

The Complexity of Fully Proportional Representation for Single-Crossing Electorates[☆]

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Abstract

We study the complexity of winner determination in single-crossing elections under two classic fully proportional representation rules—Chamberlin–Courant’s rule and Monroe’s rule. Winner determination for these rules is known to be NP-hard for unrestricted preferences. We show that for single-crossing preferences this problem admits a polynomial-time algorithm for Chamberlin–Courant’s rule, but remains NP-hard for Monroe’s rule. Our algorithm for Chamberlin–Courant’s rule can be modified to work for elections with *bounded single-crossing width*. We then consider elections that are both single-peaked and single-crossing, and develop an efficient algorithm for the egalitarian variant of Monroe’s rule for such elections. While Betzler et al. [3] have recently presented a polynomial-time algorithm for this rule under single-peaked preferences, our algorithm has considerably better worst-case running time than that of Betzler et al.

Keywords: single-crossing, winner determination, Chamberlin–Courant’s rule, Monroe’s rule, complexity

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1. Introduction

Parliamentary elections, i.e., procedures for selecting a fixed-size set of candidates that, in some sense, best represent the voters, received a lot of attention in the literature. Some well-known approaches include the first-past-the-post system (FPTP), where the voters are divided into districts and in each district a plurality election is held to find this district’s representative; party-list systems, where the voters vote for parties and later the parties distribute the seats among their members; SNTV (single nontransferable vote) and Bloc rules, where each voter picks t candidates to approve, and the rule picks k candidates with the highest number of approvals (here k is the target parliament size, and $t = 1$ for SNTV and $t = k$ for Bloc); and some variants of STV (single transferable vote). In this paper, we focus on two voting rules that for each voter explicitly define the candidate that will represent her in the parliament (such rules are said to provide *fully proportional representation*), namely, Chamberlin–Courant’s rule [6] and Monroe’s rule [18]. Winner determination algorithms for these rules can also be used for applications other than parliamentary elections, such as resource allocation [18, 25] and recommender systems [16].

Let us consider an election where n voters aim to select a k -member parliament out of m candidates. Both Chamberlin–Courant’s rule and Monroe’s rule work by finding a function Φ that maps each voter v to the candidate that is to represent v in the parliament. This function is required to output at most k candidates altogether.¹ Further, under Monroe’s rule each candidate is either assigned to about $\frac{n}{k}$ voters or to none, whereas under Chamberlin–Courant’s rule there is no such restriction (as a consequence, parliaments elected according to Chamberlin–Courant’s rule may have to use weighted voting in their proceedings).

Intuitively, under both rules, each voter should be represented by a candidate that this voter ranks as highly as possible. To specify this requirement formally, we assume that there is a global *dissatisfaction function* α ,

¹Under Monroe’s rule we are required to pick exactly k winners. Some authors also impose this requirement in the case of Chamberlin–Courant’s rule, but allowing for smaller parliaments appears to be more consistent with the spirit of this rule and is standard in its computational analysis (see, e.g., [16, 3, 25, 24, 27]). In any case, this distinction has no bite if there are at least k candidates such that each of them is ranked first by at least one voter, which is a typical situation in political elections.

$\alpha: \mathbb{N} \rightarrow \mathbb{N}$, such that $\alpha(i)$ is voter's dissatisfaction from being represented by a candidate that she views as i -th best—for instance, Borda dissatisfaction function α_B is given by $\alpha_B(i) = i - 1$. In the utilitarian variants of Chamberlin–Courant's and Monroe's rules we seek assignments that minimize the sum of voters' dissatisfactions; in the egalitarian variants (introduced recently by Betzler et al. [3]) we seek assignments that minimize the dissatisfaction of the worst-off voter.

Chamberlin–Courant's and Monroe's rules have a number of attractive properties, which distinguish them from other multiwinner rules. Indeed, they elect parliaments that (at least in some sense) proportionally represent the voters, ensure that candidates who are not individually popular do not make it to the parliament even if they come from very popular parties, and take minority candidates into account. In contrast, FPTP can provide largely disproportionate results, party-list systems cause members of parliament to feel more responsible to the parties than to the voters, SNTV and Bloc tend to disregard minority candidates, and STV is sometimes accused of putting too much emphasis on voters' top preferences. We point readers interested in the properties of Monroe's and Chamberlin–Courant's rules, as well as some other multiwinner voting rules, to the recent work of Elkind et al. [12].

Unfortunately, Chamberlin–Courant's and Monroe's rules do have one flaw that makes them impractical: It is NP-hard to compute a winning parliament with respect to these rules [21, 16, 3]. Nonetheless, these rules are so attractive that there is a growing body of research on computing their outputs either exactly (e.g., through integer linear programming formulations [20], by means of fixed-parameter tractability analysis [3], or by considering restricted preference domains [3, 27]) or approximately [16, 25, 24, 23]. We continue this line of research by considering the complexity of computing the outputs of Chamberlin–Courant's and Monroe's rules for the case where voters' preferences are single-crossing. Our results complement those of Betzler et al. [3] for single-peaked electorates.

Recall that voters are said to have single-crossing preferences if it is possible to order them so that for every pair of candidates a, b the voters who prefer a to b appear together on one side of the order and the voters who prefer b to a appear together on the other side. For example, it is quite natural to assume that the voters are aligned on the standard political left-right axis. Given two candidates a and b , where a is viewed as more left-wing and b is viewed as more right-wing, the left-leaning voters would prefer a to b and the right-leaning voters would prefer b to a . While real-life elections are

typically too noisy to have this property, it is plausible that they are often close to single-crossing, and it is important to understand the complexity of the idealized model before proceeding to study nearly single-crossing profiles. Indeed, in the context of single-peaked elections, Faliszewski et al. [14] have recently shown that many algorithmic results established for such elections can be extended to elections that are nearly single-peaked.

Our main results are as follows: for single-crossing elections winner determination under Chamberlin–Courant’s rule is in P (for every dissatisfaction function, and both for the utilitarian and for the egalitarian variant of this rule), but under Monroe’s rule it is NP-hard. Our hardness result for Monroe’s rule applies to the utilitarian setting with Borda dissatisfaction function. Our algorithm for Chamberlin–Courant’s rule extends to elections that have bounded *single-crossing width* (see [7, 8]). The proof proceeds by showing that for single-crossing elections Chamberlin–Courant’s rule admits an optimal assignment that has the *contiguous blocks property*: the set of voters assigned to an elected representative forms a contiguous block in the voters’ order witnessing that the election is single-crossing. This property can be interpreted as saying that each selected candidate represents a group of voters who are fairly similar to each other, and we believe it to be desirable in the context of proportional representation.

The NP-hardness result for winner determination under Monroe’s rule motivates us to search for further domain restrictions that may make this problem efficiently solvable. To this end, we focus on the egalitarian variant of Monroe’s rule and consider elections that are both single-peaked and single-crossing. We provide an $O(m^2n)$ algorithm for this setting (where n is the number of voters and m is the number of candidates), thus improving over the $O(n^3m^3k^3)$ algorithm (where k is the target parliament size) of Betzler et al. [3] for single-peaked preferences. Our algorithm is based on the observation that under the egalitarian variant of Monroe’s rule single-peaked single-crossing elections always admit an optimal assignment that satisfies the contiguous blocks property. The proof uses the recent characterization result of Elkind et al. [10] for this domain. We show, however, that our approach does not extend to general single-crossing elections or to the utilitarian variant of Monroe’s rule: in both cases, requiring the contiguous blocks property may rule out all optimal assignments. We then ask whether an analogue of the contiguous blocks property (obtained by considering the ordering of the voters that is induced by the axis; see Section 5 for details) can be used to design faster algorithms for all single-peaked elections (and

not just ones that are also single-crossing). Unfortunately, it appears that this is not the case: we construct a single-peaked election where no optimal assignment has the contiguous blocks property (both for Monroe’s rule and for Chamberlin–Courant’s rule, and both for the egalitarian variant and for the utilitarian variant of either rule).

The paper is organized as follows. In Section 2 we provide the required background, give the definitions of Monroe’s and Chamberlin–Courant’s rules, and define single-crossing and single-peaked elections. Then, in Sections 3 and 4, we discuss the complexity of winner determination under Chamberlin–Courant’s and Monroe’s rules, respectively. We show the limits of the contiguous blocks property approach in Section 5. We conclude the paper in Section 6 by summarizing our results and discussing future research directions.

2. Preliminaries

For every positive integer s , we let $[s]$ denote the set $\{1, \dots, s\}$. An *election* is a pair $E = (C, V)$ where $C = \{c_1, \dots, c_m\}$ is a set of candidates and $V = (v_1, \dots, v_n)$ is an ordered list of voters. Each voter $v \in V$ has a *preference order* \succ_v , i.e., a linear order over C that ranks all the candidates from the most desirable one to the least desirable one. For each voter $v \in V$ and each candidate $c \in C$, we denote by $\text{pos}_v(c)$ the position of c in v ’s preference order (the top candidate has position 1 and the last candidate has position $|C|$). We refer to the list V as the *preference profile*.

We denote the concatenation of two voter lists U and V by $U + V$. A list U is said to be a *sublist* of a list V (denoted by $U \subseteq V$) if U can be obtained from V by deleting voters. Given two disjoint sets $A, B \subset C$, we write $\dots \succ A \succ \dots \succ B \succ \dots$ to denote a vote where all candidates in A are ranked above all candidates in B .

2.1. Chamberlin–Courant’s and Monroe’s Rules

Both Chamberlin–Courant’s rule and Monroe’s rule rely on the notion of a *dissatisfaction function* (also known as a *misrepresentation function*). This function specifies, for each $i \in [m]$, a voter’s dissatisfaction from being represented by the candidate she ranks in position i .

Definition 1. For an m -candidate election, a *dissatisfaction function* is a nondecreasing function $\alpha: [m] \rightarrow \mathbb{N}$ with $\alpha(1) = 0$.

We will typically be interested in families of dissatisfaction functions, $(\alpha^m)_{m=1}^\infty$, with one function for each possible number of candidates. In particular, we will be interested in *Borda dissatisfaction function* $\alpha_B^m(i) = i - 1$. We assume that our dissatisfaction functions are computable in polynomial time with respect to m .

Let k be a positive integer. Given a mapping $\Phi: V \rightarrow C$, we set $\Phi(V) = \{\Phi(v) \mid v \in V\}$. A *k-CC-assignment function* for an election $E = (C, V)$ is a mapping $\Phi: V \rightarrow C$ such that $|\Phi(V)| \leq k$. A *k-Monroe-assignment function* for E is a *k-CC-assignment function* that additionally satisfies the following constraints: $|\Phi(V)| = k$, and for each $c \in C$ either $|\Phi^{-1}(c)| = 0$ or $\lfloor \frac{n}{k} \rfloor \leq |\Phi^{-1}(c)| \leq \lceil \frac{n}{k} \rceil$. That is, both assignment functions select (up to) k candidates, and a *k-Monroe-assignment function* additionally ensures that each selected candidate is assigned to roughly the same number of voters. For a given assignment function Φ , we say that voter $v \in V$ is *represented* (in the parliament) by candidate $\Phi(v)$. There are several ways to measure the quality of an assignment function Φ with respect to a dissatisfaction function α ; we use the following two:

1. $\ell_1(\Phi, \alpha) = \sum_{i=1, \dots, n} \alpha(\text{pos}_{v_i}(\Phi(v_i)))$, and
2. $\ell_\infty(\Phi, \alpha) = \max_{i=1, \dots, n} \alpha(\text{pos}_{v_i}(\Phi(v_i)))$.

Intuitively, $\ell_1(\Phi, \alpha)$ takes the utilitarian view of measuring the sum of voters' dissatisfactions, whereas $\ell_\infty(\Phi, \alpha)$ takes the egalitarian view of looking at the worst-off voter only.

We are now ready to define the voting rules that are the subject of this paper.

Definition 2. For every family of dissatisfaction functions $\alpha = (\alpha^m)_{m=1}^\infty$, every $R \in \{\text{CC}, \text{Monroe}\}$, and every $\ell \in \{\ell_1, \ell_\infty\}$, an α - ℓ - R *voting rule* is a mapping that takes an election $E = (C, V)$ and a positive integer k with $k \leq |C|$ as its input, and returns a *k-R-assignment function* Φ for E that minimizes $\ell(\Phi, \alpha)$. If there are several optimal assignments, the rule is free to return any of them.

Chamberlin and Courant [6] and Monroe [18] proposed the utilitarian variants of their rules and focused on Borda dissatisfaction function (though Monroe also considered the so-called *k-approval* dissatisfaction functions, where $\alpha_{k\text{-App}}^m(i) = 0$ for $i = 1, \dots, k$ and $\alpha_{k\text{-App}}^m(i) = 1$ for $i = k + 1, \dots, m$). Egalitarian variants of both rules have been recently introduced by Betzler et al. [3].

2.2. Single-Crossing and Single-Peaked Preferences

The notion of single-crossing preferences dates back to the work of Mirrlees [17]; examples of settings where single-crossing preferences arise can be found in the work of Saporiti and Tohmé [22]. Formally, single-crossing elections are defined as follows.

Definition 3. An election $E = (C, V)$, where $V = (v_1, \dots, v_n)$ is an ordered list of voters, is *single-crossing* (with respect to the given order of voters) if for each pair of candidates a, b such that $a \succ_{v_1} b$ there exists a value $t_{a,b} \in [n]$ such that $\{i \in [n] \mid a \succ_{v_i} b\} = [t_{a,b}]$.

That is, as we sweep through the list of voters from the first one towards the last one, the relative order of every pair of candidates changes at most once.

Definition 3 refers to the ordering of the voters provided by V . Alternatively, one could simply require existence of an ordering of the voters that satisfies the single-crossing property. The advantage of our approach is that it simplifies notation, yet does not affect the complexity of the problems that we study: one can compute an order of the voters that makes an election single-crossing (or decide that such an order does not exist) in time $O(nm^2)$ [5] (see also the work of Elkind et al. [11] for an alternative polynomial-time algorithm for this problem).

We also consider single-peaked elections [4].

Definition 4. Let \succ be a preference order over candidate set C , let \triangleleft be an order over C , and let a be the top alternative in \succ . We say that \succ is *single-peaked with respect to* \triangleleft if for every pair of candidates $b, c \in C$ such that $a \triangleleft b \triangleleft c$ or $c \triangleleft b \triangleleft a$ it holds that $b \succ c$. An election $E = (C, V)$ is *single-peaked with respect to an order* \triangleleft *over* C if the preference order of every voter $v \in V$ is single-peaked with respect to \triangleleft . An election $E = (C, V)$ is *single-peaked* if there exists an order \triangleleft over C with respect to which it is single-peaked.

If an election E is single-peaked with respect to some order \triangleleft then we call \triangleleft a *societal axis* for E . It is possible to decide in time $O(nm)$ whether a given election E is single-peaked and if so, compute a societal axis for it [13] (see also [2] for the first polynomial-time algorithm for this problem). Thus, just as in the case of single-crossing elections, we can freely assume that if an election is single-peaked then we are given a societal axis witnessing this.

3. Chamberlin–Courant’s Rule

We start our discussion by considering the complexity of winner determination under Chamberlin–Courant’s rule, for the case of single-crossing preferences.

3.1. Single-Crossing Preferences

A key observation in our analysis of Chamberlin–Courant’s rule is that for single-crossing preferences there always exists an optimal k -CC-assignment function where the voters matched to a given candidate form contiguous blocks within the voters’ order. In what follows, we will say that assignments of this form have the *contiguous blocks property*. We believe that this property is desirable from the social choice perspective: it means that voters who are represented by the same candidate are quite similar, which makes it easier for the candidate to act in a way that reflects the preferences of the group he represents. We will see that the contiguous blocks property also has useful algorithmic implications.

Lemma 5. *Let $E = (C, V)$ be a single-crossing election, where $C = \{c_1, \dots, c_m\}$, $V = (v_1, \dots, v_n)$, and v_1 has preference order $c_1 \succ \dots \succ c_m$. Then for every $k \in [m]$, every dissatisfaction function α for m candidates, and every $\ell \in \{\ell_1, \ell_\infty\}$, there is a k -CC-assignment Φ for E such that Φ is optimal under α - ℓ -CC and for each candidate $c \in \Phi(V)$ there are two integers, t and t' , $t \leq t'$, such that $\Phi^{-1}(c) = \{v_t, v_{t+1}, \dots, v_{t'}\}$. Moreover, for every pair of voters v_i, v_j such that $i < j$, $\Phi(v_i) = c_p$, and $\Phi(v_j) = c_q$ it holds that $p \leq q$.*

PROOF. Fix a single-crossing election $E = (C, V)$ with $C = \{c_1, \dots, c_m\}$ and $V = (v_1, \dots, v_n)$, and let Φ be an optimal k -CC-assignment function for E under α - ℓ -CC. We assume without loss of generality that for each voter v_i in V , the candidate $\Phi(v_i)$ is v_i ’s most preferred candidate in $\Phi(V)$.

Suppose for the sake of contradiction that the statement of the lemma is not true. Then (a) there exist voters v_i, v_j, v_r and candidates $c_p, c_q \in C$ such that $i < j < r$, $p \neq q$, $\Phi(v_i) = \Phi(v_r) = c_p$, and $\Phi(v_j) = c_q$, or (b) there exist voters v_i, v_j and candidates $c_p, c_q \in C$ such that $i < j$, $\Phi(v_i) = c_p$, $\Phi(v_j) = c_q$, and $p > q$. We will now argue that both of the scenarios (a) and (b) are impossible. Indeed, in scenario (a) we have $c_p \succ_{v_i} c_q$, $c_q \succ_{v_j} c_p$, and $c_p \succ_{v_r} c_q$, a contradiction with the assumption that E is single-crossing.

Similarly, in scenario (b) we have $c_p \succ_{v_1} c_q$ (by our assumption that v_1 ranks the candidates as $c_1 \succ \dots \succ c_m$), $c_q \succ_{v_i} c_p$, $c_p \succ_{v_j} c_q$. This means, in particular, that $i \neq 1$, and hence $1 < i < j$. Again, this is a contradiction with the assumption that E is single-crossing. \square

Lemma 5 suggests a dynamic programming algorithm for Chamberlin–Courant’s rule.

Theorem 6. *For every family α of polynomial-time computable dissatisfaction functions and for $\ell \in \{\ell_1, \ell_\infty\}$, there is an algorithm that given a single-crossing election E with n voters and m candidates and a positive integer k finds an optimal k -CC-assignment for E under α - ℓ -CC and runs in time $O(n^2mk)$.*

PROOF. Let $E = (C, V)$ be our input single-crossing election, where $C = \{c_1, \dots, c_m\}$, $V = (v_1, \dots, v_n)$ and v_1 has preference order $c_1 \succ \dots \succ c_m$, and let k be the target parliament size.

For every $i \in \{0\} \cup [n]$, $j \in [m]$, and $t \in [k]$ we define $A[i, j, t]$ to be the optimal ℓ -aggregated dissatisfaction that can be achieved with a t -CC-assignment function when considering election (C, V_i) , where $V_i = (v_1, \dots, v_i)$ and the assignment function can pick candidates from the set $C_j = \{c_1, \dots, c_j\}$ only. Observe that for every $j \in [m]$ and $t \in [k]$ we have $A[0, j, t] = 0$. Also, for every $i \in [n]$ and $j \in [m]$ it holds that

$$A[i, j, 1] = \min_{1 \leq j' \leq j} \ell(\alpha(\text{pos}_{v_1}(c_{j'})), \dots, \alpha(\text{pos}_{v_i}(c_{j'}))).$$

Further, for every $i \in [n]$, $t \in [k]$, and $j \in [t]$ we can compute $A[i, j, t]$ by matching each voter in V_i to her top candidate in C_j .

Finally, we claim that for every $i \in [n]$, $t \in [k] \setminus \{1\}$, and $j \in [m] \setminus [t]$ the following recursive relation holds (in the equation below, we abuse notation and treat ℓ as the respective norm on real vectors, i.e., we assume that it maps a list of values to their sum (when $\ell = \ell_1$) or their maximum (when $\ell = \ell_\infty$)):

$$A[i, j, t] = \min \left\{ A[i, j-1, t], \min_{0 \leq i^* < i} \ell \left(A[i^*, j-1, t-1], \alpha(\text{pos}_{v_{i^*+1}}(c_j)), \dots, \alpha(\text{pos}_{v_i}(c_j)) \right) \right\}.$$

Indeed, if an optimal t -CC-assignment function for (C, V_i) that picks candidates from C_j does not use c_j at all, its ℓ -aggregated dissatisfaction is given by $A[i, j-1, t]$. Otherwise, let v_{i^*+1} be the first voter to be represented by c_j ; by Lemma 5, voters v_{i^*+2}, \dots, v_i are also represented by c_j . For the first i^* voters we can then use an optimal t -CC-assignment function for (C, V_{i^*}) that picks candidates from C_{j-1} ; the dissatisfaction of such assignment is stored in $A[i^*, j-1, t-1]$. Thus, to compute $A[i, j, t]$, we need to decide whether to use c_j at all, and, if yes, who should be the first voter represented by c_j . This is exactly what our dynamic program does.

We will now establish an upper bound on the running time of our algorithm. Note first that all terms of the form $A[0, j, t]$, $j \in [m]$, $t \in [k]$ can be computed in time $O(mk)$. Further, if $i \in [n]$ and $j \in [m-1]$, we can compute $A[i, j+1, 1]$ given $A[i, j, 1]$ in time $O(n)$, and therefore the overall time needed to compute terms of the form $A[i, j, 1]$, $i \in [n]$, $j \in [m]$, is $O(n^2m)$. Next, we observe that if $i \in [n]$, $t \in [k]$, and $j \in [t]$, we can compute $A[i, j, t]$ in time $O(n+m)$. Indeed, the trivial algorithm to match each voter in V_i to her favorite candidate in C_j takes time $O(nm)$. However, we can use the fact that V_i is single-crossing, and therefore if for some $i' < i$ voter $v_{i'}$'s favorite candidate in C_j appears in position q , then voter $v_{i'+1}$'s favorite candidate in C_j appears in position q or lower. Thus, all such terms can be computed in time $O(nk^2 \max\{n, m\})$. As we can assume that $k \leq n$ and $k \leq m$, we have $nk^2 \max\{n, m\} \leq n^2mk$. Finally, having computed

$$\ell\left(A[i^*, j-1, t-1], \alpha(\text{pos}_{v_{i^*+1}}(c_j)), \dots, \alpha(\text{pos}_{v_i}(c_j))\right),$$

for some i^* , $0 < i^* < i$, we can compute

$$\ell\left(A[i^*-1, j-1, t-1], \alpha(\text{pos}_{v_{i^*}}(c_j)), \dots, \alpha(\text{pos}_{v_i}(c_j))\right)$$

in time $O(1)$, and therefore each term $A[i, j, t]$, where $i \in [n]$, $t \in [k] \setminus \{1\}$, and $j \in [m] \setminus [t]$, can be computed in time $O(n)$. Thus, the overall time needed to compute such terms is $O(n^2mk)$. It follows that, using this dynamic program, we can compute in time $O(n^2mk)$ the optimal dissatisfaction of the voters and a parliament that achieves it. \square

Note that, if the ordering of the votes witnessing that E is single-crossing is not given, the bound on the running time of our algorithm changes to $O(n^2m^2)$: we can compute the desired ordering of the votes in time $O(nm^2)$ [5], and we have $k \leq m$.

3.2. Extension to Preferences with Bounded Single-Crossing Width

Following the ideas of Cornaz, Galand, and Spanjaard [7, 8], we can extend our algorithm for Chamberlin–Courant’s rule to preferences with so-called *bounded single-crossing width*.

Definition 7 (Tideman [26]). We say that a set D , $D \subseteq C$, is a *clone set* in an election $E = (C, V)$ if each voter in V ranks the candidates from D consecutively (but not necessarily in the same order).

Definition 8 (Cornaz, Galand, and Spanjaard [7, 8]). We say that an election $E = (C, V)$ has *single-crossing width* (respectively, *single-peaked width*) at most w if there exists a partition of C into sets D_1, \dots, D_s such that (a) for each $i \in [s]$ the set D_i is a clone set in E and $|D_i| \leq w$, and (b) if we contract each D_i in each vote to a single candidate d_i , then the resulting election over $\{d_1, \dots, d_s\}$ is single-crossing (respectively, single-peaked).

Elections with small single-crossing width may arise, e.g., in parliamentary elections where the candidates are divided into (small) parties and the voters have single-crossing preferences over the parties, but not necessarily over the candidates. Using the same techniques as Cornaz et al., we obtain the following result.

Proposition 9. *For every family α of polynomial-time computable dissatisfaction functions and for every $\ell \in \{\ell_1, \ell_\infty\}$, there is an algorithm that given an election $E = (C, V)$ with $C = \{c_1, \dots, c_m\}$, $V = (v_1, \dots, v_n)$ whose single-crossing width is bounded by w , a partition of C into clone sets that witnesses this width bound, and a positive integer k , finds an optimal k -CC-assignment for E under α - ℓ -CC, and runs in time $\text{poly}(m, n, k, 2^w)$.*

PROOF. Let $E = (C, V)$ be our input election, and let D_1, \dots, D_s be a partition of C witnessing that the single-crossing width of E is at most w ; assume that the order of the sets D_1, \dots, D_s is such that the preference order of the first voter in V is of the form $D_1 \succ D_2 \succ \dots \succ D_s$. We first observe that Lemma 5 generalizes easily to elections with a given partition into clone sets. Specifically, there exists an optimal k -CC-assignment Φ for E under α - ℓ -CC where for each clone set $D \in \{D_1, \dots, D_s\}$, if $\Phi^{-1}(D) \neq \emptyset$ (that is, if at least one candidate from D is assigned to some voter) then: (a) there are two integers, t and t' , $t \leq t'$, such that $\Phi^{-1}(D) = \{v_t, v_{t+1}, \dots, v_{t'}\}$, and (b) for every pair of voters v_i, v_j such that $i < j$, $\Phi(v_i) \in D_p$, and $\Phi(v_j) \in D_q$ it

holds that $p \leq q$. That is, the voters matched to the candidates from a given clone set form a contiguous block within the voter order.

Now it is easy to modify our dynamic programming algorithm for elections with bounded single-crossing width. We guess an integer $j \in [s]$, a subset D'_j of D_j (the candidates from D_j to join the parliament), and a voter $v_i \neq v_n$ such that voters v_{i+1}, \dots, v_n are represented by the candidates from D'_j . Note that assigning the candidates from D'_j to these voters optimally is easy under Chamberlin–Courant’s rule: each voter gets her most preferred candidate from D'_j . We then use recursion to find an optimal parliament of size $k - |D'_j|$ for v_1, \dots, v_i that uses candidates from $D_1 \cup \dots \cup D_{j-1}$ only. To implement guessing, we try all possible choices of j , all possible subsets of D_j , and all possible choices of v_i , and we use dynamic programming to implement the recursive calls efficiently, just as in the perfectly single-crossing case. Since there are only $s2^w n$ possibilities to consider at each guessing step and $s \leq m$, we obtain the desired bound on the running time. \square

Naturally, for this result to be useful, we need an efficient algorithm that computes single-crossing width of an election and an appropriate division into clone sets. Fortunately, Cornaz et al. [8] provide an algorithm for this problem that runs in time $O(nm^3)$. (Interestingly, a very similar problem of finding a division into clones that results in a single-crossing election with as many candidates as possible is NP-hard [11]). As a consequence, we obtain the following fixed-parameter tractability result (see the books [19, 9] for an introduction to fixed-parameter complexity theory).

Corollary 10. *For every family α of polynomial-time computable dissatisfaction functions and for every $\ell \in \{\ell_1, \ell_\infty\}$, the problem of winner determination for α - ℓ -CC is fixed-parameter tractable with respect to the single-crossing width of the input election.*

4. Monroe’s Rule

The results of Betzler et al. [3] suggest that winner determination under Monroe’s rule tends to be harder than winner determination under Chamberlin–Courant’s rule. In this section, we show that this is also the case for single-crossing elections: we prove that for the utilitarian variant of Monroe’s rule with Borda dissatisfaction function (perhaps the most natural variant of Monroe’s rule) computing winners is NP-hard, even for single-crossing preferences. However, we then show that the egalitarian variant of

Monroe's rule admits an efficient winner determination algorithm if voters' preferences are both single-crossing and single-peaked.

4.1. Hardness for General Single-Crossing Elections

This section is devoted to proving that winner determination under Monroe's rule is NP-hard. The proof proceeds by reducing the problem of winner determination for unrestricted preferences to the case of single-crossing preferences.

Theorem 11. *Finding a set of winners under α_B - ℓ_1 -Monroe voting rule is NP-hard, even for single-crossing elections.*

The proof of this theorem is somewhat involved. We first provide the following two technical lemmas.

Lemma 12. *Consider an election $E = (C, V)$ with $C = \{c_1, \dots, c_m\}$, $V = (v_1, \dots, v_n)$. Let A and B be two sets of new candidates such that A , B , and C are pairwise disjoint, and $A = \{a_1, \dots, a_{mn}\}$, $B = \{b_1, \dots, b_{mn}\}$. For each $c \in C$, there is a single-crossing election $\text{Adj}_V(A, c, B)$ with candidate set $A \cup B \cup \{c\}$ and voter list $V' = (v'_1, \dots, v'_n)$ such that $\text{pos}_{v'_j}(c) = mn + \text{pos}_{v_j}(c)$ for each $j \in [n]$, and the election $(A \cup B \cup \{c\}, V')$, where $V'' = (v'_0, v'_1, \dots, v'_n, v'_{n+1})$, v'_0 has preference order $a_1 \succ \dots \succ a_{mn} \succ c \succ b_1 \succ \dots \succ b_{mn}$ and v'_{n+1} has preference order $b_1 \succ \dots \succ b_{mn} \succ c \succ a_1 \succ \dots \succ a_{mn}$, is also single-crossing.*

PROOF. Fix a candidate $c \in C$. We build the election $\text{Adj}_V(A, c, B)$ as follows. We set v'_1 's preference order to be

$$\begin{aligned} a_1 \succ a_2 \succ \dots \succ a_{mn} \succ b_1 \succ b_2 \succ \dots \\ \succ b_{\text{pos}_{v_1}(c)-1} \succ c \succ b_{\text{pos}_{v_1}(c)} \succ \dots \succ b_{mn}. \end{aligned}$$

For $1 \leq j \leq n-1$, we build the preference order of voter v'_{j+1} by modifying the preference order of voter v'_j . Given that the preference order of v'_j is of the form

$$\begin{aligned} a_1 \succ a_2 \succ \dots \succ a_x \succ b_1 \succ b_2 \succ \dots \succ b_y \succ c \\ \succ b_{y+1} \succ \dots \succ b_{mn} \succ a_{x+1} \succ \dots \succ a_{mn}, \end{aligned}$$

we construct the preference order of v'_{j+1} either by moving some of the candidates from B to precede c or by moving some of the candidate from A

to appear after c . Specifically, we do the following. First, we compute $d = \text{pos}_{v_{j+1}}(c) - \text{pos}_{v_j}(c)$; note that $-m < d < m$. If $d \geq 0$ then we set the preference order of v'_{j+1} to be

$$\begin{aligned} a_1 \succ a_2 \succ \dots \succ a_x \succ b_1 \succ b_2 \succ \dots \succ b_{y+d} \succ c \\ \succ b_{y+d+1} \succ \dots \succ b_{mn} \succ a_{x+1} \succ \dots \succ a_{mn}. \end{aligned}$$

If $d < 0$ then we set the preference order of v'_{j+1} to be

$$\begin{aligned} a_1 \succ a_2 \succ \dots \succ a_{x+d} \succ b_1 \succ b_2 \succ \dots \succ b_y \succ c \\ \succ b_{y+1} \succ \dots \succ b_{mn} \succ a_{x+d+1} \succ \dots \succ a_{mn}. \end{aligned}$$

Clearly, to construct each vote it suffices to shift forward or backward a block of at most m candidates; since $|A| = |B| = mn$, doing so is always possible. Finally, observe that we never change the relative order of the candidates within A and within B , and that the resulting election is single-crossing, even if we prepend v'_0 and append v'_{n+1} to it. \square

Lemma 13. *For every pair of positive integers k, n such that k divides n , every set of candidates $C = \{c_1, \dots, c_m\}$, and every order \succ_C over C there is a single-crossing election $(C, R(\succ_C))$ with $(\frac{n}{k} + 1)m$ voters such that each candidate $c_i \in C$ is ranked first by exactly $\frac{n}{k} + 1$ voters, the first voter's preference order coincides with \succ_C , and the last voter's preference order is obtained by reversing \succ_C .*

PROOF. Suppose that \succ_C is given by $c_{j_1} \succ c_{j_2} \succ \dots \succ c_{j_m}$. We build a preference profile $R(\succ_C) = V_1 + \dots + V_m$, where each V_i , $i \in [m]$, contains $\frac{n}{k} + 1$ voters. The preference order of each voter in V_i is $c_{j_i} \succ c_{j_{i+1}} \succ \dots \succ c_{j_m} \succ c_{j_{i-1}} \succ \dots \succ c_{j_2} \succ c_{j_1}$. It is clear that the resulting election is single-crossing. Also, the preference order of the last voter in $R(\succ_C)$ is the reverse of the preference order of the first voter. \square

With these lemmas and notation available, we are ready to prove Theorem 11.

PROOF OF THEOREM 11. Let $I = (C, V, k)$ be an instance of the problem of finding k winners under $\alpha_B\text{-}\ell_1$ -Monroe rule, where C , $|C| = m$, is the set of candidates, V , $|V| = n$, is the list of voters, and the goal is to elect a

$$\begin{aligned}
V_1 &: H \ E_1^\leftarrow \ \dots \ E_m^\leftarrow \ E^\leftarrow \ F_m^\leftarrow \ \dots \ F_1^\leftarrow \ c_1 \ \dots \ c_m \ D_1 \ \dots \ D_m \ G_1 \ \dots \ G_m \ G \\
V_2 &: H \ R(E_1^\leftarrow \ \dots \ E_m^\leftarrow \ E^\leftarrow \ F_m^\leftarrow \ \dots \ F_1^\leftarrow) \ c_1 \ \dots \ c_m \ D_1 \ \dots \ D_m \ G_1 \ \dots \ G_m \ G \\
v_3^1 &: H \ F_1 \ \dots \ F_m \ E \ E_m \ \dots \ E_2 \ c_1 \ E_1 \ c_2 \ \dots \ c_m \ D_1 \ \dots \ D_m \ G_1 \ \dots \ G_m \ G \\
v_3^2 &: H \ F_1 \ \dots \ F_m \ E \ D_1 \ E_m \ \dots \ E_3 \ c_2 \ E_2 \ c_3 \ \dots \ c_m \ c_1 \ E_1 \ D_2 \ \dots \ D_m \ G_1 \ \dots \ G_m \ G \\
&\vdots \\
v_3^m &: H \ F_1 \ \dots \ F_m \ E \ D_1 \ \dots \ D_{m-1} \ c_m \ E_m \ c_{m-1} \ \dots \ c_1 \ E_{m-1} \ \dots \ E_1 \ D_m \ G_1 \ \dots \ G_m \ G \\
V_4 &: H \ D_1 \ \dots \ D_m \ \text{Adj}_V(F_1, c_m, G_1) \ \dots \ \text{Adj}_V(F_m, c_1, G_m) \ E \ E_m \ \dots \ E_1 \ G \\
V_5 &: H \ R(D_1 \ \dots \ D_m \ G_1 \ \dots \ G_m \ G) \ c_m \ \dots \ c_1 \ F_1 \ \dots \ F_m \ E \ E_m \ \dots \ E_1
\end{aligned}$$

Table 1: The preference profile used in the proof of Theorem 11. For each voter list V_i , $1 \leq i \leq 5$, and for each voter v in V_i we list the (sets of) candidates in the order of v 's preference (we omit the “ \succ ” symbol for compactness). Whenever we list a set of candidates X as a part of an order, we assume that the candidates in X appear in some fixed order; for candidates in H we fix this order to be $h_1 \succ \dots \succ h_{m-k}$. We write X^\leftarrow to indicate that the candidates in X appear in the order obtained by reversing the standard order over X : a precedes b in X^\leftarrow if and only if b precedes a in X . Further, when in a line describing a preference order of an entire collection of voters $V_r = (v_1, \dots, v_s)$ (specifically, for $r \in \{2, 4, 5\}$, and $s = |V_r|$) we include a profile $V' = (v'_1, \dots, v'_s)$ (which can be an R-profile or an Adj-profile), then we mean that for each voter v_i , $i \in [s]$, in V_r , this part of this voter's preference order is the preference order of the i -th voter in V' .

parliament of size k . We assume that $n > k$ and n is divisible by k ; computing α_B - ℓ_1 -Monroe winners is known to be NP-hard under these assumptions [3, 25]. We will show how to construct in polynomial time an instance I_{sc} of the problem of winner determination under α_B - ℓ_1 -Monroe so that the election in I_{sc} is single-crossing and it is easy to extract the set of winners for I from the set of winners for I_{sc} .

We construct I_{sc} in the following way. First, we define the candidate set C_{sc} to be the union of the following disjoint sets (we provide names of the candidates only where relevant, and abbreviate $\sum_{i=1}^m$ to \sum_i):

1. $H = \{h_1, \dots, h_{m-k}\}$;
2. F_1, \dots, F_m , where $|F_i| = mn$ for each $i \in [m]$;
3. E_1, \dots, E_m , where $|E_i| = 2m^2n + m + (m - i)(2mn + 1)\frac{n}{k}$ for each $i \in [m]$;
4. E , where $|E| = m^2n + m$;
5. D_1, \dots, D_m , where $|D_i| = |E_i|$ for each $i \in [m]$;
6. G_1, \dots, G_m , where $|G_i| = |F_i| = mn$ for each $i \in [m]$;
7. G , where $|G| = (\sum_i |F_i| + |E|)$;

8. $C' = C = \{c_1, \dots, c_m\}$.

The ordered list V_{sc} of voters consists of the following five sublists (we only give names to voters that will be referred to directly later on; whenever sufficient, we only indicate the number of voters in a given list):

1. $V_1, |V_1| = |H| \frac{n}{k} = (m - k) \frac{n}{k}$;
2. $V_2, |V_2| = (\sum_i |F_i| + \sum_i |E_i| + |E|)(\frac{n}{k} + 1)$;
3. $V_3 = (v_3^1, \dots, v_3^m), |V_3| = m$;
4. $V_4, |V_4| = n$;
5. $V_5, |V_5| = (\sum_i |D_i| + \sum_i |G_i| + |G|)(\frac{n}{k} + 1)$.

We give the preferences of the voters in Table 1. It is straightforward to verify that the resulting election is single-crossing. In particular, note that, by requiring the extended election $(A \cup B \cup \{c\}, V'')$ to be single-crossing, Lemma 12 ensures that all voters in V_3, V_4 , and V_5 rank the candidates within each of the sets $F_i, G_i, i \in [m]$, in the same way. Note also that by Lemma 13 the last voter in V_2 ranks the candidates in $F_1 \cup \dots \cup F_m \cup E \cup E_m \cup \dots \cup E_1$ as $F_1 \succ \dots \succ F_m \succ E \succ E_m \succ \dots \succ E_1$.

Our goal is to find a parliament of size $k_{sc} = |C_{sc}| - (m - k)$. Consequently, each selected candidate should be assigned to $\frac{n}{k} + 1$ voters.

We claim that each optimal solution for I_{sc} satisfies the following conditions.

- (i) Each candidate $c \in F_1 \cup \dots \cup F_m \cup E \cup E_m \cup \dots \cup E_1$ is a winner and is assigned to those voters from $V_1 + V_2$ that rank c in position $|H| + 1 = m - k + 1$ (note that only one of these candidates is ranked in position $m - k + 1$ by voters in V_1).
- (ii) Each candidate $c \in D_1 \cup \dots \cup D_m \cup G_1 \cup \dots \cup G_m \cup G$ is a winner and is assigned to those voters from V_5 that rank c in position $|H| + 1 = m - k + 1$.
- (iii) Each candidate $h_i \in H$ is a winner and is assigned to $\frac{n}{k} + 1$ voters from $V_1 + V_2 + V_3$, with exactly $|H|$ voters from V_3 having some candidate from H assigned to them; each such voter ranks h_i in position i .
- (iv) Exactly k candidates from C' are winners. Each of them is assigned to $\frac{n}{k}$ voters in V_4 and to one voter in V_3 that ranks him highest.
- (v) The k winners from C' (let us call them w_1, \dots, w_k) are also α_B - ℓ_1 -Monroe winners in I ; moreover, there is an optimal solution for the original instance I where w_i is assigned to the ℓ -th voter in V if and only if w_i is assigned to the ℓ -th voter in V_4 in our solution for I_{sc} .

We will now prove that every optimal assignment Φ satisfies conditions (i)–(v). First, we make the following observations:

- (a) By a simple counting argument, $\Phi(V_{sc})$ contains at least k candidates from C' .
- (b) For each candidate h_i in H , if $h_i \in \Phi(V_{sc})$ then h_i is ranked in the i -th position in the preference order of the voters in $\Phi^{-1}(h_i)$. This is because candidates from H are always ranked first, in the order $h_1 \succ \dots \succ h_{m-k}$.
- (c) For each candidate $c \in C_{sc} \setminus (C' \cup H)$, if $c \in \Phi(V_{sc})$ then each voter in $\Phi^{-1}(c)$ ranks c in position $m - k + 1$ or lower. This is because every voter's top $m - k$ positions are taken up by the candidates from H .
- (d) Each voter in $V_1 + V_2 + V_5$ places each candidate in C' in a position lower than

$$p_1 = |H| + \sum_{i=1}^m |E_i| + \sum_{i=1}^m |F_i| + |E| = |H| + \sum_{i=1}^m |E_i| + 2m^2n + m.$$

- (e) Each voter in V_4 places each candidate in C' in a position higher than

$$p_2 = |H| + \sum_{i=1}^m |D_i| + \sum_{i=1}^m |F_i| + \sum_{i=1}^m |G_i| + m = |H| + \sum_{i=1}^m |E_i| + 2m^2n + m,$$

but lower than $p_3 = |H| + \sum_{i=1}^m |E_i|$.

- (f) $p_1 = p_2$.
- (g) For each candidate $c_j \in C'$, there is exactly one voter in V_3 that places c_j in position

$$|H| + \sum_{i=1}^m |F_i| + \sum_{i=1}^m |E_i| + |E| + 1 - |E_j| = p_1 + 1 - |E_j| \leq p_3 + 1;$$

all other voters in V_3 place c_j in a position lower than

$$|H| + \sum_{i=1}^m |E_i| + \sum_{i=1}^m |F_i| + |E| = p_2.$$

Given a set of candidates $X \subseteq C_{sc}$ and an assignment function Φ , let $\text{ds}(\Phi, X)$ denote the total dissatisfaction of voters that are assigned to candidates in X under Φ ; if $X = \{x\}$ is a singleton, we omit the curly braces

and write $\text{ds}(\Phi, x)$ in place of $\text{ds}(\Phi, \{x\})$. Note that the total dissatisfaction under Φ is given by $\text{ds}(\Phi, \Phi(V_{sc}))$.

Let Φ' be an optimal assignment function among those that use exactly k candidates from C' . We claim that Φ' satisfies conditions (i)–(iv). Note first that Φ' uses all candidates from $C_{sc} \setminus C'$. Consider an assignment function Φ that satisfies conditions (i)–(iv) and uses the same k candidates from C' (it is easy to see that such an assignment function exists); denote this set of k candidates by \hat{C} . Observe that under Φ for each candidate $c \in C_{sc} \setminus (C' \cup H)$, it holds that the dissatisfaction of each voter v with $\Phi(v) = c$ is $m - k$, and for each candidate $h_i \in H$ it holds that the dissatisfaction of each voter v with $\Phi(v) = h_i$ is $i - 1$. This means that $\text{ds}(\Phi, C_{sc} \setminus C') \leq \text{ds}(\Phi', C_{sc} \setminus C')$ for every assignment Φ' that uses all candidates from $C_{sc} \setminus C'$; in particular, $\text{ds}(\Phi, C_{sc} \setminus C') \leq \text{ds}(\Phi', C_{sc} \setminus C')$.

Moreover, by observations (d), (e), (f) and (g) we have $\text{ds}(\Phi, c) \leq \text{ds}(\Phi', c)$ for every $c \in \hat{C}$ and every assignment function Φ' . On the other hand, if Φ' assigns voters from V_4 to candidates other than those in \hat{C} , there will be at least one candidate $c' \in \hat{C}$ that is assigned to two or more voters in $V_1 \cup V_2 \cup V_3 \cup V_5$, and therefore $\text{ds}(\Phi', c') > \text{ds}(\Phi, c')$. Consequently, we have

$$\begin{aligned} \text{ds}(\Phi', \Phi'(V_{sc})) &= \text{ds}(\Phi', C_{sc} \setminus C') + \text{ds}(\Phi', \hat{C} \setminus \{c'\}) + \text{ds}(\Phi', c') \\ &> \text{ds}(\Phi, C_{sc} \setminus C') + \text{ds}(\Phi, \hat{C} \setminus \{c'\}) + \text{ds}(\Phi, c') = \text{ds}(\Phi, \Phi(V_{sc})), \end{aligned}$$

a contradiction with the optimality of Φ' . Thus, Φ' assigns all voters in V_4 to candidates in \hat{C} . By a similar argument, Φ' assigns each candidate from \hat{C} to exactly one voter from V_3 —the one that ranks this candidate highest. Consequently, Φ' assigns the remaining voters to candidates in $C_{sc} \setminus C'$, and it is immediate that the optimal way of doing so results in an assignment that satisfies conditions (i)–(iv).

We have argued that Φ' satisfies conditions (i)–(iv). We will now use this fact to prove that it also satisfies condition (v). Consider a candidate $c_j \in \hat{C}$. Let $V_4^{c_j} = \{v \in V_4 \mid \Phi'(v) = c_j\}$; we have $|V_4^{c_j}| = \frac{n}{k}$. Let V^{c_j} be the list of voters in V that correspond to those in $V_4^{c_j}$ (again, $|V^{c_j}| = \frac{n}{k}$). Let $\delta(V_4^{c_j})$ denote the dissatisfaction of voters in $V_4^{c_j}$ under Φ' , and let $\delta(V^{c_j})$ denote the dissatisfaction the voters in V^{c_j} would have if they were assigned to c_j (in I).

The total dissatisfaction of the voters assigned to c_j under Φ' is

$$\begin{aligned}
\text{ds}(\Phi', c_j) &= p_1 - |E_j| + \delta(V_4^{c_j}) \\
&= p_1 - 2m^2n - m - (m - j)(2mn + 1)\frac{n}{k} \\
&\quad + \frac{n}{k} \left(|H| + \sum_{i=1}^m |D_i| + (m - j)(2mn + 1) + mn \right) + \delta(V^{c_j}) \\
&= \left(\frac{n}{k} + 1 \right) \left(|H| + \sum_{i=1}^m |E_i| \right) + \frac{n}{k}mn + \delta(V^{c_j}),
\end{aligned}$$

which shows that the dissatisfaction of the voters in I_{sc} that are assigned to c_j under Φ' differs from the dissatisfaction of the respective voters in I , had they been assigned to c_j , by a value that only depends on n , m , and k (but not on j). Thus condition (v) holds.

It remains to show that every optimal assignment function for I_{sc} uses exactly k candidates from C' . Let Φ_{sc} be an optimal assignment function for I_{sc} that satisfies conditions (i)–(v) (and thus uses exactly k candidates from C'); let \hat{C} be the set of k candidates from C' used by Φ_{sc} . Let Φ' be an assignment function for I_{sc} that uses more than k candidates from C' . We will show that the total dissatisfaction under Φ' is higher than under Φ_{sc} .

By the same reasoning as in the proof of property (v), we note that for each candidate $c_j \in C'$ we have $\text{ds}(\Phi', c_j) \geq \text{ds}(\Phi'', c_j)$, where Φ'' is an assignment that assigns c_j to the voter from V_3 that ranks c_j highest and to $\frac{n}{k}$ voters from V_4 that rank c_j highest. By observation (g), this voter from V_3 ranks c_j in a position $p_1 + 1 - |E_j|$ and each voter in V_4 ranks c_j in position no higher than $|H| + \sum_i |D_i| + (m - j)(2mn + 1) + 1$. Thus, we obtain

$$\begin{aligned}
\text{ds}(\Phi', c_j) &\geq p_1 - |E_j| + \frac{n}{k} \left(|H| + \sum_{i=1}^m |D_i| + (m - j)(2mn + 1) \right) \\
&= p_1 + \frac{n}{k} \left(|H| + \sum_{i=1}^m |D_i| \right) - 2m^2n - m.
\end{aligned}$$

Similarly, if c_j is selected under Φ_{sc} then we can use the fact that each voter in V_4 ranks c_j no lower than $|H| + \sum_i |D_i| + (m - j)(2mn + 1) + 2mn + 1$ to

upper bound $d(\Phi_{sc}, c_j)$:

$$\begin{aligned} d(\Phi_{sc}, c_j) &\leq p_1 - |E_j| + \frac{n}{k} \left(|H| + \sum_{i=1}^m |D_i| + (m-j)(2mn+1) + 2mn \right) \\ &= p_1 + \frac{n}{k} \left(|H| + \sum_{i=1}^m |D_i| + 2mn \right) - 2m^2n - m. \end{aligned}$$

Note that neither of the bounds depends on j and that the difference between the upper bound for $ds(\Phi_{sc}, c_j)$ and the lower bound for $ds(\Phi', c_j)$ is $2m\frac{n^2}{k}$. Thus for each size- k subset of candidates C'' that are assigned under Φ' we have $ds(\Phi, C'') \geq ds(\Phi_{sc}, \hat{C}) + 2mn^2$. However, under Φ' there are at least $\frac{n}{k}$ voters outside of V_4 that are assigned to candidates in C' . Each of these voters ranks the candidate assigned to her in a position lower than $p_2 > 2m^2n + m$. On the other hand, under Φ_{sc} each of these voters is assigned to some candidate that she ranks in position $m - k + 1$ or higher. Since $\frac{n}{k}(2m^2n + m - m + k - 1) > 2mn^2$, it holds that $ds(\Phi_{sc}, C') < ds(\Phi', C')$. On the other hand, it is easy to see that $ds(\Phi_{sc}, C_{sc} \setminus C') \leq ds(\Phi', C_{sc} \setminus C')$: this follows by contrasting properties (i)–(iii) and observations (b) and (c). Thus, the total dissatisfaction under Φ_{sc} is lower than the total dissatisfaction under Φ' .

We conclude that an optimal assignment function Φ_{sc} assigns voters to exactly k candidates from C' and that the dissatisfaction of the voters in I_{sc} under Φ_{sc} is equal to the optimal dissatisfaction of the voters in I plus an easily computable value that depends on m , n , and k only. This completes the proof. \square

Betzler et al. [3] have shown a similar hardness result for single-peaked elections, but their construction relies on using a much more general notion of voter dissatisfaction. In particular, their result does not apply to the case of Borda dissatisfaction. Indeed, the complexity of winner determination under α_B - ℓ_1 -Monroe for single-peaked elections is still an open question. As our result answers this question in the case of single-crossing elections, it is tempting to ask if our proof approach could be used for single-peaked elections. Unfortunately, this does not seem to be the case. The difficulty lies in jointly implementing voters $V_3 + V_4$ in a single-peaked election (and, in particular, in positioning the candidates c_1, \dots, c_m).

4.2. ℓ_∞ -Monroe for Single-Peaked Single-Crossing Elections

We have argued that the utilitarian variant of Monroe’s rule with Borda dissatisfaction function is computationally demanding even for single-crossing preferences. In general, Monroe’s rule appears to be challenging from an algorithmic perspective: while Betzler et al. [3] provide a polynomial-time algorithm for the egalitarian variant of this rule under single-peaked preferences as well as a few fixed-parameter tractability results, they also show that, in contrast with Chamberlin–Courant’s rule, the utilitarian variant of Monroe’s rule remains NP-hard under single-peaked preferences. Moreover, the running time of Betzler et al.’s algorithm for the egalitarian variant of Monroe’s rule under single-peaked preferences is $O(n^3 m^3 k^3)$, and the algorithm itself is rather complex. We will now show that if voters’ preferences are both single-peaked and single-crossing, we can obtain a much simpler and faster algorithm for the egalitarian variant of Monroe’s rule.

The reader may wonder if the single-peaked single-crossing domain is too restrictive to be of practical interest. We believe, however, that this is not the case. In particular, this domain contains all elections that are *1-dimensional Euclidean*, i.e., ones where voters and candidates can be mapped to points on the real line so that each voter prefers a candidate that is closer to her to one that is further away [15]. It also contains all single-crossing elections that are *narcissistic*, i.e., every candidate is ranked first by at least one voter (this is implicit in the work of Barberà and Moreno [1]; see [10] for a formal proof). Intuitively, narcissistic elections arise when candidates are allowed to vote for themselves. This notion was introduced by Bartholdi and Trick [2], and was used in the context of fully proportional representation by Cornaz et al. [7]. In fact, it has recently been shown that the domain of single-peaked single-crossing elections coincides with the domain of *pre-narcissistic single-crossing elections*, i.e., single-crossing elections that can be made narcissistic single-crossing by adding voters [10]. We will now use this characterization result to argue that single-peaked single-crossing elections have the contiguous block property with respect to the egalitarian variant of Monroe’s rule, which enables us to design an efficient algorithm for this rule.

We will need a few definitions and results from the work of Elkind et al. [10].

Definition 14. An election $E = (C, V)$ is said to be *narcissistic* if for every candidate $c \in C$ there exists a voter $v \in V$ who ranks c first. An election $E = (C, V)$ is said to be *pre-narcissistic single-crossing (pre-NSC)* if there

exists a narcissistic single-crossing (NSC) election $E' = (C, V')$ such that $V \subseteq V'$.

Theorem 15 (Elkind et al. [10]). *An election is single-peaked single-crossing if and only if it is pre-NSC. Moreover, there exists an algorithm that given an election $E = (C, V)$ decides whether it is pre-NSC, and, if so, constructs an NSC election $E' = (C, V')$ such that $V \subseteq V'$, in time $O(nm^2)$.*

The following proposition establishes that pre-NSC elections have the *single-peaked trajectories property*: for each candidate his “trajectory” in the voters’ preferences has a single peak, i.e., each candidate first rises in the rankings and then descends.

Proposition 16 (Elkind et al. [10]). *For every election $E = (C, V)$ with $V = (v_1, \dots, v_n)$ that is pre-NSC with respect to the voter order (v_1, \dots, v_n) and for every candidate $c \in C$ there exists a voter $v_\ell \in V$ such that for every pair of voters v_i, v_j satisfying $j < i \leq \ell$ or $\ell \leq i < j$ it holds that $\text{pos}_{v_j}(c) \geq \text{pos}_{v_i}(c)$.*

We are now ready to prove that under the egalitarian variant of Monroe’s rule pre-NSC elections admit an optimal assignment that has the contiguous blocks property.

Lemma 17. *Consider an election $E = (C, V)$ with $C = \{c_1, \dots, c_m\}$, $V = (v_1, \dots, v_n)$ that is pre-NSC with respect to the voter order (v_1, \dots, v_n) , and let $U = (u_1, \dots, u_N)$ be a narcissistic single-crossing preference profile that witnesses this. Assume that the preference order of voter u_1 is $c_1 \succ \dots \succ c_m$. Then for every $k \in [m]$ and every dissatisfaction function α for m candidates there is a k -Monroe-assignment Φ for E such that Φ is optimal with respect to α - ℓ_∞ -Monroe and for every candidate $c \in \Phi(V)$ there are two integers, t and t' , $t \leq t'$, such that $\Phi^{-1}(c) = \{v_t, v_{t+1}, \dots, v_{t'}\}$. Moreover, for every pair of voters v_i, v_j such that $i < j$, $\Phi(v_i) = c_p$, and $\Phi(v_j) = c_q$ it holds that $p \leq q$.*

PROOF. Fix a $k \in [m]$ and a dissatisfaction function α . Let Φ be some k -Monroe-assignment for E that is optimal with respect to α - ℓ_∞ -Monroe. We will show how to transform Φ so that it satisfies the conditions in the statement of the lemma.

Let t be the smallest value in $[m]$ such that under Φ each voter is assigned to a candidate that this voter ranks in position t or higher; we have $\ell_\infty(\Phi, \alpha) = \alpha(t)$. Let $s = \max\{q \mid c_q \in \Phi(V)\}$, $n_s = |\Phi^{-1}(c_s)|$. We claim that there is a k -Monroe-assignment Φ'' that is optimal with respect to α - ℓ_∞ -Monroe and assigns c_s to the voters v_{n-n_s+1}, \dots, v_n .

If Φ itself satisfies this requirement, we are done. Otherwise we transform it as follows. Let i be the smallest index such that $\Phi(v_i) = c_s$. There must be some voter v_j , $j > i$, such that $\Phi(v_j) \neq c_s$; otherwise, Φ would have satisfied our requirement. Let $c_r = \Phi(v_j)$; note that $r < s$. Let Φ' be the k -Monroe-assignment obtained from Φ by swapping the candidates assigned to v_i and v_j , i.e.,

$$\Phi'(v) = \begin{cases} \Phi(v) & \text{if } v \neq v_i, v_j \\ c_r & \text{if } v = v_i \\ c_s & \text{if } v = v_j. \end{cases}$$

Clearly, Φ' is a k -Monroe-assignment for E , and we will now show that $\ell_\infty(\Phi', \alpha) \leq \alpha(t)$. We consider the following two cases.

v_i prefers c_s to c_r . Since E is single-crossing and v_1 prefers c_r to c_s , it must be the case that v_j prefers c_s to c_r . Thus, assigning c_s to v_j does not increase the dissatisfaction induced by the assignment. Now, consider v_i . Since v_i prefers c_s to c_r , it must be the case that the vote u with $\text{top}(u) = c_r$ precedes v_i in U and therefore by Proposition 16 we have $\text{pos}_{v_i}(c_r) \leq \text{pos}_{v_j}(c_r) \leq t$. Thus, after we assign c_r to v_i , the maximum dissatisfaction remains at most $\alpha(t)$.

v_i prefers c_r to c_s . This means that assigning c_r to v_i does not increase the dissatisfaction induced by the assignment. Further, if v_j prefers c_s to c_r then certainly $\ell_\infty(\Phi', \alpha) \leq \ell_\infty(\Phi, \alpha)$. Thus, assume that v_j prefers c_r to c_s . Then the vote u with $\text{top}(u) = c_s$ appears after v_j in U . Hence, by Proposition 16 we have $\text{pos}_{v_j}(c_s) \leq \text{pos}_{v_i}(c_s) \leq t$, which means that after we assign c_s to v_j , the maximum dissatisfaction remains at most $\alpha(t)$.

This proves that Φ' is a k -Monroe-assignment whose maximum dissatisfaction is no worse than that of Φ . By repeating the same procedure sufficiently many times, we eventually obtain an assignment Φ'' where c_s is assigned to the last n_s voters (clearly, in each iteration we get closer to this goal). The remaining candidates are handled in the same manner. That

is, we find the highest-numbered candidate in $\{c_1, \dots, c_{s-1}\} \cap \Phi(V)$ (say, $c_{s'}$), set $n_{s'} = \Phi^{-1}(c_{s'})$, and transform Φ'' into a k -Monroe-assignment where $c_{s'}$ is assigned to voters $v_{n-n_s-n_{s'}+1}, \dots, v_{n-n_s}$, and c_s is assigned to voters v_{n-n_s+1}, \dots, v_n . We repeat this procedure until we obtain a k -Monroe-assignment that satisfies the statement of the lemma. \square

Based on Lemma 17, it is easy to construct a dynamic programming algorithm for ℓ_∞ -Monroe under pre-NSC preferences.

Theorem 18. *For every family α of polynomial-time computable dissatisfaction functions there is an algorithm that, given a pre-NSC election E with n voters and m candidates and a positive integer $k \leq m$, finds an optimal k -Monroe assignment for E under α - ℓ_∞ -Monroe and runs in time $O(m^2n)$.*

PROOF. Let $E = (C, V)$ be a pre-NSC election with $C = \{c_1, \dots, c_m\}$, $V = (v_1, \dots, v_n)$, let α be a dissatisfaction function for m candidates, and let k be the target parliament size. By Theorem 15, we can compute a narcissistic single-crossing profile $U = (u_1, \dots, u_N)$ witnessing that E is pre-NSC in time $O(m^2n)$; we then renumber the candidates so that the preference order of voter u_1 is $c_1 \succ \dots \succ c_m$.

For each $i = 0, \dots, n$, $j = 0, \dots, m$, $t = 0, \dots, k$, we define $A[i, j, t]$ to be the minimum dissatisfaction that can be achieved by assigning exactly t candidates from the set $C_j = \{c_1, \dots, c_j\}$ to the voters v_1, \dots, v_i in such a way that each candidate is assigned either to $\lceil \frac{n}{k} \rceil$ consecutive voters or to $\lfloor \frac{n}{k} \rfloor$ consecutive voters; we set $A[i, j, t] = \infty$ if such an assignment does not exist. In particular, for each $j = 0, \dots, m$ we have $A[0, j, 0] = 0$ and we can compute the value $A[i, j, t]$ in time $O(1)$ whenever at least one of the arguments is 0. We also adopt the convention that $A[i, j, t] = \infty$ whenever at least one of the arguments is negative. Further, for every pair i, i' with $0 \leq i \leq i' \leq n$ and every candidate c_j , we set

$$\text{pos}_{i,i'}(c_j) = \max\{\text{pos}_{v_t}(c_j) \mid i < t \leq i'\}.$$

Let $d^- = \lfloor \frac{n}{k} \rfloor$, $d^+ = \lceil \frac{n}{k} \rceil$. The following equality holds for each $i \in [n]$, $j \in [m]$, $t \in [k]$ (we take a minimum over an empty set to be ∞):

$$\begin{aligned} A[i, j, t] = \min \Big\{ & \max \left\{ \alpha(\text{pos}_{\max\{0, i-d^-\}, i}(c_j)), A[i-d^-, j-1, t-1] \right\}, \\ & \max \left\{ \alpha(\text{pos}_{\max\{0, i-d^+\}, i}(c_j)), A[i-d^+, j-1, t-1] \right\}, \\ & A[i, j-1, t] \Big\}. \end{aligned} \quad (1)$$

Indeed, the three lines of this recurrence correspond to, respectively, assigning c_j to the last d^- voters among v_1, \dots, v_i , assigning c_j to the last d^+ voters among v_1, \dots, v_i , and not assigning c_j to any voters.

Using equality (1) and standard dynamic programming techniques, it is easy to compute $A[n, m, k]$ and the k -Monroe-assignment that achieves this dissatisfaction. To speed up computation, we first compute the values $\text{pos}_{\max\{0, i-d^-\}, i}(c_j)$ and $\text{pos}_{\max\{0, i-d^+\}, i}(c_j)$ for all $i \in [n]$, $j \in [m]$ in time $O((n+m)m)$, using the single-peaked trajectories property. After that we can compute each term $A[i, j, t]$, $i \in [n]$, $j \in [m]$, $t \in [k]$, in time $O(1)$. If k divides n , then the time needed to compute $A[n, m, k]$ and the respective assignment can be improved to $O((n+m)m)$ because we can omit the parameter t in $A[i, j, t]$. Since $k \leq m$, the overall running time of our algorithm is dominated by the first step, i.e., computing u_1 and the corresponding ordering of the candidates. The optimality of the computed assignment follows from Lemma 17. \square

5. Contiguous Blocks Property: Counterexamples

We will now explore the limitations of the algorithmic approach that is based on the contiguous blocks property.

First, we provide an example of a single-crossing election such that none of the optimal assignments for this election under Monroe's rule possesses the contiguous blocks property; in fact, the approximation ratio of every algorithm that produces assignments with this property is $\Omega(m)$. This result holds both for the utilitarian and for the egalitarian variant of Monroe's rule.

Example 19. Let $C = \{c_1, c_2\} \cup A \cup B$ be a set of candidates, where $A = \{a_1, \dots, a_{2s}\}$, $B = \{b_1, \dots, b_{2s}\}$ for some $s > 0$. Let $V = V_1 + V_2 + V_3 + V_4$ be a list of voters, where each V_i contains r voters. Voters in each V_i have identical preference orders, defined as follows.

$$\begin{aligned} V_1 : c_1 &\succ b_1 \succ \dots \succ b_{2s} \succ c_2 \succ a_1 \succ \dots \succ a_{2s}, \\ V_2 : c_1 &\succ c_2 \succ b_{2s} \succ \dots \succ b_1 \succ a_1 \succ \dots \succ a_{2s}, \\ V_3 : c_1 &\succ c_2 \succ a_1 \succ \dots \succ a_{2s} \succ b_{2s} \succ \dots \succ b_1, \\ V_4 : c_1 &\succ a_{2s} \succ \dots \succ a_1 \succ c_2 \succ b_{2s} \succ \dots \succ b_1. \end{aligned}$$

We seek a 2-member parliament, and use Borda misrepresentation function α_B .

If we partition voters into contiguous blocks, then the best we can do is to assign c_1 to $V_1 + V_2$ and a_s to $V_3 + V_4$ (or, alternatively, assign b_{s+1} to $V_1 + V_2$ and c_1 to $V_3 + V_4$), achieving dissatisfaction of $2r(s + 1)$ and $s + 1$ under $\alpha_B\text{-}\ell_1\text{-Monroe}$ and $\alpha_B\text{-}\ell_\infty\text{-Monroe}$, respectively. In contrast, if we assign c_1 to $V_1 + V_4$ and c_2 to $V_2 + V_3$, then the total dissatisfaction under $\alpha_B\text{-}\ell_1\text{-Monroe}$ and $\alpha_B\text{-}\ell_\infty\text{-Monroe}$ is $2r$ and 1 , respectively.

Further, even for narcissistic single-crossing elections, if we consider the utilitarian variant of Monroe's rule, imposing the contiguous blocks property may lead to suboptimal solutions.

Example 20. We consider the set $C = \{a, b, c, d, e, f\}$ of candidates and the following twelve voters (the profile is clearly narcissistic and it is easy to check that it is single-crossing):

$$\begin{aligned}
v_1 &: a \succ b \succ c \succ d \succ e \succ f, \\
v_2 &: b \succ a \succ c \succ d \succ e \succ f, \\
v_3 &: b \succ c \succ a \succ d \succ e \succ f, \\
v_4 &: c \succ b \succ a \succ d \succ e \succ f, \\
v_5 &: c \succ b \succ a \succ d \succ e \succ f, \\
v_6 &: c \succ b \succ a \succ d \succ e \succ f, \\
v_7 &: c \succ b \succ d \succ a \succ e \succ f, \\
v_8 &: c \succ d \succ b \succ a \succ e \succ f, \\
v_9 &: d \succ e \succ f \succ c \succ b \succ a, \\
v_{10} &: e \succ f \succ d \succ c \succ b \succ a, \\
v_{11} &: e \succ f \succ d \succ c \succ b \succ a, \\
v_{12} &: f \succ e \succ d \succ c \succ b \succ a.
\end{aligned}$$

We seek a 2-member parliament, and our voting rule is $\alpha_B\text{-}\ell_1\text{-Monroe}$. If we require the assignment function to have the contiguous blocks property, then the unique optimal solution assigns b to each voter in $V_1 = (v_1, \dots, v_6)$ and d to each voter in $V_2 = (v_7, \dots, v_{12})$. Under this assignment the total misrepresentation of the voters in V_1 and V_2 is given by 4 and 9, respectively, so the optimal total misrepresentation for assignments with the contiguous blocks property is 13.

On the other hand, without the contiguous blocks property, we can assign candidate e to voters $v_1, v_2, v_9, v_{10}, v_{11}, v_{12}$, and candidate c to voters

$v_3, v_4, v_5, v_6, v_7, v_8$, for the total misrepresentation of 11. Indeed, we will now argue that this is the unique optimal solution. This will establish that for this election the unique optimal unconstrained solution is disjoint from the unique optimal solution among those that satisfy the contiguous block property.

Let $\alpha = \alpha_B$. For each candidate x , we define $\text{lowest}(x)$ to be the following value:

$$\text{lowest}(x) = \min \left\{ \sum_{i \in I} \alpha(\text{pos}_{v_i}(x)) \mid I \subseteq [12], |I| = 6 \right\}.$$

Intuitively, $\text{lowest}(x)$ corresponds to assigning x in the best possible way. We have

$$\begin{array}{lll} \text{lowest}(a) = 9, & \text{lowest}(b) = 4, & \text{lowest}(c) = 1 \\ \text{lowest}(d) = 9, & \text{lowest}(e) = 10, & \text{lowest}(f) = 14. \end{array}$$

Thus each assignment function that gives total misrepresentation of at most 11 produces one of the following sets of winners: $\{a, c\}$, $\{b, c\}$, $\{c, d\}$, or $\{c, e\}$.

Now, if the set of winners is $\{a, c\}$ or $\{b, c\}$, the total misrepresentation is at least 12, because for each voter in $(v_9, v_{10}, v_{11}, v_{12})$ assigning any candidate from either of these sets results in a misrepresentation of at least 3.

Consider the candidates c and d . If each voter were assigned to her more preferred candidate among these two, we would get a 2-CC assignment function with misrepresentation 11. However, under this assignment 8 voters are assigned to c , so the best Monroe assignment for these two candidates has misrepresentation at least 12. Finally, it is clear that no assignment function that uses c and e can have a lower misrepresentation than 11 and we have seen that such a 2-Monroe assignment exists.

We now consider single-peaked preferences. Note that the definition of a single-peaked election does not impose any restrictions on the voter ordering, and therefore it is not immediately clear what is the correct way to extend the contiguous blocks property to this setting. However, it seems natural to order the voters according to their most preferred candidates (using the candidate order given by the axis), breaking ties according to their second most preferred candidate, etc. That is, consider an election $E = (C, V)$ that is single-peaked with respect to the order $c_1 \triangleleft \cdots \triangleleft c_m$. A voter with a preference order $c_{i_1} \succ \cdots \succ c_{i_m}$ can be identified with the string $i_1 \dots i_m$.

We reorder the voters so that if $i < j$ then the string associated with v_i is lexicographically smaller than or equal to the string associated with v_j .

We then ask if for every single-peaked election there exists an optimal assignment for Chamberlin–Courant’s rule or Monroe’s rule that satisfies the contiguous blocks property with respect to this ordering. We will now show that the answer is “no”, both for the egalitarian and the utilitarian variant of both rules. Indeed, just as in Example 19, imposing the contiguous blocks property has a cost of $\Omega(m)$.

Example 21. Let $C = \{x_1, \dots, x_m, y_1, \dots, y_m, a, b, c, d\}$ be a set of candidates and let $V = (v_1, v_2, v_3, v_4)$ be a list of voters, where

$$\begin{aligned} v_1 : a \succ x_1 \succ \dots \succ x_m \succ b \succ c \succ d \succ y_1 \succ \dots \succ y_m, \\ v_2 : b \succ c \succ d \succ y_1 \succ \dots \succ y_m \succ a \succ x_1 \succ \dots \succ x_m, \\ v_3 : c \succ b \succ a \succ x_1 \succ \dots \succ x_m \succ d \succ y_1 \succ \dots \succ y_m, \\ v_4 : d \succ y_1 \succ \dots \succ y_m \succ c \succ b \succ a \succ x_1 \succ \dots \succ x_m. \end{aligned}$$

It is easy to see that $E = (C, V)$ is single-peaked with respect to the axis

$$x_m \succ \dots \succ x_1 \succ a \succ b \succ c \succ d \succ y_1 \succ \dots \succ y_m.$$

In fact, it can be shown that all axes witnessing that E is single-peaked have the property that a, b, c , and d appear in the center of the axis, ordered as $a \triangleleft b \triangleleft c \triangleleft d$ or $d \triangleleft c \triangleleft b \triangleleft a$ (this is implied, e.g., by the analysis in [13]). Thus the ordering of the voters induced by the axis is (v_1, v_2, v_3, v_4) . We seek a 2-member parliament, and use the Borda misrepresentation function α_B .

Suppose that we assign a to v_1 and v_3 and d to v_2 and v_4 . Then the total dissatisfaction under α_B - ℓ_1 -Monroe and α_B - ℓ_∞ -Monroe is 4 and 2, respectively. However, if we impose the contiguous blocks property, then the optimal solution is to assign b to v_1 and v_2 , and c to v_3 and v_4 ; this results in total dissatisfaction of $2(m+1)$ and $m+1$ with respect to α_B - ℓ_1 -Monroe and α_B - ℓ_∞ -Monroe, respectively. For α_B - ℓ_1 -CC, we can obtain a somewhat better dissatisfaction than for α_B - ℓ_1 -Monroe, by assigning b to v_1, v_2 , and v_3 , and assigning d to v_4 . However, even for Chamberlin–Courant’s rule under every assignment that has the contiguous blocks property, at least one voter will be assigned to a candidate that she ranks in position $m+2$ or lower, so the total (egalitarian or utilitarian) dissatisfaction will be at least $m+1$.

The reader who is familiar with the algorithms of Betzler et al. [3] for Chamberlin–Courant’s rule and Monroe’s rule under single-peaked preferences may have observed that these algorithms are considerably more complex than the ones proposed in our work (for Chamberlin–Courant’s rule under single-crossing preferences and for the egalitarian variant of Monroe’s rule under single-peaked single-crossing preferences). Example 21 provides evidence that this difference in complexity is inherent: the ideas that lead to efficient algorithms for single-crossing electorates do not appear to be useful for single-peaked electorates that are not single-crossing. On the other hand, Examples 19 and 20 show that we cannot hope to extend Theorem 18 to all single-crossing elections, or to utilitarian preferences.

6. Conclusions

We have investigated the complexity of winner determination under Chamberlin–Courant’s and Monroe’s rules, for the case of single-crossing profiles. We have presented a polynomial-time algorithm for Chamberlin–Courant’s rule for single-crossing elections (and for elections that are close to being single-crossing in the sense of having bounded single-crossing width), and an NP-hardness proof for Monroe’s rule for the same setting. Our results further strengthen the intuition that Monroe’s rule is algorithmically harder than Chamberlin–Courant’s rule. Similar conclusions follow from the work of Betzler et al. [3] and Skowron et al. [25]

We complemented our negative result for Monroe’s rule by providing an efficient algorithm for the egalitarian variant of this rule under single-peaked single-crossing preferences. While this setting was known to admit a polynomial-time algorithm [3], our algorithm is considerably faster than the one described in earlier work. However, we showed that our approach does not extend to general single-crossing elections, or to the utilitarian variant of Monroe’s rule.

Perhaps the most obvious direction for future research that is suggested by our work is understanding the computational complexity of the utilitarian variant of Monroe’s rule for single-peaked single-crossing preferences and of egalitarian variant of Monroe’s rule for single-crossing preferences. While we have shown that approaches based on the contiguous blocks property are bound to fail, other approaches may be more successful. Going in another direction, perhaps it is possible to obtain efficient algorithms for our restricted domains when using dissatisfaction functions other than Borda.

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