L^2-invariants in abstract and profinite group theory

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A mis padres Ana María y Jacinto.
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Abstract

Given a group G, we denote its group von Neumann algebra by $\mathcal{N}(G)$. Briefly, the theory of L^2-invariants concerns the study of the homology groups $H_n(G; \mathcal{N}(G))$, providing a variety of interpretations and techniques in order to estimate their dimensions. An important dual aspect of the theory concerns the opposite problem. Given information on the groups $H_n(G; \mathcal{N}(G))$, what features of G do they encode?

This circle of ideas has proved to have deep implications in different branches of mathematics, including group theory and low-dimensional topology. In this exposition, building on the foundations of the theory of L^2-invariants, I will address several problems that may seem unrelated to the theory itself. To provide specific focus, we will present the story and the ideas behind the proofs of the following statements:

(i) Let G be a limit group and let U and V be two finitely generated subgroups of G. Then the *reduced Euler characteristic* $\overline{\chi}$ satisfies a submultiplicative property:

$$\overline{\chi}(U \cap V) \leqslant \overline{\chi}(U) \cdot \overline{\chi}(V).$$

Here $\overline{\chi}(H) = \max\{0, -\chi(H)\}$, where $\chi(H)$ is the usual Euler characteristic of a group H that admits a finite CW-complex as a $K(H, 1)$.

(ii) Let Σ be a closed surface and let $\pi_1(\Sigma)$ be its fundamental group. If G is a residually finite, one-relator group that has the same profinite completion as $\pi_1(\Sigma)$, then $G \cong \pi_1(\Sigma)$.

The first problem was conjectured when G is free by Hanna Neumann in the fifties, which was solved by Mineyev and Friedman in the 2010s. The second problem belongs to the area of *profinite rigidity* and presents partial progress towards the open conjecture that surface groups are determined by their profinite completion. This is a variation of a conjecture from the seventies, famously attributed to Remeslennikov, that free groups are profinitely rigid.

Statement of originality

The author declares that this thesis is based on his own original work, except where otherwise indicated.

Chapter 2 is based on standard material that is either folkloric or appropriately referenced. Regarding the main body of this thesis, Chapter 3 is based on my joint paper with Sam Fisher [FM25a], which was accepted for publication in *Compositio Mathematica*; and Chapter 4 is based on my joint paper with Andrei Jaikin–Zapirain [FM25a], which was accepted in *Selecta Mathematica, New Series.*

The three other research articles that I wrote during my DPhil studies are only mentioned during Chapter 1. The first one is [Mor24a], which was published in *Mathematische Annalen* in 2024; the second one is [Mor24b], which was published in *Communications in Algebra* in 2024; and the third one is a joint paper with Jonathan Fruchter [FM25b], which will appear in the *Israel Journal of Mathematics.*

Contents

Chapter 1

Introduction

Let $M \to \overline{M}$ be an infinite-degree covering of a compact Riemannian manifold $\overline{M}$. The L^2-invariants of this cover, as we know them today, were introduced by Atiyah [Ati76]. This paper appeared right after he had published with Singer his famous series of articles on the index theory of elliptic pseudo-differential operators of $\overline{M}$, starting in [AS68a]. The sections determined by the kernel of these operators correspond to solutions of elliptic differential equations; which, in the case of $\overline{M}$, forms a finite-dimensional real vector space. Furthermore, Atiyah and Singer had found topological expressions for their index [AS68b]. It was then the motivation of [Ati76] to develop a similar theory for the non-compact manifold M, where the space of solutions is typically infinite-dimensional. Briefly, the goal was to endow these spaces of a *real-valued dimension*, together with a topological interpretation of these numbers. To make this problem more amendable, some standard additional assumptions on the decay at infinity of these solutions had been considered previously, such as square-integrability. We denote the Deck transformation group of the covering $M \to \overline{M}$ by G. In this context, the main novelty of [Ati76] was noting that the aforementioned space of L^2-integrable solutions is a Hilbert space which is naturally acted on by the group von Neumann algebra $\mathcal{N}(G)$ of G. This permits to assign a finite real dimension to it, using the theory of normalised traces of projections in the ring $\mathcal{N}(G)$.

Consequently, these numbers were initially analytical invariants. It was Dodziuk [Dod77] who later addressed the topological side of the story, proving that they can be computed from triangulations of the manifold; an L^2-version of de Rham's theorem. In particular, these became invariants of the homotopy type of the manifold, suggesting that there could be a more general framework in which to develop the theory. This was eventually accomplished by Lück [Lüc98], who developed the foundations of L^2-invariants for covers of CW-complexes in an algebraic way. We will resume with this viewpoint in Section 1.3. Nevertheless, it is in Section 2.8 where we review the division-ring approach that we will follow in these notes. This more modern aspect of the theory emanated from Linnell's work [Lin93].

In all their different incarnations, L^2-invariants have proved to have powerful implications both in topology and geometric group theory. In this thesis, we will focus more specifically on how these homological invariants allow us to prove the Hanna Neumann conjecture on limit groups and strong versions of relative profinite rigidity for surface groups.

1.1 The Hanna Neumann conjecture

A group G has the *Howson property* if, for all finitely generated subgroups $U, V \leqslant G$, the intersection $U \cap V$ is finitely generated. The property is named after Albert G. Howson, who proved it for free groups in [How54]. Shortly thereafter, Hanna Neumann [Neu57] quantified this property by proving that

$$\mathrm{rk}(U \cap V) - 1 \leqslant 2(\mathrm{rk}(U) - 1)(\mathrm{rk}(V) - 1)$$

whenever U and V are finitely generated subgroups of a common free group, and she conjectured that the factor of 2 on the right-hand side of the inequality could be dropped. This became known as the *Hanna Neumann Conjecture*, and was the beginning of a fruitful line of research concerning these type of inequalities [Dic94, Tar96, Min11]. As we shall see, such inequalities are not limited to free groups.

Walter Neumann [Neu90] formulated a stronger version of H. Neumann's conjecture, described in Conjecture 1.1.2.

Notation 1.1.1. Given a group G that admits a finite CW-complex as a $K(G, 1)$, we denote by $\overline{\chi}(G) = \max\{-\chi(G), 0\}$ the *reduced Euler characteristic* of G, where $\chi(G)$ is the usual Euler characteristic.

Conjecture 1.1.2 (The Strengthened Hanna Neumann Conjecture (SHNC))**.** *Let U and V be finitely generated subgroups of a free group G. Let T be a complete set of representatives for the double (U, V)-cosets in G. Then*

$$\sum_{t \in T} \overline{\chi}(U \cap V^t) \leqslant \overline{\chi}(U)\overline{\chi}(V). \tag{1.1}$$

The statement of Conjecture 1.1.2 makes sense whenever G is a group such that all of its finitely generated subgroups are of finite type, since then $\overline{\chi}$ is defined for all such subgroups. We will discuss during Chapter 3 some of the ingredients, regarding L^2-homological considerations, that go into the proof of the following result.

Corollary C3. *The following groups satisfy the Strengthened Hanna Neumann conjecture (Conjecture 1.1.2):*

(a) Hyperbolic fundamental groups of graphs of free groups with cyclic edge groups.

(b) Limit groups (as introduced by Sela [Sel06]).

The proofs we know about the Hanna Neumann conjecture for free groups are fairly algebraic. Nevertheless, these do not seem to reprove the Howson property, which seems to be an essential preliminary ingredient of these algebraic considerations. Furthermore, the truly fundamental result in this sense, from which other properties on the geometry of subgroups can be established, is M. Hall's virtual retractions theorem [Hal49], which in particular has had a great influence in this thesis.

1.2 Separability and the geometry of subgroups

A classical theorem of M. Hall [Hal49] states that if H is a finitely generated subgroup of a free group F, then H appears as a retract (in fact, as a free factor) of some finite-index subgroup of F. Hall proved this theorem using combinatorial group theory, but the underlying philosophy of this result has been revisited and generalised due to its importance in algebra and topology; namely, such groups satisfy a strong form of *subgroup separability* (Definition 3.1.1). In this regard, Scott [Sco78, Sco85] geometrically extended Hall's theorem to surface groups and found the following topological implication of subgroup separability.

Proposition 1.2.1. *Let X and Y be connected, finite simplicial complexes such that $\pi_1 Y$ is subgroup separable. Let $f\colon X \to Y$ be a locally injective and π_1-injective simplicial map. Then f can be lifted to an embedding. More precisely, there exists a finite-sheeted covering $g\colon \widehat{Y} \to Y$ and an embedding $\widehat{f}\colon X \hookrightarrow \widehat{Y}$ such that $f = g \circ \widehat{f}$.*

This principle highly influenced the chain of works that culminated in Agol's solution of the virtual Haken conjecture [Ago13]. Roughly speaking, given a closed hyperbolic three-manifold M such that $\pi_1(M)$ is subgroup separable, Proposition 1.2.1 upgrades the surface subgroups of $\pi_1(M)$ obtained by Kahn and Marković [KM12] to essential, properly embedded surfaces in finite covers of M.

When it comes to quasiconvex subgroups of hyperbolic groups (in the sense of Gromov [Gro87]), the previous separability phenomenon has been established for groups coming from tessellations (e.g. all right-angled polytopes in a real-hyperbolic space

[Sco78, Sco85, ALR01]), for some arithmetic groups (e.g. Bianchi groups [ALR01]), for groups with nice Bass–Serre theoretic splittings (e.g. limit groups [Wil08]) and for many groups that act geometrically on CAT(0) cube complexes (e.g. special groups [HW08]).

Conversely, Proposition 1.2.1 can be turned into a characterisation of subgroup separability (which we recall in Proposition 3.3.5). For example, in a combinatorial setting, such as in Wilton's and Wise's work on cyclic group hierarchies [Wil08, Wis00], it is more practical to interpret this property as the ability to complete locally injective maps between compact CW-complexes to finite-sheeted coverings. We will give more details on this beautiful phrasing, which in the case of graphs goes back to Stallings [Sta83], during Section 2.1. In a joint article with Fisher [FM25a], which was mentioned above, the author revisits and modifies these explicit constructions of CW-complexes to ensure additional control on the L^2-homology of the involved spaces (see Theorem A3 & B3 below). Before giving more details in this direction, we need a few notions on the theory of L^2-invariants. We refer to Lück's book [Lüc02] for a very detailed account on this topic.

1.3 The theory of L^2-invariants

Given a group G, we denote its complex group algebra by $\mathbb{C}[G]$ and its completion with respect to the L^2-norm by $\ell^2(G)$, which is a Hilbert space. We define the *group von Neumann algebra* $\mathcal{N}(G)$ of G as follows. As a set, $\mathcal{N}(G)$ consists of the bounded operators $\ell^2(G) \to \ell^2(G)$ that commute with the right G-action. This description, together with the weak operator topology, turns $\mathcal{N}(G)$ into a Banach algebra, endowed with a structure of $(G, \mathcal{N}(G))$-bimodule. Given a subgroup H and an integer $k \geqslant 0$, one can appropriately define the k-th homology group $H_k(H; \mathcal{N}(G))$. As a right $\mathcal{N}(G)$-module, this group admits a *von Neumann dimension*, a real number that we call the *k-th L^2-Betti number* of H and that we denote by $b_k^{(2)}(H)$. This more classical approach to define L^2-Betti numbers is detailed in [Lüc02, Definition 8.30, Theorem 8.31], which will be enough for our purpose of introducing Definition 1.3.1 and Theorems A3 & B3. We will reformulate all this in more algebraic terms during Section 2.8.

Definition 1.3.1. A subgroup $H \leqslant G$ is *L^2-independent* if the kernel of the natural map $H_1(H; \mathcal{N}(G)) \to H_1(G; \mathcal{N}(G))$ has zero $\mathcal{N}(G)$-dimension. Moreover, we say that G is *L^2-Hall* if for every finitely generated subgroup $H \leqslant G$ there exists a finite-index subgroup $G_1 \leqslant G$ such that H is L^2-independent in G_1.

This property was introduced by Antolín–Jaikin-Zapirain [AJZ22], who proved that a free or surface group S is L^2-Hall as part of their proof of the *Hanna Neumann conjecture* for S. In terms of L^2-Betti numbers, this conjecture claims that, for any finitely generated $G_1, G_2 \leqslant S$, we have $b_1^{(2)}(G_1 \cap G_2) \leqslant b_1^{(2)}(G_1) \cdot b_1^{(2)}(G_2)$. Furthermore, it is proved in [AJZ22] that the previous inequality holds for many other hyperbolic cubulated groups provided that they are L^2-Hall. This was the initial motivation behind the next result, which condenses Theorems A3 and B3 from Chapter 3.

Theorem A3 & B3. *Suppose that G is a limit group or that it splits as a finite graph of free groups with cyclic edge groups that is hyperbolic relative to virtually abelian subgroups. Then G is L^2-Hall.*

The fact that limit groups are L^2-Hall, restated in Theorem B3, is proved during Section 3.3. Another application of L^2-homology relates to estimations in *homology growth*, which is very relevant to this thesis, is the pioneering approximation theorem of Lück [Lüc94]. This states that if G is a finitely presented group and $G = N_1 > N_2 > \cdots > N_m > \ldots$ is sequence of finite-index normal subgroups with trivial intersection $\bigcap_m N_m = 1$, then

$$\lim_{m\to\infty} \frac{\dim_{\mathbb{Q}} H_1(N_m; \mathbb{Q})}{|G : N_m|} = b_1^{(2)}(G).$$

In particular, the first L^2-Betti number of a finitely presented group can be computed in terms of the first Betti numbers of its finite-index subgroups. Consequently, Lück's theorem is an instance where L^2-invariants become *profinite invariants* in the following precise sense: if G and H are finitely presented, residually finite groups with the same profinite completion, then $b_1^{(2)}(G) = b_1^{(2)}(H)$. This connection with profinite rigidity, which we will expand on in the next section, portrays one of the reasons why the author became interested in L^2-homology.

1.4 Profinite rigidity

The modern, geometric approach of studying a group G via actions on spaces with fine combinatorial or metric properties is preceded by the idea of merely looking at actions on finite sets. These actions can already contain much information and sophisticated tools have been developed for their study (for instance, by means of character theory [Isa94]). The actions of G on finite sets are encoded in its finite quotients, and all of these can be merged (via a natural inverse limit construction) into its profinite completion $\widehat{G}$. Broadly, the theme of *profinite rigidity* asks to what extent G is determined by $\widehat{G}$. To make this recognition question sensible, we limit our attention exclusively to *residually finite* groups (sometimes, without explicit mention). We recall the main definition: a finitely generated group G is said to be *profinitely rigid* if, whenever H is a finitely generated group with $\widehat{G} \cong \widehat{H}$, it follows that $G \cong H$.

A foundational theorem in the area is due to Grothendieck [Gro70]. Let S be either a finitely generated free or a surface group. Suppose that a group homomorphism $\varphi\colon S \to H$, where H is finitely generated, induces an isomorphism on profinite completions $\widehat{\varphi}\colon \widehat{S} \to \widehat{H}$. Then φ itself is an isomorphism. Grothendieck in fact proved the analogous statement for any arithmetic group S and asked the following.

Question 1.4.1. *Given a homomorphism $\varphi\colon G \to H$ of finitely presented, residually finite groups that induces an isomorphism $\widehat{\varphi}\colon \widehat{G} \to \widehat{H}$, must φ be an isomorphism?*

Counterexamples to Question 1.4.1 were found in the celebrated work of Bridson–Grunewald [BG04]. Nevertheless, the search for positive answers to Question 1.4.1 has generated fruitful interactions between subgroup separability, representation rigidity and profinite rigidity.

One of the major realisations of these connections is the work of Bridson–McReynolds–Reid–Spitler [BMRS20, BMRS21], who proved that certain arithmetic lattices of $\mathrm{PSL}(2, \mathbb{R})$ and $\mathrm{PSL}(2, \mathbb{C})$ are profinitely rigid. An interesting aspect of some of these examples is that they are virtually surface groups, even though the following question remains open.

Question 1.4.2. *Are finitely generated free or surface groups profinitely rigid?*

The previous question on free groups is famously attributed to Remeslennikov (see, for example, [MK10, Question 5.48] and [NRR79, Question 12]). As suggested in the survey paper of Reid [Rei15], we include surface groups in Question 1.4.2. There are several reasons for this. The profinite rigidity of surface groups has also attracted much attention; it seems to present its own features while also proving to be very instructive to understand that of free groups. On the one hand, hyperbolic surface groups also sit as lattices of $\mathrm{PSL}(2, \mathbb{R})$ in many ways, with their representation varieties being then *too big* from the standpoint of the methods of [BMRS20], which were alluded to before.

On the other hand, surface groups admit an algebraic description that makes them resemble free groups; namely, they are amalgamated products of free groups along cyclic subgroups. Remarkably, Wilton [Wil21] proved that surface groups are profinitely rigid among groups admitting such splittings.

In my recent joint article with Jaikin-Zapirain [JZM25], the author proposes a new way to approach this problem combining various homological tools, including L^2-Betti numbers. The following result allows us to profinitely recognise surface groups among one-relator groups.

Theorem A4. *Let $\pi_1\Sigma$ be a surface group (that is, the fundamental group of a closed surface Σ). Let F be a finitely generated free group, let $w \in F$ and suppose that $G = F/\langle\langle w\rangle\rangle$ is a residually finite group such that $\widehat{G} \cong \widehat{\pi_1\Sigma}$. Then $G \cong \pi_1\Sigma$.*

In approaching Theorem A4, we discovered a connection between the profinite rigidity of surface groups and Mel'nikov's surface group conjecture (see Conjecture 4.1.2), which lead to our next theorem.

Theorem E4. *Let $n \geqslant 3$ and let F be the free group on $\{x_1, \dots, x_n\}$. Let $w \in F$ and suppose that $G = F/\langle\langle w\rangle\rangle$ is a residually finite group all of whose finite-index subgroups are one-relator groups. Then G is 2-free. Moreover, if $H_2(G;\mathbb{Z}) \neq 0$, then G is a surface group.*

We restate and prove Theorem A4 and Theorem E4 in Chapter 4.

1.4.1 Other results on residually free groups

During this and the following subsections we introduce some related results obtained by the author that will be discussed further in this thesis. Let G denote a finitely generated, residually finite group with the same profinite completion as a free or hyperbolic surface group. A surprising result of Jaikin-Zapirain [JZ23] shows that G is residually-(torsion-free nilpotent). We remark that, other than this and our next result from [Mor24a, Theorem A], not much more is known about G in this generality.

Theorem 1.4.3. *Let G be a group that satisfies the assumptions of the previous paragraph. Then the two-generated subgroups of G are free.*

Theorem 1.4.3 gives an alternative proof of Baumslag's theorem that two-generated subgroups of parafree groups are free [Bau69]. This allows us to ensure the hyperbolicity of the group in certain situations and combine results of Wilton [Wil08, Wil18, Wil21] to establish the following partial answer to Question 1.4.2.

Corollary 1.4.4. *Let G be a residually finite group that satisfies a finite abelian hierarchy (in the sense of Definition 3.4.7). If G has the same profinite completion as a free or surface group S, then it follows that $G \cong S$.*

Corollary 1.4.4 is proved in [Mor24a, Corollary 1.2]. The main ingredient of Corollary 1.4.4 is [Wil18], where Wilton proposed a novel approach to Question 1.4.2. His solution builds on partial progress on the fascinating question of Gromov on whether one-ended hyperbolic groups contain surface subgroups. In my joint paper with Fruchter [FM25b], we continue exploring this connection between surface subgroups and profinite rigidity to prove the following.

Theorem 1.4.5. *Given a group G and an integer $k \geqslant 0$, we denote by $vb_k(G)$ the virtual k-th Betti number of G; which is defined as the supremum of $b_n(H) = \dim_{\mathbb{Q}} H_n(H;\mathbb{Q})$ over finite-index subgroups $H \leqslant G$. Let G be either a finitely generated residually free group or a hyperbolic group that splits over finitely generated free groups with cyclic edge groups. Then G falls into exactly one of the following scenarios:*

1. *$vb_2(G) = 0$, and so G is free;*
2. *$vb_2(G) = 1$, and so $G \cong \pi_1(\Sigma)$ where Σ is a closed, connected surface;*
3. *$vb_2(G) = \binom{d}{2}$ for $d > 2$, and $G \cong \mathbb{Z}^d$; or*
4. *$vb_2(G) = \infty$ otherwise.*

Furthermore, if L is a limit group, then for any $n \geq 3$ we have that $vb_n(L) = \infty$ unless one of the following two holds:

1. *$\mathrm{cd}(L) < n$, in which case $vb_n(L) = 0$, or*
2. *L is free abelian of rank at least n, in which case $vb_n(L) = b_n(L) = \binom{\mathrm{rank}(L)}{n}$.*

One of the lessons we learn from the classification above is that, as appreciated by many authors, the M. Hall property on virtual retractions is not merely a way to access subgroup separability; it genuinely provides stronger conclusions. Note that Theorem 1.4.5 gives in particular a characterisation of free and surface groups by the finiteness of their second Betti number. In a slightly different fashion, it turns out that the virtual divergence between the first Betti number and the first L^2-Betti number is also a property that classifies these two exceptional classes of groups.

Theorem 1.4.6. *Let G be a non-abelian limit group that is not isomorphic to either a free or a surface group. Then for all $C > 0$ there exists a finite-index subgroup $H \leqslant G$ such that $b_1(H) - b_1^{(2)}(H) \geqslant C$.*

The proof of Theorem 1.4.6, as that of Theorem 1.4.5, does not only rely on the existence of retractions; it requires a more careful analysis of the virtual cyclic splittings of G. The key property to analyse Betti numbers from profinite completions is Serre's notion of goodness (defined in Section 2.7). With this additional ingredient, Theorem 1.4.5 and Theorem 1.4.6 lead to results on the profinite recognition of certain residually free groups. Goodness is exploited differently in [Mor24b] to study the determination of other structural properties of residually free groups; including the three properties of being a 3-manifold group [Mor24b, Corollary B], being a free-by-cyclic group [Mor24b, Theorem C] or being coherent [Mor24b, Theorem D].

1.4.2 The case of direct products

A result of Platonov and Tavgen [PT90] states that a direct product of two non-abelian free groups $\Gamma = F \times F'$ contains many finitely generated subgroups $H \leqslant \Gamma$ with $H \ncong \Gamma$ such that the inclusion map induces an isomorphism on profinite completions $\widehat{H} \to \widehat{\Gamma}$. In particular, such groups Γ are far from being profinitely rigid. On the converse direction, since finitely presented subgroups of Γ are separable [BW08], such H cannot be finitely presented. This suggested that the assumption of stronger finiteness properties may be enough to ensure some relative profinite rigidity of these groups. In this direction, Bridson conjectured [Bri22, Conjecture 7] that direct products of free groups are profinitely rigid among finitely presented, residually finite groups. Our next theorem solves this conjecture within the class of residually free groups.

Theorem 1.4.7. *Let $S_1, \dots, S_n$ be free or surface groups and let $S = S_1 \times \dots \times S_n$ be their direct product. Suppose that G is a finitely presented, residually free group such that $\widehat{G} \cong \widehat{S}$. Then $G \cong S$.*

Positive results on the profinite rigidity of direct products of surface groups have been subsequently obtained in the relative setting of Kähler groups by Hughes, Llosa-Isenrich, Py, Stover and Vidussi [HLIP$^+$25].

We conclude by commenting on the limitations of extending Theorem 1.4.7. In this regard, it should be noted that Bessa, Grunewald and Zalesskii gave examples of non-isomorphic groups G and H that are virtually direct products of free and surface groups, and hence finitely presented, such that $\widehat{G} \cong \widehat{H}$ [BGZ14, Corollary 3.10]. Bauer, Catanese and Grunewald had found similar examples using different methods in [BCG14, Section 7].

The latter examples on the lack of rigidity relate to another aspect of the theme of profinite rigidity which we have not commented on yet; the algebro-geometric formulation of Borel conjecture. More concretely, these negative results stem from the examples, due to Serre [Ser64], of smooth projective varieties that are Galois conjugate (and hence have the same algebraic group) but are not homeomorphic. For more context on this topic, and on its relation to profinite rigidity, we refer to the survey article of Bridson [Bri22].

1.5 Organisation

The purpose of Chapter 2 is to gather some basic aspects of the theory of group splittings, graph-of-spaces decompositions, homological algebra, profinite completions and L^2-Betti numbers. During Chapter 3, we discuss our proof of the L^2-Hall property for limit groups (stated above in Theorem A3 & B3); as well as some of its consequences (including the Hanna Neumann conjecture, Corollary C3). Finally, during Chapter 4, we explain the proofs of Theorem A4, on the profinite recognistion of surface groups; and Theorem E4, on Mel'nikov's surface group conjecture.

The statements and the proofs of our results suggest some natural questions and directions for further research, which we discuss at the end of Chapters 3 and 4 (more specifically, during Section 3.4 and Section 4.8).

Chapter 2

Preliminaries

2.1 Graphs of groups and spaces

The concept of a *graph of groups* was introduced as an algebraic and combinatorial object by Serre [Ser03], in order to reconstruct the structure of a group from an action on a simplicial tree. An elementary application of this theory, widely referred to as *Bass–Serre* theory, is the amalgamated product decomposition

$$\mathrm{SL}(2, \mathbb{Z}) \cong \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$$

that can be read off from a certain tiling of the hyperbolic plane $\mathbb{H}^2$.

Graphs of groups play a fundamental role in geometric group theory. They do not only allow to build a wealth of examples of groups, but they are also the language in which several classification problems are formulated. Two prominent examples of this, which we will comment on more during Section 3.3, are Sela's [Sel06] and Kharlampovich–Miasnikov's [KM98] descriptions of groups with the same first order theory as a free group.

In their influential essay, Scott and Wall [SW79] developed the theory of graph of groups topologically through the notion of a *graph of spaces*. These two viewpoints are essentially equivalent but allow for meaningfully different interpretations. Throughout these notes, we find it more convenient and natural to combine these two perspectives. For this reason, we shall now devote some time to define both graphs of groups and graphs of spaces, and their interaction.

Let Γ denote a directed graph, along with the vertex and edge sets of Γ, respectively denoted as $\mathsf{Vert}(\Gamma)$ and $\mathsf{Edge}(\Gamma)$. The graph Γ will be typically connected but it is allowed to have infinite size and valency, and some edges may be loops. For any edge $e \in \mathsf{Edge}(\Gamma)$, let $\mathsf{o}(e) \in \mathsf{Vert}(\Gamma)$ and $\mathsf{t}(e) \in \mathsf{Vert}(\Gamma)$ denote the origin and terminus of e. Hence, the functions o and t determine the direction of each edge.

Definition 2.1.1 (Graph of groups)**.** A *graph of groups* $\mathcal{G}$ consists of the following data:

1. A connected directed graph Γ, called the *underlying graph* of $\mathcal{G}$.
2. Groups G_v and G_e for every vertex $v \in \mathsf{Vert}(\Gamma)$ and edge $e \in \mathsf{Edge}(\Gamma)$.
3. Monomorphisms $\varphi_{e,\mathsf{o}} \colon G_e \to G_{\mathsf{o}(e)}$ and $\varphi_{e,\mathsf{t}} \colon G_e \to G_{\mathsf{t}(e)}$ for every edge $e \in \mathsf{Edge}(\Gamma)$.

The groups G_v and G_e are called the *vertex groups* and *edge groups* of $\mathcal{G}$. The monomorphisms $\varphi_{e,\mathsf{o}}$ and $\varphi_{e,\mathsf{t}}$ are called the *edge maps* of $\mathcal{G}$.

A graph of groups is *finite* if its underlying graph is finite. If a group G is isomorphic to the fundamental group of a graph of groups, we say that G *splits* as a graph of groups. In this situation, we will often abuse terminology and say that G *is* a graph of groups or we may refer to the graph of groups as a *splitting.* If the vertex and edge groups of a graph of groups $\mathcal{G}$ lie in classes $\mathcal{C}$ and $\mathcal{D}$, respectively, then we will say that $\mathcal{G}$ (or its fundamental group) is a graph of groups belonging to $\mathcal{C}$ with edge groups belonging to $\mathcal{D}$.

Given a graph of groups, there are at least two ways to look at its fundamental group: relative to a basepoint or relative to a spanning tree.

Definition 2.1.2 (Based fundamental group)**.** With the same notation as in Definition 2.1.1, let $v_0 \in \mathsf{Vert}(\Gamma)$ be a base vertex. For each $e \in \mathsf{Edge}(\Gamma)$, we introduce the formal symbol t_e. Let $P(\mathcal{G})$ be the group freely generated by the vertex groups G_v and the symbols t_e subject to the relations $t_e \varphi_{e,\mathsf{t}}(g) t_e^{-1} = \varphi_{e,\mathsf{o}}(g)$ for $e \in \mathsf{Edge}(\Gamma)$ and $g \in G_e$. The *fundamental group* of the graph of groups based at v_0, denoted $\pi_1(\mathcal{G}, v_0)$, is the subgroup of $P(\mathcal{G})$ consisting of the elements that can be represented as words $g_0 t_{e_1}^{\varepsilon_1} g_1 \cdots t_{e_n}^{\varepsilon_n} g_n$, where $g_i \in G_{\mathsf{t}(e_i)}$, where

$$\varepsilon_i = \pm 1, \qquad \begin{cases} g_i \in G_{\mathsf{t}(e_i)} & \text{if } \varepsilon_i = 1 \\ g_i \in G_{\mathsf{o}(e_i)} & \text{if } \varepsilon_i = -1 \end{cases}, \qquad g_0, g_n \in G_{v_0}$$

and where $(e_1, \ldots, e_n)$ forms a (not necessarily directed) loop.

An "unbased" way to look at the fundamental group is explained in Definition 2.1.3, which requires the choice of a spanning tree of the underlying graph (rather than a basepoint). This description is more flexible when considering splittings of $\mathcal{G}$ over simpler subgraphs of groups, as in Proposition 3.3.12.

Definition 2.1.3 (Fundamental group relative to a spanning tree)**.** With the notation of Definition 2.1.1, let T be a spanning tree of Γ. The *fundamental group* of $\mathcal{G}$ relative to T, denoted by $\pi_1(\mathcal{G}, T)$, is the group freely generated by the groups G_v for all $v \in \mathsf{Vert}(\Gamma)$, and the formal symbols t_e for all $e \in \mathsf{Edge}(\Gamma)$, subject to two types of relations: $t_e \varphi_{e,\mathsf{o}}(x) t_e^{-1} = \varphi_{e,\mathsf{t}}(x)$ for all $e \in \mathsf{Edge}(\Gamma)$ and $x \in G_e$; and $t_e = 1$ for all $e \in \mathsf{Edge}(T)$.

Definitions 2.1.1 and 2.1.3 coincide by [Ser03, Proposition 20, Chapter I]. An elementary argument in combinatorial group theory allows us to produce, from the presentation of $\pi_1(\mathcal{G})$ given above, a surjection $\pi_1(\mathcal{G}) \to \pi_1(\Gamma)$. In particular, these descriptions of $\pi_1(\mathcal{G})$ may be used to produce many quotients of the group. In any case, in this thesis we are more interested in studying subgroups of $\pi_1(\mathcal{G})$.

Note that if H is an arbitrary subgroup of $\pi_1(\mathcal{G})$, then H inherits a graph-of-groups structure, which comes from the action of H on the Bass–Serre tree associated to $\mathcal{G}$ (see [Ser03, Theorem 13, Chapter I]). Apart from coverings, we will see how in this context other natural kinds of immersions allow us to produce interesting subgroups.

Definition 2.1.4. Let $\mathcal{G} = (G_v, G_e; \Gamma)$ be a graph of groups. A graph of groups $\mathcal{H} = (H_v, H_e; \Upsilon)$ is a *subgraph of groups* of $\mathcal{G}$ if the following conditions hold:

1. There is an injection $\Upsilon \hookrightarrow \Gamma$, via which we think of Υ as a subgraph of Γ.
2. There are inclusions $f_v \colon H_v \hookrightarrow G_v$ and $f_e \colon H_e \hookrightarrow G_e$ for all vertices and edges of Υ, via which we think of every H_v (resp. H_e) as a subgroup of G_v (resp. G_e).
3. For every $e \in \mathsf{Edge}(\Upsilon)$, we have $H_e = H_{\mathsf{o}(e)} \cap G_e$ and $H_e = H_{\mathsf{t}(e)} \cap G_e$.
4. The diagrams

$$\begin{tikzcd} H_e \arrow[r, hook] \arrow[d, hook, "f_e"] & H_{\mathsf{o}(e)} \arrow[d, hook, "f_{\mathsf{o}(e)}"] \\ G_e \arrow[r, hook] & G_{\mathsf{o}(e)} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} H_e \arrow[r, hook] \arrow[d, hook, "f_e"] & H_{\mathsf{t}(e)} \arrow[d, hook, "f_{\mathsf{t}(e)}"] \\ G_e \arrow[r, hook] & G_{\mathsf{t}(e)} \end{tikzcd}$$

   commute for all $e \in \mathsf{Edge}(\Upsilon)$, where the horizontal maps are the edge maps of the respective graphs of groups.

Lemma 2.1.5 ([Bas93, Corollary 1.14])**.** *If $\mathcal{G}$ is a graph of groups and $\mathcal{H}$ is a subgraph of groups, then there is a canonical injective homomorphism $\pi_1(\mathcal{H}, v) \hookrightarrow \pi_1(\mathcal{G}, v)$ for any vertex v in the underlying graph of $\mathcal{H}$.*

Lemma 2.1.5 also explains why we imposed the injectivity of the edge maps in Definition 2.1.1, as it turns it into a more functional definition. A particular consequence of Lemma 2.1.5 is that the induced map from any vertex group G_v of $\mathcal{G}$ to $\pi_1(\mathcal{G})$ is injective.

We will often switch between the graph-of-groups and graph-of-spaces viewpoints, the latter of which we introduce now. The category of graphs of spaces provides a more convenient framework to name subgroups or conjugacy classes of these; without having to specify base points and avoiding the use of the analogous algebraic notion of *groupoid*.

Definition 2.1.6 (Graph of spaces)**.** A *graph of spaces* $\mathcal{X}$ consists of the following data:

1. A connected directed graph Γ, called the *underlying graph* of $\mathcal{X}$.
2. Based connected CW-complexes (X_v, x_v) and (X_e, x_e) for every vertex $v \in \mathsf{Vert}(\Gamma)$ and edge $e \in \mathsf{Edge}(\Gamma)$.
3. Based π_1-injective continuous maps $f_{e,\mathsf{o}} \colon X_e \to X_{\mathsf{o}(e)}$ and $f_{e,\mathsf{t}} \colon X_e \to X_{\mathsf{t}(e)}$ for every edge $e \in \mathsf{Edge}(\gamma)$.

The spaces X_v and X_e are called the *vertex spaces* and the *edge spaces* of $\mathcal{X}$. The maps $f_{e,\mathsf{o}}$ and $f_{e,\mathsf{t}}$ are called the edge maps. The *topological realisation* of $\mathcal{X}$ is the quotient of the space

$$X = \left(\bigsqcup_{v \in \mathsf{Vert}(\Gamma)} X_v \right) \sqcup \left(\bigsqcup_{e \in \mathsf{Edge}(\Gamma)} X_e \times [0,1] \right)$$

by the identifications $f_{e,\mathsf{o}}(x) \sim (x,0)$ and $f_{e,\mathsf{t}}(x) \sim (x,1)$ for all $x \in X_e$ and all $e \in \mathsf{Edge}(\Gamma)$. The *fundamental group* of the topological space $\mathcal{X}$ based at x_{v_0}, denoted $\pi_1(\mathcal{X}, x_{v_0})$, is defined to be $\pi_1(X, x_{v_0})$. When no confusion arises, we will usually refer to the topological realisation of $\mathcal{X}$ as a graph of spaces.

We can now make precise the correspondence between graphs of spaces and graphs of groups. Given a graph of spaces $\mathcal{X}$, we can form a graph of groups with vertex groups $\pi_1(X_v, x_v)$ and edge groups $\pi_1(X_e, x_e)$, together with the naturally induced edge maps $(f_{e,\mathsf{o}})_*$ and $(f_{e,\mathsf{t}})_*$. Conversely, to associate a graph of spaces to every graph of groups, we need the notion of an *Eilenberg–MacLane space*. Recall that a $K(G,1)$ *space* for a group G is a path-connected space X with $\pi_1(X) \cong G$ which has a contractible universal covering space. The space $K(G,1)$ was studied by Hurewicz and higher homotopical variants were introduced by Eilenberg and MacLane. Crucially, every group G admits a CW-complex $K(G,1)$, whose homotopy type depends only on G (see, for instance, [Hat02, Example 1B7 and Theorem 1B. 8]).

If $\mathcal{G}$ is a graph of groups, then a graph of spaces $\mathcal{X}$ can be constructed as follows: For each $v \in \mathsf{Vert}(\Gamma)$ and $e \in \mathsf{Edge}(\Gamma)$, let X_v (resp. X_e) be a $K(G_v,1)$ (resp. a $K(G_e,1)$); and let $f_{e,\mathsf{o}}$ and $f_{e,\mathsf{t}}$ be maps inducing $\varphi_{e,\mathsf{o}}$ and $\varphi_{e,\mathsf{t}}$, respectively. Then there is an isomorphism between $\pi_1(\mathcal{G}, v_0)$ and $\pi_1(\mathcal{X}, x_{v_0})$, which depends on the choices of v_0 and x_{v_0} up to conjugation, by the Seifert—van Kampen Theorem [Hat02, Theorem 1.20]. Even more, the injectivity of the maps $f_{e,\mathsf{o}}$ and $f_{e,\mathsf{t}}$ implies that the topological realisation X of $\mathcal{X}$ is a model of $K(\pi_1(\mathcal{X}, x_{v_0}),1)$. For a proof of this fact, we refer the reader to [Hat02, Theorem 1B.11].

Our main results of Chapter 3 concern graphs of groups where all the edge groups are either trivial or infinite cyclic; the main case of interest being where the vertex groups are free. Any such group will be referred to as a *graph of free groups with cyclic edge groups*. Such groups are realised as the fundamental group of a *graph of graphs with S^1 edge spaces*, by which we understand a graph of spaces where every vertex space is a graph and every edge space is a copy of the circle S^1. The reader may keep in mind these examples during the rest of the section.

Definition 2.1.7. Let $\mathcal{X} = (X_v, X_e; \Gamma)$ and $\mathcal{Y} = (Y_v, Y_e; \Upsilon)$ be graphs of spaces. A *map of graph of spaces* $\mathcal{Y} \to \mathcal{X}$ consists of the following data:

1. There is a simplicial map between the underlying graphs $f\colon \Upsilon \to \Gamma$, which commutes with the functions o and t.
2. There are π_1-injective based maps $f_v\colon (Y_v, y_v) \to (X_{f(v)}, x_{f(v)})$ and $f_e\colon (Y_e, y_e) \to (X_{f(e)}, x_{f(e)})$ for all $v \in \mathsf{Vert}(\Upsilon)$ and $e \in \mathsf{Edge}(\Upsilon)$.
3. The diagrams

$$\begin{tikzcd} Y_e \arrow[r] \arrow[d, "f_e"] & Y_{\mathsf{o}(e)} \arrow[d, "f_{\mathsf{o}(e)}"] \\ X_{f(e)} \arrow[r] & X_{\mathsf{o}(f(e))} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y_e \arrow[r] \arrow[d, "f_e"] & Y_{\mathsf{t}(e)} \arrow[d, "f_{\mathsf{t}(e)}"] \\ X_{f(e)} \arrow[r] & X_{\mathsf{t}(f(e))} \end{tikzcd}$$

commute for all $e \in \mathsf{Edge}(\Upsilon)$, where the horizontal maps are edge maps in the corresponding graphs of spaces.

For the sake of clarity, we note that in the diagrams of item 3 of Definition 2.1.7 we are implicitly using the fact (ensured by item 1) that $f(\mathsf{o}(e)) = \mathsf{o}(f(e))$ and $f(\mathsf{t}(e)) = \mathsf{t}(f(e))$ for all $e \in \mathsf{Edge}(\Upsilon)$.

Our main interest in this language is to study subgroups, so we will consider maps of graphs of spaces that naturally lead to π_1-injective maps between the topological realisations of the spaces. In this sense, we are only interested in two kinds of maps: subgraphs of spaces, which induce subgraphs of groups in the sense of Definition 2.1.4, and precoverings.

Definition 2.1.8. Let $\mathcal{X} = (X_v, X_e; \Gamma)$ be a graph of spaces. A graph of spaces $\mathcal{Y} = (Y_v, Y_e; \Upsilon)$ is a *subgraph of spaces* of $\mathcal{X}$ if there is a map of graph of spaces $\mathcal{Y} \to \mathcal{X}$ (as introduced in Definition 2.1.7) that satisfies the following additional properties:

1. The underlying map $\Upsilon \hookrightarrow \Gamma$ is injective, via which we think of Υ as a subgraph of Γ.
2. The π_1-injective maps $f_v \colon Y_v \to X_v$ and $f_e \colon Y_e \to X_e$ are injective for all edges and vertices of Υ; via which we think of every Y_v and Y_e as a subspace of X_v and X_e, respectively.
3. In analogy with the third item of Definition 2.1.4, we have the equalities

$$\pi_1(Y_e) = \pi_1(Y_{\mathsf{o}(e)}) \cap \pi_1(X_e) \quad \text{and} \quad \pi_1(Y_e) = \pi_1(Y_{\mathsf{t}(e)}) \cap \pi_1(X_e)$$

   for every edge $e \in \mathsf{Edge}(\Upsilon)$.

Given a subgraph of spaces $\mathcal{Y} \to \mathcal{X}$ and the induced map between their topological realisations $Y \to X$, it is clear (after choosing base-points) that $\pi_1(Y)$ can be seen as a subgraph of groups of $\pi_1(X)$. Hence, the induced map $\pi_1(Y) \to \pi_1(X)$ is injective by Lemma 2.1.5. The second class of π_1-injective maps we will consider are precoverings.

Definition 2.1.9. A map of graphs of spaces $\mathcal{X}' \to \mathcal{X}$ is a *precovering* if the following conditions hold:

1. The map between the underlying graphs $f \colon \Gamma' \to \Gamma$ is locally injective.
2. For every $v \in \mathsf{Vert}(\Gamma')$ and $e \in \mathsf{Edge}(\Gamma')$, the maps $X'_v \to X_{f(v)}$ and $X'_e \to X_{f(e)}$ are covering maps.

We refer to the domain X' (the realisation of $\mathcal{X}'$) as a *precover* of X. Moreover, we say that the precovering is *finite-sheeted* if every covering map $X'_v \to X_{f(v)}$ is finite-sheeted in the usual sense (that is, if preimages of points are finite).

The fact that covering maps are π_1-injective generalises to precoverings.

Lemma 2.1.10 ([Wil08, Proposition 2.19]). *A precovering $X' \to X$ induces an injection of fundamental groups $\pi_1(X') \to \pi_1(X)$.*

Observe that if X decomposes as a graph of spaces and $Y \to X$ is a covering space, then Y inherits a graph-of-spaces structure where every vertex (resp. edge) space of Y covers some vertex (resp. edge) space of X. The key concept when it comes to understand how far is a precovering from an actual covering is that of an *elevation,* which we define below. In this regard, we anticipate that a precovering $X' \to X$ is a covering if and only if all the elevations of edge maps of X to X' are edge maps of X'.

Definition 2.1.11. Given a based map $f\colon (X, x) \to (Y, y)$ and a based covering space $(\widehat{Y}, \widehat{y}) \to (Y, y)$, an *elevation* of f to $\widehat{Y}$ based at $\widetilde{y}$ is a based covering $(\widehat{X}, \widehat{x}) \to (X, x)$ and a map $\widehat{f}\colon (\widehat{X}, \widehat{x}) \to (\widehat{Y}, \widehat{y})$ such that:

(i) The following diagram commutes

$$\begin{tikzcd}
(\widehat{X}, \widehat{x}) \arrow[r, "\widehat{f}"] \arrow[d, "p_X"] & (\widehat{Y}, \widehat{y}) \arrow[d, "p_Y"] \\
(X, x) \arrow[r, "f"] & (Y, y).
\end{tikzcd} \tag{2.1}$$

(ii) As a subgroup of $\pi_1(X, x)$, the fundamental group $\pi_1(\widehat{X}, \widehat{x})$ is equal to the preimage of $\pi_1(\widehat{Y}, \widehat{y})$ under the induced map $f_*\colon \pi_1(X, x) \to \pi_1(Y, y)$.

Recall that given the spaces of (2.1) and the maps p_X, p_Y, and f; the map $\widehat{f}$ exists if and only if $\pi_1(\widehat{X}, \widehat{x}) \leqslant f_*^{-1}(\pi_1(\widehat{Y}, \widehat{y}))$. This is an elementary lifting criterion in covering space theory (see [Hat02, Proposition 1.33]). With this in mind, it is clear that there is a lift of f that factors through the universal cover $\widetilde{f}\colon \widetilde{X} \to \widehat{Y}$. In this sense, what condition (ii) of Definition 2.1.11 indicates is that an elevation of f to $\widehat{Y}$ is not any lift of f, but a minimal one.

In the context of Definition 2.1.11, as we vary the preimage $\widehat{y}$ of y under the covering map $\widehat{Y} \to Y$, we recover what we call the *unbased elevations* (or, simply, elevations) of f to $\widehat{Y}$. We now specify the sense in which two elevations are the same.

Definition 2.1.12. With the notation of Definition 2.1.11, we say that two elevations $\widehat{f}_1\colon (\widehat{X}_1, \widehat{x}_1) \to (\widehat{Y}, \widehat{y}_1)$ and $\widehat{f}_2\colon (\widehat{X}_2, \widehat{x}_2) \to (\widehat{Y}, \widehat{y}_2)$ are *equivalent* (or *isomorphic*) if there exists a based homeomorphism $h\colon (\widehat{X}_1, \widehat{x}_1) \to (\widehat{X}_2, \widehat{x}_2)$ covering the identity map on X such that $\widehat{f}_1 = \widehat{f}_2 \circ h$.

We stress that elevations can be used to measure the extend to which a precover fails to be a cover. In fact, a precovering map $\widehat{X} \to X$ of graphs of spaces is a covering map if and only if no elevation of an edge space of X is *hanging* in $\widehat{X}$, which we shall define now.

Remark 2.1.13. A precovering $Y \to X$ factors through a subgraph of spaces $Y \to \overline{Y}$, inducing an isomorphism of fundamental groups $\pi_1(Y) \to \pi_1(\overline{Y})$, and a covering $\overline{Y} \to X$. For this, we simply take $\overline{Y}$ to represent the subgroup $\pi_1(Y) \leqslant \pi_1(X)$, pulling back the graph-of-spaces structure of X. The cover $\overline{Y}$ is referred to as the *canonical completion* of the precover Y in [Wil07, Proposition 2.9]. Note that even if X and Y are compact, $\overline{Y}$ has an infinite underlying graph (and so it is non-compact) when the image of $\pi_1(Y)$ in $\pi_1(X)$ has infinite index. In particular, this notion of canonical completion of precoverings between graphs of spaces defers radically from the notion of *canonical completion of local isometries between special cube complexes*, introduced in [HW08, Section 6].

The canonical completion $\overline{Y}$ defined above allows us to define the elevations of a precover, a particular kind of which is highlighted in the next definition from [Wil07, Remark 2.8] and [Wil08, Remark 2.18].

Definition 2.1.14. Given a precovering map of graphs of spaces $Y \to X$, we consider an edge space X_e of X and an elevation $\overline{Y}_f$ (in the sense of Definition 2.1.11) to $\overline{Y}$ (with the notation of Remark 2.1.13). By construction, $\overline{Y}_f$ is an edge space of $\overline{Y}$. We say that $\overline{Y}_f$ is a *hanging elevation* of X_e to the precover Y if it is contained in Y but is not an edge space of Y.

We make some last comments on our conventions and on the definitions that we decided to present in this section. Our choice of merely looking at two kinds of maps of graphs of spaces, namely subgraphs of spaces and precoverings, is not arbitrary. To explain this, let us focus on a simplicial graph X and let $G = \pi_1(X)$. When viewing X as a graph of points, a precover Y of X is just an immersion $Y \to X$, which is always π_1-injective [Sta83], and every finitely generated subgroup $H \leqslant G$ is realised by an immersion. So precovers are just a graph-of-spaces version of graph immersions and they serve similar purposes. In fact, given a graph of spaces X, it is not hard to check that the precovers of X with finite underlying graph describe all the finitely generated subgroups of $\pi_1(X)$. We will give more details on these sorts of arguments during Chapter 3, but we anticipate the crucial fact that in our context such precovers can be described by a composition of subgraphs of spaces and finite-sheeted coverings. Recall that this is the case for graphs, since Stallings

[Sta83] proved that every immersion of finite graphs factors through an embedding and a finite-sheeted covering; a strong topological incarnation of the M. Hall property. Analogous statements were proved by Scott [Sco78, Sco85] for compact surfaces; by Gitik, Wise and Wilton [Git97, Wis00, Wil07] for certain graphs of spaces (where edge spaces are homeomorphic to points or circles) and by Haglund and Wise [HW08] for special cube complexes.

We conclude this section reflecting on how the notion of precovering relates more modern technology based on cube complexes. The articles discussed in the previous paragraph were already mentioned in Section 1.2 but here we can state a crucial aspect that distinguishes them; namely, that the arguments of Chapter 3 rely on the local structure of the corresponding graphs of groups, just as in [Git97, Wis00, Wil07, Wil08]. Even if the M. Hall property for quasiconvex subgroups is known to hold in the greater generality of special groups [HW08], the strengthening treated in Chapter 3 (that is, the L^2-Hall property) does not carry over into the class of special hyperbolic groups (as discussed in Section 3.2.1). So even if the groups of Chapter 3 are virtually special, which already provides important separability consequences, we cannot forget the structure of their cyclic splittings. In particular, the notion of precovering, tailored to our cyclic splittings, cannot be simply replaced by the more general notion of local isometries of cube complexes [HW08, Definition 2.9], since from this viewpoint we cannot analyse the induced maps in L^2-homology at the level of accuracy that is required in Section 3.3.

2.2 Homology and cohomology of groups

Unless stated otherwise, all modules are assumed to be left modules. However, in many natural situations the consideration of both left and right modules is required. Let R be a ring. Given a right R-module N and a left R-module M, we can define the abelian group $\operatorname{Tor}_i^R(N,M)$. Almost by definition, $\operatorname{Tor}_0^R(N,M) \cong N \otimes_R M$ as an abelian group. In general, the functors $\operatorname{Tor}_n^R(M,-)$ are the *derived functors* of $M \otimes_R -$. We recall the following concrete description.

Definition 2.2.1. Given a left R-module M and a right R-module N, together with a projective resolution $P_\bullet \to N \to 0$, we define

$$\operatorname{Tor}_n^R(N,M) = H_n(P_\bullet \otimes_R M), \tag{2.2}$$

which does not depend on the choice of the resolution $P_\bullet$.

Our previous definition does not emphasize the symmetry of the Tor functor. In fact, one can alternative take a projective resolution of right modules $Q_\bullet \to M \to 0$ and compute

$$\mathrm{Tor}_n^R(N, M) = H_n(N \otimes_R Q_\bullet).$$

Let S be another ring. If N is additionally an (S, R)-bimodule, then $\mathrm{Tor}_n^R(N, M)$ is naturally a left S-module for all n. Similarly, if M is an (R, S)-bimodule, then $\mathrm{Tor}_n^R(N, M)$ is naturally a right S-module.

Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of left R-modules. There are several situations in which we are given a extension of unital rings $R \hookrightarrow S$ and we want to understand the extension of scalars $S \otimes_R M_i$; for instance, when we look at completions of modules in the profinite and L^2-sense in Chapter 4. However, the corresponding sequence $0 \to S \otimes_R M_1 \to S \otimes_R M_2 \to S \otimes_R M_3 \to 0$ may no longer be exact, and requires a correction term. This is how the functor $\mathrm{Tor}_i^R(S, -)$ shows up and this is why it is named the *derived functor* of $S \otimes_R (-)$. All this is made precise in the long exact sequence in Tor associated to a short exact sequence of modules $0 \to M_1 \to M_2 \to M_3 \to 0$. Given an (S, R)-bimodule N, there is an exact sequence of left S-modules of the form

$$\begin{tikzcd}
& \cdots \arrow[r] & \mathrm{Tor}_{n+1}^R(N, M_3) \arrow[dll, rounded corners] & \\
\mathrm{Tor}_n^R(N, M_1) \arrow[r] & \mathrm{Tor}_n^R(N, M_2) \arrow[r] & \mathrm{Tor}_n^R(N, M_3) \arrow[r] & \cdots .
\end{tikzcd} \tag{2.3}$$

A standard reference for this material is [Wei94, Chapters 2 and 3]. Given two left R-modules M and N and a projective resolution $P_\bullet \to N \to 0$, we define

$$Ext_R^n(N, M) := H_n(\mathrm{Hom}_R(P_\bullet, M)), \tag{2.4}$$

which, again, does not depend on the choice of the resolution $P_\bullet$.

The homology and cohomology of the group G can now be expressed in terms of the Tor and Ext functors as follows. Let M be a $\mathbb{Z}[G]$-module. As in [Bro94, Chapter III, Section 2], the n-dimensional homology group of G with coefficients in M is given by

$$H_n(G; M) = \mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, M),$$

where $\mathbb{Z}$ denotes the trivial right $\mathbb{Z}[G]$-module. Analogously, the n-dimensional cohomology group of G with coefficients in M is

$$H^n(G; M) = \mathrm{Ext}_{\mathbb{Z}[G]}^n(\mathbb{Z}, M).$$

When M is p-torsion, it naturally becomes an $\mathbb{F}_p[G]$-module and we can equivalently define

$$H_n(G;M) = \operatorname{Tor}_n^{\mathbb{F}_p[G]}(\mathbb{F}_p, M) \quad \text{and} \quad H^n(G;M) = \operatorname{Ext}^n_{\mathbb{F}_p[G]}(\mathbb{F}_p, M).$$

Another fundamental long exact sequence is the Chiswell's Mayer–Vietoris exact sequence, used for computing the homology of a graph of groups.

Theorem 2.2.2 ([Chi76, Theorem 2])**.** *Let R be a ring, let $\mathcal{G}$ be a graph of groups with underlying graph Γ and $G = \pi_1(\mathcal{G})$, and let M be an $R[G]$-module. Then there is a long exact sequence*

$$\begin{tikzcd}
 & & \cdots \arrow[r] & H_{n+1}(G;M) \arrow[dlll, rounded corners, hook] \\
\bigoplus_{e\in\mathsf{Edge}(\Gamma)} H_n(G_e;M) \arrow[r] & \bigoplus_{v\in\mathsf{Vert}(\Gamma)} H_n(G_v;M) \arrow[r] & H_n(G;M) \arrow[r] & \cdots .
\end{tikzcd}$$

We conclude this section explaining the role of the augmentation ideal of a group in the study of the algebraic and homological properties of the group itself. The discussion here will be quite superficial but the importance of this ideal will become apparent in the proofs of Chapter 3 and Chapter 4. Given a field K and a group G, denote by $I_{K[G]}$ the *augmentation ideal* of the group ring $K[G]$; which, as a subset of $K[G]$, is described as

$$I_{K[G]} = \left\{ \sum_{g\in G} \lambda_g \cdot g : \lambda_g \in K, \sum \lambda_g = 0 \right\}.$$

In other words, $I_{K[G]}$ is the kernel of the *augmentation map*; that is, the obvious surjection of $K[G]$-modules $K[G] \to K$. Recall that, as usual, K here is endowed with the trivial G-action. Later on, the ground field K will be fixed and the notation I_G, instead of $I_{K[G]}$, will present no ambiguity. Given a subgroup $H \leqslant G$, we denote by I_H^G the left $K[G]$-submodule of I_G generated by I_H. In addition, even if H is not normal in G, we will write $K[G/H]$ to refer to the left $K[G]$-module of left cosets of H in G.

A basic idea that we shall exploit during Chapters 3 and 4 and that has its root in the so-called *dimension shifting* (for instance, see [Wei94, Application 6.2.5]). From the short exact sequence $0 \to I_G \to K[G] \to K$ and the long exact sequence from (2.3), we obtain natural isomorphisms of left S-modules

$$\operatorname{Tor}_n^{K[G]}(M,K) \cong \operatorname{Tor}_{n-1}^{K[G]}(M, I_G)$$

for every $(S, K[G])$-bimodule M. This anticipates the important role played by the augmentation ideal I_G in studying the homology of G. For this and other reasons, the following isomorphisms will become very useful later on.

Lemma 2.2.3 ([JZ24b], Lemma 2.1)**.** *Let $T \leqslant H \leqslant G$ be subgroups. Then the following hold.*

1. *The canonical map $K[G] \otimes_{K[H]} I_H \to I_H^G$ that sends $a \otimes b$ to $a \cdot b$ for all $a \in K[G]$ and $b \in I_H$ is an isomorphism of left $K[G]$-modules.*
2. *The canonical map $K[G] \otimes_{K[H]} \left(I_H / I_T^H\right) \to I_H^G / I_T^G$ that sends $a \otimes (b + I_T^H)$ to $ab + I_T^G$ for all $a \in K[G]$ and $b \in I_H$ is an isomorphism of left $K[G]$-modules.*
3. *The kernel of the canonical map of $K[G]$-modules $K[G/T] \to K[G/H]$ is naturally isomorphic to I_H^G / I_T^G.*

2.3 Pseudovarieties of finite groups: their topologies and completions

We say that a non-empty class of finite groups $\mathcal{C}$ is a *pseudovariety* if it is closed under subgroups, quotients and finite direct products. We denote by $\mathcal{N}_p$ the pseudovariety of finite p-groups (where p denotes a prime number) and by $\mathcal{S}$ the pseudovariety of all finite solvable groups. Given two pseudovarieties $\mathcal{C}$ and $\mathcal{B}$, we denote by $\mathcal{CB}$ the pseudovariety consisting of isomorphic classes of group G having a normal subgroup $N \in \mathcal{C}$ such that $G/N \in \mathcal{B}$.

Given a group G and a pseudovariety $\mathcal{C}$ of finite groups we denote by

$$G_{\widehat{\mathcal{C}}} = \varprojlim_{G/N \in \mathcal{C}} G/N$$

the pro-$\mathcal{C}$ completion of G. Given an $\mathbb{F}_p[G]$-module M, we denote by

$$St_G(M) = \bigcap_{m \in M} \operatorname{stab}_G(m)$$

the intersection of all stabilisers of elements of M. Hence, $St_G(M)$ acts trivially on M. Taking this notation into account, we write $L \leq_{\mathcal{C}} M$ if L is a submodule of M of finite index and $G/St_G(M/L) \in \mathcal{C}$. In other words, L is open in the pro-$\mathcal{C}$ topology of M, although we will rarely use this language in the context of modules.

Given an $\mathbb{F}_p[G]$-module M, its *pro-$\mathcal{C}$ completion* $M_{\widehat{\mathcal{C}}}$ is defined as the inverse limit

$$M_{\widehat{\mathcal{C}}} = \varprojlim_{L \leqslant_{\mathcal{C}} M} M/L.$$

If M is a finitely generated $\mathbb{F}_p[G]$-module, it is not hard to check that the modules $I_{\mathbb{F}_p[N]}M \leqslant_{\mathcal{C}} M$ form a basis of neighbourhoods of the zero element of M. From this, we deduce the isomorphisms

$$M_{\widehat{\mathcal{C}}} \cong \varprojlim_{G/N \in \mathcal{C}} M/\left(I_{\mathbb{F}_p[N]}M\right) \cong \varprojlim_{G/N \in \mathcal{C}} \left(\mathbb{F}_p[G/N] \otimes_{\mathbb{F}_p[G]} M\right).$$

This completion construction can also be performed for rings, to obtain profinite rings as an inverse limit of finite rings. The most important example in this direction is the *completed group algebra* of the profinite group $\mathbf{G}$ with $\mathbb{F}_p$-coefficients. This $\mathbb{F}_p$-algebra is defined as

$$\mathbb{F}_p[\![\mathbf{G}]\!] = \varprojlim_{\mathbf{N} \trianglelefteq_o \mathbf{G}} \mathbb{F}_p[\mathbf{G}/\mathbf{N}],$$

where $\mathbf{N} \trianglelefteq_o \mathbf{G}$ denotes that $\mathbf{N}$ is an open normal subgroup of $\mathbf{G}$. Observe that if M is an $\mathbb{F}_p[G]$-module then $M_{\widehat{\mathcal{C}}}$ becomes naturally an $\mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]$-module. We denote by $\delta_{M,\widehat{\mathcal{C}}}$ the canonical map

$$M \xrightarrow{\delta_{M,\widehat{\mathcal{C}}}} M_{\widehat{\mathcal{C}}}. \tag{2.5}$$

We note that a right $\mathbb{F}_p[G]$-module has a pro-$\mathcal{C}$ completion in a similar way. If M is an $\mathbb{F}_p[G]$-bimodule, then M has left and right pro-$\mathcal{C}$ completions. In this case, to avoid ambiguity, we denote its right pro-$\mathcal{C}$-completion by $M_{\widehat{\mathcal{C}},r}$. The following result can be proved arguing as in [GJZ23, Lemma 4.3].

Proposition 2.3.1. *Let M be a $\mathbb{F}_p[G]$-module. Consider the natural map α_M of $\mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]$-modules*

$$\mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!] \otimes_{\mathbb{F}_p[G]} M \xrightarrow{\alpha_M} M_{\widehat{\mathcal{C}}}$$

defined by extending linearly the map $r \otimes m \mapsto r \cdot \delta_{M,\widehat{\mathcal{C}}}(m)$. Then the following holds.

1. *If M is finitely generated, then α_M is onto.*

2. *If M is finitely presented, then α_M is an isomorphism.*

3. *Given finitely presented $\mathbb{F}_p[G]$-modules M_1 and M_2 and an $\mathbb{F}_p[G]$-module homomorphism $\gamma\colon M_1 \to M_2$, we obtain the following natural commutative diagram of $\mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]$-modules.*

$$\begin{tikzcd}
\mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!] \otimes_{\mathbb{F}_p[G]} M_1 \arrow[r, "\mathrm{Id} \otimes \gamma"] \arrow[d, "\alpha_{M_1}"] & \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!] \otimes_{\mathbb{F}_p[G]} M_2 \arrow[d, "\alpha_{M_2}"] \\
(M_1)_{\widehat{\mathcal{C}}} \arrow[r, "\gamma_{\widehat{\mathcal{C}}}"] & (M_2)_{\widehat{\mathcal{C}}}.
\end{tikzcd}$$

4. Given an exact sequence of finitely presented $\mathbb{F}_p[G]$-modules

$$M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \longrightarrow 0,$$

we obtain the following exact sequence of $\mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]$-modules.

$$(M_1)_{\widehat{\mathcal{C}}} \xrightarrow{\alpha_{\widehat{\mathcal{C}}}} (M_2)_{\widehat{\mathcal{C}}} \xrightarrow{\beta_{\widehat{\mathcal{C}}}} (M_3)_{\widehat{\mathcal{C}}} \longrightarrow 0.$$

Let Γ be a finitely generated residually-$\mathcal{C}$ group. The *$\mathcal{C}$-genus* of Γ, denoted by $\mathcal{G}_{\mathcal{C}}(\Gamma)$, is the set of isomorphism classes of finitely generated residually-$\mathcal{C}$ groups G having the same quotients in $\mathcal{C}$ as Γ. It is well-known (see, for example, [RZ10, Corollary 3.2.8]) that $G \in \mathcal{G}_{\mathcal{C}}(\Gamma)$ if and only if the pro-$\mathcal{C}$ completions of G and Γ are isomorphic: $G_{\widehat{\mathcal{C}}} \cong \Gamma_{\widehat{\mathcal{C}}}$.

In the next proposition, we should think of the inclusion $R \hookrightarrow S$ as the inclusion $\mathbb{F}_p[G] \hookrightarrow \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]$ for a group G that is good in a pseudovariety of finite groups $\mathcal{C}$.

Proposition 2.3.2. *Let $R \hookrightarrow S$ be an extension of unital rings. Suppose that A is an R-module such that $\operatorname{Tor}_i^R(S, A) = 0$ for all $i \geqslant 1$ that admits a finite resolution by finitely generated projective modules P_i of the form:*

$$0 \longrightarrow P_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} A \longrightarrow 0. \tag{2.6}$$

Then there exists an isomorphism of right S-modules

$$\operatorname{Ext}^n_R(A, R) \otimes_R S \longrightarrow \operatorname{Ext}^n_S(S \otimes_R A, S).$$

To prove Proposition 2.3.2, we will need the following lemma, whose proof we leave as an exercise for the reader.

Lemma 2.3.3. *Let P be a finitely generated projective R-module. Then there is a canonical isomorphism F_P^S of S-modules*

$$\operatorname{Hom}_R(P, R) \otimes_R S \xrightarrow{F_P^S} \operatorname{Hom}_S(S \otimes_R P, S),$$

that consists on extending linearly the map $\left(F_P^S(\psi \otimes s_1)\right)(s_2 \otimes x) = s_2 \cdot \psi(x) \cdot s_1$ for all $s_1, s_2 \in S$, $x \in P$ and $\psi \in \operatorname{Hom}_R(P, R)$.

Proof of Proposition 2.3.2. After applying the functor $\operatorname{Hom}_R(-, R)$ to the resolution (2.6), we get a cochain complex with coboundary maps

$$\partial^i \colon \operatorname{Hom}_R(P_{i-1}, R) \to \operatorname{Hom}_R(P_i, R)$$

and the abelian group $\operatorname{Ext}^n_R(A, R)$ is isomorphic to $\operatorname{Hom}_R(P_n, R)/\operatorname{im}\partial^n$. On the other hand, since $\operatorname{Tor}^R_i(S, A) = 0$, we can apply the functor $(S \otimes_R -)$ to get a projective resolution of $S \otimes_R A$ of the form

$$0 \longrightarrow S \otimes_R P_n \xrightarrow{\partial^S_n} \cdots \xrightarrow{\partial^S_1} S \otimes_R P_0 \xrightarrow{\partial^S_0} S \otimes_R A \longrightarrow 0.$$

After applying the functor $\operatorname{Hom}_S(-, S)$ to this resolution, the induced cochain complex has maps $\partial^i_S \colon \operatorname{Hom}_S(S \otimes_R P_{i-1}, S) \to \operatorname{Hom}_S(S \otimes_R P_i, S)$. As before, the abelian group $\operatorname{Ext}^n_S(S \otimes_R A, S)$ is isomorphic to $\operatorname{Hom}_S(S \otimes_R P_n, S)/\operatorname{im}\partial^n_S$. The situation can be summarised with the following commutative diagram:

$$\begin{tikzcd}
\operatorname{Hom}_R(P_{n-1}, R) \otimes_R S \arrow[r] \arrow[d, "F^S_{P_{n-1}}"] & \operatorname{Hom}_R(P_n, R) \otimes_R S \arrow[r] \arrow[d, "F^S_{P_n}"] & \operatorname{Ext}^n_R(A, R) \otimes_R S \arrow[r] \arrow[d] & 0 \\
\operatorname{Hom}_S(S \otimes_R P_{n-1}, S) \arrow[r] & \operatorname{Hom}_S(S \otimes_R P_n, S) \arrow[r] & \operatorname{Ext}^n_S(S \otimes_R A, S) \arrow[r] & 0.
\end{tikzcd}$$

The maps $F^S_{P_i}$ are isomorphisms by Lemma 2.3.3 and they naturally induce the required isomorphism $\operatorname{Ext}^n_R(A, R) \otimes_R S \cong \operatorname{Ext}^n_S(S \otimes_R A, S)$. □

2.4 Finiteness properties

Let R be a ring and let n be either an integer or $n = \infty$. An R-module M is $\mathtt{FP}_n$ if it admits a resolution by projective R-modules $P_* \to M \to 0$, where P_i is finitely generated for all $0 \leqslant i \leqslant n$. A group G is $\mathtt{FP}_n(\mathbb{F}_p)$ (for an integer $n \geqslant 0$ or for $n = \infty$) if the trivial $\mathbb{F}_p[G]$-module $\mathbb{F}_p$ is $\mathtt{FP}_n$. A group G is said to be $\mathtt{FP}(\mathbb{F}_p)$ if the trivial $\mathbb{F}_p[G]$-module $\mathbb{F}_p$ admits a finite projective resolution. We recall the standard observation that if a group G admits a finite CW-complex X as a $K(G, 1)$, then the contractibility of $\widetilde{X}$ leads directly to a finite resolution of the trivial $\mathbb{F}_p[G]$-module $\mathbb{F}_p$ by finitely generated free $\mathbb{F}_p[G]$-modules.

We remind the reader that a group G is $\mathtt{FP}_\infty(\mathbb{F}_p)$ if and only if it is $\mathtt{FP}_n(\mathbb{F}_p)$ for all n by [Bro82, Chapter VIII, Proposition 4.5]; and also that a group G of finite cohomological dimension is of type $\mathtt{FP}(\mathbb{F}_p)$ if and only if it is $\mathtt{FP}_\infty(\mathbb{F}_p)$ by [Bro82, Chapter VIII, Proposition 6.1]. We also want to recall the following well-known result from [Wat60, Theorem 1] (see also [Bro82, Proposition 6.8]).

Proposition 2.4.1. *Let G be an $\mathtt{FP}(\mathbb{F}_p)$ group with $\operatorname{cd}(G) \leqslant n$; let R and S be two rings; and let A be an $(\mathbb{F}_p[G], R)$-bimodule and B an (R, S)-bimodule. Then the canonical map*

$$H^n(G; A) \otimes_R B \cong H^n(G; A \otimes_R B).$$

is an isomorphism of right S-modules

We will use the following consequence in the context of one-relator groups, which essentially says that the top dimensional homology with finite coefficients is always virtually remembered.

Corollary 2.4.2. *Let G be a group of cohomological dimension two and let $U_1 \leqslant U_2 \leqslant G$ be subgroups of finite index in G. Then the corestriction map $H^2(U_1; \mathbb{F}_p) \to H^2(U_2; \mathbb{F}_p)$ is onto.*

2.5 Poincaré duality in dimension two

Recall that a *surface group* we will mean the fundamental group of a compact closed surface Σ. We are mainly interested in the (generic) case when the Euler characteristic $\chi(\Sigma)$ is negative. Equivalently, this is the case when Σ admits a Riemannian metric of constant sectional curvature equal to -1, and so its universal cover $\widetilde{\Sigma}$ is $\pi_1(\Sigma)$-equivariantly isometric to the hyperbolic plane $\mathbb{H}^2$. We will not explicitly use this viewpoint; although it indirectly appears, through the work of Scott [Sco78, Sco85], in Proposition 3.2.7. In any case, one key consequence that we shall exploit later on is that $\pi_1(\Sigma)$ is *hyperbolic* in the sense of Gromov [Gro87] and that $\pi_1(\Sigma)$ admits many cyclic splittings; which can be found, for instance, by choosing *pair-of-pants decompositions* of Σ.

We review the one-relator presentations of the fundamental groups of the closed surfaces Σ with $\chi(\Sigma) < 0$. In the orientable case, surface groups admit presentations of the form

$$S_g = \langle x_1, \ldots, x_g, y_1, \ldots, y_g | \ [x_1 y_1] \cdots [x_g, y_g] = 1 \rangle \text{ for genus } g \geqslant 2;$$

and in the non-orientable closed case it is

$$N_g = \langle x_1, \ldots, x_g | \ x_1^2 \cdots x_g^2 = 1 \rangle \text{ for genus } g \geqslant 3.$$

Although free groups arise as fundamental groups of non-closed surfaces of negative Euler characteristic, we will not consider free groups as surface groups. There are two key differences between the closed and non-closed case: the groups $\pi_1(\Sigma)$, as opposed to free groups, are one-ended; and they satisfy the Poincaré duality. To expand on this, we first recall this definition using Farrell's approach [Far75].

Definition 2.5.1. Let K be a field and let G be an $\mathtt{FP}_2(K)$ group of cohomological dimension two. We say that G is *Poincaré Duality of dimension two over K* if $H^1(G; K[G]) = 0$ and $H^2(G; K[G]) \cong K$ as K-vector spaces. If, in addition, the action of G on $H^2(G; K[G])$ is trivial, we say that G is *orientable.*

Hence, one way we recognise if a group G is a surface group is by studying the corresponding cohomology groups $H^i(G, \mathbb{F}_p[G])$. In the case of integer coefficients, Eckmann–Linnell–Müller [EM80, EL83] showed that any group satisfying the conditions of Definition 2.5.1 must in fact be a surface group. Bowditch [Bow04] extended this principle to any coefficient field, obtaining in particular the following.

Theorem 2.5.2. *Suppose that* G *is a torsion-free Poincaré duality group of dimension two over the field* $\mathbb{F}_p$*. Then* G *is a surface group.*

We have only stated the version over $\mathbb{F}_p$ for concreteness. Even more, this is the version that we shall use in Theorem D4 because it provides stronger conclusions than the analogous statement on rational coefficients.

Let F be a free group on $\{x_1, \ldots, x_n\}$. We say that $w \in F$ is an *orientable* (resp. *non-orientable) surface word* if there exists an automorphism $\varphi \in \mathrm{Aut}(F)$ such that $\varphi(w) = [x_1, x_2][x_3, x_4] \cdots [x_{n-1}, x_n]$ (resp. $\varphi(w) = x_1^2 x_2^2 \cdots x_n^2$). We shall also need the following standard result. For the convenience of the reader, we sketch its proof.

Proposition 2.5.3. *Let* F *be a finitely generated group of rank* k *and* $u \in F$*. Assume that* $F *_{u=\overline{u}} \overline{F}$ *is a surface group. Then* u *is a surface word.*

Proof. Suppose that there exists a closed surface Σ such that $F *_{u=\overline{u}} \overline{F} \cong \pi_1\Sigma$. It is well-known that group-theoretic cyclic splittings of $\pi_1\Sigma$ must come from a geometric splitting of the surface Σ. This is a two-dimensional analogue of the so-called Stallings–Epstein–Waldhausen construction for 3-manifolds (see, for instance, the exposition of Shalen [Sha01, Section 2]). Hence, there exists an essential simple closed curve γ in Σ satisfying the following properties:

- The surface Σ_γ that results from cutting Σ along γ has exactly two connected components Σ_1 and Σ_2 (and so the boundaries of both Σ_1 and Σ_2 have exactly one connected component).
- There are isomorphisms $\psi_1 \colon \pi_1\Sigma_1 \to F$ and $\psi_2 \colon \pi_1\Sigma_2 \to \overline{F}$ such that $\psi_1(\partial\Sigma_1) = u$ and $\psi_2(\partial\Sigma_2) = \overline{u}$.

If Σ_1 is a genus g surface with one boundary component, then $\pi_1\Sigma_1$ is free in generators $a_1, \cdots, a_k$, where $k = 2g$ (resp. $k = g + 1$) if Σ is orientable (resp. non-orientable). Now we note that from [Cul81, Theorem 1.5] that the isomorphism of group pairs $(\pi_1\Sigma_1; \pi_1\partial\Sigma_1) \cong (F; u)$ implies that the word $u \in F$ belongs to the $\mathrm{Aut}(F)$-orbit of the word $[a_1, a_2] \cdots [a_{k-1}, a_k]$ if Σ_1 is orientable (resp. $a_1^2 a_2^2 \cdots a_k^2$ if Σ_1 is non-orientable). Hence, u is a surface word. $\square$

2.6 Cohomology of profinite groups

A *topological* $\mathbf{G}$*-module* $\mathbf{M}$ is an abelian Hausdorff topological group $\mathbf{M}$ which is endowed with the structure of an abstract $\mathbf{G}$-module such that the action $\mathbf{G}\times\mathbf{M} \to \mathbf{M}$ is continuous. Given a profinite group $\mathbf{G}$ and a topological $\mathbf{G}$-module $\mathbf{M}$, we denote by $H^k_{\mathrm{cts}}(\mathbf{G};\mathbf{M})$ the k-th continuous cochain cohomology group of $\mathbf{G}$ with coefficients in $\mathbf{M}$ as in [NSW08, Definition 2.7.1]. For our computations, we are only interested in profinite modules. Recall that a *profinite* $\mathbf{G}$*-module* $\mathbf{M}$ is a topological $\mathbf{G}$-module which is additionally compact and totally disconnected. Equivalently, we can simply define a profinite $\mathbf{G}$-module $\mathbf{M}$ to be an inverse limit of finite discrete $\mathbf{G}$-modules. From this point of view, we can give the following practical recipe from [NSW08, Corollary 2.7.6] to compute the cohomology of $\mathbf{G}$ with coefficients in $\mathbf{M}$.

Proposition 2.6.1. *Let* $\mathbf{M} = \varprojlim_i M_i$ *be an inverse limit of finite discrete* $\mathbf{G}$*-modules* $\{M_i\}_{i\in\mathcal{N}}$*. Assume that* $H^k(\mathbf{G};M_i)$ *is finite for all* $i\in\mathcal{N}$*. Then the canonical map*

$$H^{k+1}_{cts}(\mathbf{G},\mathbf{M}) \longrightarrow \varprojlim_i H^{k+1}_{cts}(\mathbf{G},M_i)$$

is an isomorphism.

The cohomology groups $H^n_{\mathrm{cts}}(\mathbf{G},\mathbf{M})$ can also be computed from finite resolutions of profinite modules, in direct analogy with the way cohomology was introduced in Section 2.2 for abstract groups, by appropriately taking into account the topology of the underlying modules and the continuity of the maps. Nevertheless, in the presence of finite generation, which is the assumption we will work with in practice, the topology is not so relevant. This is illustrated by the following two results.

(i) If $\mathbf{G}$ is a profinite group and $\mathbf{M}$ is a finitely generated and projective abstract $\mathbb{F}_p[\![\mathbf{G}]\!]$-module, then $\mathbf{M}$ is naturally a profinite projective $\mathbb{F}_p[\![\mathbf{G}]\!]$-module (with the same action).

(ii) Every abstract homomorphism between finitely generated profinite $\mathbb{F}_p[\![\mathbf{G}]\!]$-modules is continuous.

We refer the reader to [KM18, Lemma 1.1] for proofs and for more consequences of this sort of principle. We can review this discussion as follows: given a profinite group $\mathbf{G}$ and a profinite $\mathbf{G}$-module $\mathbf{M}$ which is of type `FP` as an abstract $\mathbb{F}_p[\![\mathbf{G}]\!]$-module, we have the isomorphism

$$H^i_{\mathrm{cts}}(\mathbf{G};\mathbf{M}) \cong \mathrm{Ext}^i_{\mathbb{F}_p[\![\mathbf{G}]\!]}(\mathbb{F}_p,\mathbf{M}) \tag{2.7}$$

for every non-negative integer i. Crucially, the right-hand side of (2.7), which was explained in (2.4), involves no topology. At first, there is no apparent gain in this isomorphism, given that the left-hand side of (2.7) can be explicitly defined; while the right-hand side does not seem functional in this context, where the topological machinery of classifying spaces of abstract groups is not available. In the following section we clarify this matter and explain how the right-hand side of (2.7) may be computed when $\mathbf{G}$ arises as the profinite completion of an abstract group.

2.7 On Serre's cohomological goodness

Let G be an abstract group and let $\mathcal{C}$ be a pseudovariety of finite groups. We say that G is *cohomologically p-good in $\mathcal{C}$* (or just p-good in $\mathcal{C}$) if for any finite discrete $\mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]$-module M with $|M| = p^k$ and $G/St_G(M) \in \mathcal{C}$, the induced map

$$H^i_{\mathrm{cts}}(G_{\widehat{\mathcal{C}}}; M) \longrightarrow H^i(G; M) \tag{2.8}$$

is an isomorphism. If, for all primes p, G is p-good in the variety of all finite groups, then we just say that G is *good* (recovering Serre's original notion [Ser97]).

We should remark that the property of cohomological goodness has been rechristened by Wilkes [Wil24] as *cohomological separability.* Even if this name is more informative about the nature of the property, we chose for our purposes to stick to the classical terminology.

There are several ways in which the property of goodness appears naturally as an useful tool in the study of profinite completions of groups. We briefly comment on some instances of this:

- If G and H are good and $\widehat{G} \cong \widehat{H}$, then $H^i(G; \mathbb{F}_p) \cong H^i(H; \mathbb{F}_p)$. Hence, in certain relative settings, goodness allows us to view the Betti numbers of a group G with finite coefficients (that is, the dimensions $\dim_{\mathbb{F}_p} H^i(G; \mathbb{F}_p)$) as profinite invariants. This observation will be exploited multiple times when studying the profinite rigidity of surface groups in Chapter 4.

- Goodness allows us to produce residually finite groups via group extensions of residually finite groups, as noted in the work of Grunewald–Jaikin-Zapirain–Zalesskii [GJZZ08, Proposition 6.1]. In this sense, the presence of goodness prevents the appearance of the famous non-residually finite examples of Deligne [Del78], which are central extensions of the symplectic group $\mathrm{Sp}(4, \mathbb{Z})$.

- Lastly, goodness allows us to transmit structural properties of G to $\widehat{G}$, such as admitting a certain description as an extension of groups. Again, this fact is appreciated as part of [GJZZ08, Proposition 6.1].

Furthermore, as we shall explain, goodness allows us to relate resolutions of G to resolutions of $\widehat{G}$ (and so their finiteness properties in the sense of Section 2.4).

Proposition 2.7.1. *Let $\mathcal{C}$ be a pseudovariety of finite groups such that $\mathcal{N}_p\mathcal{C} = \mathcal{C}$. Let G be a $\mathtt{FP}(\mathbb{F}_p)$ group that is cohomologically good in $\mathcal{C}$, then* $\operatorname{Tor}_i^{\mathbb{F}_p[G]}(\mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!], \mathbb{F}_p) = 0$ *if $i \geqslant 1$.*

Proof. The same proof as of [JZ20b, Proposition 3.1] applies in this case. $\square$

The previous proposition implies that, given a projective resolution of the trivial $\mathbb{F}_p[G]$-module $\mathbb{F}_p$

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow \mathbb{F}_p \longrightarrow 0,$$

we can apply the functor $(\mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!] \otimes_{\mathbb{F}_p[G]} -)$ to naturally obtain a projective resolution of the trivial $\mathbb{F}_p[\![G_{\widehat{\mathcal{S}}}]\!]$-module $\mathbb{F}_p$ of the form

$$0 \longrightarrow \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!] \otimes_{\mathbb{F}_p[G]} P_n \longrightarrow \cdots \longrightarrow \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!] \otimes_{\mathbb{F}_p[G]} P_0 \longrightarrow \mathbb{F}_p \longrightarrow 0.$$

This implies that if G is a good $\mathtt{FP}$ group then, in the appropriate sense, $\widehat{G}$ is also $\mathtt{FP}$. We explore these sorts of ideas to develop a novel use of goodness in Chapter 4. This consists in producing maps between certain derived functors of G-modules and the corresponding derived functors of their profinite completions, viewed as $\widehat{G}$-modules. We should remark that, in general, there is no way to make such comparison.

We finish this section by recalling the definition of Poincaré Duality for profinite groups. Farrell's approach to this notion, described in Definition 2.5.1, naturally carries over to this setting.

Definition 2.7.2. Let $\mathbf{G}$ be a profinite group of type $\mathtt{FP}_2(\mathbb{F}_p)$. We say that $\mathbf{G}$ is a *Poincaré duality group of dimension* 2 *at* p if $\operatorname{cd}_p(\mathbf{G}) = 2$, $H^1_{\text{cts}}(\mathbf{G}, \mathbb{F}_p[\![\mathbf{G}]\!]) = 0$ and $H^2_{\text{cts}}(\mathbf{G}; \mathbb{F}_p[\![\mathbf{G}]\!]) \cong \mathbb{F}_p$ as abelian groups. If, in addition, the action of $\mathbf{G}$ in $H^2_{\text{cts}}(\mathbf{G}; \mathbb{F}_p[\![\mathbf{G}]\!])$ is trivial, we will say that $\mathbf{G}$ is an *orientable* Poincaré duality group of dimension 2 at p.

2.8 The Hughes-free properties of division rings

A group is said to be *locally indicable* if for every finitely generated subgroup non-trivial subgroup $H \leqslant G$ there exists a surjection $H \to \mathbb{Z}$. We denote by G a locally indicable group and by K a field. An embedding $\varphi\colon K[G] \hookrightarrow \mathcal{D}$ of the group algebra $K[G]$ into a division ring $\mathcal{D}$ is called *Hughes-free* if the following conditions are satisfied:

(i) The image $\varphi(K[G])$ generates $\mathcal{D}$ as a division ring.

(ii) Let $H \leqslant G$ be a finitely generated subgroup and let $f\colon H \to \mathbb{Z}$ be an epimorphism with kernel N, and let $t \in H$ map to a generator of $\mathbb{Z}$ under f. Let $\mathcal{D}_N$ denote the division closure of $\varphi(K[N])$. Then $\{\varphi(t^i) : i \in \mathbb{Z}\} \subseteq \mathcal{D}$ is linearly independent over $\mathcal{D}_N$.

We give two major reasons for the importance of this definition.

- Extrinsic and foundational reason: By a theorem of Hughes [Hug70], if a Hughes-free embedding of $K[G]$ exists, then it is unique up to $K[G]$-isomorphism. Thus, if $K[G]$ has a Hughes-free embedding, then we denote the division ring by $\mathcal{D}_{K[G]}$ and think of $K[G]$ as a subset of $\mathcal{D}_{K[G]}$. We will call $\mathcal{D}_{K[G]}$ *the Hughes-free division ring* of $K[G]$. In particular, given a subgroup $H \leqslant G$, the division closure of $K[H]$ in $\mathcal{D}_{K[G]}$ is isomorphic to the Hughes-free division ring $\mathcal{D}_{K[H]}$. This will allow us to relate the L^2-homology of G with that of its subgroups.
- Intrinsic and practical reason: Once we know that $\mathcal{D}_{K[G]}$ really encodes features of G, the Hughes-free property is a vital tool to analyse the structure of $\mathcal{D}_{K[G]}$ in its own right, even when we are no longer interested in looking at more embeddings of $K[G]$ into division rings.

The existence of Hughes-free division rings has been established for many classes of locally indicable groups, and in particular for all locally indicable groups when the ground field K has characteristic zero. We collect some of these results in the following proposition, which does not pretend to be exhaustive by any means. For a more complete account of the status of this problem we refer to [JZ20a].

Proposition 2.8.1. *A Hughes-free embedding $K[G] \hookrightarrow \mathcal{D}_{K[G]}$ is guaranteed to exist under any of the following assumptions:*

1. *The group G is locally indicable and the field K has characteristic zero.*

2. The group G residually-(locally indicable and amenable) and K is any field.

Proof. The first assertion is a consequence of Jaikin-Zapirain–López-Álvarez's resultion of the strong Atiyah conjecture for locally indicable groups [JZLÁ20, Corollary 1.4] and the second assertion is [JZ21, Corollary 1.3]. □

Hughes-free division rings provide powerful homological invariants. The algebraic approach to the computation of L^2-invariants that unifies this exposition consists on recovering this homology by considering $\mathcal{D}_{K[G]}$ as our module. It is convenient to recall that modules over a division ring are automatically free modules and that they have a well-defined dimension in the usual sense of linear algebra. This avoids the more subtle notion of dimension we referred to in 1.3 (more specifically, we refer to Definition 1.3.1 and the preceeding discussion).

An homomorphism of unital rings $R \to S$ will sometimes referred to as an R-ring. If $R \to \mathcal{D}$ is a division R-ring, the $\mathcal{D}$-Betti numbers of a left R-module M are defined as

$$b_i^{\mathcal{D}}(M) := \dim_{\mathcal{D}} \operatorname{Tor}_i^R(\mathcal{D}, M). \tag{2.9}$$

Note that in degree zero this corresponds to taking the dimension over $\mathcal{D}$:

$$b_0^{\mathcal{D}}(M) = \dim_{\mathcal{D}} \mathcal{D} \otimes_R M.$$

Setting K to be the trivial $K[G]$-module, we obtain homological numerical invariants of the group G:

$$b_n^{\mathcal{D}_{K[G]}}(G) := b_n^{\mathcal{D}_{K[G]}}(K) = \dim_{\mathcal{D}_{K[G]}} \operatorname{Tor}_n^{K[G]}(\mathcal{D}_{K[G]}, K). \tag{2.10}$$

We will refer to these as the $\mathcal{D}_{K[G]}$-Betti numbers of G. We refer to the survey paper of Jaikin-Zapirain [JZ19] for a more complete account of this algebraic perspective in L^2-invariants.

We conclude by listing several properties that will be used throughout the thesis. We emphasize point (1) below, which states that, when $K = \mathbb{C}$, the $\mathcal{D}_{K[G]}$-Betti numbers coincide with the L^2-Betti numbers of G.

Proposition 2.8.2. *Let G be a locally indicable group and let K be a field such that a Hughes-free embedding $K[G] \hookrightarrow \mathcal{D}_{K[G]}$ exists.*

(1) If $K = \mathbb{C}$, then $b_n^{\mathcal{D}_{K[G]}}(G) = b_n^{(2)}(G)$ for all integers $n \geqslant 0$.

(2) If G is nontrivial, then $b_0^{\mathcal{D}_{K[G]}}(G) = 0$, otherwise $b_0^{\mathcal{D}_{K[G]}}(G) = 1$.

(3) If G is of finite type, then $\chi(G) = \sum_{i=0}^{\infty}(-1)^i b_i^{\mathcal{D}_{K[G]}}(G)$.

(4) Let $H \leqslant G$ be a subgroup of finite index. Then $\mathcal{D}_{K[H]} \otimes_{K[H]} K[G] \cong \mathcal{D}_{K[G]}$ as $(\mathcal{D}_{K[H]}, K[G])$-bimodules. Consequently, for every $n \geqslant 0$ we have

$$b_n^{\mathcal{D}_{K[H]}}(H) = |G : H| \cdot b_n^{\mathcal{D}_{K[G]}}(G).$$

(5) Let $k \geqslant 0$ be an integer, let H be a subgroup of G and let N be a left $K[H]$-module. There is an isomorphism of left $\mathcal{D}_{K[G]}$-modules of the form

$$\operatorname{Tor}_k^{K[H]}(\mathcal{D}_{K[G]}, N) \cong \operatorname{Tor}_k^{K[G]}(\mathcal{D}_{K[G]}, K[G] \otimes_{K[H]} N).$$

In particular, if $N = K$ is the trivial $K[H]$-module, then

$$b_k^{K[H]}(H) = \dim_{\mathcal{D}_{K[G]}} \operatorname{Tor}_k^{K[G]}(\mathcal{D}_{K[G]}, K[G/H]).$$

(6) Let $k \geqslant 0$ be an integer, let H be a finite-index subgroup of G and let M be a left $K[G]$-module. There is an isomorphism of left $\mathcal{D}_{K[H]}$-modules of the form

$$\operatorname{Tor}_k^{K[H]}(\mathcal{D}_{K[H]}, M) \cong \operatorname{Tor}_k^{K[G]}(\mathcal{D}_{K[G]}, M).$$

(7) If G is free on $m \geqslant 1$ generators, then $b_1^{\mathcal{D}_{K[G]}}(G) = m - 1$ and $b_n^{\mathcal{D}_{K[G]}}(G) = 0$ for all $n \neq 1$. If G is amenable, then $b_n^{\mathcal{D}_{K[G]}}(G) = 0$ for all n.

Proof. Statement (1) follows from [JZLÁ20, Theorem 1.1], while (2) can easily be proved directly from the definitions. To prove (3), let

$$0 \to K[G]^{r_d} \to K[G]^{r_{d-1}} \to \ldots \to K[G]^{r_0} \to K \to 0$$

be a resolution of the trivial $K[G]$-module K by finitely generated free $K[G]$-modules. By definition, $\chi(G) = \sum_{i=0}^{d}(-1)^i r_i$. After tensoring with $\mathcal{D}_{K[G]}$ and omitting the rightmost term, we obtain the chain complex

$$0 \to \mathcal{D}_{K[G]}^{r_d} \to \mathcal{D}_{K[G]}^{r_{d-1}} \to \ldots \to \mathcal{D}_{K[G]}^{r_0} \to 0$$

whose boundary maps we denote by $\partial_i \colon \mathcal{D}_{K[G]}^{r_i} \to \mathcal{D}_{K[G]}^{r_{i-1}}$. Since $\mathcal{D}_{K[G]}$ is a division ring, the rank-nullity theorem holds, and therefore there is a decomposition

$$\mathcal{D}_{K[G]}^{r_i} \cong \ker \partial_i \oplus \operatorname{im} \partial_i \cong \operatorname{Tor}_i^{K[G]}(\mathcal{D}_{K[G]}, K) \oplus \operatorname{im} \partial_{i+1} \oplus \operatorname{im} \partial_i.$$

Since, by definition, $b_i^{\mathcal{D}_{K[G]}}(G) = \dim_{\mathcal{D}_{K[G]}} \operatorname{Tor}_i^{K[G]}(\mathcal{D}_{K[G]}, K)$, we obtain

$$\chi(G) = \sum_{i=0}^{\infty}(-1)^i b_i^{\mathcal{D}_{K[G]}}(G).$$

Statement (4) is a direct consequence of [Grä20, Corollary 8.3]. The isomorphism of (5) follows from a standard application of Shapiro's lemma on the second entry of the Tor functor. The second equation of (6) follows from setting N to be the trivial $K[H]$-module K and from noting that, as left $K[G]$-modules, $K[G] \otimes_{K[H]} K \cong K[G/H]$. Similarly, we apply Shapiro's lemma to the first entry of Tor to obtain the isomorphism

$$\operatorname{Tor}_k^{K[H]}(\mathcal{D}_{K[H]}, M) \cong \operatorname{Tor}_k^{K[G]}(\mathcal{D}_{K[H]} \otimes_{K[H]} K[G], M).$$

Now (6) follows from (4) and the previous isomorphism. Finally, for (7), the claim about free groups can be proved easily using (2) and (3). The claim about amenable groups follows from [HK21, Theorem 3.9(6)]. Note that only the case $K = \mathbb{Q}$ is treated there, but general case admits the same proof. $\square$

Chapter 3

The Hanna Neumann conjecture for limit groups

In this chapter we address the Hanna Neumann conjecture for limit groups. We first recall the stronger formulation of this conjecture due to Walter Neumann [Neu90]. For this, first remind that given a group G which admits a finite CW-complex as a $K(G,1)$, we denote by $\overline{\chi}(G) = \max\{-\chi(G), 0\}$ the reduced Euler characteristic of G.

Conjecture 3.0.1 (The Strengthened Hanna Neumann Conjecture (SHNC))**.** *Let U and V be finitely generated subgroups of a free group G. Let T be a complete set of representatives for the double (U, V)-cosets in G. Then*

$$\sum_{t \in T} \overline{\chi}(U \cap V^t) \leqslant \overline{\chi}(U)\overline{\chi}(V). \tag{3.1}$$

The statement of Conjecture 3.0.1 makes sense whenever G is a group such that all of its finitely generated subgroups are of finite type (so that $\overline{\chi}$ is defined for all such subgroups). Conjecture 3.0.1 was resolved independently by Mineyev [Min12] and Friedman [Fri15]. More recently, Jaikin-Zapirain [JZ17] gave an alternative proof which also applies to free pro-p groups G. Later on, some two-dimensional groups where shown to satisfy Conjecture 3.0.1, such as Demushkin groups by Jaikin-Zapirain–Shusterman [JZS19] and surface groups by Antolín–Jaikin-Zapirain [AJZ22].

We will denote by K a field and by G a group that has a Hughess-free division ring $K[G] \hookrightarrow \mathcal{D}_{K[G]}$ (an object that was introduced in Section 2.8). The definition of L^2-independent subgroups and L^2-Hall groups was given in Definition 1.3.1 in terms of the group von Neumann algebra of G. We reformulate this definition in a way that naturally extends to all ground fields.

Definition 3.0.2. Let H be a subgroup of G. Consider the natural surjection of left $K[G]$-modules $K[G/H] \to K$. This induces a natural map

$$\operatorname{Tor}_1^{K[G]}(\mathcal{D}_{K[G]}, K[G/H]) \to \operatorname{Tor}_1^{K[G]}(\mathcal{D}_{K[G]}, K). \tag{3.2}$$

We say that H is $\mathcal{D}_{K[G]}$*-independent* if the map is injective. When K is a subfield of $\mathbb{C}$, we will say that H is L^2*-independent* in G; recovering Definition 1.3.1.

By [AJZ22, Proposition 4.2], H is $\mathcal{D}_{K[G]}$-independent in G if and only if the co-restriction map $H_1(H; \mathcal{D}_{K[G]}) \to H_1(G; \mathcal{D}_{K[G]})$ is injective. So Definition 3.0.2 is the natural extension of Antolín–Jaikin-Zapirain's definition of L^2-independence given in [AJZ22, Section 4]. Working over different ground fields will uniformly include various cases of interest while adding no technical difficulty. On the other hand, an advantage of working with the $\mathcal{D}_{K[G]}$-Hall property is that the condition that $H_1(H; \mathcal{D}_{K[G]}) \to H_1(G; \mathcal{D}_{K[G]})$ be injective is somewhat less awkward than the condition in Definition 3.0.2.

We chose to state Definition 3.0.2 in terms of Tor because it gives the flexibility of playing with the two entries of the functor and exploit the isomorphisms listed in Proposition 2.8.2. Definition 3.0.2 heavily relies on the underlying embedding of H into G, not only on H and G abstractly.

Definition 3.0.3. Given a monomorphism $f \colon H \hookrightarrow G$, we will say that f is $\mathcal{D}_{K[G]}$*-injective* if $f(H)$ is $\mathcal{D}_{K[G]}$-independent in G (or L^2*-injective* when $K = \mathbb{C}$).

The reason we gave the previous definition is that the injectivity of map (3.2) depends on the choice of embedding of H into G. For example, the embedding $f \colon F(a, b, c) \to G = F(x, y, z)$ defined by $f(a) = x^2$, $f(b) = y$ and $f(c) = y^x$ does not lead to an L^2-independent subgroup of G.

It should also be noted that the relation between all the notions of $\mathcal{D}_{K[G]}$-independence, as K varies, is not well-understood. We refer to Section 3.4 for a more thorough discussion.

Definition 3.0.4. We say that a group G is $\mathcal{D}_{K[G]}$*-Hall* or that it has the $\mathcal{D}_{K[G]}$*-Hall property* if for every finitely generated subgroup $H \leqslant G$ there exists a finite-index subgroup $G_1 \leqslant G$ such that H is $\mathcal{D}_{K[G]}$-independent in G_1. If we specialise to $K = \mathbb{C}$, we say that G is L^2*-Hall* or has the L^2*-Hall property*; recovering Definition 1.3.1.

Antolín–Jaikin-Zapirain proved that free and surface groups have the L^2-Hall property [AJZ22, Theorem 4.4] and showed that if G is a hyperbolic limit group that has the L^2-Hall property, then Conjecture 3.0.1 holds for G [AJZ22, Theorem 1.3]. The motivation of [FM25a] was to find a wealth of L^2-Hall groups to give more positive examples to the latter conjecture. Recently, Brown and Kharlampovich [BK23, Corollary 28] proved that the L^2-Hall property holds for limit groups and hence that Conjecture 3.0.1 holds for hyperbolic limit groups G. We will later discuss how this extends to all limit groups and other groups enjoying similar cyclic hierarchies.

3.1 Main results

One of our results from [FM25a] establishes the L^2-Hall property for toral relatively hyperbolic graphs of free groups with cyclic edge groups, and hence Conjecture 3.0.1 for these groups (Corollary C3).

Theorem A3. *Let G be a group splitting as a finite graph of finitely generated free groups with cyclic edge groups. If G is hyperbolic relative to virtually abelian subgroups, then G satisfies the L^2-Hall property.*

We can also prove the L^2-Hall property for the class of limit groups. Perhaps the most famous characterisation of this class is the one confirmed by Sela [Sel06] and Kharlampovich–Miasnikov [KM98] in their solutions of Tarski's problem on classifying finitely generated groups with the same existential theory as a free group. We will use one of the theorems of Kharlampovich–Miasnikov [KM98] that limit groups are exactly the finitely generated subgroups of ICE groups, which we describe in Definition 3.3.1. Wilton [Wil08] used this hierarchy in his proof of the local retractions property for limit groups, and we build on his methods to prove the following result [FM25a, Theorem B].

Theorem B3 (Theorem 3.3.13). *Limit groups satisfy the L^2-Hall property.*

Theorem B3 gives an alternative proof of [BK23, Corollary 28]. One of the potential interests in revisiting the L^2-Hall property for limit groups in [FM25a] was to give an inductive argument that could work for more general finite abelian hierarchies (see Conjecture 3.4.8 for a precise statement and for further discussion in this direction).

In [FM25a], we prove the $\mathcal{D}_{K[G]}$-Hall property for the groups of Theorem A3 and Theorem B3. However, for expository reasons, we chose to present Theorem B3 for $K = \mathbb{C}$; to not loose sight of the main geometric ideas.

Antolín and Jaikin-Zapirain's proof [AJZ22] that the L^2-Hall property implies the SHNC for hyperbolic limit groups also applies to toral relatively hyperbolic graphs of free groups with cyclic edge groups and all limit groups. To see this, one needs to incorporate recent results of Minasyan [Min23] and Minasyan–Mineh [MM22] on the Wilson–Zalesskii property and double coset separability, which were not available to Antolín–Jaikin-Zapirain.

Corollary C3. *Suppose that G is a limit group or that it splits as a finite graph of free groups with cyclic edge groups that is hyperbolic relative to virtually abelian subgroups. Then G satisfies the Strengthened Hanna Neumann conjecture.*

We will not delve into the proofs of Theorem A3 and Corollary C3 in this exposition. The proof of Theorem A3 is similar to that of Theorem B3 in spirit. Nevertheless, the proof of Corollary C3 from the two previous theorems is a long and tricky argument of Antolín–Jaikin-Zapirain [AJZ22]. We refer to [FM25a, Section 6] for a brief account of this corollary. Even if we do not demonstrate Theorem A3 here, we introduce and prove one more result (Theorem D3) that is needed in this proof, a result that is concerned with extending the L^2-Hall property by finite-index overgroups.

Definition 3.1.1. We say that a group G is *subgroup separable* if for every finitely generated subgroup $H \leqslant G$ and every $g \in G \setminus H$, there exists a finite quotient $f\colon G \to Q$ such that $g \notin f(H)$.

Theorem D3 is inspired by Wise's description of subgroup separable graphs of free groups with cyclic edge groups [Wis00]. One fundamental step in this proof is ensuring that the groups of Theorem A3 (among many other ones) have virtually *clean* splittings. In plain terms, this means that these groups virtually admit a classifying space with a graph-of-spaces structure where the vertex spaces are simplicial graphs, the edge spaces are simplicial subdivisions of circles and the edge maps are combinatorial embeddings. For the purpose of obtaining subgroup separability, Wise reduces the problem to the previous scenario; given the crucial fact that virtually subgroup separable groups are, themselves, subgroup separable. However, in general, it is unclear whether the L^2-Hall property passes to finite-index overgroups (see Question 3.4.4). This virtual flexibility required in the proof of Theorem B3 is guaranteed by the following statement.

Theorem D3 (Theorem 3.2.11)**.** *Let G be a finitely generated locally indicable group with $\operatorname{cd}(G) = 2$ and $b_2^{(2)}(G) = 0$. Suppose that G has a finite-index subgroup that satisfies the L^2-Hall property. Then G satisfies the L^2-Hall property.*

Note that not all subgroup separable graphs of free groups with cyclic edge groups are L^2-Hall; for instance, $F_2 \times \mathbb{Z}$ is not L^2-Hall. In this regard, Corollary 3.2.16 shows that this is the only potential obstruction for some graphs of free groups to be L^2-Hall. We present a explicit prediction in [FM25a, Conjecture 4.11].

Remark 3.1.2. There are conjectures that relate the classes of groups of Theorems A3 and B3. Wise asked whether graphs of free groups with cyclic edge groups are virtually limit groups if and only if they do not contain $F_2 \times \mathbb{Z}$ (see [Wis18, Problem 1.5]). If this was true, then Theorem D3, together with the L^2-Hall property for limit groups (as proved in [BK23] or Theorem B3) would imply Theorem A3. This is the case for the hyperbolic one-relator group $G = \langle a, b, c \mid a^2b^2c^3 \rangle$, which is a non-limit group (as proved in Example 3.2.17) that falls under the assumptions of Theorem A3, while it happens to be virtually a limit group by [Wis18].

3.2 On the class of L^2-Hall groups

In this section we discuss several methods to produce L^2-Hall groups. We first study various combinatorial situations (in terms of graphs of groups) that provide L^2-independent subgroups (which we shall need in the proofs of Theorems A3 and B3); and finally in Theorem 3.2.11 we prove that the L^2-Hall property passes to finite-index overgroups in our setting (as anticipated in Theorem D3).

Notation 3.2.1. In this section, K always denotes a field. Unless otherwise stated, we assume that all groups are locally indicable and that their group algebras over K have Hughes-free embeddings, as introduced in Section 2.8. We recall that this is the case when $\operatorname{char} K = 0$ by Proposition 2.8.1.

The augmentation ideal corresponding to a subgroup captures a lot of structure of the subgroup and hence Proposition 3.2.2 provides a useful reformulation of the notion of $\mathcal{D}_{K[G]}$-independence.

Proposition 3.2.2 ([AJZ22, Corollary 4.3]). *Let $H \leqslant U \leqslant G$ be finitely generated subgroups and suppose that $b_2^{\mathcal{D}_{K[G]}}(G) = 0$. Then H is $\mathcal{D}_{K[G]}$-independent in U if and only if $b_1^{\mathcal{D}_{K[G]}}(I_U^G/I_H^G) = 0$.*

The following hereditary feature of the L^2-Hall property will be useful later. Recall that a ring homomorphism $R \to S$ is *(right) faithfully flat* if for every morphism $M \to N$ of (right) R-modules, $M \to N$ is injective if and only if $M \otimes_R S \to N \otimes_R S$ is injective. There is the corresponding concept of left faithful flatness, which is defined analogously. If $\mathcal{D}_1 \to \mathcal{D}_2$ is a morphism of division rings, then it is necessarily injective and (left and right) faithfully flat. Indeed, consider a morphism of $\mathcal{D}_1$-modules $M \to N$. Since $\mathcal{D}_1$ is a division ring, $\mathcal{D}_2 \cong \oplus_I \mathcal{D}_1$ for some index set I. From the commutative diagram

$$\begin{tikzcd}
M \otimes_{\mathcal{D}_1} \mathcal{D}_2 \arrow[r] \arrow[d, "\cong"] & N \otimes_{\mathcal{D}_1} \mathcal{D}_2 \arrow[d, "\cong"] \\
\bigoplus_I M \arrow[r] & \bigoplus_I N\,,
\end{tikzcd}$$

it follows at once that $M \to N$ is injective if and only if $M \otimes_{\mathcal{D}_1} \mathcal{D}_2 \to N \otimes_{\mathcal{D}_1} \mathcal{D}_2$ is.

Lemma 3.2.3. *The $\mathcal{D}_{K[G]}$-Hall property passes to subgroups.*

Proof. Let G be a $\mathcal{D}_{K[G]}$-Hall group and let $H \leqslant G$ be a subgroup. Let $U \leqslant H$ be a finitely generated subgroup. Then there is a subgroup $G_0 \leqslant G$ of finite index such that the horizontal map in the diagram

$$\begin{tikzcd}
H_1(U;\mathcal{D}_{K[G_0]}) \arrow[rr] \arrow[dr] & & H_1(G_0;\mathcal{D}_{K[G_0]}) \\
& H_1(G_0 \cap H;\mathcal{D}_{K[G_0]}) \arrow[ur] &
\end{tikzcd}$$

is injective. But then $H_1(U;\mathcal{D}_{K[G_0]}) \to H_1(G_0 \cap H;\mathcal{D}_{K[G_0]})$ is injective. Since extensions of division rings are faithfully flat, the commutative diagram

$$\begin{tikzcd}
H_1(U;\mathcal{D}_{K[G_0]}) \arrow[r] \arrow[d, "\cong"] & H_1(G_0 \cap H;\mathcal{D}_{K[G_0]}) \arrow[d, "\cong"] \\
\mathcal{D}_{K[G_0]} \underset{\mathcal{D}_{K[G_0 \cap H]}}{\otimes} H_1(U;\mathcal{D}_{K[G_0\cap H]}) \arrow[r] & \mathcal{D}_{K[G_0]} \underset{\mathcal{D}_{K[G_0 \cap H]}}{\otimes} H_1(G_0 \cap H;\mathcal{D}_{K[G_0\cap H]})
\end{tikzcd}$$

implies that $H_1(U;\mathcal{D}_{k[G_0\cap H]}) \to H_1(G_0 \cap H;\mathcal{D}_{k[G_0\cap H]})$ is injective. Hence, H has the L^2-Hall property. $\square$

We conclude by collecting the first of several instances where we understand L^2-independent subgraphs of groups.

Lemma 3.2.4. *Let $\mathcal{Y}$ be a subgraph of groups of $\mathcal{Z}$ with the same underlying graph Γ and let $Z := \pi_1(\mathcal{Z})$. Suppose that the following conditions hold:*

1. *For every vertex $v \in \mathsf{Vert}(\Gamma)$, the map $Y_v \to Z_v$ is $\mathcal{D}_{K[Z]}$-injective.*

2. *For every $e \in \mathsf{Edge}(\Gamma)$, the map $Y_e \to Z_e$ is an isomorphism.*

Then the natural injection $\pi_1(\mathcal{Y}) \to \pi_1(\mathcal{Z})$ is also $\mathcal{D}_{K[Z]}$-injective.

Proof. We view $\pi_1(\mathcal{Y})$ as a subgroup of Z via the canonical inclusion. The subgraph of groups $\mathcal{Y}$ of $\mathcal{Z}$ induces a map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{e\in\mathsf{Edge}(\Gamma^{\mathcal{Y}})} K[Y/Y_v] \arrow[r] \arrow[d] & \bigoplus_{v\in\mathsf{Vert}(\Gamma^{\mathcal{Y}})} K[Y/Y_v] \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus_{e\in\mathsf{Edge}(\Gamma^{\mathcal{Z}})} K[Z/Z_e] \arrow[r] & \bigoplus_{v\in\mathsf{Edge}(\Gamma^{\mathcal{Z}})} K[Z/Z_v] \arrow[r] & K \arrow[r] & 0\,.
\end{tikzcd}$$

Since Chiswell's Mayer–Vietoris exact sequence is induced by applying a Tor functor to the short exact sequences of the above form (see the proof of [Chi76, Theorem 2]), the long exact sequences are automatically natural and thus we obtain maps between Chiswell's exact sequences for $\mathcal{Y}$ and $\mathcal{Z}$:

$$\begin{tikzcd}
\bigoplus_{e\in\mathsf{Edge}(\Gamma^{\mathcal{Y}})} H_1(Y_e) \arrow[r] \arrow[d, "\cong"] & \bigoplus_{v\in\mathsf{Vert}(\Gamma^{\mathcal{Y}})} H_1(Y_v) \arrow[r] \arrow[d, hook] & H_1(\pi_1(\mathcal{Y})) \arrow[r] \arrow[d] & \bigoplus_{ve\in\mathsf{Edge}(\Gamma^{\mathcal{Y}})} H_0(Y_e) \arrow[d, "\cong"] \\
\bigoplus_{e\in\mathsf{Edge}(\Gamma^{\mathcal{Z}})} H_1(Z_e) \arrow[r] & \bigoplus_{v\in\mathsf{Vert}(\Gamma^{\mathcal{Z}})} H_1(Z_v) \arrow[r] & H_1(Z) \arrow[r] & \bigoplus_{e\in\mathsf{Edge}(\Gamma^{\mathcal{Z}})} H_0(Z_e)\,,
\end{tikzcd}$$

where $H_i(-)$ stands for $H_i(-;\mathcal{D}_{K[Z]})$ (for $i = 0, 1$). By the Four Lemma, the map $H_1(\pi_1(\mathcal{Y})) \to H_1(Z)$ is injective. ☐

3.2.1 Surface groups and other examples

We are already in a position to establish the $\mathcal{D}_{K[G]}$-Hall property for some classes of groups.

Example 3.2.5 (Amenable groups)**.** Let G be a group with the property that $b_1^{\mathcal{D}_{K[G]}}(H) = 0$ for all subgroups $H \leqslant G$. Then G is trivially $\mathcal{D}_{K[G]}$-Hall. Since amenable groups have vanishing L^2-Betti numbers above degree 0 and amenability passes to subgroups, this shows that amenable groups are L^2-Hall. If G is amenable and $K[G]$ is a domain (which is the case for us, since we are assuming Notation 3.2.1), then the same reasoning shows that G is $\mathcal{D}_{K[G]}$-Hall.

There are also non-amenable groups which are L^2-Hall for the reason discussed above. As an example, let T be a Tarski monster of prime order p and let $G = T \times \mathbb{Z}$. Since all the proper subgroups of T are isomorphic to $\mathbb{Z}/p$, it follows that every finitely generated subgroup H of G has $b_1^{(2)}(H) = 0$ and therefore G is L^2-Hall. However,

T is non-amenable, and therefore so is G. Note that G is not locally indicable (or even torsion-free) and therefore $\mathcal{D}_{K[G]}$ does not exist. However, it still makes sense to discuss the L^2-Hall property for this group since L^2-invariants are defined for all groups.

Example 3.2.6 (Free groups)**.** Let F be a finitely generated free group and let $H \leqslant F$ be a finitely generated subgroup. A theorem of M. Hall [Hal49], which we have already referred to several times, states that H is a free factor in some finite-index subgroup $F' \leqslant F$. By Lemma 3.2.4, H is $\mathcal{D}_{K[F]}$-independent in F', showing that F is $\mathcal{D}_{K[F]}$-Hall.

As a direct consequence of the aforementioned M.Hall's theorem, free groups are subgroup separable. Scott [Sco78, Sco85] established this property more generally for surface groups using hyperbolic geometry. We refer to [Wil07, Corollary 3.8] for a Bass-Serre theoretic argument. As proved in [LR08], Scott's argument can be upgraded to get virtual retractions. In this regard, importance of virtual retractions in obtaining subgroup separability was first appreciated in the works of Wise [Wis02], Haglund–Wise [HW08] and Long–Reid [LR08].

We now discuss how virtual retractions relate to the L^2-Hall property. Antolín–Jaikin-Zapirain proved that surface groups are L^2-Hall in [AJZ22, Theorem 4.4] using these virtual retractions combined with other algebraic ideas, such as the theory of Demushkin groups and the cohomological goodness of surface groups. We will now explain how Scott's argument can be turned into a topological proof of the L^2-Hall property of surface groups.

Proposition 3.2.7. *Torsion-free surface groups satisfy the $\mathcal{D}_{K[G]}$-Hall property.*

The real projective plane is the only closed surface with torsion in its fundamental group, which is isomorphic to $\mathbb{Z}/2$. One could easily prove the L^2-Hall property in this case. However, to even state the $\mathcal{D}_{K[G]}$-Hall property we required the existence of $\mathcal{D}_{K[G]}$, which is not guaranteed in this case.

Proof of Proposition 3.2.7. For simplicity, we will suppose that our surfaces are orientable. We say that a compact connected subsurface X of a connected surface S is *incompressible* if no component of the closure of the complement $S \smallsetminus X$ is a disc. If $\pi_1(X) \neq 1$, then X is incompressible if and only if the induced map $\pi_1(X) \to \pi_1(S)$ is injective.

Let G be the fundamental group of a closed connected surface Σ with $\chi(\Sigma) \leqslant 0$ (the case when $\chi(\Sigma) > 0$ is trivial). Let $H \leqslant G$ be a non-trivial finitely generated subgroup. Let $\Sigma' \to \Sigma$ be the covering space corresponding to H. Then Σ' is a (possibly non-compact) surface with fundamental group H. Let Σ_c be a compact core for Σ', that is, $\Sigma_c \subseteq \Sigma'$ is a compact, connected, incompressible subsurface such that the natural map $\pi_1(\Sigma_c) \to \pi_1(\Sigma')$ is an isomorphism. The existence of Σ_c is ensured by [Sco78, Lemma 1.5]. Scott also showed in [Sco78, Lemma 1.4 and Theorem 3.3] that there is a commutative diagram

$$\begin{tikzcd}
& \widehat{\Sigma} \arrow[dr] & \\
\Sigma_c \arrow[ur, hook] \arrow[rr] & & \Sigma
\end{tikzcd}$$

where $\widehat{\Sigma} \to \Sigma$ is an intermediate finite-sheeted covering into which Σ_c projects homeomorphically. Since $\pi_1(\Sigma_c) \cong H \neq 1$, the boundary $\partial\Sigma_c$ is incompressible in Σ_c. Consequently, Σ_c is an incompressible subsurface of $\widehat{\Sigma}$ and every connected component $\widehat{\Sigma}_i$ of the closure of the complement $\widehat{\Sigma} \setminus \Sigma_c$ has the property that its boundary is incompressible. It follows that $\widehat{\Sigma}$ admits a decomposition as a finite graph of spaces where the vertex spaces are $\{\widehat{\Sigma}_i, \Sigma_c\}$, various of which are glued along some of their boundary components, so the edge spaces are circles and the edge maps are π_1-injective. This produces a splitting for the fundamental group $\pi_1(\widehat{\Sigma})$ where one vertex is $\pi_1(\Sigma_c)$, the other vertices are $\pi_1(\widehat{\Sigma}_i)$ and the edge groups are infinite cyclic. By Lemma 3.2.4, the group $H = \pi_1(\Sigma_c)$ is $\mathcal{D}_{K[G]}$-independent in $\pi_1(\widehat{\Sigma})$, and therefore G is $\mathcal{D}_{K[G]}$-Hall. □

The ideas of Proposition 3.2.7, such as the construction of a compact core for a subgroup H and the reconstruction of H from cyclic splittings, motivates the strategy that we follow in Theorem B3 for limit groups. In this sense, another instructive, preliminary situation is the case when the edge groups of the splitting are trivial; that is, the case of free products.

Proposition 3.2.8. *The class of finitely generated subgroup separable $\mathcal{D}_{K[G]}$-Hall groups is closed under free products.*

Note that this generalises the proof that free groups are $\mathcal{D}_{K[G]}$-Hall.

Proof of Proposition 3.2.8. Let A and B be finitely generated subgroup separable groups with the $\mathcal{D}_{K[G]}$-Hall property. Let X_A and X_B be classifying spaces for A and B, respectively, and let X be the space obtained from X_A, X_B, and an edge $I = [0,1]$ by gluing the point $0 \in I$ to a basepoint in X_A and the point $1 \in I$ to a basepoint in X_B. Then X is a $K(A*B,1)$, and has a natural graph-of-spaces structure, where the underlying graph has two vertices and one edge.

Let $H \leqslant A*B$ be a finitely generated subgroup and let $Y \to X$ be the covering space corresponding to H. Let Z be a finite core for Y, i.e. Z is a connected subgraph of groups $Z \hookrightarrow Y$ inducing a π_1-isomorphism, whose underlying graph Γ_Z is finite, and with $Z_v = Y_v$ for all vertices v in Γ_Z. Denote the fundamental groups of the A-vertices (i.e. those vertex spaces in Z covering X_A) by X_{A_i}, where $\pi_1(X_{A_i}) = A_i \leqslant A$. Similarly, denote the B-vertices by X_{B_j}, where $\pi_1(X_{B_j}) = B_j \leqslant B$. For each i (resp. j), let $A_i' \leqslant A$ (resp. $B_j' \leqslant B$) be a finite-index subgroup containing A_i (resp. B_j) such that $A_i \hookrightarrow A_i'$ (resp. $B_j \hookrightarrow B_j'$) is $\mathcal{D}_{K[G]}$-injective (recall Definition 3.0.3).

Let $X_{A_i} \to X_{A_i'}$ be the covering map associated to $A_i \leqslant A_i'$ and let $P_i \subseteq X_{A_i}$ be the set of points that are the endpoints of edges Z. By the topological characterisation of subgroup separability of Scott (stated in Proposition 3.3.5), there is a finite-index subgroup A_i'' of A_i' with $A_i \leqslant A_i'' \leqslant A_i'$ and such that the induced covering map $X_{A_i} \to X_{A_i''}$ is injective on P_i. Note that A_i is still $\mathcal{D}_{K[G]}$-injective in A_i''. A similar discussion applies to the B-vertices, where we obtain new groups B_j'' and spaces $X_{B_j''}$ satisfying the analogous conditions.

Let $\overline{Z}$ be the following graph of spaces: it has the same underlying as Z, the vertex spaces X_{A_i} (resp. B_j) are replaced with $X_{A_i''}$ (resp. $X_{B_j''}$), and there is an edge joining the points $x \in X_{A_i''}$ and $y \in X_{B_j''}$ if and only if they are the images of points x' and y' under the coverings $X_{A_i} \to X_{A_i''}$ and $X_{B_j} \to X_{B_j''}$, respectively, and x' and y' were joined by an edge in Z. From the construction, the covering spaces of the vertices induce a map of graphs of spaces $Z \to \overline{Z}$, which is an isomorphism on underlying graphs. Then $\pi_1(Z) \hookrightarrow \pi_1(\overline{Z})$ is $\mathcal{D}_{K[G]}$-injective by Lemma 3.2.4.

The process of completing $\overline{Z}$ to a finite-sheeted cover $\widehat{Z}$ of X is standard. This is detailed, for instance, in [Wil07, Theorem 3.2]. For this, one adds various disjoint copies of the vertices X_A and X_B to the precover $\overline{Z}$ until the resulting space satisfies *Stallings' principle* (see [Wil07, Proposition 3.1]). Then, certain pairs of the hanging elevations of edge maps can be glued together along additional trivial edge spaces to produce the finite-sheeted cover $\overline{Z} \to X$. As before, the inclusion $\overline{Z} \hookrightarrow \widehat{Z}$ induces a $\mathcal{D}_{K[G]}$-injection on fundamental groups, which proves the claim. □

It is natural to ask whether subgroup separability is needed in Proposition 3.2.8, but it is unclear to the author if, for instance, the free product of finitely generated and residually finite L^2-Hall groups is L^2-Hall (see Question 3.4.3). As the following remark suggests, residual finiteness may be necessary in this kind of considerations.

Remark 3.2.9. The L^2-Hall property is not closed under free products in general. Let A be an infinite, simple, amenable group. Note that finitely generated examples of such groups exist by [JM13]. Then A has the L^2-Hall property but $A * A$ does not. To see this, let $F \leqslant A * A$ be a free subgroup of rank $d(F) > 2$. Then $b_1^{(2)}(F) > 1 = b_1^{(2)}(A * A)$ and hence F is not L^2-independent in $A * A$. Moreover, A is simple and therefore $A * A$ has no nontrivial finite-index subgroups. We conclude that $A * A$ does not have the L^2-Hall property.

We conclude with more non-examples of L^2-Hall groups, now making emphasis on classes that appear very frequently in geometric group theory.

Example 3.2.10. If G is either the fundamental group of a closed hyperbolic 3-manifold or is of the form (finitely generated nonabelian free)-by-cyclic then $b_1^{(2)}(G) = 0$. In the former case, this is a theorem of Dodziuk [Dod79]; and in latter case this follows, for instance, from [Lüc02, Theorem 1.39]. Moreover, G contains nonabelian free subgroups. When G is the fundamental group of a closed hyperbolic 3-manifold, G is word-hyperbolic and this fact is due to a folkloric ping-pong argument. If the free group of rank two F was L^2-independent in some G_1, with G_1 of finite index in G, then we would have $1 = b_1^{(2)}(F) \leqslant b_1^{(2)}(G)$. However, $b_1^{(2)}(G_1) = |G : G_1|\, b_1^{(2)}(G) = 0$ by item (4) of Proposition 2.8.2. This proves that G is not L^2-Hall. With a simple Bass–Serre theoretic argument, we can also embed F into any non-solvable generalised Baumslag–Solitar group; and no such group will be L^2-Hall either.

3.2.2 Passing to finite-index overgroups

In this subsection we prove Theorem 3.2.11. This will be crucial when establishing the L^2-Hall property for graphs of free groups with cyclic edge groups. Theorem D3 from the introduction will follow from Corollary 3.2.14 and Lemma 3.2.15.

Theorem 3.2.11. *Let G be a finitely generated and suppose that $G_1 \leqslant G$ is a finite-index subgroup and that $H \leqslant G$ is a finitely generated subgroup such that $b_2^{\mathcal{D}_{K[H]}}(H) = 0$. Then the following statements are true.*

(1) If H is $\mathcal{D}_{K[G]}$-independent in G, then $H \cap G_1$ is $\mathcal{D}_{K[G]}$-independent in G_1.

(2) If there exists a finite-index subgroup $H_0 \leqslant H$ such that H_0 is $\mathcal{D}_{K[G]}$-independent in G_1, then there exists a finite-index subgroup $G_0 \leqslant G$ containing H as a $\mathcal{D}_{K[G]}$-independent subgroup.

The first statement (1) is essentially [AJZ22, Proposition 5.2], whose argument is followed to additionally prove statement (2). We first prove the following simple lemma.

Lemma 3.2.12. *Let G be a finitely generated group and suppose that $H \leqslant T \leqslant G$ are subgroups such that $|T : H| < \infty$. If H is $\mathcal{D}_{K[G]}$-independent in G, then T is $\mathcal{D}_{K[G]}$-independent in G.*

Proof. Consider the short exact sequence of $K[G]$-modules

$$0 \to I_T^G/I_H^G \to I_G/I_H^G \to I_G/I_T^G \to 0.$$

The induced long exact sequence in $\operatorname{Tor}_\bullet^{K[G]}(\mathcal{D}_{K[G]}, -)$ contains the following sequence of $\mathcal{D}_{K[G]}$-modules:

$$\operatorname{Tor}_1^{K[G]}(\mathcal{D}_{K[G]}, I_G/I_H^G) \to \operatorname{Tor}_1^{K[G]}(\mathcal{D}_{K[G]}, I_G/I_T^G) \to \operatorname{Tor}_0^{K[G]}(\mathcal{D}_{K[G]}, I_T^G/I_H^G).$$

By Proposition 3.2.2 and the assumption that H is $\mathcal{D}_{K[G]}$-independent in G, it follows that the left-most term $\operatorname{Tor}_1^{K[G]}(\mathcal{D}_{K[G]}, I_G/I_H^G)$ is zero. Moreover, since H is finite index in T, it is not hard to see that I_T^G/I_H^G is a finite-dimensional K-vector space, so $\operatorname{Tor}_0^{K[G]}(\mathcal{D}_{K[G]}, I_T^G/I_H^G) = 0$. It follows directly from the short exact sequence above that $\operatorname{Tor}_1^{K[G]}(\mathcal{D}_{K[G]}, I_G/I_T^G) = 0$. This implies, again by Proposition 3.2.2, that T is $\mathcal{D}_{K[G]}$-independent in G. □

We are now ready to explain the proof of Theorem 3.2.11.

Proof of Theorem 3.2.11. Let $H_1 = H \cap G_1$. We begin by proving statement (1). By Proposition 3.2.2, it is enough to show that

$$\operatorname{Tor}_1^{K[G_1]}(\mathcal{D}_{K[G_1]}, I_{G_1}/I_{H_1}^{G_1}) = 0.$$

Claim 3.2.13. *As subsets of $K[G]$, we have the equality $I_{H_1}^{G_1} = I_{G_1} \cap I_H^G$.*

Proof. Consider the following commutative diagram of natural maps

$$\begin{tikzcd}
K[G_1] \arrow[r, hook, "\iota_1"] \arrow[d, "p_{H_1}^{G_1}"] & K[G] \arrow[d, "p_H^G"] \\
K[G_1/H_1] \arrow[r, hook, "\iota_2"] & K[G/H].
\end{tikzcd}$$

The horizontal arrows ι_1 and ι_2 are injective. It is clear that $I_{H_1}^{G_1} \subseteq I_{G_1} \cap I_H^G$. For the reverse inclusion, we will use the above diagram. If $x \in I_{G_1} \cap I_H^G$, then $x \in K[G_1]$ and x belongs to the kernel of $p_H^G \circ \iota_1$. By the commutativity of the diagram and the injectivity of ι_2, the element x must belong to the kernel of $p_{H_1}^{G_1}$, which equals $I_{H_1}^{G_1}$ by Lemma 2.2.3. ◇

Claim 3.2.13 implies that the natural map of $K[G_1]$-modules $I_{G_1}/I_{H_1}^{G_1} \to I_G/I_H^G$ is injective. Furthermore, since $I_{G_1}/I_{H_1}^{G_1}$ (resp. I_G/I_H^G) is the kernel of the augmentation $K[G_1/H_1] \to K$ (resp. $K[G/H] \to K$) by Lemma 2.2.3, there is an exact sequence of $K[G_1]$-modules of the form

$$0 \to I_{G_1}/I_{H_1}^{G_1} \to I_G/I_H^G \to K[G/H]/K[G_1/H_1] \to 0.$$

Let $T \subseteq G$ be a set of representatives for the double (G_1, H)-cosets in G such that $1 \in T$. Denote by M_t the $K[G_1]$-module $K[G_1/(H^t \cap G_1)]$. Then $K[G/H] \cong \oplus_{t \in T} M_t$ as $K[G_1]$-modules. Let $\mathcal{D} = \mathcal{D}_{K[G_1]}$. Notice that $\mathrm{Tor}_2^{K[G_1]}(\mathcal{D}, M_t) = 0$ for all $t \in T$. The reason is that, by Proposition 2.8.2(5), its $\mathcal{D}$-dimension equals

$$b_2^{\mathcal{D}_{K[H^t \cap G_1]}}(H^t \cap G_1) = b_2^{\mathcal{D}_{K[H]}}(H) \cdot |H^t : H^t \cap G_1| = 0. \tag{3.3}$$

Note that, to obtain (3.3), we have also used the fact that $H^t \cap G_1$ is finite index in $H^t \cong H$, as well as the multiplicativity of $\mathcal{D}_{K[G]}$-Betti numbers (Proposition 2.8.2(4)). By the additivity of the $\mathcal{D}$-dimension function, it follows from Proposition 2.8.2(5) and Eq. (3.3) that the $K[G_1]$-module $N = K[G/H]/K[G_1/H_1] \cong \oplus_{t \in T \smallsetminus \{1\}} M_t$ has $\mathrm{Tor}_2^{K[G_1]}(\mathcal{D}, N) = 0$.

The long exact sequence in Tor gives us an exact sequence of $\mathcal{D}$-modules of the form

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{Tor}_2^{K[G_1]}(\mathcal{D}, N) \arrow[dll, rounded corners] \\
\mathrm{Tor}_1^{K[G_1]}(\mathcal{D}, I_{G_1}/I_{H_1}^{G_1}) \arrow[r] & \mathrm{Tor}_1^{K[G_1]}(\mathcal{D}, I_G/I_H^G) \arrow[r] & \mathrm{Tor}_1^{K[G_1]}(\mathcal{D}, N).
\end{tikzcd} \tag{3.4}$$

We have already proved that $\mathrm{Tor}_2^{K[G_1]}(\mathcal{D}, N) = 0$. So statement (1) will follow from diagram (3.4) if we prove that $\mathrm{Tor}_1^{K[G_1]}(\mathcal{D}, I_G/I_H^G) = 0$. We know from Proposition 2.8.2(6) that

$$\mathrm{Tor}_1^{K[G_1]}(\mathcal{D}, I_G/I_H^G) \cong \mathrm{Tor}_1^{K[G]}(\mathcal{D}_{K[G]}, I_G/I_H^G).$$

Furthermore, the right-hand side vanishes by Proposition 3.2.2 and the assumption that H is $\mathcal{D}_{K[G]}$-independent in G. This completes the proof of the first statement (1).

We now prove (2). The subgroups $H_0 \leqslant H \cap G_1 \leqslant G_1$ have the property that $|H \cap G_1 : H_0| < \infty$ and that H_0 is $\mathcal{D}_{K[G]}$-independent in G_1. By Lemma 3.2.12, $H \cap G_1$ is $\mathcal{D}_{K[G]}$-independent in G_1. Let $G_2 \trianglelefteqslant G_1$ be a normal subgroup of finite index; since $b_2^{\mathcal{D}_{K[H]}}(H) = 0$, it follows from part (1) that $H \cap G_2$ is $\mathcal{D}_{K[G]}$-independent in G_2. Thus, by replacing G_1 by G_2, we may assume that G_1 is normal in G and that $H \cap G_1$ is $\mathcal{D}_{K[G]}$-independent in G_1.

We claim that H is $\mathcal{D}_{K[G]}$-independent in $G_0 = \langle G_1, H \rangle = G_1 \cdot H$. For this, we first observe that $T = \{1\}$ is a set of representatives of the double (G_1, H)-cosets in G_0. So the argument given in (1) shows that the canonical map

$$I_{G_1}/I^{G_1}_{H \cap G_1} \xrightarrow{\cong} I_{G_0}/I^{G_0}_{H} \tag{3.5}$$

is an isomorphism of $K[G_1]$-modules. Using that $H \cap G_1$ is $\mathcal{D}_{K[G]}$-independent in G_1, we can argue as before to deduce from (3.5), Proposition 2.8.2(6) and Proposition 3.2.2 that

$$\operatorname{Tor}_1^{K[G_0]}(\mathcal{D}_{K[G_0]}, I_{G_0}/I^{G_0}_H) \cong \operatorname{Tor}_1^{K[G_1]}(\mathcal{D}_{K[G_1]}, I_{G_1}/I^{G_1}_{H \cap G_1}) = 0.$$

Thus, again by Proposition 3.2.2, H is $\mathcal{D}_{K[G]}$-independent in G_0 □

The following is a direct consequence of Theorem 3.2.11.

Corollary 3.2.14. *Let G be a finitely generated group and suppose that all finitely generated subgroups $H \leqslant G$ have the property that $b_2^{\mathcal{D}_{K[H]}}(H) = 0$ and that there exist finite-index subgroups $H_1 \leqslant H$ and $G_1 \leqslant G$ such that H_1 is $\mathcal{D}_{K[G]}$-independent in G_1. Then G is $\mathcal{D}_{K[G]}$-Hall.*

The vanishing of $b_2^{\mathcal{D}_{K[H]}}(H) = 0$ imposed in Corollary 3.2.14 is a local condition on G that sometimes can be better condensed, as in the following lemma.

Lemma 3.2.15. *Let G be a group of cohomological dimension $\operatorname{cd}_K(G) = n$ with $b_n^{K[G]}(G) = 0$. Then $b_n^{K[H]}(H) = 0$ for every subgroup $H \leqslant G$.*

Proof. Note that the natural map

$$\mathcal{D}_{K[H]} \otimes_{K[H]} F \to \mathcal{D}_{K[G]} \otimes_{K[G]} F \tag{3.6}$$

is injective, where F is a free left $K[G]$-module. To see this, it is enough to prove the claim when $F = K[G]$. Let T be a right transversal for H in G. The Hughes-freeness of $\mathcal{D}_{K[G]}$ implies that the map $\oplus_{t \in T} \mathcal{D}_{K[H]} \cdot t \to \mathcal{D}_{K[G]}$ induced by the inclusions $\mathcal{D}_{K[H]} \cdot t \hookrightarrow \mathcal{D}_{K[G]}$ is injective [Grä20, Corollary 8.3]. The map of (3.6) when $F = K[G]$ equals the composition

$$\mathcal{D}_{K[H]} \otimes_{K[H]} K[G] \xrightarrow{\cong} \mathcal{D}_{K[H]} \otimes_{K[H]} \left(\bigoplus_{t \in T} K[H] \cdot t \right) \xrightarrow{\cong} \bigoplus_{t \in T} \mathcal{D}_{K[H]} \cdot t \hookrightarrow \mathcal{D}_{K[G]}$$

and is therefore injective, as desired.

The claim now follows easily. Let $0 \to F_n \to \cdots \to F_0 \to K \to 0$ be a free resolution of the trivial $K[G]$-module K. This resolution exists because G has a classifying space of dimension at most n (we do not claim the modules F_i to be finitely generated). If $b_n^{\mathcal{D}_{K[H]}}(H) \neq 0$, then there is a non-trivial element z in the kernel of $\mathcal{D}_{K[H]} \otimes_{K[H]} F_n \to \mathcal{D}_{K[H]} \otimes_{K[H]} F_{n-1}$. Then z is also a nonzero element of the kernel of $\mathcal{D}_{K[G]} \otimes_{K[G]} F_n \to \mathcal{D}_{K[G]} \otimes_{K[G]} F_{n-1}$ and therefore $b_n^{\mathcal{D}_{K[G]}}(G) \neq 0$. ☐

While we only have a conjectural characterisation of which general graphs of free groups with cyclic edge are L^2-Hall, the case of an amalgam is entirely understood.

Corollary 3.2.16. *If G is an amalgam of free groups over a cyclic subgroup, then G has the L^2-Hall property if and only if it does not contain a subgroup isomorphic to $F_2 \times \mathbb{Z}$.*

Proof. First note that $F_2 \times \mathbb{Z}$ is not L^2-Hall and so it cannot be a subgroup of an L^2-Hall group by Lemma 3.2.3. Conversely, assume that G does not contain a copy of $F_2 \times \mathbb{Z}$. Then [Wis18, Theorem 1.2] implies that G has a finite-index subgroup that is a limit group. Limit groups are L^2-Hall by [BK23] (or, alternatively, by Theorem B3) and have vanishing second L^2-Betti number [BK17a]. Thus, G is L^2-Hall by Corollary 3.2.14. ☐

We conclude this section with a detailed example that justifies all our effort put into showing that the L^2-Hall property can sometimes pass to finite-index overgroups. We give examples of one-relator groups that do not immediately admit a sufficiently nice cyclic splitting, although they do virtually.

Example 3.2.17. Given integers $n, m, q \geqslant 2$, the family of one-relator groups

$$G_{n,m,q} = \langle a, b, c \mid a^n b^m c^q \rangle$$

includes one of the first families of parafree groups given by Baumslag [Bau67]. They are hyperbolic by the Bestvina–Feighn combination theorem [BF92], virtually special [HW10] and virtually limit groups [Wis18]. Nevertheless, they are neither a limit nor a special group on the nose. For a more careful treatment of this kind of argument we refer the reader to the author's master thesis [Mor21, Chapters 4-5], although we shall briefly prove here the latter assertion.

(a) It is a consequence of the work of Sela [Sel01, Sel06] that a limit group is also a fully residually free group (and conversely). In particular, any non-abelian limit group surjects F_2. Suppose that any of the groups $G = G_{n,m,q}$

defined above is a limit group, and consider a surjection $\varphi\colon G \to F$, where F is the free group of rank 2. Let p denote a prime $p > \max\{|m|, |n|, |q|\}$. Since G has free pro-p completion of rank 2, the corresponding surjection on pro-p completions $\varphi_{\widehat{p}}\colon G_{\widehat{p}} \to F_{\widehat{p}}$ would be an isomorphism, by the Hopfian property satisfied by all (topologically) finitely generated profinite groups. Now, given that G is residually-p [Bau67], the natural map $G \to G_{\widehat{p}}$ is injective, and so is the map φ. This would prove that φ gives an isomorphism $G \cong F$. Nevertheless, there are many ways to prove that G is not free. Probably the most immediate and elementary argument to certify this claim goes by checking that $G_{n,m,q}$ has virtual second Betti number. To ease the notation, we focus on $G = G_{2,2,3} = \langle a, b, c \,|\, a^2b^2c^3\rangle$. The map $f\colon G \to \mathbb{Z}/3$ given by $f(a) = f(b) = 0$ and $f(c) = 1 + 3\mathbb{Z}$ has a kernel K of index 3 with a 4-generator 2-relator aspherical presentation of the form

$$K \cong \langle a_0, b_0, a_1, b_1, a_2, b_2 \,|\, a_0^2b_0^2a_1^2b_1^2, a_0^2b_0^2a_2^2b_2^2\rangle.$$

From this, we compute $H_2(K;\mathbb{F}_2) \cong (\mathbb{Z}/2)^2$ and conclude that K is not free. Alternatively, one can in fact prove using Whitehead's algorithm [Whi36] that G is freely indecomposable. As a curiosity, the latter stronger conclusion (on free indecomposability) can also ensured by our argument above. Since G has two-generated p-abelianisation, if G splits as a free product of two non-trivial groups $G = A * B$, then both have cyclic p-abelianisation, implying that their pro-p completions satisfy $A_{\widehat{p}} \cong B_{\widehat{p}} \cong \mathbb{Z}_p$ are abelian. This way, it would follow that $G \cong F_2$. In conclusion, once we know that G is a non-free parafree group, it follows that G is, in fact, freely indecomposable.

(b) Antolín–Minasyan [AM15, Corollary 1.6] showed, among other things, that RAAGs satisfy a strong Tits alternative; namely, a finitely generated subgroup H of a RAAG is either abelian or admits a surjection to a non-abelian free group. Hence, if G was special in the sense of Haglund–Wise [HW08], $G_{n,m,p}$ would be a non-abelian subgroup of a RAAG and so it would surject F_2, which we have just proved to be impossible in part (a).

3.3 The L^2-Hall property for limit groups

Wilton [Wil08] proved that limit groups have the local retractions property, and hence that they are subgroup separable, using Kharlampovich and Miasnikov's [KM98] characterisation of limit groups in terms of *ICE groups* (see Definition 3.3.1 below). More precisely, limit groups are exactly the finitely generated groups that arise as subgroups of ICE groups (see Theorem 3.3.3). Since the local retractions property passes to subgroups, Wilton only needs to deal with ICE groups in [Wil08]. Analogously, we certified that the L^2-Hall property passes to subgroups in Lemma 3.2.3, so it also sufficient to deal with ICE groups in order to prove Theorem B3. Our argument is different from that of [BK23] and we expect it to be flexible enough to include more general finite abelian hierarchies of relatively hyperbolic groups as in Conjecture 3.4.8.

Definition 3.3.1. Let H be a group and let $Z \leqslant H$ be the centraliser of an element. For an integer $n \geqslant 1$, the group $H *_Z (Z \times \mathbb{Z}^n)$ is an *extension of H by a centraliser*. A group is an *ICE group* (standing for "iterated centraliser extension") if it can be obtained from a finitely generated free group by a finite sequence of extensions by centralisers.

In particular, an ICE group G admits a nice finite hierarchy, which endows it with a simple and explicit $K(G,1)$. We call the length of this hierarchy (from Definition 3.3.1) the *complexity* of the G. If this complexity is zero, the group G is free, and a classifying space X can be taken to be a bouquet of circles. Otherwise, we can write $G = H *_Z (Z \times \mathbb{Z}^n)$ for simpler (that is, lower complexity) ICE group H. It is not hard to show that we may assume at each step that Z is infinite cyclic (see [Wil08, Remark 1.14]). In this case, we take X to be the graph of Y and T^{n+1} with edge group S^1, where Y is the classifying space of H constructed by induction, and S^1 maps to a loop representing the centralised element in H and to a coordinate circle in T^{n+1}. The spaces obtained in this way will be called *ICE spaces*. For the purpose of proving statements about ICE groups (or ICE spaces) by induction, it is important to make the following distinction between their elements (resp. between their loops).

Definition 3.3.2. Given an ICE group G and a fixed splitting $G = H *_Z (Z \times \mathbb{Z}^n)$ for some ICE group H (as in Definition 3.3.1), we will say that an element $g \in G$ is *elliptic* (resp. *hyperbolic*) if it acts as an elliptic isometry (resp. hyperbolic isometry) of the Bass–Serre tree associated to such splitting. Similarly, we will say that a loop of the ICE space X, the $K(G,1)$ defined after Definition 3.3.1, is elliptic or hyperbolic depending on whether it represents an elliptic or hyperbolic element of $\pi_1(X)$, respectively.

In more concrete terms, given an ICE group H and an element g in the ICE group $G = H *_Z (Z \times \mathbb{Z}^n)$; we have that g is elliptic if it is conjugated into one of the vertex groups H or $Z \times \mathbb{Z}^n$. This dichotomy can be used to compute the centraliser of g. If

g is hyperbolic, then $C_G(g) \cong \mathbb{Z}$. However, if g is elliptic, we distinguish two cases: either $g = tht^{-1}$ for some $t \in G$ and $h \in H$, in which case $C_G(g) = tC_H(h)t^{-1}$; or $g = tat^{-1}$, for some $t \in G$ and $a \in Z \times \mathbb{Z}^n$, in which case $C_G(g) = t(Z \times \mathbb{Z}^n)t^{-1}$. This computation has the consequence that non-cyclic centralisers are maximal abelian and elliptic subgroups, which closely relates to the fact that these splittings are 2-acylindrical (see [Sel01, Lemmas 2.1 and 2.3] and [Wil08, Lemma 1.12] for precise statements in this direction, as well as their converse).

We refer the reader to [Wil08, Section 1.6] for a concise survey of this material. We emphasise the following important theorem of Kharlampovich and Miasnikov [KM98] which was alluded to above, because it gives the powerful characterisation of limit groups that we shall work with.

Theorem 3.3.3. *A finitely generated group G is a limit group if and only if it is a subgroup of an ICE group.*

Definition 3.3.4. A collection of elements $\mathcal{L}$ in a group G is *independent* if g commutes with no conjugate of h for all pairs of distinct elements $g, h \in \mathcal{L}$. Given a CW-complex X and a collection $\mathcal{L}$ of essential immersed loops in the 1-skeleton of X (that is, they are locally embedded); we say that $\mathcal{L}$ is independent if it represents in $\pi_1(X)$ an independent collection in the previous sense.

3.3.1 The L^2-tame property

We proved in Proposition 3.2.8 that the L^2-Hall property is closed under free products, provided that the factors are subgroup separable. It is the purpose of this subsection to extend this to some cyclic splittings (including ICE groups). The standard argument from Proposition 3.2.8 illustrates how additional separability properties seem to be required on the vertex groups. In particular, residual finiteness seems very weak in this context (see Question 3.4.3). Hence, we will only focus on how to obtain L^2-Hall subgroup separable groups, since this is now a more geometrically meaningful class of groups to play with. To have a better appreciation of how geometry comes into the picture, we state Scott's topological reformulation of subgroup separability [Sco78] in the form of [Wil08, Lemma 1.3].

Proposition 3.3.5. *Let X be a connected finite CW-complex and let $G = \pi_1(X)$. The following properties are equivalent:*

(i) The group G is subgroup separable.

(ii) *For every covering $X' \to X$, with finitely generated $\pi_1(X')$, and every compact subset $\Delta \subseteq X'$, there exists an intermediate finite cover $X' \to \widehat{X} \to X$ such that Δ projects homeomorphically into $\widehat{X}$ via the map $X' \to \widehat{X}$.*

We already know that graphs of subgroup separable groups may not be subgroup separable. In fact, non-balanced Baumslag–Solitar groups provide counterexamples to this claim. Nevertheless, there are natural conditions to impose to a splitting to ensure its separability.

Remark 3.3.6 (Subgroup separability from splittings)*.* Suppose that we are given a graph of CW-complexes X such that every vertex group $\pi_1(X_v)$ is subgroup separable. We informally discuss how Proposition 3.3.5 could be potentially used to show that $\pi_1(X)$ is subgroup separable. Given a finitely generated subgroup $H \leqslant \pi_1(X)$, we consider the corresponding cover $Y_0 \to X$. Viewing Y_0 as a graph of spaces, we can take $Y \subseteq Y_0$ to be a connected subgraph of spaces (now with finite underlying graph) such that $\pi_1(Y) \to \pi_1(Y_0)$ is surjective. By Lemma 2.1.5, $\pi_1(Y) \to \pi_1(Y_0)$ is an isomorphism. We want to complete the precover $Y \to X$ to a finite-sheeted cover $\widehat{Y} \to X$. For this, the most difficult step is to build an intermediate finite-sheeted precovering $\overline{Y} \to X$. To go from such $\overline{Y}$ to a finite cover one could similarly as in Proposition 3.2.8, hence the main difficulty lies in building the intermediate finite-sheeted precover.

Note that every vertex space Y_v of Y is a cover of some vertex X_w. Given this, one obvious attempt to build $\overline{Y}$ would be to replace each vertex Y_v with a finite-sheeted cover $\overline{Y_v} \to X_w$. Nevertheless, for this to lead to a precover of X, we need to be able to glue all the edge spaces incident to $\overline{Y_v}$. It could happen that the elevations of an edge space X_e to two incident copies of $\overline{Y_v}$ are not isomorphic. In conclusion, one requires a stronger version of item (ii) of Proposition 3.3.5 for each vertex space X_v, relative to the incident edge spaces, to ensure the subgroup separability of X from this splitting. This is noted by Wilton in [Wil08, Section 3], and to fix this he strengthens the induction hypothesis; introducing the property of tameness [Wil08, Definition 3.1].

During Remark 3.3.6 we explained the problems that arise when trying to prove, geometrically, that a graph of subgroup separable groups is subgroup separable. We now delve into the additional problems that appear when we also take into account the L^2-homology. Firstly, to identify the L^2-homology of the spaces and their fundamental groups, we assume that all vertex and edge spaces are aspherical CW-complexes.

Secondly, following Scott's philosophy (Proposition 3.3.5) the L^2-Hall property concerns the ability to complete precovers of X to finite-sheeted covers preserving the L^2-homology in the process. In this regard, even if we could build the covers $\overline{Y}$ described in Remark 3.3.6, we would still have to ensure that the π_1-injective map $Y \to \overline{Y}$ is also L^2-injective. This requires, again, a stronger assumption on the edge spaces, as the following example shows.

Example 3.3.7. Consider $G = \pi_1(\Sigma_2) = \langle a, b, c, d \mid [a, b] = [c, d]\rangle$, which splits as $F(a, b) *_{[a,b]=[c,d]} F(c, d)$. We consider the L^2-independent subgroups $H \leqslant F(a, b)$ and $K \leqslant F(c, d)$ given by $H = F(a^2, b^2)$ and $K = F(c^2, d^2)$. It is clear that the induced map $H * K \to G$ is injective, although it is not L^2-injective for the obvious reason that $b_1^{(2)}(H * K) = 3 > 2 = b_1^{(2)}(G)$.

Example 3.3.7 illustrates that subgraphs of groups that are L^2-injective on vertex groups need not be L^2-injective overall. This becomes a problem when using the strategy of Remark 3.3.6 to prove the L^2-Hall property, as one needs some control on the non-trivial L^2-classes that have non-trivial support on multiple vertex spaces. Wilton's notion of tameness [Wil08, Definition 3.1] and Example 3.3.7 motivates the following notion of L^2-tameness that allows us to inductively have such control.

Definition 3.3.8. Consider a complex X, a covering $X' \to X$ and a finite (possibly empty) collection of independent (Definition 3.3.4), essential loops $\mathcal{L} = \{\delta_i \colon C_i \to X\}$. The cover X' is *L^2-tame over* $\mathcal{L}$ if the following holds. Let $\Delta \subseteq X'$ be a finite subcomplex and let $\mathcal{L}' = \{\delta'_j \colon C'_j \to X'\}$ be a finite collection of pairwise non-isomorphic infinite-degree elevations, where each δ'_j is an elevation of some $\delta_i \in \mathcal{L}$. Then for all sufficiently large positive integers d, there exists an intermediate finite-sheeted covering $X' \to \widehat{X} \to X$ that satisfies the following.

1. Every δ'_j descends to an elevation $\widehat{\delta}_j \colon \widehat{C_j} \to \widehat{X}$ of degree d.
2. The elevations $\widehat{\delta}_j$ are pairwise non-isomorphic.
3. The subcomplex Δ injects into $\widehat{X}$.
4. The natural map $\pi_1(X') \to \pi_1(\widehat{X})$ extends to an injective and L^2-injective map

$$\pi_1(X') * \left(\coprod_{\mathcal{L}'} \mathbb{Z}\right) \to \pi_1(\widehat{X}),$$

   defined as follows: if the copy of $\mathbb{Z}$ is labelled by $\delta'_j \in \mathcal{L}'$ then the element $1 \in \mathbb{Z}$ is mapped to the class of the image of $\widehat{\delta'_j}$ in $\pi_1(\widehat{X})$.

The subscripts i and j in Definition 3.3.8 are different, indicating that there may be several elevations δ'_j in $\mathcal{L}'$ for each δ_i in $\mathcal{L}$.

Remark 3.3.9. The L^2-tameness of all coverings $X' \to X$ with finitely generated $\pi_1(X')$ and empty $\mathcal{L}$ implies the L^2-Hall property for $\pi_1(X)$.

As anticipated, the idea is that the strengthened version described in Definition 3.3.8 (with additional prescribed data relative to $\mathcal{L}$) admits a proof by induction and avoids bad embeddings as the one described in Example 3.3.7.

3.3.2 Achieving L^2-independence from cyclic splittings

This subsection is merely devoted to prove a technical, yet elementary, result (Proposition 3.3.12) that is used in proving the L^2-tame property (Definition 3.3.8) by induction. This part may be skipped in a first reading. Contrary to what is possibly suggested by the structure of our exposition, it is the specifics of Proposition 3.3.12 which did really shape, a posteriori, the definition of L^2-tameness. Its main role is to fix the lack of L^2-independence presented in Example 3.3.7. We believe that the following example will help to motivate the statement of Proposition 3.3.12.

Example 3.3.10 (L^2-independent subgroups of cyclic amalgamated products)**.** Let F_1 and F_2 be two finitely generated free groups. Let $u_i \in F_i$ be non-trivial elements for all $i \in \{1, 2\}$. We consider the corresponding amalgamated product $G = F_1 *_{u_1=u_2} F_2$. Let $H_1 \leqslant F_1$ and $H_2 \leqslant F_2$ be two finitely generated free subgroups. It is an standard Bass–Serre theoretic argument (implicit to Lemma 2.1.5) that, when viewing H_1 and H_2 as subgroups of G, the natural map $H_1 *_{H_1 \cap H_2} H_2 \to G$ is injective. Moreover, if $u_1 \in H_1$ and $u_2 \in H_2$, it is a simple Mayer–Vietoris argument (explained in Proposition 3.2.2) that the map $H_1 *_{H_1 \cap H_2} H_2 \to G$ is also L^2-injective. As we saw in Example 3.3.7, this L^2-injectivity is not always ensured. Nevertheless, there is a natural condition to impose to H_1 and H_2 to ensure L^2-independence of $\langle H_1, H_2 \rangle$ while allowing the possibility that $H_1 \cap \langle u_1 \rangle = \{1\}$ and $H_2 \cap \langle u_2 \rangle = \{1\}$. Suppose that the natural maps $H_1 * \langle u_1 \rangle \to F_1$ and $H_2 * \langle u_2 \rangle$ are injective and L^2-injective. Then the natural map $H_1 * H_2 * \langle u_1 \rangle \to G$ is injective and L^2-injective. The reason is that the previous map factors through an isomorphism

$$H_1 * H_2 * \langle u_1 \rangle \to (H_1 * \langle u_1 \rangle) \underset{u_1=u_2}{*} (H_2 * \langle u_2 \rangle)$$

and through the following injective and L^2-injective map (by Lemma 3.2.4):

$$(H_1 * \langle u_1 \rangle) \underset{u_1=u_2}{*} (H_2 * \langle u_2 \rangle) \to F_1 \underset{u_1=u_2}{*} F_2 = G.$$

We adapt Example 3.3.10 to HNN extensions.

Example 3.3.11 (L^2-independent subgroups of cyclic HNN extensions)**.** Let F be a finitely generated free group and let $u_1, u_2 \in F$ be non-trivial elements. We consider the injective map $\theta\colon \langle u_1 \rangle \to F$ described by $\theta(u_1) = u_2$ and its corresponding HNN extension $G = F_{u_1,\theta} = \langle F, t \,|\, t u_1 t^{-1} = u_2 \rangle$. Suppose that the natural map $H * \langle u_1 \rangle * \langle u_2 \rangle \to F$ is injective and L^2-injective. Then we claim that the natural map $H * \langle u_1 \rangle * \langle t \rangle \to G$ is injective and L^2-injective. In fact, the latter map factors through an isomorphism

$$H * \langle u_1 \rangle * \langle t \rangle \to \left(H * \langle u_1 \rangle * \langle u_2 \rangle\right)_{u_1,\theta}$$

and through the following injective and L^2-injective map (again, by Lemma 3.2.4):

$$\left(H * \langle u_1 \rangle * \langle u_2 \rangle\right)_{u_1,\theta} \to F_{u_1,\theta} = G.$$

Proposition 3.3.12 generalises the L^2-independent examples of Example 3.3.10 and Example 3.3.11 to the graphs of groups that appear in the proof of Theorem 3.3.13. We also consider Proposition 3.3.12 to be of potential interest for proving that relatively hyperbolic groups with a finite abelian hierarchy have the L^2-Hall property (see Conjecture 3.4.8 for a more precise statement).

Proposition 3.3.12. *Let $\mathcal{W}$ be a subgraph of groups of $\mathcal{Z}$ (in the sense of Definition 2.1.4) that have the same underlying graph Γ, where all the edge groups of $\mathcal{Z}$ are infinite cyclic. Let $G = \pi_1(\mathcal{Z})$ and suppose that there is a bipartite structure $\mathsf{Vert}(\Gamma) = \mathsf{Vert}_\mathsf{o} \sqcup \mathsf{Vert}_\mathsf{t}$ of Γ so that no two different edges of $\mathsf{Edge}(\Gamma)$ have the same endpoints and so that Z_v is free abelian for all $v \in \mathsf{Vert}_\mathsf{t}$. We moreover assume that the orientation on Γ is such that $\mathsf{o}(e) \in \mathsf{Vert}_\mathsf{o}$ and $\mathsf{t}(e) \in \mathsf{Vert}_\mathsf{t}$ for all $e \in \mathsf{Edge}(\Gamma)$. We denote by z_e a generator of the infinite cyclic group Z_e and by T a spanning tree of Γ. Consider the presentation of G relative to T (as described in Definition 2.1.3) and, for every $e \in \mathsf{Edge}(\Gamma) \smallsetminus \mathsf{Edge}(T)$, denote by t_e the formal letter associated to e. For all $v \in \mathsf{Vert}_\mathsf{o}$, we suppose that there are finite subsets $\mathcal{L}_v^{(0)} \subseteq \mathcal{L}_v \subseteq Z_v$ that satisfy the following two properties:*

1. *We have the inclusions $\mathcal{L}_v \setminus \mathcal{L}_v^{(0)} \subseteq \bigcup_{\mathsf{o}(e)=v} \varphi_{\mathsf{o},e}(z_e) \subseteq W_v \cup \left(\mathcal{L}_v \setminus \mathcal{L}_v^{(0)}\right)$.*
2. *For all $v \in \mathsf{Vert}_\mathsf{o}$, the natural map*

$$W_v * \left(\coprod_{\mathcal{L}_v} \mathbb{Z}\right) \to Z_v$$

   is injective and $\mathcal{D}_{K[G]}$-injective.

We denote by E_T an intermediate subset $\mathsf{Edge}(T) \subseteq E_T \subseteq \mathsf{Edge}(\Gamma)$ such that $\varphi_{e,\mathsf{o}}(z_e) \in \mathcal{L}_{\mathsf{o}(e)} \setminus \mathcal{L}^{(0)}_{\mathsf{o}(e)}$ for all $e \in \mathsf{Edge}(\Gamma) \setminus E_T$. If we name $\mathcal{L}^{(0)} = \bigcup_{v \in \mathsf{Vert}(\Gamma)} \mathcal{L}^{(0)}_v$ and $\mathcal{L}^{(t)} = \{t_e : e \in \mathsf{Edge}(\Gamma) \setminus E_T\}$, then the natural map

$$\pi_1(\mathcal{W}) * \left(\coprod_{\mathcal{L}^{(0)} \bigcup \mathcal{L}^{(t)}} \mathbb{Z} \right) \to \pi_1(\mathcal{Z}) \tag{3.7}$$

is injective and $\mathcal{D}_{K[G]}$-injective (in the sense of Definition 3.0.3).

We make some preliminary comments on the proof of Proposition 3.3.12:

1. We will consider several intermediate maps of graphs of groups $\mathcal{W} \to \mathcal{W}^{(1)} \to \mathcal{W}^{(2)} \to \mathcal{W}^{(3)} \to \mathcal{W}^{(4)} \to \mathcal{Z}$, which will be π_1-injective and $\mathcal{D}_{K[G]}$-injective, to obtain the desired conclusion.
2. Recall from Lemma 2.1.5 that, given a subgraph of groups $\mathcal{H}$ of $\mathcal{G}$, the canonical map $\pi_1(\mathcal{H}) \to \pi_1(\mathcal{G})$ injective.
3. Not all of the $\mathcal{W}^{(i)}$ will be subgraphs of groups of $\mathcal{Z}$ in the strict sense of Definition 2.1.4. Nevertheless, the corresponding maps at each vertex group $W^{(i)}_v \to Z_v$ and edge group $W^{(i)}_e \to Z_e$ will be injective, which will be enough to understand their maps in L^2-homology (recall Example 3.3.10).
4. In order to define the graphs $\mathcal{W}^{(i)}$, we will only specify their vertex and edge groups; as the corresponding edge maps will be assumed to be the restrictions of the edge maps of $\mathcal{Z}$.

Proof of Proposition 3.3.12. We define a graph of groups $\mathcal{W}^{(1)}$ as follows. For every $v \in \mathsf{Vert}_\mathsf{t}$, we set $W^{(1)}_v = Z_v$. For every $v \in \mathsf{Vert}_\mathsf{o}$ and $e \in \mathsf{Edge}(\Gamma)$, we leave $W^{(1)}_v = W_v$ and $W^{(1)}_e = W_e$ untouched. The graph of groups $\mathcal{W}^{(1)}$ is not quite a subgraph of groups of $\mathcal{Z}$ but $\mathcal{W}$ is a subgraph of groups of $\mathcal{W}^{(1)}$. In addition, the canonical injective map

$$\pi_1(\mathcal{W}) \to \pi_1(\mathcal{W}^{(1)}) \tag{3.8}$$

is $\mathcal{D}_{K[G]}$-injective by Lemma 3.2.4.

We split $\mathcal{L}_v \setminus \mathcal{L}^{(0)}_v$ as a disjoint union of $\mathcal{L}^{(1)}_v$ and $\mathcal{L}^{(2)}_v$, where $\mathcal{L}^{(1)}_v$ consists exactly of the elements $\varphi_{\mathsf{o},e}(z_e) \in \mathcal{L}_v$ such that $e \in E_T$. Consider another intermediate graph of groups $\mathcal{W}^{(1)} \to \mathcal{W}^{(2)} \to \mathcal{Z}$ defined as follows:

- $W^{(2)}_v = W^{(1)}_v * \left(\coprod_{\mathcal{L}^{(1)}_v} \mathbb{Z}\right)$ for $v \in \mathsf{Vert}_\mathsf{o}$;

- $W_v^{(2)} = W_v^{(1)}$ for $v \in \mathsf{Vert}_{\mathsf{t}}$;
- $W_e^{(2)} = Z_e$ for $e \in E_T$;
- $W_e^{(2)} = W_e^{(1)}$ for $e \in \mathsf{Edge}(\Gamma) \smallsetminus E_T$.

Letting $E^{(1)} = \{t_e : e \in E_T \smallsetminus E(T), \varphi_{e,\mathsf{o}}(z_e) \in \mathcal{L}_v^{(1)}\}$, the canonical map

$$\pi_1(\mathcal{W}^{(1)}) * \left(\coprod_{E^{(1)}} \mathbb{Z} \right) \to \pi_1(\mathcal{W}^{(2)}) \tag{3.9}$$

is an isomorphism, so $\pi_1(\mathcal{W}^{(1)}) \to \pi_1(\mathcal{W}^{(2)})$ is $\mathcal{D}_{K[G]}$-injective.

We move on to define $\mathcal{W}^{(2)} \to \mathcal{W}^{(3)} \to \mathcal{Z}$ as follows:

- $W_v^{(3)} = W_v^{(2)} * \left(\coprod_{\mathcal{L}_v^{(0)}} \mathbb{Z}\right)$ for $v \in \mathsf{Vert}_{\mathsf{o}}$;
- $W_v^{(3)} = W_v^{(2)}$ for $v \in \mathsf{Vert}_{\mathsf{t}}$;
- $W_e^{(3)} = W_v^{(2)}$ for $e \in E_T$;
- $W_e^{(3)} = W_v^{(2)}$ for $e \in \mathsf{Edge}(\Gamma) \smallsetminus E_T$.

Similarly as we argued in Example 3.3.10, we see from the presentation of $\pi_1(\mathcal{W}^{(3)})$ that the canonical map

$$\pi_1(\mathcal{W}^{(2)}) * \left(\coprod_{\mathcal{L}^{(0)}} \mathbb{Z} \right) \to \pi_1(\mathcal{W}^{(3)}) \tag{3.10}$$

is an isomorphism. Finally, we define $\mathcal{W}^{(3)} \to \mathcal{W}^{(4)} \to \mathcal{Z}$ as follows:

- $W_v^{(4)} = W_v^{(3)} * \left(\coprod_{\mathcal{L}_v^{(2)}} \mathbb{Z}\right)$ for $v \in \mathsf{Vert}_{\mathsf{o}}$;
- $W_v^{(4)} = W_v^{(3)}$ for $v \in \mathsf{Vert}_{\mathsf{t}}$;
- $W_e^{(4)} = Z_e$ for $e \in \mathsf{Edge}(\Gamma)$.

As we argued in Example 3.3.11, here we also conclude that the natural map

$$\pi_1(\mathcal{W}^{(3)}) * \left(\coprod_{\mathcal{L}^{(t)}} \mathbb{Z} \right) \to \pi_1(\mathcal{W}^{(4)}) \tag{3.11}$$

is an isomorphism. Observe that $\mathcal{W}^{(4)} \leqslant \mathcal{Z}$ admits the following description:

- $W_v^{(4)} = W_v * \left(\coprod_{\mathcal{L}_v} \mathbb{Z}\right)$ for $v \in \mathsf{Vert}_{\mathsf{o}}$;

- $W_v^{(4)} = Z_v$ for $v \in \mathsf{Vert}_{\mathsf{t}}$;
- $W_e^{(4)} = Z_e$ for $e \in \mathsf{Edge}(\Gamma)$.

By construction, $\mathcal{W}^{(4)}$ is a subgraph of groups of $\mathcal{Z}$. Even more, by our assumption on (3.7) and by Lemma 3.2.4, the canonical injective map

$$\pi_1(\mathcal{W}^{(4)}) \to \pi_1(\mathcal{Z}) \tag{3.12}$$

is $\mathcal{D}_{K[G]}$-injective. From the chain of injections and $\mathcal{D}_{K[G]}$-injections described in (3.8), (3.9), (3.10), (3.11) and (3.12); we conclude that the canonical map

$$\pi_1(\mathcal{W}) * \left(\coprod_{\mathcal{L}^{(0)} \bigcup \mathcal{L}^{(t)}} \mathbb{Z} \right) \to \pi_1(\mathcal{Z})$$

is injective and $\mathcal{D}_{K[G]}$-injective. The proof is complete. □

3.3.3 The proof of Theorem B3

As discussed in Section 3.3.1, the following theorem implies Theorem B3 from the introduction, and this subsection is devoted to its proof.

Theorem 3.3.13. *Let X be an ICE space, let $H \leqslant \pi_1(X)$ be a finitely generated subgroup and let $X_H \to X$ be the corresponding covering. Suppose that $\mathcal{L}$ is a (possibly empty) finite set of independent loops, each of which generates a maximal abelian subgroup of $\pi_1(X)$. Then X_H is L^2-tame over $\mathcal{L}$ (in the sense of Definition 3.3.8).*

Proof. We proceed by induction on the complexity of the ICE space (the length of the hierarchy). The base of the induction corresponds to X being a graph, which requires a slight modification of the M. Hall theorem (we refer to [Wil08, Corollary 1.8] for a precise proof in this language). We now give more details on how this induction base differs from the classical M. Hall and how the assumption of independence of loops is used, since the induction step will later be performed by applying similar ideas.

The base case of the induction: Assume that X is a finite graph. Since the fundamental group of the cover X_H is finitely generated, we can build a finite core of X_H; that is, a finite, connected subgraph $X' \subseteq X_H$ such that the induced map $\pi_1(X') \to \pi_1(X_H)$ is an isomorphism. The elevations of $\mathcal{L}$ to X_H are locally injective combinatorial paths, and so proper maps (preimages of compact subsets are compact). In particular, since each of the elevations from $\mathcal{L}'$ has infinite degree, their images are bi-infinite embedded paths that eventually scape the core X'.

Fix a loop δ_i from $\mathcal{L}$. The condition that δ_i generates a maximally abelian subgroup implies that the pairwise intersections of elevations of δ_i are either empty or compact. Similarly, the fact that the loops of $\mathcal{L}$ is independent implies that if δ'_{j_1} and δ'_{j_2} are elevations of δ_{i_1} and δ_{i_2} (resp.), with $i_1 \neq i_2$, then the intersection of δ'_{j_1} and δ'_{j_2} is either empty or compact.

We remove the interior of X' from X_H and denote the resulting graph by $X_H \setminus X'$. Let L'_j be the intersection of $X_H \setminus X'$ with the image of δ'_j. By the previous paragraph, we can enlarge X' so that it is a core that additionally satisfies that the spaces L'_j are homeomorphic to $\mathbb{R}$ and pairwise disjoint. It is now easy to see that there is an intermediate cover $X_H \to \widehat{X} \to X$ such that:

1. The covering $\widehat{X} \to X$ is finite-sheeted and contains $X' \subseteq \widehat{X}$.
2. If we denote by $\widehat{L}_j$ the projection of each L'_j to $\widehat{X}$, the spaces $\widehat{L}_j$ are pairwise disjoint compact intervals.

It is now clear that the map from item 4 from Definition 3.3.8, of the form

$$\pi_1(X') * \left(\coprod_{\mathcal{L}'} \mathbb{Z} \right) \to \pi_1(\widehat{X}),$$

represents a free factor of $\pi_1(\widehat{X})$; and so it is injective and L^2-injective.

Induction step: Now assume X is an ICE space that decomposes as a graph of spaces with two vertices (a lower complexity ICE space Y and a torus T^n) and one edge space homeomorphic to S^1. This naturally induces a graph-of-spaces structure for X_H, whose underlying graph we denote by $\Gamma(X_H)$. The construction of the appropriate finite-sheeted covering of X is divided in several steps. Recall that each vertex space of the splitting of X_H (and of all splittings from now on) is either a covering space of Y or a covering space of the torus T^n (possibly of infinite degree).

Denote by $\{\varepsilon_i \colon E_i \to X\}$ and $\{\delta_i \colon D_i \to X\}$ the elliptic and hyperbolic loops of $\mathcal{L}$, respectively, in the sense of Definition 3.3.2. We partition $\mathcal{L}'$ accordingly into $\{\varepsilon'_j\}$ and $\{\delta^H_j\}$, which denote the elevations of elliptic and hyperbolic loops, respectively. We anticipate that we will handle the elevations $\{\varepsilon'_j\}$ using the induction hypothesis, but the analysis of the other elevations $\{\delta^H_j\}$ is carried out separately and similarly as in the base of the induction. We should remark a key property satisfied by these hyperbolic elevations relative to the splitting of G, which was trivially satisfied in the base of the induction. If we denote by Γ_H the (typically infinite) underlying graph associated to the splitting of X_H as a cover of X, the composition map $\delta^H_j \colon \mathbb{R} \to$

$X_H \to \Gamma_H$ is proper. We refer to [Wil08, Lemma 2.16] for a proof. This is an incarnation of the 2-acylindricity of the splitting of G, which was hinted at by the discussion on centralisers of elements given after Definition 3.3.2.

Step 1 (The precovers X' and X'')**.** Let $\Delta \subseteq X_H$ be a finite subcomplex. Note that any connected subgraph of spaces of X_H is naturally a precover of X. We take a subcomplex $X' \subseteq X_H$ that satisfies the following conditions:

1. The subcomplex X' is a core for H, i.e. X' is a subgraph of spaces with finite underlying graph such that the induced map $\pi_1(X') \to \pi_1(H)$ is an isomorphism.
2. The subcomplex Δ is contained in X'.
3. The image of each ε'_j is contained in X'.
4. Each infinite-degree elevation $\delta_j^H\colon \mathbb{R} \to X_H$ restricts to a map $\delta'_j\colon D'_j \to X'$, where $D'_j \subseteq \mathbb{R}$ is a finite union of compact intervals.

Even if, strictly speaking, the maps δ'_j from item 4 right above may not be elevations; we may think of δ'_j as a *non-full elevation*, following Wilton's terminology [Wil08, Section 2.6]. In this sense, our next objective is extending X', similarly as we argued in the base case, so each δ'_j becomes an actual elevation. For this, we introduce intermediate precovers $X' \subseteq X'' \subseteq X''' \subseteq \overline{X} \subseteq X_H$ of X. The fact that the edge spaces of our splitting are not points (as in the base case of the induction), allows for a new phenomenon that was alluded to above in item 4; namely, that the domains D'_j of the maps δ'_j may be disconnected.

From [Wil08, Lemma 2.24], we can enlarge X' to $X'' \subseteq X_H$ so that X'' still enjoys properties (1)—(4) listed above while additionally satisfying that the corresponding elevations $\delta''_j\colon D''_j \to X''$ are *disparate* (in the sense of [Wil08, Definition 2.22]). In particular, the induced map

$$\pi_1(X'') \to \pi_1(X_H) \tag{3.13}$$

is an isomorphism. The role of disparity is essentially ensuring that the domains D''_j of δ''_j differ (in particular, that the map $\coprod \partial D''_j \to X''$ is injective). This is necessary if we expect (among other things) to be able to complete D''_j to circles in a precover $X'' \subseteq \overline{X}$, where they generate a free subgroup of $\pi_1(\overline{X})$ that is also freely independent from $\pi_1(X'')$, in analogy with the base of the induction.

Step 2 (The precover $\overline{X}$)**.** Recall that each D''_j is the union of finitely many compact intervals and that D''_j fits in the following commutative diagram:

$$\begin{tikzcd}
X'' \arrow[d, hook] & D''_j \arrow[l, "\delta''_j"'] \arrow[d, hook] \\
X_H \arrow[d] & \widetilde{D_i} \arrow[l, "\delta^H_j"'] \arrow[d] \\
X & D_i \arrow[l, "\delta_i"'] ,
\end{tikzcd}$$

where $\widetilde{D_i} \cong \mathbb{R}$ embeds into X_H via δ^H_j by assumption. Now we fold $\widetilde{D_i}$ into a big circle. More precisely, for all sufficiently large positive integers d, there exists $\overline{D}_j \cong S^1$ such that $D''_j \to D_i$ factors through an embedding $D''_j \hookrightarrow \overline{D}_j$ and a d-sheeted covering map $\overline{D}_j \to D_i$. By [Wil08, Lemma 2.23], we can extend X'' to a precover $\overline{X}$ such that each δ''_j extends to a full elevation $\overline{\delta}_j \colon \overline{D}_j \to \overline{X}$ and the diagram

$$\begin{tikzcd}
X'' \arrow[d] & D''_j \arrow[l, "\delta''_j"'] \arrow[d, hook] & \\
\overline{X} \arrow[d] & \overline{D}_j \arrow[l, "\overline{\delta}_j"'] \arrow[d] \arrow[r, "\cong"] & S^1 \arrow[d, "\deg\ d"] \\
X & D_i \arrow[l, "\delta_i"'] \arrow[r, "\cong"] & S^1
\end{tikzcd}$$

commutes. By possibly enlarging Δ, we can assume that the images of the $\overline{\delta}_j$ are contained in Δ.

We shall give some details on the construction of $\overline{X}$ from [Wil08, Lemma 2.23], to ensure the required control in the L^2-homology. To build $\overline{X}$, one first enlarges X'' to X''' by adding some simply connected vertex spaces of X_H to obtain $X'' \hookrightarrow X''' \hookrightarrow X_H$. In particular, the induced map $\pi_1(X'') \to \pi_1(X''')$ is an isomorphism. Then, one considers a collection of pairs $(\varphi^H_{k,\mathsf{o}}, \varphi^H_{k,\mathsf{t}})$ of edge maps $\varphi^H_{k,\mathsf{o}} \colon \mathbb{R}_\mathsf{o} \to X_H$ and $\varphi^H_{k,\mathsf{t}} \colon \mathbb{R}_\mathsf{t} \to X_H$ which are elevations of the incident and terminal edge maps of some edge space S^1_k of X. Furthermore, these pairs $(\varphi^H_{k,\mathsf{o}}, \varphi^H_{k,\mathsf{t}})$ will have the property that these are not edge maps of X'''. Recall that such elevations are called *hanging elevations* (as defined in Definition 2.1.14). For each k, we denoted by $\mathbb{R}_\mathsf{o}$ and $\mathbb{R}_\mathsf{t}$ the domains of $\varphi^H_{k,\mathsf{o}}$ and $\varphi^H_{k,\mathsf{t}}$, respectively (which are the universal covers of S^1_k). We fix a deck transformation $\tau\colon \mathbb{R}_{\mathsf{o}} \to \mathbb{R}_{\mathsf{t}}$ so that the natural diagram

$$\begin{tikzcd}
\mathbb{R}_{\mathsf{o}} \arrow[rr, "\tau"] \arrow[dr] & & \mathbb{R}_{\mathsf{t}} \arrow[dl] \\
& S^1_k &
\end{tikzcd}$$

commutes. Then $\overline{X}$ is constructed from X''' by adding the edge space $\mathbb{R}_{\mathsf{o}}$ with incident and terminal edge maps given by $\varphi^H_{k,\mathsf{o}}$ and $\varphi^H_{k,\mathsf{t}} \circ \tau$ for each k.

We denote by Γ the underlying graph of the splitting of $\overline{X}$. Notice that the underlying graph of X'' may be smaller. We set $E_T \subseteq \mathsf{Edge}(\Gamma)$ to be the edges of $\Gamma(X''')$. We enlarge the splitting of X'' by adding trivial vertex groups and just assume that its underlying graph is also Γ. So we view $\pi_1(X'')$ as the fundamental group of a graph of groups $\mathcal{W}$ whose underlying graph is Γ. We denote by T a spanning tree of the underlying graph $\Gamma(X''')$ of X''', which is also a spanning tree of Γ.

Step 3 (The finite-sheeted precover $\widehat{X}$)**.** By [Wil08, Proposition 3.4], there exists an intermediate finite-sheeted precovering $\overline{X} \to \widehat{X} \to X$ satisfying the following properties for all sufficiently large positive integers d:

1. The underlying graph of $\widehat{X}$ is Γ.
2. The subcomplex Δ projects homeomorphically into $\widehat{X}$.
3. Each ε'_j descends to a full elevation $\widehat{\varepsilon}_j\colon \widehat{E}_j \to \widehat{X}$ with $\widehat{E}_j \to E_i$ being a covering of degree d.

Since Δ injects into $\widehat{X}$, we already know that $\overline{\delta}_j$ descends to a full elevation $\widehat{\delta}_j\colon \overline{D}_j \to \widehat{X}$. We want to apply Proposition 3.3.12 and prove that the natural map

$$\pi_1(X'') * \left(\coprod_{\mathcal{L}} \mathbb{Z} \right) \to \pi_1(\widehat{X})$$

is injective and L^2-injective. Before this, we need to introduce more notation. There is a natural bipartite structure of Γ given by the bipartite structure of the splittings of ICE groups. More precisely, $\mathsf{Vert}(\Gamma)$ is split into disjoint sets $\mathsf{Vert}_{\mathsf{o}}$ and $\mathsf{Vert}_{\mathsf{t}}$ so that, for all $e \in \mathsf{Edge}(\Gamma)$, $\mathsf{o}(e) \in \mathsf{Vert}_{\mathsf{o}}$, and $X''_{\mathsf{o}(e)}$ is a covering of Y; and, similarly, $\mathsf{t}(e) \in \mathsf{Vert}_{\mathsf{t}}$ and $X''_{\mathsf{t}(e)}$ is a covering of the torus T^n. We denote by $\mathcal{Z}$ the graph of groups corresponding to $\pi_1(\widehat{X})$, whose underlying graph is Γ. At the end of Step 2, we defined the graphs of groups $\mathcal{W}$ and $\mathcal{Z}$, the spanning tree $T \subseteq \Gamma$ and the subset of edges $\mathsf{Edge}(T) \subseteq E_T \subseteq \mathsf{Edge}(\Gamma)$. Denote by $\mathcal{L}^{(0)}$ the collection of elements of $\pi_1(\widehat{X})$

that are represented by the images of the elliptic loops $\{\widehat{\varepsilon}_j\}$. For each $v \in \mathsf{Vert}_{\mathsf{o}}$, we define $\mathcal{L}_v$ to be the subset of Z_v that contains $\mathcal{L}_v^{(0)}$ and the elements $\varphi_{\mathsf{o},e}(z_e)$ such that $\varphi_{\mathsf{o},e}(z_e) \notin W_v$. By construction, it is not hard to see that, up to a homotopy of $\widehat{X}$, we have that:

(a) The subset of $\pi_1(\widehat{X})$ represented by the images of $\widehat{\varepsilon_j}$ is exactly $\bigcup_{v\in\mathsf{Vert}_{\mathsf{o}}} \mathcal{L}_v^{(0)}$.

(b) The subset of $\pi_1(\widehat{X})$ represented by the images of $\widehat{\delta_j}$ is

$$\{t_e : e \in \mathsf{Edge}(\Gamma) \setminus E_T\},$$

where we view $\pi_1(\widehat{X})$ as in Definition 2.1.3 (relative to the spanning tree T).

Before applying Proposition 3.3.12, we observe that we can ensure that $\widehat{X}$ satisfies an additional property, on top of the three listed above. Our inductive hypothesis implies that the complex $\overline{X}_v$ is L^2-tame relative to $\mathcal{L}_v$ for each $v \in \mathsf{Vert}_{\mathsf{o}}$. The construction of [Wil08, Proposition 3.4] replaces each vertex space $\overline{X}_v$ by a finite cover $\widehat{X}_v$ of one of the two vertex spaces of X (either Y or the torus T^n) compatibly along the edge spaces. If we perform this construction of $\widehat{X}$ by adequately replacing the notion of tameness by our notion of L^2-tameness, we could have ensured that the finite-sheeted precover $\widehat{X}$ satisfies the following additional property:

4. For each $v \in \mathsf{Vert}_{\mathsf{o}}$, the natural map

$$\pi_1(\overline{X}_v) * \left(\coprod_{\mathcal{L}_v} \mathbb{Z}\right) \to \pi_1(\widehat{X}_v)$$

is injective and L^2-injective.

By applying Proposition 3.3.12 to the subgraph of groups $\mathcal{W} \leqslant \mathcal{Z}$ introduced above (and keeping in mind remarks (a) and (b) above), the induced map

$$\pi_1(X'') * \left(\coprod_{\mathcal{L}} \mathbb{Z}\right) \to \pi_1(\widehat{X}), \tag{3.14}$$

is injective and L^2-injective.

Step 4 (The finite-sheeted cover $\widehat{X}^+$)**.** Finally, $\widehat{X}$ can be extended to a finite-sheeted covering $\widehat{X}^+ \to X$ by adding additional vertex spaces glued along cylinders by [Wil08, Proposition 3.7]. Hence, $\pi_1(\widehat{X})$ is the vertex group of a cyclic splitting of $\pi_1(\widehat{X}^+)$ and, by Lemma 3.2.4, the injective map

$$\pi_1(\widehat{X}) \to \pi_1(\widehat{X}^+) \tag{3.15}$$

is L^2-injective.

We have gathered all the ingredients to prove that the finite-sheeted cover $\widehat{X}^+$ satisfies the fourth point of the L^2-tame property, namely that the induced map

$$\pi_1(X_H) * \left(\coprod_{\mathcal{L}} \mathbb{Z}\right) \to \pi_1(\widehat{X}^+),$$

is injective and L^2-injective. This is a direct consequence of the fact that the maps described in Equations (3.13), (3.14) and (3.15) are injective and L^2-injective. □

3.4 Questions and conjectures

Our arguments to prove that limit groups are L^2-Hall suggest various questions, which exhibit the limitations of the current methods. First of all, it should be noted that with our arguments we can prove that a limit group L is $\mathcal{D}_{K[L]}$-Hall for any field K. In order to achieve this, we had to slightly modify Wilton's constructions from his proof of the M. Hall property of limit groups [Wil08]. Nevertheless, it is unclear if these modifications were necessary on the first place. More concretely, a confirmation of the following conjecture would establish the L^2-Hall property for limit groups directly from [Wil08].

Conjecture 3.4.1. *Let L be a limit group and let $H \leqslant L$ be a retract. Then H is $\mathcal{D}_{K[L]}$-independent for every field K.*

We should note that a geometric description of all retracts, even when the ambient group is a finitely generated free group, seems to be out of reach. On a different note, our methods do not only rely on all the machinery developed to ensure the existence of retracts, but also on the fact that they have finite cyclic hierarchies. We believe there should be L^2-Hall examples outside this class, as we suggest in our next question.

Question 3.4.2. *Are there examples of hyperbolic L^2-Hall groups (or $\mathcal{D}_{K[G]}$-Hall groups for some field K) that do not virtually have a finite cyclic hierarchy?*

One reason why it is hard to answer the previous question is that we have no way to ensure the L^2-Hall property without having, among other features, a strong verification of subgroup separability. The class of L^2-Hall subgroup separable groups is closed by free products Proposition 3.2.8. A good starting point would be to weaken the assumptions in this claim.

Question 3.4.3. *Is the class of finitely generated residually finite L^2-Hall groups closed under free products?*

In a related fashion, residual finiteness may help to produce more examples L^2-Hall groups via finite extensions. For concreteness, we state the following (which should be compared to Theorem D3).

Question 3.4.4. *If a finitely generated group G is virtually $\mathcal{D}_{K[G]}$-Hall, is it $\mathcal{D}_{K[G]}$-Hall?*

The matter of whether the characteristic of the ground field K is any relevant to these questions is also poorly understood.

Question 3.4.5. *Are there examples of L^2-Hall groups that are not $\mathcal{D}_{K[G]}$-Hall for some field K of positive characteristic?*

An even more fundamental question, close to Question 3.4.5, is the following.

Question 3.4.6. *If a finitely generated subgroup $H \leqslant G$ is $\mathcal{D}_{K[G]}$-independent for some field K, is it also $\mathcal{D}_{K[G]}$-independent for every K?*

The previous question is still open for free groups, although Jaikin-Zapirain proved in [JZ24a] that, when F is free, $\mathcal{D}_{\mathbb{F}_2[F]}$-independence is equivalent to $\mathcal{D}_{\mathbb{Q}[F]}$-independence. We finish this section with one last conjecture which should be much more approachable than the previous ones. It is desirable to have a class of groups satisfying the SHNC containing both the graphs of free groups under consideration and limit groups, as this would provide a unifying framework for our results. We now propose such a class.

Definition 3.4.7. Let $\mathcal{C}_0$ be the class of groups containing only the trivial group. Inductively, we define $\mathcal{C}_{n+1}$ to be the class of groups G such that either G is virtually in $\mathcal{C}_n$ or G has the form $H*_A$ (resp. $H *_A K$), where H (resp. H and K) belong to $\mathcal{C}_n$ and A is a finitely generated free abelian group. We say that G admits a *finite abelian hierarchy* if it lies in $\mathcal{C}_n$ for some n.

Our inductive arguments on the hierarchy of limit groups suggest that they may carry over into the more general setting of the following conjecture.

Conjecture 3.4.8. *Let G be a group that admits a finite abelian hierarchy. Suppose that G is torsion-free and hyperbolic relative to virtually abelian subgroups. Then G is L^2-Hall and satisfies the Strengthened Hanna Neumann Conjecture.*

Chapter 4

On the profinite rigidity of surface groups

A old and compelling question in group theory which seems to attract more attention over the years asks to what extend a group is determined by its finite quotients. We recall that this subject, which was introduced in Section 1.4, is known by the name of *profinite rigidity* and has been explored with a rich variety of techniques ranging from algebra to geometry and topology. One of the most intriguing open questions is whether finitely generated free groups and surface groups are profinitely rigid (Question 1.4.2). In this chapter, we address this and the related questions for surface groups. Our results are already quite illustrative when applied to one-relator groups, where many of these questions remain unanswered. By a *one-relator group* we mean a group that admits a presentation with finitely many generators and with exactly one relation. Recall from Section 2.5 that surface groups are one-relator groups. This is, then, a natural class in which to test the profinite rigidity of surface groups.

Theorem A4. *Let S be a surface group and let G be a residually finite, one-relator group. If $\widehat{G} \cong \widehat{S}$, then $G \cong S$.*

Theorem A4 is a consequence of Theorem C4, which involves more general assumptions that we move on to discuss now.

4.1 The Main results

The criterion we use to recognise surface groups from their profinite completion is the well-known fact that these are the Poincaré duality groups of cohomological dimension two by the work of Eckmann–Linnell–Müller [EM80, EL83] (subsequently generalised by Bowditch [Bow04]). Theorem B4 is a prosolvable extension of this principle that will allow us to recognise surface groups from their finite quotients in various situations.

Theorem B4. *Let p be a prime and let G be a RFRS group of cohomological dimension two and type $\mathtt{FP}_2(\mathbb{F}_p)$. Suppose that G is prosolvable p-good and that its prosolvable completion $G_{\widehat{\mathcal{S}}}$ is Poincaré Duality of dimension two at p. Then G is a surface group.*

The notion of prosolvable p-goodness is an analogue of Serre's notion of cohomological goodness [Ser97] that we reviewed in Section 2.7. We also recalled the notion of Poincaré duality in Definition 2.7.2. A simple criterion to produce these groups is offered by Kochloukova–Zalesskii [KZ08]. They showed that the pro-solvable completion of an abstract Poincaré duality group Γ of dimension n over $\mathbb{F}_p$ that is p-good is also Poincaré duality of dimension n at p. One of the motivations of the paper [JZM25] was to get a two-dimensional converse of this fact (as we do in Theorem B4, strengthened in Theorem 4.4.1 for other varieties).

We refer the reader to Section 4.3 for the definition of RFRS groups. Theorem B4 suggests that this class of groups may also be very useful to handle questions on profinite rigidity. In fact, we can prove the following.

Theorem C4. *Let S be a surface group and let G be a finitely generated group with $\operatorname{cd}(G) = 2$ and $b_2^{(2)}(G) = 0$. Assume that either*

1. *G is residually finite and has the same finite quotients as S; or*
2. *G is residually-(finite solvable) and has the same finite solvable quotients as S.*

Then $G \cong S$.

Theorem C4 is applicable to many classes of groups. For example, it includes the class of one-relator groups (because they have $\operatorname{cd}(G) \leqslant 2$ if they are torsion-free by Lyndon [Lyn50] and they have $b_2^{(2)}(G) = 0$ by Dicks–Linnell [DL07]). So Theorem A4 follows from Theorem C4. On the other hand, it also allows us to handle limit groups for the following reason. If L is a limit group then $b_2^{(2)}(L) = 0$ by [BK17b]. If, in addition, L has the same prosolvable completion as a non-abelian free or surface group S, it follows from [Mor24a, Theorem A] that L does not contain $\mathbb{Z}^2$ and hence $\operatorname{cd}(L) \leqslant 2$ by Sela's finite cyclic hierarchy [Sel01]. So Theorem C4 proves that surface groups can be recognised among limit groups by uniquely looking at their finite solvable quotients, which strengthens previous results obtained in [Wil21, Mor24a, FM25b]. The analogue of Theorem C4 for the case when S is free remains open.

4.1.1 Measure equivalence

The fact that Theorem C4 applies to torsion-free one-relator groups allows us to obtain in Theorem D4 new results in the subject of measure equivalent words. Before discussing them, we recall a few definitions. Let F be the free group on d free generators $x_1, \ldots, x_d$ and let $w \in F$. Given a finite group G, we define a homomorphism $w_G \colon G^d \to G$ that maps $(g_1, \ldots, g_d)$ to the image of w under the homomorphism $F \to G$ that sends x_i to g_i. We say that two words $w, u \in F$ are *measure equivalent in* G if for every $g \in G$, $|w_G^{-1}(g)| = |u_G^{-1}(g)|$. If $\mathcal{C}$ is a non-empty family of finite groups, we say that $w, u \in F$ are *measure equivalent in* $\mathcal{C}$ if they are measure equivalent in every $G \in \mathcal{C}$. For example if $u = \varphi(w)$ for some $\varphi \in \operatorname{Aut}(F)$ then u and w are measure equivalent in the class of all finite groups. The main conjecture in the subject is the following.

Conjecture 4.1.1. *Let $u, w \in F$ be two words of a finitely generated free group. Assume that u and w are measure equivalent in all finite groups. Then there exists $\varphi \in \operatorname{Aut}(F)$ such that $u = \varphi(w)$.*

The conjecture is known when one of the words is primitive [PP15] (see also [Wil18, GJZ23]) and when one of the words is a surface word [Wil21] (see also [MP21]). We can reprove the conjecture for surface words and, moreover, we show that it is enough to look at the finite solvable quotients.

Theorem D4. *Let F be the free group on generators $x_1, \ldots, x_k$. Suppose we lie under one of the following scenarios:*

1. *We can write $k = 2d$ and w has the form of the orientable surface word $[x_1, x_2] \cdots [x_{2d-1}, x_{2d}]$.*
2. *The word w is the non-orientable surface word $x_1^2 \cdots x_k^2$.*

Suppose that $u \in F$ is measure equivalent with w in the class of finite solvable groups. Then there exists $\varphi \in \operatorname{Aut}(F)$ such that $u = \varphi(w)$.

In fact, we will prove that it is enough to consider the quotients in the pseudovariety $\mathcal{N}_p(\mathcal{A}_f \mathcal{N}_q)$ for some prime p and q (see Subsection 2.3 for definitions).

4.1.2 The surface group conjecture

Our last result represents some progress towards the solution of Mel'nikov's conjecture. A finitely generated group G is called *Mel'nikov* if all its subgroups of finite index are one-relator groups. The following problem appears in [KM18, Problem 7.36] and also in [BFR19, Conjecture 2.17].

Conjecture 4.1.2 (Mel'nikov)**.** *An infinite and residually finite Mel'nikov group is either a free group, a surface group or a solvable Baumslag–Solitar group.*

The two-generated case of the conjecture is proved by Gardam–Kielak–Logan [GKL23]. In this paper we consider n-generator one-relator groups with $n \geqslant 3$. We say that a group is *2-free* if all its two-generated subgroups are free.

Theorem E4. *Let $n \geqslant 3$ and let F be the free group on $\{x_1, \ldots, x_n\}$. Let $w \in F$ and suppose that $G = F/\langle\langle w \rangle\rangle$ is a residually finite Mel'nikov group. Then G is 2-free. Moreover, if there exists a finite-index subgroup $H \leqslant G$ such that $H_2(H; \mathbb{Z}) \neq 0$, then G is a surface group.*

This settles Conjecture 4.1.2 in the case of non-trivial second integral homology. Theorem E4 also answers a question of Gardam–Kielak–Logan [GKL23, Question 3.3] on whether residually finite Mel'nikov groups have negative immersions. In what follows, we shall not require the exact definition but we will use the fact that a one-relator group G has negative immersions if and only if G is 2-free by a result of Louder and Wilton [LW22, Theorem 1.3 and Remark 1.7].

4.1.3 More details on our methods

For expository reasons, it may be instructive to devote this subsection to discuss informally the proofs of Theorems B4, C4, D4 and E4. This will allow us to emphasise which are the novel ingredients of our proofs and how they are structured.

Firstly, we make one observation about the assumptions of Theorem C4. Instead of finite generation and vanishing of the second L^2-Betti number, the ingredients that we really require in the proof of Theorem C4 are G being $\mathtt{FP}_2(\mathbb{F}_p)$ and p-good in the variety of finite solvable groups (defined in Section 2.7) to then conclude it from Theorem B4. However, it is in general very difficult to obtain prosolvable goodness, so we consider that the assumptions imposed in the statement of Theorem C4 are not just sufficient, but also more natural and practical.

From Theorem B4 to Theorem C4:. Jaikin-Zapirain [JZ23, Theorem 1.1] showed that a group G as in part (1) or part (2) of Theorem C4 must be RFRS.

So part (2) implies part (1) and to deduce part (2) from Theorem B4 we only have to show that G is prosolvable good and of type $\mathtt{FP}_2(\mathbb{F}_p)$. Recall that if a finitely presented group G is cohomologically good and has the same profinite completion as a surface group S then, by Lück's approximation [Lüc94], $b_2^{(2)}(G) = 0$. Interestingly, in Claim 4.5.2 we conversely prove that one can recover goodness from $\mathrm{cd}(G) = 2$ and $b_2^{(2)}(G) = 0$. A new ingredient involved in this step, and which we consider of independent interest, is that the Euler characteristic of the group will equal its L^2 Euler characteristic in this setting (mod-p versions of this are stated in Proposition 4.3.1). This summarises how we prove Theorem C4 from Theorem B4.

The proof of Theorem B4: The ultimate goal is to prove that G is Poincaré Duality of dimension two over the field $\mathbb{F}_p$ and then conclude from Bowditch [Bow04]. During the proof we work with a smaller pseudovariety $\mathcal{C}$ than the variety of solvable groups $\mathcal{S}$ to get a stronger formulation of Theorem B4, but both equally work and for simplicity we sketch the argument in terms of $\mathcal{S}$.

As we argue in Claim 4.4.2, since $G_{\widehat{\mathcal{S}}}$ is a freely indecomposable profinite group, G is also freely indecomposable and hence $H^1(G; \mathbb{F}_p[G]) = 0$. Without loss of generality, we can also suppose that $G_{\widehat{\mathcal{S}}}$ is orientable. Given this, it remains to show that $H^2(G; \mathbb{F}_p[G]) \cong \mathbb{F}_p$ as trivial G-modules to prove that G is an orientable Poincaré duality group. There is a natural surjective map $H^2(G; \mathbb{F}_p[G]) \to H^2(G; \mathbb{F}_p) \cong \mathbb{F}_p$ with kernel K. The remainder of the proof consists on understanding the cohomological, finiteness and profinite properties of K to show that $K = 0$. For example, using ideas from Sections 2.6 and 2.7, we can prove that the prosolvable completion $K_{\widehat{\mathcal{S}}}$ of K is zero. Later on, using some tools from Section 4.3, we deduce that K is L^2-acyclic. Another fundamental ingredient is to ensure that K has projective dimension at most one, i.e. that $\mathrm{Ext}^2_{\mathbb{F}_p[G]}(K, \mathbb{F}_p[G]) = 0$. For this, we compare $\mathrm{Ext}^2_{\mathbb{F}_p[G]}(K, \mathbb{F}_p[G])$ with $\mathrm{Ext}^2_{\mathbb{F}_p[\![G_{\widehat{\mathcal{S}}}]\!]}(K_{\widehat{\mathcal{S}}}, \mathbb{F}_p[\![G_{\widehat{\mathcal{S}}}]\!])$, which we already know to be zero because $K_{\widehat{\mathcal{S}}} = 0$. Note that, in general, there is no way to make this comparison, since there is no canonical map between derived functors of G-modules and $G_{\widehat{\mathcal{S}}}$-modules (as discussed during Section 2.7). Nevertheless, after deriving more information on the finiteness properties and on the cohomological goodness of the G-module K, we can prove that $\mathrm{Ext}^2_{\mathbb{F}_p[G]}(K, \mathbb{F}_p[G]) \cong \mathrm{Ext}^2_{\mathbb{F}_p[\![G_{\widehat{\mathcal{S}}}]\!]}(K_{\widehat{\mathcal{S}}}, \mathbb{F}_p[\![G_{\widehat{\mathcal{S}}}]\!])$ using the tools from Section 2.3. Once we have proven the above, we know that there is a short exact sequence of right $\mathbb{F}_p[G]$-modules of the form

$$0 \to P_1 \to P_0 \to K \to 0,$$

where P_0 and P_1 are finitely generated projective $\mathbb{F}_p[G]$-modules with the same $\mathcal{D}_{\mathbb{F}_p[G]}$-dimension. Furthermore, since $K_{\widehat{\mathcal{S}}} = 0$, the map $P_1 \to P_0$ is dense in the prosolvable topology. It follows from [JZ23, Theorem 4.3] that the map $P_1 \to P_0$ is an isomorphism and so $K = 0$. This completes the proof of Theorem B4.

The proof of Theorem D4: This result is an application of Theorem C4 applied to the double $G = F *_{u=u'} F'$ of F along the corresponding word u.

Lastly, we discuss the proof of Theorem E4.

Achieving 2-freeness: The first part of the proof of Theorem E4 is ensuring that the one-relator group $G = F/\langle\langle w \rangle\rangle$ is 2-free. With this purpose, we use a criterion of Louder–Wilton [LW22, Theorem 1.5 and Definition 6.5]. This says that it suffices to rule out the existence of a subgroup K of F of rank two containing w such that the canonical homomorphism $P = K/\langle\langle w \rangle\rangle \to G$ is an embedding of a non-free 2-generator one-relator group P into G. We begin by showing that G is prosolvable good (which requires a different argument as the one in the proof of Theorem C4). This will lead to the closure $\overline{P}$ of P in $\widehat{G}$ being a projective profinite group. Then, by estimating the Betti numbers of the open subgroups of the closure $\overline{P}$ of P in $\widehat{G}$, we obtain that the p-Sylows of $\overline{P}$ are pro-p cyclic. This implies that $\overline{P}$ is meta-abelian by a result of Zassenhaus [Rib17, Lemma 4.2.5] and hence that $P \cong B(1, m)$. A similar analysis on the virtual Betti numbers of $B(1, m)$ is needed to rule out this subgroup of G.

Recognising the surface group: The second part of Theorem E4 consists on showing that G is a surface group if there is a finite-index subgroup $H < G$ with $H^2(H; \mathbb{Z}) \neq 0$. For this we apply Theorem B4, although it is not immediate to see that the required assumptions on G are fulfilled. The group G is a 2-free one-relator group and hence a virtually compact special group by Louder–Wilton [LW22] and Linton [Lin25]. So G is virtually $\mathcal{N}_p$-RFRS. It remains to prove that $G_{\widehat{\mathcal{S}}}$ is Poincaré Duality of dimension two at some prime p. For this, we build on the fact that G is prosolvable good. More precisely, we apply to the maximal p-quotients of $G_{\widehat{\mathcal{S}}}$ the result of Andožskiǐ [And73] that a non-free one-relator pro-p group that satisfies that all its open subgroups are one-relator groups must be a Demushkin group (recall that these are the Poincaré duality pro-p groups of dimension two [Ser97, Section I.4.5, Example 2]). Interestingly, this was the original motivation of Mel'nikov to formulate Conjecture 4.1.2.

4.2 The Euler characteristic and its variants

Recall that L^2-Betti numbers are multiplicative under finite covers (Proposition 2.8.2). Another well-known invariant known to satisfy this useful property is the Euler characteristic of a space. We shall describe different versions of this invariant both for groups and modules. The main purpose is to find scenarios, beyond strong finiteness assumptions, where all these coincide.

Given a division R-ring $\mathcal{D}$ and an R-module M of type $\mathtt{FP}$, we define

$$\chi^{\mathcal{D}}(M) := \sum_i (-1)^i \, b_i^{\mathcal{D}}(M). \tag{4.1}$$

Importantly, this invariant is additive in the following sense.

Proposition 4.2.1. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of R-modules. If any two of the modules M_1, M_2 or M_3 are $\mathtt{FP}$, then the third one is also $\mathtt{FP}$. Moreover, if $\mathcal{D}$ is a division R-ring, then*

$$\chi^{\mathcal{D}}(M_2) = \chi^{\mathcal{D}}(M_1) + \chi^{\mathcal{D}}(M_3).$$

Proof. The first part of the proposition is proved in [Bie81, Proposition 1.4 and Proposition 4.1b], and the second part is a consequence of the long exact sequence for Tor functors [Wei94, Chapter 2-3]. □

Let $f\colon R \to S$ be an R-ring. We say that f is *epic* if for every ring Q and homomorphisms $\alpha, \beta\colon S \to Q$, the equality $\alpha \circ f = \beta \circ f$ implies $\alpha = \beta$. As in [Coh06, Chap. 7.2], we define an *epic division R-ring* as a division ring $\mathcal{D}$ together with an epic homomorphism $\varphi\colon R \to \mathcal{D}$. The condition on φ to be epic is equivalent to the condition that the division closure of $\varphi(R)$ is equal to $\mathcal{D}$.

Let $\varphi'\colon R \to \mathcal{D}'$ be another epic division R-ring. A subhomomorphism of epic division R-rings $\mathcal{D} \to \mathcal{D}'$ is a homomorphism $\psi\colon K \to \mathcal{D}'$, where K is a local subring of $\mathcal{D}$ containing $\varphi(R)$ with maximal ideal $\ker \psi$, such that $\psi \circ \varphi = \varphi'$. Two subhomomorphisms are equivalent if their restriction to the intersection of their domains coincide and are again subhomomorphisms. A *specialisation* $\mathcal{D} \to \mathcal{D}'$ of epic division R-rings is an equivalence class of subhomomorphisms in the previous sense.

Proposition 4.2.2. *Let R be a ring, $\varphi\colon R \to \mathcal{D}$ and $\varphi'\colon R \to \mathcal{D}'$ two epic division R-rings and M an $\mathtt{FP}(R)$-module. If there is a specialisation $\mathcal{D} \to \mathcal{D}'$, then $\chi^{\mathcal{D}}(M) = \chi^{\mathcal{D}'}(M)$.*

Proof. In view of Proposition 4.2.1, it is enough to prove the claim when $M = P$ is a finitely generated projective R-module, in which case $\chi^{R,\mathcal{D}_0}(M) = \dim_{\mathcal{D}_0} D_0 \otimes_R P$ for any division R-ring $R \to \mathcal{D}_0$.

Let $\psi\colon K \to \mathcal{D}'$ be a homomorphism, where K is a local subring of $\mathcal{D}$ containing $\varphi(R)$ with maximal ideal $\ker\psi$, such that $\psi \circ \varphi = \varphi'$. Since projective modules over a local ring are free [Kap58], $K \otimes_R P \cong K^k$ as K-modules for some $k \in \mathcal{N}$. Hence, $k = \dim_{\mathcal{D}} \mathcal{D} \otimes_R P$. Moreover, if we denote by $\mathcal{Q} \subseteq \mathcal{D}'$ the image of ψ (which is the division quotient ring of K), we have $\mathcal{Q} \otimes_R P \cong \mathcal{Q} \otimes_K K^k \cong \mathcal{Q}^k$ and hence $k = \dim_{\mathcal{D}'} \mathcal{D}' \otimes_R P$ as well. ☐

4.3 On RFRS groups and their L^2-Betti numbers

As part of our group cohomological tools, we require some estimations about L^2-Betti numbers. They allow us to relate cohomology with rational and finite coefficients to the profinite completion of the group (for example, they are used in a criterion to recognise goodness in Claim 4.5.2). Since we are going to consider only RFRS groups, we recall the usual Betti numbers and the L^2-Betti numbers (together with its mod-p versions) for these groups in a convenient and uniform algebraic way.

We first recall the definition of RFRS groups. A group G is called *residually finite rationally solvable* or simply *RFRS* if there exists a chain $G = H_0 > H_1 > \cdots$ of finite index normal subgroups of G with trivial intersection such that H_{i+1} contains a normal subgroup K_{i+1} of H_i such that H_i/K_{i+1} is torsion-free abelian. A chain $\{H_i\}$ satisfying this property is called *witnessing*.

The class of RFRS groups played a fundamental role in Agol's solution of the virtual fibring theorem [Ago13] (see also [Kie20]). We will see later how this class is very useful in establishing results on profinite rigidity. For this purpose, it will be more convenient to work with mod-p variants of this class, which we now define uniformly.

Let $\mathcal{V}$ be an extension-closed pseudovariety of finite solvable groups. We say that a group G is *$\mathcal{V}$-RFRS* if G has a witnessing chain $\{H_i\}$ such that H_i is normal in G and $G/H_i \in \mathcal{V}$ for all i. Observe that in this case G is residually-$\mathcal{V}$. It is clear that a RFRS group is also $\mathcal{S}$-RFRS (recall that $\mathcal{S}$ is the pseudovariety of finite solvable groups). We will pay special attention to the property of $\mathcal{N}_p$-RFRS (recall that $\mathcal{N}_p$ is the pseudovariety of all finite p-groups). The class of $\mathcal{N}_p$-RFRS groups are also named RFRp groups in [KS20].

The usual Betti numbers of a group G are defined by

$$b_i(G) = b_i^{\mathbb{Q}}(\mathbb{Z}) = b_i^{\mathbb{Q}}(\mathbb{Q}) \quad \text{and} \quad b_{i,p}(G) = b_i^{\mathbb{F}_p}(\mathbb{Z}) = b_i^{\mathbb{F}_p}(\mathbb{F}_p).$$

Implicit in the previous equations are the $\mathbb{Z}[G]$-rings $\mathbb{Z}[G] \to \mathbb{Q}$ and $\mathbb{Z}[G] \to \mathbb{F}_p$. We refer the reader to the definition given in equation (2.9). These Betti numbers have L^2-variants over different ground fields, which we introduced in 2.9.

Let G be a RFRS group and K a field. It is proved in [JZ21, Proposition 4.4] that G is residually-(poly-$\mathbb{Z}$). It then follows from Proposition 2.8.1 that there exists a Hughes-free embedding $K[G] \hookrightarrow \mathcal{D}_{K[G]}$ and the *L^2-Betti* and *mod-p L^2-Betti numbers* of G can be defined, following Equation (2.9), as

$$b_i^{\mathcal{D}_{\mathbb{Q}[G]}}(\mathbb{Q}) = \dim_{\mathcal{D}_{\mathbb{Q}[G]}} H_i(G; \mathcal{D}_{\mathbb{Q}[G]}) \quad \text{and} \quad b_i^{\mathcal{D}_{\mathbb{F}_p[G]}}(\mathbb{F}_p) = \dim_{\mathcal{D}_{\mathbb{F}_p[G]}} H_i(G; \mathcal{D}_{\mathbb{F}_p[G]}).$$

Proposition 4.3.1. *Let G be a RFRS group of cohomological dimension 2, let K be a field and let M be a $K[G]$-module of type* `FP`*.*

1. *For any two $K[G]$-division rings $\mathcal{D}_1$ and $\mathcal{D}_2$ we have $\chi^{\mathcal{D}_1}(M) = \chi^{D_2}(M)$.*
2. *We have the inequality*
$$b_i^{\mathcal{D}_{K[G]}}(M) \leqslant b_i^K(M).$$
3. *If $\beta_2^{(2)}(G) = 0$, then G is of type* $\mathtt{FP}_2(\mathbb{Z})$*.*

Proof. By [JZ21, Corollary 1.3], $\mathcal{D}_{K[G]}$ is universal division $K[G]$-ring of fractions. Thus, there exist specialisations $\mathcal{D}_{K[G]} \to \mathcal{D}_i$ $(i = 1, 2)$. Therefore, by Proposition 4.2.2, $\chi^{\mathcal{D}_1}(M) = \chi^{\mathcal{D}_2}(M)$, proving (1). Again, item (2) follows from the universality of $\mathcal{D}_{K[G]}$. Finally, item (3) follows from [JZL25, Corollary 3.4]. □

The following theorem is the well-known Lück approximation. It shows, in particular, that $b_1^{(2)}$ is a profinite invariant among finitely presented groups.

Theorem 4.3.2 ([Lüc94])**.** *Let G be residually finite* $\mathtt{FP}_{n+1}(\mathbb{Q})$ *group and let $G = N_1 > N_2 > \cdots > N_m > \ldots$ be any sequence of finite-index normal subgroups with $\bigcap_m N_m = 1$. Then*

$$\lim_{m\to\infty} \frac{b_n(N_m)}{|G : N_m|} = b_n^{(2)}(G).$$

The following result is a slight variation of [JZ23, Theorem 4.3].

Theorem 4.3.3. *Let $\mathcal{V}$ be a pseudovariety of finite solvable groups, $\mathcal{E} = \mathcal{A}\mathcal{V}$ and G a $\mathcal{V}$-RFRS group. Let L and M be two finitely generated right projective $\mathbb{F}_p[G]$-modules with*

$$b_0^{\mathcal{D}_{\mathbb{F}_p[G]}}(L) = b_0^{\mathcal{D}_{\mathbb{F}_p[G]}}(M).$$

If $f\colon L \to M$ is an $\mathbb{F}_p[G]$-homomorphism such that the induced map

$$L \otimes_{\mathbb{F}_p[G]} \mathbb{F}_p[\![G_{\widehat{\mathcal{E}}}]\!] \xrightarrow{f_{\widehat{\mathcal{E}}}} M \otimes_{\mathbb{F}_p[G]} \mathbb{F}_p[\![G_{\widehat{\mathcal{E}}}]\!]$$

is surjective, then f is an isomorphism.

Proof. There exists L' such that $L \oplus L'$ is a free finitely generated $\mathbb{F}_p[G]$-module. Now apply [JZ23, Theorem 4.3] to the map $L \oplus L' \xrightarrow{f\oplus \mathrm{Id}} M \oplus L'$. □

4.4 From profinite to abstract Poincaré duality

Theorem B4 can be strengthened to the following result.

Theorem 4.4.1. *Let p be a prime, let $\mathcal{V}$ be an extension closed pseudovariety of finite solvable groups and let $\mathcal{C} = \mathcal{N}_p(\mathcal{A}_f\mathcal{V})$. Let G be a $\mathcal{V}$-RFRS group of type $\mathtt{FP}_2(\mathbb{F}_p)$ that is p-good in $\mathcal{C}$ and has cohomological dimension 2. Suppose that $G_{\widehat{\mathcal{C}}}$ is a Poincaré Duality of dimension two at p. Then G is a surface group.*

Theorem B4 corresponds to the case when $\mathcal{V}$ (and hence $\mathcal{C}$) is the variety of finite solvable groups. Even if the condition of being $\mathcal{V}$-RFRS for a small $\mathcal{V}$ (such as $\mathcal{N}_p$) is stronger than that of being RFRS in the usual sense, the assumption on $G_{\widehat{\mathcal{C}}}$ being Poincaré Duality only involves quotients in $\mathcal{C}$. This lead to stronger consequences not only on the profinite rigidity of surface groups (Theorem 4.5.1), but also on the topic of measure equivalent words (Theorem 4.6.1).

Proof of Theorem 4.4.1. After replacing $G_{\widehat{\mathcal{C}}}$ by the kernel of the action of $G_{\widehat{\mathcal{C}}}$ in the cohomology group $H^2_{\mathrm{cts}}(G_{\widehat{\mathcal{C}}}; \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!])$, we can assume that $G_{\widehat{\mathcal{C}}}$ is orientable; that is, that $H^2_{\mathrm{cts}}(G_{\widehat{\mathcal{C}}}; \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]) \cong \mathbb{F}_p$ is the trivial $G_{\widehat{\mathcal{C}}}$-module. Observe that this does not interfere the conclusion of Theorem 4.4.1 because, by Kerckhoff's realisation theorem [Ker83], a torsion-free virtually surface group is a surface group itself.

Our strategy is now to show that G is an orientable $\mathtt{PD}_2$ group over $\mathbb{F}_p$ and conclude from Theorem 2.5.2 that G is a surface group. According to Definition 2.5.1, this amounts to checking that $H^1(G; \mathbb{F}_p[G]) = 0$ and that $H^2(G; \mathbb{F}_p[G]) \cong \mathbb{F}_p$ as G-modules.

Claim 4.4.2. *The group G satisfies $H^1(G; \mathbb{F}_p[G]) = 0$.*

Proof. Notice that $G_{\widehat{\mathcal{C}}}$ does not split as a free pro-$\mathcal{C}$ product because, otherwise, if we had $G_{\widehat{\mathcal{C}}} = \mathbf{U} \coprod \mathbf{V}$ for non-trivial $\mathbf{U}$ and $\mathbf{V}$, it would follow that $\mathrm{cd}_p(\mathbf{U}) \leqslant 1$ and $\mathrm{cd}_p(\mathbf{V}) \leqslant 1$ by a result of Serre [Ser97, Section 1.4]. This would contradict the fact that $\mathrm{cd}_p(G_{\widehat{\mathcal{C}}}) = 2$. So both $G_{\widehat{\mathcal{C}}}$ and G are freely indecomposable. In particular, since G is not virtually cyclic, we deduce from Stallings theorem [Sta68] that G is one-ended. Therefore, $H^1(G; \mathbb{F}_p[G])$ vanishes (as proved in the book of Dicks and Dunwoody [DD89, Theorem 6.10]). ◇

Since G if of type $\mathtt{FP}_2(\mathbb{F}_p)$, we have an exact sequence.

$$0 \longrightarrow P_2 \xrightarrow{\alpha} P_1 \xrightarrow{\beta} P_0 \longrightarrow \mathbb{F}_p \longrightarrow 0, \tag{4.2}$$

where the P_i are finitely generated projective $\mathbb{F}_p[G]$-modules. We name

$$M := H^2(G; \mathbb{F}_p[G]).$$

After applying the right-exact contravariant functor $\mathrm{Hom}(-, \mathbb{F}_p[G])$, we obtain the following exact sequence of right $\mathbb{F}_p[G]$-modules:

$$0 \longrightarrow Q_0 \xrightarrow{\beta^*} Q_1 \xrightarrow{\alpha^*} Q_2 \longrightarrow M \longrightarrow 0, \tag{4.3}$$

where $Q_i = \mathrm{Hom}(P_i, \mathbb{F}_p[G])$. It is exact by Claim 4.4.2. In particular, M is $\mathtt{FP}$ as a right $\mathbb{F}_p[G]$-module by Proposition 4.2.1.

Claim 4.4.3. *Let A be a $\mathbb{F}_p[G]$-bimodule that is finitely generated as a right $\mathbb{F}_p[G]$-module. Then $H^2(G; A)_{\widehat{\mathcal{C}},r}$ and $H^2_{cts}(G_{\widehat{\mathcal{C}}}; A_{\widehat{\mathcal{C}},r})$ are isomorphic as right $\mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]$-modules (the notation for $A_{\widehat{\mathcal{C}},r}$ was introduced after (2.5)).*

Proof. First observe that since A is finitely generated as right $\mathbb{F}_p[G]$-module, the right $\mathbb{F}_p[G]$-module $H^2(G; A)$, which is isomorphic by Proposition 2.4.1 to $M \otimes_{\mathbb{F}_p[G]} A$ and so is also finitely generated. Hence,

$$A_{\widehat{\mathcal{C}},r} \cong \varprojlim_{G/N \in \mathcal{C}} A \otimes_{\mathbb{F}_p[G]} \mathbb{F}_p[G/N]$$

and

$$H^2(G; A)_{\widehat{\mathcal{C}}} \cong \varprojlim_{G/N \in \mathcal{C}} H^2(G; A) \otimes_{\mathbb{F}_p[G]} \mathbb{F}_p[G/N].$$

Again using that A is finitely generated as a right $\mathbb{F}_p[G]$-module, we obtain that $A \otimes_{\mathbb{F}_p[G]} \mathbb{F}_p[G/N]$ is finite if N is a normal open subgroup of G. Thus, $H^1_{\mathrm{cts}}(G_{\widehat{\mathcal{C}}}, A \otimes_{\mathbb{F}_p[G]} \mathbb{F}_p[G/N])$ is finite. Hence, by Proposition 2.6.1, the canonical map

$$H^2_{\mathrm{cts}}(G_{\widehat{\mathcal{C}}}; A_{\mathcal{C},r}) \longrightarrow \varprojlim_{G/N\in\mathcal{C}} H^2_{\mathrm{cts}}(G_{\widehat{\mathcal{C}}}; A\otimes_{\mathbb{F}_p[G]}\mathbb{F}_p[G/N]) \tag{4.4}$$

is an isomorphism.

From the p-goodness of G in $\mathcal{C}$ we get that the restriction map

$$H^2_{\mathrm{cts}}(G_{\widehat{\mathcal{C}}}; A\otimes_{\mathbb{F}_p[G]}\mathbb{F}_p[G/N]) \longrightarrow H^2(G; A\otimes_{\mathbb{F}_p[G]}\mathbb{F}_p[G/N]) \tag{4.5}$$

is an isomorphism.

Finally, by Proposition 2.4.1, the canonical map

$$H^2(G;A)\otimes_{\mathbb{F}_p[G]}\mathbb{F}_p[G/N] \longrightarrow H^2(G; A\otimes_{\mathbb{F}_p[G]}\mathbb{F}_p[G/N]) \tag{4.6}$$

is also an isomorphism. The claim follows from (4.4), (4.5) and (4.6). $\diamond$

We can compute the pro-$\mathcal{C}$ completion of M directly from Claim 4.4.3.

Claim 4.4.4. *We have that $M_{\widehat{\mathcal{C}}} \cong \mathbb{F}_p$, as trivial right G-modules.*

Proof. This follows from Claim 4.4.3, because $H^2_{\mathrm{cts}}(G_{\widehat{\mathcal{C}}}; \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]) \cong \mathbb{F}_p$. $\diamond$

By the p-goodness of G at $\mathcal{C}$, we know that $H^2(G,\mathbb{F}_p) \cong H^2_{\mathrm{cts}}(G_{\widehat{\mathcal{C}}},\mathbb{F}_p) \cong \mathbb{F}_p$. Consider the kernel K of the natural surjective map $H^2(G;\mathbb{F}_p[G]) \to H^2(G;\mathbb{F}_p)$. Observe that, since M and $\mathbb{F}_p$ are `FP`, K is `FP` by Proposition 4.2.1.

Claim 4.4.5. *The pro-$\mathcal{C}$ completion $K_{\widehat{\mathcal{C}}}$ is zero.*

Proof. By construction, we have the short exact sequence $0 \to K \to M \to \mathbb{F}_p \to 0$, from which we get the exact sequence $K_{\widehat{\mathcal{C}}} \to M_{\widehat{\mathcal{C}}} \to \mathbb{F}_p \to 0$ by item 4 of Proposition 2.3.1. Moreover, the map $M_{\widehat{\mathcal{C}}} \to \mathbb{F}_p$ was proved to be an isomorphism in Claim 4.4.4. Hence, to prove the claim, it suffices to check that the canonical map $K_{\widehat{\mathcal{C}}} \to M_{\widehat{\mathcal{C}}}$ is injective. For this, let L be a submodule of K of finite index such that $G/St_G(K/L) \in \mathcal{C}$. It remains to prove that L is open in M, namely, that $G/St_G(M/L) \in \mathcal{C}$. Recall that $St_G(K/L)$ is the biggest subgroup of G that acts trivially in K/L. We name $U = St_G(K/L)$ and $V = St_G(M/L)$. From the fact that $M/K \cong \mathbb{F}_p$, we obtain an exact sequence $0 \to K/L \to M/L \to \mathbb{F}_p \to 0$, and so the inclusion $[U,U]U^p \leqslant V$. Since $\mathcal{C}$ is a pseudovariety satisfying that $\mathcal{N}_p\mathcal{C} = \mathcal{C}$, it follows that $G/V \in \mathcal{C}$, as we wanted. $\diamond$

This is the first of a sequence of claims where we aim to show that the module K is zero. Since G is cohomologically p-good in $\mathcal{C}$, by Proposition 2.7.1,

$$\operatorname{Tor}_i^{\mathbb{F}_p[G]}(\mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!], \mathbb{F}_p) = 0 \text{ for all } i \geqslant 1.$$

Hence, we can apply the functor $(\mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!] \otimes -)$ to the $\mathbb{F}_p[G]$-resolution of the trivial module $\mathbb{F}_p$ of equation (4.2) to get the coresponding projective resolution of the trivial $\mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]$-module $\mathbb{F}_p$:

$$0 \longrightarrow (P_2)_{\widehat{\mathcal{C}}} \xrightarrow{\widehat{\alpha}} (P_1)_{\widehat{\mathcal{C}}} \xrightarrow{\widehat{\beta}} (P_0)_{\widehat{\mathcal{C}}} \longrightarrow \mathbb{F}_p \longrightarrow 0. \tag{4.7}$$

Claim 4.4.6. *We have* $\operatorname{Tor}_i^{\mathbb{F}_p[G]}(M, \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]) = 0$ *for all* $i \geqslant 1$.

Proof. Since $G_{\widehat{\mathcal{C}}} \cong S_{\widehat{\mathcal{C}}}$,

$$\operatorname{Ext}^2_{\mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]}(\mathbb{F}_p, \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]) \cong H^2_{\mathrm{cts}}(G_{\widehat{\mathcal{C}}}; \mathbb{F}_p) \cong \mathbb{F}_p.$$

We can use the projective resolution C_* of M given by (4.3) to compute the group $\operatorname{Ext}^2_{\mathbb{F}_p[G]}(M, \mathbb{F}_p[G])$. It will be isomorphic to

$$H^2\left(\operatorname{Hom}_{\mathbb{F}_p[G]}(C_*, \mathbb{F}_p[G])\right) \cong P_0/\operatorname{im}\beta \cong \mathbb{F}_p.$$

Similarly, we will compute $\operatorname{Ext}^2_{\mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]}(M_{\widehat{\mathcal{C}}}, \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!])$. For this purpose, we recall the isomorphisms $H^0_{\mathrm{cts}}(G_{\widehat{\mathcal{C}}}, \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]) = 0$ and $H^1_{\mathrm{cts}}(G_{\widehat{\mathcal{C}}}, \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]) = 0$, as well as

$$H^2_{\mathrm{cts}}(G_{\widehat{\mathcal{C}}}, \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]) \cong M_{\widehat{\mathcal{C}}} \cong \mathbb{F}_p.$$

The equation (4.7) is a projective resolution of $\mathbb{F}_p$, so after applying the functor $\operatorname{Hom}_{\mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]}(-, \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!])$ to (4.7), we get the following exact sequence of right $\mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]$-modules.

$$0 \longrightarrow (Q_0)_{\widehat{\mathcal{C}}} \xrightarrow{\beta^*_{\widehat{\mathcal{C}}}} (Q_1)_{\widehat{\mathcal{C}}} \xrightarrow{\alpha^*_{\widehat{\mathcal{C}}}} (Q_2)_{\widehat{\mathcal{C}}} \longrightarrow M_{\widehat{\mathcal{C}}} \longrightarrow 0. \tag{4.8}$$

The resolution of $M_{\widehat{\mathcal{C}}}$ described in (4.8) was obtained by applying the functor $(\mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!] \otimes -)$ to (4.2) and then the functor $\operatorname{Hom}_{\mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]}(-, \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!])$. By the commutation of functors illustrated in Lemma 2.3.3, we can also recover (4.8) by first applying the functor $\operatorname{Hom}(-, \mathbb{F}_p[G])$ to (4.2), obtaining the projective resolution of M described in (4.3), and then applying $- \otimes_{\mathbb{F}_p[G]} \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]$. In particular, this implies that if we start with the resolution (4.3) of M and we apply $- \otimes_{\mathbb{F}_p[G]} \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]$ we get an exact sequence. By definition, this means that $\operatorname{Tor}_i^{\mathbb{F}_p[G]}(M, \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]) = 0$ for all $i \geqslant 1$, as we wanted. ◇

Claim 4.4.7. *We have that* $\operatorname{Ext}^2_{\mathbb{F}_p[G]}(K, \mathbb{F}_p[G]) = 0$.

Proof. Since $\operatorname{cd}(G) = 2$, it follows that $\operatorname{Ext}^3_{\mathbb{F}_p[G]}(\mathbb{F}_p, \mathbb{F}_p[G]) = 0$. In particular, from the long exact sequence obtained when applying the derived functor $\operatorname{Ext}(-, \mathbb{F}_p[G])$ to the short exact sequence of $\mathbb{F}_p[G]$-modules $0 \to K \to M \to \mathbb{F}_p \to 0$, we can extract an exact sequence of right $\mathbb{F}_p[G]$-modules of the form

$$\operatorname{Ext}^2_{\mathbb{F}_p[G]}(\mathbb{F}_p, \mathbb{F}_p[G]) \longrightarrow \operatorname{Ext}^2_{\mathbb{F}_p[G]}(M, \mathbb{F}_p[G]) \longrightarrow \operatorname{Ext}^2_{\mathbb{F}_p[G]}(K, \mathbb{F}_p[G]) \longrightarrow 0.$$

We showed in the proof of Claim 4.4.6 that $\operatorname{Ext}^2_{\mathbb{F}_p[G]}(M, \mathbb{F}_p[G])$ is isomorphic to the trivial $\mathbb{F}_p[G]$-module $\mathbb{F}_p$. In particular, its quotient $\operatorname{Ext}^2_{\mathbb{F}_p[G]}(K, \mathbb{F}_p[G])$ is a finite $\mathbb{F}_p[G]$-module with a trivial action. This immediately gives the isomorphism

$$\operatorname{Ext}^2_{\mathbb{F}_p[G]}(K, \mathbb{F}_p[G]) \cong \operatorname{Ext}^2_{\mathbb{F}_p[G]}(K, \mathbb{F}_p[G]) \otimes_{\mathbb{F}_p[G]} \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]. \tag{4.9}$$

We know that $\operatorname{Tor}_i^{\mathbb{F}_p[G]}(\mathbb{F}_p, \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]) = 0$ by the p-goodness of G at $\mathcal{C}$ and that $\operatorname{Tor}_i^{\mathbb{F}_p[G]}(M, \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]) = 0$ by Claim 4.4.6. Consequently, by the long exact sequence in Tor obtained after applying the functor $- \otimes_{\mathbb{F}_p[G]} \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]$ to the short exact sequence $0 \to K \to M \to \mathbb{F}_p \to 0$, which we reviewed in (2.3) for left modules, we get that $\operatorname{Tor}_i^{\mathbb{F}_p[G]}(K, \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]) = 0$. By Proposition 2.3.2 (which is stated and proved for left modules, simply to match with the convention of Section 2.3), we deduce that

$$\operatorname{Ext}^2_{\mathbb{F}_p[G]}(K, \mathbb{F}_p[G]) \otimes_{\mathbb{F}_p[G]} \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!] \cong \operatorname{Ext}^2_{\mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]}(K_{\widehat{\mathcal{C}}}, \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]). \tag{4.10}$$

The right-hand side of (4.10) vanishes because $K_{\widehat{\mathcal{C}}} = 0$ by Claim 4.4.5. Thus, it follows that $\operatorname{Ext}^2_{\mathbb{F}_p[G]}(K, \mathbb{F}_p[G]) = 0$ from the isomorphisms (4.9) and (4.10). ◇

Claim 4.4.8. *The* $\mathbb{F}_p[G]$*-modules* $\mathbb{F}_p$ *and* M *satisfy that*

$$\chi^{\mathcal{D}_{\mathbb{F}_p[G]}}(\mathbb{F}_p) = \chi^{\mathcal{D}_{\mathbb{F}_p[G]}}(M).$$

Proof. This follows from the equations (4.2) and (4.3). ◇

Claim 4.4.9. *We have that* $b_i^{\mathcal{D}_{\mathbb{F}_p[G]}}(K) = 0$ *for all* $i \geqslant 0$.

Proof. On the one hand, Claim 4.4.5 implies that $K \otimes_{\mathbb{F}_p[G]} \mathbb{F}_p = 0$. By Proposition 4.3.1(2),

$$b_i^{\mathcal{D}_{\mathbb{F}_p[G]}}(K) \leqslant b_i^{\mathbb{F}_p}(K)$$

and hence $b_0^{\mathcal{D}_{\mathbb{F}_p[G]}}(K) = 0$.

On the other hand, from the additivity of $\chi^{\mathcal{D}_{\mathbb{F}_p[G]}}$ (Proposition 4.2.1) and the short exact sequence

$$0 \longrightarrow K \longrightarrow M \longrightarrow \mathbb{F}_p \longrightarrow 0,$$

we get that $\chi^{\mathcal{D}_{\mathbb{F}_p[G]}}(K) = \chi^{\mathcal{D}_{\mathbb{F}_p[G]}}(M) - \chi^{\mathcal{D}_{\mathbb{F}_p[G]}}(\mathbb{F}_p)$. Moreover, $\chi^{\mathcal{D}_{\mathbb{F}_p[G]}}(\mathbb{F}_p) = \chi^{\mathcal{D}_{\mathbb{F}_p[G]}}(M)$ by Claim 4.4.8, and so $\chi^{\mathcal{D}_{\mathbb{F}_p[G]}}(K) = 0$.

Since K has projective dimension at most one by Claim 4.4.7, we have $b_i^{\mathcal{D}_{\mathbb{F}_p[G]}}(K) = 0$ for all $i \geqslant 2$. In addition, by formula (4.1),

$$0 = \chi^{\mathcal{D}_{\mathbb{F}_p[G]}}(K) = b_0^{\mathcal{D}_{\mathbb{F}_p[G]}}(K) - b_1^{\mathcal{D}_{\mathbb{F}_p[G]}}(K).$$

So it also follows that $b_1^{\mathcal{D}_{\mathbb{F}_p[G]}}(K) = 0$, completing the proof of Claim 4.4.9. $\diamond$

The previous chain of claims tell us that the module K vanishes from the point of view of its pro-$\mathcal{C}$ completion and L^2-Betti numbers, culminating in the following statement.

Claim 4.4.10. *The $\mathbb{F}_p[G]$-module K is zero.*

Proof. Since $\operatorname{Ext}^2_{\mathbb{F}_p[G]}(K, \mathbb{F}_p[G]) = 0$, the finitely generated right $\mathbb{F}_p[G]$-module K has projective dimension at most one. Hence, we have a short exact sequence

$$0 \longrightarrow T_1 \xrightarrow{\gamma} T_0 \longrightarrow K \longrightarrow 0,$$

where T_0 and T_1 are finitely generated projective $\mathbb{F}_p[G]$-modules. We know that $b_0^{\mathcal{D}_{\mathbb{F}_p[G]}}(K) = b_1^{\mathcal{D}_{\mathbb{F}_p[G]}}(K) = 0$ by Claim 4.4.9, so it follows that the induced map

$$T_1 \otimes_{\mathbb{F}_p[G]} \mathcal{D}_{\mathbb{F}_p[G]} \longrightarrow T_0 \otimes_{\mathbb{F}_p[G]} \mathcal{D}_{\mathbb{F}_p[G]}$$

is an isomorphism. Hence, $b_0^{\mathcal{D}_{\mathbb{F}_p[G]}}(T_1) = b_0^{\mathcal{D}_{\mathbb{F}_p[G]}}(T_0)$. Furthermore, $K_{\widehat{\mathcal{C}}} = 0$ by Claim 4.4.5, so the induced map

$$T_1 \otimes_{\mathbb{F}_p[G]} \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!] \longrightarrow T_0 \otimes_{\mathbb{F}_p[G]} \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]$$

is surjective. Applying Theorem 4.3.3, we obtain that γ is an isomorphism. Thus K is trivial. $\diamond$

We have shown that the kernel K of the natural map $H^2(G; \mathbb{F}_p[G]) \to H^2(G; \mathbb{F}_p)$ is zero and hence $H^2(G; \mathbb{F}_p[G]) \cong \mathbb{F}_p$. By Theorem 2.5.2, this implies that G is a surface group. $\square$

4.5 Reading off goodness from profinite surfaces

We briefly surveyed in Section 2.7 the definition and main applications of the property of goodness. We also introduced the more general definition of p-goodness in $\mathcal{C}$. This condition is used in the proof of Theorem B3 in an essential way. However, there are not many ways to build groups with this property. In fact, most groups we know to be good satisfy a kind of group hierarchy, such as Right-Angled Artin groups, or appear as retracts of good groups, such as special groups.

It is therefore a challenge to ensure the goodness of a group without the use of splittings. We will now see how to prove this condition for some groups in the profinite genus of a free or surface group in Theorem 4.5.1, as part of our proof of Theorem C4.

It was shown in [JZ23, Theorem 1.1 and Proposition 4.1] that a group G satisfying the assumptions of Theorem C4 is $\mathcal{N}_q$-RFRS and residually-$\mathcal{N}_p$ for all primes p and q if S is an orientable surface group and $q > 2$ if S is a non-orientable surface group. So Theorem C4 is implied by the following stronger result.

Theorem 4.5.1. *Assume that either*

1. *S is an orientable surface group and q and p are arbitrary primes or*
2. *S is a non-orientable surface group, $p = 2$ and $q \neq 2$ is prime.*

Let G be an $\mathcal{N}_q$-RFRS group and assume also that G is residually-$\mathcal{N}_p$, finitely generated, of cohomological dimension 2 and has trivial second L^2-Betti number. Let $\mathcal{C} = \mathcal{N}_p(\mathcal{A}_f\mathcal{N}_q)$. If $G_{\widehat{\mathcal{C}}} \cong S_{\widehat{\mathcal{C}}}$, then $G \cong S$.

We want to use Theorem 4.4.1. First observe that G is of type $\mathtt{FP}_2(\mathbb{Z})$ by Proposition 4.3.1(3). The next main ingredient is the following.

Claim 4.5.2. *The group G is cohomologically p-good in $\mathcal{C}$.*

Proof. Weigel and Zalesskii [WZ04, Proposition 3.1] showed that an abstract group G all of whose finite-index subgroups $H < G$ are *pro-p good* (i.e. the canonical map $H^i_{\mathrm{cts}}(H_{\widehat{p}}; \mathbb{F}_p) \to H^i(H; \mathbb{F}_p)$ is an isomorphism for all integers $i \geqslant 0$) must be cohomologically p-good. In our setting, the same argument works to show that G is cohomologically p-good in $\mathcal{C}$ if, for all i and for all finite-index subgroups $H < G$ that are open in the pro-$\mathcal{C}$ topology, the natural map $H^i_{\mathrm{cts}}(H_{\widehat{p}}; \mathbb{F}_p) \to H^i(H; \mathbb{F}_p)$ is an isomorphism.

The previous map is always an isomorphism for $i \leqslant 1$ and, under our assumptions on G and $G_{\widehat{\mathcal{C}}}$, both $H^i_{\mathrm{cts}}(H_{\widehat{p}}; \mathbb{F}_p)$ and $H^i(H; \mathbb{F}_p)$ vanish if $i \geqslant 3$. Furthermore, the natural map $H^2_{\mathrm{cts}}(H_{\widehat{p}};\mathbb{F}_p) \to H^2(H;\mathbb{F}_p)$ is always injective by [Ser97, Section 2.6]. So it suffices to show that $H^2_{\mathrm{cts}}(H_{\widehat{p}};\mathbb{F}_p) \cong H^2(H;\mathbb{F}_p)$.

There exists a surface group S' such that $H_{\widehat{\mathcal{N}_p}} \cong S'_{\widehat{\mathcal{N}_p}}$. In particular, $b_{1,p}(H) = b_{1,p}(S')$. By Theorem 4.3.2, $b_1^{(2)}(H) = b_1^{(2)}(S')$, and so (4.1) implies that $\chi(H) = \chi(S')$.

Thus, taking again into account the formula (4.1), we obtain

$$\begin{aligned} b_{2,p}(H) =& \chi(H) + b_{1,p}(H) - b_{0,p}(H) = \chi(S') + b_{1,p}(S') - b_{0,p}(S') \\ =& b_{2,p}(S') = \dim_{\mathbb{F}_p} H^2_{\mathrm{cts}}(S'_{\widehat{p}};\mathbb{F}_p) = \dim_{\mathbb{F}_p} H^2_{\mathrm{cts}}(H_{\widehat{p}};\mathbb{F}_p). \end{aligned}$$

This shows that the natural injective map $H^2_{\mathrm{cts}}(H_{\widehat{p}};\mathbb{F}_p) \to H^2(H;\mathbb{F}_p)$ is an isomorphism. □

Since S is p-good in $\mathcal{C}$, its pro-$\mathcal{C}$ completion $S_{\widehat{\mathcal{C}}}$ is Poincaré duality of dimension two at p. So G lies under the assumptions of Theorem 4.4.1 and hence G is a surface group. Surface groups are distinguished from each other by their abelianisation. Thus $G \cong S$ and the proof of Theorem 4.5.1 is complete.

4.6 The proof of Theorem D4

Theorem D4 is a consequence of the following stronger result.

Theorem 4.6.1. *Assume that either*

1. *F is a free group freely generated by generators $x_1, y_1, \dots, x_d, y_d$, $w = [x_1, y_1] \cdots [x_d, y_d]$ and q and p are arbitrary primes or*
2. *F is a free group freely generated by generators $x_1, \dots, x_k$, $w = x_1^2 \cdots x_k^2$, $p = 2$ and q is a prime different from 2.*

Let $u \in F$. Assume that w and u are measure equivalent in the pseudovariety $\mathcal{N}_p(\mathcal{A}_f\mathcal{N}_q)$. Then there exists $\varphi \in \mathrm{Aut}(F)$ such that $u = \varphi(w)$.

Proof of Theorem 4.6.1. We want to apply Theorem 4.5.1 and the following two claims allow us to do so.

Claim 4.6.2. *The element u is not a proper power in F.*

Proof. Assume that u is an r-power for some prime r. Define the pseudovariety $\mathcal{U} = \mathcal{A}_r\mathcal{N}_q$. Then by [GJZ23, Proposition 3.2], w is a r-power in $\mathbf{G} = F_{\widehat{\mathcal{U}}}$. It is clear that $r \neq q$. Thus, we assume that $r \neq q$.

Observe that since w is an r-power, it is a pro-q element of $\mathbf{G}$. But this is not the case. Let us prove it, for example, if $w = [x_1, y_1] \cdots [x_d, y_d]$. In this case consider the normal subgroup $\mathbf{N}$ of $\mathbf{G}$ generated by $x_1^q, y_1, x_2, \dots, y_d$. Then $w[\mathbf{N}, \mathbf{N}] = [x_1, y_1][\mathbf{N}, \mathbf{N}]$ is not a pro-q element in $\mathbf{G}/[\mathbf{N}, \mathbf{N}]$. ◇

Consider the doubles $G = F *_u F$ and $S = F *_w F$. The group G is an one-relator group without torsion. So G has cohomological dimension 2 by [Lyn50] and trivial second L^2-Betti number by [DL07]. In addition, by [Bau62], G is residually-$\mathcal{N}_r$ for any prime r. Since

$$G_{\widehat{\mathcal{N}_q}} \cong (F *_w F)_{\widehat{\mathcal{N}_q}},$$

arguing as in the proof of [JZ23, Proposition 4.1] or [Rei15, Theorem 9.2], we obtain that G is $\mathcal{N}_q$-RFRS. Let $\mathcal{C} = \mathcal{N}_p(\mathcal{A}_f \mathcal{N}_q)$.

Claim 4.6.3. *There is an isomorphism $G_{\widehat{\mathcal{C}}} \cong S_{\widehat{\mathcal{C}}}$.*

Proof. In view of [RZ10, Corollary 3.2.8], we have to show that G and S have the same finite quotients in $\mathcal{C}$. Let $P \in \mathcal{C}$. Denote by $h(G, P)$ (resp. $e(G, P)$) the number of homomorphims (resp. epimorphisms) $H \to P$. Then we have

$$h(G, P) = \sum_{a \in P} |u_P^{-1}(a)|^2 = \sum_{a \in P} |w_P^{-1}(a)|^2 = h(S, P).$$

Taking into account that

$$h(G, P) = \sum_{T \leqslant P} e(G, T) \text{ and } h(S, P) = \sum_{T \leqslant P} e(S, T),$$

and arguing by induction on $|P|$, we obtain that $e(G, P) = e(S; P)$ for every $P \in \mathcal{C}$. Thus, G and S have the same finite quotients in $\mathcal{C}$. ◇

By Theorem 4.5.1, we obtain that $G \cong S$. Thus, by Proposition 2.5.3, u is a surface word. □

4.7 Mel'nikov's groups

In this section we prove Theorem E4 from the introduction, but we first restate a more complete formulation.

Theorem 4.7.1. *Let $n \geqslant 3$ and F the free group on $\{x_1, \dots, x_n\}$. Let $w \in F$ and let $G = F/\langle\langle w \rangle\rangle$ be a residually finite Mel'nikov group. Then G is 2-free. Moreover, the following statements hold.*

(I) If $H^2(H;\mathbb{F}_p) = 0$ for all finite-index subgroups $H \leqslant G$ and all primes p, then $\widehat{G}$ is a projective profinite group.

(II) Otherwise, if $H^2(H;\mathbb{F}_p) \neq 0$ for some finite-index subgroup $H \leqslant G$ and some prime p, then G is a surface group.

Proof of Theorem 4.7.1. We divide the proof in various claims. We fix a prime p for the rest of the discussion. Without loss of generality we will assume that $w \neq 1$ and w is not primitive. It is clear also that w is not a proper power. Lyndon's asphericity theorem [Lyn50] proves that the presentation 2-complex of $G = F/\langle\langle w \rangle\rangle$ is a $K(G, 1)$ and that $\chi(G) = 2 - n$.

Claim 4.7.2. *For any subgroup $H \leqslant G$ of finite index we have*

$$\dim_{\mathbb{F}_p} H^1(H;\mathbb{F}_p) \leqslant (n-2)|G : H| + 2.$$

Proof. Recall that $\chi(H) = \dim_{\mathbb{F}_p} H^2(H;\mathbb{F}_p) - \dim_{\mathbb{F}_p} H^1(H;\mathbb{F}_p) + 1$ for any finite-index subgroup $H \leqslant G$. Moreover, it is clear that $\dim_{\mathbb{F}_p} H^2(H;\mathbb{F}_p) \leqslant 1$ (see [Lyn50]). The multiplicativity of the Euler characteristic $\chi(H) = (2-n)|G : H|$ gives the desired conclusion. ◇

Our next aim consists on showing that G is p-good in $\mathcal{C}$. Contrary to the assumptions of Claim 4.5.2, we do not have any strong input on the pro-$\mathcal{C}$ completion of G, so we argue slightly differently.

Claim 4.7.3. *Let $U_1 \leqslant U_2 \leqslant G$ be two subgroups of finite index in G, such that p divides $|U_2 : U_1|$. Then the restriction map $H^2(U_2, \mathbb{F}_p) \to H^2(U_1, \mathbb{F}_p)$ is trivial.*

Proof. If one of the groups $H^2(U_2, \mathbb{F}_p)$ and $H^2(U_1, \mathbb{F}_p)$ is trivial, then the claim is obvious. Suppose that both are non-trivial. So both are isomorphic to $\mathbb{F}_p$. By Corollary 2.4.2, the corestriction map $\mathrm{Cor}\colon H^2(U_1, \mathbb{F}_p) \to H^2(U_2, \mathbb{F}_p)$ is onto and so it is an isomorphism. Since the composition $\mathrm{Cor} \circ \mathrm{Res}$ is equal to the endomorphism that consists on multiplication by $|U_2 : U_1|$, $\mathrm{Cor} \circ \mathrm{Res} = 0$, and so Res should be the trivial map. ◇

We are ready to prove in Claim 4.7.4 that G is p-good in $\mathcal{C}$. Notice that the second conclusion of Claim 4.7.4 is inspired by Serre's profinite analogue [Ser97, Exercise 5(b)] of Strebel's theorem [Str77] on infinite-index subgroups of $\mathtt{PD}_n$ groups.

Claim 4.7.4. *Let $\mathcal{C}$ be pseudovariety of finite groups such that $\mathcal{N}_p\mathcal{C} = \mathcal{C}$. Then any subgroup H of G of finite index is cohomologically p-good in $\mathcal{C}$. In particular,* $\mathrm{cd}_p(G_{\widehat{\mathcal{C}}}) \leqslant 2$. *Moreover, for any closed subgroup* $\mathbf{H}$ *of* $G_{\widehat{\mathcal{C}}}$ *such that* p^∞ *divides the index* $|G_{\widehat{\mathcal{C}}} : \mathbf{H}|$, *it follows that* $\mathrm{cd}_p(\mathbf{H}) \leqslant 1$.

Proof. Since H is finitely generated and of cohomological dimension 2, we only have to check that the map (2.8) for $H \to H_{\widehat{\mathcal{C}}}$ is an isomorphism in degree 2. By Claim 4.7.3, the condition (D_2) of [Ser97, Section I.2.6] holds for the embedding $H \hookrightarrow H_{\widehat{\mathcal{C}}}$. Hence, H is cohomologically p-good in $\mathcal{C}$. For the second part, we observe that, if $k \geqslant 2$ and if $\mathbf{U}$ is an open subgroup of $\mathbf{H}$, then

$$H^k_{\mathrm{cts}}(\mathbf{U}, \mathbb{F}_p) = \varinjlim_{\mathbf{U} \leqslant \mathbf{V} \leq_o G_{\widehat{\mathcal{C}}}} H^k_{\mathrm{cts}}(\mathbf{V}, \mathbb{F}_p) = \varinjlim_{\mathbf{U} \leqslant \mathbf{V} \leq_o G_{\widehat{\mathcal{C}}}} H^k(\mathbf{V} \cap G, \mathbb{F}_p) = 0,$$

where the second equality is a consequence of cohomological p-goodness and the rightmost equality is due to Claim 4.7.3. Thus, $\mathrm{cd}_p(\mathbf{H}) \leqslant 1$. ◇

Claim 4.7.4 implies that if we lied under the assumptions of part (I) of Theorem 4.7.1, then its profinite completion $\widehat{G}$ would satisfy $\mathrm{cd}_p(\widehat{G}) = 1$ for all primes p, and hence $\widehat{G}$ would be projective [RZ10, Theorem 7.6.7]. This completes the proof of part (I). Before proving part (II) of Theorem 4.7.1, we first show that G is 2-free. We proceed by contradiction. Let us suppose that G is not 2-free. Then, by [LW22, Theorem 1.5 and Definition 6.5], there exists a subgroup K of F of rank 2 containing w such that the canonical homomorphism $P = K/\langle\langle w\rangle\rangle \to G$ is an embedding of a non-free group P into G. From this assumption, we will now derive several claims that will lead to a contradiction.

Claim 4.7.5. *For any normal subgroup $H \leqslant G$ of finite index and every prime p, we have that* $\dim_{\mathbb{F}_p} H^2(H \cap P; \mathbb{F}_p) \leqslant 1$.

Proof. Let $\langle\langle w\rangle\rangle \leqslant U \lhd F$ be such that $H = U/\langle\langle w\rangle\rangle$. Since the latter normal closure is taken inside F, we denote $\langle\langle w\rangle\rangle = \langle w^F\rangle$, were w^F is the collection of conjugates of w in F. We take a transversal $\{t_1, t_2, \dots, t_{|K:U\cap K|}\}$ of $U \cap K$ in K, with $t_1 = 1$, and then we complete it to a transversal $T = \{t_1, \dots, t_{|F:U|}\}$ of U in F. We denoted by n the rank of F. By the Nielsen–Schreier formula, $d(U) = 1 + (n-1)|F : U|$. By assumption, the group $H = U/\langle\langle w^{t_1}, \dots, w^{t_{|F:U|}}\rangle\rangle$ is one-relator. Let m be such that $U/\langle\langle w^{t_1}, \dots, w^{t_{|F:U|}}\rangle\rangle$ admits an m-generator one-relator presentation. By the multiplicativity of the Euler characteristic we have that $m = 2 + (n-2)|F : U|$. From the fact that the p-abelianisation of H is m-generated, it follows that the images of $w^{t_1}, \dots, w^{t_{|F:U|}}$ in $U/[U,U]U^p$ generate a subspace of dimension at least $d(U) - m =$

$|F : U| - 1$. Hence, the images of $w^{t_1}, \dots, w^{t_{|K:U\cap K|}}$ in $U/[U,U]U^p$ generate a subspace of dimension at least $|K : U \cap K| - 1$. In particular, the images of $w^{t_1}, \dots, w^{t_{|K:U\cap K|}}$ in $(U \cap K)/[U \cap K, U \cap K](U \cap K)^p$ generate a subspace of dimension at least $|K : U\cap K| - 1$. Recall that the free group K has rank two and so $d(U\cap K) = |K : U\cap K| + 1$. We also observe that $H \cap P = U \cap (K\langle w^F \rangle)/\langle w^F \rangle = (U\cap K)\langle w^F\rangle/\langle w^F\rangle = U\cap K/\langle w^U\rangle$. From this, we obtain that $\dim_{\mathbb{F}_p} H^1(H \cap P; \mathbb{F}_p) \leqslant 2$. Again, by the multiplicativity of χ, $\chi(H \cap P) = |P : H \cap P|\, \chi(P) = 0$. Thus $\dim_{\mathbb{F}_p} H^2(H \cap P; \mathbb{F}_p) \leqslant 1$. ◇

Claim 4.7.6. *The group P is isomorphic to $B(1, m)$.*

Proof. Let $\overline{P}$ be the closure of P in $\widehat{G}$. Let $\mathbf{W}$ be an open subgroup of $\overline{P}$ and let $\mathbf{U}$ be an open normal subgroup of $\widehat{G}$ such that $\mathbf{U} \cap \overline{P} \leqslant \mathbf{W}$. We put $U = \mathbf{U} \cap G$ and $W = \mathbf{W} \cap P$. Fix a prime p. Observe that $U \cap P$ is a subgroup of W and so, by Corollary 2.4.2, the corestriction map $H^2(U \cap P; \mathbb{F}_p) \to H^2(W; \mathbb{F}_p)$ is onto. Thus, we have that

$$\begin{aligned} \dim_{\mathbb{F}_p} H^1_{\text{cts}}(\mathbf{W}; \mathbb{F}_p) &\leqslant \dim_{\mathbb{F}_p} H^1(W; \mathbb{F}_p) = 1 + \dim_{\mathbb{F}_p} H^2(W; \mathbb{F}_p) \\ &\leqslant 1 + \dim_{\mathbb{F}_p} H^2(U \cap P; \mathbb{F}_p) \overset{\text{Claim 4.7.5}}{\leqslant} 2. \end{aligned}$$

Hence, the pro-p Sylow subgroups of $\overline{P}$ are of bounded rank. We know from Claim 4.7.4 that $\mathrm{cd}_p(\overline{P}) = 1$ and hence that the pro-p Sylows of $\overline{P}$ are free pro-p groups [RZ10, Section 7.7]. Thus, in fact, the pro-p Sylow subgroups of $\overline{P}$ are pro-p cyclic. A profinite group all of whose pro-p Sylows are pro-p cyclic must be meta-cyclic (see [Rib17, Lemma 4.2.5], this follows from the analogous result for finite groups due to Zassenhaus).

Since P is embedded in $\overline{P}$, then P is meta-abelian. So we know that P is non-cyclic, that it contains no non-abelian free subgroups and that it splits as a HNN extension $P = V *_{E,\theta}$ of a one-relator group V along an injective map of a non-trivial free Magnus subgroup $\theta\colon E \to V$. Hence, $E \cong \mathbb{Z}$. Moreover, if E was a proper subgroup of V and θ was not surjective, then we could produce a non-abelian free subgroup of P using Bass–Serre theory. It follows that $P \cong B(1, m)$. ◇

Claim 4.7.7. *For any prime p, there exists $\langle\!\langle K \rangle\!\rangle \leqslant U \lhd F$ such that $|F : U| = p$.*

Proof. In the proof of Claim 4.7.6 we have shown that the pro-p Sylow subgroups of $\overline{P}$ are pro-p cyclic. Hence, p^∞ divides $|\widehat{G}/\overline{\langle\!\langle P \rangle\!\rangle}[\widehat{G}, \widehat{G}]|$. ◇

In order to reach a contradiction, we consider two possible cases about P. Suppose first that $P \cong B(1, m)$ with $m \neq 2$. Then, for any prime p dividing $m - 1$, $w \in [K, K]K^p$. Consider the subgroup U from Claim 4.7.7 and name $H = U/\langle\langle w \rangle\rangle$, which has finite index in G. Since $w \in [U, U]U^p$,

$$\dim_{\mathbb{F}_p} H^1(H; \mathbb{F}_p) = (n-1)p + 1 > (n-2)p + 2.$$

But this contradicts Claim 4.7.2. Secondly, suppose that $P \cong B(1, 2)$, there exists a subgroup G_1 of G of finite index such that $P \cap G_1 \cong B(1, m)$ with $m \neq 2$. Applying a similar argument as before we also reach a contradiction. We have proved the following.

Claim 4.7.8. *The group G is 2-free.*

Now we move on to show part (II) of Theorem E4. We assume that there exists a subgroup H of G of finite index such that $H^2(H; \mathbb{F}_p) \cong \mathbb{F}_p$. Recall that a torsion-free group that is virtually isomorphic to a surface group must be a surface group itself by Kerckhoff's realisation theorem [Ker83]. So we can assume that $H = G$.

By [Lin25], it follows from Claim 4.7.8 that G is hyperbolic and virtually special (in the sense of Haglund–Wise [HW08]). Again, by [HW08], G virtually embeds into a right-angled Artin group, which is $\mathcal{N}_q$-RFRS by [KS20]. Hence, G is virtually $\mathcal{N}_q$-RFRS. Again, for notational convenience, we can assume that G itself is $\mathcal{N}_q$-RFRS. Put $\mathcal{C} = \mathcal{N}_p(\mathcal{A}_f \mathcal{N}_q)$ and assume the notation of Section 4.4. We already have some of the assumptions that are required to apply Theorem 4.4.1 to G and hence conclude that G is a surface group. We proved that G is p-good in $\mathcal{C}$ in Claim 4.7.4. It remains to show that $G_{\widehat{\mathcal{C}}}$ is Poincaré Duality of dimension two at p.

Claim 4.7.9. *We have that $H^2_{cts}(G_{\widehat{\mathcal{C}}}, \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]) \cong \mathbb{F}_p$.*

Proof. By Proposition 2.6.1, we have an isomorphism

$$H^2_{\text{cts}}(G_{\widehat{\mathcal{C}}}; \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]) \longrightarrow \varprojlim_{G/N \in \mathcal{C}} H^2_{\text{cts}}(G_{\widehat{\mathcal{C}}}; \mathbb{F}_p[G/N])$$

Using Shapiro's lemma and the p-goodness of G in $\mathcal{C}$, we can reformulate the above to get the isomorphism

$$H^2_{\text{cts}}(G_{\widehat{\mathcal{C}}}; \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]) \longrightarrow \varprojlim_{G/N \in \mathcal{C}} H^2(N; \mathbb{F}_p), \tag{4.11}$$

where, given two open subgroups $N_1 \leqslant N_2$ of G in the pro-$\mathcal{C}$ topology, the corresponding map $H^2(N_1; \mathbb{F}_p) \to H^2(N_2; \mathbb{F}_p)$ that defines the inverse limit above is the corestriction map. Now we gather three observations.

- The corestriction map is always onto (Corollary 2.4.2).
- Each group $H^2(N;\mathbb{F}_p)$ is either trivial or $\mathbb{F}_p$ (because N is a one-relator group).
- By assumption, $H^2(G;\mathbb{F}_p) \cong \mathbb{F}_p$.

From these, it is not hard to see that every $H^2(N;\mathbb{F}_p)$ must be isomorphic to $\mathbb{F}_p$, that the correstriction maps are isomorphisms and hence, by (4.11), that $H^2_{\text{cts}}(G_{\widehat{\mathcal{C}}};\mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]) \cong \mathbb{F}_p$. $\diamond$

The last ingredient will be to prove that $H^1_{\text{cts}}(G_{\widehat{\mathcal{C}}},\mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!])$ vanishes. This will be done in our last two claims.

Claim 4.7.10. *For any subgroup $H \leqslant G$ of finite index, $H_{\widehat{\mathcal{N}_p}}$ is Demushkin.*

Proof. By the first part of Claim 4.7.4, $H_{\widehat{\mathcal{N}_p}}$ is non-free one-relator pro-p group. By Claim 4.7.2, any open subgroup $\mathbf{U} \leqslant_o H_{\widehat{\mathcal{N}_p}}$ has the property that

$$\dim_{\mathbb{F}_p} H^1_{\text{cts}}(\mathbf{U};\mathbb{F}_p) - 2 \leqslant (\dim_{\mathbb{F}_p} H^1_{\text{cts}}(H_{\widehat{\mathcal{N}_p}};\mathbb{F}_p) - 2)|H_{\widehat{\mathcal{N}_p}} : \mathbf{U}|.$$

It follows from [And73] (and also from [DL83]) that $H_{\widehat{\mathcal{N}_p}}$ is Demushkin. $\diamond$

We do not really need to delve into the definition of Demushkin groups. However, for the purpose of Claim 4.7.11, we recall that Demushkin groups are the Poincaré Duality pro-p groups of dimension two (see [Ser97, Section I.4.5, Example 2]).

Claim 4.7.11. *We have that $H^1_{cts}(G_{\widehat{\mathcal{C}}},\mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]) = 0$.*

Proof. Let $\mathbf{H}$ be an open normal subgroup of $G_{\widehat{\mathcal{C}}}$. Then $\mathbf{H} = \overline{H}$, where $H = \mathbf{H} \cap G$. Let $\mathbf{N}_H$ the kernel of the canonical map $\mathbf{H} \to H_{\widehat{\mathcal{N}_p}}$. Then we can write $G_{\widehat{\mathcal{C}}}$ as the following inverse limit

$$G_{\widehat{\mathcal{C}}} \cong \varprojlim_{\mathbf{H} \lhd_o G_{\widehat{\mathcal{C}}}} G_{\widehat{\mathcal{C}}}/\mathbf{N}_H.$$

This implies, by Proposition 2.6.1, that

$$H^1_{\text{cts}}(G_{\widehat{\mathcal{C}}},\mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]) \cong \varprojlim_{\mathbf{H} \lhd_o G_{\widehat{\mathcal{C}}}} H^1_{\text{cts}}(G_{\widehat{\mathcal{C}}},\mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}/\mathbf{N}_H]\!]).$$

Using Shapiro's lemma and the fact that H has finite index in G, we obtain the isomorphisms

$$H^1_{\text{cts}}(G_{\widehat{\mathcal{C}}},\mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}/\mathbf{N}_H]\!]) \cong H^1_{\text{cts}}(\mathbf{H},\mathbb{F}_p[\![\mathbf{H}/\mathbf{N}_H]\!]) = H^1_{\text{cts}}(H_{\widehat{\mathcal{N}_p}},\mathbb{F}_p[\![H_{\widehat{\mathcal{N}_p}}]\!]).$$

Lastly, by Claim 4.7.10, $H_{\widehat{\mathcal{N}_p}}$ is Demushkin and hence the right-most cohomology group is trivial. It follows that $H^1_{\text{cts}}(G_{\widehat{\mathcal{C}}},\mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]) = 0$. $\diamond$

By Claims 4.7.4, 4.7.9 and 4.7.11; we have that $\mathrm{cd}_p(G_{\widehat{\mathcal{C}}}) \leqslant 2$, $H^2_{\mathrm{cts}}(G_{\widehat{\mathcal{C}}}, \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]) \cong \mathbb{F}_p$ and $H^1_{\mathrm{cts}}(G_{\widehat{\mathcal{C}}}, \mathbb{F}_p[\![G_{\widehat{\mathcal{C}}}]\!]) = 0$. Hence, $G_{\widehat{\mathcal{C}}}$ is Poincaré Duality of dimension two over $\mathbb{F}_p$ at p (recall Definition 2.7.2). Finally, by Theorem 4.4.1, G is a surface group and the proof of Theorem 4.7.1 is complete. □

4.8 Questions and conjectures

We state a number of questions in the direction of establishing Conjecture 4.1.1 and Conjecture 4.1.2. Conjecture 4.1.1 predicts that if two words of a finitely generated free group are measure equivalent, then they belong to the same Aut-orbit. Our methods suggest two steps in achieving this, the second of which has nothing to do with profinite rigidity.

Question 4.8.1. *Given a finitely generated free group F and $u \in F$, is the corresponding double $F *_u F$ profinitely rigid among graphs of free groups with cyclic edge groups?*

Arguing as in Section 4.6, a positive answer to Question 4.8.1 for $u \in F$ would imply that if $v \in F$ is measure equivalent to u, then the doubles $F *_v F$ and $F *_u F$ are abstractly isomorphic. Given this, it is natural to ask how strong is the latter condition.

Question 4.8.2. *If two words $u, v \in F$ lead to isomorphic doubles, must u and v be in the same orbit under the action of* $\mathrm{Aut}(F)$*?*

We conclude stating some other questions and conjectures that, in addition to their independent interest, show up naturally in connection to Mel'nikov's surface group conjecture (Conjecture 4.1.2).

Conjecture 4.8.3. *Let G be a parafree group all of whose finite-index subgroups are one-relator groups. Then G is free.*

The parasurface case of Conjecture 4.8.3 is solved in Theorem E4, but in the case of parafree groups there seems to be less structure to play with. For instance, we recall that all parasurface groups are one-ended, while the number of ends of a non-abelian parafree group can be one or infinite. This observation had the important cohomological implication reflected in Claim 4.4.2. We suggest another question that would be enough to solve Conjecture 4.1.2.

Question 4.8.4. *Which one-relator groups G admit a finite-index subgroup H with $b_2(H) \neq 0$? Is this true for all one-ended hyperbolic one-relator groups?*

We want to go one step further and conjecture this kind of phenomenon for another large class of hyperbolic groups.

Conjecture 4.8.5. *Let G be a one-ended hyperbolic special group. Then G contains a finite-index subgroup $H < G$ such that $b_2(H) \neq 0$.*

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