \\
&\to \int_B \langle \Omega_A, \phi \rangle.
\end{aligned}$$

Step 2: we now wish to globalise the argument. To do, we cover M by local trivialisable subsets B_i, and consider trivialisations $P|_{B_i} \to B_i$, and similarly for the bundle $\mathbf{Ad}(P) \otimes T^*M$. By compactness, we can extract a finite subcover of local trivialisations. The local argument applies on each of them.

Let $1 = \sum_{i=1}^m \chi_i$ be a partition of unity adapted to the local trivialisations, such that $\operatorname{supp} \chi_i \subset B_i$. For a smooth $\phi \in \Gamma(B, \mathbf{Ad}(P) \otimes \wedge^2 T^*M)$, we have

$$\begin{aligned}\int_M \langle \Omega^\varepsilon, \phi \rangle &= \sum_{i=1}^m \int_{B_i} \chi_i \langle \Omega^\varepsilon, \phi \rangle \\ &= \sum_{i=1}^m \int_B \chi_i \left(-\langle \tilde{A}^\varepsilon, \delta\phi \rangle + \frac{1}{2} \langle [\tilde{A}^\varepsilon \wedge \tilde{A}^\varepsilon], \phi \rangle \right) \\ &\to \sum_{i=1}^m \int_{B_i} \chi_i \langle \Omega_A, \phi \rangle = \int_M \langle \Omega_A, \phi \rangle .\end{aligned}$$

Q.E.D.

We note immediately a corollary.

Corollary 6.7. *Let M, $P \to M$, ∇ as in Theorem 6.1, and let $(\nabla^\varepsilon) \subset \mathcal{A}(P)$ be a sequence of flat connections on P. Assume that $\sup_\varepsilon \|A^\varepsilon\|_{\mathrm{L}^p} < \infty$, and up to passing to a subsequence, $A^\varepsilon \rightharpoonup A$. Then the connection $\nabla + A$ is flat.*

Proof. By hypothesis, ∇^1 is a flat connection. Moreover, $\nabla^\varepsilon - \nabla^1 = \tilde{A}^\varepsilon$ is bounded in L^p, and therefore weakly convergent in L^p, up to a non-relabelled sequence.

As the connection ∇^1 is flat, we have $\Omega^1 = \nabla^1 \circ \nabla^1 = 0$ and so the complex

$$\Gamma(M, \mathbf{Ad}(P) \otimes \wedge^\bullet T^*M) \xrightarrow{\nabla^1} \Gamma(M, \mathbf{Ad}(P) \otimes \wedge^{\bullet+1} T^*M)$$

is exact. The rest of the proof follows as in the proof of Theorem 6.1, Step 1. Q.E.D.

Theorem 6.1 does not extend to the borderline case $p = 2$. However, under the additional assumption that the sequence $[A^\varepsilon \wedge A^\varepsilon]$ is equi-integrable, one may repeat the proof of Theorem 6.1, substituting Lemma 6.6 for Lemma 6.5 in Step 2 of the proof, thus proving the following statement:

Corollary 6.8. *Let (M, g), $P \to M$ and $\nabla \in \mathcal{A}(P)$ be as in Theorem 6.1, and let $\nabla^\varepsilon = \nabla + A^\varepsilon \subset \mathcal{A}(P)$ be a sequence of connection forms such that*

$$\sup_{\varepsilon > 0} \|A^\varepsilon\|_{\mathrm{L}^2} < \infty,$$

$$\sup_{\varepsilon > 0} \|\Omega_\varepsilon\|(M) < \infty,$$

and assume in addition that $[A^\varepsilon \wedge A^\varepsilon]$ *is equi-integrable. Finally let* $\nabla_A = \nabla + A$ *be an* L^p *connection such that* $A^\varepsilon \rightharpoonup A$ *in* L^p *up to a non-relabelled subsequence. Then* $\Omega_\varepsilon \overset{*}{\rightharpoonup} \Omega_A$ *in the sense of distributions*

6.3 Yang–Mills connections

Assuming an L^p bound on a set of connections, even with no assumption about the first derivatives, yields compactness results for non-linear PDE. Our first application of Theorem 6.1 is a compactness theorem for Yang–Mills equations.

6.3.1 The Yang–Mills functional

Recall (see *e.g.* [61]) that a connection $\nabla_A \in \mathcal{A}(P)$ is a Yang–Mills connection if it is a critical point of the Yang–Mills action integral $\mathcal{YM}(\nabla_A)$, defined by

$$\mathcal{YM}(\nabla_A) := \|\Omega_A\|^2_{\mathrm{L}^2} = \int_M |\Omega_A|^2. \tag{6.7}$$

The Euler–Lagrange equation of this functional is the Yang–Mills equation,

$$D_A^* \Omega_A = 0, \tag{6.8}$$

where D_A^* refers to the adjoint of D_A, the covariant derivative associated with ∇_A. Recall that, for any k-form $\omega \in \Gamma(M, \mathfrak{g} \otimes \wedge^k T^* M)$, its covariant derivative is given by

$$D_A \omega = d\omega + [A \wedge \omega].$$

Thus, the adjoint covariant derivative is given by

$$D_A^* \omega = \delta\omega - (-1)^{(n-k)(k-1)} * [A \wedge *\omega]$$

for any arbitrary k-form $\omega \in \Gamma(M, \mathfrak{g} \otimes \wedge^k T^* M)$, where δ is the co-differential operator on k-forms, and $*$ is the Hodge duality operator for $n = \dim(M)$.

On a compact Riemannian manifold without boundary, the Yang–Mills functional (6.7) is finite for connections $\nabla_A \in \mathcal{A}^{1,2}(P) \cap \mathcal{A}^{0,4}(P)$. By Sobolev embeddings, $W^{1,p}$ connections for $p \geqslant \max(\frac{4n}{n+4}, 2)$ are in this class. Indeed, such connections are in L^4 so that $[A \wedge A] \in \mathrm{L}^2$ and $\Omega_A \in \mathrm{L}^2$. In such a space, the critical points of (6.7) satisfy equation (6.8) in the weak sense; that is, for all $\phi \in \Lambda^1(M, \mathbf{Ad}(P))$,

$$\int_M \langle \Omega_A, D_A \phi \rangle = 0. \tag{6.9}$$

This is the weak form (or distributional form) of equation (6.8).

We now state our theorem.

THEOREM 6.9. *Let P be a principal G-bundle over M, and let $\widetilde{\nabla}$ be a fixed connection. Consider a sequence of weak Yang-Mills connections $\nabla_{A_\varepsilon} \in \mathcal{A}^{0,4}(P)$, which can be written as $\nabla_{A_\varepsilon} = \widetilde{\nabla} + A_\varepsilon$ for the $\mathbf{Ad}(P)$-valued one-form A_ε, so that $D^*_{A_\varepsilon}\Omega_{A_\varepsilon} = 0$. Assume that*

$$A_\varepsilon \rightharpoonup A \text{ weakly in } \mathrm{L}^4_{loc}(M, \mathbf{Ad}(P) \otimes T^*M), \tag{6.10a}$$

$$\sup_{\varepsilon>0} \|\Omega_{A_\varepsilon}\|_{\mathrm{L}^2(E)} < \infty \qquad \text{for any bounded domain } E \subset M. \tag{6.10b}$$

Then $\widetilde{\nabla} + A$ is a weak Yang–Mills connection.

The first hypothesis (6.10a) is the weak convergence hypothesis. The second hypothesis (6.10b) states that the sequence considered has local finite energy. We recall that, as a distribution, the curvature form makes sense for $\mathrm{L}^2_{\mathrm{loc}}$ connections. Hypothesis (6.10b) states that this distribution is in fact an $\mathrm{L}^2_{\mathrm{loc}}$ function, for each $\varepsilon > 0$. This does not imply that the connection itself is bounded in $\mathrm{W}^{1,2}_{\mathrm{loc}}$ in general.

We remark that the same result in Theorem 6.9 holds if the sequence of Yang-Mills connections ∇_{A_ε} is relaxed as a sequence of approximate Yang–Mills connections that satisfy

$$D^*_{A_\varepsilon}\Omega_{A_\varepsilon} \overset{*}{\rightharpoonup} 0 \qquad \text{in } \mathcal{D}',$$

beside conditions (6.10a)–(6.10b); that is, the weak limit is still a weak Yang–Mills connection.

Remark 6.2. *In dimension $n = 4$, much attention has been devoted to anti-self-dual connections, which form a special class of solutions to the Yang–Mills equations. A connection A is anti-self-dual if it satisfies $*\Omega_A = \omega \wedge \Omega_A$. Applying Theorem 6.1 to the anti-self-duality equation, it is straightforward to check that anti-self-dual connections are stable under weak L^p-convergence for $p > 2$.*

Similarly, in higher dimensions, $n > 4$, the anti-self-dual condition $*\Omega_A = \Omega_A$ generalises in the following way. Let ω be a closed $(n-4)$-form. A connection A is said to be ω-anti-self-dual if it satisfies

$$*\Omega_A = -\omega \wedge \Omega_A.$$

Theorem 6.1 thus shows the weak stability of the class of ω-anti-self-dual connections.

6.3.2 Proof of Theorem 6.9

The main point in the proof of Theorem 6.9 is to verify the weak continuity of quadratic terms appearing in the Yang–Mills equations. Those terms are of the form

$$[\Omega_{A_\varepsilon} \wedge A_\varepsilon],$$

i.e. the non-linear terms pass to the limit in the sense of distributions. To this effect, we use a weak version of the Bianchi identity (Step 1 of the proof).

Proof of Theorem 6.9. We divide the proof into four steps.

1. Let $p \geqslant \frac{3n}{n+2}$ and $\nabla_A = \widetilde{\nabla} + A \in \mathcal{A}^{1,p}_{\mathrm{loc}}(P)$, and let Ω_A be the curvature form. It is easy to see that

$$D_A \Omega_A = 0 \qquad \text{in the sense of distributions;} \tag{6.11}$$

that is, for any smooth, compactly supported $\phi \in \Lambda^3(M, \mathbf{Ad}(P))$,

$$\int_M \langle \Omega_A, \delta\phi - *[A \wedge *\phi]\rangle = 0. \tag{6.12}$$

Indeed, recall that the curvature form Ω_A of a smooth connection $A \in \mathcal{A}(P)$ satisfies the Bianchi identity $D_A\Omega_A = 0$. For $A \in \mathcal{A}^{1,p}_{\mathrm{loc}}(P)$, $\Omega_A \in \mathrm{L}^{\frac{p^*}{2}}_{\mathrm{loc}}(M, \mathbf{Ad}(P) \otimes \wedge^2 T^*M)$ so that, by Young's inequality,

$$[\Omega_A \wedge A] \in \mathrm{L}^q_{\mathrm{loc}}(M, \mathbf{Ad}(P) \otimes \wedge^3 T^*M) \qquad \text{for } \tfrac{1}{q} = \tfrac{1}{p} + \tfrac{2}{p^*},$$

where $p^* = \frac{np}{n-p}$ is the Sobolev conjugate of p.

To ensure the integrability of the product, we need $q \geqslant 1$, which is equivalent to the condition

$$p \geqslant \frac{3n}{n+2}.$$

Thus, taking a sequence of smooth connections approximating A in $\mathcal{A}^{1,p}_{\mathrm{loc}}$, we can pass to the limit in equation (6.12).

2. Consider the sequence, $\{A_\varepsilon\}_{\varepsilon>0}$, satisfying $D^*_{A_\varepsilon}\Omega_{A_\varepsilon} = 0$ in the sense of distributions. Then, for any smooth, compactly supported $\phi \in \Lambda^1(M, \mathbf{Ad}(P))$, A_ε satisfy

$$\int_M \langle \Omega_{A_\varepsilon}, d\phi\rangle = -\int_M \langle \Omega_{A_\varepsilon}, [A_\varepsilon \wedge \phi]\rangle \tag{6.13}$$

(also see equation (6.9)). By Proposition 6.1, the left-hand side of (6.13) passes to the limit in the sense of distributions.

3. We now focus on the right-hand side term. Let us fix a bounded domain E such that ϕ is compactly supported inside E. By the triple product identity, for $\phi \in \Lambda^1(M, \mathbf{Ad}(P))$, we have

$$\int_E \langle \Omega_{A_\varepsilon}, [A_\varepsilon \wedge \phi] \rangle = \int_E \langle *[*\Omega_{A_\varepsilon} \wedge A_\varepsilon], \phi \rangle.$$

Then the question reduces to the weak continuity of the non-linear term $[*\Omega_{A_\varepsilon} \wedge A_\varepsilon]$.

By the Bianchi identity (see equation (6.11)), we have

$$\int_M \langle \Omega_{A_\varepsilon}, \delta\psi \rangle = \int_M \langle \Omega_{A_\varepsilon}, *[A_\varepsilon \wedge *\psi] \rangle$$

for any compactly supported $\psi \in \Lambda^3(M, \mathbf{Ad}(P))$. By Young's inequality, we find that $d\Omega_{A_\varepsilon} \in \mathrm{L}^1(E)$ and

$$\|d\Omega_{A_\varepsilon}\|_{\mathrm{L}^1(E)} \leqslant \|\Omega_{A_\varepsilon}\|_{\mathrm{L}^2(E)} \|A_\varepsilon\|_{\mathrm{L}^2(E)} \leqslant C \|\Omega_{A_\varepsilon}\|_{\mathrm{L}^2(E)} \|A_\varepsilon\|_{\mathrm{L}^4(E)},$$

so that $\{d\Omega_{A_\varepsilon}\}$ is uniformly bounded in $\mathrm{L}^1(E)$, which implies that

$$\{d\Omega_{A_\varepsilon}\} \quad \text{is compactly contained in } \mathrm{W}^{-1,r}(E) \text{ for } r < 1^* = \tfrac{n}{n-1}, \tag{6.14}$$

by the Rellich–Kondrachov theorem. On the other hand, the curvature form sequence is uniformly bounded in L^2, so that $d\Omega_{A_\varepsilon}$ is uniformly bounded in $\mathrm{W}^{-1,q}_{\mathrm{loc}}$ for $q \leqslant 2$. By interpolation, $d\Omega_{A_\varepsilon}$ is compactly contained in $\mathrm{W}^{-1,\frac{3}{2}}_{\mathrm{loc}}$.

Furthermore, by definition, we have

$$dA_\varepsilon = \Omega_{A_\varepsilon} - \frac{1}{2}[A_\varepsilon \wedge A_\varepsilon],$$

so that dA_ε is in a bounded subset of $\mathrm{L}^{\frac{3}{2}}(E)$ which implies that

$$\{dA_\varepsilon\} \quad \text{is compact in } \mathrm{W}^{-1,s}(E) \text{ for } s < \tfrac{3n}{2n-3}. \tag{6.15}$$

As A_ε is uniformly bounded in $\mathrm{L}^4_{\mathrm{loc}}$,

$$\{dA_\varepsilon\} \quad \text{is bounded in } \mathrm{W}^{-1,4}_{\mathrm{loc}}. \tag{6.16}$$

By interpolation, we combine (6.15) with (6.16) to obtain

$$\{dA_\varepsilon\} \Subset \mathrm{W}^{-1,3}(E). \tag{6.17}$$

4. We are now in a position to apply Lemma 6.5 to the product $[\Omega_{A_\varepsilon} \wedge A_\varepsilon]$ with $p_1 = \frac{3}{2}$, $p_2 = 3$, and $k = 2$. Using (6.14) and (6.17), we conclude the proof. Q.E.D.

Remark 6.3. *The proof carries in the same way if one considers approximate solutions of the Yang–Mills equations,* i.e. *connections such that there exists a sequence of one-forms η_ε such that $d^*_{A_\varepsilon}\Omega_{A_\varepsilon} = \eta_\varepsilon$ and $\eta_\varepsilon \to 0$ in $\mathcal{D}'$.*

Chapter 7

Weak Convergence of Isometric Immersions

We study the stability of the isometry and the curvature constraints for weakly converging sequences of isometric immersions. Our basic setting in this chapter is a sequence of isometric immersions u_ε converging weakly in $\mathrm{W}^{2,p}$, for $p \geqslant 1$. We investigate whether the limit is still isometric, and whether it satisfies the Gauß–Codazzi–Ricci equations.

We first define a notion of *weak immersion*. For $p \geqslant 1$, we say that a map $u : M \to \mathbf{R}^N$ is a $\mathrm{W}^{2,p}_{\mathrm{loc}}$ immersion if it is both a $\mathrm{W}^{2,p}_{\mathrm{loc}}(M, \mathbf{R}^N)$ map and a bi-Lipschitz homeomorphism onto its image. d It is clear that, under the above hypothesis, $u(M)$ is a Lipschitz submanifold of $\mathbf{R}^N$. If $p \leqslant n$, u is not C^1 in general, so that u may fail to be a differentiable immersion in the classical sense. Moreover, when u is an isometric immersion in the classical sense, u is necessarily a homeomorphism onto its image and is also Lipschitz by the isometry condition.

The notion of weak immersions arises naturally in the study of Sobolev immersions whenever the immersions are not be assumed to be C^1 *a priori*; see *e.g.* [151, 152]. In the case that M is a two-dimensional surface given as the graph of a $\mathrm{W}^{2,2}$ $\mathbf{R}$-valued function (so that we take $N = 3$), it is known [127, 166] that the manifold induced by the graph admits a bi-Lipschitz parametrization. The definition given above generalizes this fact; see *e.g.* [166] for examples of weak immersions of surfaces in $\mathbf{R}^3$ that are not C^1.

A weak immersion of a Riemannian manifold (M, g) is isometric if it satisfies $\langle du, du \rangle = g$ pointwise almost everywhere; this is equivalently written as $u^* e = g$. We note that the induced metric $g_{ij} = \langle \partial_i u, \partial_j u \rangle$ is an $\mathrm{L}^\infty_{\mathrm{loc}}$ function on M. Similarly, it is easy to see that, for a local basis of the normal bundle $\{\nu_1, \ldots, \nu_{N-n}\}$,

$$\mathbf{II}^k_{ij} = \langle \partial_{ij} u, \nu_k \rangle, \qquad 1 \leqslant k \leqslant N - n, \quad 1 \leqslant i, j \leqslant n,$$

which are $\mathrm{L}^p_{\mathrm{loc}}$ functions.

A sequence of immersions $u_\varepsilon : M \to \mathbf{R}^N$ is said to be weakly convergent in $\mathrm{W}^{2,p}_{\mathrm{loc}} \cap W^{1,\infty}_{\mathrm{loc}}$ if there exists a $\mathrm{W}^{2,p}_{\mathrm{loc}}$ immersion u (as above) such that

$$u_\varepsilon \overset{*}{\rightharpoonup} u \qquad \text{in the weak* topology of } \mathrm{W}^{1,\infty}_{\mathrm{loc}},$$

and

$$\nabla^2 u_\varepsilon \rightharpoonup \nabla^2 u \qquad \text{weakly in } \mathrm{L}^p_{\mathrm{loc}},$$

which will be written as $u_\varepsilon \rightharpoonup u$ in $\mathrm{W}^{2,p}_{\mathrm{loc}} \cap \mathrm{W}^{1,\infty}_{\mathrm{loc}}$ from now on. It is straightforward to verify that if u_ε converges weakly to an immersion u in $\mathrm{W}^{2,p}$ for $p > 2$, then the first fundamental forms of u_ε converge in $\mathrm{L}^\infty_{\mathrm{loc}}$ to that of u (see Section 7.2 for a further discussion on this question). We note that the notions defined above for immersions into $\mathbf{R}^N$ may be extended in the usual way to the case where $(\mathbf{R}^N, e)$ is replaced by an arbitrary N-dimensional Riemannian manifold $(\widetilde{M}, \tilde{g})$, by isometrically embedding $(\widetilde{M}, \tilde{g})$ into a larger Euclidean space $\mathbf{R}^{\widetilde{N}}$, by Nash's theorem [135].

We give weak continuity results for the Gauß–Codazzi–Ricci equations along such sequences of weakly converging isometric immersions.

THEOREM 7.1. *Let (M, g) be a Riemannian manifold. Let u_ε be a sequence of weak $\mathrm{W}^{2,p}_{loc}$ isometric immersions of (M, g) into a Riemannian manifold $(\widetilde{M}, \tilde{g})$, and let $\mathbf{II}_\varepsilon \in \mathrm{L}^p_{loc}(M, \mathbf{Hom}(TM \times TM, NM))$ be the second fundamental forms associated to each u_ε.*

Assume that, for any bounded domain $E \subset M$,

$$\sup_{\varepsilon > 0} \|\mathbf{II}_\varepsilon\|_{\mathrm{L}^p(E)} < \infty \qquad \textit{for } p > 2.$$

Then, up to a subsequence, $u_\varepsilon \rightharpoonup u$ weakly in $\mathrm{W}^{2,p}_{loc}$ such that u is a weak $\mathrm{W}^{2,p}_{loc}$ isometric immersion and any orthonormal coframe on (M, g) may be extended into a Darboux coframe on $(\widetilde{M}, \tilde{g})$, adapted to u and satisfying the Gauß–Codazzi–Ricci equations (3.11) *in the sense of distributions.*

We remark that the Cartan equations are always verified by a sufficiently regular coframe ($\theta \in \mathrm{W}^{1,2}$ is sufficient, see Chapter 4); the point of the theorem is to construct the Darboux coframes along the sequence u_ε so as to pass to limits in the non-linear term of the Gauß equation, which depends only on the intrinsic geometry of M. Thus it represents a higher-order obstruction to the existence of isometric immersions.

This result is related to the convergence theorems for Sobolev immersions with bounded second fundamental form and volume in [14, 114], whose main result is the following: consider a sequence u_ε of $\mathrm{W}^{2,p}$ immersions with $p > n$ from closed Riemannian manifolds M_ε into $(\mathbf{R}^N, e)$ such that

$$\sup_{\varepsilon>0} \|\mathbf{II}_\varepsilon\|_{\mathrm{L}^p(M_\varepsilon)} + \sup_{\varepsilon>0} \mathrm{Vol}(M_\varepsilon) < \infty,$$

fixing a common point $q \in u_\varepsilon(M_\varepsilon) \subset \mathbf{R}^N$. Then there exist a manifold M, C^1 diffeomorphisms $\Psi_\varepsilon : M \to M_\varepsilon$, and a $\mathrm{W}^{2,p}$ immersion u such that, up to a subsequence, $u_\varepsilon \circ \Psi_\varepsilon \rightharpoonup u$ in the C^1–topology. As an immediate consequence, the metric sequence $(u_\varepsilon \circ \Psi_\varepsilon)^* e$ induced by the Euclidean structure on M converges uniformly.

In our context, if the manifold is closed, given a sequence of isometric $\mathrm{W}^{2,p}$ immersions, the isometry condition clearly implies volume constancy. Thus, when $p > n$, the C^1–convergence of the immersions is clear from the main theorem of [14]; so is the fact that the limiting immersion is isometric. Therefore, the key point of Theorem 7.1 is about the convergence of *curvatures* along sequences of isometric immersions for $p > 2$ ($p > n$ is not necessary).

A stronger result, including the case $p = 2$, may be derived in the case of codimension 1.

THEOREM 7.2. *Let (M, g) be a smooth closed, Riemannian n-manifold, $u_\varepsilon : M \to \mathbf{R}^N$, $N = n+1$, be a sequence of weak isometric immersions, and assume that $u_\varepsilon \rightharpoonup u$ weakly in $\mathrm{W}^{2,2}(M, \mathbf{R}^N)$, u satisfies $\langle du, du \rangle = g$ and on an arbitrary parallelisable subset of M, any orthonormal coframe on (M, g) may be extended into an orthonormal Darboux coframe satisfying the Gauß–Codazzi–Ricci equations.*

Compared to Theorem 7.1, the main difficulty for the proof of Theorem 7.2 is to verify that the quadratic term ${}^t\omega^{\mathbf{II}} \wedge \omega^{\mathbf{II}}$, which is only bounded in $\mathrm{L}^1_{\mathrm{loc}}$ *a priori*, is in fact weakly continuous in the sense of distributions. In this case, Lemma 6.5 fails to provide the desired conclusion. Instead, we rely on Lemma 6.6 above, motivated by Conti-Dolzmann-Müller [44] for the classical case in Euclidean spaces.

Theorems 7.1 and 7.2 may be interpreted as compactness theorems for weak isometric immersions. Before stating the next corollary, let us recall that a sequence of immersions $u_\varepsilon : M \to \mathbf{R}^N$ is said to fix a point of $\mathbf{R}^N$ if there exists $\xi \in M$ and $q \in \mathbf{R}^N$ such that for all indices ε, $u_\varepsilon(\xi) = q$.

Corollary 7.3. *Let (M, g) be a closed Riemannian manifold, and let u_ε be a sequence of weak isometric immersions in $\mathbf{R}^N$ such that the u_ε fix a point of $\mathbf{R}^N$, and moreover*

$$\sup_{\varepsilon>0} \|\nabla^2 u_\varepsilon\|_{\mathrm{L}^p(M)} < \infty, \qquad p > 2. \tag{7.1}$$

or $p \geqslant 2$ *if* $N = n+1$. *Then all accumulation points of the sequence* u_ε *are isometric and satisfy the Gauß–Codazzi–Ricci equations.*

Such bounds as (7.1) are commonplace. For instance, if the second fundamental forms of a sequence of isometric immersions are uniformly bounded in some L^p space, by definition this implies a uniform L^p bound on $\nabla^2 u$.

Proof of Corollary 7.3. As the u_ε are isometric, there holds $\langle du_\varepsilon, du_\varepsilon\rangle = g$, and so $\|du_\varepsilon\|_{\mathrm{L}^\infty} \leqslant \|g\|_{\mathrm{L}^\infty}^{1/2}$. By the Banach–Alaoglu theorem and estimate (7.1), up to a subsequence, $u_\varepsilon \rightharpoonup u$ weakly in $\mathrm{W}^{2,p}$ and weakly* in $\mathrm{W}^{1,\infty}$. Theorem 7.1 shows that u satisfies $\langle du, du\rangle = g$ and the Gauß–Codazzi–Ricci equations. Q.E.D.

Theorems 7.1 and 7.2 are closely related to [33, 36]. These works studied the weak continuity properties of the Gauß–Codazzi–Ricci equations for a sequence of solutions with a uniform bound in L^p. Our result extends and generalises the point of view of [33], but instead of considering the weak continuity of the equations themselves, we investigate weak continuity of *immersions*. Note that as exposed in Chapter 3, the Gauß–Codazzi–Ricci equations are a consequence of the curvature equation in $\mathbf{R}^N$, so that weak continuity of the *equations* is a corollary of Theorem 6.1 (or specifically of Corollary 6.7).

A variety of extensions and generalisations of Theorems 7.1 and 7.2 may easily be obtained, treating *e.g.* the case of manifolds with boundaries, or semi-Riemannian manifolds. For the sake of brevity, these arguments have been left out of the body of this Chapter.

When $p < 2$, curvature may fail to be integrable, but the metric remains weakly continuous. The second main result of the chapter, Theorem 7.4, provides a general argument for the weak continuity of the isometry constraint under a mild boundedness assumption on the mean curvature.

THEOREM 7.4. *Let* u_ε *be a sequence of weak* $\mathrm{W}^{2,p}$ *isometric immersions converging in the weak* topology of* $\mathrm{W}^{1,\infty}$ *to* u. *Finally, let* A^ε *be the generalised mean curvatures of* u_ε *in* $\mathbf{R}^N$, *and assume that for any bounded domain* $E \subset M$,

$$\sup_{\varepsilon>0} \|A^\varepsilon\|(E) < \infty.$$

Then u *is isometric.*

Such bounds occur naturally in several problems related to immersions of surfaces: as an example, we apply Theorem 7.4 to obtain weak compactness results for the Weyl problem (see Theorem 7.10).

Theorem 7.4 contrasts with Nash's theorems [134, 135], which show that the isometry constraint is not continuous with respect to the weak* topology of $\mathrm{W}^{1,\infty}$—see a more precise discussion in §7.2.

This chapter is organised in two main sections. In Section 7.1, we establish the weak continuity of curvatures along weakly convergent sequences in $\mathrm{W}^{2,p}$. The reader will find the proofs of Theorems 7.1 and 7.2 in §7.1.4–§7.1.5, respectively §7.1.6. Section 7.2 then focuses on the isometry constraint and proves the weak continuity of the constraint under a mild boundedness hypothesis (see Theorem 7.4). In §7.2.2, we give an application of Theorem 7.4 to isometric embeddings of spheres with non-negative curvature. Finally, Appendix 7.A concludes this chapter by surveying some parallels between the Gauß–Codazzi equations of surfaces in $\mathbf{R}^3$ and the problem of transonic flow. In particular, we show how the results of this chapter mirror results of the theory of transonic flow.

7.1 Weak continuity of curvatures

7.1.1 Tangent bundle and normal bundle

Let (M, g) and $(\widetilde{M}, \tilde{g})$ be Riemannian manifolds of dimensions n and N respectively, with $N > n$, and let $u : M \to \widetilde{M}$ be a weak isometric immersion, *i.e.* a local bi-Lipschitz homeomorphism onto its image.

We recall (see Chapter 3) that the normal bundle is defined as the quotient bundle $NM = T\widetilde{M}|_{u(M)}/TM$. By orthogonality, at every point $x \in M$, the fiber $\big(T\widetilde{M}|_{u(M)}\big)_x$ decomposes into an orthogonal sum

$$\big(T\widetilde{M}|_{u(M)}\big)_x = \big(T\widetilde{M}|_{u(M)}\big)_x^{\top} \oplus \big(T\widetilde{M}|_{u(M)}\big)_x^{\perp} \simeq TM_x \oplus NM_x.$$

Additionally, we recall that the normal bundle is independent of the metric on M.

Let $E \subset M$ be an open domain, and $X \in \Gamma(E, TM)$. The tangent mapping u_* defines u_*X as an element of $(T\widetilde{M})^{\top} \subset T\widetilde{M}$, as the vector u_*X is clearly tangent to $u(M) \subset \widetilde{M}$. By isometry

$$\tilde{g}(u_*X, u_*Y) = g(X, Y),$$

and hence $|u_*X| = |X|$, where the norm is taken with respect to $T\widetilde{M}$, respectively TM.

7.1.2 Weak continuity of the metric tensor

First we deal with the isometry condition under the hypotheses of Theorems 7.1, and 7.2. Indeed, it is immediate that the isometry condition passes to limits, by Sobolev embeddings (one may also directly apply Theorem 7.4).

Lemma 7.5. *Let (M, g) be a smooth closed Riemannian manifold, and let u^ε be a sequence of weak immersions such that*

$$\sup_{\varepsilon>0} \left(\|u^\varepsilon\|_{\mathrm{W}^{1,\infty}} + \|\nabla^2 u^\varepsilon\|_{\mathrm{L}^p}\right) < \infty, \qquad p \geqslant 1. \tag{7.2}$$

Finally assume that up to subsequence, u^ε converges weakly in $\mathrm{W}^{2,p}$ to u. Then $\langle du^\varepsilon, du^\varepsilon\rangle \to \langle du, du\rangle$ almost everywhere.

Hypothesis (7.2) holds under the weak convergence hypotheses of Theorems 7.1 and 7.2. The result of the Lemma follows simply from standard compactness arguments. $\mathrm{W}^{2,p}$ embeds compactly in $\mathrm{W}^{1,q}$ for $q < 1^* = n/(n-1)$, and hence $du^\varepsilon \to du$ strongly in L^q. As $du^\varepsilon \overset{*}{\rightharpoonup} du$, it is clear that $du^\varepsilon \to du$ almost everywhere and hence the convergence is strong in L^2. In particular, if we are given a sequence of weak isometric immersions, any weak $\mathrm{W}^{2,p}$ limit point of the sequence is also isometric.

7.1.3 Construction of the Darboux coframes

Recall that a choice of Riemannian metrics fixes an identification between the tangent and cotangent bundle. In this manner, to each tangent vector X is associated a dual one-form θ via the Riemannian metric g; similarly, to each vector $u_* X$ is associated an extension of θ, denoted by $\tilde{\theta}$. Then, at a point $x \in M$, we have

$$\begin{aligned} \theta(Y) &= g(X, Y) \qquad \text{for any } Y \in TM_x, \\ \tilde{\theta}(Z) &= \tilde{g}(u_* X, Z) \qquad \text{for any } Z \in (T\widetilde{M})_{u(x)}. \end{aligned}$$

The isometry condition on u implies that $|\theta| = |\tilde{\theta}|$. Note in particular that $\tilde{\theta} \in \left(T^*\widetilde{M}\right)^\top$ since, by definition,

$$\tilde{\theta}(Z) = \tilde{g}(u_* X, Z) \; = 0 \qquad \text{for any } Z \in \left(T\widetilde{M}\right)^\perp.$$

Let

$$\begin{aligned} \tau^* \,:\, T^*M &\to T^*\widetilde{M}|_{u(M)}, \\ \theta &\mapsto \tilde{\theta}. \end{aligned}$$

Similarly, we denote its dual map $\tau : T\widetilde{M}|_{u(M)} \to TM$, which is the orthogonal projection onto the tangent space, satisfying

$$\theta(X^\top) = \theta(\tau X) = (\tau^*\theta)(X) \qquad \text{for } \theta \in T^*M_x \text{ and } X \in (T\widetilde{M})_{u(x)}.$$

Standard properties of the quotients imply that there exist bundle morphisms $\nu : NM \to T\widetilde{M}$ and $\pi : T\widetilde{M} \to NM$ (the latter being the quotient map) such that the following short sequences are exact:

$$0 \to TM \xrightarrow{u_*} TN|_{u(M)} \xrightarrow{\pi} NM \simeq T\widetilde{M}|_{u(M)}/TM \to 0, \tag{7.3}$$

$$0 \to NM \xrightarrow{\nu} T\widetilde{M}|_{u(M)} \xrightarrow{\tau} TM \to 0. \tag{7.4}$$

If $u : (M, g) \to (\widetilde{M}, \tilde{g})$ is isometric (*i.e.*, u_* is an isometry onto its image) and NM is equipped with the pull-back metric $\nu^*\tilde{g}$, then all four maps ν, τ, u_*, and π are isometries.

Finally, for notational reference, we write out the dualized versions of the sequences (7.3) and (7.4):

$$0 \to T^*M \xrightarrow{\tau^*} T^*\widetilde{M}|_{u(M)} \xrightarrow{\nu^*} N^*M \to 0,$$

$$0 \to N^*M \xrightarrow{\pi^*} T^*\widetilde{M}|_{u(M)} \xrightarrow{u^*} T^*M \to 0.$$

We now describe how to obtain bounds on mappings τ and ν.

Lemma 7.6. *Let $u : (M, g) \to (\widetilde{M}, \tilde{g})$ be a weak $W^{2,p}$ isometric immersion, $p \geqslant 1$, and let E be a bounded parallelisable domain of M. Then there exists a constant $C > 0$ depending only on the $C^{0,1}$ norm of g and the $C^{1,1}$ norm of $\tilde{g}$ such that, for any one-form $\theta \in T^*M$ on E,*

$$\|\tau^*\theta\|_{\mathrm{W}^{1,p}(E)} \leqslant C\big(\|\nabla\theta\|_{\mathrm{L}^p(E)}\|\nabla u\|_{\mathrm{L}^\infty(E)} + \|\theta\|_{\mathrm{L}^\infty(E)}\|\nabla^2 u\|_{\mathrm{L}^p(E)}\big).$$

Similar bounds also hold for ν^*.

Proof. Let $\theta^\sharp = X \in TM$, let $\tilde{\theta} = \tau^*\theta$, and let Z and W be local vector fields defined in a neighborhood of $u(M)$. We calculate

$$\nabla_W\tilde{\theta}(Z) = \tilde{g}(u_*X, \nabla_W Z) + \tilde{g}(\nabla_W(u_*X), Z),$$

where ∇ denotes here the Levi-Civita connection induced by $\tilde{g}$. By a crude application of Young's inequality, recalling that u is a bi-Lipschitz homeomorphism, we obtain

$$\begin{aligned}\|\nabla\tilde{\theta}\|_{\mathrm{L}^p} &\leqslant C(|\tilde{g}|_{C^{1,1}})\big(\|\nabla X\|_{\mathrm{L}^p}\|\nabla u\|_{\mathrm{L}^\infty} + \|X\|_{\mathrm{L}^\infty}\|\nabla^2 u\|_{\mathrm{L}^p}\big)\\ &\leqslant C(|\tilde{g}|_{C^{1,1}}, |g|_{C^{0,1}})\big(\|\nabla\theta\|_{\mathrm{L}^p}\|\nabla u\|_{\mathrm{L}^\infty} + \|\theta\|_{\mathrm{L}^\infty}\|\nabla^2 u\|_{\mathrm{L}^p}\big),\end{aligned}$$

where C depends only on g and the first derivative of g. Q.E.D.

We now extend these operators to frame bundles. Let $X \in \mathscr{D}(u(M))$ be an adapted Darboux frame. In a local trivialization, we may write $X = (X^1, \ldots, X^N)$. Then $\tau X = (X^1, \ldots, X^n) = X^\top$ is a tangent frame on TM. Similarly, $\pi X = (X^{n+1}, \ldots, X^N) = X^\perp$ is a normal frame in NM. As usual, we abuse notation and denote by the same symbol τ (respectively π) the map on frames $\mathscr{F}\tau$ (respectively $\mathscr{F}\pi$).

By duality, for a tangent coframe $\alpha = (\alpha^1, \ldots, \alpha^n)$ in T^*M and a normal coframe $\eta = (\eta^1, \ldots, \eta^{N-n})$ in N^*M,

$$\tau^*\alpha + \pi^*\eta = (\tau^*\alpha^1, \ldots, \tau^*\alpha^n, \pi^*\eta^1, \ldots, \pi^*\eta^{N-n})$$

is an adapted coframe.

Lemma 7.7. *Let (M, g), $(\widetilde{M}, \tilde{g})$ and u be as in Lemma 7.6, and let TM and NM be equipped with the metrics induced by $\tilde{g}$. Then any choice of local orthonormal coframe α on T^*M and η on N^*M yields a local Darboux coframe $\theta \in \mathscr{D}^*(u(M))$. Moreover, for any bounded domain $E \subset M$:*

$$\|\theta\|_{\mathrm{W}^{1,p}(E)} \leqslant C\big((\|\alpha\|_{\mathrm{W}^{1,p}(E)} + \|\eta\|_{\mathrm{W}^{1,p}(E)})\|\nabla u\|_{\mathrm{L}^\infty(E)} + \|\nabla u\|_{\mathrm{W}^{1,p}(E)}\big). \tag{7.5}$$

Proof. The coframe θ, when viewed as a coframe on $T^*\widetilde{M}$, is given by $\theta = \tau^*\alpha + \pi^*\eta$. Then θ is a Darboux coframe; in particular,

$$|\alpha^i| = |\tau^*\alpha^i| = 1 = |\eta^j| = |\pi^*\eta^j| \qquad \text{for } i = 1, \ldots, n \text{ and } j = 1, \ldots, N-n.$$

Moreover, Lemma 7.6 applied to τ^* and ν^* implies (7.5). Q.E.D.

7.1.4 Proof of Theorem 7.1 in the flat case

We can now provide the proof of Theorem 7.1. For simplicity of notation, we first consider the case $(\widetilde{M}, \tilde{g}) = (\mathbf{R}^N, e)$; the general case is presented in §7.1.5.

Proof. We divide the proof into four steps.

1. Let u_ε be a sequence of weak $\mathrm{W}^{2,p}_{\mathrm{loc}}$ isometric immersions of (M, g_ε) in $\mathbf{R}^N$ for $p > 2$. Then there is a sequence of maps $\nu_\varepsilon : NM \to T\mathbf{R}^N$, which induces a sequence of metrics $\hat{g}_\varepsilon = \nu_\varepsilon^* e \in \mathrm{L}^\infty_{\mathrm{loc}} \cap \mathrm{W}^{1,p}_{\mathrm{loc}}$ on the normal bundle NM.

As $u_\varepsilon \rightharpoonup u$ in $\mathrm{W}^{2,p}_{\mathrm{loc}} \cap \mathrm{W}^{1,\infty}_{\mathrm{loc}}$, there exists a map $\nu \in \mathrm{W}^{1,p}_{\mathrm{loc}} \cap \mathrm{L}^\infty_{\mathrm{loc}}$ such that $\nu_\varepsilon \rightharpoonup \nu$ in $\mathrm{W}^{1,p}_{\mathrm{loc}} \cap \mathrm{L}^\infty_{\mathrm{loc}}$, which induces a metric $\hat{g} = \nu^* e$ on NM.

Let $E \subset M$ be a bounded domain, and let α be an orthonormal tangent coframe on $TM|_E$ with respect to g. By hypothesis, $g_\varepsilon \to g$. Then orthonormalizing α with respect to each g_ε yields a sequence α_ε of coframes on TM such that α_ε is orthonormal with respect to g_ε and $\alpha_\varepsilon \to \alpha$.

If necessary by restricting E to a smaller domain, we fix a trivialization of NM over a neighborhood $E \subset M$. This trivialization fixes coordinate vectors $\{\partial^{n+1}, \dots, \partial^N\}$ on NM. Orthonormalizing this basis with respect to $\hat{g}_\varepsilon$ yields a sequence of orthonormal coframes in NM, say η_ε, such that $\eta_\varepsilon \rightharpoonup \eta$ in $\mathrm{W}^{1,p}_{\mathrm{loc}} \cap \mathrm{L}^\infty_{\mathrm{loc}}$.

2. We now define the following sequence of the Darboux coframes. For $\varepsilon > 0$, let

$$\theta_\varepsilon = \tau_\varepsilon^* \alpha_\varepsilon + \pi_\varepsilon^* \eta_\varepsilon, \qquad \theta = \tau^* \alpha + \pi^* \eta.$$

By Lemma 7.7, it is clear that $\theta_\varepsilon \rightharpoonup \theta$ in $\mathrm{W}^{1,p}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$. Let

$$d\theta_\varepsilon = \omega_\varepsilon \wedge \theta_\varepsilon \qquad \text{for } \varepsilon > 0. \tag{7.6}$$

Recall that, for the Darboux coframes, the connection form ω_ε splits into a tangential connection form $\omega_\varepsilon^\top$, a normal connection form $\omega_\varepsilon^\perp$, and a second fundamental form $\omega_\varepsilon^{\mathbf{II}}$. Then we claim that, for any bounded domain $E \subset M$,

$$\sup_{\varepsilon > 0} \|(\omega_\varepsilon^\top, \omega_\varepsilon^{\mathbf{II}}, \omega_\varepsilon^\perp)\|_{\mathrm{L}^p(E)} < \infty.$$

This follows from the general fact that a sequence of uniformly bounded orthonormal coframes in $\mathrm{W}^{1,p}$ locally generates a sequence of uniformly bounded connection forms in $\mathrm{L}^p_{\mathrm{loc}}$. More precisely, let $\theta_\varepsilon = (\theta_\varepsilon^\top, \theta_\varepsilon^\perp)$ be a sequence of the Darboux coframes indexed by $\varepsilon > 0$. Then system (7.6) is linear in ω_ε. Inverting and recalling that $|\theta_\varepsilon| = 1$ since the coframe is orthonormal by definition, we conclude that, for any bounded domain $E \subset M$,

$$\|\omega_\varepsilon\|_{\mathrm{L}^p(E)} \leqslant C \|\theta_\varepsilon\|_{\mathrm{W}^{1,p}(E)}.$$

3. From Step 2, we conclude from the weak convergence and linearity that

$$\omega_\varepsilon \rightharpoonup \omega \qquad \text{weakly in } \mathrm{L}^p_{\mathrm{loc}}.$$

Moreover, for each $\varepsilon > 0$,

$$d\omega_\varepsilon + \omega_\varepsilon \wedge \omega_\varepsilon = 0.$$

We conclude by the weak continuity of curvature (Proposition 6.1) that

$$d\omega + \omega \wedge \omega = 0. \tag{7.7}$$

Recall that

$$\omega = \begin{pmatrix} \omega^\top & {}^t\omega^{\mathbf{II}} \\ -\omega^{\mathbf{II}} & \omega^\perp \end{pmatrix} \in \mathfrak{so}(N);$$

as the Cartan equations (7.7) are satisfied, the Codazzi and Ricci equations are thus satisfied in the limit.

4. We now claim that $\omega^\top = \tau^*\beta$, where β is the connection form of coframe α, *i.e.*, $d\alpha = \beta \wedge \alpha$, which follows from a uniqueness argument. Then we have

$$d\theta^\top|_{TM} = d(\tau^*\alpha) = \tau^* d\alpha = \tau^*(\beta \wedge \alpha) = \tau^*\beta \wedge \theta^\top|_{TM}.$$

Recall that α is a coframe on (M, g) and g is a C^2 metric, so that the connection form β satisfies the curvature equation

$$d\beta + \beta \wedge \beta = \mathbf{Rm}_g(\alpha \wedge \alpha) \qquad \text{in the sense of distributions,}$$

where the right-hand side is understood as the curvature endomorphism acting on two-forms. Thus, we recover the Gauß equation:

$${}^t\omega^{\mathbf{II}} \wedge \omega^{\mathbf{II}} = d\omega^\top + \omega^\top \wedge \omega^\top = \tau^*(d\beta + \beta \wedge \beta) = \tau^*\,\mathbf{Rm}_g(\alpha \wedge \alpha).$$

This completes the proof. Q.E.D.

7.1.5 Proof of Theorem 7.1

As mentioned earlier, there is nothing special about $\mathbf{R}^N$ in Theorem 7.1, and thus the proof may be given for a general ambient Riemannian manifold $(\widetilde{M}, \tilde{g})$.

As before, we denote by $\mathbf{Rm}$ the Riemann curvature tensor of g, and $\widetilde{\mathbf{Rm}}$ that of $\tilde{g}$. Similarly, the curvature form of an orthonormal coframe on (M, g) is denoted by Ω, and that of an orthonormal coframe on $(\widetilde{M}, \tilde{g})$ by $\widetilde{\Omega}$.

Proof. Let u_ε be a sequence of isometric immersions $u_\varepsilon : (M, g) \to (\widetilde{M}, \tilde{g})$. Let $E \subset M$ be any bounded domain, and let α be an orthonormal coframe on $TM|_E$. Restricting domain E if necessary, we may consider a local trivialization of $NM|_E$. As in the proof of Theorem 7.1, maps ν_ε induce the metrics on NM by pull-back, namely $\widehat{g_\varepsilon} = \nu_\varepsilon^* \tilde{g}$. Thus, we find as earlier that there is a sequence of coframes η_ε defined on N^*M, orthonormal with respect to $\widehat{g}_\varepsilon$.

Moreover, by weak convergence, there is a map $\nu : M \to \widetilde{M}$, a metric $\hat{g} = \nu^*\tilde{g}$, and a coframe η orthonormal with respect to $\hat{g}$ such that $\eta_\varepsilon \rightharpoonup \eta$ weakly in $\mathrm{W}^{1,p}(E)$ and weak-* in L^∞.

We define the Darboux coframe

$$\theta_\varepsilon = \tau_\varepsilon^* \alpha + \pi_\varepsilon^* \eta_\varepsilon \qquad \text{for } \varepsilon > 0.$$

Let $d\theta_\varepsilon = \omega_\varepsilon \wedge \theta_\varepsilon$ for $\varepsilon > 0$. The weak convergence of $\theta_\varepsilon \rightharpoonup \theta$ and linearity imply that $\omega_\varepsilon \rightharpoonup \omega$ weakly in $\mathrm{L}^p(E)$.

By construction, θ_ε is a $\mathrm{W}^{1,p}(E)$ coframe on $(\widetilde{M}, \tilde{g})$ so that it must satisfy the curvature equation:

$$d\omega_\varepsilon + \omega_\varepsilon \wedge \omega_\varepsilon = \widetilde{\mathbf{Rm}}(\theta_\varepsilon \wedge \theta_\varepsilon).$$

Recalling that θ_ε is orthonormal and uniformly bounded in $\mathrm{L}^\infty(E)$, the curvature term $\widetilde{\mathbf{Rm}}(\theta_\varepsilon \wedge \theta_\varepsilon)$ is uniformly bounded in $\mathrm{L}^\infty(E)$. Thus, we find that, for $\varepsilon > 0$, the term

$$-\omega_\varepsilon \wedge \omega_\varepsilon + \widetilde{\mathbf{Rm}}(\theta_\varepsilon \wedge \theta_\varepsilon)$$

is uniformly bounded in $\mathrm{L}^{\frac{p}{2}}(E)$, which implies that

$$\{d\omega_\varepsilon\} \qquad \text{embeds compactly into } \mathrm{W}^{-1,q}(E) \text{ for } q < \left(\tfrac{p}{2}\right)^* = \tfrac{np}{2n-p}.$$

As ω_ε is uniformly bounded in $\mathrm{L}^p(E)$,

$$\{d\omega_\varepsilon\} \qquad \text{is uniformly bounded in } \mathrm{W}^{-1,p}(E).$$

Then, by interpolation, $d\omega_\varepsilon$ is compactly contained in $\mathrm{W}^{-1,2}(E)$. We conclude, as in Proposition 6.1, that

$$d\omega + \omega \wedge \omega = \widetilde{\mathbf{Rm}}(\theta \wedge \theta).$$

Decomposing the connection form ω along its tangential, normal, and **II** part yields the Gauß–Codazzi–Ricci equations, which are therefore satisfied in the limit as $\varepsilon \to 0$, as in the earlier proof. Q.E.D.

7.1.6 Proof of Theorem 7.2

In order to prove Theorem 7.2, we rely on the second div-curl lemma proved in Chapter 6, namely, Lemma 6.6, instead of Lemma 6.5.

Proof of Theorem 7.2. Let u_ε be a sequence of weak isometric immersions of (M, g) in $\mathbf{R}^N = \mathbf{R}^{n+1}$ for $p \geqslant 2$. We recall that, in the codimension one case, the normal bundle NM is a line bundle. Hence, we may view $\nu_\varepsilon : NM \to T\mathbf{R}^{n+1}|_{u(M)}$ simply as the Gauß map. We divide the proof into four steps.

1. Let $E \subset M$ be any bounded domain, and let α be an orthonormal coframe on $TM|_E$ with respect to g. As in the proof of Theorem 7.1, we now have the following sequence of the Darboux coframes for $\varepsilon > 0$:

$$\theta_\varepsilon = \tau_\varepsilon^* \alpha + \eta_\varepsilon,$$

where η_ε is the co-vector dual to ν_ε for each fixed $\varepsilon > 0$. Similarly, let $\theta = \tau^* \alpha + \eta$. It is clear that $\theta_\varepsilon \rightharpoonup \theta$ weakly in $\mathrm{W}^{1,p}(E)$ and weak-star in $\mathrm{L}^\infty(E)$.

2. Let ω_ε be the connection form of θ_ε for $\varepsilon > 0$. As usual, we decompose the connection form into its tangential, normal, and **II** parts; however, in codimension one, the normal connection is always zero, so that

$$\omega_\varepsilon = \begin{pmatrix} \omega_\varepsilon^\top & {}^t\omega_\varepsilon^{\mathbf{II}} \\ -\omega_\varepsilon^{\mathbf{II}} & 0 \end{pmatrix} \in \mathfrak{so}(n+1).$$

3. We now claim that $\omega_\varepsilon^\top = \tau_\varepsilon^* \beta$ for $\varepsilon > 0$, where β is the connection form associated to the tangent coframe α, *i.e.*, $d\alpha = \beta \wedge \alpha$.

Indeed, we have

$$d\theta_\varepsilon^\top|_{TM} = d(\tau_\varepsilon^* \alpha) = \tau_\varepsilon^* d\alpha = \tau_\varepsilon^*(\beta \wedge \alpha)$$

By uniqueness, this leads to the claim.

As a consequence, we conclude that $\omega_\varepsilon^\top$ converges to $\omega^\top$ in $\mathrm{W}^{1,p}(E)$ and strongly in $\mathrm{L}^2(E)$. In addition, we have

$$d\omega_\varepsilon^\top + \omega_\varepsilon^\top \wedge \omega_\varepsilon^\top = \tau_\varepsilon^*(d\beta + \beta \wedge \beta) = \tau_\varepsilon^* \Omega = \tau_\varepsilon^* \,\mathbf{Rm}(\alpha \wedge \alpha),$$

where $\mathbf{Rm}$ is the Riemann curvature endomorphism associated to g.

Furthermore, recall that θ_ε is weakly convergent in $\mathrm{W}^{1,2}(E)$. Hence, the connection form ω_ε is uniformly bounded in $\mathrm{L}^2(E)$, so that $\omega_\varepsilon^{\mathbf{II}}$ is uniformly bounded in $\mathrm{L}^2(E)$. This implies that the week limit still obeys the Codazzi equation.

4. We now prove that ${}^t\omega_\varepsilon^{\mathbf{II}} \wedge \omega_\varepsilon^{\mathbf{II}}$ is equi-integrable. Fix $\varepsilon > 0$ and an arbitrary $F \subset E$. By the Gauß equation, we have

$$\int_F |{}^t\omega_\varepsilon^{\mathbf{II}} \wedge \omega_\varepsilon^{\mathbf{II}}| = \int_F |d\omega_\varepsilon^\top + \omega_\varepsilon^\top \wedge \omega_\varepsilon^\top| = \int_F |\mathbf{Rm}(\alpha \wedge \alpha)|,$$

which is an L^1 function that is independent of ε, so that

$$\limsup_{F \subset E, |F| \to 0} \sup_{\varepsilon > 0} \int_F |{}^t\omega_\varepsilon^{\mathbf{II}} \wedge \omega_\varepsilon^{\mathbf{II}}| = 0;$$

that is, ${}^t\omega_\varepsilon^{\mathbf{II}} \wedge \omega_\varepsilon^{\mathbf{II}}$ is equi-integrable.

The conclusion now follows from Lemma 6.6, as in the proof of Theorem 7.1.

Q.E.D.

Remark 7.1. *The calculation for showing the equi-integrability of* ${}^t\omega^{\mathbf{II}} \wedge \omega^{\mathbf{II}}$ *holds for arbitrary codimension* $N - n \geqslant 1$*, and hence the Gauß equation is always weakly continuous along a sequence of isometric immersions converging weakly in* $\mathrm{W}^{2,2}_{loc}$*. However, if* $N > n+1$*, the Ricci equation contains a further quadratic term* $\omega^\perp \wedge \omega^\perp$*, for which the argument of the proof is unavailable, as the "normal curvature" is not controlled a priori.*

7.2 Weak continuity and the isometry constraint

We continue our study of weak continuity of isometric immersions in Sobolev spaces. Motivated by the results of the previous section, one would like understand the $\mathrm{W}^{2,p}$ regime for $p < 2$. Theorem 7.4 gives a very mild sufficient condition for weak continuity of the metric tensor.

Before giving a proof, we first recall that the isometry constraint is not continuous for the weak* topology of $\mathrm{W}^{1,\infty}$.

Let $u : M \to \mathbf{R}^N$ be a smooth short immersion of (M, g) in $\mathbf{R}^N$, i.e. $\langle du, du \rangle < g$ in the sense of bilinear forms. As M is compact, this is always possible provided the codimension $N - n$ is large enough. By the Nash–Kuiper theorem (see Chapter 2), u may be uniformly approximated by a sequence of immersions $u_\varepsilon \in C^1(M, \mathbf{R}^N)$, such that

$$
\begin{aligned}
&u_\varepsilon \xrightarrow{C^0} u, \\
&(u_\varepsilon)^* e = \langle du_\varepsilon, du_\varepsilon \rangle = g.
\end{aligned}
$$

Taking sup norms on both sides, we have $\|du_\varepsilon\|_{\mathrm{L}^\infty} \leqslant \|g\|^{1/2}_{\mathrm{L}^\infty}$. Therefore, up to a subsequence we have $u_\varepsilon \overset{*}{\rightharpoonup} u$ in the weak-* topology of $\mathrm{W}^{1,\infty}$. Overall this shows that the isometry constraint is not continuous with respect to the weak* topology of $\mathrm{W}^{1,\infty}$.

Remark 7.2. *This follows from the more general framework of convex integration. For simplicity, let us assume that we are on a bounded domain* $E \subset \mathbf{R}^n$*, unknowns* $Z : E \to \mathbf{R}^N$*, and a constraint set* $K \subset \mathbf{R}^N$*. We consider systems of the form*

$$
\begin{cases}
\sum_{i=1}^n A_i \partial_i Z(x) = 0, & \\
Z(x) \in K \subset \mathbf{R}^N & \textit{for almost every } x \in E.
\end{cases}
\tag{7.8}
$$

where the A_i *are constant coefficient matrices.*

Convex integration is a technique to construct solutions to such a system by deforming or iterating from a set of subsolutions. To be more precise, let us fix $X_0 \neq \varnothing$

to be a bounded space of subsolutions, and X its closure in some topology. A problem is said to satisfy the analytical h-principle if the set

$$\left\{ Z \in X \,\middle|\, \sum_{i=1}^{n} A_i \partial_i Z(x) = 0, Z(x) \in K \ a.e. \right\}$$

is dense in the topology of X (see [162]—note carefully that this notion of h-principle, though related, is distinct from Gromov's h-principle, see [77]). In other words, every element of X_0 can be obtained as the limit of solutions of the problem in the topology of X. In particular, if X_0 is formed of functions that are not solutions to (7.8), we see that the problem (7.8) is not continuous with respect to the topology of X, as solutions may converge in X to non-solutions. This may be achieved under quite general hypotheses on (7.8) and X_0 (see e.g. *[45, 128, 129, 162]).*

In the particular case of isometric immersions $u : M \to \mathbf{R}^N$, we write $Z = du$ and the setting is

$$\begin{cases} dZ & = 0, \\ Z(x) & \in K := \{\theta \in T_xM \otimes T^*_{u(x)}\mathbf{R}^N \,|\, \langle \theta, \theta \rangle = g\}, \quad x \in M, \end{cases} \tag{7.9}$$
$$X_0 = \{u \in C^\infty(M, \mathbf{R}^N) \,|\, u \text{ is an immersion and } \langle du, du \rangle < g\}.$$

The reader is referred to Chapter 3 for the meaning of the system (7.9). Here we consider only *the isometry constraint. Note that if $N \geqslant 2n$, $X_0 \neq \varnothing$ by Whitney's embedding theorem (see Chapter 2), and we define X to be the weak* closure of X_0 in* L^∞.

7.2.1 Proof of Theorem 7.4

To prove Theorem 7.4, two lemmas are required.

Lemma 7.8. *Let (M, g) be a Riemannian manifold, and $u : (M, g) \to \mathbf{R}^N$ be a weak isometric immersion. Then*

$$\triangle_g u = A(\nabla u, \nabla u), \tag{7.10}$$

where A is the trace of **II**. *In coordinates, $A = g^{ij} \mathbf{II}^m_{ij} \nu_m$, where the ν_m are $\mathbf{R}^N$-valued orthornormal vectors, normal to $u(M)$.*

Lemma 7.8 is classical if u is C^2. The reader will recognise equation (7.10) as the harmonic map equation.

Proof of Lemma 7.8. It is easy to verify that in any local coordinate system on M, a weak $\mathrm{W}^{2,p}$ isometric immersion u satisfies

$$\partial_{ij}^2 u = \Gamma_{ij}^k \partial_k u + \mathbf{II}_{ij}^m \nu_m,$$

where the indices range over $1 \leqslant i, j, k \leqslant n$ and $1 \leqslant m \leqslant N - n$, and the ν_m are $\mathbf{R}^N$-valued orthonormal vectors, normal to $u(M)$. Recalling that in any local coordinates, the Laplace–Beltrami operator is given by

$$\triangle_g = g^{ij} \partial_{ij}^2 - g^{ij} \Gamma_{ij}^k \partial_k,$$

it follows immediately from tracing by g that

$$\triangle_g u = g^{ij} \mathbf{II}_{ij}^m \nu_m.$$

In particular, as the ν_m are normal to $u(M)$, we have $\triangle_g u \perp (T\mathbf{R}^N_{u(\cdot)})^\top$ almost everywhere, in the sense that for almost every $x \in M$,

$$\triangle_g u(x) \perp (T\mathbf{R}^N_{u(x)})^\top.$$

Q.E.D.

Lemma 7.9. *Let (M, g) be a smooth Riemannian manifold, let $E \subset M$ be a bounded smooth domain, and let $p > 2$. Let $\{u_\varepsilon\} \subset \mathrm{W}^{1,p}(E)$ be a bounded sequence of functions, and assume that*

$$\sup_{\varepsilon > 0} \| \triangle_g u_\varepsilon \|(E) < \infty.$$

Then the u_ε are pre-compact in $\mathrm{W}^{1,2}(E)$.

Proof of Lemma 7.9. Let us fix $\eta > 0$. Up to a non-relabelled subsequence, by Egorov's theorem there exists a measurable set $E_\eta \subset E$ such that

- $|E \setminus E_\eta| \leqslant \eta$,
- $u_\varepsilon - u$ converges to 0 uniformly on E_η.

We can now choose k such that $|u_\varepsilon - u| \leqslant \frac{\gamma}{2}$ for some arbitrary $\gamma > 0$.

Let τ_γ be the truncation function defined for all $x \in \mathbf{R}$ by

$$\tau_\gamma(x) = \min(\gamma, \max(-\gamma, x)).$$

Finally, we pick a test function ζ such that $\zeta|_{E_\eta} = 1$. We calculate

$$\begin{aligned}\int_{E_\eta} |du_\varepsilon - du|^2 \leqslant \int_E \zeta\langle du_\varepsilon - du, d\tau_\gamma(u_\varepsilon - u)\rangle \\ = \int_E \langle du_\varepsilon, d(\zeta\tau_\gamma(u_\varepsilon - u))\rangle \\ - \int_E \langle d\zeta, du_\varepsilon\rangle\tau_\gamma(u_\varepsilon - u) \\ - \int_E \zeta\langle du, d\tau_\gamma(u_\varepsilon - u)\rangle\end{aligned}$$

Note that $\tau_\gamma(u_\varepsilon - u) \rightharpoonup$ weakly in $\mathrm{W}^{1,p}(E)$, so that the third term vanishes in the limit $\varepsilon \to 0$. For the second term, as the u_ε are bounded in $\mathrm{L}^p(E)$ and $\tau_\gamma(u_\varepsilon - u)$ converges uniformly to 0, the second integral vanishes as well. For the first term, recall that by hypothesis, there exists measures m^ε of finite total variation on M, namely $\sup_\varepsilon \|m^\varepsilon\|(E) < \infty$, and such that

$$\triangle\, u_\varepsilon = m^\varepsilon,$$

in the sense that for any $\phi \in C_c^\infty(E)$,

$$\int_E \langle du, d\phi\rangle = \int_E \phi dm^\varepsilon.$$

As $\tau_\gamma(u_\varepsilon - u) \in \mathrm{W}^{1,p} \cap \mathrm{L}^\infty$, we have

$$\begin{aligned}\int_E \langle du_\varepsilon, d(\zeta\tau_\gamma(u_\varepsilon - u))\rangle &= \int_E \zeta\tau_\gamma(u_\varepsilon - u)dm^\varepsilon \\ &\leqslant m^\varepsilon(E)\sup|\zeta\tau_\gamma(u_\varepsilon - u)| \\ &= \gamma\sup_\varepsilon \|m^\varepsilon\|(E) \to 0\end{aligned}$$

as $k \to \infty$. Overall we have gained the estimate

$$\int_{E_\eta} |du_\varepsilon - du|^2 \leqslant \gamma\sup_\varepsilon \|m^\varepsilon\|(E) \to 0.$$

Hence on E_η, du_ε converges pointwise almost everywhere to du in E_η. This holds for arbitrary η, and hence pointwise almost everywhere in M; as du_ε is bounded in $\mathrm{L}^p(E)$, we conclude that du is pre-compact in $\mathrm{W}^{1,q}(E)$ for $1 \leqslant q < p$. As by hypothesis $p > 2$, we have the result. Q.E.D.

Once these two lemmas are established, the proof of Theorem 7.4 follows immediately.

Proof of Theorem 7.4. Let $u_\varepsilon : (M, g) \to \mathbf{R}^N$ be a sequence of isometric immersions with L^1 second fundamental form. By Lemma 7.8, the $\triangle_g u_\varepsilon$ have uniformly bounded total mass. Applying Lemma 7.9 componentwise, they are pre-compact in $\mathrm{W}^{1,2}$, and in fact in $\mathrm{W}^{1,p}$ for all $p < \infty$. This is sufficient to pass to limits in the isometry constraint. Q.E.D.

7.2.2 The Weyl Problem with continuous, non-negative, Gauß curvature

As an illustration of Theorem 7.4, we construct convex isometric immersions of spheres with non-negative curvature.

THEOREM 7.10. *Let g be a $C^{2,\alpha}$ metric on $\mathbf{S}^2$, satisfying $K \geqslant 0$. Then there exists a convex isometric immersion $u : \mathbf{S}^2 \to \mathbf{R}^3$ such that $u^*e = g$, where e is the Eucliean metric on $\mathbf{R}^3$.*

This theorem is closely related to a celebrated theorem due to Nirenberg [137], which gave a complete answer to the Weyl Problem [176]. Where other approaches require a C^4 bound [81, 84, 97] on the metric g, the proof of Theorem 7.10 requires only $C^{2,\alpha}$ control of the conformal factor.

When $K > 0$, the Weyl problem reduces to solving a non-linear uniformly elliptic PDE. The ellipticity depends on a lower bound on the Gauß curvature. When $K \geqslant 0$ is allowed to vanish somewhere, ellipticity is not uniform and may be lost.

To illustrate this fact, we note that both Weyl's and Nirenberg's approaches rely on a C^2 estimate [137, 176] (see also [40]) giving a uniform bound on the mean curvature

$$\|H\|_{\mathrm{L}^\infty} \leqslant C \left(\sup_{\mathbf{S}^2} (K - \frac{\Delta K}{4K}) \right)^{1/2}. \tag{7.11}$$

Here the notation Δ refers to the Laplace–Beltrami operator with respect to the metric g on $\mathbf{S}^2$. The bound was subsequently improved in [81] to

$$\|H\|_{\mathrm{L}^\infty} \leqslant C \left(\sup_{\mathbf{S}^2} (K^2 - \frac{3}{2} \Delta K) \right)^{1/2}. \tag{7.12}$$

The bound (7.12) allowed the authors to deal with the case of metrics on $\mathbf{S}^2$ with non-negative curvature. Note that both (7.11) and (7.12) require $\Delta K \in \mathrm{L}^\infty(\mathbf{S}^2)$. By contrast, the arguments used in the proof of Theorem 7.10 requires no higher-order estimates.

The idea of the proof of Theorem 7.10 is an approximation argument, where we substitue Theorem 7.4 for Weyl's estimate (7.11). We construct smooth metrics

g^ε approximating g, apply Nirenberg's theorem 7.12 and seek the desired isometric immersion as a limit.

The first lemma constructs a sequence of approximate solutions for the problem.

Lemma 7.11. *Let* $(\mathbf{S}^2, g)$ *be as in Theorem 7.10. Then there exists a sequence of smooth metrics* g^ε *such that* $g^\varepsilon \to g$ *in* $C^{2,\alpha}$ *and a sequence of smooth convex embeddings* u_ε *such that* $\langle du_\varepsilon, du_\varepsilon \rangle = g^\varepsilon$.

The proof relies on Nirenberg's theorem for strictly convex isometric immersions (see also Section 2.3):

THEOREM 7.12 (Nirenberg)**.** *Let* g *be a smooth Riemannian metric on* $\mathbf{S}^2$ *with positive Gauß curvature* $K_g > 0$. *Then there exists a smooth isometric embedding* u *of* $(\mathbf{S}^2, g)$ *in* $\mathbf{R}^3$ *such that* $u(\mathbf{S}^2)$ *is the boundary of a convex body.*

Recall that if g is a smooth Riemannian metric on $\mathbf{S}^2$, there exists a smooth function ϕ such that $g = e^{2\phi}\sigma$, where σ is the standard round metric on the sphere. ϕ can be obtained by calculating the Gauß curvature of g and σ (recall that $K_\sigma = 1$): ϕ must satisfy the equation

$$K_g = e^{-2\phi}(1 - \Delta\phi).$$

Recall also that if g is a smooth metric on $\mathbf{S}^2$, its Gauß curvature satisfies the Gauß–Bonnet formula

$$\int_{\mathbf{S}^2} K dA = 2\pi\chi(\mathbf{S}^2) = 4\pi.$$

Proof of Lemma 7.11. Let ϕ be the conformal factor such that $g = e^{2\phi}\sigma$. As ϕ satisfies $\Delta\phi = 1 - Ke^{2\phi}$, using elliptic regularity, we obtain $\phi \in C^{2,\alpha}(\mathbf{S}^2)$.

We construct approximations of g by strictly positively curved smooth metrics. To do so, we consider a sequence ϕ^ε approximating ϕ.

First consider a function w satisfying $\Delta w = -1$ on the set $Z := K^{-1}(0)$. Note that this set is closed, and by the Gauß–Bonnet theorem, there exists at least a point of positive curvature, so that we can find such a w by solving Poisson's equation with Dirichlet boundary condition on an open set U such that $Z \subset U \subset \mathbf{S}^2$.

Then we take a sequence ϕ^ε of smooth functions converging in $C^{2,\alpha}$ to $\phi + \eta w$, where $\eta > 0$ is a small number such that as $\varepsilon \to 0$, $\eta \to 0$. Then the metrics $e^{2\phi^\varepsilon}$ are smooth, positively curved metrics converging in $C^{2,\alpha}$ norm to g.

We now apply Nirenberg's theorem to each g^ε. Then there exists a sequence of embeddings $u_\varepsilon : \mathbf{S}^2 \to \mathbf{R}^3$ such that $\langle du_\varepsilon, du_\varepsilon \rangle = g^\varepsilon$, and moreover the u_ε are convex and smooth. Q.E.D.

To prove Theorem 7.10, we now obtain a priori bounds on the u_ε, which are given in Lemma 7.13 below. The C^1 estimate is standard. Note that as $(\mathbf{S}^2, e^{2\phi}\sigma)$ is a complete compact Riemannian manifold, by the Hopf–Rinow theorem it must have bounded intrinsic diameter. Therefore we see that

$$c := \sup_{\varepsilon} \min_{Y \in \mathbf{R}^3} \max_{x \in \mathbf{S}^2} |u_\varepsilon(x) - Y| < \infty.$$

Assuming from now on and without loss of generality that the barycenter of the convex spheres $u_\varepsilon(\mathbf{S}^2)$ is the origin in $\mathbf{R}^3$, we have a uniform bound on each map u_ε.

Observe that as $(u_\varepsilon)^* e = \langle du_\varepsilon, du_\varepsilon \rangle = g^\varepsilon = e^{2\phi_\varepsilon}\sigma$, we have

$$\|\nabla u_\varepsilon\|_{\mathrm{L}^\infty} \leqslant e^{\phi^\varepsilon} < \infty.$$

Overall we have shown part (a) of the following lemma:

Lemma 7.13. *Let us consider a sequence of u_ε obtained as in Lemma 7.11. Then*

1. *there holds*

$$\sup_{\varepsilon > 0} \|u_\varepsilon\|_{C^1} < \infty.$$

2. *there holds:*

$$\sup_{\varepsilon > 0} \|\Delta u_\varepsilon\|_{\mathrm{L}^1} < \infty$$

We prove part (b), which is a Laplace estimate. The key estimate is due to Minkowksi. Let M be a closed Riemannian manifold, u be a smooth embedding $u : M \to \mathbf{R}^3$, and $\nu : M \to \mathbb{G}_{2,3}$ the Gauß map. Then

$$\int_M H dA = \int_M K \langle u, \nu \rangle dA.$$

For a proof, the reader is referred to [96, 161].

Proof. Let $\nu_\varepsilon : \mathbf{S}^2 \to \mathbb{G}_{2,3} \simeq \mathbf{S}^2 \subset \mathbf{R}^3$ be the Gauß map. Writing $\vec{H} = H\nu$, we choose the orientation of ν consistent with the sign of K^ε, so that $K^\varepsilon > 0$ and $H^\varepsilon > 0$. Since $u_\varepsilon(\mathbf{S}^2)$ is convex, the support function is positive $\langle \nu_\varepsilon, u_\varepsilon \rangle > 0$,

Recall that σ and g^ε are conformally related, i.e.

$$g^\varepsilon = e^{2\phi^\varepsilon}\sigma$$

Thus the Laplacians $\not\triangle$ and $\triangle_{g^\varepsilon}$ are conformally related by $\not\triangle = e^{-2\phi^\varepsilon} \triangle_{g^\varepsilon}$. Letting dA^ε be the volume element associated to g^ε, we have

$$\begin{aligned}
\int_{\mathbf{S}^2} |\not\triangle u_\varepsilon| dVol_\sigma &= \int_{E_i} e^{-2\phi^\varepsilon} |H^\varepsilon| dVol_\sigma \\
&= \int_{\mathbf{S}^2} H^\varepsilon dA^\varepsilon \\
&= \int_{\mathbf{S}^2} K^\varepsilon \langle u_\varepsilon, \nu_\varepsilon \rangle dA^\varepsilon \\
&\leqslant \sup |u_\varepsilon| \int_{\mathbf{S}^2} K^\varepsilon dA^\varepsilon \\
&= 4\pi \sup |u_\varepsilon|,
\end{aligned}$$

by the Gauß–Bonnet formula. The conclusion follows from Lemma 7.13. Q.E.D.

Proof of Theorem 7.10. For each g^ε, Nirenberg's theorem guarantees the existence of a smooth convex isometric immersion of $(\mathbf{S}^2, g^\varepsilon) \hookrightarrow \mathbf{R}^3$.

Applying Lemmas 7.13 and 7.9, there exist a convex map u such that $u_\varepsilon \to u$ uniformly; moreover the u_ε are pre-compact in $\mathrm{W}^{1,2}$. Because of the C^1 bound (Lemma 7.13) they are therefore pre-compact in any $\mathrm{W}^{1,p}$ space for $p < \infty$. Thus up to a subsequence the limit u satisfies $u^*e = \langle du, du \rangle = e^{2\phi}\sigma = g$ almost everywhere, and hence is a convex isometric immersion of $(\mathbf{S}^2, g)$. Q.E.D.

Remark 7.3. *Lemma 7.13 relies on the fact that the total mean curvature is a conformal invariant. The proof may be adapted to show that the first fundamental form is converging along weakly* convergent sequences of conformal immersions. For brevity, such considerations are left out of this document.*

7.A A parallel with continuum mechanics

In the works of Chen–Slemrod–Wang [35, 37], a parallel was drawn between the equations of balance of momentum in two and three dimensions, and the compatibility equations for isometric immersions for the cases $M^2 \hookrightarrow \mathbf{R}^3$ and $M^3 \hookrightarrow \mathbf{R}^6$. We highlight this parallel in light of the results of this chapter.

7.A.1 Transonic flow

For simplicity, let us fix a domain $E \subset \mathbf{R}^2$. The equations of steady compressible flow on E ask for a pair $(\rho, \mathbf{u})$ solving

$$\operatorname{div}(\rho \mathbf{u}) = 0, \tag{7.13}$$

$$\operatorname{rot} \mathbf{u} = 0. \tag{7.14}$$

Equation (7.13) expresses the conservation of mass along the flow; equation (7.14) is the additional potential flow condition. In other words, we assume that the flow is irrotational.

Furthermore, in isentropic flow ρ and $|\mathbf{u}| =: q$ are linked by the Bernoulli relation $\rho = \rho(q)$, the exact form of which depends on the precise choice of gas dynamics. For instance, if one studies γ-law gas, i.e. gas where the pressure depends on the density as $p(\rho) = \frac{1}{\gamma}\rho^\gamma$, then the Bernoulli relation takes the form

$$\rho(q) = \left(1 - \frac{\gamma - 1}{2} q^2\right)^{\frac{1}{\gamma-1}}.$$

The local speed of sound is given as

$$c^2(\rho) = p'(\rho) = \rho^{\gamma-1}.$$

When $q = |\mathbf{u}| < c$, system (7.13), (7.14) is elliptic; when $q > c$, (7.13), (7.14) is hyperbolic.

In addition to conservation of mass, classically one must preserve momentum along the flow. This gives rise to the equation

$$\operatorname{div}\left(\rho \mathbf{u} \otimes \mathbf{u} + p\mathbb{I}\right) = 0. \tag{7.15}$$

The system (7.13), (7.14) has been studied by Morawetz [120], where it is shown that smooth flows past profiles are not stable in general and hence will develop shocks, even in the elliptic regime. More precisely, we assume that there is a region $P \subset E$

(*e.g.* an airfoil profile) and consider solutions of the system (7.13), (7.14) in $E \setminus P$. A transonic flow past P is not continuous in general.

In [4, 68, 157], it is showed that if the prescribed flow on the boundary ∂E is sufficiently subsonic $\sup_{\partial E} |\mathbf{u}| < q_*$, then there exists a regular flow past the profile P. However, in the regime $q_* < \sup_{\partial E} |\mathbf{u}| \leqslant c$, one already expect a supersonic flow region and the presence of shocks. This is referred to as transonic flow. The critical flow problem refers to solving the boundary-value problem when $\sup_{\partial E} |\mathbf{u}_0| = q_*$.

In [121, 122], Morawetz proposed to study transonic flow through weak compactness methods, and specifically the compensated compactness approach of Tartar [163] and DiPerna [57]. Equation (7.15) is considered as an entropy condition which may relaxed to an inequality to account for shocks, while equations (7.13) and (7.13) are solved exactly in L^∞. For some polytropic gas and under the assumption of non-stagnation, the paper [34] obtains L^∞ entropy solutions, based on Morawetz' framework.

The critical flow problem was solved in [31] using a different compensated compactness framework. Consider a sequence of approximate subsonic boundary data $\mathbf{u}_0^\varepsilon$ on ∂E such that $\mathbf{u}_0^\varepsilon \to \mathbf{u}_0$. By the existence theorem for subsonic flow [4, 68, 157], there exists a sequence of solutions $\mathbf{u}^\varepsilon$ such that $\sup_{E \setminus P} |\mathbf{u}^\varepsilon| < c$. In [31], the authors showed that the conservation of momentum acts as entropies, which may be used to show the pointwise convergence of the $\mathbf{u}^\varepsilon$ to a function $\mathbf{u}$ which is a weak L^∞ solution of the boundary-value problem. The method was generalised to the n-dimensional case in [32]. A celebrated theorem due to Uhlenbeck [167] shows that this solution has interior Hölder regularity.

Minimal surfaces in $\mathbf{R}^3$ solve a non-linear elliptic equation which has many formal similarities with transonic flow, and many of the theorems that apply to one translate to the other: *e.g.* in [4, Appendix III], a Bernstein-type theorem is proved for transonic flow. We now describe a more recent correspondence with another geometric PDE system, the Gauß–Codazzi system, due to [35].

7.A.2 Gauß–Codazzi equations

Let us now assume that $E \subset \mathbf{R}^2$ is equipped with a Riemannian metric $g = g_{ij} dx^i \otimes dx^j$. The Gauß–Codazzi equations on E take the following form.

$$\begin{aligned} \nabla_i \mathbf{II}_{jk} &= \nabla_j \mathbf{II}_{ik}, \qquad (7.16) \\ \det \mathbf{II} &= K|g| = K \det g. \end{aligned}$$

Defining the correspondence $\mathbf{II} = \rho\mathbf{u} \otimes \mathbf{u} + p\mathbb{I}$, it is easily seen (see [35]) that $\rho, \mathbf{u}$ solves a system of the form

$$\begin{cases} \operatorname{rot} \mathbf{u} = \mathcal{R}_1, \\ \operatorname{div} \rho\mathbf{u} = \mathcal{R}_2, \end{cases} \tag{7.17}$$

where the $\mathcal{R}_i$ are zero-th-order non-linear terms in ρ and $\mathbf{u}$ (compare (7.16) and (7.15)). The Gauß equation translates into a Bernoulli-type relation

$$\rho = \frac{1}{\sqrt{q^2 + K|g|}},$$

and a Chaplyagin-type gas law $p(\rho) = -\sqrt{q^2 + K|g|}$. This yields a "speed of sound" given by

$$c^2(\rho) = p'(\rho) = q^2 + K|g|,$$

so that the system is (indeed) elliptic when $K > 0$, i.e. "subsonic", and hyperbolic when $K < 0$, i.e. "supersonic". With respect to system (7.17), the Codazzi equations play the same role as the momentum equations (7.15) for Euler's equations.

From this point of view, the Weyl problem (for non-negative curvature $K \geqslant 0$) and the critical flow problem have many formal similarities, as both are concerned with the degenerate case of an elliptic problem. Thus, Nirenberg's theorem about the Weyl problem [137] mirrors the existence theorems of Bers [4] and Gilbarg–Finn [68]. Theorem 7.10, which generalises [81], parallels the critical flow theorem of [31]. In both cases, the proof consists in exploiting additional conservation laws to show pointwise convergence almost everywhere.

We note that by Theorem 7.2, the Gauß–Codazzi equations are weakly continuous along sequences of L^2-weakly converging distributional solutions, independently of the type of the equation (i.e., the sign of the Gauß curvature). The picture is sensibly different in the case of transonic flow: the equations are known to be continuous in the weak* topology of L^∞ in the subsonic-sonic region [31], or across the transonic line if the flow excludes both cavitation and stagnation [122]. It is an open question whether Morawetz' weak* continuity result extends to include flows with either stagnation or cavitation.

Chapter 8

Weak Compactness of Approximate Solutions to the Gauß–Codazzi–Ricci Equations

A general strategy for constructing solutions of a non-linear partial differential equation or system of equations is the weak compactness method. Given a system of PDE, one seeks a solution as a limit (in a suitable topology) of a sequence of "approximate solutions" of the system. Such approximate solutions may be constructed by a variety of ways; a typical example is the vanishing viscosity method.

In this chapter, we develop parts of such a programme for isometric immersions. A possible approach is to obtain isometric immersions through the realisation theorem (see §8.1), that is, to find solutions of the Gauß–Codazzi–Ricci equations. This approach was initiated by Chen–Slemrod–Wang [35]. The main contribution of this chapter is a weak compactness theorem for approximate solutions of the Gauß–Codazzi–Ricci system (Theorem 8.1) which lets us construct an isometric immersion out of a sequence of approximate solutions with mild boundedness assumptions.

Recall that a solution to the Gauß–Codazzi–Ricci equations is a triplet $(\omega^\top, \omega^{\mathbf{II}}, \omega^\perp)$, where $\omega^\top$ and $\omega^\perp$ are connection forms, and $\omega^{\mathbf{II}}$ is formed from the second fundamental form. From the work of Chapter 6, if one can construct uniform L^p bounds on a sequence of such solution triplets, then their weak limits will also solve the equations.

From the point of view of the viscosity method, it is natural to assume to be able to construct approximate solutions with a priori uniform L^p bound on their second fundamental form, but not on the tangential or normal connections. Were this possible, one could use the result of Chapter 6 to obtain a weak compactness result (see also [36] where a local weak continuity result is proved, [33] for the global case). In addition, were we able to construct directly *exact* solutions, uniform L^p bounds on the second fundamental form would then imply uniform $\mathrm{W}^{2,p}$ bounds, as in Chapter 7 (see also [14, 114]). However, in the viscosity method, dealing with merely *approximate* solutions prevents such reasoning.

Informally, our main theorem (Theorem 8.1) states that a sequence of weak approximate solutions of the Gauß–Codazzi–Ricci equations with uniform L^p bounds on the second fundamental, for $p > \dim(M)$, is weakly pre-compact, and that there exists a $\mathrm{W}^{2,p}$ isometric immersion realising the limit point of the sequence of second fundamental forms. The proof combines a div-curl argument, as in Chapters 6 and 7, with Uhlenbeck's gauge theorem [168] to recover suitable bounds on the sequence of approximate solutions.

The method of proof of Theorem 8.1 may be generalised and adapted to a number of related cases: manifolds with boundaries, non-compact manifolds with compact exhaustions, manifolds with varying metrics, or even immersions into curved targets (instead of $\mathbf{R}^N$). For the sake of brevity, these arguments are not part of the present chapter.

The novelty of our approach in Theorem 8.1 is the introduction of elliptic techniques to deal with the lack of control on a mixed-type system, as in general, the Gauß–Codazzi–Ricci system has no fixed type. Interestingly, related phenomena have recently been discovered in the context of Lorentzian manifolds, where the curvature system is also not elliptic a priori (see [147, 148]).

This chapter is organised as follows. In Section 8.1, we present in detail the viscosity framework considered in this chapter, and state the main theorem (Theorem 8.1). The proof of Theorem 8.1 is then broken down into four steps. The first step, which is a local construction, is presented in details in Section 8.2. In Section 8.3, this local construction is then patched together on the manifold M. Finally, Section 8.4 completes the proof of the main theorem.

8.1 A weak compactness framework for isometric immersions

Let us first recall the broader framework of weak convergence methods. A non-linear partial differential equation is a relation of the form

$$\mathscr{R}[u, f] = 0, \tag{8.1}$$

where f is some data (which, to fix ideas, we can assume smooth), u is the unknown, and $\mathscr{R}$ is a relation involving derivatives of u. Weak convergence methods propose to study these equations by means of approximations: one considers a sequence of modified problems of the form

$$\mathscr{R}^\varepsilon[u^\varepsilon, f^\varepsilon] = 0, \tag{8.2}$$

indexed by a parameter $\varepsilon \to 0$, where $f^\varepsilon \to f$ and $\mathscr{R}^\varepsilon \to \mathscr{R}$ in some reasonable topology, and then hope to obtain u as a limit of u^ε. Typically, the modified problems are designed so that it is easy—or easier—to produce solutions u^ε satisfying (8.2): *e.g.* $\mathscr{R}^\varepsilon$ is an approximation procedure such as a numerical scheme, or a minimising sequence, if the problem (8.1) is variational. While there are many ways of constructing such approximate problems, this approach pre-supposes a choice of topology for the convergence of the u^ε, and some continuity properties of the operator $\mathscr{R}$ with respect to this chosen topology.

This description is both too abstract and too vague to encompass all problems one might hope to tackle with such a method; the proposed procedure requires careful tailoring to each problem. The reader is referred to the survey [64], which presents an overview of some of the main techniques and applications.

In the case of isometric immersions, we seek to apply this method to construct solutions of the Gauß–Codazzi–Ricci equations from approximate solutions through weak compactness in Sobolev spaces.

8.1.1 Approximate solutions of the Gauß–Codazzi–Ricci equations

Let (M, g) be a closed Riemannian manifold. Recall (see Chapter 3) that an isometric immersion of (M, g) in $\mathbf{R}^N$ may be described in terms of a connection on the tangent bundle (the Levi–Civita connection), a connection on the normal bundle, and a second fundamental form $\mathbf{II} : TM \times TM \to NM$.

In terms of coframes, the existence of an isometric immersion is equivalent with the existence of a flat connection

$$\omega = \begin{pmatrix} \omega^\top & {}^t\omega^{\mathbf{II}} \\ -\omega^{\mathbf{II}} & \omega^\perp \end{pmatrix} \tag{8.3}$$

on the bundle $TM \oplus NM$. Here $\omega^{\mathbf{II}}$ is a contraction of $\mathbf{II}$, and $\omega^\top$ and $\omega^\perp$ are the tangential and normal connection forms. The flatness condition $d\omega + \omega \wedge \omega = 0$ gives rise to the Gauß–Codazzi–Ricci equations when written in terms of the components of ω (see Chapter 3).

A sequence of approximate solutions ω_k is a sequence of connections of the form (8.3) on the bundle $TM \oplus NM$ such that

$$d\omega_k + \omega_k \wedge \omega_k = \eta_k.$$

As η_k is a $\mathfrak{so}$-valued two-form, we may decompose it into three components

$$\eta_k = \begin{pmatrix} \eta_k^G & {}^t\eta_k^C \\ -\eta_k^C & \eta_k^R \end{pmatrix}.$$

Concretely, given a vector bundle and a sequence of approximate solutions of the Gauß–Codazzi–Ricci equations (i.e. such that η_k vanishes in some reasonable sense), we seek to construct an exact solution.

In this chapter, we shall assume the following "approximate solutions" framework whenever referring to approximate solutions of the Gauß–Codazzi–Ricci system.

(A) $\omega_k^\top$ is a sequence of connections on $\mathscr{F}T^*M$;

(B) $V \to M$ is a metric vector bundle on M of rank $m \in \mathbf{N}$;

(C) $\omega_k^\perp$ is a sequence of connections on $\mathscr{F}V^*$, the coframe bundle of V, which is a principal $\mathbf{SO}(m)$ bundle;

(D) $\mathbf{II}_k : TM \times TM \to V$ is a sequence of symmetric tensors, which is uniformly bounded in L^p for $p > n = \dim(M)$;

(E) η_k^C is pre-compact in $\mathrm{W}^{-1,q}$, $q > 2$, and $\eta_k^C \overset{*}{\rightharpoonup} 0$ in the sense of distributions. Moreover $\eta_k^G \rightharpoonup 0$ and $\eta_k^R \rightharpoonup 0$ weakly in L^p.

The vector bundle $V \to M$ is the putative normal bundle, with $\omega_k^\perp$ being putative normal connections. The sequence of tensors $\mathbf{II}_k$ are approximate second fundamental forms. The terms η_k^C, η_k^G and η_k^R are error terms, or viscosity terms.

We emphasize that we do not assume any boundedness a priori on $\omega_k^\top$ and $\omega_k^\perp$; our only boundedness assumptions are (D), on the sequence of approximate second fundamental forms, and (E), on the viscosity terms. Assumption (E) is the vanishing viscosity assumption. It is possible that in applications, one can take $\eta_k^G = 0$ and $\eta_k^R = 0$, in which case the proof simplifies mildly.

Our main theorem is as follows.

THEOREM 8.1. *Let* (M,g) *be a closed Riemannian manifold, and assume that the framework (A)—(E) above holds. Let* $\mathbf{II}$ *be an accumulation point of* $\mathbf{II}_k$*. Then there exists a* $\mathrm{W}^{2,p}$ *isometric immersion* $u:(M,g)\to\mathbf{R}^N$ *realising* $\mathbf{II}$ *as its second fundamental form, which is unique up to rigid motions of* $\mathbf{R}^N$*.*

For the case of immersing surfaces in three-dimensional space, such a framework was fully implemented in [25, 35]. These works are limited to constructing isometric immersions of surfaces of negative curvature. L^∞ bounds are obtained on approximate second fundamental forms by the method of invariant regions. In this case, the Ricci equation is trivial and one takes $\eta_k^G=0$. The L^∞ bound obtained in these papers ensures that the viscous terms η_k^C are compactly contained in $\mathrm{W}^{-1,q}$, as in (E). Thus the framework (A)—(E) presented above may be considered a generalisation of the compactness theorem of [35].

8.1.2 Method of proof

The proof may be decomposed in four steps. First, one obtains local bounds on the tangential and normal connections. Second, these are patched globally to obtain global bounds. Third, one passes to limits (up to subsequences) in the triple $(\omega_k^\top,\omega_k^{\mathbf{II}},\omega_k^\perp)$. Finally, one uses the realisation theorem to deduce the existence of an isometric immersion realising the above triple.

The first step relies on putting both the tangential and the normal connection in Coulomb gauge on suitably small domains of M. Our proof employs Uhlenbeck's gauge theorem [168]. We shall show that locally, both the tangential and normal connections admit Coulomb gauges—which we refer to as adapted Uhlenbeck gauge, in the context of the Gauß–Codazzi–Ricci equations.

THEOREM 8.2. *Let* $E\subset M$ *be an open trivialising domain,* $p>n$ *and* $\omega=(\omega^\top,\omega^{\mathbf{II}},\omega^\perp)\in\mathrm{L}^p(E,\mathfrak{so}(N)\otimes T^*M|_E)$ *a weak approximate solution of the Gauß–Codazzi–Ricci equations, i.e.* $d\omega+\omega\wedge\omega=\eta$*. Then if* $\|\eta^G\|_{\mathrm{L}^{p/2}}+\|\eta^R\|_{\mathrm{L}^{p/2}}+\|\mathbf{II}\|_{\mathrm{L}^p}$ *is sufficiently small,* ω *is gauge-equivalent to a weak approximate solution* $\tilde\omega=(\tilde\omega^\top,\tilde\omega^{\mathbf{II}},\tilde\omega^\perp)$ *such that* $\delta\tilde\omega^\top=0$ *and* $\delta\tilde\omega^\perp=0$*.*

The reader is referred to Theorem 8.7 for a more precise statement of this theorem.

The second step follows Uhlenbeck's strategy for patching together local Coulomb gauges. Using elliptic regularity, we may use the previous theorem to obtain uniform bounds on the approximate solutions. This leads to a global pre-compactness result:

THEOREM 8.3. *Let $p > n$ and $\omega_k = (\omega_k^\top, \omega_k^{\mathbf{II}}, \omega_k^\perp) \in \mathrm{L}^p$ be a sequence of global weak approximate solution of the Gauß–Codazzi–Ricci equation, and assume that $\mathbf{II}_k$, respectively η_k^G and η_k^R, are uniformly bounded in L^p spaces, respectively in $\mathrm{L}^{p/2}$, for $p > n$. Then the ω_k are weakly pre-compact in L^p.*

The reader is referred to Theorem 8.16 for a more precise statement of this theorem.

The third step of this proof outline is a corollary of the work of Chapter 6. We shall need a div-curl lemma for $\mathfrak{g}$-valued one-forms, where $\mathfrak{g}$ is a Lie algebra (which we will take to be $\mathfrak{so}$). The following statement is a specific case of the main result of Chapter 6 (see Lemma 6.5)

THEOREM 8.4. *Let $E \subset M$ be an open trivialising domain of a smooth closed Riemannian manifold, $p > 2$ and $\alpha^\varepsilon \in \mathrm{L}^p(E, \mathfrak{g} \otimes T^*M)$ be a sequence of one-forms satisfying*

$$\begin{aligned} &d\alpha^\varepsilon \Subset \mathrm{W}^{-1,p}, \\ &\alpha^\varepsilon \rightharpoonup \alpha \quad \text{in} \quad \mathrm{L}^p. \end{aligned}$$

Then $[\alpha^\varepsilon \wedge \alpha^\varepsilon] \overset{*}{\rightharpoonup} [\alpha \wedge \alpha]$ in the sense of distributions.

As a corollary of this theorem, and following similar lines of argumentation as in Chapter 7, we shall verify that the accumulation points of the sequence ω_k satisfies the Gauß–Codazzi–Ricci equations.

The last step of the aforementioned proof outline was essentially completed by Mardare [117], who proved that the classical solvability theorems for Pfaff systems remained true in L^p spaces ($p > n$), and as a corollary obtained the following statement (see also [33], which follows the proof of the realisation theorem given in [165], using Mardare's theorem).

THEOREM 8.5 (Realisation Theorem)**.** *Let (M, g) be a closed, simply connected, smooth Riemannian manifold and let $\nabla^\top$ be its Levi-Civita connection. Let $V \to M$ be a vector bundle, together with a bundle metric and a compatible L^p connection $\nabla^\perp$. Finally let $\mathbf{II} : TM \times TM \to V$ be a symmetric L^p tensor, such that the Gauß–Codazzi–Ricci equations are satisfied for $(\nabla^\top, \mathbf{II}, \nabla^\perp)$. Then there exists a $\mathrm{W}^{2,p}$ isometric immersion of (M, g) into $\mathbf{R}^N$ such that $NM \simeq V$, and moreover that immersion is unique up to rigid motions of $\mathbf{R}^N$.*

The realisation theorem is a corollary of the following more general fact (see *e.g.* [61]):

THEOREM 8.6. *Let $P \to M$ be a principal G-bundle. If there exists a connection D with flat curvature $\Omega_D = 0$ on P, then P is the trivial bundle, i.e. there exists a bundle isomorphism taking $P \to M$ to the trivial bundle $G \times M \to M$, and moreover D is taken to the product connection.*

This theorem extends verbatim to connections $D \in \mathcal{A}^{0,p}(P)$, for $p > n = \dim(M)$.

The remainder of this chapter is dedicated to giving a proof of Theorem 8.1 following this proof outline. Section 8.2 gives a proof of Theorem 8.2, and in Section 8.3, the local construction is used to obtain global bounds on the sequence of connections. Applying Theorems 8.4 and 8.5 lets us complete the proof of the main result of the chapter in Section 8.4.

8.2 The local result: proof of Theorem 8.2

Let us fix a local trivialisation of the tangent bundle TM and of the putative normal bundle V, over a domain $B^n \simeq E \subset M$, so that $TM|_E \simeq \mathbf{R}^n \times B^n$ and $V|_E \simeq \mathbf{R}^m \times B^n$. Let $\mathbf{II}$ be the approximate second fundamental form, which is a symmetric two-tensor $\mathbf{II} : TM \times TM \to V$. Finally, let $\theta^\top$ and $\theta^\perp$ be coframes on TM and V respectively, realising the connection forms $\omega^\top$ and $\omega^\perp$. The local trivialisations of TM and V induce local trivialisations of their coframe bundles, so that $\mathscr{F}T^*M \simeq \mathbf{SO}(n) \times B^n$ and $\mathscr{F}V^* \simeq \mathbf{SO}(m) \times B^n$.

We say that the matrix-valued one-form

$$\omega = \begin{pmatrix} \omega^\top & {}^t\omega^{\mathbf{II}} \\ -\omega^{\mathbf{II}} & \omega^\perp \end{pmatrix} \in \mathfrak{so}(n+m) \tag{8.4}$$

is an approximate solution of the Gauß–Codazzi–Ricci equations if there exists a matrix-valued two-form

$$\eta = \begin{pmatrix} \eta^G & {}^t\eta^C \\ -\eta^C & \eta^R \end{pmatrix} \in \mathfrak{so}(n+m), \tag{8.5}$$

such that $d\omega + \omega \wedge \omega = \eta$.

This section is dedicated to proving Theorem 8.2, of which we now give a more precise statement.

THEOREM 8.7 (Adapted Uhlenbeck gauge)**.** *Let us consider $\omega = (\omega^\top, \omega^{\mathbf{II}}, \omega^\perp) \in \mathrm{L}^p(B^n, \mathfrak{so}(n+m) \otimes T^*M|_{B^n})$, $p > n$, to be a weak approximate solution of the Gauß–Codazzi–Ricci equations. Then there exists $\gamma > 0$ such that if $\|\mathbf{II}\|_{\mathrm{L}^p} + \|\eta^G\|_{\mathrm{L}^{p/2}} + \|\eta^R\|_{\mathrm{L}^{p/2}} < \gamma$, then there exists an adapted Uhlenbeck transformation $s = (s^\top, s^\perp) \in \mathrm{W}^{2,p/2}(B^n, \mathbf{SO}(n) \oplus \mathbf{SO}(m))$ such that $\tilde{\omega} = s^*\omega$ satisfies*

1. $\delta\tilde{\omega}^\top = 0 = \delta\tilde{\omega}^\perp$.

2. $\langle x, \tilde{\omega}^\top\rangle|_{\partial B^n} = 0 = \langle x, \tilde{\omega}^\perp\rangle|_{\partial B^n}$.

3. *We have*

$$\|(\tilde{\omega}^\top, \tilde{\omega}^\perp)\|_{\mathrm{W}^{1,p/2}} \leqslant C_n(\|\mathbf{II}\|_{\mathrm{L}^p} + \|\eta^G\|_{\mathrm{L}^{p/2}} + \|\eta^R\|_{\mathrm{L}^{p/2}}).$$

The proof consists of seeking a suitable transformation s by means of the Uhlenbeck local gauge theorem. In [168], local Coulomb gauges are constructed for $\mathrm{W}^{1,p}$ connections with L^p bounded curvature [168, Theorem 1.3]. This is then employed to show weak gauge-convergence of a sequence of Sobolev connections with L^p-bounded curvature [168, Theorem 1.5] (commonly referred to as "weak Uhlenbeck compactness", see *e.g.* [175]).

In §8.2.1, we introduce Darboux transformations, among which we hope to select a suitable transformation s in which to obtain additional bounds on the tangential and normal connections. Then, §8.2.2 introduces a regularity lemma necessary in order to apply Uhlenbeck's theorem, which is done in 8.2.3 to complete the proof of the Theorem. Finally, §8.2.4 gives a first local weak compactness result.

8.2.1 Darboux transformations

Recall from Chapter 3 that a coframe $\theta^1, \ldots, \theta^N \in \mathscr{F}(T^*\mathbf{R}^N)$ is said to be adapted to an isometric immersion $u : M \to \mathbf{R}^N$ if $\theta^1, \ldots, \theta^n$ is cotangent to M and is a coframe on M. In a local trivialisation of the Darboux bundle, we write as usual $\theta = (\theta^1, \ldots, \theta^n; \theta^{n+1}, \ldots, \theta^N) =: (\theta^\top; \theta^\perp)$. Such coframes transform into one another through elements of $\mathbf{SO}(n) \oplus \mathbf{SO}(N-n) \subset \mathbf{SO}(N)$. This group acts matricially as

$$s = \begin{pmatrix} s^\top & 0 \\ 0 & s^\perp \end{pmatrix} \in \mathbf{SO}(N), \quad s^\top \in \mathbf{SO}(n), s^\perp \in \mathbf{SO}(N-n),$$

so that if $\theta = (\theta^\top, \theta^\perp)$ is an adapted coframe, so is $s^*\theta = s\theta$, where the last product is understood as a matrix product in a local trivialisation of the Darboux bundle (see Chapter 3). Indeed, assume that θ is a Darboux coframe. Viewing $s^\top$ and $s^\perp$ as matrix operators with respect to this fixed trivialisation as we may, we have by linearity $u^*(s^\top\theta^\top) = s^\top u^*\theta^\top$, while $u^*(s^\perp\theta^\perp) = s^\perp u^*\theta^\perp = 0$, so that the coframe $s\theta$ is adapted.

By analogy with gauge theory, we shall refer to the group $\mathbf{SO}(n) \oplus \mathbf{SO}(N-n)$ as being the Darboux group, or "gauge group" of the Darboux bundle associated with a particular isometric immersion, and s as a Darboux transformation, or "adapted gauge".

Thus a Darboux transformation is a (local) section of the bundle $\mathbf{Aut}(\mathscr{D}^*(u(M)))$ of automorphisms of the Darboux bundle. The fiber of this bundle is $\mathbf{SO}(n)\oplus\mathbf{SO}(N-n)$.

Let us consider an adapted coframe θ, and let ω be the associated connection form, i.e. $d\theta = \omega \wedge \theta$. Recall that we may write

$$\omega = \begin{pmatrix} \omega^\top & {}^t\omega^{\mathbf{II}} \\ -\omega^{\mathbf{II}} & \omega^\perp \end{pmatrix} \in \mathfrak{so}(N), \quad \omega^\top \in \mathfrak{so}(n), \omega^\perp \in \mathfrak{so}(N-n).$$

A Darboux transformation s acts on the connection form by conjugation, namely

$$\begin{aligned} s^*\omega &= s^{-1} \circ (d+\omega) \circ s \\ &= s^{-1}ds + s^{-1}\omega s \\ &= \begin{pmatrix} (s^\top)^{-1}ds^\top & 0 \\ 0 & (s^\perp)^{-1}ds^\perp \end{pmatrix} + \begin{pmatrix} (s^\top)^{-1}\omega^\top s^\top & (s^\top)^{-1}{}^t\omega^{\mathbf{II}}s^\perp \\ -(s^\perp)^{-1}\omega^{\mathbf{II}}s^\top & (s^\perp)^{-1}\omega^\perp s^\perp \end{pmatrix} \\ &= \begin{pmatrix} (s^\top)^{-1}ds^\top + (s^\top)^{-1}\omega^\top s^\top & (s^\top)^{-1}{}^t\omega^{\mathbf{II}}s^\perp \\ -(s^\perp)^{-1}\omega^{\mathbf{II}}s^\top & (s^\perp)^{-1}ds^\perp + (s^\perp)^{-1}\omega^\perp s^\perp \end{pmatrix}. \end{aligned}$$

Recall that the Gauß–Codazzi–Ricci equations are but the curvature equations for connections forms on $\mathbf{R}^N$. As curvature is tensorial, it is immediate that the equations are invariant under the action of the Darboux group. It also follows from the realisation theorem that the isometric immersion generated by a triplet $\omega^\top, \omega^{\mathbf{II}}, \omega^\perp$ is unique up to rigid motion along an orbit of the Darboux group.

We now consider coframes on TM and V respectively, where $V \to M$ is the putative normal bundle. In the local trivialisation considered in this section, we have $TM \simeq \mathbf{R}^n \times B^n$ and $V \simeq \mathbf{R}^m \times B^n$. The gauge group of the cotangent frame bundle is $\mathbf{SO}(n)$, while that of the coframe bundle of V is $\mathbf{SO}(m)$. For $s^\top$ a gauge on the cotangent frame bundle, and $s^\perp$ a gauge of the coframe bundle of V, we form

$$s = \begin{pmatrix} s^\top & 0 \\ 0 & s^\perp \end{pmatrix}, \tag{8.6}$$

to be an adapted transformation on $TM \oplus V$. As before, if ω is a connection as in (8.4), and η is an error term of the form (8.5), s acts by conjugation, so that

$$s^*\omega = \begin{pmatrix} (s^\top)^{-1}ds^\top + (s^\top)^{-1}\omega^\top s^\top & (s^\top)^{-1}{}^t\omega^{\mathbf{II}}s^\perp \\ -(s^\perp)^{-1}\omega^{\mathbf{II}}s^\top & (s^\perp)^{-1}ds^\perp + (s^\perp)^{-1}\omega^\perp s^\perp \end{pmatrix}.$$

Lemma 8.8 (Regularity)**.** *Let ω and $\tilde{\omega}$ be two approximate solutions of the the Gauß–Codazzi–Ricci equations, as in* (8.4)*, in a fixed trivialisation. Let s be a local adapted transformation of the form* (8.6) *such that $s^*\omega = \tilde{\omega}$. Then for all $1 \leqslant p \leqslant \infty$,*

$$\|ds\|_{\mathrm{L}^p} \leqslant \|\omega^\top\|_{\mathrm{L}^p} + \|\tilde{\omega}^\top\|_{\mathrm{L}^p} + \|\omega^\perp\|_{\mathrm{L}^p} + \|\tilde{\omega}^\perp\|_{\mathrm{L}^p}.$$

Moreover, if $\omega^{\top,\perp}$ and $\tilde{\omega}^{\top,\perp}$ are $\mathrm{W}^{1,p}$ connection forms, then s is a $\mathrm{W}^{2,p}$ transformation.

Proof. We start from the equation

$$ds = s\omega - \tilde{\omega}s,$$

which is simply the conjugacy relation re-written symmetrically. Writing out the components of ω and $\tilde{\omega}$, we have

$$\begin{pmatrix} ds^{\top} & 0 \\ 0 & ds^{\perp} \end{pmatrix} = \begin{pmatrix} s^{\top}\omega^{\top} & s^{\top}{}^{t}\omega^{\mathbf{II}} \\ -s^{\perp}\omega^{\mathbf{II}} & s^{\perp}\omega^{\perp} \end{pmatrix} - \begin{pmatrix} \omega^{\top}s^{\top} & {}^{t}\omega^{\mathbf{II}}s^{\perp} \\ -\omega^{\mathbf{II}}s^{\top} & \omega^{\perp}s^{\perp} \end{pmatrix},$$

whence it follows that

$$\|ds^{\top}\|_{\mathrm{W}^{k,p}} \leqslant \|\omega^{\top}\|_{\mathrm{W}^{k,p}} + \|\omega^{\perp}\|_{\mathrm{W}^{k,p}},$$

for $k = 0, 1$, and similarly for $s^{\perp}$. Here we have used crucially that $s^{\top}$ is orthogonal. Using the multiplication theorems in Sobolev spaces over Lie groups, the result follows, as in [168, Lemma 1.2]. Q.E.D.

8.2.2 $\mathrm{W}^{1,p/2}$ regularity

In order to be able to apply Uhlenbeck's theorem (quoted below as Theorem 8.12), we need to construct gauge-equivalent connections with higher Sobolev regularity. To do so, we apply a slice theorem on the space of connections. The procedure described here is more general than is needed (see *e.g.* [174]).

The first lemma concerns the regularity of connections with curvature in L^p.

Lemma 8.9. *Let $G \subset \mathbf{SO}(N)$ be a Lie group, $P \to M$ a principal G-bundle, $p > \max(n, 2)$, and $D \in \mathcal{A}^{0,p}(P)$ be such that the associated curvature verifies $\Omega_D \in \mathrm{L}^{p/2}$. Then there exists a smooth connection $D_0 \in \mathcal{A}(P)$ and a global gauge transformation s such that $s^*D \in \mathcal{A}^{1,p/2}(P)$ and moreover*

$$\|s^*D - D_0\|_{\mathrm{L}^p} \leqslant C\|D - D_0\|_{\mathrm{L}^p}.$$

The idea of the proof is to turn the curvature equation into an elliptic system so that elliptic regularity gives the claimed regularity on D. To do so, we seek a Coulomb gauge for D. As the connection is only L^p, the Coulomb condition has to be understood in the weak sense.

We rely on the following local slice theorem. The reader is referred to [175] for the proof.

THEOREM 8.10 (Local L^p-slice)**.** *Let $D_0 \in \mathcal{A}^{0,p}(P)$ be a fixed connection. Then there exists a $\eta > 0$ such that for all $D \in \mathcal{A}^{0,p}(P)$ with $\|D - D_0\|_{\mathrm{L}^p} < \eta$, there exists a global $\mathrm{W}^{1,p}$ gauge transformation s such that*

$$\|s^*D - D_0\|_{\mathrm{L}^p} \leqslant C\|D - D_0\|_{\mathrm{L}^p}.$$

and in addition, $D_0^*(s^*D - D_0) = 0$ in the sense of distributions, i.e.

$$\int_M \langle s^*D - D_0, D_0\phi\rangle = 0 \qquad \forall \phi \in C^\infty(M, \mathbf{Ad}(P)). \tag{8.7}$$

This theorem states that in a small enough L^p-neighbourhood, every connection admits a representative in Coulomb gauge.

We are now ready to prove Lemma 8.9.

Proof of Lemma 8.9. We apply the local slice theorem to the connection D. There exists a connection $D_0 \in \mathcal{A}^{0,p}(P)$ such that $\|D - D_0\|_{\mathrm{L}^p} \leqslant \eta$ and a global $\mathrm{W}^{1,p}$ gauge transformation s such that (8.7) holds.

The connection form ω satisfies the following equation: for all $\phi \in \Omega^2(M, \mathbf{Ad}(P))$, there holds

$$\int_M \langle \omega, \delta\phi - \frac{1}{2}(-1)^n * [\omega \wedge *\phi]\rangle = \int_M \langle \Omega, \phi\rangle, \quad \phi \in \Omega^2(M, \mathbf{Ad}(P)),$$

which is simply the weak formulation of the curvature equation $\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega]$. We apply the local slice theorem. Let s be a $\mathrm{W}^{1,p}$ gauge transformation for ω such that (8.7) holds, and let $\tilde{\omega} = s^*D - D_0$. As s is continuous, the curvature form satisfies $\tilde{\Omega} = s^*\Omega \in \mathrm{L}^{p/2}$. Then $\tilde{\omega}$ solves, in a local trivialisation,

$$\begin{aligned} &\int_M \langle \tilde{\omega}, \delta\phi - *[\tilde{\omega} \wedge *\phi]\rangle = \int_M \langle \tilde{\Omega}, \phi\rangle, && \phi \in \Omega^2(M, \mathbf{Ad}(P)) \\ &\int_M \langle \tilde{\omega}, d\psi\rangle = 0, && \psi \in C^\infty(M, \mathbf{Ad}(P)) \end{aligned}$$

This system of equations is the weak formulation of the problem

$$\begin{aligned} &d\tilde{\omega} + \tilde{\omega} \wedge \tilde{\omega} = \tilde{\Omega} \in \mathrm{L}^{p/2}, \\ &\delta\tilde{\omega} = 0. \end{aligned}$$

This problem is elliptic. Note that $\tilde{\omega} \wedge \tilde{\omega} \in \mathrm{L}^{p/2}$; thus the problem takes the form of a Cauchy–Riemann system

$$\begin{cases} d\tilde{\omega} &= \alpha \in \mathrm{L}^{p/2}, \\ \delta\tilde{\omega} &= 0. \end{cases}$$

As $p > 2$ and M is closed, elliptic regularity implies that $\tilde{\omega} \in \mathrm{W}^{1,p/2}$. Q.E.D.

Applying Lemma 8.9 to the tangential and normal connection forms, we shall from now on assume that the sequence of connections $\omega_k^\top$ on TM and $\omega_k^\perp$ on $V \to M$ are $\mathrm{W}^{1,p/2}$ connections:

Lemma 8.11. *Let $\omega = (\omega^\top, \omega^{\mathbf{II}}, \omega^\perp)$ be an approximate solution of the Gauß–Codazzi–Ricci equations. Then there exists an adapted transformation s such that $s^*\omega =: \tilde{\omega}$ satisfies $\tilde{\omega}^\top \in \mathrm{W}^{1,p/2}$ and $\tilde{\omega}^\perp \in \mathrm{W}^{1,p/2}$.*

Proof. We apply Lemma 8.9 to the Gauß equation

$$d\omega^\top + \omega^\top \wedge \omega^\top = {}^t\omega^{\mathbf{II}} \wedge \omega^{\mathbf{II}} + \eta^G.$$

As $\mathbf{II}$ is in L^p, ${}^t\omega^{\mathbf{II}} \wedge \omega^{\mathbf{II}} \in \mathrm{L}^{p/2}$. Lemma 8.9 gives a global gauge transformation s such that $s^*\omega^\top \in \mathrm{W}^{1,p/2}$.

For $\omega^\perp$, we proceed similarly with the Ricci equation

$$d\omega^\perp + \omega^\perp \wedge \omega^\perp = \omega^{\mathbf{II}} \wedge {}^t\omega^{\mathbf{II}} + \eta^R.$$

Q.E.D.

8.2.3 Uhlenbeck's construction

Recall Uhlenbeck's theorem on the existence of local Coulomb gauge [168, Theorem 2.1]:

THEOREM 8.12 (Uhlenbeck gauge)**.** *Let $B^n \subset M$, a closed Riemannian manifold. Let $d + \omega \in \mathcal{A}^{1,p}(B)$ be a connection on B, $\omega \in \mathrm{W}^{1,p}(B, T^*M \otimes \mathfrak{so}(N))$ for $p > n/2$ and $\Omega := d\omega + \omega \wedge \omega$. Then there exists $\kappa = \kappa(n)$ such that if*

$$\|\Omega\|_{\mathrm{L}^p(B)} < \kappa,$$

then there exists a gauge transform s_U such that the connection form $\tilde{\omega} := s_U^{-1} ds_U + s_U^{-1}\omega s_U$ satisfies

$$\begin{aligned} \delta\tilde{\omega} &= 0, \\ \langle x, \tilde{\omega}\rangle|_{\partial B} &= 0, \\ \|\tilde{\omega}\|_{\mathrm{W}^{1,p}(B)} &\leqslant C(n)\|\Omega\|_{\mathrm{L}^p(B)}, \\ s_U &\in \mathrm{W}^{2,p}(M, G). \end{aligned}$$

We are now ready to prove Theorem 8.7. The idea is clear: we apply Uhlenbeck's theorem to both the tangential and normal connection.

Proof of Theorem 8.7. Recall the setting: we consider a local trivialisation of the tangent bundle $TM \simeq \mathbf{R}^n \times B^n$ and the putative normal bundle $V \simeq \mathbf{R}^m \times B^n$, and an approximate solution $\omega = (\omega^\top, \omega^{\mathbf{II}}, \omega^\perp) \in \mathrm{L}^p(B^n, \mathfrak{so}(n+m) \otimes T * M)$ of the Gauß–Codazzi–Ricci equations, for $p > n$.

By Lemma 8.11, we may assume that $\omega^\top \in \mathrm{W}^{1,p/2}(B^n, \mathbf{SO}(n) \otimes T^*M)$, and similarly, $\omega^\perp \in \mathrm{W}^{1,p/2}(B^n, \mathbf{SO}(m) \otimes T^*M)$.

The Gauß equation expresses the curvature of $\omega^\top$ in terms of the second fundamental form $\omega^{\mathbf{II}}$, and the error term η^G:

$$d\omega^\top + \omega^\top \wedge \omega^\top = {}^t\omega^{\mathbf{II}} \wedge \omega^{\mathbf{II}} + \eta^G.$$

Taking $\mathrm{L}^{p/2}$ norms on both sides, we have

$$\begin{aligned}\|\Omega^\top\|^{p/2}_{\mathrm{L}^{p/2}} &\leqslant \int_B |{}^t\omega^{\mathbf{II}} \wedge \omega^{\mathbf{II}}|^{p/2}(*1) + \int_B |\eta^G|^{p/2}(*1)\\ &\leqslant \left(\int_B |\omega^{\mathbf{II}}|^p(*1)\right)^2 + \int_B |\eta^G|^{p/2}(*1)\\ &\leqslant C\left(\int_B |\mathbf{II}|^p(*1)\right)^2 + \int_B |\eta^G|^{p/2}(*1).\end{aligned}$$

If we choose $C\|\mathbf{II}\|_{\mathrm{L}^p} + \|\eta^G\|_{\mathrm{L}^{p/2}} < \kappa(n)$, where κ is given by Theorem 8.12, then there exists a $\mathrm{W}^{2,p}$ gauge transformation $s_U^\top$ of T^*M such that $s^*\omega^\top$ is in Coulomb gauge, in the sense of Theorem 8.12.

Applying the same reasoning to the Ricci equation, one finds a $\mathrm{W}^{2,p}$ gauge transformation $s_U^\perp$ of the (putative) normal bundle such that $(s_U^\perp)^*\omega^\perp$ is in Coulomb gauge, provided $C\|\mathbf{II}\|_{\mathrm{L}^p} + \|\eta^R\|_{\mathrm{L}^{p/2}} < \kappa(n)$.

Thus, selecting $\gamma = \kappa(n)$ and taking

$$s = \begin{pmatrix} s_U^\top & 0 \\ 0 & s_U^\perp \end{pmatrix} \in \mathbf{SO}(n) \oplus \mathbf{SO}(m)$$

yields the desired adapted gauge transformation. Q.E.D.

8.2.4 A local existence theorem

Theorem 8.7 is already sufficient to state and prove a local version of the theorem, working in a fixed trivialisation of the tangent bundle and the putative normal bundle, up to possibly shrinking the domain.

THEOREM 8.13 (Local statement)**.** *Let (M, g) be a closed Riemannian manifold, $E \subset M$ a trivialising open domain, $p > n = \dim(M)$, and for all $k \in \mathbf{N}$ let*

$\omega_k = (\omega_k^\top, \omega_k^{\mathbf{II}}, \omega_k^\perp) \in \mathrm{L}^p(E, \mathfrak{so}(N) \otimes T^*M|_E)$ be a sequences of approximate solutions of the Gauß–Codazzi–Ricci equations, i.e. assume that there exists a sequence of two-forms $\eta_k = (\eta_k^G, \eta_k^C, \eta_k^R) \in \mathrm{L}^{p/2}(E, \mathfrak{so}(N) \otimes \wedge^2 T^*M)$ such that

$$d\omega_k + \omega_k \wedge \omega_k = \eta_k,$$

in the sense of distributions, and moreover

$$\begin{aligned} \eta_k^G &\rightharpoonup 0, \\ \eta_k^R &\rightharpoonup 0, \\ \omega_k^{\mathbf{II}} &\rightharpoonup \omega^{\mathbf{II}}, \end{aligned}$$

in the weak topology of L^p, and $\eta_k^C \Subset W^{-1,q}$ for some $q > 2$, $\eta_k^C \overset{*}{\rightharpoonup} 0$ in the sense of distributions.

Then there exists a domain $E' \subset E$ and a $\mathrm{W}^{2,p}$ isometric immersion $u : E' \to \mathbf{R}^N$ which is unique up to rigid motions.

The proof consists in verifying that there exists an exact solution of the Gauß–Codazzi–Ricci equations on E, which we extract as a weak limit of the ω_k.

Proof. Up to shrinking the domain B^n, we may assume that $\|\omega_k^{\mathbf{II}}\|_{\mathrm{L}^p} + \|\eta_k^G\|_{\mathrm{L}^{p/2}} + \|\eta_k^R\|_{\mathrm{L}^{p/2}} < \gamma$, where γ is given by Theorem 8.7. Then there exists a sequence of gauge transform $s_k^\top$ on TM and $s_k^\perp$ on V such that $(s_k^\top)^*\omega_k^\top$ satisfies $\sup_{k\in\mathbf{N}} \|(s_k^\top)^*\omega_k^\top\|_{\mathrm{W}^{1,p}} < \infty$, and similarly for $\omega_k^\perp$. By compact Sobolev embeddings, $(s_k^\top)^*\omega_k^\top$ converges strongly in L^q for all $q < p^*$, and similarly for the normal connection.

Let us denote

$$s_k = \begin{pmatrix} s_k^\top & 0 \\ 0 & s_k^\perp \end{pmatrix}.$$

As $\omega_k^{\mathbf{II}}$ are bounded in L^p and the s_k are orthogonal transformations, $s_k^*\omega_k$ is bounded in L^p. Let ω be its weak limit, up to a possible non-relabelled subsequence.

Recall that as ω_k are approximate solutions of the Gauß–Codazzi–Ricci equations, so are $s_k^*\omega_k$. As $d\omega_k + \omega_k \wedge \omega_k = \eta_k$, we have

$$d(s_k^*\omega_k) = -s_k^*\omega_k \wedge s_k^*\omega_k + s_k^*\eta_k.$$

and as $s^*\omega_k$ is bounded in $\mathrm{L}^{p/2}$, which embeds compactly into $\mathrm{W}^{-1,r}$ for $r < (p/2)^*$, the RHS is compactly contained in $\mathrm{W}^{-1,q}$. Applying the div-curl lemma (Theorem 8.4), $s_k^*\omega_k \wedge s_k^*\omega_k \overset{*}{\rightharpoonup} \omega \wedge \omega$, where ω is the weak limit of $s_k^*\omega_k$, up to a subsequence, by the Banach–Alaoglu theorem. By hypothesis $\eta_k^G \rightharpoonup 0$, $\eta_k^R \rightharpoonup 0$, and $\eta_k^C \overset{*}{\rightharpoonup} 0$ in the sense of distributions, so that ω is a flat connection. The L^p realisation theorem, Theorem 8.5, then applies, which yields the conclusion. Q.E.D.

8.3 Global weak compactness: proof of Theorem 8.3

We now wish to apply the construction of the previous section to control the full connection form, and thus to be able to pass to limits. The basic idea is to choose a cover of M by sufficiently small local trivialisations in adapted Uhlenbeck gauge. This is possible thanks to the uniform bound on $\mathbf{II}^k$ and the compactness of M.

8.3.1 Good covers of M

The basis of our argument follows Uhlenbeck's [168, Theorem 3.6]. First, by considering geodesic balls and shrinking them if necessary, the following holds, as M is a compact manifold:

Lemma 8.14. *Let M be a closed Riemannian manifold, and let $F \in \mathrm{L}^p(M)$, where $p > n/2$. Then M may be covered by finitely many geodesic balls U_α of M such that the metric on U_α is equivalent to the Euclidean metric and $\int_{U_\alpha} |F|^p < \varepsilon$ for any arbitrary $\varepsilon > 0$. The cover depends on p and ε.*

Uhlenbeck [168] proved that such a cover may be refined in such a way that any sets of co-cycles may be related by a family of transformation maps on the subcover.

More precisely, recall that a principal G-bundle on M may be described by a set of co-cycles, or overlap maps over a cover $\{U_\alpha \subset M \,|\, \alpha \in \mathscr{A}\}$ of M, which are maps

$$h_{\alpha\beta} : U_\alpha \cap U_\beta \to G,$$

satisfying the co-cycle conditions

$$h_{\alpha\beta} h_{\beta\alpha} = 1 \quad \text{on } U_\alpha \cap U_\beta, \tag{8.8}$$

$$h_{\alpha\beta} h_{\beta\gamma} = h_{\alpha\gamma} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma. \tag{8.9}$$

Here $\alpha, \beta, \gamma \in \mathscr{A}$.

The main case of interest for us will be the case where the co-cycles are given as product of trivilisations of the bundle: let τ_α be a family of trivialisations over U_α. Letting $h_{\alpha\beta} := \tau_\alpha (\tau_\beta)^{-1}$, the h_α are a family of co-cycles associated to the cover U_α.

One needs to relate two families of such co-cycles on M. Uhlenbeck proved [168]:

THEOREM 8.15. *Let M be a closed manifold, G a compact Lie group, $(U_\alpha)_{\alpha\in\mathscr{A}}$ a finite cover of M, and $g_{\alpha\beta}$ and $h_{\alpha\beta}$ two families of $\mathrm{W}^{2,p}(U_\alpha, G)$ co-cycles (i.e.*

satisfying equations (8.8) and (8.9)). Then there exists a subcover V_α of U_α and a family of functions $\rho_\alpha \in \mathrm{W}^{2,p}(V_\alpha, G)$ such that on $V_\alpha \cap V_\beta$, there holds

$$g_{\alpha\beta} = \rho_\alpha h_{\alpha\beta} \rho_\beta^{-1}.$$

Moreover, if for all $\beta \neq \alpha$ such that $V_\alpha \cap V_\beta \neq \varnothing$, the $\mathrm{W}^{2,p}$ norms of $h_{\alpha\beta}$ and $g_{\alpha\beta}$ are bounded by $\mu > 0$, then there exists a $k(\mu)$ such that $\| \exp^{-1}(\rho_\alpha)\|_{\mathrm{W}^{2,p}(V_\alpha, G)} < k(\mu)$.

We now describe the proof strategy of Theorem 8.3 (reformulated below as Theorem 8.16). We start with a putative normal bundle on M, and a sequence of approximate solutions of the Gauß–Codazzi–Ricci equations. We want to leverage the uniform bounds on the second fundamental form in order to construct a suitable cover of M by subset U_α such that on each U_α, one may apply the local adapted gauge theorem, Theorem 8.7. This gives, for each $k \in \mathbf{N}$ and each $\alpha \in \mathscr{A}$, a local trivialisation σ_α^k. We then use the machinery of Theorem 8.15 in order to construct a global limiting tangent and normal connections on M.

8.3.2 Proof of Theorem 8.3

We first give a precise statement for Theorem 8.3.

THEOREM 8.16. *Let (M, g) be a closed Riemannian manifold, $V \to M$ a metric vector bundle, and $\omega_k = (\omega_k^\top, \omega_k^{\mathbf{II}}, \omega_k^\perp)$ a sequence of approximate solutions of the Gauß–Codazzi–Ricci equations, such that*

$$\sup_{k\in\mathbf{N}} \|\mathbf{II}_k\|_{\mathrm{L}^p} < \infty, \tag{8.10}$$

$$\sup_{k\in\mathbf{N}} \|\eta_k^G\|_{\mathrm{L}^{p/2}} < \infty, \tag{8.11}$$

$$\sup_{k\in\mathbf{N}} \|\eta_k^R\|_{\mathrm{L}^{p/2}} < \infty. \tag{8.12}$$

Then there exists a sequence s_k of $\mathrm{W}^{2,p/2}$ adapted gauge transformations such that $s_k^*\omega_k$ is pre-compact in the weak topology of L^p.

The proof will yield slightly more than the claimed weak L^p compactness. We shall show that there exists a sequence of $\mathrm{W}^{2,p/2}$ adapted gauge transformations s_k such that up to subsequences, $(s_k^\top)^*\omega_k^\top$ converges weakly in $\mathrm{W}^{1,p/2}$ to a global connection $\omega^\top$ on $\mathscr{F}T^*M$, and similarly, $(s_k^\perp)^*\omega_k^\perp$ converges weakly in $\mathrm{W}^{1,p/2}$ to a global connection $\omega^\perp$ on $\mathscr{F}V^*$.

Proof of Theorem 8.3. **Step 1:** selection of a good cover.

By virtue of the bounds (8.10), (8.11) and (8.12), and Lemma 8.14, we cover M by geodesic balls U_α such that for each $\alpha \in \mathscr{A}$, we have

$$\left(\int_{U_\alpha} |\mathbf{II}_k|^p (*1)\right)^{1/p} + \left(\int_{U_\alpha} |\eta_k^G|^{p/2}(*1)\right)^{2/p} + \left(\int_{U_\alpha} |\eta_k^R|^{p/2}(*1)\right)^{2/p} < \gamma,$$

where $\gamma > 0$ is the constant given by Theorem 8.7. Note that the sets U_α do not depend on k but do depend on the uniform bound assumed in (8.10),(8.11) and (8.12).

We may now apply Theorem 8.7 to each U_α and each $k \in \mathbf{N}$, putting our sequence of connections forms into adapted Uhlenbeck gauge. Thus on each U_α, we have a trivialisation $\sigma_\alpha^k : (T^*M \oplus V)|_{U_\alpha} \simeq U_\alpha \times \mathbf{R}^{n+m}$ and such that in this trivialisation, $(\sigma_{k,\alpha}^{-1})^*\omega^k$ is in adapted Uhlenbeck gauge, in the sense of Theorem 8.7. More precisely, we have

$$(\sigma_\alpha^k)^*(\tilde{\omega}_k)|_{U_\alpha} = \omega_{k,\alpha},$$

where $\delta\tilde{\omega}_{k,\alpha}^\top = 0 = \delta\tilde{\omega}_{k,\alpha}^\top$.

Let us write the overlap functions as

$$g_{\alpha\beta}^k = \sigma_\alpha^k(\sigma_\beta^k)^{-1}. \tag{8.13}$$

For $\alpha \neq \beta$, we have

$$(g_{\alpha\beta}^k)^*\tilde{\omega}_k|_{U_\alpha} = \tilde{\omega}_k|_{U_\beta}.$$

By Lemma 8.8 and the bounds on $\tilde{\omega}^k$ given by Theorem 8.7, we have

$$\begin{aligned} \|dg_{\alpha\beta}^k\|_{\mathrm{L}^{p/2}} &\leqslant \|\tilde{\omega}_k|_{U_\alpha}\|_{\mathrm{L}^{p/2}} + \|\tilde{\omega}_k|_{U_\beta}\|_{\mathrm{L}^{p/2}} \\ &\leqslant C\left(\|\tilde{\omega}_k^\top|_{U_\alpha}\|_{\mathrm{W}^{1,p}} + \|\tilde{\omega}_k^\top|_{U_\beta}\|_{\mathrm{W}^{1,p}} + \|\tilde{\omega}_k^\perp|_{U_\alpha}\|_{\mathrm{W}^{1,p}} + \|\tilde{\omega}_k^\perp|_{U_\beta}\|_{\mathrm{W}^{1,p}}\right) \\ &\leqslant 2C \sup_{k\in\mathbf{N}}\left(\|\mathbf{II}_k\|_{\mathrm{L}^p} + \|\eta_k^G\|_{\mathrm{L}^{p/2}} + \|\eta_k^R\|_{\mathrm{L}^{p/2}}\right), \end{aligned}$$

and similarly for the $\mathrm{L}^{p/2}$ norm of the second derivatives of $g_{\alpha\beta}^k$. Thus the family $(g_{\alpha\beta}^k)$ are bounded uniformly in $k \in \mathbf{N}$ in $\mathrm{W}^{2,p/2}(U_\alpha \cap U_\beta, G)$, where G is the group of adapted Darboux transformations.

Step 2: Definition of a global gauge transformation.

First we note that up to non-relabelled sequences, we may assume that the $(\tilde{\omega}^k)^\top|_{U_\alpha}$ converge weakly in $\mathrm{W}^{1,p/2}$ to a limit $(\tilde{\omega})^\top|_{U_\alpha}$ for each fixed α, and similarly for the normal parts $(\tilde{\omega}^k)^\perp|_{U_\alpha} \rightharpoonup (\tilde{\omega})^\perp|_{U_\alpha}$, while the second fundamental forms converge weakly in L^p in any trivialisation $\tilde{\omega}_k^{\mathbf{II}}|_{U_\alpha} \rightharpoonup \tilde{\omega}^{\mathbf{II}}|_{U_\alpha}$. Moreover, we may also assume that the overlap transformations $g_{\alpha\beta}^k \rightharpoonup g_{\alpha\beta}$ weakly in $\mathrm{W}^{2,p/2}(U_\alpha \cap U_\beta, G)$. By the compact embedding of $\mathrm{W}^{2,p/2}$ into continuous functions, the $g_{\alpha\beta}^k$ converge uniformly to $g_{\alpha\beta}$.

We now apply Theorem 8.15. This yields a cover of M by open sets $V_\alpha \subset U_\alpha$ and functions $\rho_\alpha^k \in \mathrm{W}^{2,p/2}(V_\alpha, G)$, for all $k > m$, where m is fixed but sufficiently large, such that the ρ_α^k are uniformly bounded in $\mathrm{W}^{2,p/2}$ and moreover, for all $k \geqslant m$,

$$g_{\alpha\beta}^k = \rho_\alpha^k g_{\alpha\beta}^m (\rho_\beta^k)^{-1}. \tag{8.14}$$

Philosophically, for $k \geqslant m$, the sequence of overlap functions $g_{\alpha\beta}^k$ stabilises.

As before, we shall assume that up to a non-relabelled sequence, the ρ_α^k converge uniformly and weakly in $\mathrm{W}^{2,p/2}$ to limits $\rho_\alpha \in \mathrm{W}^{2,p/2}(V_\alpha, G)$.

We may now define global gauge transformations for each $k > m$, given on each V_α by

$$s_k|_{V_\alpha} = \sigma_{\alpha,k}^{-1} \rho_{\alpha,k} \sigma_{\alpha,m}. \tag{8.15}$$

Our gauge transforms are precisely chosen to be well-defined globally. Indeed, on $V_\alpha \cap V_\beta$ for $\alpha \neq \beta$, we have

$$\begin{aligned} s_k|_{V_\alpha \cap V_\beta} &= \sigma_{\alpha,k}^{-1} \rho_{\alpha,k} \sigma_{\alpha,m}|_{V_\alpha \cap V_\beta} \\ &= \sigma_{\alpha,k}^{-1} \rho_{\alpha,k} g_{\alpha\beta}^m \sigma_{\beta,m}|_{V_\alpha \cap V_\beta} \\ &= \sigma_{\alpha,k}^{-1} g_{\alpha\beta}^k \rho_\beta^k \sigma_{\beta,m}|_{V_\alpha \cap V_\beta} \\ &= \sigma_{\beta,k}^{-1} \rho_{\beta,k} \sigma_{\beta,m}|_{V_\alpha \cap V_\beta} \\ &= s_k|_{V_\beta \cap V_\alpha}, \end{aligned}$$

where we use (8.13) and (8.14). Thus the s_k are globally defined. Moreover, by construction, $s_k \in \mathrm{W}^{2,p/2}(M, G)$. Note carefully that we do not have a uniform bound on the s_k though, as there are no uniform bounds on the local trivialisations $\sigma_{k,\alpha}$.

We claim that $s_k^* \omega_k$ is weakly convergent. This is a calculation using the definition of the s_k and the weak convergences of the $\tilde{\omega}_{k,\alpha}$ and the $\rho_{k,\alpha}$ obtained above. Noting (8.15), we have $s_k|_{V_\alpha} \sigma_{\alpha,m}^{-1} = \sigma_{\alpha,k}^{-1} \rho_{\alpha,k}$ and so

$$\begin{aligned} (\sigma_{\alpha,m}^{-1})^* s_k^* \omega_k|_{V_\alpha} &= \rho_{\alpha,k}^* (\sigma_{\alpha,k}^{-1})^* \omega_k \\ &= \rho_{\alpha,k}^* \tilde{\omega}_k|_{V_\alpha} \\ &\rightharpoonup \rho_\alpha^* \tilde{\omega}|_{V_\alpha}. \end{aligned}$$

The last step follows from the fact that $\tilde{\omega}_k \rightharpoonup \tilde{\omega}$ and $\rho_{k,\alpha} \rightharpoonup \rho_\alpha$. Pre-composing with $\sigma_{\alpha,m}$, we have

$$\begin{aligned} s_k^* \omega_k|_{V_\alpha} &= \sigma_{\alpha,m}^*(\rho_{\alpha,k}^* \tilde{\omega}_k) \\ &\rightharpoonup \sigma_{\alpha,m} \rho_\alpha^* \tilde{\omega}|_{V_\alpha}, \end{aligned}$$

weakly in L^p, and weakly in $\mathrm{W}^{1,p/2}$ for the tangential and normal parts. Q.E.D.

8.4 Construction of isometric immersions: proof of Theorem 8.1

Having constructed uniform $\mathrm{W}^{1,p}$ bounds on the tangential and normal connections, we may now verify the validity of the Gauß–Codazzi–Ricci equations in the limit.

Proposition 8.17. *Let $p > n$ and $\omega_k = (\omega_k^\top, \omega_k^{\mathbf{II}}, \omega_k^\perp) \in \mathrm{L}^p$ be a sequence of global weak approximate solution of the Gauß–Codazzi–Ricci equations, and let $\omega = (\omega^\top, \omega^{\mathbf{II}}, \omega^\perp)$ be an accumulation point of the ω_k given by Theorem 8.16. Then ω solves the Gauß–Codazzi–Ricci equations.*

Proof. By Theorem 8.16, there exists a (non-relabelled) subsequence of indices, a sequence of adapted gauge transforms s_k, and a limit $\omega = (\omega^\top, \omega^{\mathbf{II}}, \omega^\perp)$ such that $s_k^* \omega_k \rightharpoonup \omega$. Locally, one may write each s_k as

$$s_k = \begin{pmatrix} s_k^\top & 0 \\ 0 & s_k^\perp \end{pmatrix},$$

and we have

$$\begin{aligned} (s_k^\top)^* \omega_k^\top &\rightharpoonup \omega^\top, \\ (s_k^\perp)^* \omega_k^\perp &\rightharpoonup \omega^\perp, \end{aligned}$$

weakly in $\mathrm{W}^{1,p}$, while $s_k^* \omega^{\mathbf{II}} \rightharpoonup \omega^{\mathbf{II}}$ weakly in L^p.

Recall that for every $k \in \mathbf{N}$ the Gauß–Codazzi–Ricci equations hold approximately for $(\omega_k^\top, \omega_k^{\mathbf{II}}, \omega_k^\perp)$, in the sense that there exist two-forms η_k such that locally, one may write

$$\eta_k = \begin{pmatrix} \eta_k^G & {}^t\eta_k^C \\ -\eta_k^C & \eta_k^R \end{pmatrix},$$

and $d\omega_k + \omega_k \wedge \omega_k = \eta_k$. By hypothesis, we have $(s_k^\top)^* \eta_k^G \rightharpoonup 0$ up to a subsequence, and similarly $(s_k^\perp)^* \eta_k^R \rightharpoonup 0$.

Let us write $s_k^*\omega := \tilde{\omega}_k$ and $s_k^*\eta_k = \tilde{\eta}_k$. We now show that the Gauß–Codazzi–Ricci equations hold in the limit as an application of Lemma 6.5. This follows closely the argument of Theorem 7.2.

Recall that Gauß–Codazzi–Ricci equations

$$\begin{aligned}
&d\omega_k^\top + \omega_k^\top \wedge \omega_k^\top - {}^t\omega_k^{\mathbf{II}} \wedge \omega_k^{\mathbf{II}} = \eta_k^G,\\
&d\omega_k^{\mathbf{II}} + \omega_k^{\mathbf{II}} \wedge \omega_k^\top + \omega_k^\perp \wedge \omega_k^{\mathbf{II}} = \eta_k^C,\\
&d\omega_k^\perp + \omega_k^\perp \wedge \omega_k^\perp - \omega_k^{\mathbf{II}} \wedge {}^t\omega_k^{\mathbf{II}} = \eta_k^R,
\end{aligned}$$

are invariant under the action (by conjugation) of adapted Darboux transformations.

We have uniform $\mathrm{W}^{1,p/2}$ bounds on $\tilde{\omega}_k^\top$ and $\tilde{\omega}_k^\perp$; moreover, we have uniform L^p bounds on $\tilde{\omega}_k^{\mathbf{II}}$. Moreover, η_k^C is compactly contained in $\mathrm{W}^{-1,q}$ for $q > 2$, and η_k^G and η_k^R are bounded in L^p. Hence the whole RHS is compactly contained in $\mathrm{W}^{-1,q}$ for $q > 2$.

Apply the div-curl lemma (Theorem 8.4), we see that $\tilde{\omega}_k \wedge \tilde{\omega}_k \overset{*}{\rightharpoonup} \omega \wedge \omega$ in the sense of distributions. In particular, ${}^t\tilde{\omega}_k^{\mathbf{II}} \wedge \tilde{\omega}_k^{\mathbf{II}} \overset{*}{\rightharpoonup} {}^t\omega^{\mathbf{II}} \wedge \omega^{\mathbf{II}}$, and similarly $\tilde{\omega}_k^{\mathbf{II}} \wedge {}^t\tilde{\omega}_k^{\mathbf{II}} \overset{*}{\rightharpoonup} \omega^{\mathbf{II}} \wedge {}^t\omega^{\mathbf{II}}$. As $\tilde{\eta}_k \overset{*}{\rightharpoonup} 0$ in the sense of distributions, we conclude that $d\omega + \omega \wedge \omega = 0$, that is, $\omega = (\omega^\top, \omega^{\mathbf{II}}, \omega^\perp)$ is a weak solution of the Gauß–Codazzi–Ricci equations.

Q.E.D.

This concludes the proof of the main result of the chapter.

Proof of Theorem 8.1. By Proposition 8.17, there exists a global L^p solution of the Gauß–Codazzi–Ricci equations on M. By the realisation theorem (*cf.* Theorem 8.5), there exists a global isometric immersion of M into $\mathbf{R}^N$ with prescribed second fundamental form $\mathbf{II}$, tangential connection $\omega^\top$ and normal connection $\omega^\perp$, which is unique up to rigid motions of $\mathbf{R}^N$. Q.E.D.

Appendices

Appendix A

Sobolev Spaces

Sobolev spaces are standard spaces for the analysis of partial differential equations, see *e.g.* [56, 66, 73]. The purpose of this appendix is to summarise basic facts about Sobolev spaces in various contexts. These facts are used throughout the thesis without further reference. For the ACL characterisation, the reader is referred to [124]; Sobolev spaces on vector bundles are reviewed in *e.g.* [175, Appendix C]; for a detailed exposition on Sobolev spaces on manifolds, see *e.g.* [89]. Finally, for fractional Sobolev spaces, see [56].

A.1 ACL characterisation of Sobolev spaces

We recall the characterisation of Sobolev spaces through restrictions on lines in a domain of $\mathbf{R}^n$.

Let $I \subset \mathbf{R}$ be an interval. Recall that a function $f \in \mathrm{L}^1(I)$ is said to be absolutely continuous if for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $a_1 < b_1 < a_2 < \cdots < b_k$ such that $a_i, b_i \in I$ and $\sum_{i=1}^k (b_i - a_i) < \delta$, there holds $\sum_{i=1}^k |f(b_i) - f(a_i)| < \varepsilon$.

An absolutely continuous function f on I is differentiable almost everywhere and its derivative f' is $\mathrm{L}^1(I)$.

Let $E \subset \mathbf{R}^n$ be a domain. A function $f \in \mathrm{L}^1(E)$ is said to be absolutely continuous on lines if for almost every (one-dimensional) line L parallel to the axes of $\mathbf{R}^n$, $f|_L$ is absolutely continuous. The next theorem gives a characterisation of Sobolev functions in terms of absolute continuity on lines.

THEOREM A.1. *Let* $1 \leqslant p \leqslant \infty$ *and let* $L_i = \{(x^1, \dots, x^{i-1}, t, x^{i+1}, \dots, x^n) \,\big|\, t \in \mathbf{R}\} \cap E$ *be a line parallel to the* i*-th axis of* $\mathbf{R}^n$*. . Then* $f \in \mathrm{W}^{1,p}(E) \Leftrightarrow$ *there exists a measurable function* $\tilde{f}$ *such that* $f = \tilde{f}$ *almost everywhere, and for all* $i =$

$1, \ldots, n$ and almost every line L_i, $\tilde{f}|_{L_i}$ is absolutely continuous, and moreover $(\tilde{f}|_{L_i})' \in \mathrm{L}^p(E)$. Furthermore, in this case, its partial derivatives $(\tilde{f}|_{L_i})'$ coincide with the weak derivatives $\partial_i f$ almost everywhere.

For the proof of this exact statement, we refer the reader to [124].

This theorem seems to have several attributions—to Nikodym for the one-dimensional case, and Morrey and Calkin, and sometimes also to Gagliardo (though his work [71] appeared 18 year after Morrey's and Calkin's [24]).

A.2 Sobolev spaces $\mathrm{W}^{s,p}(M, V)$, $s \in \mathbf{N}$, on vector bundles

We consider a closed Riemannian manifold M and a vector bundle $V \to M$. Assume that V is equipped with an inner product of the fibers and a smooth covariant derivative ∇. For $E \subset M$ let $\Gamma(E, V)$ be the set of smooth sections of $V|_E \to E$.

For $1 \leqslant p < \infty$, $s \in \mathbf{N}$, the L^p and $\mathrm{W}^{s,p}$ norms of $\zeta \in \Gamma(\bar{E}, V)$ are given by

$$\|\zeta\|_{\mathrm{L}^p}^p = \int_E |\zeta|_g^p dV_g, \qquad \|\zeta\|_{\mathrm{W}^{s,p}}^p = \sum_{j=0}^{s} \|\nabla^{(j)}\zeta\|_{\mathrm{L}^p}^p,$$

respectively, for $p = \infty$,

$$\|\zeta\|_{\mathrm{L}^\infty} = \text{ess sup}\, |\zeta|_g, \qquad \|\zeta\|_{\mathrm{W}^{s,\infty}} = \sum_{j=0}^{s} \|\nabla^{(j)}\zeta\|_{\mathrm{L}^\infty}.$$

The Sobolev spaces $\mathrm{W}^{s,p}(E, V)$ are defined as the completion of $\Gamma(\bar{E}, V)$ with respect to the norm $\|\cdot\|_{\mathrm{W}^{s,p}}$. Similarly, the $\mathrm{W}_0^{s,p}(E; V)$ spaces are defined as the completion of $\Gamma(E, V)$ with respect to the norm $\|\cdot\|_{\mathrm{W}^{s,p}}$. The spaces $C(E, V)$ and $C^{m,\alpha}(E, V)$ for $m \in \mathbf{N}$ and $0 < \alpha \leqslant 1$ are the spaces of continuous, respectively $C^{m,\alpha}$, sections over E.

Negative-order Sobolev spaces are defined as dual spaces. For $1 < p, q < \infty$, p and $q = p/(p-1)$ being Hölder conjugate exponents, we define

$$\mathrm{W}^{-s,p}(E, V|_E) = (\mathrm{W}_0^{s,q}(E, V|_E))^*.$$

As an example, we consider the case of k-forms over M. Let $V = \wedge^k T^* M$. M being a Riemannian manifold, V is equipped with a scalar product, inherited from the Riemannian metric, and a covariant derivative, defined by extending the Levi-Civita connection to covariant tensors on M. The above shows that one can define Sobolev spaces $\mathrm{W}^{s,p}(M, \wedge^k T^* M)$ of k-forms.

Remark A.1. *If the base manifold M is compact, the Sobolev spaces $\mathrm{W}^{k,p}(M,V)$ do not in fact depend on the choice of inner product on V, as on a compact manifold, any two continuous metrics g and h are equivalent,* i.e. *there exists a constant $C > 0$ such that for any point $p \in M$ and any vector $X \in V_p$, we have $Cg_p(X,X) \leqslant h_p(X,X) \leqslant C^{-1}g_p(X,X)$.*

A.3 Fractional Sobolev spaces $\mathrm{W}^{s,p}(E)$, $s \in \mathbf{R}_+ \setminus \mathbf{N}$

We now restrict our attention to the case of real-valued functions. Let $0 < s < 1$ and $1 \leqslant p < \infty$. For a smooth, compactly supported function $v \in C_c^\infty(E)$, we define the fractional Sobolev semi-norm as

$$[v]^p_{\mathrm{W}^{s,p}} := \int_E \int_E \frac{|v(x) - v(y)|^p}{|x-y|^{n+sp}} dxdy.$$

The Sobolev space $\mathrm{W}^{s,p}_0(E)$ is defined as the completion of smooth compactly supported functions under the norm

$$\|v\|^p_{\mathrm{W}^{s,p}(E)} = \|v\|^p_{\mathrm{L}^p(E)} + [v]^p_{\mathrm{W}^{s,p}}.$$

If E is bounded and $v \in C^{1,\alpha}(\bar{E})$ then $v \in \mathrm{W}^{s,p}(E)$ for all $s \leqslant \alpha$ and all $1 \leqslant p \leqslant \infty$.

We recall the following form of the Gagliardo–Nirenberg inequality (see *e.g.* [10, Appendix D] and the references therein for a proof of this particular inequality). Let $v \in \mathrm{W}^{s,p} \cap \mathrm{L}^\infty$, for $0 < s < \infty$ and $1 \leqslant p < \infty$ such that $(s,p) \neq (1,1)$. Then for all $0 < t < 1$,

$$\|v\|_{\mathrm{W}^{st,p/t}} \leqslant C\|v\|^{1-t}_{\mathrm{L}^\infty}\|v\|^t_{\mathrm{W}^{s,p}}.$$

If E is a Lipschitz domain, the Morrey embedding theorem states that when $sp > n = \dim(E)$, there holds $\mathrm{W}^{s,p}(E) \hookrightarrow C^{0,\alpha}_b(E)$ for $\alpha = s - n/p$. The space $C^{0,\alpha}_b(E)$ is the space of bounded, continuous functions on E such that $\forall x, y \in E$,

$$|u(x) - u(y)| \leqslant [u]_{C^{0,\alpha}}|x-y|.$$

A.3.1 Regularisation estimates

The following lemma gives estimates on mollifications of functions in Sobolev spaces.

Lemma A.2. *Let $1 \leqslant p < \infty$, $E \subset \mathbf{R}^n$ an open domain, $\phi \in \mathrm{L}^p(D)$. Let $\varepsilon > 0$; we consider the domain $D_\varepsilon = \{x \in D \,|\, d(x, \partial D) > \varepsilon\}$ and the mollification $\phi^\varepsilon = \phi * \eta^\varepsilon$,* *where* $\eta_\varepsilon = \varepsilon^{-n}\eta(\cdot/\varepsilon)$ *a non-negative mollifier with* $\int \eta = 1$ *and* $\operatorname{supp}\eta \subset B_1(0)$*. Then the following holds:*

$$\|\phi^\varepsilon\|_{\mathrm{L}^p(D_\varepsilon)} \leqslant \|\phi\|_{\mathrm{L}^p(D)}. \tag{A.1}$$

Moreover, if $0 < s < 1$ *and* $\phi \in \mathrm{W}^{s,p}(D)$*, there holds.*

$$\|\nabla\phi^\varepsilon\|_{\mathrm{L}^p(D_\varepsilon)} \leqslant C\varepsilon^{s-1}[\phi]_{\mathrm{W}^{s,p}(D)}, \tag{A.2}$$

$$\|\phi - \phi^\varepsilon\|_{\mathrm{L}^p(D_\varepsilon)} \leqslant \varepsilon^s[\phi]_{\mathrm{W}^{s,p}(D)}. \tag{A.3}$$

Proof. **Proof of** (A.1)**:** This estimate is standard [73]. Noting that $\eta_\varepsilon = \eta_\varepsilon^{1/p}\eta_\varepsilon^{1-1/p}$ and using Hölder's inequality,

$$\begin{aligned}
|\phi^\varepsilon| &= \left|\int_{B_\varepsilon} \phi(y)\eta_\varepsilon(x-y)dy\right| \\
&\leqslant \left(\int_{B_\varepsilon} \eta_\varepsilon(x-y)dy\right)^{1-1/p}\left(\int_{B_\varepsilon} |\phi(x)|^p\eta_\varepsilon(x-y)dy\right)^{1/p} \\
&= \left(\int_{B_\varepsilon} |\phi(x)|^p\eta_\varepsilon(x-y)dy\right)^{1/p}.
\end{aligned}$$

Integrating over D_ε and using Fubini's theorem, we have

$$\begin{aligned}
\int_{D_\varepsilon} |\phi^\varepsilon(x)|^p dx &\leqslant \int_{D_\varepsilon}\int_{B_\varepsilon} |\phi(x)|^p\eta_\varepsilon(x-y)dydx \\
&\leqslant \int_D |\phi(x)|^p dx.
\end{aligned}$$

Proof of (A.2)**:** We first note that as in the proof of Lemma 5.8, we have

$$\nabla\phi^\varepsilon(x) = -\int_D (\phi(x) - \phi(y))\nabla\eta_\varepsilon(x-y)dy.$$

Integrating by parts and using Hölder's inequality one has

$$\begin{aligned}
|\nabla\phi^\varepsilon(x)| &\leqslant \left|\int_{B_\varepsilon} \phi(y)\nabla\eta_\varepsilon(x-y)dy\right| \\
&= \left|\int_{B_\varepsilon} (\phi(y) - \phi(x))\nabla\eta_\varepsilon(x-y)dy\right| \\
&\leqslant \left(\int_{B_\varepsilon} |\nabla\eta_\varepsilon(x-y)|dy\right)^{1-1/p}\left(\int_{B_\varepsilon} |\phi(y)-\phi(x)|^p|\nabla\eta_\varepsilon(x-y)|dy\right)^{1/p}.
\end{aligned}$$

We note that

$$\left(\int_{B_\varepsilon} |\nabla\eta_\varepsilon(x-y)|dy\right)^{1-1/p} \leqslant \left(\int_{B_\varepsilon} \varepsilon^{-n-1}\left|\nabla\eta\left(\frac{x-y}{\varepsilon}\right)\right|dy\right)^{1-1/p} = C\varepsilon^{\frac{1-p}{p}}.$$

Integrating in x and using the definition of fractional Sobolev spaces, we have

$$\begin{aligned}\int_D |\nabla\phi^\varepsilon(x)|^p dx &\leqslant C\varepsilon^{1-p}\int_D\int_{B_\varepsilon}\frac{|\phi(x)-\phi(y)|^p}{|x-y|^{n+sp}}\varepsilon^{n+sp}|\nabla\eta_\varepsilon(x-y)|dydx\\ &\leqslant C\varepsilon^{-p}\varepsilon^{sp}[\phi]^p_{\mathrm{W}^{s,p}(D)}.\end{aligned}$$

Proof of (A.3): As for the proof of (A.1),

$$\begin{aligned}|\phi(x)-\phi^\varepsilon(x)| &= \left|\int_{B_\varepsilon}(\phi(x)-\phi(y))\eta_\varepsilon(x-y)dy\right|\\ &\leqslant \left(\int_{B_\varepsilon}|\phi(x)-\phi(y)|^p\eta_\varepsilon(x-y)dy\right)^{1/p}\end{aligned}$$

Integrating in x, we have

$$\begin{aligned}\int_{D_\varepsilon}|\phi(x)-\phi^\varepsilon(x)|^p dx &\leqslant \int_{D_\varepsilon}\int_{B_\varepsilon}|\phi(x)-\phi(y)|^p\eta_\varepsilon(x-y)dydx\\ &\leqslant \int_{D_\varepsilon}\int_{B_\varepsilon}\frac{|\phi(x)-\phi(y)|^p}{|x-y|^{n+sp}}\varepsilon^{n+sp}\eta_\varepsilon(x-y)dydx\\ &\leqslant \varepsilon^{sp}[\phi]^p_{\mathrm{W}^{s,p}}.\end{aligned}$$

Q.E.D.

Appendix B

Principal Bundles and Connections

This appendix establishes background material on connections; standard references on the subject are [70, 161, 175].

B.1 Vector bundles and principal bundles

Let M be a smooth manifold. A fiber bundle on M is a topological space X together with a surjective map $X \xrightarrow{\pi} M$ such that $\forall p \in M$, $\pi^{-1}(p) \simeq F$ where F is a fixed topological space. The space F is the fiber space, while $\pi^{-1}(p)$ is termed the fiber over p. When F is a vector space, the bundle is called a vector bundle.

Let G be a topological group, *e.g.* a Lie group. A principal G-bundle is a fiber bundle with fiber G together with a continuous, free and transitive action of G onto the fibers. If G is a matrix group, G acts by multiplication on the fibers.

Finite dimensional vector bundles and principal G-bundles, where G is a matrix group, are in some sense equivalent structures. To a vector bundle $V \to M$, one may construct its frame bundle $\mathscr{F}V \to M$, which is a principal $\mathbf{GL}(N)$-bundle, where $N = \dim V$ (the dimension of the fiber). Similarly, to a principal G-bundle, a choice of linear representation of G generates a vector bundle.

B.2 Lie algebras

Throughout this thesis we assume M to be a compact smooth Riemannian manifold without boundary, $\partial M = \varnothing$. Let $P \to M$ be a principal bundle, and assume that the structure group G of P is a matrix group. We shall assume that G is a subgroup of the special orthogonal group preserving the prescribed inner product on the fiber. $\mathbf{Aut}(P)$ is the automorphism bundle, with fiber G; finally $\mathbf{Ad}(P)$ is the adjoint bundle, whose fiber is the Lie Algebra $\mathfrak{g}$ generated by G. By definition $\mathfrak{g} \simeq T_1 G$. The notation $\Gamma(M, P)$ refers to the sheaf of sections of the bundle $P \to M$ (*cf.* list of notations).

Furthermore, as $\mathfrak{g}$ is a Lie algebra and G is a compact Lie group, there exists a unique G-equivariant scalar product on $\mathbf{Ad}(P)$, denoted by $\langle \cdot, \cdot \rangle$. The scalar product is compatible with the Lie bracket, in the sense that the triple product identity $\langle [\cdot, \cdot], \cdot \rangle = \langle \cdot, [\cdot, \cdot] \rangle$ holds. In addition, we rescale it in such a way that for arbitrary $\xi, \zeta \in \mathfrak{g}$, there holds $|[\xi, \zeta]| \leqslant |\xi||\zeta|$.

B.3 Differential forms

Let $A, B \in \Omega^k(M, \mathbf{Ad}(P)) := \Gamma(M, \mathbf{Ad}(P) \otimes \wedge^k T^*M)$ be $\mathbf{Ad}(P)$-valued k-forms. We denote by $[\cdot, \cdot]$ the Lie bracket on the Lie algebra $\mathfrak{g}$, which induces a Lie bracket defined fiberwise on $\mathbf{Ad}(P)$. The notation $[A \wedge B]$ denotes the wedge product of two k-forms, the values of which are combined through the Lie bracket.

Recall that for two $\mathfrak{g}$-valued one-forms $\alpha = A_i dx^i$ and $\beta = B_i dx^i$, their product is defined as

$$[\alpha \wedge \beta] := \sum_{i,j=1}^{n} [A_i, B_j] dx^i \wedge dx^j = \sum_{i<j} \left([A_i, B_j] - [A_j, B_i] \right) dx^i \wedge dx^j.$$

We note in particular that $[\alpha \wedge \alpha] = 2 \sum_{i<j} [A_i, A_j] dx^i \wedge dx^j \neq 0$, unless the Lie bracket is trivial (i.e. the Lie group is abelian).

More generally, this defines a wedge product on $\mathfrak{g}$-valued forms. Let $\alpha \in \Omega^p(M, \mathfrak{g})$ and $\beta \in \Omega^q(M, \mathfrak{g})$. We define $[\alpha \wedge \beta] \in \Omega^{p+q}(M, \mathfrak{g})$ by

$$[\alpha \wedge \beta](X_1, \ldots, X_{p+q}) = \sum_{\sigma \in \mathrm{Sh}_{p+q}} \varepsilon(\sigma) [\alpha(X_{\sigma(1)}, \ldots, X_{\sigma(p)}), \beta(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)})],$$

where $\varepsilon(\sigma)$ is the signature of the shuffle $\sigma \in \mathrm{Sh}_{p+q}$, i.e. monotone bijections of $\mathbf{Z}_{p+q}$. There holds

$$\begin{aligned}
&[\alpha \wedge \beta] = (-1)^{pq+1} [\beta \wedge \alpha], \\
&[[\alpha, \alpha], \alpha] = 0, \\
&d[\alpha \wedge \beta] = [d\alpha \wedge \beta] + (-1)^{pq} [\alpha \wedge d\beta].
\end{aligned}$$

Finally, for three $\mathfrak{g}$-valued one-forms α, β and γ, the Jacobi identity implies

$$[\alpha \wedge [\beta \wedge \gamma]] = [[\gamma \wedge \alpha] \wedge \beta] + [[\alpha \wedge \beta] \wedge \gamma].$$

The Jacobi identity gives a quantitive statement of the non-associativity of $(\mathfrak{g}, [\cdot, \cdot])$, and thus of $(\mathfrak{g} \otimes \wedge^{\bullet} T^*M, [\cdot \wedge \cdot])$.

Recall that there is a scalar product on $\mathfrak{g}$, which induces a scalar product on the space of sections $\Gamma(M, \mathbf{Ad}(P) \otimes \wedge^k T^*M)$, defined as $\langle \cdot, \cdot \rangle \otimes g(\cdot, \cdot)$. Here $g(\cdot, \cdot) = \langle \cdot, \cdot \rangle_g$ is the scalar product induced on T^*M by the Riemannian metric on M (and given componentwise by the inverse of the metric tensor g). For one-forms $A = A_i dx^i, B = B_j dy^j$ as before, we have

$$\begin{aligned} \langle A, B \rangle &= \langle A_i, B_j \rangle \otimes \langle dx^i, dy^j \rangle_g \\ &= \langle A_i, B_j \rangle \otimes *(dx^i \wedge (*dy^j)) \\ &\in \Gamma(M, \mathbf{R}) = C^\infty(M). \end{aligned}$$

This defines a norm $|A|^2 = \langle A, A \rangle$ on k-forms.

B.4 Connections

A connection is a way of relating the fibers of a bundle intrinsically. There are several equivalent ways of defining a connection (see *e.g.* [161, 175]). Recall that for a fiber bundle $P \overset{\pi}{\to} M$, the vertical bundle of P is the bundle $U \to M$ defined as $U = \mathrm{Ker}(d\pi : TP \to TM)$. Any subbundle H of P such that $P = U \oplus H$ is called horizontal. An Ehresmann connection, or horizontal connection, is simply a choice of horizontal space together with the projection $P \to H$. A connection on a principal G-bundle is a G-equivariant horizontal connection. We call $\mathcal{A}(P)$ the space of smooth connections on the bundle $P \to M$. Note that $\mathcal{A}(P)$ is an affine space.

Let us fix a smooth base connection D_0. Then any connection in $\mathcal{A}(P)$ differs from D_0 by a $\mathbf{Ad}(P)$-valued one-form. Up to choosing a local trivialisation of P, we can therefore view any connection $D \in \mathcal{A}(P)$ as being given by a potential

$$D = D_0 + A, \quad A \in \Gamma(M, \mathfrak{g} \otimes T^*M).$$

The description of the connection D in terms of potential depends on the trivialisation of $\mathbf{Ad}(P)$, and D_0. Any other choice leads to a different gauge potential $\tilde{A}$. Equivalently, we may consider a covering of $P \to M$ by trivialising sets U_α; a connection D is then given as a list of $\mathfrak{g}$-valued one-forms A_α.

Recall that a gauge transformation is an equivariant fiber-preserving automorphism of P, i.e. a section of the bundle $\mathbf{Aut}(P)$, which, fiberwise, is the group G. The gauge transformation group acts on connections in the following way: letting as before $D = D_0 + A$ locally, and s a gauge transformation, we have

$$s^* D = s^{-1} D_0 s + s^{-1} A s,$$

where s is a gauge transformation, and $s^*D = s^{-1} \circ D \circ s$ is the pull-back of D under s, namely, for a section $\sigma \in \Gamma(M, P)$, $(s^*D)\sigma := s^{-1}D(s\sigma)$. In a local trivialisation, s is a G-valued map. We view D and s^*D as being equivalent as connections, and we say that they are gauge-equivalent.

The main invariant of a connection is its curvature. For a connection $D \in \mathcal{A}(P)$, its curvature is the two-form $\Omega_D \in \Gamma(M, \mathbf{Ad}(P) \otimes \wedge^2 T^*M)$ representing the action of the mapping $D \circ D$. In a local trivialisation U_α, we may write $D = d + A_\alpha$ and its curvature is

$$\Omega_D|_{U_\alpha} = dA_\alpha + \frac{1}{2}[A_\alpha \wedge A_\alpha].$$

Note that contrary to connections, Ω_D is a tensor: under a gauge transformation, its transforms as

$$s^*\Omega_D = s^{-1}\Omega_D s.$$

In this thesis, we consider weak continuity properties of Ω_D for sequences of connections in Sobolev spaces, which we now define. Let $k \in \mathbf{N}$ and $1 \leqslant p \leqslant \infty$; $\mathcal{A}^{k,p}(P)$ denotes the the spaces of connections on T, defined by

$$\begin{aligned} \mathcal{A}^{k,p}(P) &= D_0 + \mathrm{W}^{k,p}(M, \mathbf{Ad}(P) \otimes T^*M) \\ &= \{D_0 + A \,|\, A \in \mathrm{W}^{k,p}(M, \mathbf{Ad}(P) \otimes T^*M)\}. \end{aligned}$$

Recall that $\mathbf{Ad}(P) \otimes T^*M$ is a vector bundle equipped with a scalar product. Thus for all $k \in \mathbf{N}$ and $1 \leqslant p \leqslant \infty$, the Sobolev space of $\mathrm{W}^{k,p}$ sections of the bundle $\mathbf{Ad}(P) \otimes T^*M \to M$ is well-defined (see Appendix A). Spaces of connections are affine. Thus all L^p norms of connections are with respect to a base connection D_0 as above, so that we can write

$$\|D_0 + A\|_{\mathrm{L}^p} := \|A\|_{\mathrm{L}^p},$$

where we identify the one-form A with the connection $D_0 + A$ in a convenient abuse of notation. One defines Sobolev norms of connections in a similar manner. Note carefully that although the Sobolev norms depend on a choice of origin D_0 (and a metric), the Sobolev spaces $\mathcal{A}^{k,p}(P)$ do not.

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